

Lecture 11
Tue. 9.27.2016

Poincare, Lagrange, Hamiltonian, and Jacobi mechanics

(Unit 1 Ch. 12, Unit 2 Ch. 2-7, Unit 3 Ch. 1-3, Unit 7 Ch. 1-2)

Parabolic and 2D-IHO orbital envelopes (Review of Lecture 9 p.56-81 and a generalization.)

Clues for future assignments ([Web Simulation: CouIt](#))

Examples of Hamiltonian mechanics in phase plots

1D Pendulum and phase plot ([Web Simulations: Pendulum, Cycloidulum, JerkIt \(Vert Driven Pendulum\)](#))

1D-HO phase-space control ([Old Mac OS & Web Simulations of "Catcher in the Eye"](#))

Exploring phase space and Lagrangian mechanics more deeply

A weird "derivation" of Lagrange's equations

*Poincare identity and Action, **Jacobi**-Hamilton equations*

How Classicists might have "derived" quantum equations

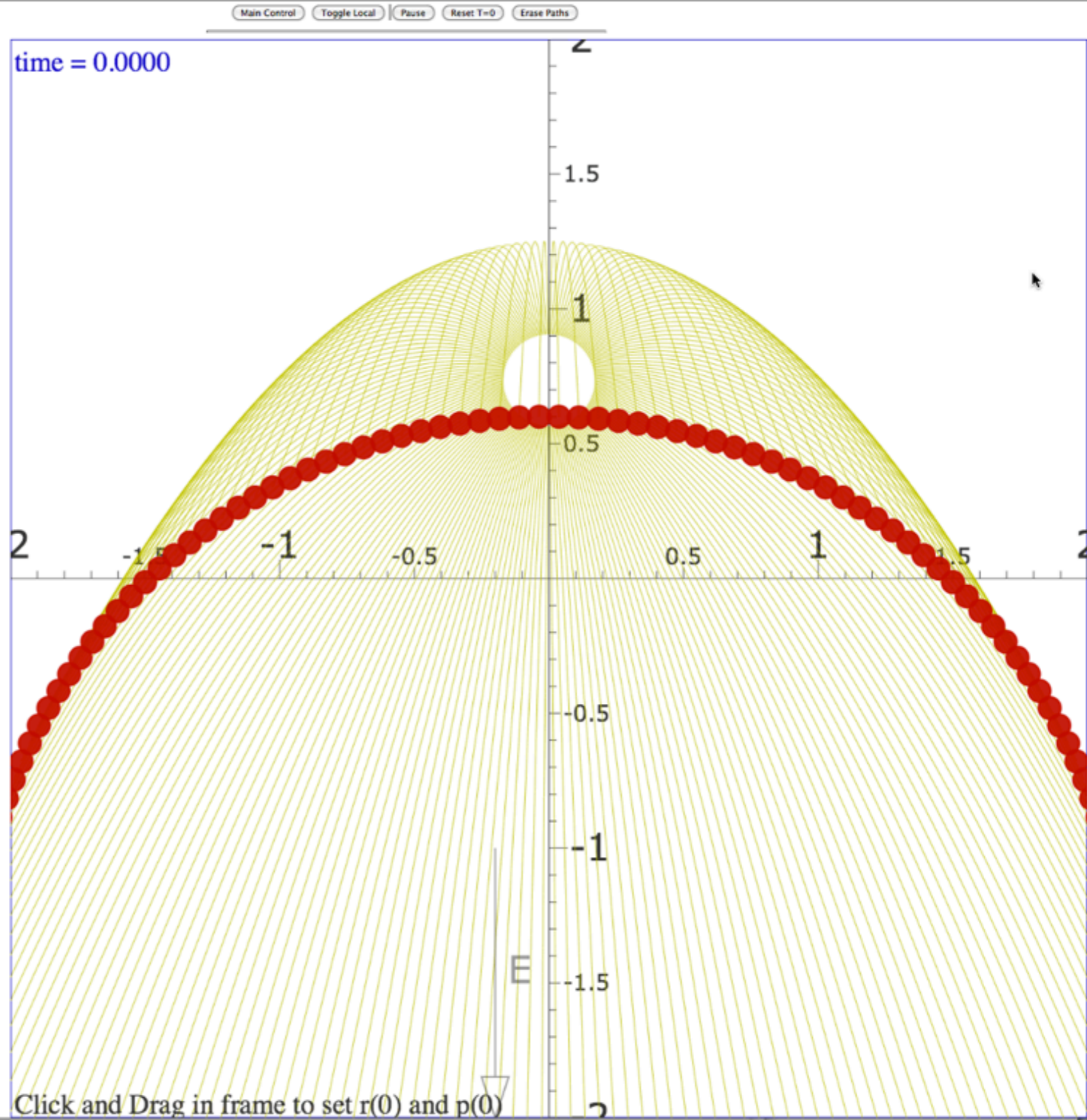
Huygen's contact transformations enforce minimum action

How to do quantum mechanics if you only know classical mechanics

(["Color-Quantization" simulations](#): Davis-Heller "Color-Quantization" or "Classical Chromodynamics")

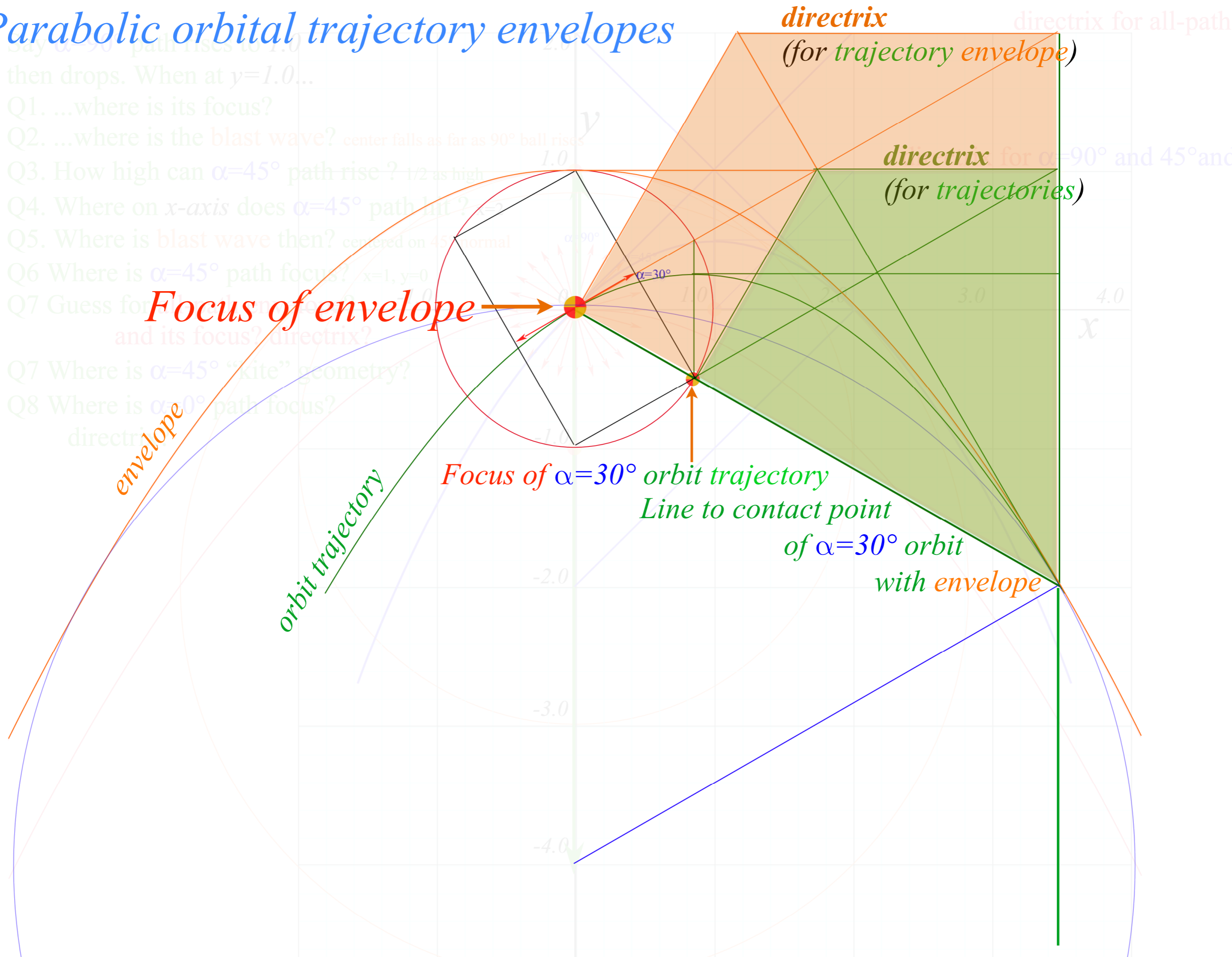
→ *Parabolic ~~1D~~ ~~III~~ orbital envelopes (Review of Lecture 9 p.56-81 and a generalization.)*
Some clues for future assignments ([Web Simulation: CouII](#))

Plot r(t) Plot p(t) Fix r(0) Fix p(0)
 Do swarm Beam
 Color action No stops Field vectors Info
 Draw masses Axes Coordinates Lenz
 Set p by phi Elastic 2 Free



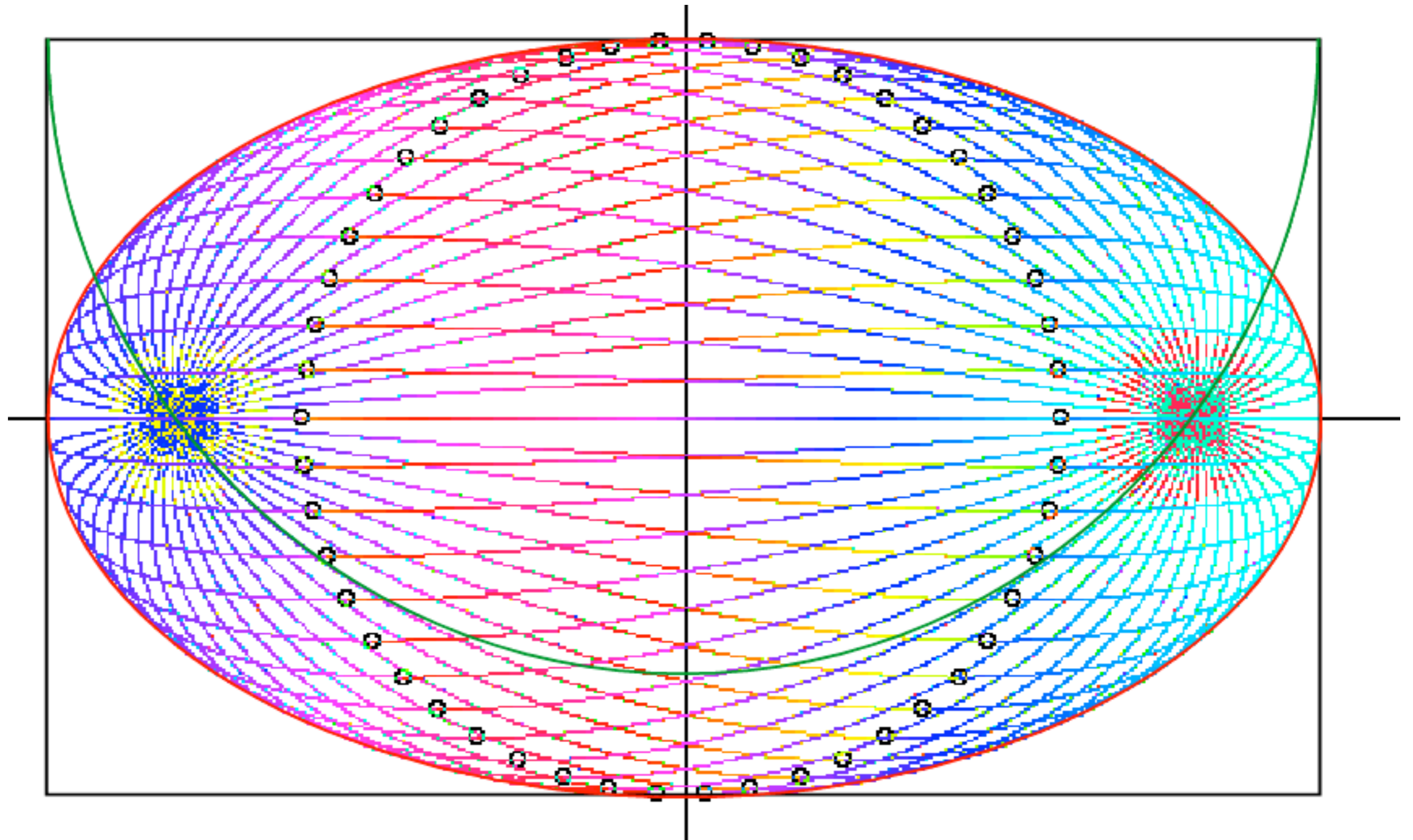
[Link ⇒ Coult - Simulation of the Volcanoes of Io](#)

Parabolic orbital trajectory envelopes



Parabolic and 2D-IHO orbital envelopes (Review of Lectures 0, 1, 5, 6, 8, 10 and 11 a generalization.)
Some clues for future assignments ([Web Simulation: CouIIIt](#))

Exploding-starlet elliptical envelope and contacting elliptical trajectories



*(Web Simulation: CouItt - Exploding*Starlet {IHO Potential})*

Initial position $x(0) = 1$

Initial position $y(0) = 0$

Initial momentum $p_x(0) = 0$

Initial momentum $p_y(0) = 1$

Terminal time $t(\text{off}) = 3.45$

Maximum step size $dt = 0.01$

Start launch angle $\phi_1 = -180$

Start launch angle $\phi_2 = 180$

Number of burst paths = 51

Charge of Nucleus 1 = 0

Charge of Nucleus 2 = 0

Coulomb $(k_{12}) = 0$

Core thickness $r = 0.000001$

x-Stark field $E_x = 0$

y-Stark field $E_y = 0$

Zeeman field $B_z = 0$

Diamagnetic strength $k = -0.638$

Plank constant $\hbar = 2$

Color quantization hues = 64

Color quantization bands = 2

Fractional Error (e^{-x}) , $x = 8$

Particle Size = 2

Fix $r(0)$ Fix $p(0)$ Do swarm Beam

Plot $r(t)$ Plot $p(t)$

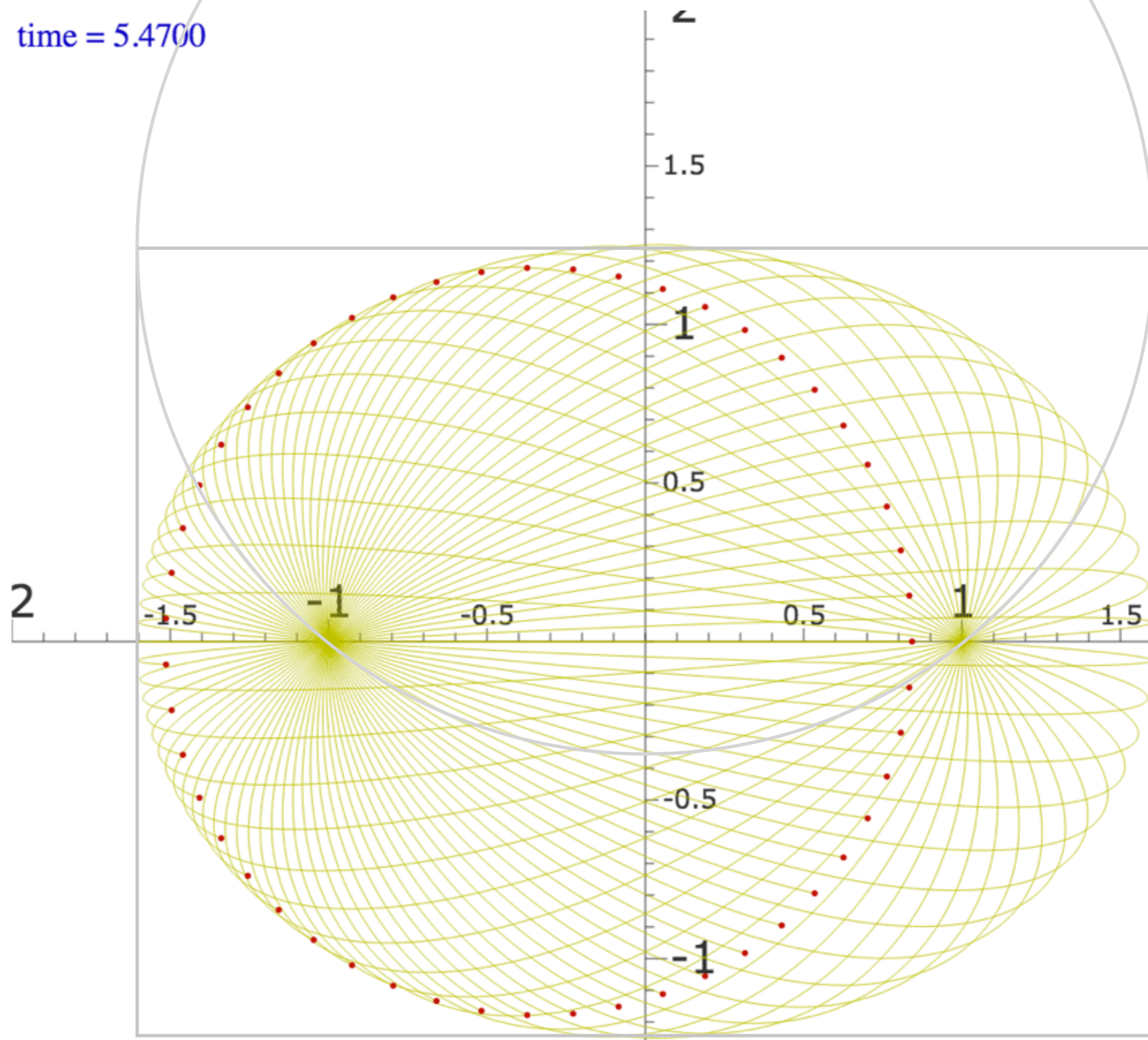
Color action No stops Field vectors Info

Draw masses Axes Coordinates Lenz

Set p by ϕ Elastic 2 Free

Save to GIF

time = 5.4700



*(Web Simulation: CouIt - Exploding*Starlet {IHO Potential})*

Examples of Hamiltonian mechanics in phase plots



1D Pendulum and phase plot (Web Simulations: [Pendulum](#), [Cycloidulum](#), [JerkIt](#) (Vert Driven Pendulum))

Circular pendulum dynamics and elliptic functions

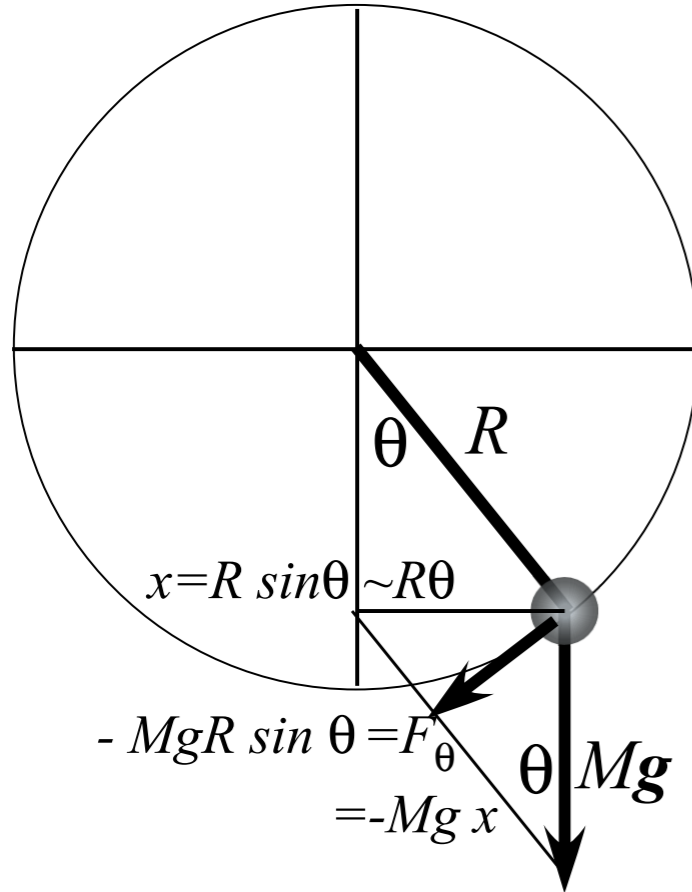
Cycloid pendulum dynamics and “sawtooth” functions

1D-HO phase-space control (Old Mac OS & [Web Simulations](#) of “Catcher in the Eye”)

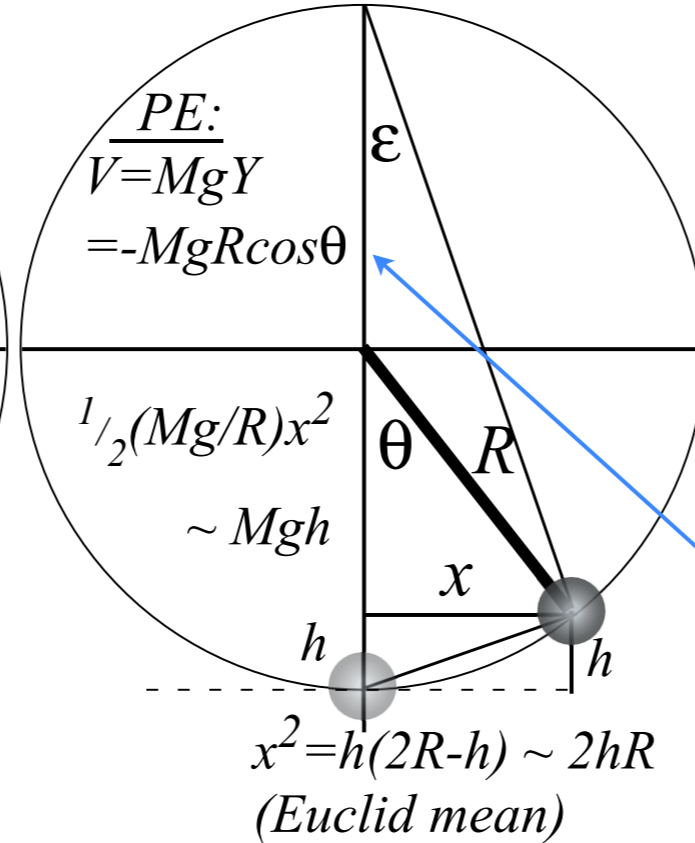
1D Pendulum and phase plot

(Unit 2 Chapter 7 Fig. 2)

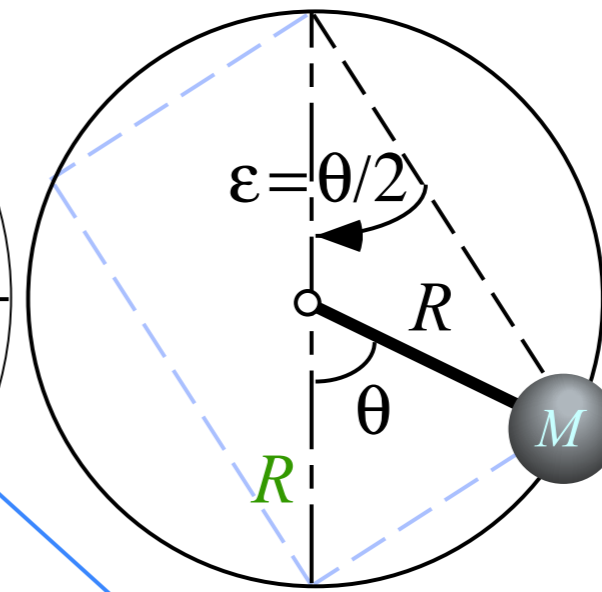
(a) Force geometry



(b) Energy geometry



(c) Time geometry



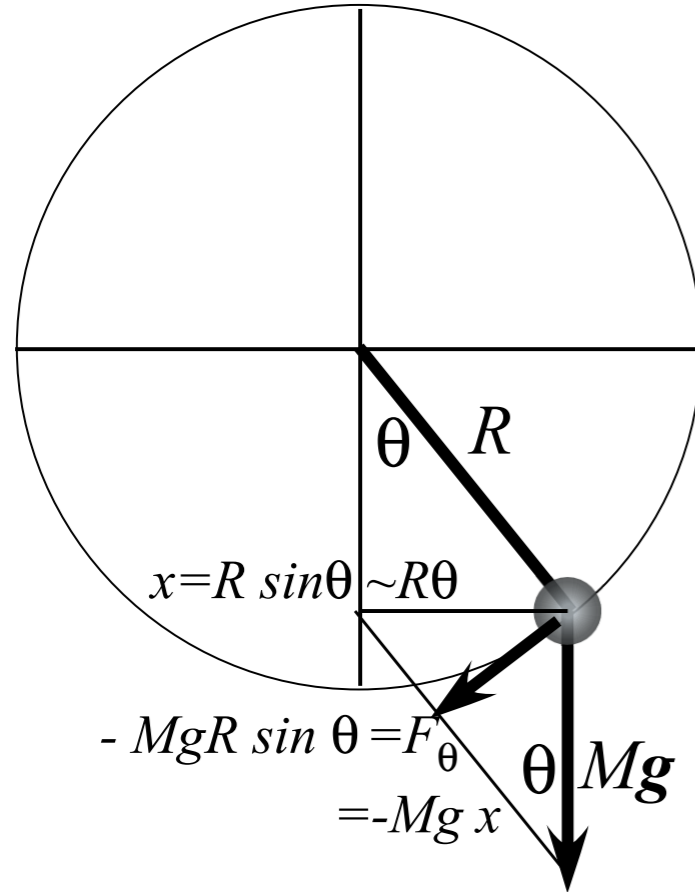
NOTE: Very common loci of \pm sign blunders

Lagrangian function $L = KE - PE = T - U$ where potential energy is $U(\theta) = -MgR \cos \theta$

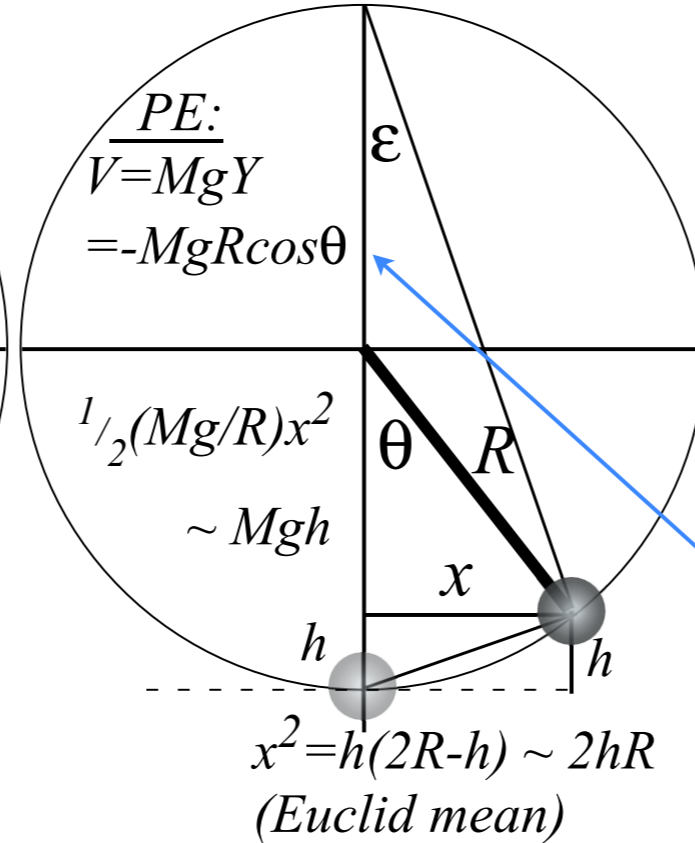
$$L(\dot{\theta}, \theta) = \frac{1}{2} I \dot{\theta}^2 - U(\theta) = \frac{1}{2} I \dot{\theta}^2 + MgR \cos \theta$$

1D Pendulum and phase plot

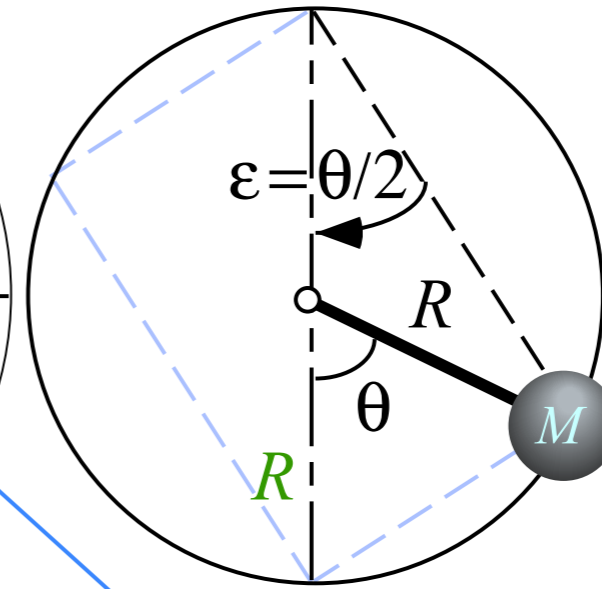
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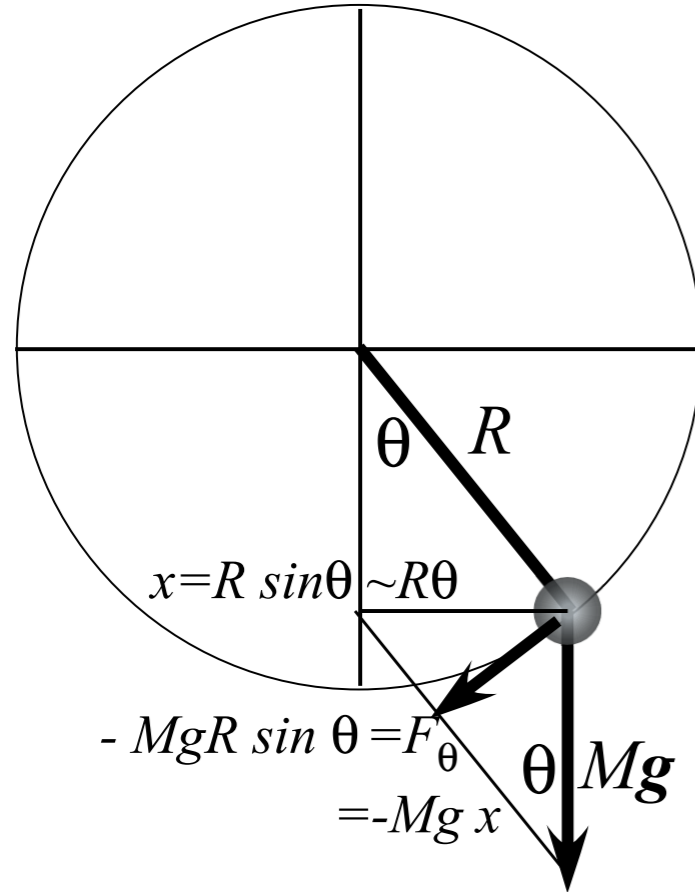
$$L(\dot{\theta}, \theta) = \frac{1}{2} I \dot{\theta}^2 - U(\theta) = \frac{1}{2} I \dot{\theta}^2 + MgR \cos \theta$$

Hamiltonian function $H = KE + PE = T + U$ where potential energy is $U(\theta) = -MgR \cos \theta$

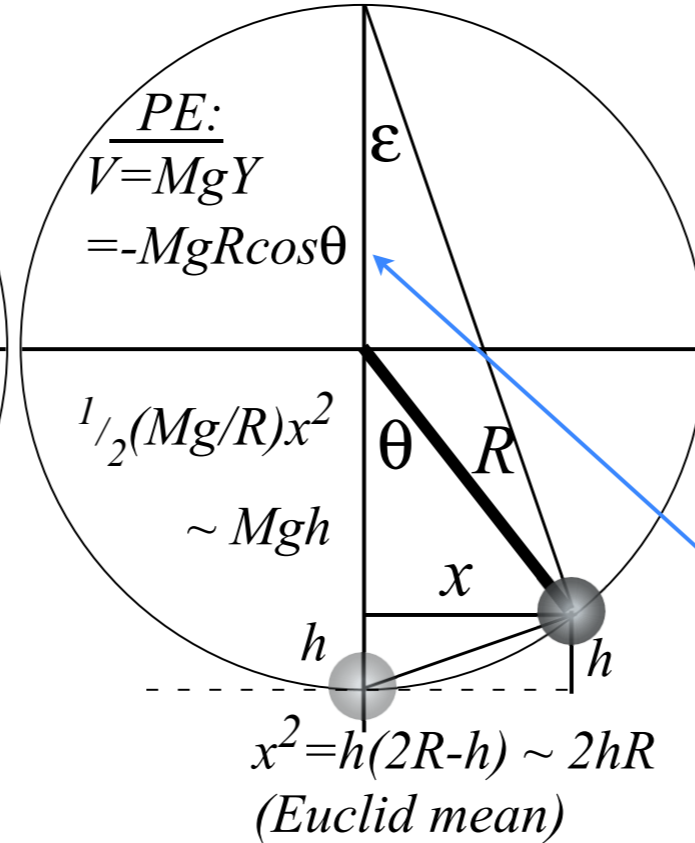
$$H(p_\theta, \theta) = \frac{1}{2I} p_\theta^2 + U(\theta) = \frac{1}{2I} p_\theta^2 - MgR \cos \theta = E = \text{const.}$$

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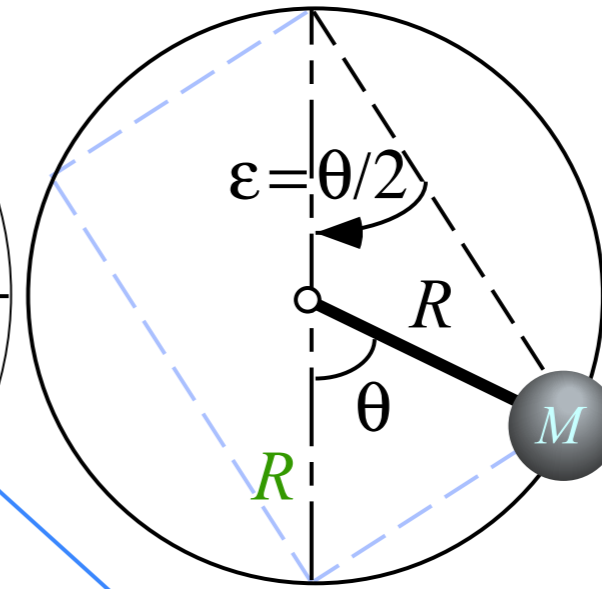
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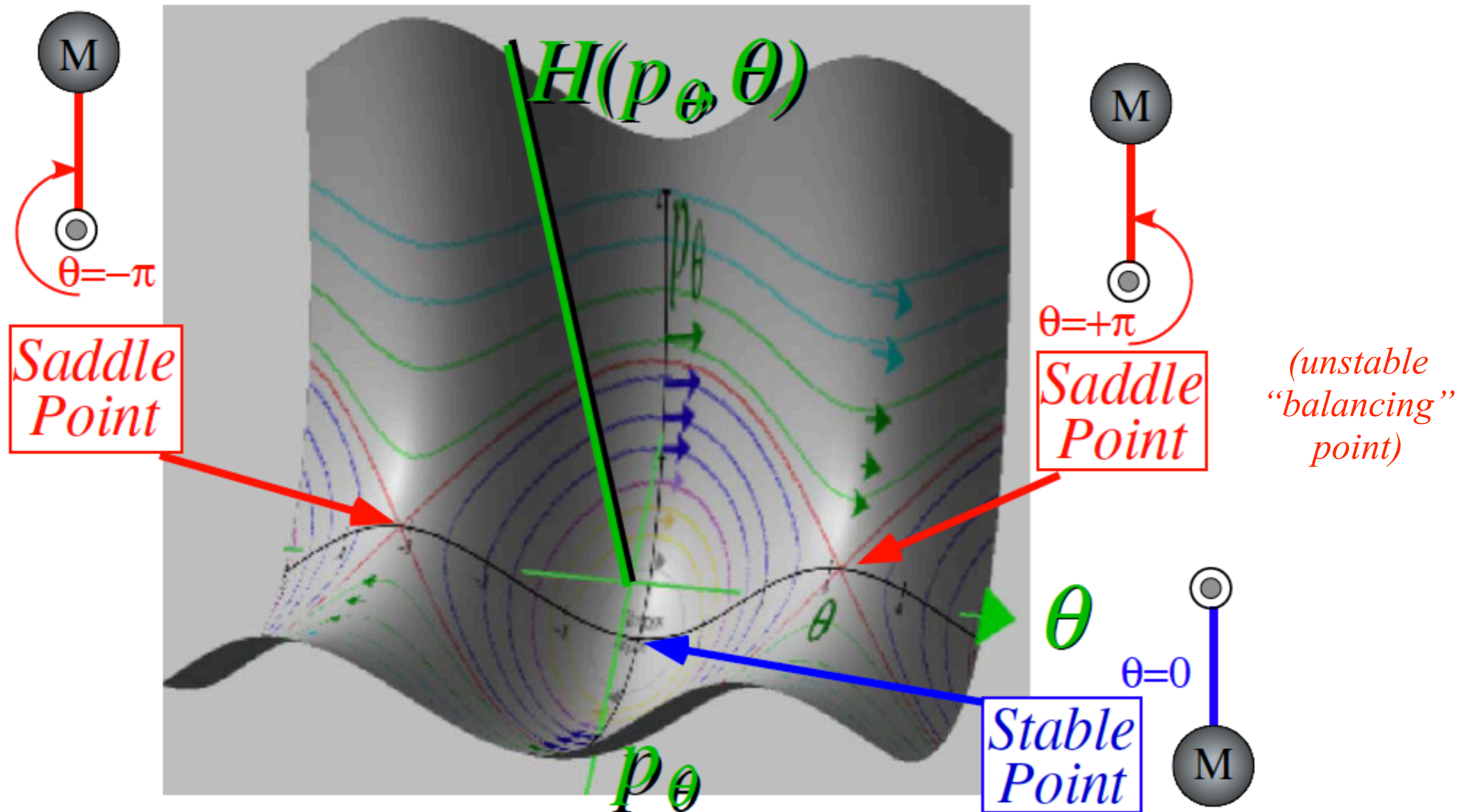
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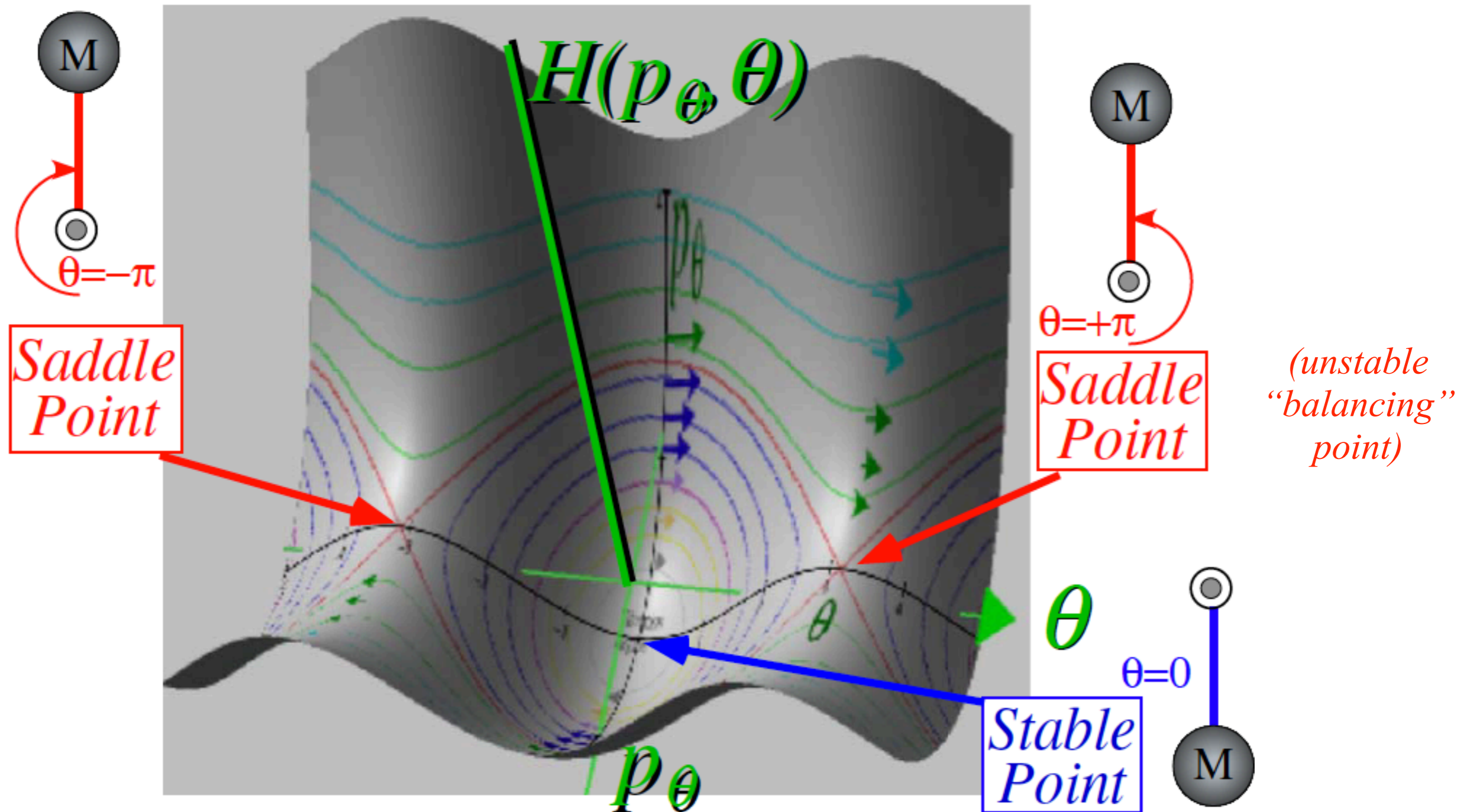
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implies: $p_\theta = \sqrt{2I(E + MgR \cos \theta)}$



Example of plot of Hamilton for 1D-solid pendulum in its Phase Space (θ, p_θ)

$$H(p_\theta, \theta) = E = \frac{1}{2I} p_\theta^2 - MgR \cos \theta, \quad \text{or: } p_\theta = \sqrt{2I(E + MgR \cos \theta)}$$



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Funny way to look at Hamilton's equations:

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \partial_p H \\ -\partial_q H \end{pmatrix} = \mathbf{e}_H \times (-\nabla H) = (\overrightarrow{\text{H-axis}}) \times (\overrightarrow{\text{fall line}}), \quad \text{where: } \begin{cases} (\overrightarrow{\text{H-axis}}) = \mathbf{e}_H = \mathbf{e}_q \times \mathbf{e}_p \\ (\overrightarrow{\text{fall line}}) = -\nabla H \end{cases}$$

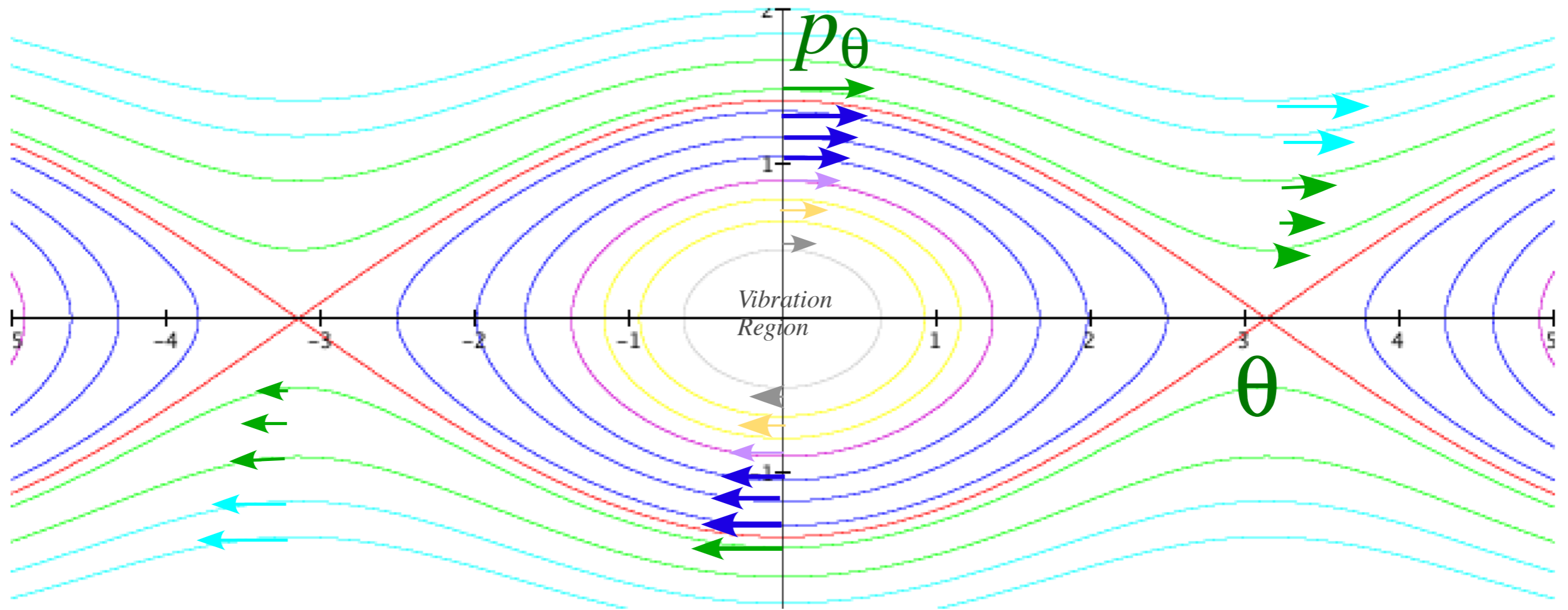


Fig. 2.7.2 Phase portrait or topography map for simple pendulum

(Unit 2 Chapter 7 Fig. 2)

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Circular pendulum dynamics and elliptic functions

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Circular pendulum dynamics and elliptic functions

Hamiltonian function $H = KE + PE = T + U$ where potential energy is $U(\theta) = -MgR \cos \theta$

$$H(p_\theta, \theta) = \frac{1}{2I} p_\theta^2 + U(\theta) = \frac{1}{2I} p_\theta^2 - MgR \cos \theta = E = \text{const.} \quad \text{implies: } p_\theta = \sqrt{2I(E + MgR \cos \theta)}$$

Let $E = MgY = -MgR \cos \theta_0$ be potential energy where $KE = 0$ or $p_\theta = 0$

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$$\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \frac{d\theta}{dt} = p_\theta / I = \sqrt{2I(E + MgR \cos \theta)} / I \quad \text{where: } I = MR^2$$

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$E = MgY = -MgR \cos \theta_0$
↓

Quadrature integral gives quarter-period $\tau_{1/4}$:

$$\sqrt{\frac{I}{2MgR}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos \theta - \cos \theta_0}} = \int_0^{\theta_0} dt = (\text{Travel time } 0 \text{ to } \theta_0) = \tau_{1/4}$$

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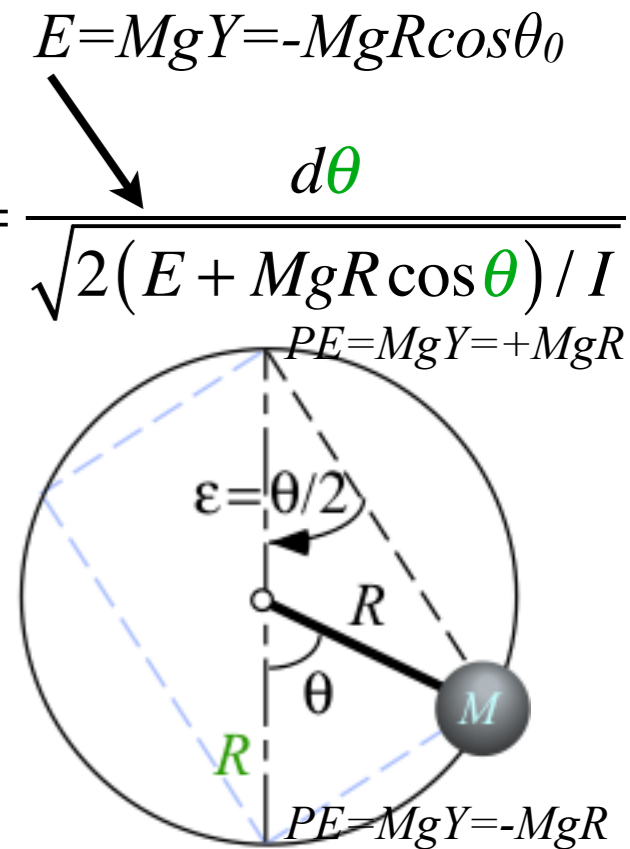
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Uses a half-angle coordinate $\varepsilon = \theta/2$

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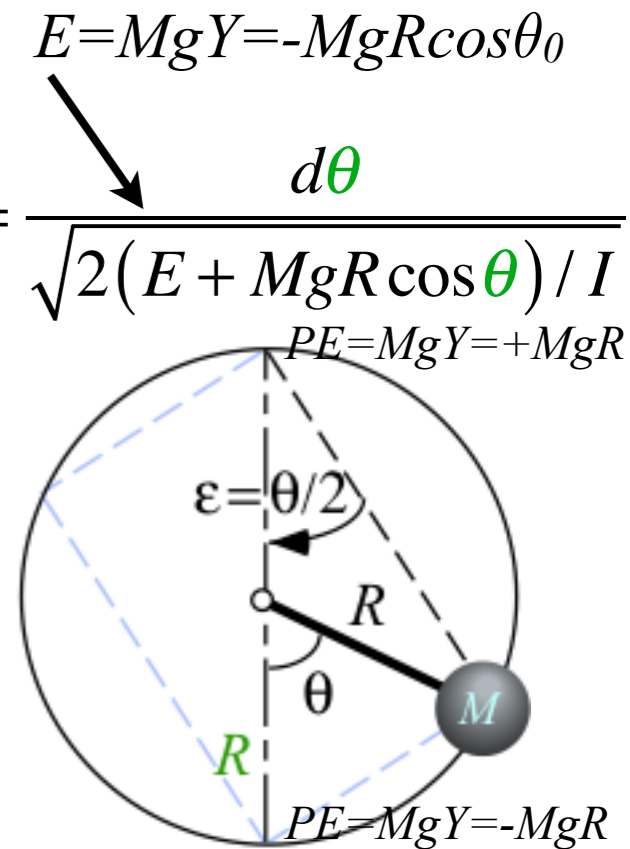
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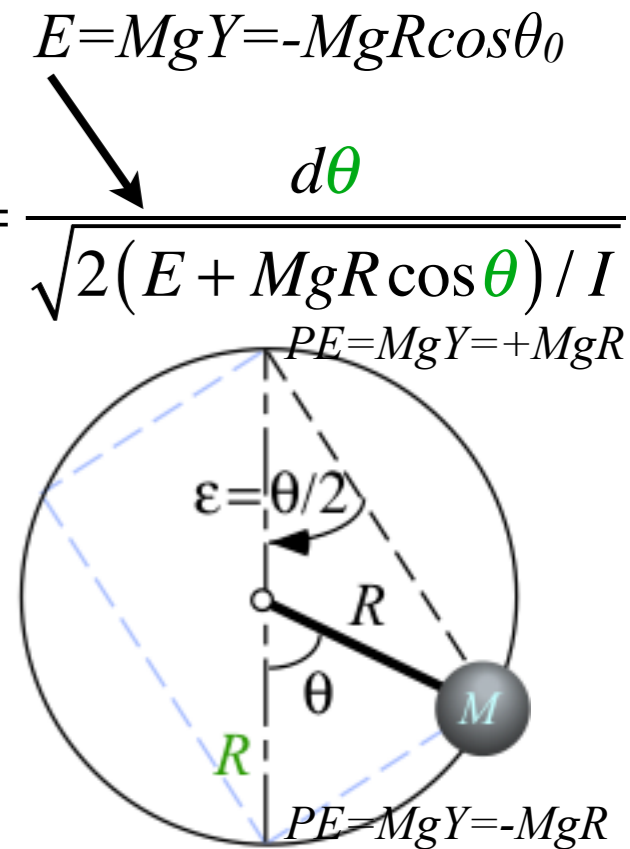
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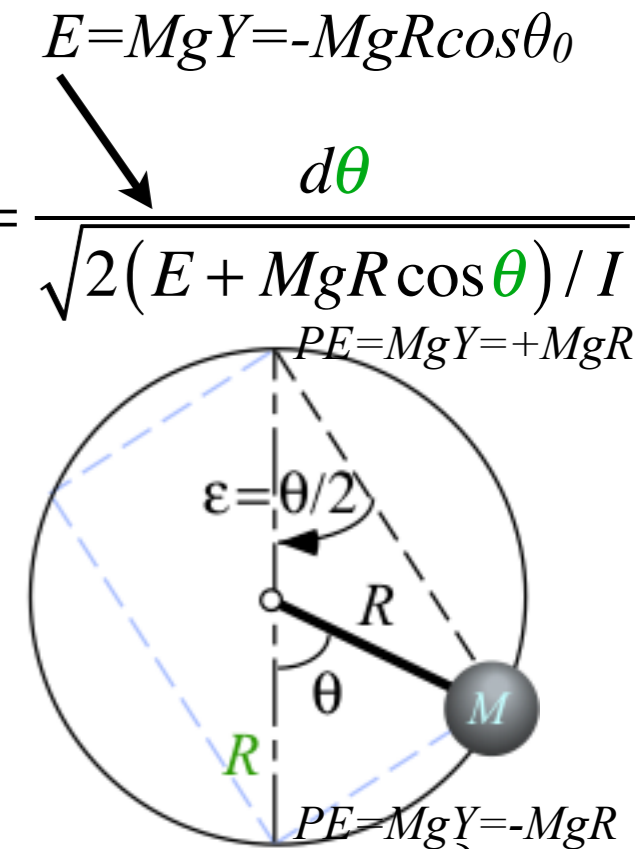
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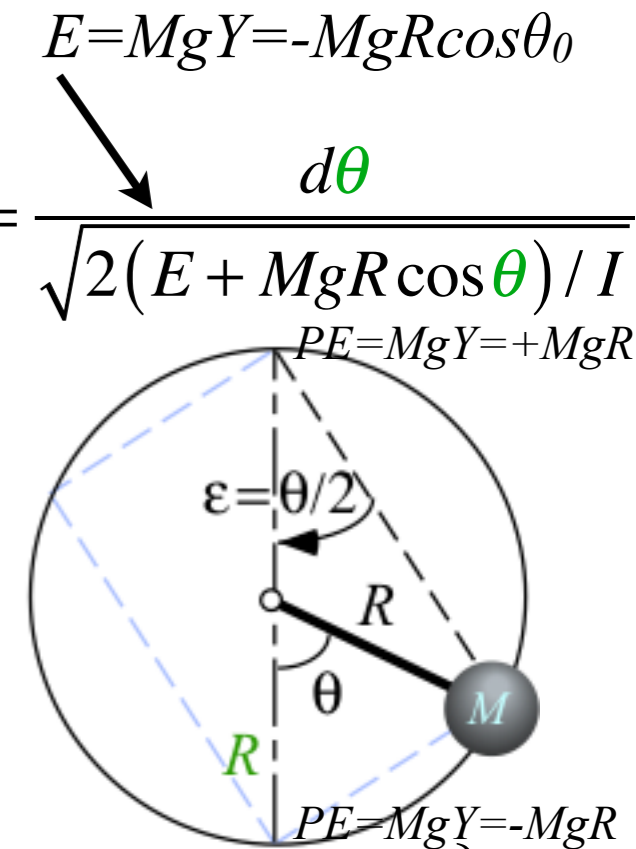
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The integral is an *elliptic integral of the first kind*: $F(k, \varepsilon_0) = am^{-1}$ or the "inverse amu" function.

$$F(k, \varepsilon_0) \equiv \int_0^{\varepsilon_0} \frac{d\varepsilon}{\sqrt{1 - k^2 \sin^2 \varepsilon}} \equiv am^{-1}(k, \varepsilon_0)$$



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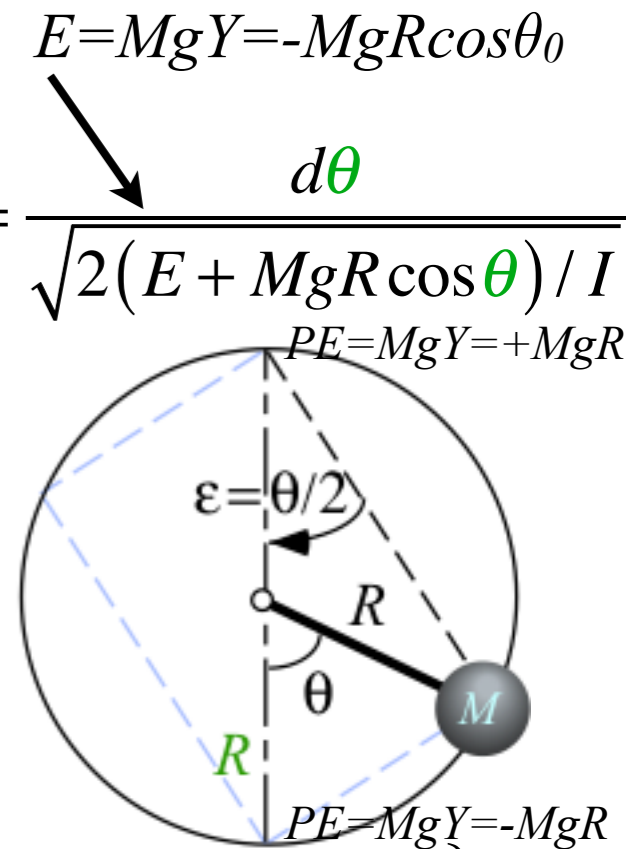
$$\cos \theta = 1 - 2 \sin^2 \frac{\theta}{2} = 1 - 2 \sin^2 \varepsilon, \quad \cos \theta - \cos \theta_0 = 2 \sin^2 \varepsilon_0 - 2 \sin^2 \varepsilon$$

$$\tau_{1/4} = \sqrt{\frac{I}{MgR}} \int_0^{\varepsilon_0} \frac{d\varepsilon}{\sqrt{\sin^2 \varepsilon_0 - \sin^2 \varepsilon}} = \sqrt{\frac{R}{g}} \int_0^{\varepsilon_0} \frac{k d\varepsilon}{\sqrt{1 - k^2 \sin^2 \varepsilon}}, \quad \text{where: } \left\{ \begin{array}{l} 1/k = \sin \varepsilon_0 = \sin \frac{\theta_0}{2} \\ I = MR^2 \end{array} \right.$$

The integral is an *elliptic integral of the first kind*: $F(k, \varepsilon_0) = am^{-1}$ or the "inverse amu" function.

$$F(k, \varepsilon_0) \equiv \int_0^{\varepsilon_0} \frac{d\varepsilon}{\sqrt{1 - k^2 \sin^2 \varepsilon}} \equiv am^{-1}(k, \varepsilon_0) \quad \tau_{1/4} = \sqrt{\frac{R}{g}} \int_0^{\varepsilon_0} \frac{d\varepsilon}{\sqrt{\varepsilon_0^2 - \varepsilon^2}} = \sqrt{\frac{R}{g}} \sin^{-1} \frac{\varepsilon}{\varepsilon_0} \Big|_0^{\varepsilon_0} = \sqrt{\frac{R}{g}} \frac{\pi}{2} = \tau \frac{2\pi}{4}$$

For low amplitude $\varepsilon \ll 1$: $\sin \varepsilon_0 \simeq \varepsilon_0$ reduces $\tau_{1/4}$ to $\tau \frac{2\pi}{4}$



Circular pendulum dynamics and elliptic functions

Hamiltonian function $H = KE + PE = T + U$ where potential energy is $U(\theta) = -MgR \cos \theta$

$$H(p_\theta, \theta) = \frac{1}{2I} p_\theta^2 + U(\theta) = \frac{1}{2I} p_\theta^2 - MgR \cos \theta = E = \text{const.} \quad \text{implies: } p_\theta = \sqrt{2I(E + MgR \cos \theta)}$$

Let $E = MgY = -MgR \cos \theta_0$ be potential energy where $KE = 0$ or $p_\theta = 0$

$$\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \frac{d\theta}{dt} = p_\theta / I = \sqrt{2I(E + MgR \cos \theta)} / I \quad \text{where: } I = MR^2 \quad \text{or: } dt = \frac{d\theta}{\sqrt{2(E + MgR \cos \theta)} / I}$$

Quadrature integral gives quarter-period $\tau_{1/4}$:

$$\sqrt{\frac{I}{2MgR}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos \theta - \cos \theta_0}} = \int_0^{\theta_0} dt = (\text{Travel time } 0 \text{ to } \theta_0) = \tau_{1/4}$$

Uses a half-angle coordinate $\varepsilon = \theta/2$

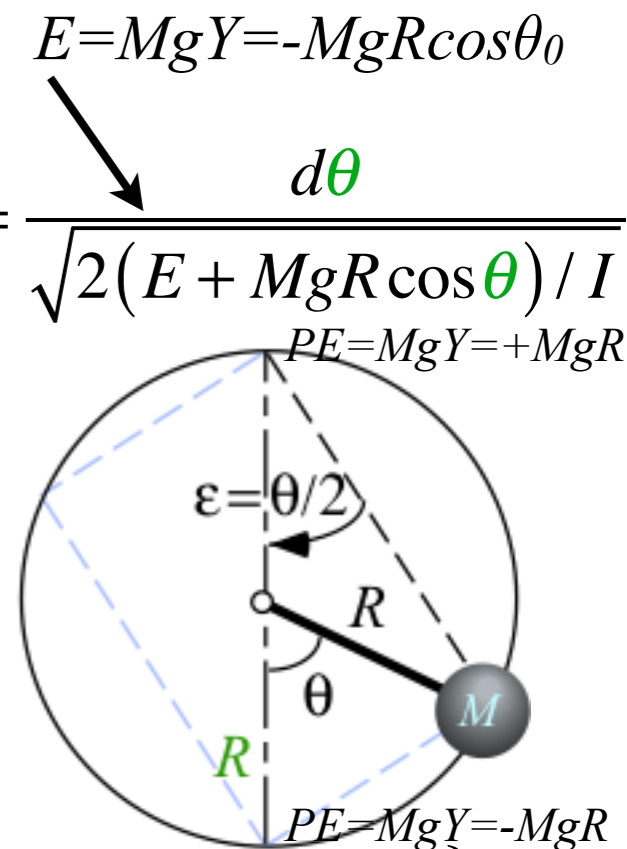
$$\cos \theta = 1 - 2 \sin^2 \frac{\theta}{2} = 1 - 2 \sin^2 \varepsilon, \quad \cos \theta - \cos \theta_0 = 2 \sin^2 \varepsilon_0 - 2 \sin^2 \varepsilon$$

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$$F(k, \varepsilon_0) \equiv \int_0^{\varepsilon_0} \frac{d\varepsilon}{\sqrt{1 - k^2 \sin^2 \varepsilon}} \equiv am^{-1}(k, \varepsilon_0) \quad \tau_{1/4} = \sqrt{\frac{R}{g}} \int_0^{\varepsilon_0} \frac{d\varepsilon}{\sqrt{\varepsilon_0^2 - \varepsilon^2}} = \sqrt{\frac{R}{g}} \sin^{-1} \frac{\varepsilon}{\varepsilon_0} \Big|_0^{\varepsilon_0} = \sqrt{\frac{R}{g}} \frac{\pi}{2} = \tau \frac{2\pi}{4}$$

$$\text{low } \varepsilon \ll 1: t = \sqrt{\frac{R}{g}} \int_0^{\varepsilon(t)} \frac{d\varepsilon}{\sqrt{\varepsilon_0^2 - \varepsilon^2}} = \sqrt{\frac{R}{g}} \sin^{-1} \frac{\varepsilon}{\varepsilon_0} \Big|_0^{\varepsilon(t)} = \sqrt{\frac{R}{g}} \sin^{-1} \frac{\varepsilon(t)}{\varepsilon_0} \quad \text{For low amplitude } \varepsilon \ll 1: \sin \varepsilon_0 \simeq \varepsilon_0 \text{ reduces } \tau_{1/4} \text{ to } \tau \frac{2\pi}{4}$$



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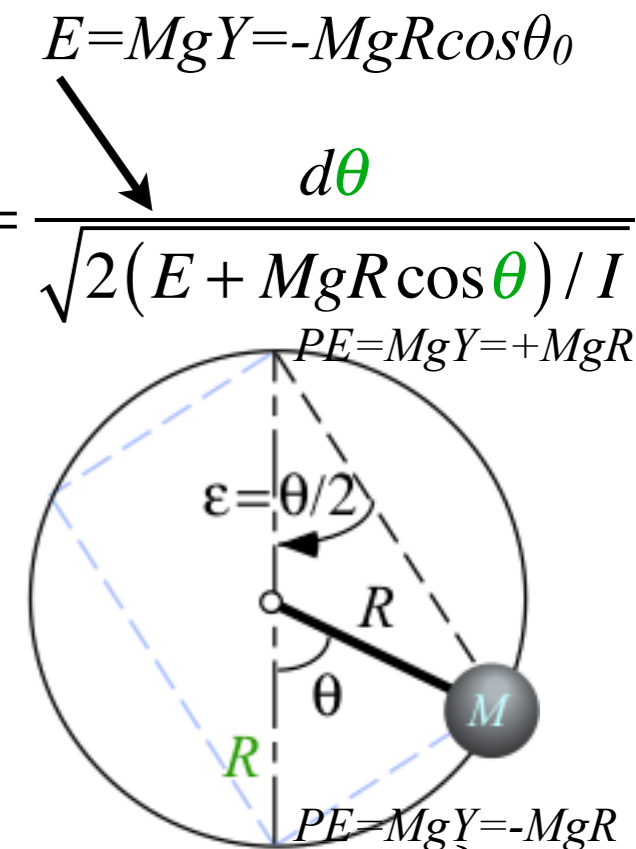
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..reduces to sine...

$$\epsilon(t) = \epsilon_0 \sin \sqrt{\frac{g}{R}} t = \epsilon_0 \sin \omega t, \quad \text{where: } \omega = \sqrt{\frac{g}{R}}$$

For low amplitude $\epsilon \ll 1$: $\sin \epsilon_0 \simeq \epsilon_0$ reduces $\tau_{1/4}$ to $\tau \frac{2\pi}{4}$



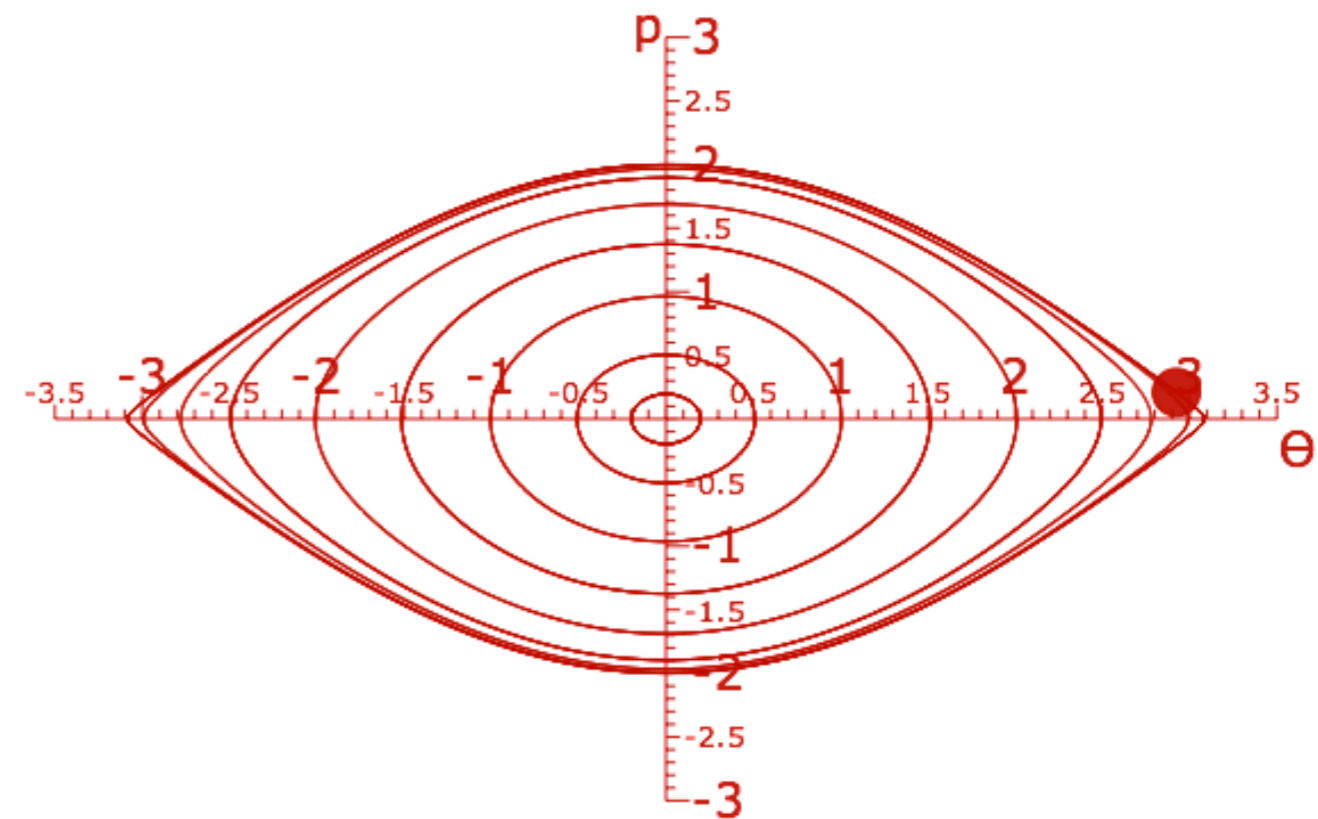
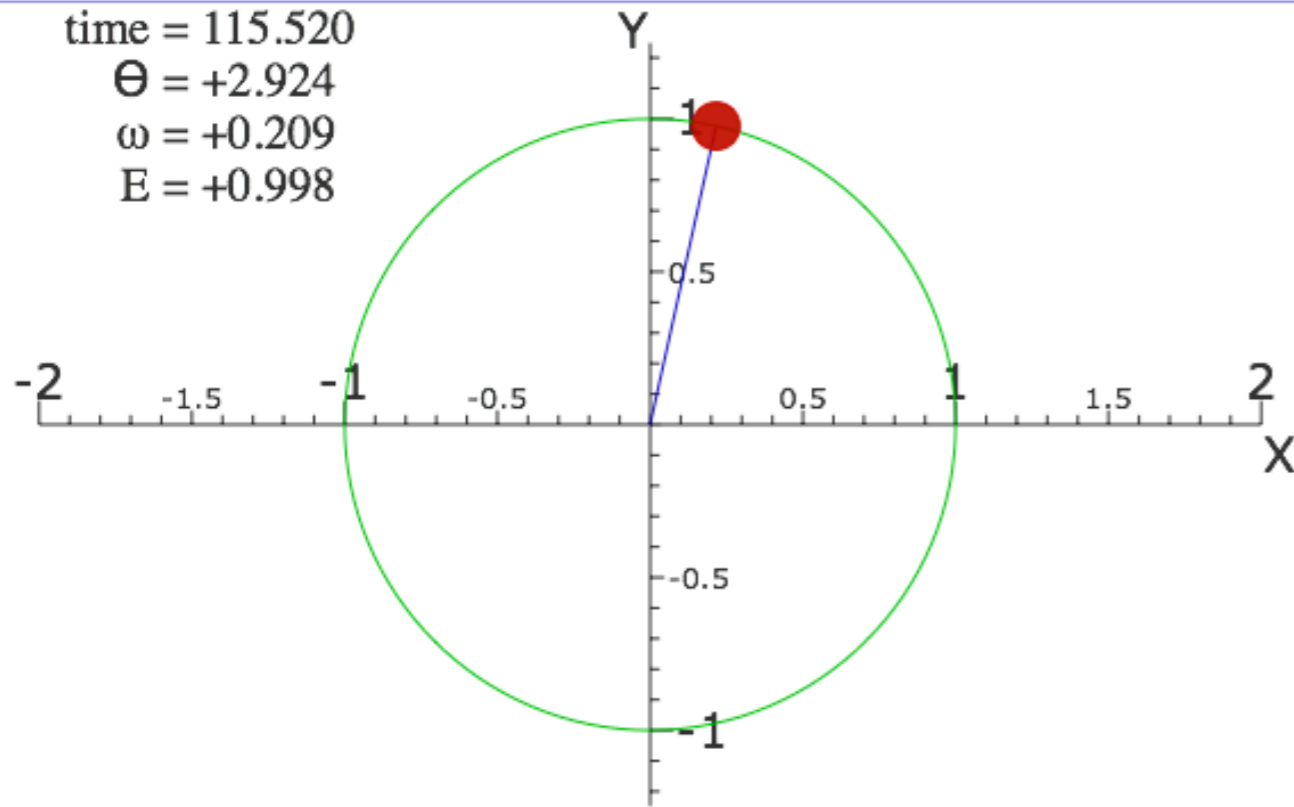
Circular pendulum dynamics and elliptic functions

time = 115.520

$\Theta = +2.924$

$\omega = +0.209$

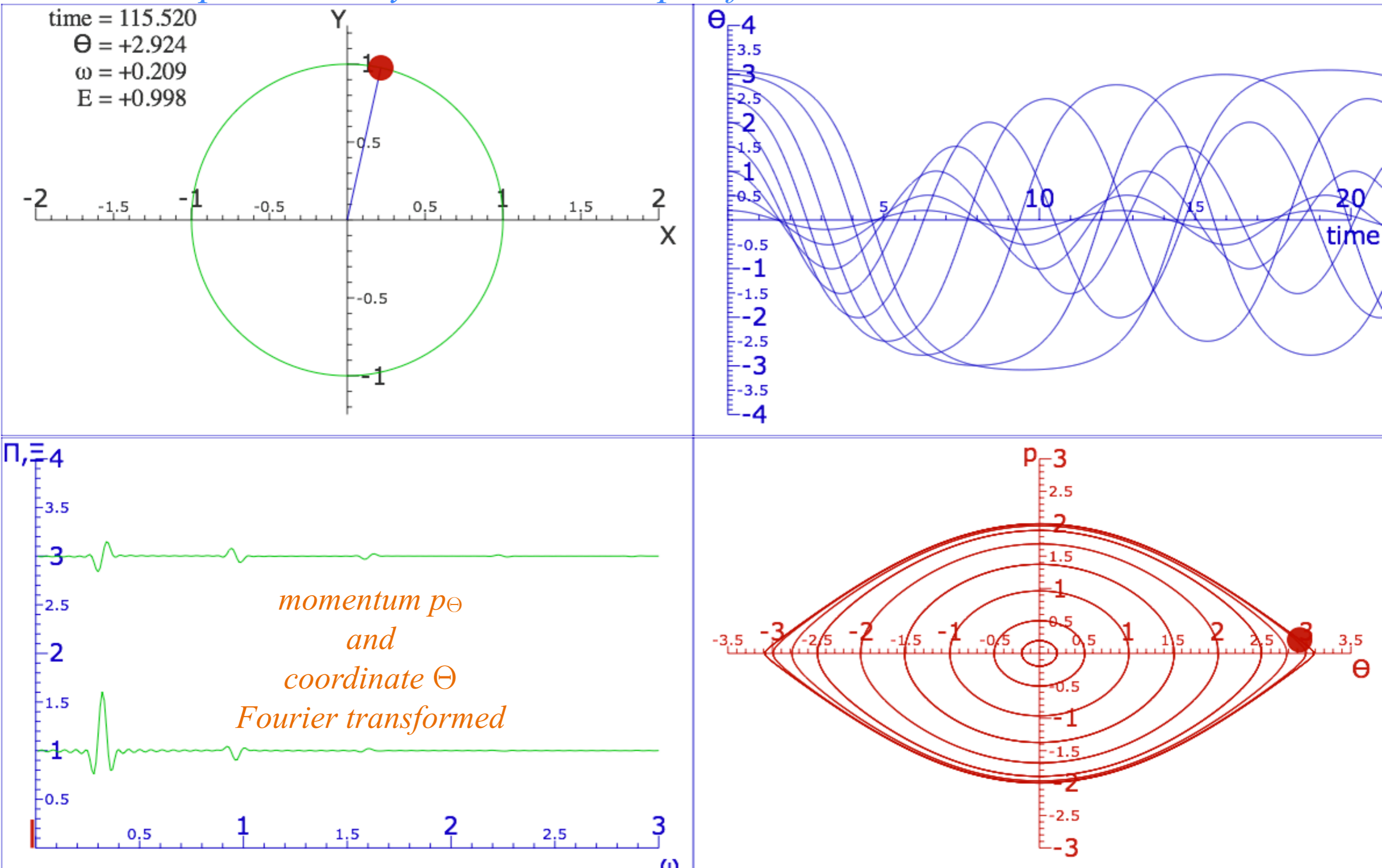
$E = +0.998$



(Simulations of pendulum)

(See also: Simulation of cycloidally constrained pendulum)

Circular pendulum dynamics and elliptic functions



(Simulations of pendulum)

(See also: Simulation of cycloidally constrained pendulum)

Examples of Hamiltonian mechanics in phase plots

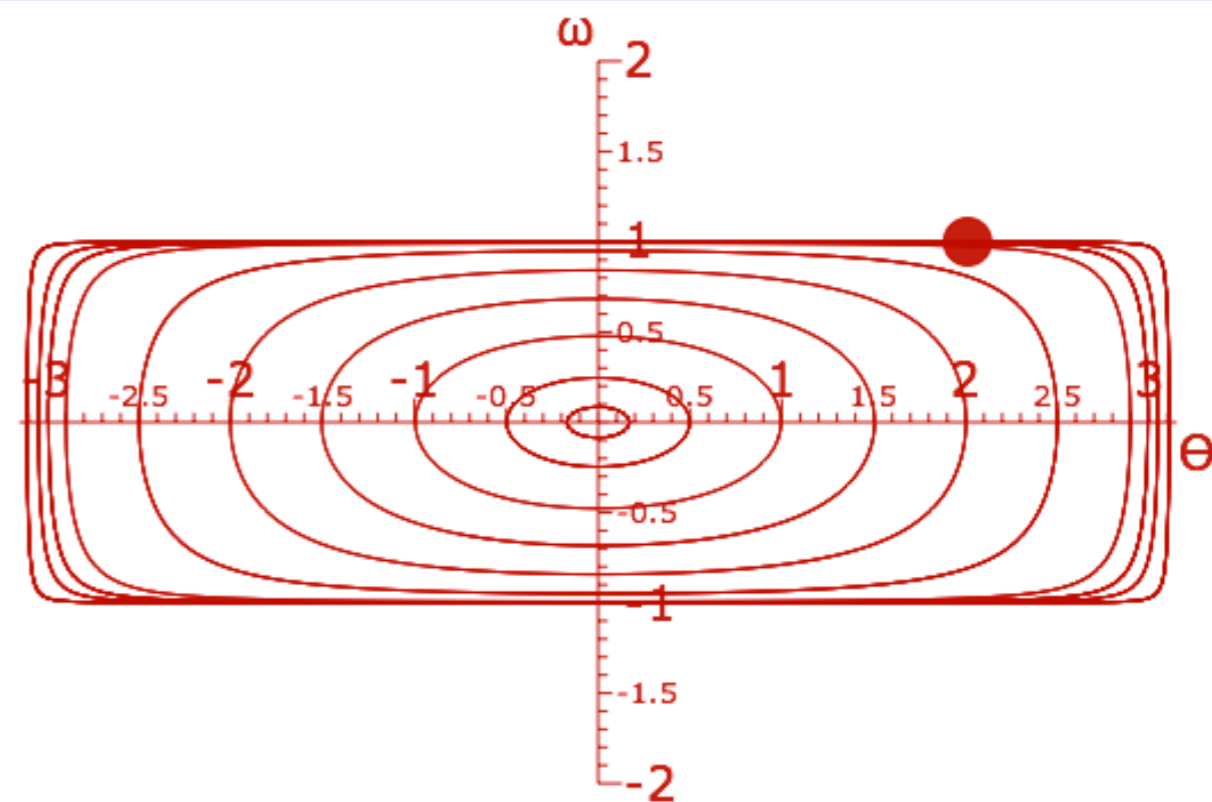
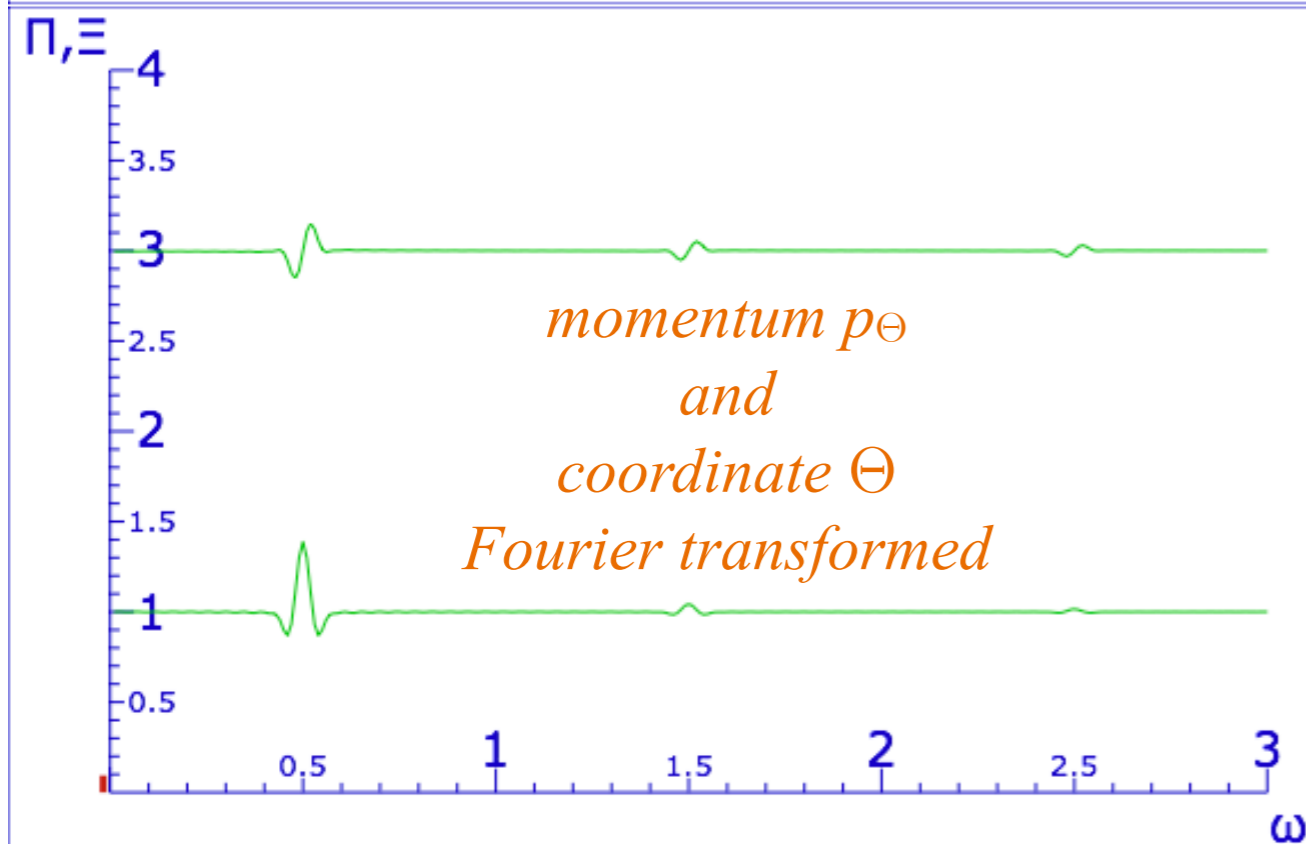
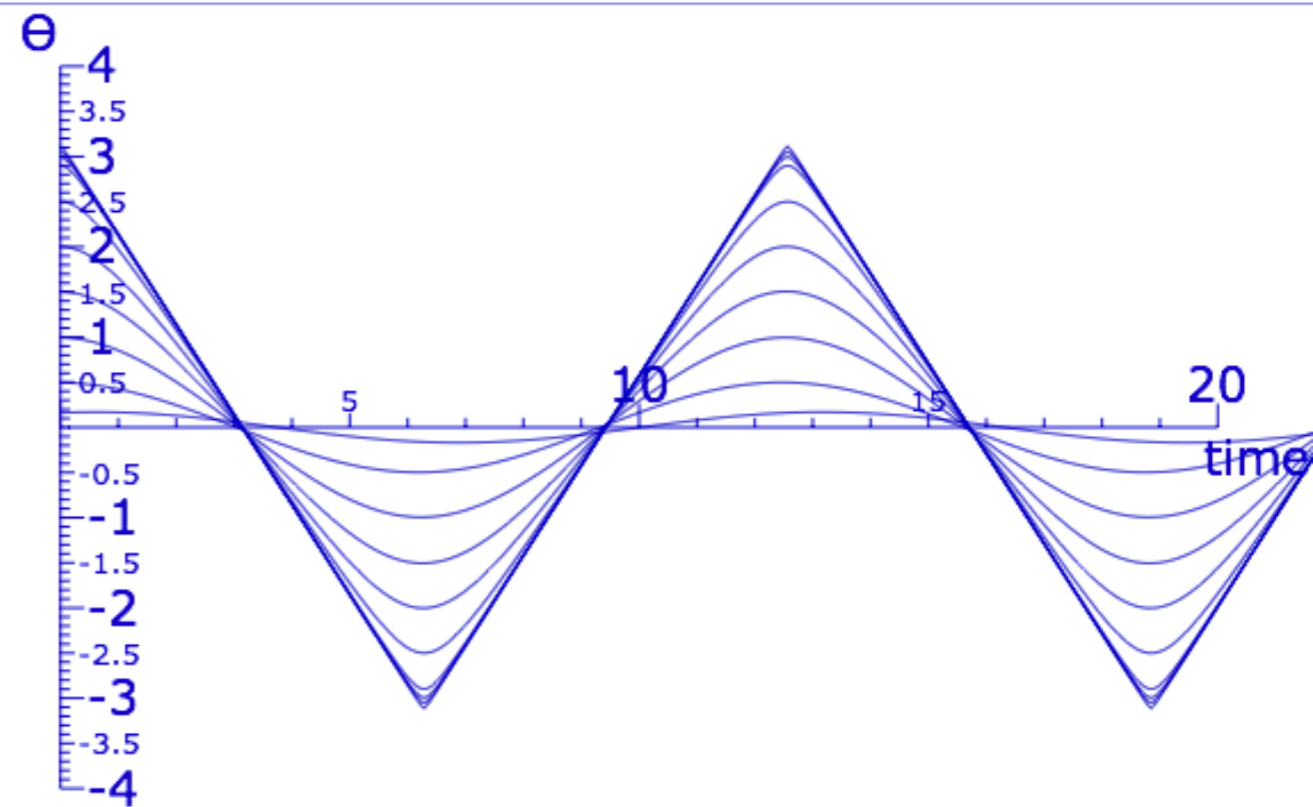
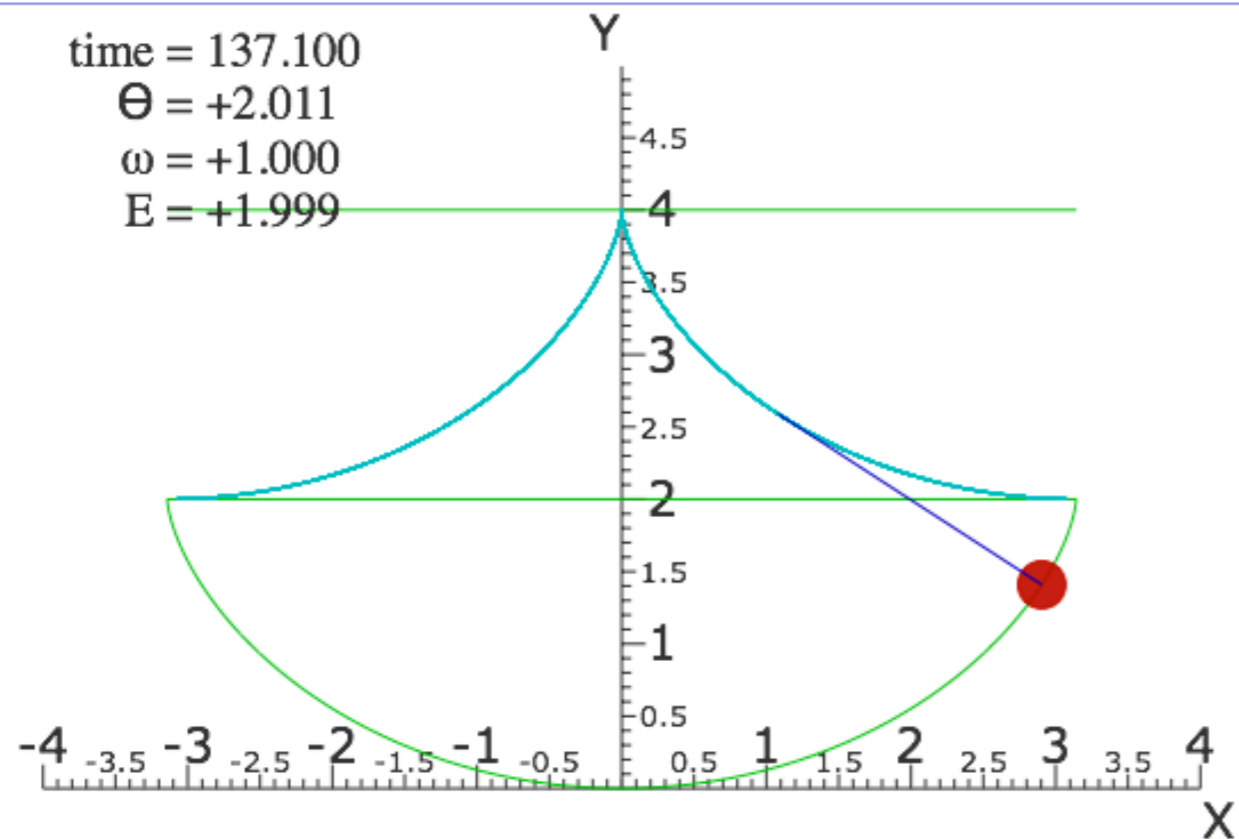
1D Pendulum and phase plot (Web Simulations: [Pendulum](#), [Cycloidulum](#), [JerkIt \(Vert Driven Pendulum\)](#))

Circular pendulum dynamics and elliptic functions

 *Cycloid pendulum dynamics and “sawtooth” functions*

1D-HO phase-space control (Old Mac OS & [Web Simulations of “Catcher in the Eye”](#))

Cycloid pendulum dynamics and "sawtooth" functions



(Simulations of cycloidally constrained pendulum)

Examples of Hamiltonian mechanics in phase plots

1D Pendulum and phase plot (Web Simulations: [Pendulum](#), [Cycloidulum](#), [JerkIt \(Vert Driven Pendulum\)](#))

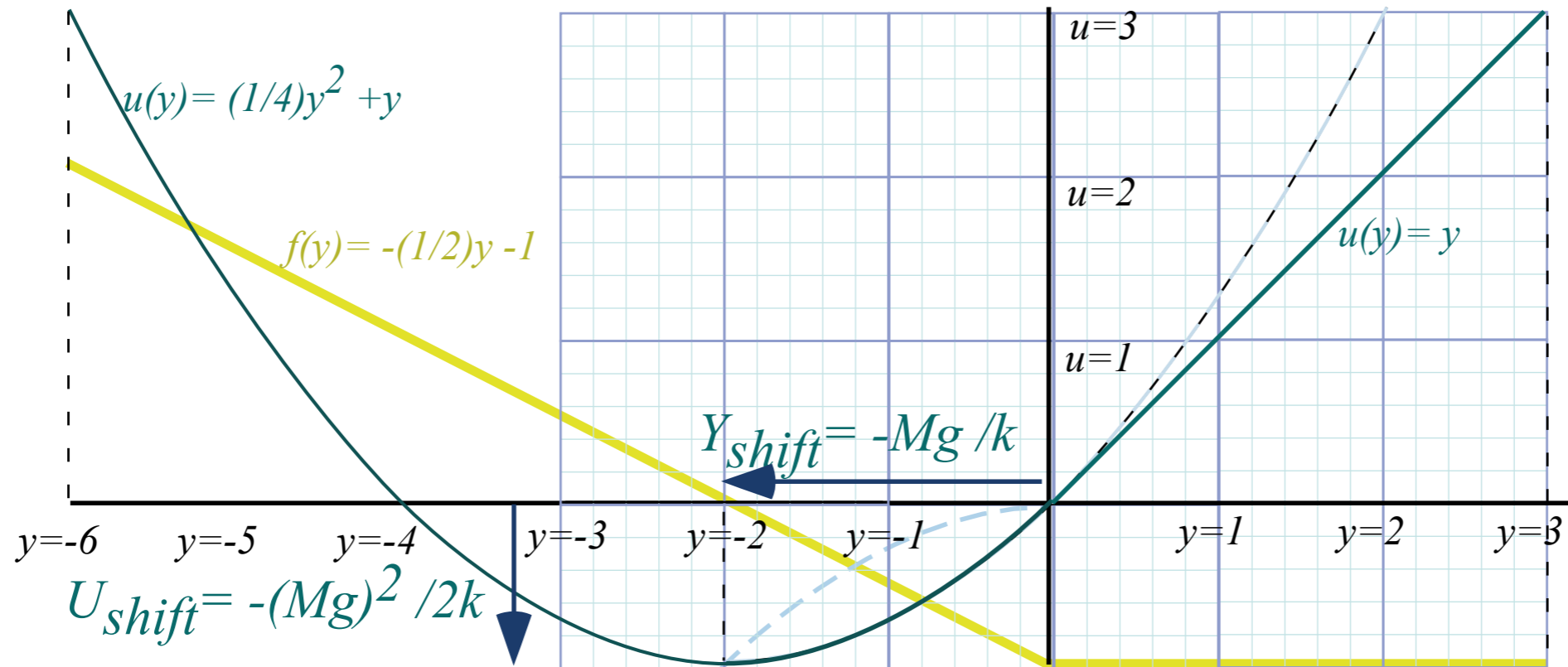
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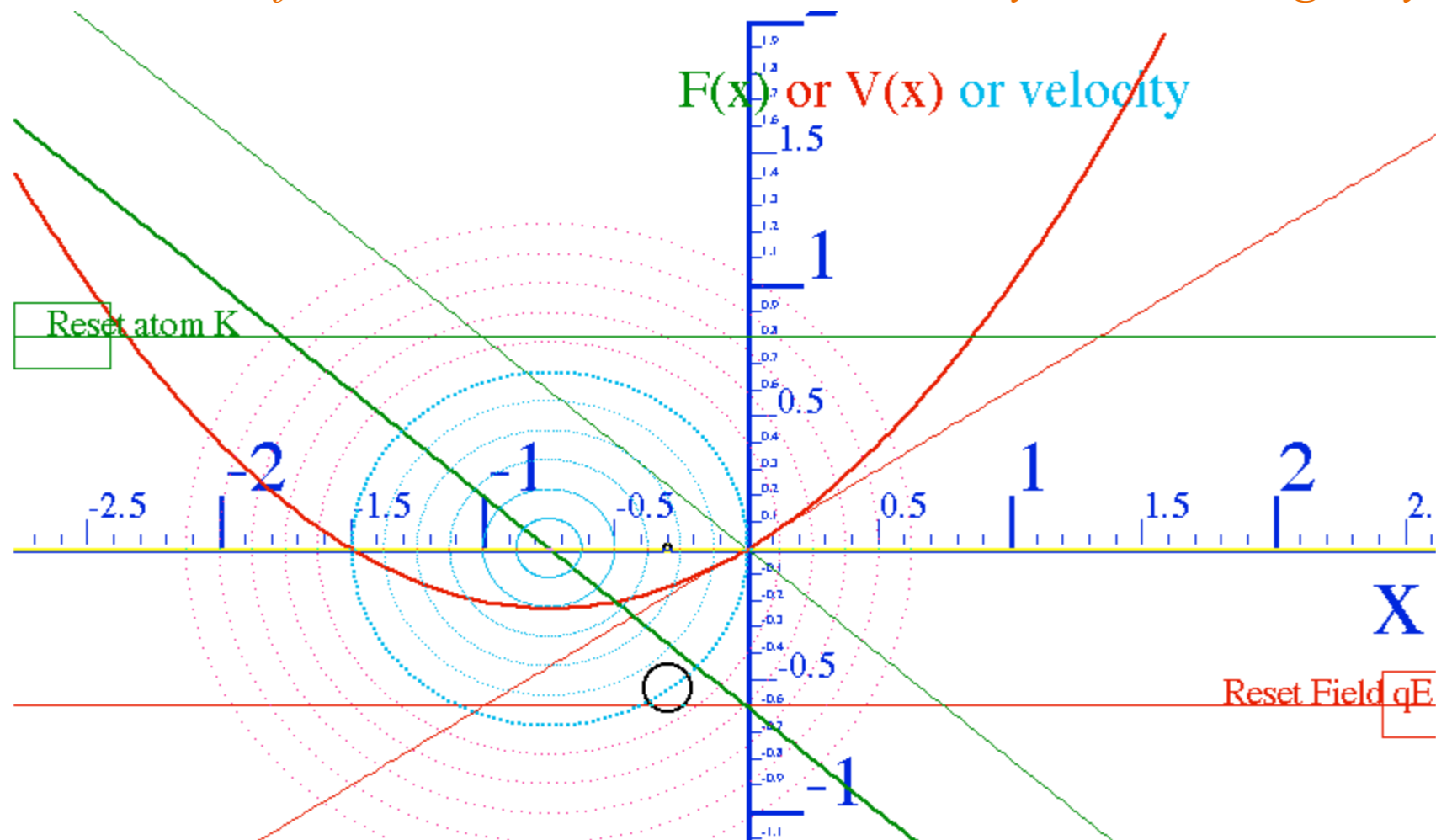
$$F(Y) = -kY - Mg$$

$$U(Y) = (1/2)kY^2 + MgY$$



Unit 1
Fig. 7.4

Web Simulation of atomic classical (or semi-classical) dynamics using varying phase control



Exploring phase space and Lagrangian mechanics more deeply

 *A weird “derivation” of Lagrange’s equations*

Poincare identity and Action, Jacobi-Hamilton equations

How Classicists might have “derived” quantum equations

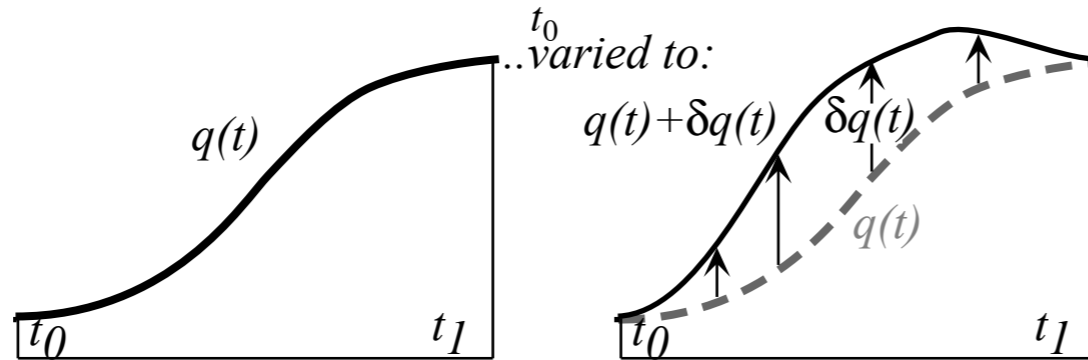
Huygen’s contact transformations enforce minimum action

How to do quantum mechanics if you only know classical mechanics

A strange “derivation” of Lagrange’s equations by Calculus of Variation

Variational calculus finds extreme (minimum or maximum) values to entire integrals

Minimize (or maximize): $S(q) = \int_{t_0}^{t_1} dt L(q(t), \dot{q}(t), t).$



An arbitrary but small variation function $\delta q(t)$ is allowed at every point t in the figure along the curve except at the end points t_0 and t_1 . There we demand it not vary at all. (1)

$$\delta q(t_0) = 0 = \delta q(t_1) \quad (1)$$

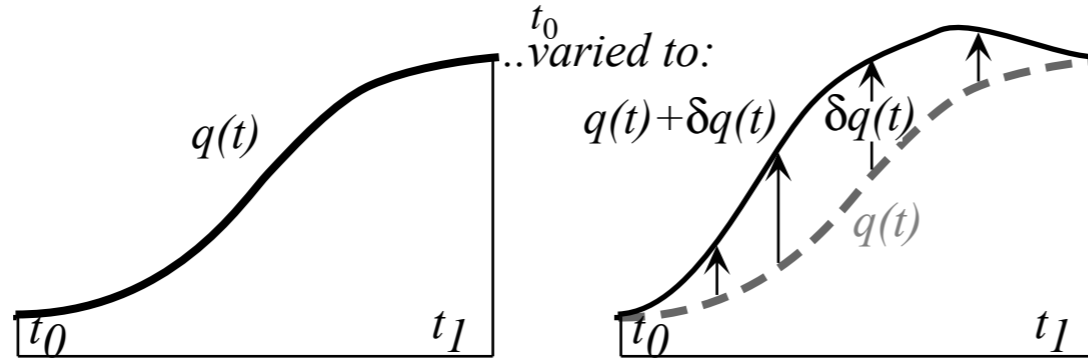
1st order $L(q + \delta q)$ approximate:

$$S(q + \delta q) = \int_{t_0}^{t_1} dt \left[L(q, \dot{q}, t) + \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right] \quad \text{where: } \delta \dot{q} = \frac{d}{dt} \delta q$$

A weird "derivation" of Lagrange's equations

Variational calculus finds extreme (minimum or maximum) values to entire integrals

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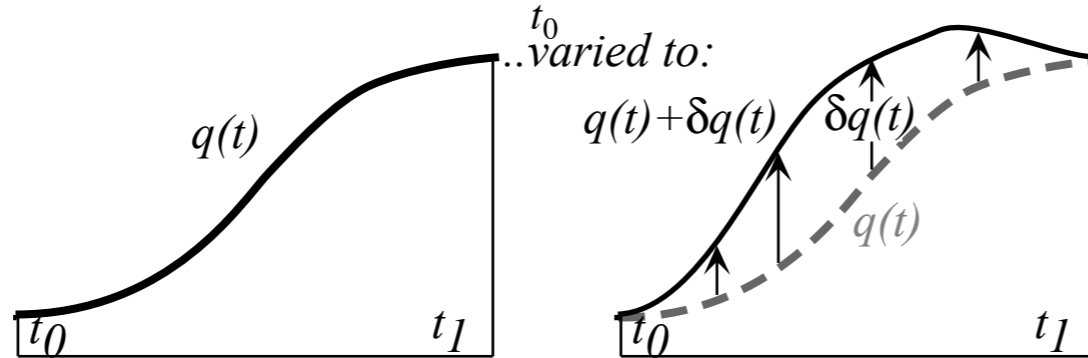
Replace $\frac{\partial L}{\partial \dot{q}} \delta \dot{q}$ with $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \delta q \right) - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \delta q$

Diagram showing the product rule: $u \cdot \frac{dv}{dt} = \frac{d}{dt}(uv) - \frac{du}{dt}v$. Arrows point from this rule to the replacement step above.

A weird "derivation" of Lagrange's equations

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$$S(q) = \int_{t_0}^{t_1} dt L(q(t), \dot{q}(t), t).$$



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1st order $L(q+\delta q)$ approximate: $\delta q(t_0) = 0 = \delta q(t_1)$ (1)

Replace $\frac{\partial L}{\partial \dot{q}} \delta \dot{q}$ with $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \delta q \right) - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \delta q$

Derivation: $u \cdot \frac{dv}{dt} = \frac{d}{dt}(uv) - \frac{du}{dt}v$

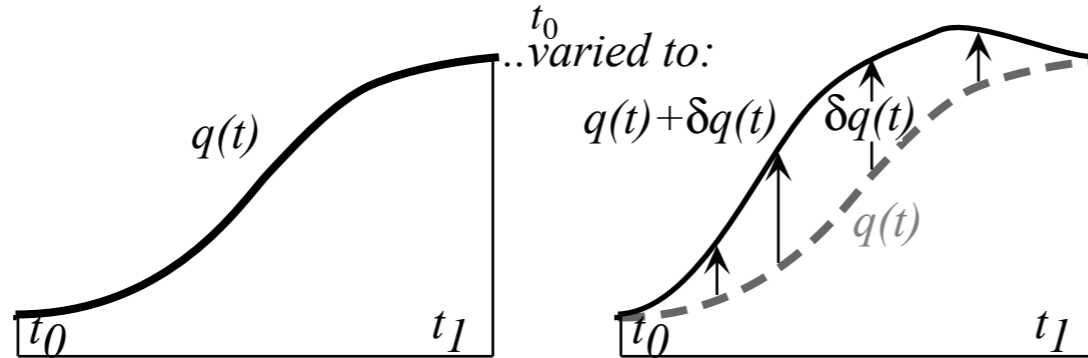
$$S(q + \delta q) = \int_{t_0}^{t_1} dt \left[L(q, \dot{q}, t) + \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right]$$

$$S(q + \delta q) = \int_{t_0}^{t_1} dt \left[L(q, \dot{q}, t) + \frac{\partial L}{\partial q} \delta q - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \delta q \right] + \int_{t_0}^{t_1} dt \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \delta q \right)$$

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Variational calculus finds extreme (minimum or maximum) values to entire integrals

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Mathematical identity: $u \cdot \frac{dv}{dt} = \frac{d}{dt}(uv) - \frac{du}{dt}v$

$$S(q + \delta q) = \int_{t_0}^{t_1} dt \left[L(q, \dot{q}, t) + \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right]$$

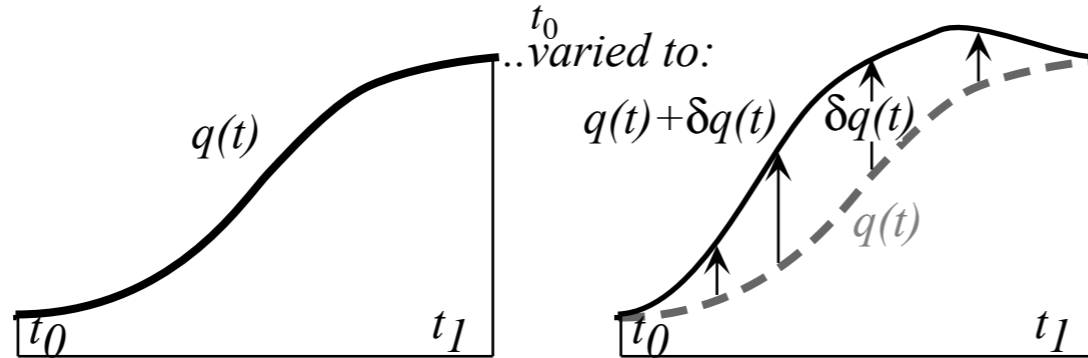
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$$= \int_{t_0}^{t_1} dt L(q, \dot{q}, t) + \int_{t_0}^{t_1} dt \left[\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right] \delta q + \left(\frac{\partial L}{\partial \dot{q}} \delta q \right) \Big|_{t_0}^{t_1}$$

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due to requiring (1)

Third term vanishes by (1). This leaves first order variation: $\delta S = S(q + \delta q) - S(q) = \int_{t_0}^{t_1} dt \left[\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right] \delta q$

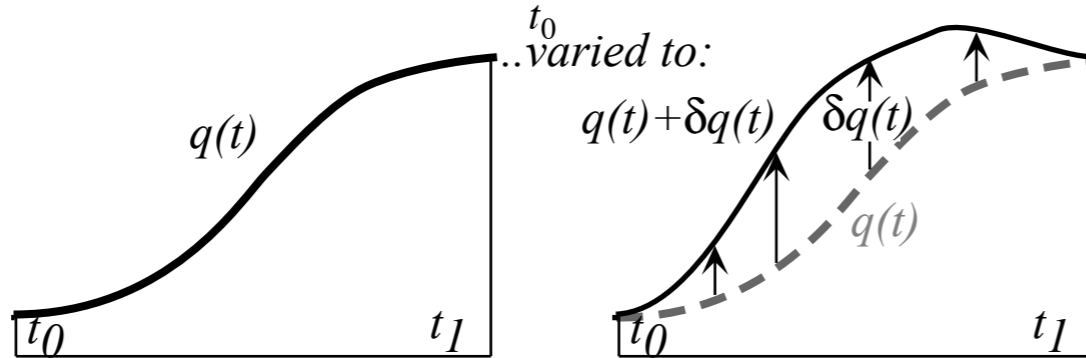
Extreme value (actually *minimum* value) of $S(q)$ occurs *if and only if* Lagrange equation is satisfied!

$$\delta S = 0 \Rightarrow \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0 \quad \text{Euler-Lagrange equation(s)}$$

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$$S(q + \delta q) = \int_{t_0}^{t_1} dt \left[L(q, \dot{q}, t) + \frac{\partial L}{\partial q} \delta q - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \delta q \right] + \int_{t_0}^{t_1} dt \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \delta q \right)$$

$$= \int_{t_0}^{t_1} dt L(q, \dot{q}, t) + \int_{t_0}^{t_1} dt \left[\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right] \delta q + \left(\frac{\partial L}{\partial \dot{q}} \delta q \right) \Big|_{t_0}^{t_1}$$

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$$\delta S = 0 \Rightarrow \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0 \quad \text{Euler-Lagrange equation(s)}$$

But, WHY is nature so inclined to fly JUST SO as to minimize the Lagrangian $L = T - U$???

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Legendre-Poincare identity and Action

Legendre transform $L(\mathbf{v}) = \mathbf{p} \cdot \mathbf{v} - H(\mathbf{p})$ becomes *Poincare's invariant differential* if dt is cleared.

$$L \cdot dt = \mathbf{p} \cdot \mathbf{v} \cdot dt - H \cdot dt = \mathbf{p} \cdot d\mathbf{r} - H \cdot dt \quad \left(\mathbf{v} = \frac{d\mathbf{r}}{dt} \text{ implies: } \mathbf{v} \cdot dt = d\mathbf{r} \right)$$

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This is the time differential dS of *action* $S = \int L \cdot dt$ whose time derivative is rate L of *quantum phase*.

$$dS = L \cdot dt = \mathbf{p} \cdot d\mathbf{r} - H \cdot dt \quad \text{where: } L = \frac{dS}{dt}$$

Legendre-Poincare identity and Action

Legendre transform $L(\mathbf{v}) = \mathbf{p} \cdot \mathbf{v} - H(\mathbf{p})$ becomes *Poincare's invariant differential* if dt is cleared.

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This is the time differential dS of *action* $S = \int L \cdot dt$ whose time derivative is rate L of *quantum phase*.

$$dS = L \cdot dt = \mathbf{p} \cdot d\mathbf{r} - H \cdot dt \quad \text{where: } L = \frac{dS}{dt}$$

Unit 8 shows *DeBroglie law* $\mathbf{p} = \hbar \mathbf{k}$ and *Planck law* $H = \hbar \omega$ make *quantum plane wave phase* Φ :

$$\Phi = S/\hbar = \int L \cdot dt / \hbar$$

Legendre-Poincare identity and Action

Legendre transform $L(\mathbf{v}) = \mathbf{p} \cdot \mathbf{v} - H(\mathbf{p})$ becomes *Poincare's invariant differential* if dt is cleared.

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Q: When is the *Action*-differential dS integrable?

A: A differential $dW = f_x(x, y)dx + f_y(x, y)dy$ is *integrable* to a $W(x, y)$ if: $f_x = \frac{\partial W}{\partial x}$ and: $f_y = \frac{\partial W}{\partial y}$

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Similar to conditions for integrating work differential $dW = \mathbf{f} \cdot d\mathbf{r}$ to get potential $W(\mathbf{r})$. That condition is **no curl allowed**: $\nabla \times \mathbf{f} = \mathbf{0}$ or ∂ -symmetry of W :

$$\frac{\partial f_x}{\partial y} = \frac{\partial^2 W}{\partial y \partial x} = \frac{\partial^2 W}{\partial x \partial y} = \frac{\partial f_y}{\partial x}$$

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These conditions are known as *Jacobi-Hamilton equations*

Exploring phase space and Lagrangian mechanics more deeply

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How to do quantum mechanics if you only know classical mechanics



How Jacobi-Hamilton could have “derived” Schrodinger equations

(Given “quantum wave”)

$$\psi(\mathbf{r}, t) = e^{iS/\hbar} = e^{i(\mathbf{p}\cdot\mathbf{r} - H\cdot t)/\hbar} = e^{i(\mathbf{k}\cdot\mathbf{r} - \omega\cdot t)}$$

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Momentum Operator
or \mathbf{p} -op in \mathbf{r} -basis
 $\mathbf{p} \Rightarrow \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{r}}$

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Try 1st t -derivative of wave ψ

$$\frac{\partial}{\partial t} \psi(\mathbf{r}, t) = \frac{\partial}{\partial t} e^{iS/\hbar} = \frac{\partial(iS/\hbar)}{\partial t} e^{iS/\hbar} = (i/\hbar) \frac{\partial S}{\partial t} \psi(\mathbf{r}, t)$$

$$= (i/\hbar)(-H) \psi(\mathbf{r}, t) \text{ or: } i\hbar \frac{\partial}{\partial t} \psi(\mathbf{r}, t) = H \psi(\mathbf{r}, t)$$

How Jacobi-Hamilton could have “derived” Schrodinger equations

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Schrodinger time equation
 $i\hbar \psi(\mathbf{r}, t) = H \psi(\mathbf{r}, t)$

Exploring phase space and Lagrangian mechanics more deeply

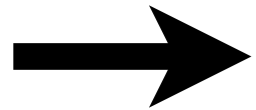
A weird “derivation” of Lagrange’s equations

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Huygen’s contact transformations enforce minimum action

How to do quantum mechanics if you only know classical mechanics



Huygen's contact transformations enforce minimum action

Each point \mathbf{r}_k on a wavefront "broadcasts" in all directions.

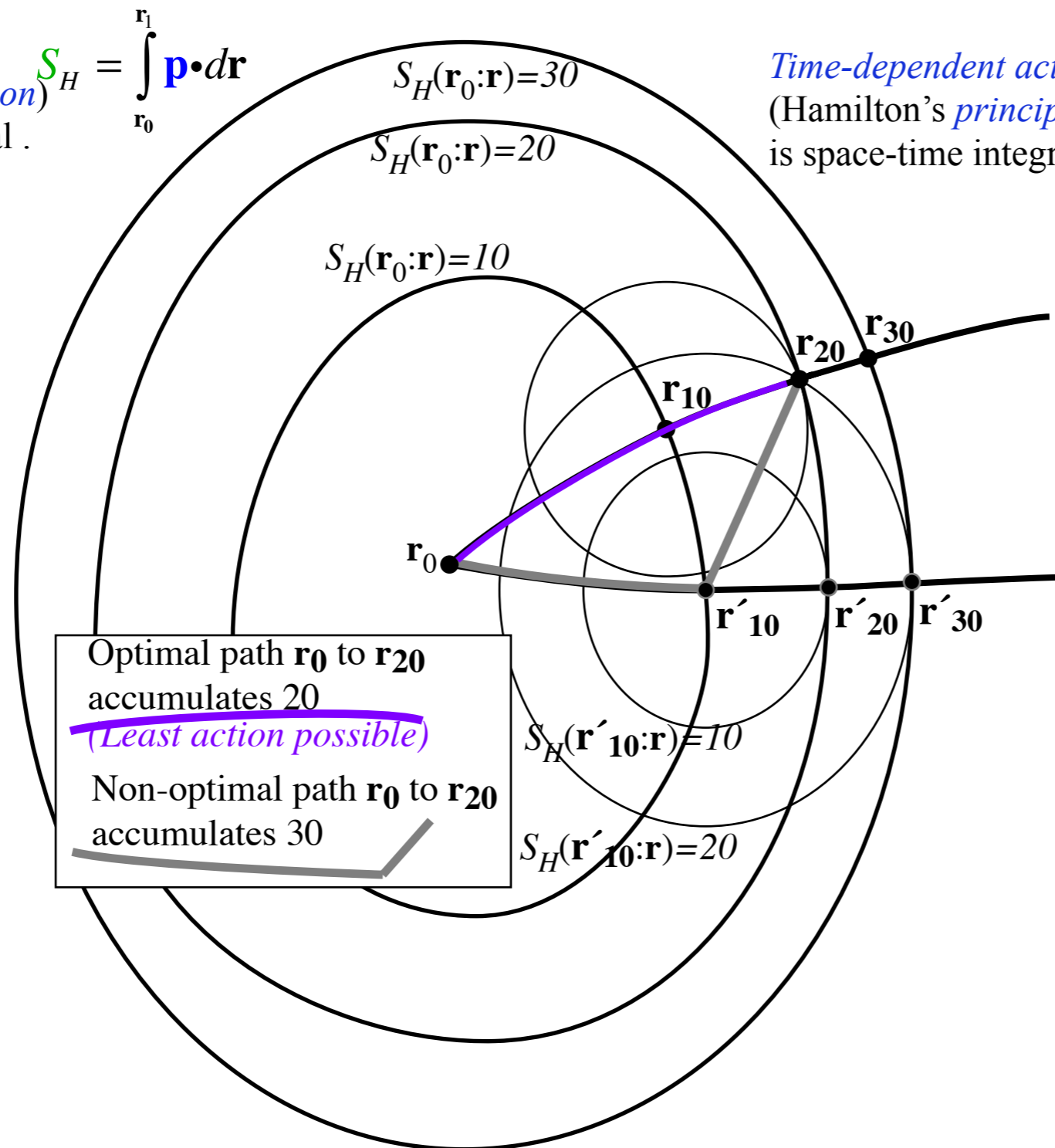
Only **minimum action** path interferes constructively

Time-independent action
(Hamilton's *reduced action*)
is a purely spatial integral .

$$S_H = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{p} \cdot d\mathbf{r}$$

Time-dependent action
(Hamilton's *principle action*)
is space-time integral .

$$S_p = \int_{\mathbf{r}_0 t_0}^{\mathbf{r}_1 t_1} (\mathbf{p} \cdot d\mathbf{r} - H \cdot dt)$$



Optimal path \mathbf{r}_0 to \mathbf{r}_{20}
accumulates 20
(Least action possible)

Non-optimal path \mathbf{r}_0 to \mathbf{r}_{20}
accumulates 30

Unit 1
Fig. 12.12

Huygen's contact transformations enforce minimum action

Each point \mathbf{r}_k on a wavefront "broadcasts" in all directions.

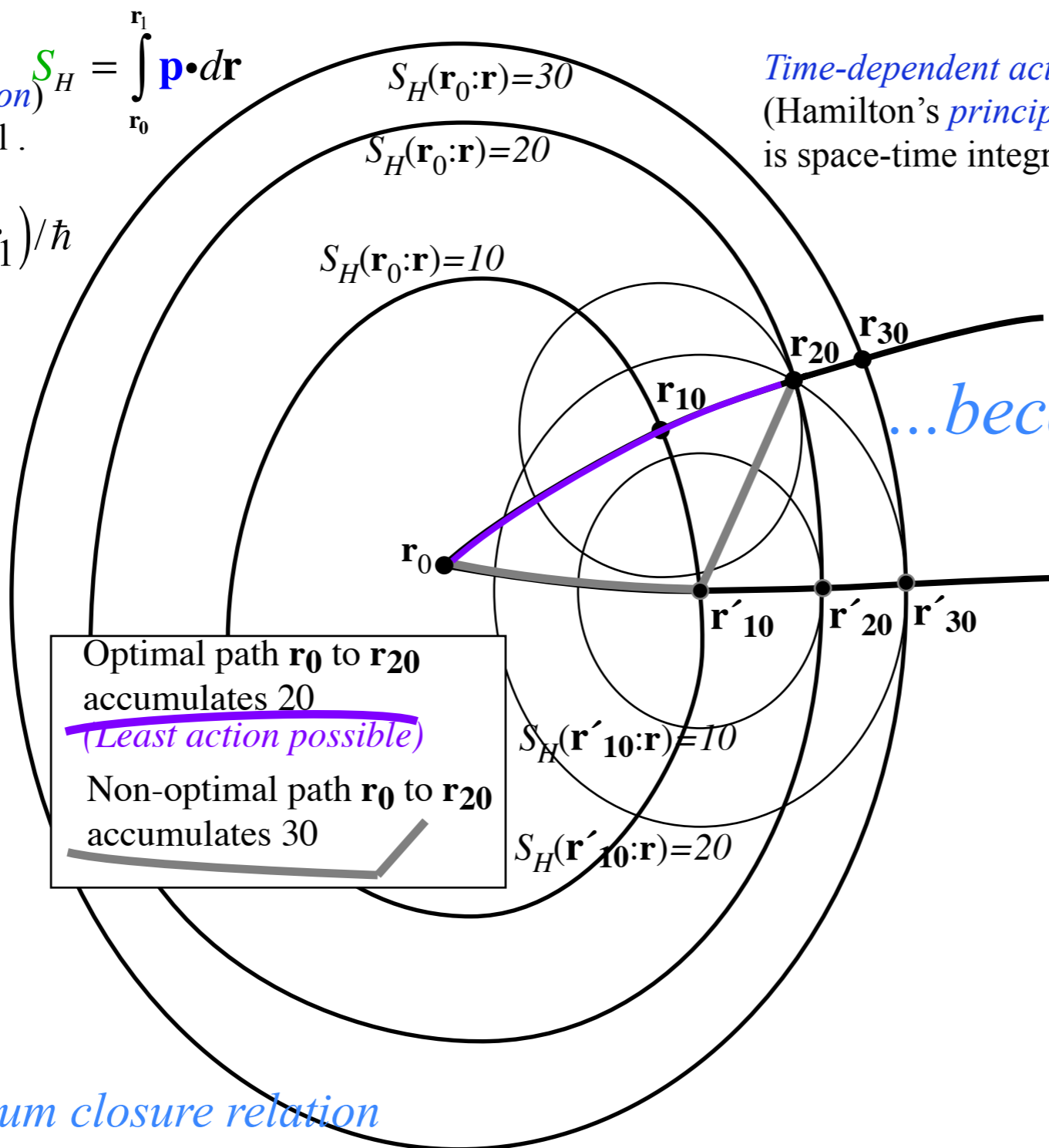
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Time-dependent action $S_p = \int_{\mathbf{r}_0, t_0}^{\mathbf{r}_1, t_1} (\mathbf{p} \cdot d\mathbf{r} - H \cdot dt)$ is space-time integral.

$$\langle \mathbf{r}_1 | \mathbf{r}_0 \rangle = e^{i S_H(\mathbf{r}_0 : \mathbf{r}_1) / \hbar}$$

$$\langle \mathbf{r}_1, t_1 | \mathbf{r}_0, t_0 \rangle = e^{i S(\mathbf{r}_0, t_0 : \mathbf{r}_1, t_1) / \hbar}$$



...because action is quantum wave phase

Unit 1
Fig. 12.12

Feynman's path-sum closure relation

$$\sum_{\mathbf{r}'} \langle \mathbf{r}_1 | \mathbf{r}' \rangle \langle \mathbf{r}' | \mathbf{r}_0 \rangle \equiv \sum_{\mathbf{r}'} e^{i(S_H(\mathbf{r}_0 : \mathbf{r}') + S_H(\mathbf{r}' : \mathbf{r}_1)) / \hbar} = e^{i S_H(\mathbf{r}_0 : \mathbf{r}_1) / \hbar} = \langle \mathbf{r}_1 | \mathbf{r}_0 \rangle$$

Exploring phase space and Lagrangian mechanics more deeply

A weird “derivation” of Lagrange’s equations

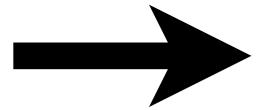
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How to do quantum mechanics if you only know classical mechanics

Davis-Heller “Color-Quantization” or “Classical Chromodynamics”



How to do quantum mechanics if you only know classical mechanics

Bohr quantization requires quantum phase S_H/\hbar in amplitude to be an integral multiple n of 2π after a closed loop integral $S_H(\mathbf{r}_0:\mathbf{r}_0) = \int_{r_0}^{r_0} \mathbf{p} \cdot d\mathbf{r}$. The integer n ($n = 0, 1, 2, \dots$) is a *quantum number*.

$$1 = \langle \mathbf{r}_0 | \mathbf{r}_0 \rangle = e^{i S_H(\mathbf{r}_0:\mathbf{r}_0)/\hbar} = e^{i \Sigma_H/\hbar} = 1 \quad \text{for: } \Sigma_H = 2\pi \hbar n = h n$$

Numerically integrate Hamilton's equations and Lagrangian L . Color the trajectory according to the current accumulated value of action $S_H(\mathbf{0} : \mathbf{r})/\hbar$. Adjust energy to quantized pattern (if closed system*)

$$S_H(\mathbf{0} : \mathbf{r}) = S_p(\mathbf{0}, 0 : \mathbf{r}, t) + Ht = \int_0^t L dt + Ht .$$

How to do quantum mechanics if you only know classical mechanics

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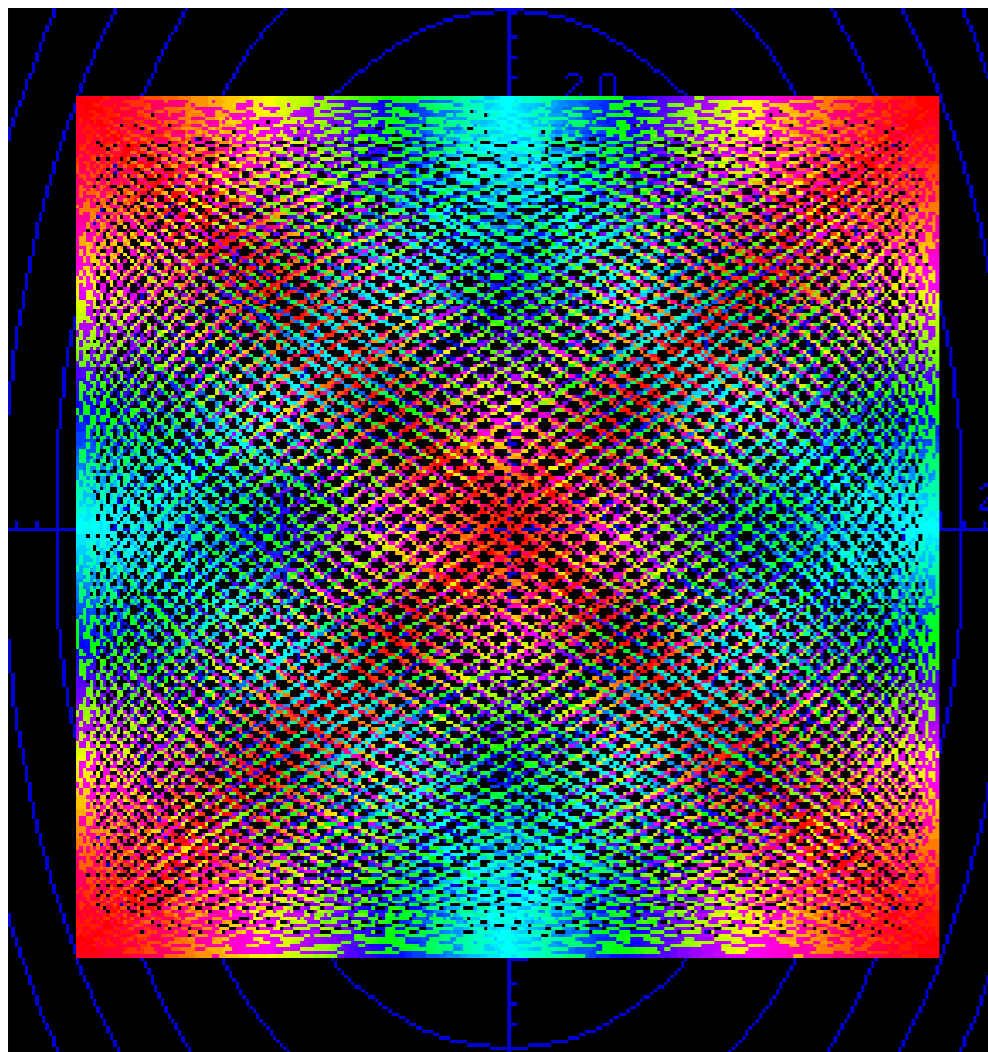
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$$S_H(\mathbf{0} : \mathbf{r}) = S_p(\mathbf{0}, 0 : \mathbf{r}, t) + Ht = \int_0^t L dt + Ht.$$

The hue should represent the phase angle $S_H(\mathbf{0} : \mathbf{r})/\hbar \text{ modulo } 2\pi$ as, for example,

$0=\text{red}$, $\pi/4=\text{orange}$, $\pi/2=\text{yellow}$, $3\pi/4=\text{green}$, $\pi=\text{cyan}$ (opposite of red), $5\pi/4=\text{indigo}$, $3\pi/2=\text{blue}$, $7\pi/4=\text{purple}$, and $2\pi=\text{red}$ (full color circle).

Interpolating action on a palette of 32 colors is enough precision for low quanta.



*simulation
by
"Color U(2)"*

Unit 1
Fig.
12.13

*closed system
has quantized E.
Standing wave has
only two phases(\pm)
cyan and *red*

Wavepacket and Color-quantization:

M. J. Davis and E. J. Heller, J. Chem. Phys. 75, 246 (1981)

[The Semiclassical Way to Molecular Spectroscopy:
Eric J. Heller, Acc. Chem. Res. 1981, 14, 368-375](#)

How to do quantum mechanics if you only know classical mechanics

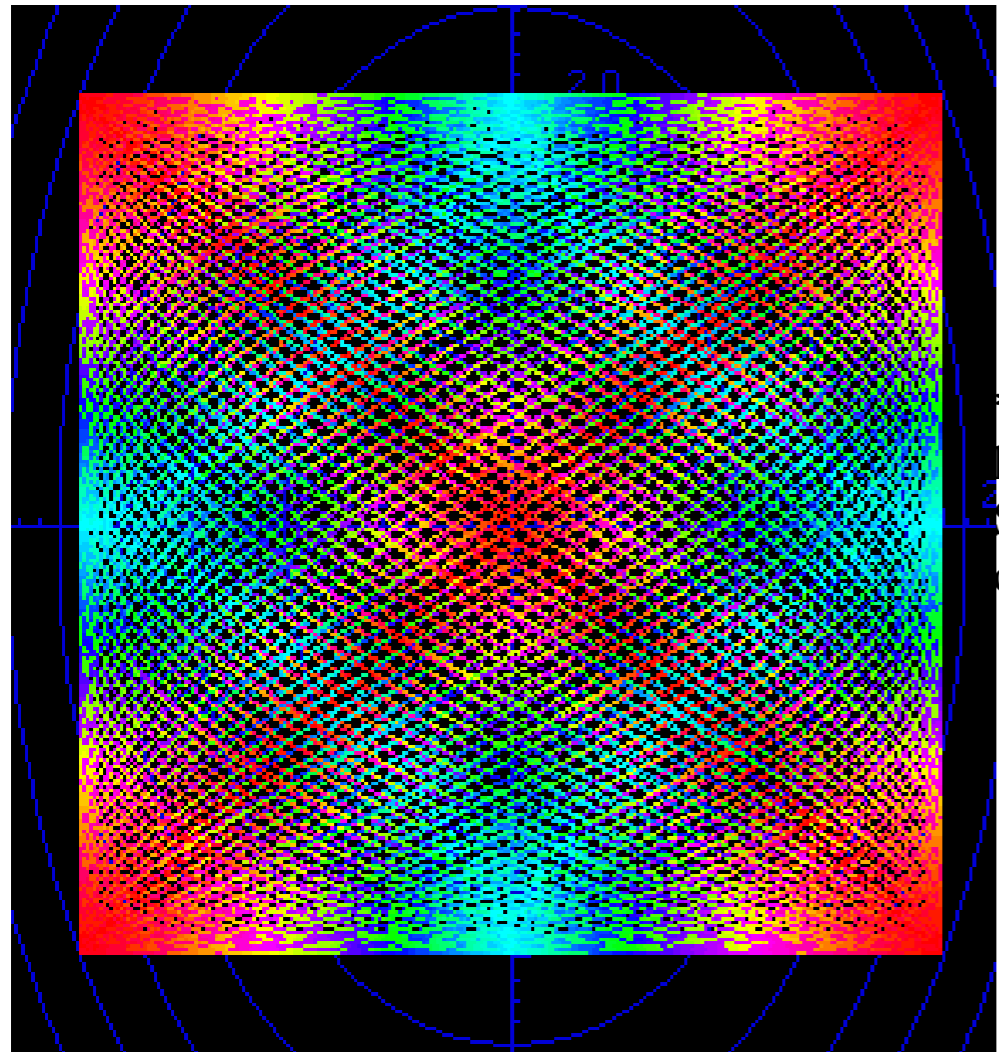
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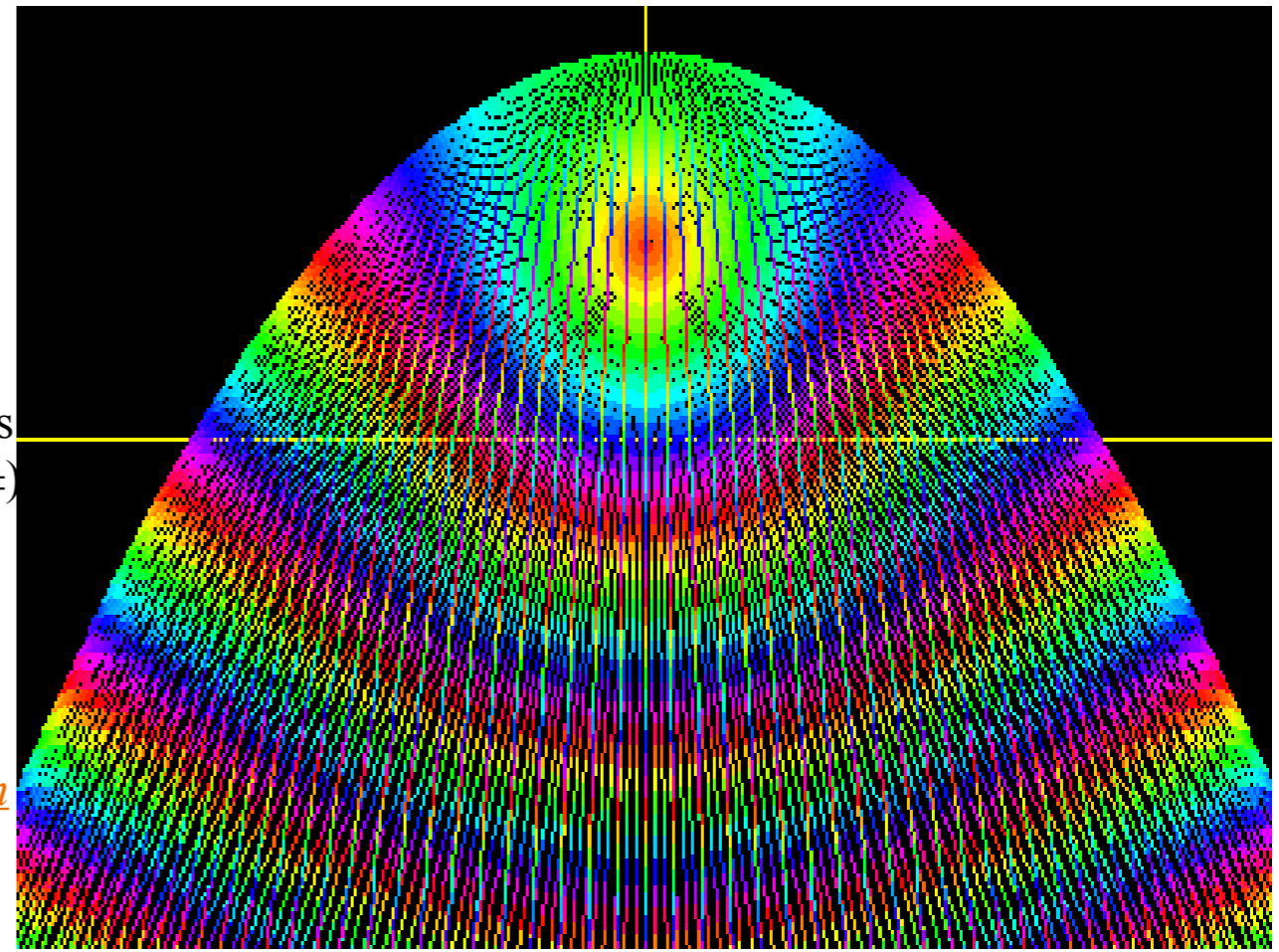
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Simulation by "Color U(2)"
 Unit 1 Fig. 12.13
 *closed system has quantized E
 Standing wave has only two phases(\pm) cyan and red
 Unit 1 Fig. 12.14
Web Simulation by "CoulIt"

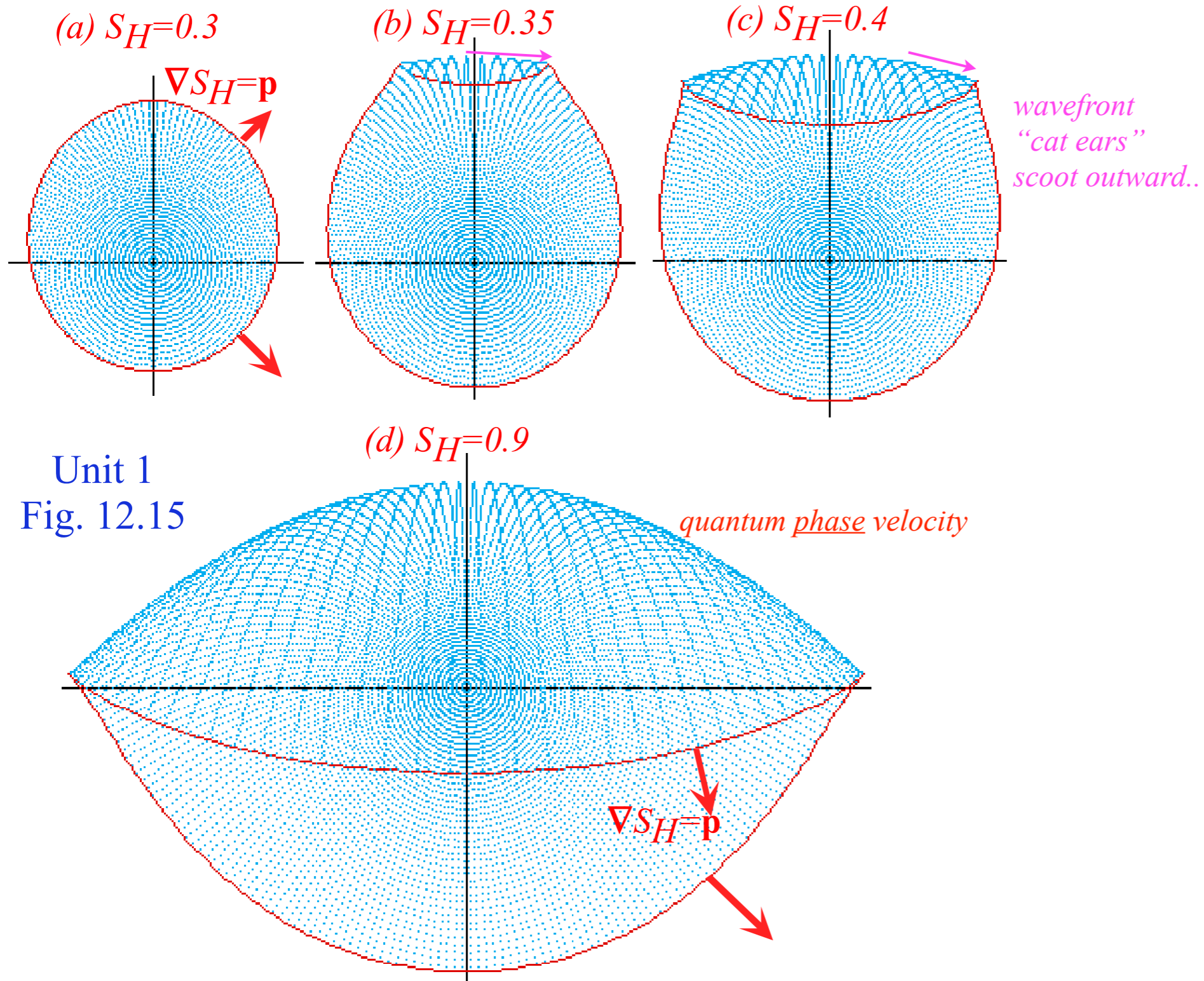
*open system has continuous energy



A moving wave has a *quantum phase velocity* found by setting $S = \text{const.}$ or $dS(0,0:r,t) = 0 = \mathbf{p} \cdot d\mathbf{r} - H dt$.

$$\mathbf{v}_{\text{phase}} = \frac{d\mathbf{r}}{dt} = \frac{H}{\mathbf{p}} = \frac{\omega}{\mathbf{k}}$$

Quantum "phase wavefronts"



Unit 1
Fig. 12.15

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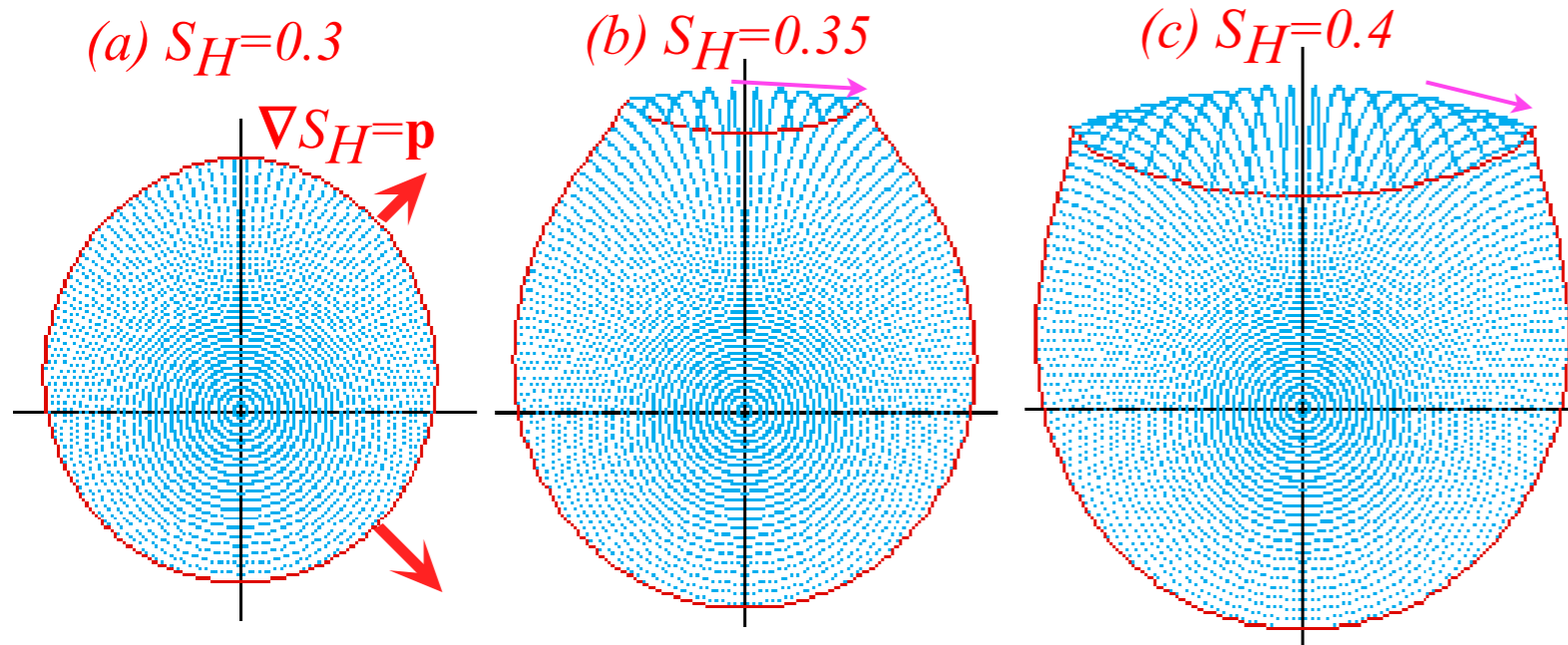
$$\mathbf{V}_{phase} = \frac{d\mathbf{r}}{dt} = \frac{H}{\mathbf{p}} = \frac{\omega}{\mathbf{k}}$$

This is quite the opposite of classical particle velocity which is *quantum group velocity*.

$$\mathbf{V}_{group} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\partial \omega}{\partial \mathbf{k}}$$

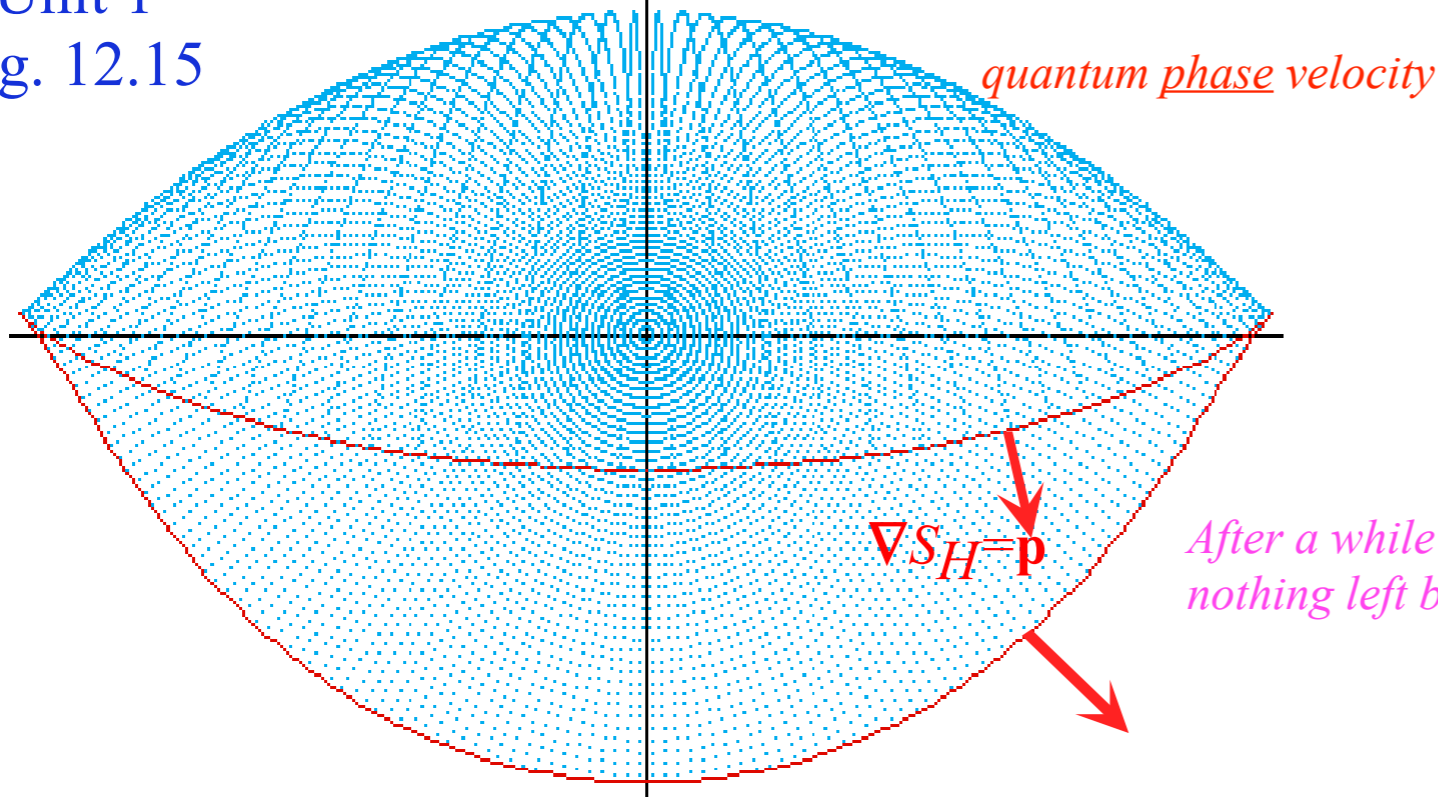
Note: This is Hamilton's 1st Equation

Quantum "phase wavefronts"



wavefront
"cat ears"
scoot outward..

(d) $S_H=0.9$



After a while ...
nothing left but a smile!

Unit 1
Fig. 12.15

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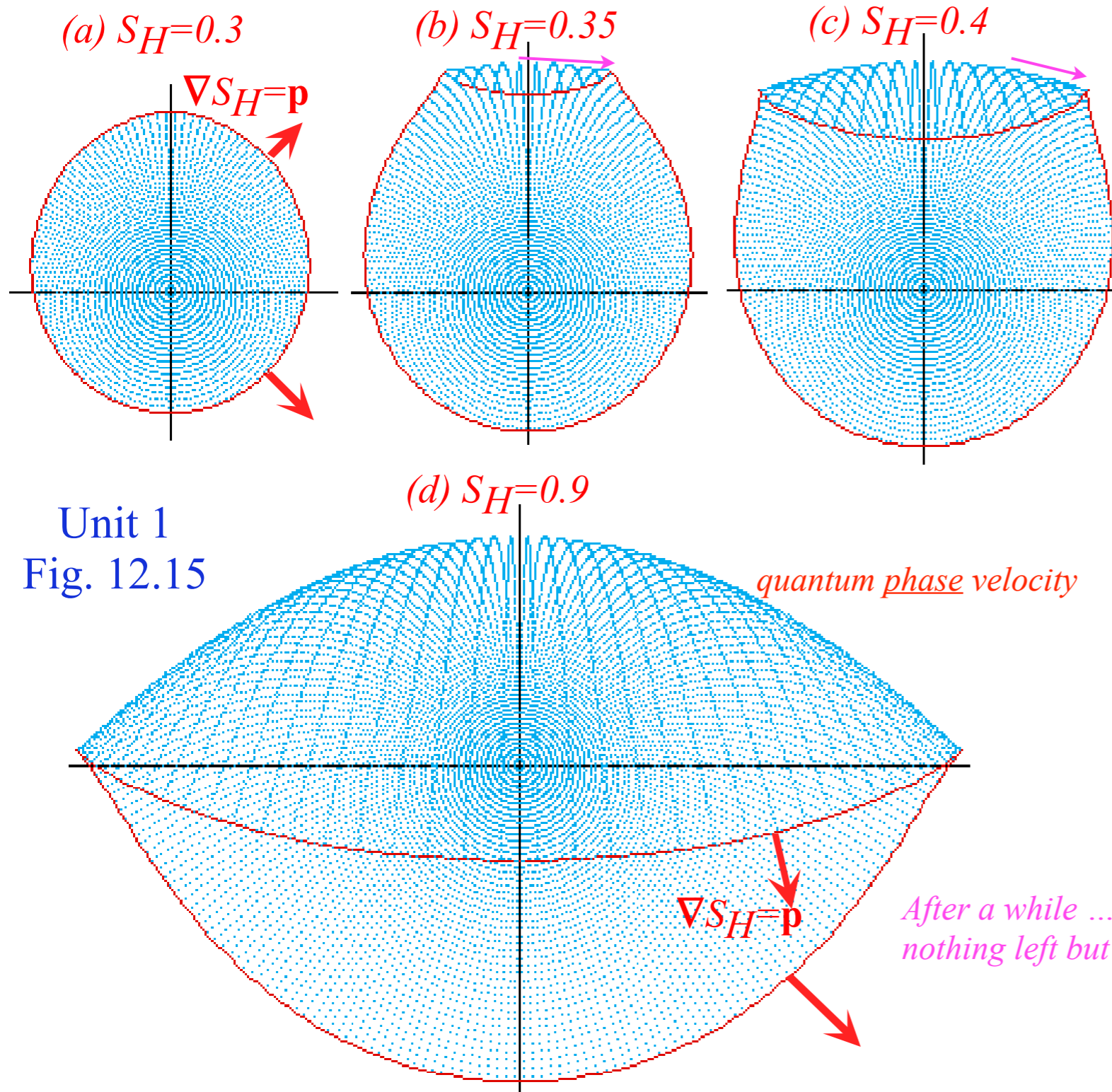
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Quantum "phase wavefronts"



wavefront
"cat ears"
scoot outward..



16th Century carving on St. Wifred's in Grappenhall



...on St. Nicolas



From *Alice's Adventures in Wonderland* by Lewis Carroll (1865)

Unit 1
Fig. 12.15

After a while ...
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This is quite the opposite of classical particle velocity which is *quantum group velocity*.

$$\mathbf{V}_{group} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\partial \omega}{\partial \mathbf{k}}$$

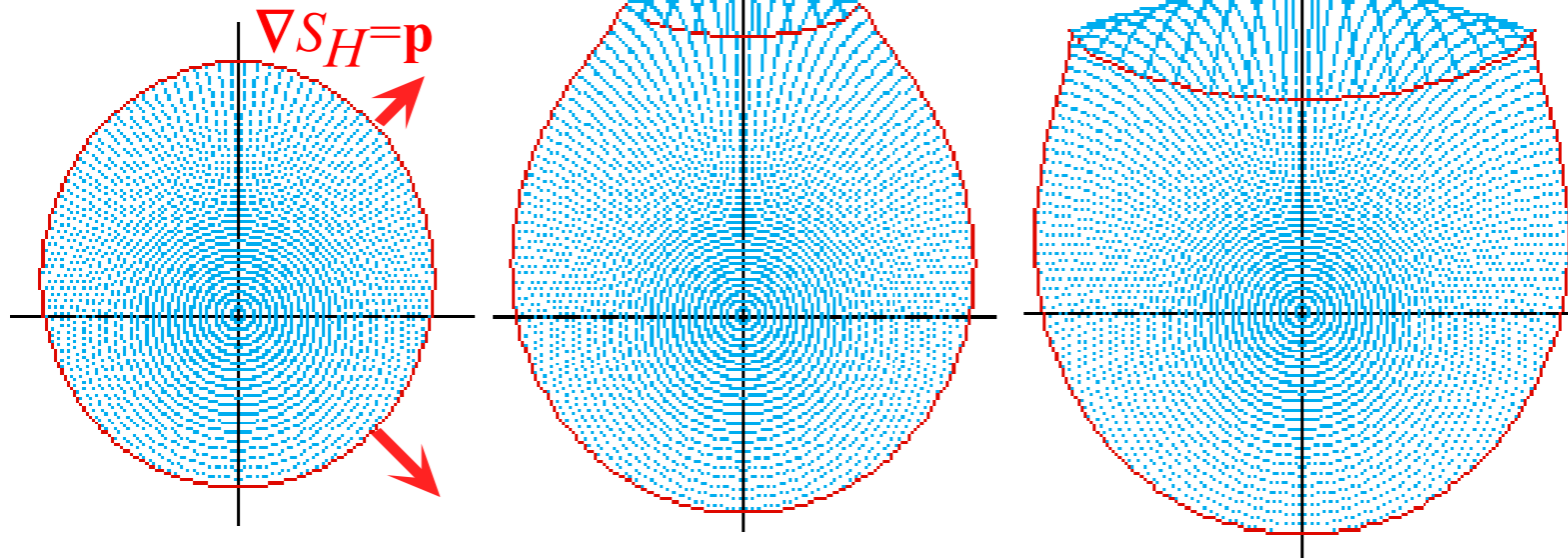
Note: This is Hamilton's 1st Equation

Quantum "phase wavefronts"

(a) $S_H=0.3$

(b) $S_H=0.35$

(c) $S_H=0.4$



(d) $S_H=0.9$

higher V_{phase} up here

quantum phase velocity

Unit 1
Fig. 12.15

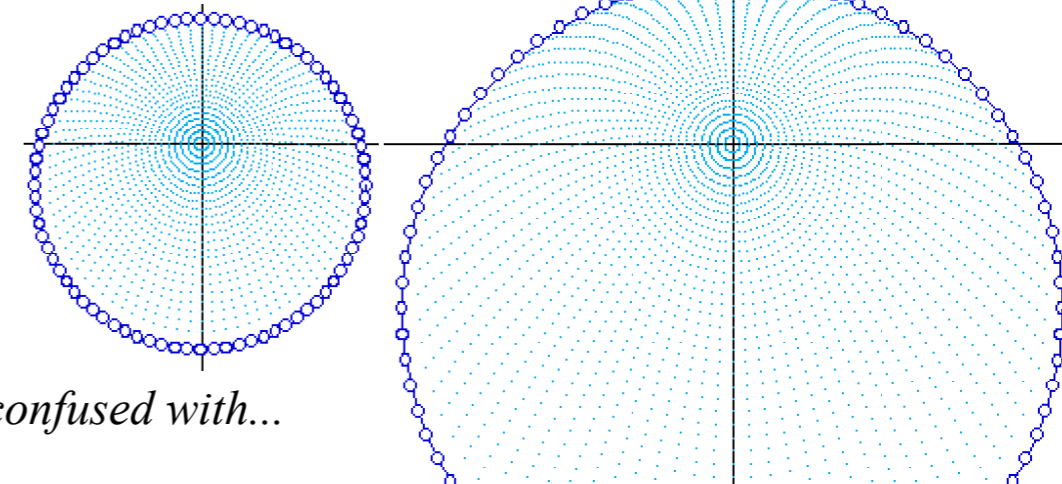
lower V_{phase} down here

$\nabla S_H = \mathbf{p}$

Classical "blast wavefronts"

(a) $T=0.4$

(b) $T=1.0$



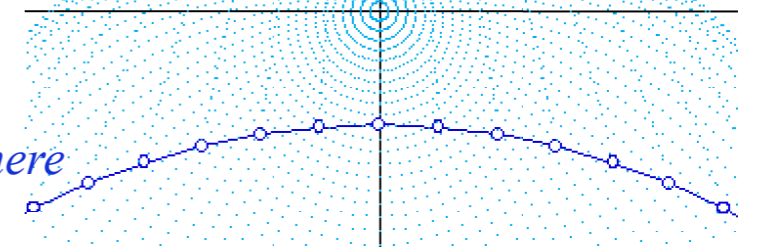
...not to be confused with...

...quantum group velocity...
that is classical particle velocity

(c) $T=2.3$

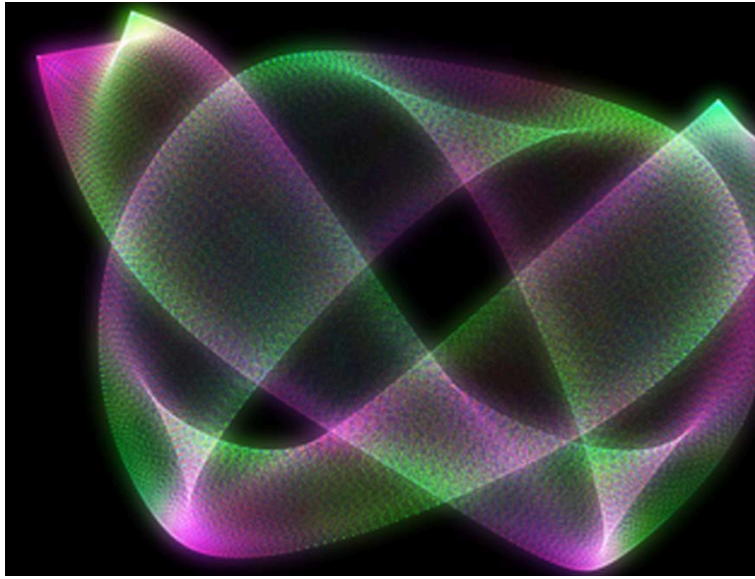
lower V_{group} up here

higher V_{group} down here



Check out the Heller Galleries

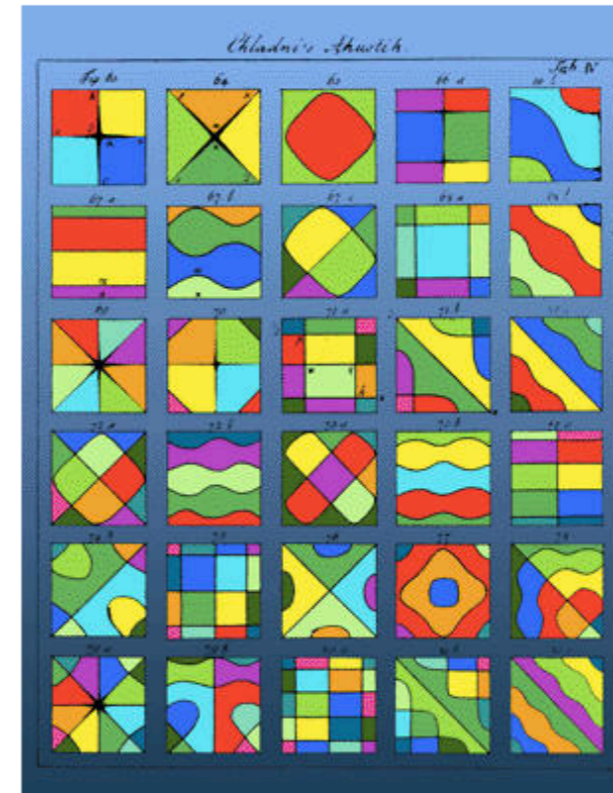
<http://jalbum.net/en/browse/user/album/1696720>



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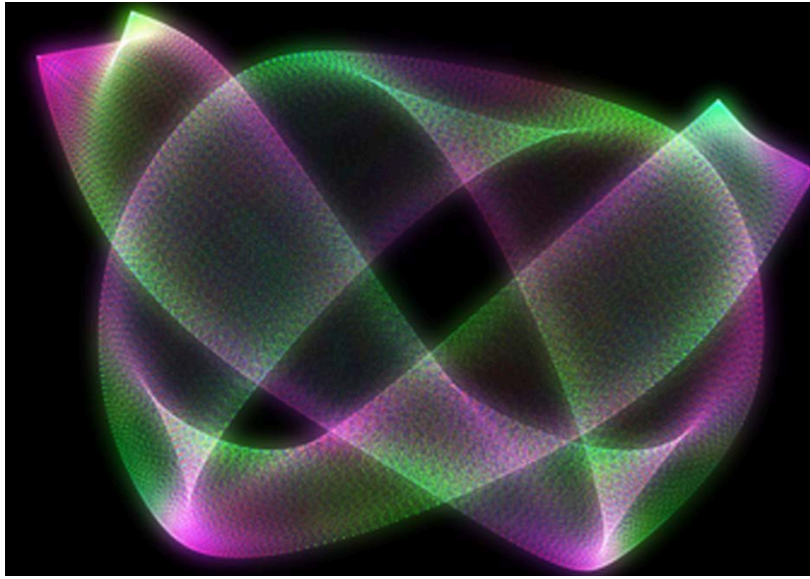
[Chladni](#)



The diagrams of Ernst Chladni (1756-1827) are the scientific, artistic, and even the sociological birthplace of the modern field of wave physics and quantum chaos. Educated in Law at the University of Leipzig, and an amateur musician, Chladni soon followed his love of science and wrote one of the first treatises on acoustics, "Discovery of the Theory of Pitch". Chladni had an inspired idea: to make waves in a solid material visible. This he did by getting metal plates to vibrate, stroking them with a violin bow. Sand or a similar substance spread on the surface of the plate naturally settles to the places where the metal vibrates the least, making such places visible. These places are the so-called nodes, which are wavy lines on the surface. The plates vibrate at pure, audible pitches, and each pitch has a unique nodal pattern. Chladni took the trouble to carefully diagram the patterns, which helped to popularize his work. Then he hit the lecture circuit, fascinating audiences in Europe with live demonstrations. This culminated with a command performance for Napoleon, who was so impressed that he offered a prize to anyone who could explain the patterns. More than that, according to Chladni himself, Napoleon remarked that irregularly shaped plate would be much harder to understand! While this was surely also known to Chladni, it is remarkable that Napoleon had this insight. Chladni received a sum of 6000 francs from Napoleon, who also offered 3000 francs to anyone who could explain the patterns. The mathematician Sophie Germain took the prize in 1816, although her solutions were not completed until the work of Kirchoff thirty years later. Even so, the patterns for irregular shapes remained (and to some extent remains) unexplained. Government funding of waves research goes back a long way! (Chladni was also the first to maintain that meteorites were extraterrestrial; before that, the popular theory was that they were of volcanic origin.) One of his diagrams is the basis for image, which is a playfully colored version of Chladni's original line drawing. Chladni's original work on waves confined to a region was followed by equally remarkable progress a few years later.

Check out the Heller Galleries

<http://jalbum.net/en/browse/user/album/1696720>



National Science Foundation (NSF)
Arlington, VA

September-November 2002

Selected images.

http://search.nsf.gov/search?ie=&site=nsf&output=xml_no_dtd&proxyreload=1&client=nsf&lr=&proxystylesheet=http%3A%2F%2Fwww.nsf.gov%2Fsearch%2Fnsf_new.xslt&oe=&btnG.x=0&btnG.y=0&q=eric+heller

University Museum, University of Arkansas, Fayetteville, AK

October 2002 - December 2002

"Approaching Chaos: Visions from the Quantum Frontier"

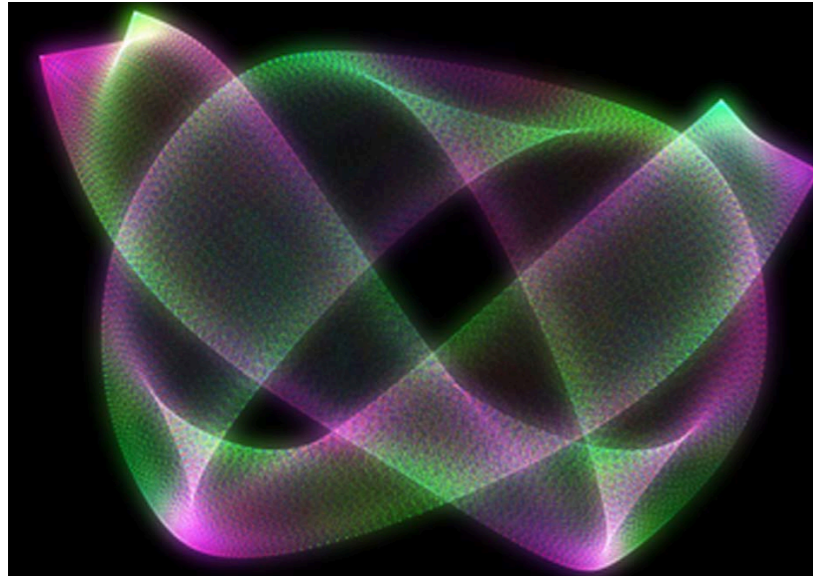
Approaching Chaos is supported by a grant from the National Science Foundation and by MIT Museum and the Center for Theoretical Physics at the Massachusetts Institute of Technology.



[Bessel 21](#)

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University Museum, University of Arkansas, Fayetteville, AK**

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"Approaching Chaos: Visions from the Quantum Frontier"

Approaching Chaos is supported by a grant from the National Science Foundation and by MIT Museum and the Center for Theoretical Physics at the Massachusetts Institute of Technology.

**UAF Museum closed after this exhibit*



Lecture 11 ends here
Thur. 9.27.2016

[Bessel 21](#)