Lecture 16
Thur. 10.13.2016

Hamilton Equations for Trebuchet and Other Things
(Ch. 5-9 of Unit 2)

Review of Hamiltonian equation derivation (Elementary trebuchet)
- Hamiltonian definition from Lagrangian and γ_{mn} tensor
- Hamilton's equations and Poincare invariant relations
- Hamiltonian expression and contravariant γ^{mn} tensor

Hamiltonian energy and momentum conservation and symmetry coordinates
- Coordinate transformation helps reduce symmetric Hamiltonian
- Free-space trebuchet kinematics by symmetry
  - Algebraic approach
  - Direct approach and Superball analogy
- Trebuchet vs Flinger and sports kinematics
  - The multiple approaches to Mechanics

Chapter 1. The Trebuchet: A dream problem for Galileo?

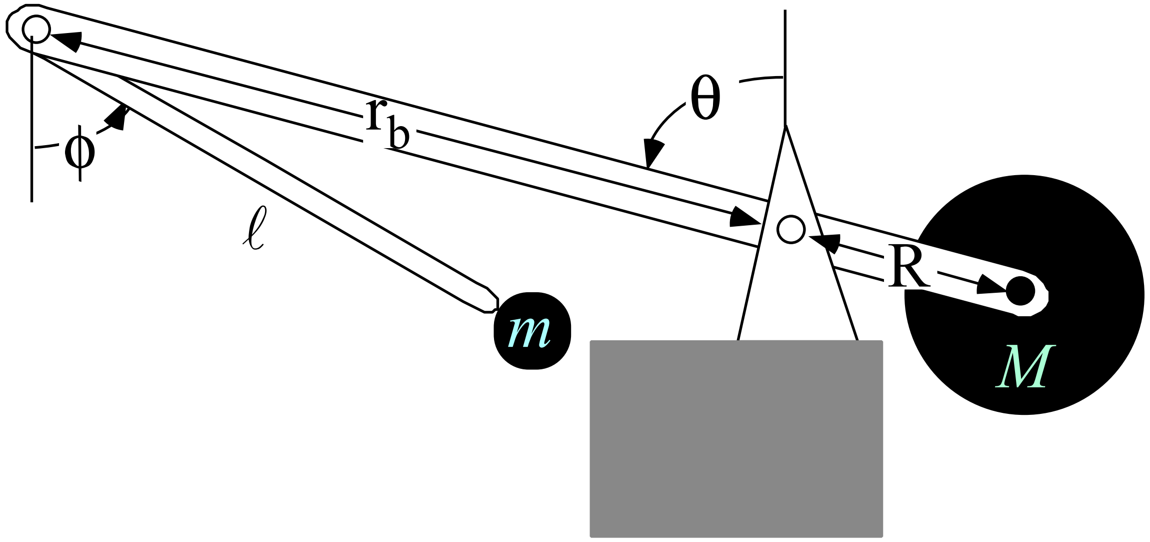
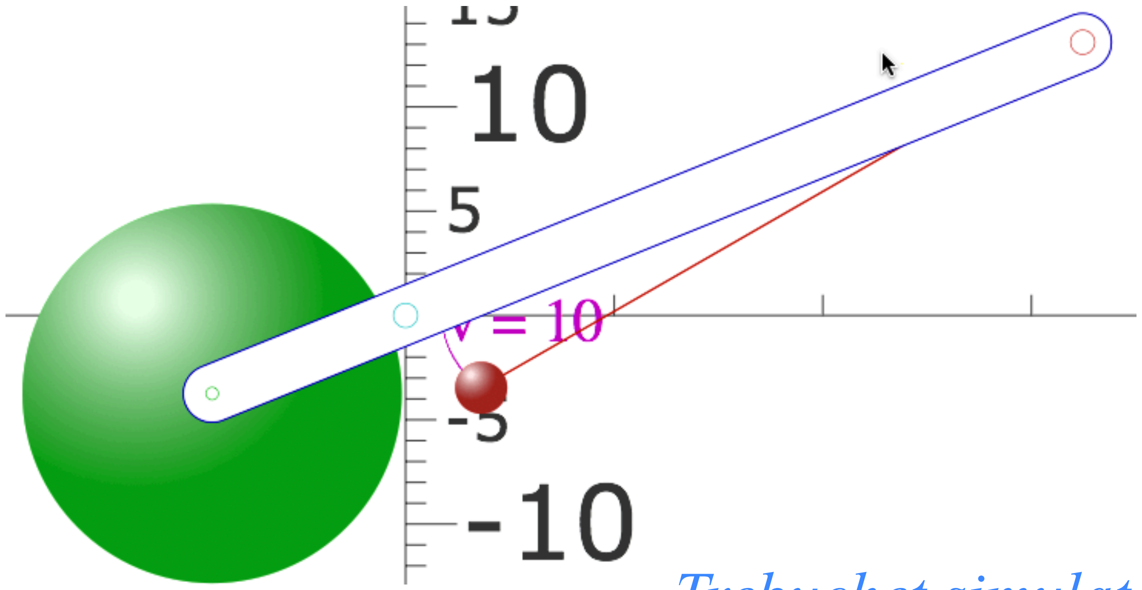
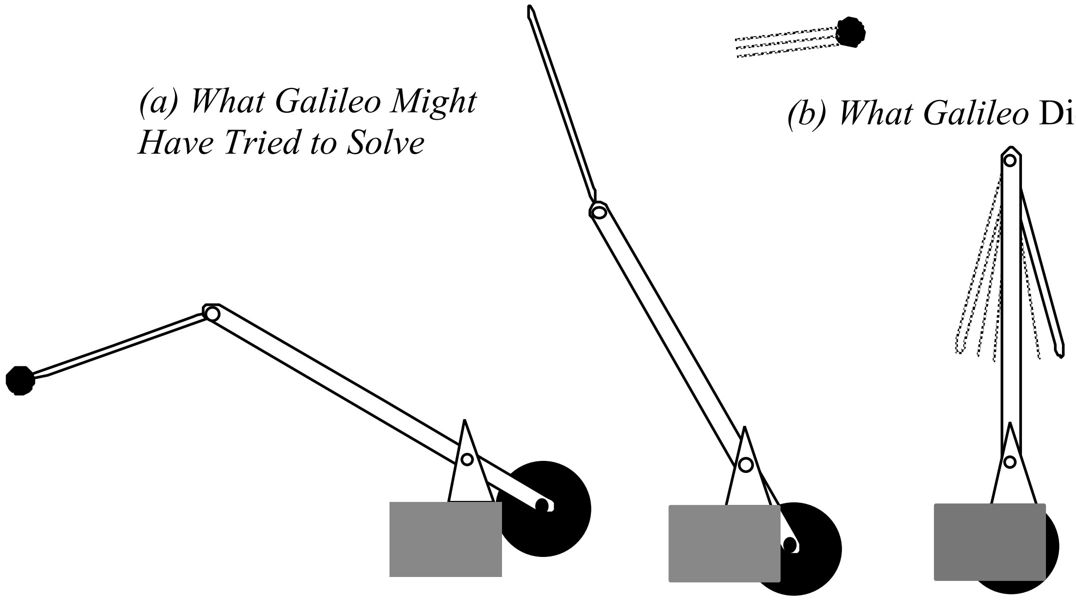


Fig. 2.1.1 An elementary ground-fixed trebuchet



Trebuchet simulator

http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html

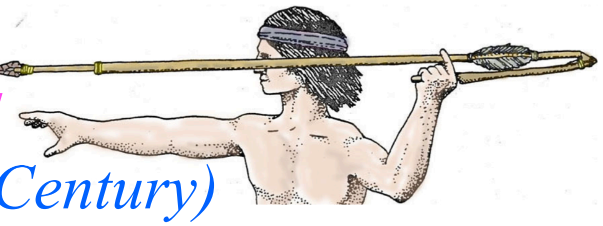


(a) What Galileo Might Have Tried to Solve

(b) What Galileo Did *Solve*

(Simple pendulum dynamics)

Fig. 2.1.2 Galileo's (supposed) problem



The Atlatl

(Cahokia, IL 12th Century)

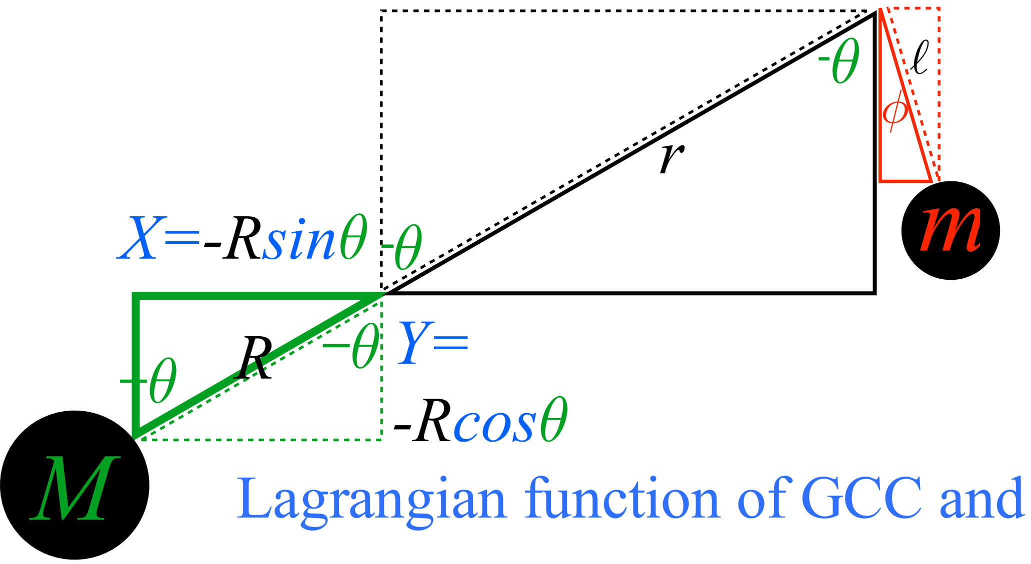
Review of Hamiltonian equation derivation (Elementary trebuchet)

→ *Hamiltonian definition from Lagrangian and* γ_{mn} *tensor*

Hamilton's equations and Poincare invariant relations

Hamiltonian expression and contravariant γ^{mn} *tensor*

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2 \right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta - \phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$



$$T = \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta - \phi) \\ -mr\ell\cos(\theta - \phi) & m\ell^2 \end{pmatrix}}_{} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Dynamic metric tensor

γ_{mn}

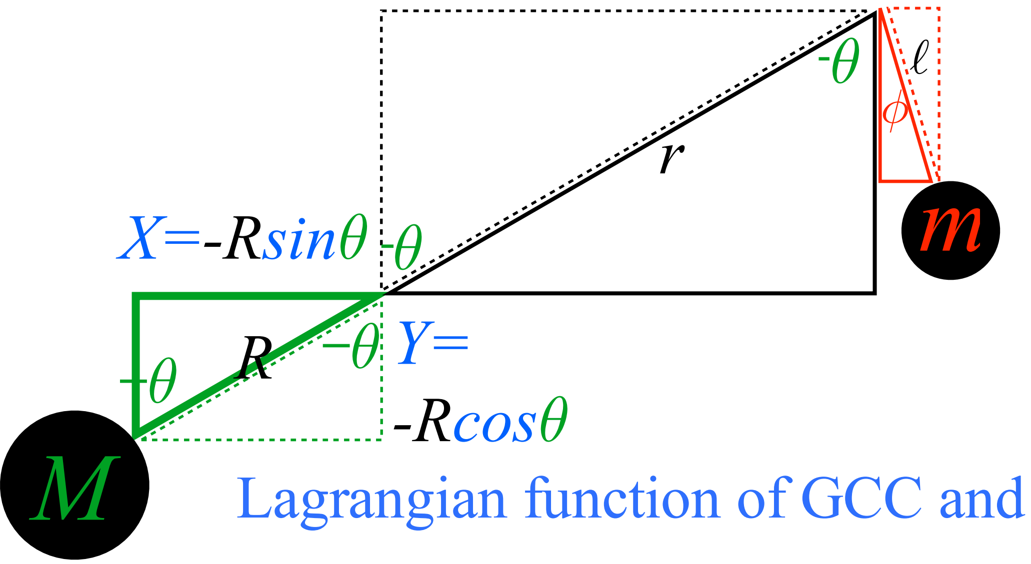
in GCC θ and ϕ

Lagrangian function of GCC and velocities: $L(\theta,\phi,\dot{\theta},\dot{\phi},t) = T(\theta,\phi,\dot{\theta},\dot{\phi},t) - V(\theta,\phi,t)$

$$dL(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{\partial L}{\partial \theta}d\theta + \frac{\partial L}{\partial \phi}d\phi + \frac{\partial L}{\partial \dot{\theta}}d\dot{\theta} + \frac{\partial L}{\partial \dot{\phi}}d\dot{\phi} + \frac{\partial L}{\partial t}dt$$

1st differential chain

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2 \right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$



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velocity chain

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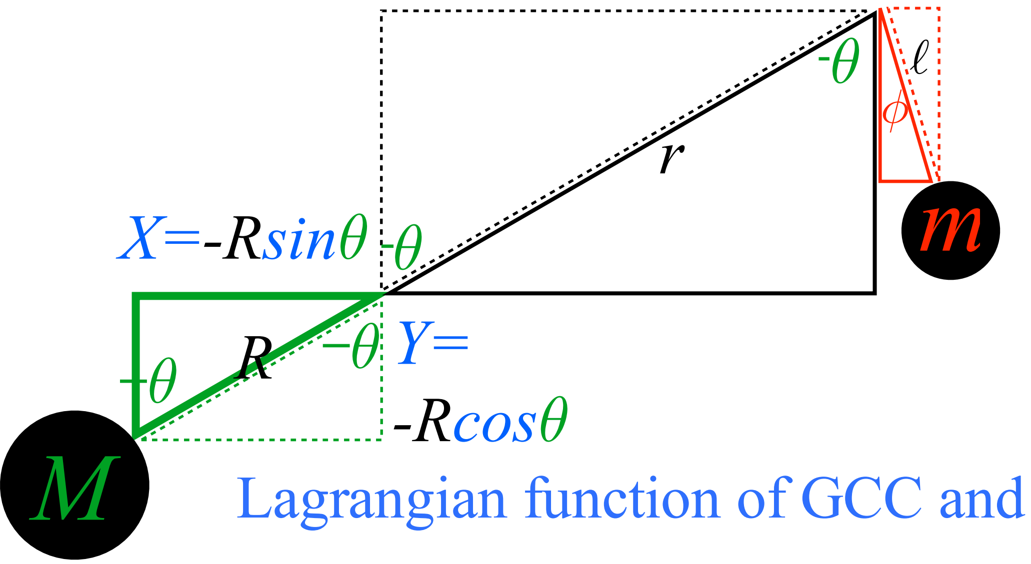

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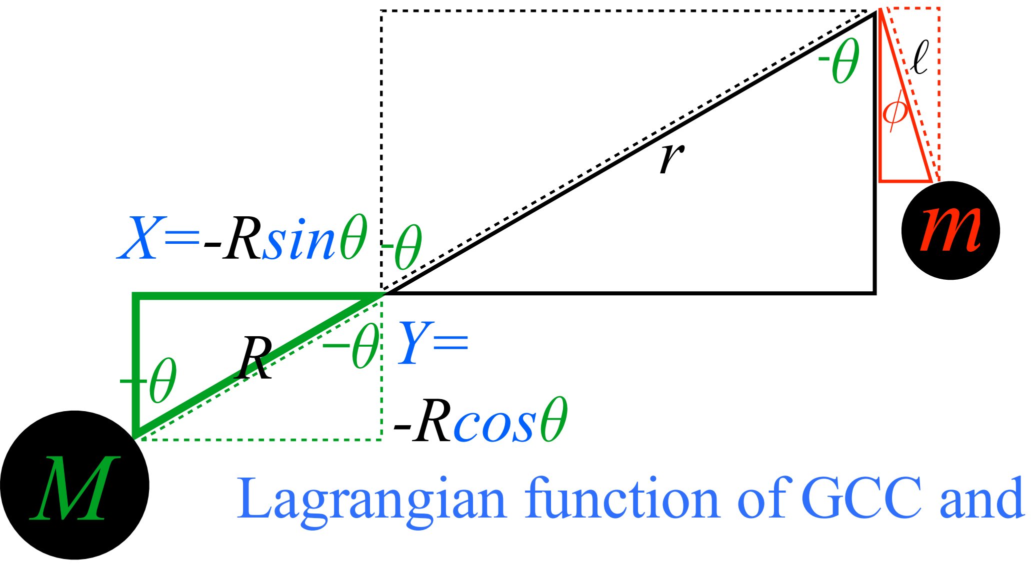

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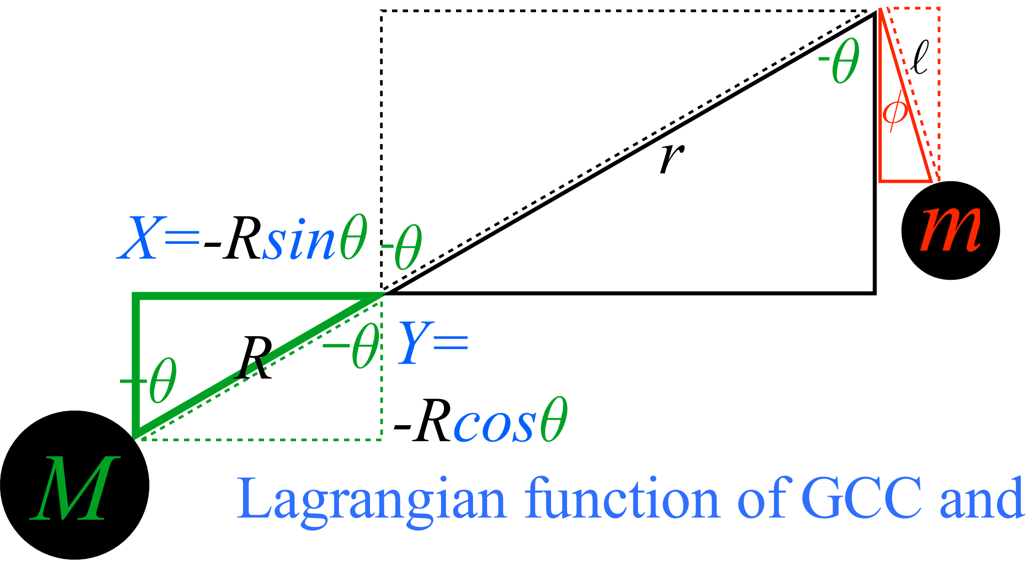

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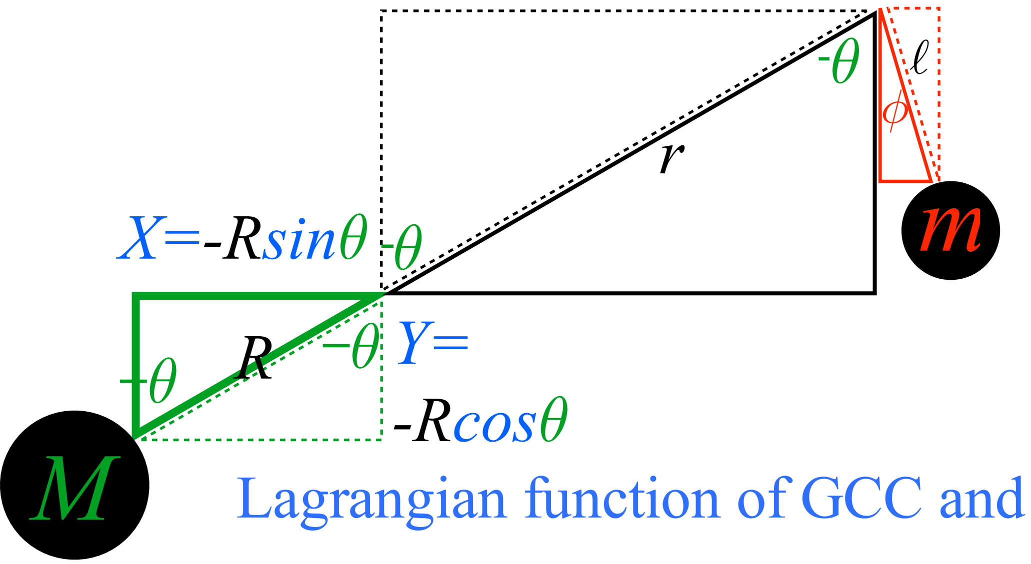

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(Consolidating)

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$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

Defining the Hamiltonian function

Hamiltonian function of GCC and momenta: $H(\theta,\phi,p_\theta,p_\phi,t) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$

Review of Hamiltonian equation derivation (Elementary trebuchet)

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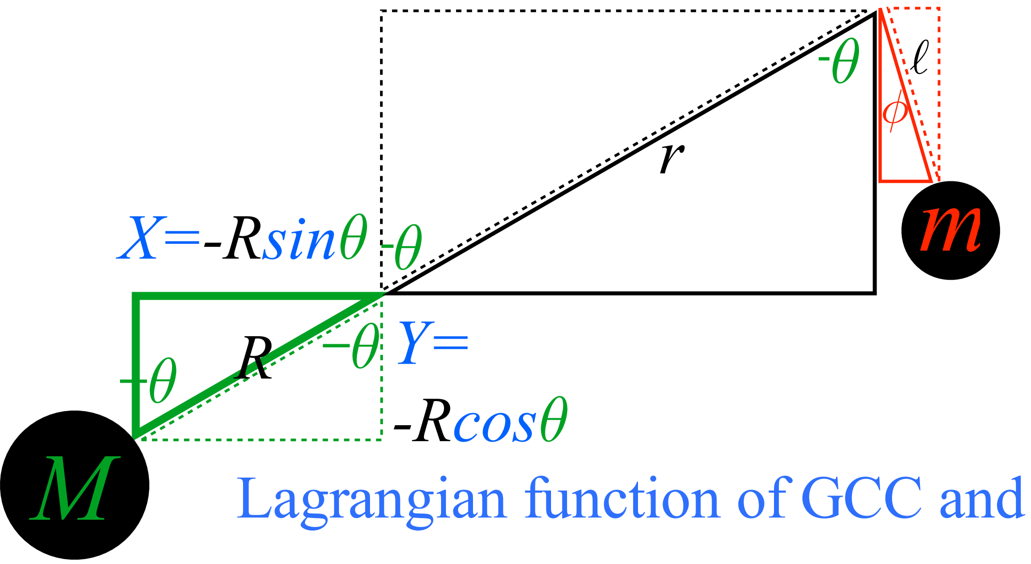

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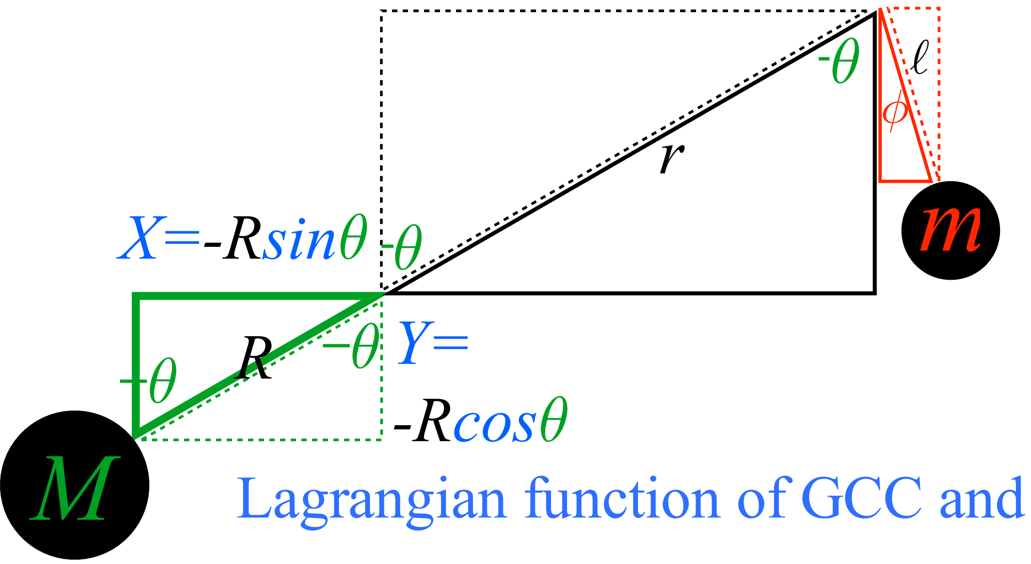

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by assumed Lagrange functionality

$$\frac{\partial H}{\partial\theta} = -\frac{\partial L}{\partial\theta} = -\dot{p}_\theta$$

by Lagrange equations

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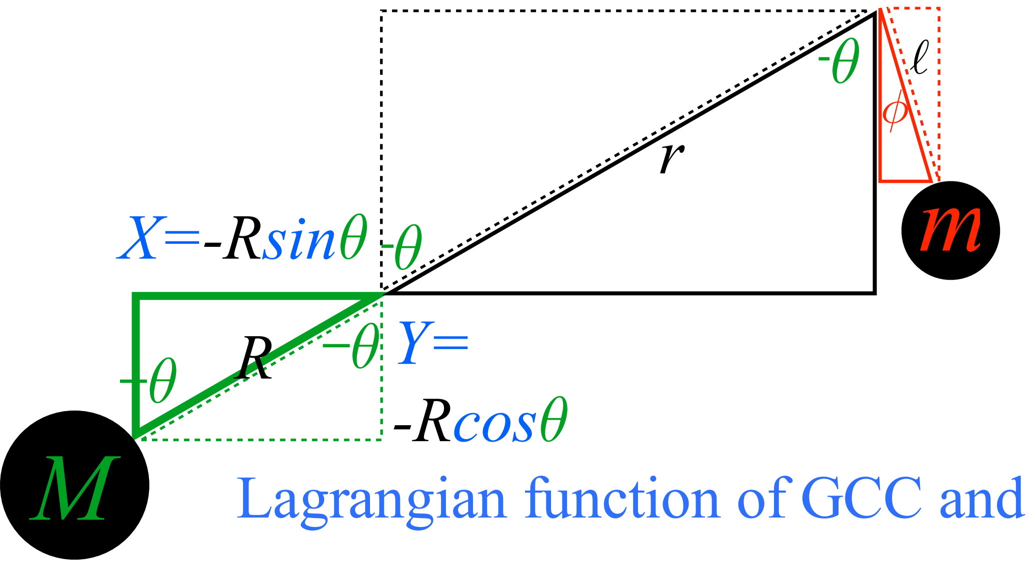


$$T = \frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix}$$

Dynamic metric tensor

γ_{mn}

in GCC θ and ϕ

Lagrangian function of GCC and velocities: $L(\theta,\phi,\dot{\theta},\dot{\phi},t) = T(\theta,\phi,\dot{\theta},\dot{\phi},t) - V(\theta,\phi,t)$

$$dL(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{\partial L}{\partial\theta}d\theta + \frac{\partial L}{\partial\phi}d\phi + \frac{\partial L}{\partial\dot{\theta}}d\dot{\theta} + \frac{\partial L}{\partial\dot{\phi}}d\dot{\phi} + \frac{\partial L}{\partial t}dt$$

$$\dot{L}(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{dL}{dt} = \frac{\partial L}{\partial\theta}\frac{d\theta}{dt} + \frac{\partial L}{\partial\phi}\frac{d\phi}{dt} + \frac{\partial L}{\partial\dot{\theta}}\frac{d\dot{\theta}}{dt} + \frac{\partial L}{\partial\dot{\phi}}\frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

velocity chain

$$\dot{L}(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{dL}{dt} = \overset{\downarrow}{\dot{p}_\theta}\frac{d\theta}{dt} + \overset{\downarrow}{\dot{p}_\phi}\frac{d\phi}{dt} + \overset{\downarrow}{p_\theta}\frac{d\dot{\theta}}{dt} + \overset{\downarrow}{p_\phi}\frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

Lagrange equations

$$= \frac{dL}{dt} = \frac{d}{dt}\left(p_\theta\dot{\theta} + p_\phi\dot{\phi}\right) + \frac{\partial L}{\partial t}$$

(Consolidating)

$$\frac{d}{dt}\underbrace{\left(p_\theta\dot{\theta} + p_\phi\dot{\phi} - L\right)} = -\frac{\partial L}{\partial t}$$

(Rearranging)

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

Defining the Hamiltonian function

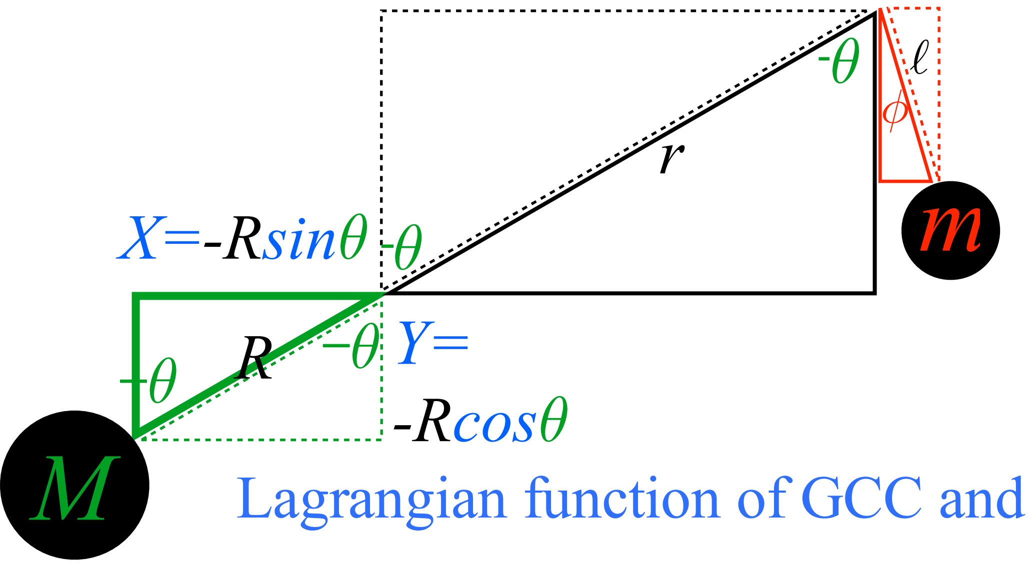
Hamiltonian function of GCC and momenta: $H(\theta,\phi,p_\theta,p_\phi,t) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$

by assumed Lagrange functionality

$$\frac{\partial H}{\partial\theta} \overset{\downarrow}{=} -\frac{\partial L}{\partial\theta} \overset{\uparrow}{=} -\dot{p}_\theta \qquad \frac{\partial H}{\partial p_\theta} \overset{\downarrow}{=} \dot{\theta} - \cancel{\frac{\partial L}{\partial p_\theta}} \overset{\uparrow}{=} \dot{\theta}$$

by Lagrange equations

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2\right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$



$$T = \frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

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Dynamic metric tensor

γ_{mn}

in GCC θ and ϕ

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velocity chain

$$\dot{L}(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{dL}{dt} = \frac{\partial L}{\partial\theta}\frac{d\theta}{dt} + \frac{\partial L}{\partial\phi}\frac{d\phi}{dt} + \frac{\partial L}{\partial\dot{\theta}}\frac{d\dot{\theta}}{dt} + \frac{\partial L}{\partial\dot{\phi}}\frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

Lagrange equations

$$\dot{L}(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{dL}{dt} = \overset{\downarrow}{\dot{p}_\theta}\frac{d\theta}{dt} + \overset{\downarrow}{\dot{p}_\phi}\frac{d\phi}{dt} + \overset{\downarrow}{p_\theta}\frac{d\dot{\theta}}{dt} + \overset{\downarrow}{p_\phi}\frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

(Consolidating)

$$= \frac{dL}{dt} = \frac{d}{dt}\left(p_\theta\dot{\theta} + p_\phi\dot{\phi}\right) + \frac{\partial L}{\partial t}$$

(Rearranging)

$$\frac{d}{dt}\underbrace{\left(p_\theta\dot{\theta} + p_\phi\dot{\phi} - L\right)} = -\frac{\partial L}{\partial t}$$

Defining the Hamiltonian function

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

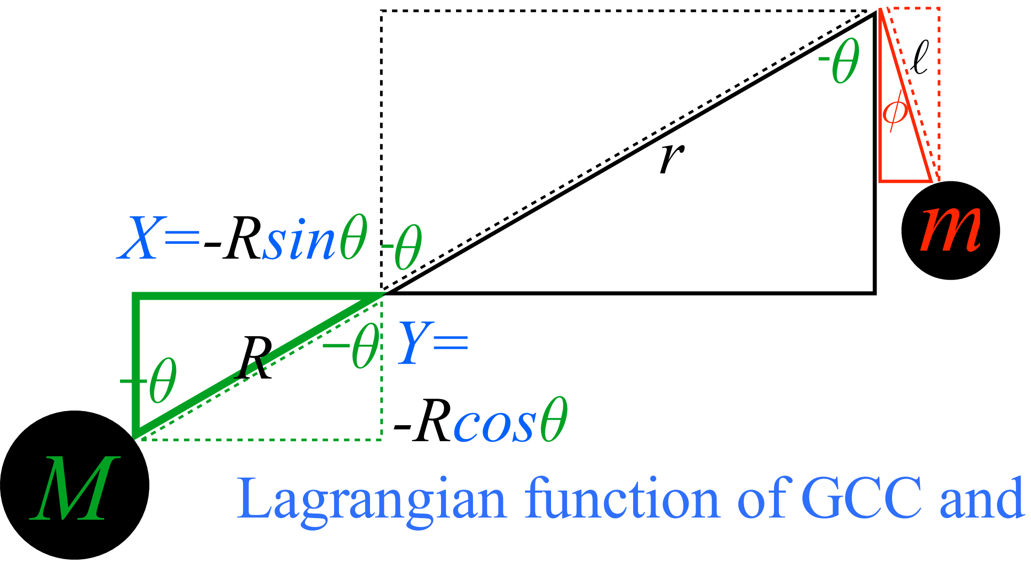
Hamiltonian function of GCC and momenta: $H(\theta,\phi,p_\theta,p_\phi,t) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$

by assumed Lagrange functionality

$$\frac{\partial H}{\partial\theta} \overset{\downarrow}{=} -\frac{\partial L}{\partial\theta} = -\dot{p}_\theta \qquad \frac{\partial H}{\partial p_\theta} \overset{\downarrow}{=} \dot{\theta} - \cancel{\frac{\partial L}{\partial p_\theta}} = \dot{\theta} \qquad \frac{\partial H}{\partial\dot{\theta}} \overset{\downarrow}{=} p_\theta - \frac{\partial L}{\partial\dot{\theta}} = 0$$

by Lagrange equations

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2\right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$



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in GCC θ and φ

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velocity chain

$$\dot{L}\left(\theta,\phi,\dot{\theta},\dot{\phi},t\right) = \frac{dL}{dt} = \overset{\downarrow}{\dot{p}_\theta}\frac{d\theta}{dt} + \overset{\downarrow}{\dot{p}_\phi}\frac{d\phi}{dt} + \overset{\downarrow}{p_\theta}\frac{d\dot{\theta}}{dt} + \overset{\downarrow}{p_\phi}\frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

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(Consolidating)

$$\frac{d}{dt}\underbrace{\left(p_\theta\dot{\theta} + p_\phi\dot{\phi} - L\right)} = -\frac{\partial L}{\partial t}$$

(Rearranging)

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

Defining the Hamiltonian function

Hamiltonian function of GCC and momenta: $H\left(\theta,\phi,p_\theta,p_\phi,t\right) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$

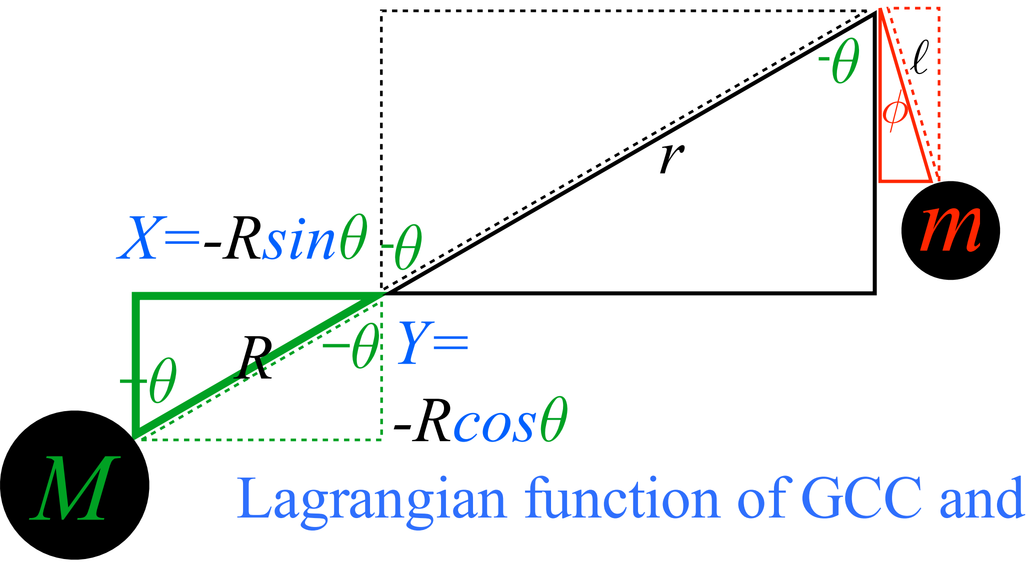
by assumed Lagrange functionality

$$\frac{\partial H}{\partial\theta} \overset{\downarrow}{=} -\dot{p}_\theta \qquad \frac{\partial H}{\partial p_\theta} \overset{\downarrow}{=} \dot{\theta} \qquad \cancel{\frac{\partial H}{\partial\dot{\theta}} \overset{\downarrow}{\equiv} 0} \qquad \frac{\partial H}{\partial\phi} = -\dot{p}_\phi \qquad \frac{\partial H}{\partial p_\phi} = \dot{\phi} \qquad \cancel{\frac{\partial H}{\partial\dot{\phi}} \equiv 0}$$

Hamilton's equations

by Lagrange equations

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2 \right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$



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Dynamic metric tensor

γ_{mn}

in GCC θ and ϕ

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Lagrange equations

$$= \frac{dL}{dt} = \frac{d}{dt}\left(p_\theta\dot{\theta} + p_\phi\dot{\phi}\right) + \frac{\partial L}{\partial t}$$

(Consolidating)

$$\frac{d}{dt}\underbrace{\left(p_\theta\dot{\theta} + p_\phi\dot{\phi} - L\right)}_{} = -\frac{\partial L}{\partial t}$$

(Rearranging)

$$\boxed{\frac{dH}{dt} = -\frac{\partial L}{\partial t}}$$

Defining the Hamiltonian function

Hamiltonian function of GCC and momenta: $\boxed{H(\theta,\phi,p_\theta,p_\phi,t) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L}$ *Poincare-Legendre relation*

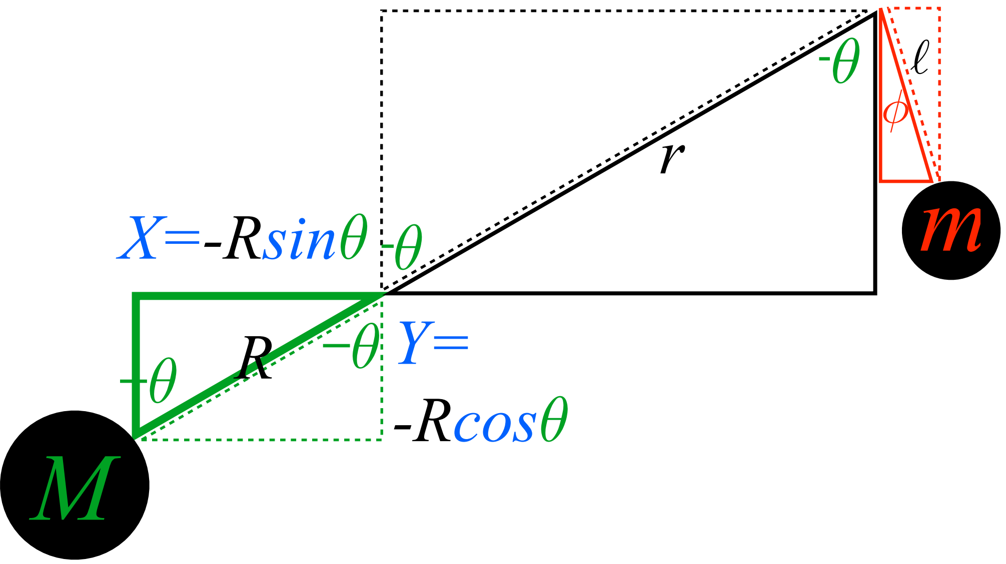
$$\boxed{\frac{\partial H}{\partial \theta} = -\dot{p}_\theta \qquad \frac{\partial H}{\partial p_\theta} = \dot{\theta} \qquad \cancel{\frac{\partial H}{\partial \dot{\theta}}} \equiv 0 \qquad \frac{\partial H}{\partial \phi} = -\dot{p}_\phi \qquad \frac{\partial H}{\partial p_\phi} = \dot{\phi} \qquad \cancel{\frac{\partial H}{\partial \dot{\phi}}} \equiv 0}$$

Hamilton's equations

Review of Hamiltonian equation derivation (Elementary trebuchet)

- *Hamiltonian definition from Lagrangian and* γ_{mn} *tensor*
- *Hamilton's equations and Poincare invariant relations*
- ➡ *Hamiltonian expression and contravariant* γ^{mn} *tensor*

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2 \right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$



$$T = \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta\theta} & \gamma_{\theta\phi} \\ \gamma_{\phi\theta} & \gamma_{\phi\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Covariant metric tensor

γ_{mn}

$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

Contravariant metric tensor

γ^{mn}

$$T = \frac{1}{2}\frac{\begin{pmatrix} p_\theta & p_\phi \end{pmatrix} \overbrace{\begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2 \end{pmatrix}} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi) \right]} = \frac{1}{2}\gamma^{mn}p_m p_n$$

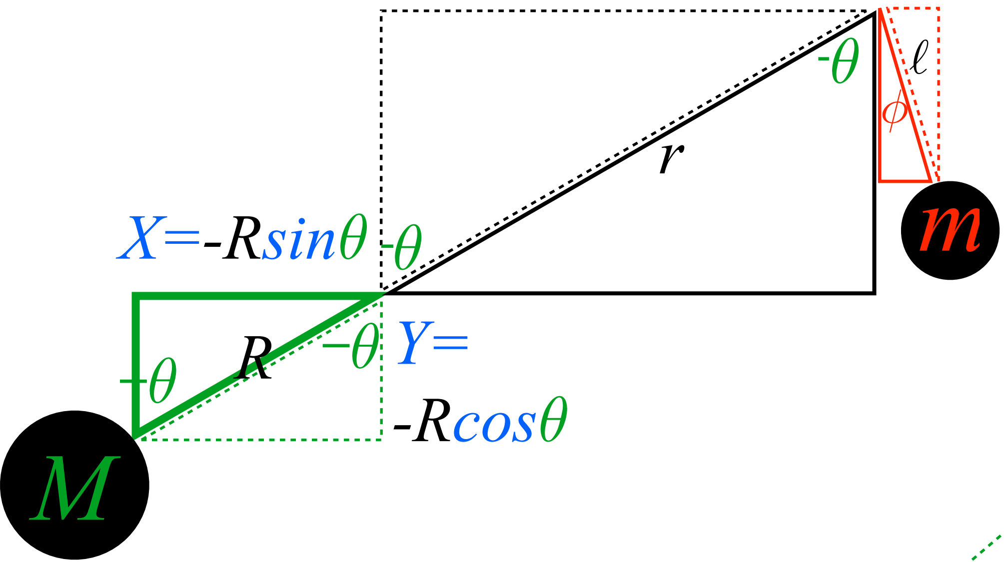
Lagrangian function of GCC and velocities: $L\left(\theta,\phi,\dot{\theta},\dot{\phi},t\right) = T\left(\theta,\phi,\dot{\theta},\dot{\phi},t\right) - V\left(\theta,\phi,t\right)$

Hamiltonian function of GCC and momenta: $H\left(\theta,\phi,p_\theta,p_\phi,t\right) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$ *Poincare-Legendre relation*

$$H = \quad p_\theta\dot{\theta} \quad + \quad p_\phi\dot{\phi} \quad - T \quad + V$$

$$H = \left(\gamma_{\theta\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\right)\dot{\theta} + \left(\gamma_{\phi\theta}\dot{\theta} + \gamma_{\phi\phi}\dot{\phi}\right)\dot{\phi} - \frac{1}{2}\left(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}\right) + V$$

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2 \right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$



$$T = \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta\theta} & \gamma_{\theta\phi} \\ \gamma_{\phi\theta} & \gamma_{\phi\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Covariant metric tensor γ_{mn}

$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

Contravariant metric tensor γ^{mn}

$$T = \frac{1}{2}\frac{\begin{pmatrix} p_\theta & p_\phi \end{pmatrix} \overbrace{\begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2 \end{pmatrix}} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi) \right]} = \frac{1}{2}\gamma^{mn}p_m p_n$$

Lagrangian function of GCC and velocities: $L\left(\theta,\phi,\dot{\theta},\dot{\phi},t\right) = T\left(\theta,\phi,\dot{\theta},\dot{\phi},t\right) - V\left(\theta,\phi,t\right)$

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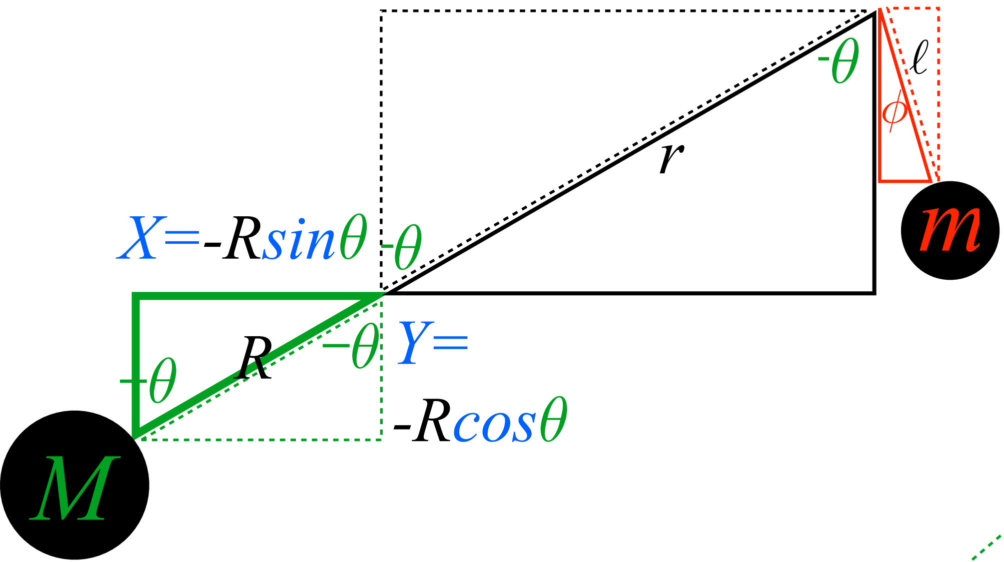
$$H = \quad p_\theta\dot{\theta} \quad + \quad p_\phi\dot{\phi} \qquad - T \qquad + V$$

$$H = \left(\gamma_{\theta\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\right)\dot{\theta} + \left(\gamma_{\phi\theta}\dot{\theta} + \gamma_{\phi\phi}\dot{\phi}\right)\dot{\phi} - \frac{1}{2}\left(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}\right) \; + V$$

$$H = \frac{1}{2}\left(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}\right) + V \qquad = T + V$$

(Only correct numerically!)

Total KE $= T=\frac{1}{2}\left[M\dot{X}^2+M\dot{Y}^2+m\dot{x}^2+m\dot{y}^2\right]=\frac{1}{2}\left[(MR^2+mr^2)\dot{\theta}^2-2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi}+m\ell^2\dot{\phi}^2\right]$



$$T=\frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\underbrace{\begin{pmatrix}MR^2+mr^2 & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}=\frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}=\begin{pmatrix}\gamma_{\theta\theta} & \gamma_{\theta\phi}\\ \gamma_{\phi\theta} & \gamma_{\phi\phi}\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

Covariant metric tensor γ_{mn}

$$\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}=\underbrace{\begin{pmatrix}\gamma^{\theta\theta} & \gamma^{\theta\phi}\\ \gamma^{\phi\theta} & \gamma^{\phi\phi}\end{pmatrix}}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}$$

Contravariant metric tensor γ^{mn}

$$T=\frac{1}{2}\frac{\begin{pmatrix}p_\theta & p_\phi\end{pmatrix}\begin{pmatrix}m\ell^2 & mr\ell\cos(\theta-\phi)\\ mr\ell\cos(\theta-\phi) & MR^2+mr^2\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}}{m\ell^2\left[MR^2+mr^2\sin^2(\theta-\phi)\right]}=\frac{1}{2}\gamma^{mn}p_m p_n$$

Lagrangian function of GCC and velocities: $L\left(\theta,\phi,\dot{\theta},\dot{\phi},t\right)=T\left(\theta,\phi,\dot{\theta},\dot{\phi},t\right)-V\left(\theta,\phi,t\right)$

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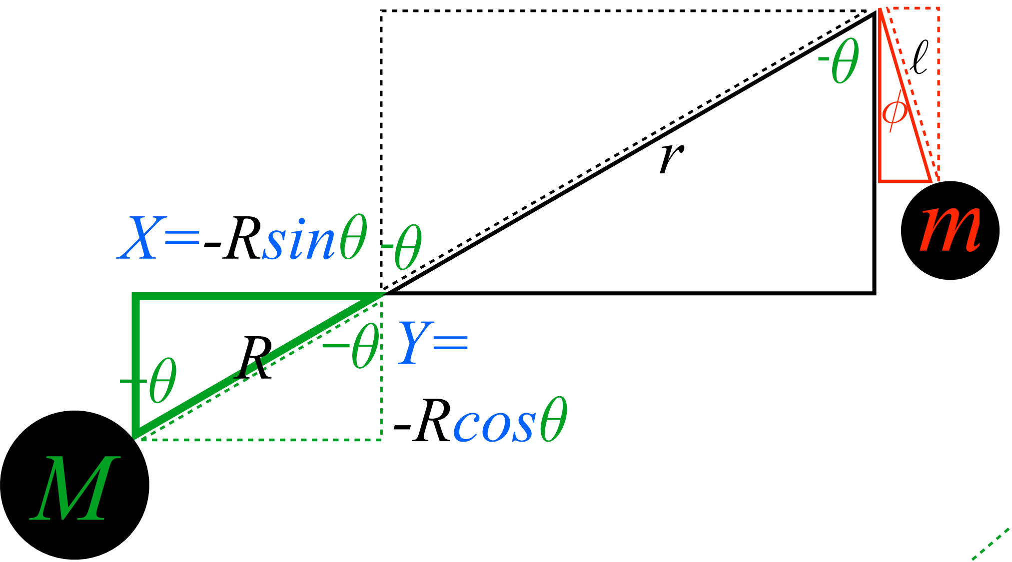
$$H=\left(\gamma_{\theta\theta}\dot{\theta}+\gamma_{\theta\phi}\dot{\phi}\right)\dot{\theta}+\left(\gamma_{\phi\theta}\dot{\theta}+\gamma_{\phi\phi}\dot{\phi}\right)\dot{\phi}-\frac{1}{2}\left(\gamma_{\theta\theta}\dot{\theta}\dot{\theta}+\gamma_{\theta\phi}\dot{\phi}\dot{\theta}+\gamma_{\phi\theta}\dot{\theta}\dot{\phi}+\gamma_{\phi\phi}\dot{\phi}\dot{\phi}\right)\;+V$$

$$H=\frac{1}{2}\left(\gamma_{\theta\theta}\dot{\theta}\dot{\theta}+\gamma_{\theta\phi}\dot{\phi}\dot{\theta}+\gamma_{\phi\theta}\dot{\theta}\dot{\phi}+\gamma_{\phi\phi}\dot{\phi}\dot{\phi}\right)+V\qquad=T+V\equiv E$$

(Only correct numerically!)

Hamiltonian must be explicit in momenta p_m

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2\right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$



$$T = \frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta\theta} & \gamma_{\theta\phi} \\ \gamma_{\phi\theta} & \gamma_{\phi\phi} \end{pmatrix}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix}$$

Covariant metric tensor

γ_{mn}

$$\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix}\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

Contravariant metric tensor

γ^{mn}

$$T = \frac{1}{2}\frac{\begin{pmatrix} p_\theta & p_\phi \end{pmatrix}\overbrace{\begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2 \end{pmatrix}}\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi)\right]} = \frac{1}{2}\gamma^{mn}p_m p_n$$

Lagrangian function of GCC and velocities: $L\left(\theta,\phi,\dot{\theta},\dot{\phi},t\right) = T\left(\theta,\phi,\dot{\theta},\dot{\phi},t\right) - V\left(\theta,\phi,t\right)$

Hamiltonian function of GCC and momenta: $H\left(\theta,\phi,p_\theta,p_\phi,t\right) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$ *Poincare-Legendre relation*

$$H = \quad p_\theta\dot{\theta} \quad + \quad p_\phi\dot{\phi} \quad - T \quad + V$$

$$H = \left(\gamma_{\theta\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\right)\dot{\theta} + \left(\gamma_{\phi\theta}\dot{\theta} + \gamma_{\phi\phi}\dot{\phi}\right)\dot{\phi} - \frac{1}{2}\left(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}\right) + V$$

$$H = \frac{1}{2}\left(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}\right) + V \qquad = T + V \equiv E$$

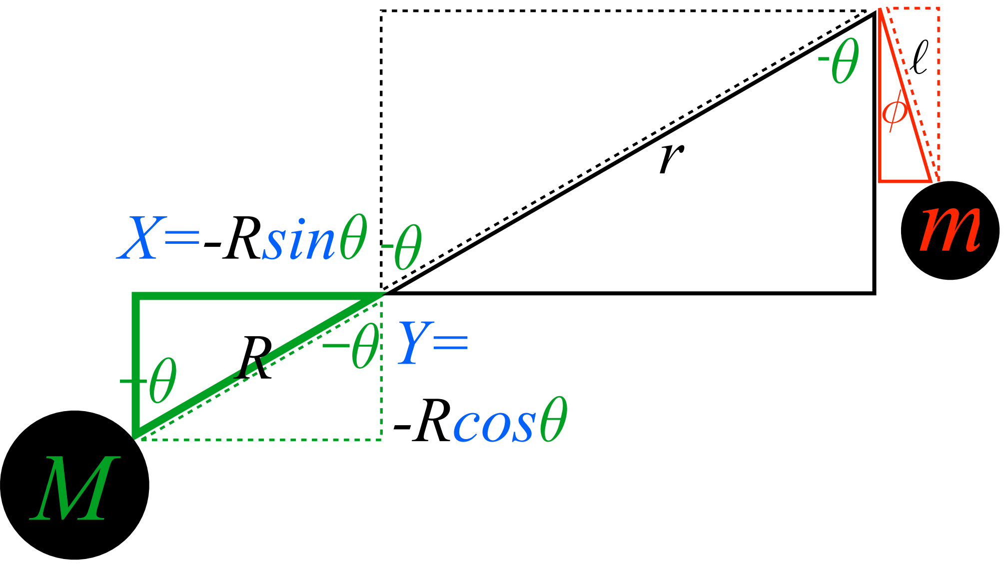
(Only correct numerically!)

Hamiltonian must be explicit in momenta p_m

$$H = \frac{1}{2}\left(\gamma^{\theta\theta}p_\theta p_\theta + \gamma^{\theta\phi}p_\theta p_\phi + \gamma^{\phi\theta}p_\phi p_\theta + \gamma^{\phi\phi}p_\phi p_\phi\right) + V = T + V$$

(Correct formally and numerically)

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2 \right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$



$$T = \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta\theta} & \gamma_{\theta\phi} \\ \gamma_{\phi\theta} & \gamma_{\phi\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Covariant metric tensor

γ_{mn}

$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

Contravariant metric tensor

γ^{mn}

$$T = \frac{1}{2}\frac{\begin{pmatrix} p_\theta & p_\phi \end{pmatrix} \overbrace{\begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2 \end{pmatrix}} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi) \right]} = \frac{1}{2}\gamma^{mn}p_m p_n$$

Lagrangian function of GCC and velocities: $L(\theta,\phi,\dot{\theta},\dot{\phi},t) = T(\theta,\phi,\dot{\theta},\dot{\phi},t) - V(\theta,\phi,t)$

Hamiltonian function of GCC and momenta: $H(\theta,\phi,p_\theta,p_\phi,t) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$ *Poincare-Legendre relation*

$$H = \quad p_\theta\dot{\theta} \qquad + \qquad p_\phi\dot{\phi} \qquad\qquad -T \qquad\qquad +V$$

$$H = \left(\gamma_{\theta\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\right)\dot{\theta} + \left(\gamma_{\phi\theta}\dot{\theta} + \gamma_{\phi\phi}\dot{\phi}\right)\dot{\phi} - \frac{1}{2}\left(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}\right) + V$$

$$H = \frac{1}{2}\left(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}\right) + V \qquad = T + V \equiv E$$

(Only correct numerically!)

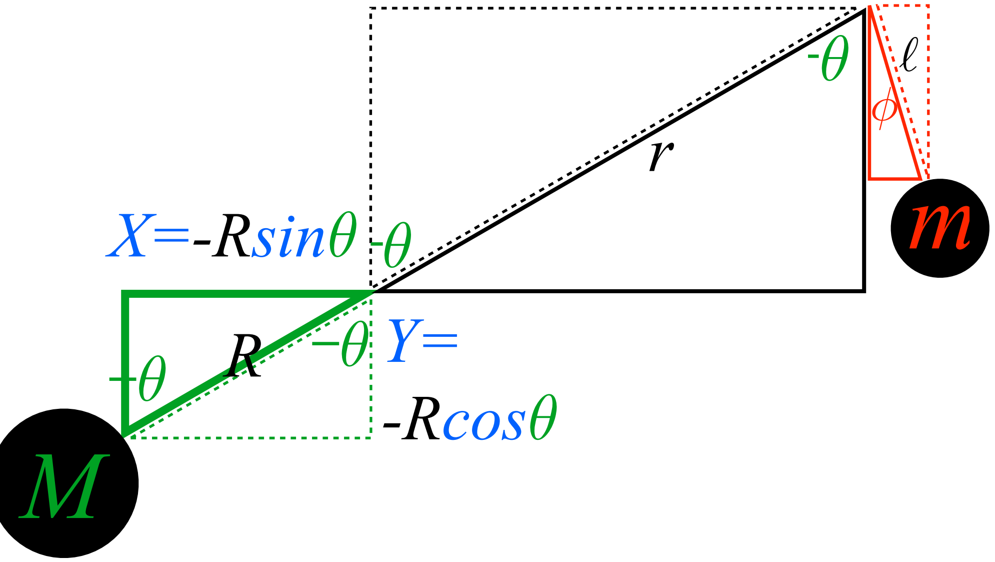
Hamiltonian must be explicit in momenta p_m

$$H = \frac{1}{2}\left(\gamma^{\theta\theta}p_\theta p_\theta + \gamma^{\theta\phi}p_\theta p_\phi + \gamma^{\phi\theta}p_\phi p_\theta + \gamma^{\phi\phi}p_\phi p_\phi\right) + V \quad = T + V$$

(Correct formally and numerically)

$$H = \frac{m\ell^2 p_\theta p_\theta + 2mr\ell\cos(\theta-\phi)p_\theta p_\phi + (MR^2 + mr^2)p_\phi p_\phi}{2m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi) \right]} + V$$

Hamilton equations for elementary trebuchet



$$\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}=\begin{pmatrix}\gamma^{\theta\theta} & \gamma^{\theta\phi}\\ \gamma^{\phi\theta} & \gamma^{\phi\phi}\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}$$

Contravariant metric tensor

γ^{mn}

$$T=\frac{1}{2}\frac{\begin{pmatrix}p_\theta & p_\phi\end{pmatrix}\overbrace{\begin{pmatrix}m\ell^2 & mr\ell\cos(\theta-\phi)\\ mr\ell\cos(\theta-\phi) & MR^2+mr^2\end{pmatrix}}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}}{m\ell^2\left[MR^2+mr^2\sin^2(\theta-\phi)\right]}=\frac{1}{2}\gamma^{mn}p_m p_n$$

$$\boxed{\begin{aligned}\dot{\theta}&=\gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi\\ \dot{\phi}&=\gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi\end{aligned}}$$

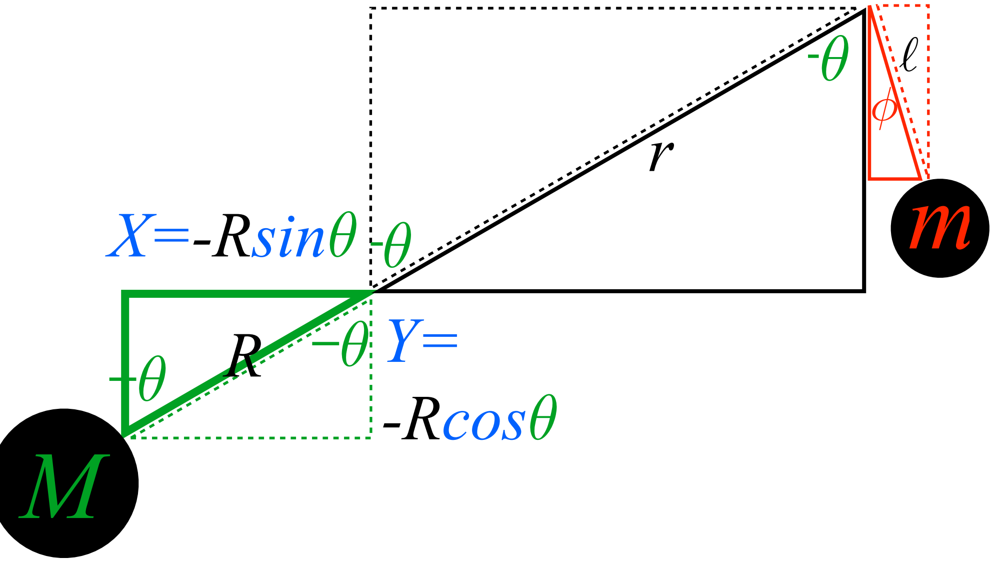
$$\boxed{\begin{aligned}p_\theta&=\gamma_{\theta\theta}\dot{\theta}+\gamma_{\theta\phi}\dot{\phi}\\ p_\phi&=\gamma_{\phi\theta}\dot{\theta}+\gamma_{\phi\phi}\dot{\phi}\end{aligned}}$$

Coordinate equations

$$\frac{\partial H}{\partial p_\theta}=\dot{\theta}=\ \gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi$$

$$\frac{\partial H}{\partial p_\phi}=\dot{\phi}=\ \gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi$$

Hamilton equations for elementary trebuchet



$$\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix}=\begin{pmatrix}\gamma^{\theta\theta} & \gamma^{\theta\phi}\\ \gamma^{\phi\theta} & \gamma^{\phi\phi}\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}$$

Contravariant metric tensor

γ^{mn}

$$T=\frac{1}{2}\frac{\begin{pmatrix}p_\theta & p_\phi\end{pmatrix}\begin{pmatrix}m\ell^2 & mr\ell\cos(\theta-\phi)\\ mr\ell\cos(\theta-\phi) & MR^2+mr^2\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}}{m\ell^2\left[MR^2+mr^2\sin^2(\theta-\phi)\right]}=\frac{1}{2}\gamma^{mn}p_m p_n$$

$$\boxed{\begin{aligned}\dot\theta&=\gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi\\ \dot\phi&=\gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi\end{aligned}}$$

$$\boxed{\begin{aligned}p_\theta&=\gamma_{\theta\theta}\dot\theta+\gamma_{\theta\phi}\dot\phi\\ p_\phi&=\gamma_{\phi\theta}\dot\theta+\gamma_{\phi\phi}\dot\phi\end{aligned}}$$

Coordinate equations

$$\frac{\partial H}{\partial p_\theta}=\dot\theta=\ \gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi$$

$$\frac{\partial H}{\partial p_\phi}=\dot\phi=\ \gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi$$

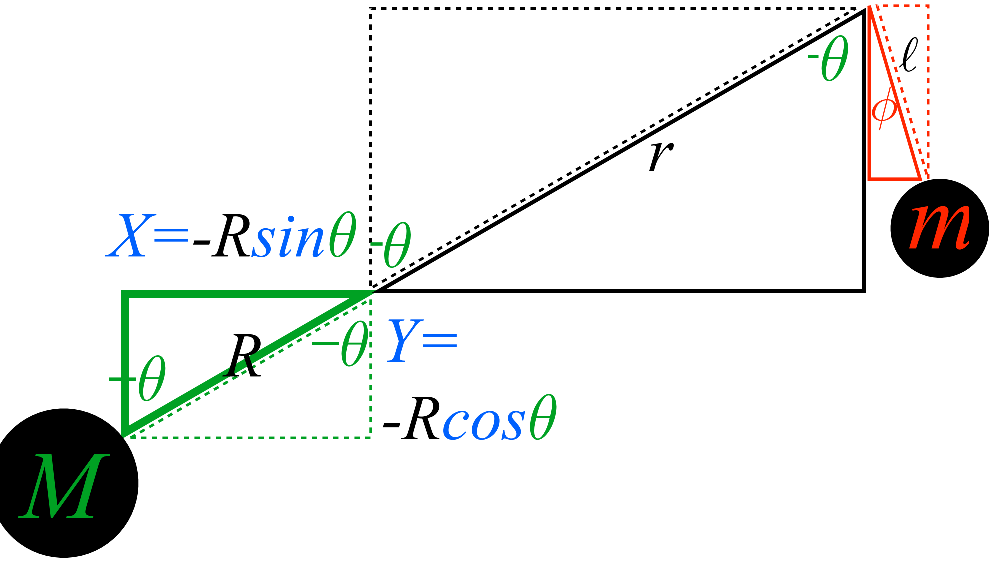
Momentum/force equations

$$\dot p_\theta=-\frac{\partial H}{\partial\theta}=\frac{\partial L}{\partial\theta}=\frac{\partial T}{\partial\theta}-\frac{\partial V}{\partial\theta}$$
$$=mr\ell\dot\theta\dot\phi\sin(\theta-\phi)+F_\theta$$

(May just use Lagrange results…
…but to be formally correct…
…must convert contra-velocities
to covariant momenta!)

$$\dot p_\phi=-\frac{\partial H}{\partial\phi}=\frac{\partial L}{\partial\phi}=\frac{\partial T}{\partial\phi}-\frac{\partial V}{\partial\phi}$$
$$=\ \ \text{-}mr\ell\dot\theta\dot\phi\sin(\theta-\phi)+F_\phi$$

Hamilton equations for elementary trebuchet



$$\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}=\begin{pmatrix}\gamma^{\theta\theta} & \gamma^{\theta\phi}\\ \gamma^{\phi\theta} & \gamma^{\phi\phi}\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}$$

Contravariant metric tensor

$$\gamma^{mn}$$

$$T=\frac{1}{2}\frac{\begin{pmatrix}p_\theta & p_\phi\end{pmatrix}\begin{pmatrix}m\ell^2 & mr\ell\cos(\theta-\phi)\\ mr\ell\cos(\theta-\phi) & MR^2+mr^2\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}}{m\ell^2\left[MR^2+mr^2\sin^2(\theta-\phi)\right]}=\frac{1}{2}\gamma^{mn}p_m p_n$$

$$\boxed{\begin{aligned}\dot{\theta}&=\gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi\\ \dot{\phi}&=\gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi\end{aligned}}$$

$$\boxed{\begin{aligned}p_\theta&=\gamma_{\theta\theta}\dot{\theta}+\gamma_{\theta\phi}\dot{\phi}\\ p_\phi&=\gamma_{\phi\theta}\dot{\theta}+\gamma_{\phi\phi}\dot{\phi}\end{aligned}}$$

Coordinate equations

$$\frac{\partial H}{\partial p_\theta}=\dot{\theta}=\gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi$$

$$\frac{\partial H}{\partial p_\phi}=\dot{\phi}=\gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi$$

Momentum/force equations

$$\dot{p}_\theta=-\frac{\partial H}{\partial\theta}=\frac{\partial L}{\partial\theta}=\frac{\partial T}{\partial\theta}-\frac{\partial V}{\partial\theta}$$

$$=mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)+F_\theta$$

$$=mr\ell(\gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi)(\gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi)\sin(\theta-\phi)+F_\theta$$

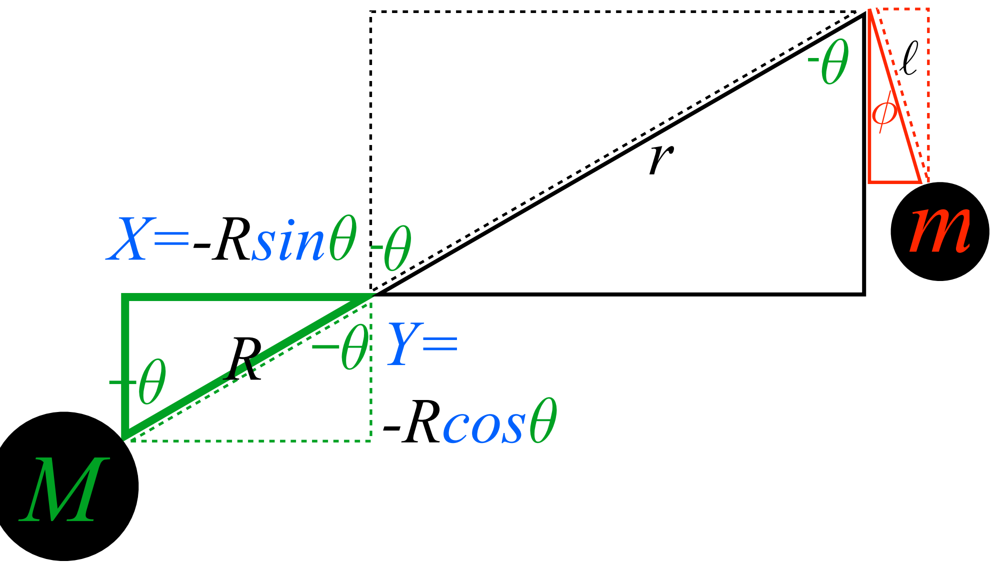
(May just use Lagrange results…
…but to be formally correct…
…must convert contra-velocities
to covariant momenta!)

$$\dot{p}_\phi=-\frac{\partial H}{\partial\phi}=\frac{\partial L}{\partial\phi}=\frac{\partial T}{\partial\phi}-\frac{\partial V}{\partial\phi}$$

$$=\ \text{-}mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)+F_\phi$$

$$=\text{-}mr\ell(\gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi)(\gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi)\sin(\theta-\phi)+F_\phi$$

Hamilton equations for elementary trebuchet



$$\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix}=\begin{pmatrix}\gamma^{\theta\theta} & \gamma^{\theta\phi}\\ \gamma^{\phi\theta} & \gamma^{\phi\phi}\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}$$

Contravariant metric tensor

γ^{mn}

$$T=\frac{1}{2}\frac{\begin{pmatrix}p_\theta & p_\phi\end{pmatrix}\overbrace{\begin{pmatrix}m\ell^2 & mr\ell\cos(\theta-\phi)\\ mr\ell\cos(\theta-\phi) & MR^2+mr^2\end{pmatrix}}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}}{m\ell^2\left[MR^2+mr^2\sin^2(\theta-\phi)\right]}=\frac{1}{2}\gamma^{mn}p_m p_n$$

$$\boxed{\begin{aligned}\dot\theta&=\gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi\\ \dot\phi&=\gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi\end{aligned}}$$

$$\boxed{\begin{aligned}p_\theta&=\gamma_{\theta\theta}\dot\theta+\gamma_{\theta\phi}\dot\phi\\ p_\phi&=\gamma_{\phi\theta}\dot\theta+\gamma_{\phi\phi}\dot\phi\end{aligned}}$$

Coordinate equations

$$\frac{\partial H}{\partial p_\theta}=\dot\theta=\ \gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi$$

$$\frac{\partial H}{\partial p_\phi}=\dot\phi=\ \gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi$$

Momentum/force equations

$$\dot p_\theta=-\frac{\partial H}{\partial\theta}=\frac{\partial L}{\partial\theta}=\frac{\partial T}{\partial\theta}-\frac{\partial V}{\partial\theta}$$

$$=mr\ell\dot\theta\dot\phi\sin(\theta-\phi)+F_\theta$$

$$=mr\ell(\gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi)(\gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi)\sin(\theta-\phi)+F_\theta$$

$$=mr\ell(\gamma^{\theta\theta}\gamma^{\phi\theta}p_\theta^2+[\gamma^{\theta\theta}\gamma^{\phi\phi}+(\gamma^{\theta\phi})^2]p_\phi p_\theta+\gamma^{\theta\phi}\gamma^{\phi\phi}p_\phi^2)\sin(\theta-\phi)+F_\theta$$

(May just use Lagrange results…
…but to be formally correct…
…must convert contra-velocities
to covariant momenta!)

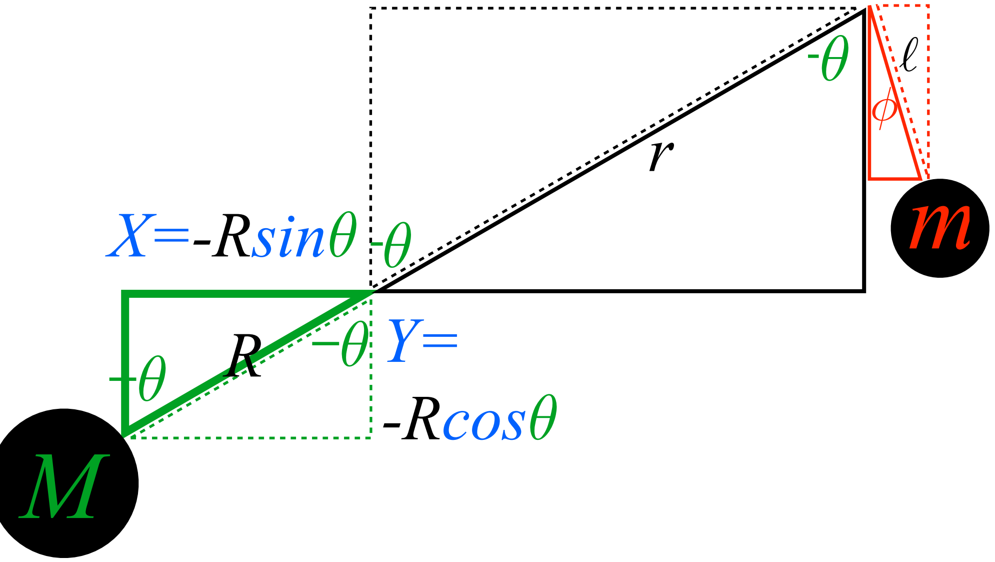
$$\dot p_\phi=-\frac{\partial H}{\partial\phi}=\frac{\partial L}{\partial\phi}=\frac{\partial T}{\partial\phi}-\frac{\partial V}{\partial\phi}$$

$$=\ \text{-}mr\ell\dot\theta\dot\phi\sin(\theta-\phi)+F_\phi$$

$$=\text{-}mr\ell(\gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi)(\gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi)\sin(\theta-\phi)+F_\phi$$

$$=\text{-[messy factor]}\sin(\theta-\phi)+F_\phi$$

Hamilton equations for elementary trebuchet



$$\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}=\begin{pmatrix}\gamma^{\theta\theta} & \gamma^{\theta\phi}\\ \gamma^{\phi\theta} & \gamma^{\phi\phi}\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}$$

Contravariant metric tensor

γ^{mn}

$$T=\frac{1}{2}\frac{\begin{pmatrix}p_\theta & p_\phi\end{pmatrix}\begin{pmatrix}m\ell^2 & mr\ell\cos(\theta-\phi)\\ mr\ell\cos(\theta-\phi) & MR^2+mr^2\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}}{m\ell^2\left[MR^2+mr^2\sin^2(\theta-\phi)\right]}=\frac{1}{2}\gamma^{mn}p_m p_n$$

$$\boxed{\begin{aligned}\dot{\theta}&=\gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi\\ \dot{\phi}&=\gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi\end{aligned}}$$

$$\boxed{\begin{aligned}p_\theta&=\gamma_{\theta\theta}\dot{\theta}+\gamma_{\theta\phi}\dot{\phi}\\ p_\phi&=\gamma_{\phi\theta}\dot{\theta}+\gamma_{\phi\phi}\dot{\phi}\end{aligned}}$$

Coordinate equations

$$\frac{\partial H}{\partial p_\theta}=\dot{\theta}=\gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi \qquad \frac{\partial H}{\partial p_\phi}=\dot{\phi}=\gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi$$

Momentum/force equations

$$\begin{aligned}\dot{p}_\theta&=-\frac{\partial H}{\partial\theta}=\frac{\partial L}{\partial\theta}=\frac{\partial T}{\partial\theta}-\frac{\partial V}{\partial\theta}\\ &=mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)+F_\theta\\ &=mr\ell(\gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi)(\gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi)\sin(\theta-\phi)+F_\theta\\ &=mr\ell(\gamma^{\theta\theta}\gamma^{\phi\theta}p_\theta^2+[\gamma^{\theta\theta}\gamma^{\phi\phi}+(\gamma^{\theta\phi})^2]p_\phi p_\theta+\gamma^{\theta\phi}\gamma^{\phi\phi}p_\phi^2)\sin(\theta-\phi)+F_\theta\end{aligned}$$

$$\begin{aligned}\dot{p}_\phi&=-\frac{\partial H}{\partial\phi}=\frac{\partial L}{\partial\phi}=\frac{\partial T}{\partial\phi}-\frac{\partial V}{\partial\phi}\\ &=\text{-}mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)+F_\phi\\ &=\text{-}mr\ell(\gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi)(\gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi)\sin(\theta-\phi)+F_\phi\\ &=\text{-[messy factor]}\sin(\theta-\phi)+F_\phi\end{aligned}$$

A lesson on Hamiltonian “elegance”...
...may be very elegant formally...but may ***not*** *be so elegant algebraically!*

Hamiltonian energy and momentum conservation and symmetry coordinates

➡ *Coordinate transformation helps reduce symmetric Hamiltonian*

Free-space trebuchet kinematics by symmetry

- *Algebraic approach*
- *Direct approach and Superball analogy*

Trebuchet vs Flinger and sports kinematics

- *Many approaches to Mechanics*

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \qquad +\pi/2$

Jacobian Lemma-1 definition: $\phi_B = -\theta + \phi - \pi/2$

$$\begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta_B}{\partial \theta} & \frac{\partial \theta_B}{\partial \phi} \\ \frac{\partial \phi_B}{\partial \theta} & \frac{\partial \phi_B}{\partial \phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Previous lab absolute trebuchet coordinate angles θ and φ

compared to

new angles θ_B and ϕ_B.

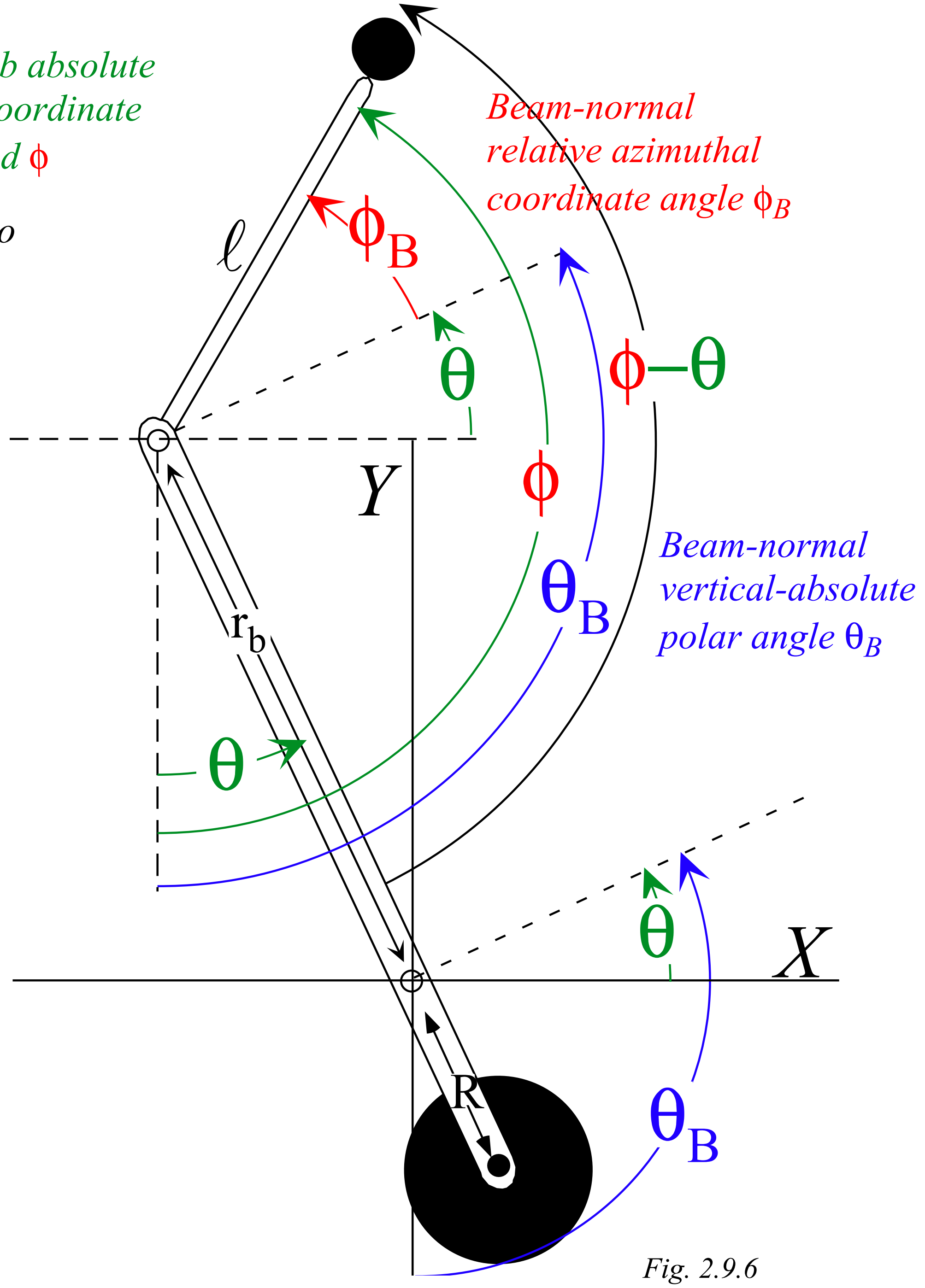


Fig. 2.9.6
Lab (θ,φ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.
(Each value is positive.)

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \qquad +\pi/2$

Jacobian Lemma-1 definition: $\phi_B = -\theta + \phi - \pi/2$

$$\begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta_B}{\partial \theta} & \frac{\partial \theta_B}{\partial \phi} \\ \frac{\partial \phi_B}{\partial \theta} & \frac{\partial \phi_B}{\partial \phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Kajobian of inverse transform $\phi_B = \phi - \theta - \pi/2$ and $\theta = \theta_B \ -\pi/2$

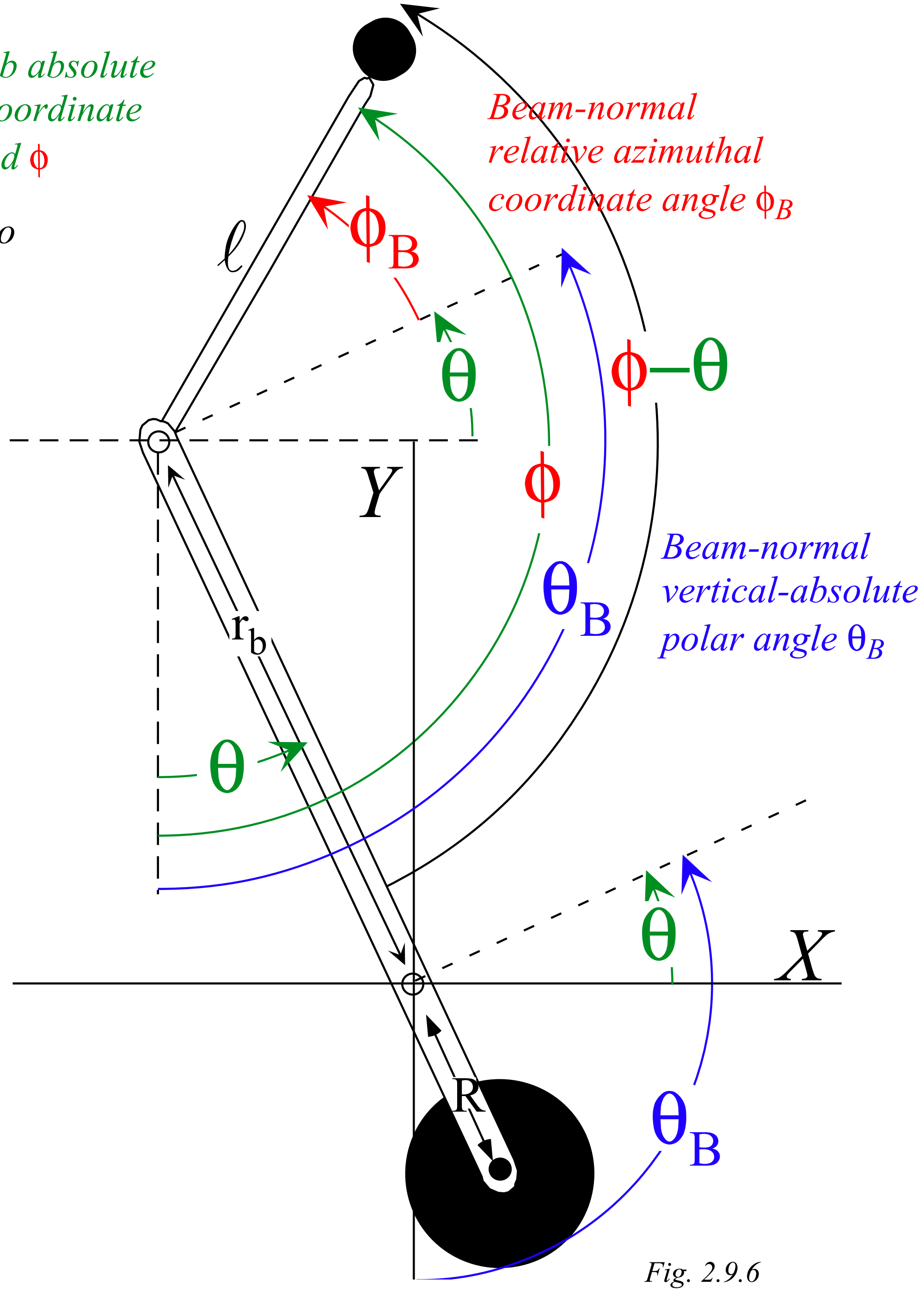
$\phi = \theta_B + \phi_B$

$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta}{\partial \theta_B} & \frac{\partial \theta}{\partial \phi_B} \\ \frac{\partial \phi}{\partial \theta_B} & \frac{\partial \phi}{\partial \phi_B} \end{pmatrix} \begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix}$$

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B.



Beam-normal relative azimuthal coordinate angle ϕ_B

Beam-normal vertical-absolute polar angle θ_B

Fig. 2.9.6

Lab (θ,ϕ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.

(Each value is positive.)

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \qquad +\pi/2$

Jacobian Lemma-1 definition: $\phi_B = -\theta + \phi - \pi/2$

$$\begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta_B}{\partial\theta} & \frac{\partial\theta_B}{\partial\phi} \\ \frac{\partial\phi_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\phi} \end{pmatrix} \begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix}$$

Kajobian of inverse transform $\phi_B = \phi - \theta - \pi/2$ and $\theta = \theta_B \;\; -\pi/2$

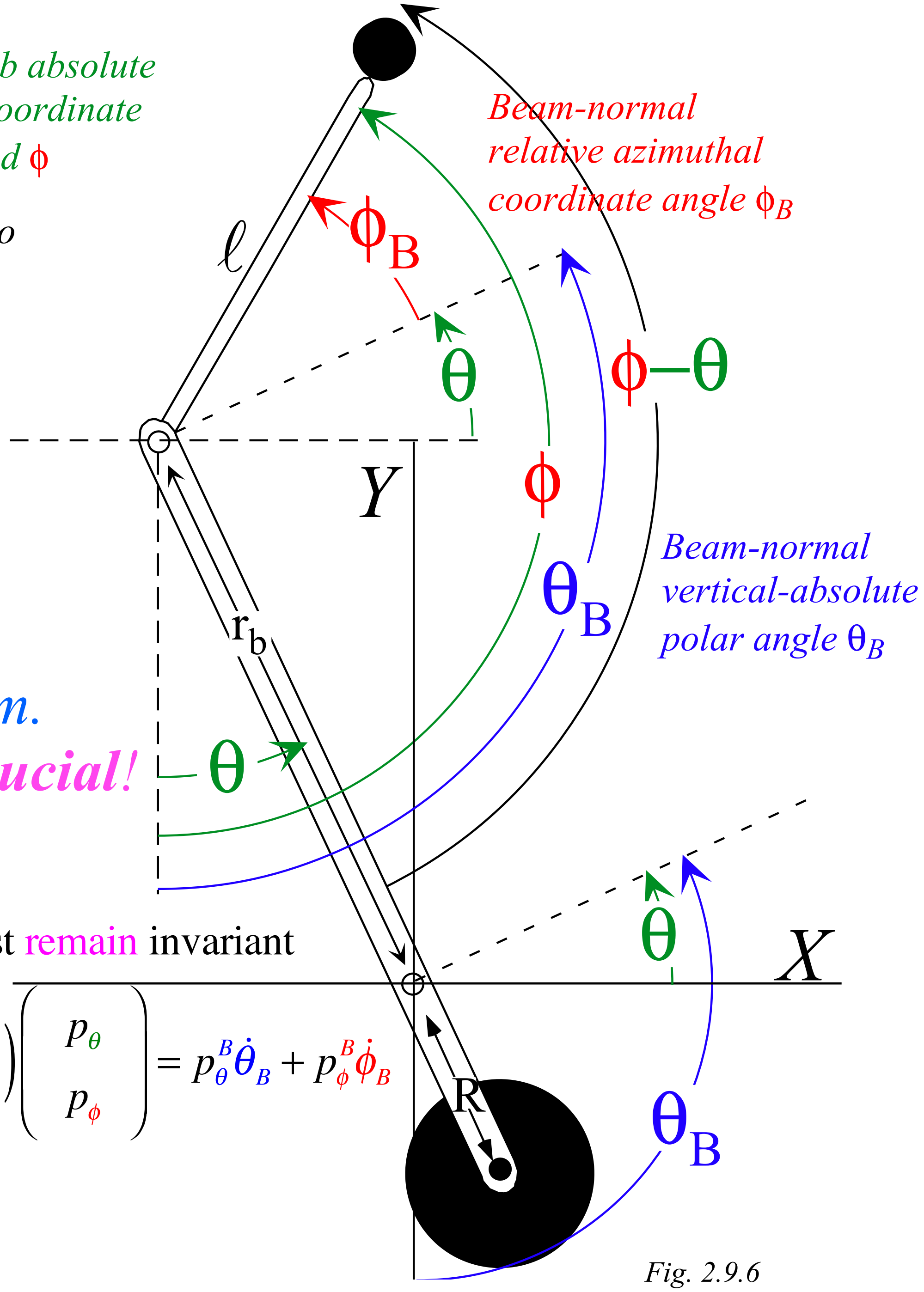
$\phi = \theta_B + \phi_B$

$$\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\theta}{\partial\phi_B} \\ \frac{\partial\phi}{\partial\theta_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix}$$

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B.



Beam-normal relative azimuthal coordinate angle ϕ_B

Beam-normal vertical-absolute polar angle θ_B

Be careful with momentum.
*Poincare invariance is **crucial**!*

Poincare invariant must remain invariant

$$p_\theta\dot\theta + p_\phi\dot\phi = \begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = p^B_\theta\dot\theta_B + p^B_\phi\dot\phi_B$$

Fig. 2.9.6

Lab (θ,ϕ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.

(Each value is positive.)

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \quad +\pi/2$

Jacobian Lemma-1 definition: ϕ_B $=-\theta+\phi-\pi/2$

$$\begin{pmatrix}\dot{\theta}_B\\ \dot{\phi}_B\end{pmatrix}=\begin{pmatrix}\frac{\partial\theta_B}{\partial\theta} & \frac{\partial\theta_B}{\partial\phi}\\ \frac{\partial\phi_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\phi}\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}=\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

Kajobian of inverse transform $\phi_B = \phi - \theta - \pi/2$ and $\theta = \theta_B \ -\pi/2$

$\phi = \theta_B + \phi_B$

$$\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}=\begin{pmatrix}\frac{\partial\theta}{\partial\theta_B} & \frac{\partial\theta}{\partial\phi_B}\\ \frac{\partial\phi}{\partial\theta_B} & \frac{\partial\phi}{\partial\phi_B}\end{pmatrix}\begin{pmatrix}\dot{\theta}_B\\ \dot{\phi}_B\end{pmatrix}=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}\dot{\theta}_B\\ \dot{\phi}_B\end{pmatrix}$$

p_m *transform is* TRANSPOSE INVERSE *to* q^m

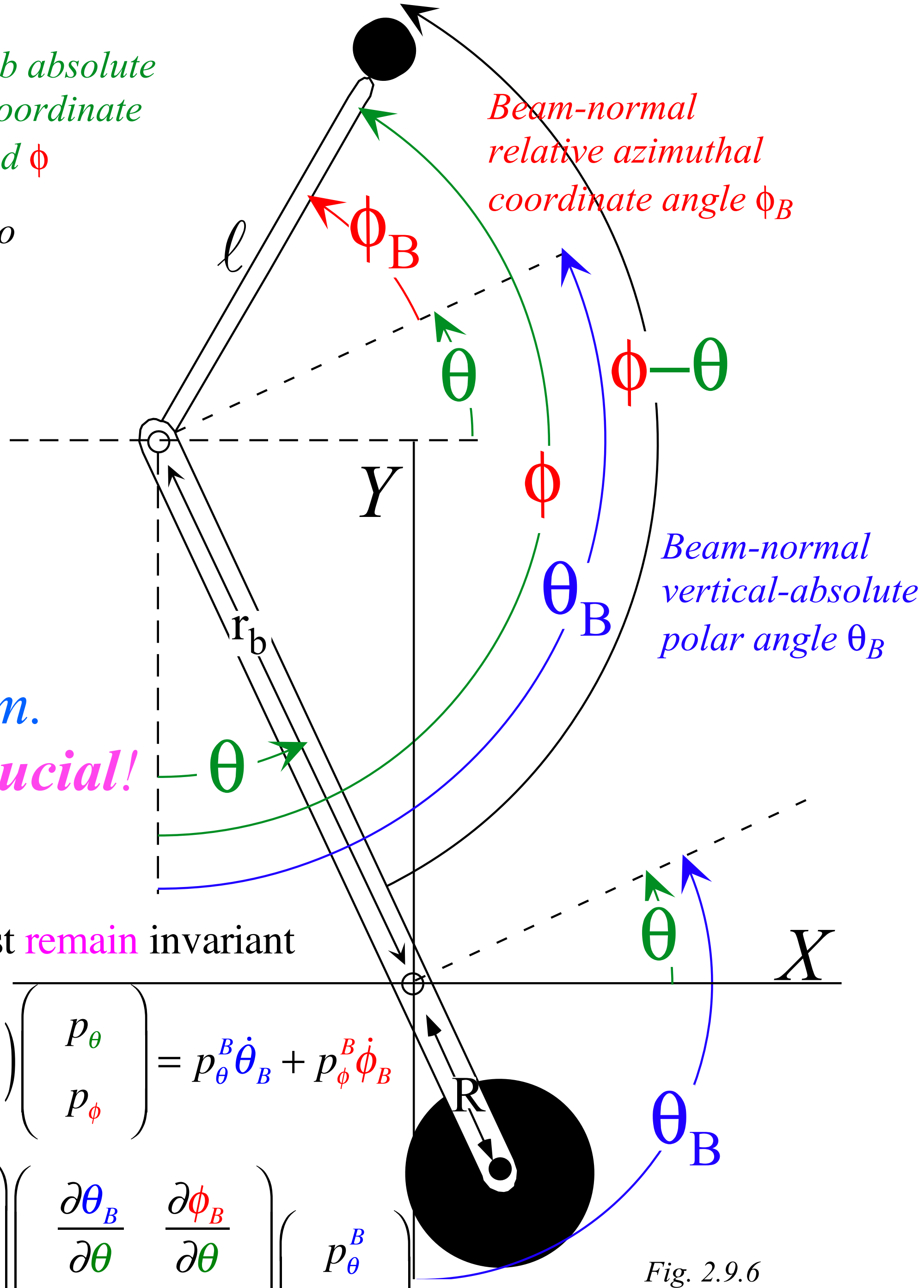
$$\begin{pmatrix}p^B_\theta\\ p^B_\phi\end{pmatrix}=\begin{pmatrix}\frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B}\\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B}\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}$$

$$\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}=\begin{pmatrix}\frac{\partial\theta_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\theta}\\ \frac{\partial\theta_B}{\partial\phi} & \frac{\partial\phi_B}{\partial\phi}\end{pmatrix}\begin{pmatrix}p^B_\theta\\ p^B_\phi\end{pmatrix}=\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}\begin{pmatrix}p^B_\theta\\ p^B_\phi\end{pmatrix}$$

Previous lab absolute trebuchet coordinate angles θ and φ

compared to

new angles θ_B and ϕ_B.



Beam-normal relative azimuthal coordinate angle ϕ_B

Beam-normal vertical-absolute polar angle θ_B

Be careful with momentum.
*Poincare invariance is **crucial**!*

Poincare invariant must remain invariant

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p_m transform is TRANSPOSE INVERSE *to q^m*

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Resulting momentum transform: $p_\theta = p^B_\theta - p^B_\phi$

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Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

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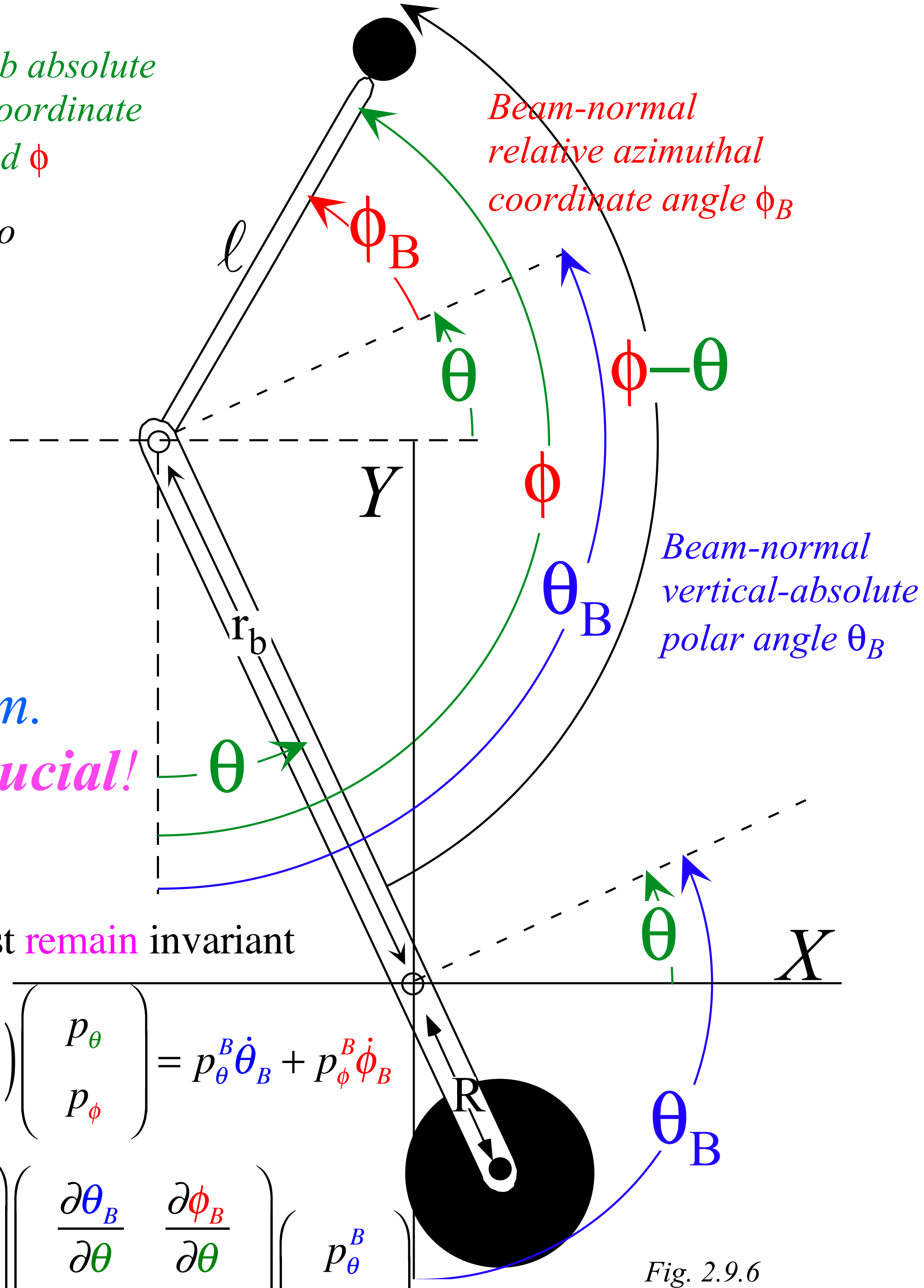

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p_m transform is TRANSPOSE INVERSE to q^m

$$\begin{pmatrix}p^B_\theta\\ p^B_\phi\end{pmatrix} = \begin{pmatrix}\frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B}\\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B}\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix} = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}$$

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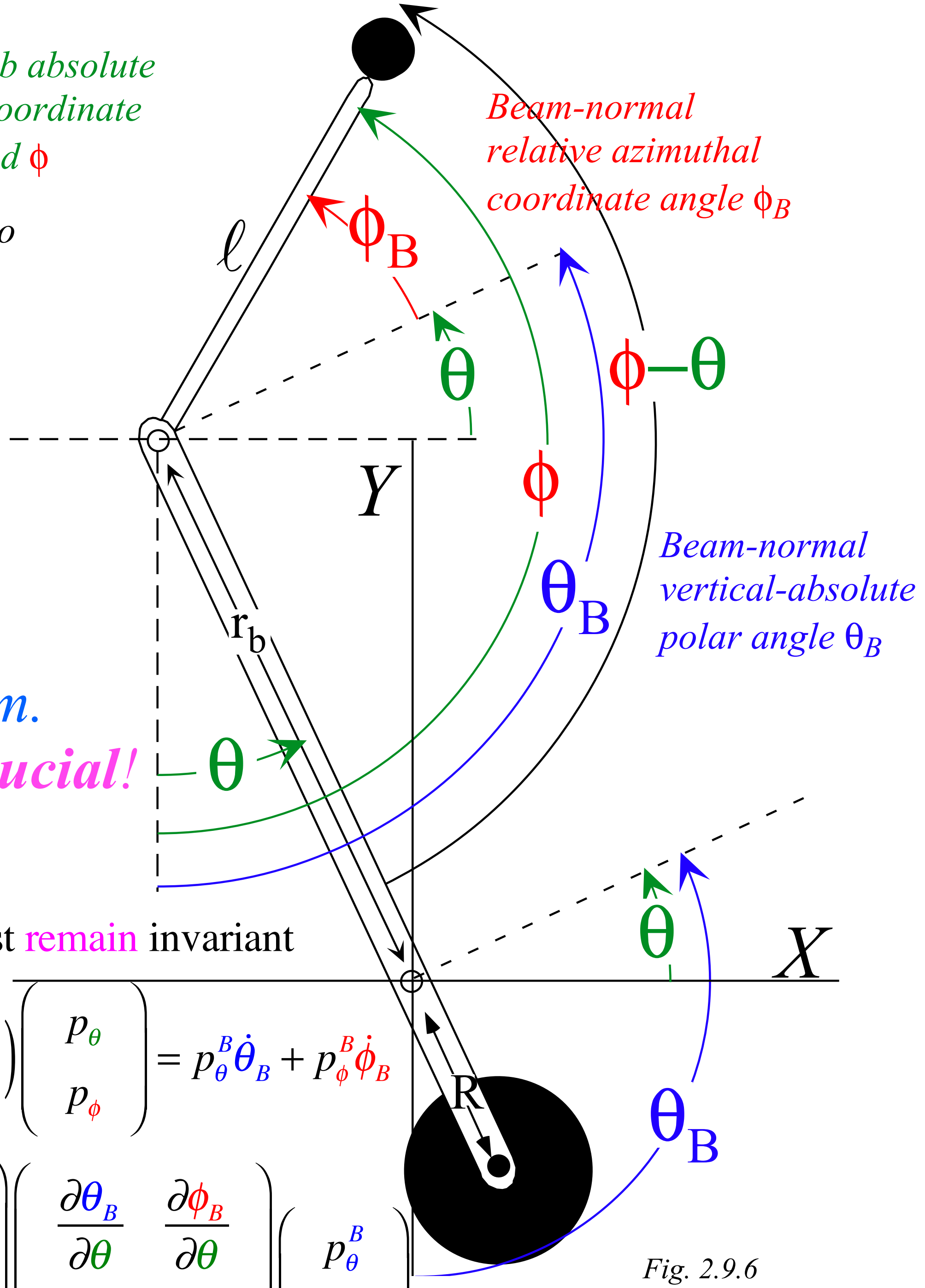

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$$H = \frac{m\ell^2 p_\theta p_\theta + (MR^2 + mr^2)p_\phi p_\phi + 2mr\ell\, p_\theta p_\phi \cos(\theta-\phi)}{2m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi)\right]} \qquad +V$$

Original (ϕ,θ) Hamiltonian

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \quad +\pi/2$

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Kajobian of inverse transform $\phi_B = \phi - \theta - \pi/2$ and $\theta = \theta_B - \pi/2$

$\phi = \theta_B + \phi_B$

$$\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix} = \begin{pmatrix}\frac{\partial\theta}{\partial\theta_B} & \frac{\partial\theta}{\partial\phi_B}\\ \frac{\partial\phi}{\partial\theta_B} & \frac{\partial\phi}{\partial\phi_B}\end{pmatrix}\begin{pmatrix}\dot\theta_B\\ \dot\phi_B\end{pmatrix} = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}\dot\theta_B\\ \dot\phi_B\end{pmatrix}$$

p_m transform is TRANSPOSE INVERSE *to q^m*

$$\begin{pmatrix}p^B_\theta\\ p^B_\phi\end{pmatrix} = \begin{pmatrix}\frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B}\\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B}\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix} = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}$$

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Resulting momentum transform: $p_\theta = p^B_\theta - p^B_\phi$

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Previous lab absolute trebuchet coordinate angles θ and φ

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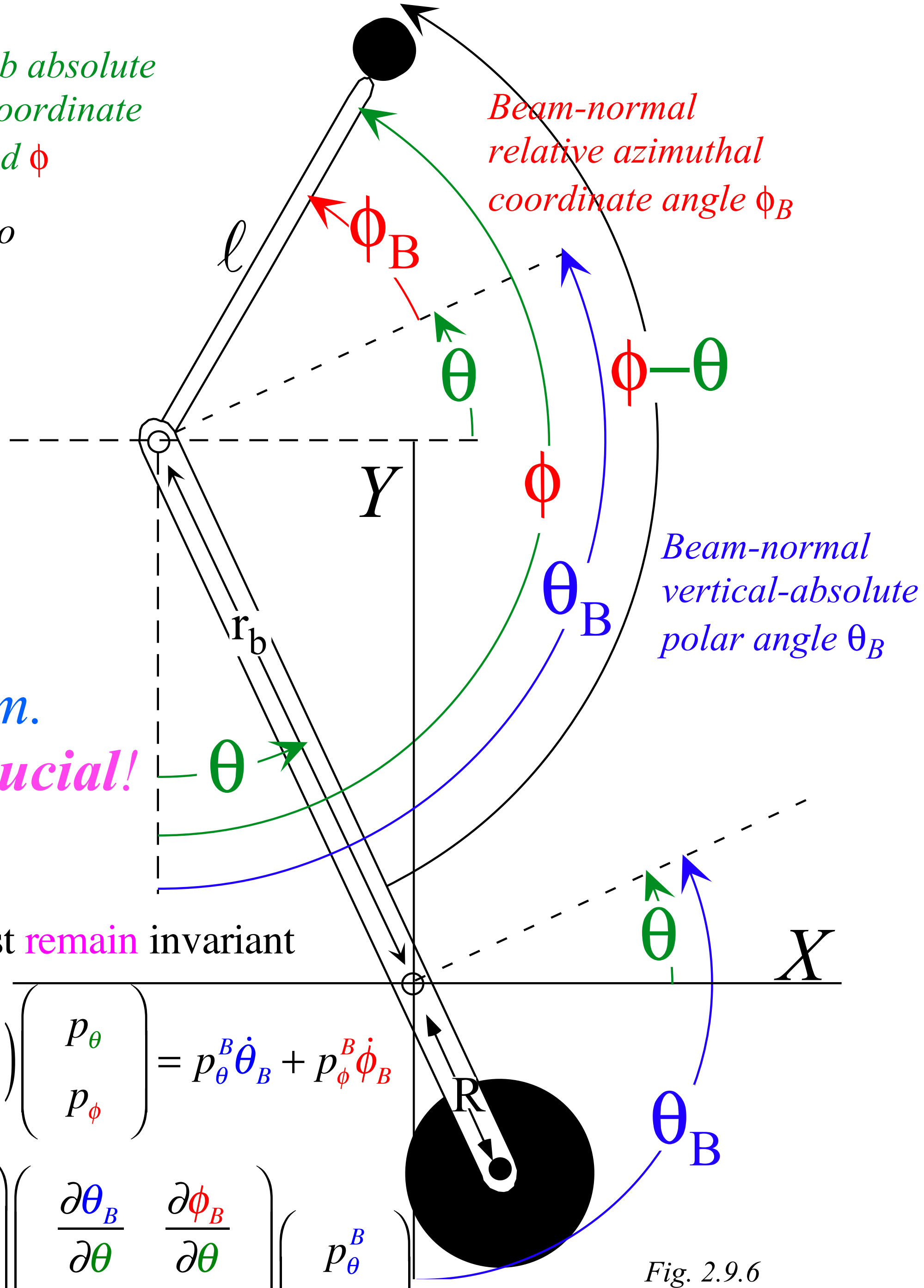

Be careful with momentum.
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Original (φ,θ) Hamiltonian

(Use $\phi_B = \pi/2 - (\theta - \phi)$)

$$H = \frac{m\ell^2\left(p^B_\theta - p^B_\phi\right)^2 + \left(MR^2 + mr^2\right)\left(p^B_\phi\right)^2 - 2mr\ell p^B_\phi\left(p^B_\theta - p^B_\phi\right)\sin\phi_B}{m\ell^2\left[MR^2 + mr^2\cos^2\phi_B\right]} + V$$

Transformed (ϕ_B,θ_B) Hamiltonian

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \quad +\pi/2$

Jacobian Lemma-1 definition: $\phi_B = -\theta + \phi - \pi/2$

$$\begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta_B}{\partial\theta} & \frac{\partial\theta_B}{\partial\phi} \\ \frac{\partial\phi_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

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p_m transform is TRANSPOSE INVERSE to q^m

$$\begin{pmatrix} p_\theta^B \\ p_\phi^B \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B} \\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

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Resulting momentum transform: $p_\theta = p_\theta^B - p_\phi^B$

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Be careful with momentum.
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Previous lab absolute trebuchet coordinate angles θ and ϕ

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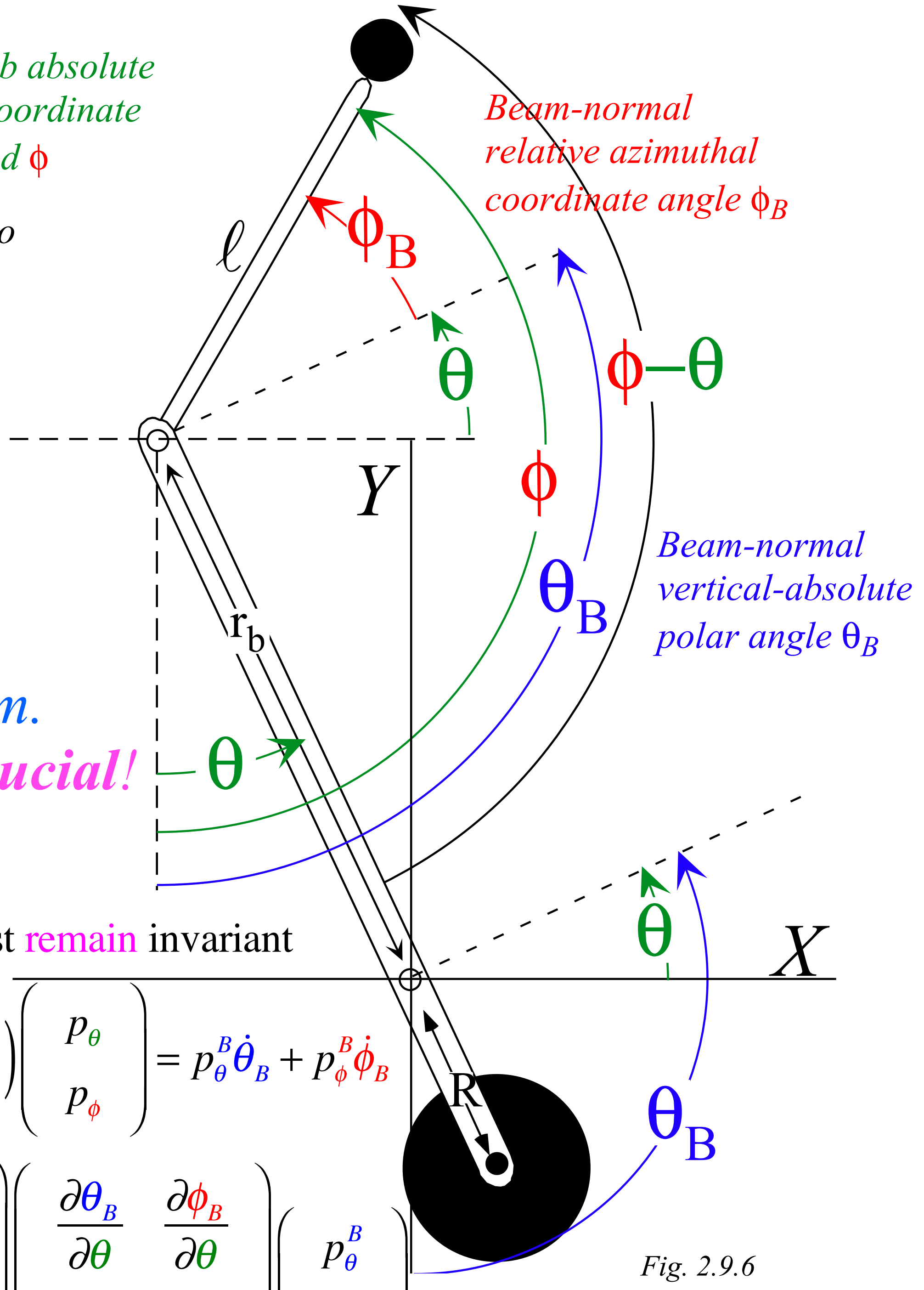

Poincare invariant must remain invariant

$$p_\theta\dot{\theta} + p_\phi\dot{\phi} = \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = p_\theta^B\dot{\theta}_B + p_\phi^B\dot{\phi}_B$$

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Lab (θ,ϕ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.
(Each value is positive.)

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$$\boxed{\theta - \phi = -\pi/2 - \phi_B}$$

$F_\theta = -MgR\sin\theta + mgr\sin\theta$

$F_\phi = -mg\ell\sin\phi$

$$H = \frac{m\ell^2\left(p_\theta^B - p_\phi^B\right)^2 + \left(MR^2 + mr^2\right)\left(p_\phi^B\right)^2 - 2mr\ell p_\phi^B\left(p_\theta^B - p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2 + mr^2\cos^2\phi_B\right]} - \left(MR - mr\right)g\sin\theta_B \ - mg\ell\cos\left(\phi_B + \theta_B\right)$$

Hamiltonian energy and momentum conservation and symmetry coordinates

- *Coordinate transformation helps reduce symmetric Hamiltonian*
- *Free-space trebuchet kinematics by symmetry*
  - → *Algebraic approach*
  - *Direct approach and Superball analogy*
- *Trebuchet vs Flinger and sports kinematics*
  - *Many approaches to Mechanics*

Free-space trebuchet kinematics by symmetry: Algebraic approach

(using last line of p. 36)

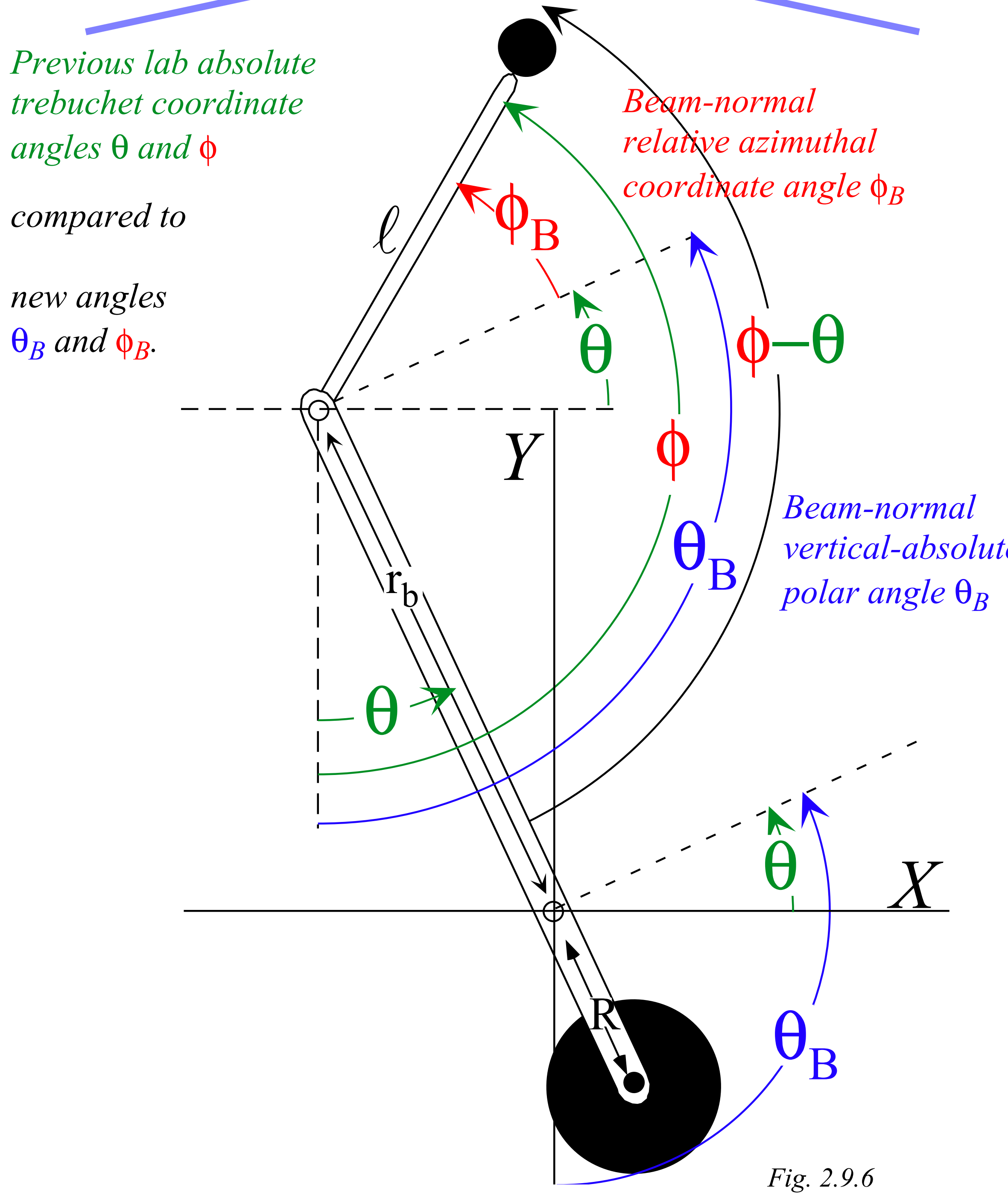
(Assume zero-gravity)

$$H=\frac{m\ell^2\left(p_\theta^B-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(p_\theta^B-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}\cancel{-\left(MR-mr\right)g\sin\theta_B-mg\ell\cos\left(\phi_B+\theta_B\right)}$$

For zero-gravity H is not a function of θ_B

so : $\dot{p}_\theta^B=\frac{\partial H}{\partial\theta_B}=0$ and : $p_\theta^B=\Lambda=const.$

H is not an explicit function of t so : $H=const.=E$



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H is not an explicit function of t so : $H=const.=E$

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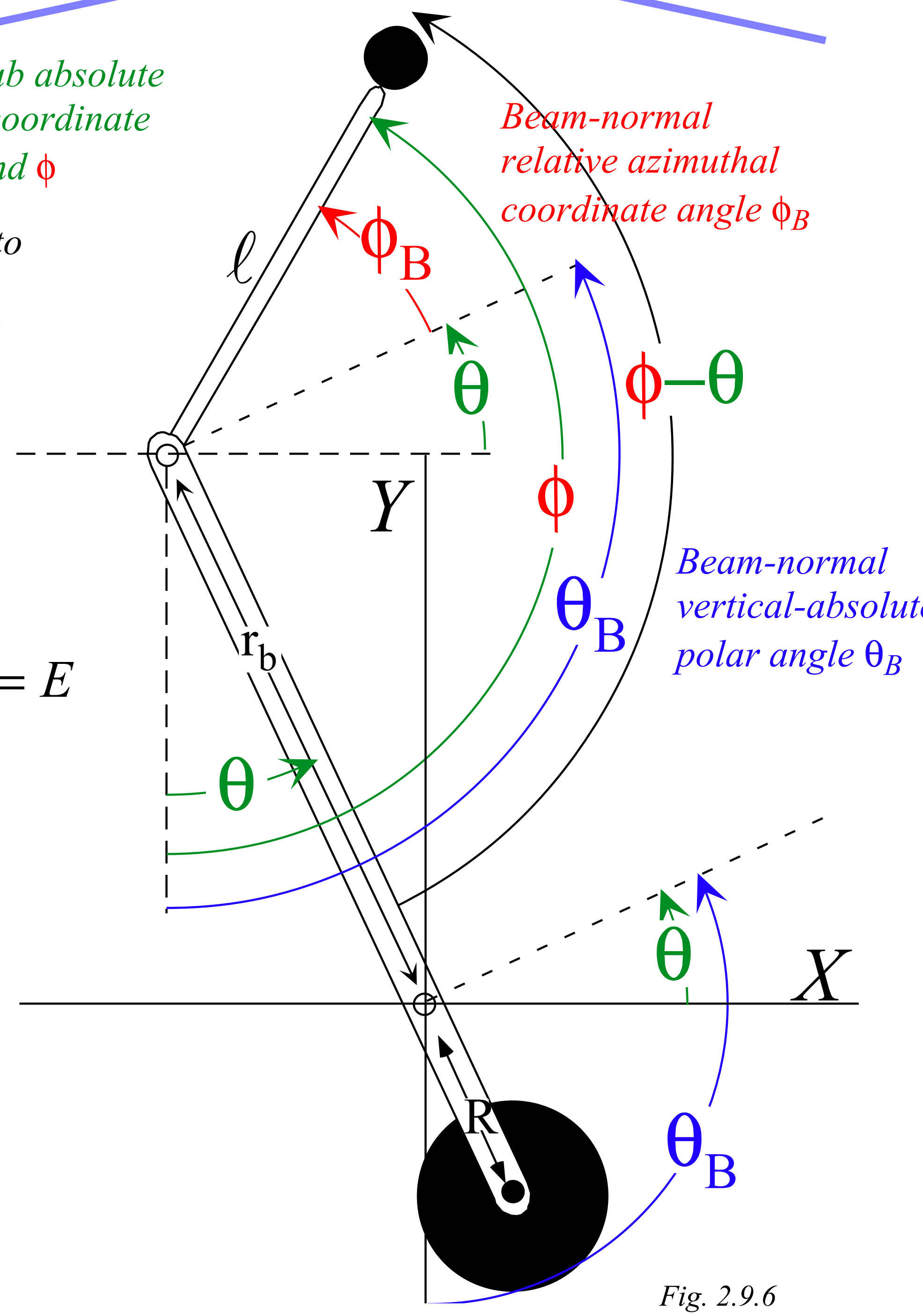

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Beam-normal relative azimuthal coordinate angle ϕ_B

Beam-normal vertical-absolute polar angle θ_B

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$$H=\frac{m\ell^2\left(p_\theta^B-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(p_\theta^B-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}\xcancel{-\left(MR-mr\right)g\sin\theta_B-mg\ell\cos\left(\phi_B+\theta_B\right)}$$

For zero-gravity H is not a function of θ_B

so : $\dot{p}_\theta^B=\frac{\partial H}{\partial\theta_B}=0$ and : $p_\theta^B=\Lambda=const.$

H is not an explicit function of t so : $H=const.=E$

$$H=\frac{m\ell^2\left(\Lambda-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(\Lambda-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}=const.=E$$

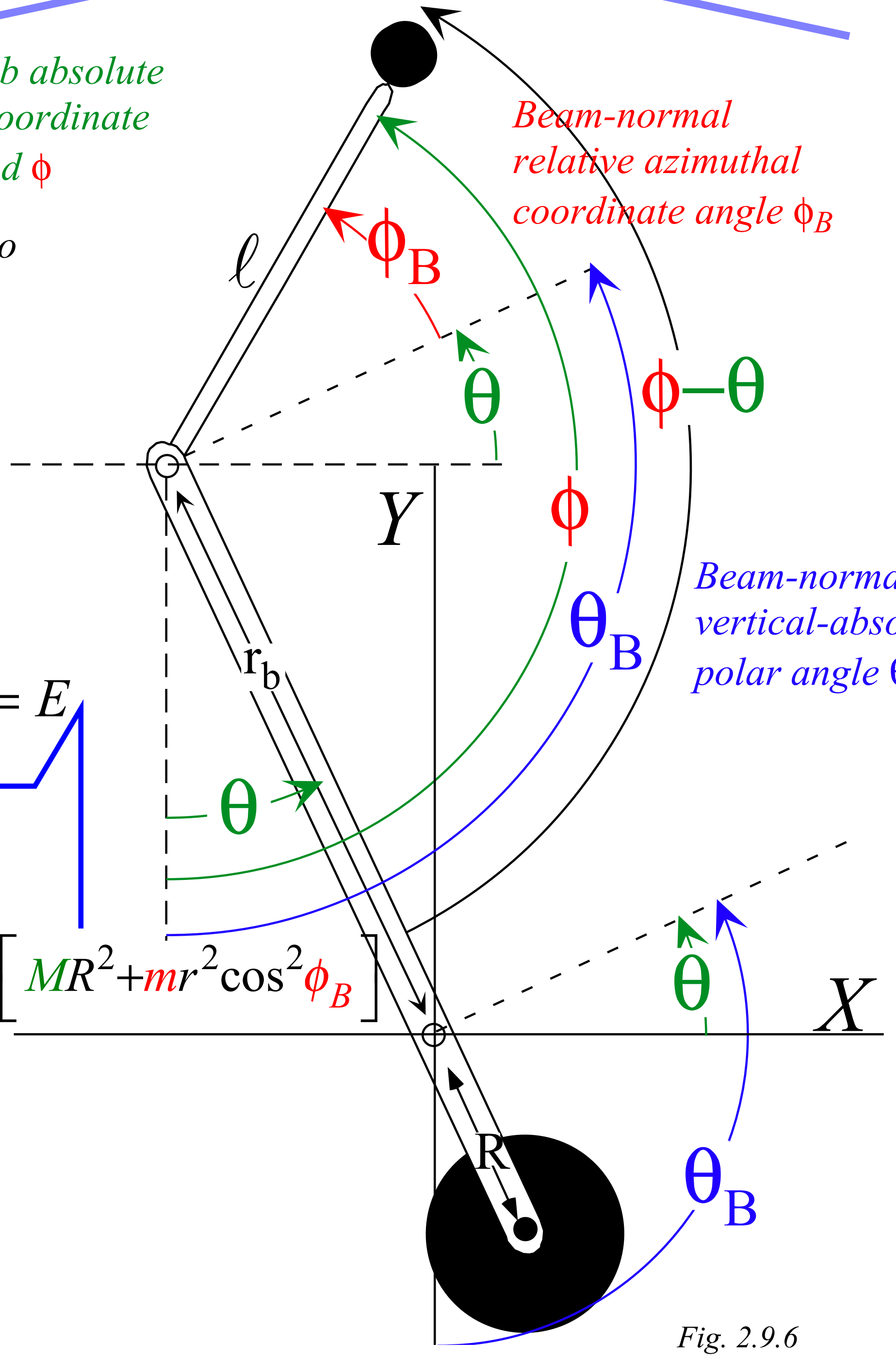
Rewrite $H=E$ as a quadratic equation in p_ϕ :

$$m\ell^2\left(\Lambda^2-2\Lambda(p_\phi^B)+(p_\phi^B)^2\right)+\left(MR^2+mr^2\right)(p_\phi^B)^2-2mr\ell(p_\phi^B)\left(\Lambda-p_\phi^B\right)\sin\phi_B=Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]$$

Previous lab absolute trebuchet coordinate angles θ and φ

compared to

new angles θ_B and ϕ_B.



Beam-normal relative azimuthal coordinate angle ϕ_B

Beam-normal vertical-absolute polar angle θ_B

Fig. 2.9.6

Lab (θ,φ) and beam-normal (θ

Throwing-momentum p^B_ϕ is a function of beam-relative angle ϕ_B, total E, and $\Lambda=p^B_\theta$.

Free-space trebuchet kinematics by symmetry: Algebraic approach

(using last line of p. 36)

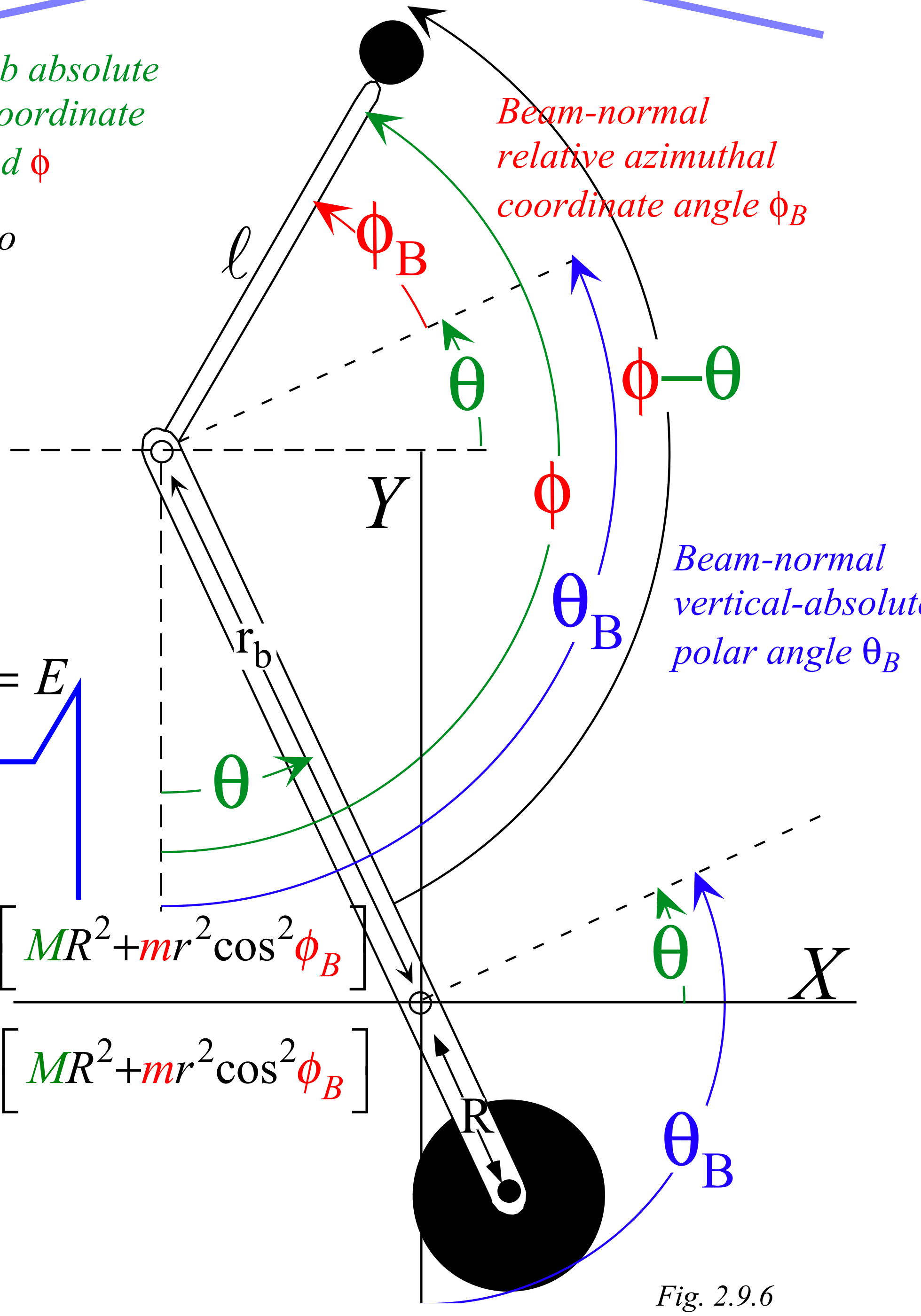
(Assume zero-gravity)

$$H=\frac{m\ell^2\left(p_\theta^B-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(p_\theta^B-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}\xcancel{-\left(MR-mr\right)g\sin\theta_B-mg\ell\cos\left(\phi_B+\theta_B\right)}$$

For zero-gravity H is not a function of θ_B

so : $\dot{p}_\theta^B=\frac{\partial H}{\partial\theta_B}=0$ *and :* $p_\theta^B=\Lambda=const.$

H is not an explicit function of t so : H=const.=E



Previous lab absolute trebuchet coordinate angles θ and φ

compared to

new angles θ_B and ϕ_B.

$$H=\frac{m\ell^2\left(\Lambda-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(\Lambda-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}=const.=E$$

Rewrite H=E as a quadratic equation in p_ϕ :

$$m\ell^2\left(\Lambda^2-2\Lambda(p_\phi^B)+(p_\phi^B)^2\right)+\left(MR^2+mr^2\right)(p_\phi^B)^2-2mr\ell(p_\phi^B)\left(\Lambda-p_\phi^B\right)\sin\phi_B=Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]$$

$$m\ell^2\Lambda^2-2m\ell^2\Lambda(p_\phi^B)+\left(m\ell^2+2mr\ell\sin\phi_B+MR^2+mr^2\right)(p_\phi^B)^2-2mr\ell\Lambda\sin\phi_B(p_\phi^B)=Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]$$

Fig. 2.9.6

Lab (θ,φ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.

(Each value is positive.)

Throwing-momentum p^B_ϕ is a function of beam-relative angle ϕ_B, total E, and $\Lambda=p^B_\theta$.

Free-space trebuchet kinematics by symmetry: Algebraic approach

(using last line of p. 36)

(Assume zero-gravity)

$$H = \frac{m\ell^2\left(p_\theta^B - p_\phi^B\right)^2 + \left(MR^2+mr^2\right)\left(p_\phi^B\right)^2 - 2mr\ell p_\phi^B\left(p_\theta^B - p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]} \xcancel{-\left(MR-mr\right)g\sin\theta_B - mg\ell\cos\left(\phi_B+\theta_B\right)}$$

For zero-gravity H is not a function of θ_B

so : $\dot{p}_\theta^B = \frac{\partial H}{\partial \theta_B} = 0$ *and :* $p_\theta^B = \Lambda = const.$

H is not an explicit function of t so : H=const.=E

$$H = \frac{m\ell^2\left(\Lambda - p_\phi^B\right)^2 + \left(MR^2+mr^2\right)\left(p_\phi^B\right)^2 - 2mr\ell p_\phi^B\left(\Lambda - p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]} = const. = E$$

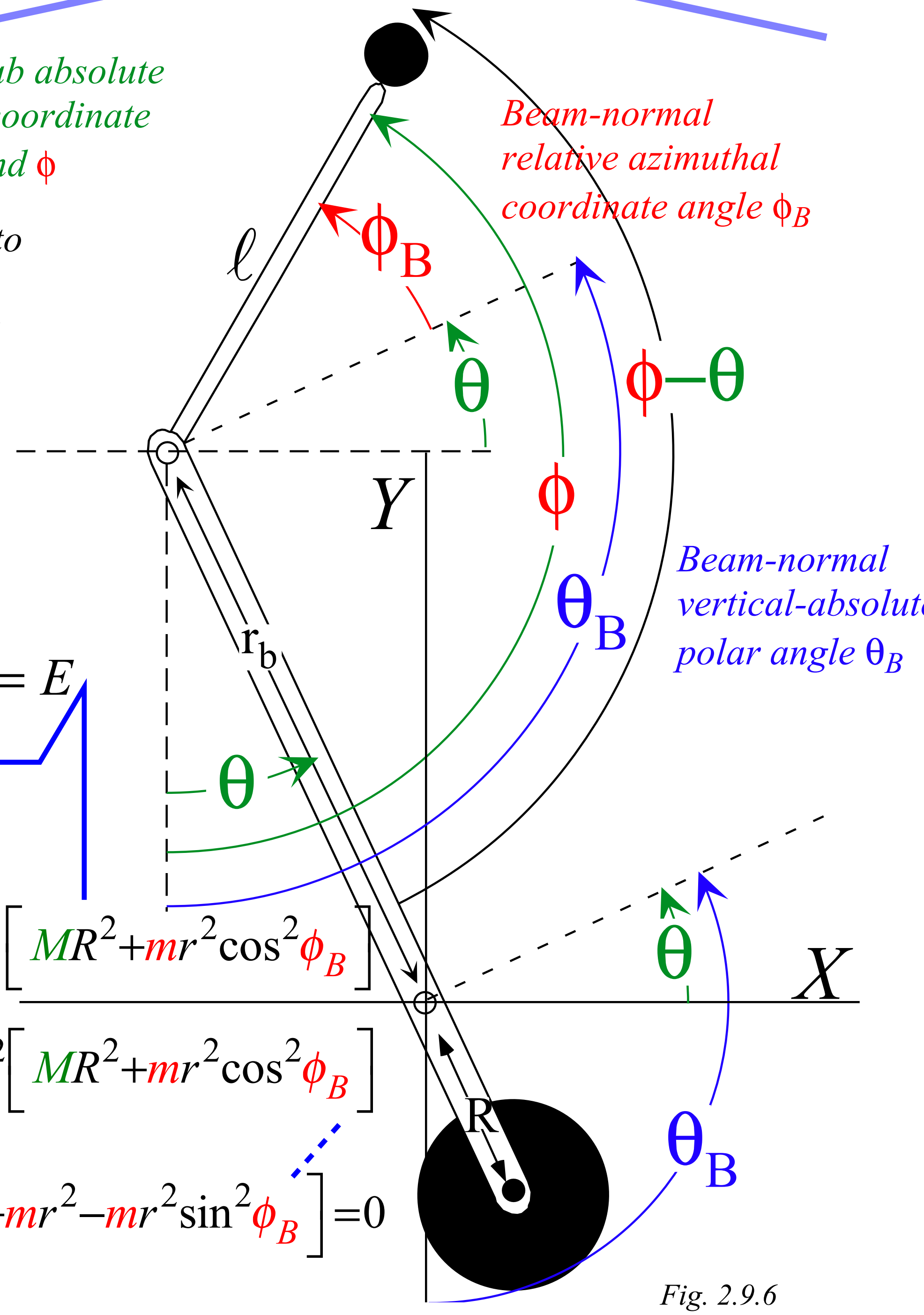
Rewrite H=E as a quadratic equation in p_ϕ :

$$m\ell^2\left(\Lambda^2 - 2\Lambda(p_\phi^B) + (p_\phi^B)^2\right) + \left(MR^2+mr^2\right)(p_\phi^B)^2 - 2mr\ell(p_\phi^B)\left(\Lambda - p_\phi^B\right)\sin\phi_B = Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]$$

$$m\ell^2\Lambda^2 - 2m\ell^2\Lambda(p_\phi^B) + \left(m\ell^2+2mr\ell\sin\phi_B+MR^2+mr^2\right)(p_\phi^B)^2 - 2mr\ell\Lambda\sin\phi_B(p_\phi^B) = Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]$$

$$\left(m\ell^2+2mr\ell\sin\phi_B+MR^2+mr^2\right)(p_\phi^B)^2 - 2\Lambda\left(mr\ell\sin\phi_B+m\ell^2\right)(p_\phi^B) + m\ell^2\Lambda^2 - Em\ell^2\left[MR^2+mr^2-mr^2\sin^2\phi_B\right] = 0$$

Throwing-momentum p^B_ϕ is a function of beam-relative angle ϕ_B, total E, and $\Lambda = p^B_\theta$.



Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B.

Beam-normal relative azimuthal coordinate angle ϕ_B

Beam-normal vertical-absolute polar angle θ_B

Fig. 2.9.6

Lab (θ,ϕ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.

(Each value is positive.)

Free-space trebuchet kinematics by symmetry: Algebraic approach

(using last line of p. 36)

(Assume zero-gravity)

$$H=\frac{m\ell^2\left(p_\theta^B-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(p_\theta^B-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}\xcancel{-\left(MR-mr\right)g\sin\theta_B-mg\ell\cos\left(\phi_B+\theta_B\right)}$$

For zero-gravity H is not a function of θ_B

so : $\dot{p}_\theta^B=\frac{\partial H}{\partial\theta_B}=0$ *and :* $p_\theta^B=\Lambda=const.$

H is not an explicit function of t so : H=const.=E

$$H=\frac{m\ell^2\left(\Lambda-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(\Lambda-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}=const.=E$$

Rewrite H=E as a quadratic equation in p_ϕ :

$$m\ell^2\left(\Lambda^2-2\Lambda(p_\phi^B)+(p_\phi^B)^2\right)+\left(MR^2+mr^2\right)(p_\phi^B)^2-2mr\ell(p_\phi^B)\left(\Lambda-p_\phi^B\right)\sin\phi_B=Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]$$

$$m\ell^2\Lambda^2-2m\ell^2\Lambda(p_\phi^B)+\left(m\ell^2+2mr\ell\sin\phi_B+MR^2+mr^2\right)(p_\phi^B)^2-2mr\ell\Lambda\sin\phi_B(p_\phi^B)=Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]$$

$$\left(m\ell^2+2mr\ell\sin\phi_B+MR^2+mr^2\right)(p_\phi^B)^2-2\Lambda\left(mr\ell\sin\phi_B+m\ell^2\right)(p_\phi^B)+m\ell^2\Lambda^2-Em\ell^2\left[MR^2+mr^2-mr^2\sin^2\phi_B\right]=0$$

$$\left(1+2(r/\ell)\sin\phi_B+J\right)(p_\phi^B)^2-2\Lambda\left((r/\ell)\sin\phi_B+1\right)(p_\phi^B)+\Lambda^2-E\left[I-mr^2\sin^2\phi_B\right]=0$$

Throwing-momentum $p^B{}_\phi$ is a function of beam-relative angle ϕ_B, total E, and $\Lambda=p^B{}_\theta$.

with: $J=\frac{MR^2+mr^2}{m\ell^2}$, $I=MR^2+mr^2$

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B.

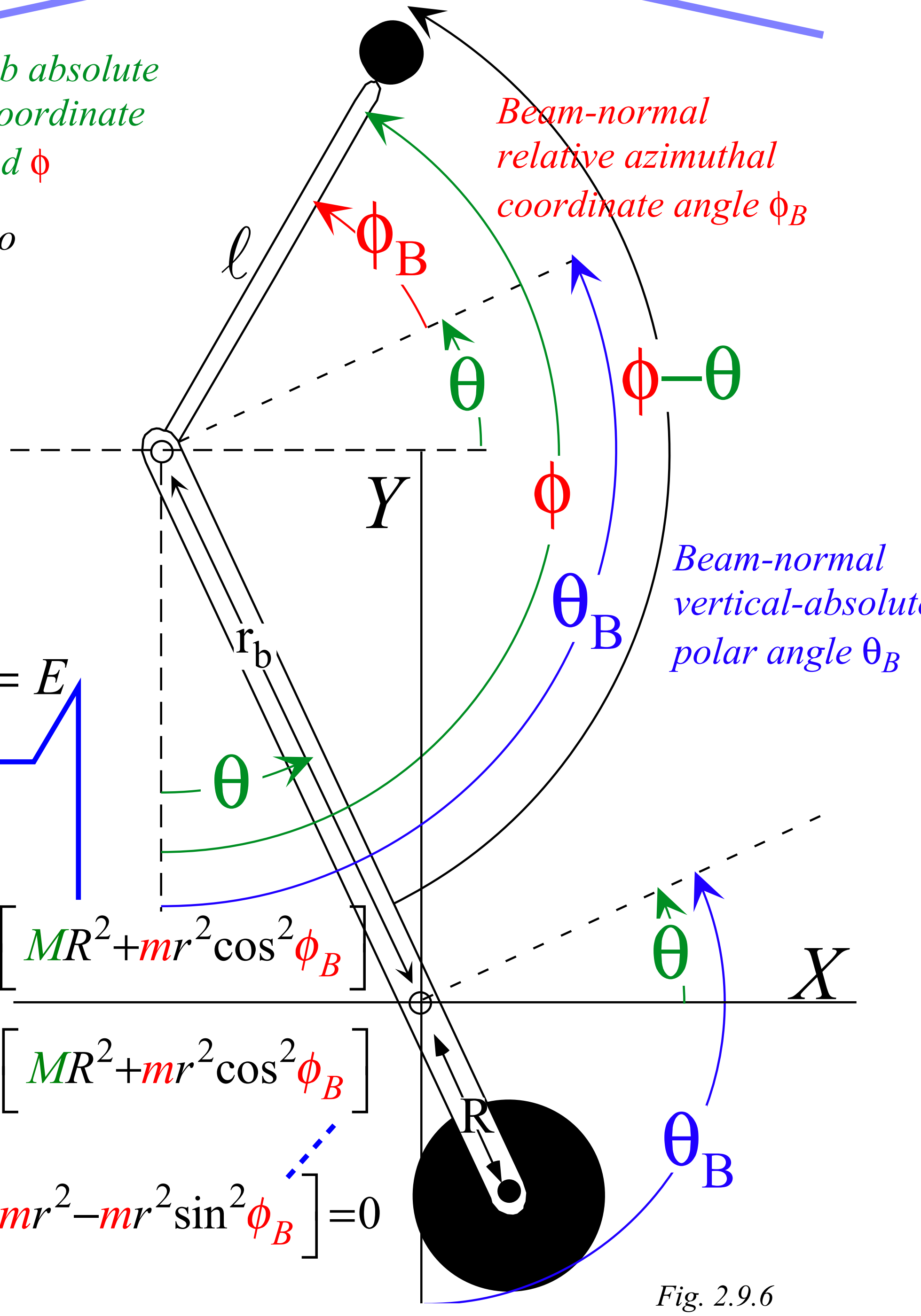


Fig. 2.9.6

Lab (θ,ϕ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.

(Each value is positive.)

Free-space trebuchet kinematics by symmetry: Algebraic approach

(using last line of p. 36)

(Assume zero-gravity)

$$H=\frac{m\ell^2\left(p_\theta^B-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(p_\theta^B-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}\xcancel{-\left(MR-mr\right)g\sin\theta_B-mg\ell\cos\left(\phi_B+\theta_B\right)}$$

For zero-gravity H is not a function of θ_B

so : $\dot{p}_\theta^B=\dfrac{\partial H}{\partial\theta_B}=0$ and : $p_\theta^B=\Lambda=const.$

H is not an explicit function of t so : $H=const.=E$

$$H=\frac{m\ell^2\left(\Lambda-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(\Lambda-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}=const.=E$$

Rewrite $H=E$ as a quadratic equation in p_ϕ :

$$m\ell^2\left(\Lambda^2-2\Lambda(p_\phi^B)+(p_\phi^B)^2\right)+\left(MR^2+mr^2\right)(p_\phi^B)^2-2mr\ell(p_\phi^B)\left(\Lambda-p_\phi^B\right)\sin\phi_B=Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]$$

$$m\ell^2\Lambda^2-2m\ell^2\Lambda(p_\phi^B)+\left(m\ell^2+2mr\ell\sin\phi_B+MR^2+mr^2\right)(p_\phi^B)^2-2mr\ell\Lambda\sin\phi_B(p_\phi^B)=Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]$$

$$\left(m\ell^2+2mr\ell\sin\phi_B+MR^2+mr^2\right)(p_\phi^B)^2-2\Lambda\left(mr\ell\sin\phi_B+m\ell^2\right)(p_\phi^B)+m\ell^2\Lambda^2-Em\ell^2\left[MR^2+mr^2-mr^2\sin^2\phi_B\right]=0$$

$$\left(1+2(r/\ell)\sin\phi_B+J\right)(p_\phi^B)^2-2\Lambda\left((r/\ell)\sin\phi_B+1\right)(p_\phi^B)+\Lambda^2-E\left[I-mr^2\sin^2\phi_B\right]=0$$

(using quadratic solution: $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$)

Throwing-momentum $p^B{}_\phi$ is a function of beam-relative angle ϕ_B, total E, and $\Lambda=p^B{}_\theta$.

$$p_\phi^B=\frac{2\Lambda\left((r/\ell)\sin\phi_B+1\right)\pm\sqrt{4\Lambda^2\left((r/\ell)\sin\phi_B+1\right)^2-4\left(1+2(r/\ell)\sin\phi_B+J\right)\left(\Lambda^2-E\left[I-mr^2\sin^2\phi_B\right]\right)}}{2\left(1+2(r/\ell)\sin\phi_B+J\right)}$$

with: $J=\dfrac{MR^2+mr^2}{m\ell^2}$, $I=MR^2+mr^2$

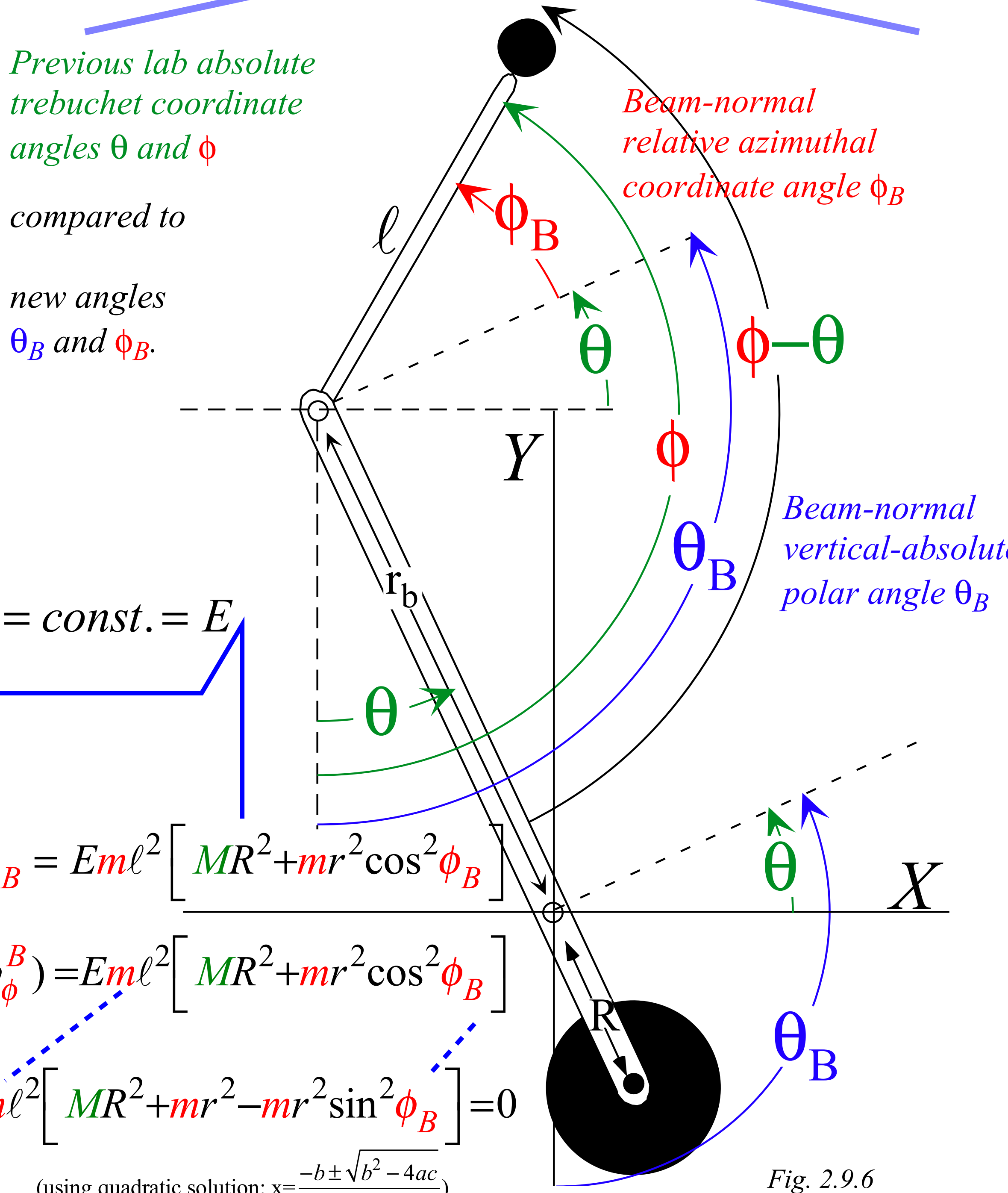


Fig. 2.9.6
Lab (θ,ϕ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.
(Each value is positive.)

Hamiltonian energy and momentum conservation and symmetry coordinates

- *Coordinate transformation helps reduce symmetric Hamiltonian*
- *Free-space trebuchet kinematics by symmetry*
  - *Algebraic approach*
  - → *Direct approach and Superball analogy*
- *Trebuchet vs Flinger and sports kinematics*
  - *Many approaches to Mechanics*

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

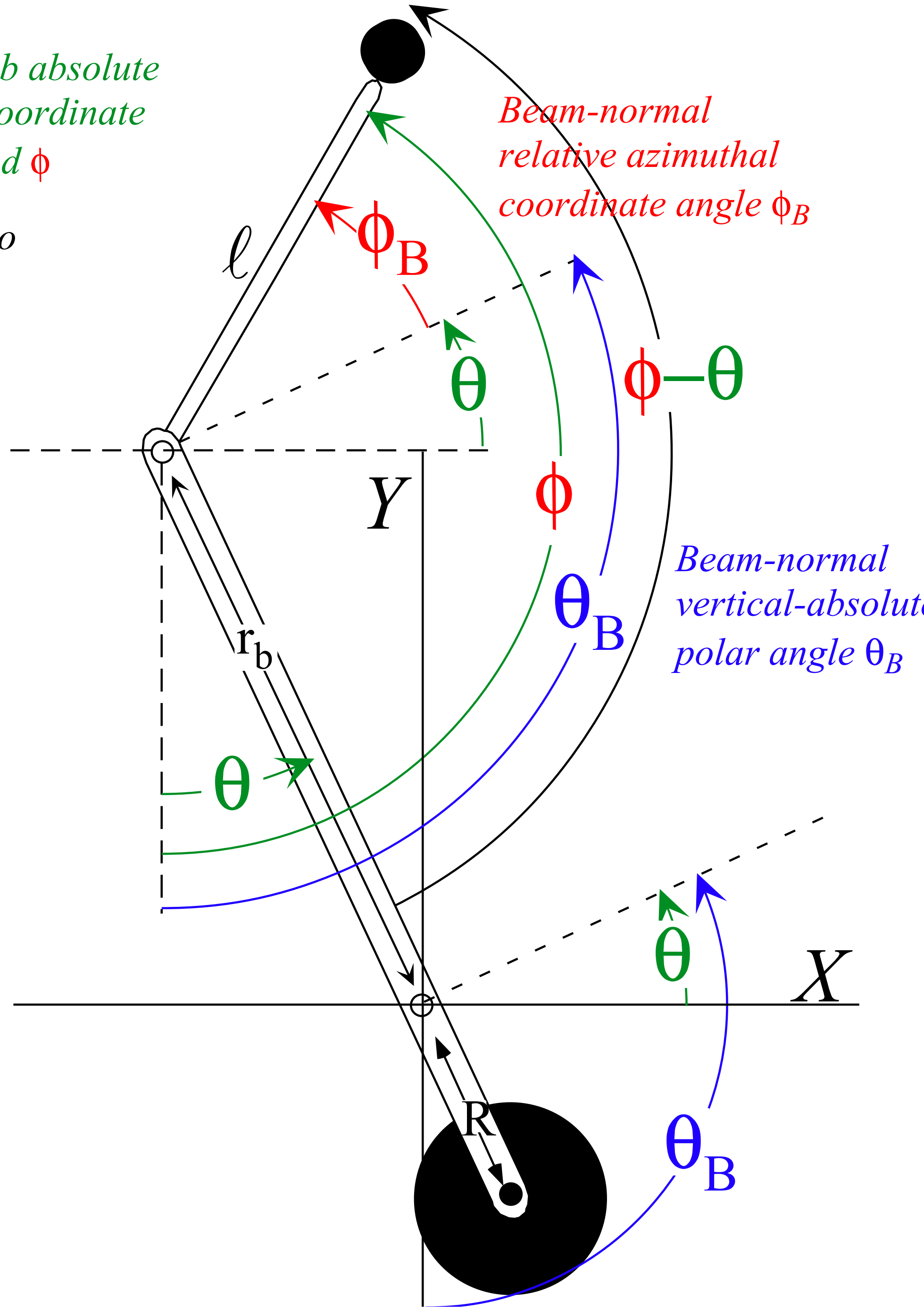
(Assume zero-gravity)

Energy for zero-gravity

$$\text{Total } KE = T = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$

$$p_\theta = \frac{\partial T}{\partial\dot{\theta}} = (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

$$p_\phi = \frac{\partial T}{\partial\dot{\phi}} = m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$



Previous lab absolute trebuchet coordinate angles θ *and* ϕ

compared to

new angles θ_B *and* ϕ_B.

Beam-normal relative azimuthal coordinate angle ϕ_B

Beam-normal vertical-absolut polar angle θ_B

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

(Assume zero-gravity)

Energy for zero-gravity

$$\text{Total KE} = T = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta - \phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta - \phi)$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta - \phi)$$

Transform to beam-relative coordinates and momenta

$$\theta = \theta_B \quad -\pi/2$$
$$\phi = \theta_B + \phi_B$$

$$\theta - \phi = -\phi_B - \pi/2$$

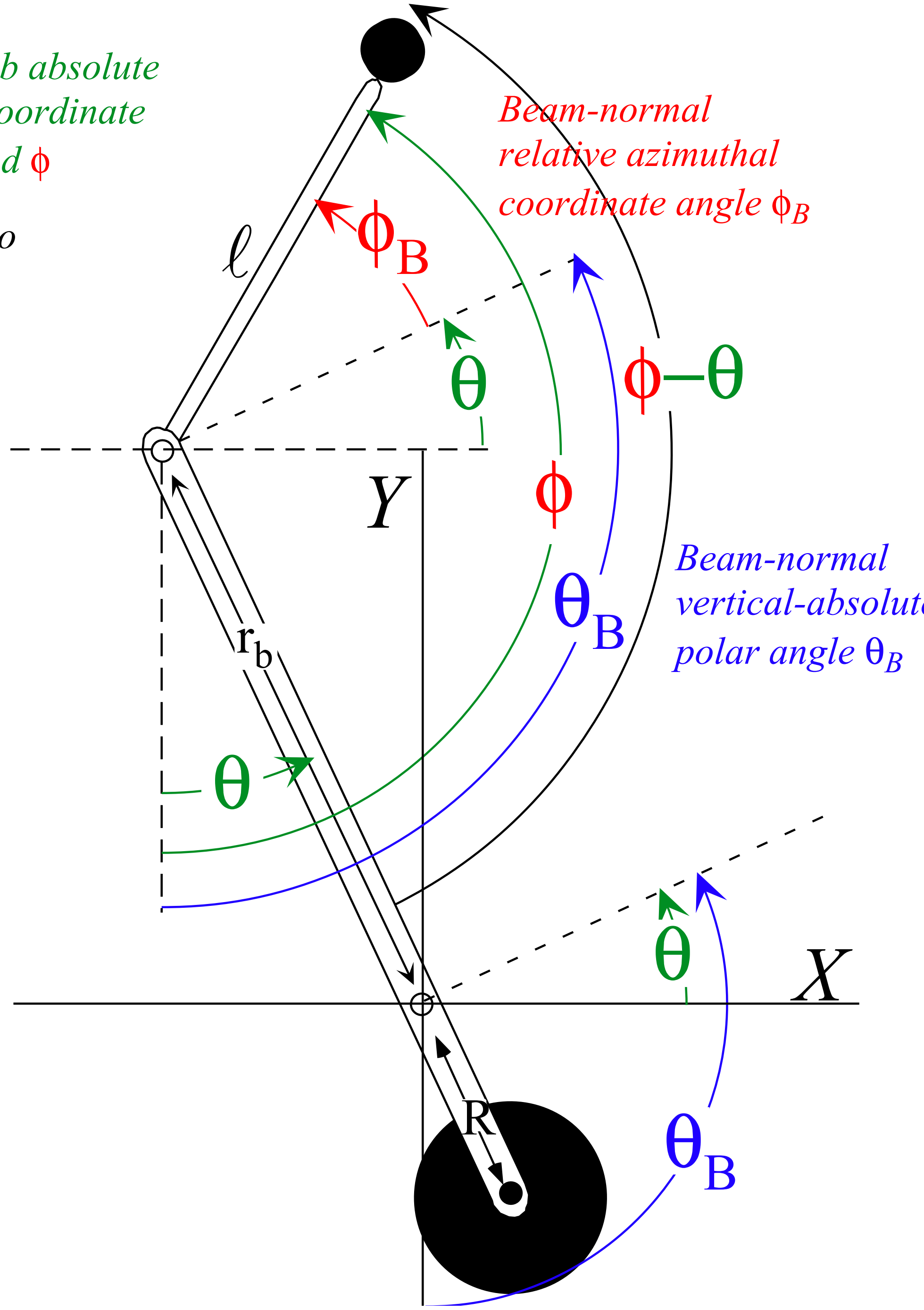
$$\theta_B = \theta \quad +\pi/2$$
$$\phi_B = -\theta + \phi - \pi/2$$

$$p_\theta = p_\theta^B - p_\phi^B$$
$$p_\phi = \quad p_\phi^B$$

Previous lab absolute trebuchet coordinate angles θ and φ

compared to

new angles θ_B and ϕ_B.



Beam-normal relative azimuthal coordinate angle ϕ_B

Beam-normal vertical-absolut polar angle θ_B

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

(Assume zero-gravity)

Energy for zero-gravity

$$\text{Total KE} = T = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

Transform to beam-relative coordinates and momenta

$$\boxed{\begin{aligned} \theta &= \theta_B \quad -\pi/2 \\ \phi &= \theta_B + \phi_B \end{aligned}} \qquad \boxed{\begin{aligned} \theta_B &= \theta \quad +\pi/2 \\ \phi_B &= -\theta + \phi - \pi/2 \end{aligned}} \qquad \boxed{\begin{aligned} p_\theta &= p_\theta^B - p_\phi^B \\ p_\phi &= \quad p_\phi^B \end{aligned}}$$

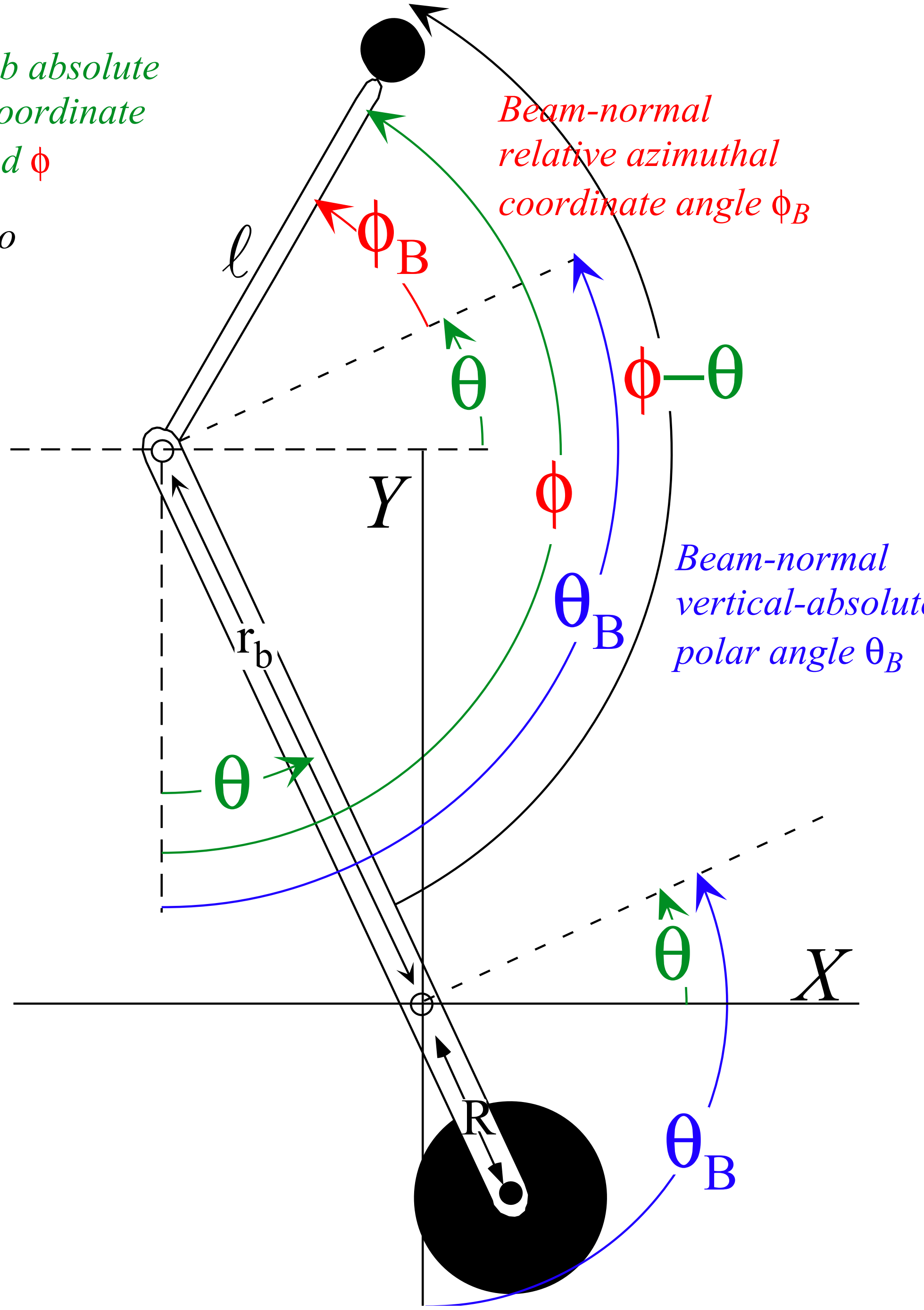
$$\theta - \phi = -\phi_B - \pi/2$$

$$2E = (MR^2 + mr^2)\dot{\theta}^2 + 2mr\ell\dot{\phi}\dot{\theta}\sin\phi_B + m\ell^2\dot{\phi}^2 = const.$$

Previous lab absolute trebuchet coordinate angles θ and φ

compared to

new angles θ_B and ϕ_B.



Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

(Assume zero-gravity)

Energy for zero-gravity

$$\text{Total } KE = T = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$

$$p_\theta = \frac{\partial T}{\partial\dot{\theta}} = (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

$$p_\phi = \frac{\partial T}{\partial\dot{\phi}} = m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

Transform to beam-relative coordinates and momenta

$\theta = \theta_B \quad -\pi/2$

$\phi = \theta_B + \phi_B$

$\theta_B = \theta \quad +\pi/2$

$\phi_B = -\theta + \phi - \pi/2$

$p_\theta = p_\theta^B - p_\phi^B$

$p_\phi = \quad p_\phi^B$

$\theta - \phi = -\phi_B - \pi/2$

$$2E = (MR^2 + mr^2)\dot{\theta}^2 + 2mr\ell\dot{\phi}\dot{\theta}\sin\phi_B + m\ell^2\dot{\phi}^2 = const.$$

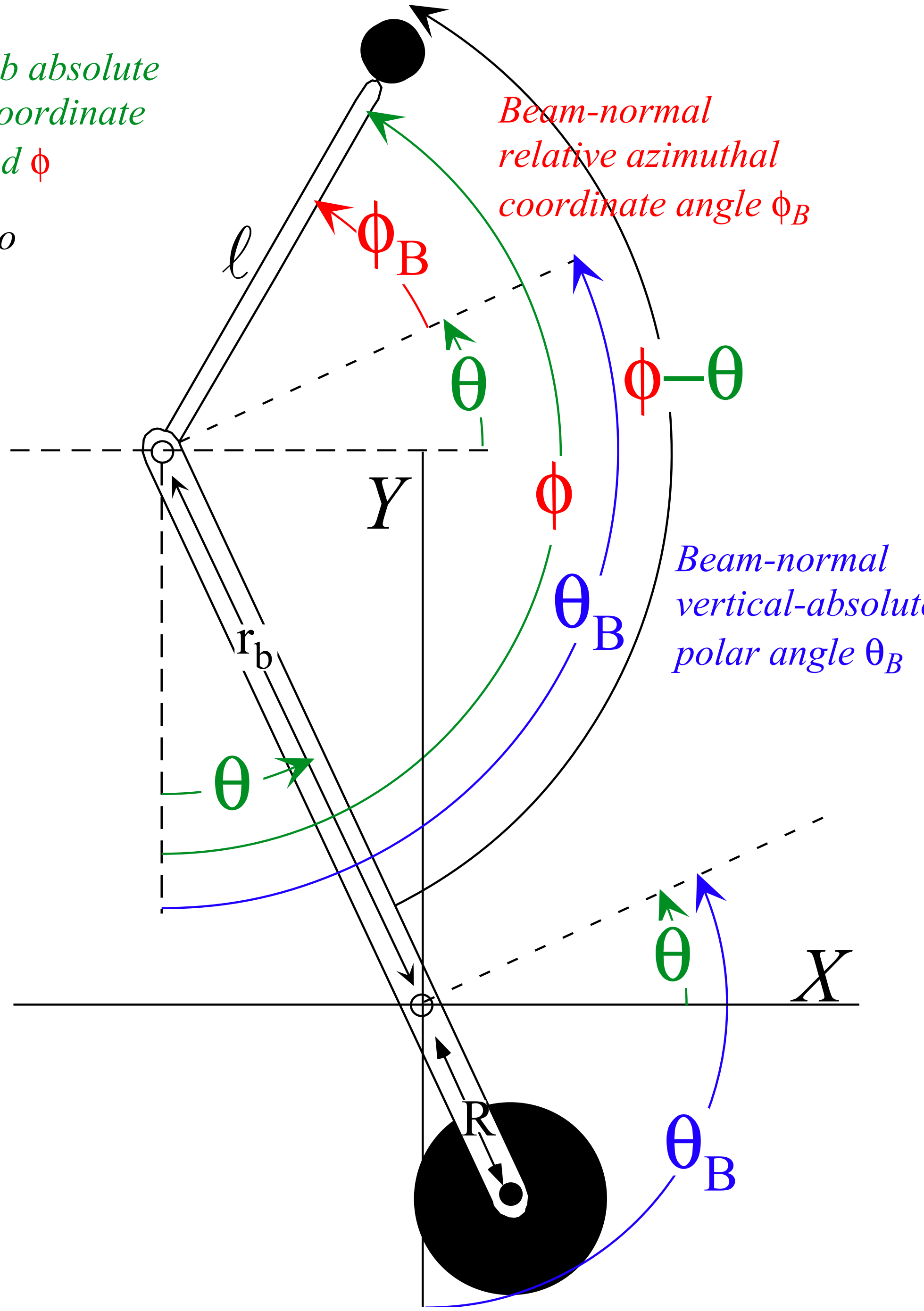
$$p_\theta^B = \Lambda = const. = \quad p_\theta \quad + \quad p_\phi$$

$$= \left((MR^2 + mr^2)\dot{\theta} + mr\ell\dot{\phi}\sin\phi_B \right) + \left(m\ell^2\dot{\phi} + mr\ell\dot{\theta}\sin\phi_B \right)$$

Previous lab absolute trebuchet coordinate angles θ and φ

compared to

new angles θ_B and ϕ_B.



Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Energy for zero-gravity

(Assume zero-gravity)

$$\text{Total } KE = T = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

Transform to beam-relative coordinates and momenta

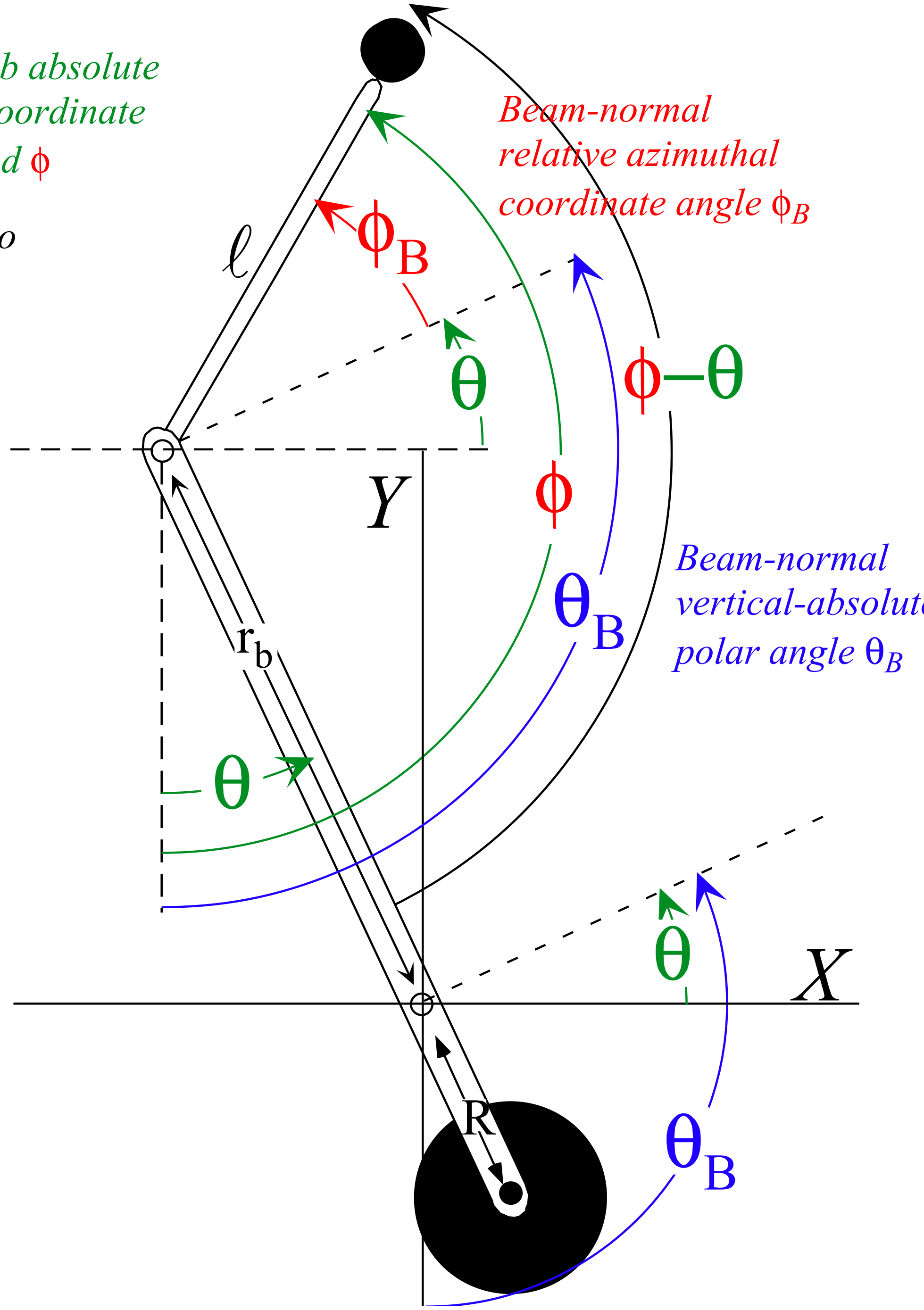
$$\boxed{\begin{aligned} \theta &= \theta_B - \pi/2 \\ \phi &= \theta_B + \phi_B \end{aligned}} \qquad \boxed{\begin{aligned} \theta_B &= \theta + \pi/2 \\ \phi_B &= -\theta + \phi - \pi/2 \end{aligned}} \qquad \boxed{\begin{aligned} p_\theta &= p_\theta^B - p_\phi^B \\ p_\phi &= p_\phi^B \end{aligned}}$$

$$\theta - \phi = -\phi_B - \pi/2$$

Previous lab absolute trebuchet coordinate angles θ and φ

compared to

new angles θ_B and ϕ_B.



$$2E = (MR^2 + mr^2)\dot{\theta}^2 + 2mr\ell\dot{\phi}\dot{\theta}\sin\phi_B + m\ell^2\dot{\phi}^2 = const.$$

$$p_\theta^B = \Lambda = const. = \quad p_\theta \quad + \quad p_\phi$$

$$= \left((MR^2 + mr^2)\dot{\theta} + mr\ell\dot{\phi}\sin\phi_B \right) + \left(m\ell^2\dot{\phi} + mr\ell\dot{\theta}\sin\phi_B \right)$$

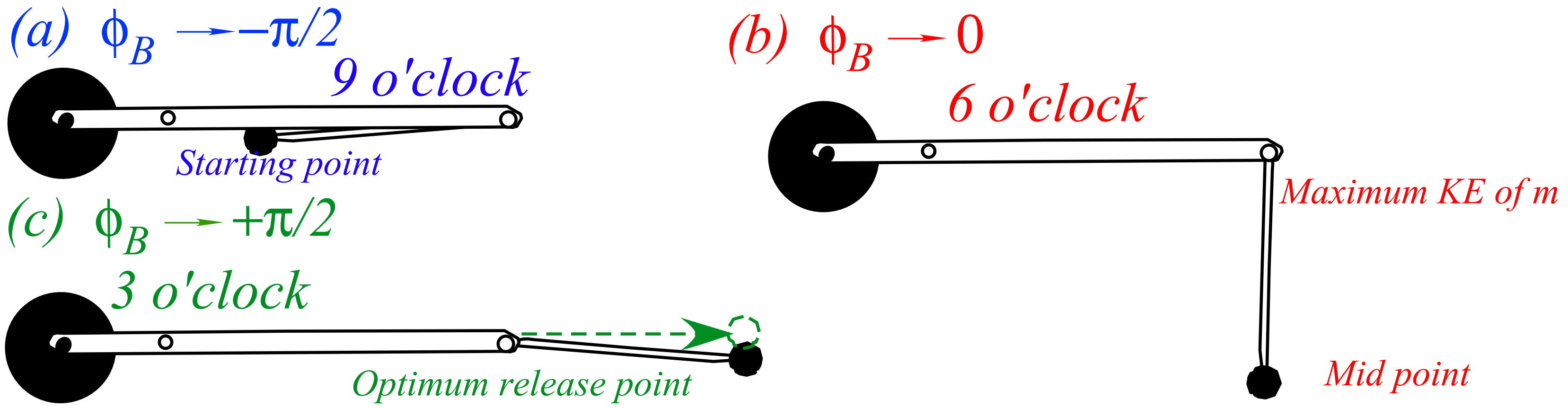
Case of equal arms $r = \ell$ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot{\theta}^2 + mr^2\left(\dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B + \dot{\phi}^2\right) \\ \Lambda &= MR^2\dot{\theta} + mr^2\left(1 + \sin\phi_B\right)\left(\dot{\theta} + \dot{\phi}\right) \end{aligned}\right\} \text{(For: } r = \ell\text{)}$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r = \ell$ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot{\theta}^2 + mr^2\left(\dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B + \dot{\phi}^2\right) \\ \Lambda &= MR^2\dot{\theta} + mr^2\left(1+\sin\phi_B\right)\left(\dot{\theta}+\dot{\phi}\right) \end{aligned}\right\} \begin{aligned}&\text{For:}\\ &r=\ell\end{aligned}$$

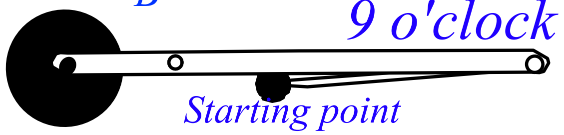


Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

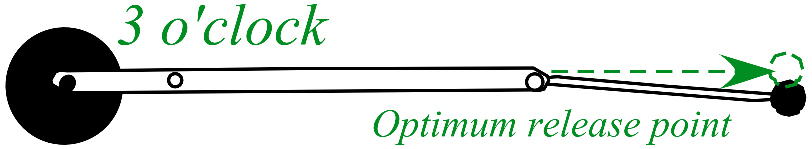
Case of equal arms r =ℓ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot\theta^2 + mr^2\left(\dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B + \dot\phi^2\right) \\ \Lambda &= MR^2\dot\theta + mr^2\left(1+\sin\phi_B\right)\left(\dot\theta+\dot\phi\right) \end{aligned}\right\} \text{For: } r=\ell$$

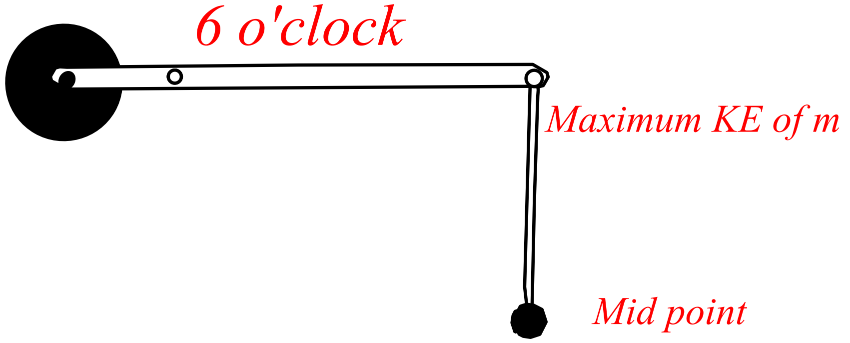
(a) $\phi_B \rightarrow -\pi/2$



(c) $\phi_B \rightarrow +\pi/2$



(b) $\phi_B \rightarrow 0$



Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

$$\phi_B = \frac{-\pi}{2}: \quad \sin\phi_B = -1 \quad \left\{\begin{aligned} 2E &= MR^2\dot\theta^2_{-\pi/2} + mr^2\left(\dot\phi_{-\pi/2} - \dot\phi_{-\pi/2}\right)^2 \\ \Lambda &= MR^2\dot\theta_{-\pi/2} \end{aligned}\right.$$

Conserved initial 2E

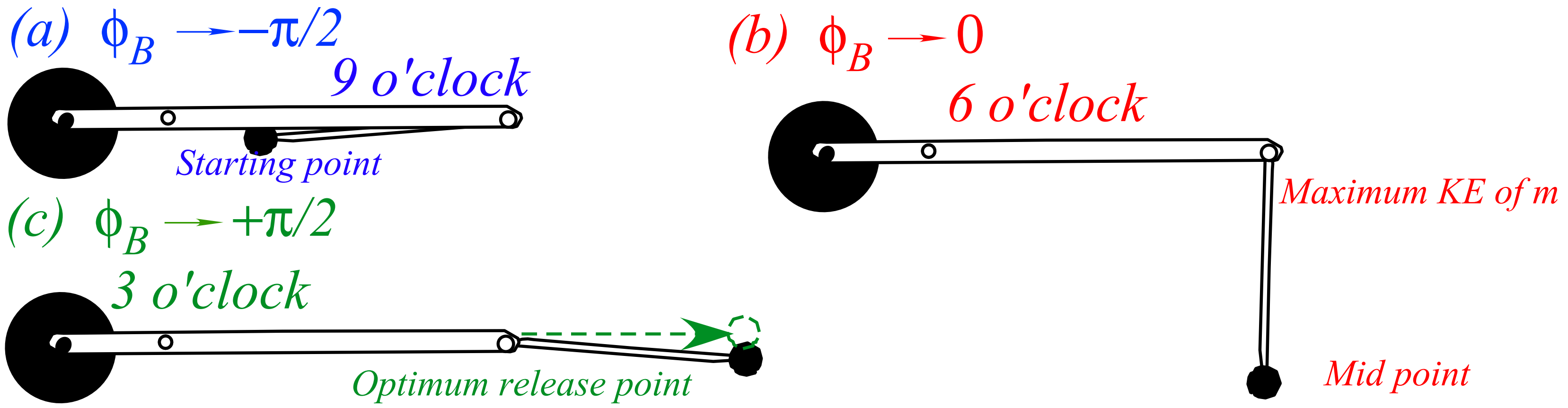
$$or: \left\{\begin{aligned} 2E &= MR^2\omega^2 \\ \Lambda &= MR^2\omega \end{aligned}\right.$$

initial Λ

For: $\dot\theta_{-\pi/2} = \omega = \dot\phi_{-\pi/2}$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r = \ell$ (easier algebra)



$$\left.\begin{aligned} 2E &= MR^2\dot{\theta}^2 + mr^2\left(\dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B + \dot{\phi}^2\right) \\ \Lambda &= MR^2\dot{\theta} + mr^2\left(1 + \sin\phi_B\right)\left(\dot{\theta} + \dot{\phi}\right) \end{aligned}\right\} \begin{matrix}\text{For:} \\ r = \ell\end{matrix}$$

Start at 9 o'clock with $\phi_B \sim -90°$ (beam r and throwing arm ℓ rotating together)

$$\begin{matrix}\phi_B = \dfrac{-\pi}{2}: \\ \sin\phi_B = -1\end{matrix} \left\{\begin{aligned} 2E &= MR^2\dot{\theta}^2_{-\pi/2} + mr^2\left(\dot{\phi}_{-\pi/2} - \dot{\phi}_{-\pi/2}\right)^2 \\ \Lambda &= MR^2\dot{\theta}_{-\pi/2} \end{aligned}\right. \quad or: \overset{\text{Conserved}}{\left\{\begin{aligned} &\text{initial } 2E \\ &2E = MR^2\omega^2 \\ &\Lambda = MR^2\omega \\ &\text{initial } \Lambda \end{aligned}\right.} \quad \text{For: } \dot{\theta}_{-\pi/2} = \omega = \dot{\phi}_{-\pi/2}$$

Move to 6 o'clock with $\phi_B \sim 0°$ (beam r slowing, throwing arm ℓ accelerating)

$$\begin{matrix}\phi_B = 0: \\ \sin\phi_B = 0\end{matrix} \left\{\begin{aligned} 2E &= MR^2\dot{\theta}_0^2 + mr^2\left(\dot{\phi}_0^2 + \dot{\theta}_0^2\right) \\ \Lambda &= MR^2\dot{\theta}_0 + mr^2\left(\dot{\phi}_0 + \dot{\theta}_0\right) \end{aligned}\right.$$

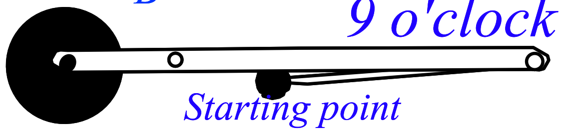
Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms r =ℓ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot{\theta}^2 + mr^2\left(\dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B + \dot{\phi}^2\right) \\ \Lambda &= MR^2\dot{\theta} + mr^2\left(1+\sin\phi_B\right)\left(\dot{\theta}+\dot{\phi}\right) \end{aligned}\right\} \text{For: } r=\ell$$

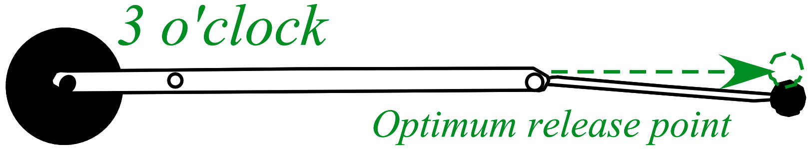
(a) $\phi_B \rightarrow -\pi/2$

9 o'clock



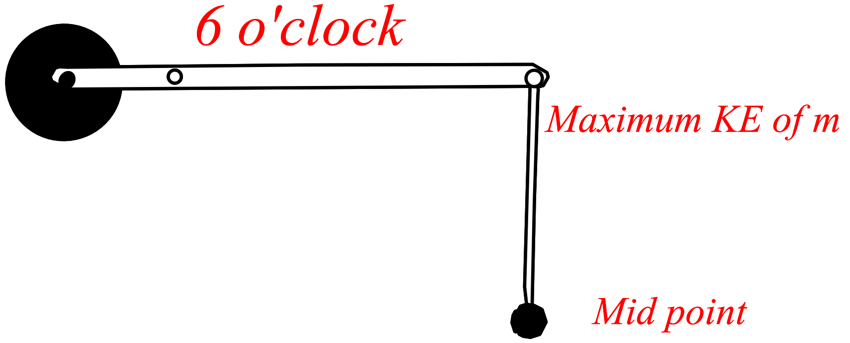
(c) $\phi_B \rightarrow +\pi/2$

3 o'clock



(b) $\phi_B \rightarrow 0$

6 o'clock



Start at 9 o’clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

$$\phi_B = \frac{-\pi}{2} : \left\{ \begin{aligned} 2E &= MR^2\dot{\theta}^2_{-\pi/2} + mr^2\left(\dot{\phi}_{-\pi/2} - \dot{\phi}_{-\pi/2}\right)^2 \\ \Lambda &= MR^2\dot{\theta}_{-\pi/2} \end{aligned} \right.$$

$\sin\phi_B$=-1

Conserved

$$or: \left\{ \begin{aligned} &\text{initial } 2E \\ &2E = MR^2\omega^2 \\ &\Lambda = MR^2\omega \\ &\text{initial } \Lambda \end{aligned} \right.$$

For: $\dot{\theta}_{-\pi/2} = \omega = \dot{\phi}_{-\pi/2}$

Move to 6 o’clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B = 0 : \left\{ \begin{aligned} 2E &= MR^2\dot{\theta}_0^2 + mr^2\left(\dot{\phi}_0^2 + \dot{\theta}_0^2\right) \\ \Lambda &= MR^2\dot{\theta}_0 + mr^2\left(\dot{\phi}_0 + \dot{\theta}_0\right) \end{aligned} \right.$$

$\sin\phi_B$=0

Move to 3 o’clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

Conserved

initial 2E

$$\phi_B = \pi/2 : \left\{ \begin{aligned} 2E &= MR^2\dot{\theta}^2_{\pi/2} + mr^2\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right)^2 = MR^2\omega^2 \\ \Lambda &= MR^2\dot{\theta}_{\pi/2} + 2mr^2\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right) = MR^2\omega \end{aligned} \right.$$

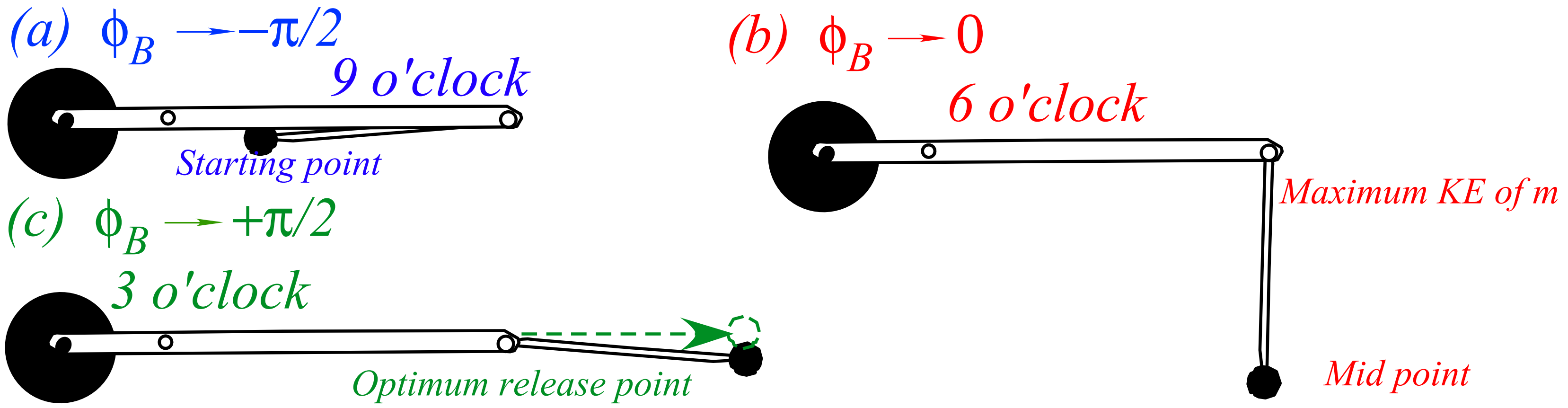
initial Λ

$\sin\phi_B$=+1

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms r =ℓ (easier algebra)

$$\left.\begin{array}{c} 2E=MR^2\dot\theta^2+mr^2\left(\dot\theta^2+2\dot\phi\dot\theta\sin\phi_B+\dot\phi^2\right) \\ \Lambda = MR^2\dot\theta + mr^2\left(1+\sin\phi_B\right)\left(\dot\theta+\dot\phi\right) \end{array}\right\} \begin{array}{l}\text{For:}\\ r=\ell\end{array}$$



(a) $\phi_B \to -\pi/2$

9 o'clock

Starting point

(c) $\phi_B \to +\pi/2$

3 o'clock

Optimum release point

(b) $\phi_B \to 0$

6 o'clock

Maximum KE of m

Mid point

Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

$$\begin{array}{l}\phi_B = \dfrac{-\pi}{2}: \\ \sin\phi_B\text{=-1}\end{array}\left\{\begin{array}{c} 2E = MR^2\dot\theta^2_{-\pi/2} + mr^2\left(\dot\phi_{-\pi/2} - \dot\phi_{-\pi/2}\right)^2 \\ \Lambda = MR^2\dot\theta_{-\pi/2}\end{array}\right. \quad or: \left\{\begin{array}{c} \text{initial } 2E \\ 2E = MR^2\omega^2 \\ \Lambda = MR^2\omega \\ \text{initial } \Lambda \end{array}\right.$$

Conserved

For: $\dot\theta_{-\pi/2} = \omega = \dot\phi_{-\pi/2}$

$$KE(m) = \frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B\right) = \left\{\begin{array}{ll} \dfrac{mr^2}{2}\left(\dot\phi-\dot\theta\right)^2 & \left(\text{For: } \phi_B = -\dfrac{\pi}{2}\right) \\ \dfrac{mr^2}{2}\left(\dot\phi^2+\dot\theta^2\right) & \left(\text{For: } \phi_B = 0\right) \\ \dfrac{mr^2}{2}\left(\dot\phi+\dot\theta\right)^2 & \left(\text{For: } \phi_B = \dfrac{\pi}{2}\right) \end{array}\right.$$

Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\begin{array}{l}\phi_B = 0: \\ \sin\phi_B\text{=0}\end{array}\left\{\begin{array}{c} 2E = MR^2\dot\theta_0^2 + mr^2\left(\dot\phi_0^2 + \dot\theta_0^2\right) \\ \Lambda = MR^2\dot\theta_0 + mr^2\left(\dot\phi_0 + \dot\theta_0\right)\end{array}\right.$$

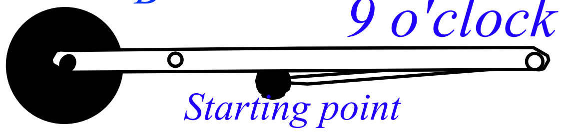
Move to 3 o'clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

Conserved

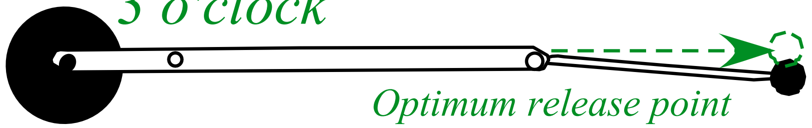
$$\begin{array}{l}\phi_B = \pi/2: \\ \sin\phi_B\text{=+1}\end{array}\left\{\begin{array}{l} 2E = MR^2\dot\theta^2_{\pi/2} + mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2 \overset{\text{initial } 2E}{= MR^2\omega^2} \\ \Lambda = MR^2\dot\theta_{\pi/2} + 2mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right) \underset{\text{initial } \Lambda}{= MR^2\omega}\end{array}\right.$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

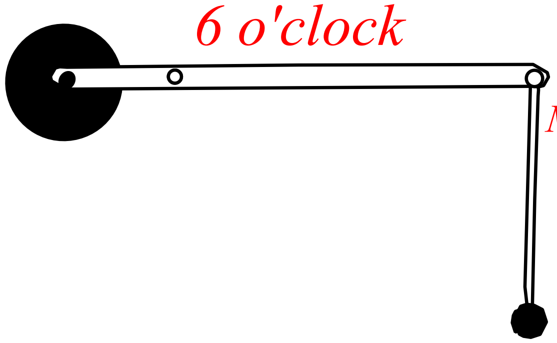
(a) $\phi_B \rightarrow -\pi/2$



(c) $\phi_B \rightarrow +\pi/2$



(b) $\phi_B \rightarrow 0$



Case of equal arms r =ℓ (easier algebra)

$$\left.\begin{array}{l} 2E=MR^2\dot{\theta}^2+mr^2\left(\dot{\theta}^2+2\dot{\phi}\dot{\theta}\sin\phi_B+\dot{\phi}^2\right) \\ \Lambda = MR^2\dot{\theta}+mr^2\left(1+\sin\phi_B\right)\left(\dot{\theta}+\dot{\phi}\right) \end{array}\right\} \text{For: } r=\ell$$

Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

$$\phi_B = \frac{-\pi}{2} : \left\{ \begin{array}{c} 2E = MR^2\dot{\theta}^{\,2}_{-\pi/2} + mr^2\left(\dot{\phi}_{-\pi/2} - \dot{\phi}_{-\pi/2}\right)^2 \\ \Lambda = MR^2\dot{\theta}_{-\pi/2} \end{array}\right.$$

$\sin\phi_B$=-1

Conserved

$$or : \left\{ \begin{array}{l} \text{initial } 2E \\ 2E = MR^2\omega^2 \\ \Lambda = MR^2\omega \\ \text{initial } \Lambda \end{array}\right.$$

For: $\dot{\theta}_{-\pi/2} = \omega = \dot{\phi}_{-\pi/2}$

Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B = 0 : \left\{ \begin{array}{c} 2E = MR^2\dot{\theta}_0^2 + mr^2\left(\dot{\phi}_0^2 + \dot{\theta}_0^2\right) \\ \Lambda = MR^2\dot{\theta}_0 + mr^2\left(\dot{\phi}_0 + \dot{\theta}_0\right) \end{array}\right.$$

$\sin\phi_B$=0

Move to 3 o'clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

Conserved

$$\phi_B = \pi/2 : \left\{ \begin{array}{l} 2E = MR^2\dot{\theta}^2_{\pi/2} + mr^2\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right)^2 \;\; \overset{\text{initial } 2E}{= MR^2\omega^2} \\ \Lambda = MR^2\dot{\theta}_{\pi/2} + 2mr^2\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right) \;\; \underset{\text{initial } \Lambda}{= MR^2\omega} \end{array}\right.$$

$\sin\phi_B$=+1

$$\rightarrow \left(\omega^2 - \dot{\theta}^2_{\pi/2}\right) = \frac{mr^2}{MR^2}\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right)^2$$

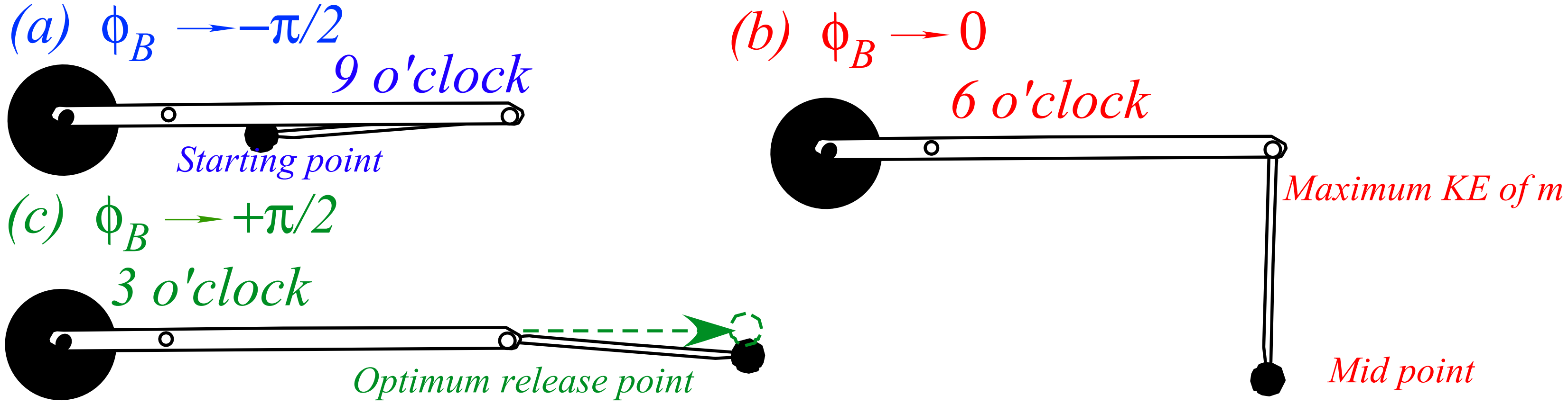
$$\rightarrow \left(\omega - \dot{\theta}_{\pi/2}\right) = \frac{2mr^2}{MR^2}\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right)$$

$$KE(m) = \frac{mr^2}{2}\left(\dot{\phi}^2 + \dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B\right)$$

$$= \left\{ \begin{array}{ll} \frac{mr^2}{2}\left(\dot{\phi} - \dot{\theta}\right)^2 & \left(\text{For: } \phi_B = -\frac{\pi}{2}\right) \\ \frac{mr^2}{2}\left(\dot{\phi}^2 + \dot{\theta}^2\right) & \left(\text{For: } \phi_B = 0\right) \\ \frac{mr^2}{2}\left(\dot{\phi} + \dot{\theta}\right)^2 & \left(\text{For: } \phi_B = \frac{\pi}{2}\right) \end{array}\right.$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r=\ell$ (easier algebra)



$$\left.\begin{aligned} 2E&=MR^2\dot\theta^2+mr^2\left(\dot\theta^2+2\dot\phi\dot\theta\sin\phi_B+\dot\phi^2\right)\\ \Lambda&=MR^2\dot\theta+mr^2(1+\sin\phi_B)(\dot\theta+\dot\phi)\end{aligned}\right\}\ \text{For: } r=\ell$$

Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

Conserved

$$\begin{matrix}\phi_B=\dfrac{-\pi}{2}:\\ \sin\phi_B=-1\end{matrix}\left\{\begin{aligned}2E&=MR^2\dot\theta^2_{-\pi/2}+mr^2\left(\dot\phi_{-\pi/2}-\dot\phi_{-\pi/2}\right)^2\\ \Lambda&=MR^2\dot\theta_{-\pi/2}\end{aligned}\right.\quad or:\left\{\begin{aligned}&\text{initial }2E\\ &2E=MR^2\omega^2\\ &\Lambda=MR^2\omega\\ &\text{initial }\Lambda\end{aligned}\right.\quad \text{For: }\dot\theta_{-\pi/2}=\omega=\dot\phi_{-\pi/2}$$

Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\begin{matrix}\phi_B=0:\\ \sin\phi_B=0\end{matrix}\left\{\begin{aligned}2E&=MR^2\dot\theta_0^2+mr^2\left(\dot\phi_0^2+\dot\theta_0^2\right)\\ \Lambda&=MR^2\dot\theta_0+mr^2\left(\dot\phi_0+\dot\theta_0\right)\end{aligned}\right.$$

$$KE(m)=\frac{mr^2}{2}\left(\dot\phi^2+\dot\theta^2+2\dot\phi\dot\theta\sin\phi_B\right)$$

$$=\begin{cases}\dfrac{mr^2}{2}\left(\dot\phi-\dot\theta\right)^2 & \left(\text{For: }\phi_B=-\dfrac{\pi}{2}\right)\\ \dfrac{mr^2}{2}\left(\dot\phi^2+\dot\theta^2\right) & \left(\text{For: }\phi_B=0\right)\\ \dfrac{mr^2}{2}\left(\dot\phi+\dot\theta\right)^2 & \left(\text{For: }\phi_B=\dfrac{\pi}{2}\right)\end{cases}$$

Move to 3 o'clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

Conserved

$$\begin{matrix}\phi_B=\pi/2:\\ \sin\phi_B=+1\end{matrix}\left\{\begin{aligned}2E&=MR^2\dot\theta^2_{\pi/2}+mr^2\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)^2 &&\overset{\text{initial }2E}{=MR^2\omega^2}\\ \Lambda&=MR^2\dot\theta_{\pi/2}+2mr^2\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right) &&\underset{\text{initial }\Lambda}{=MR^2\omega}\end{aligned}\right.$$

$$\rightarrow\ (\omega^2-\dot\theta^2_{\pi/2})=\frac{mr^2}{MR^2}\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)^2$$

$$\rightarrow\ (\omega-\dot\theta_{\pi/2})=\frac{2mr^2}{MR^2}\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)$$

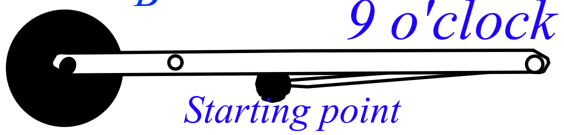
$$\xrightarrow{\text{divide }2E\text{ by }\Lambda}\ (\omega+\dot\theta_{\pi/2})=\frac{1}{2}\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

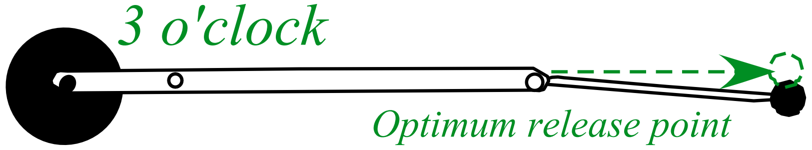
Case of equal arms r =ℓ (easier algebra)

$$\left.\begin{aligned}2E&=MR^2\dot\theta^2+mr^2\left(\dot\theta^2+2\dot\phi\dot\theta\sin\phi_B+\dot\phi^2\right)\\ \Lambda&=MR^2\dot\theta+mr^2\left(1+\sin\phi_B\right)\left(\dot\theta+\dot\phi\right)\end{aligned}\right\}\quad \text{For: } r=\ell$$

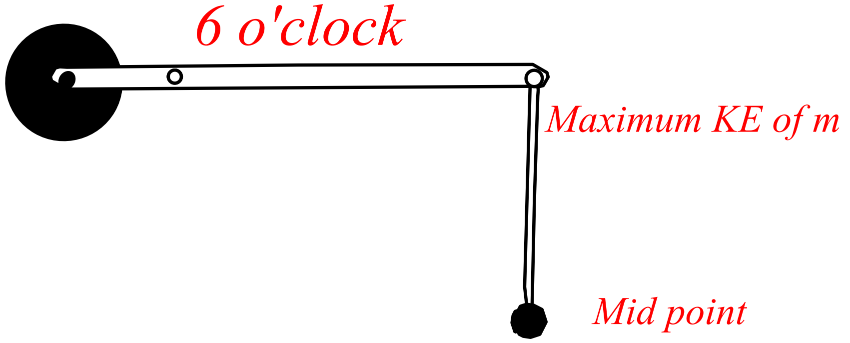
(a) $\phi_B \rightarrow -\pi/2$



(c) $\phi_B \rightarrow +\pi/2$



(b) $\phi_B \rightarrow 0$



Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

$$\phi_B=\frac{-\pi}{2}: \left\{\begin{aligned}2E&=MR^2\dot\theta^2_{-\pi/2}+mr^2\left(\dot\phi_{-\pi/2}-\dot\phi_{-\pi/2}\right)^2\\ \Lambda&=MR^2\dot\theta_{-\pi/2}\end{aligned}\right. \quad or: \left\{\begin{aligned}2E&=MR^2\omega^2\\ \Lambda&=MR^2\omega\end{aligned}\right. \quad \text{For: } \dot\theta_{-\pi/2}=\omega=\dot\phi_{-\pi/2}$$

$\sin\phi_B$=-1

Conserved initial 2E

initial Λ

Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B=0: \left\{\begin{aligned}2E&=MR^2\dot\theta_0^2+mr^2\left(\dot\phi_0^2+\dot\theta_0^2\right)\\ \Lambda&=MR^2\dot\theta_0+mr^2\left(\dot\phi_0+\dot\theta_0\right)\end{aligned}\right.$$

$\sin\phi_B$=0

$$KE(m)=\frac{mr^2}{2}\left(\dot\phi^2+\dot\theta^2+2\dot\phi\dot\theta\sin\phi_B\right)=\left\{\begin{aligned}&\frac{mr^2}{2}\left(\dot\phi-\dot\theta\right)^2 &&\left(\text{For: } \phi_B=-\frac{\pi}{2}\right)\\ &\frac{mr^2}{2}\left(\dot\phi^2+\dot\theta^2\right) &&\left(\text{For: } \phi_B=0\right)\\ &\frac{mr^2}{2}\left(\dot\phi+\dot\theta\right)^2 &&\left(\text{For: } \phi_B=\frac{\pi}{2}\right)\end{aligned}\right.$$

Move to 3 o'clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

Conserved

$$\phi_B=\pi/2: \left\{\begin{aligned}2E&=MR^2\dot\theta^2_{\pi/2}+mr^2\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)^2 &&= MR^2\omega^2 \text{ initial 2E}\\ \Lambda&=MR^2\dot\theta_{\pi/2}+2mr^2\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right) &&= MR^2\omega \text{ initial } \Lambda\end{aligned}\right.$$

$\sin\phi_B$=+1

$$\left(\omega^2-\dot\theta^2_{\pi/2}\right)=\frac{mr^2}{MR^2}\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)^2$$

$$\left(\omega-\dot\theta_{\pi/2}\right)=\frac{2mr^2}{MR^2}\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)$$

divide 2E by Λ

$$\left(\omega+\dot\theta_{\pi/2}\right)=\frac{1}{2}\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)$$

$$\downarrow$$

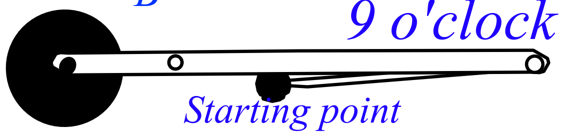
$$\boxed{\dot\phi_{\pi/2}=\dot\theta_{\pi/2}+2\omega}$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

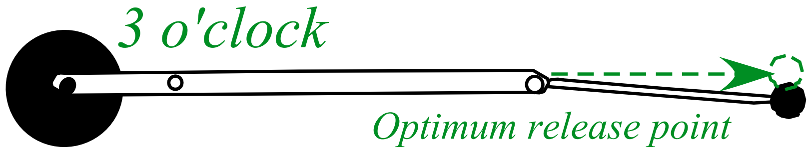
Case of equal arms $r=\ell$ (easier algebra)

$$\left.\begin{aligned} 2E&=MR^2\dot\theta^2+mr^2\left(\dot\theta^2+2\dot\phi\dot\theta\sin\phi_B+\dot\phi^2\right)\\ \Lambda&=MR^2\dot\theta+mr^2\left(1+\sin\phi_B\right)\left(\dot\theta+\dot\phi\right)\end{aligned}\right\}\ \text{For: } r=\ell$$

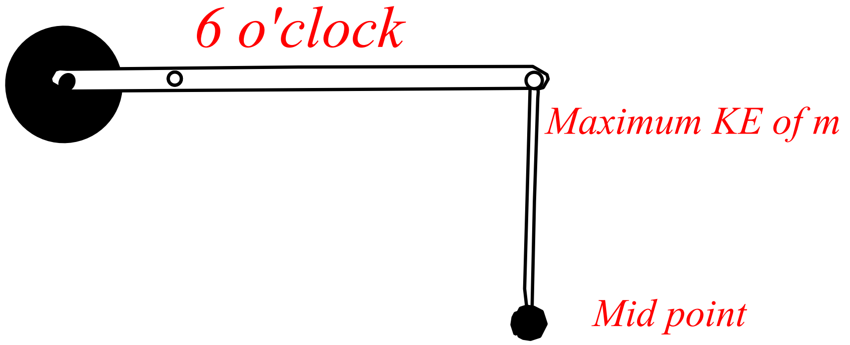
(a) $\phi_B \rightarrow -\pi/2$



(c) $\phi_B \rightarrow +\pi/2$



(b) $\phi_B \rightarrow 0$



Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

$$\begin{array}{l}\phi_B=\dfrac{-\pi}{2}:\\ \sin\phi_B\text{=-1}\end{array}\left\{\begin{aligned} 2E&=MR^2\dot\theta^2_{-\pi/2}+mr^2\left(\dot\phi_{-\pi/2}-\dot\phi_{-\pi/2}\right)^2\\ \Lambda&=MR^2\dot\theta_{-\pi/2}\end{aligned}\right.\quad or:\left\{\begin{aligned} &\text{Conserved initial } 2E\\ &2E=MR^2\omega^2\\ &\Lambda=MR^2\omega\\ &\text{initial }\Lambda\end{aligned}\right.\quad \text{For: } \dot\theta_{-\pi/2}=\omega=\dot\phi_{-\pi/2}$$

$$KE(m)=\frac{mr^2}{2}\left(\dot\phi^2+\dot\theta^2+2\dot\phi\dot\theta\sin\phi_B\right)=\left\{\begin{array}{ll}\dfrac{mr^2}{2}\left(\dot\phi-\dot\theta\right)^2 & \left(\text{For: } \phi_B=-\dfrac{\pi}{2}\right)\\ \dfrac{mr^2}{2}\left(\dot\phi^2+\dot\theta^2\right) & \left(\text{For: } \phi_B=0\right)\\ \dfrac{mr^2}{2}\left(\dot\phi+\dot\theta\right)^2 & \left(\text{For: } \phi_B=\dfrac{\pi}{2}\right)\end{array}\right.$$

Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\begin{array}{l}\phi_B=0:\\ \sin\phi_B\text{=0}\end{array}\left\{\begin{aligned} 2E&=MR^2\dot\theta_0^2+mr^2\left(\dot\phi_0^2+\dot\theta_0^2\right)\\ \Lambda&=MR^2\dot\theta_0+mr^2\left(\dot\phi_0+\dot\theta_0\right)\end{aligned}\right.$$

Move to 3 o'clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

$$\begin{array}{l}\phi_B=\pi/2:\\ \sin\phi_B\text{=+1}\end{array}\left\{\begin{aligned} 2E&=MR^2\dot\theta^2_{\pi/2}+mr^2\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)^2 &&\overset{\text{Conserved initial 2E}}{=MR^2\omega^2}\\ \Lambda&=MR^2\dot\theta_{\pi/2}+2mr^2\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right) &&\underset{\text{initial }\Lambda}{=MR^2\omega}\end{aligned}\right.$$

$$\longrightarrow\ (\omega^2-\dot\theta^2_{\pi/2})=\frac{mr^2}{MR^2}\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)^2$$

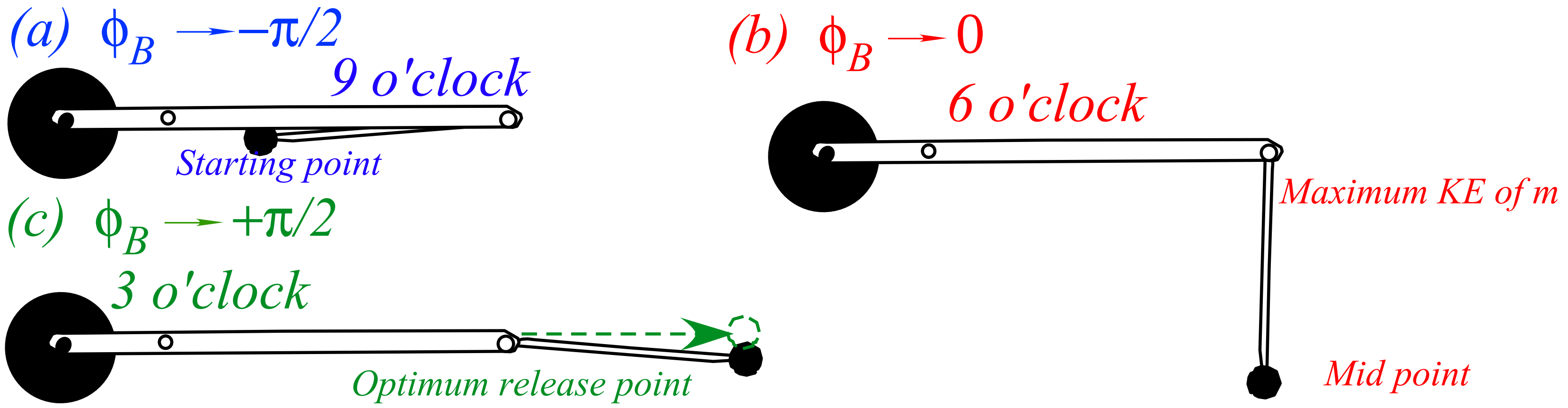
$$\longrightarrow\ (\omega-\dot\theta_{\pi/2})=\frac{2mr^2}{MR^2}\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)$$

$$\xrightarrow[\text{by }\Lambda]{\text{divide 2E}}\ (\omega+\dot\theta_{\pi/2})=\frac{1}{2}\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)$$

$$\downarrow$$

$$\boxed{\dot\phi_{\pi/2}=\dot\theta_{\pi/2}+2\omega}$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy



Case of equal arms $r = \ell$ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot\theta^2 + mr^2\left(\dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B + \dot\phi^2\right) \\ \Lambda &= MR^2\dot\theta + mr^2\left(1+\sin\phi_B\right)\left(\dot\theta + \dot\phi\right) \end{aligned}\right\} \text{For: } r=\ell$$

Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

$$\begin{matrix}\phi_B = \dfrac{-\pi}{2}: \\ \sin\phi_B\text{=-1}\end{matrix}\left\{\begin{aligned} 2E &= MR^2\dot\theta^2_{-\pi/2} + mr^2\left(\dot\phi_{-\pi/2} - \dot\phi_{-\pi/2}\right)^2 \\ \Lambda &= MR^2\dot\theta_{-\pi/2} \end{aligned}\right. \quad or: \left\{\begin{aligned} 2E &= MR^2\omega^2 \\ \Lambda &= MR^2\omega \end{aligned}\right.$$

Conserved

initial 2E

initial Λ

For: $\dot\theta_{-\pi/2} = \omega = \dot\phi_{-\pi/2}$

Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\begin{matrix}\phi_B = 0: \\ \sin\phi_B\text{=0}\end{matrix}\left\{\begin{aligned} 2E &= MR^2\dot\theta_0^2 + mr^2\left(\dot\phi_0^2 + \dot\theta_0^2\right) \\ \Lambda &= MR^2\dot\theta_0 + mr^2\left(\dot\phi_0 + \dot\theta_0\right) \end{aligned}\right.$$

$$KE(m) = \frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B\right)$$

$$= \begin{cases} \dfrac{mr^2}{2}\left(\dot\phi - \dot\theta\right)^2 & \left(\text{For: } \phi_B = -\dfrac{\pi}{2}\right) \\ \dfrac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2\right) & \left(\text{For: } \phi_B = 0\right) \\ \dfrac{mr^2}{2}\left(\dot\phi + \dot\theta\right)^2 & \left(\text{For: } \phi_B = \dfrac{\pi}{2}\right) \end{cases}$$

Move to 3 o'clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

Conserved

$$\begin{matrix}\phi_B = \pi/2: \\ \sin\phi_B\text{=+1}\end{matrix}\left\{\begin{aligned} 2E &= MR^2\dot\theta^2_{\pi/2} + mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2 &= MR^2\omega^2 \\ \Lambda &= MR^2\dot\theta_{\pi/2} + 2mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right) &= MR^2\omega \end{aligned}\right.$$

initial 2E

initial Λ

$$(\omega^2 - \dot\theta^2_{\pi/2}) = \frac{mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2$$

$$(\omega - \dot\theta_{\pi/2}) = \frac{2mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

divide 2E by Λ

$$(\omega + \dot\theta_{\pi/2}) = \frac{1}{2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

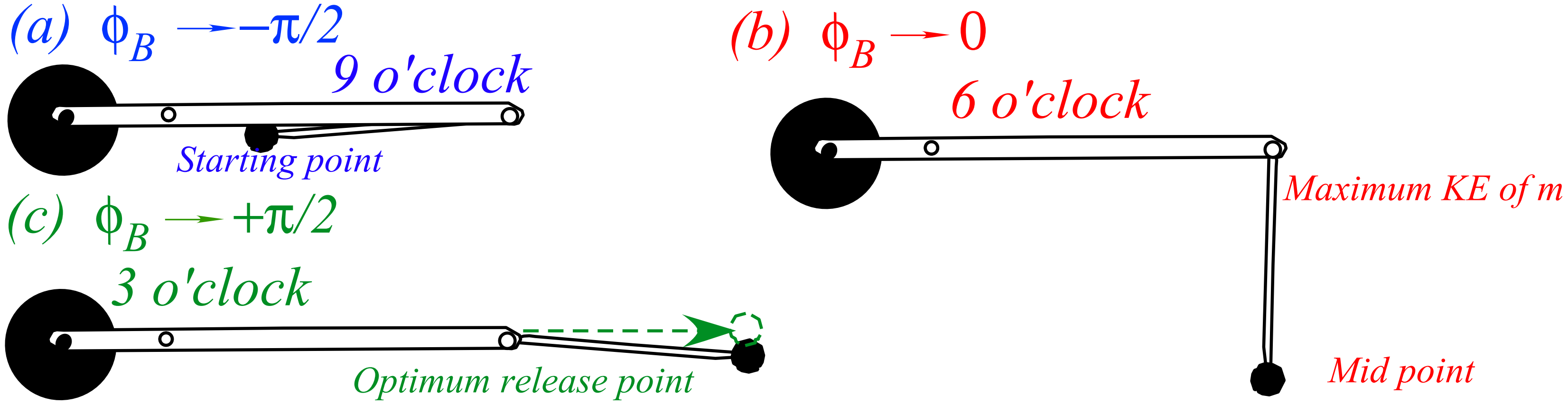
$$\boxed{\dot\phi_{\pi/2} = \dot\theta_{\pi/2} + 2\omega}$$

substitute

$$\omega - \dot\theta_{\pi/2} = \frac{2mr^2}{MR^2}\left(2\omega + 2\dot\theta_{\pi/2}\right)$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r=\ell$ (easier algebra)



$$\left.\begin{aligned} 2E&=MR^2\dot\theta^2+mr^2\left(\dot\theta^2+2\dot\phi\dot\theta\sin\phi_B+\dot\phi^2\right)\\ \Lambda&=MR^2\dot\theta+mr^2\left(1+\sin\phi_B\right)\left(\dot\theta+\dot\phi\right)\end{aligned}\right\}\text{For: } r=\ell$$

Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

Conserved

$$\phi_B=\frac{-\pi}{2}: \quad \sin\phi_B=-1 \quad \left\{\begin{aligned} 2E&=MR^2\dot\theta^2_{-\pi/2}+mr^2\left(\dot\phi_{-\pi/2}-\dot\phi_{-\pi/2}\right)^2\\ \Lambda&=MR^2\dot\theta_{-\pi/2}\end{aligned}\right. \quad or: \left\{\begin{aligned} &\text{initial } 2E\\ &2E=MR^2\omega^2\\ &\Lambda=MR^2\omega\\ &\text{initial } \Lambda\end{aligned}\right. \quad \text{For: } \dot\theta_{-\pi/2}=\omega=\dot\phi_{-\pi/2}$$

Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B=0: \quad \sin\phi_B=0 \quad \left\{\begin{aligned} 2E&=MR^2\dot\theta_0^2+mr^2\left(\dot\phi_0^2+\dot\theta_0^2\right)\\ \Lambda&=MR^2\dot\theta_0+mr^2\left(\dot\phi_0+\dot\theta_0\right)\end{aligned}\right.$$

$$KE(m)=\frac{mr^2}{2}\left(\dot\phi^2+\dot\theta^2+2\dot\phi\dot\theta\sin\phi_B\right)=\left\{\begin{aligned} &\frac{mr^2}{2}\left(\dot\phi-\dot\theta\right)^2 && \left(\text{For: } \phi_B=-\frac{\pi}{2}\right)\\ &\frac{mr^2}{2}\left(\dot\phi^2+\dot\theta^2\right) && \left(\text{For: } \phi_B=0\right)\\ &\frac{mr^2}{2}\left(\dot\phi+\dot\theta\right)^2 && \left(\text{For: } \phi_B=\frac{\pi}{2}\right)\end{aligned}\right.$$

Move to 3 o'clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

Conserved

$$\phi_B=\pi/2: \quad \sin\phi_B=+1 \quad \left\{\begin{aligned} 2E&=MR^2\dot\theta^2_{\pi/2}+mr^2\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)^2 = MR^2\omega^2 \quad \text{initial } 2E\\ \Lambda&=MR^2\dot\theta_{\pi/2}+2mr^2\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right) = MR^2\omega \quad \text{initial } \Lambda\end{aligned}\right.$$

$$\rightarrow \left(\omega^2-\dot\theta^2_{\pi/2}\right)=\frac{mr^2}{MR^2}\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)^2$$

$$\rightarrow \left(\omega-\dot\theta_{\pi/2}\right)=\frac{2mr^2}{MR^2}\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)$$

divide 2E by Λ

$$\left(\omega+\dot\theta_{\pi/2}\right)=\frac{1}{2}\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)$$

$$\downarrow$$

$$\boxed{\dot\phi_{\pi/2}=\dot\theta_{\pi/2}+2\omega}$$

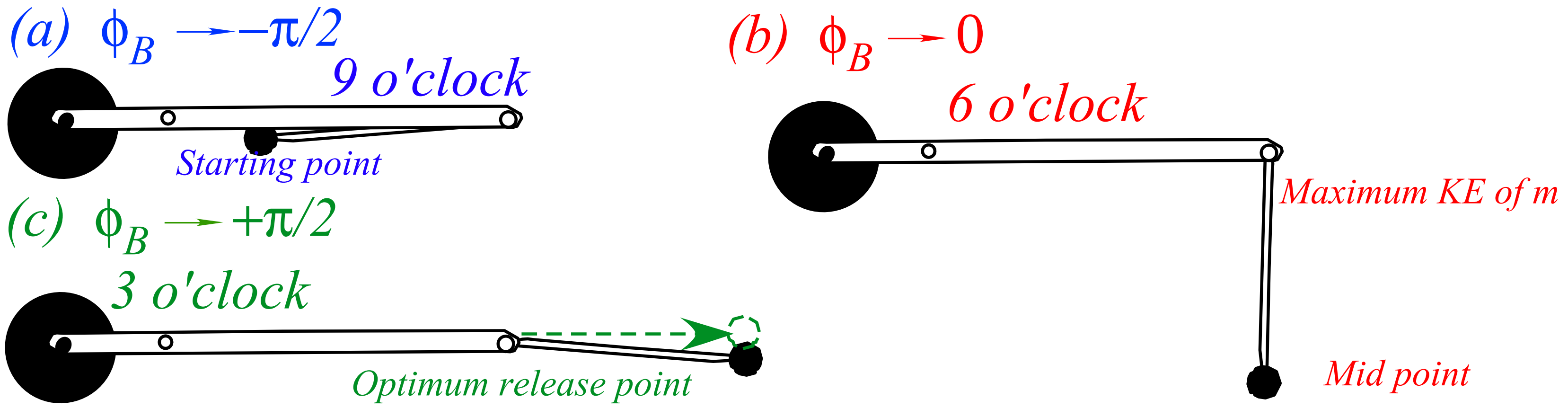
substitute

$$\omega-\dot\theta_{\pi/2}=\frac{2mr^2}{MR^2}\left(2\omega+2\dot\theta_{\pi/2}\right)$$

$$\omega-\frac{4mr^2}{MR^2}\omega=\dot\theta_{\pi/2}+\frac{4mr^2}{MR^2}\dot\theta_{\pi/2}$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r = \ell$ (easier algebra)



$$\left.\begin{array}{l} 2E = MR^2\dot\theta^2 + mr^2\left(\dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B + \dot\phi^2\right) \\ \Lambda = MR^2\dot\theta + mr^2\left(1+\sin\phi_B\right)\left(\dot\theta+\dot\phi\right) \end{array}\right\} \begin{array}{l}\text{For:}\\ r=\ell\end{array}$$

Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

$$\begin{array}{l}\phi_B = \frac{-\pi}{2} \\ \sin\phi_B\text{=-1}\end{array} : \left\{\begin{array}{l} 2E = MR^2\dot\theta^{\,2}_{-\pi/2} + mr^2\left(\dot\phi_{-\pi/2} - \dot\phi_{-\pi/2}\right)^2 \\ \Lambda = MR^2\dot\theta_{-\pi/2} \end{array}\right. \quad or: \left\{\begin{array}{l} \text{Conserved} \\ \text{initial } 2E \\ 2E = MR^2\omega^2 \\ \Lambda = MR^2\omega \\ \text{initial } \Lambda \end{array}\right. \quad \text{For: } \dot\theta_{-\pi/2} = \omega = \dot\phi_{-\pi/2}$$

Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\begin{array}{l}\phi_B = 0 \\ \sin\phi_B\text{=0}\end{array} : \left\{\begin{array}{l} 2E = MR^2\dot\theta_0^2 + mr^2\left(\dot\phi_0^2 + \dot\theta_0^2\right) \\ \Lambda = MR^2\dot\theta_0 + mr^2\left(\dot\phi_0 + \dot\theta_0\right) \end{array}\right.$$

$$KE(m) = \frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B\right)$$

$$= \left\{\begin{array}{ll} \frac{mr^2}{2}\left(\dot\phi - \dot\theta\right)^2 & \left(\text{For: } \phi_B = -\frac{\pi}{2}\right) \\ \frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2\right) & \left(\text{For: } \phi_B = 0\right) \\ \frac{mr^2}{2}\left(\dot\phi + \dot\theta\right)^2 & \left(\text{For: } \phi_B = \frac{\pi}{2}\right) \end{array}\right.$$

Move to 3 o'clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

$$\begin{array}{l}\phi_B = \pi/2 \\ \sin\phi_B\text{=+1}\end{array} : \left\{\begin{array}{ll} 2E = MR^2\dot\theta^2_{\pi/2} + mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2 & \begin{array}{l}\text{Conserved}\\ \text{initial } 2E \\ = MR^2\omega^2\end{array} \\ \Lambda = MR^2\dot\theta_{\pi/2} + 2mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right) & \begin{array}{l} = MR^2\omega \\ \text{initial } \Lambda\end{array} \end{array}\right.$$

$$\left(\omega^2 - \dot\theta^2_{\pi/2}\right) = \frac{mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2$$

$$\left(\omega - \dot\theta_{\pi/2}\right) = \frac{2mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

$$\xrightarrow[\text{by } \Lambda]{\text{divide } 2E} \left(\omega + \dot\theta_{\pi/2}\right) = \frac{1}{2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

$$\boxed{\dot\phi_{\pi/2} = \dot\theta_{\pi/2} + 2\omega}$$

substitute

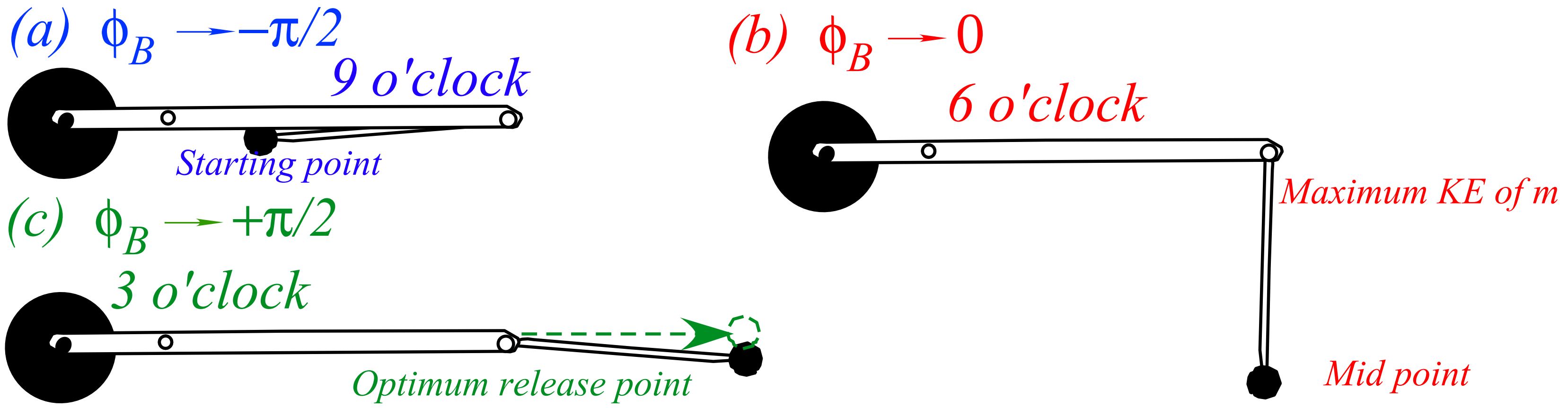
$$\omega - \dot\theta_{\pi/2} = \frac{2mr^2}{MR^2}\left(2\omega + 2\dot\theta_{\pi/2}\right)$$

$$\omega - \frac{4mr^2}{MR^2}\omega = \dot\theta_{\pi/2} + \frac{4mr^2}{MR^2}\dot\theta_{\pi/2} \xrightarrow{\text{solve}} \boxed{\dot\theta_{\pi/2} = \frac{1 - \frac{4mr^2}{MR^2}}{1 + \frac{4mr^2}{MR^2}}\omega}$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms r =ℓ (easier algebra)

$$\left.\begin{aligned} 2E&=MR^2\dot\theta^2+mr^2\left(\dot\theta^2+2\dot\phi\dot\theta\sin\phi_B+\dot\phi^2\right)\\ \Lambda&=MR^2\dot\theta+mr^2\left(1+\sin\phi_B\right)\left(\dot\theta+\dot\phi\right)\end{aligned}\right\}\ \text{For: } r=\ell$$



Start at 9 o’clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

$$\begin{matrix}\phi_B=\dfrac{-\pi}{2}:\\ \sin\phi_B\text{=-1}\end{matrix}\left\{\begin{aligned} 2E&=MR^2\dot\theta_{-\pi/2}^2+mr^2\left(\dot\phi_{-\pi/2}-\dot\phi_{-\pi/2}\right)^2\\ \Lambda&=MR^2\dot\theta_{-\pi/2}\end{aligned}\right.\quad or:\left\{\begin{aligned} &\text{initial }2E\\ &2E=MR^2\omega^2\\ &\Lambda=MR^2\omega\\ &\text{initial }\Lambda\end{aligned}\right.$$

Conserved

For: $\dot\theta_{-\pi/2}=\omega=\dot\phi_{-\pi/2}$

Move to 6 o’clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\begin{matrix}\phi_B=0:\\ \sin\phi_B\text{=0}\end{matrix}\left\{\begin{aligned} 2E&=MR^2\dot\theta_0^2+mr^2\left(\dot\phi_0^2+\dot\theta_0^2\right)\\ \Lambda&=MR^2\dot\theta_0+mr^2\left(\dot\phi_0+\dot\theta_0\right)\end{aligned}\right.$$

$$KE(m)=\frac{mr^2}{2}\left(\dot\phi^2+\dot\theta^2+2\dot\phi\dot\theta\sin\phi_B\right)=\left\{\begin{matrix}\dfrac{mr^2}{2}\left(\dot\phi-\dot\theta\right)^2 & \left(\text{For: }\phi_B=-\dfrac{\pi}{2}\right)\\ \dfrac{mr^2}{2}\left(\dot\phi^2+\dot\theta^2\right) & \left(\text{For: }\phi_B=0\right)\\ \dfrac{mr^2}{2}\left(\dot\phi+\dot\theta\right)^2 & \left(\text{For: }\phi_B=\dfrac{\pi}{2}\right)\end{matrix}\right.$$

Move to 3 o’clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

$$\begin{matrix}\phi_B=\pi/2:\\ \sin\phi_B\text{=+1}\end{matrix}\left\{\begin{aligned} 2E&=MR^2\dot\theta_{\pi/2}^2+mr^2\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)^2 &&=MR^2\omega^2\ \text{initial }2E\\ \Lambda&=MR^2\dot\theta_{\pi/2}+2mr^2\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right) &&=MR^2\omega\ \text{initial }\Lambda\end{aligned}\right.$$

Conserved

$$\longrightarrow (\omega^2-\dot\theta_{\pi/2}^2)=\frac{mr^2}{MR^2}\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)^2$$

$$\longrightarrow (\omega-\dot\theta_{\pi/2})=\frac{2mr^2}{MR^2}\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)$$

$$\xrightarrow{\text{divide }2E\text{ by }\Lambda}(\omega+\dot\theta_{\pi/2})=\frac{1}{2}\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)$$

$$\downarrow$$

$$\boxed{\dot\phi_{\pi/2}=\dot\theta_{\pi/2}+2\omega}$$

substitute

$$\omega-\dot\theta_{\pi/2}=\frac{2mr^2}{MR^2}\left(2\omega+2\dot\theta_{\pi/2}\right)$$

$$\omega-\frac{4mr^2}{MR^2}\omega=\dot\theta_{\pi/2}+\frac{4mr^2}{MR^2}\dot\theta_{\pi/2}$$

$$\xrightarrow{\text{solve}}\boxed{\dot\theta_{\pi/2}=\frac{1-\dfrac{4mr^2}{MR^2}}{1+\dfrac{4mr^2}{MR^2}}\omega}$$

Large M≫m case

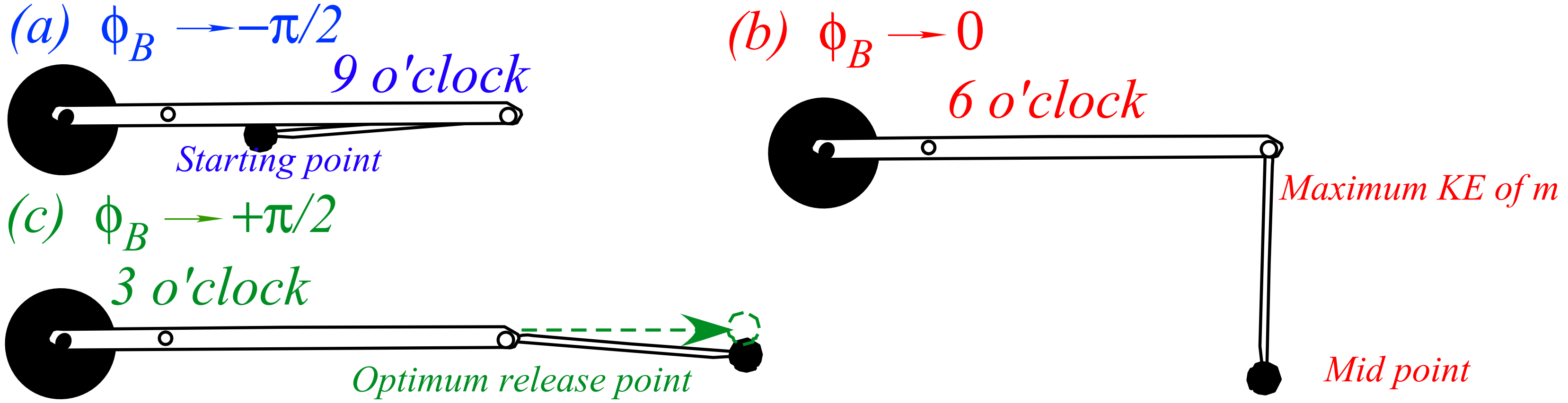
$$\boxed{\dot\phi_{\pi/2}=\omega+2\omega=3\omega}$$

$$\boxed{\dot\theta_{\pi/2}=\frac{1-0}{1+0}\omega=\omega}$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r=\ell$ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot\theta^2 + mr^2\left(\dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B + \dot\phi^2\right) \\ \Lambda &= MR^2\dot\theta + mr^2\left(1+\sin\phi_B\right)\left(\dot\theta + \dot\phi\right)\end{aligned}\right\} \text{For: } r=\ell$$



Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

$$\begin{matrix}\phi_B = \frac{-\pi}{2}: \\ \sin\phi_B = -1\end{matrix}\left\{\begin{aligned} 2E &= MR^2\dot\theta^2_{-\pi/2} + mr^2\left(\dot\phi_{-\pi/2} - \dot\phi_{-\pi/2}\right)^2 \\ \Lambda &= MR^2\dot\theta_{-\pi/2}\end{aligned}\right. \quad or: \left\{\begin{aligned} &\text{Conserved initial } 2E \\ &2E = MR^2\omega^2 \\ &\Lambda = MR^2\omega \\ &\text{initial } \Lambda\end{aligned}\right. \quad \text{For: } \dot\theta_{-\pi/2} = \omega = \dot\phi_{-\pi/2}$$

$$KE(m) = \frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B\right) = \left\{\begin{aligned} &\frac{mr^2}{2}\left(\dot\phi - \dot\theta\right)^2 \quad \left(\text{For: } \phi_B = -\frac{\pi}{2}\right) \\ &\frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2\right) \quad \left(\text{For: } \phi_B = 0\right) \\ &\frac{mr^2}{2}\left(\dot\phi + \dot\theta\right)^2 \quad \left(\text{For: } \phi_B = \frac{\pi}{2}\right)\end{aligned}\right.$$

Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\begin{matrix}\phi_B = 0: \\ \sin\phi_B = 0\end{matrix}\left\{\begin{aligned} 2E &= MR^2\dot\theta^2_0 + mr^2\left(\dot\phi^2_0 + \dot\theta^2_0\right) \\ \Lambda &= MR^2\dot\theta_0 + mr^2\left(\dot\phi_0 + \dot\theta_0\right)\end{aligned}\right.$$

Move to 3 o'clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

$$\begin{matrix}\phi_B = \pi/2: \\ \sin\phi_B = +1\end{matrix}\left\{\begin{aligned} 2E &= MR^2\dot\theta^2_{\pi/2} + mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2 = MR^2\omega^2 \quad \text{Conserved initial } 2E \\ \Lambda &= MR^2\dot\theta_{\pi/2} + 2mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right) = MR^2\omega \quad \text{initial } \Lambda\end{aligned}\right.$$

$$\left(\omega^2 - \dot\theta^2_{\pi/2}\right) = \frac{mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2$$

$$\left(\omega - \dot\theta_{\pi/2}\right) = \frac{2mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

divide 2E by Λ

$$\left(\omega + \dot\theta_{\pi/2}\right) = \frac{1}{2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

$$\boxed{\dot\phi_{\pi/2} = \dot\theta_{\pi/2} + 2\omega}$$

substitute

$$\omega - \dot\theta_{\pi/2} = \frac{2mr^2}{MR^2}\left(2\omega + 2\dot\theta_{\pi/2}\right)$$

$$\omega - \frac{4mr^2}{MR^2}\omega = \dot\theta_{\pi/2} + \frac{4mr^2}{MR^2}\dot\theta_{\pi/2}$$

solve

$$\boxed{\dot\theta_{\pi/2} = \frac{1 - \frac{4mr^2}{MR^2}}{1 + \frac{4mr^2}{MR^2}}\omega}$$

Large M≫m case

$$\boxed{\dot\phi_{\pi/2} = \omega + 2\omega = 3\omega}$$

$$\boxed{\dot\theta_{\pi/2} = \frac{1-0}{1+0}\omega = \omega}$$

Optimum $MR^2 = 4mr^2$ case

$$\boxed{\dot\phi_{\pi/2} = 0 + 2\omega = 2\omega}$$

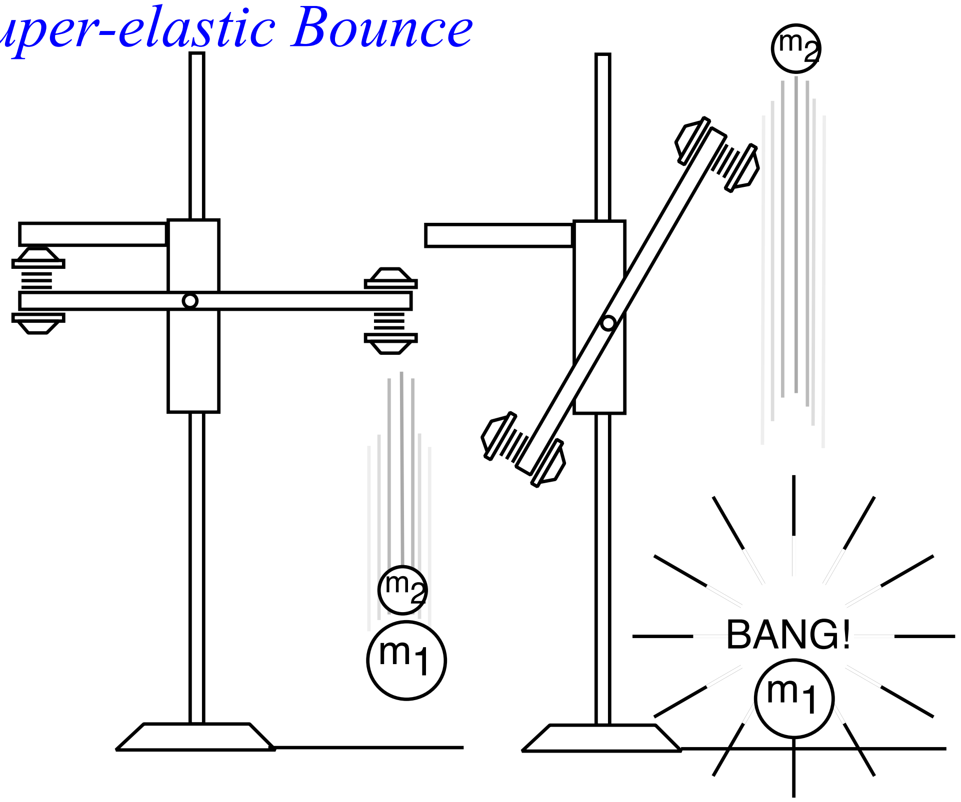
$$\boxed{\dot\theta_{\pi/2} = \frac{1-1}{1+1}\omega = 0}$$

Analogous Superball Models

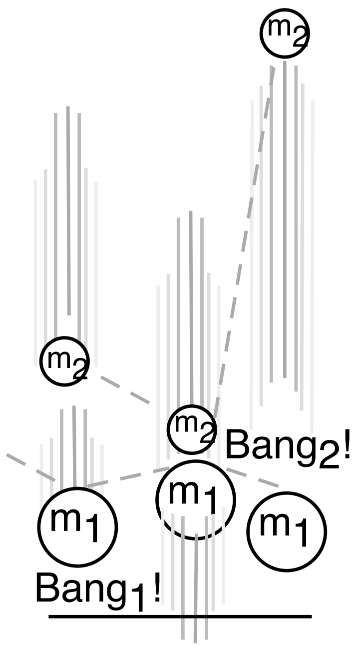
Similar in some ways to trebuchet models

Class of W. G. Harter, "Velocity Amplification in Collision Experiments Involving Superballs," Am. J. Phys. **39**, 656 (1971)
(A class project)

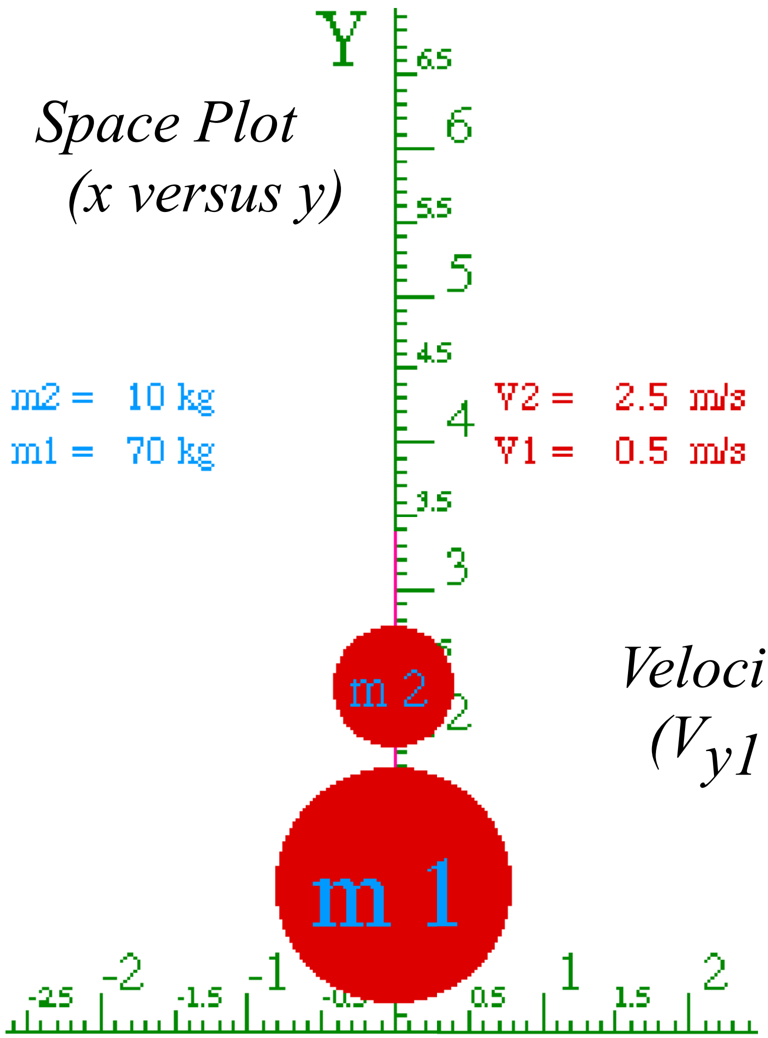
Super-elastic Bounce



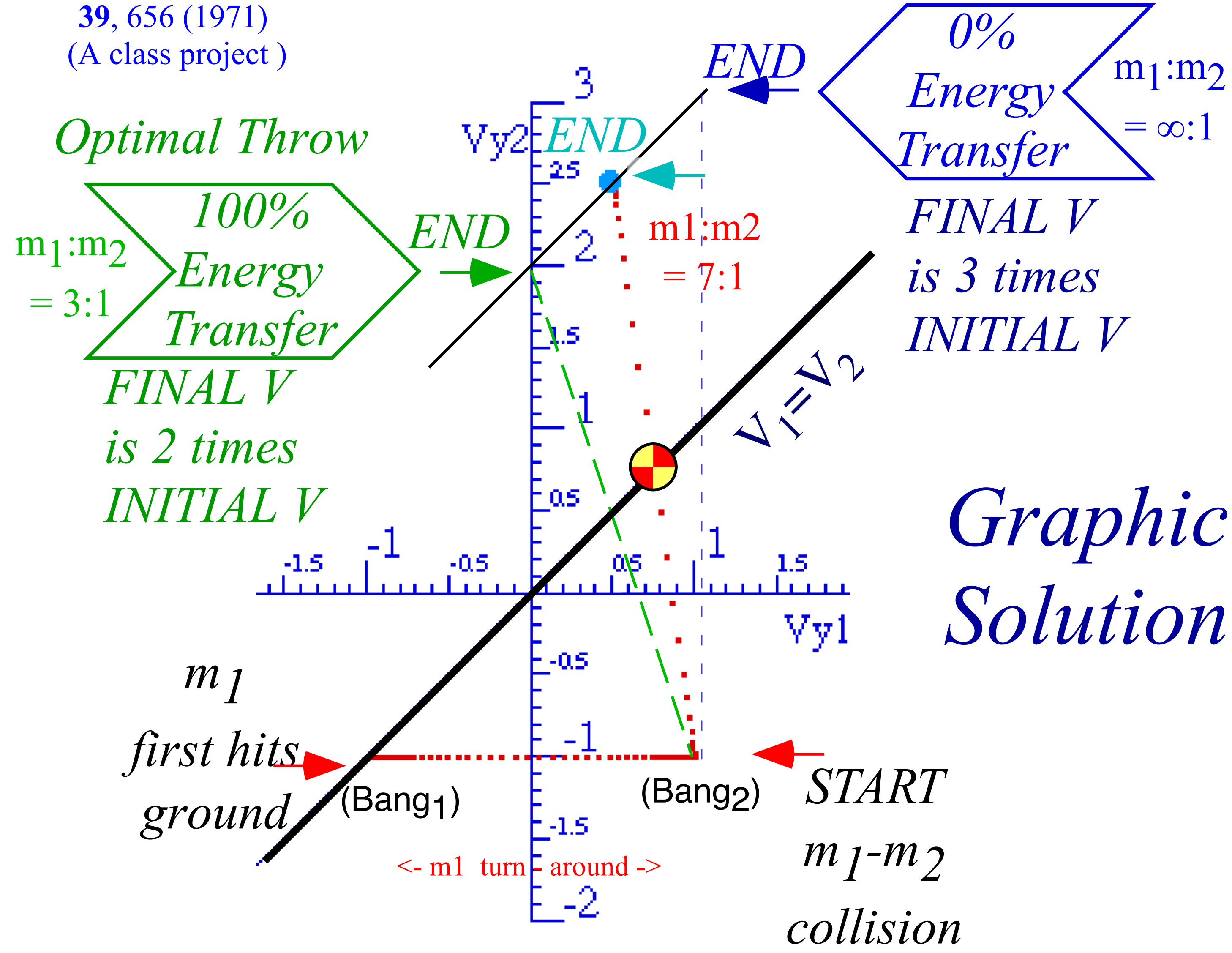
2-Bang Model



Space Plot (x versus y)



Velocity Plot → *(V_{y1} versus V_{y2})*



Optimal Throw

m_1:m_2 = 3:1

100% Energy Transfer

FINAL V is 2 times INITIAL V

m_1:m_2 = 7:1

Fastest Throw

0% Energy Transfer

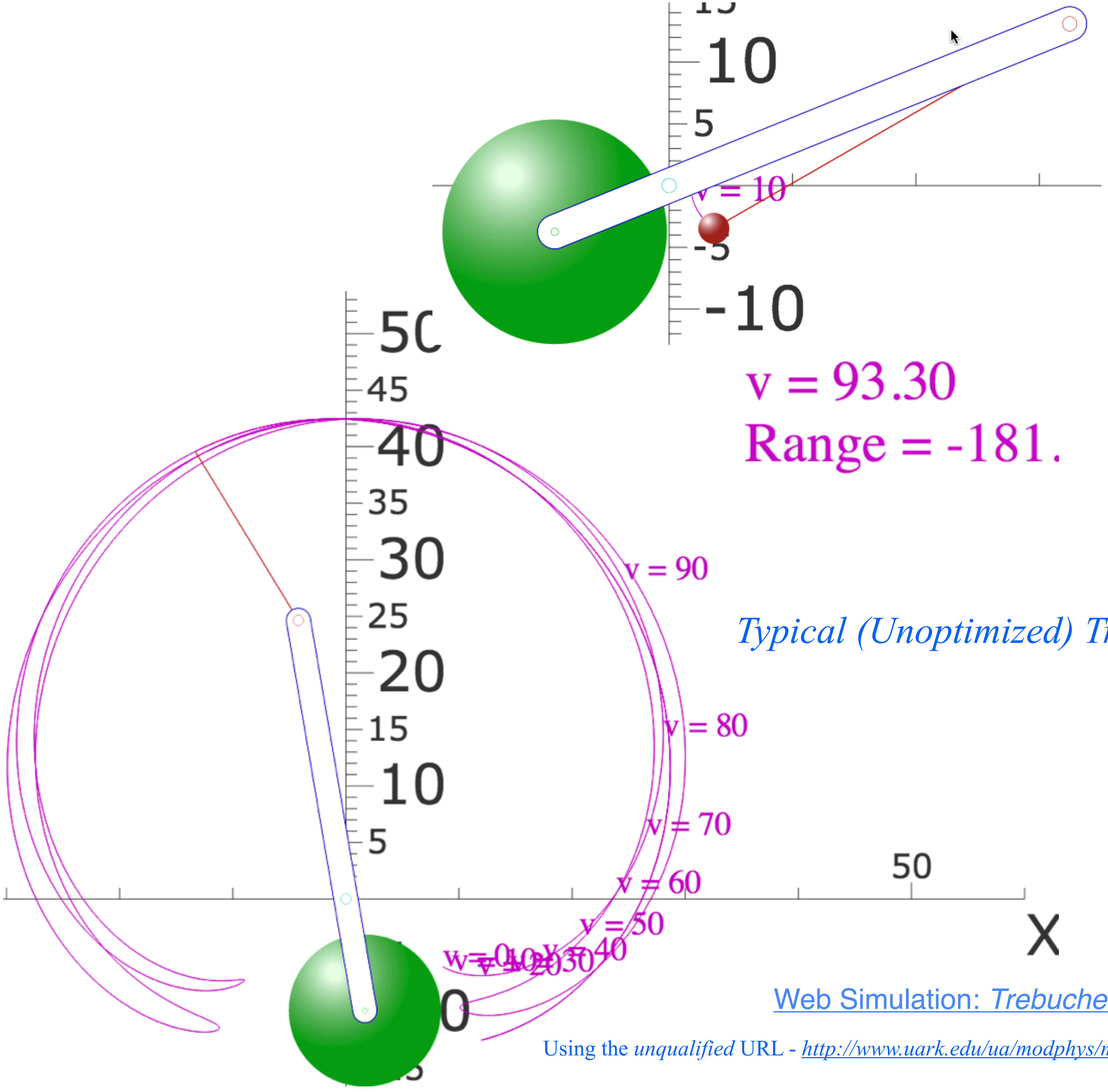
m_1:m_2 = ∞:1

FINAL V is 3 times INITIAL V

m_1 first hits ground

START m_1-m_2 collision

Graphic Solution

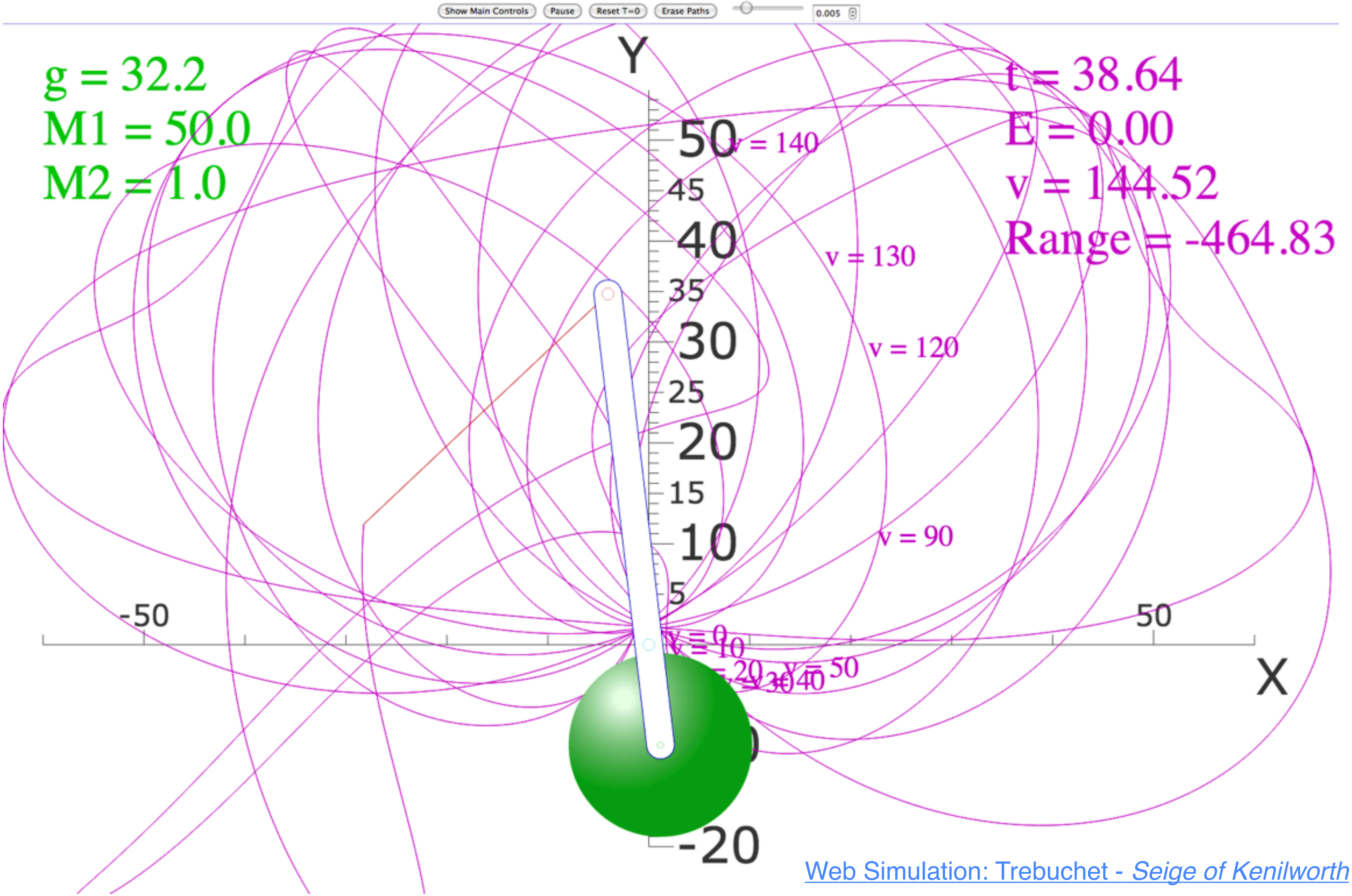


Typical (Unoptimized) Trebuchet

[Web Simulation: *Trebuchet*](http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html)

Using the *unqualified* URL - *http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html*

Trebuchet in Siege of Kenilworth 1215 ACE
(Re-enactment shown on NOVA-TV 2005)

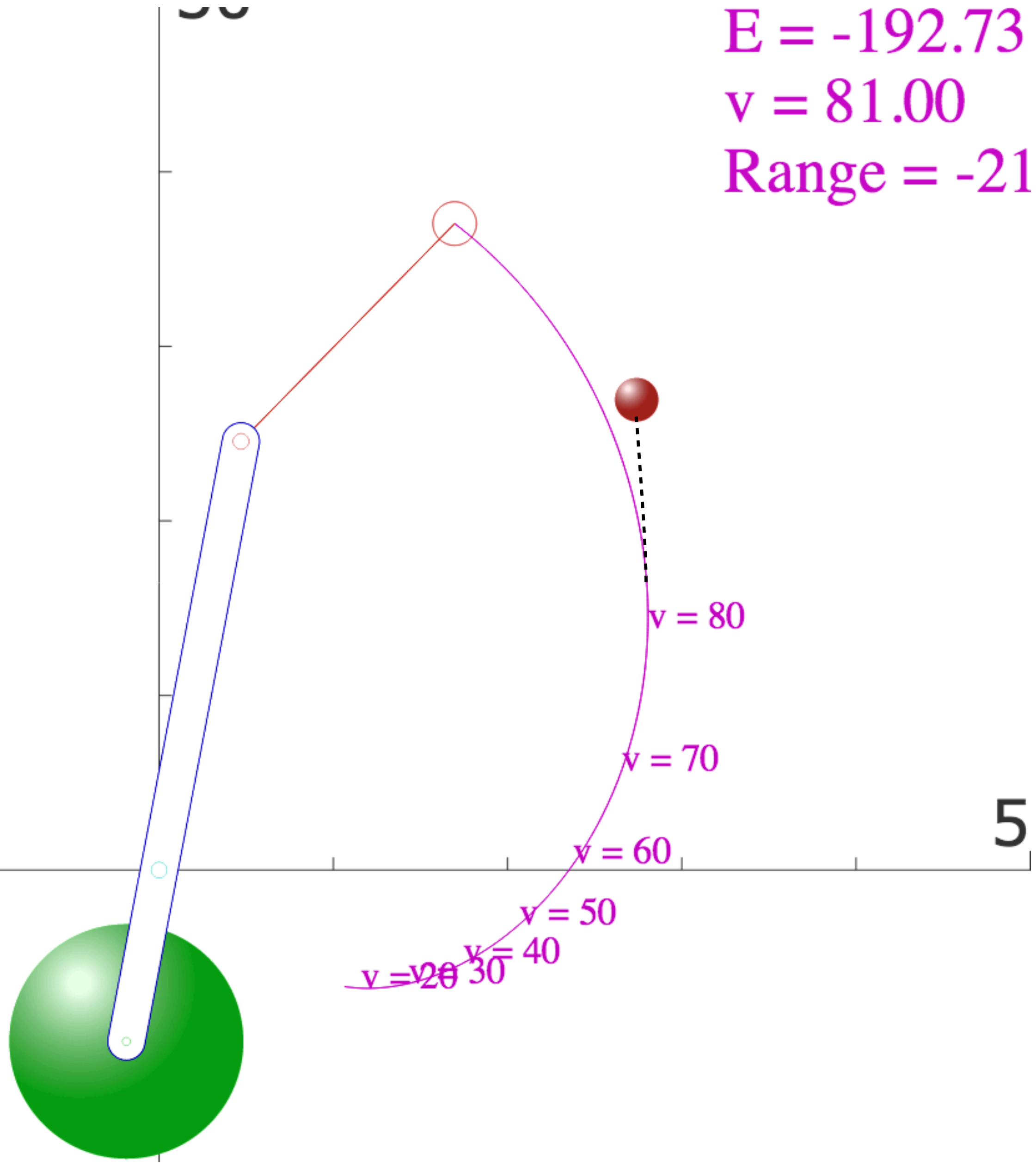


g = 32.2
M1 = 50.0
M2 = 1.0

t = 38.64
E = 0.00
v = 144.52
Range = -464.83

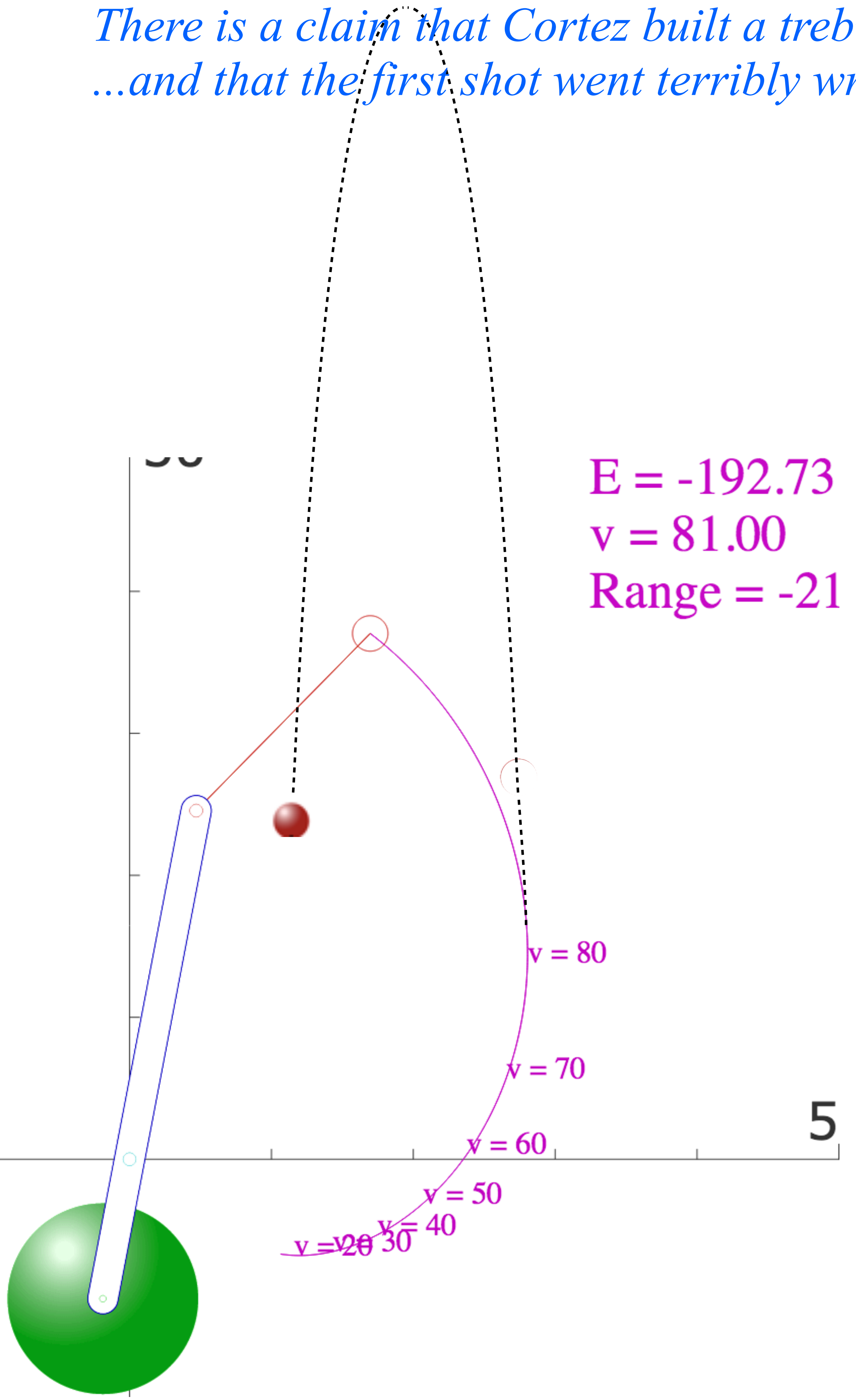
Web Simulation: Trebuchet - *Seige of Kenilworth*

There is a claim that Cortez built a trebuchet to lay siege on Montezuma around 1500...



Web Simulation: Trebuchet - *Montezuma's Revenge*

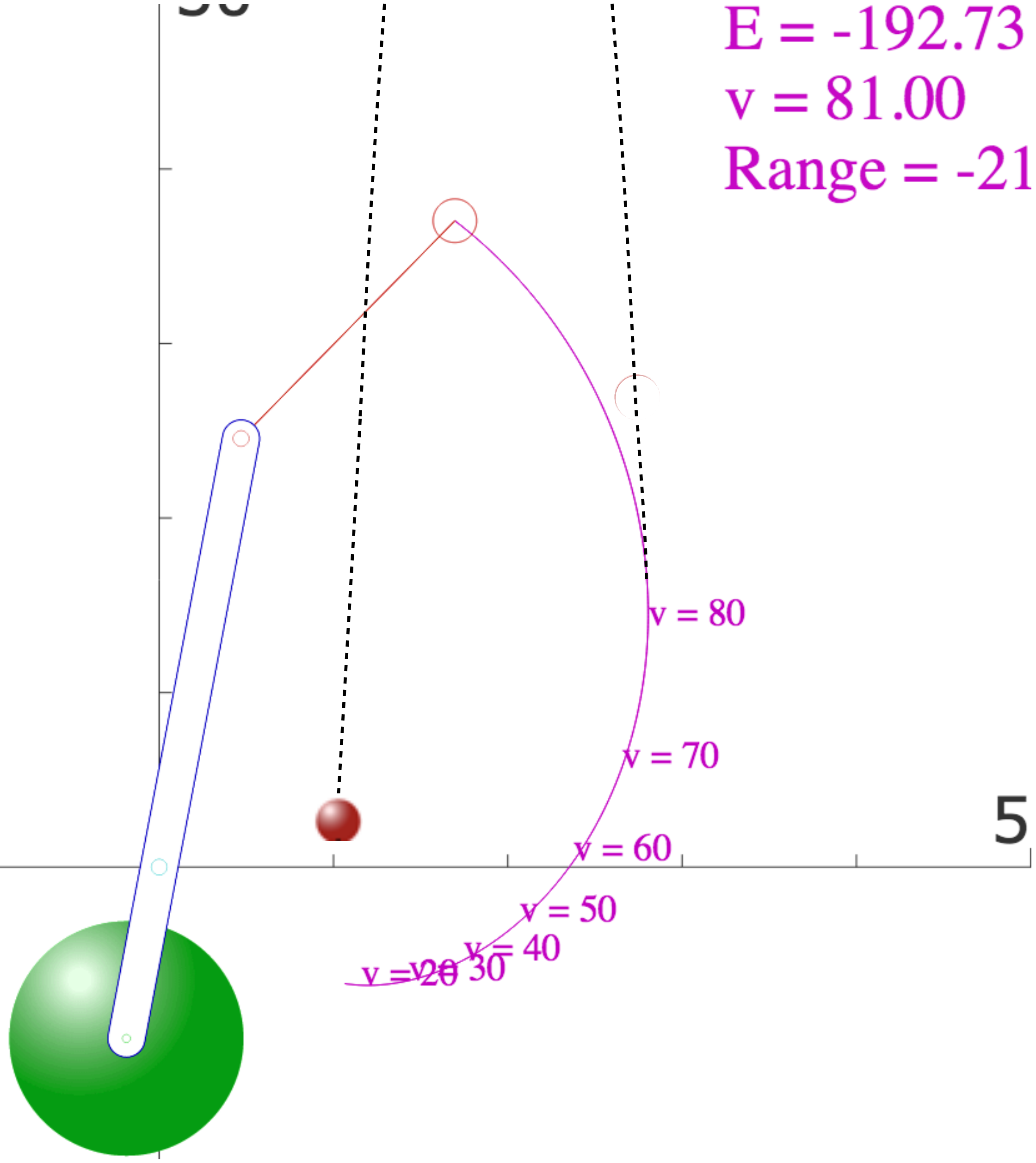
There is a claim that Cortez built a trebuchet to lay siege on Montezuma around 1500...
...and that the first shot went terribly wrong...

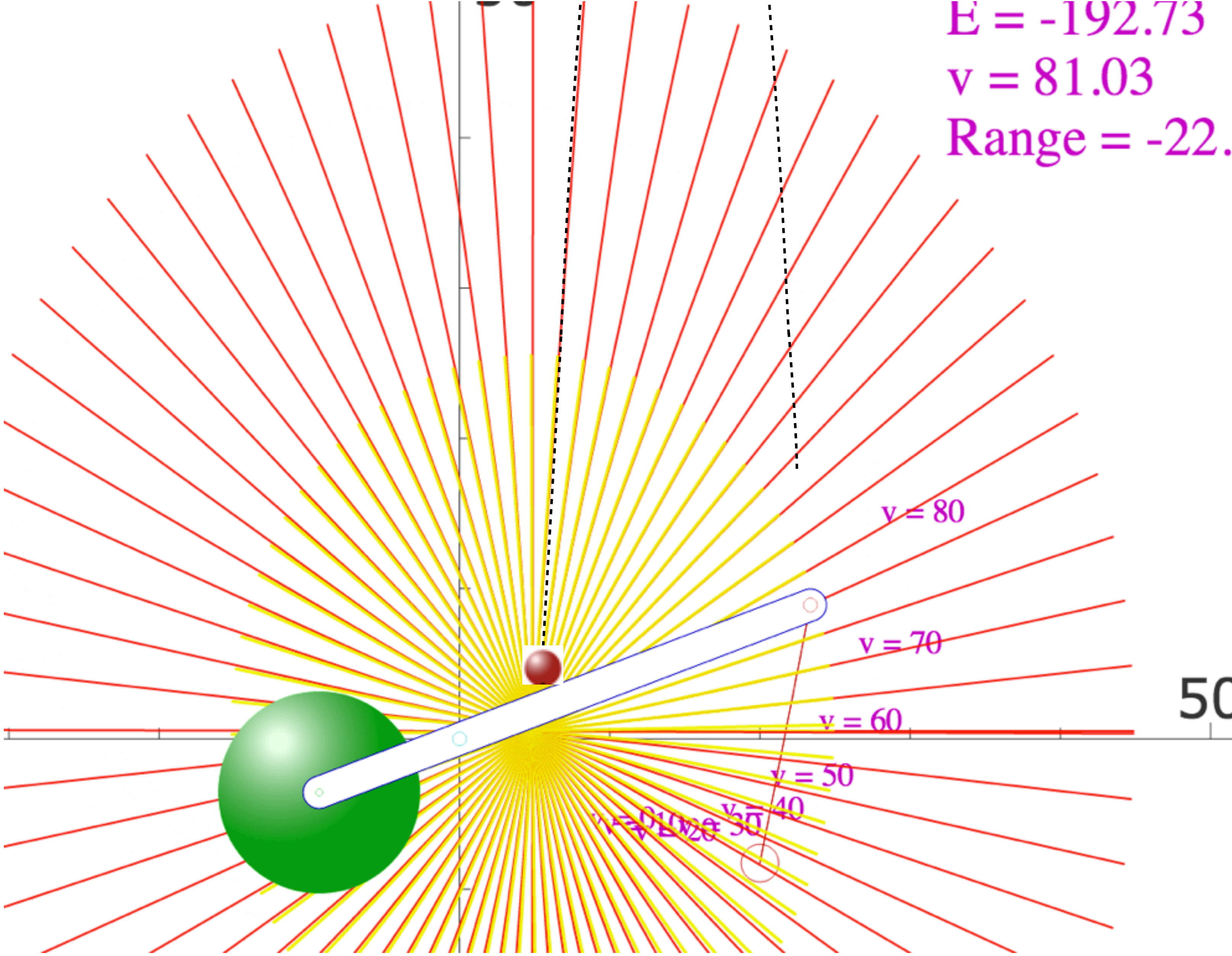


[Web Simulation: Trebuchet - *Montezuma's Revenge*]

There is a claim that Cortez built a trebuchet to lay siege on Montezuma around 1500...
...and that the first shot went terribly wrong...

...if this story is true, then it gives new meaning
to the expression "Montezuma's Revenge"...



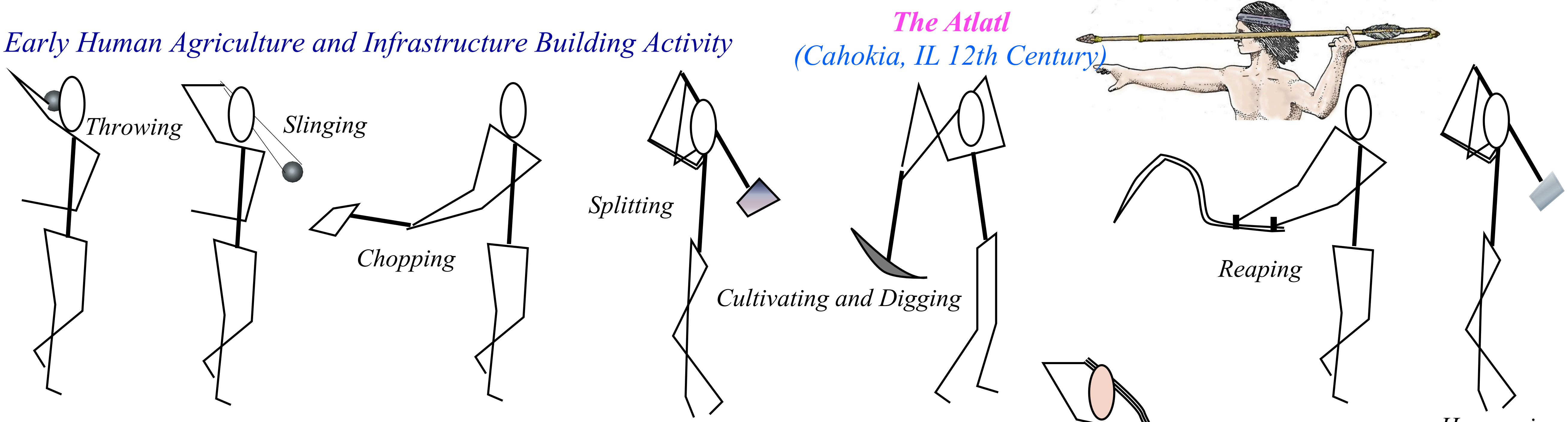


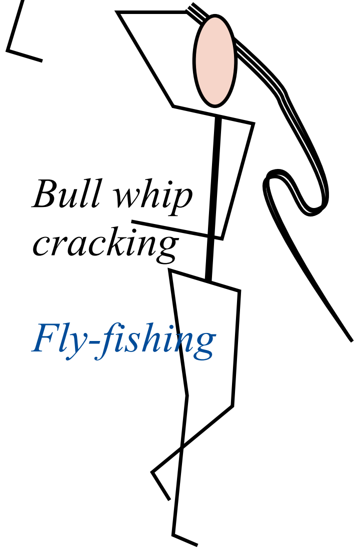
[Web Simulation: Trebuchet - *Montezuma's Revenge*]

Hamiltonian energy and momentum conservation and symmetry coordinates

- *Coordinate transformation helps reduce symmetric Hamiltonian*
- *Free-space trebuchet kinematics by symmetry*
  - *Algebraic approach*
  - *Direct approach and Superball analogy*
- → *Trebuchet vs Flinger and sports kinematics*
  - *Many approaches to Mechanics*

Early Human Agriculture and Infrastructure Building Activity

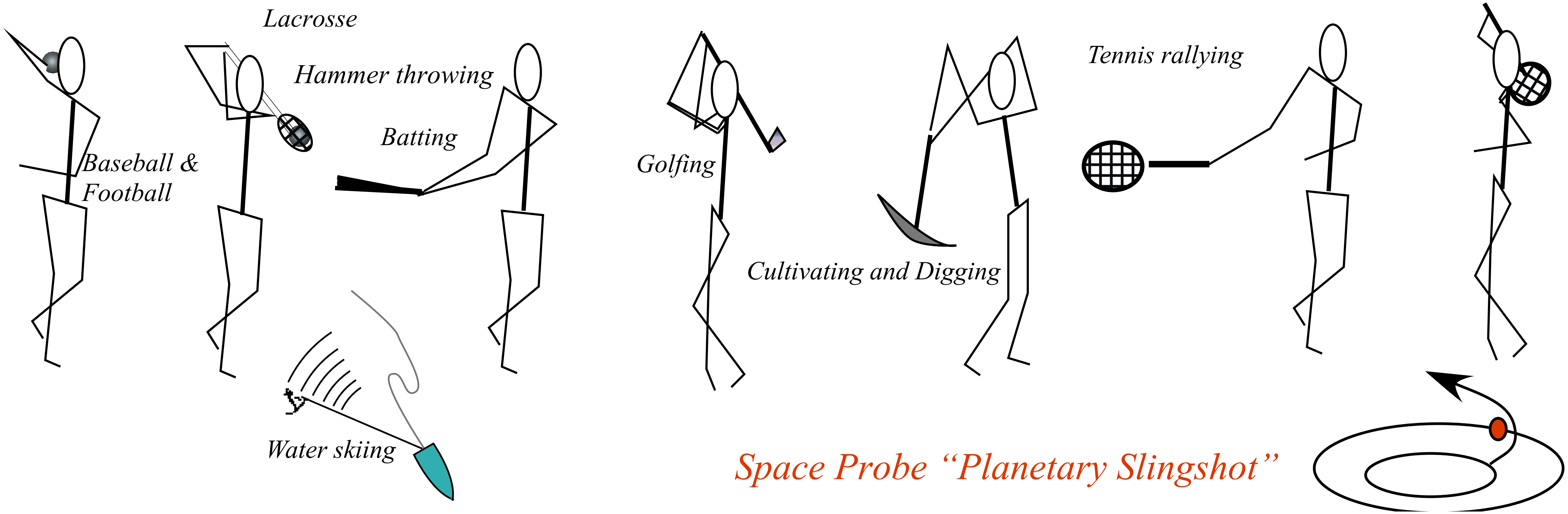




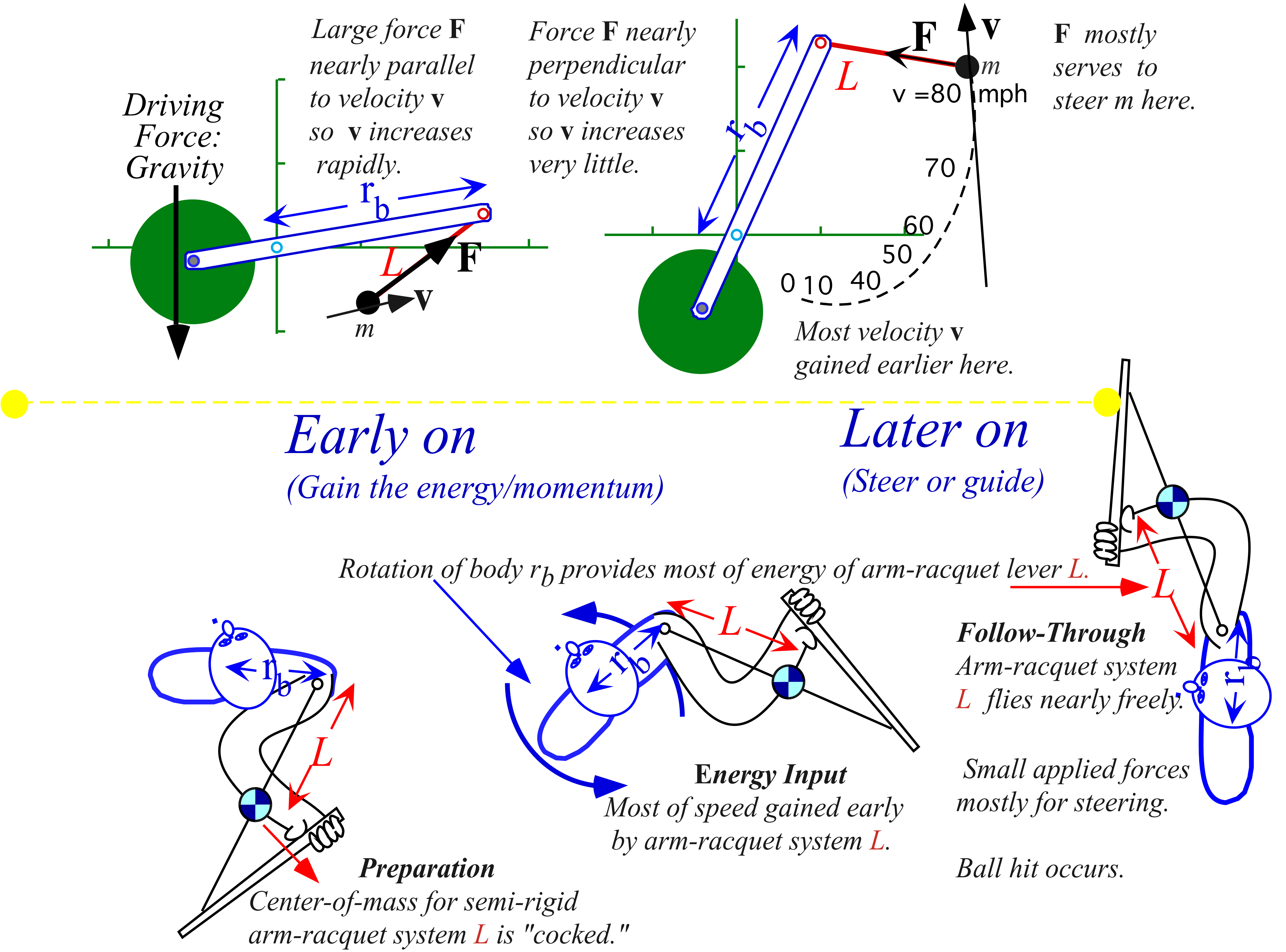
What Trebuchet mechanics is really good for...

"Ring-The-Bell"
(at the Fair)

Later Human Recreational Activity



Trebuchet analogy with racquet swing - What we learn



Driving Force: Gravity

*Large force **F** nearly parallel to velocity **v** so **v** increases rapidly.*

*Force **F** nearly perpendicular to velocity **v** so **v** increases very little.*

***F** mostly serves to steer m here.*

*Most velocity **v** gained earlier here.*

Early on

(Gain the energy/momentum)

Later on

(Steer or guide)

Rotation of body r$_b$ provides most of energy of arm-racquet lever L.

Preparation
Center-of-mass for semi-rigid arm-racquet system L is "cocked."

Energy Input
Most of speed gained early by arm-racquet system L.

Follow-Through
Arm-racquet system L flies nearly freely.

Small applied forces mostly for steering.

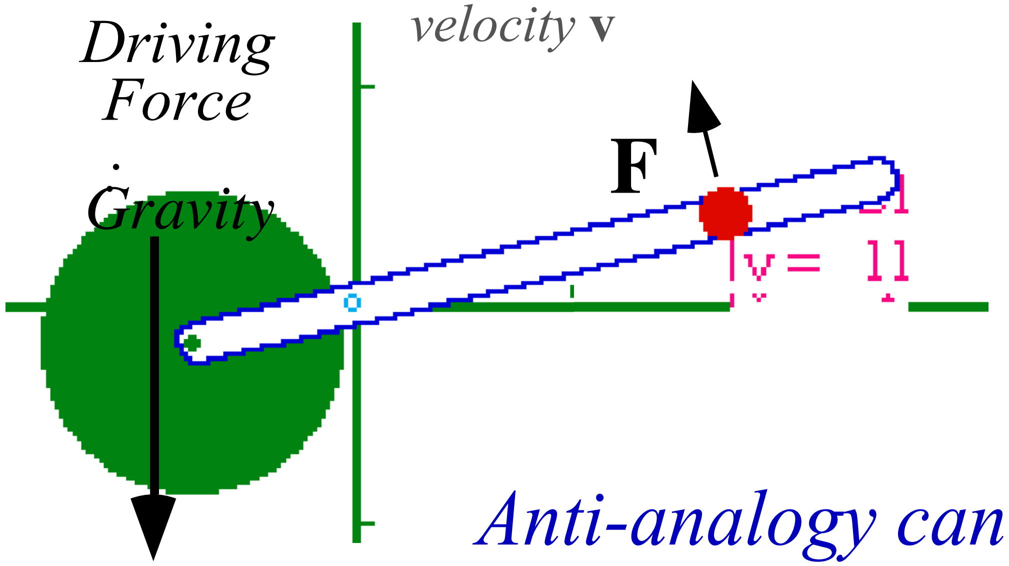
Ball hit occurs.

An Opposite to Trebuchet Mechanics- The "Flinger"

Web Simulation: *Trebuchet - "Flinger"*

Early on

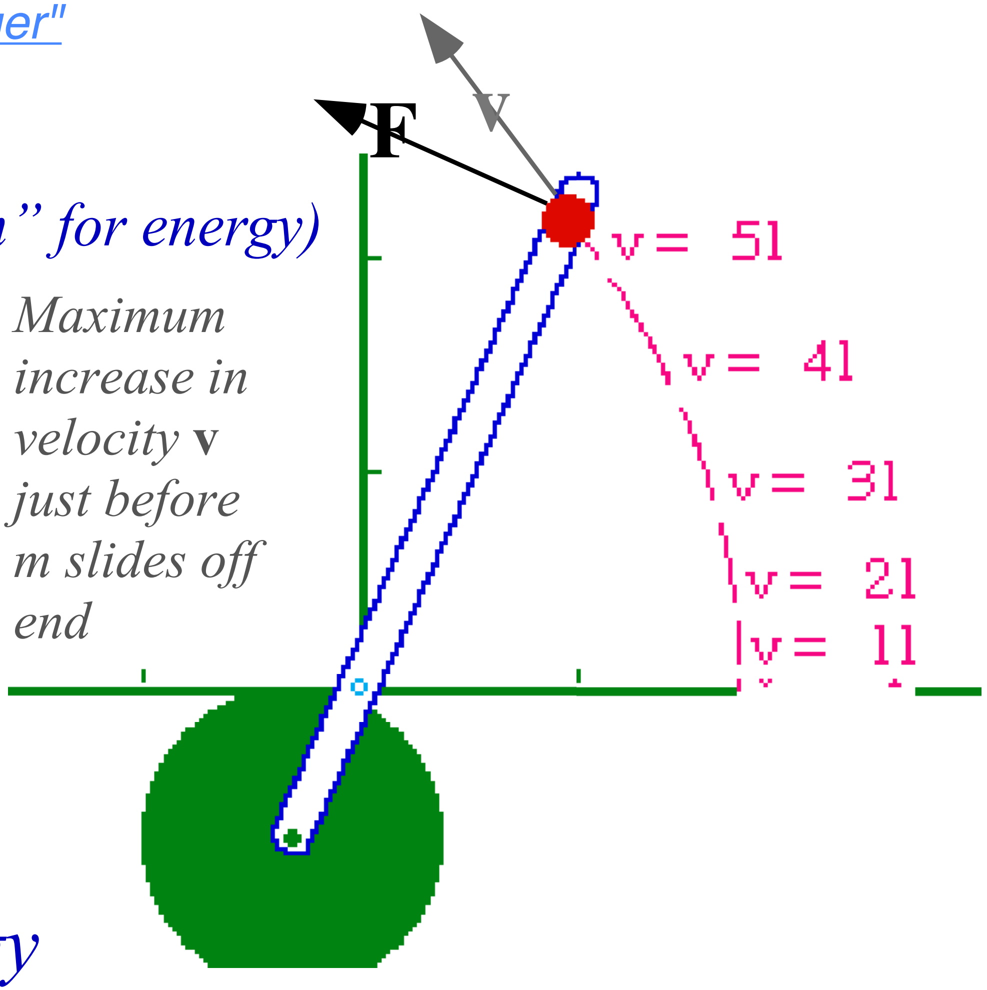
(Not much happening)



Not much increase in velocity **v**

Later on

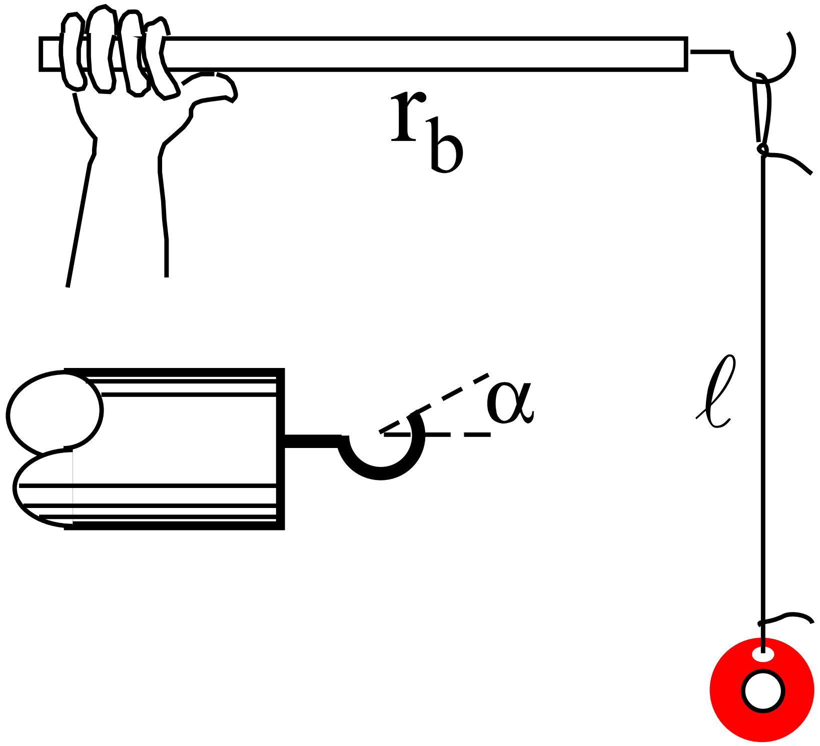
(Last-minute "cram" for energy)



Maximum increase in velocity **v** *just before m slides off end*

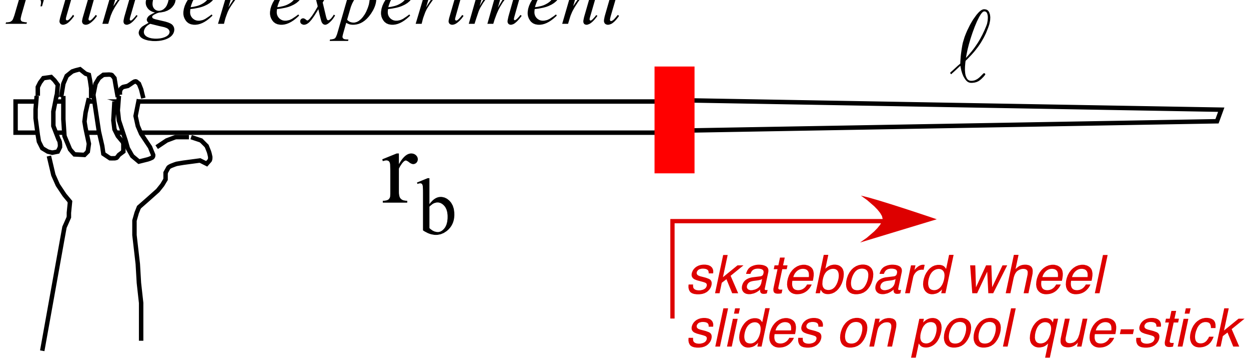
Anti-analogy can be useful pedagogy

Trebuchet-like experiment



skateboard wheel swings

Flinger experiment

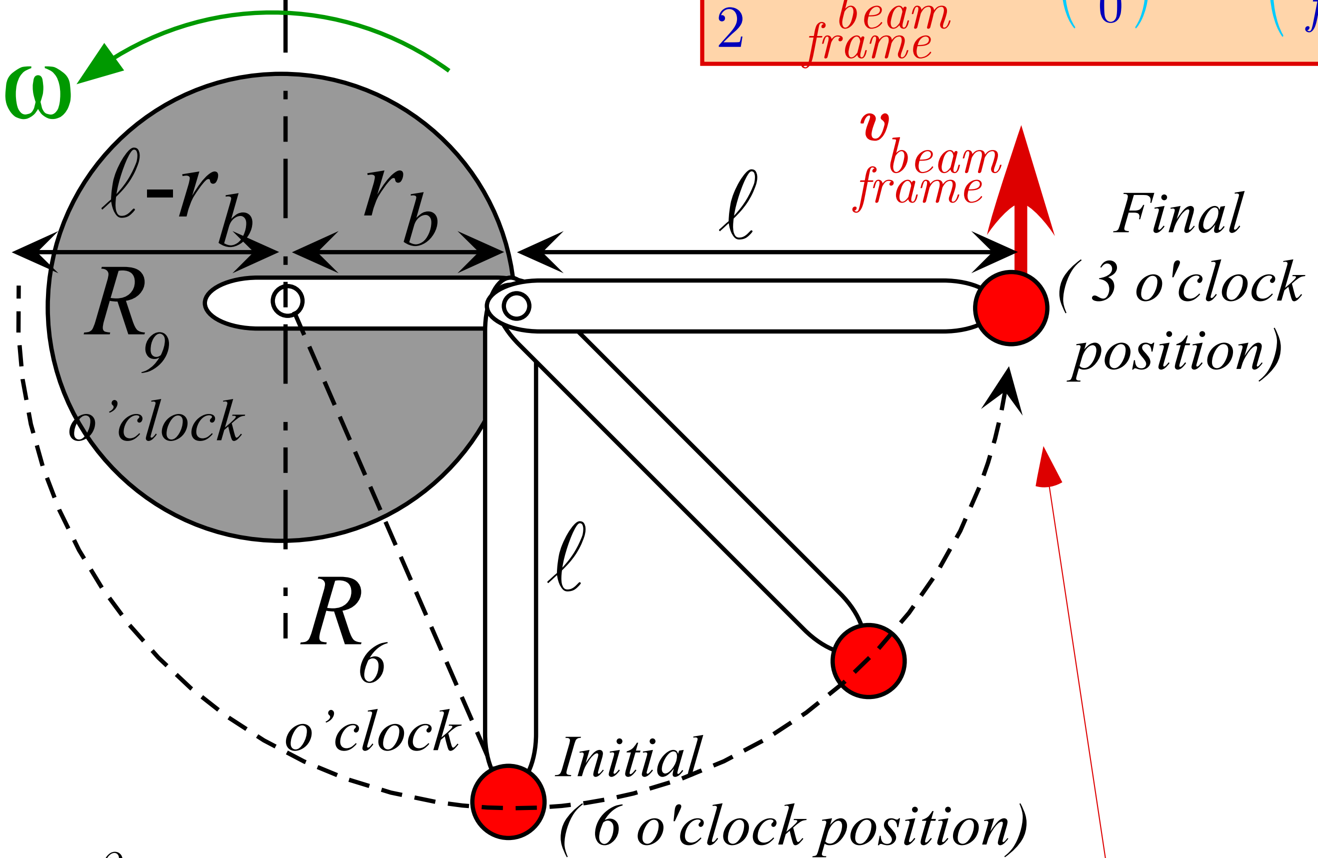


Generic URL - http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html

Trebuchet model in rotating beam frame

Assume: Constant beam ω

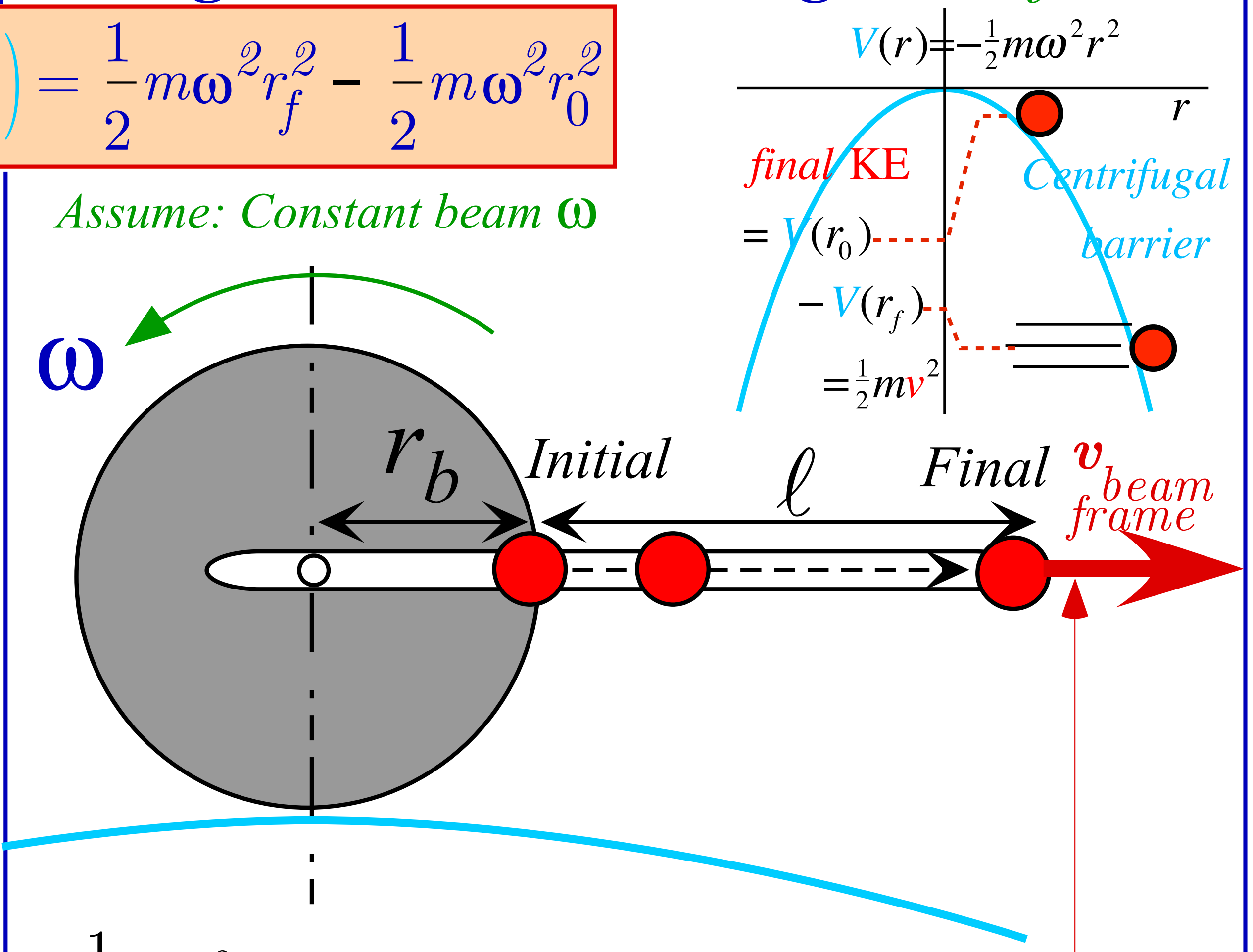
$$\frac{1}{2}mv^2_{\substack{beam\\frame}} = V(r_0) - V(r_f) = \frac{1}{2}m\omega^2 r_f^2 - \frac{1}{2}m\omega^2 r_0^2$$



$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(\text{trebuchet}) =$$ *Final Trebuchet* $KE_{\substack{beam\\frame}}$

Flinger model in rotating beam frame

Assume: Constant beam ω



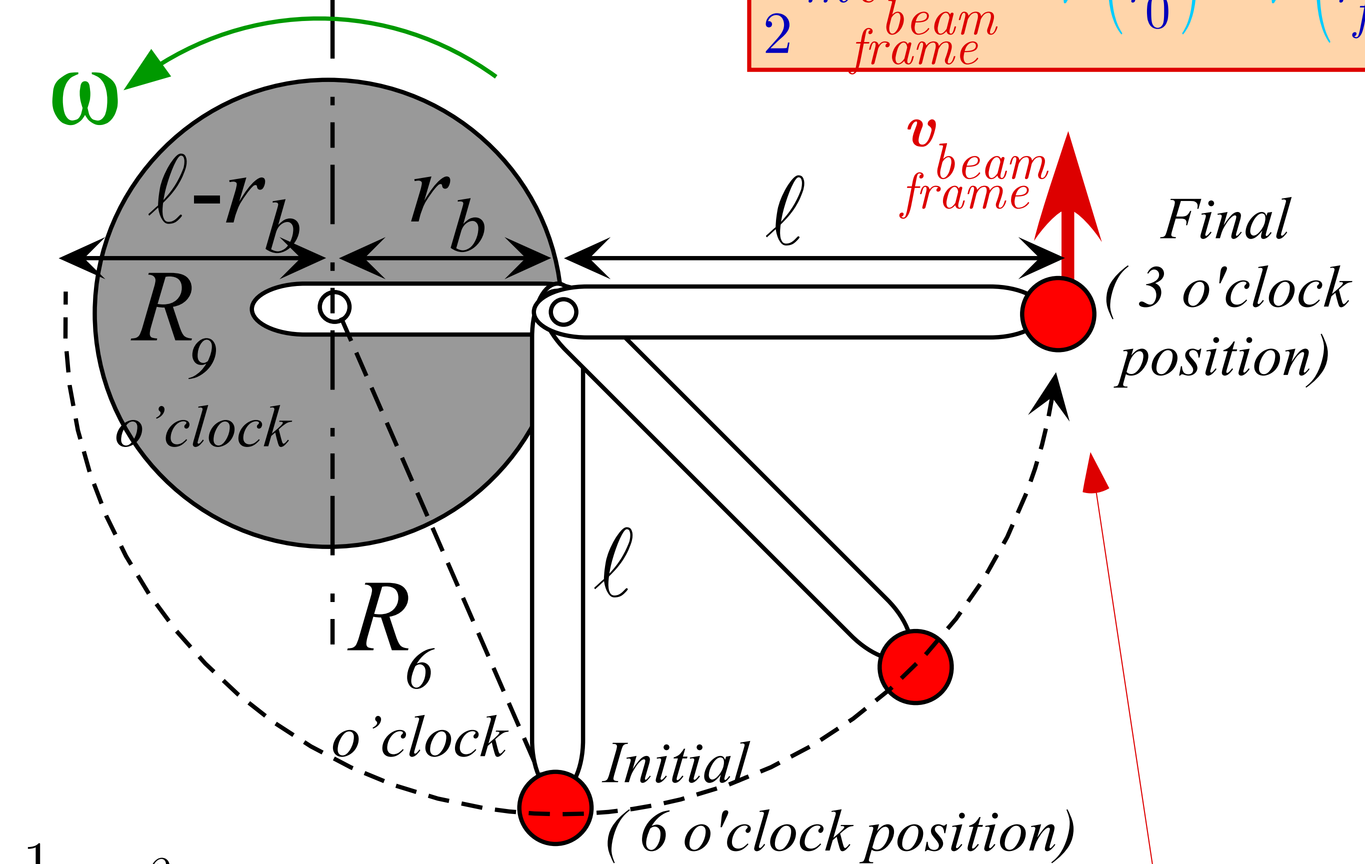
$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(\text{flinger}) =$$ *Final Flinger* $KE_{\substack{beam\\frame}}$

Trebuchet model in rotating beam frame

Flinger model in rotating beam frame

$$\frac{1}{2}mv^2_{\substack{beam\\frame}} = V(r_0) - V(r_f) = \frac{1}{2}m\omega^2 r_f^2 - \frac{1}{2}m\omega^2 r_0^2$$

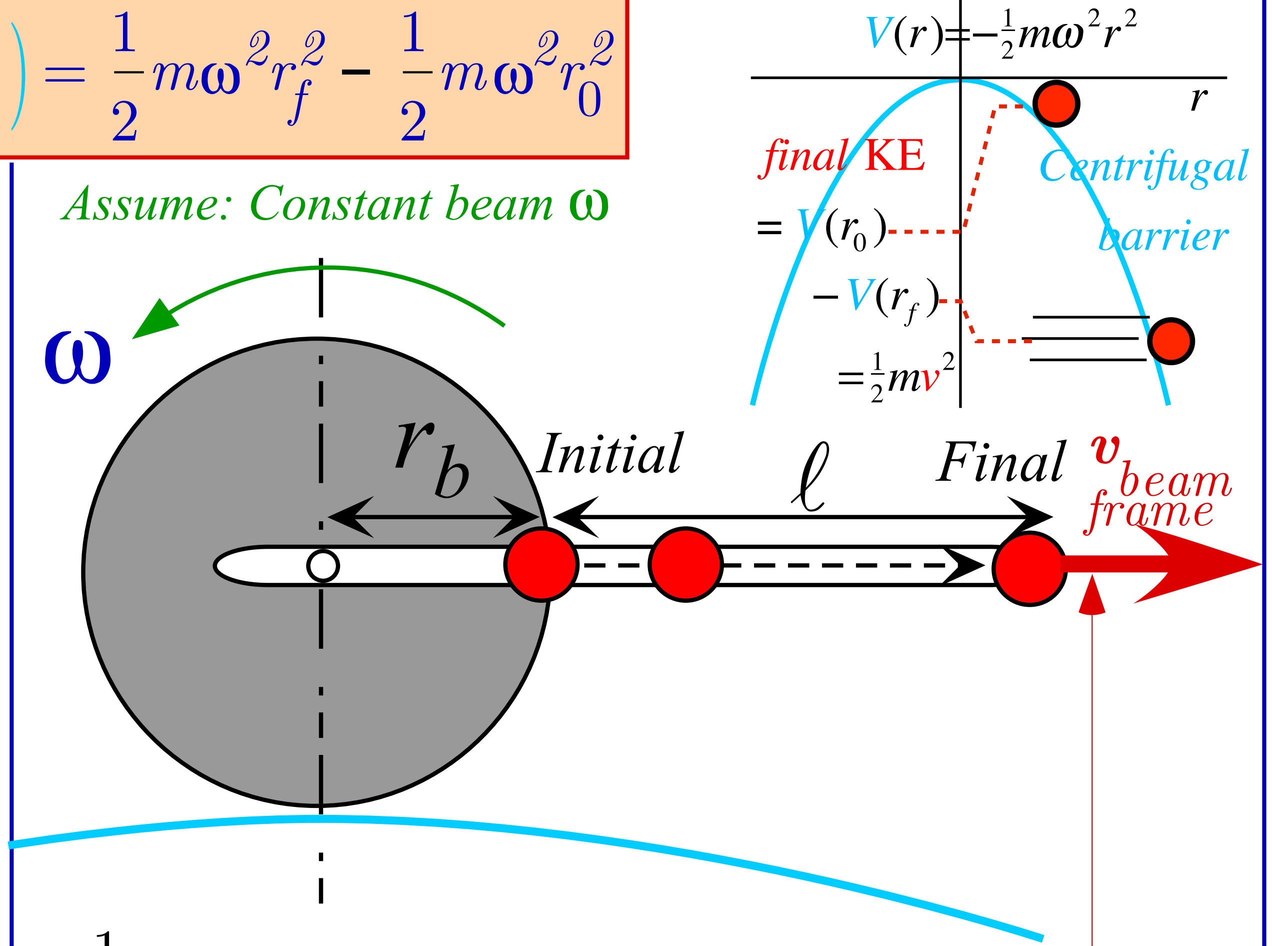
Assume: Constant beam ω



$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(\text{trebuchet}) = \text{Final Trebuchet KE}_{\substack{beam\\frame}}$$

$$\underbrace{\frac{1}{2}m\omega^2\left(r_b + \ell\right)^2}_{Final} - \underbrace{\frac{1}{2}m\omega^2\left(r_b^2 + \ell^2\right)}_{Initial} = \frac{1}{2}m\omega^2\left(2r_b\ \ell\right)$$

Assume: Constant beam ω



$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(\text{flinger}) = \text{Final Flinger KE}_{\substack{beam\\frame}}$$

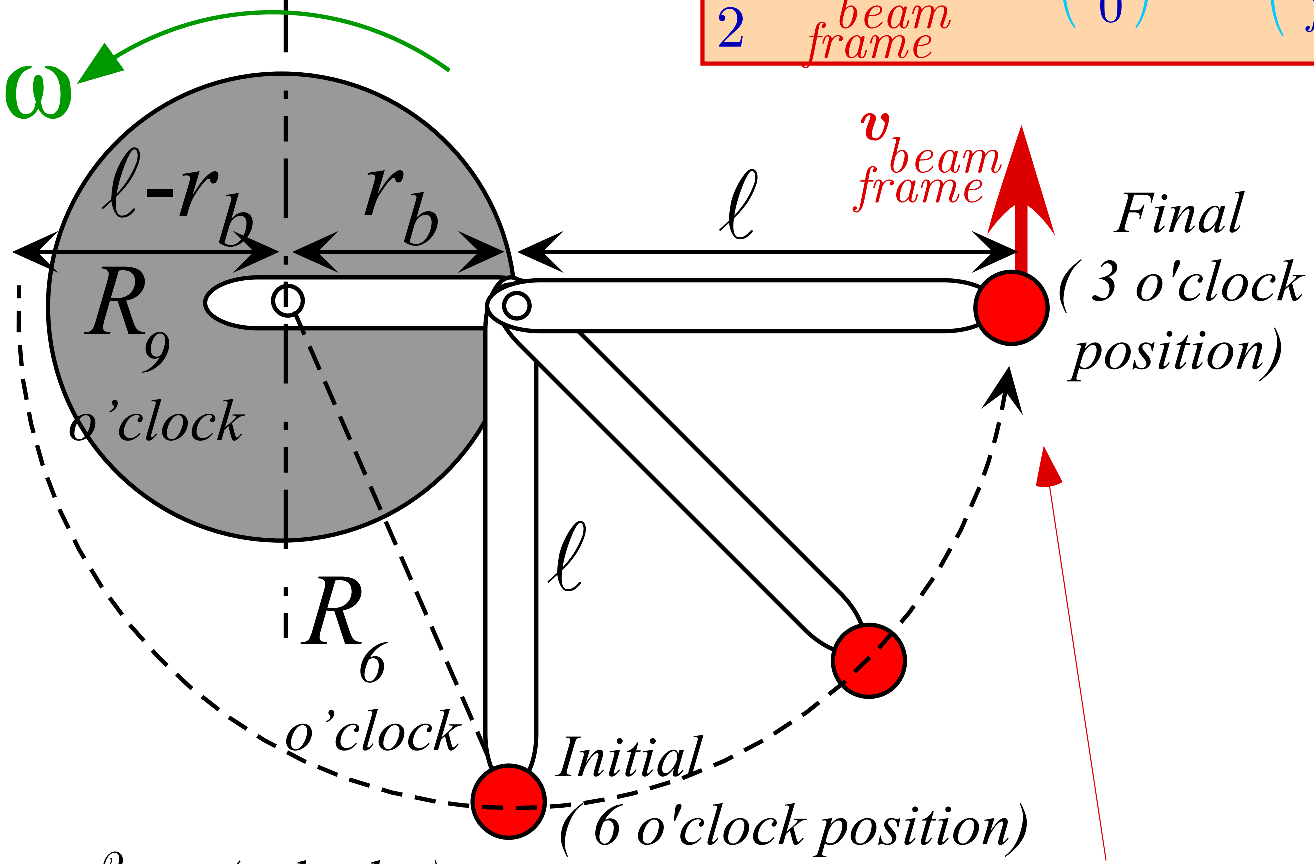
$$\underbrace{\frac{1}{2}m\omega^2\left(r_b + \ell\right)^2}_{Final} - \underbrace{\frac{1}{2}m\omega^2 r_b^2}_{Initial} = \frac{1}{2}m\omega^2\ell\left(2r_b + \ell\right)$$

Trebuchet model in rotating beam frame

Flinger model in rotating beam frame

$$\frac{1}{2}mv^2_{\substack{beam\\frame}} = V(r_0) - V(r_f) = \frac{1}{2}m\omega^2 r_f^2 - \frac{1}{2}m\omega^2 r_0^2$$

Assume: Constant beam ω

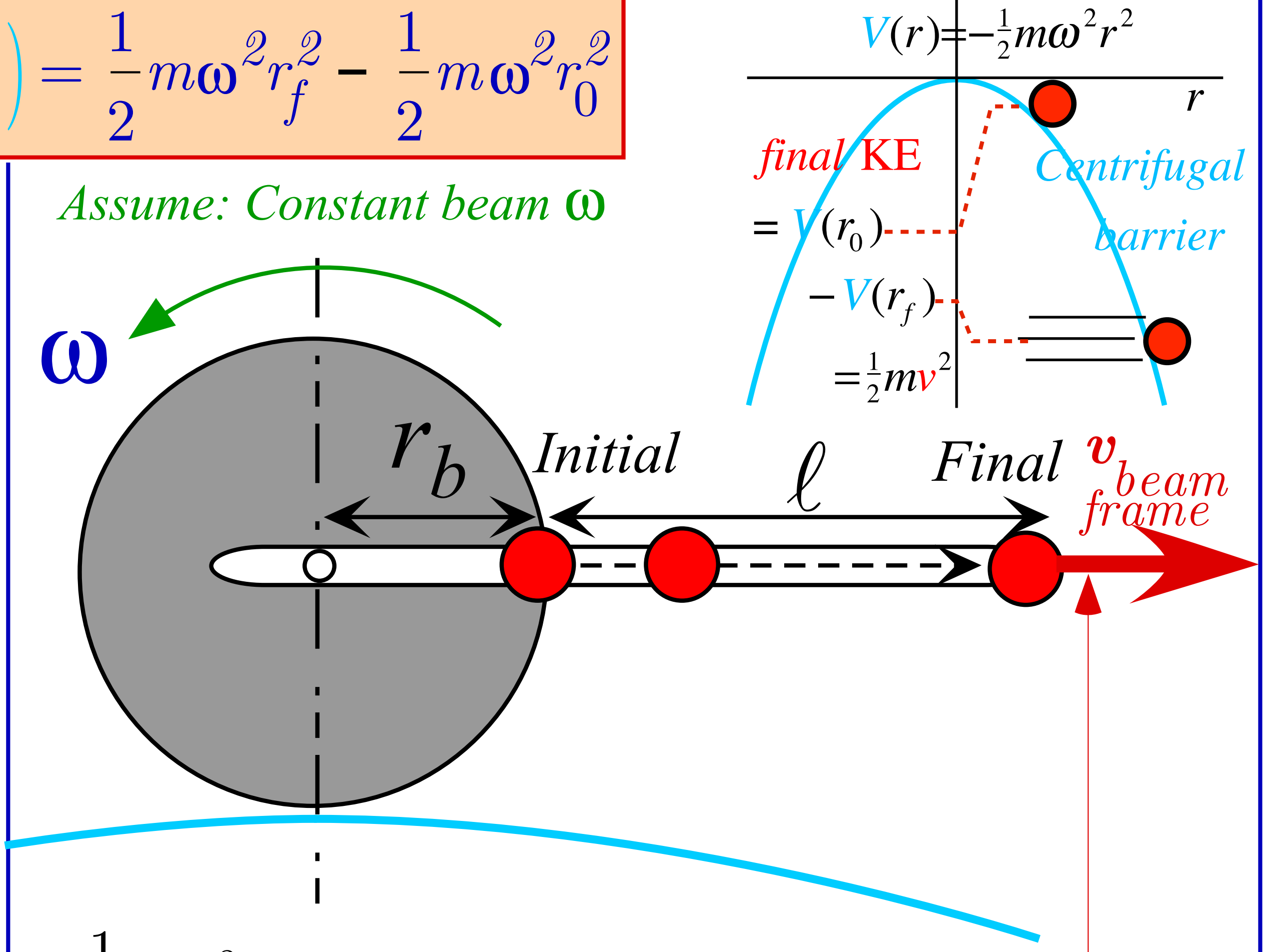


$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(trebuchet) = \text{Final Trebuchet } KE_{\substack{beam\\frame}}$$

$$\underset{\substack{Final\\3\ o'clock}}{\frac{1}{2}m\omega^2\left(r_b + \ell\right)^2} - \underset{\substack{Initial\\6\ o'clock}}{\frac{1}{2}m\omega^2\left(r_b^2 + \ell^2\right)} = \frac{1}{2}m\omega^2\left(2r_b\ \ell\right)$$

$$R^2_{\substack{6\\o'clock}} = r_b^2 + \ell^2$$

Assume: Constant beam ω



$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(flinger) = \text{Final Flinger } KE_{\substack{beam\\frame}}$$

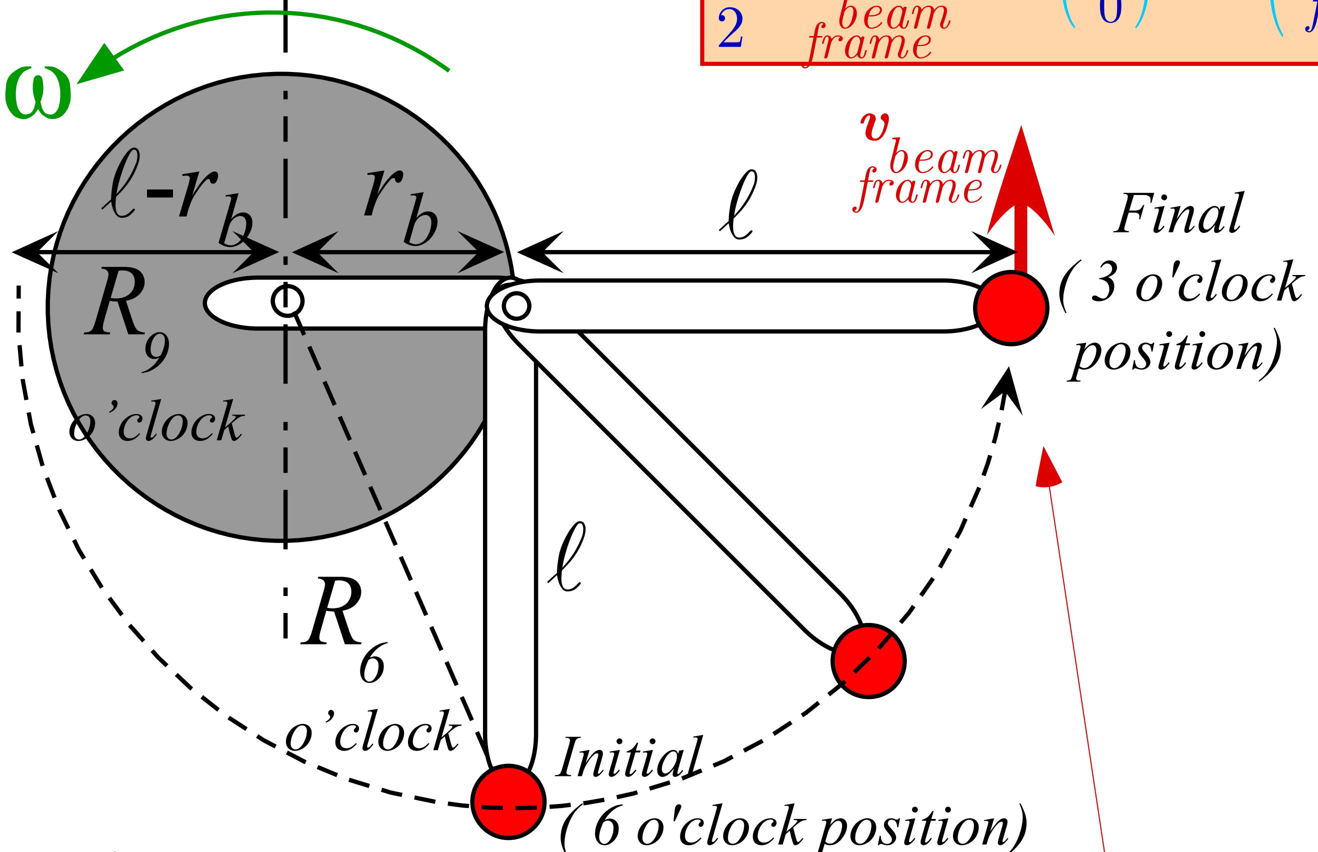
$$\underset{\substack{Final\\3\ o'clock}}{\frac{1}{2}m\omega^2\left(r_b + \ell\right)^2} - \underset{\substack{Initial\\3\ o'clock}}{\frac{1}{2}m\omega^2 r_b^2} = \frac{1}{2}m\omega^2\ell\left(2r_b + \ell\right)$$

Flinger KE is $\frac{m\omega^2}{2}\ell^2$ more than 6 o'clock trebuchet but misdirected

Trebuchet model in rotating beam frame

$$\frac{1}{2}mv^2_{\substack{beam\\frame}} = V(r_0) \qquad V(r_f) = \frac{1}{2}m\omega^2 r_f^2 \qquad \frac{1}{2}m\omega^2 r_0^2$$

Assume: Constant beam ω



$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(\text{trebuchet}) = \text{Final Trebuchet KE}_{\substack{beam\\frame}}$$

$$\frac{1}{2}m\omega^2\left(r_b + \ell\right)^2 - \frac{1}{2}m\omega^2\left(r_b^2 + \ell^2\right) = \frac{1}{2}m\omega^2\left(2r_b\ \ell\right)$$

Final
3 o'clock

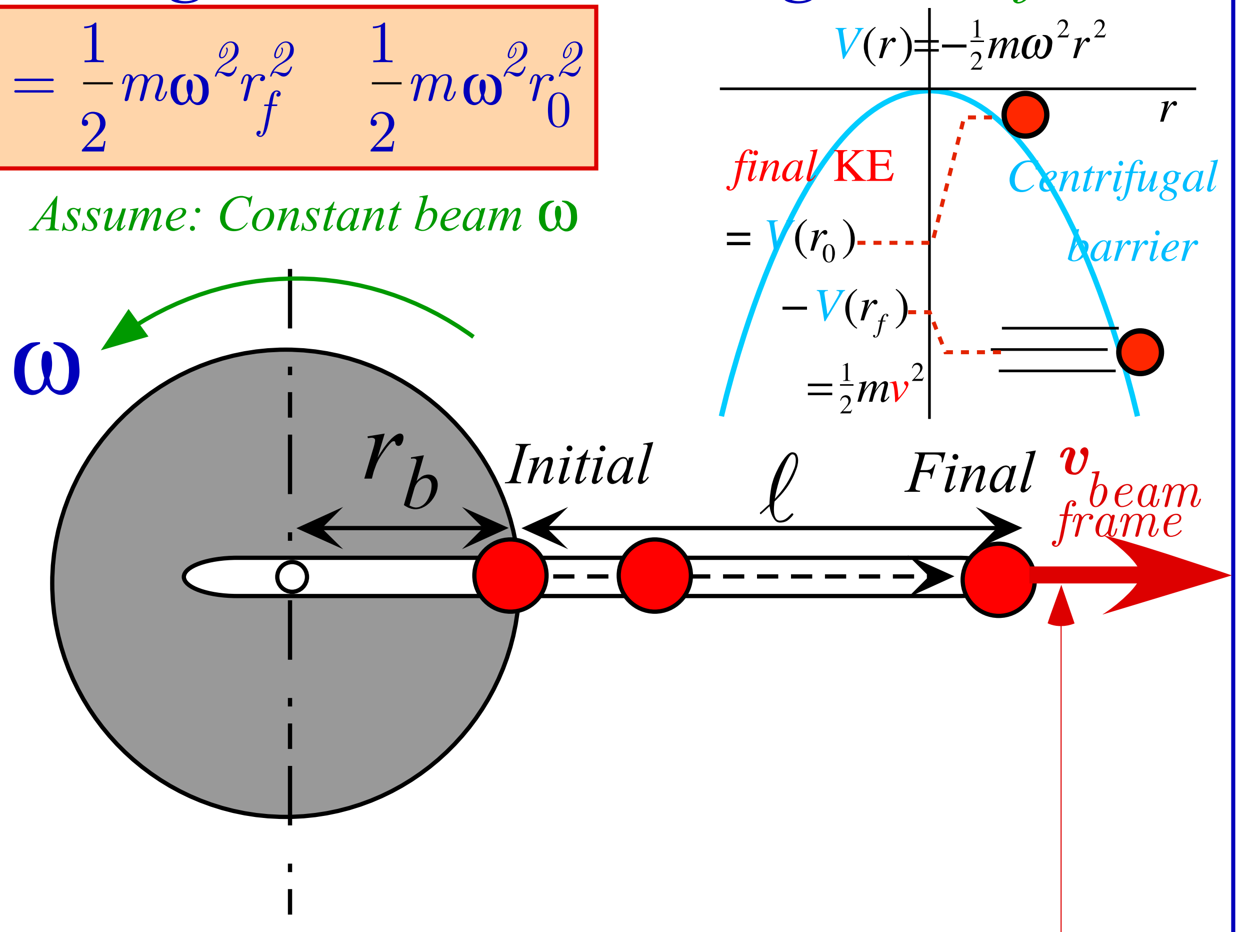
Initial
6 o'clock

$$R^2_{\substack{6\\o'clock}} = r_b^2 + \ell^2$$

Initial
9 o'clock
$$= \frac{1}{2}m\omega^2\left(4r_b\ \ell\right)$$

$$R^2_{\substack{9\\o'clock}} = r_b^2 + \ell^2 - 2r_b\ell$$

Flinger model in rotating beam frame



Assume: Constant beam ω

$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(\text{flinger}) = \text{Final Flinger KE}_{\substack{beam\\frame}}$$

$$\frac{1}{2}m\omega^2\left(r_b + \ell\right)^2 - \frac{1}{2}m\omega^2 r_b^2 = \frac{1}{2}m\omega^2\ell\left(2r_b + \ell\right)$$

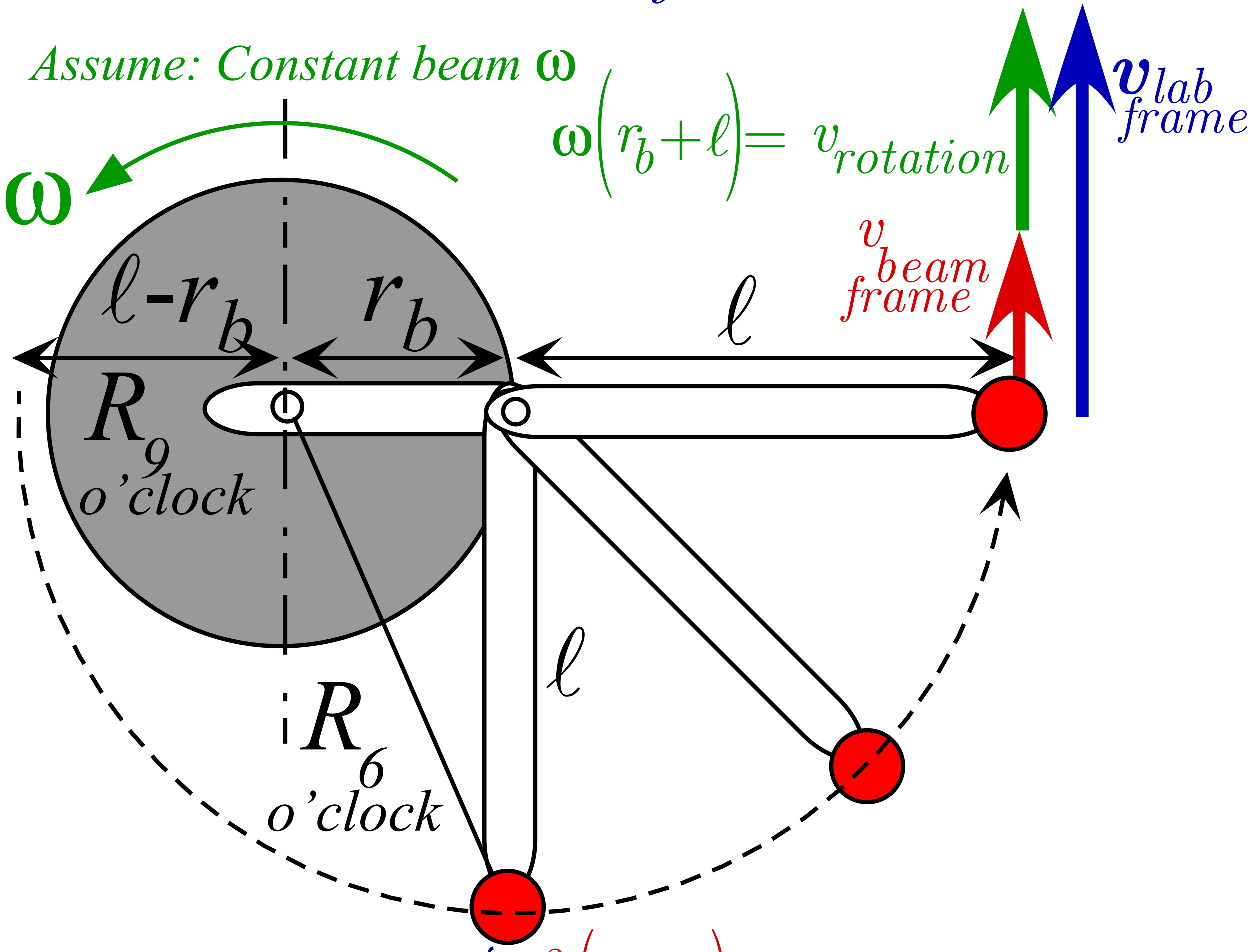
Final
3 o'clock

Initial
3 o'clock

Flinger KE is $\frac{m\omega^2}{2}\ell^2$ more than 6 o'clock trebuchet
but misdirected

Flinger KE is $\frac{m\omega^2}{2}\left(2r_b\ell - \ell^2\right)$ less than 9 o'clock trebuchet
and misdirected

Trebuchet model in lab frame



$$v^2_{\substack{beam\\frame}}(\text{trebuchet}) = \begin{cases} \omega^2\left(2r_b\ \ell\right) & \textit{half-cocked 6 o'clock} \\ \omega^2\left(4r_b\ \ell\right) & \textit{fully-cocked 9 o'clock} \end{cases}$$

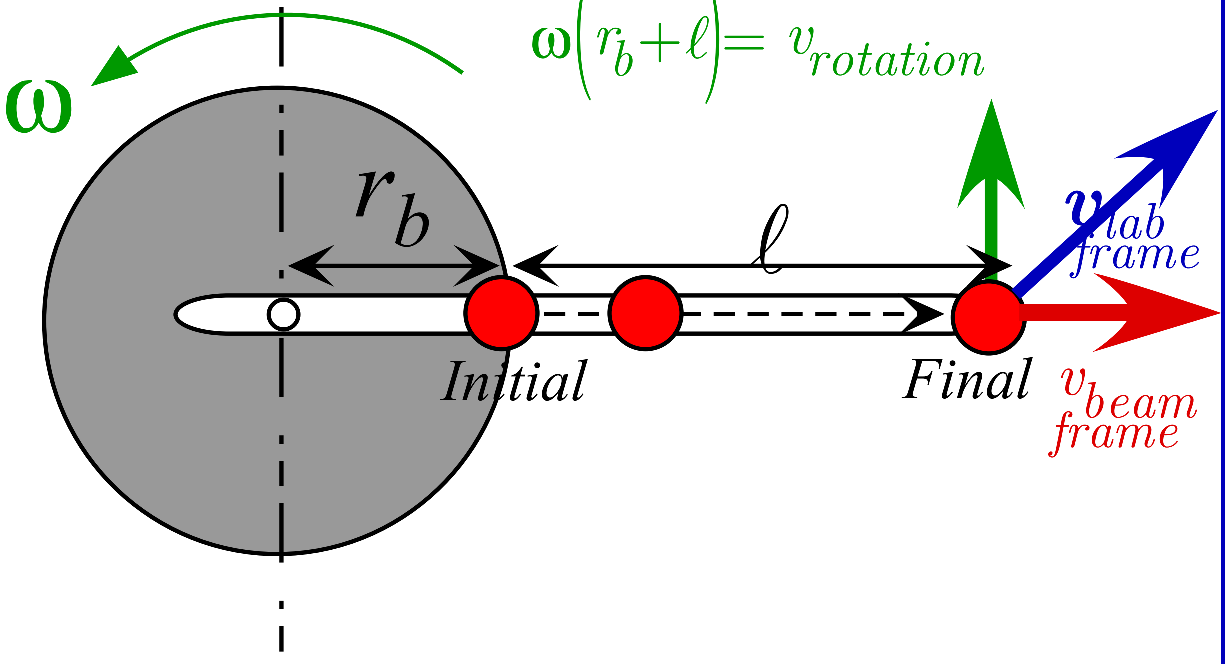
$$v_{lab\ \ frame}\left(trebuchet\right) =$$

$$\begin{cases} \omega\left(r_b + \ell + \sqrt{2\ell r_b}\right) & \textit{half-cocked 6 o'clock} \\ \omega\left(r_b + \ell + 2\sqrt{\ell r_b}\right) & \textit{fully-cocked 9 o'clock} \end{cases}$$

$$= \begin{cases} 5.00\omega \\ 5.82\omega \end{cases} \qquad = \begin{cases} 5.16\omega \\ 6.00\omega \end{cases} \qquad = \begin{cases} 5.00\omega \\ 5.82\omega \end{cases}$$

$$\left(r_b = 2\,,\ell = 1\right),\ \left(r_b = 1.5\,,\ell = 1.5\right),\left(r_b = 1\,,\ell = 2\right)$$

Flinger model in lab frame



$$v^2_{\substack{beam\\frame}}(\text{flinger}) \ = \ \omega^2\ell\left(2r_b + \ell\right)$$

$$v_{lab\ \ frame}\left(flinger\right) =$$

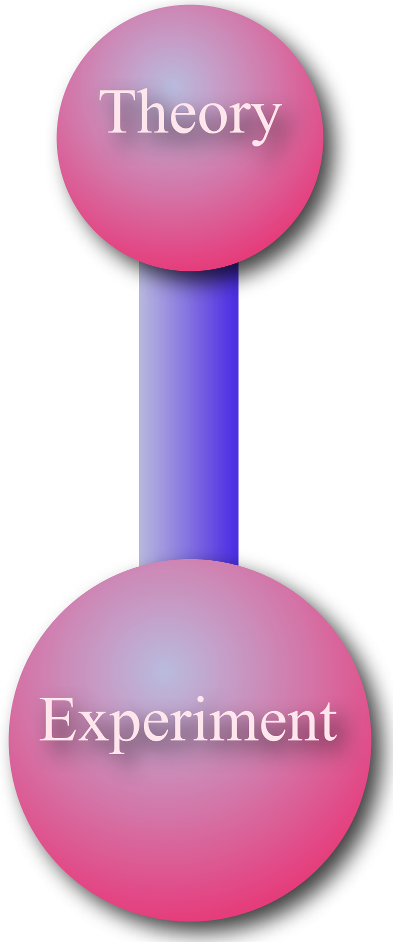
$$= \omega\sqrt{\left(r_b + \ell\right)^2 + \ell\left(2r_b + \ell\right)} = \omega\sqrt{2\left(r_b + \ell\right)^2 - r_b{}^2}$$

(compare)

$$= 3.74\omega \qquad = 3.96\omega \qquad = 4.12\omega$$

$$\left(r_b = 2\,,\ell = 1\right),\ \left(r_b = 1.5\,,\ell = 1.5\right),\left(r_b = 1\,,\ell = 2\right)$$

Physics used to be pretty much bi-polar...



Now that situation is changing...

Many Approaches to Mechanics (Trebuchet Equations)

Each has advantages and disadvantages

- U.S. Approach

Quick'n dirty
Newton F=Ma Equations
Cartesian coordinates

- French Approach

Tres elegant
Lagrange Equations
in Generalized Coordinates

$$F_\ell = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell}$$

- German Approach

Pride and Precision
Riemann Christoffel Equations
in Differential Manifolds

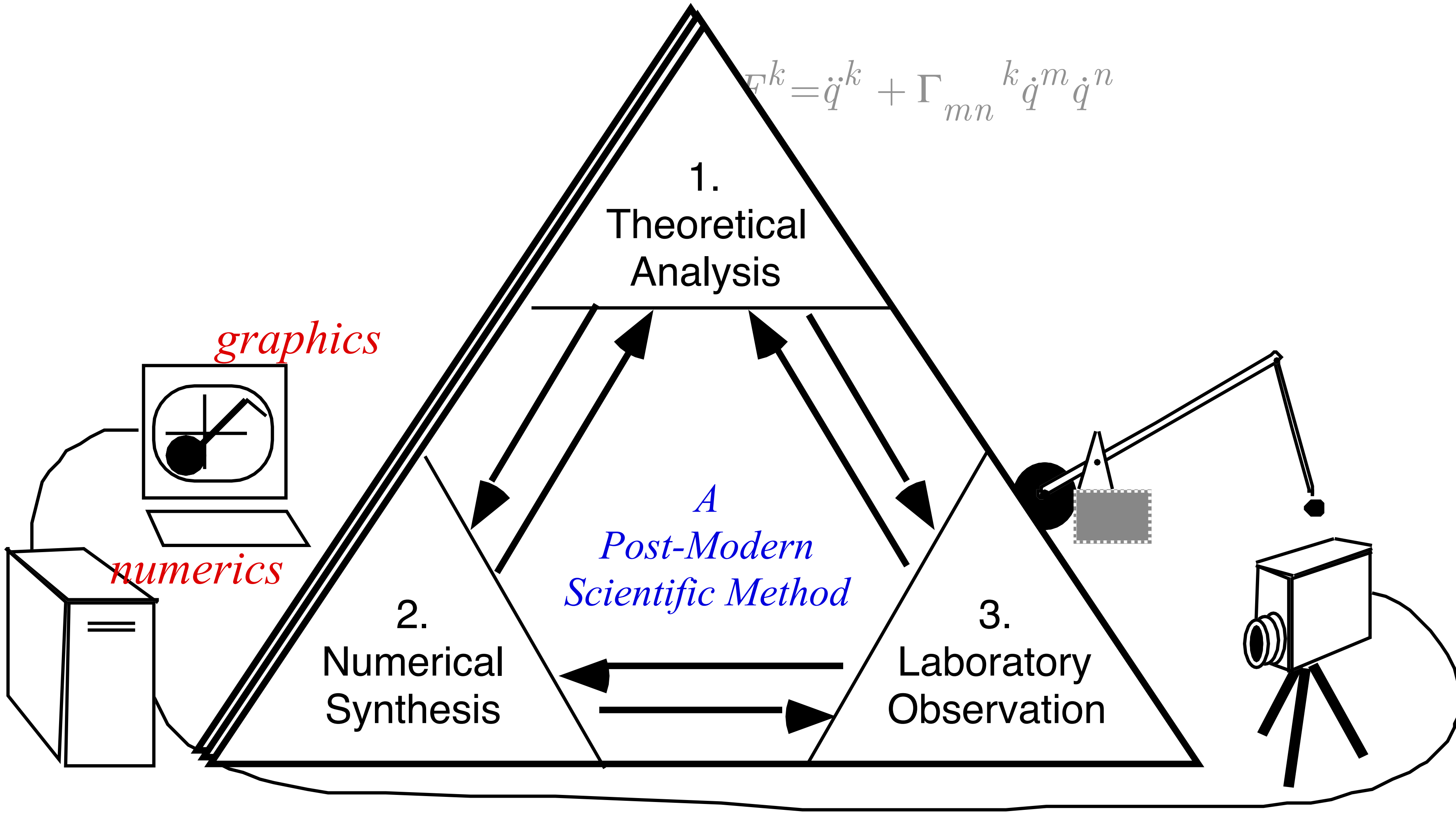
$$F^k = \ddot{q}^k + \Gamma_{mn}{}^k \dot{q}^m \dot{q}^n$$

- Anglo-Irish Appproach

Powerfully Creative
Hamilton's Equations
Phase Space

$$\dot{p}_j = -\frac{\partial H}{\partial q^j}, \qquad \dot{q}^k = \frac{\partial H}{\partial p^k}.$$

- Unified Approach



All approaches have one thing in common:
The *Art of Approximation*
Physics lives and dies by the art of
approximate models and analogs.

$$dS = Ldt = p_\mu dq^\mu - Hdt$$

Hamilton-Jacobi-Poincare: $p_\mu = \frac{\partial S}{\partial q^\mu}$, $-H = \frac{\partial S}{\partial t}$

Force, Work, and Acceleration

$$dW = F_X\ dX + F_Y\ dY + F_x\ dx + F_x\ dy$$

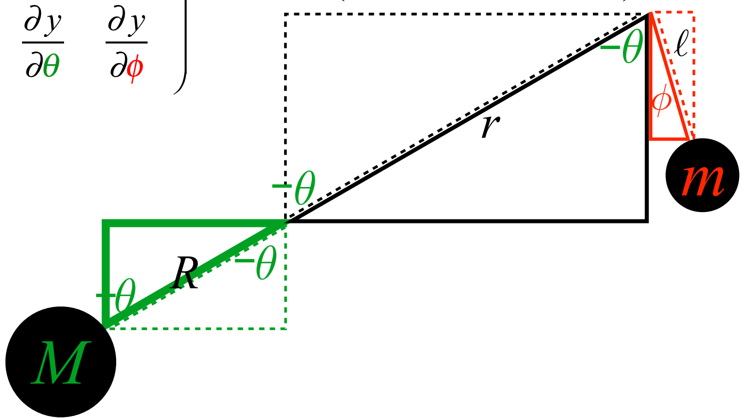
$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$\begin{aligned}
dW = F_X\ dX &= M\ddot{X}\,dX &&= F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi &&= M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi \\
+F_Y\ dY &\quad + M\ddot{Y}\,dY &&\quad + F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi &&\quad + M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi \\
+F_x\ dx &\quad + m\ddot{x}\,dx &&\quad + F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi &&\quad + m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi \\
+F_y\ dy &\quad + m\ddot{y}\,dy &&\quad + F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi &&\quad + m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi
\end{aligned}$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Lagrange trickery:

STEP **D** Add up first and last columns for each variable θ and ϕ for: $T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$

$$Let: F_X\frac{\partial X}{\partial\theta} + F_Y\frac{\partial Y}{\partial\theta} + F_x\frac{\partial x}{\partial\theta} + F_y\frac{\partial y}{\partial\theta}$$

$$\equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta}$$

$$Let: F_X\frac{\partial X}{\partial\phi} + F_Y\frac{\partial Y}{\partial\phi} + F_x\frac{\partial x}{\partial\phi} + F_y\frac{\partial y}{\partial\phi}$$

$$\equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi}$$

Completes derivation of Lagrange covariant-force equation for each GCC variable θ and ϕ .

$$F_X R\cos\theta + F_Y R\sin\theta - F_x r\cos\theta - F_y r\sin\theta$$

$$\equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta}$$

Add F_θ gravity given

$(F_X = 0\ ,\ F_Y = -Mg)$

$(F_x = 0\ ,\ F_y = -mg)$

$$F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta} = -MgR\sin\theta + mgr\sin\theta$$

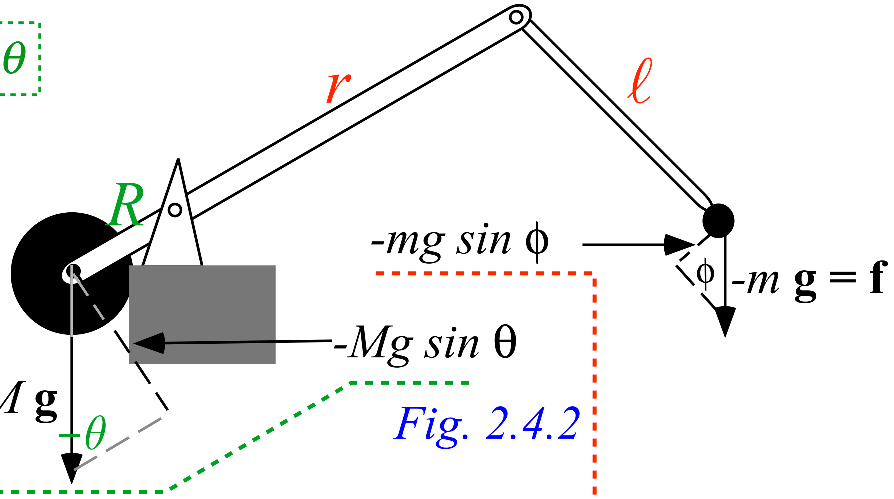


Fig. 2.4.2

$$F_X\cdot 0 + F_Y\cdot 0 + F_x\ell\cos\phi + F_y\ell\sin\phi$$

$$\equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi}$$

Add F_ϕ gravity given

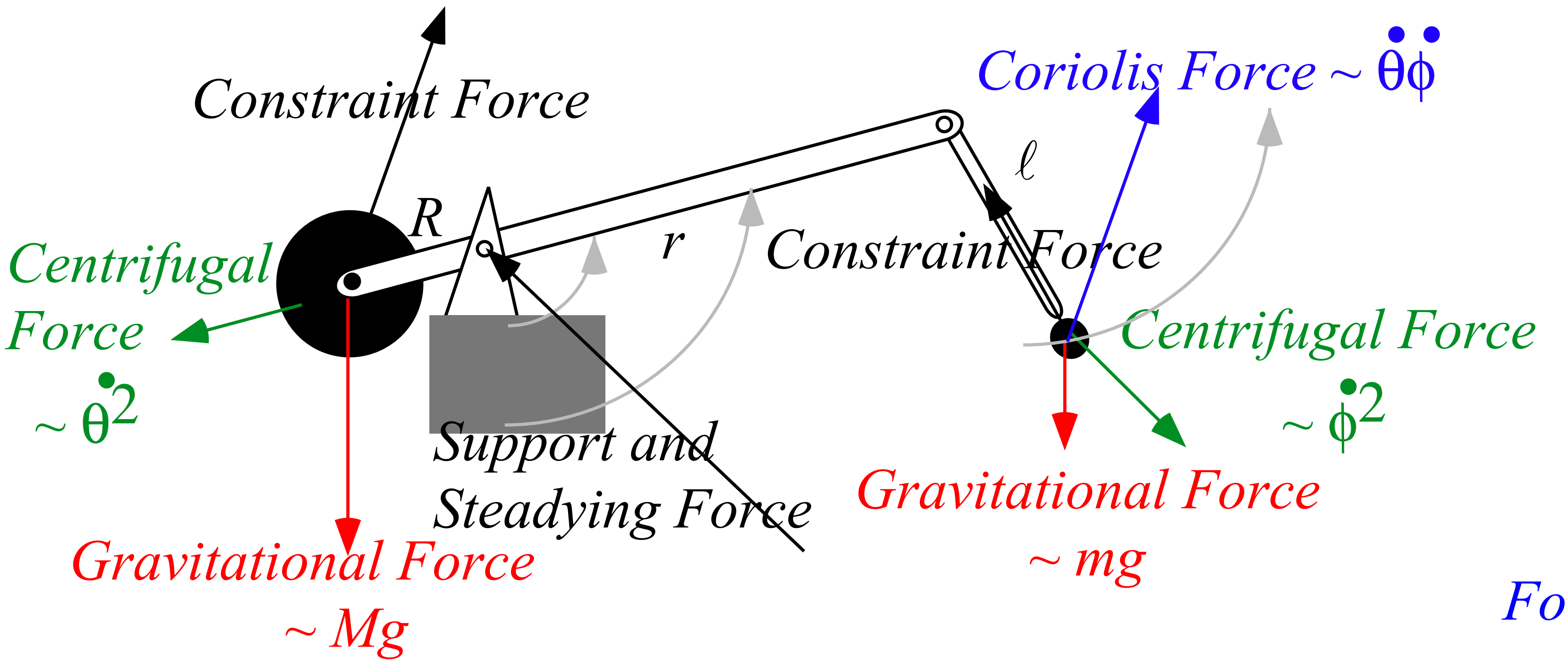
$(F_X = 0\ ,\ F_Y = -Mg)$

$(F_x = 0\ ,\ F_y = -mg)$

$$F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi} = -mg\ell\sin\phi$$

These are competing torques on main beam R... ... and a torque on throwing lever ℓ

Forces: total, genuine, potential, and/or fictitious



Acceleration and 'Fictitious' Forces:
Coriolis
Centrifugal

Applied 'Real' Forces:
Gravity
Stimuli
Friction...

Constraint 'Internal' Forces:
Stresses
Support...
(Do not contribute. Do no work.)

$$\dot{p}_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} = \frac{\partial T}{\partial \theta} + F_\theta + 0$$

$$\dot{p}_\phi = \frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} = \frac{\partial T}{\partial \phi} + F_\phi + 0$$

Lagrange Force equations
(See also derivation Eq. (2.4.7) on p. 23 , Unit 2)

Fig. 2.5.2 (modified)

For conservative forces

where: $F_\theta = -\frac{\partial V}{\partial \theta}$ and: $\frac{\partial V}{\partial \dot{\theta}} = 0$

$F_\phi = -\frac{\partial V}{\partial \phi}$ and: $\frac{\partial V}{\partial \dot{\phi}} = 0$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} \qquad \dot{p}_\theta = \frac{\partial L}{\partial \theta}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} \qquad \dot{p}_\phi = \frac{\partial L}{\partial \phi}$$

Lagrange Potential equations

$L=T-V$

Compare to derivation Eq (12.25a) in Ch. 12 of Unit 1 and Eq. (3.5.10) in Unit 3.