Lecture 18
Tue. 10.25.2016

Electromagnetic Lagrangian and charge-field mechanics (Ch. 2.8 of Unit 2)

Charge mechanics in electromagnetic fields

- *Vector analysis for particle-in-(**A**,Φ)-potential*
- *Lagrangian for particle-in-(**A**,Φ)-potential*
- *Hamiltonian for particle-in-(**A**,Φ)-potential*
  - *Canonical momentum in (**A**,Φ) potential*
  - *Hamiltonian formulation*
  - *Hamilton's equations*

Crossed E and B field mechanics

- *Classical Hall-effect and cyclotron orbit orbit equations*
  - *Vector theory vs. complex variable theory*
- *Mechanical analog of cyclotron and FBI rule*
  - *Cycloid and epicycloid ruler&compass geometry*
  - *Cycloid geometry of flying levers*
    - *Practical poolhall application*

*This mechanical analog of (E_x,B_z) field mimics **A**-field with tabletop **v**-field*

Charge mechanics in electromagnetic fields

- → *Vector analysis for particle-in-(**A**,Φ)-potential*
- *Lagrangian for particle-in-(**A**,Φ)-potential*
- *Hamiltonian for particle-in-(**A**,Φ)-potential*
  - *Canonical momentum in (**A**,Φ) potential*
  - *Hamiltonian formulation*
  - *Hamilton's equations*

Vector analysis for particle-in-($\mathbf{A}$,Φ)-potential

So-called ***pondermotive*** form for Newton's *F=ma* equation for a mass *m* of charge *e*.

electronic charge:
e =-1.602176·10^{-19}Coulombs

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e(\mathbf{E}+\mathbf{v}\times\mathbf{B})$$

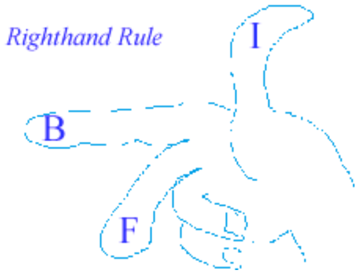
Electric field **E** and magnetic field **B**

scalar potential field $\mathbf{\Phi}=\mathbf{\Phi}(\mathbf{r},t)$

vector potential field $\mathbf{A}=\mathbf{A}(\mathbf{r},t)$

$$\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B}=\nabla\times\mathbf{A}$$



$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

*Vector analysis for particle-in-(**A**,Φ)-potential*

So-called ***pondermotive*** form for Newton's $F=ma$ equation for a mass m of charge e.

electronic charge:
e =-1.602176·10^{-19}Coulombs

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e(\mathbf{E}+\mathbf{v}\times\mathbf{B})$$

Electric field **E** and magnetic field **B**

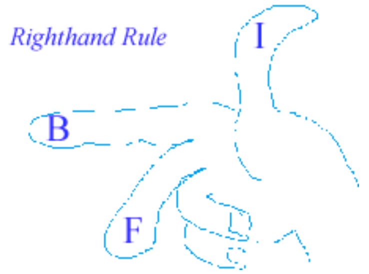
scalar potential field $\mathbf{\Phi}=\mathbf{\Phi}(\mathbf{r},t)$

vector potential field $\mathbf{A}=\mathbf{A}(\mathbf{r},t)$

$$\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B}=\nabla\times\mathbf{A}$$

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e\left[-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}+\mathbf{v}\times\left(\nabla\times\mathbf{A}\right)\right]$$



$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

Vector analysis for particle-in-(A,Φ)-potential

So-called *pondermotive* form for Newton's $F=ma$ equation for a mass m of charge e.

electronic charge:
e =-1.602176·10^{-19}Coulombs

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e(\mathbf{E}+\mathbf{v}\times\mathbf{B})$$

Electric field **E** and magnetic field **B**

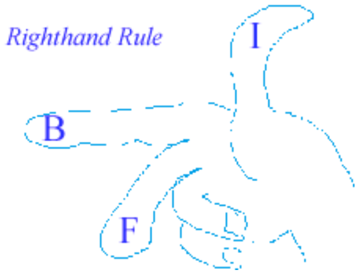
scalar potential field $\boldsymbol{\Phi}=\boldsymbol{\Phi}(\mathbf{r},t)$

vector potential field $\mathbf{A}=\mathbf{A}(\mathbf{r},t)$

$$\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B}=\nabla\times\mathbf{A}$$

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e\left[-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}+\mathbf{v}\times\left(\nabla\times\mathbf{A}\right)\right]$$



$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

Doing a double-cross

ε_{ijk}-Tensor analysis of $\mathbf{v}\times(\nabla\times\mathbf{A})$

$$\left[\mathbf{v}\times(\nabla\times\mathbf{A})\right]_k=\varepsilon_{kij}v_i(\nabla\times\mathbf{A})_j$$

*Vector analysis for particle-in-(**A**,Φ)-potential*

So-called ***pondermotive*** form for Newton's *F=ma* equation for a mass *m* of charge *e*.

electronic charge:
e =-1.602176·10^{-19}Coulombs

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e(\mathbf{E}+\mathbf{v}\times\mathbf{B})$$

Electric field **E** and magnetic field **B**

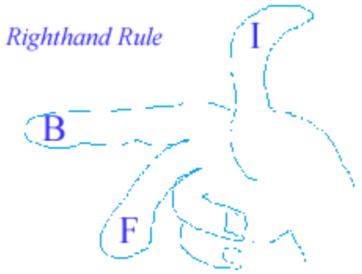
scalar potential field **Φ=Φ(r**,*t)*

vector potential field **A=A(r**,*t)*

$$\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B}=\nabla\times\mathbf{A}$$

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e\left[-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}+\mathbf{v}\times\left(\nabla\times\mathbf{A}\right)\right]$$



F=*q*v×B=I×B

Doing a double-cross

ε_{ijk}-Tensor analysis of $\mathbf{v}\times(\nabla\times\mathbf{A})$

$$\left[\mathbf{v}\times(\nabla\times\mathbf{A})\right]_k=\varepsilon_{kij}v_i\left(\nabla\times\mathbf{A}\right)_j$$

$$\varepsilon_{ijk}=\begin{cases}+1 \text{ for } \textit{even} \text{ permutaion of } i<j<k\\ 0 \quad \text{if any of } i,j,k \text{ are equal}\\ -1 \text{ for } \textit{odd} \text{ permutaion of } i<j<k\end{cases}$$

$$\varepsilon_{ijk}=\quad\varepsilon_{ikj}=\quad\varepsilon_{kij}=1$$

$$=-\varepsilon_{jik}=-\varepsilon_{jki}=-\varepsilon_{kji}$$

Vector analysis for particle-in-($\mathbf{A}$,Φ)-potential

So-called ***pondermotive*** form for Newton's *F=ma* equation for a mass *m* of charge *e*.

electronic charge:
e =-1.602176·10^{-19}Coulombs

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F} = e(\mathbf{E}+\mathbf{v}\times\mathbf{B})$$

Electric field **E** and magnetic field **B**

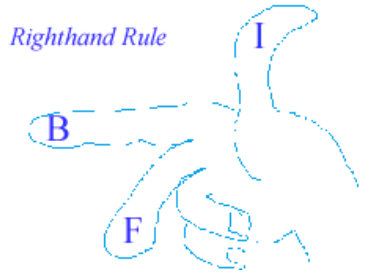
scalar potential field $\boldsymbol{\Phi}$=$\boldsymbol{\Phi}$($\mathbf{r}$,*t*)

vector potential field $\mathbf{A}$=$\mathbf{A}$($\mathbf{r}$,*t*)

$$\mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla\times\mathbf{A}$$

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F} = e\left[-\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t} + \mathbf{v}\times\left(\nabla\times\mathbf{A}\right)\right]$$



F=*q***v**×**B**=**I**×**B**

Doing a double-cross

ε_{ijk}-Tensor analysis of $\mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{A})$

$$\left[\mathbf{v}\times(\nabla\times\mathbf{A})\right]_k = \varepsilon_{kij}v_i\left(\nabla\times\mathbf{A}\right)_j$$

$$= \varepsilon_{kij}v_i\left(\varepsilon_{abj}\left(\partial_a A_b\right)\right)$$

$$\varepsilon_{ijk} = \begin{cases} +1 \text{ for } \textit{even} \text{ permutaion of } i<j<k \\ 0 \quad \text{if any of } i,j,k \text{ are equal} \\ -1 \text{ for } \textit{odd} \text{ permutaion of } i<j<k \end{cases}$$

$$\varepsilon_{ijk} = \quad \varepsilon_{ikj} = \quad \varepsilon_{kij} = 1$$

$$= -\varepsilon_{jik} = -\varepsilon_{jki} = -\varepsilon_{kji}$$

Vector analysis for particle-in-(A,Φ)-potential

So-called ***pondermotive*** form for Newton's *F=ma* equation for a mass *m* of charge *e*.

electronic charge:
e =-1.602176·10^{-19}Coulombs

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e(\mathbf{E}+\mathbf{v}\times\mathbf{B})$$

Electric field **E** and magnetic field **B**

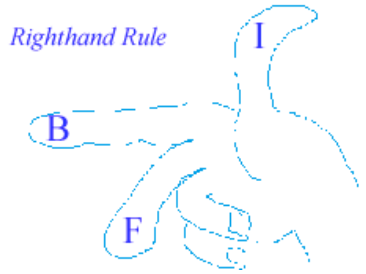
scalar potential field **Φ=Φ(r**,*t)*

vector potential field **A=A(r**,*t)*

$$\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B}=\nabla\times\mathbf{A}$$

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e\left[-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}+\mathbf{v}\times\left(\nabla\times\mathbf{A}\right)\right]$$



F=*q*v×B=I×B

Doing a double-cross

ε_{ijk}-Tensor analysis of $\mathbf{v}\times(\nabla\times\mathbf{A})$

$$\left[\mathbf{v}\times(\nabla\times\mathbf{A})\right]_k=\varepsilon_{kij}v_i\left(\nabla\times\mathbf{A}\right)_j$$

$$=\varepsilon_{kij}v_i\left(\varepsilon_{abj}\left(\partial_a A_b\right)\right)$$

$$=\quad\varepsilon_{kij}\varepsilon_{abj}\quad v_i\left(\partial_a A_b\right)$$

$$=(\delta_{ka}\delta_{ib}-\delta_{kb}\delta_{ia})v_i\left(\partial_a A_b\right)$$

$$\varepsilon_{ijk}=\begin{cases}+1 \text{ for } \mathit{even} \text{ permutaion of } i<j<k\\ 0 \quad \text{if any of } i,j,k \text{ are equal}\\ -1 \text{ for } \mathit{odd} \text{ permutaion of } i<j<k\end{cases}$$

$$\varepsilon_{ijk}=\quad\varepsilon_{ikj}=\quad\varepsilon_{kij}=1$$

$$=-\varepsilon_{jik}=-\varepsilon_{jki}=-\varepsilon_{kji}$$

Applying Levi-Civita ε-identity:

$$\varepsilon_{kij}\varepsilon_{abj}\;=\delta_{ka}\delta_{ib}-\delta_{kb}\delta_{ia}$$

*Vector analysis for particle-in-(**A**,Φ)-potential*

So-called ***pondermotive*** form for Newton's *F=ma* equation for a mass *m* of charge *e*.

electronic charge:
e =-1.602176·10^{-19}*Coulombs*

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e(\mathbf{E}+\mathbf{v}\times\mathbf{B})$$

Electric field **E** and magnetic field **B**

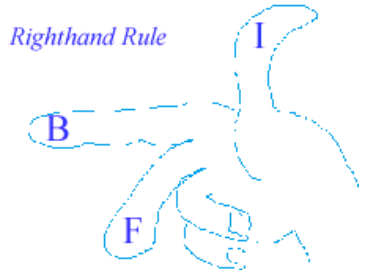
scalar potential field **Φ=Φ(r**,*t)*

vector potential field **A=A(r**,*t)*

$$\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B}=\nabla\times\mathbf{A}$$

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e\left[-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}+\mathbf{v}\times\left(\nabla\times\mathbf{A}\right)\right]$$



F=*q*v×B=I×B

Doing a double-cross

ε_{ijk}-Tensor analysis of $\mathbf{v}\times(\nabla\times\mathbf{A})$

$$\varepsilon_{ijk}=\begin{cases}+1 \text{ for } \textit{even} \text{ permutaion of } i<j<k\\ 0 \quad \text{if any of } i,j,k \text{ are equal}\\ -1 \text{ for } \textit{odd} \text{ permutaion of } i<j<k\end{cases}$$

$$\varepsilon_{ijk}=\quad\varepsilon_{ikj}=\quad\varepsilon_{kij}=1$$

$$=-\varepsilon_{jik}=-\varepsilon_{jki}=-\varepsilon_{kji}$$

$$\left[\mathbf{v}\times(\nabla\times\mathbf{A})\right]_k=\varepsilon_{kij}v_i\left(\nabla\times\mathbf{A}\right)_j$$

$$=\varepsilon_{kij}v_i\left(\varepsilon_{abj}\left(\partial_a A_b\right)\right)$$

$$=\quad\varepsilon_{kij}\varepsilon_{abj}\qquad v_i\left(\partial_a A_b\right)$$

$$=(\delta_{ka}\delta_{ib}-\delta_{kb}\delta_{ia})v_i\left(\partial_a A_b\right)$$

$$=\delta_{ka}\delta_{ib}v_i\left(\partial_a A_b\right)-\delta_{kb}\delta_{ia}v_i\left(\partial_a A_b\right)$$

Applying Levi-Civita ε-identity:

$$\varepsilon_{kij}\varepsilon_{abj}\;=\delta_{ka}\delta_{ib}-\delta_{kb}\delta_{ia}$$

Vector analysis for particle-in-(**A**,Φ)-potential

So-called ***pondermotive*** form for Newton's *F=ma* equation for a mass *m* of charge *e*.

electronic charge:
e =-1.602176·10^-19^Coulombs

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e(\mathbf{E}+\mathbf{v}\times\mathbf{B})$$

Electric field **E** and magnetic field **B**

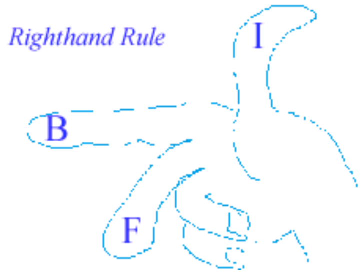
scalar potential field **Φ=Φ(r**,*t)*

vector potential field **A=A(r**,*t)*

$$\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B}=\nabla\times\mathbf{A}$$

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e\left[-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}+\mathbf{v}\times\left(\nabla\times\mathbf{A}\right)\right]$$



F=*q*v×B=I×B

Doing a double-cross

ε_{ijk}-Tensor analysis of **v**×(**∇**×**A**)

$$\varepsilon_{ijk}=\begin{cases}+1 \text{ for } \textit{even} \text{ permutaion of } i<j<k\\ 0 \quad \text{if any of } i,j,k \text{ are equal}\\ -1 \text{ for } \textit{odd} \text{ permutaion of } i<j<k\end{cases}$$

$$\varepsilon_{ijk}=\quad\varepsilon_{ikj}=\quad\varepsilon_{kij}=1$$

$$=-\varepsilon_{jik}=-\varepsilon_{jki}=-\varepsilon_{kji}$$

$$\begin{aligned}\left[\mathbf{v}\times(\nabla\times\mathbf{A})\right]_k &=\varepsilon_{kij}v_i\left(\nabla\times\mathbf{A}\right)_j\\ &=\varepsilon_{kij}v_i\left(\varepsilon_{abj}\left(\partial_a A_b\right)\right)\\ &=\quad\varepsilon_{kij}\varepsilon_{abj}\quad v_i\left(\partial_a A_b\right)\\ &=(\delta_{ka}\delta_{ib}-\delta_{kb}\delta_{ia})v_i\left(\partial_a A_b\right)\\ &=\delta_{ka}\delta_{ib}v_i\left(\partial_a A_b\right)-\delta_{kb}\delta_{ia}v_i\left(\partial_a A_b\right)\\ &=\quad v_b\left(\partial_k A_b\right)\quad -v_a\left(\partial_a A_k\right)\\ &=\quad\left(\partial_k A_b\right)v_b\quad -v_a\left(\partial_a A_k\right)\end{aligned}$$

Applying Levi-Civita ε-identity:

$$\varepsilon_{kij}\varepsilon_{abj}\ =\delta_{ka}\delta_{ib}-\delta_{kb}\delta_{ia}$$

Vector analysis for particle-in-(A,Φ)-potential

So-called ***pondermotive*** form for Newton's $F=ma$ equation for a mass m of charge e.

electronic charge:
$e = -1.602176 \cdot 10^{-19}$Coulombs

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Electric field **E** and magnetic field **B**

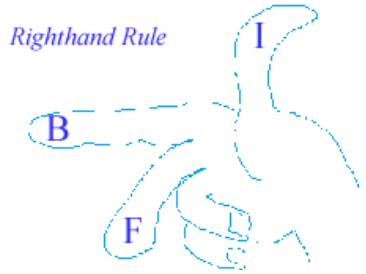
scalar potential field $\mathbf{\Phi}=\mathbf{\Phi}(\mathbf{r},t)$

vector potential field $\mathbf{A}=\mathbf{A}(\mathbf{r},t)$

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F} = e\left[-\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \times \left(\nabla \times \mathbf{A}\right)\right]$$



$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

Doing a double-cross

ε_{ijk}-Tensor analysis of $\mathbf{v}\times(\nabla\times\mathbf{A})$

$$\varepsilon_{ijk} = \begin{cases} +1 \text{ for } \textit{even} \text{ permutaion of } i<j<k \\ 0 \quad \text{if any of } i,j,k \text{ are equal} \\ -1 \text{ for } \textit{odd} \text{ permutaion of } i<j<k \end{cases}$$

$$\varepsilon_{ijk} = \quad \varepsilon_{ikj} = \quad \varepsilon_{kij} = 1$$

$$= -\varepsilon_{jik} = -\varepsilon_{jki} = -\varepsilon_{kji}$$

$$\begin{aligned}
\left[\mathbf{v}\times(\nabla\times\mathbf{A})\right]_k &= \varepsilon_{kij} v_i \left(\nabla\times\mathbf{A}\right)_j \\
&= \varepsilon_{kij} v_i \left(\varepsilon_{abj}\left(\partial_a A_b\right)\right) \\
&= \quad \varepsilon_{kij}\varepsilon_{abj} \quad v_i\left(\partial_a A_b\right) \\
&= (\delta_{ka}\delta_{ib} - \delta_{kb}\delta_{ia}) v_i\left(\partial_a A_b\right) \\
&= \delta_{ka}\delta_{ib} v_i\left(\partial_a A_b\right) - \delta_{kb}\delta_{ia} v_i\left(\partial_a A_b\right) \\
&= \quad v_b\left(\partial_k A_b\right) \quad - v_a\left(\partial_a A_k\right) \\
&= \quad \left(\partial_k A_b\right) v_b \quad - v_a\left(\partial_a A_k\right) \\
&= \partial_k\left(A_b v_b\right) - \left(\partial_k v_b\right) A_b - v_a\left(\partial_a A_k\right)
\end{aligned}$$

Applying Levi-Civita ε-identity:

$$\varepsilon_{kij}\varepsilon_{abj} \; = \delta_{ka}\delta_{ib} - \delta_{kb}\delta_{ia}$$

*Vector analysis for particle-in-(**A**,Φ)-potential*

So-called *pondermotive* form for Newton's *F=ma* equation for a mass *m* of charge *e*.

electronic charge:
e =-1.602176·10^{-19}*Coulombs*

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F} = e(\mathbf{E} + \mathbf{v}\times\mathbf{B})$$

Electric field **E** and magnetic field **B**

scalar potential field **Φ=Φ(r**,*t)*

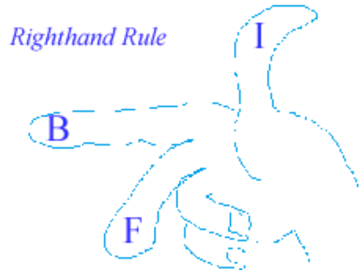
vector potential field **A=A(r**,*t)*

$$\mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla\times\mathbf{A}$$

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F} = e\left[-\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t} + \mathbf{v}\times\left(\nabla\times\mathbf{A}\right)\right]$$

Doing a double-cross



$$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$$

ε_{ijk}-Tensor analysis of $\mathbf{v}\times(\nabla\times\mathbf{A})$

$$\varepsilon_{ijk} = \begin{cases} +1 \text{ for } \textit{even} \text{ permutaion of } i<j<k \\ 0 \quad \text{if any of } i,j,k \text{ are equal} \\ -1 \text{ for } \textit{odd} \text{ permutaion of } i<j<k \end{cases}$$

$$\varepsilon_{ijk} = \quad \varepsilon_{ikj} = \quad \varepsilon_{kij} = 1$$

$$= -\varepsilon_{jik} = -\varepsilon_{jki} = -\varepsilon_{kji}$$

$$\begin{aligned}
\left[\mathbf{v}\times(\nabla\times\mathbf{A})\right]_k &= \varepsilon_{kij}v_i\left(\nabla\times\mathbf{A}\right)_j \\
&= \varepsilon_{kij}v_i\left(\varepsilon_{abj}\left(\partial_a A_b\right)\right) \\
&= \quad \varepsilon_{kij}\varepsilon_{abj} \quad v_i\left(\partial_a A_b\right) \\
&= (\delta_{ka}\delta_{ib} - \delta_{kb}\delta_{ia})v_i\left(\partial_a A_b\right) \\
&= \delta_{ka}\delta_{ib}v_i\left(\partial_a A_b\right) - \delta_{kb}\delta_{ia}v_i\left(\partial_a A_b\right) \\
&= \quad v_b\left(\partial_k A_b\right) \quad - v_a\left(\partial_a A_k\right) \\
&= \quad \left(\partial_k A_b\right)v_b \quad - v_a\left(\partial_a A_k\right) = \left(\nabla\mathbf{A}\right)\bullet\mathbf{v} - \mathbf{v}\bullet\nabla\mathbf{A} \\
&= \partial_k\left(A_b v_b\right) - \left(\partial_k v_b\right)A_b - v_a\left(\partial_a A_k\right) = \nabla(\mathbf{A}\bullet\mathbf{v}) - (\nabla\mathbf{v})\bullet\mathbf{A} - \mathbf{v}\bullet\nabla\mathbf{A}
\end{aligned}$$

Applying Levi-Civita ε-identity:

$$\varepsilon_{kij}\varepsilon_{abj} \; = \delta_{ka}\delta_{ib} - \delta_{kb}\delta_{ia}$$

Converting back to Gibbs's **bold** notation involves *tensors* like ∇A and ∇**v**.

*Vector analysis for particle-in-(**A**,Φ)-potential*

So-called *pondermotive* form for Newton's *F=ma* equation for a mass *m* of charge *e*.

electronic charge:
e =-1.602176·10^{-19}Coulombs

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e(\mathbf{E}+\mathbf{v}\times\mathbf{B})$$

Electric field **E** and magnetic field **B**

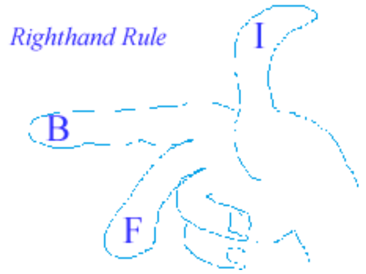
*scalar potential field **Φ=Φ(r**,t)*

*vector potential field **A=A(r**,t)*

$$\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B}=\nabla\times\mathbf{A}$$

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e\left[-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}+\mathbf{v}\times\left(\nabla\times\mathbf{A}\right)\right]$$



F=*q*v×B=I×B

Doing a double-cross

ε_{ijk}-Tensor analysis of **v**×(**∇**×**A**)

$$\varepsilon_{ijk}=\begin{cases}+1 \text{ for } \textit{even} \text{ permutaion of } i<j<k\\ 0 \quad \text{if any of } i,j,k \text{ are equal}\\ -1 \text{ for } \textit{odd} \text{ permutaion of } i<j<k\end{cases}$$

$$\varepsilon_{ijk}=\quad\varepsilon_{ikj}=\quad\varepsilon_{kij}=1$$

$$=-\varepsilon_{jik}=-\varepsilon_{jki}=-\varepsilon_{kji}$$

$$\begin{aligned}\left[\mathbf{v}\times(\nabla\times\mathbf{A})\right]_k &= \varepsilon_{kij}v_i\left(\nabla\times\mathbf{A}\right)_j\\ &=\varepsilon_{kij}v_i\left(\varepsilon_{abj}\left(\partial_a A_b\right)\right)\\ &=\quad\varepsilon_{kij}\varepsilon_{abj}\quad v_i\left(\partial_a A_b\right)\\ &=(\delta_{ka}\delta_{ib}-\delta_{kb}\delta_{ia})v_i\left(\partial_a A_b\right)\\ &=\delta_{ka}\delta_{ib}v_i\left(\partial_a A_b\right)-\delta_{kb}\delta_{ia}v_i\left(\partial_a A_b\right)\\ &=\quad v_b\left(\partial_k A_b\right)\quad -v_a\left(\partial_a A_k\right)\\ &=\quad\left(\partial_k A_b\right)v_b\quad -v_a\left(\partial_a A_k\right)=\left(\nabla\mathbf{A}\right)\bullet\mathbf{v}-\mathbf{v}\bullet\nabla\mathbf{A}\\ &=\partial_k\left(A_b v_b\right)-\left(\partial_k v_b\right)A_b-v_a\left(\partial_a A_k\right)=\nabla(\mathbf{A}\bullet\mathbf{v})-(\nabla\mathbf{v})\bullet\mathbf{A}-\mathbf{v}\bullet\nabla\mathbf{A}\end{aligned}$$

Applying Levi-Civita ε-identity:

$$\varepsilon_{kij}\varepsilon_{abj}\ =\delta_{ka}\delta_{ib}-\delta_{kb}\delta_{ia}$$

Converting back to Gibbs's **bold** notation involves *tensors* like ∇A and ∇**v**.

Newtonian mechanics has *no explicit dependence* of position **r** and velocity **v**.

r-partial derivative of **v** (or vice-versa) is identically zero.

$$\partial_k v^j\equiv 0\quad \textit{iff}:\quad \nabla\mathbf{v}=\frac{\partial\mathbf{v}}{\partial\mathbf{r}}=\mathbf{0}$$

Vector analysis for particle-in-(A,Φ)-potential

So-called *pondermotive* form for Newton's $F=ma$ equation for a mass m of charge e.

electronic charge:
e =-1.602176·10^{-19}Coulombs

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e(\mathbf{E}+\mathbf{v}\times\mathbf{B})$$

Electric field **E** and magnetic field **B**

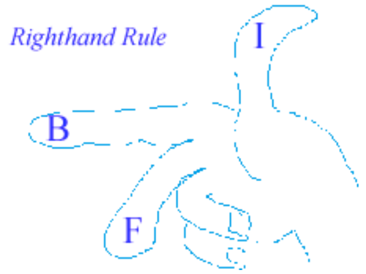
scalar potential field $\boldsymbol{\Phi}=\boldsymbol{\Phi}(\mathbf{r},t)$

vector potential field $\mathbf{A}=\mathbf{A}(\mathbf{r},t)$

$$\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B}=\nabla\times\mathbf{A}$$

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e\left[-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}+\mathbf{v}\times\left(\nabla\times\mathbf{A}\right)\right]$$



$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

Doing a double-cross

ε_{ijk}-Tensor analysis of $\mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{A})$

$$\left[\mathbf{v}\times(\nabla\times\mathbf{A})\right]_k=\varepsilon_{kij}v_i\left(\nabla\times\mathbf{A}\right)_j$$
$$=\varepsilon_{kij}v_i\left(\varepsilon_{abj}\left(\partial_a A_b\right)\right)$$
$$=\quad\varepsilon_{kij}\varepsilon_{abj}\quad v_i\left(\partial_a A_b\right)$$
$$=(\delta_{ka}\delta_{ib}-\delta_{kb}\delta_{ia})v_i\left(\partial_a A_b\right)$$
$$=\delta_{ka}\delta_{ib}v_i\left(\partial_a A_b\right)-\delta_{kb}\delta_{ia}v_i\left(\partial_a A_b\right)$$
$$=\quad v_b\left(\partial_k A_b\right)\quad -v_a\left(\partial_a A_k\right)$$
$$=\quad\left(\partial_k A_b\right)v_b\quad -v_a\left(\partial_a A_k\right)=\left(\nabla\mathbf{A}\right)\bullet\mathbf{v}-\mathbf{v}\bullet\nabla\mathbf{A}$$
$$=\partial_k\left(A_b v_b\right)-\left(\partial_k v_b\right)A_b-v_a\left(\partial_a A_k\right)=\nabla(\mathbf{A}\bullet\mathbf{v})-(\nabla\mathbf{v})\bullet\mathbf{A}-\mathbf{v}\bullet\nabla\mathbf{A}$$

$$\varepsilon_{ijk}=\begin{cases}+1 \text{ for } \textit{even} \text{ permutaion of } \ i<j<k\\ 0 \quad \text{if any of } \ i,j,k \text{ are equal}\\ -1 \text{ for } \textit{odd} \text{ permutaion of } \ i<j<k\end{cases}$$

$$\varepsilon_{ijk}=\quad\varepsilon_{ikj}=\quad\varepsilon_{kij}=1$$
$$=-\varepsilon_{jik}=-\varepsilon_{jki}=-\varepsilon_{kji}$$

Applying Levi-Civita ε-identity:

$$\varepsilon_{kij}\varepsilon_{abj}\ =\delta_{ka}\delta_{ib}-\delta_{kb}\delta_{ia}$$

Converting back to Gibbs's **bold** notation involves *tensors* like ∇**A** and ∇**v**.

Newtonian mechanics has *no explicit dependence* of position **r** and velocity **v**.

r-partial derivative of **v** (or vice-versa) is identically zero.

$$\partial_k v^j\equiv 0\ \ \textit{iff}:\ \ \nabla\mathbf{v}=\frac{\partial\mathbf{v}}{\partial\mathbf{r}}=\mathbf{0}$$

$$\mathbf{v}\times\left(\nabla\times\mathbf{A}\right)=\nabla(\mathbf{A}\bullet\mathbf{v})-\quad 0\quad-\mathbf{v}\bullet\nabla\mathbf{A}$$ *for particle mechanics*

Summary of Vector analysis for particle-in-($\boldsymbol{A}$,Φ)-potential

Tensor index notation helps to distinguish $(\nabla\mathbf{A})\bullet\mathbf{v}$, $\mathbf{v}\bullet(\nabla\mathbf{A})$, and $\nabla(\mathbf{A}\bullet\mathbf{v})=(\nabla\mathbf{A})\bullet\mathbf{v}+(\nabla\mathbf{v})\bullet\mathbf{A}$.

$$\left[(\nabla\mathbf{A})\bullet\mathbf{v}\right]_k=\frac{\partial A_j}{\partial x_k}v_j$$

$$=(\partial_k A_j)v_j$$

$$\left[\mathbf{v}\bullet(\nabla\mathbf{A})\right]_k=v_j\frac{\partial A_k}{\partial x_j}$$

$$=(v_j\partial_j A_k)$$

$$\left[\nabla(\mathbf{A}\bullet\mathbf{v})\right]_k=\left[(\nabla\mathbf{A})\bullet\mathbf{v}+(\nabla\mathbf{v})\bullet\mathbf{A}\right]_k$$

$$\partial_k\left(A_b v_b\right)=\left(\partial_k v_b\right)A_b-\left(\partial_k v_a\right)A_a$$

$$\mathbf{v}\times\left(\nabla\times\mathbf{A}\right)=\nabla(\mathbf{A}\bullet\mathbf{v})-\quad 0\quad-\mathbf{v}\bullet\nabla\mathbf{A}$$ *for particle mechanics*

Charge mechanics in electromagnetic fields

*Vector analysis for particle-in-(**A**,Φ)-potential*

→ *Lagrangian for particle-in-(**A**,Φ)-potential*

*Hamiltonian for particle-in-(**A**,Φ)-potential*

- *Canonical momentum in (**A**,Φ) potential*
- *Hamiltonian formulation*
- *Hamilton's equations*

Lagrangian for particle-in-(A,Φ)-potential

So-called *pondermotive* form for Newton's *F=ma* equation for a mass *m* of charge *e*.

electronic charge:
e =-1.602176·10^{-19}Coulombs

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e(\mathbf{E}+\mathbf{v}\times\mathbf{B})$$

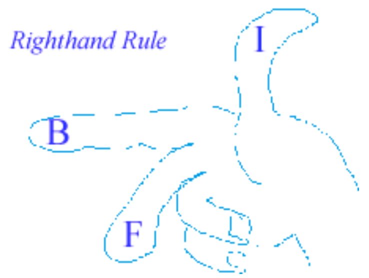
Electric field **E** and magnetic field **B**

scalar potential field $\mathbf{\Phi}=\mathbf{\Phi}(\mathbf{r},t)$

vector potential field $\mathbf{A}=\mathbf{A}(\mathbf{r},t)$

$$\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B}=\nabla\times\mathbf{A}$$



$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e\left[-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}+\mathbf{v}\times\left(\nabla\times\mathbf{A}\right)\right]$$

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e\left[-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}+\nabla(\mathbf{v}\bullet\mathbf{A})-(\mathbf{v}\bullet\nabla)\mathbf{A}\right]$$

$$\mathbf{v}\times\left(\nabla\times\mathbf{A}\right)=\nabla(\mathbf{A}\bullet\mathbf{v})-\quad 0 \quad-\mathbf{v}\bullet\nabla\mathbf{A}$$ *for particle mechanics*

*Lagrangian for particle-in-(**A**,Φ)-potential*

So-called ***pondermotive*** form for Newton's *F=ma* equation for a mass *m* of charge *e*.

electronic charge:
e =-1.602176·10^{-19}*Coulombs*

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e(\mathbf{E}+\mathbf{v}\times\mathbf{B})$$

Electric field **E** and magnetic field **B**

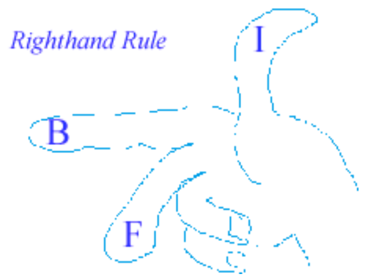
scalar potential field **Φ=Φ(r**,*t)*

vector potential field **A=A(r**,*t)*

$$\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B}=\nabla\times\mathbf{A}$$

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e\left[-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}+\mathbf{v}\times\left(\nabla\times\mathbf{A}\right)\right]$$



F=*q*v×B=I×B

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e\left[-\nabla\Phi-\boxed{\frac{\partial\mathbf{A}}{\partial t}}+\nabla(\mathbf{v}\bullet\mathbf{A})-\boxed{(\mathbf{v}\bullet\nabla)\mathbf{A}}\right]$$

Chain rule expansion of vector potential total *t*-derivative: $\frac{d\mathbf{A}}{dt}=\frac{\partial\mathbf{A}}{\partial x}\dot{x}+\frac{\partial\mathbf{A}}{\partial y}\dot{y}+\frac{\partial\mathbf{A}}{\partial z}\dot{z}+\frac{\partial\mathbf{A}}{\partial t}=\boxed{\frac{\partial\mathbf{A}}{\partial t}}+\boxed{(\mathbf{v}\bullet\nabla)\mathbf{A}}$

*Lagrangian for particle-in-(**A**,Φ)-potential*

So-called ***pondermotive*** form for Newton's *F=ma* equation for a mass *m* of charge *e*.

electronic charge:
e =-1.602176·10^{-19}Coulombs

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e(\mathbf{E}+\mathbf{v}\times\mathbf{B})$$

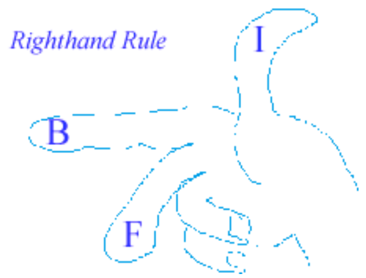
Electric field **E** and magnetic field **B**

scalar potential field **Φ=Φ(r**,*t)*

vector potential field **A=A(r**,*t)*

$$\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B}=\nabla\times\mathbf{A}$$



F=*q*v×B=I×B

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e\left[-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}+\mathbf{v}\times\left(\nabla\times\mathbf{A}\right)\right]$$

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e\left[-\nabla\Phi-\boxed{\frac{\partial\mathbf{A}}{\partial t}}+\nabla(\mathbf{v}\bullet\mathbf{A})-\boxed{(\mathbf{v}\bullet\nabla)\mathbf{A}}\right]$$

Chain rule expansion of vector potential total *t*-derivative:
$$\frac{d\mathbf{A}}{dt}=\frac{\partial\mathbf{A}}{\partial x}\dot{x}+\frac{\partial\mathbf{A}}{\partial y}\dot{y}+\frac{\partial\mathbf{A}}{\partial z}\dot{z}+\frac{\partial\mathbf{A}}{\partial t}=\boxed{\frac{\partial\mathbf{A}}{\partial t}}+(\mathbf{v}\bullet\nabla)\mathbf{A}$$

$$m\frac{d\mathbf{v}}{dt}=e\left[-\nabla\Phi+\nabla(\mathbf{v}\bullet\mathbf{A})-\frac{\partial\mathbf{A}}{\partial t}-(\mathbf{v}\bullet\nabla)\mathbf{A}\right]$$

Lagrangian for particle-in-($\mathbf{A}$,Φ)-potential

So-called ***pondermotive*** form for Newton's $F=ma$ equation for a mass m of charge e.

electronic charge:
$e=-1.602176\cdot 10^{-19}$Coulombs

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e(\mathbf{E}+\mathbf{v}\times\mathbf{B})$$

Electric field **E** and magnetic field **B**

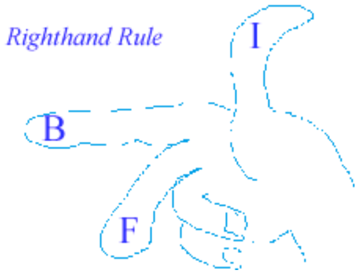
scalar potential field $\mathbf{\Phi}=\mathbf{\Phi}(\mathbf{r},t)$

vector potential field $\mathbf{A}=\mathbf{A}(\mathbf{r},t)$

$$\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B}=\nabla\times\mathbf{A}$$

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e\left[-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}+\mathbf{v}\times\left(\nabla\times\mathbf{A}\right)\right]$$



$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e\left[-\nabla\Phi-\boxed{\frac{\partial\mathbf{A}}{\partial t}}+\nabla(\mathbf{v}\bullet\mathbf{A})-\boxed{(\mathbf{v}\bullet\nabla)\mathbf{A}}\right]$$

Chain rule expansion of vector potential total t-derivative: $\frac{d\mathbf{A}}{dt}=\frac{\partial\mathbf{A}}{\partial x}\dot{x}+\frac{\partial\mathbf{A}}{\partial y}\dot{y}+\frac{\partial\mathbf{A}}{\partial z}\dot{z}+\frac{\partial\mathbf{A}}{\partial t}=\boxed{\frac{\partial\mathbf{A}}{\partial t}+(\mathbf{v}\bullet\nabla)\mathbf{A}}$

$$m\frac{d\mathbf{v}}{dt}=e\left[-\nabla\Phi+\nabla(\mathbf{v}\bullet\mathbf{A})-\frac{\partial\mathbf{A}}{\partial t}-(\mathbf{v}\bullet\nabla)\mathbf{A}\right]=e\left[-\nabla(\Phi-\mathbf{v}\bullet\mathbf{A})-\frac{d\mathbf{A}}{dt}\right]$$

Lagrangian for particle-in-(A,Φ)-potential

So-called ***pondermotive*** form for Newton's *F=ma* equation for a mass *m* of charge *e*.

electronic charge:
e =-1.602176·10^{-19}Coulombs

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e(\mathbf{E}+\mathbf{v}\times\mathbf{B})$$

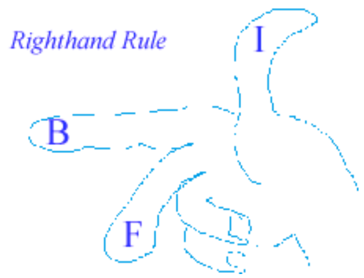
Electric field **E** and magnetic field **B**

scalar potential field **Φ=Φ(r**,*t*)

vector potential field **A=A(r**,*t*)

$$\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B}=\nabla\times\mathbf{A}$$



F=*q*v×B=I×B

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e\left[-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}+\mathbf{v}\times\left(\nabla\times\mathbf{A}\right)\right]$$

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e\left[-\nabla\Phi-\boxed{\frac{\partial\mathbf{A}}{\partial t}}+\nabla(\mathbf{v}\bullet\mathbf{A})-\boxed{(\mathbf{v}\bullet\nabla)\mathbf{A}}\right]$$

Chain rule expansion of vector potential total *t*-derivative:

$$\frac{d\mathbf{A}}{dt}=\frac{\partial\mathbf{A}}{\partial x}\dot{x}+\frac{\partial\mathbf{A}}{\partial y}\dot{y}+\frac{\partial\mathbf{A}}{\partial z}\dot{z}+\frac{\partial\mathbf{A}}{\partial t}=\boxed{\frac{\partial\mathbf{A}}{\partial t}+(\mathbf{v}\bullet\nabla)\mathbf{A}}$$

$$m\frac{d\mathbf{v}}{dt}=e\left[-\nabla\Phi+\nabla(\mathbf{v}\bullet\mathbf{A})-\frac{\partial\mathbf{A}}{\partial t}-(\mathbf{v}\bullet\nabla)\mathbf{A}\right]=e\left[-\nabla(\Phi-\mathbf{v}\bullet\mathbf{A})-\frac{d\mathbf{A}}{dt}\right]$$

$$m\frac{d\mathbf{v}}{dt}=\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}$$

$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}=\frac{d}{dt}\left(-e\mathbf{A}\right)\qquad-\nabla(e\Phi-\mathbf{v}\bullet e\mathbf{A})\qquad-e\frac{d\mathbf{A}}{dt}$$

*Lagrangian for particle-in-(**A**,Φ)-potential*

So-called ***pondermotive*** form for Newton's *F=ma* equation for a mass *m* of charge *e*.

electronic charge:
e =-1.602176·10^{-19}Coulombs

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F} = e(\mathbf{E}+\mathbf{v}\times\mathbf{B})$$

Electric field **E** and magnetic field **B**

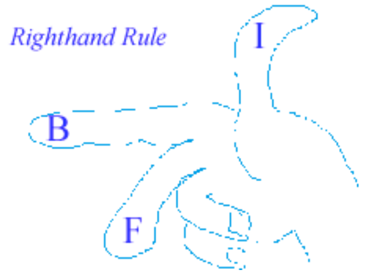
scalar potential field **Φ=Φ(r**,*t)*

vector potential field **A=A(r**,*t)*

$$\mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla\times\mathbf{A}$$

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F} = e\left[-\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t} + \mathbf{v}\times\left(\nabla\times\mathbf{A}\right)\right]$$



F=*q*v×B=I×B

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F} = e\left[-\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t} + \nabla(\mathbf{v}\bullet\mathbf{A}) - (\mathbf{v}\bullet\nabla)\mathbf{A}\right]$$

Chain rule expansion of vector potential total *t*-derivative:
$$\frac{d\mathbf{A}}{dt} = \frac{\partial\mathbf{A}}{\partial x}\dot{x} + \frac{\partial\mathbf{A}}{\partial y}\dot{y} + \frac{\partial\mathbf{A}}{\partial z}\dot{z} + \frac{\partial\mathbf{A}}{\partial t} = \frac{\partial\mathbf{A}}{\partial t} + (\mathbf{v}\bullet\nabla)\mathbf{A}$$

$$m\frac{d\mathbf{v}}{dt} = e\left[-\nabla\Phi + \nabla(\mathbf{v}\bullet\mathbf{A}) - \frac{\partial\mathbf{A}}{\partial t} - (\mathbf{v}\bullet\nabla)\mathbf{A}\right] = e\left[-\nabla(\Phi - \mathbf{v}\bullet\mathbf{A}) - \frac{d\mathbf{A}}{dt}\right]$$

$$m\frac{d\mathbf{v}}{dt} = \frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}$$

$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\frac{1}{2}m\mathbf{v}\bullet\mathbf{v} = \frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\left(e\Phi - \mathbf{v}\bullet e\mathbf{A}\right) - \nabla\left(e\Phi - \mathbf{v}\bullet e\mathbf{A}\right)$$

$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\left(e\Phi - \mathbf{v}\bullet e\mathbf{A}\right) = -e\frac{d\mathbf{A}}{dt}$$

Inserting Φ-term that $\partial_{\mathbf{v}}$ zeros :

$$\left(\textit{This step requires that}: \frac{\partial}{\partial\mathbf{v}}\left(e\Phi\right) = 0\right) \quad \left(\textit{and}: \frac{\partial}{\partial\mathbf{v}}\left(\mathbf{v}\bullet e\mathbf{A}\right) = e\mathbf{A}\right)$$

Lagrangian for particle-in-(A,Φ)-potential

So-called *pondermotive* form for Newton's *F=ma* equation for a mass *m* of charge *e*.

electronic charge:
e =-1.602176·10^{-19}*Coulombs*

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e(\mathbf{E}+\mathbf{v}\times\mathbf{B})$$

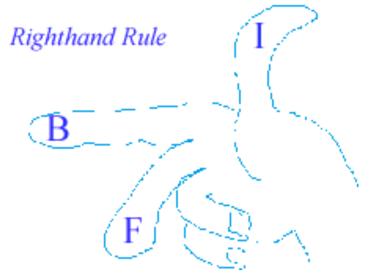
Electric field **E** and magnetic field **B**

scalar potential field **Φ=Φ(r**,*t)*

vector potential field **A=A(r**,*t)*

$$\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B}=\nabla\times\mathbf{A}$$



F=*q*v×B=I×B

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e\left[-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}+\mathbf{v}\times\left(\nabla\times\mathbf{A}\right)\right]$$

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e\left[-\nabla\Phi-\boxed{\frac{\partial\mathbf{A}}{\partial t}}+\nabla(\mathbf{v}\bullet\mathbf{A})-\boxed{(\mathbf{v}\bullet\nabla)\mathbf{A}}\right]$$

Chain rule expansion of vector potential total *t*-derivative:
$$\frac{d\mathbf{A}}{dt}=\frac{\partial\mathbf{A}}{\partial x}\dot{x}+\frac{\partial\mathbf{A}}{\partial y}\dot{y}+\frac{\partial\mathbf{A}}{\partial z}\dot{z}+\frac{\partial\mathbf{A}}{\partial t}=\frac{\partial\mathbf{A}}{\partial t}+(\mathbf{v}\bullet\nabla)\mathbf{A}$$

$$m\frac{d\mathbf{v}}{dt}=e\left[-\nabla\Phi+\nabla(\mathbf{v}\bullet\mathbf{A})-\frac{\partial\mathbf{A}}{\partial t}-(\mathbf{v}\bullet\nabla)\mathbf{A}\right]=e\left[-\nabla(\Phi-\mathbf{v}\bullet\mathbf{A})-\frac{d\mathbf{A}}{dt}\right]$$

$$m\frac{d\mathbf{v}}{dt}=\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}$$

$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}=\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\left(e\Phi-\mathbf{v}\bullet e\mathbf{A}\right)-\nabla\left(e\Phi-\mathbf{v}\bullet e\mathbf{A}\right)$$

$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\left(e\Phi-\mathbf{v}\bullet e\mathbf{A}\right)=-e\frac{d\mathbf{A}}{dt}$$

$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}-(e\Phi-\mathbf{v}\bullet e\mathbf{A})\right)=\frac{\partial}{\partial\mathbf{r}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}-(e\Phi-\mathbf{v}\bullet e\mathbf{A})\right)$$

This step requires that :
$$\nabla\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}\right)\equiv\frac{\partial}{\partial\mathbf{r}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}\right)=0$$

Inserting **v•v**-term that $\partial_{\mathbf{r}}$ zeros :

Lagrangian for particle-in-(A,Φ)-potential

So-called *pondermotive* form for Newton's $F=ma$ equation for a mass m of charge e.

electronic charge:
$e = -1.602176 \cdot 10^{-19}$ *Coulombs*

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F} = e(\mathbf{E} + \mathbf{v}\times\mathbf{B})$$

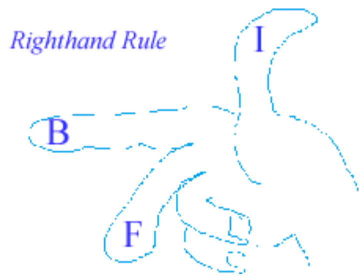
Electric field **E** and magnetic field **B**

scalar potential field $\boldsymbol{\Phi}=\boldsymbol{\Phi}(\mathbf{r},t)$

vector potential field $\mathbf{A}=\mathbf{A}(\mathbf{r},t)$

$$\mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla\times\mathbf{A}$$



$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F} = e\left[-\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t} + \mathbf{v}\times\left(\nabla\times\mathbf{A}\right)\right]$$

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F} = e\left[-\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t} + \nabla(\mathbf{v}\bullet\mathbf{A}) - (\mathbf{v}\bullet\nabla)\mathbf{A}\right]$$

Chain rule expansion of vector potential total *t*-derivative:

$$\frac{d\mathbf{A}}{dt} = \frac{\partial\mathbf{A}}{\partial x}\dot{x} + \frac{\partial\mathbf{A}}{\partial y}\dot{y} + \frac{\partial\mathbf{A}}{\partial z}\dot{z} + \frac{\partial\mathbf{A}}{\partial t} = \frac{\partial\mathbf{A}}{\partial t} + (\mathbf{v}\bullet\nabla)\mathbf{A}$$

$$m\frac{d\mathbf{v}}{dt} = e\left[-\nabla\Phi + \nabla(\mathbf{v}\bullet\mathbf{A}) - \frac{\partial\mathbf{A}}{\partial t} - (\mathbf{v}\bullet\nabla)\mathbf{A}\right] = e\left[-\nabla(\Phi - \mathbf{v}\bullet\mathbf{A}) - \frac{d\mathbf{A}}{dt}\right]$$

$$m\frac{d\mathbf{v}}{dt} = \frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}$$

$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\frac{1}{2}m\mathbf{v}\bullet\mathbf{v} = \frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\left(e\Phi - \mathbf{v}\bullet e\mathbf{A}\right) - \nabla\left(e\Phi - \mathbf{v}\bullet e\mathbf{A}\right)$$

$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\left(e\Phi - \mathbf{v}\bullet e\mathbf{A}\right) = -e\frac{d\mathbf{A}}{dt}$$

$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\left(\underbrace{\frac{1}{2}m\mathbf{v}\bullet\mathbf{v} - (e\Phi - \mathbf{v}\bullet e\mathbf{A})}\right) = \frac{\partial}{\partial\mathbf{r}}\left(\underbrace{\frac{1}{2}m\mathbf{v}\bullet\mathbf{v} - (e\Phi - \mathbf{v}\bullet e\mathbf{A})}\right)$$

$$\frac{\partial}{\partial\mathbf{r}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}\right) = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial\mathbf{v}} = \frac{\partial L}{\partial\mathbf{r}}$$

Lagrangian for particle-in-(A,Φ)-potential

So-called *pondermotive* form for Newton's *F=ma* equation for a mass *m* of charge *e*.

electronic charge:
e =-1.602176·10⁻¹⁹Coulombs

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e(\mathbf{E}+\mathbf{v}\times\mathbf{B})$$

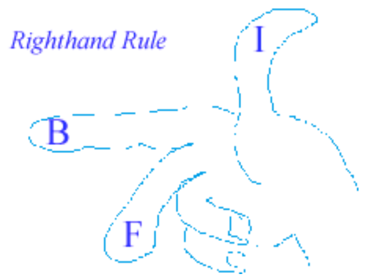
Electric field **E** and magnetic field **B**

scalar potential field **Φ=Φ(r**,*t)*

vector potential field **A=A(r**,*t)*

$$\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B}=\nabla\times\mathbf{A}$$



F=*q*v×B=I×B

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e\left[-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}+\mathbf{v}\times\left(\nabla\times\mathbf{A}\right)\right]$$

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e\left[-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}+\nabla(\mathbf{v}\bullet\mathbf{A})-(\mathbf{v}\bullet\nabla)\mathbf{A}\right]$$

Chain rule expansion of vector potential total *t*-derivative:
$$\frac{d\mathbf{A}}{dt}=\frac{\partial\mathbf{A}}{\partial x}\dot{x}+\frac{\partial\mathbf{A}}{\partial y}\dot{y}+\frac{\partial\mathbf{A}}{\partial z}\dot{z}+\frac{\partial\mathbf{A}}{\partial t}=\frac{\partial\mathbf{A}}{\partial t}+(\mathbf{v}\bullet\nabla)\mathbf{A}$$

$$m\frac{d\mathbf{v}}{dt}=e\left[-\nabla\Phi+\nabla(\mathbf{v}\bullet\mathbf{A})-\frac{\partial\mathbf{A}}{\partial t}-(\mathbf{v}\bullet\nabla)\mathbf{A}\right]=e\left[-\nabla(\Phi-\mathbf{v}\bullet\mathbf{A})-\frac{d\mathbf{A}}{dt}\right]$$

$$m\frac{d\mathbf{v}}{dt}=\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}$$

$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}=\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\left(e\Phi-\mathbf{v}\bullet e\mathbf{A}\right)-\nabla\left(e\Phi-\mathbf{v}\bullet e\mathbf{A}\right)$$

$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\left(e\Phi-\mathbf{v}\bullet e\mathbf{A}\right)=-e\frac{d\mathbf{A}}{dt}$$

$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\left(\underbrace{\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}-(e\Phi-\mathbf{v}\bullet e\mathbf{A})}\right)=\frac{\partial}{\partial\mathbf{r}}\left(\underbrace{\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}-(e\Phi-\mathbf{v}\bullet e\mathbf{A})}\right)$$

$$\frac{\partial}{\partial\mathbf{r}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}\right)=0$$

$$\frac{d}{dt}\frac{\partial L}{\partial\mathbf{v}}=\frac{\partial L}{\partial\mathbf{r}}$$

Lagrangian has a *linear* velocity term *e***v•A** in addition to the usual quadratic *KE=mv²/2* and *PE=e*Φ.

$$L=L(\mathbf{r},\mathbf{v},t)=\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}-\left(e\Phi(\mathbf{r},t)-\mathbf{v}\bullet e\mathbf{A}(\mathbf{r},t)\right)$$

Charge mechanics in electromagnetic fields

*Vector analysis for particle-in-(**A**,Φ)-potential*

*Lagrangian for particle-in-(**A**,Φ)-potential*

*Hamiltonian for particle-in-(**A**,Φ)-potential*

→ *Canonical momentum in (**A**,Φ) potential*

Hamiltonian formulation

Hamilton's equations

*Hamiltonian for particle-in-(**A**,Φ)-potential*

Lagrangian has a *linear* velocity term $e\mathbf{v}\bullet\mathbf{A}$ in addition to the usual quadratic $KE=mv^2/2$ and $PE=e\Phi$.

$$L = L(\mathbf{r},\mathbf{v},t) = \frac{1}{2}m\mathbf{v}\bullet\mathbf{v} - \left(e\Phi(\mathbf{r},t) - \mathbf{v}\bullet e\mathbf{A}(\mathbf{r},t)\right)$$

*Canonical momentum in (**A**,Φ) potential*

Canonical momentum is defined by *L*'s **v**-derivative

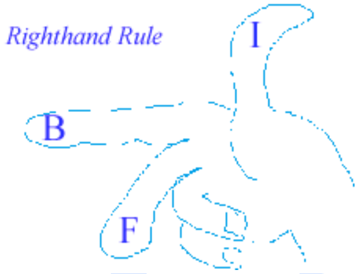
$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v} - \left(e\Phi(\mathbf{r},t) - \mathbf{v}\bullet e\mathbf{A}(\mathbf{r},t)\right)\right)$$

$$\mathbf{p} = m\mathbf{v} + e\mathbf{A}(\mathbf{r},t)$$

Hamiltonian for particle-in-($\boldsymbol{A}$,Φ)-potential

Lagrangian has a *linear* velocity term $e\mathbf{v}\bullet\mathbf{A}$ in addition to the usual quadratic $KE=mv^2/2$ and $PE=e\Phi$.

$$L = L(\mathbf{r},\mathbf{v},t) = \frac{1}{2}m\mathbf{v}\bullet\mathbf{v} - \big(e\Phi(\mathbf{r},t) - \mathbf{v}\bullet e\mathbf{A}(\mathbf{r},t)\big)$$



$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

Canonical momentum in ($\boldsymbol{A}$,Φ) potential

Canonical momentum is defined by L's $\mathbf{v}$-derivative

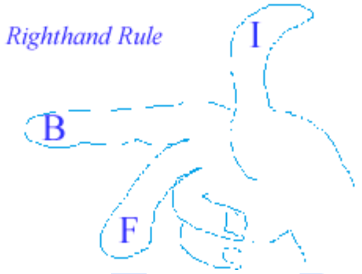
$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v} - \big(e\Phi(\mathbf{r},t) - \mathbf{v}\bullet e\mathbf{A}(\mathbf{r},t)\big)\right) = \frac{\partial}{\partial \mathbf{v}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v} - e\Phi(\mathbf{r},t)\right)_{For\ \mathbf{A}(\mathbf{r},t)=0}$$

$$\mathbf{p} = m\mathbf{v} + e\mathbf{A}(\mathbf{r},t) \qquad\qquad = \quad m\mathbf{v} \qquad For\ \mathbf{A}(\mathbf{r},t)=0$$

Hamiltonian for particle-in-(A,Φ)-potential

Lagrangian has a *linear* velocity term $e\mathbf{v}\bullet\mathbf{A}$ in addition to the usual quadratic $KE=mv^2/2$ and $PE=e\Phi$.

$$L = L(\mathbf{r},\mathbf{v},t) = \frac{1}{2}m\mathbf{v}\bullet\mathbf{v} - \left(e\Phi(\mathbf{r},t) - \mathbf{v}\bullet e\mathbf{A}(\mathbf{r},t)\right)$$



$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

Canonical momentum in (A,Φ) potential

Canonical momentum is defined by L's $\mathbf{v}$-derivative

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v} - \left(e\Phi(\mathbf{r},t) - \mathbf{v}\bullet e\mathbf{A}(\mathbf{r},t)\right)\right) = \frac{\partial}{\partial \mathbf{v}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v} - e\Phi(\mathbf{r},t)\right)_{For\ \mathbf{A}(\mathbf{r},t)=0}$$

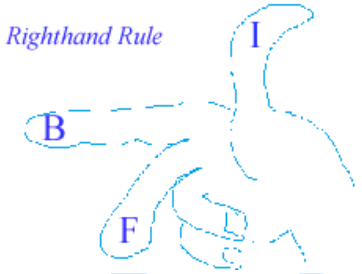
$$\mathbf{p} = m\mathbf{v} + e\mathbf{A}(\mathbf{r},t) \qquad = \quad m\mathbf{v} \qquad For\ \mathbf{A}(\mathbf{r},t)=0$$

Lagrangian is usual form $L= T - V$ with electric (scalar) potential $V=\boldsymbol{\Phi}(\mathbf{r},t)$
<u>if</u> magnetic (vector) potential $\mathbf{A}=\mathbf{A}(\mathbf{r},t)$ is zero everywhere.

Hamiltonian for particle-in-(A,Φ)-potential

Lagrangian has a *linear* velocity term $e\mathbf{v}\bullet\mathbf{A}$ in addition to the usual quadratic $KE=mv^2/2$ and $PE=e\Phi$.

$$L = L(\mathbf{r},\mathbf{v},t) = \frac{1}{2}m\mathbf{v}\bullet\mathbf{v} - \left(e\Phi(\mathbf{r},t) - \mathbf{v}\bullet e\mathbf{A}(\mathbf{r},t)\right)$$



$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

Canonical momentum in (A,Φ) potential

Canonical momentum is defined by L's $\mathbf{v}$-derivative

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v} - \left(e\Phi(\mathbf{r},t) - \mathbf{v}\bullet e\mathbf{A}(\mathbf{r},t)\right)\right) = \frac{\partial}{\partial \mathbf{v}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v} - e\Phi(\mathbf{r},t)\right)_{For\ \mathbf{A}(\mathbf{r},t)=0}$$

$$\mathbf{p} = m\mathbf{v} + e\mathbf{A}(\mathbf{r},t) \qquad = \quad m\mathbf{v} \qquad For\ \mathbf{A}(\mathbf{r},t)=0$$

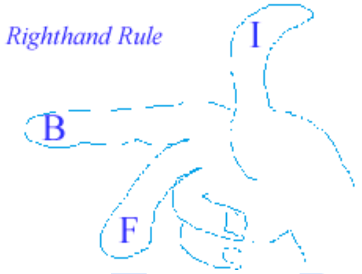
Lagrangian is usual form $L= T - V$ with electric (scalar) potential $V=\mathbf{\Phi}(\mathbf{r},t)$
<u>if</u> magnetic (vector) potential $\mathbf{A}=\mathbf{A}(\mathbf{r},t)$ is zero everywhere.

Then canonical momentum is usual form: $\mathbf{p} = m\mathbf{v}$ $\quad (For\ \mathbf{A}(\mathbf{r},t)=0)$

*Hamiltonian for particle-in-(**A**,Φ)-potential*

Lagrangian has a *linear* velocity term *e***v•A** in addition to the usual quadratic *KE=mv²/2* and *PE=e*Φ.

$$L = L(\mathbf{r},\mathbf{v},t) = \frac{1}{2} m\mathbf{v} \bullet \mathbf{v} - \left(e\Phi(\mathbf{r},t) - \mathbf{v} \bullet e\mathbf{A}(\mathbf{r},t) \right)$$



F=*q***v**×**B**=**I**×**B**

*Canonical momentum in (**A**,Φ) potential*

Canonical momentum is defined by *L*'s **v**-derivative

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \left(\frac{1}{2} m\mathbf{v} \bullet \mathbf{v} - \left(e\Phi(\mathbf{r},t) - \mathbf{v} \bullet e\mathbf{A}(\mathbf{r},t) \right) \right) = \frac{\partial}{\partial \mathbf{v}} \left(\frac{1}{2} m\mathbf{v} \bullet \mathbf{v} - e\Phi(\mathbf{r},t) \right)_{For\ \mathbf{A}(\mathbf{r},t)=0}$$

$$\mathbf{p} = m\mathbf{v} + e\mathbf{A}(\mathbf{r},t) \qquad = \quad m\mathbf{v} \qquad For\ \mathbf{A}(\mathbf{r},t)=0$$

Lagrangian is usual form *L= T - V* with electric (scalar) potential *V*=**Φ**(**r**,*t)*
<u>if</u> magnetic (vector) potential **A**=**A**(**r**,*t)* is zero everywhere.

Then canonical momentum is usual form: $\mathbf{p} = m\mathbf{v}$ $(For\ \mathbf{A}(\mathbf{r},t)=0)$

Otherwise vector potential term **-v**·*e***A** leads to an extraordinary *canonical momentum:* **p** =*m***v**+*e***A**(**r**,*t)*.
*Particle momentum m***v** is not canonical, but related to *canonical* **p** as follows: *m***v**=**p** - *e***A**(**r**,*t)*

Charge mechanics in electromagnetic fields

Vector analysis for particle-in-($\boldsymbol{A}$,Φ)-potential

Lagrangian for particle-in-($\boldsymbol{A}$,Φ)-potential

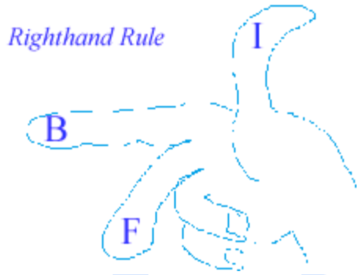
Hamiltonian for particle-in-($\boldsymbol{A}$,Φ)-potential

- *Canonical momentum in ($\boldsymbol{A}$,Φ) potential*
- → *Hamiltonian formulation*
- *Hamilton's equations*

Hamiltonian for charged particle in fields

The Hamiltonian function of the Legendre-Poincare form is the following.

$$H=\sum_{\mu}\dot{q}^{\mu}p_{\mu}-L=\mathbf{v}\bullet\mathbf{p}-L=\mathbf{v}\bullet\left(m\mathbf{v}+e\mathbf{A}(\mathbf{r},t)\right)-\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}-\left(e\Phi(\mathbf{r},t)-\mathbf{v}\bullet e\mathbf{A}(\mathbf{r},t)\right)\right)$$



$$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$$

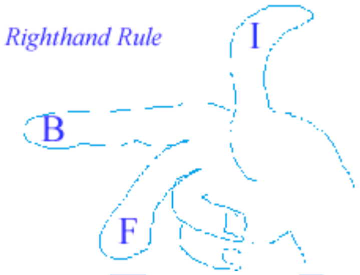
Hamiltonian for charged particle in fields

The Hamiltonian function of the Legendre-Poincare form is the following.

$$H=\sum_{\mu}\dot{q}^{\mu}p_{\mu}-L=\mathbf{v}\bullet\mathbf{p}-L=\mathbf{v}\bullet\left(m\mathbf{v}+\xcancel{e\mathbf{A}(\mathbf{r},t)}\right)-\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}-\left(e\Phi(\mathbf{r},t)-\mathbf{v}\bullet\xcancel{e\mathbf{A}(\mathbf{r},t)}\right)\right)$$

$$H=\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}+e\Phi(\mathbf{r},t)$$

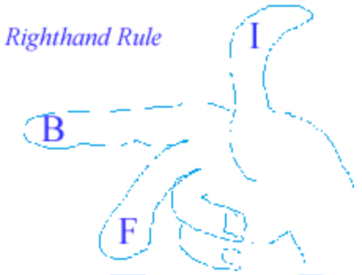
(Only correct numerically!)


Righthand Rule

$$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$$

Hamiltonian for charged particle in fields

The Hamiltonian function of the Legendre-Poincare form is the following.



F=q**v**×**B**=**I**×**B**

$$H=\sum_{\mu}\dot{q}^{\mu}p_{\mu}-L=\mathbf{v}\bullet\mathbf{p}-L=\mathbf{v}\bullet\left(m\mathbf{v}+\cancel{e\mathbf{A}(\mathbf{r},t)}\right)-\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}-\left(e\Phi(\mathbf{r},t)-\mathbf{v}\bullet\cancel{e\mathbf{A}(\mathbf{r},t)}\right)\right)$$

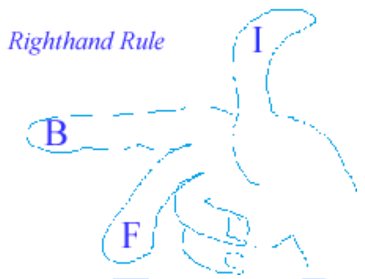
$$H=\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}+e\Phi(\mathbf{r},t)$$

(Only correct numerically!)

Vector potential **A** seems to cancel out completely, leaving a familiar $H=T+V$ with only scalar $V=e\Phi$.

Hamiltonian for charged particle in fields

The Hamiltonian function of the Legendre-Poincare form is the following.



F=q**v**×**B**=**I**×**B**

$$H=\sum_{\mu}\dot{q}^{\mu}p_{\mu}-L=\mathbf{v}\bullet\mathbf{p}-L=\mathbf{v}\bullet\left(m\mathbf{v}+\xcancel{e\mathbf{A}(\mathbf{r},t)}\right)-\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}-\left(e\Phi(\mathbf{r},t)-\mathbf{v}\bullet\xcancel{e\mathbf{A}(\mathbf{r},t)}\right)\right)$$

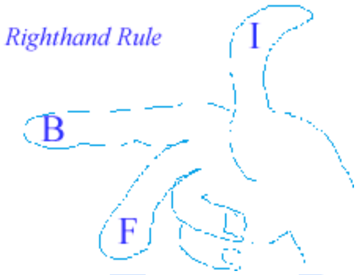
$$H=\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}+e\Phi(\mathbf{r},t)$$

(Only correct numerically!)

Vector potential **A** seems to cancel out completely, leaving a familiar $H=T+V$ with only scalar $V=e\Phi$.

But Hamiltonian is explicit function of *momentum* **p**. Must replace velocity **v** using $m\mathbf{v}=\mathbf{p}-e\mathbf{A}(\mathbf{r},t)$.

Hamiltonian for charged particle in fields



F=q**v**×**B**=**I**×**B**

The Hamiltonian function of the Legendre-Poincare form is the following.

$$H = \sum_{\mu} \dot{q}^{\mu} p_{\mu} - L = \mathbf{v} \bullet \mathbf{p} - L = \mathbf{v} \bullet \left(m\mathbf{v} + \cancel{e\mathbf{A}(\mathbf{r},t)} \right) - \left(\frac{1}{2} m\mathbf{v} \bullet \mathbf{v} - \left(e\Phi(\mathbf{r},t) - \mathbf{v} \bullet \cancel{e\mathbf{A}(\mathbf{r},t)} \right) \right)$$

$$H = \frac{1}{2} m\mathbf{v} \bullet \mathbf{v} + e\Phi(\mathbf{r},t)$$

(Only correct numerically!)

Vector potential **A** seems to cancel out completely, leaving a familiar $H=T+V$ with only scalar $V=e\Phi$.

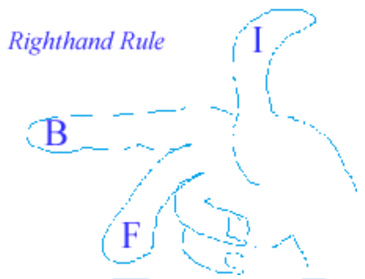
But Hamiltonian is explicit function of *momentum* **p**. Must replace velocity **v** using m**v**=**p** - e**A***(***r***,t)*.

$$H = \frac{1}{2m} \left(\mathbf{p} - e\mathbf{A}(\mathbf{r},t) \right) \bullet \left(\mathbf{p} - e\mathbf{A}(\mathbf{r},t) \right) + e\Phi(\mathbf{r},t)$$

(Correct formally and numerically)

Hamiltonian for charged particle in fields

The Hamiltonian function of the Legendre-Poincare form is the following.



F=q**v**×**B**=**I**×**B**

$$H=\sum_{\mu}\dot{q}^{\mu}p_{\mu}-L=\mathbf{v}\bullet\mathbf{p}-L=\mathbf{v}\bullet\left(m\mathbf{v}+\cancel{e\mathbf{A}(\mathbf{r},t)}\right)-\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}-\left(e\Phi(\mathbf{r},t)-\mathbf{v}\bullet\cancel{e\mathbf{A}(\mathbf{r},t)}\right)\right)$$

$$H=\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}+e\Phi(\mathbf{r},t)$$

(Only correct numerically!)

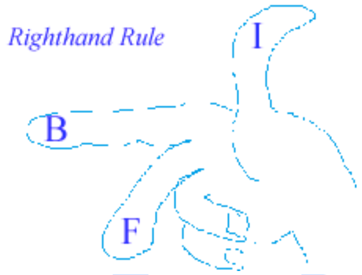
Vector potential **A** seems to cancel out completely, leaving a familiar $H=T+V$ with only scalar $V=e\Phi$.

But Hamiltonian is explicit function of *momentum* **p**. Must replace velocity **v** using $m\mathbf{v}=\mathbf{p}-e\mathbf{A}(\mathbf{r},t)$.

$$H=\frac{1}{2m}\left(\mathbf{p}-e\mathbf{A}(\mathbf{r},t)\right)\bullet\left(\mathbf{p}-e\mathbf{A}(\mathbf{r},t)\right)+e\Phi(\mathbf{r},t)$$

(Correct formally and numerically)

$$H=\frac{\mathbf{p}\bullet\mathbf{p}}{2m}-\frac{e}{2m}\left(\mathbf{p}\bullet\mathbf{A}+\mathbf{A}\bullet\mathbf{p}\right)+\frac{e^2}{2m}\mathbf{A}\bullet\mathbf{A}+e\Phi(\mathbf{r},t)$$



F=q**v**×**B**=**I**×**B**

Charge mechanics in electromagnetic fields

- *Vector analysis for particle-in-(**A**,Φ)-potential*
- *Lagrangian for particle-in-(**A**,Φ)-potential*
- *Hamiltonian for particle-in-(**A**,Φ)-potential*
  - *Canonical momentum in (**A**,Φ) potential*
  - *Hamiltonian formulation*
  - → *Hamilton's equations*

Hamilton's equations for charged particle in fields

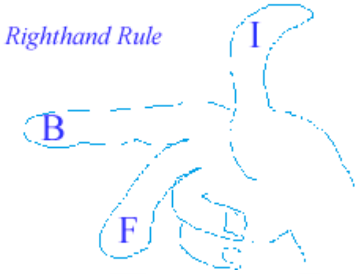
Hamiltonian is explicit function of *momentum* **p**. Must replace velocity **v** using $m\mathbf{v}=\mathbf{p} - e\mathbf{A}(\mathbf{r},t)$.

$$H=\frac{1}{2m}\left(\mathbf{p}-e\mathbf{A}(\mathbf{r},t)\right)\bullet\left(\mathbf{p}-e\mathbf{A}(\mathbf{r},t)\right)+e\Phi(\mathbf{r},t)$$

(Correct formally and numerically)

$$H=\frac{\mathbf{p}\bullet\mathbf{p}}{2m}-\frac{e}{2m}\left(\mathbf{p}\bullet\mathbf{A}+\mathbf{A}\bullet\mathbf{p}\right)+\frac{e^2}{2m}\mathbf{A}\bullet\mathbf{A}+e\Phi(\mathbf{r},t)$$

(Expanded)



$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

Hamiltonian for charged particle in fields

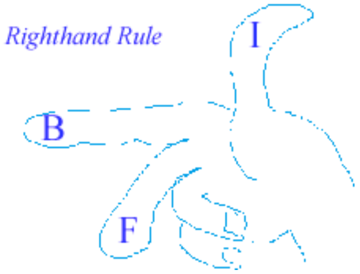
Hamiltonian is explicit function of *momentum* **p**. Must replace velocity **v** using $m\mathbf{v}=\mathbf{p} - e\mathbf{A}(\mathbf{r},t)$.

$$H=\frac{1}{2m}\left(\mathbf{p}-e\mathbf{A}(\mathbf{r},t)\right)\bullet\left(\mathbf{p}-e\mathbf{A}(\mathbf{r},t)\right)+e\Phi(\mathbf{r},t)$$

(Correct formally and numerically)

$$H=\frac{\mathbf{p}\bullet\mathbf{p}}{2m}-\frac{e}{2m}\left(\mathbf{p}\bullet\mathbf{A}+\mathbf{A}\bullet\mathbf{p}\right)+\frac{e^2}{2m}\mathbf{A}\bullet\mathbf{A}+e\Phi(\mathbf{r},t)$$

(Expanded)



$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

Hamilton's **v** equation: $\mathbf{v}=\dot{\mathbf{r}}=\frac{\partial H}{\partial \mathbf{p}}=\frac{\mathbf{p}-e\mathbf{A}(\mathbf{r},t)}{m}$

Hamilton's equations for charged particle in fields

Hamiltonian is explicit function of *momentum* **p**. Must replace velocity **v** using $m\mathbf{v}=\mathbf{p} - e\mathbf{A}(\mathbf{r},t)$.

$$H = \frac{1}{2m}\left(\mathbf{p} - e\mathbf{A}(\mathbf{r},t)\right)\bullet\left(\mathbf{p} - e\mathbf{A}(\mathbf{r},t)\right) + e\Phi(\mathbf{r},t)$$

(Correct formally and numerically)

$$H = \frac{\mathbf{p}\bullet\mathbf{p}}{2m} - \frac{e}{2m}\left(\mathbf{p}\bullet\mathbf{A} + \mathbf{A}\bullet\mathbf{p}\right) + \frac{e^2}{2m}\mathbf{A}\bullet\mathbf{A} + e\Phi(\mathbf{r},t)$$

(Expanded)

Hamilton's **v** equation: $\mathbf{v} = \dot{\mathbf{r}} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\mathbf{p} - e\mathbf{A}(\mathbf{r},t)}{m}$ *(Just copies particle velocity relation.)*

Hamilton's equations for charged particle in fields

Hamiltonian is explicit function of *momentum* **p**. Must replace velocity **v** using $m\mathbf{v}=\mathbf{p} - e\mathbf{A}(\mathbf{r},t)$.

$$H = \frac{1}{2m}\left(\mathbf{p} - e\mathbf{A}(\mathbf{r},t)\right)\bullet\left(\mathbf{p} - e\mathbf{A}(\mathbf{r},t)\right) + e\Phi(\mathbf{r},t)$$

(Correct formally and numerically)

$$H = \frac{\mathbf{p}\bullet\mathbf{p}}{2m} - \frac{e}{2m}\left(\mathbf{p}\bullet\mathbf{A} + \mathbf{A}\bullet\mathbf{p}\right) + \frac{e^2}{2m}\mathbf{A}\bullet\mathbf{A} + e\Phi(\mathbf{r},t)$$

(Expanded)

Hamilton's **v** equation:

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\mathbf{p} - e\mathbf{A}(\mathbf{r},t)}{m}$$

(Just copies particle velocity relation.)

Hamilton's *d***p***/dt* equation:

(In index notation.)

$$\dot{p}_a = -\frac{\partial H}{\partial x_a} = -\frac{\partial}{\partial x_a}\frac{\left(p_\mu - eA_\mu\right)\left(p_\mu - eA_\mu\right)}{2m} - e\frac{\partial \Phi}{\partial x_a}$$

Hamilton's equations for charged particle in fields

Hamiltonian is explicit function of *momentum* **p**. Must replace velocity **v** using $m\mathbf{v}=\mathbf{p} - e\mathbf{A}(\mathbf{r},t)$.

$$H=\frac{1}{2m}\left(\mathbf{p}-e\mathbf{A}(\mathbf{r},t)\right)\bullet\left(\mathbf{p}-e\mathbf{A}(\mathbf{r},t)\right)+e\Phi(\mathbf{r},t)$$

(Correct formally and numerically)

$$H=\frac{\mathbf{p}\bullet\mathbf{p}}{2m}-\frac{e}{2m}\left(\mathbf{p}\bullet\mathbf{A}+\mathbf{A}\bullet\mathbf{p}\right)+\frac{e^2}{2m}\mathbf{A}\bullet\mathbf{A}+e\Phi(\mathbf{r},t)$$

(Expanded)

Hamilton's **v** equation:

$$\mathbf{v}=\dot{\mathbf{r}}=\frac{\partial H}{\partial \mathbf{p}}=\frac{\mathbf{p}-e\mathbf{A}(\mathbf{r},t)}{m}$$

(Just copies particle velocity relation.)

Hamilton's $d\mathbf{p}/dt$ equation:

(In index notation.)

$$\dot{p}_a=-\frac{\partial H}{\partial x_a}=-\frac{\partial}{\partial x_a}\frac{\left(p_\mu-eA_\mu\right)\left(p_\mu-eA_\mu\right)}{2m}-e\frac{\partial\Phi}{\partial x_a}$$

$m\mathbf{v}+e\mathbf{A}(\mathbf{r},t)=\mathbf{p}$ ....

$$\dot{p}_a=m\dot{v}_a+e\dot{A}_a=+\frac{\left(p_\mu-eA_\mu\right)}{m}e\frac{\partial A_\mu}{\partial x_a}-e\frac{\partial\Phi}{\partial x_a}$$

Hamilton's equations for charged particle in fields

Hamiltonian is explicit function of *momentum* **p**. Must replace velocity **v** using $m\mathbf{v}=\mathbf{p}-e\mathbf{A}(\mathbf{r},t)$.

$$H=\frac{1}{2m}\left(\mathbf{p}-e\mathbf{A}(\mathbf{r},t)\right)\bullet\left(\mathbf{p}-e\mathbf{A}(\mathbf{r},t)\right)+e\Phi(\mathbf{r},t)$$

(Correct formally and numerically)

$$H=\frac{\mathbf{p}\bullet\mathbf{p}}{2m}-\frac{e}{2m}\left(\mathbf{p}\bullet\mathbf{A}+\mathbf{A}\bullet\mathbf{p}\right)+\frac{e^2}{2m}\mathbf{A}\bullet\mathbf{A}+e\Phi(\mathbf{r},t)$$

(Expanded)

Hamilton's **v** equation: $\mathbf{v}=\dot{\mathbf{r}}=\frac{\partial H}{\partial \mathbf{p}}=\frac{\mathbf{p}-e\mathbf{A}(\mathbf{r},t)}{m}$ *(Just copies particle velocity relation.)*

Hamilton's $d\mathbf{p}/dt$ equation:
(In index notation.)

$$\dot{p}_a=-\frac{\partial H}{\partial x_a}=-\frac{\partial}{\partial x_a}\frac{\left(p_\mu-eA_\mu\right)\left(p_\mu-eA_\mu\right)}{2m}-e\frac{\partial\Phi}{\partial x_a}$$

$$\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}$$

$$E_a=-\frac{\partial\Phi}{\partial x^a}-\frac{\partial A_a}{\partial t}$$

$m\mathbf{v}+e\mathbf{A}(\mathbf{r},t)=\mathbf{p}$

$$\dot{p}_a=m\dot{v}_a+e\dot{A}_a=+\underbrace{\frac{\left(p_\mu-eA_\mu\right)}{m}}e\frac{\partial A_\mu}{\partial x_a}\quad-\underbrace{e\frac{\partial\Phi}{\partial x_a}}$$

$$-\frac{\partial\Phi}{\partial x^a}=\frac{\partial A_a}{\partial t}+E_a$$

$$\dot{p}_a=m\dot{v}_a+e\dot{A}_a=e\left(v_\mu\frac{\partial A_\mu}{\partial x_a}\quad+\overbrace{\frac{\partial A_a}{\partial t}+E_a}\right)$$

Hamilton's equations for charged particle in fields

Hamiltonian is explicit function of *momentum* **p**. Must replace velocity **v** using $m\mathbf{v}=\mathbf{p} - e\mathbf{A}(\mathbf{r},t)$.

$$H = \frac{1}{2m}\left(\mathbf{p}-e\mathbf{A}(\mathbf{r},t)\right)\bullet\left(\mathbf{p}-e\mathbf{A}(\mathbf{r},t)\right)+e\Phi(\mathbf{r},t)$$

(Correct formally and numerically)

$$H = \frac{\mathbf{p}\bullet\mathbf{p}}{2m} - \frac{e}{2m}\left(\mathbf{p}\bullet\mathbf{A}+\mathbf{A}\bullet\mathbf{p}\right) + \frac{e^2}{2m}\mathbf{A}\bullet\mathbf{A}+e\Phi(\mathbf{r},t)$$

(Expanded)

Hamilton's **v** equation:

$$\mathbf{v}=\dot{\mathbf{r}}=\frac{\partial H}{\partial\mathbf{p}}=\frac{\mathbf{p}-e\mathbf{A}(\mathbf{r},t)}{m}$$

(Just copies particle velocity relation.)

Hamilton's *d***p***/dt* equation:
(In index notation.)

$$\dot p_a = -\frac{\partial H}{\partial x_a} = -\frac{\partial}{\partial x_a}\frac{\left(p_\mu-eA_\mu\right)\left(p_\mu-eA_\mu\right)}{2m} - e\frac{\partial\Phi}{\partial x_a}$$

$m\mathbf{v}+e\mathbf{A}(\mathbf{r},t)=\mathbf{p}$

$$\dot p_a = m\dot v_a + e\dot A_a = +\underbrace{\frac{\left(p_\mu-eA_\mu\right)}{m}} e\frac{\partial A_\mu}{\partial x_a} - e\frac{\partial\Phi}{\partial x_a}$$

$$\dot p_a = m\dot v_a + e\dot A_a = e\left(v_\mu \frac{\partial A_\mu}{\partial x_a} + \overbrace{\frac{\partial A_a}{\partial t}+E_a}\right)$$

$$\dot p_a = m\dot v_a + e\dot A_a = e\left(v_\mu \frac{\partial A_\mu}{\partial x_a} + \overbrace{\boxed{\dot A_a - v_\mu\frac{\partial A_a}{\partial x_\mu}}} + E_a\right)$$

$$\mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t}$$

$$E_a = -\frac{\partial\Phi}{\partial x^a} - \frac{\partial A_a}{\partial t}$$

$$-\frac{\partial\Phi}{\partial x^a} = \frac{\partial A_a}{\partial t} + E_a$$

$$\frac{\partial\mathbf{A}}{\partial t} = \frac{d\mathbf{A}}{dt} - (\mathbf{v}\bullet\nabla)\mathbf{A}$$

$$\frac{\partial A_a}{\partial t} = \boxed{\dot A_a - v_\mu\frac{\partial A_a}{\partial x_\mu}}$$

Hamilton's equations for charged particle in fields

Hamiltonian is explicit function of *momentum* **p**. Must replace velocity **v** using $m\mathbf{v}=\mathbf{p}-e\mathbf{A}(\mathbf{r},t)$.

$$H=\frac{1}{2m}\left(\mathbf{p}-e\mathbf{A}(\mathbf{r},t)\right)\bullet\left(\mathbf{p}-e\mathbf{A}(\mathbf{r},t)\right)+e\Phi(\mathbf{r},t)$$

(Correct formally and numerically)

$$H=\frac{\mathbf{p}\bullet\mathbf{p}}{2m}-\frac{e}{2m}\left(\mathbf{p}\bullet\mathbf{A}+\mathbf{A}\bullet\mathbf{p}\right)+\frac{e^2}{2m}\mathbf{A}\bullet\mathbf{A}+e\Phi(\mathbf{r},t)$$

(Expanded)

Hamilton's **v** equation: $\mathbf{v}=\dot{\mathbf{r}}=\frac{\partial H}{\partial\mathbf{p}}=\frac{\mathbf{p}-e\mathbf{A}(\mathbf{r},t)}{m}$ *(Just copies particle velocity relation.)*

Hamilton's *d***p***/dt* equation:
(In index notation.)

$$\dot{p}_a=-\frac{\partial H}{\partial x_a}=-\frac{\partial}{\partial x_a}\frac{\left(p_\mu-eA_\mu\right)\left(p_\mu-eA_\mu\right)}{2m}-e\frac{\partial\Phi}{\partial x_a}$$

$m\mathbf{v}+e\mathbf{A}(\mathbf{r},t)=\mathbf{p}$

$$\dot{p}_a=m\dot{v}_a+e\dot{A}_a=+\underbrace{\frac{\left(p_\mu-eA_\mu\right)}{m}}e\frac{\partial A_\mu}{\partial x_a}-e\frac{\partial\Phi}{\partial x_a}$$

$$\dot{p}_a=m\dot{v}_a+e\dot{A}_a=e\left(v_\mu\frac{\partial A_\mu}{\partial x_a}+\overbrace{\frac{\partial A_a}{\partial t}+E_a}\right)$$

$$\dot{p}_a=m\dot{v}_a+\cancel{e\dot{A}_a}=e\left(v_\mu\frac{\partial A_\mu}{\partial x_a}+\overbrace{\boxed{\cancel{\dot{A}_a}-v_\mu\frac{\partial A_a}{\partial x_\mu}}}+E_a\right)$$

$$\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}$$

$$E_a=-\frac{\partial\Phi}{\partial x^a}-\frac{\partial A_a}{\partial t}$$

$$-\frac{\partial\Phi}{\partial x^a}=\frac{\partial A_a}{\partial t}+E_a$$

$$\frac{\partial\mathbf{A}}{\partial t}=\frac{d\mathbf{A}}{dt}-(\mathbf{v}\bullet\nabla)\mathbf{A}$$

$$\frac{\partial A_a}{\partial t}=\boxed{\dot{A}_a-v_\mu\frac{\partial A_a}{\partial x_\mu}}$$

Hamilton's equations for charged particle in fields

Hamiltonian is explicit function of ***momentum*** **p**. Must replace velocity **v** using $m\mathbf{v}=\mathbf{p} - e\mathbf{A}(\mathbf{r},t)$.

$$H=\frac{1}{2m}\left(\mathbf{p}-e\mathbf{A}(\mathbf{r},t)\right)\bullet\left(\mathbf{p}-e\mathbf{A}(\mathbf{r},t)\right)+e\Phi(\mathbf{r},t)$$

(Correct formally and numerically)

$$H=\frac{\mathbf{p}\bullet\mathbf{p}}{2m}-\frac{e}{2m}\left(\mathbf{p}\bullet\mathbf{A}+\mathbf{A}\bullet\mathbf{p}\right)+\frac{e^2}{2m}\mathbf{A}\bullet\mathbf{A}+e\Phi(\mathbf{r},t)$$

(Expanded)

Hamilton's **v** equation: $\mathbf{v}=\dot{\mathbf{r}}=\frac{\partial H}{\partial \mathbf{p}}=\frac{\mathbf{p}-e\mathbf{A}(\mathbf{r},t)}{m}$ *(Just copies particle velocity relation.)*

Hamilton's $d\mathbf{p}/dt$ equation: *(In index notation.)*

$$\dot p_a=-\frac{\partial H}{\partial x_a}=-\frac{\partial}{\partial x_a}\frac{\left(p_\mu-eA_\mu\right)\left(p_\mu-eA_\mu\right)}{2m}-e\frac{\partial\Phi}{\partial x_a}$$

$m\mathbf{v}+e\mathbf{A}(\mathbf{r},t)=\mathbf{p}$

$$\dot p_a=m\dot v_a+e\dot A_a=+\underbrace{\frac{\left(p_\mu-eA_\mu\right)}{m}}_{v_\mu}e\frac{\partial A_\mu}{\partial x_a}-e\frac{\partial\Phi}{\partial x_a}$$

$$\dot p_a=m\dot v_a+e\dot A_a=e\left(v_\mu\frac{\partial A_\mu}{\partial x_a}+\overbrace{\frac{\partial A_a}{\partial t}+E_a}\right)$$

$$\dot p_a=m\dot v_a+\cancel{e\dot A_a}=e\left(v_\mu\frac{\partial A_\mu}{\partial x_a}+\overbrace{\boxed{\cancel{\dot A_a}-v_\mu\frac{\partial A_a}{\partial x_\mu}}}+E_a\right)$$

$$m\dot v_a=e\left(v_\mu\frac{\partial A_\mu}{\partial x_a}-v_\mu\frac{\partial A_a}{\partial x_\mu}+E_a\right)$$

$$\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}$$

$$E_a=-\frac{\partial\Phi}{\partial x^a}-\frac{\partial A_a}{\partial t}$$

$$-\frac{\partial\Phi}{\partial x^a}=\frac{\partial A_a}{\partial t}+E_a$$

$$\frac{\partial\mathbf{A}}{\partial t}=\frac{d\mathbf{A}}{dt}-(\mathbf{v}\bullet\nabla)\mathbf{A}$$

$$\frac{\partial A_a}{\partial t}=\boxed{\dot A_a-v_\mu\frac{\partial A_a}{\partial x_\mu}}$$

Hamilton's equations for charged particle in fields

Hamiltonian is explicit function of *momentum* **p**. Must replace velocity **v** using $m\mathbf{v}=\mathbf{p}-e\mathbf{A}(\mathbf{r},t)$.

$$H=\frac{1}{2m}\left(\mathbf{p}-e\mathbf{A}(\mathbf{r},t)\right)\bullet\left(\mathbf{p}-e\mathbf{A}(\mathbf{r},t)\right)+e\Phi(\mathbf{r},t)$$

(Correct formally and numerically)

$$H=\frac{\mathbf{p}\bullet\mathbf{p}}{2m}-\frac{e}{2m}\left(\mathbf{p}\bullet\mathbf{A}+\mathbf{A}\bullet\mathbf{p}\right)+\frac{e^2}{2m}\mathbf{A}\bullet\mathbf{A}+e\Phi(\mathbf{r},t)$$

(Expanded)

Hamilton's **v** equation:

$$\mathbf{v}=\dot{\mathbf{r}}=\frac{\partial H}{\partial\mathbf{p}}=\frac{\mathbf{p}-e\mathbf{A}(\mathbf{r},t)}{m}$$

(Just copies particle velocity relation.)

Hamilton's $d\mathbf{p}/dt$ equation:
(In index notation.)

$$\dot{p}_a=-\frac{\partial H}{\partial x_a}=-\frac{\partial}{\partial x_a}\frac{\left(p_\mu-eA_\mu\right)\left(p_\mu-eA_\mu\right)}{2m}-e\frac{\partial\Phi}{\partial x_a}$$

$$\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}$$

$$E_a=-\frac{\partial\Phi}{\partial x^a}-\frac{\partial A_a}{\partial t}$$

$m\mathbf{v}+e\mathbf{A}(\mathbf{r},t)=\mathbf{p}$

$$\dot{p}_a=m\dot{v}_a+e\dot{A}_a=+\underbrace{\frac{\left(p_\mu-eA_\mu\right)}{m}}_{}e\frac{\partial A_\mu}{\partial x_a}-e\frac{\partial\Phi}{\partial x_a}$$

$$-\frac{\partial\Phi}{\partial x^a}=\frac{\partial A_a}{\partial t}+E_a$$

$$\dot{p}_a=m\dot{v}_a+e\dot{A}_a=e\left(v_\mu\frac{\partial A_\mu}{\partial x_a}+\overbrace{\frac{\partial A_a}{\partial t}+E_a}\right)$$

$$\frac{\partial\mathbf{A}}{\partial t}=\frac{d\mathbf{A}}{dt}-(\mathbf{v}\bullet\nabla)\mathbf{A}$$

$$\frac{\partial A_a}{\partial t}=\boxed{\dot{A}_a-\sum_\mu v_\mu\frac{\partial A_a}{\partial x_\mu}}$$

$$\dot{p}_a=m\dot{v}_a+e\dot{A}_a=e\left(v_\mu\frac{\partial A_\mu}{\partial x_a}+\overbrace{\boxed{\dot{A}_a-v_\mu\frac{\partial A_a}{\partial x_\mu}}}+E_a\right)$$

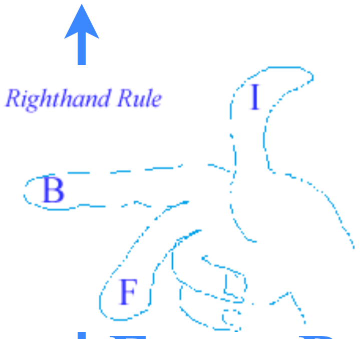
$$m\dot{v}_a=e\left(\underbrace{v_\mu\frac{\partial A_\mu}{\partial x_a}-v_\mu\frac{\partial A_a}{\partial x_\mu}}+E_a\right)$$

$$m\dot{\mathbf{v}}=e\left(\mathbf{v}\times\left(\nabla\times\mathbf{A}\right)+\mathbf{E}\right)=e(\mathbf{v}\times\mathbf{B}+\mathbf{E})$$

$$\mathbf{B}=\nabla\times\mathbf{A}$$

Hamilton's equations for charged particle in fields

Hamiltonian is explicit function of ***momentum*** **p**. Must replace velocity **v** using $m\mathbf{v}=\mathbf{p}-e\mathbf{A}(\mathbf{r},t)$.



$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

$$H=\frac{1}{2m}\left(\mathbf{p}-e\mathbf{A}(\mathbf{r},t)\right)\bullet\left(\mathbf{p}-e\mathbf{A}(\mathbf{r},t)\right)+e\Phi(\mathbf{r},t)$$

(Correct formally and numerically)

$$H=\frac{\mathbf{p}\bullet\mathbf{p}}{2m}-\frac{e}{2m}\left(\mathbf{p}\bullet\mathbf{A}+\mathbf{A}\bullet\mathbf{p}\right)+\frac{e^2}{2m}\mathbf{A}\bullet\mathbf{A}+e\Phi(\mathbf{r},t)$$

(Expanded)

Hamilton's **v** equation: $\mathbf{v}=\dot{\mathbf{r}}=\frac{\partial H}{\partial\mathbf{p}}=\frac{\mathbf{p}-e\mathbf{A}(\mathbf{r},t)}{m}$ *(Just copies particle velocity relation.)*

$$\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}$$

Hamilton's *d***p**/*dt* equation:
(In index notation.)

$$\dot p_a=-\frac{\partial H}{\partial x_a}=-\frac{\partial}{\partial x_a}\frac{\left(p_\mu-eA_\mu\right)\left(p_\mu-eA_\mu\right)}{2m}-e\frac{\partial\Phi}{\partial x_a}$$

$$E_a=-\frac{\partial\Phi}{\partial x^a}-\frac{\partial A_a}{\partial t}$$

$m\mathbf{v}+e\mathbf{A}(\mathbf{r},t)=\mathbf{p}$

$$\dot p_a=m\dot v_a+e\dot A_a=+\underbrace{\frac{\left(p_\mu-eA_\mu\right)}{m}}e\frac{\partial A_\mu}{\partial x_a}-e\frac{\partial\Phi}{\partial x_a}$$

$$-\frac{\partial\Phi}{\partial x^a}=\frac{\partial A_a}{\partial t}+E_a$$

$$\dot p_a=m\dot v_a+e\dot A_a=e\left(v_\mu\frac{\partial A_\mu}{\partial x_a}+\overbrace{\frac{\partial A_a}{\partial t}+E_a}\right)$$

$$\frac{\partial\mathbf{A}}{\partial t}=\frac{d\mathbf{A}}{dt}-(\mathbf{v}\bullet\nabla)\mathbf{A}$$

$$\dot p_a=m\dot v_a+e\dot A_a=e\left(v_\mu\frac{\partial A_\mu}{\partial x_a}+\overbrace{\boxed{\dot A_a-v_\mu\frac{\partial A_a}{\partial x_\mu}}}+E_a\right)$$

$$\frac{\partial A_a}{\partial t}=\boxed{\dot A_a-\sum_\mu v_\mu\frac{\partial A_a}{\partial x_\mu}}$$

...and now

we come back

full circle...

$$m\dot v_a=e\left(\underbrace{v_\mu\frac{\partial A_\mu}{\partial x_a}-v_\mu\frac{\partial A_a}{\partial x_\mu}}+E_a\right)$$

$$m\dot{\mathbf{v}}=e\left(\mathbf{v}\times\left(\nabla\times\mathbf{A}\right)+\mathbf{E}\right)=e(\mathbf{v}\times\mathbf{B}+\mathbf{E})$$

$$\mathbf{B}=\nabla\times\mathbf{A}$$

$\mathbf{v}\times\left(\nabla\times\mathbf{A}\right)=\mathbf{v}\bullet\left(\nabla\mathbf{A}\right)-\left(\mathbf{v}\bullet\nabla\right)\mathbf{A}$ *for particle mechanics*

Crossed E and B field mechanics

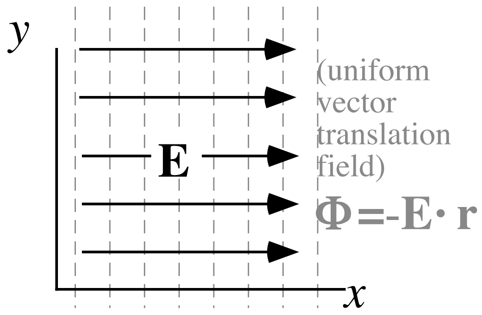
→ *Classical Hall-effect and cyclotron orbit orbit equations*
Vector theory vs. complex variable theory
Mechanical analog of cyclotron and FBI rule
Cycloid and epicycloid ruler&compass geometry
Cycloid geometry of flying levers
Practical poolhall application

Crossed E and B field mechanics

A constant **E** field has a scalar potential field **Φ** with constant gradient.

$$\Phi(\mathbf{r}) = -\mathbf{E}\bullet\mathbf{r}, \qquad -\nabla\Phi(\mathbf{r}) = \nabla\left(-\mathbf{E}\bullet\mathbf{r}\right) = \mathbf{E} = const.$$

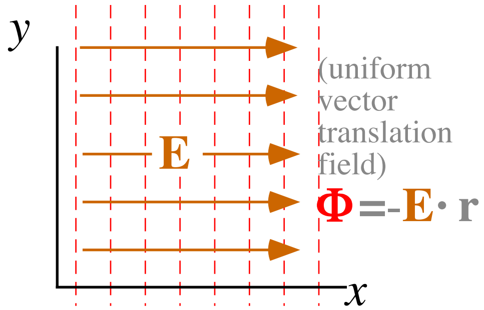
Fig. 2.4.1.



Crossed E and B field mechanics

A constant **E** field has a scalar potential field **Φ** with constant gradient.

$$\Phi(\mathbf{r}) = -\mathbf{E} \bullet \mathbf{r}, \qquad -\nabla\Phi(\mathbf{r}) = -\nabla\left(-\mathbf{E} \bullet \mathbf{r}\right) = \mathbf{E} = const.$$



Crossed E and B field mechanics

A constant **E** field has a scalar potential field **Φ** with constant gradient.

$$\boldsymbol{\Phi}(\mathbf{r}) = -\mathbf{E}\bullet\mathbf{r}, \qquad -\nabla\boldsymbol{\Phi}(\mathbf{r}) = -\nabla\left(-\mathbf{E}\bullet\mathbf{r}\right) = \mathbf{E} = const.$$

A constant **B** field has a vector potential field **A** that resembles a disc spinning counter-clockwise around the **B** axis.

$$\mathbf{A}(\mathbf{r}) = \frac{1}{2}\mathbf{B}\times\mathbf{r}, \qquad \nabla\times\mathbf{A}(\mathbf{r}) = \nabla\times\left(\frac{1}{2}\mathbf{B}\times\mathbf{r}\right) = \mathbf{B} = const.$$

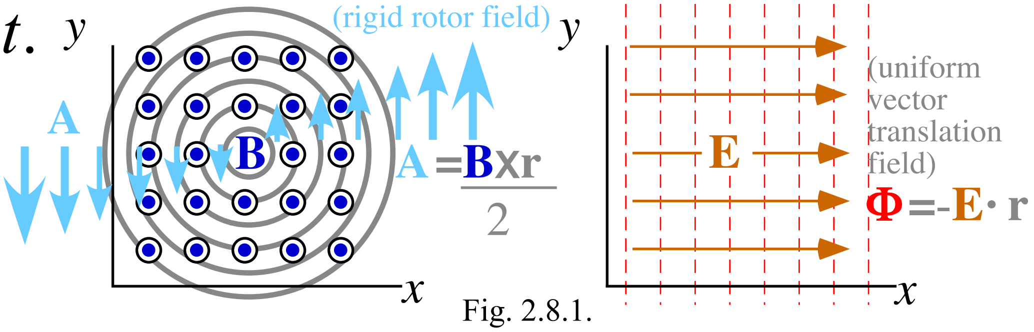


Fig. 2.8.1.

This mechanical analog of (E_x,B_z) field mimics ***A****-field with tabletop* ***v****-field*

Crossed E and B field mechanics

A constant **E** field has a scalar potential field **Φ** with constant gradient.

$$\Phi(\mathbf{r}) = -\mathbf{E} \bullet \mathbf{r}, \qquad -\nabla\Phi(\mathbf{r}) = -\nabla\left(-\mathbf{E}\bullet\mathbf{r}\right) = \mathbf{E} = const.$$

A constant **B** field has a vector potential field **A** that resembles a disc spinning counter-clockwise around the **B** axis.

$$\mathbf{A}(\mathbf{r}) = \frac{1}{2}\mathbf{B}\times\mathbf{r}, \qquad \nabla\times\mathbf{A}(\mathbf{r}) = \nabla\times\left(\frac{1}{2}\mathbf{B}\times\mathbf{r}\right) = \mathbf{B} = const.$$

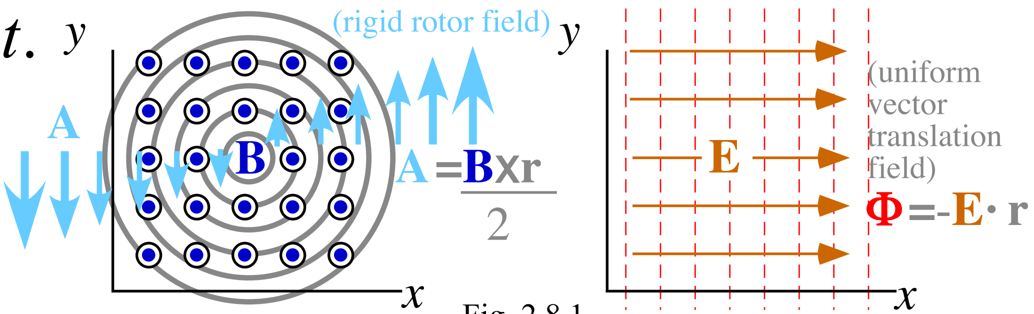
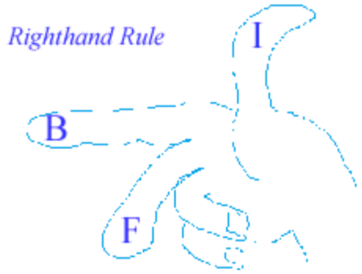


Fig. 2.8.1.

Newtonian electromagnetic equations of motion:

$$m\dot{\mathbf{v}} = e(\mathbf{E} + \mathbf{v}\times\mathbf{B})$$

$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B}.$$



$\mathbf{F} = q\mathbf{v}\times\mathbf{B} = \mathbf{I}\times\mathbf{B}$

Crossed E and B field mechanics

A constant **E** field has a scalar potential field **Φ** with constant gradient.

$$\Phi(\mathbf{r}) = -\mathbf{E} \bullet \mathbf{r}, \qquad -\nabla\Phi(\mathbf{r}) = -\nabla\left(-\mathbf{E} \bullet \mathbf{r}\right) = \mathbf{E} = const.$$

A constant **B** field has a vector potential field **A** that resembles a disc spinning counter-clockwise around the **B** axis.

$$\mathbf{A}(\mathbf{r}) = \frac{1}{2}\mathbf{B} \times \mathbf{r}, \qquad \nabla \times \mathbf{A}(\mathbf{r}) = \nabla \times \left(\frac{1}{2}\mathbf{B} \times \mathbf{r}\right) = \mathbf{B} = const.$$

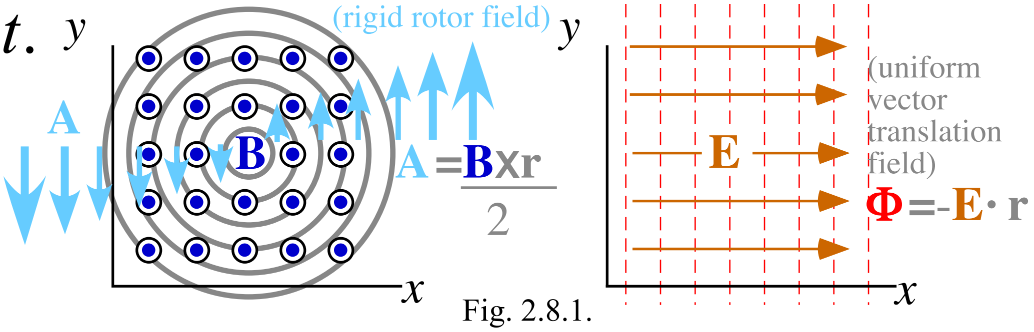


Fig. 2.8.1.

Newtonian electromagnetic equations of motion: $m\dot{\mathbf{v}} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$

$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v} \times \frac{e}{m}\mathbf{B} = \varepsilon + \mathbf{v} \times \frac{e}{m}B\hat{\mathbf{e}}_Z$$

$$\varepsilon_x = \frac{e}{m}E_x \qquad \varepsilon_y = \frac{e}{m}E_y \qquad B = \frac{e}{m}B_z$$

Shorthand Labeling

Crossed E and B field mechanics

A constant **E** field has a scalar potential field **Φ** with constant gradient.

$$\Phi(\mathbf{r}) = -\mathbf{E}\bullet\mathbf{r}, \qquad -\nabla\Phi(\mathbf{r}) = -\nabla\left(-\mathbf{E}\bullet\mathbf{r}\right) = \mathbf{E} = const.$$

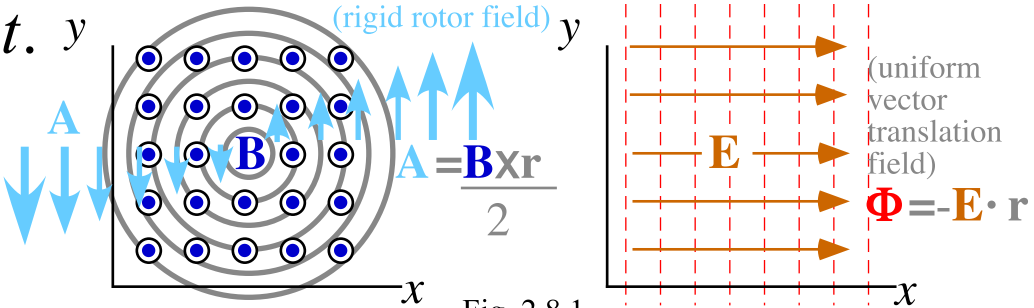


Fig. 2.8.1.

A constant **B** field has a vector potential field **A** that resembles a disc spinning counter-clockwise around the **B** axis.

$$\mathbf{A}(\mathbf{r}) = \frac{1}{2}\mathbf{B}\times\mathbf{r}, \qquad \nabla\times\mathbf{A}(\mathbf{r}) = \nabla\times\left(\frac{1}{2}\mathbf{B}\times\mathbf{r}\right) = \mathbf{B} = const.$$

Newtonian electromagnetic equations of motion: $m\dot{\mathbf{v}} = e(\mathbf{E}+\mathbf{v}\times\mathbf{B})$

$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \varepsilon + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z$$

$$\varepsilon_x = \frac{e}{m}E_x \qquad \varepsilon_y = \frac{e}{m}E_y \qquad B = \frac{e}{m}B_z$$

Shorthand Labeling

Gibb's notation:

$$\dot{\mathbf{v}} \quad = \quad \boldsymbol{\varepsilon} \quad + \quad \mathbf{v} \quad \times B\hat{\mathbf{e}}_\mathbf{z}$$

$$\dot{v}_x\hat{\mathbf{e}}_\mathbf{x} + \dot{v}_y\hat{\mathbf{e}}_\mathbf{y} = \varepsilon_x\hat{\mathbf{e}}_\mathbf{x} + \varepsilon_y\hat{\mathbf{e}}_\mathbf{y} + (v_x\hat{\mathbf{e}}_\mathbf{x} + v_y\hat{\mathbf{e}}_\mathbf{y})\times B\hat{\mathbf{e}}_\mathbf{z}$$

$$= \varepsilon_x\hat{\mathbf{e}}_\mathbf{x} + \varepsilon_y\hat{\mathbf{e}}_\mathbf{y} - Bv_x\hat{\mathbf{e}}_\mathbf{y} + Bv_y\hat{\mathbf{e}}_\mathbf{x}$$

where: $\hat{\mathbf{e}}_\mathbf{x}\times\hat{\mathbf{e}}_\mathbf{z} = -\hat{\mathbf{e}}_\mathbf{x}$ and: $\hat{\mathbf{e}}_\mathbf{y}\times\hat{\mathbf{e}}_\mathbf{z} = \hat{\mathbf{e}}_\mathbf{x}$

Crossed E and B field mechanics

Classical Hall-effect and cyclotron orbit orbit equations

→ *Vector theory vs. complex variable theory*

Mechanical analog of cyclotron and FBI rule

Cycloid and epicycloid ruler&compass geometry

Cycloid geometry of flying levers

Practical poolhall application

Crossed E and B field mechanics

A constant **E** field has a scalar potential field **Φ** with constant gradient.

$$\boldsymbol{\Phi}(\mathbf{r}) = -\mathbf{E}\bullet\mathbf{r}, \qquad -\nabla\boldsymbol{\Phi}(\mathbf{r}) = -\nabla\left(-\mathbf{E}\bullet\mathbf{r}\right) = \mathbf{E} = const.$$

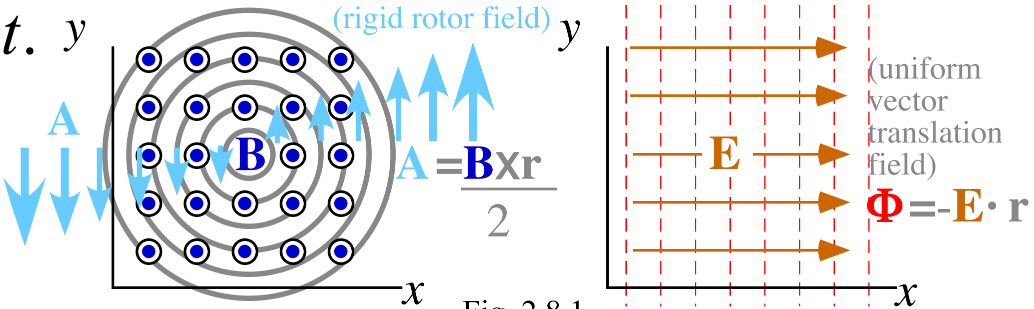


Fig. 2.8.1.

A constant **B** field has a vector potential field **A** that resembles a disc spinning counter-clockwise around the **B** axis.

$$\mathbf{A}(\mathbf{r}) = \frac{1}{2}\mathbf{B}\times\mathbf{r}, \qquad \nabla\times\mathbf{A}(\mathbf{r}) = \nabla\times\left(\frac{1}{2}\mathbf{B}\times\mathbf{r}\right) = \mathbf{B} = const.$$

Newtonian electromagnetic equations of motion: $m\dot{\mathbf{v}} = e(\mathbf{E}+\mathbf{v}\times\mathbf{B})$

$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \boldsymbol{\varepsilon} + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z$$

$$\varepsilon_x = \frac{e}{m}E_x \qquad \varepsilon_y = \frac{e}{m}E_y \qquad B = \frac{e}{m}B_z$$

Shorthand Labeling

Gibb's notation:

$$\dot{\mathbf{v}} \quad = \quad \boldsymbol{\varepsilon} \quad + \quad \mathbf{v} \quad \times B\hat{\mathbf{e}}_\mathbf{z}$$

$$\dot{v}_x\hat{\mathbf{e}}_\mathbf{x} + \dot{v}_y\hat{\mathbf{e}}_\mathbf{y} = \varepsilon_x\hat{\mathbf{e}}_\mathbf{x} + \varepsilon_y\hat{\mathbf{e}}_\mathbf{y} + (v_x\hat{\mathbf{e}}_\mathbf{x} + v_y\hat{\mathbf{e}}_\mathbf{y})\times B\hat{\mathbf{e}}_\mathbf{z}$$

$$= \varepsilon_x\hat{\mathbf{e}}_\mathbf{x} + \varepsilon_y\hat{\mathbf{e}}_\mathbf{y} - Bv_x\hat{\mathbf{e}}_\mathbf{y} + Bv_y\hat{\mathbf{e}}_\mathbf{x}$$

where: $\hat{\mathbf{e}}_\mathbf{x}\times\hat{\mathbf{e}}_\mathbf{z} = -\hat{\mathbf{e}}_\mathbf{x}$ and: $\hat{\mathbf{e}}_\mathbf{y}\times\hat{\mathbf{e}}_\mathbf{z} = \hat{\mathbf{e}}_\mathbf{x}$

Complex variable velocity: $v=v_x+iv_y$ and electric field: $\varepsilon=\varepsilon_x+i\varepsilon_y$

$$\dot{v}_x + i\dot{v}_y = \varepsilon_x + i\varepsilon_y - \; iBv_x + Bv_y = \varepsilon_x + i\varepsilon_y - \; iB(v_x + iv_y)$$

$$\dot{v} \quad = \quad \varepsilon \quad - \quad iBv$$

with replacements : $\hat{\mathbf{e}}_\mathbf{x}\rightarrow 1$ and : $\hat{\mathbf{e}}_\mathbf{y}\rightarrow i = \sqrt{-1}$

Crossed E and B field mechanics

A constant **E** field has a scalar potential field **Φ** with constant gradient.

$$\Phi(\mathbf{r}) = -\mathbf{E}\bullet\mathbf{r}, \qquad -\nabla\Phi(\mathbf{r}) = -\nabla\left(-\mathbf{E}\bullet\mathbf{r}\right) = \mathbf{E} = const.$$

A constant **B** field has a vector potential field **A** that resembles a disc spinning counter-clockwise around the **B** axis.

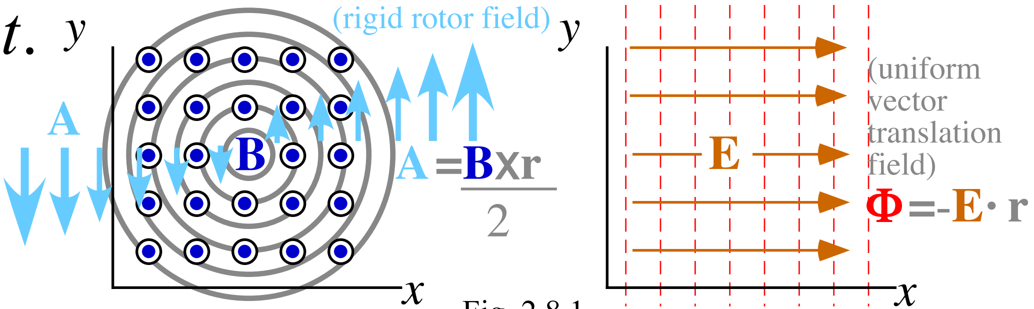


Fig. 2.8.1.

$$\mathbf{A}(\mathbf{r}) = \frac{1}{2}\mathbf{B}\times\mathbf{r}, \qquad \nabla\times\mathbf{A}(\mathbf{r}) = \nabla\times\left(\frac{1}{2}\mathbf{B}\times\mathbf{r}\right) = \mathbf{B} = const.$$

Newtonian electromagnetic equations of motion: $m\dot{\mathbf{v}} = e(\mathbf{E} + \mathbf{v}\times\mathbf{B})$

$$\boxed{\begin{array}{l} \dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \boldsymbol{\varepsilon} + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z \\ \varepsilon_x = \frac{e}{m}E_x \qquad \varepsilon_y = \frac{e}{m}E_y \qquad B = \frac{e}{m}B_z \\ \textit{Shorthand Labeling} \end{array}}$$

Gibb's notation:

$$\dot{\mathbf{v}} \quad = \quad \boldsymbol{\varepsilon} \quad + \quad \mathbf{v} \quad \times B\hat{\mathbf{e}}_{\mathbf{z}}$$

$$\dot{v}_x\hat{\mathbf{e}}_{\mathbf{x}} + \dot{v}_y\hat{\mathbf{e}}_{\mathbf{y}} = \varepsilon_x\hat{\mathbf{e}}_{\mathbf{x}} + \varepsilon_y\hat{\mathbf{e}}_{\mathbf{y}} + (v_x\hat{\mathbf{e}}_{\mathbf{x}} + v_y\hat{\mathbf{e}}_{\mathbf{y}})\times B\hat{\mathbf{e}}_{\mathbf{z}}$$

$$= \varepsilon_x\hat{\mathbf{e}}_{\mathbf{x}} + \varepsilon_y\hat{\mathbf{e}}_{\mathbf{y}} - Bv_x\hat{\mathbf{e}}_{\mathbf{y}} + Bv_y\hat{\mathbf{e}}_{\mathbf{x}}$$

where: $\hat{\mathbf{e}}_{\mathbf{x}}\times\hat{\mathbf{e}}_{\mathbf{z}} = -\hat{\mathbf{e}}_{\mathbf{x}}$ and: $\hat{\mathbf{e}}_{\mathbf{y}}\times\hat{\mathbf{e}}_{\mathbf{z}} = \hat{\mathbf{e}}_{\mathbf{x}}$

Complex variable velocity: $v=v_x+iv_y$ and electric field: $\varepsilon=\varepsilon_x+i\varepsilon_y$

$$\dot{v}_x + i\dot{v}_y = \varepsilon_x + i\varepsilon_y - \ iBv_x + Bv_y = \varepsilon_x + i\varepsilon_y - \ iB(v_x + iv_y)$$

$$\dot{v} \quad = \quad \varepsilon \quad - \quad iBv$$

with replacements : $\hat{\mathbf{e}}_{\mathbf{x}}\rightarrow 1$ and : $\hat{\mathbf{e}}_{\mathbf{y}}\rightarrow i = \sqrt{-1}$

A velocity transformation $V(t)=v(t)+\beta$ cancels constant ε-field to give an equation: $\boxed{\dot{V}=(const.)V}$

$$\boxed{\dot{V}(t)=\dot{v}(t)+\dot{\beta} = \varepsilon - iBv = \varepsilon - iB(V(t)-\beta) = -iBV(t)}$$

Pick β so: $iB\beta = -\varepsilon$

Then : $\boxed{\beta = -\frac{\varepsilon}{iB} = i\frac{\varepsilon}{B}}$

Crossed E and B field mechanics

A constant **E** field has a scalar potential field **Φ** with constant gradient.

$$\Phi(\mathbf{r}) = -\mathbf{E}\bullet\mathbf{r}, \qquad -\nabla\Phi(\mathbf{r}) = -\nabla\left(-\mathbf{E}\bullet\mathbf{r}\right) = \mathbf{E} = const.$$

A constant **B** field has a vector potential field **A** that resembles a disc spinning counter-clockwise around the **B** axis.

$$\mathbf{A}(\mathbf{r}) = \frac{1}{2}\mathbf{B}\times\mathbf{r}, \qquad \nabla\times\mathbf{A}(\mathbf{r}) = \nabla\times\left(\frac{1}{2}\mathbf{B}\times\mathbf{r}\right) = \mathbf{B} = const.$$

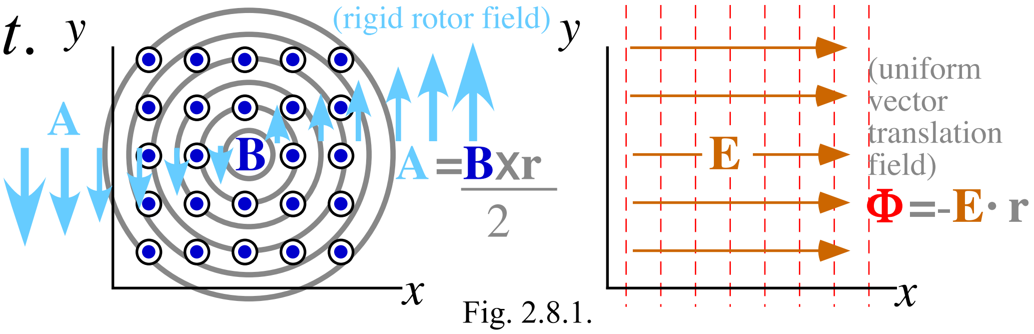


Fig. 2.8.1.

Newtonian electromagnetic equations of motion: $m\dot{\mathbf{v}} = e(\mathbf{E} + \mathbf{v}\times\mathbf{B})$

$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \boldsymbol{\varepsilon} + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z$$

$$\varepsilon_x = \frac{e}{m}E_x \qquad \varepsilon_y = \frac{e}{m}E_y \qquad B = \frac{e}{m}B_z$$

Shorthand Labeling

Gibb's notation:

$$\dot{\mathbf{v}} \quad = \quad \boldsymbol{\varepsilon} \quad + \quad \mathbf{v} \quad \times B\hat{\mathbf{e}}_\mathbf{z}$$

$$\dot{v}_x\hat{\mathbf{e}}_\mathbf{x} + \dot{v}_y\hat{\mathbf{e}}_\mathbf{y} = \varepsilon_x\hat{\mathbf{e}}_\mathbf{x} + \varepsilon_y\hat{\mathbf{e}}_\mathbf{y} + (v_x\hat{\mathbf{e}}_\mathbf{x} + v_y\hat{\mathbf{e}}_\mathbf{y})\times B\hat{\mathbf{e}}_\mathbf{z}$$

$$= \varepsilon_x\hat{\mathbf{e}}_\mathbf{x} + \varepsilon_y\hat{\mathbf{e}}_\mathbf{y} - Bv_x\hat{\mathbf{e}}_\mathbf{y} + Bv_y\hat{\mathbf{e}}_\mathbf{x}$$

where: $\hat{\mathbf{e}}_\mathbf{x}\times\hat{\mathbf{e}}_\mathbf{z} = -\hat{\mathbf{e}}_\mathbf{x}$ and: $\hat{\mathbf{e}}_\mathbf{y}\times\hat{\mathbf{e}}_\mathbf{z} = \hat{\mathbf{e}}_\mathbf{x}$

Complex variable velocity: $v=v_x+iv_y$ and electric field: $\varepsilon=\varepsilon_x+i\varepsilon_y$

$$\dot{v}_x + i\dot{v}_y = \varepsilon_x + i\varepsilon_y - iBv_x + Bv_y = \varepsilon_x + i\varepsilon_y - iB(v_x + iv_y)$$

$$\dot{v} \quad = \quad \varepsilon \quad - \quad iBv$$

with replacements : $\hat{\mathbf{e}}_\mathbf{x}\rightarrow 1$ and : $\hat{\mathbf{e}}_\mathbf{y}\rightarrow i = \sqrt{-1}$

A velocity transformation $V(t)=v(t)+\beta$ cancels constant ε-field to give an equation: $\boxed{\dot{V}=(const.)V}$

$$\boxed{\dot{V}(t)=\dot{v}(t)+\dot{\beta} = \varepsilon - iBv = \varepsilon - iB(V(t)-\beta) = -iBV(t)}$$

Pick β so: $iB\beta = -\varepsilon$

Then : $\boxed{\beta = -\frac{\varepsilon}{iB} = i\frac{\varepsilon}{B}}$

Move last part of this calculation UP↑

Crossed E and B field mechanics (Solution by complex variables)

$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \varepsilon + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z$$

$$\varepsilon_x = \frac{e}{m}E_x \qquad \varepsilon_y = \frac{e}{m}E_y \qquad B = \frac{e}{m}B_z$$

Shorthand Labeling

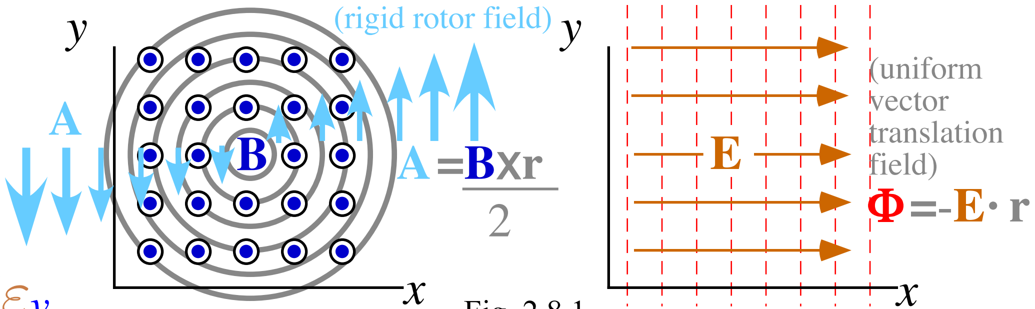


Fig. 2.8.1.

Complex variable velocity: $v=v_x+iv_y$ and electric field: $\varepsilon=\varepsilon_x+i\varepsilon_y$

$$\dot{v}_x + i\dot{v}_y = \varepsilon_x + i\varepsilon_y - iBv_x + Bv_y = \varepsilon_x + i\varepsilon_y - iB(v_x + iv_y)$$

$$\dot{v} \quad = \quad \varepsilon \quad - \quad iBv \qquad \text{with replacements} : \hat{\mathbf{e}}_\mathbf{x} \to 1 \qquad \text{and} : \hat{\mathbf{e}}_\mathbf{y} \to i = \sqrt{-1}$$

A velocity transformation $V(t)=v(t)+\beta$ cancels constant ε-field to give an equation: $\dot{V}=(const.)V$

$$\dot{V}(t) = \dot{v}(t) + \dot{\beta} = \varepsilon - iBv = \varepsilon - iB(V(t) - \beta) = -iBV(t) \qquad \text{where} : \beta = -\frac{\varepsilon}{iB} = i\frac{\varepsilon}{B}$$

Crossed E and B field mechanics (Solution by complex variables)

$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \varepsilon + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z$$

$$\varepsilon_x = \frac{e}{m}E_x \qquad \varepsilon_y = \frac{e}{m}E_y \qquad B = \frac{e}{m}B_z$$

Shorthand Labeling

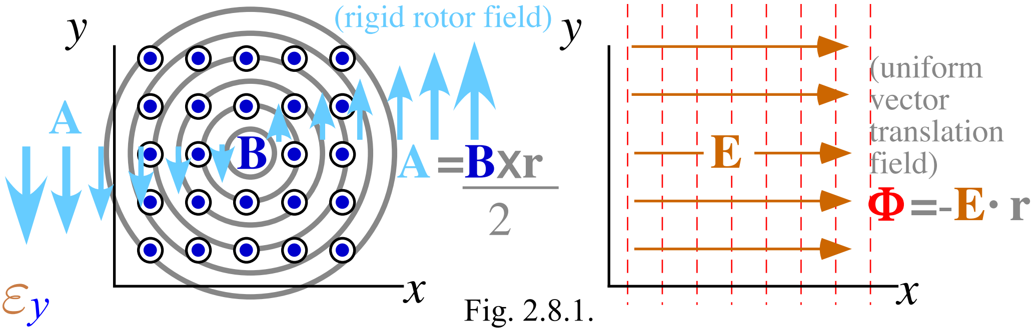


Fig. 2.8.1.

Complex variable velocity: $v=v_x+iv_y$ and electric field: $\varepsilon=\varepsilon_x+i\varepsilon_y$

$$\dot v_x + i\dot v_y = \varepsilon_x + i\varepsilon_y - iBv_x + Bv_y = \varepsilon_x + i\varepsilon_y - iB(v_x + iv_y)$$

$$\dot v = \varepsilon - iBv \qquad \text{with replacements : } \hat{\mathbf{e}}_\mathbf{x} \to 1 \qquad \text{and : } \hat{\mathbf{e}}_\mathbf{y} \to i = \sqrt{-1}$$

A velocity transformation $V(t)=v(t)+\beta$ cancels constant ε-field to give an equation: $\boxed{\dot V=(const.)V}$

$$\dot V(t)=\dot v(t)+\dot\beta = \varepsilon - iBv = \varepsilon - iB(V(t)-\beta) = -iBV(t) \qquad \text{where : } \boxed{\beta = -\frac{\varepsilon}{iB} = i\frac{\varepsilon}{B}}$$

An exponential $V(t)=e^{-iBt}V(0)$ solution results: e^{-iBt} is a clockwise 2D rotation.

Crossed E and B field mechanics (Solution by complex variables)

$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \varepsilon + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z$$

$$\varepsilon_x = \frac{e}{m}E_x \qquad \varepsilon_y = \frac{e}{m}E_y \qquad B = \frac{e}{m}B_z$$

Shorthand Labeling

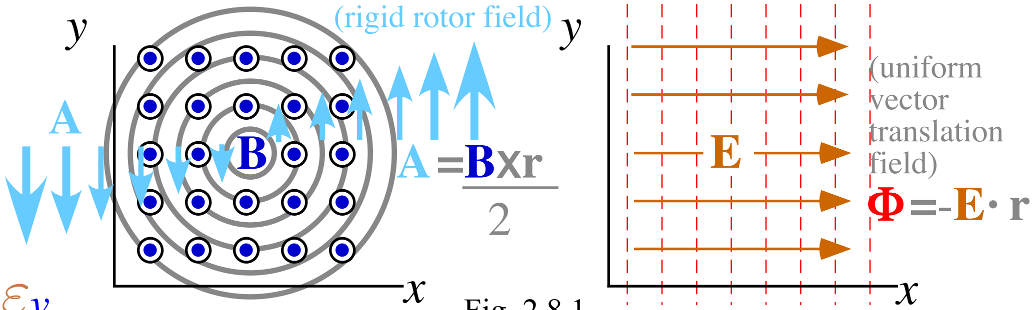


Fig. 2.8.1.

Complex variable velocity: $v=v_x+iv_y$ and electric field: $\varepsilon=\varepsilon_x+i\varepsilon_y$

$$\dot{v}_x + i\dot{v}_y = \varepsilon_x + i\varepsilon_y - iBv_x + Bv_y = \varepsilon_x + i\varepsilon_y - iB(v_x + iv_y)$$

$$\dot{v} = \varepsilon - iBv \qquad \text{with replacements : } \hat{\mathbf{e}}_{\mathbf{x}} \to 1 \qquad \text{and : } \hat{\mathbf{e}}_{\mathbf{y}} \to i = \sqrt{-1}$$

A velocity transformation $V(t)=v(t)+\beta$ cancels constant ε-field to give an equation: $\dot{V}=(const.)V$

$$\dot{V}(t)=\dot{v}(t)+\dot{\beta} = \varepsilon - iBv = \varepsilon - iB(V(t)-\beta) = -iBV(t) \qquad \text{where : } \beta = -\frac{\varepsilon}{iB} = i\frac{\varepsilon}{B}$$

An exponential $V(t)=e^{-iBt}V(0)$ solution results: e^{-iBt} is a clockwise 2D rotation.

$$v(t)+\beta = V(t) = e^{-iB\cdot t}V(0) = e^{-iB\cdot t}(v(0)+\beta)$$

Crossed E and B field mechanics (Solution by complex variables)

$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \varepsilon + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z$$

$$\varepsilon_x = \frac{e}{m}E_x \qquad \varepsilon_y = \frac{e}{m}E_y \qquad B = \frac{e}{m}B_z$$

Shorthand Labeling

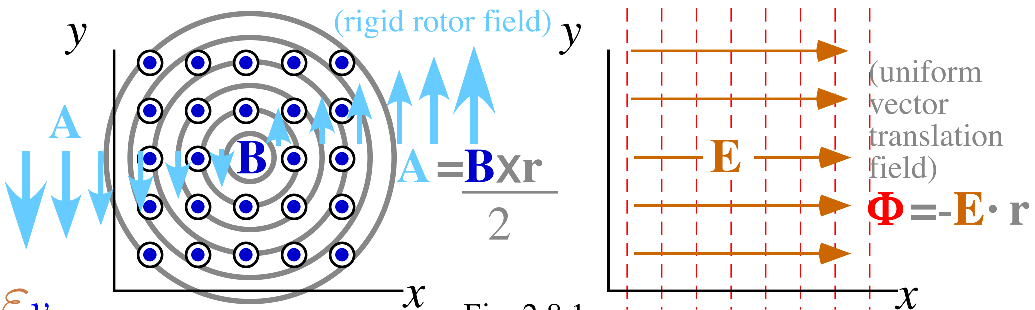


Fig. 2.8.1.

Complex variable velocity: $v=v_x+iv_y$ and electric field: $\varepsilon=\varepsilon_x+i\varepsilon_y$

$$\dot{v}_x + i\dot{v}_y = \varepsilon_x + i\varepsilon_y - iBv_x + Bv_y = \varepsilon_x + i\varepsilon_y - iB(v_x + iv_y)$$

$$\dot{v} = \varepsilon - iBv \qquad \text{with replacements : } \hat{\mathbf{e}}_{\mathbf{x}} \rightarrow 1 \qquad \text{and : } \hat{\mathbf{e}}_{\mathbf{y}} \rightarrow i = \sqrt{-1}$$

A velocity transformation $V(t)=v(t)+\beta$ cancels constant ε-field to give an equation: $\dot{V}=(const.)V$

$$\dot{V}(t) = \dot{v}(t) + \dot{\beta} = \varepsilon - iBv = \varepsilon - iB(V(t) - \beta) = -iBV(t) \qquad \text{where : } \beta = -\frac{\varepsilon}{iB} = i\frac{\varepsilon}{B}$$

An exponential $V(t)=e^{-iBt}V(0)$ solution results: e^{-iBt} is a clockwise 2D rotation.

complex form

$$v(t) + \beta = V(t) = e^{-iB\cdot t}V(0) = e^{-iB\cdot t}(v(0) + \beta) \quad \text{or:} \quad v(t) = e^{-iB\cdot t}(v(0) + \beta) - \beta = e^{-iB\cdot t}(v(0) + i\frac{\varepsilon}{B}) - i\frac{\varepsilon}{B}$$

Crossed E and B field mechanics (Solution by complex variables)

$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \varepsilon + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z$$

$$\varepsilon_x = \frac{e}{m}E_x \qquad \varepsilon_y = \frac{e}{m}E_y \qquad B = \frac{e}{m}B_z$$

Shorthand Labeling

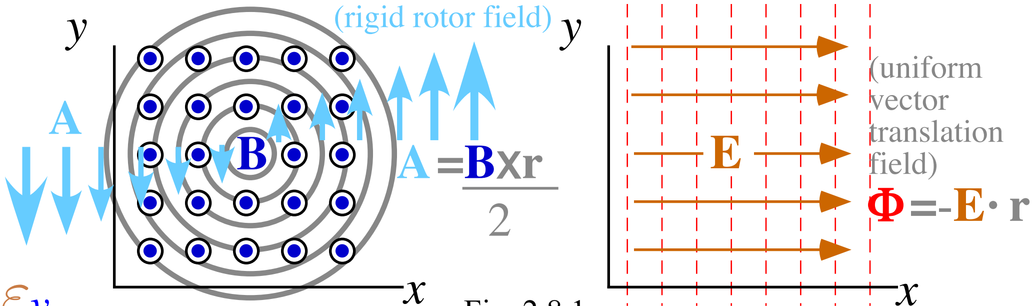


Fig. 2.8.1.

Complex variable velocity: $v=v_x+iv_y$ *and electric field:* $\varepsilon=\varepsilon_x+i\varepsilon_y$

$$\dot{v}_x + i\dot{v}_y = \varepsilon_x + i\varepsilon_y - iBv_x + Bv_y = \varepsilon_x + i\varepsilon_y - iB(v_x + iv_y)$$

$$\dot{v} = \varepsilon - iBv \qquad \text{with replacements : } \hat{\mathbf{e}}_\mathbf{x} \rightarrow 1 \qquad \text{and : } \hat{\mathbf{e}}_\mathbf{y} \rightarrow i = \sqrt{-1}$$

A velocity transformation $V(t)=v(t)+\beta$ cancels constant ε-field to give an equation: $\boxed{\dot{V}=(const.)V}$

$$\dot{V}(t)=\dot{v}(t)+\dot{\beta} = \varepsilon - iBv = \varepsilon - iB(V(t)-\beta) = -iBV(t) \qquad \text{where : } \boxed{\beta = -\frac{\varepsilon}{iB} = i\frac{\varepsilon}{B}}$$

An exponential $V(t)=e^{-iBt}V(0)$ solution results: e^{-iBt} is a clockwise 2D rotation.

complex form

$$v(t)+\beta = V(t) = e^{-iB\cdot t}V(0) = e^{-iB\cdot t}(v(0)+\beta) \quad \text{or:} \quad v(t) = e^{-iB\cdot t}(v(0)+\beta)-\beta = e^{-iB\cdot t}\left(v(0)+i\frac{\varepsilon}{B}\right)-i\frac{\varepsilon}{B}$$

Expanding e^{-iBt}, $v=v_x+iv_y$, and $\varepsilon=\varepsilon_x+i\varepsilon_y$ reveals x (Real) and y (Imaginary) components

$$\begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix} = \begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix} \begin{pmatrix} v_x(0) - \frac{\varepsilon_y}{B} \\ v_y(0) + \frac{\varepsilon_x}{B} \end{pmatrix} + \begin{pmatrix} \frac{\varepsilon_y}{B} \\ -\frac{\varepsilon_x}{B} \end{pmatrix}$$

vector form

Crossed E and B field mechanics (Solution by complex variables)

$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \varepsilon + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z$$

$$\varepsilon_x = \frac{e}{m}E_x \qquad \varepsilon_y = \frac{e}{m}E_y \qquad B = \frac{e}{m}B_z$$

Shorthand Labeling

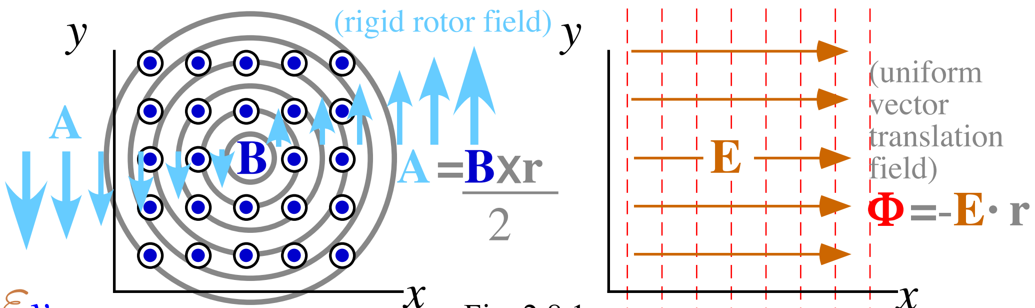


Fig. 2.8.1.

Complex variable velocity: $v=v_x+iv_y$ *and electric field:* $\varepsilon=\varepsilon_x+i\varepsilon_y$

$$\dot v_x + i\dot v_y = \varepsilon_x + i\varepsilon_y - iBv_x + Bv_y = \varepsilon_x + i\varepsilon_y - iB(v_x + iv_y)$$

$$\dot v = \varepsilon - iBv \qquad \text{with replacements}: \hat{\mathbf{e}}_\mathbf{x}\to 1 \qquad \text{and}: \hat{\mathbf{e}}_\mathbf{y}\to i = \sqrt{-1}$$

A velocity transformation $V(t)=v(t)+\beta$ cancels constant ε-field to give an equation: $\boxed{\dot V=(const.)V}$

$$\dot V(t) = \dot v(t) + \dot\beta = \varepsilon - iBv = \varepsilon - iB(V(t)-\beta) = -iBV(t) \qquad \text{where}: \boxed{\beta = -\frac{\varepsilon}{iB} = i\frac{\varepsilon}{B}}$$

An exponential $V(t)=e^{-iBt}V(0)$ solution results: e^{-iBt} is a clockwise 2D rotation.

complex form

$$v(t)+\beta = V(t) = e^{-iB\cdot t}V(0) = e^{-iB\cdot t}(v(0)+\beta) \qquad \text{or:} \qquad v(t) = e^{-iB\cdot t}(v(0)+\beta)-\beta = e^{-iB\cdot t}\left(v(0)+i\frac{\varepsilon}{B}\right)-i\frac{\varepsilon}{B}$$

Expanding e^{-iBt}, $v=v_x+iv_y$, and $\varepsilon=\varepsilon_x+i\varepsilon_y$ reveals x (Real) and y (Imaginary) components

$$\begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix} = \begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix} \begin{pmatrix} v_x(0) - \frac{\varepsilon_y}{B} \\ v_y(0) + \frac{\varepsilon_x}{B} \end{pmatrix} + \begin{pmatrix} \frac{\varepsilon_y}{B} \\ -\frac{\varepsilon_x}{B} \end{pmatrix}$$

vector form

Integrating $v(t)$ yields complex coordinate $q=x+iy$ affected by both ε_x and ε_y.

Crossed E and B field mechanics (Solution by complex variables)

$$\boxed{\begin{array}{l}\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \varepsilon + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z \\ \varepsilon_x = \frac{e}{m}E_x \qquad \varepsilon_y = \frac{e}{m}E_y \qquad B = \frac{e}{m}B_z \\ \textit{Shorthand Labeling}\end{array}}$$

Complex variable velocity: $v=v_x+iv_y$ *and electric field:* $\varepsilon=\varepsilon_x+i\varepsilon_y$

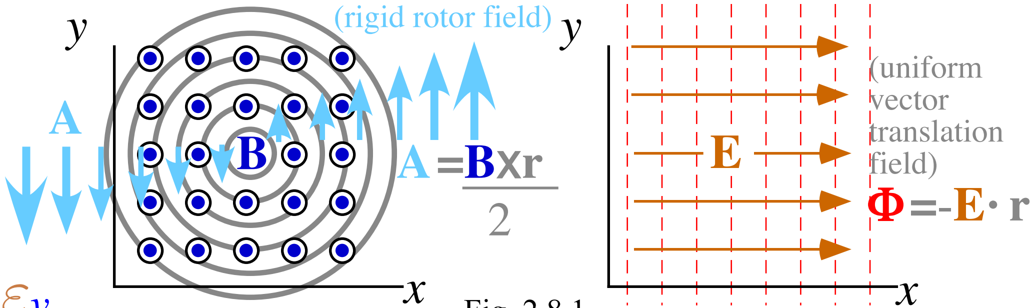


Fig. 2.8.1.

$$\dot{v}_x + i\dot{v}_y = \varepsilon_x + i\varepsilon_y - \ iBv_x + Bv_y = \varepsilon_x + i\varepsilon_y - \ iB(v_x + iv_y)$$

$$\dot{v} \quad = \quad \varepsilon \quad - \quad iBv \qquad \text{with replacements : } \hat{\mathbf{e}}_{\mathbf{x}} \rightarrow 1 \quad \text{and : } \hat{\mathbf{e}}_{\mathbf{y}} \rightarrow i = \sqrt{-1}$$

A velocity transformation $V(t)=v(t)+\beta$ cancels constant ε-field to give an equation: $\boxed{\dot{V}=(const.)V}$

$$\boxed{\dot{V}(t) = \dot{v}(t) + \dot{\beta} = \varepsilon - iBv = \varepsilon - iB(V(t) - \beta) = -iBV(t)} \qquad \text{where : } \boxed{\beta = -\frac{\varepsilon}{iB} = i\frac{\varepsilon}{B}}$$

An exponential $V(t)=e^{-iBt}V(0)$ solution results: e^{-iBt} is a clockwise 2D rotation.

complex form

$$v(t)+\beta = V(t) = e^{-iB\cdot t}V(0) = e^{-iB\cdot t}(v(0)+\beta) \quad \text{or:} \quad v(t) = e^{-iB\cdot t}(v(0)+\beta) - \beta = e^{-iB\cdot t}(v(0)+i\frac{\varepsilon}{B}) - i\frac{\varepsilon}{B}$$

Expanding e^{-iBt}, $v=v_x+iv_y$, and $\varepsilon=\varepsilon_x+i\varepsilon_y$ reveals x (Real) and y (Imaginary) components

$$\begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix} = \begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix} \begin{pmatrix} v_x(0) - \frac{\varepsilon_y}{B} \\ v_y(0) + \frac{\varepsilon_x}{B} \end{pmatrix} + \begin{pmatrix} \frac{\varepsilon_y}{B} \\ -\frac{\varepsilon_x}{B} \end{pmatrix}$$

vector form

Integrating $v(t)$ yields complex coordinate $q=x+iy$ affected by both ε_x and ε_y.

$$q(t) = \int v(t)\,dt = \frac{e^{-iB\cdot t}}{-iB}(v(0)+i\frac{\varepsilon}{B}) - i\frac{\varepsilon}{B}\cdot t + Const. \quad \text{where: } Const. = q(0) - (\frac{v(0)}{-iB} - \frac{\varepsilon}{B^2})$$

complex form

Crossed E and B field mechanics (Solution by complex variables)

$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \varepsilon + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z$$

$$\varepsilon_x = \frac{e}{m}E_x \qquad \varepsilon_y = \frac{e}{m}E_y \qquad B = \frac{e}{m}B_z$$

Shorthand Labeling

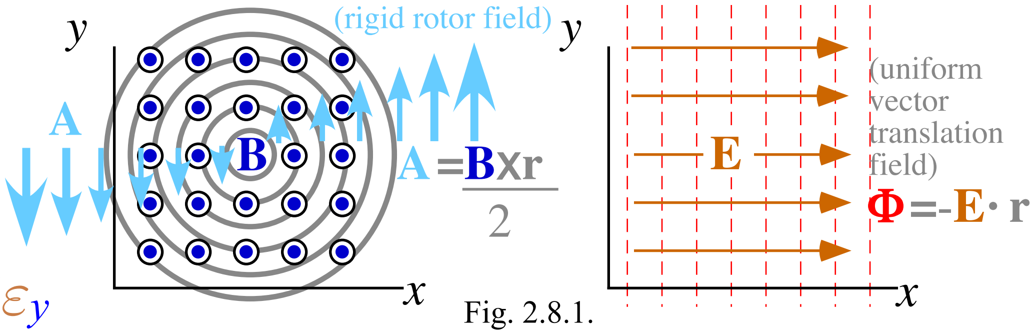


Fig. 2.8.1.

Complex variable velocity: $v=v_x+iv_y$ and electric field: $\varepsilon=\varepsilon_x+i\varepsilon_y$

$$\dot{v}_x + i\dot{v}_y = \varepsilon_x + i\varepsilon_y - \; iBv_x + Bv_y = \varepsilon_x + i\varepsilon_y - \; iB(v_x + iv_y)$$

$$\dot{v} \quad = \quad \varepsilon \quad - \quad iBv \qquad \text{with replacements : } \hat{\mathbf{e}}_\mathbf{x} \rightarrow 1 \qquad \text{and : } \hat{\mathbf{e}}_\mathbf{y} \rightarrow i = \sqrt{-1}$$

A velocity transformation $V(t)=v(t)+\beta$ cancels constant ε-field to give an equation: $\dot{V}=(const.)V$

$$\dot{V}(t) = \dot{v}(t) + \dot{\beta} = \varepsilon - iBv = \varepsilon - iB(V(t)-\beta) = -iBV(t) \qquad \text{where : } \beta = -\frac{\varepsilon}{iB} = i\frac{\varepsilon}{B}$$

An exponential $V(t)=e^{-iBt}V(0)$ solution results: e^{-iBt} is a clockwise 2D rotation.

complex form

$$v(t)+\beta = V(t) = e^{-iB\cdot t}V(0) = e^{-iB\cdot t}(v(0)+\beta) \quad \text{or:} \quad v(t) = e^{-iB\cdot t}(v(0)+\beta)-\beta = e^{-iB\cdot t}(v(0)+i\frac{\varepsilon}{B}) - i\frac{\varepsilon}{B}$$

Expanding e^{-iBt}, $v=v_x+iv_y$, and $\varepsilon=\varepsilon_x+i\varepsilon_y$ reveals x (Real) and y (Imaginary) components

$$\begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix} = \begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix} \begin{pmatrix} v_x(0) - \frac{\varepsilon_y}{B} \\ v_y(0) + \frac{\varepsilon_x}{B} \end{pmatrix} + \begin{pmatrix} \frac{\varepsilon_y}{B} \\ -\frac{\varepsilon_x}{B} \end{pmatrix}$$

vector form

Integrating $v(t)$ yields complex coordinate $q=x+iy$ affected by both ε_x and ε_y.

$$q(t) = \int v(t)\,dt = \frac{e^{-iB\cdot t}}{-iB}(v(0)+i\frac{\varepsilon}{B}) - i\frac{\varepsilon}{B}\cdot t + Const. \quad \text{where: } Const. = q(0) - (\frac{v(0)}{-iB} - \frac{\varepsilon}{B^2})$$

complex form

$$x(t)+iy(t) = e^{-iB\cdot t}(i\frac{v(0)}{B} - \frac{\varepsilon}{B^2}) - i\frac{\varepsilon}{B}\cdot t + x(0) + iy(0) - i\frac{v(0)}{B} + \frac{\varepsilon}{B^2}$$

Crossed E and B field mechanics (Solution by complex variables)

$$\boxed{\begin{array}{l} \dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \boldsymbol{\varepsilon} + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z \\ \varepsilon_x = \frac{e}{m}E_x \qquad \varepsilon_y = \frac{e}{m}E_y \qquad B = \frac{e}{m}B_z \\ \text{\textit{Shorthand Labeling}} \end{array}}$$

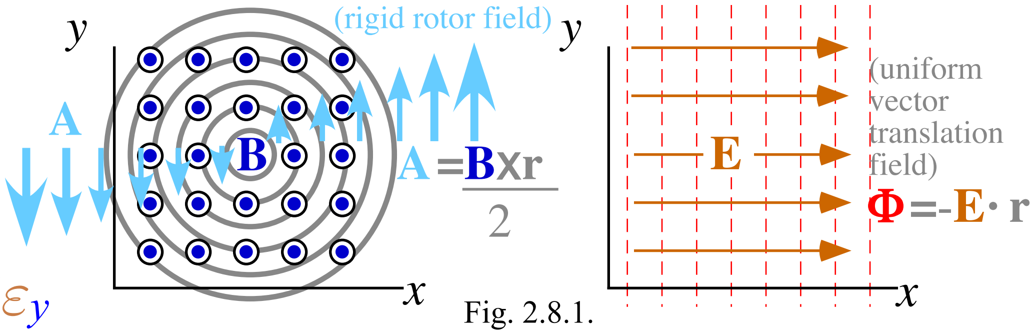


Fig. 2.8.1.

Complex variable velocity: $v=v_x+iv_y$ *and electric field:* $\varepsilon=\varepsilon_x+i\varepsilon_y$

$$\dot v_x + i\dot v_y = \varepsilon_x + i\varepsilon_y - \; iBv_x + Bv_y = \varepsilon_x + i\varepsilon_y - \; iB(v_x + iv_y)$$

$$\dot v \quad = \quad \varepsilon \quad - \quad iBv \qquad \text{with replacements : } \hat{\mathbf{e}}_{\mathbf{x}} \to 1 \qquad \text{and : } \hat{\mathbf{e}}_{\mathbf{y}} \to i = \sqrt{-1}$$

A velocity transformation $V(t)=v(t)+\beta$ cancels constant ε-field to give an equation: $\boxed{\dot V=(const.)V}$

$$\boxed{\dot V(t) = \dot v(t) + \dot\beta = \varepsilon - iBv = \varepsilon - iB(V(t)-\beta) = -iBV(t)} \qquad \text{where : } \boxed{\beta = -\frac{\varepsilon}{iB} = i\frac{\varepsilon}{B}}$$

An exponential $V(t)=e^{-iBt}V(0)$ solution results: e^{-iBt} is a clockwise 2D rotation.

complex form

$$v(t)+\beta = V(t) = e^{-iB\cdot t}V(0) = e^{-iB\cdot t}(v(0)+\beta) \quad \text{or:} \quad v(t) = e^{-iB\cdot t}(v(0)+\beta)-\beta = e^{-iB\cdot t}(v(0)+i\frac{\varepsilon}{B}) - i\frac{\varepsilon}{B}$$

Expanding e^{-iBt}, $v=v_x+iv_y$, and $\varepsilon=\varepsilon_x+i\varepsilon_y$ reveals x (Real) and y (Imaginary) components

$$\begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix} = \begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix} \begin{pmatrix} v_x(0) - \frac{\varepsilon_y}{B} \\ v_y(0) + \frac{\varepsilon_x}{B} \end{pmatrix} + \begin{pmatrix} \frac{\varepsilon_y}{B} \\ -\frac{\varepsilon_x}{B} \end{pmatrix}$$

vector form

Integrating $v(t)$ yields complex coordinate $q=x+iy$ affected by both ε_x and ε_y.

$$q(t) = \int v(t)\,dt = \frac{e^{-iB\cdot t}}{-iB}(v(0)+i\frac{\varepsilon}{B}) - i\frac{\varepsilon}{B}\cdot t + Const. \quad \text{where: } Const. = q(0) - (\frac{v(0)}{-iB} - \frac{\varepsilon}{B^2})$$

complex form

$$x(t)+iy(t) = e^{-iB\cdot t}(i\frac{v(0)}{B} - \frac{\varepsilon}{B^2}) - i\frac{\varepsilon}{B}\cdot t + x(0) + iy(0) - i\frac{v(0)}{B} + \frac{\varepsilon}{B^2}$$

Move last part of this calculation UP↑

Crossed E and B field mechanics (Solution by complex variables)

$$\dot{V}(t)=\dot{v}(t)+\dot{\beta}=\varepsilon-iBv=\varepsilon-iB(V(t)-\beta)=-iBV(t) \qquad \text{where :}\ \boxed{\beta=-\frac{\varepsilon}{iB}=i\frac{\varepsilon}{B}}$$

An exponential $V(t)=e^{-iBt}V(0)$ solution results: e^{-iBt} is a clockwise 2D rotation.

complex form

$$v(t)+\beta=V(t)=e^{-iB\cdot t}V(0)=e^{-iB\cdot t}(v(0)+\beta) \quad \text{or:} \quad v(t)=e^{-iB\cdot t}(v(0)+\beta)-\beta=e^{-iB\cdot t}(v(0)+i\frac{\varepsilon}{B})-i\frac{\varepsilon}{B}$$

Expanding e^{-iBt}, $v=v_x+iv_y$, and $\varepsilon=\varepsilon_x+i\varepsilon_y$ reveals x (Real) and y (Imaginary) components

$$\begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix}=\begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix}\begin{pmatrix} v_x(0)-\frac{\varepsilon_y}{B} \\ v_y(0)+\frac{\varepsilon_x}{B} \end{pmatrix}+\begin{pmatrix} \frac{\varepsilon_y}{B} \\ -\frac{\varepsilon_x}{B} \end{pmatrix}$$

vector form

Integrating $v(t)$ yields complex coordinate $q=x+iy$ affected by both ε_x and ε_y.

$$q(t)=\int v(t)\,dt=\frac{e^{-iB\cdot t}}{-iB}(v(0)+i\frac{\varepsilon}{B})-i\frac{\varepsilon}{B}\cdot t+Const. \quad \text{where: } Const.=q(0)-(\frac{v(0)}{-iB}-\frac{\varepsilon}{B^2})$$

complex form

$$x(t)+iy(t)=e^{-iB\cdot t}(i\frac{v(0)}{B}-\frac{\varepsilon}{B^2})-i\frac{\varepsilon}{B}\cdot t+x(0)+iy(0)-i\frac{v(0)}{B}+\frac{\varepsilon}{B^2}$$

complex form

Crossed E and B field mechanics (Solution by complex variables)

$$\boxed{\dot{V}(t)=\dot{v}(t)+\dot{\beta}=\varepsilon-iBv=\varepsilon-iB(V(t)-\beta)=-iBV(t)} \quad \text{where :} \boxed{\beta=-\frac{\varepsilon}{iB}=i\frac{\varepsilon}{B}}$$

An exponential $V(t)=e^{-iBt}V(0)$ solution results: e^{-iBt} is a clockwise 2D rotation.

complex form

$$v(t)+\beta=V(t)=e^{-iB\cdot t}V(0)=e^{-iB\cdot t}(v(0)+\beta) \quad \text{or:} \quad v(t)=e^{-iB\cdot t}(v(0)+\beta)-\beta=e^{-iB\cdot t}(v(0)+i\frac{\varepsilon}{B})-i\frac{\varepsilon}{B}$$

Expanding e^{-iBt}, $v=v_x+iv_y$, and $\varepsilon=\varepsilon_x+i\varepsilon_y$ reveals x (Real) and y (Imaginary) components

$$\begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix}=\begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix}\begin{pmatrix} v_x(0)-\frac{\varepsilon_y}{B} \\ v_y(0)+\frac{\varepsilon_x}{B} \end{pmatrix}+\begin{pmatrix} \frac{\varepsilon_y}{B} \\ -\frac{\varepsilon_x}{B} \end{pmatrix}$$

vector form

Integrating $v(t)$ yields complex coordinate $q=x+iy$ affected by both ε_x and ε_y.

$$q(t)=\int v(t)\,dt=\frac{e^{-iB\cdot t}}{-iB}(v(0)+i\frac{\varepsilon}{B})-i\frac{\varepsilon}{B}\cdot t+Const. \quad \text{where: } Const.=q(0)-(\frac{v(0)}{-iB}-\frac{\varepsilon}{B^2})$$ *complex form*

$$x(t)+iy(t)=e^{-iB\cdot t}(i\frac{v(0)}{B}-\frac{\varepsilon}{B^2})-i\frac{\varepsilon}{B}\cdot t+x(0)+iy(0)-i\frac{v(0)}{B}+\frac{\varepsilon}{B^2}$$ *complex form*

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}=\begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix}\begin{pmatrix} -\frac{v_y(0)}{B}-\frac{\varepsilon_x}{B^2} \\ \frac{v_x(0)}{B}-\frac{\varepsilon_y}{B^2} \end{pmatrix}+\begin{pmatrix} \frac{\varepsilon_y}{B}t \\ -\frac{\varepsilon_x}{B}t \end{pmatrix}+\begin{pmatrix} x(0)+\frac{v_y(0)}{B}+\frac{\varepsilon_x}{B^2} \\ y(0)-\frac{v_x(0)}{B}+\frac{\varepsilon_y}{B^2} \end{pmatrix}$$ *vector form*

Crossed E and B field mechanics (Solution by complex variables)

$$\dot{V}(t)=\dot{v}(t)+\dot{\beta}=\varepsilon-iBv=\varepsilon-iB(V(t)-\beta)=-iBV(t) \qquad \text{where :} \boxed{\beta=-\frac{\varepsilon}{iB}=i\frac{\varepsilon}{B}}$$

An exponential $V(t)=e^{-iBt}V(0)$ solution results: e^{-iBt} is a clockwise 2D rotation.

$$v(t)+\beta=V(t)=e^{-iB\cdot t}V(0)=e^{-iB\cdot t}(v(0)+\beta) \quad \text{or:} \quad v(t)=e^{-iB\cdot t}(v(0)+\beta)-\beta=e^{-iB\cdot t}(v(0)+i\frac{\varepsilon}{B})-i\frac{\varepsilon}{B}$$

complex form

Expanding e^{-iBt}, $v=v_x+iv_y$, and $\varepsilon=\varepsilon_x+i\varepsilon_y$ reveals x (Real) and y (Imaginary) components

$$\begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix}=\begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix}\begin{pmatrix} v_x(0)-\frac{\varepsilon_y}{B} \\ v_y(0)+\frac{\varepsilon_x}{B} \end{pmatrix}+\begin{pmatrix} \frac{\varepsilon_y}{B} \\ -\frac{\varepsilon_x}{B} \end{pmatrix}$$

vector form

Integrating $v(t)$ yields complex coordinate $q=x+iy$ affected by both ε_x and ε_y.

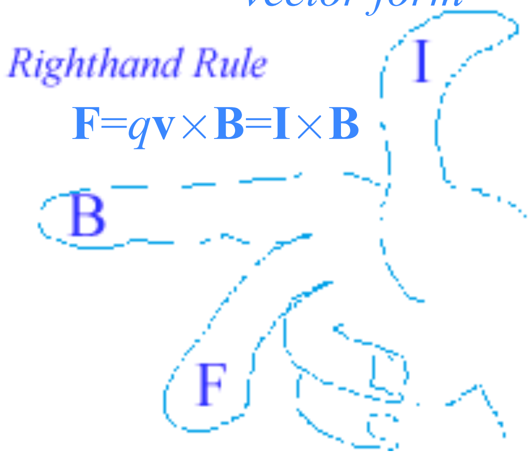
$$q(t)=\int v(t)\,dt=\frac{e^{-iB\cdot t}}{-iB}(v(0)+i\frac{\varepsilon}{B})-i\frac{\varepsilon}{B}\cdot t+Const. \quad \text{where: } Const.=q(0)-(\frac{v(0)}{-iB}-\frac{\varepsilon}{B^2})$$

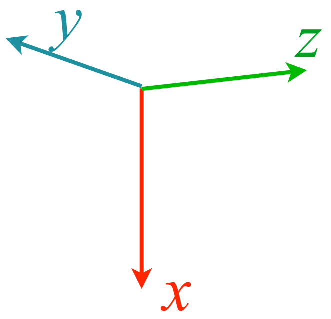
complex form

$$x(t)+iy(t)= \quad e^{-iB\cdot t} \quad (i\frac{v(0)}{B}-\frac{\varepsilon}{B^2}) \quad -i\frac{\varepsilon}{B}\cdot t \quad +x(0)+iy(0)-i\frac{v(0)}{B}+\frac{\varepsilon}{B^2}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}=\begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix}\begin{pmatrix} -\frac{v_y(0)}{B}-\frac{\varepsilon_x}{B^2} \\ \frac{v_x(0)}{B}-\frac{\varepsilon_y}{B^2} \end{pmatrix}+\begin{pmatrix} \frac{\varepsilon_y}{B}t \\ -\frac{\varepsilon_x}{B}t \end{pmatrix}+\begin{pmatrix} x(0)+\frac{v_y(0)}{B}+\frac{\varepsilon_x}{B^2} \\ y(0)-\frac{v_x(0)}{B}+\frac{\varepsilon_y}{B^2} \end{pmatrix}$$

vector form





Crossed E and B field mechanics (Solution by complex variables)

$$\boxed{\dot{V}(t)=\dot{v}(t)+\dot{\beta}=\varepsilon-iBv=\varepsilon-iB(V(t)-\beta)=-iBV(t)} \quad \text{where :} \boxed{\beta=-\frac{\varepsilon}{iB}=i\frac{\varepsilon}{B}}$$

An exponential $V(t)=e^{-iBt}V(0)$ solution results: e^{-iBt} is a clockwise 2D rotation.

$$v(t)+\beta=V(t)=e^{-iB\cdot t}V(0)=e^{-iB\cdot t}(v(0)+\beta) \quad \text{or:} \quad v(t)=e^{-iB\cdot t}(v(0)+\beta)-\beta=e^{-iB\cdot t}(v(0)+i\frac{\varepsilon}{B})-i\frac{\varepsilon}{B}$$

complex form

Expanding e^{-iBt}, $v=v_x+iv_y$, and $\varepsilon=\varepsilon_x+i\varepsilon_y$ reveals x (Real) and y (Imaginary) components

$$\begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix}=\begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix}\begin{pmatrix} v_x(0)-\frac{\varepsilon_y}{B} \\ v_y(0)+\frac{\varepsilon_x}{B} \end{pmatrix}+\begin{pmatrix} \frac{\varepsilon_y}{B} \\ -\frac{\varepsilon_x}{B} \end{pmatrix}$$

vector form

Integrating $v(t)$ yields complex coordinate $q=x+iy$ affected by both ε_x and ε_y.

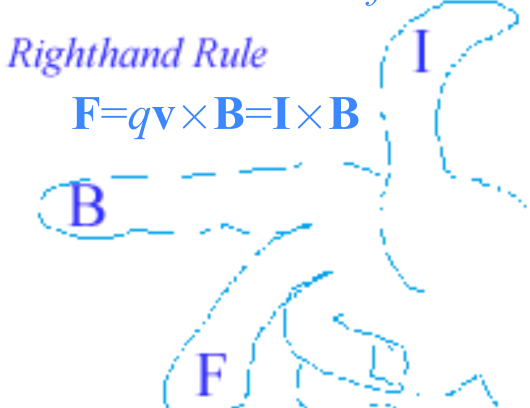
$$q(t)=\int v(t)\,dt=\frac{e^{-iB\cdot t}}{-iB}(v(0)+i\frac{\varepsilon}{B})-i\frac{\varepsilon}{B}\cdot t+Const. \quad \text{where: } Const.=q(0)-(\frac{v(0)}{-iB}-\frac{\varepsilon}{B^2})$$

complex form

$$x(t)+iy(t)= \qquad e^{-iB\cdot t} \qquad (i\frac{v(0)}{B}-\frac{\varepsilon}{B^2}) \qquad -i\frac{\varepsilon}{B}\cdot t \qquad +x(0)+iy(0)-i\frac{v(0)}{B}+\frac{\varepsilon}{B^2}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}=\begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix}\begin{pmatrix} -\frac{v_y(0)}{B}-\frac{\varepsilon_x}{B^2} \\ \frac{v_x(0)}{B}-\frac{\varepsilon_y}{B^2} \end{pmatrix}+\begin{pmatrix} \frac{\varepsilon_y}{B}t \\ -\frac{\varepsilon_x}{B}t \end{pmatrix}+\begin{pmatrix} x(0)+\frac{v_y(0)}{B}+\frac{\varepsilon_x}{B^2} \\ y(0)-\frac{v_x(0)}{B}+\frac{\varepsilon_y}{B^2} \end{pmatrix}$$

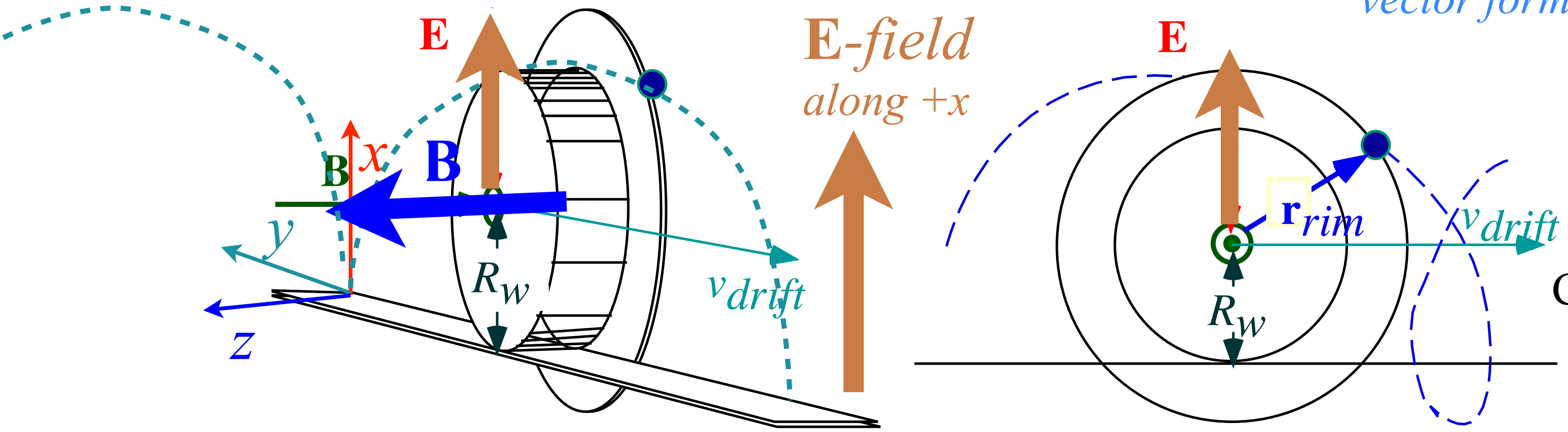
vector form



Cycloid example:
initial $(x(0), y(0))=(0,0)$
and $(v_x(0), v_y(0))=(0,0)$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}= \text{is on rim of a wheel of radius } R_W=E/B^2$$

$$\begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix}\begin{pmatrix} -\frac{E}{B^2} \\ 0 \end{pmatrix}+\begin{pmatrix} 0 \\ -\frac{E}{B}t \end{pmatrix}+\begin{pmatrix} \frac{E}{B^2} \\ 0 \end{pmatrix}$$



Constant v_{drift} in -y-direction

$v_{drift}=-E/B$

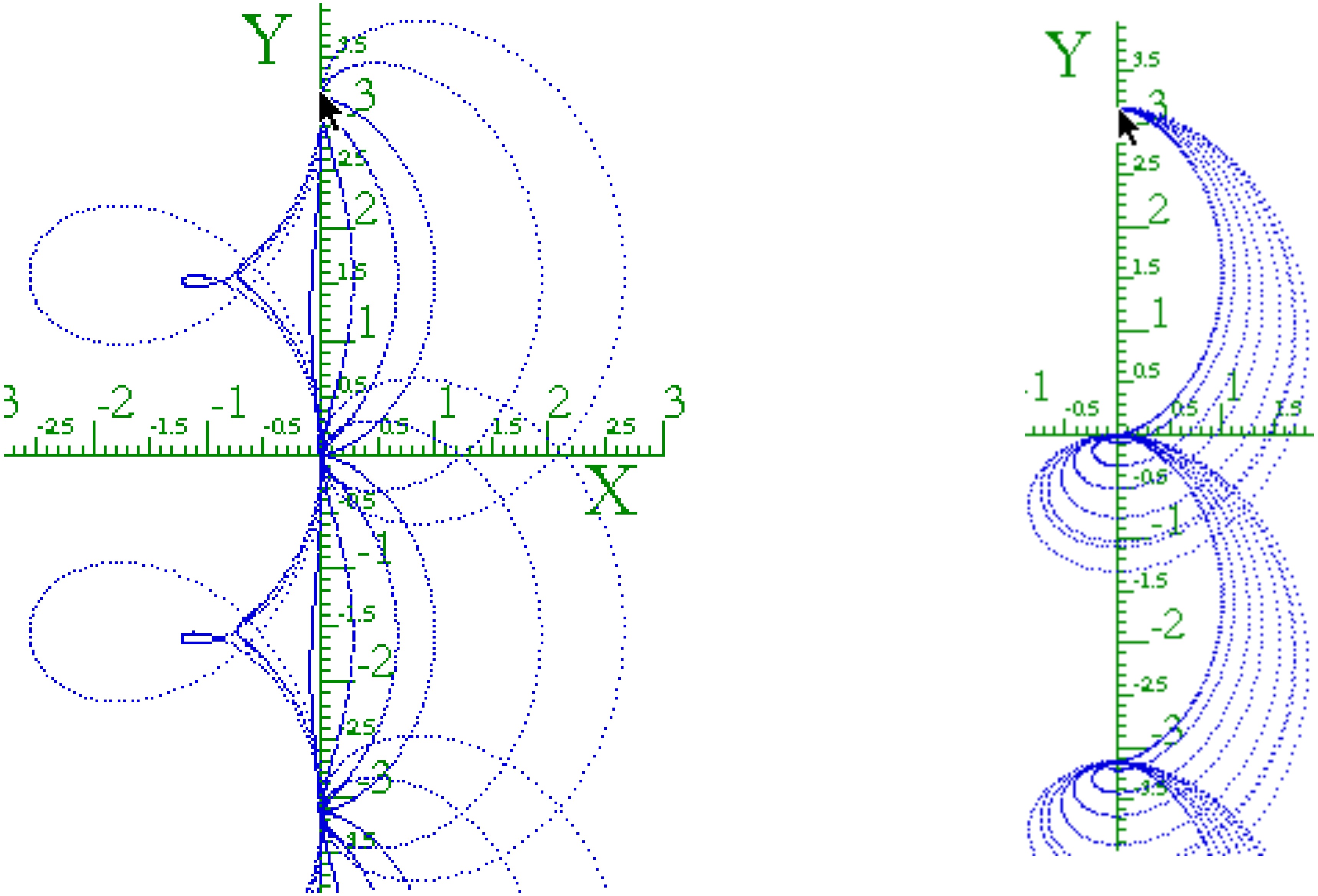
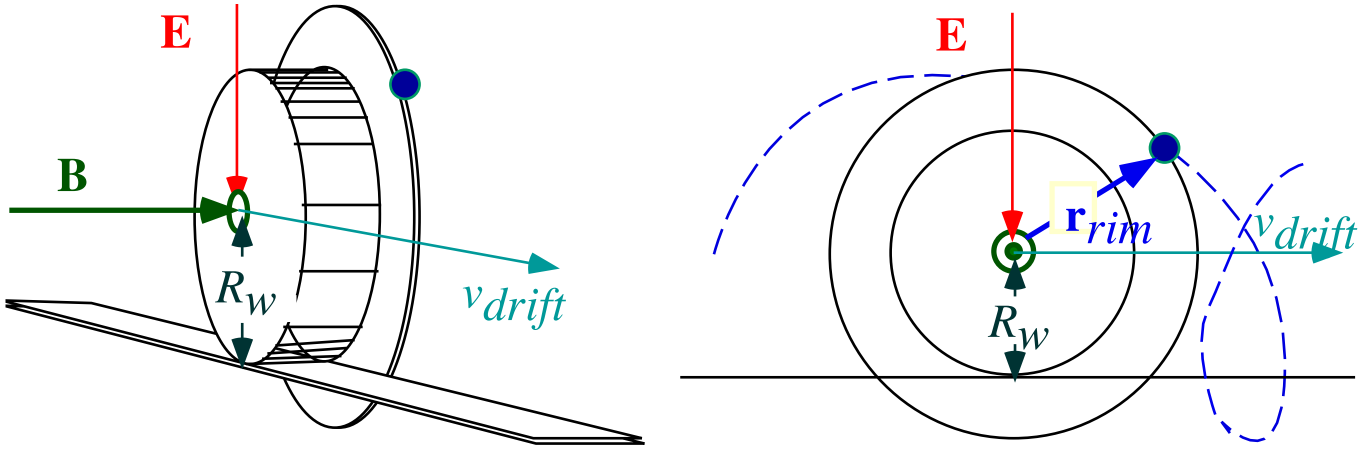
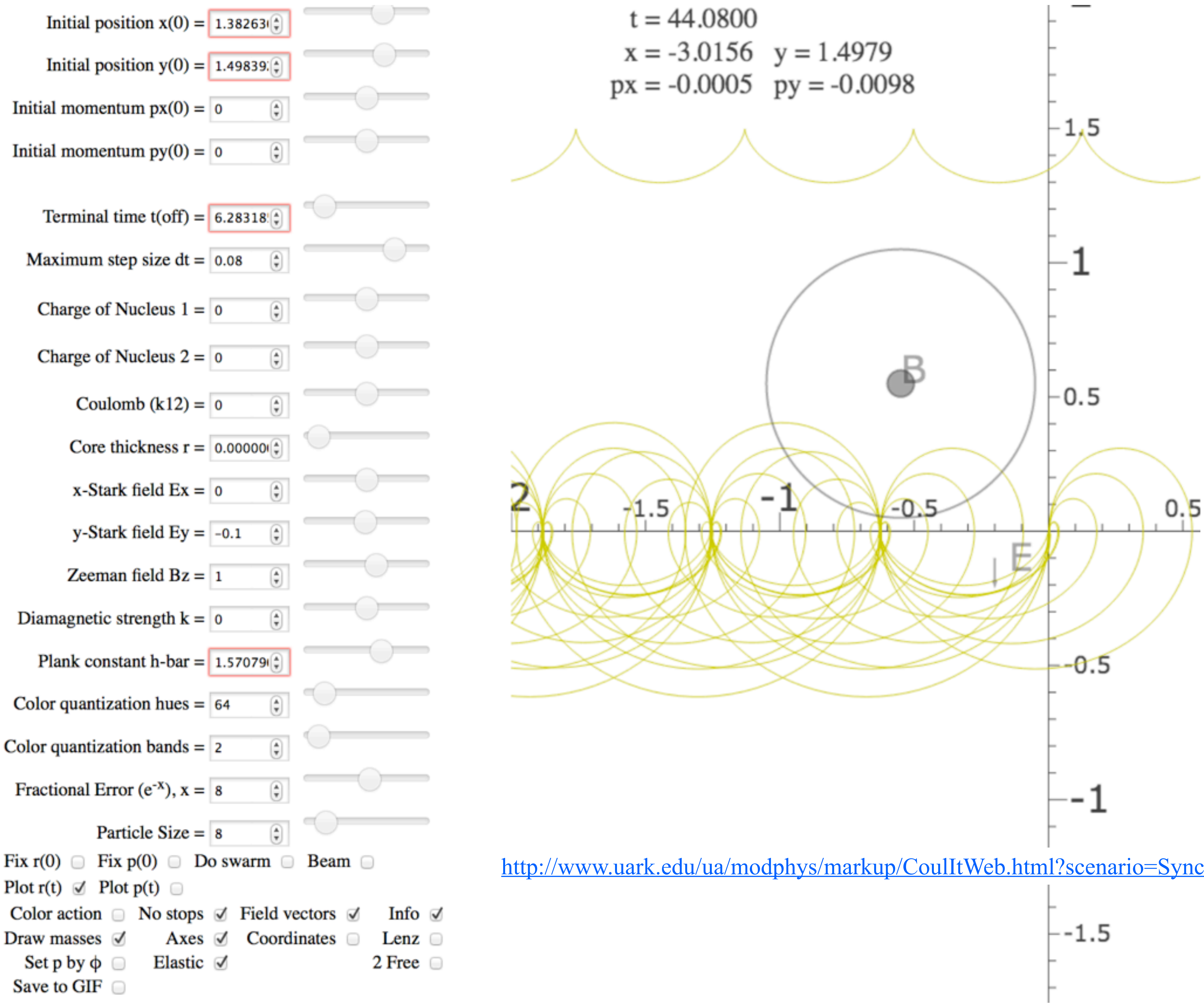


Fig. 2.8.2 Trajectories of unit charge and mass in magnetic and electric fields (*E=1/2, B=1*)

Fig. 2.8.3 Rolling railroad wheel and rim analogy for cyclotron orbits





Initial position x(0) = 1.38263

Initial position y(0) = 1.49839

Initial momentum px(0) = 0

Initial momentum py(0) = 0

Terminal time t(off) = 6.28318

Maximum step size dt = 0.08

Charge of Nucleus 1 = 0

Charge of Nucleus 2 = 0

Coulomb (k12) = 0

Core thickness r = 0.00000

x-Stark field Ex = 0

y-Stark field Ey = -0.1

Zeeman field Bz = 1

Diamagnetic strength k = 0

Plank constant h-bar = 1.57079

Color quantization hues = 64

Color quantization bands = 2

Fractional Error (e^{-x}), x = 8

Particle Size = 8

Fix r(0) ☐ Fix p(0) ☐ Do swarm ☐ Beam ☐

Plot r(t) ☑ Plot p(t) ☐

Color action ☐ No stops ☑ Field vectors ☑ Info ☑

Draw masses ☑ Axes ☑ Coordinates ☐ Lenz ☐

Set p by ϕ ☐ Elastic ☑ 2 Free ☐

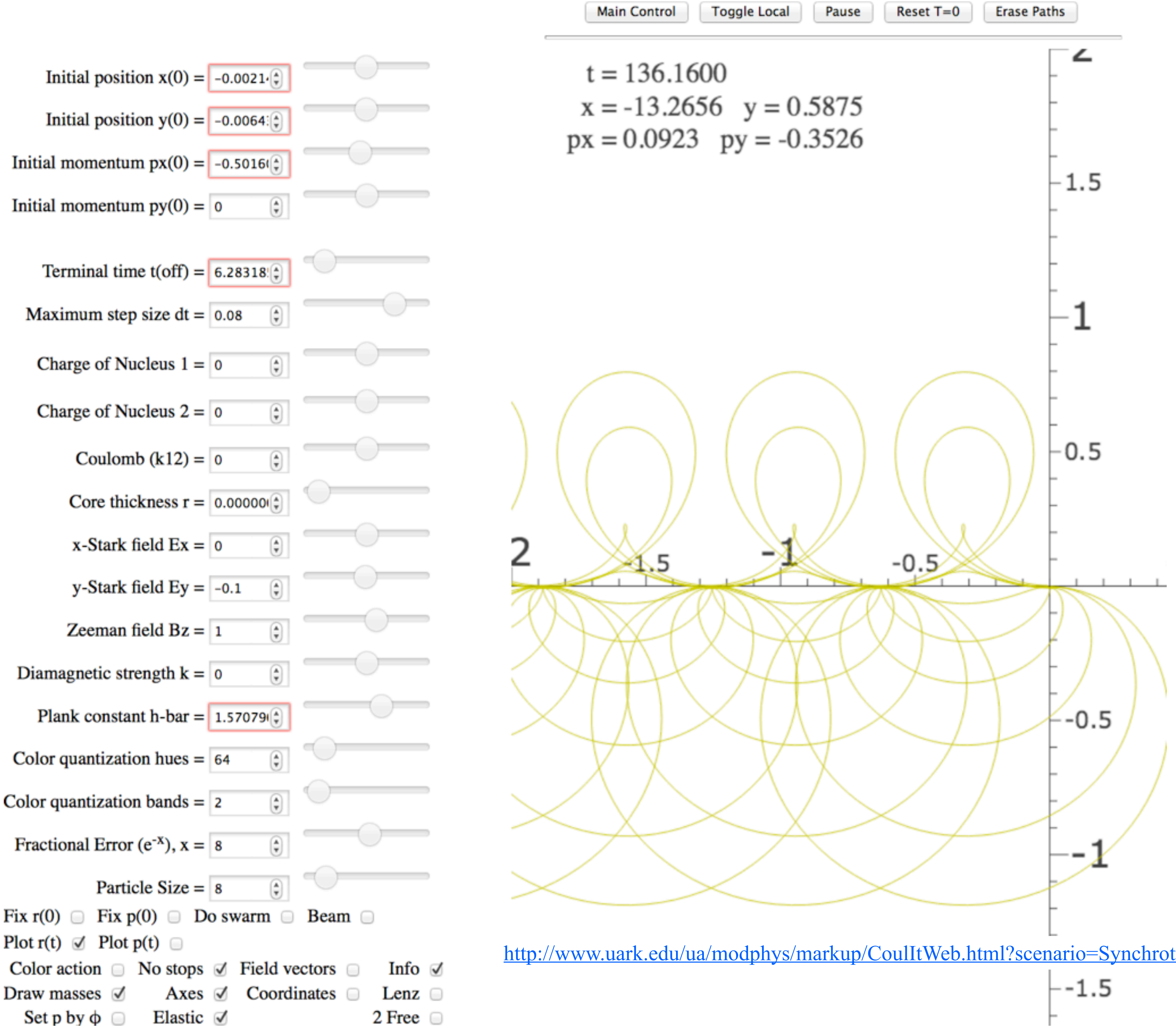
Save to GIF ☐

t = 44.0800

x = -3.0156 y = 1.4979

px = -0.0005 py = -0.0098

http://www.uark.edu/ua/modphys/markup/CoulItWeb.html?scenario=SynchrotronMotion



Initial position x(0) = -0.0021

Initial position y(0) = -0.0064

Initial momentum px(0) = -0.5016

Initial momentum py(0) = 0

Terminal time t(off) = 6.28318

Maximum step size dt = 0.08

Charge of Nucleus 1 = 0

Charge of Nucleus 2 = 0

Coulomb (k12) = 0

Core thickness r = 0.00000

x-Stark field Ex = 0

y-Stark field Ey = -0.1

Zeeman field Bz = 1

Diamagnetic strength k = 0

Plank constant h-bar = 1.57079

Color quantization hues = 64

Color quantization bands = 2

Fractional Error (e^{-x}), x = 8

Particle Size = 8

Fix r(0) ☐ Fix p(0) ☐ Do swarm ☐ Beam ☐

Plot r(t) ☑ Plot p(t) ☐

Color action ☐ No stops ☑ Field vectors ☐ Info ☑

Draw masses ☑ Axes ☑ Coordinates ☐ Lenz ☐

Set p by φ ☐ Elastic ☑ 2 Free ☐

t = 136.1600

x = -13.2656 y = 0.5875

px = 0.0923 py = -0.3526

http://www.uark.edu/ua/modphys/markup/CoulItWeb.html?scenario=SynchrotronMotion2

Crossed E and B field mechanics

Classical Hall-effect and cyclotron orbit orbit equations

Vector theory vs. complex variable theory

→*Mechanical analog of cyclotron and FBI rule*

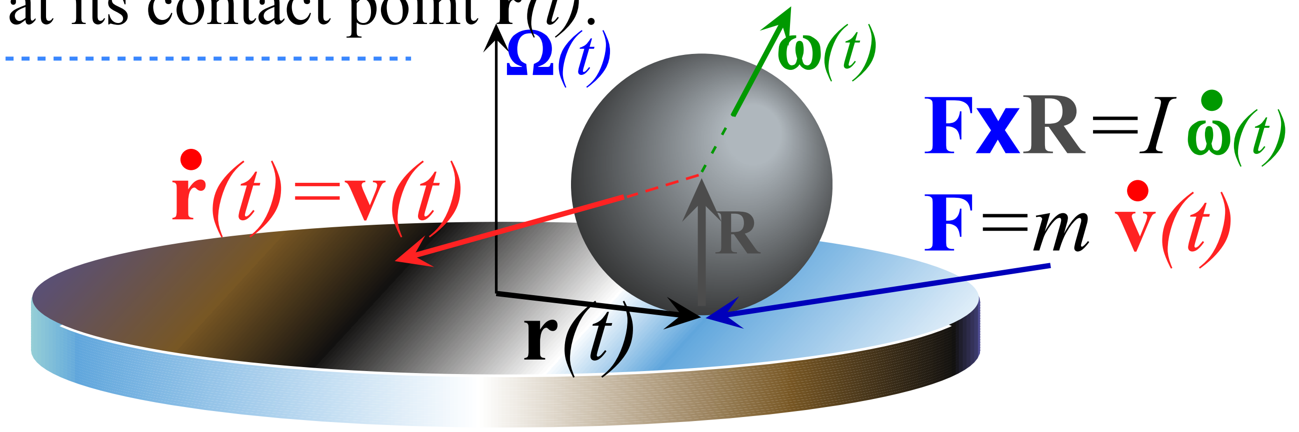
Cycloid and epicycloid ruler&compass geometry

Cycloid geometry of flying levers

Practical poolhall application

Mechanical analog of cyclotron and FBI rule

Velocity vector of the ball contact point ($\mathbf{v}(t)-\omega(t)\times\mathbf{R}$) equals table surface velocity $\boldsymbol{\Omega}\times\mathbf{r}(t)$ at its contact point $\mathbf{r}(t)$.

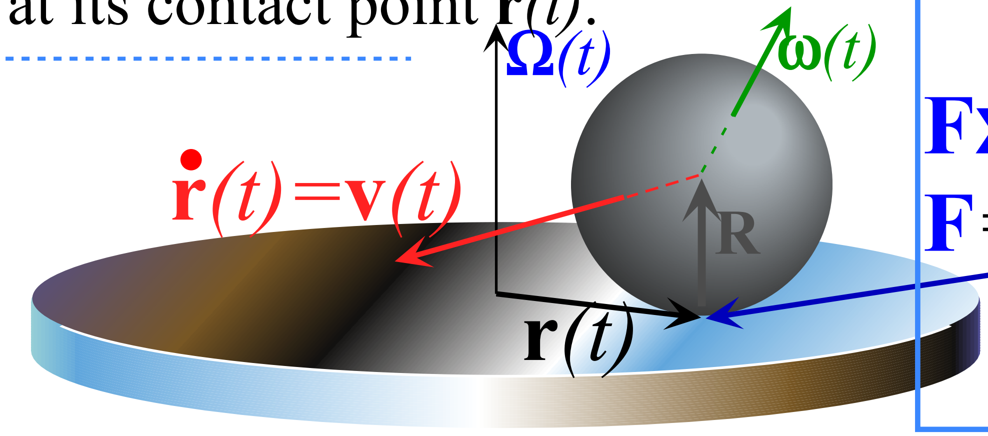


Turntable turning at constant angular velocity $\boldsymbol{\Omega}=\Omega\hat{\mathbf{z}}$.

[YouTube Video of Analog to Syncrotron Motion]

Mechanical analog of cyclotron and FBI rule

Velocity vector of the ball contact point ($\mathbf{v}(t)$-$\omega(t)\times\mathbf{R}$) equals table surface velocity $\mathbf{\Omega}\times\mathbf{r}(t)$ at its contact point $\mathbf{r}(t)$.



Equations of Motion:

rotation $Torque = \mathbf{F}\times\mathbf{R} = I\dot{\omega}$

$$\mathbf{F}\mathbf{x}\mathbf{R}=I\,\dot{\omega}(t)$$

$$\mathbf{F}=m\,\dot{\mathbf{v}}(t)$$

translation $Force=\mathbf{F} = m\dot{\mathbf{v}}$

Turntable turning at constant angular velocity $\mathbf{\Omega}=\Omega\hat{\mathbf{z}}$.

*Torque-and-*F=ma
equations of motion:

$$I\dot{\omega}(t)=\mathbf{F}(t)\times\mathbf{R}$$

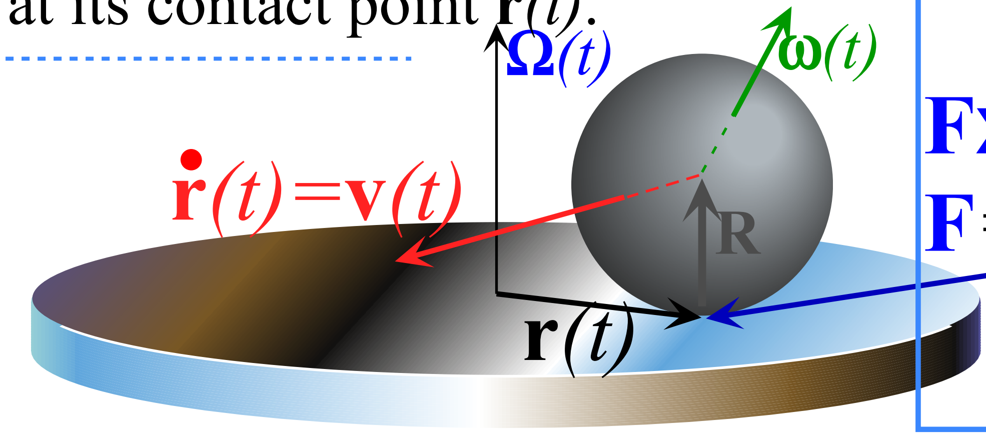
$$=m\dot{\mathbf{v}}(t)\times\mathbf{R}$$

$$=m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R$$

Mechanical analog of cyclotron and FBI rule

Velocity vector of the ball contact point ($\mathbf{v}(t)$-$\boldsymbol{\omega}(t)\times\mathbf{R}$) equals table surface velocity $\boldsymbol{\Omega}\times\mathbf{r}(t)$ at its contact point $\mathbf{r}(t)$.

Rolling Constraint



Equations of Motion:

rotation $Torque = \mathbf{F}\times\mathbf{R} = I\dot{\boldsymbol{\omega}}$

$\mathbf{F}\mathbf{x}\mathbf{R}=I\,\dot{\boldsymbol{\omega}}(t)$

$\mathbf{F}=m\,\dot{\mathbf{v}}(t)$

translation $Force=\mathbf{F} = m\dot{\mathbf{v}}$

Turntable turning at constant angular velocity $\boldsymbol{\Omega} = \Omega\hat{\mathbf{z}}$.

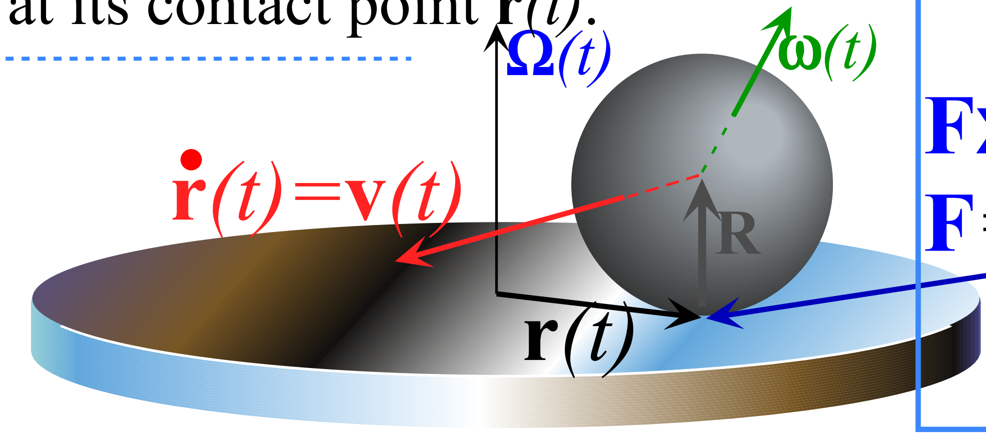
No-slipping: $\mathbf{v}(t)$-$\boldsymbol{\omega}(t)\times\mathbf{R} = \boldsymbol{\Omega}\times\mathbf{r}(t)$ *(where:* $\mathbf{R}=R\hat{\mathbf{z}}$ and $\boldsymbol{\Omega} = \Omega\hat{\mathbf{z}}$ *are constant.)*

*Torque-and-*F=ma
equations of motion:

$$
\begin{aligned}
I\dot{\boldsymbol{\omega}}(t) &= \mathbf{F}(t)\times\mathbf{R} \\
&= m\dot{\mathbf{v}}(t)\times\mathbf{R} \\
&= m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R
\end{aligned}
$$

Mechanical analog of cyclotron and FBI rule

Velocity vector of the ball contact point ($\mathbf{v}(t)$-$\boldsymbol{\omega}(t)\times\mathbf{R}$) equals table surface velocity $\boldsymbol{\Omega}\times\mathbf{r}(t)$ at its contact point $\mathbf{r}(t)$.



Equations of Motion:

rotation *Torque* $= \mathbf{F}\times\mathbf{R} = I\dot{\boldsymbol{\omega}}$

$\mathbf{F}\mathbf{x}\mathbf{R}=I\,\dot{\boldsymbol{\omega}}(t)$

$\mathbf{F}=m\,\dot{\mathbf{v}}(t)$

translation *Force*$=\mathbf{F} = m\dot{\mathbf{v}}$

Rolling Constraint

Turntable turning at constant angular velocity $\boldsymbol{\Omega} = \Omega\hat{\mathbf{z}}$.

No-slipping: $\mathbf{v}(t)$-$\boldsymbol{\omega}(t)\times\mathbf{R} = \boldsymbol{\Omega}\times\mathbf{r}(t)$ *(where:* $\mathbf{R}=R\hat{\mathbf{z}}$ and $\boldsymbol{\Omega} = \Omega\hat{\mathbf{z}}$ *are constant.)*

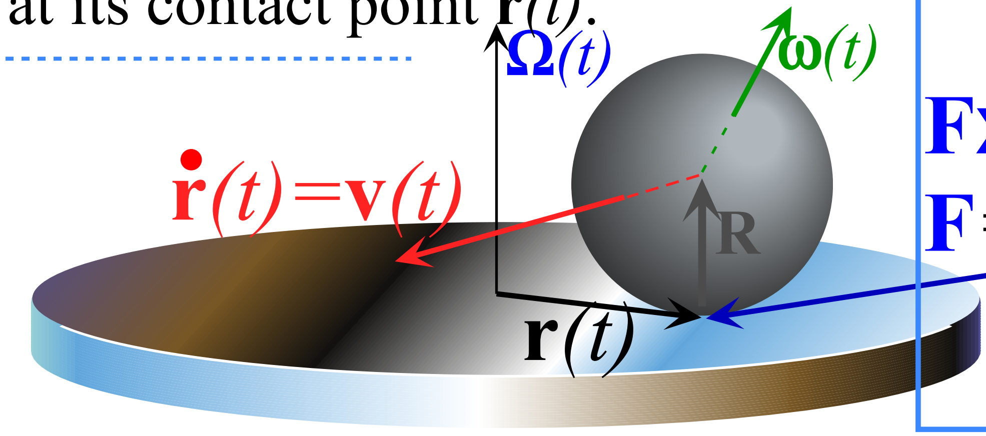
$$\mathbf{v}(t)=\boldsymbol{\Omega}\times\mathbf{r}(t)+\boldsymbol{\omega}(t)\times\mathbf{R} \quad =\boldsymbol{\Omega}\times\mathbf{r}(t)+\boldsymbol{\omega}(t)\times\hat{\mathbf{z}}R$$

*Torque-and-*F=ma
equations of motion:

$$I\dot{\boldsymbol{\omega}}(t)=\mathbf{F}(t)\times\mathbf{R}$$
$$=m\dot{\mathbf{v}}(t)\times\mathbf{R}$$
$$=m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R$$

Mechanical analog of cyclotron and FBI rule

Velocity vector of the ball contact point ($\mathbf{v}(t)$-$\boldsymbol{\omega}(t)\times\mathbf{R}$) equals table surface velocity $\boldsymbol{\Omega}\times\mathbf{r}(t)$ at its contact point $\mathbf{r}(t)$.



Equations of Motion:

rotation $Torque = \mathbf{F}\times\mathbf{R} = I\dot{\boldsymbol{\omega}}$

$\mathbf{F}\mathbf{x}\mathbf{R}=I\,\dot{\boldsymbol{\omega}}(t)$

$\mathbf{F}=m\,\dot{\mathbf{v}}(t)$

translation $Force=\mathbf{F} = m\dot{\mathbf{v}}$

Rolling Constraint

Turntable turning at constant angular velocity $\boldsymbol{\Omega} = \Omega\hat{\mathbf{z}}$.

No-slipping: $\mathbf{v}(t)$-$\boldsymbol{\omega}(t)\times\mathbf{R} = \boldsymbol{\Omega}\times\mathbf{r}(t)$ *(where:* $\mathbf{R}=R\hat{\mathbf{z}}$ and $\boldsymbol{\Omega} = \Omega\hat{\mathbf{z}}$ *are constant.)*

$$\mathbf{v}(t)=\boldsymbol{\Omega}\times\mathbf{r}(t)+\boldsymbol{\omega}(t)\times\mathbf{R} \quad =\boldsymbol{\Omega}\times\mathbf{r}(t)+\boldsymbol{\omega}(t)\times\hat{\mathbf{z}}R$$

Do time-derivative

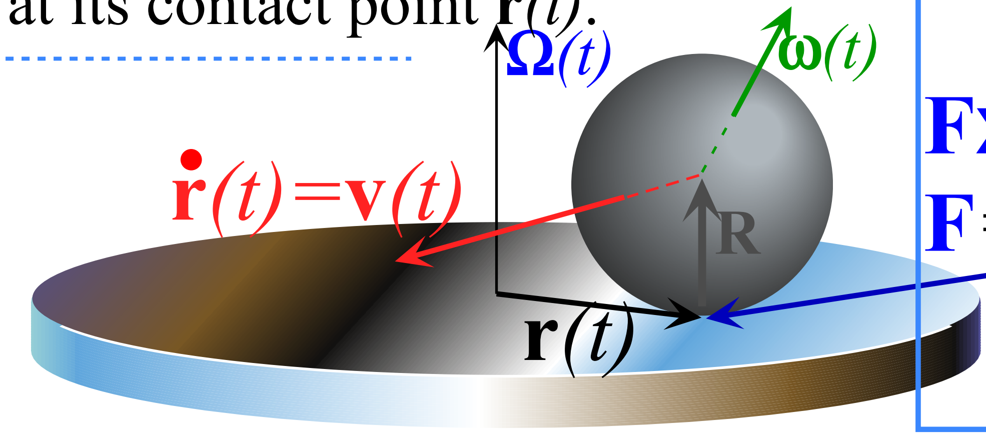
$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\dot{\mathbf{r}}(t)+\dot{\boldsymbol{\omega}}(t)\times\hat{\mathbf{z}}R=\boldsymbol{\Omega}\times\mathbf{v}(t)+\dot{\boldsymbol{\omega}}(t)\times\hat{\mathbf{z}}R$$

*Torque-and-*F=ma
equations of motion:

$$I\dot{\boldsymbol{\omega}}(t)=\mathbf{F}(t)\times\mathbf{R}$$
$$=m\dot{\mathbf{v}}(t)\times\mathbf{R}$$
$$=m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R$$

Mechanical analog of cyclotron and FBI rule

Velocity vector of the ball contact point ($\mathbf{v}(t)-\boldsymbol{\omega}(t)\times\mathbf{R}$) equals table surface velocity $\boldsymbol{\Omega}\times\mathbf{r}(t)$ at its contact point $\mathbf{r}(t)$.



Equations of Motion:

rotation $Torque = \mathbf{F}\times\mathbf{R} = I\dot{\boldsymbol{\omega}}$

$$\mathbf{F}\mathbf{x}\mathbf{R}=I\,\dot{\boldsymbol{\omega}}(t)$$

$$\mathbf{F}=m\,\dot{\mathbf{v}}(t)$$

translation $Force=\mathbf{F} = m\dot{\mathbf{v}}$

Rolling Constraint

Turntable turning at constant angular velocity $\boldsymbol{\Omega}=\Omega\hat{\mathbf{z}}$.

No-slipping: $\mathbf{v}(t)-\boldsymbol{\omega}(t)\times\mathbf{R}=\boldsymbol{\Omega}\times\mathbf{r}(t)$ *(where:* $\mathbf{R}=R\hat{\mathbf{z}}$ and $\boldsymbol{\Omega}=\Omega\hat{\mathbf{z}}$ *are constant.)*

$$\mathbf{v}(t)=\boldsymbol{\Omega}\times\mathbf{r}(t)+\boldsymbol{\omega}(t)\times\mathbf{R}\quad=\boldsymbol{\Omega}\times\mathbf{r}(t)+\boldsymbol{\omega}(t)\times\hat{\mathbf{z}}R$$

Do time-derivative

$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\dot{\mathbf{r}}(t)+\dot{\boldsymbol{\omega}}(t)\times\hat{\mathbf{z}}R=\boldsymbol{\Omega}\times\mathbf{v}(t)+\dot{\boldsymbol{\omega}}(t)\times\hat{\mathbf{z}}R$$

$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\dot{\mathbf{r}}(t)\;+\;\dot{\boldsymbol{\omega}}(t)\quad\times\hat{\mathbf{z}}R$$

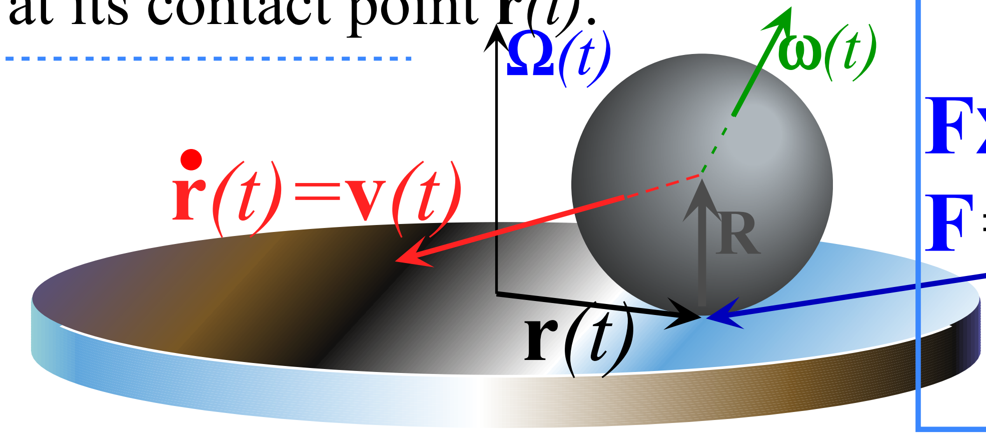
use: $\dot{\boldsymbol{\omega}}(t)=\dfrac{m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R}{I}$

*Torque-and-*F=ma
equations of motion:

$$I\dot{\boldsymbol{\omega}}(t)=\mathbf{F}(t)\times\mathbf{R}$$
$$=m\dot{\mathbf{v}}(t)\times\mathbf{R}$$
$$=m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R$$

Mechanical analog of cyclotron and FBI rule

Velocity vector of the ball contact point ($\mathbf{v}(t)$-$\boldsymbol{\omega}(t)\times\mathbf{R}$) equals table surface velocity $\boldsymbol{\Omega}\times\mathbf{r}(t)$ at its contact point $\mathbf{r}(t)$.



Equations of Motion:

rotation *Torque* $=\mathbf{F}\times\mathbf{R}=I\dot{\boldsymbol{\omega}}$

$$\mathbf{F}\mathbf{x}\mathbf{R}=I\,\dot{\boldsymbol{\omega}}(t)$$

$$\mathbf{F}=m\,\dot{\mathbf{v}}(t)$$

translation *Force*$=\mathbf{F}=m\dot{\mathbf{v}}$

Rolling Constraint

Turntable turning at constant angular velocity $\boldsymbol{\Omega}=\Omega\hat{\mathbf{z}}$.

No-slipping: $\mathbf{v}(t)$-$\boldsymbol{\omega}(t)\times\mathbf{R}=\boldsymbol{\Omega}\times\mathbf{r}(t)$ *(where:* $\mathbf{R}=R\hat{\mathbf{z}}$ and $\boldsymbol{\Omega}=\Omega\hat{\mathbf{z}}$ *are constant.)*

$$\mathbf{v}(t)=\boldsymbol{\Omega}\times\mathbf{r}(t)+\boldsymbol{\omega}(t)\times\mathbf{R}\quad=\boldsymbol{\Omega}\times\mathbf{r}(t)+\boldsymbol{\omega}(t)\times\hat{\mathbf{z}}R$$

Do time-derivative

$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\dot{\mathbf{r}}(t)+\dot{\boldsymbol{\omega}}(t)\times\hat{\mathbf{z}}R=\boldsymbol{\Omega}\times\mathbf{v}(t)+\dot{\boldsymbol{\omega}}(t)\times\hat{\mathbf{z}}R$$

*Torque-and-*F=ma *equations of motion:*

$$I\dot{\boldsymbol{\omega}}(t)=\mathbf{F}(t)\times\mathbf{R}$$

$$=m\dot{\mathbf{v}}(t)\times\mathbf{R}$$

$$=m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R$$

$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\dot{\mathbf{r}}(t)\;+\quad\dot{\boldsymbol{\omega}}(t)\quad\times\hat{\mathbf{z}}R$$

use: $\dot{\boldsymbol{\omega}}(t)=\dfrac{m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R}{I}$

$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\mathbf{v}(t)+\frac{m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R}{I}\times\hat{\mathbf{z}}R$$

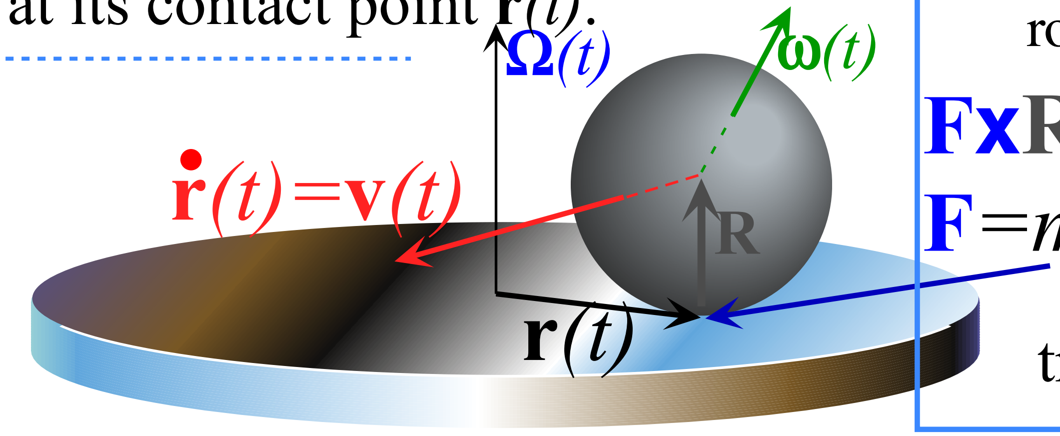
use: $(\mathbf{B}\times\mathbf{C})\times\mathbf{A}=(\mathbf{A}\bullet\mathbf{B})\mathbf{C}-(\mathbf{A}\bullet\mathbf{C})\mathbf{B}$

with: $\mathbf{B}=\dfrac{m\dot{\mathbf{v}}(t)}{I}$ and: $\mathbf{A}=\hat{\mathbf{z}}R=\mathbf{C}$

Mechanical analog of cyclotron and FBI rule

Velocity vector of the ball contact point ($\mathbf{v}(t)$-$\boldsymbol{\omega}(t)\times\mathbf{R}$) equals table surface velocity $\boldsymbol{\Omega}\times\mathbf{r}(t)$ at its contact point $\mathbf{r}(t)$.

Rolling Constraint



Equations of Motion:

rotation $Torque = \mathbf{F}\times\mathbf{R} = I\dot{\boldsymbol{\omega}}$

$$\mathbf{F}\mathbf{x}\mathbf{R}=I\,\dot{\boldsymbol{\omega}}(t)$$

$$\mathbf{F}=m\,\dot{\mathbf{v}}(t)$$

translation $Force=\mathbf{F} = m\dot{\mathbf{v}}$

Turntable turning at constant angular velocity $\boldsymbol{\Omega}=\Omega\hat{\mathbf{z}}$.

No-slipping: $\mathbf{v}(t)$-$\boldsymbol{\omega}(t)\times\mathbf{R}=\boldsymbol{\Omega}\times\mathbf{r}(t)$ *(where:* $\mathbf{R}=R\hat{\mathbf{z}}$ and $\boldsymbol{\Omega}=\Omega\hat{\mathbf{z}}$ *are constant.)*

$$\mathbf{v}(t)=\boldsymbol{\Omega}\times\mathbf{r}(t)+\boldsymbol{\omega}(t)\times\mathbf{R}\quad=\boldsymbol{\Omega}\times\mathbf{r}(t)+\boldsymbol{\omega}(t)\times\hat{\mathbf{z}}R$$

Do time-derivative

*Torque-and-*F=ma *equations of motion:*

$$I\dot{\boldsymbol{\omega}}(t)=\mathbf{F}(t)\times\mathbf{R}$$
$$=m\dot{\mathbf{v}}(t)\times\mathbf{R}$$
$$=m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R$$

$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\dot{\mathbf{r}}(t)+\dot{\boldsymbol{\omega}}(t)\times\hat{\mathbf{z}}R=\boldsymbol{\Omega}\times\mathbf{v}(t)+\dot{\boldsymbol{\omega}}(t)\times\hat{\mathbf{z}}R$$

$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\dot{\mathbf{r}}(t)\;+\quad\dot{\boldsymbol{\omega}}(t)\quad\times\hat{\mathbf{z}}R$$

use: $\dot{\boldsymbol{\omega}}(t)=\dfrac{m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R}{I}$

$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\mathbf{v}(t)+\frac{m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R}{I}\times\hat{\mathbf{z}}R$$

use: $(\mathbf{B}\times\mathbf{C})\times\mathbf{A}=(\mathbf{A}\bullet\mathbf{B})\mathbf{C}-(\mathbf{A}\bullet\mathbf{C})\mathbf{B}$

with: $\mathbf{B}=\dfrac{m\dot{\mathbf{v}}(t)}{I}$ and: $\mathbf{A}=\hat{\mathbf{z}}R=\mathbf{C}$

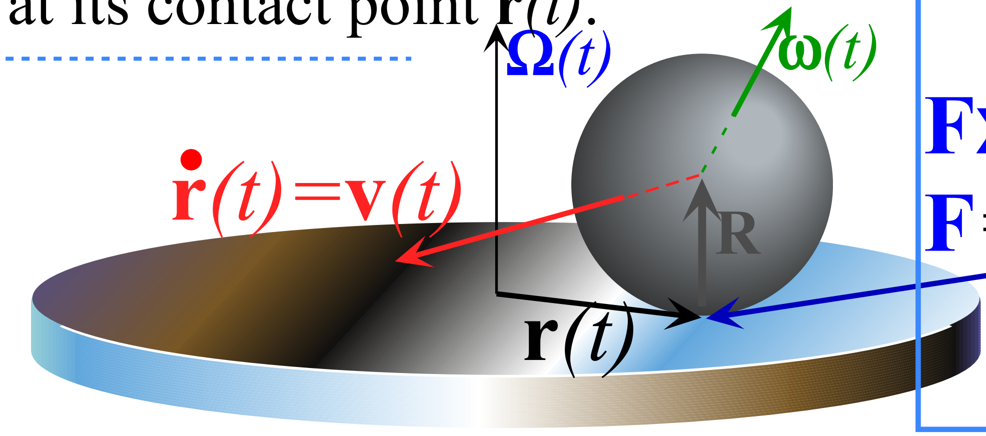
$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\mathbf{v}(t)+\frac{m\dot{\mathbf{v}}(t)\bullet\hat{\mathbf{z}}R}{I}\hat{\mathbf{z}}R-\frac{mR^2}{I}\dot{\mathbf{v}}(t)$$

($\mathbf{v}(t)$ *always normal to* $\hat{\mathbf{z}}$)

Mechanical analog of cyclotron and FBI rule

Velocity vector of the ball contact point ($\mathbf{v}(t)$-$\omega(t)\times\mathbf{R}$) equals table surface velocity $\boldsymbol{\Omega}\times\mathbf{r}(t)$ at its contact point $\mathbf{r}(t)$.

Rolling Constraint



Equations of Motion:

rotation *Torque* $=\mathbf{F}\times\mathbf{R}=I\dot{\omega}$

$$\mathbf{F}\mathbf{x}\mathbf{R}=I\,\dot{\omega}(t)$$

$$\mathbf{F}=m\,\dot{\mathbf{v}}(t)$$

translation *Force* $=\mathbf{F}=m\dot{\mathbf{v}}$

Turntable turning at constant angular velocity $\boldsymbol{\Omega}=\Omega\hat{\mathbf{z}}$.

No-slipping: $\mathbf{v}(t)$-$\omega(t)\times\mathbf{R}=\boldsymbol{\Omega}\times\mathbf{r}(t)$ *(where:* $\mathbf{R}=R\hat{\mathbf{z}}$ and $\boldsymbol{\Omega}=\Omega\hat{\mathbf{z}}$ *are constant.)*

$$\mathbf{v}(t)=\boldsymbol{\Omega}\times\mathbf{r}(t)+\omega(t)\times\mathbf{R}=\boldsymbol{\Omega}\times\mathbf{r}(t)+\omega(t)\times\hat{\mathbf{z}}R$$

Do time-derivative

*Torque-and-*F=ma *equations of motion:*

$$I\dot{\omega}(t)=\mathbf{F}(t)\times\mathbf{R}$$
$$=m\dot{\mathbf{v}}(t)\times\mathbf{R}$$
$$=m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R$$

$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\dot{\mathbf{r}}(t)+\dot{\omega}(t)\times\hat{\mathbf{z}}R=\boldsymbol{\Omega}\times\mathbf{v}(t)+\dot{\omega}(t)\times\hat{\mathbf{z}}R$$

$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\dot{\mathbf{r}}(t)+\quad\dot{\omega}(t)\quad\times\hat{\mathbf{z}}R$$

use: $\dot{\omega}(t)=\dfrac{m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R}{I}$

$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\mathbf{v}(t)+\frac{m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R}{I}\times\hat{\mathbf{z}}R$$

use: $(\mathbf{B}\times\mathbf{C})\times\mathbf{A}=(\mathbf{A}\bullet\mathbf{B})\mathbf{C}-(\mathbf{A}\bullet\mathbf{C})\mathbf{B}$

$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\mathbf{v}(t)+\frac{m\dot{\mathbf{v}}(t)\bullet\hat{\mathbf{z}}R}{I}\hat{\mathbf{z}}R-\frac{mR^2}{I}\dot{\mathbf{v}}(t)$$

with: $\mathbf{B}=\dfrac{m\dot{\mathbf{v}}(t)}{I}$ and: $\mathbf{A}=\hat{\mathbf{z}}R=\mathbf{C}$

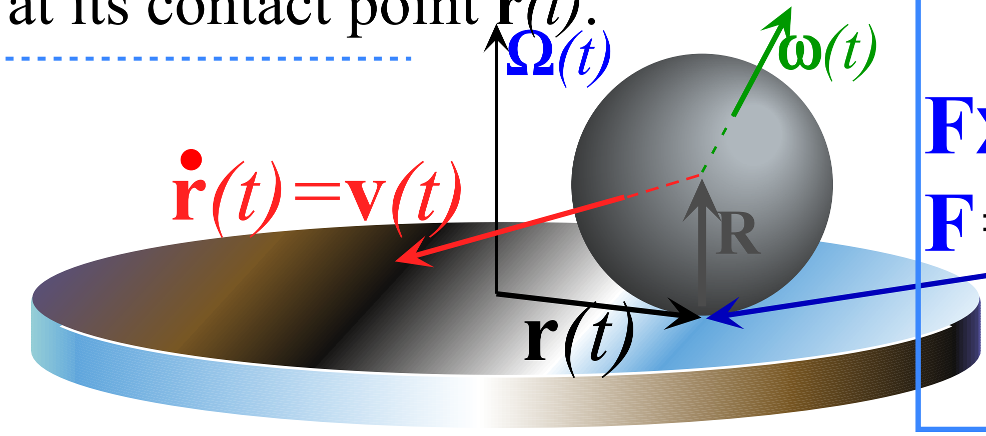
($\mathbf{v}(t)$ *always normal to* $\hat{\mathbf{z}}$)

since $\dot{\mathbf{v}}(t)$ always in table plane

$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\mathbf{v}(t)+\quad 0\quad-\frac{mR^2}{I}\dot{\mathbf{v}}(t)$$

Mechanical analog of cyclotron and FBI rule

Velocity vector of the ball contact point ($\mathbf{v}(t)$-$\boldsymbol{\omega}(t)\times\mathbf{R}$) equals table surface velocity $\boldsymbol{\Omega}\times\mathbf{r}(t)$ at its contact point $\mathbf{r}(t)$.



Equations of Motion:

rotation $Torque = \mathbf{F}\times\mathbf{R} = I\dot{\boldsymbol{\omega}}$

$\mathbf{F}\mathbf{x}\mathbf{R}=I\,\dot{\boldsymbol{\omega}}(t)$

$\mathbf{F}=m\,\dot{\mathbf{v}}(t)$

translation $Force=\mathbf{F} = m\dot{\mathbf{v}}$

Rolling Constraint

Turntable turning at constant angular velocity $\boldsymbol{\Omega}=\Omega\hat{\mathbf{z}}$.

No-slipping: $\mathbf{v}(t)-\boldsymbol{\omega}(t)\times\mathbf{R}=\boldsymbol{\Omega}\times\mathbf{r}(t)$ *(where:* $\mathbf{R}=R\hat{\mathbf{z}}$ and $\boldsymbol{\Omega}=\Omega\hat{\mathbf{z}}$ *are constant.)*

$$\mathbf{v}(t)=\boldsymbol{\Omega}\times\mathbf{r}(t)+\boldsymbol{\omega}(t)\times\mathbf{R}\quad=\boldsymbol{\Omega}\times\mathbf{r}(t)+\boldsymbol{\omega}(t)\times\hat{\mathbf{z}}R$$

Do time-derivative

*Torque-and-*F=ma *equations of motion:*

$$I\dot{\boldsymbol{\omega}}(t)=\mathbf{F}(t)\times\mathbf{R}$$
$$=m\dot{\mathbf{v}}(t)\times\mathbf{R}$$
$$=m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R$$

$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\dot{\mathbf{r}}(t)+\dot{\boldsymbol{\omega}}(t)\times\hat{\mathbf{z}}R=\boldsymbol{\Omega}\times\mathbf{v}(t)+\dot{\boldsymbol{\omega}}(t)\times\hat{\mathbf{z}}R$$

$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\dot{\mathbf{r}}(t)\ +\quad\dot{\boldsymbol{\omega}}(t)\quad\times\hat{\mathbf{z}}R$$

use: $\dot{\boldsymbol{\omega}}(t)=\dfrac{m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R}{I}$

$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\mathbf{v}(t)+\frac{m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R}{I}\times\hat{\mathbf{z}}R$$

use: $(\mathbf{B}\times\mathbf{C})\times\mathbf{A}=(\mathbf{A}\bullet\mathbf{B})\mathbf{C}-(\mathbf{A}\bullet\mathbf{C})\mathbf{B}$

with: $\mathbf{B}=\dfrac{m\dot{\mathbf{v}}(t)}{I}$ and: $\mathbf{A}=\hat{\mathbf{z}}R=\mathbf{C}$

$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\mathbf{v}(t)+\frac{m\dot{\mathbf{v}}(t)\bullet\hat{\mathbf{z}}R}{I}\hat{\mathbf{z}}R-\frac{mR^2}{I}\dot{\mathbf{v}}(t)$$

($\mathbf{v}(t)$ *always normal to* $\hat{\mathbf{z}}$)

since $\dot{\mathbf{v}}(t)$ always in table plane

$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\mathbf{v}(t)+\quad 0\quad-\frac{mR^2}{I}\dot{\mathbf{v}}(t)$$

$$\left(1+\frac{mR^2}{I}\right)\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\mathbf{v}(t)$$

or :

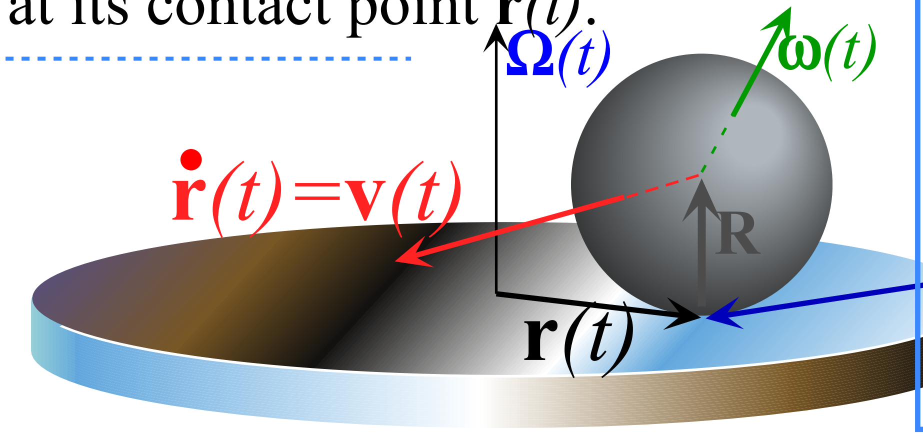
F=B×v *mechanical analog:*

$$\dot{\mathbf{v}}(t)=\frac{\boldsymbol{\Omega}}{1+\frac{mR^2}{I}}\times\mathbf{v}(t)$$

Mechanical analog of cyclotron and FBI rule

Velocity vector of the ball contact point ($\mathbf{v}(t)$-$\boldsymbol{\omega}(t)\times\mathbf{R}$) equals table surface velocity $\boldsymbol{\Omega}\times\mathbf{r}(t)$ at its contact point $\mathbf{r}(t)$.

Rolling Constraint



Equations of Motion:

rotation $Torque = \mathbf{F}\times\mathbf{R} = I\dot{\boldsymbol{\omega}}$

$\mathbf{F}\mathbf{x}\mathbf{R}=I\,\dot{\boldsymbol{\omega}}(t)$

$\mathbf{F}=m\,\dot{\mathbf{v}}(t)$

translation $Force=\mathbf{F} = m\dot{\mathbf{v}}$

Turntable turning at constant angular velocity $\boldsymbol{\Omega}=\Omega\hat{\mathbf{z}}$.

No-slipping: $\mathbf{v}(t)-\boldsymbol{\omega}(t)\times\mathbf{R}=\boldsymbol{\Omega}\times\mathbf{r}(t)$ *(where:* $\mathbf{R}=R\hat{\mathbf{z}}$ and $\boldsymbol{\Omega}=\Omega\hat{\mathbf{z}}$ *are constant.)*

$$\mathbf{v}(t)=\boldsymbol{\Omega}\times\mathbf{r}(t)+\boldsymbol{\omega}(t)\times\mathbf{R} \quad =\boldsymbol{\Omega}\times\mathbf{r}(t)+\boldsymbol{\omega}(t)\times\hat{\mathbf{z}}R$$

Do time-derivative

$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\dot{\mathbf{r}}(t)+\dot{\boldsymbol{\omega}}(t)\times\hat{\mathbf{z}}R=\boldsymbol{\Omega}\times\mathbf{v}(t)+\dot{\boldsymbol{\omega}}(t)\times\hat{\mathbf{z}}R$$

*Torque-and-*F=ma *equations of motion:*

$$I\dot{\boldsymbol{\omega}}(t)=\mathbf{F}(t)\times\mathbf{R}$$
$$=m\dot{\mathbf{v}}(t)\times\mathbf{R}$$
$$=m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R$$

$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\dot{\mathbf{r}}(t) \; + \quad \dot{\boldsymbol{\omega}}(t) \quad \times\hat{\mathbf{z}}R$$

use: $\dot{\boldsymbol{\omega}}(t)=\dfrac{m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R}{I}$

$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\mathbf{v}(t)+\frac{m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R}{I}\times\hat{\mathbf{z}}R$$

use: $(\mathbf{B}\times\mathbf{C})\times\mathbf{A}=(\mathbf{A}\bullet\mathbf{B})\mathbf{C}-(\mathbf{A}\bullet\mathbf{C})\mathbf{B}$

with: $\mathbf{B}=\dfrac{m\dot{\mathbf{v}}(t)}{I}$ and: $\mathbf{A}=\hat{\mathbf{z}}R=\mathbf{C}$

$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\mathbf{v}(t)+\frac{m\dot{\mathbf{v}}(t)\bullet\hat{\mathbf{z}}R}{I}\hat{\mathbf{z}}R-\frac{mR^2}{I}\dot{\mathbf{v}}(t)$$

($\mathbf{v}(t)$ *always normal to* $\hat{\mathbf{z}}$)

since $\dot{\mathbf{v}}(t)$ always in table plane

$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\mathbf{v}(t)+ \quad 0 \quad -\frac{mR^2}{I}\dot{\mathbf{v}}(t)$$

$$\left(1+\frac{mR^2}{I}\right)\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\mathbf{v}(t)$$

or :

***ma**=e**B**×**v** mechanical analog:*

$$\dot{\mathbf{v}}(t)=\frac{\boldsymbol{\Omega}}{1+\dfrac{mR^2}{I}}\times\mathbf{v}(t)$$

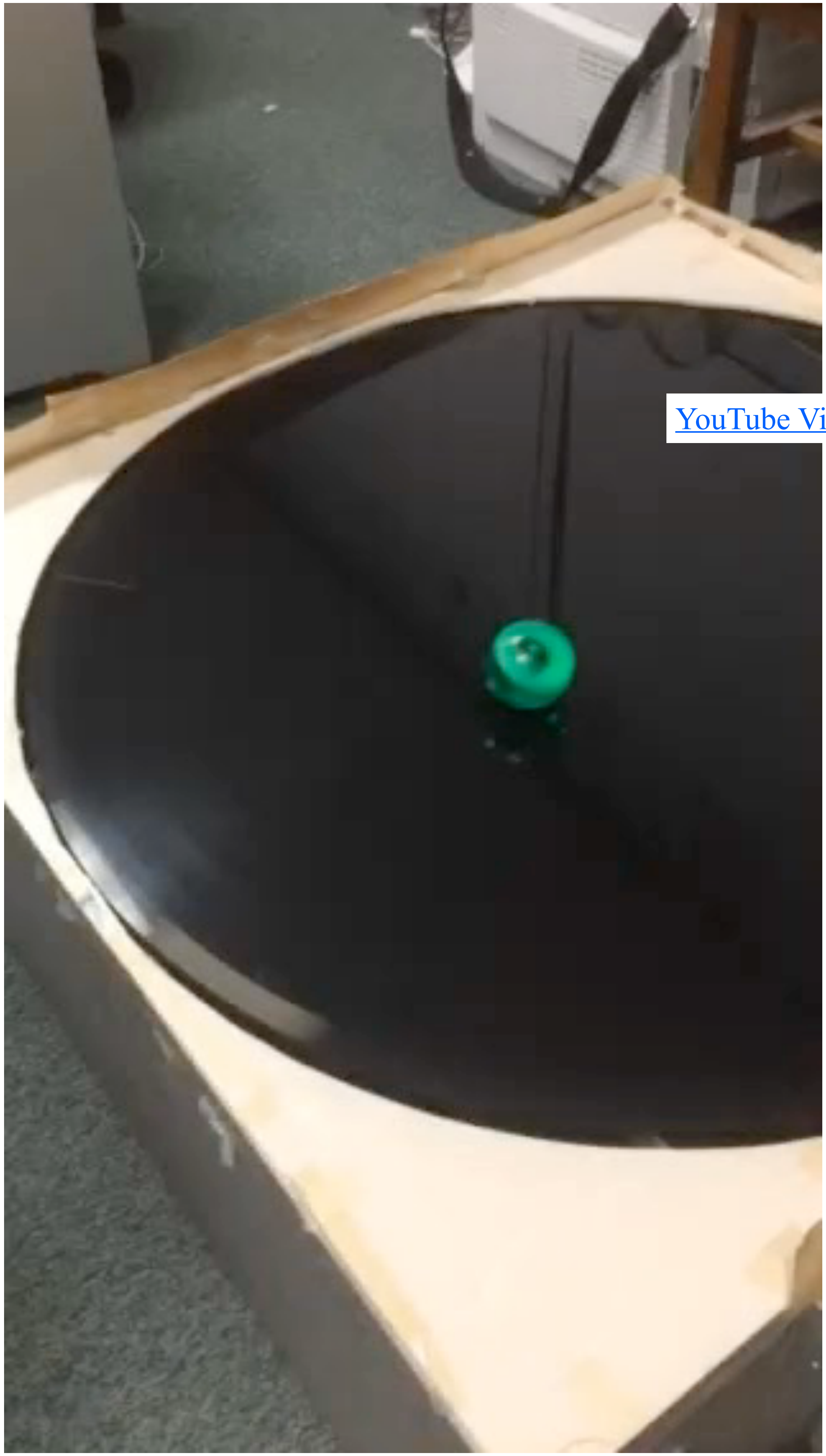
Mechanical analog cyclotron frequency

$$\omega=\frac{e}{m}B=\frac{\boldsymbol{\Omega}}{1+\dfrac{mR^2}{I}}$$

$\omega=\frac{2}{7}\boldsymbol{\Omega}$ for: $\frac{I}{mR^2}=\frac{2}{5}$ ●

$=\frac{2}{5}\boldsymbol{\Omega}$ for: $\frac{I}{mR^2}=\frac{2}{3}$ ○

YouTube Video of Analog to Syncrotron Motion



[YouTube Video of Analog to Syncrotron Motion]

Solid ball has 2 orbits
as table turns 7 rotations

Mechanical analog
cyclotron frequency

$$\omega = \frac{e}{m} B = \frac{\Omega}{1 + \frac{mR^2}{I}}$$

$$\boxed{\omega = \tfrac{2}{7}\Omega} \text{ for: } \frac{I}{mR^2} = \frac{2}{5} \quad ●$$

$$= \tfrac{2}{5}\Omega \text{ for: } \frac{I}{mR^2} = \frac{2}{3} \quad ○$$

Crossed E and B field mechanics

Classical Hall-effect and cyclotron orbit orbit equations

Vector theory vs. complex variable theory

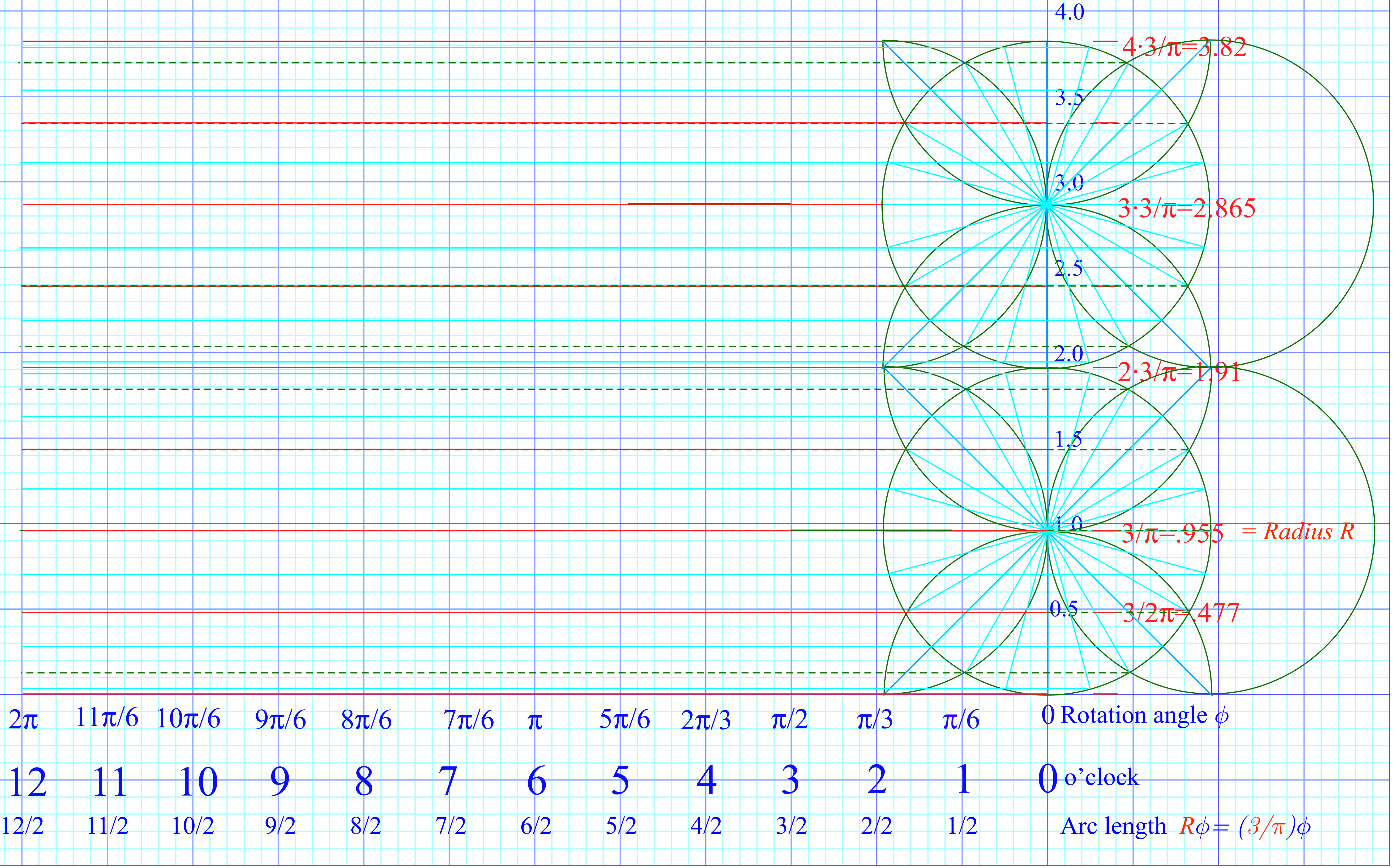
Mechanical analog of cyclotron and FBI rule

→ *Cycloid and epicycloid ruler&compass geometry*

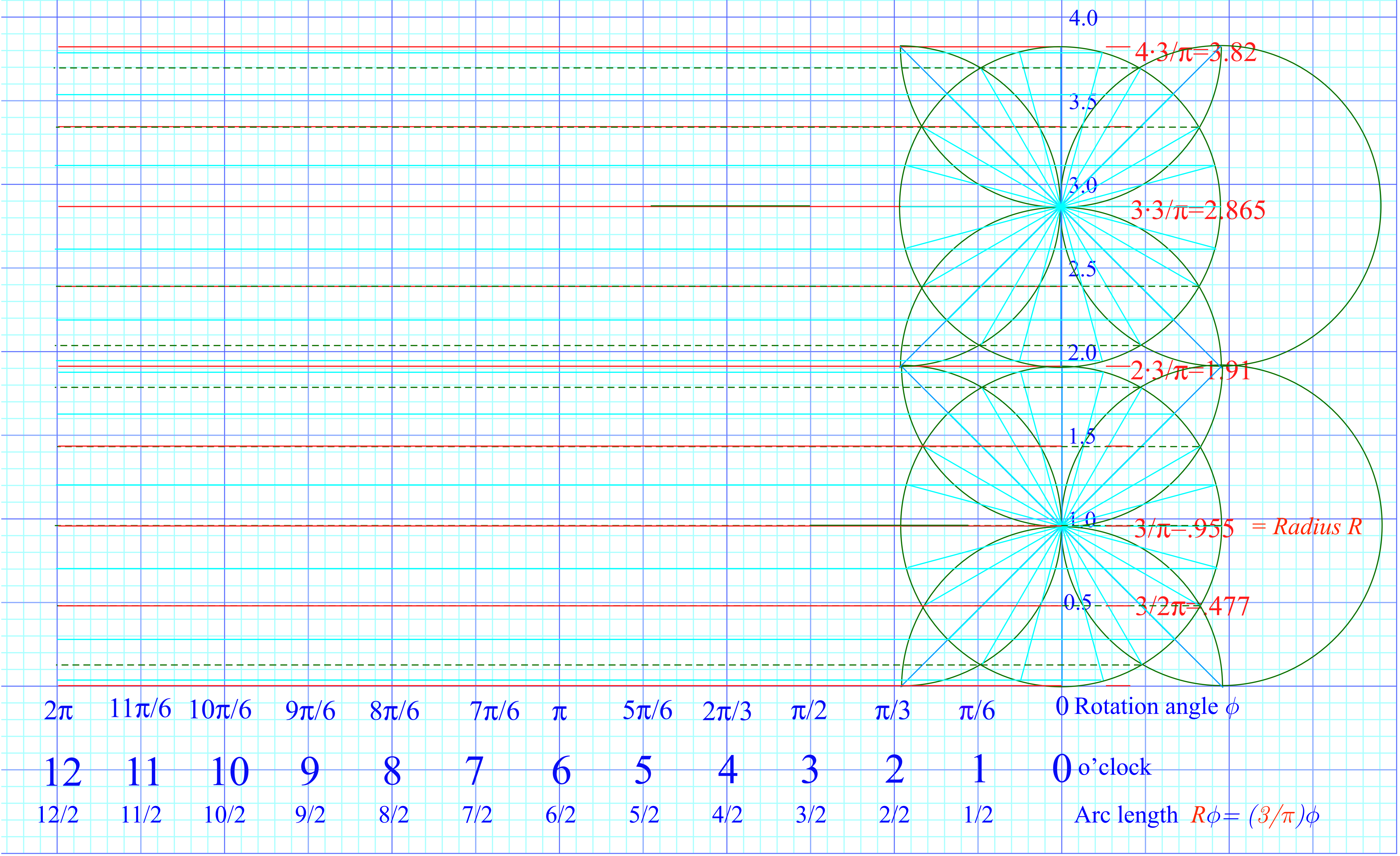
Cycloid geometry of flying levers

Practical poolhall application

Here the radius is plotted as an irrational $R=3/\pi=0.955$ length so rolling by rational angle $\phi= m\pi/n$ is a rational length of rolled -out circumference $R\phi= (3/\pi)m\pi/n=3m/n$.

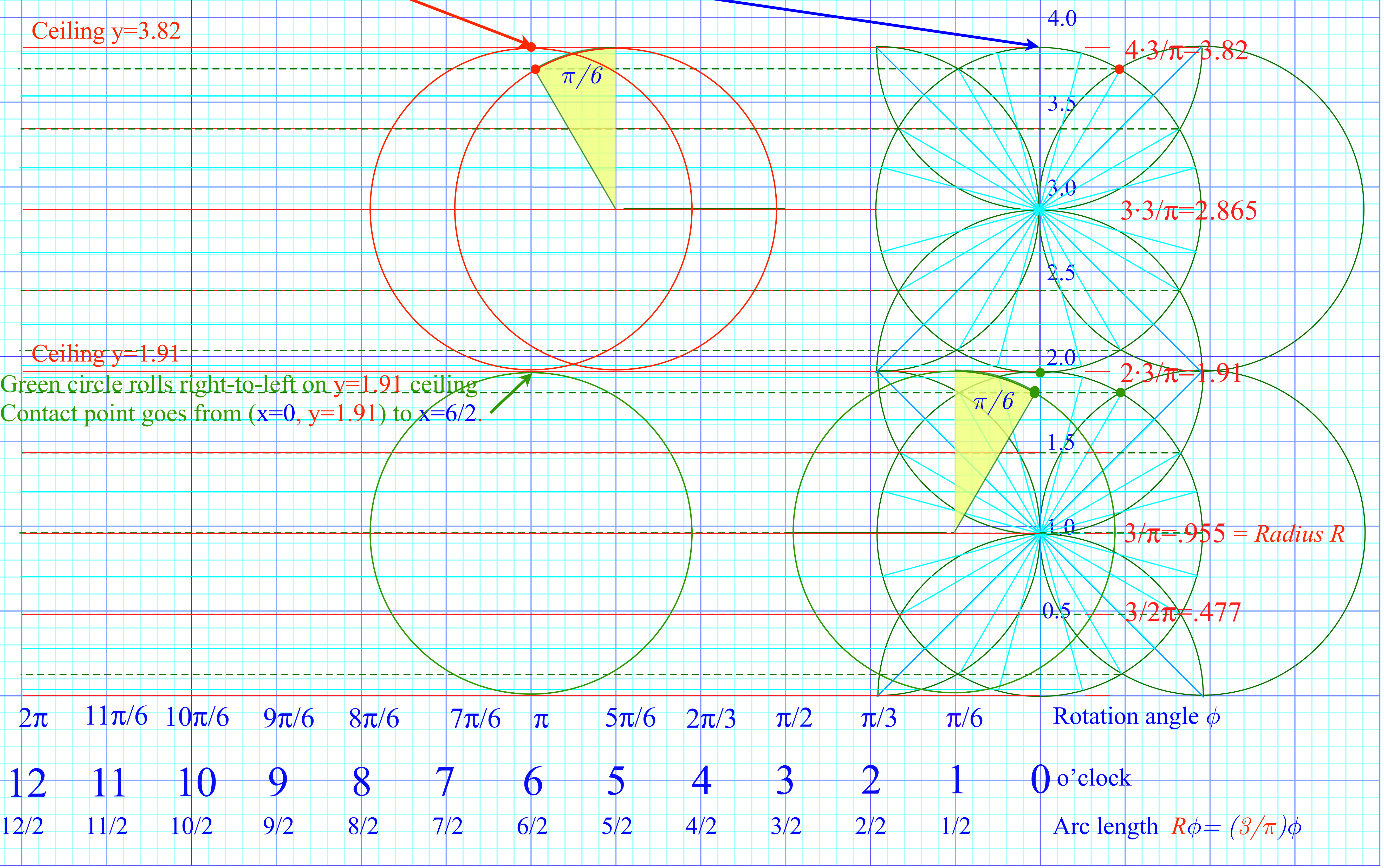


Here the radius is plotted as an irrational $R=3/\pi=0.955$ length so rolling by rational angle $\phi= m\pi/n$ is a rational length of rolled-out circumference $R\phi=(3/\pi)m\pi/n=3m/n$. Diameter is $2R=6/\pi=1.91$



Here the radius is plotted as an irrational $R=3/\pi=0.955$ length so rolling by rational angle $\phi=m\pi/n$ is a rational length of rolled -out circumference $R\phi=(3/\pi)m\pi/n=3m/n$. Diameter is $2R=6/\pi=1.91$

Red circle rolls left-to-right on y=3.82 ceiling
Contact point goes from (x=6/2, y=3.82) to x=0.

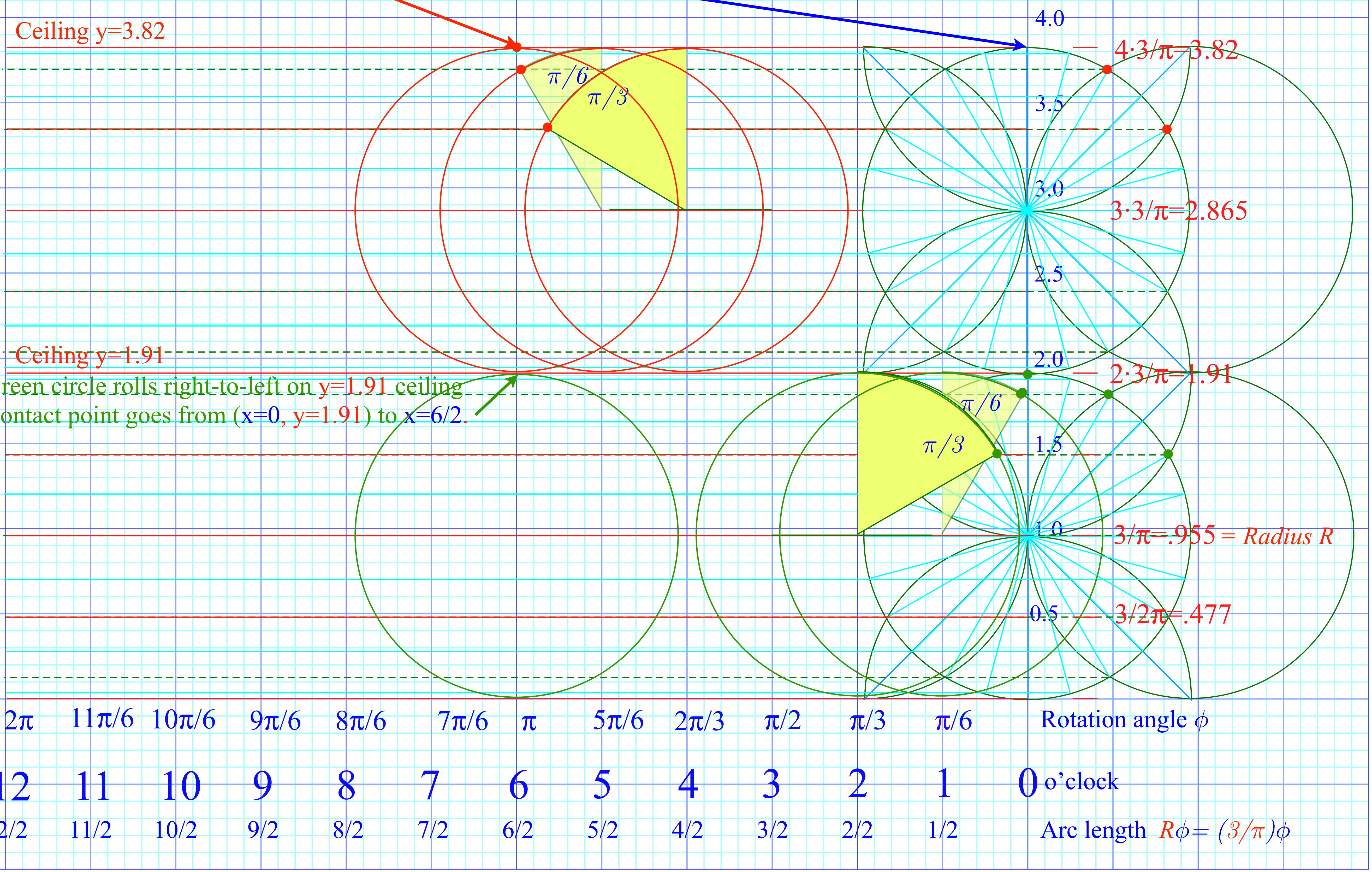


Green circle rolls right-to-left on y=1.91 ceiling
Contact point goes from (x=0, y=1.91) to x=6/2.

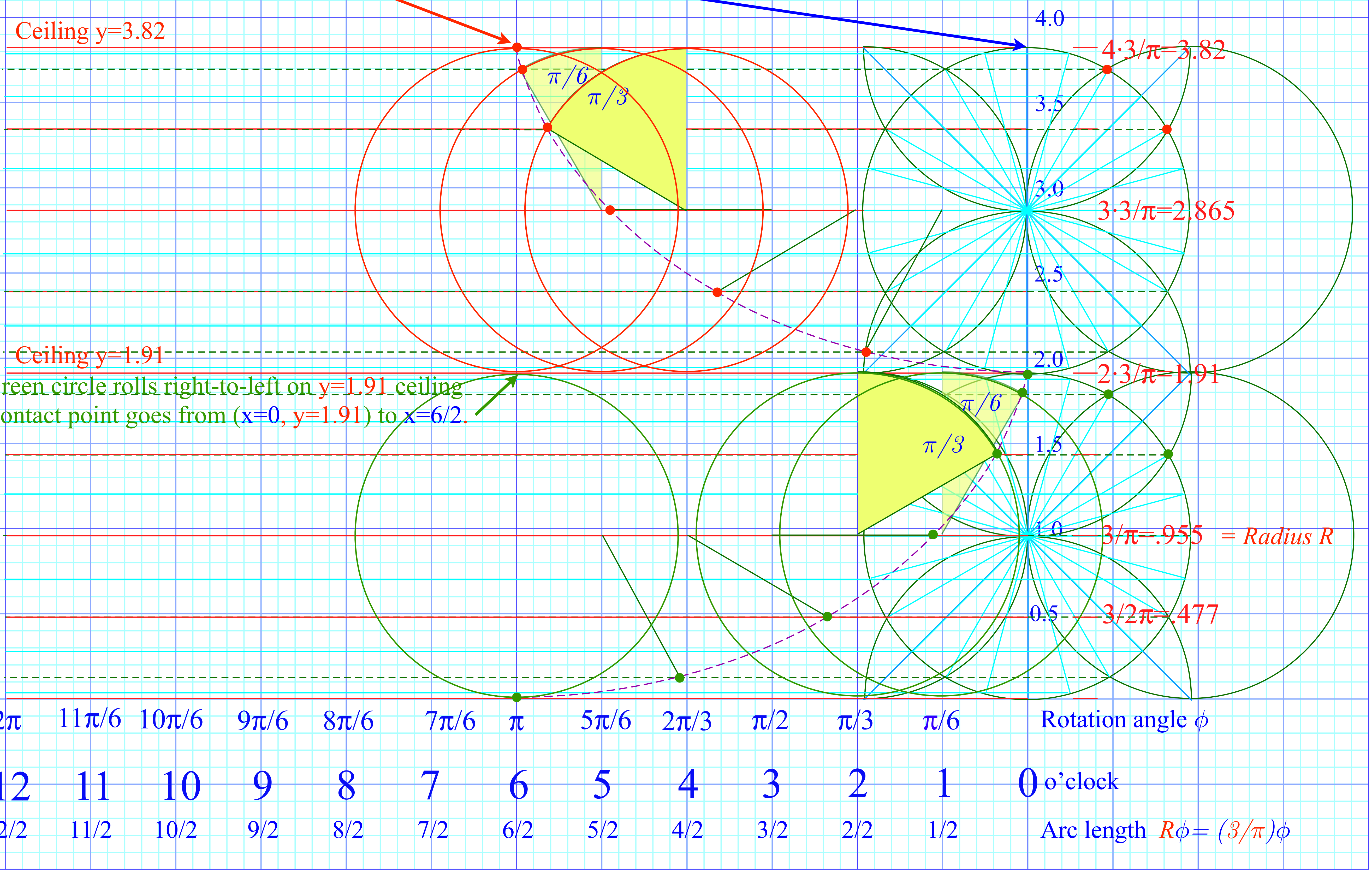
Here the radius is plotted as an irrational $R=3/\pi=0.955$ length so rolling by rational angle $\phi= m\pi/n$ is a rational length of rolled -out circumference $R\phi= (3/\pi)m\pi/n=3m/n$. Diameter is $2R=6/\pi=1.91$

Red circle rolls left-to-right on y=3.82 ceiling
Contact point goes from (x=6/2, y=3.82) to x=0.

Green circle rolls right-to-left on y=1.91 ceiling
Contact point goes from (x=0, y=1.91) to x=6/2.



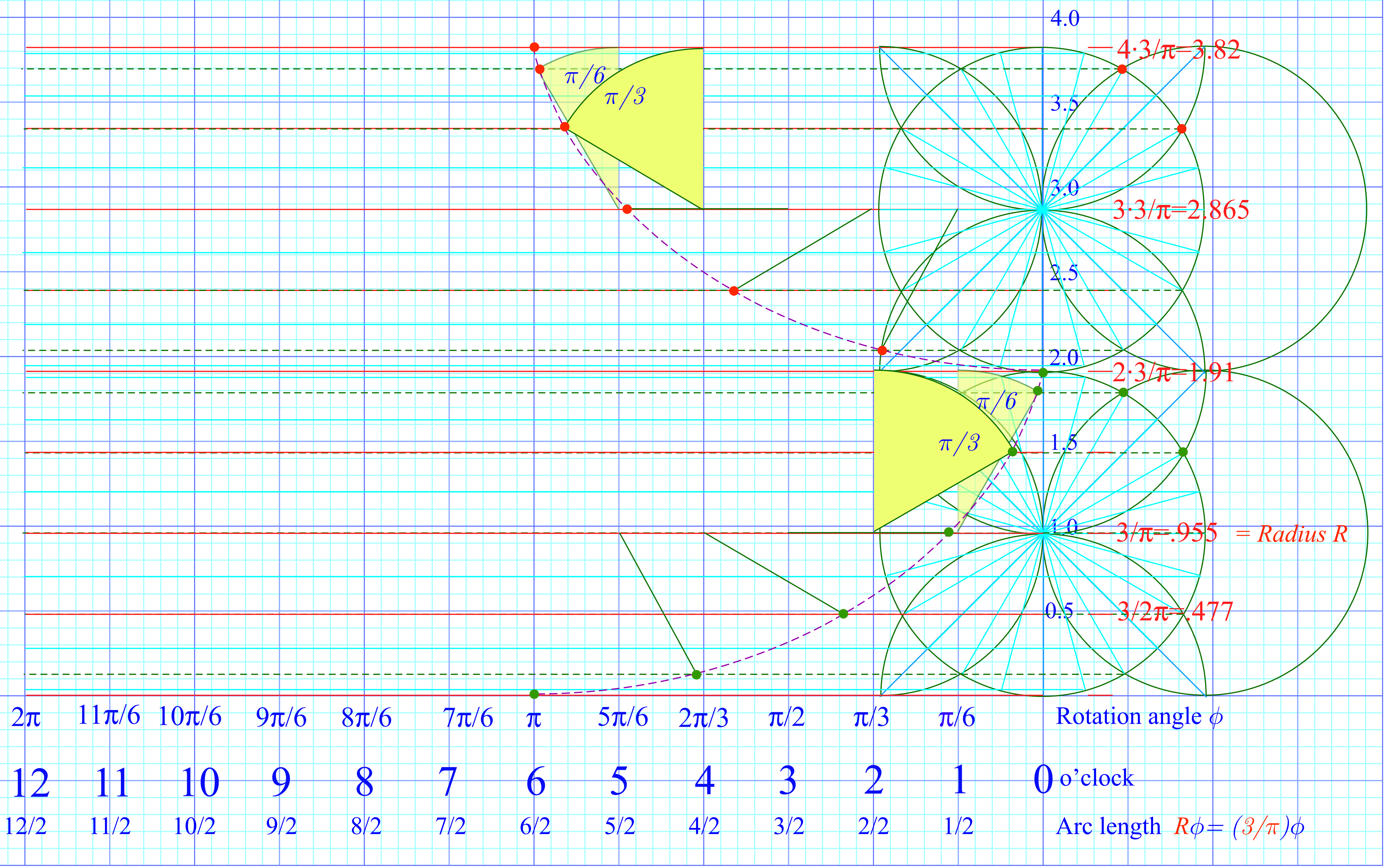
Here the radius is plotted as an irrational $R=3/\pi=0.955$ length so rolling by rational angle $\phi= m\pi/n$ is a rational length of rolled -out circumference $R\phi= (3/\pi)m\pi/n=3m/n$. Diameter is $2R=6/\pi=1.91$



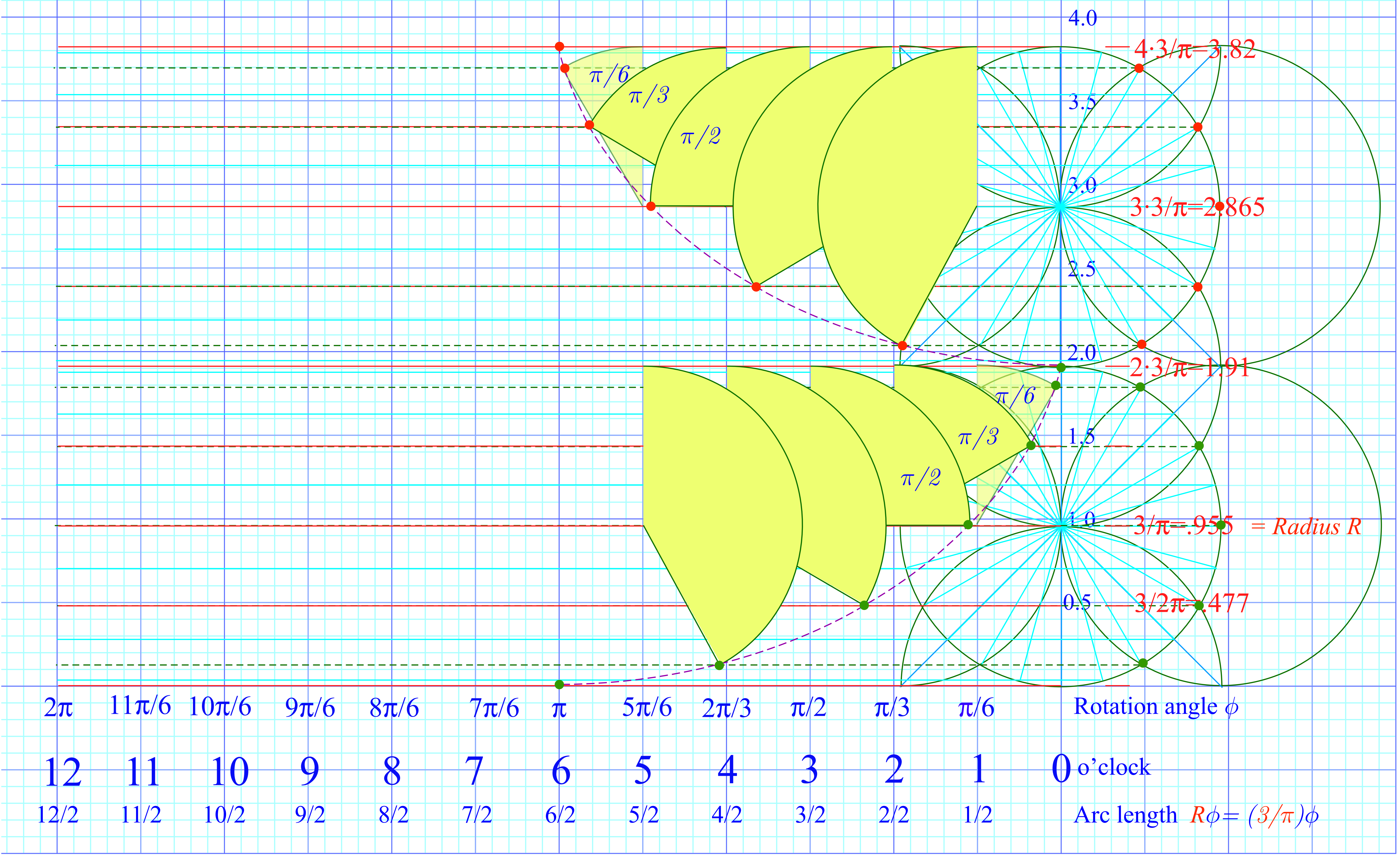
Red circle rolls left-to-right on y=3.82 ceiling
Contact point goes from (x=6/2, y=3.82) to x=0.

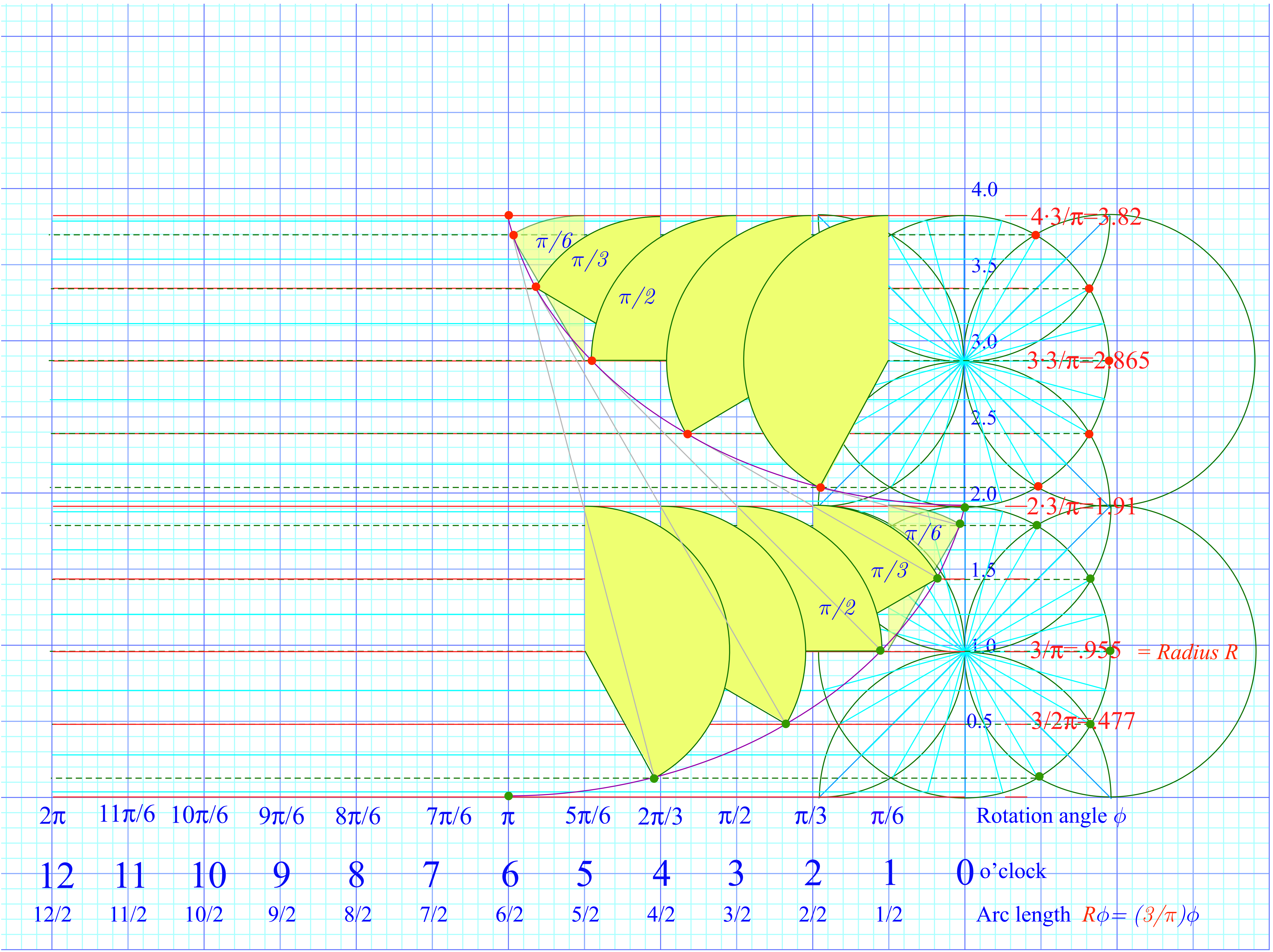
Green circle rolls right-to-left on y=1.91 ceiling
Contact point goes from (x=0, y=1.91) to x=6/2.

Here the radius is plotted as an irrational $R=3/\pi=0.955$ length so rolling by rational angle $\phi=m\pi/n$ is a rational length of rolled-out circumference $R\phi=(3/\pi)m\pi/n=3m/n$. Diameter is $2R=6/\pi=1.91$

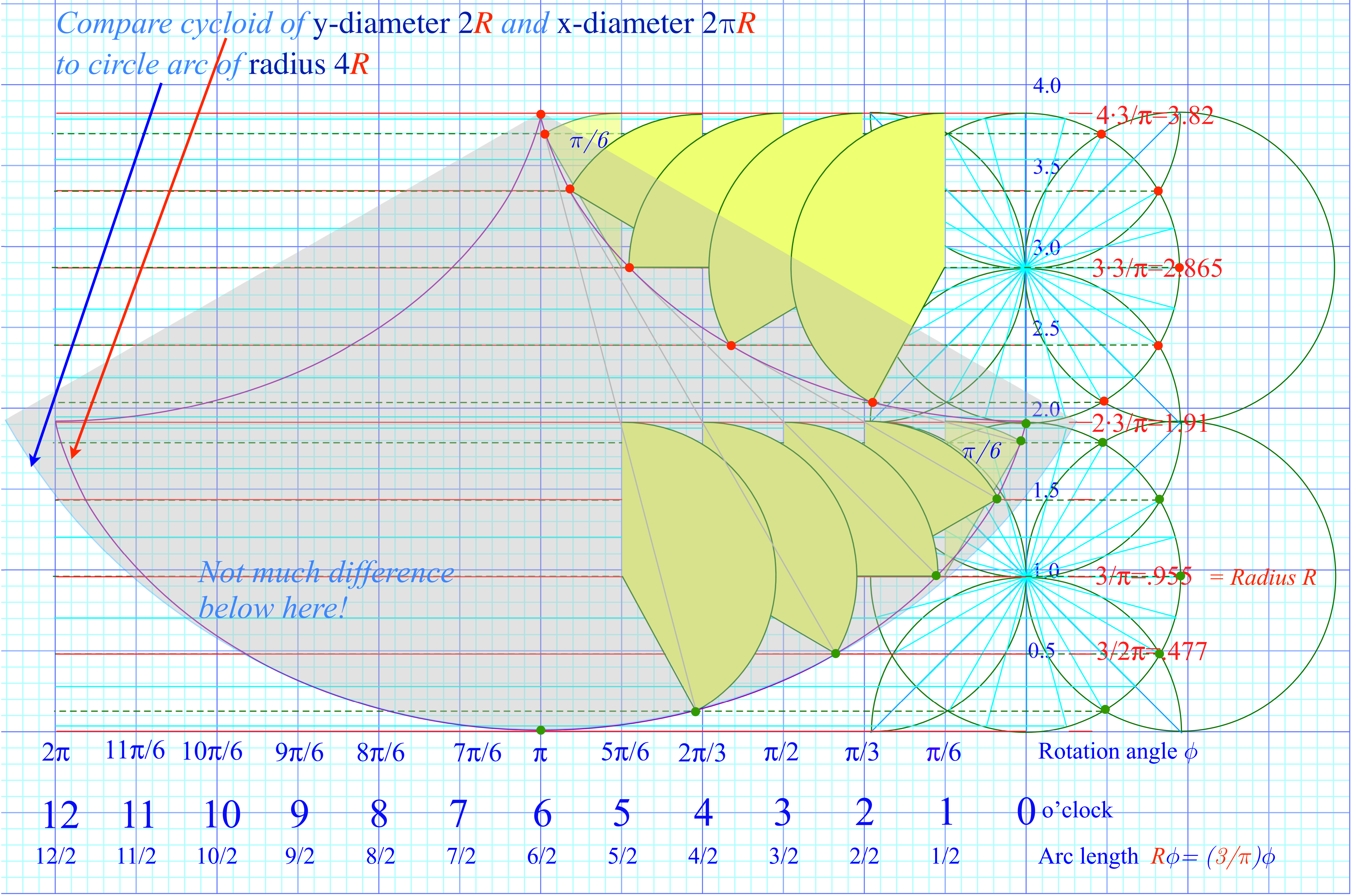


Here the radius is plotted as an irrational $R=3/\pi=0.955$ length so rolling by rational angle $\phi=m\pi/n$ is a rational length of rolled-out circumference $R\phi=(3/\pi)m\pi/n=3m/n$. Diameter is $2R=6/\pi=1.91$




4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
4·3/π=3.82
3·3/π=2.865
2·3/π=1.91
3/π=.955 = Radius R
3/2π=.477
π/6
π/3
π/2
π/6
π/3
π/2
2π
11π/6
10π/6
9π/6
8π/6
7π/6
π
5π/6
2π/3
π/2
π/3
π/6
Rotation angle φ
12
11
10
9
8
7
6
5
4
3
2
1
0 o'clock
12/2
11/2
10/2
9/2
8/2
7/2
6/2
5/2
4/2
3/2
2/2
1/2
Arc length Rφ= (3/π)φ

Here the radius is plotted as an irrational $R=3/\pi=0.955$ length so rolling by rational angle $\phi= m\pi/n$ is a rational length of rolled-out circumference $R\phi=(3/\pi)m\pi/n=3m/n$. Diameter is $2R=6/\pi=1.91$



Compare cycloid of y-diameter $2R$ *and* x-diameter $2\pi R$ *to circle arc of* radius $4R$

Not much difference below here!

Crossed E and B field mechanics

Classical Hall-effect and cyclotron orbit orbit equations

Vector theory vs. complex variable theory

Mechanical analog of cyclotron and FBI rule

Cycloid and epicycloid ruler&compass geometry

Cycloid geometry of flying levers

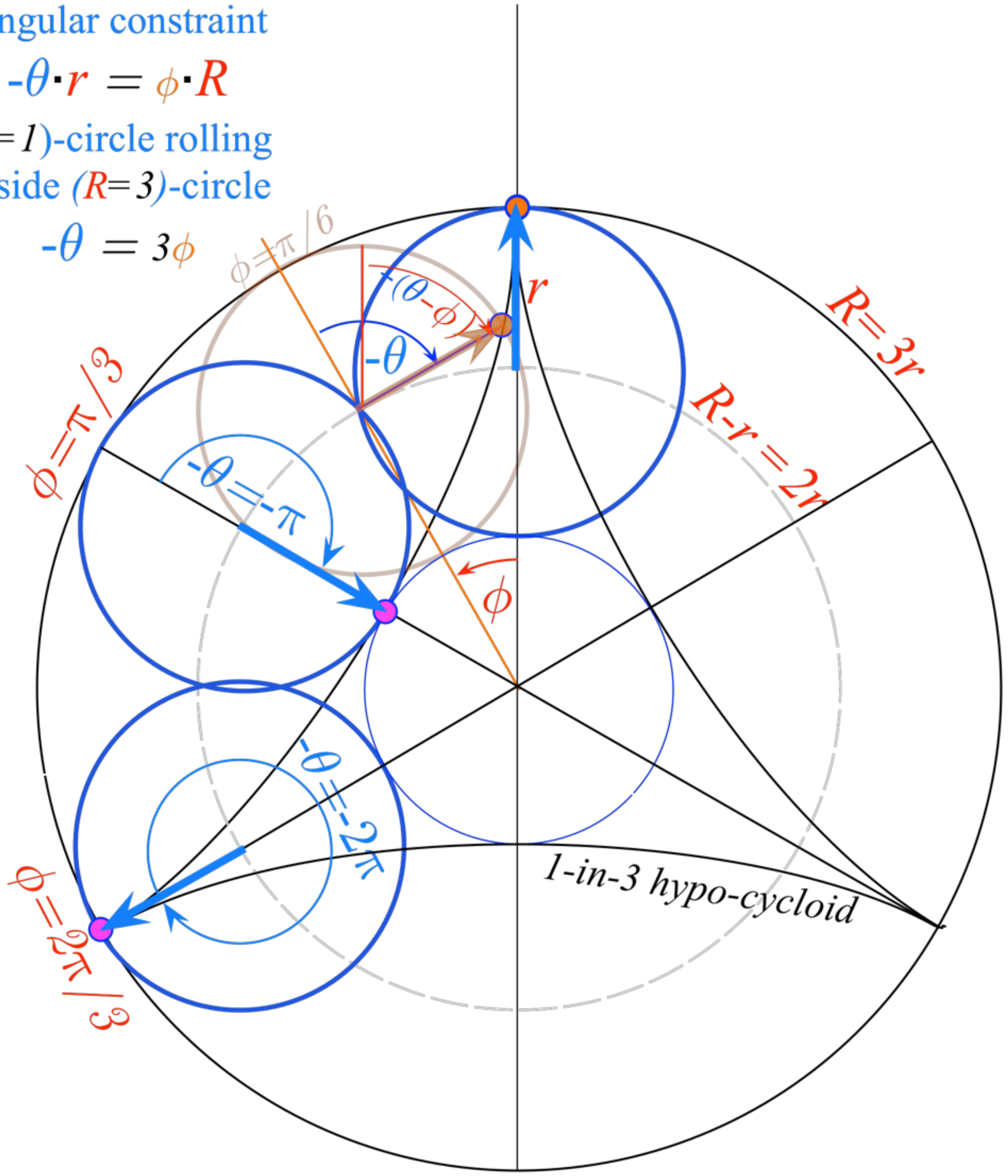
Practical poolhall application

Angular constraint

$-\theta \cdot r = \phi \cdot R$

(r=1)-circle rolling inside (R=3)-circle

$-\theta = 3\phi$



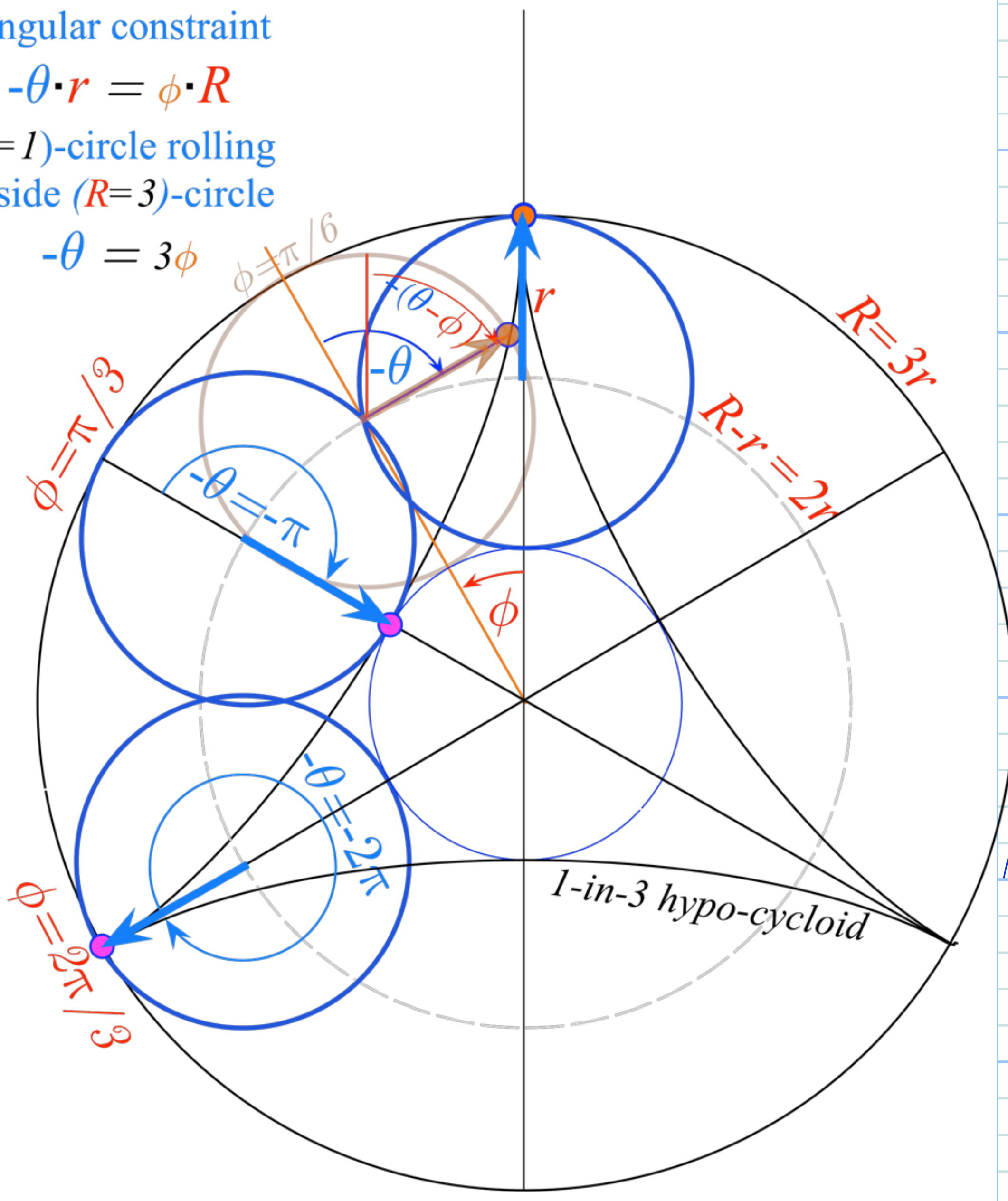
Hyper-and-Hypo-Cycloidal coordinate geometry and dynamics

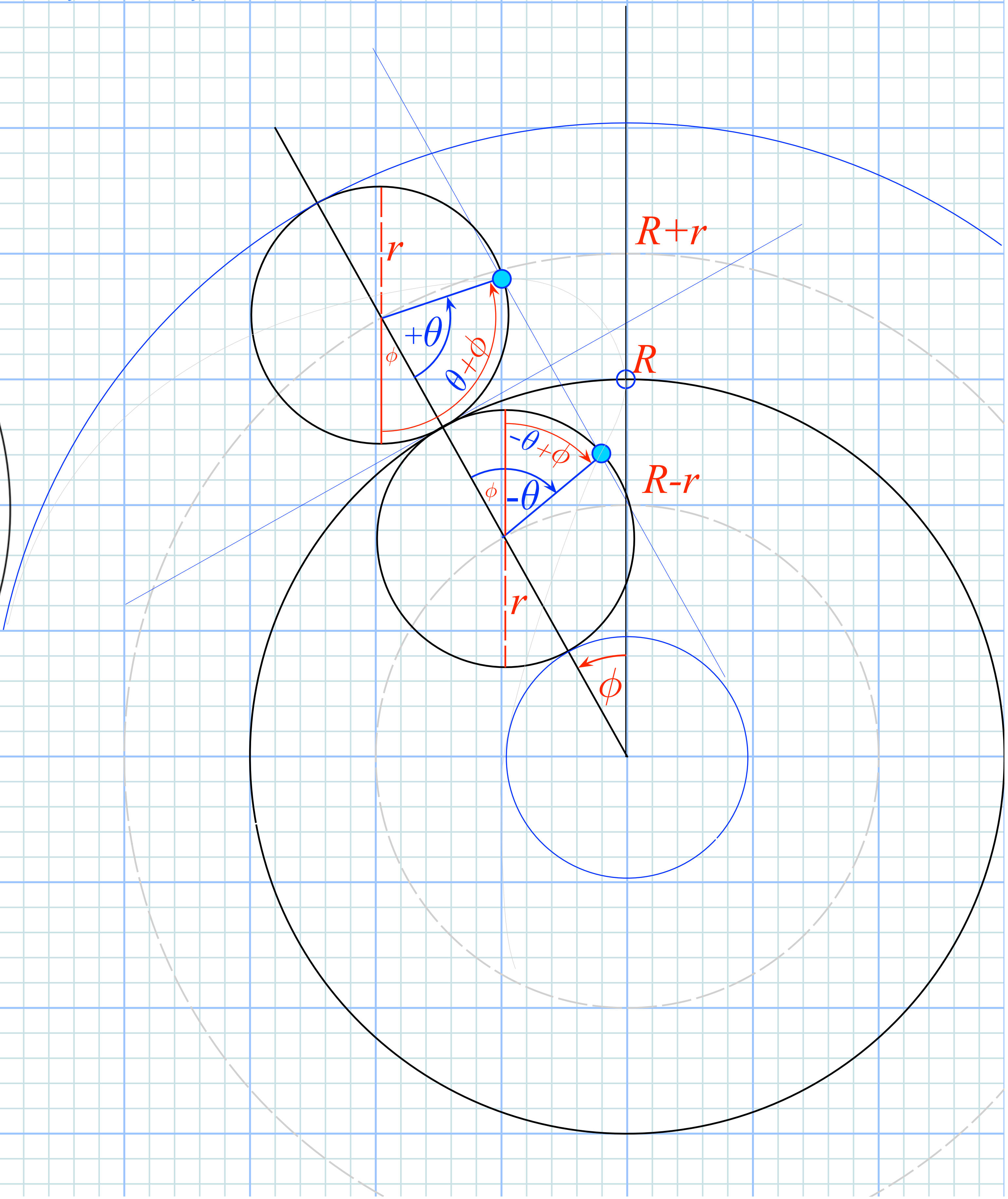
Angular constraint

$-\theta \cdot r = \phi \cdot R$

(r=1)-circle rolling inside (R=3)-circle

$-\theta = 3\phi$





Hyper-and-Hypo-Cycloidal coordinate geometry and dynamics

Hyper-cycloid constrained by: $\theta r = R\phi$ or: $\theta = \frac{R}{r}\phi$

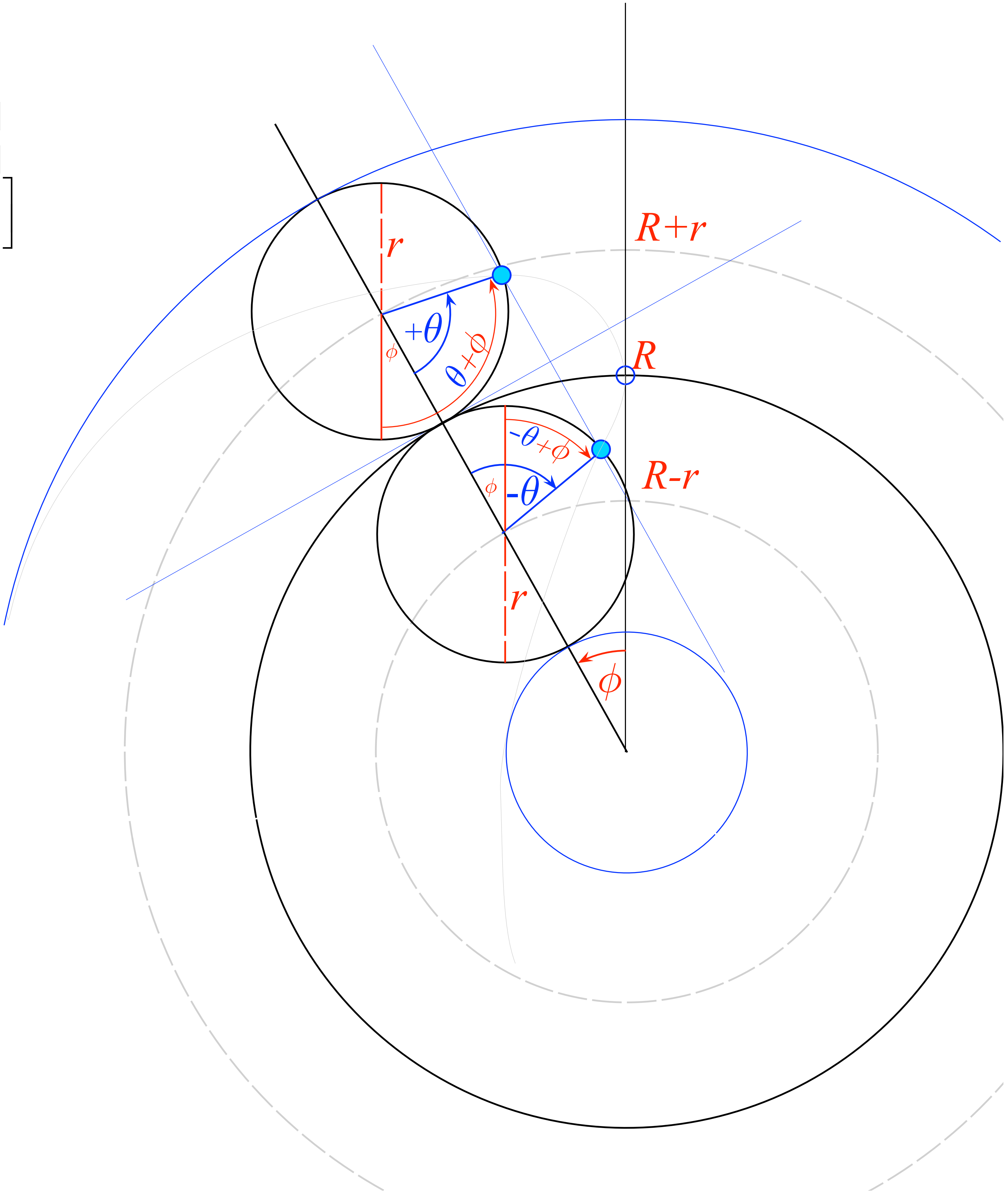
$$x = -(R+r)\sin\phi + r\sin(\theta+\phi) = r\left[-\left(\frac{R}{r}+1\right)\sin\phi + \sin\left(\frac{R}{r}+1\right)\phi\right]$$

$$y = \ (R+r)\cos\phi - r\cos(\theta+\phi) = r\left[\ \left(\frac{R}{r}+1\right)\cos\phi - \cos\left(\frac{R}{r}+1\right)\phi\right]$$

Hypo-cycloid constrained by: $-\theta r = -R\phi$ or: $\theta = \frac{R}{r}\phi$

$$x = -(R-r)\sin\phi + r\sin(\theta\text{-}\phi) = r\left[-\left(\frac{R}{r}-1\right)\sin\phi + \sin\left(\frac{R}{r}-1\right)\phi\right]$$

$$y = \ (R-r)\cos\phi + r\cos(\theta\text{-}\phi) = r\left[\ \left(\frac{R}{r}-1\right)\cos\phi + \cos\left(\frac{R}{r}-1\right)\phi\right]$$



Hyper-and-Hypo-Cycloidal coordinate geometry and dynamics

Hyper-cycloid constrained by: $\theta r = R\phi$ or: $\theta = \frac{R}{r}\phi$

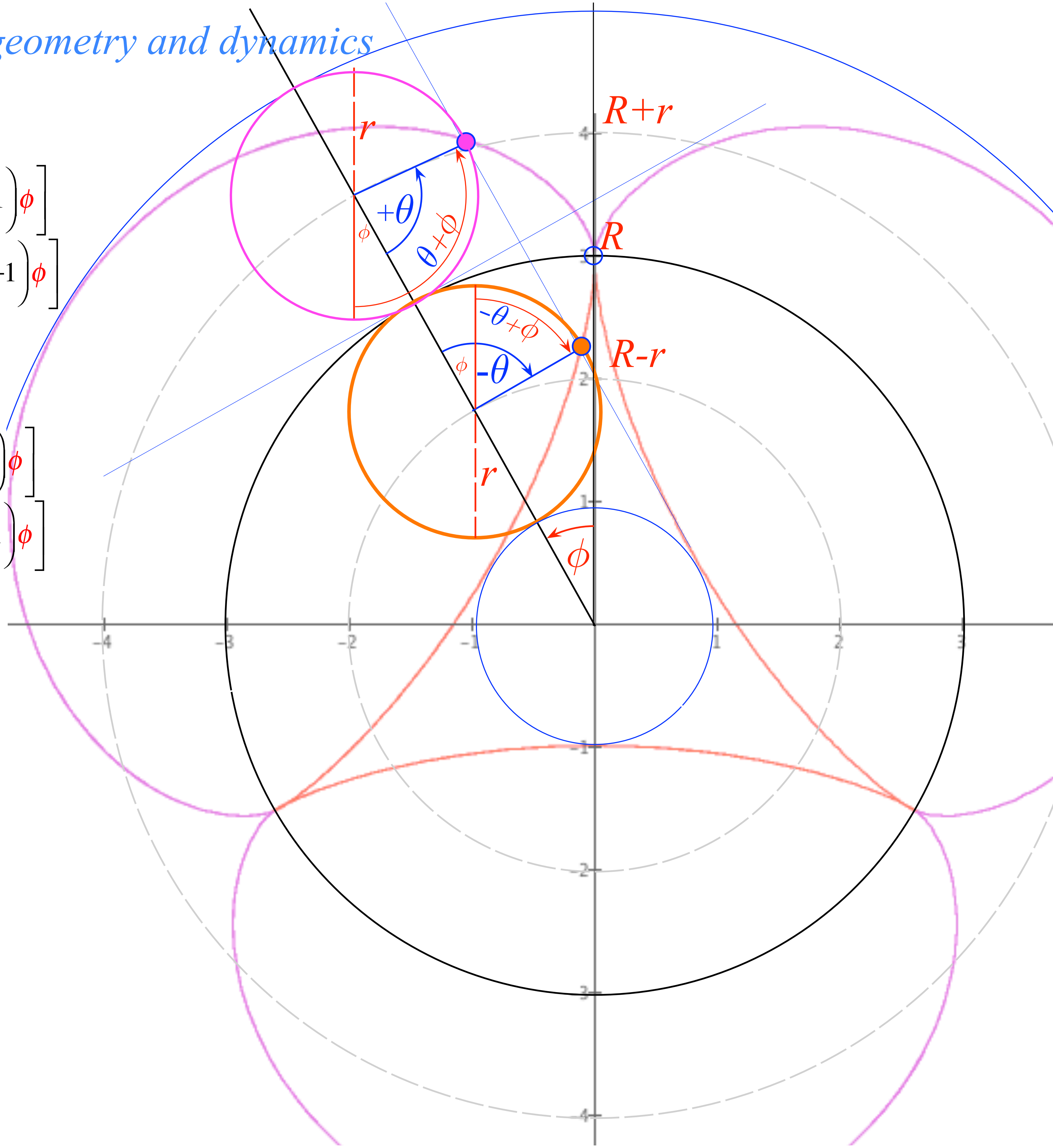
$$x = -(R+r)\sin\phi + r\sin(\theta+\phi) = r\left[-\left(\frac{R}{r}+1\right)\sin\phi + \sin\left(\frac{R}{r}+1\right)\phi\right]$$

$$y = \ (R+r)\cos\phi - r\cos(\theta+\phi) = r\left[\ \left(\frac{R}{r}+1\right)\cos\phi - \cos\left(\frac{R}{r}+1\right)\phi\right]$$

Hypo-cycloid constrained by: $-\theta r = -R\phi$ or: $\theta = \frac{R}{r}\phi$

$$x = -(R-r)\sin\phi + r\sin(\theta-\phi) = r\left[-\left(\frac{R}{r}-1\right)\sin\phi + \sin\left(\frac{R}{r}-1\right)\phi\right]$$

$$y = \ (R-r)\cos\phi + r\cos(\theta-\phi) = r\left[\ \left(\frac{R}{r}-1\right)\cos\phi + \cos\left(\frac{R}{r}-1\right)\phi\right]$$



Hyper-and-Hypo-Cycloidal coordinate geometry and dynamics

Hyper-cycloid constrained by: $\theta r = R\phi$ or: $\theta = \frac{R}{r}\phi$

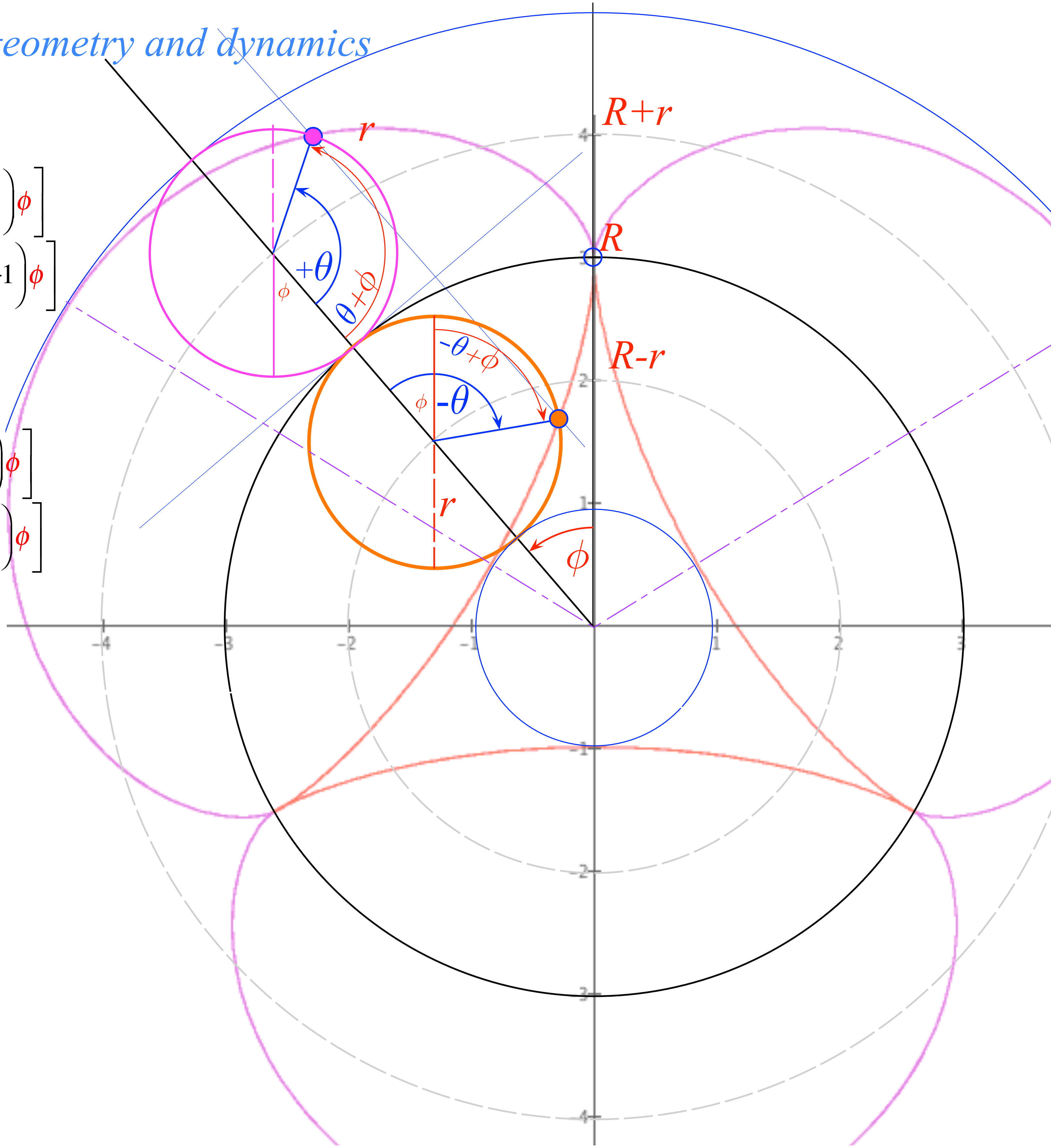
$$x = -(R+r)\sin\phi + r\sin(\theta+\phi) = r\left[-\left(\frac{R}{r}+1\right)\sin\phi + \sin\left(\frac{R}{r}+1\right)\phi\right]$$

$$y = \ (R+r)\cos\phi - r\cos(\theta+\phi) = r\left[\ \left(\frac{R}{r}+1\right)\cos\phi - \cos\left(\frac{R}{r}+1\right)\phi\right]$$

Hypo-cycloid constrained by: $-\theta r = -R\phi$ or: $\theta = \frac{R}{r}\phi$

$$x = -(R-r)\sin\phi + r\sin(\theta\text{-}\phi) = r\left[-\left(\frac{R}{r}-1\right)\sin\phi + \sin\left(\frac{R}{r}-1\right)\phi\right]$$

$$y = \ (R-r)\cos\phi + r\cos(\theta\text{-}\phi) = r\left[\ \left(\frac{R}{r}-1\right)\cos\phi + \cos\left(\frac{R}{r}-1\right)\phi\right]$$



Hypo-Cycloidal coordinate geometry and dynamics

Hypo-cycloid constrained by: $-\theta r = -R\phi$ or: $\theta = \frac{R}{r}\phi$

$$x = -(R-r)\sin\phi + r\sin(\theta\text{-}\phi) = r\left[-\left(\frac{R}{r}-1\right)\sin\phi + \sin\left(\frac{R}{r}-1\right)\phi\right]$$

$$y = (R-r)\cos\phi + r\cos(\theta\text{-}\phi) = r\left[\left(\frac{R}{r}-1\right)\cos\phi + \cos\left(\frac{R}{r}-1\right)\phi\right]$$

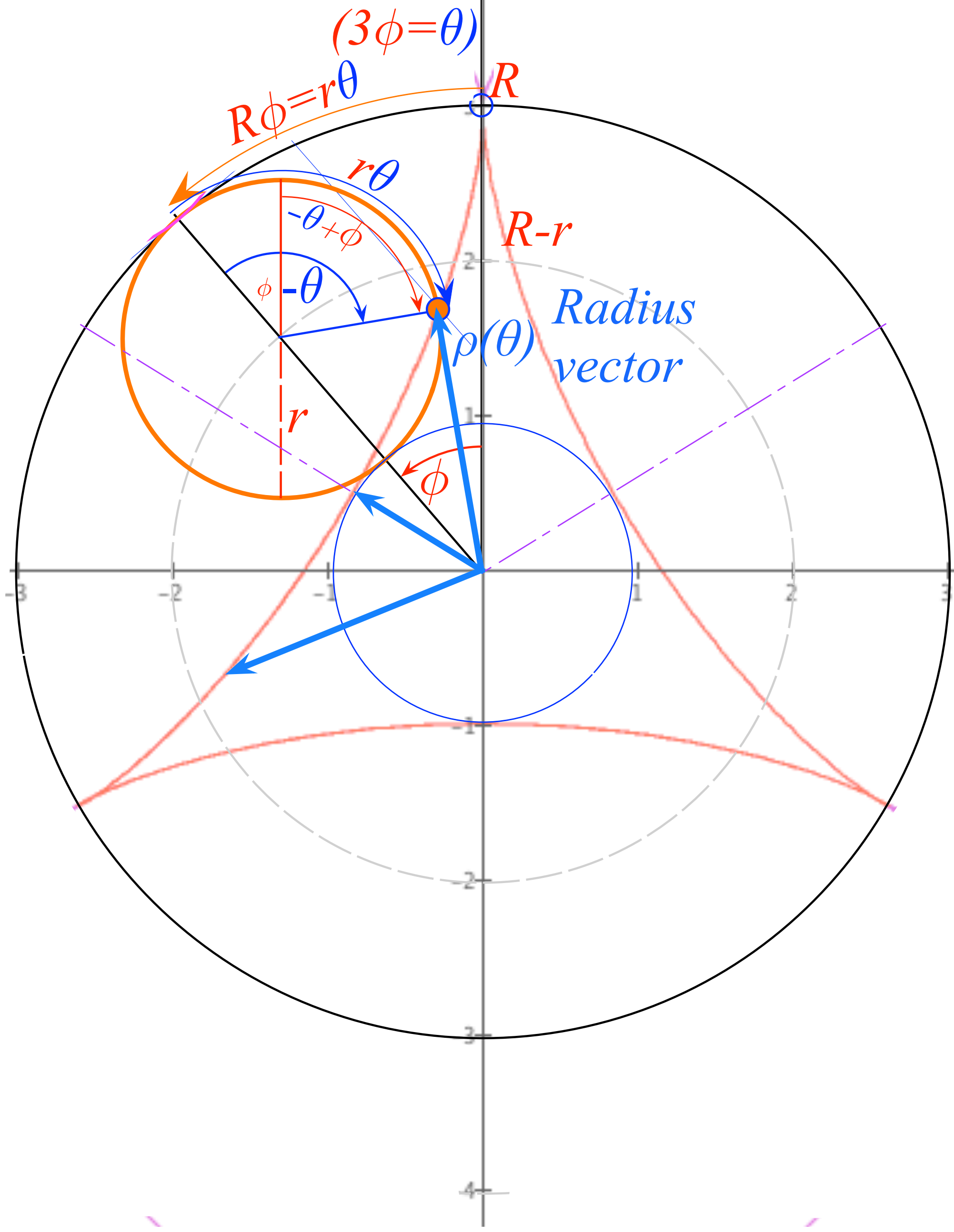
Hypo-cycloid trajectory radius ρ for: $r = 1$

$$x = -(R-1)\sin\phi + \sin(R-1)\phi \qquad x^2 = (R-1)^2\sin^2\phi - 2(R-1)\sin\phi\sin(R-1)\phi + \sin^2(R-1)\phi$$

$$y = (R-1)\cos\phi + \cos(R-1)\phi \qquad y^2 = (R-1)^2\cos^2\phi + 2(R-1)\cos\phi\cos(R-1)\phi + \cos^2(R-1)\phi$$

$$\rho^2 = x^2 + y^2 = (R-1)^2 + 2(R-1)\left[\cos\phi\cos(R-1)\phi - \sin\phi\sin(R-1)\phi\right] + 1$$

$$\rho^2 = x^2 + y^2 = (R-1)^2 + 2(R-1)\cos(R\phi) + 1$$



Hypo-Cycloidal coordinate geometry and dynamics

Hypo-cycloid constrained by: $-\theta r = -R\phi$ or: $\theta = \frac{R}{r}\phi$

$$x = -(R-r)\sin\phi + r\sin(\theta\text{-}\phi) = r\left[-\left(\frac{R}{r}-1\right)\sin\phi + \sin\left(\frac{R}{r}-1\right)\phi\right]$$

$$y = \ (R-r)\cos\phi + r\cos(\theta\text{-}\phi) = r\left[\ \left(\frac{R}{r}-1\right)\cos\phi + \cos\left(\frac{R}{r}-1\right)\phi\right]$$

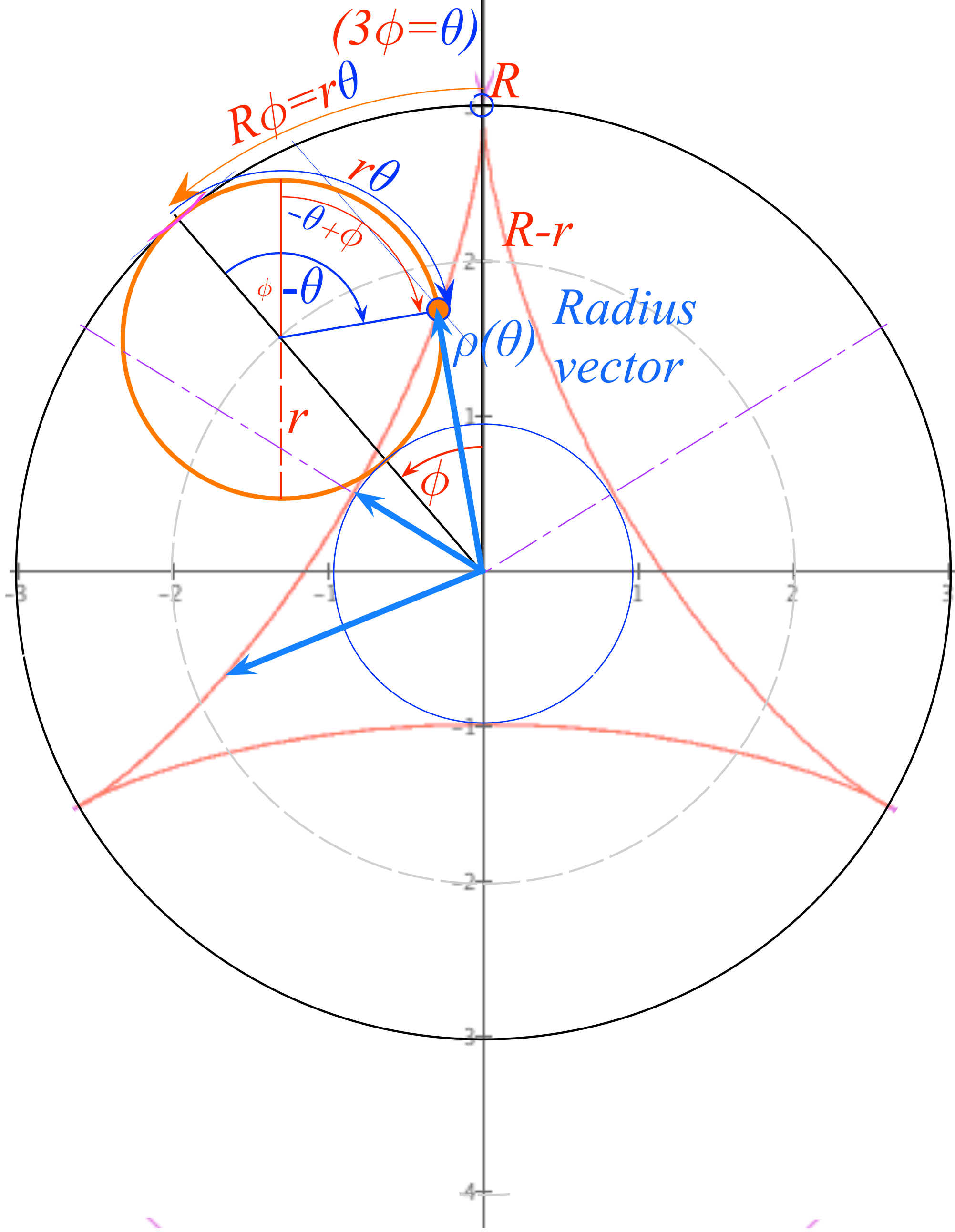
Hypo-cycloid trajectory radius ρ for: $r = 1$

$$x = -(R-1)\sin\phi + \sin(R-1)\phi \qquad x^2 = (R-1)^2\sin^2\phi - 2(R-1)\sin\phi\sin(R-1)\phi + \sin^2(R-1)\phi$$

$$y = \ (R-1)\cos\phi + \cos(R-1)\phi \qquad y^2 = (R-1)^2\cos^2\phi + 2(R-1)\cos\phi\cos(R-1)\phi + \cos^2(R-1)\phi$$

$$\rho^2 = x^2 + y^2 = (R-1)^2 + 2(R-1)\left[\cos\phi\cos(R-1)\phi - \sin\phi\sin(R-1)\phi\right] + 1$$

$$\rho^2 = x^2 + y^2 = (R-1)^2 + 2(R-1)\cos(R\phi) + 1 = R^2 - 4(R-1)\sin^2\frac{R\phi}{2}$$



Crossed E and B field mechanics

Classical Hall-effect and cyclotron orbit orbit equations

Vector theory vs. complex variable theory

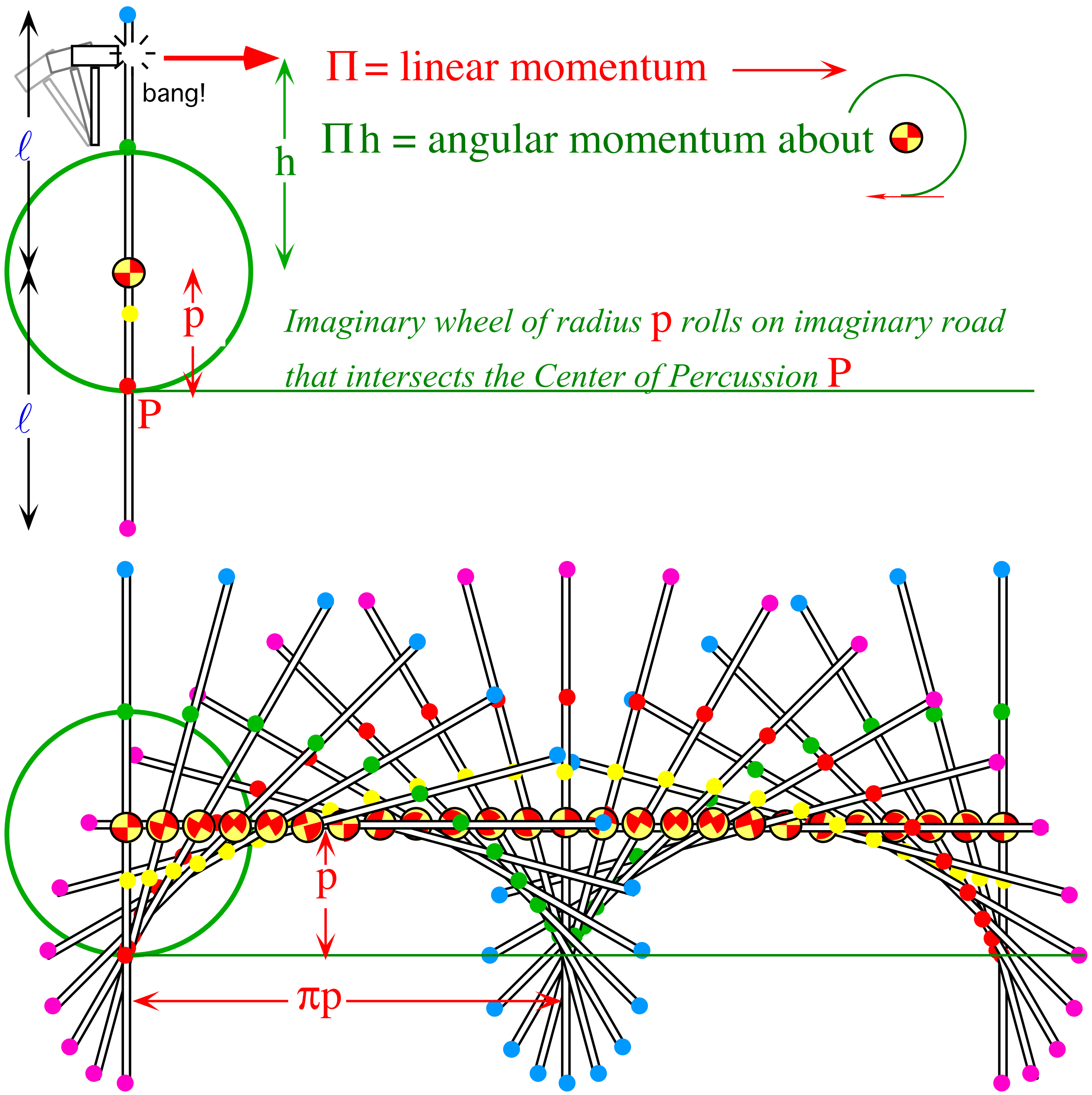
Mechanical analog of cyclotron and FBI rule

Cycloid and epicycloid ruler&compass geometry

→ Cycloid geometry of flying levers

Practical poolhall application

If you hammer a stick at a point h meters from its center
you give it some linear momentum Π
and some angular momentum $\Lambda = h\cdot\Pi$



Π = linear momentum

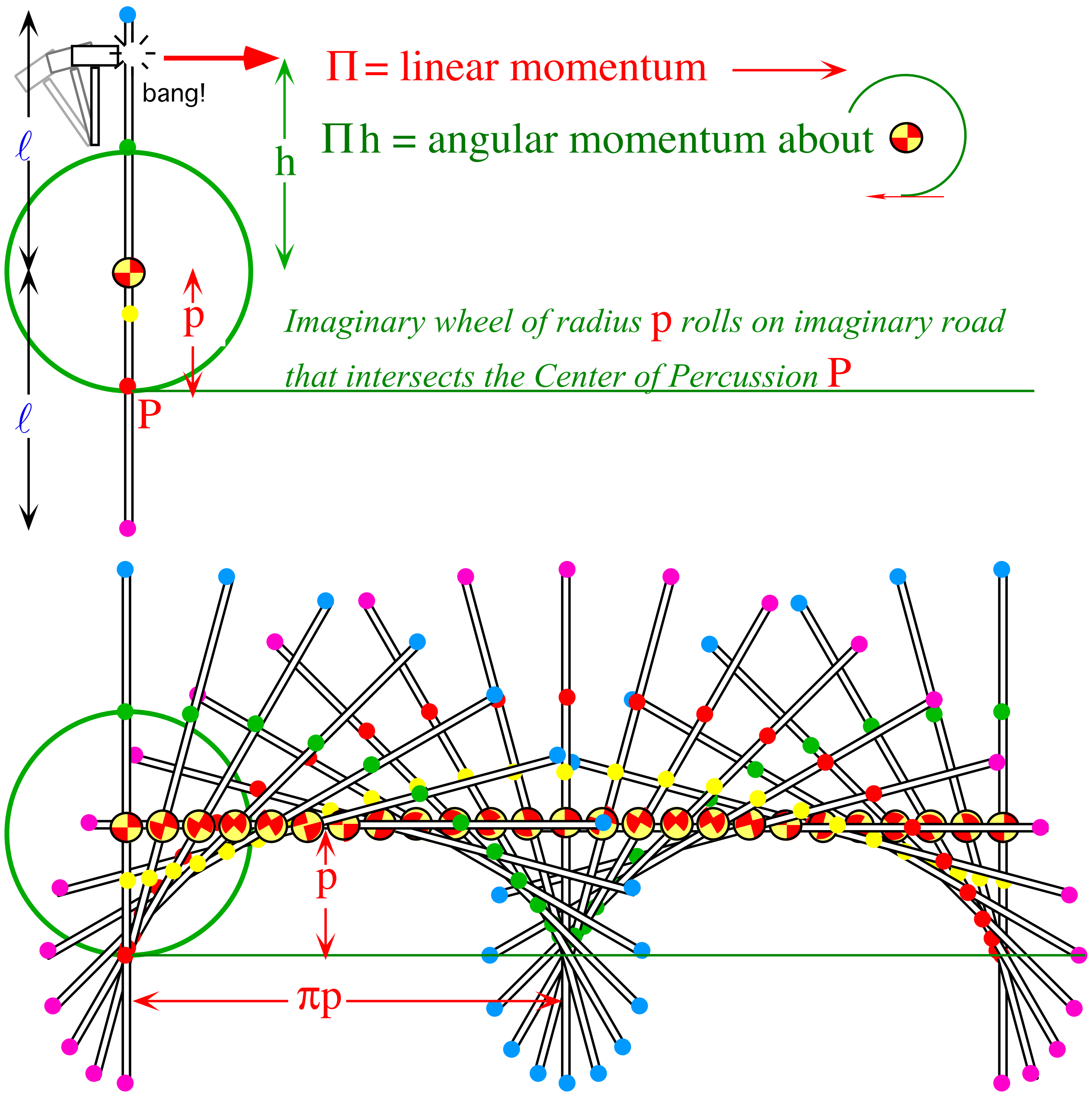
Πh = angular momentum about

Imaginary wheel of radius p *rolls on imaginary road*
that intersects the Center of Percussion P

Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

If you hammer a stick at a point h meters from its center
you give it some linear momentum Π
and some angular momentum $\Lambda = h \cdot \Pi$

Resulting angular velocity ω about the center
is angular momentum Λ divided by
moment of inertia $I = M\,\ell^2/3$ of the stick.



Π = linear momentum →

Πh = angular momentum about

Imaginary wheel of radius p *rolls on imaginary road*
that intersects the Center of Percussion P

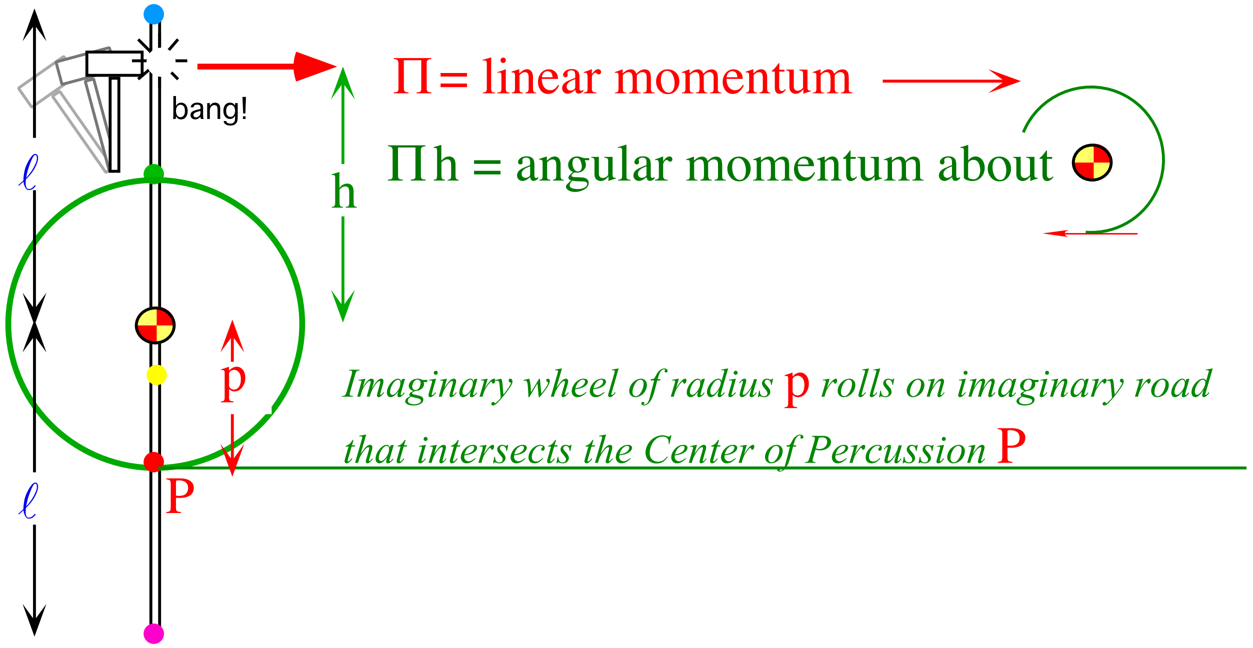
Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

If you hammer a stick at a point h meters from its center
you give it some linear momentum Π
and some angular momentum $\Lambda = h \cdot \Pi$

Resulting angular velocity ω about the center
is angular momentum Λ divided by
moment of inertia $I = M\,\ell^2/3$ of the stick.

$$\omega = \Lambda / I \qquad (=3\Lambda/(M\,\ell^2)\ \text{for stick})$$

$$= h\Pi / I \quad (=3h\Pi/(M\,\ell^2)\text{for stick})$$



Π = linear momentum

Πh = angular momentum about

Imaginary wheel of radius p *rolls on imaginary road that intersects the Center of Percussion* P

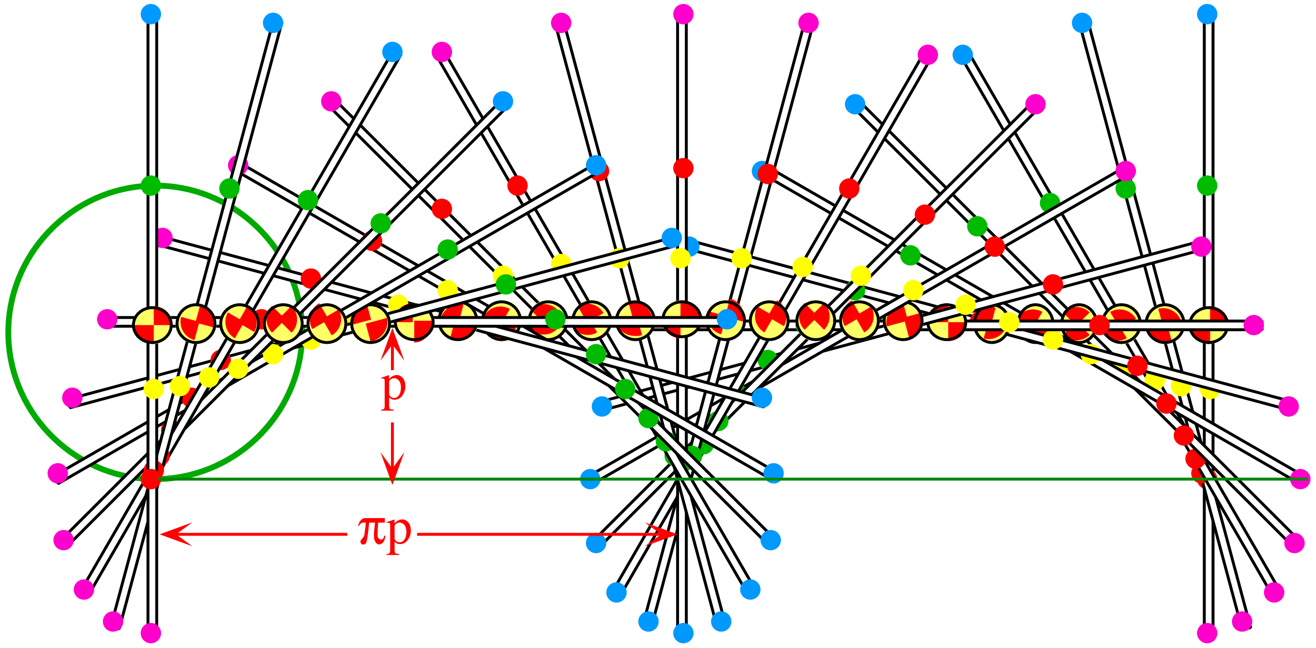
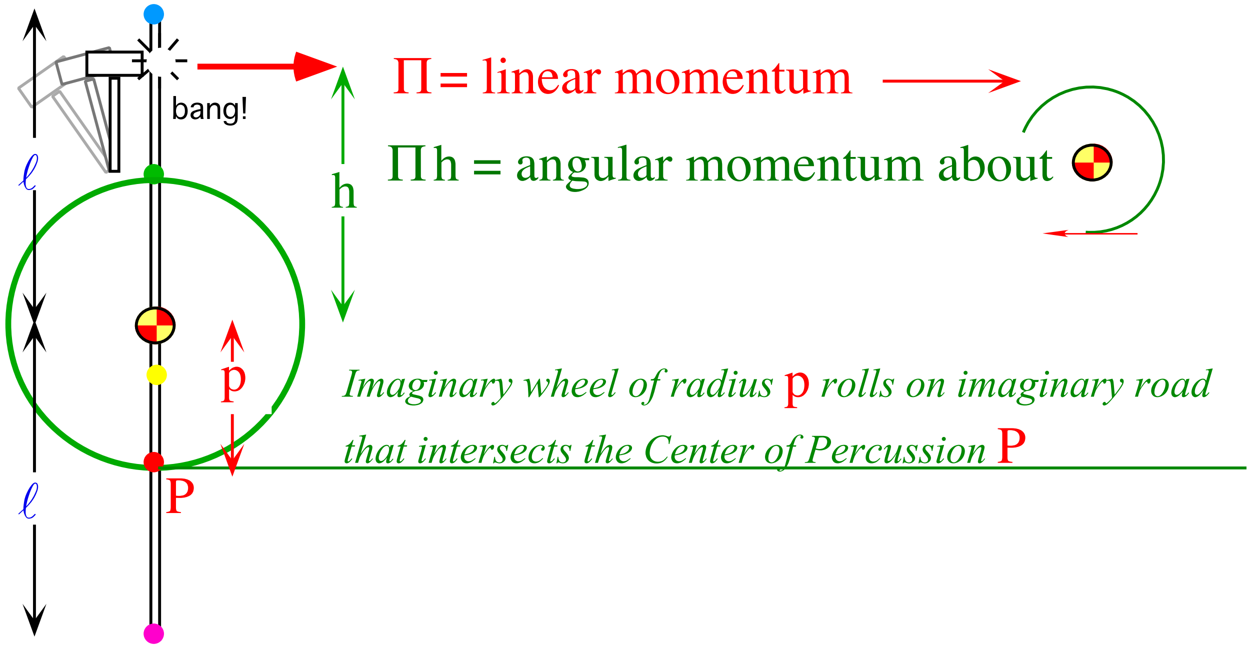


Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

If you hammer a stick at a point h meters from its center
you give it some linear momentum Π
and some angular momentum $\Lambda = h \cdot \Pi$

Resulting angular velocity ω about the center
is angular momentum Λ divided by
moment of inertia $I = M\,\ell^2/3$ of the stick.



Π= linear momentum →

Πh = angular momentum about

Imaginary wheel of radius p *rolls on imaginary road*
that intersects the Center of Percussion P

$$\omega = \Lambda\,/\,I \qquad (=3\Lambda\,/(M\,\ell^2)\ \text{ for stick})$$
$$= h\Pi\,/\,I \quad (=3h\Pi/(M\,\ell^2)\text{for stick})$$

One point P, or *center of percussion* (CoP), is
on the wheel where speed $p\omega$ due to rotation
just cancels translational speed V_{Center} of stick.

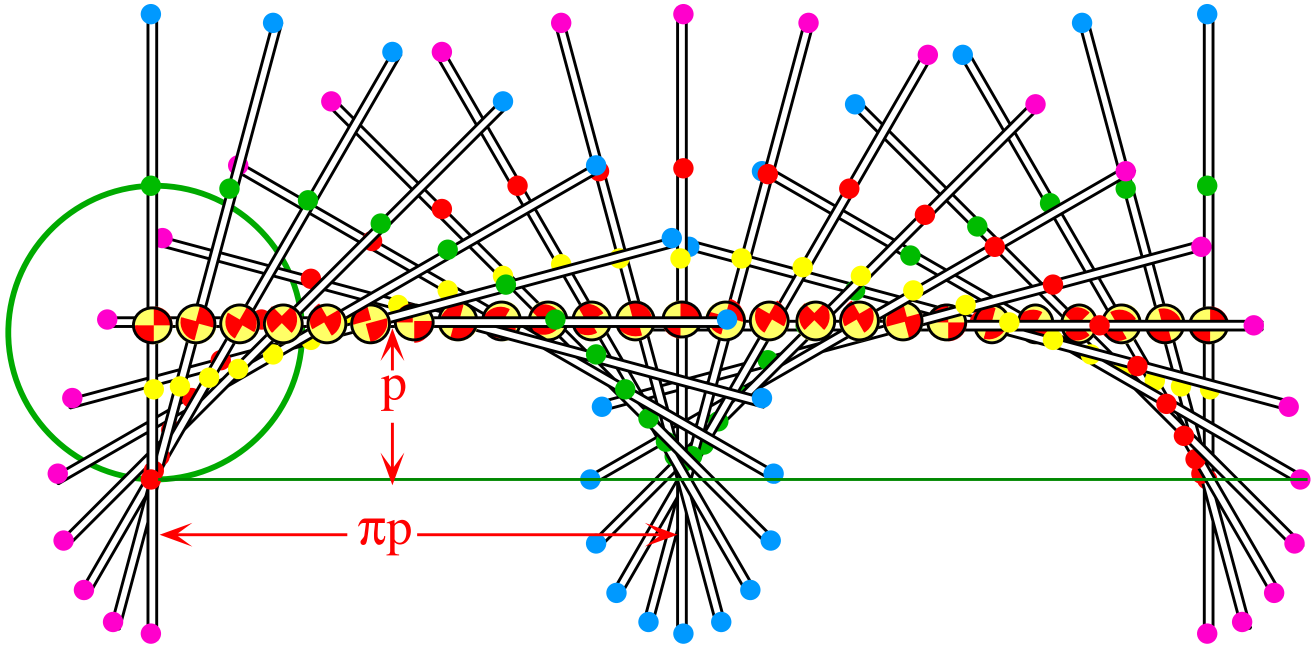
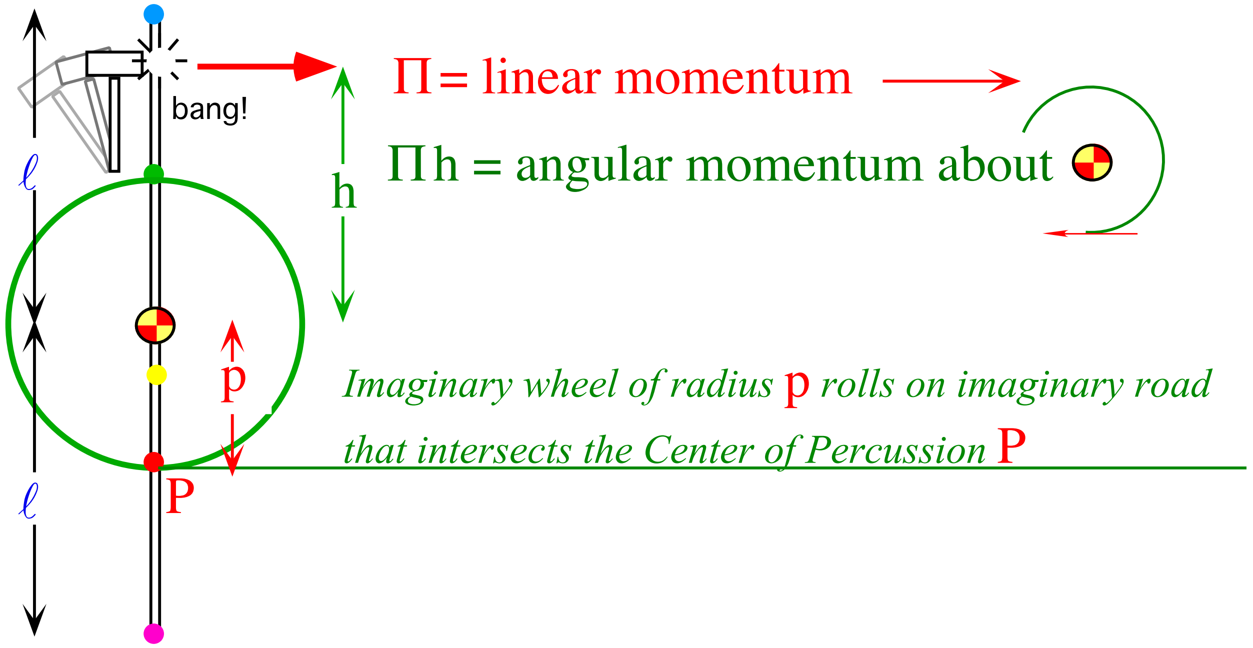


Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

If you hammer a stick at a point h meters from its center
you give it some linear momentum Π
and some angular momentum $\Lambda = h\cdot\Pi$



Π = linear momentum →

Πh = angular momentum about

Imaginary wheel of radius p *rolls on imaginary road that intersects the Center of Percussion* P

Resulting angular velocity ω about the center is angular momentum Λ divided by moment of inertia $I = M\,\ell^2/3$ of the stick.

$$\omega = \Lambda / I \qquad (=3\Lambda/(M\,\ell^2)\ \text{for stick})$$
$$= h\Pi / I \qquad (=3h\Pi/(M\,\ell^2)\ \text{for stick})$$

One point P, or *center of percussion* (CoP), is on the wheel where speed $p\omega$ due to rotation just cancels translational speed V_{Center} of stick.

$$\Pi / M = V_{Center} = |p\omega| = p\cdot h\Pi/I$$

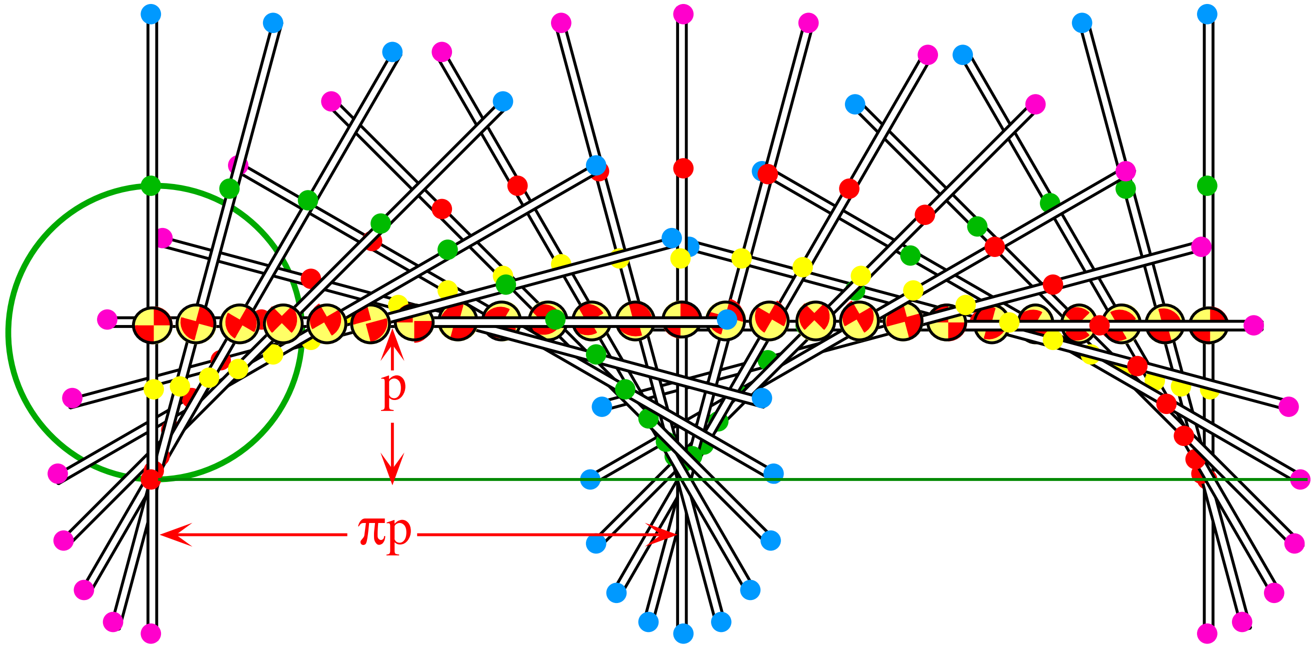
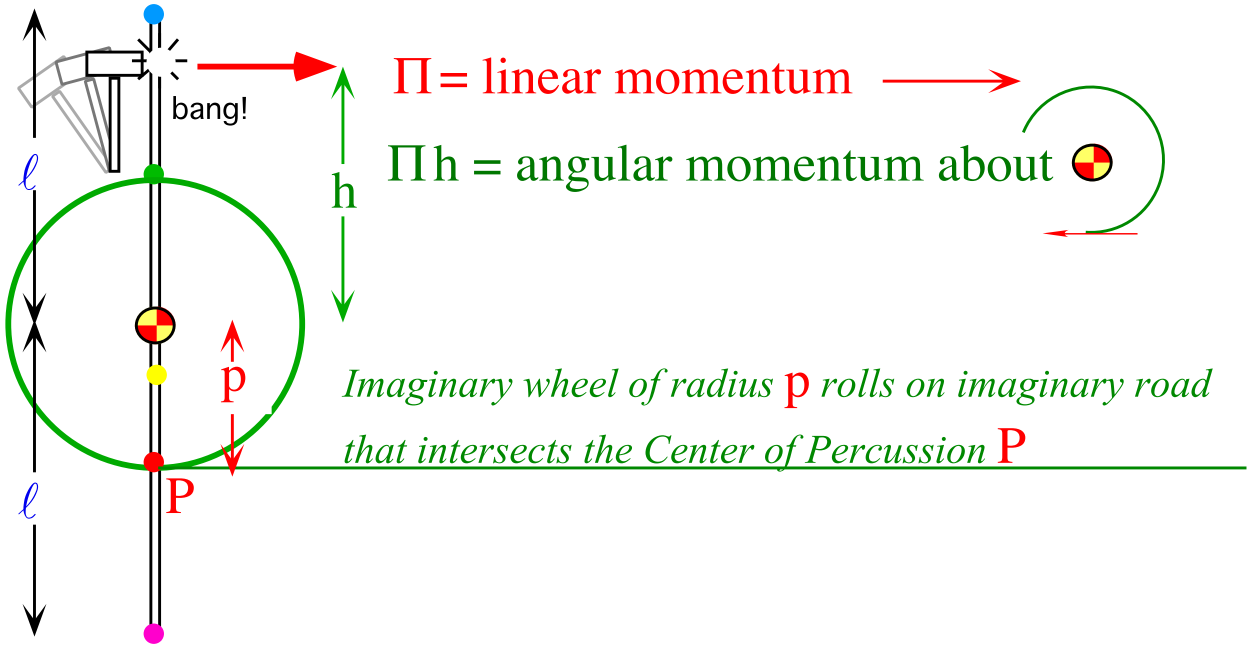


Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

If you hammer a stick at a point h meters from its center you give it some linear momentum Π and some angular momentum $\Lambda = h\cdot\Pi$



Π = linear momentum

Πh = angular momentum about

Imaginary wheel of radius p *rolls on imaginary road that intersects the Center of Percussion* P

Resulting angular velocity ω about the center is angular momentum Λ divided by moment of inertia $I = M\,\ell^2/3$ of the stick.

$$\omega = \Lambda / I \qquad (=3\Lambda/(M\,\ell^2)\ \text{for stick})$$

$$= h\Pi / I \qquad (=3h\Pi/(M\,\ell^2)\text{for stick})$$

One point P, or *center of percussion* (CoP), is on the wheel where speed $p\omega$ due to rotation just cancels translational speed V_{Center} of stick.

$$\Pi/M = V_{Center} = |p\omega| = p\cdot h\Pi/I$$

$$I/M = \qquad = \qquad = p\cdot h$$

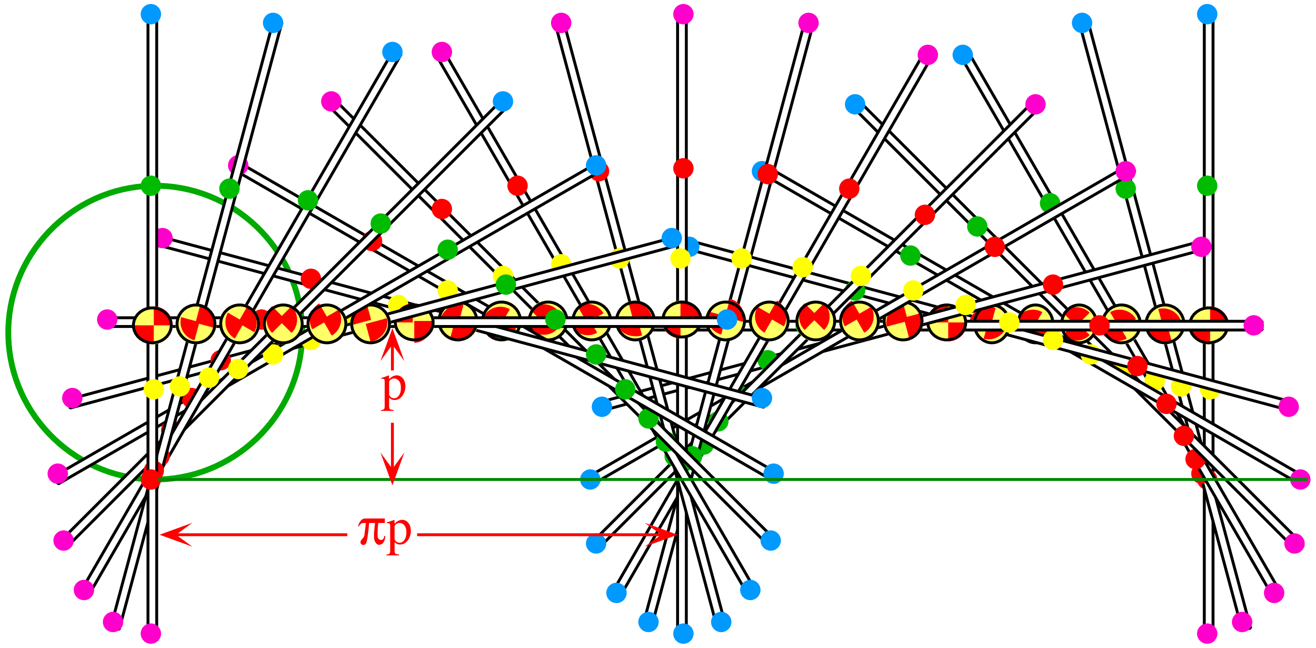


Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

If you hammer a stick at a point h meters from its center
you give it some linear momentum Π
and some angular momentum $\Lambda = h\cdot\Pi$

Resulting angular velocity ω about the center
is angular momentum Λ divided by
moment of inertia $I = M\,\ell^2/3$ of the stick.

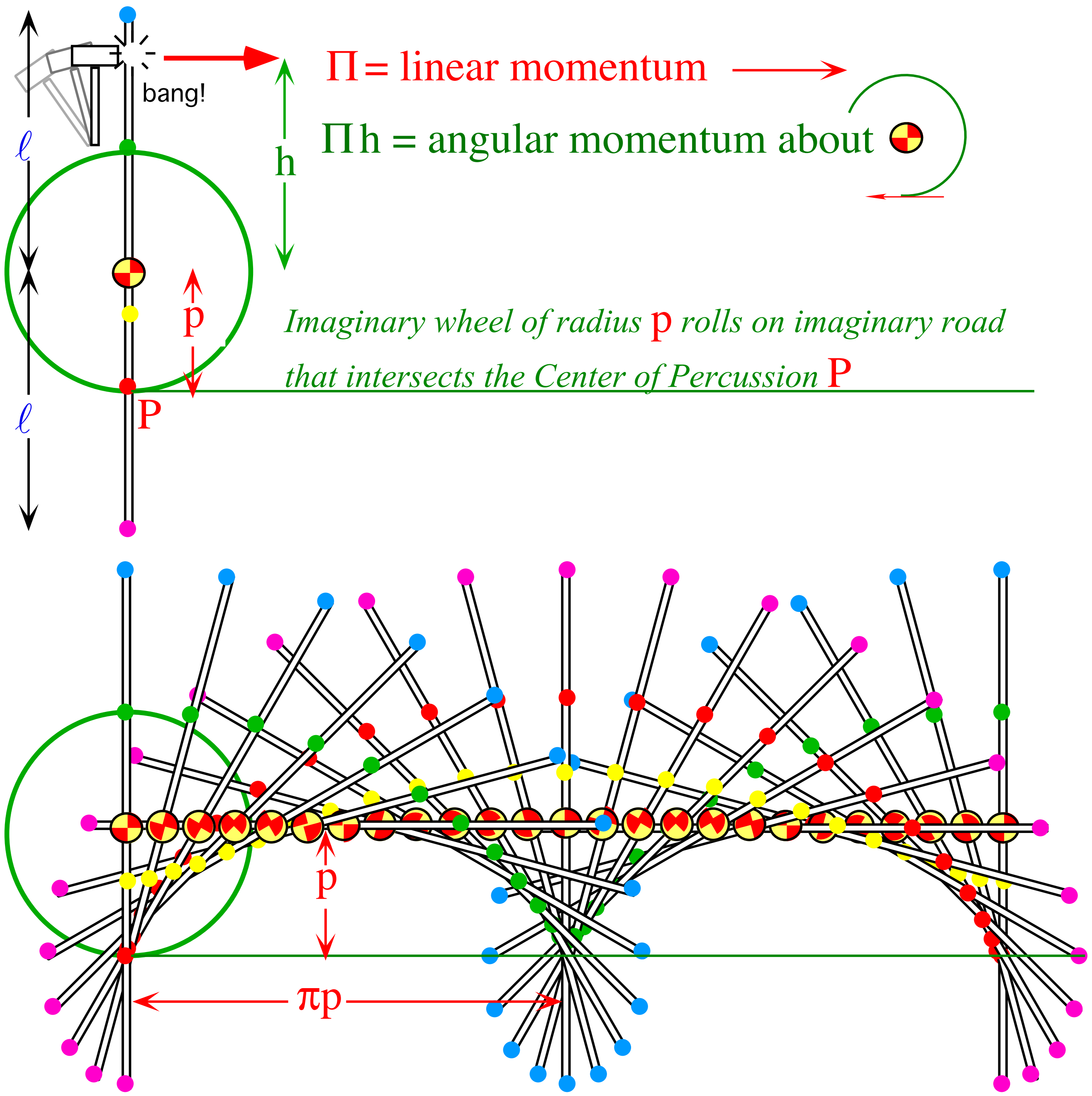
$$\omega = \Lambda / I \qquad (=3\Lambda/(M\,\ell^2)\ \text{for stick})$$

$$= h\Pi / I \quad (=3h\Pi/(M\,\ell^2)\text{for stick})$$

One point P, or *center of percussion* (CoP), is
on the wheel where speed $p\omega$ due to rotation
just cancels translational speed V_{Center} of stick.

$$\Pi/M = V_{Center} = |p\omega| = p\cdot h\Pi/I$$

$$I/M = \qquad = \qquad = p\cdot h \qquad \textit{or: } p=I/(Mh)$$



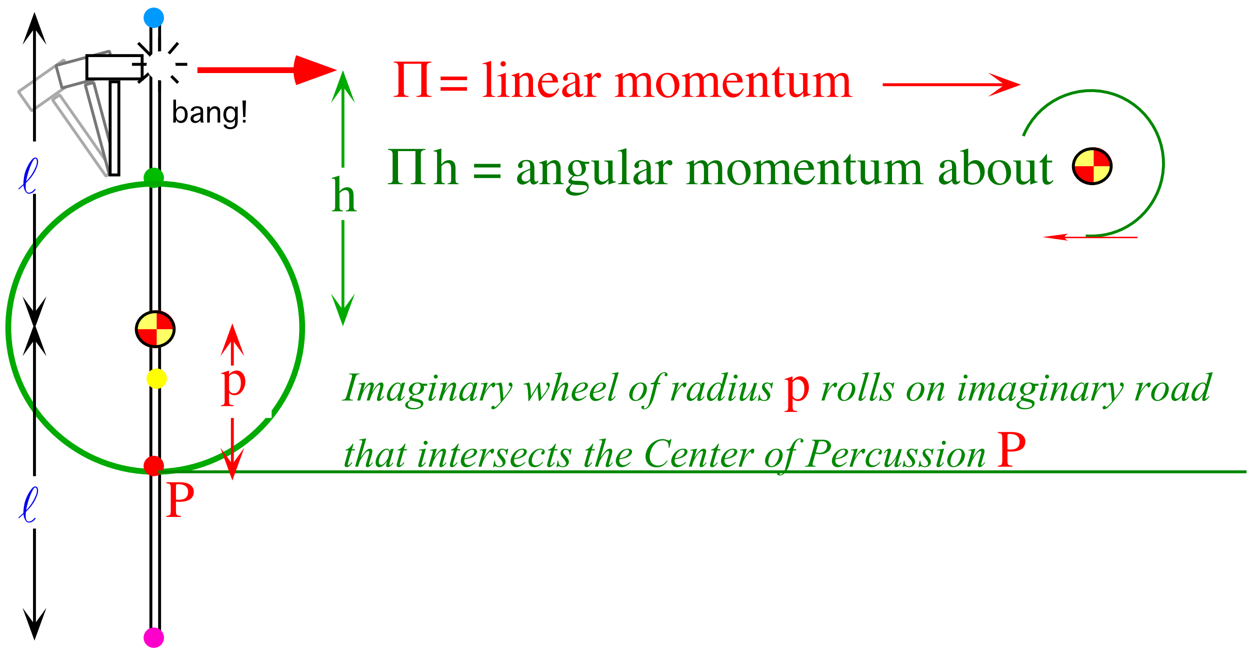
Π= linear momentum

Πh = angular momentum about

Imaginary wheel of radius p *rolls on imaginary road that intersects the Center of Percussion* P

Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

If you hammer a stick at a point h meters from its center
you give it some linear momentum Π
and some angular momentum $\Lambda = h\cdot\Pi$



Π = linear momentum →

Πh = angular momentum about

Imaginary wheel of radius p *rolls on imaginary road that intersects the Center of Percussion* P

Resulting angular velocity ω about the center
is angular momentum Λ divided by
moment of inertia $I = M\,\ell^2/3$ of the stick.

$$\omega = \Lambda / I \qquad (=3\Lambda/(M\,\ell^2)\ \text{for stick})$$
$$= h\Pi / I \quad (=3h\Pi/(M\,\ell^2)\text{for stick})$$

One point P, or *center of percussion* (CoP), is
on the wheel where speed $p\omega$ due to rotation
just cancels translational speed V_{Center} of stick.

$$\Pi/M = V_{Center} = |p\omega| = p\cdot h\Pi/I$$

$$I/M = \qquad = \qquad = p\cdot h \qquad or:\ p = I/(Mh)$$

P follows a normal cycloid made by a circle
of radius $p=I/(Mh)$ rolling on an imaginary road
thru point P in direction of Π.

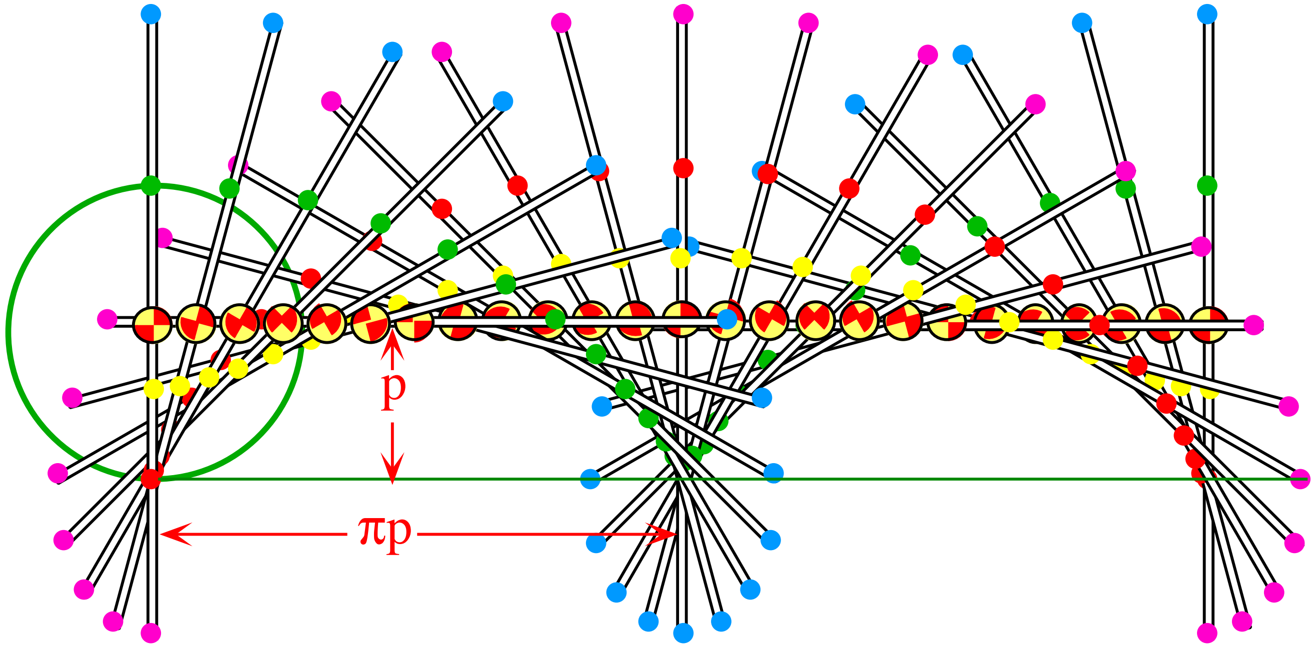
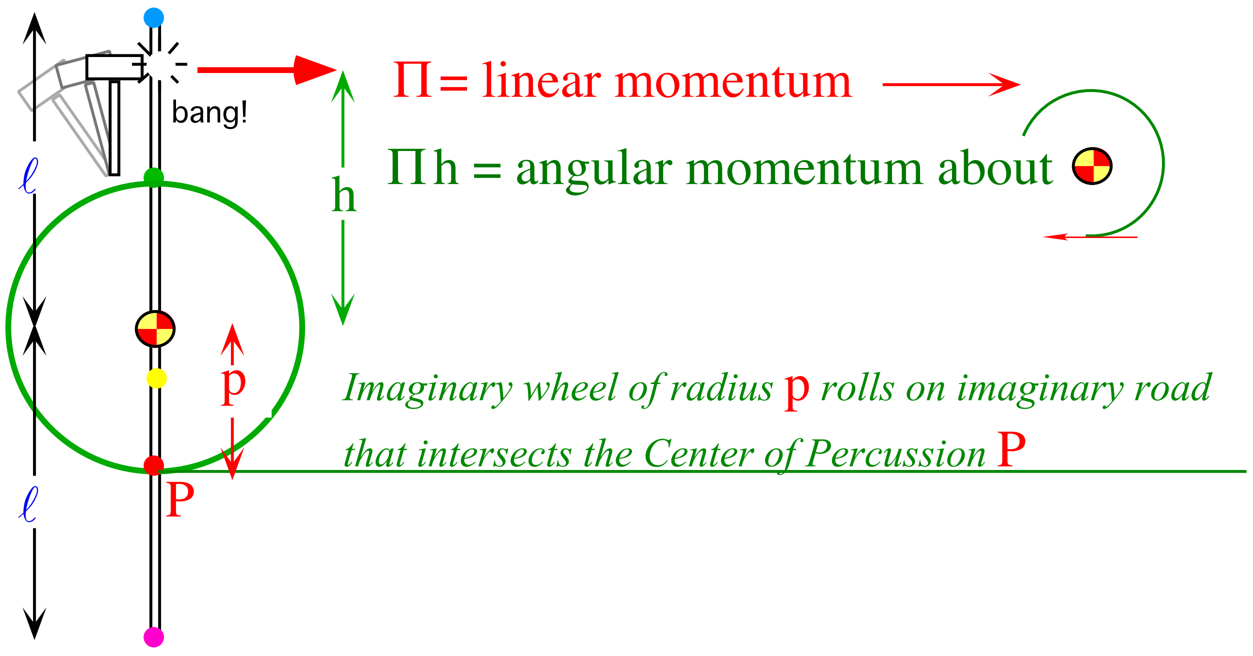


Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

If you hammer a stick at a point h meters from its center
you give it some linear momentum Π
and some angular momentum $\Lambda = h\cdot\Pi$



Π = linear momentum →

Πh = angular momentum about

Imaginary wheel of radius p *rolls on imaginary road*
that intersects the Center of Percussion P

Resulting angular velocity ω about the center
is angular momentum Λ divided by
moment of inertia $I = M\,\ell^2/3$ of the stick.

$$\omega = \Lambda / I \qquad (=3\Lambda/(M\,\ell^2) \text{ for stick)}$$
$$= h\Pi / I \quad (=3h\Pi/(M\,\ell^2)\text{for stick)}$$

One point P, or *center of percussion* (CoP), is
on the wheel where speed $p\omega$ due to rotation
just cancels translational speed V_{Center} of stick.

$$\Pi/M = V_{Center} = |p\omega| = p\cdot h\Pi/I$$

$$I/M = \qquad = \qquad = p\cdot h \qquad \textit{or: } p = I/(Mh)$$

P follows a normal cycloid made by a circle
of radius $p=I/(Mh)$ rolling on an imaginary road
thru point P in direction of Π.

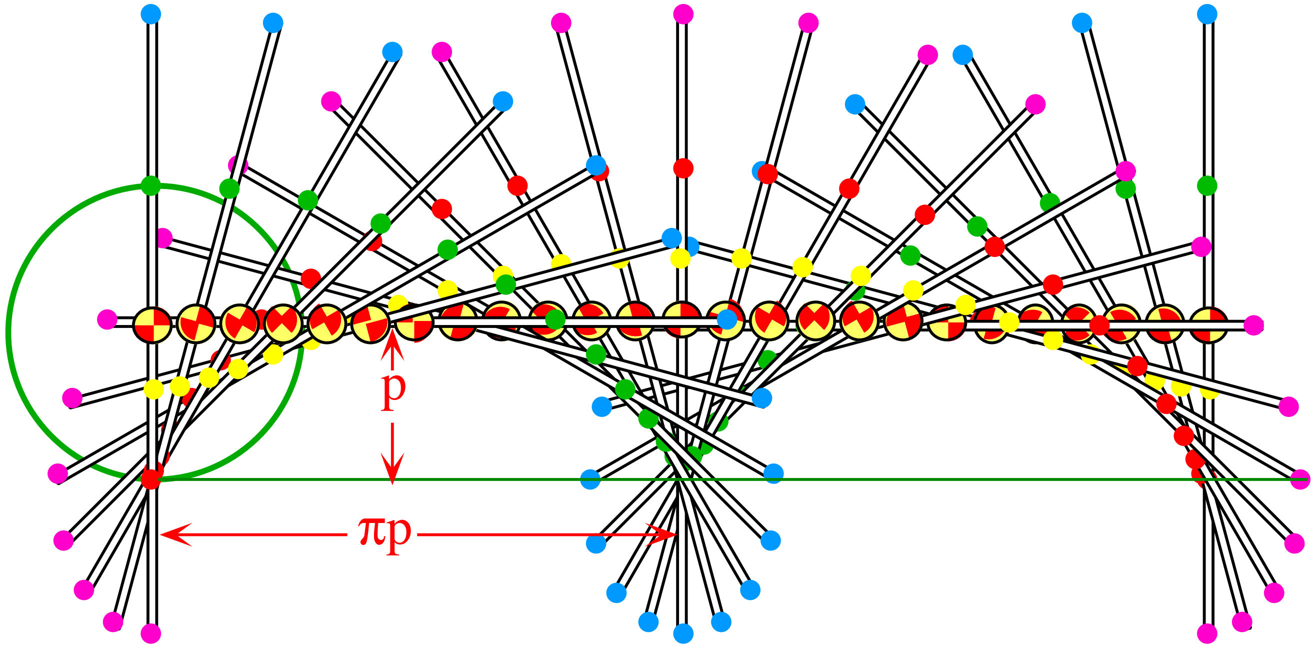


Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

The *percussion radius* $p = \ell^2/3h$ is of the CoP point
that has no velocity just after hammer hits at h.

Crossed E and B field mechanics

Classical Hall-effect and cyclotron orbit orbit equations

Vector theory vs. complex variable theory

Mechanical analog of cyclotron and FBI rule

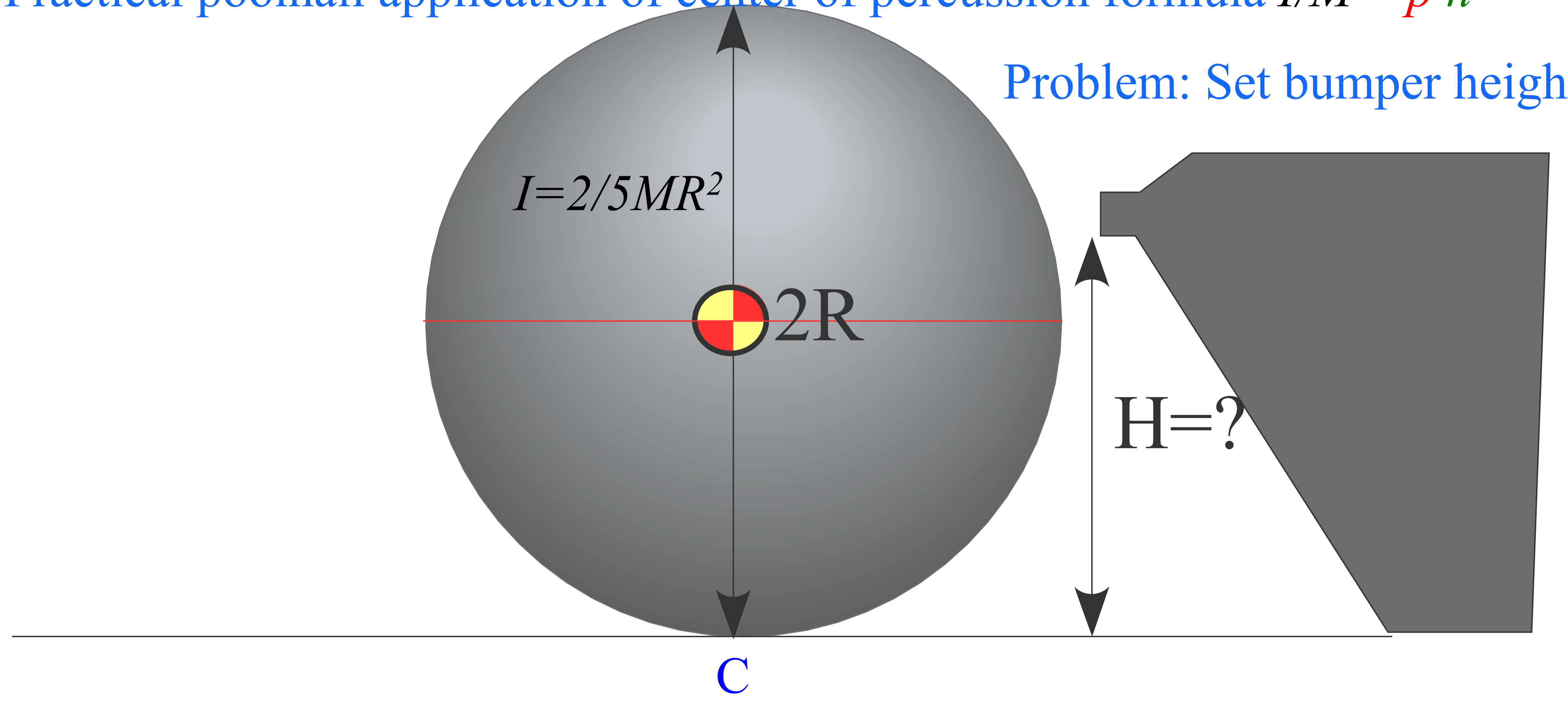
Cycloid and epicycloid ruler&compass geometry

Cycloid geometry of flying levers

→ *Practical poolhall application*

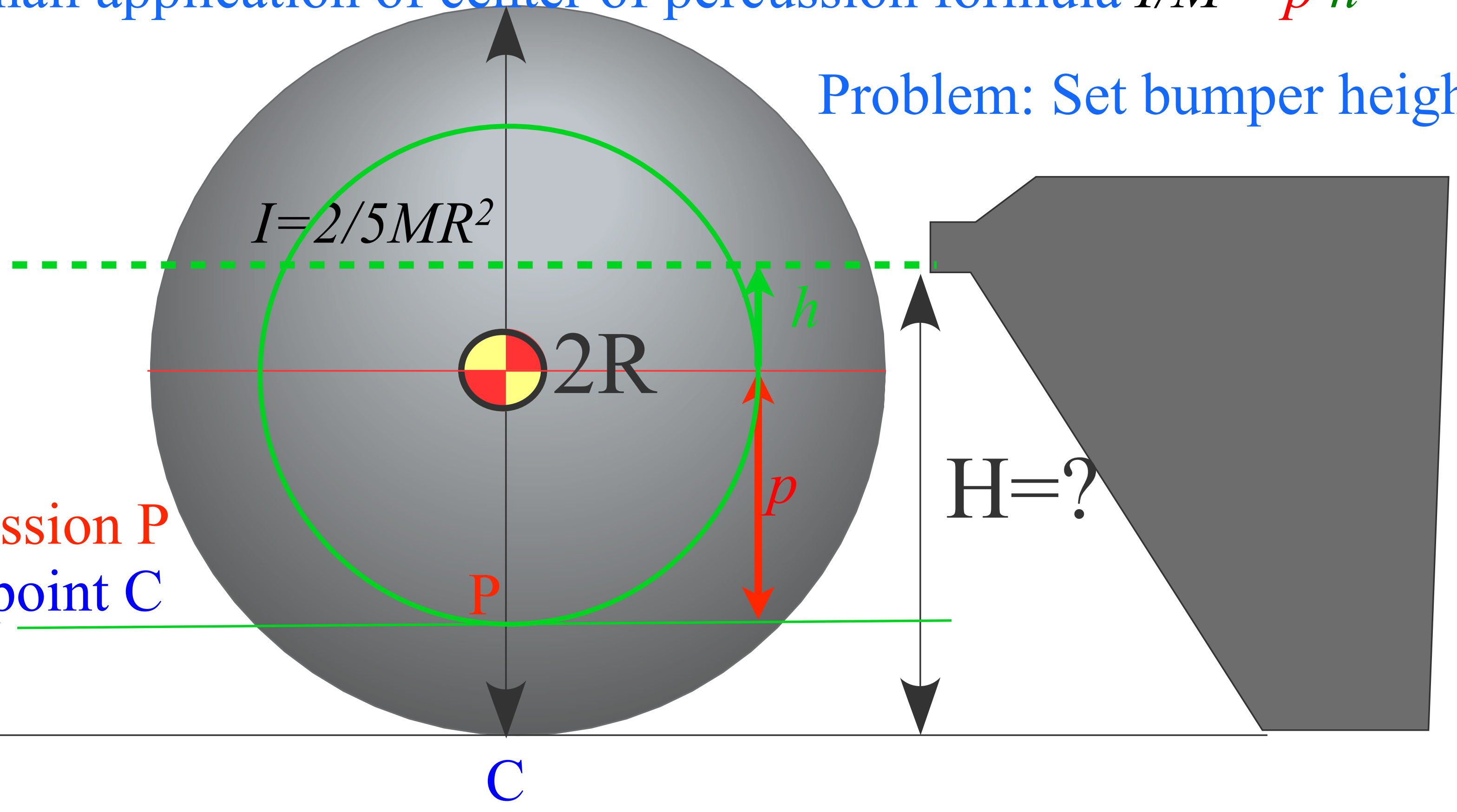
Practical poolhall application of center of percussion formula $I/M = p \cdot h$

Problem: Set bumper height H so ball does not skid.



Practical poolhall application of center of percussion formula $I/M = p\cdot h$

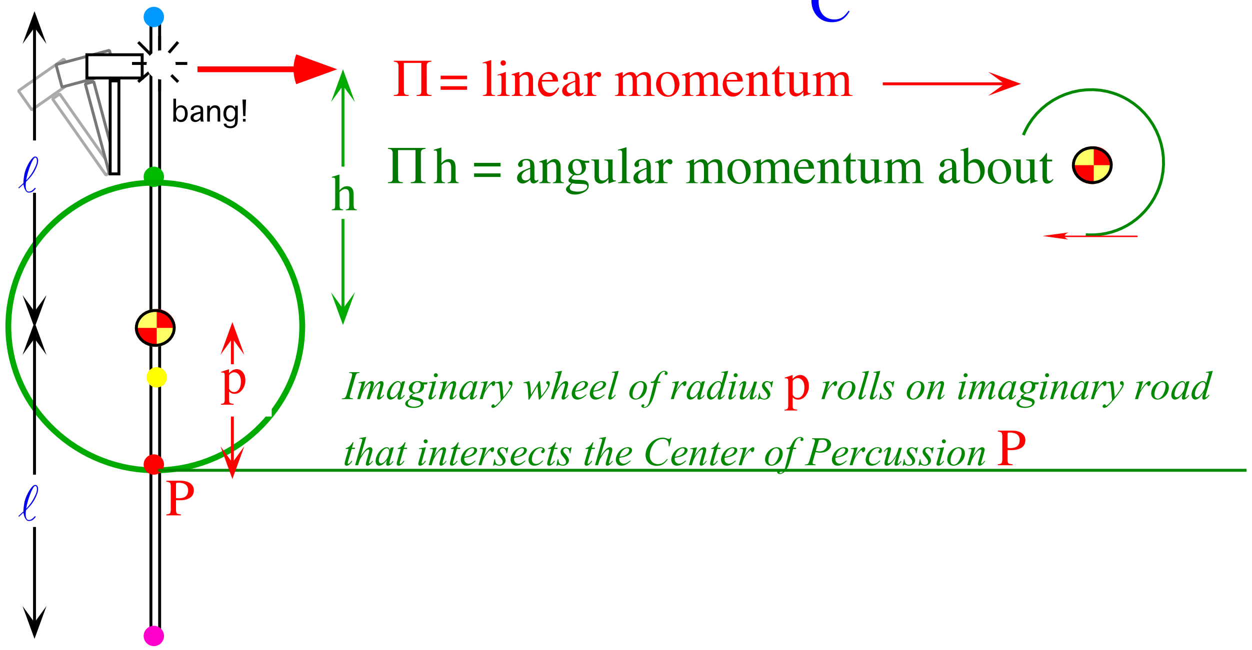
Problem: Set bumper height H so ball does not skid.



Where should bumper height H be set to make ball contact point C at the center of percussion P?

center of percussion P above contact point C

$I/M = p\cdot h$



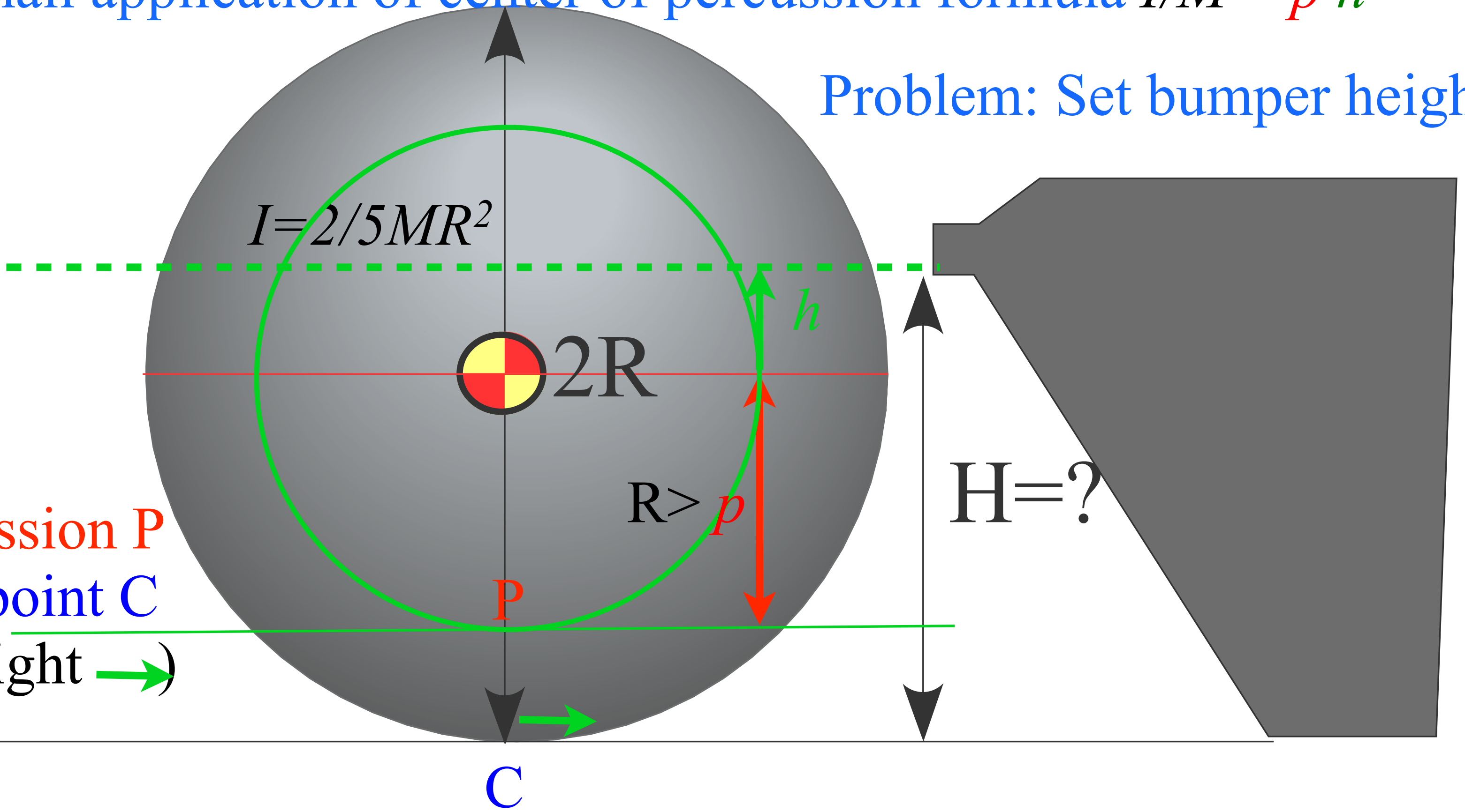
Π = linear momentum

Πh = angular momentum about

Imaginary wheel of radius p *rolls on imaginary road that intersects the Center of Percussion* P

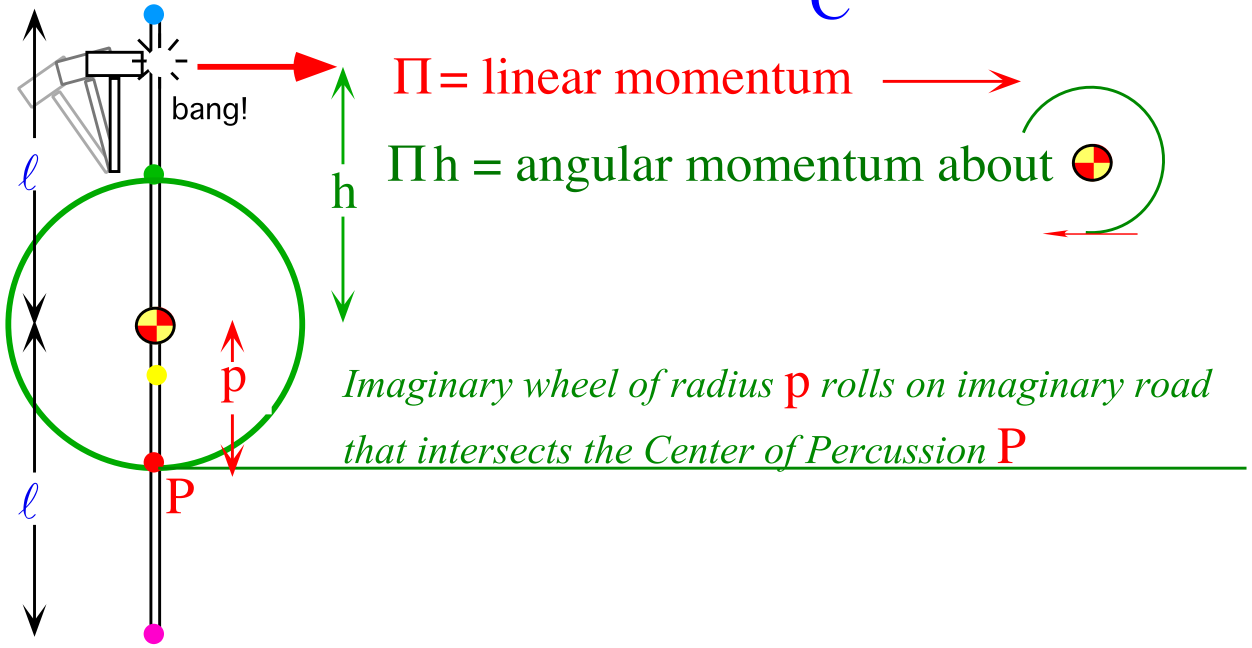
Practical poolhall application of center of percussion formula $I/M = p\cdot h$

Problem: Set bumper height H so ball does not skid.



Where should bumper height H be set to make ball contact point C at the center of percussion P?

center of percussion P
above contact point C
(Ball skids to right →)



Π = linear momentum →

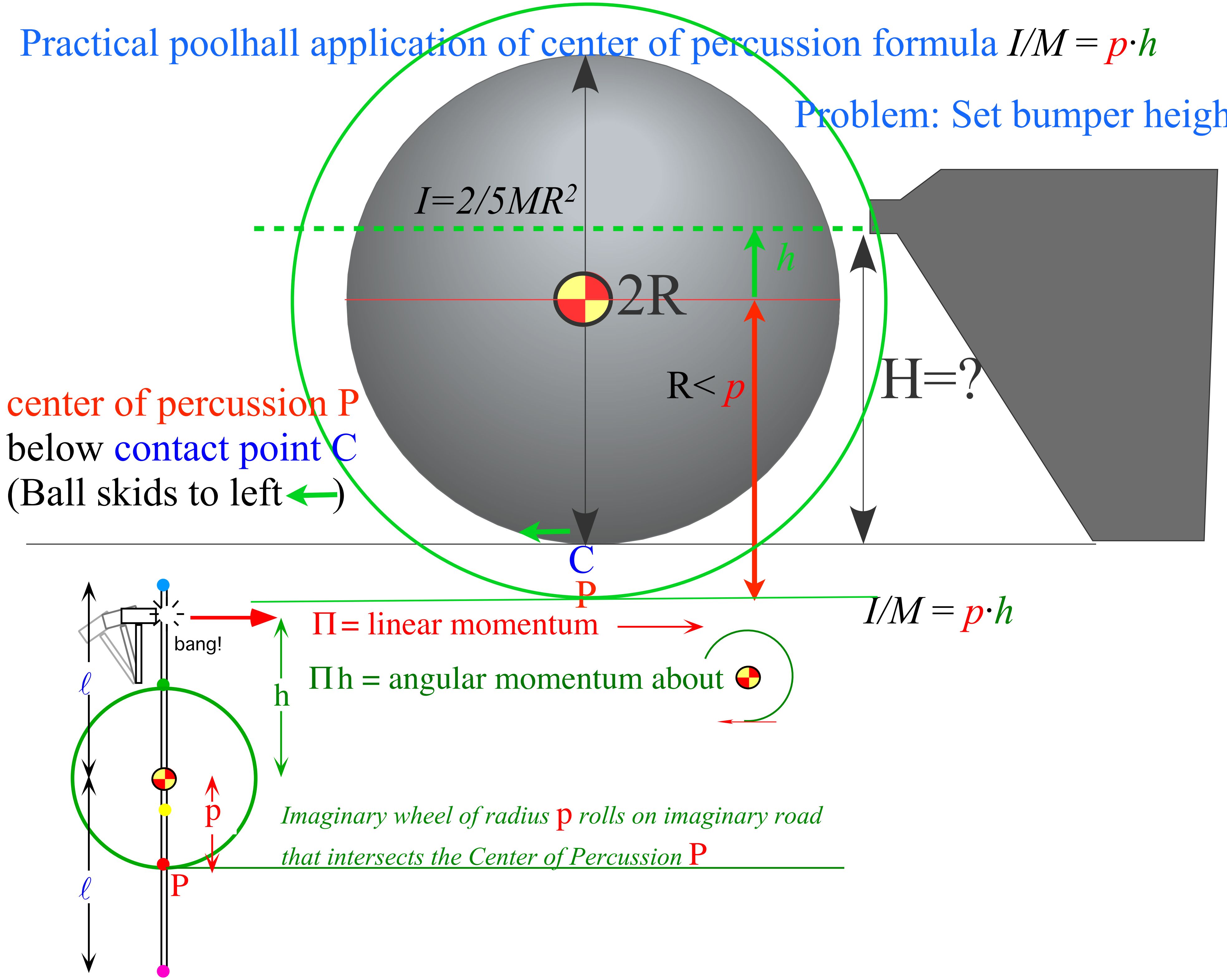
Πh = angular momentum about ⊕

Imaginary wheel of radius p *rolls on imaginary road that intersects the Center of Percussion* P

$I/M = p\cdot h$

Practical poolhall application of center of percussion formula $I/M = p\cdot h$

Problem: Set bumper height H so ball does not skid.



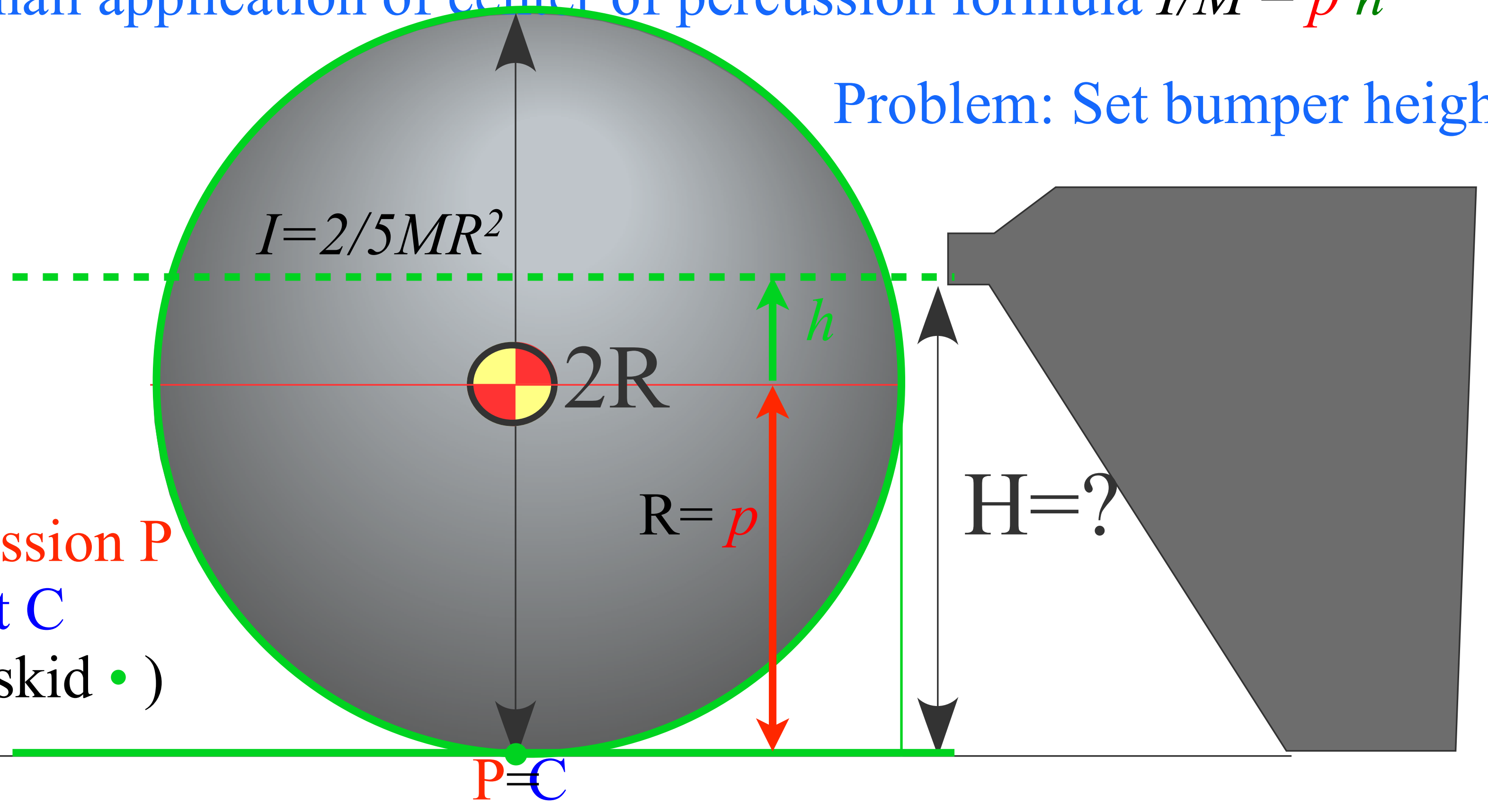
Where should bumper height H be set to make ball contact point C at the center of percussion P?

center of percussion P
below contact point C
(Ball skids to left ←)

Imaginary wheel of radius p *rolls on imaginary road*
that intersects the Center of Percussion P

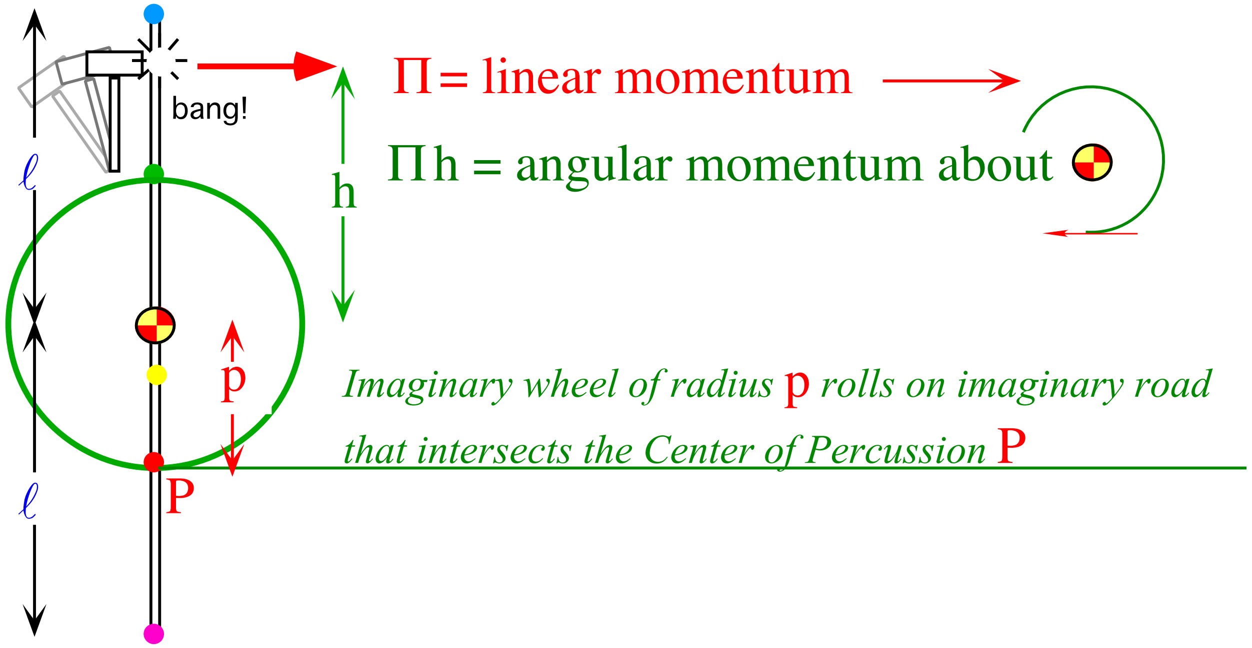
Practical poolhall application of center of percussion formula $I/M = p\cdot h$

Problem: Set bumper height H so ball does not skid.



Where should bumper height H be set to make ball contact point C at the center of percussion P?

center of percussion P
at contact point C
(Ball does not skid •)



Π = linear momentum

Πh = angular momentum about ◐

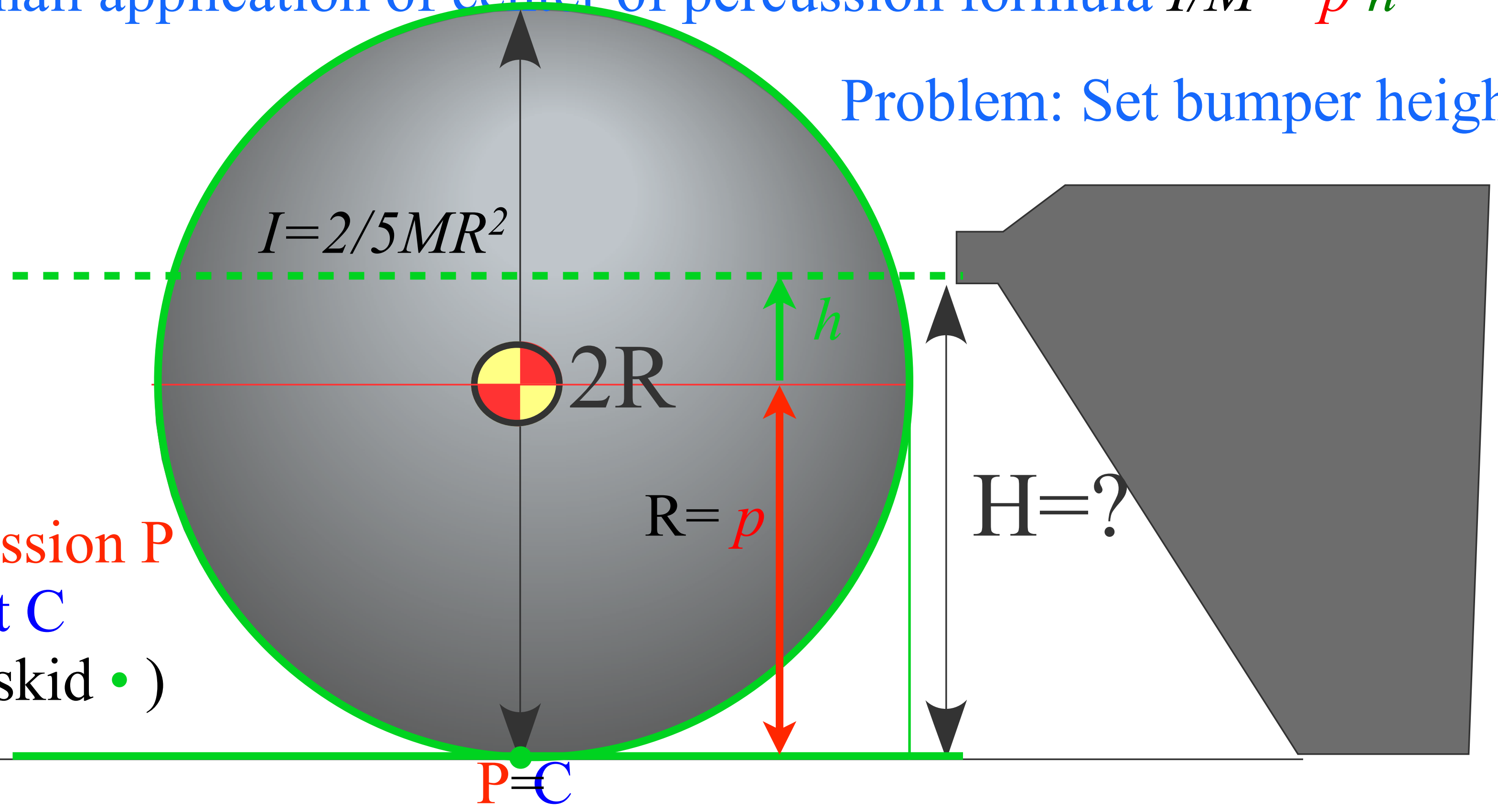
Imaginary wheel of radius p *rolls on imaginary road that intersects the Center of Percussion* P

$I/M = p\cdot h$

$h = I/Mp = I/MR$ (For R= p)

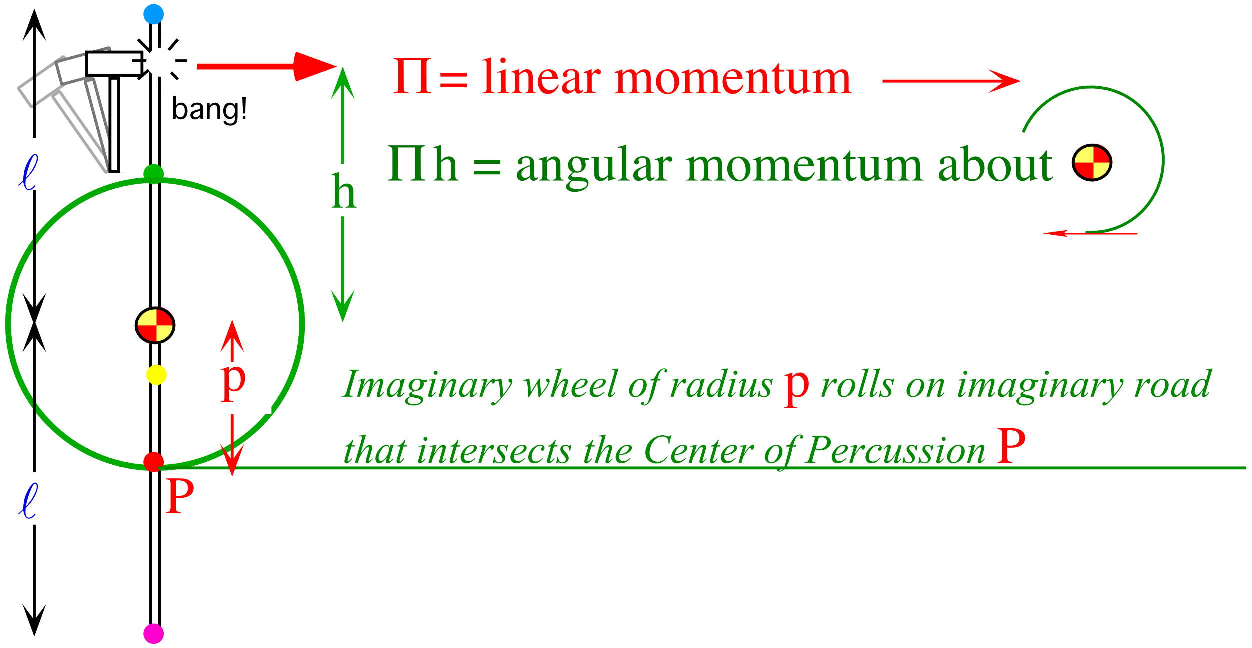
Practical poolhall application of center of percussion formula $I/M = p \cdot h$

Problem: Set bumper height H so ball does not skid.



center of percussion P
at contact point C
(Ball does not skid •)

Where should bumper
height H be set to make
ball contact point C at the
center of percussion P?



Π = linear momentum →

Πh = angular momentum about

Imaginary wheel of radius p *rolls on imaginary road*
that intersects the Center of Percussion P

$$I/M = p \cdot h$$

$$h = I/Mp = I/MR \qquad \text{(For } R = p\text{)}$$

$$= 2/5MR^2/MR$$

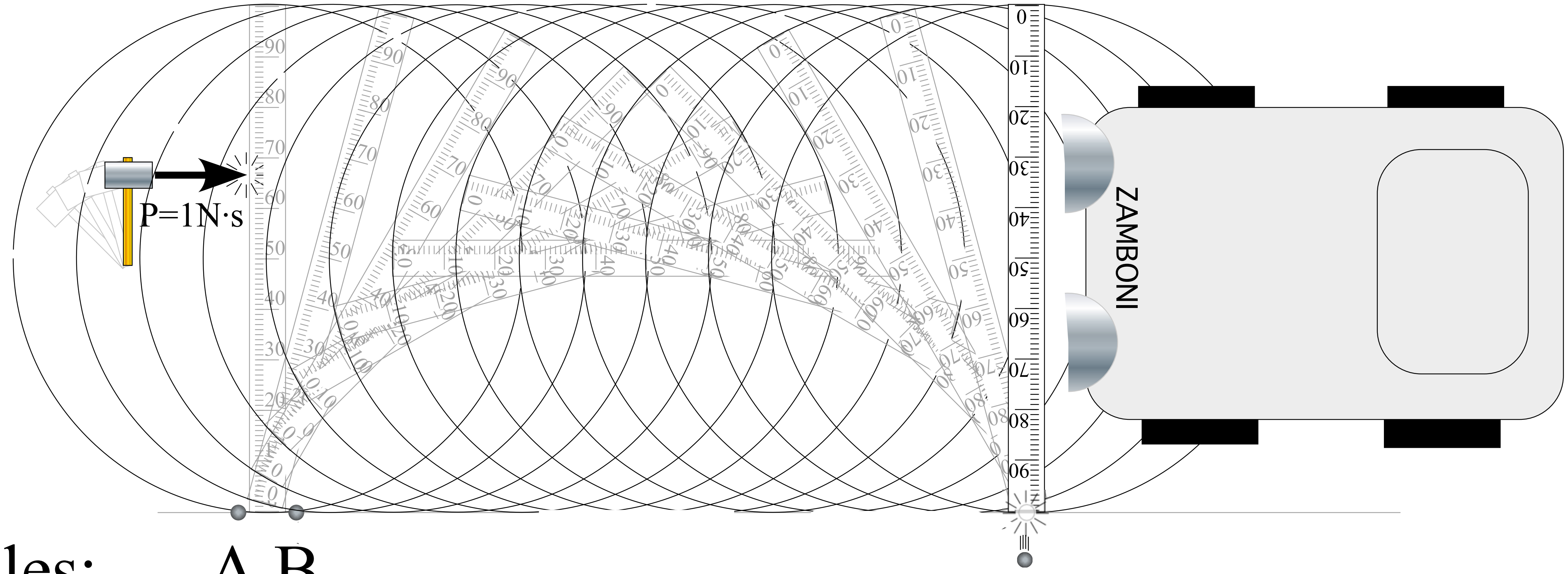
$$= ⅖(R)$$

For: $H = R+h = 7/5(R)$ ball does not skid.

$$= R+h = 7/10(2R)$$

The Zamboni-Ice-Shot problem

(Assumes frictionless ice rink)



Marbles: A B

C

Where on a meter-stick do you hit it so as to not disturb marbles A or B and...

...knock marble C down as shown.