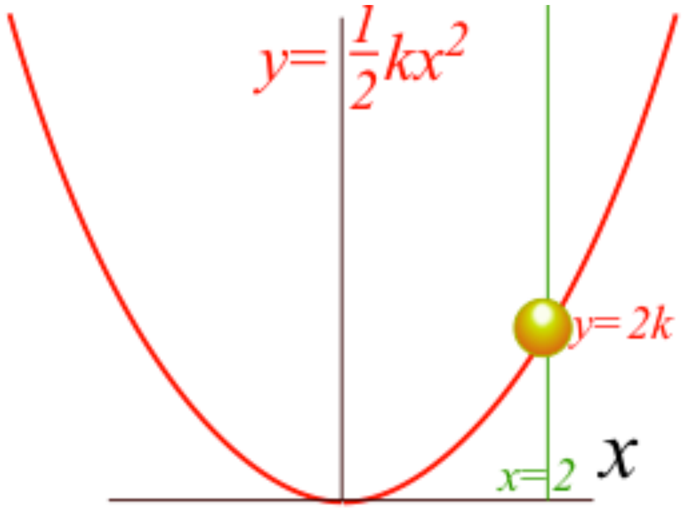
Lecture 19
Thu. 10.27.2016

Classical Constraints: Comparing various methods (Ch. 9 of Unit 3)

Simple constrained problem...



...and a variety of solutions

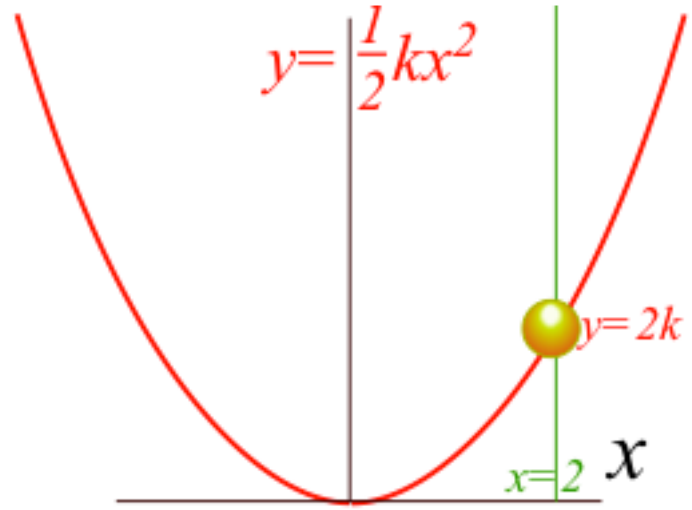
Some Ways to do constraint analysis

- *Way 1. Simple constraint insertion*
- *Way 2. GCC constraint webs*
  - *Find covariant force equations*
  - *Compare covariant vs. contravariant forces*

Other Ways to do constraint analysis

- *Way 3. OCC constraint webs*
  - *Preview of atomic-Stark orbits*
    - *Classical Hamiltonian separability*
- *Way 4. Lagrange multipliers*
  - *Lagrange multiplier as eigenvalues*
  - *Multiple multipliers*
  - *"Non-Holonomic" multipliers*

Simple constrained problem...



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Some Ways to do constraint analysis

→ *Way 1. Simple constraint insertion*

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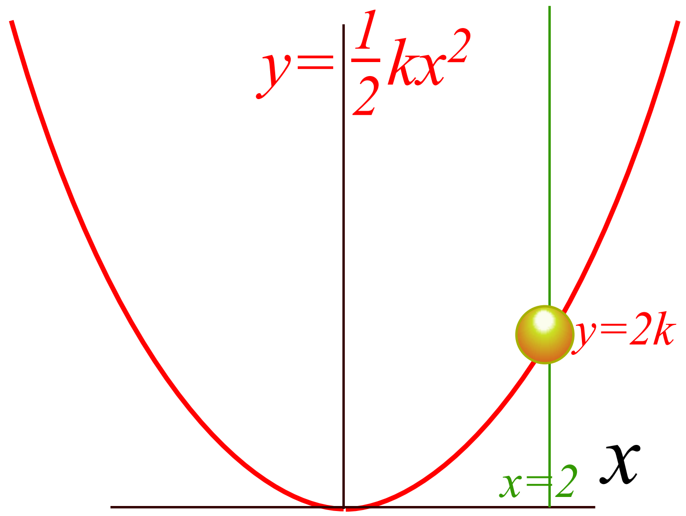
Find covariant force equations

Compare covariant vs. contravariant forces

Ways to analyze a particle m constrained to parabola $y=1/2kx^2$
on (x,y)-plane with gravitational potential $V(\mathbf{r})=mgy$.

(a) Constrained motion



Way 1. Lagrangian has the constraint(s) simply inserted.

$$L=\frac{1}{2}(m\dot{x}^2+m\dot{y}^2)-mgy$$

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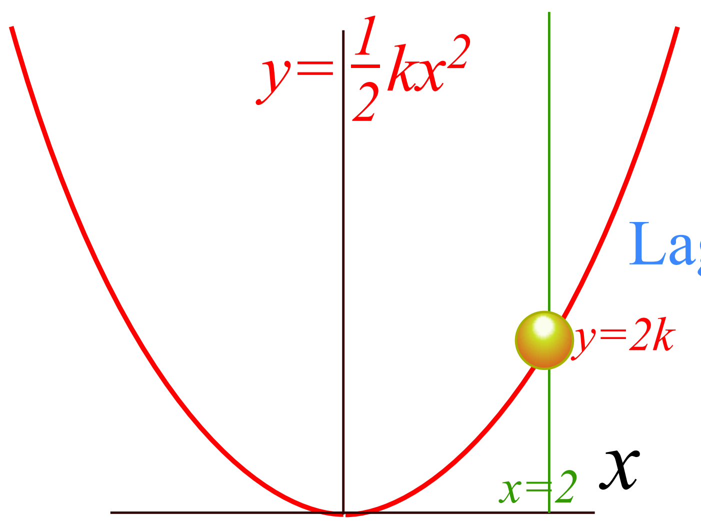

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Lagrangian then has one dimension x, one momentum p_x, and one force f_x.

$$L=\frac{1}{2}(m\dot{x}^2+m(kx\dot{x})^2)-m\frac{1}{2}gkx^2 \qquad p_x=\frac{\partial L}{\partial \dot{x}} \qquad f_x=\frac{\partial L}{\partial x}$$

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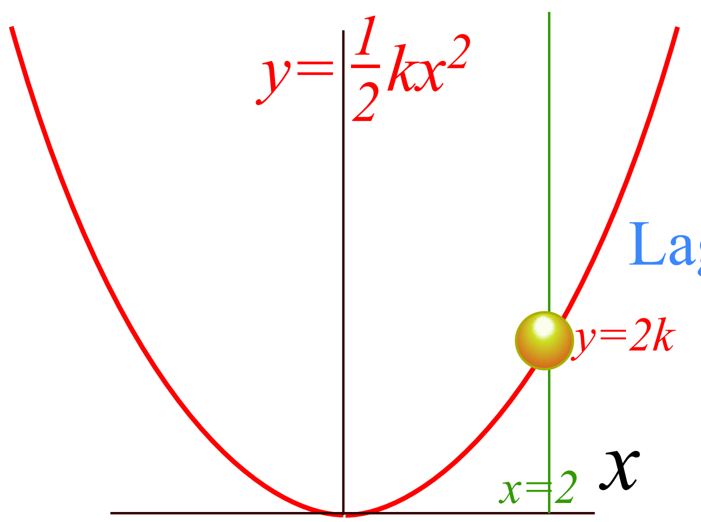

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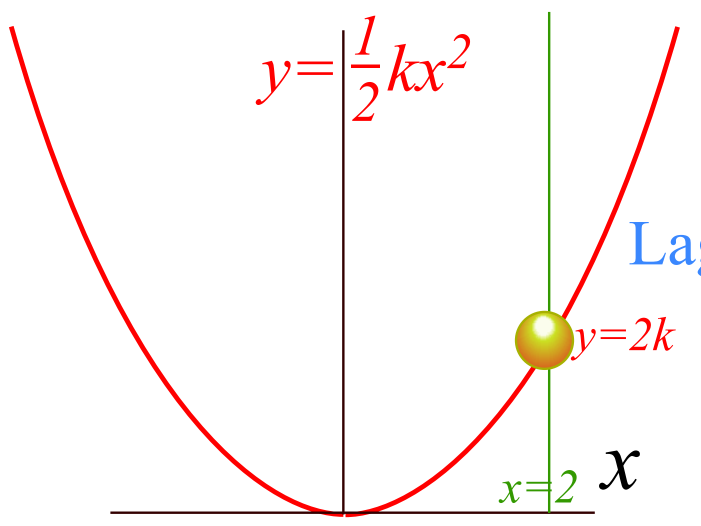

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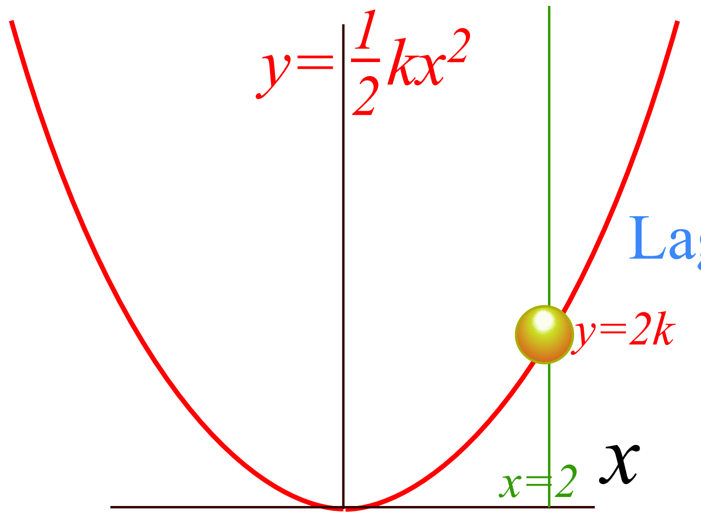

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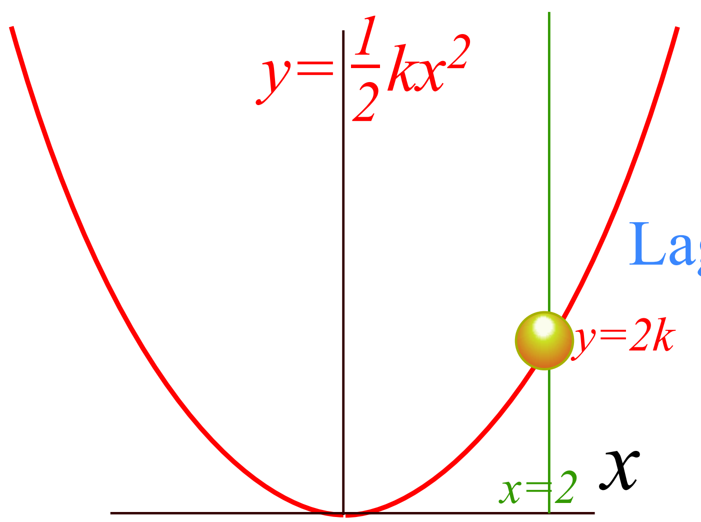

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$$p_x = \frac{\partial L}{\partial \dot{x}} = m(\dot{x} + k^2x^2\dot{x})$$

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Lagrange equation $\dot{p}_x = f_x = \frac{\partial L}{\partial x}$

$$\dot{p}_x = m(\ddot{x} + k^2x^2\ddot{x} + 2k^2x\dot{x}^2) = \frac{\partial L}{\partial x}$$

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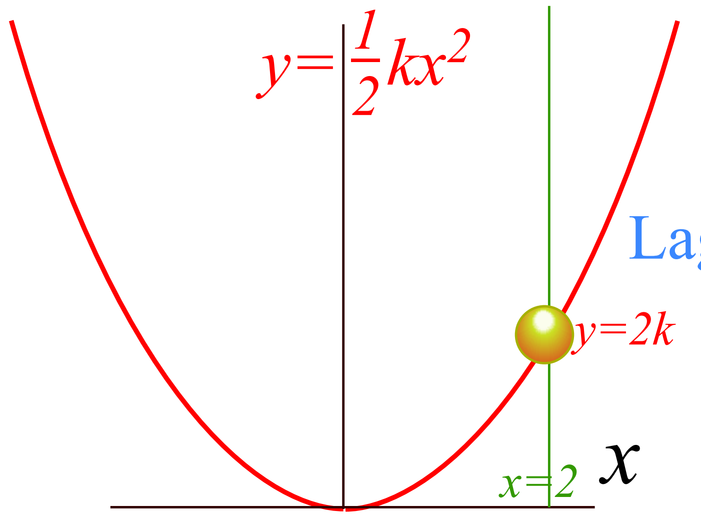

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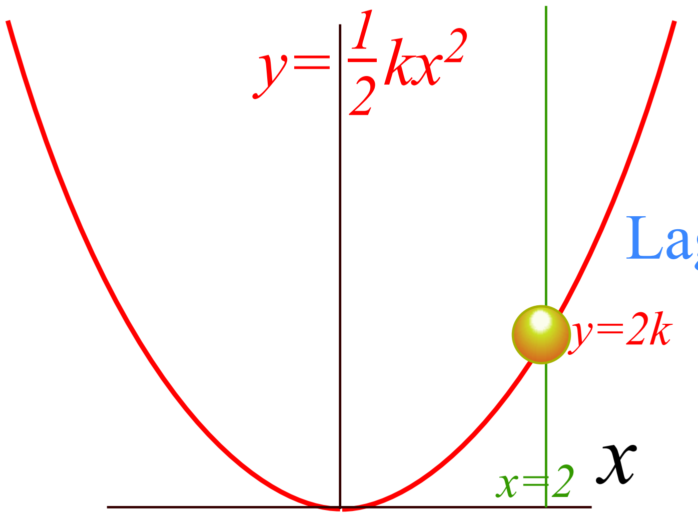

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$$\dot{p}_x=m(1+k^2x^2)\ddot{x} \qquad =-mk^2x\dot{x}^2-mgkx$$

Ways to analyze a particle m constrained to parabola $y=1/2kx^2$ on (x,y)-plane with gravitational potential $V(\mathbf{r})=mgy$.

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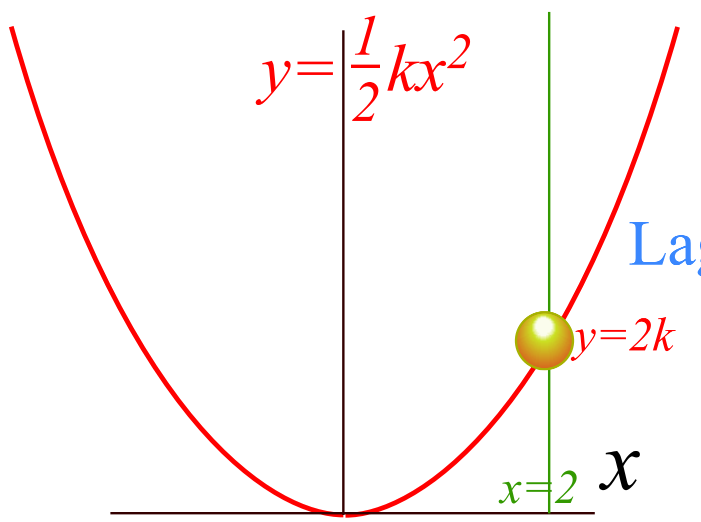

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$$=\frac{m}{2}(\dot{x}^2+k^2x^2\dot{x}^2-gkx^2) \qquad =m(\dot{x}+k^2x^2\dot{x}) \qquad =m(k^2x\dot{x}^2-gkx)$$

Lagrange equation $\dot{p}_x=f_x=\frac{\partial L}{\partial x}$ gives oscillator $\ddot{x}=-K(x,\dot{x})x$

$$\dot{p}_x=m(\ddot{x}+k^2x^2\ddot{x}+2k^2x\dot{x}^2)=\frac{\partial L}{\partial x}=m(k^2x\dot{x}^2-gkx)$$

$$m(1+k^2x^2)\ddot{x} \qquad =-mk^2x\dot{x}^2-mgkx=-m(k\dot{x}^2-g)kx$$

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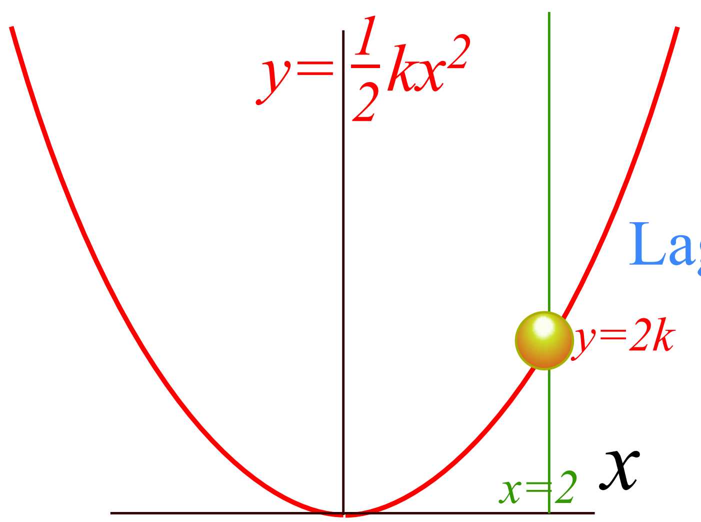

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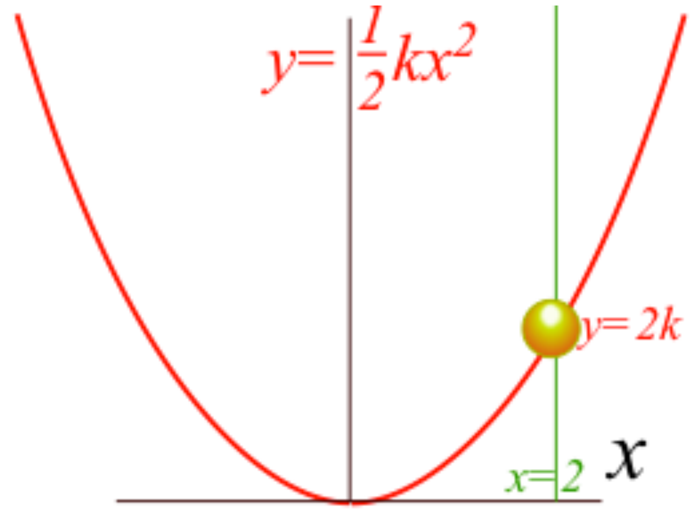
Lagrange equation $\dot{p}_x=f_x=\frac{\partial L}{\partial x}$ gives oscillator $\ddot{x}=-K(x,\dot{x})x$ with "spring factor" K:

$$\dot{p}_x=m(\ddot{x}+k^2x^2\ddot{x}+2k^2x\dot{x}^2)=\frac{\partial L}{\partial x}=m(k^2x\dot{x}^2-gkx)$$

$$m(1+k^2x^2)\ddot{x}=-mk^2x\dot{x}^2-mgkx=-m(k\dot{x}^2-g)kx$$

$$\boxed{\ddot{x}=\frac{-k\dot{x}^2-g}{1+k^2x^2}kx}$$

Simple constrained problem...



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Some Ways to do constraint analysis

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→ *Way 2. GCC constraint webs*

Find covariant force equations

Compare covariant vs. contravariant forces

Way 2. *GCC constraint webs.*

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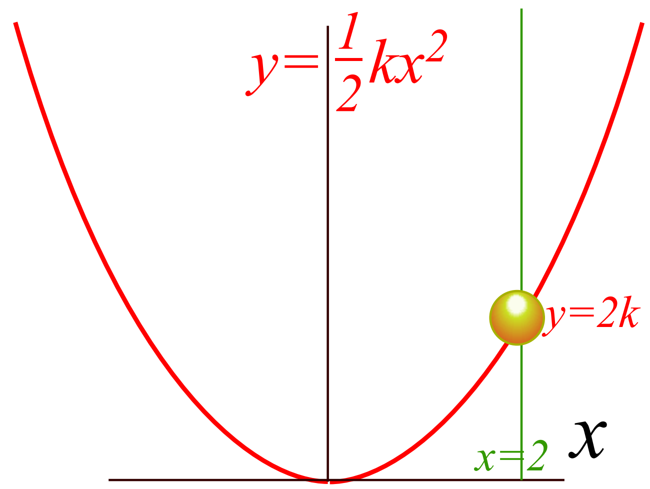
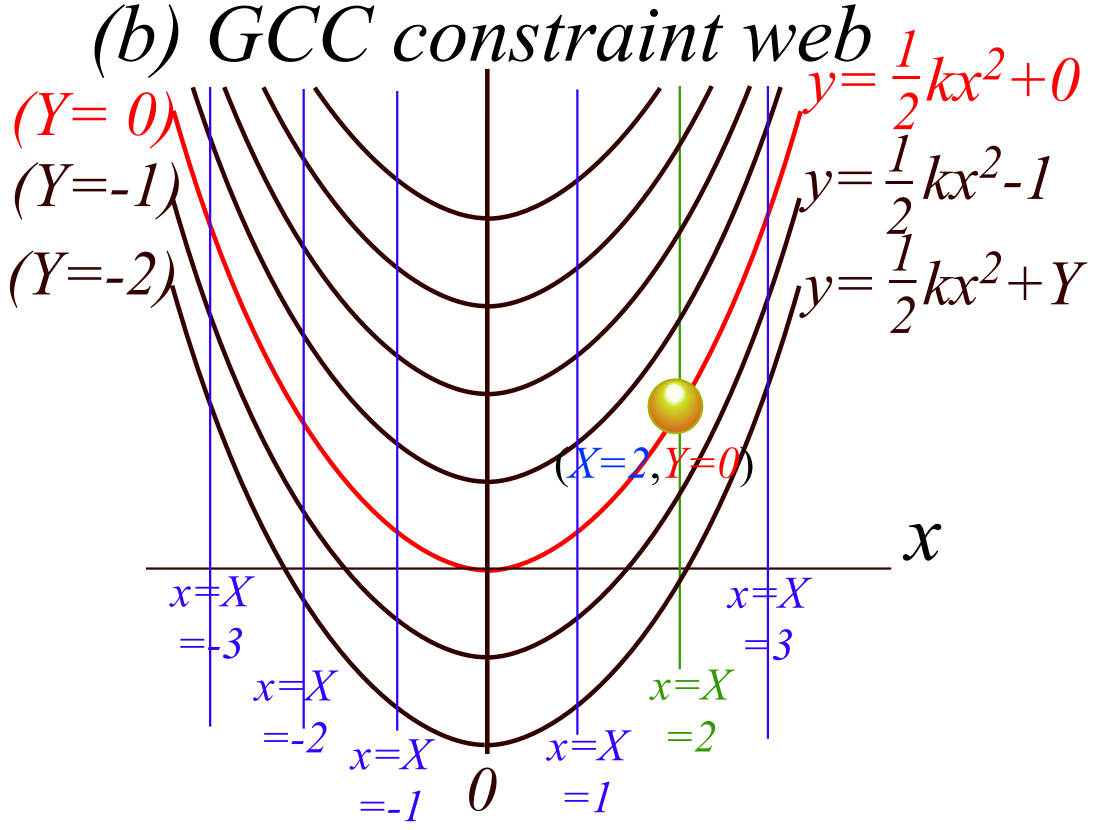


$$x = X$$
$$y=\frac{1}{2}kx^2+Y$$

Cartesian (x,y) transform to GCC (X,Y)

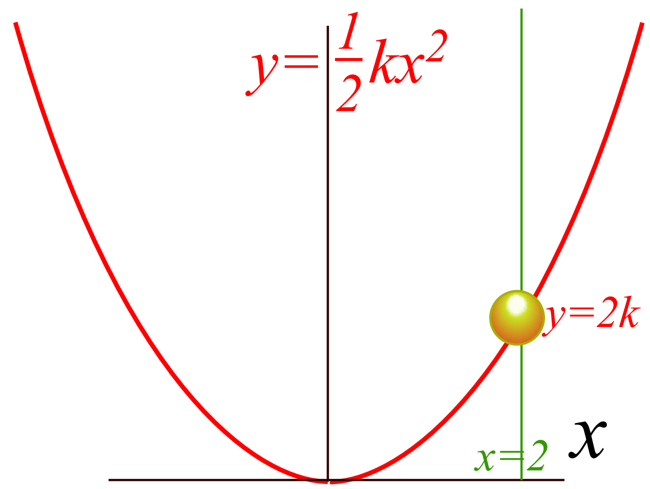
(b) GCC constraint web



Incorporate the constraint curve $y=1/2kx^2$ into any matching GCC web.

Way 2. GCC constraint webs.

(a) Constrained motion



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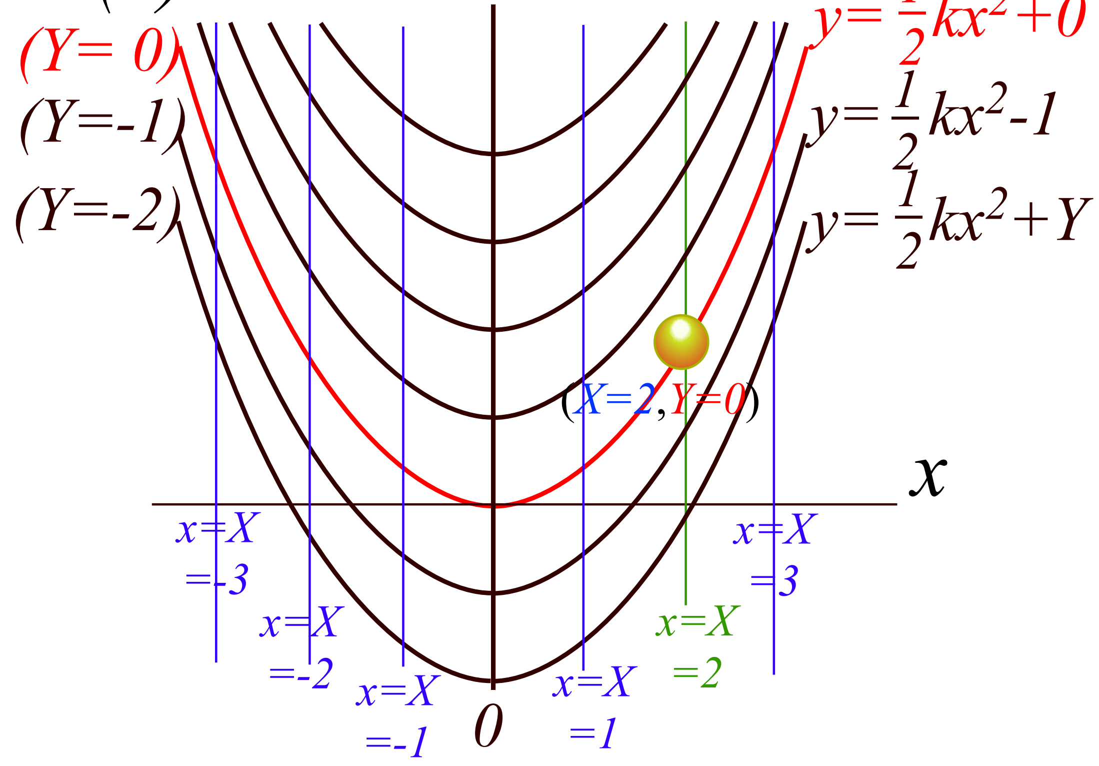

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$x=q^1=X$ $\qquad$ $y=1/2kx^2 +q^2= kX^2/2+Y$

we define shorthand:

$X\equiv q^1$ and $Y\equiv q^2$ to avoid writing $q_{ueer}{}^{Indices}$

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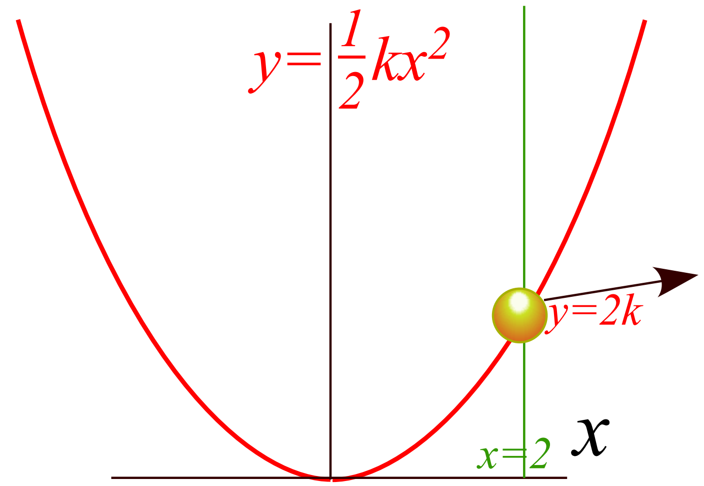

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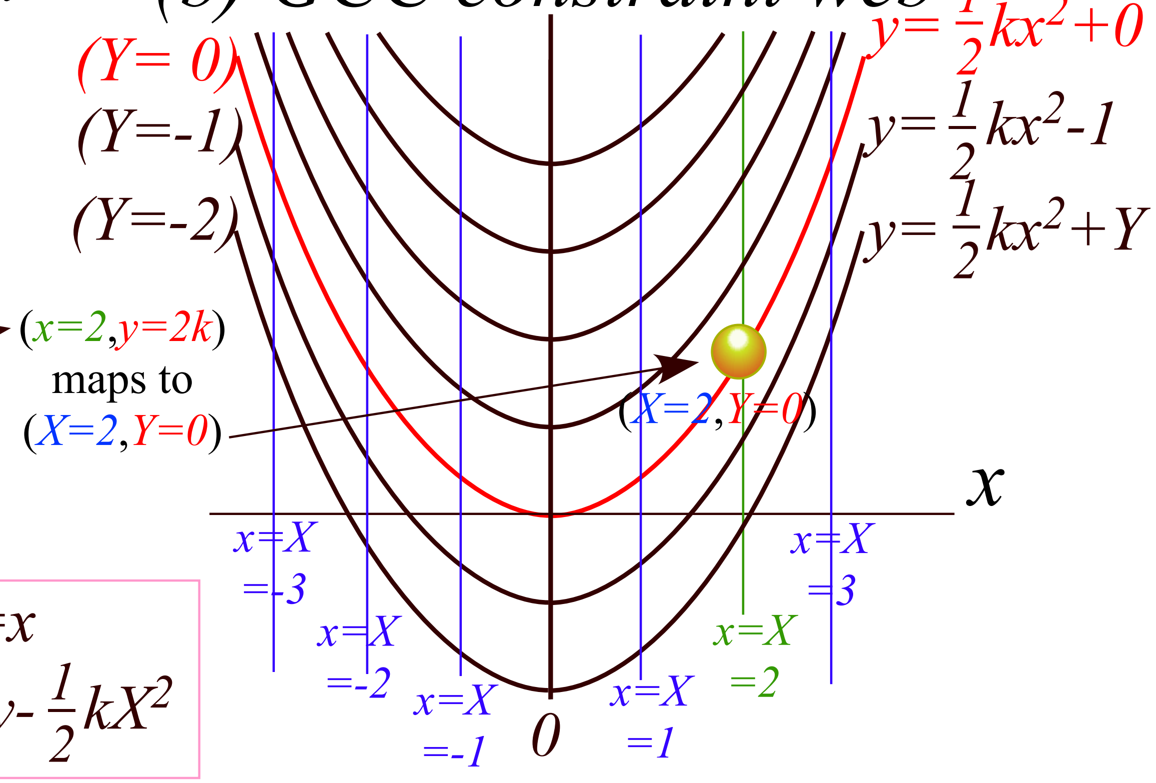
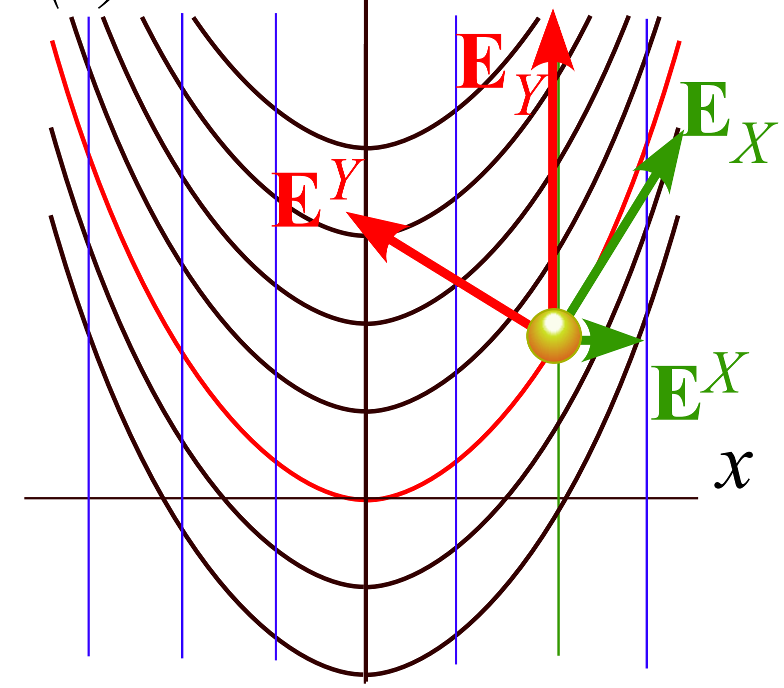


(c) GCC **E**-vectors



$x = X$
$y=\frac{1}{2}kx^2+Y$

Cartesian (x,y) transform to GCC (X,Y)

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$Y=y-\frac{1}{2}kX^2$

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Incorporate the constraint curve $y=1/2kx^2$ into any matching GCC web.

$x=q^1=X \qquad y=1/2kx^2+q^2= kX^2/2+Y$

Find: Covariant $\mathbf{E}_k$ in columns of Jacobian J matrix

$$J=\begin{pmatrix} \frac{\partial x}{\partial X}=1 & \frac{\partial x}{\partial Y}=0 \\ \frac{\partial y}{\partial X}=+kx & \frac{\partial y}{\partial Y}=1 \end{pmatrix} \qquad \mathbf{E}_X=\begin{pmatrix} 1 \\ kx \end{pmatrix},\ \mathbf{E}_Y=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

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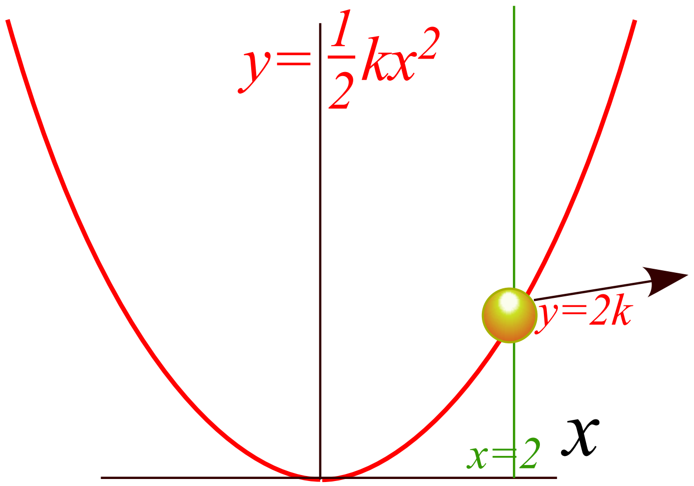

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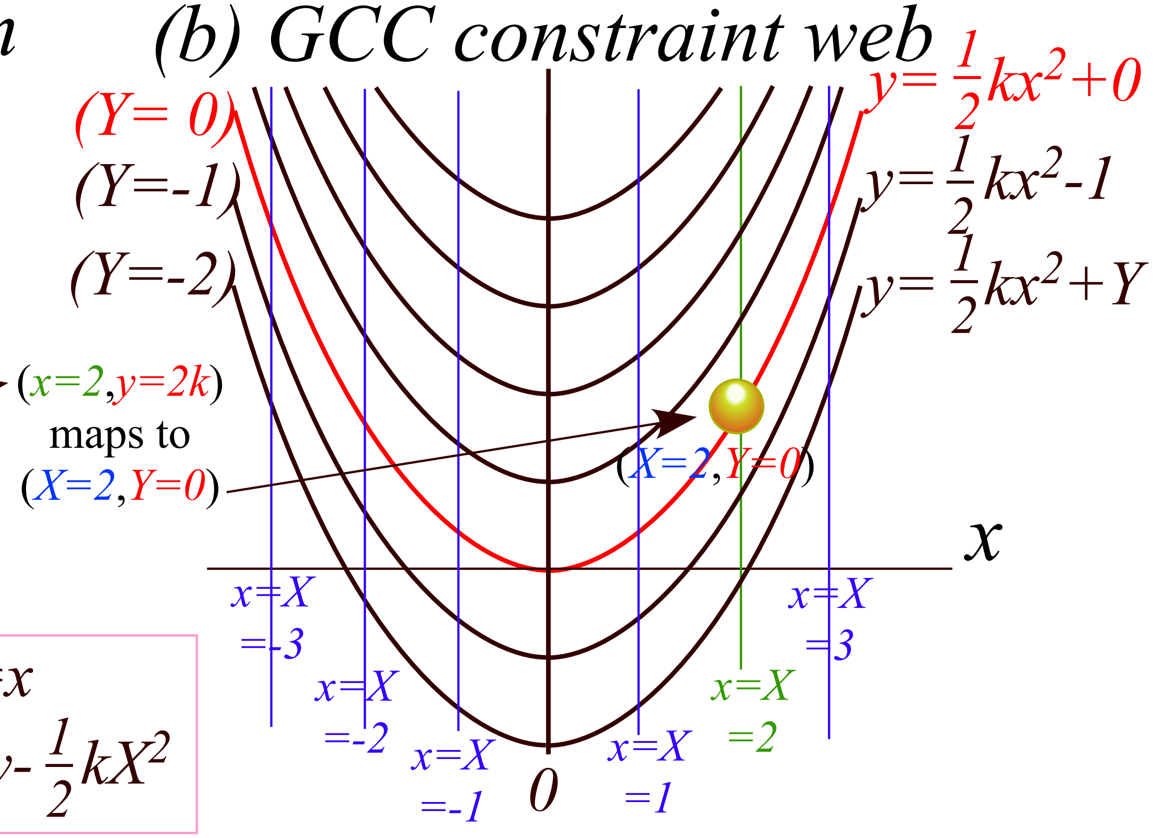

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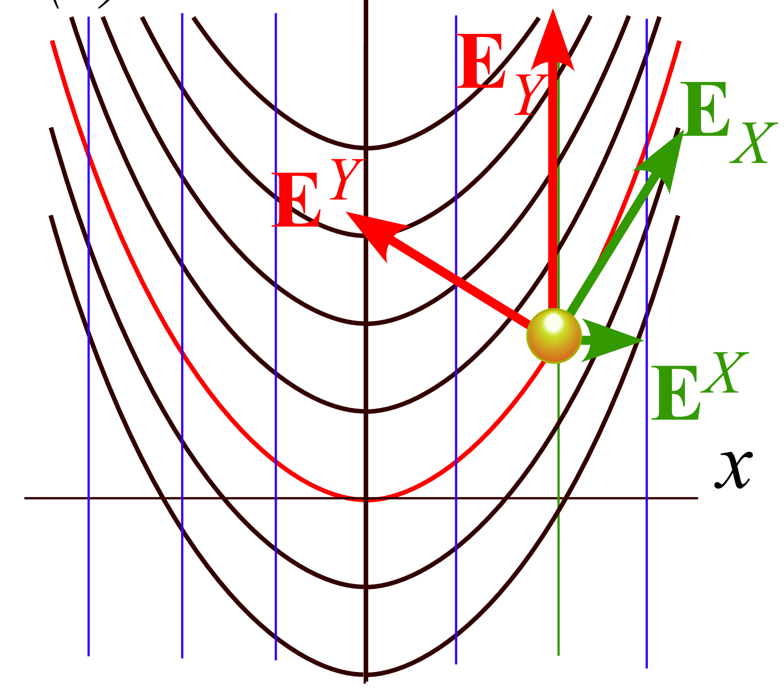

$x = X$

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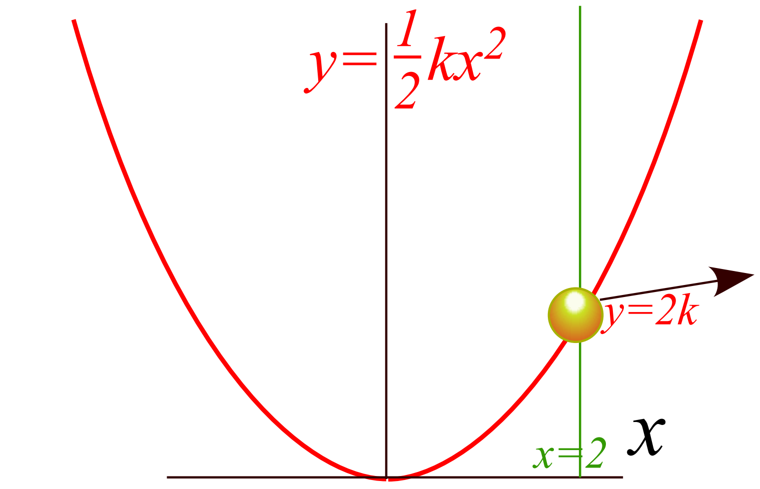
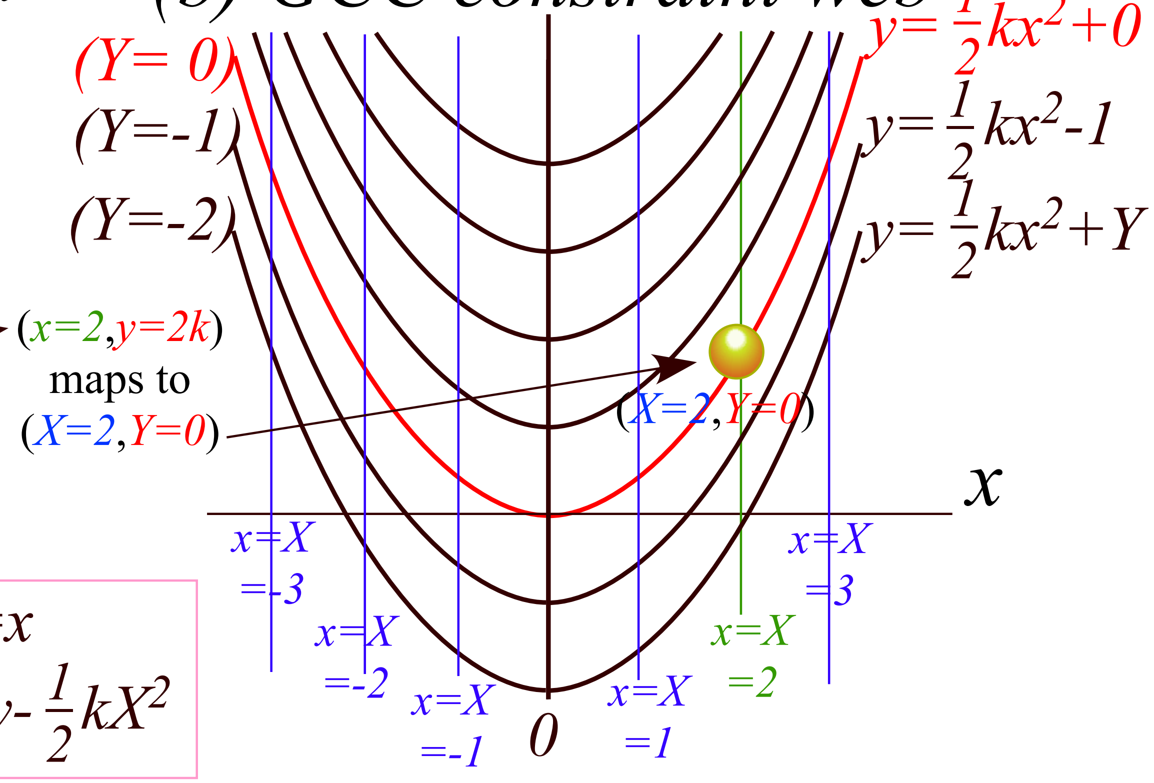
Contravariant $\mathbf{E}^k$ in rows of Kajobian K

$$\begin{pmatrix} \frac{\partial X}{\partial x} = 1 & \frac{\partial X}{\partial y} = 0 \\ \frac{\partial Y}{\partial x} = -kx & \frac{\partial Y}{\partial y} = 1 \end{pmatrix} = K \qquad \mathbf{E}^X = \begin{pmatrix} 1 & 0 \end{pmatrix} \quad \mathbf{E}^Y = \begin{pmatrix} -kx & 1 \end{pmatrix}$$

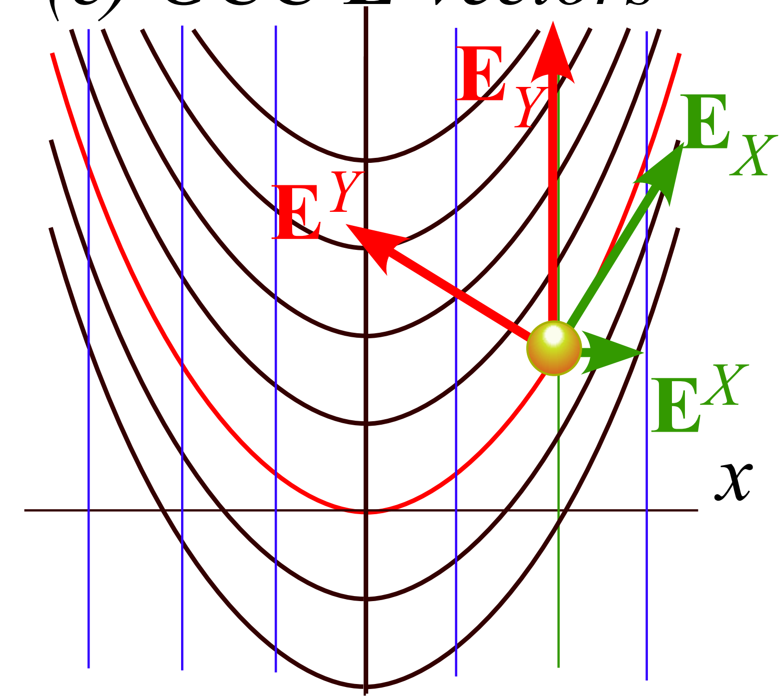
(a) Constrained motion



(b) GCC constraint web



(c) GCC E-vectors



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$y = \frac{1}{2}kx^2 + Y$

Cartesian (x,y) transform to GCC (X,Y)

$X = x$

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Find: 1st coordinate differentials and velocity relations:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ +kx & 1 \end{pmatrix} \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} \qquad \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -kx & 1 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$

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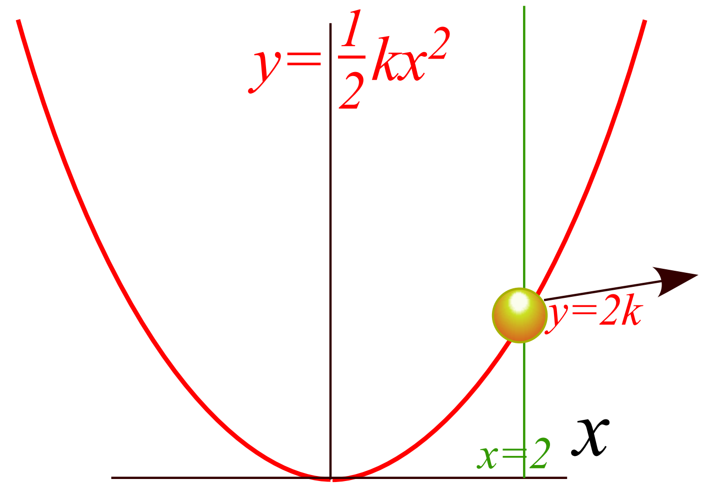
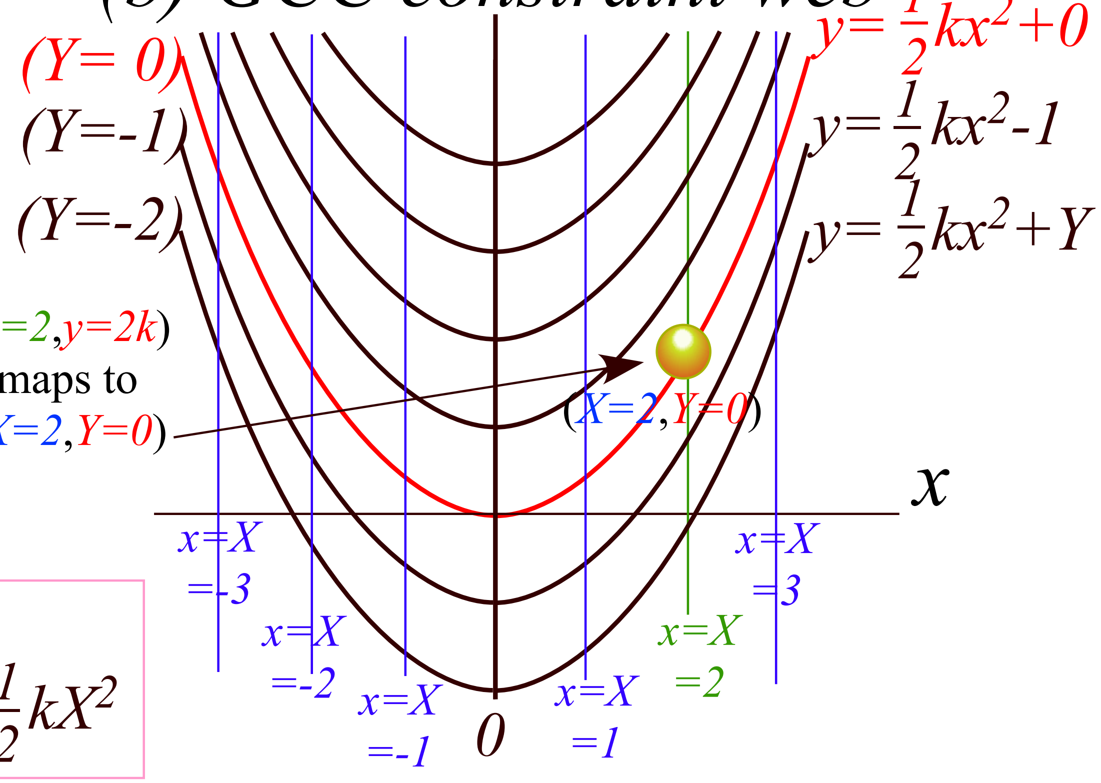
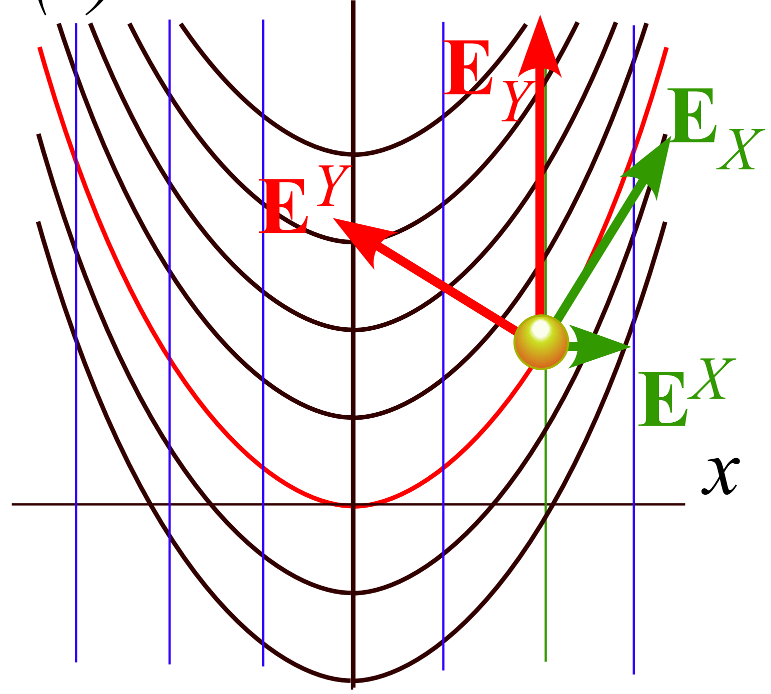


$(x=2, y=2k)$ maps to $(X=2, Y=0)$

(b) GCC constraint web



(c) GCC **E**-vectors



$x = X$
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Cartesian (x,y) transform to GCC (X,Y)

$X = x$
$Y = y-\frac{1}{2}kX^2$

we define shorthand:

$X\equiv q^1$ and $Y\equiv q^2$ to avoid writing $q_{queer}^{Indices}$

Incorporate the constraint curve $y=1/2kx^2$ into any matching GCC web.

$x=q^1=X$ $\quad$ $y=1/2kx^2+q^2=kX^2/2+Y$

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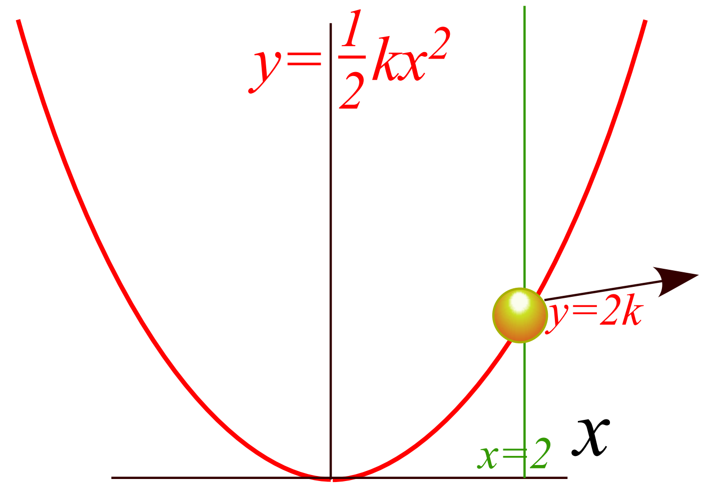
Way 2. GCC constraint webs.

(a) Constrained motion



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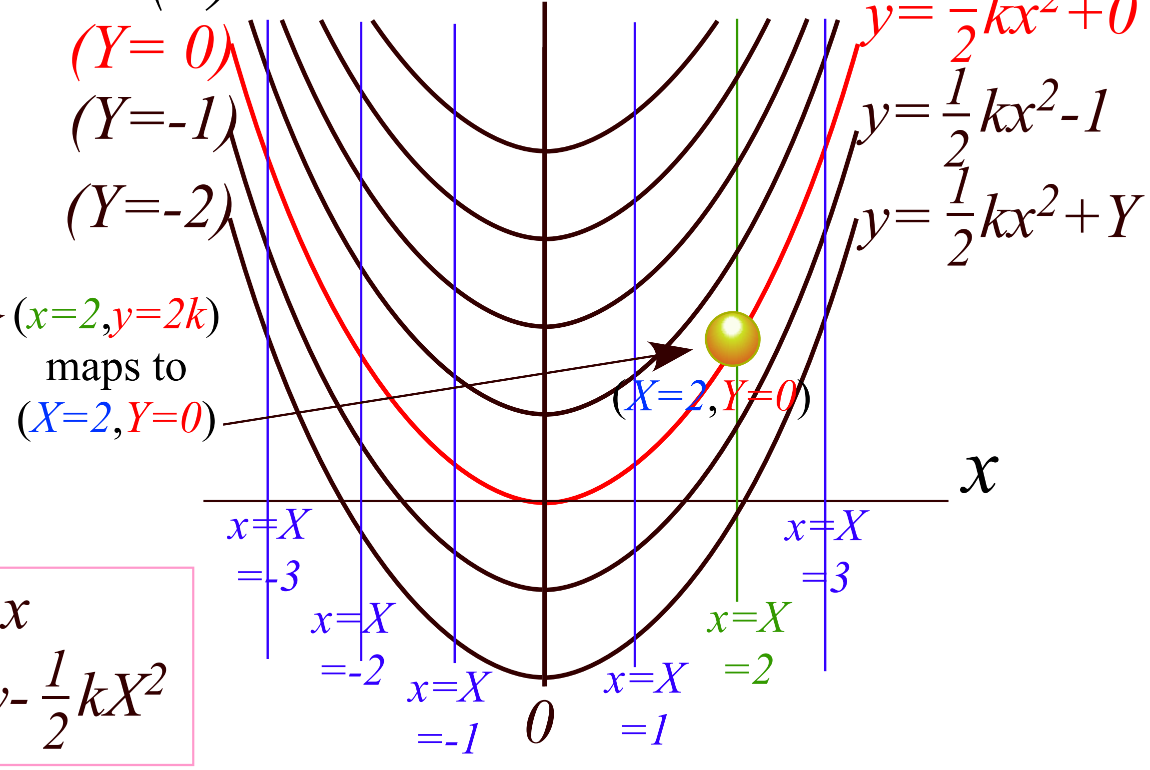

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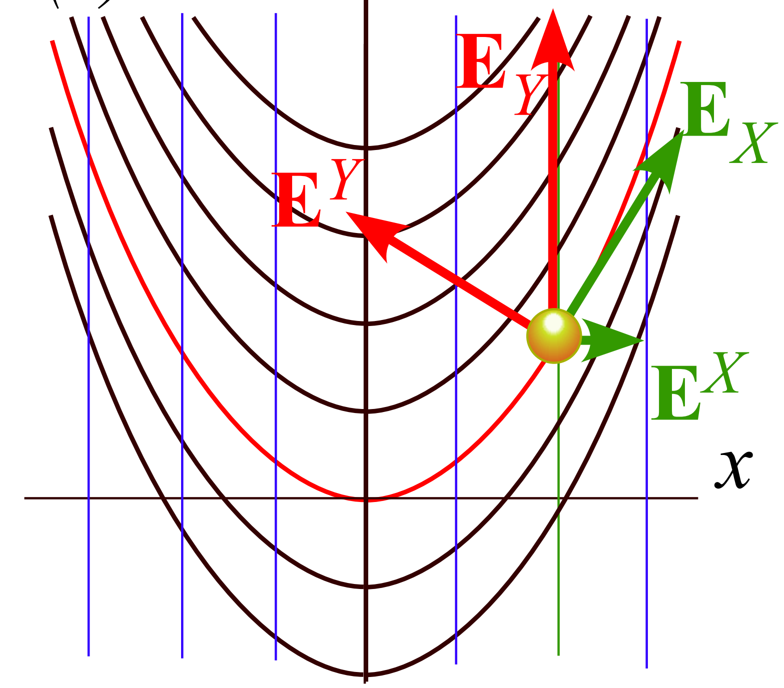

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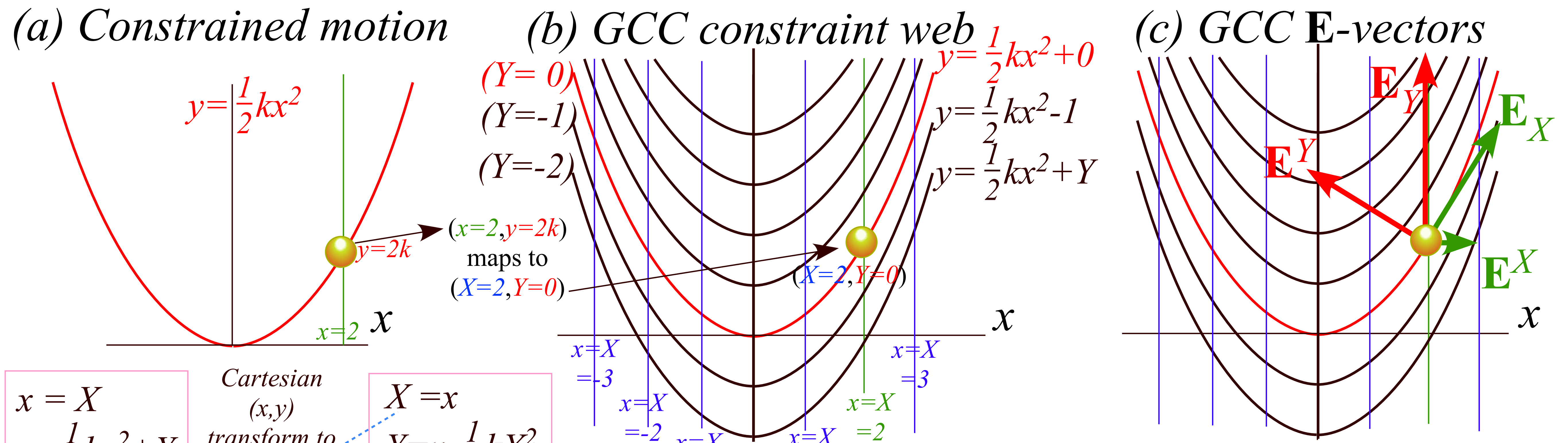

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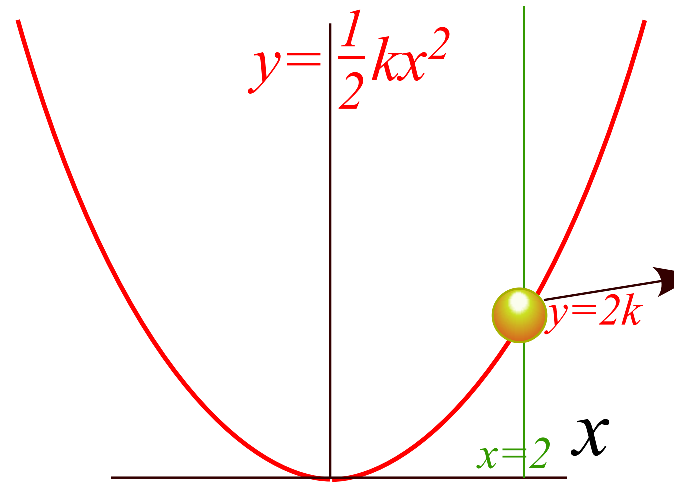
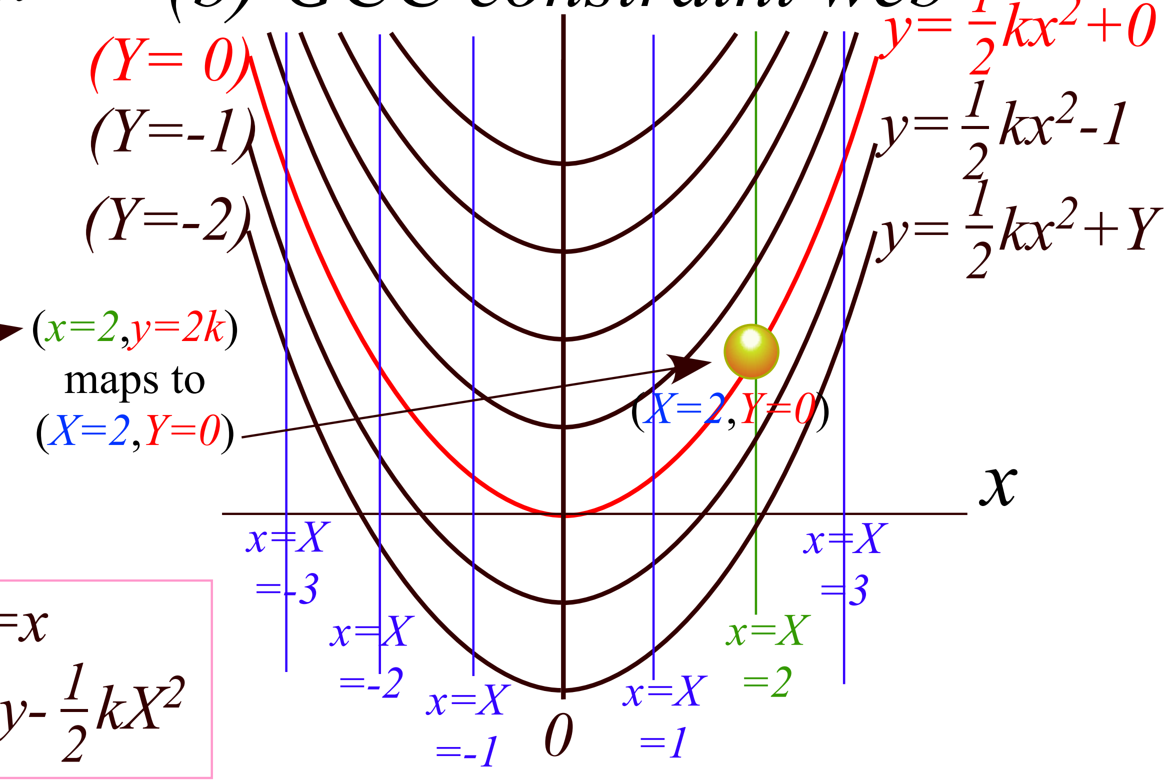


(b) GCC constraint web



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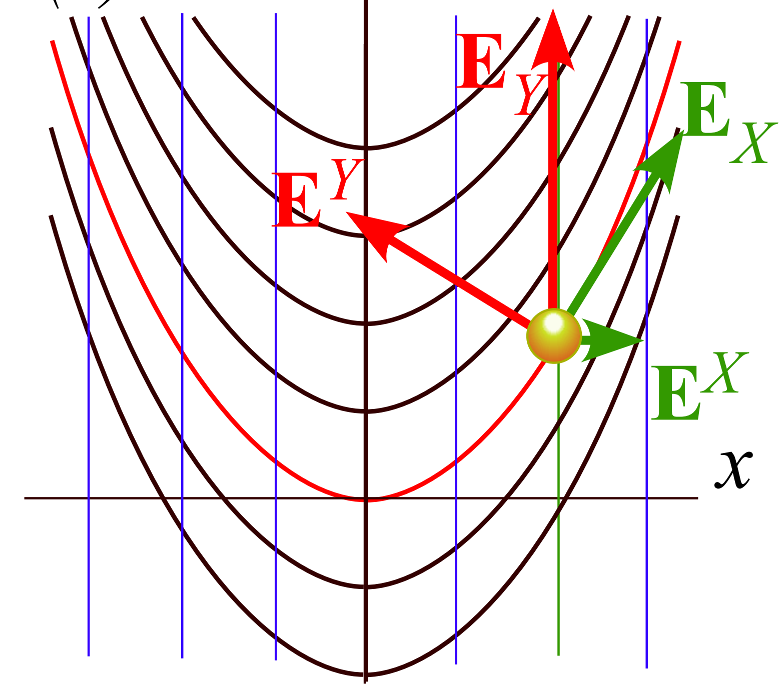

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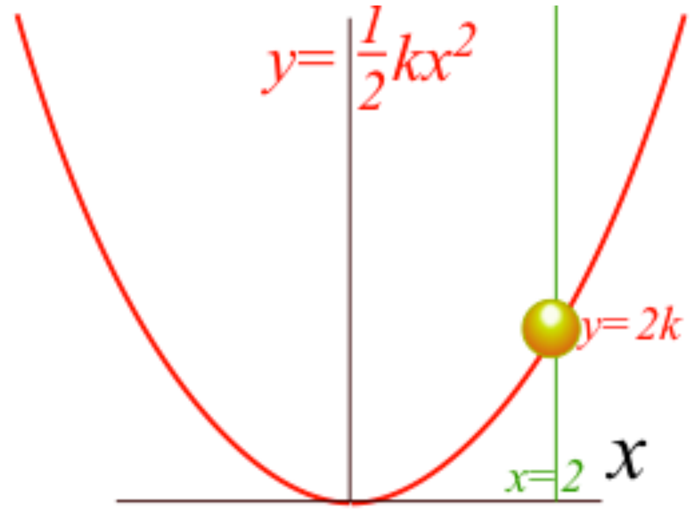
...and Lagrangian:

$$L = T - V = m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2 - gY - \frac{gk}{2}X^2\right] \qquad V = mgy = mg(Y+kX^2/2)$$

Simple constrained problem...



...and a variety of solutions

Some Ways to do constraint analysis

Way 1. Simple constraint insertion

Way 2. GCC constraint webs

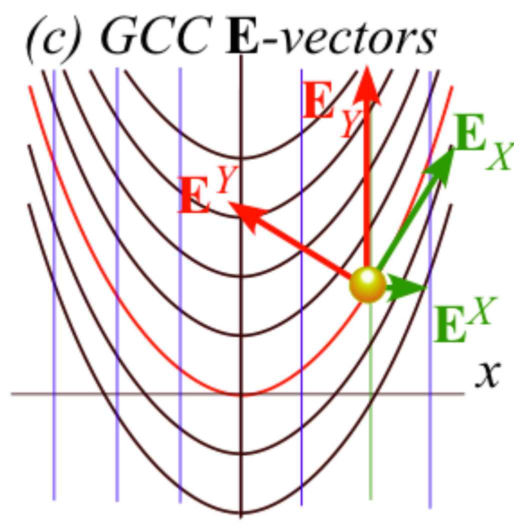
→ *Find covariant force equations*

Compare covariant vs. contravariant forces

Find: Lagrange equations from Lagrangian $L = T - V = m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2 - gY - \frac{gk}{2}X^2\right]$

$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix} = m \overset{(metric\ \gamma_{AB})}{\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}} \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \end{pmatrix}$$

(1st Lagrange equations) $p_m = \frac{\partial L}{\partial \dot{q}^m}$



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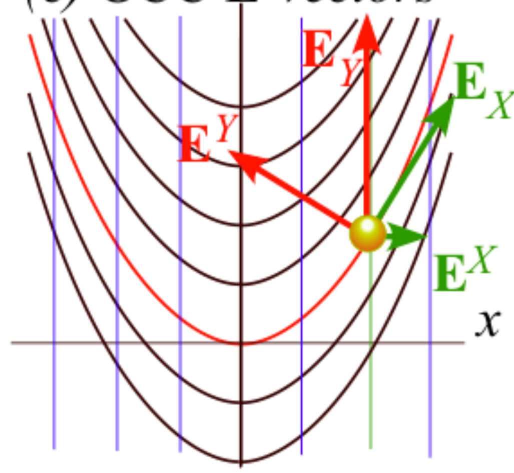

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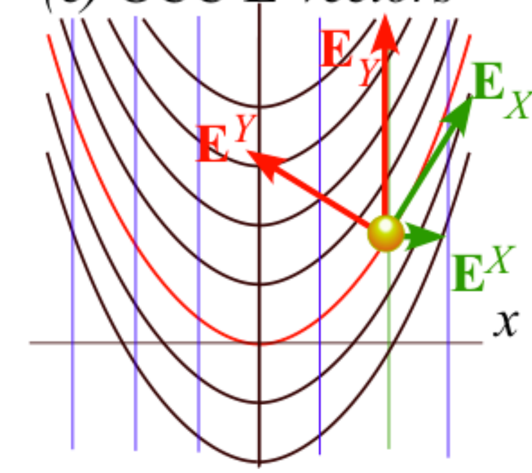

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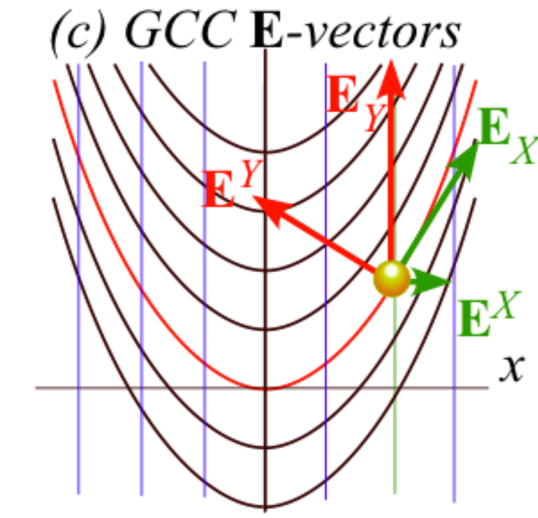

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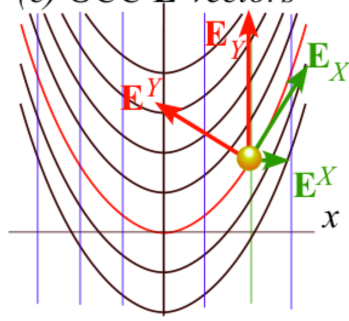
(c) GCC **E**-vectors



Find: Lagrange equations from Lagrangian $L = T - V = m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2 - gY - \frac{gk}{2}X^2\right]$

(metric γ_{AB})

$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \end{pmatrix}$$

(1st Lagrange equations) $p_m = \frac{\partial L}{\partial \dot{q}^m}$

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$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = \frac{d}{dt}\left[m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}\right] = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix}$$

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(cancel)

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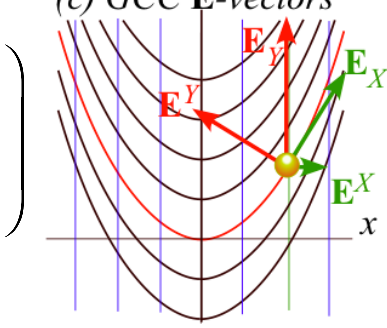

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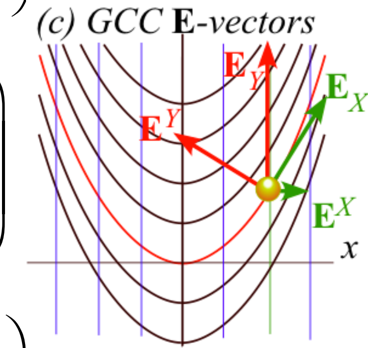

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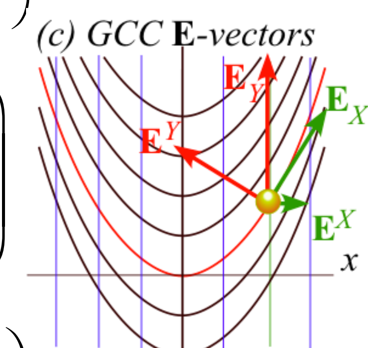

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Use γ^{AB} to get contra-(Riemann) equations. (Contra-force F^m_{con} is zero until we turn on constraint $Y =_{const.}$)

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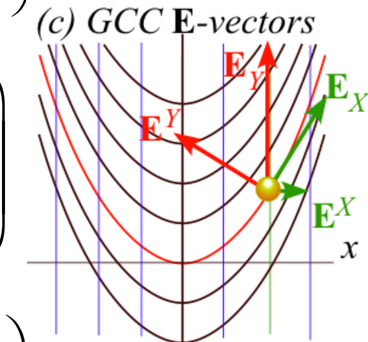

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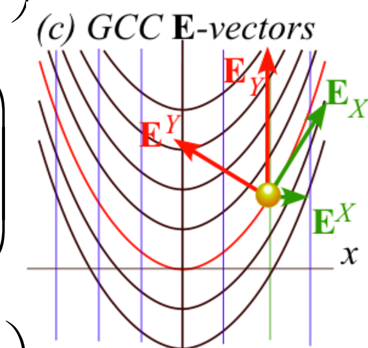

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(cancel)

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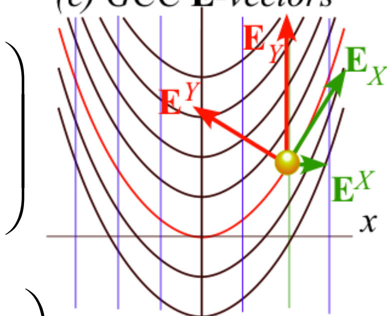
(c) GCC **E**-vectors



$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

Use γ^{AB} to get contra-(Riemann) equations. (Contra-force F_{con}^m is zero until we turn on constraint $Y=_{const.}$)

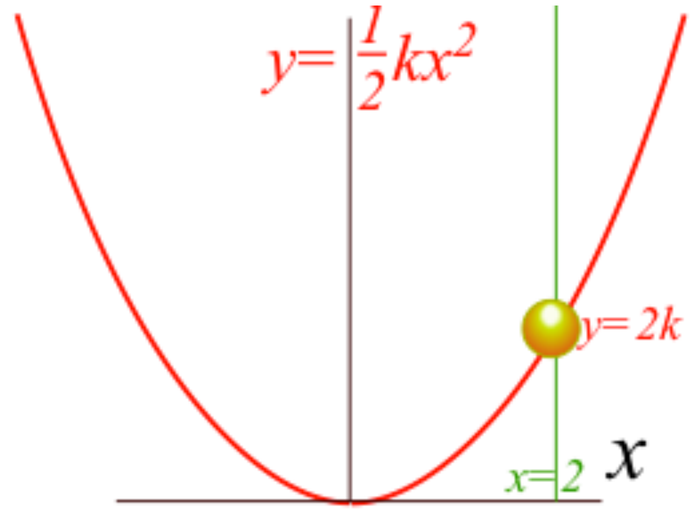
$$\frac{1}{m}\underbrace{\begin{pmatrix} 1 & -kX \\ -kX & 1+k^2X^2 \end{pmatrix}}_{(inverse\ of\ \gamma_{AB})}\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = \begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + \underbrace{\begin{pmatrix} 1 & -kX \\ -kX & 1+k^2X^2 \end{pmatrix}}_{(inverse\ of\ \gamma_{AB})}\begin{pmatrix} kX(k\dot{X}^2+g) \\ k\dot{X}^2+g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_{con}^X \\ F_{con}^Y \end{pmatrix}$$

(cancelations)

$$\frac{1}{m}\begin{pmatrix} 1 & -kX \\ -kX & 1+k^2x^2 \end{pmatrix}\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = \begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + \begin{pmatrix} 0 \\ k\dot{X}^2+g \end{pmatrix} = \begin{pmatrix} \ddot{X} \\ \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_{con}^X \\ F_{con}^Y \end{pmatrix}$$

$\ddot{x} = 0 = \ddot{X}$

Simple constrained problem...



...and a variety of solutions

Some Ways to do constraint analysis

Way 1. Simple constraint insertion

Way 2. GCC constraint webs

Find covariant force equations

→ *Compare covariant vs. contravariant forces*

Constraint force components are covariant

Frictionless constraint forces have *covariant* components F_B^{cov}

$$\mathbf{F} = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y = F_X^{cov}\nabla X + F_Y^{cov}\nabla Y$$

*(F_A are coefficients of **normal** vectors $\mathbf{E}^A$)*

Frictional force components are contravariant

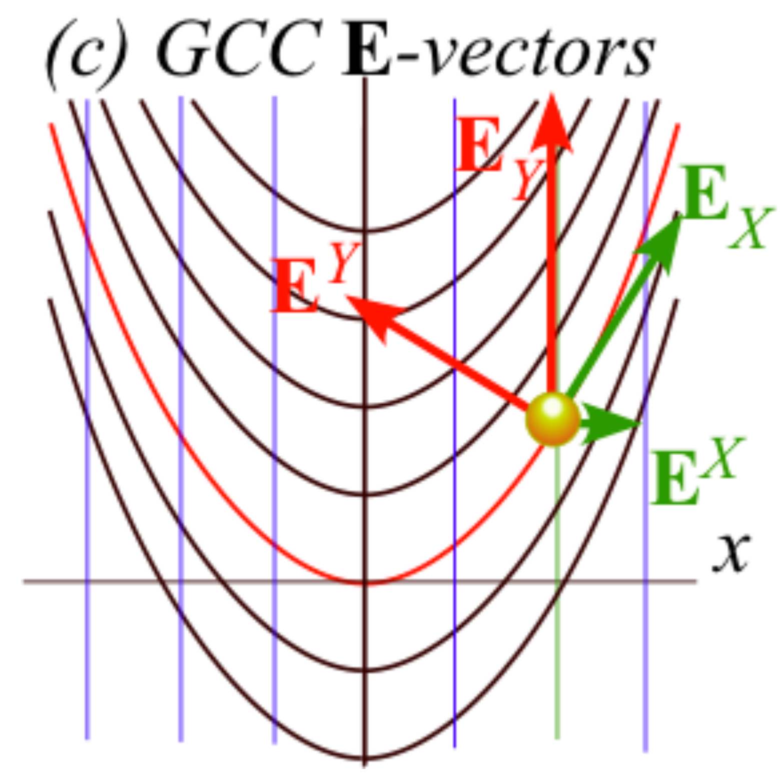
Frictional or driving forces have *contravariant* components F_{con}^A

$$\mathbf{F} = F_{con}^X\mathbf{E}_X + F_{con}^Y\mathbf{E}_Y = F_{con}^X\frac{\partial \mathbf{r}}{\partial X} + F_{con}^Y\frac{\partial \mathbf{r}}{\partial Y}$$

*(F^A are coefficients of **tangent** vectors $\mathbf{E}_A$)*

General case repeated from p.34

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$



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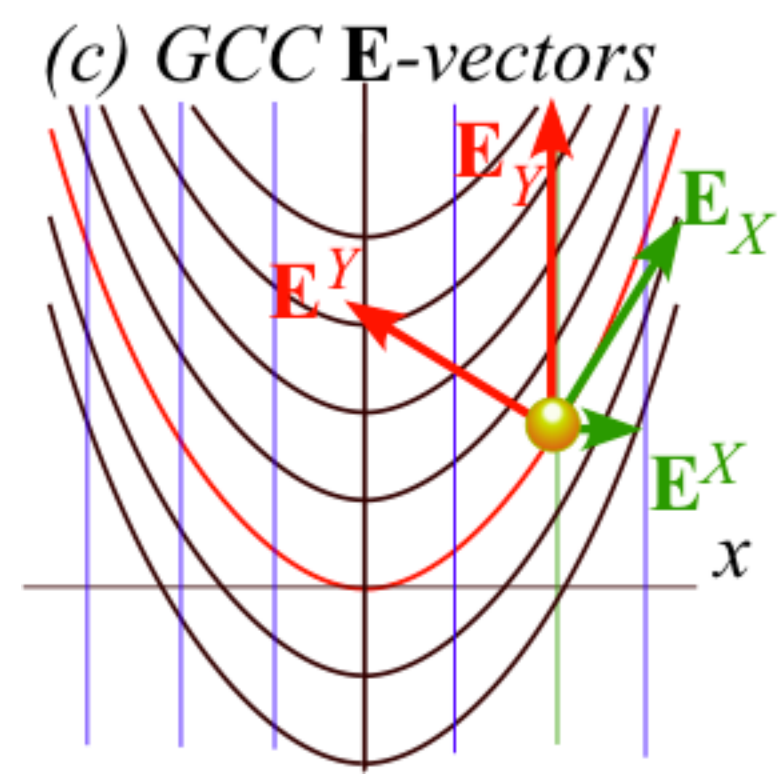

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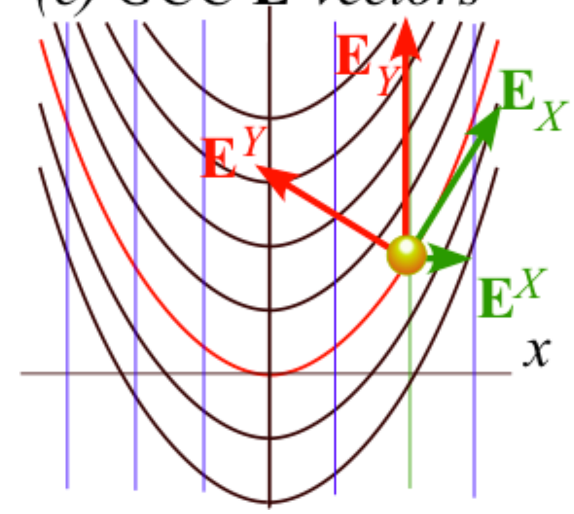
(F^A are coefficients of ***tangent*** *vectors $\mathbf{E}_A$)*



$\dot{Y} = 0 = \ddot{Y}$

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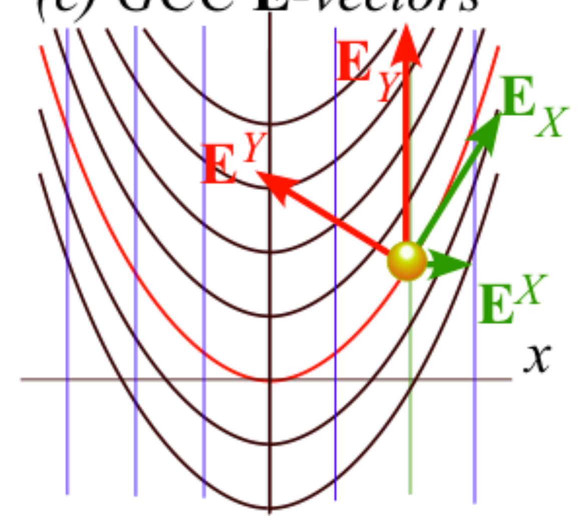

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$$\ddot{X} = -\frac{k^2X\dot{X}^2 + gkX}{1+k^2X^2} = -\frac{k\dot{X}^2 + g}{1+k^2X^2}kX$$

FINALLY ! We get the Way 1. solution of p.12

Recall: $x \equiv X$

$$\ddot{X} \equiv \ddot{x} = \frac{-k\dot{x}^2 - g}{1+k^2x^2}kx$$

General case repeated from p.34

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$$\mathbf{F} = F_Y^{cov}\,\mathbf{E}^Y = m(kX\ddot{X}+0+k\dot{X}^2+g)\begin{pmatrix}-kX\\ 1\end{pmatrix}$$

General case repeated from p.34

$$\begin{pmatrix}\dot{p}_X-\frac{\partial L}{\partial X}\\ \dot{p}_Y-\frac{\partial L}{\partial Y}\end{pmatrix}=m\begin{pmatrix}(1+k^2X^2)\ddot{X}+kX\ddot{Y}+k^2X\dot{X}^2+gkX\\ kX\ddot{X}+\ddot{Y}+k\dot{X}^2+g\end{pmatrix}=\begin{pmatrix}F_X^{cov}\\ F_Y^{cov}\end{pmatrix}$$

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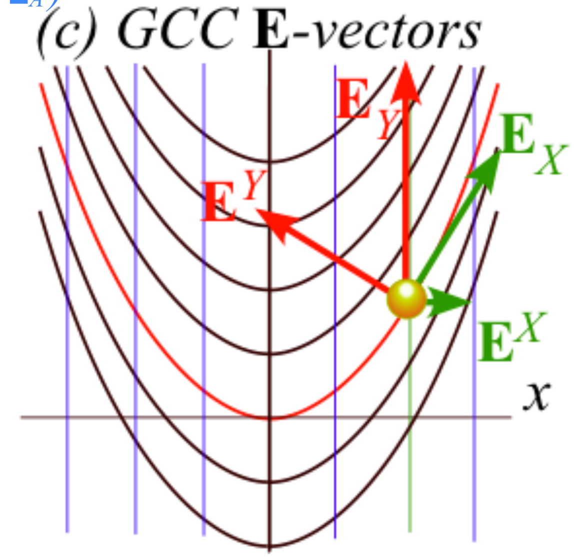

$\dot{Y}=0=\ddot{Y}$

Recall: $x\equiv X$

$$\ddot{X}\equiv\ddot{x}=\frac{-k\dot{x}^2-g}{1+k^2x^2}kx$$

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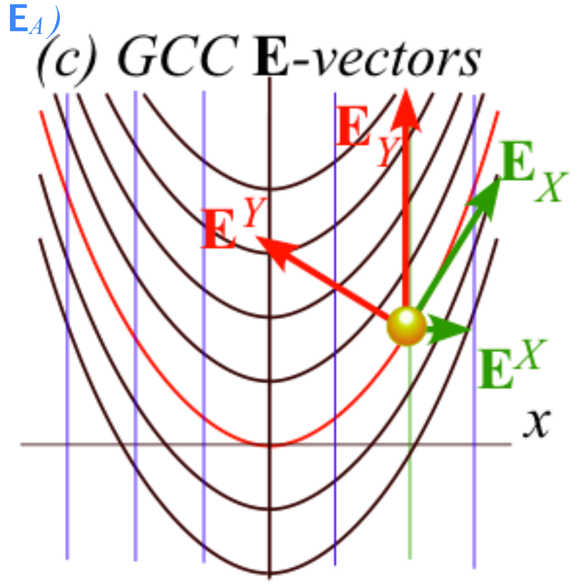

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$$\mathbf{F} = F_Y^{cov}\,\mathbf{E}^Y$$
$$= m(kX\ddot{X} + 0 + k\dot{X}^2 + g)\begin{pmatrix}-kX \\ 1\end{pmatrix}$$
$$= m\left(\frac{-kX(k\dot{X}^2+g)}{1+k^2X^2} + \frac{(k\dot{X}^2+g)(1+k^2X^2)}{1+k^2X^2}\right)\begin{pmatrix}-kX \\ 1\end{pmatrix}$$

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$$\begin{pmatrix}F_x\\F_y\end{pmatrix} = \left(=\begin{pmatrix}0\\ mk\dot{X}^2+mg\end{pmatrix}\right)_{at:X=0}$$

Centripetal force mkv^2+mg *(what roller-coaster rider feels at bottom)*

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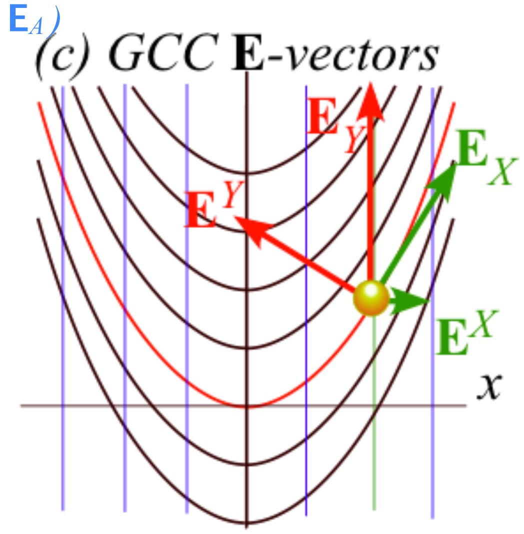


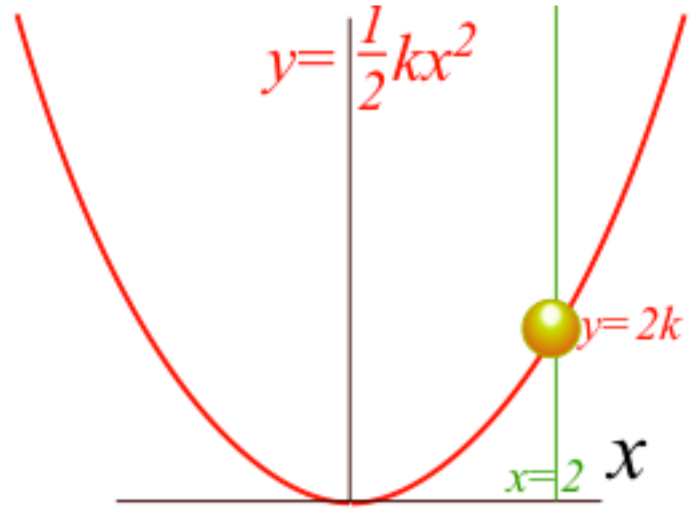
$$\dot{Y}=0=\ddot{Y}$$

Recall: $x\equiv X$

$$\ddot{X}\equiv\boxed{\ddot{x}=\frac{-k\dot{x}^2-g}{1+k^2x^2}kx}$$

$$-g=\ddot{y}=\tfrac{d^2}{dt^2}\left(\tfrac{1}{2}kX^2+Y\right)$$
$$=k\dot{X}^2+kX\ddot{X}+\ddot{Y}\,(=k\dot{X}^2+\ddot{Y}\text{ for }\ddot{X}=0)$$

Simple constrained problem...



...and a variety of solutions

Other Ways to do constraint analysis

→ *Way 3. OCC constraint webs*

Preview of atomic-Stark orbits

Classical Hamiltonian separability

Way 4. Lagrange multipliers

Lagrange multiplier as eigenvalues

Multiple multipliers

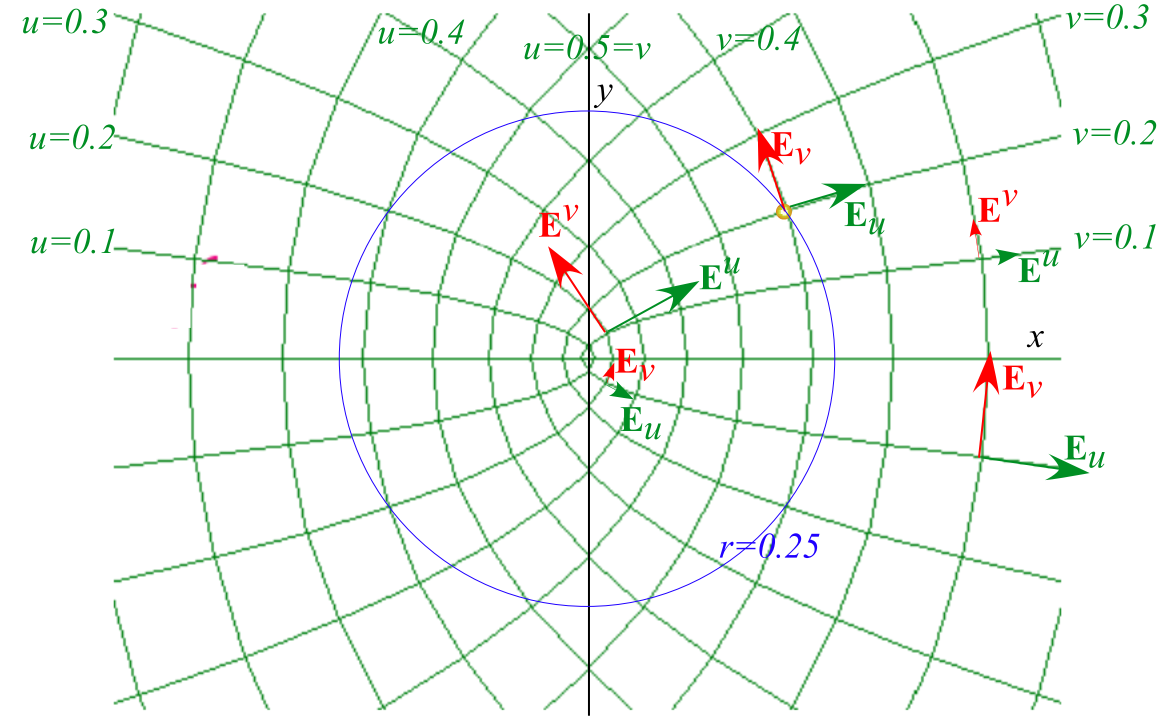
"Non-Holonomic" multipliers

Way 3. Parabolic OCC approach

Complex function $z=w^2$ or its inverse $w=z^{1/2}$ of complex variables $z=x+iy$ and $w=u+iv$.

Expansion of z and then absolute square $|z|^2$ give relations between Cartesian (x,y) and OCC(u,v)

$$z = x + iy = (u + iv)^2 = u^2 - v^2 + i2uv \qquad r^2 = z * z = x^2 + y^2 = (u^2 + v^2)^2 = u^4 + v^4 + 2u^2v^2$$



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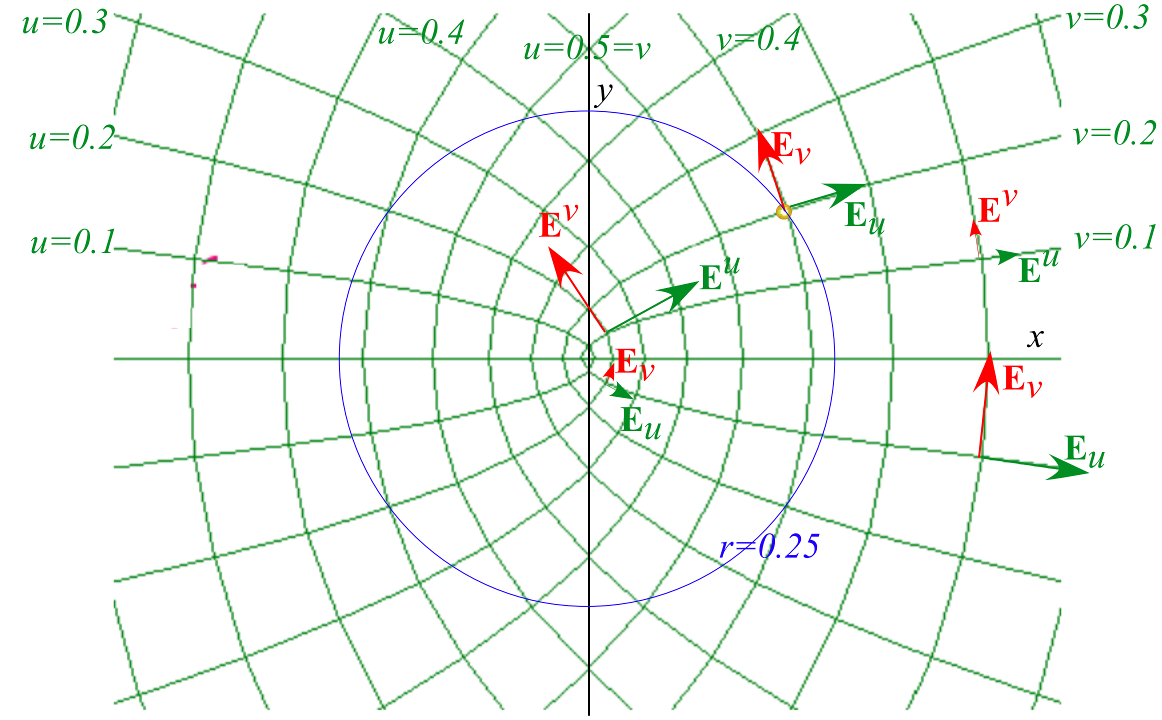
$$x = u^2 - v^2$$

$$y = 2uv$$

$$r = u^2 + v^2$$



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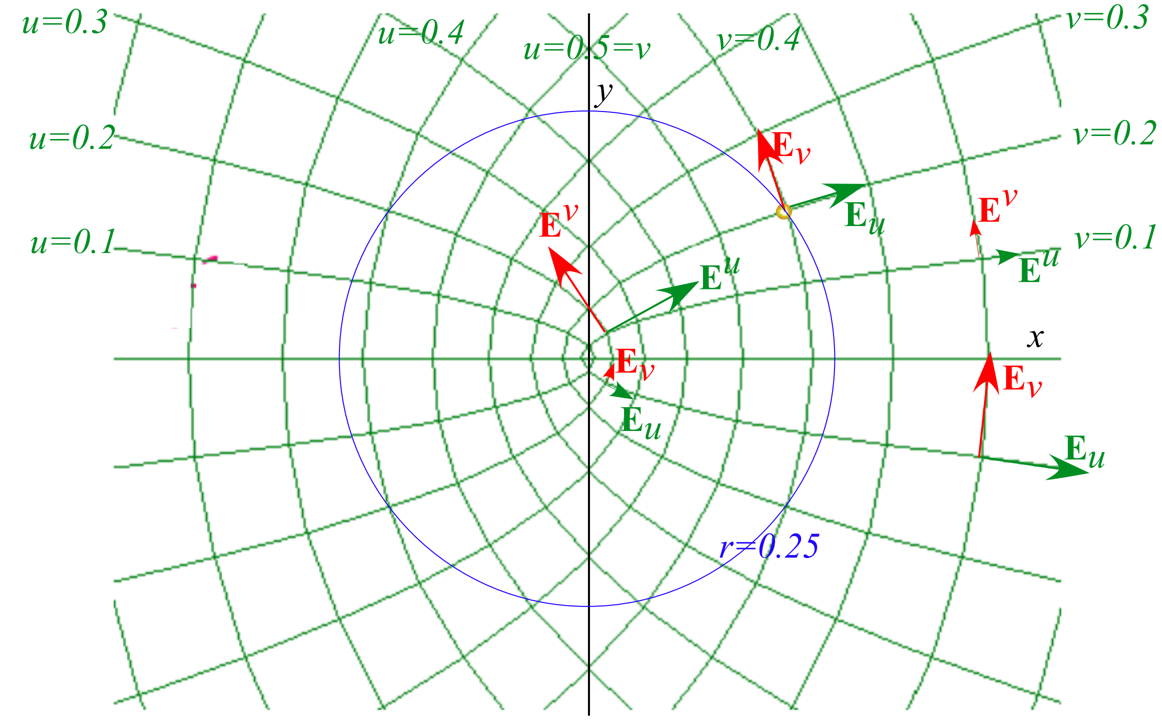
$$2u^2 = r + x = \sqrt{x^2 + y^2} + x$$

$$2v^2 = r - x = \sqrt{x^2 + y^2} - x$$



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$$x = u^2 - v^2$$

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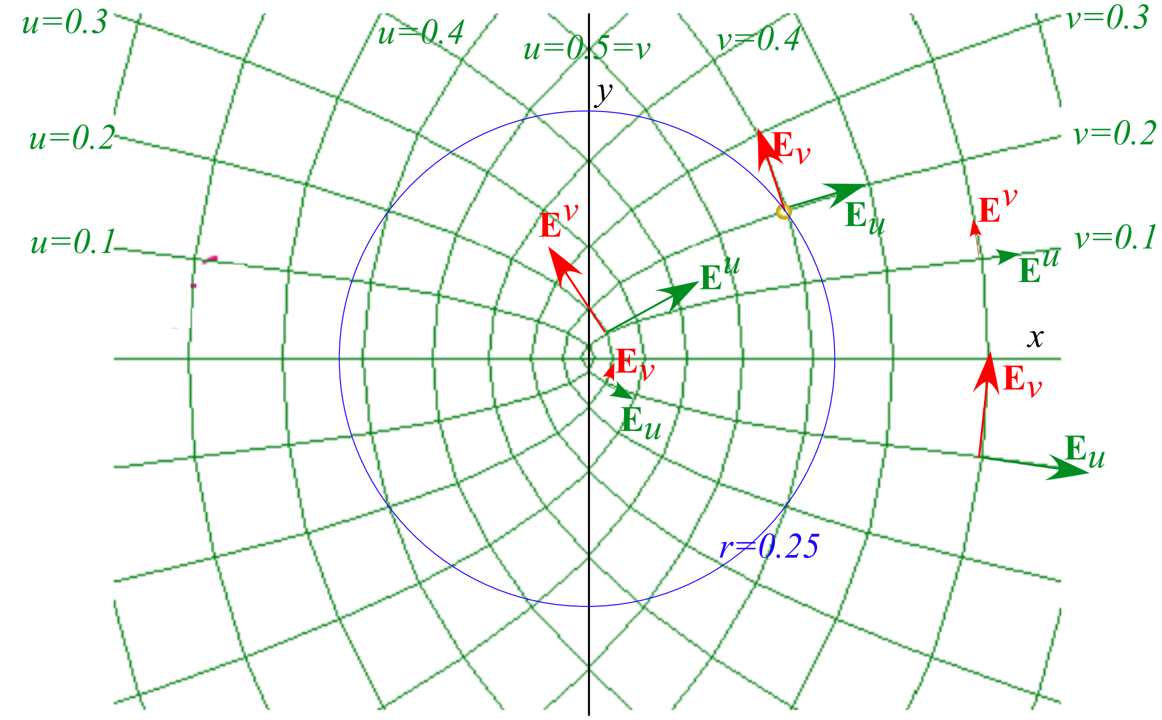
$$r = u^2 + v^2 \qquad 2v^2 = r - x = \sqrt{x^2+y^2} - x$$

$$y^2 = 4u^2v^2 = 4u^2(u^2 - x)$$

$$y^2 = 4v^2u^2 = 4v^2(v^2 + x)$$

Gives confocal parabolics



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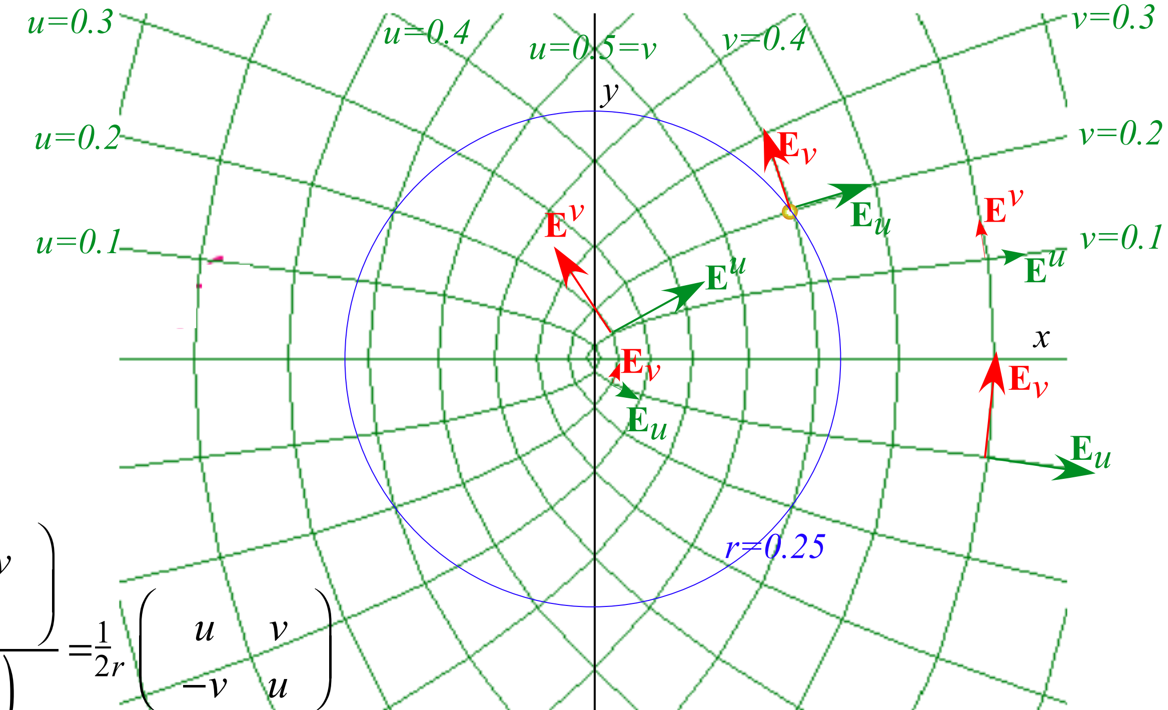

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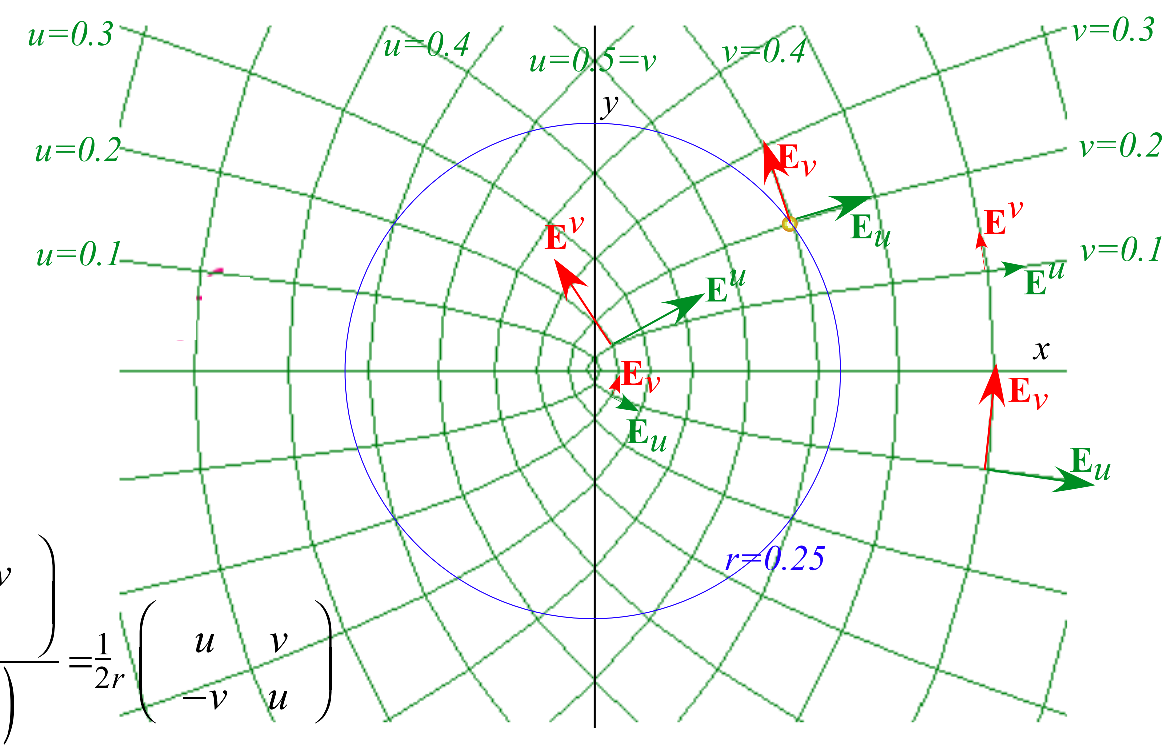

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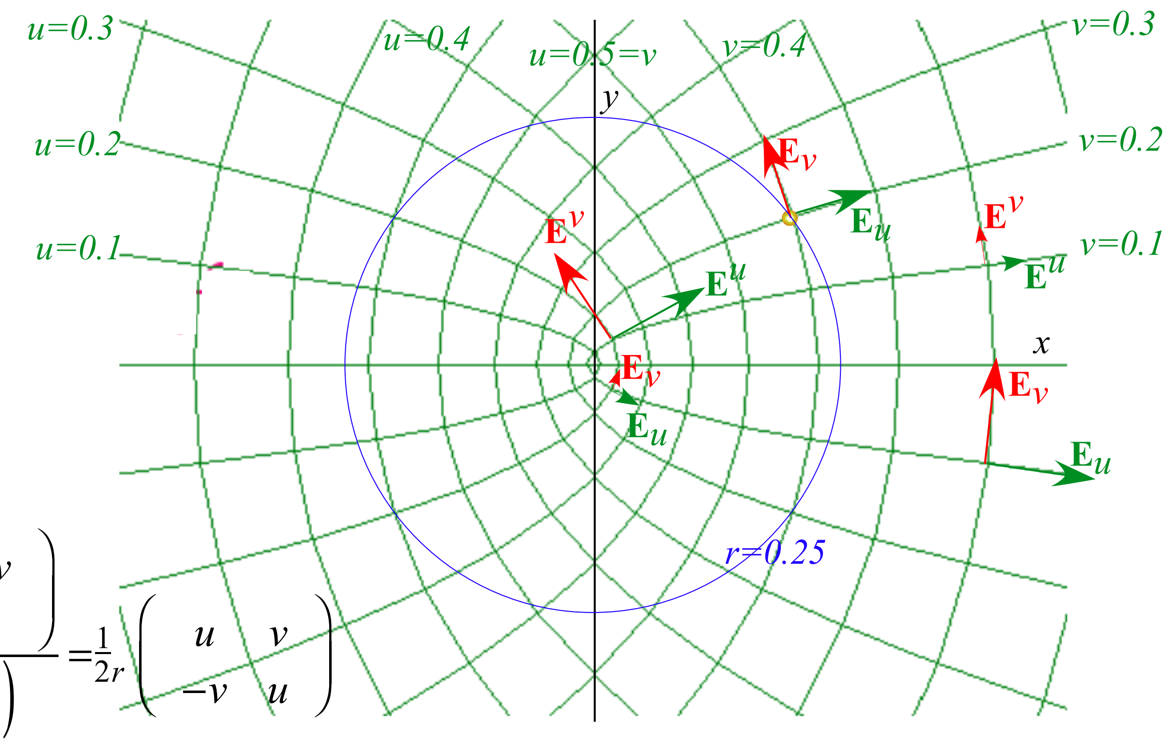

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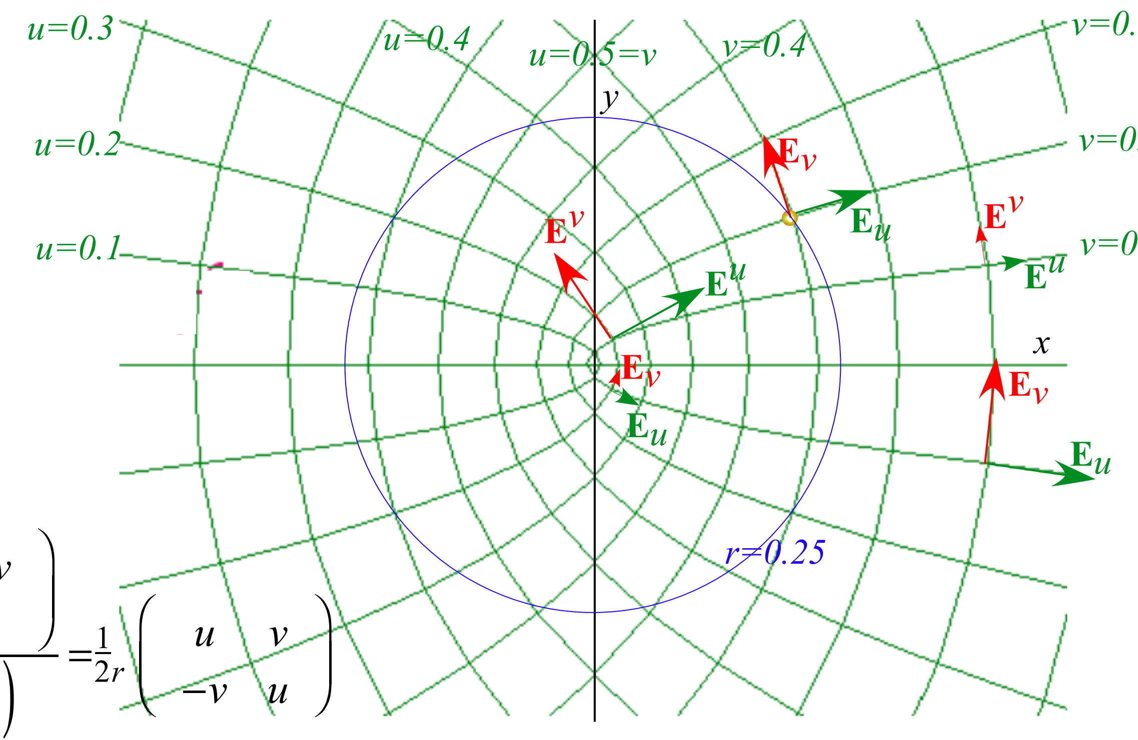

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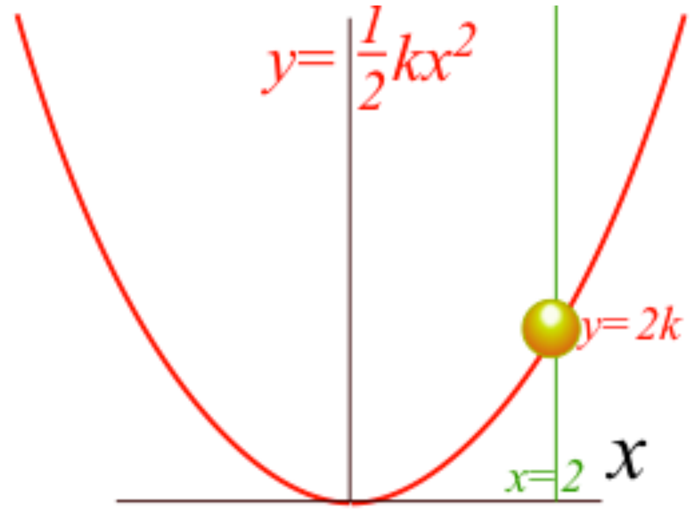
Simple constrained problem...



...and a variety of solutions

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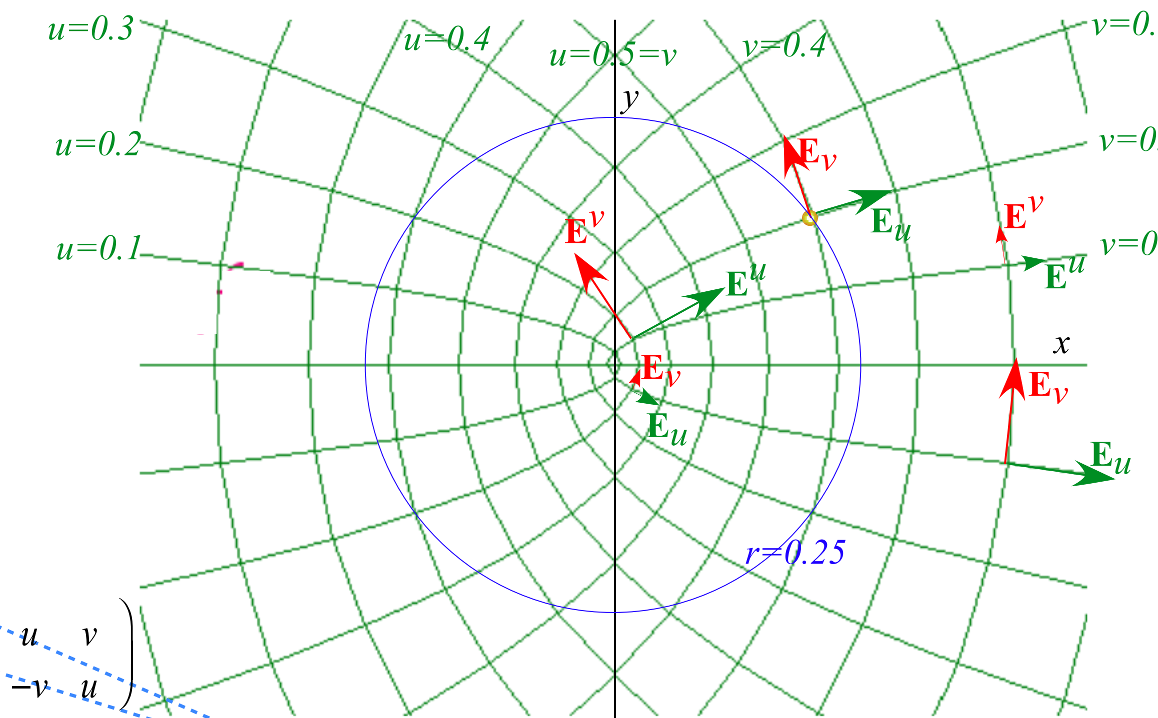

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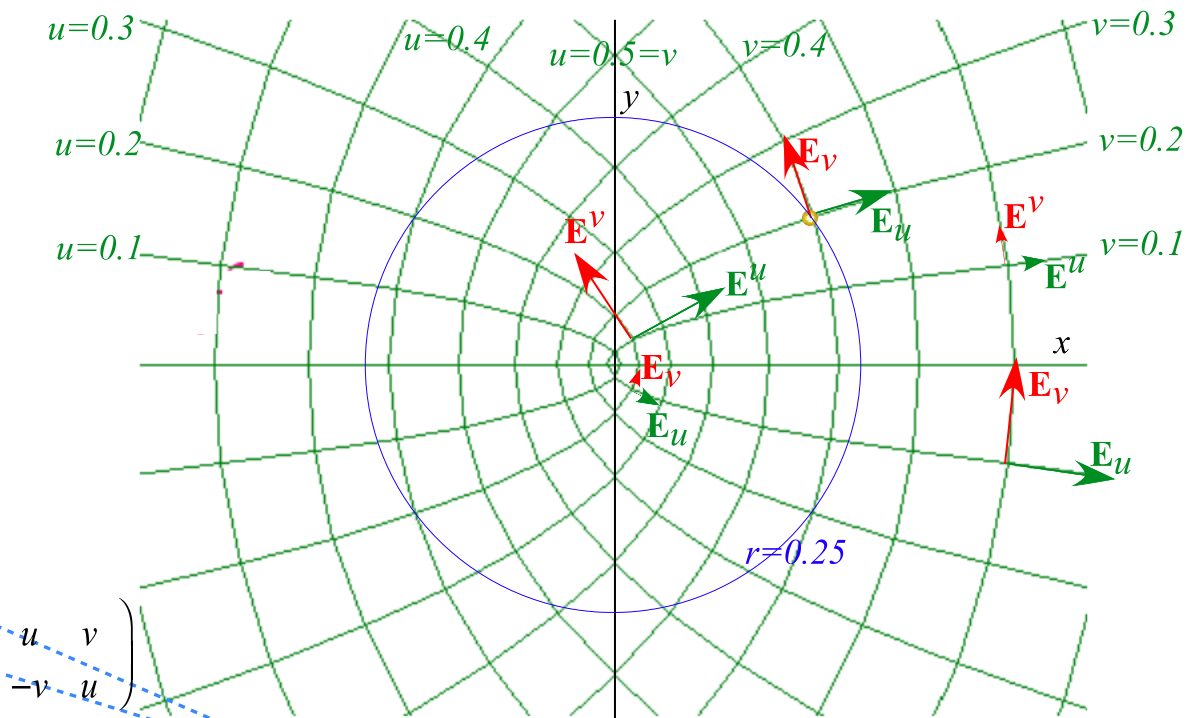

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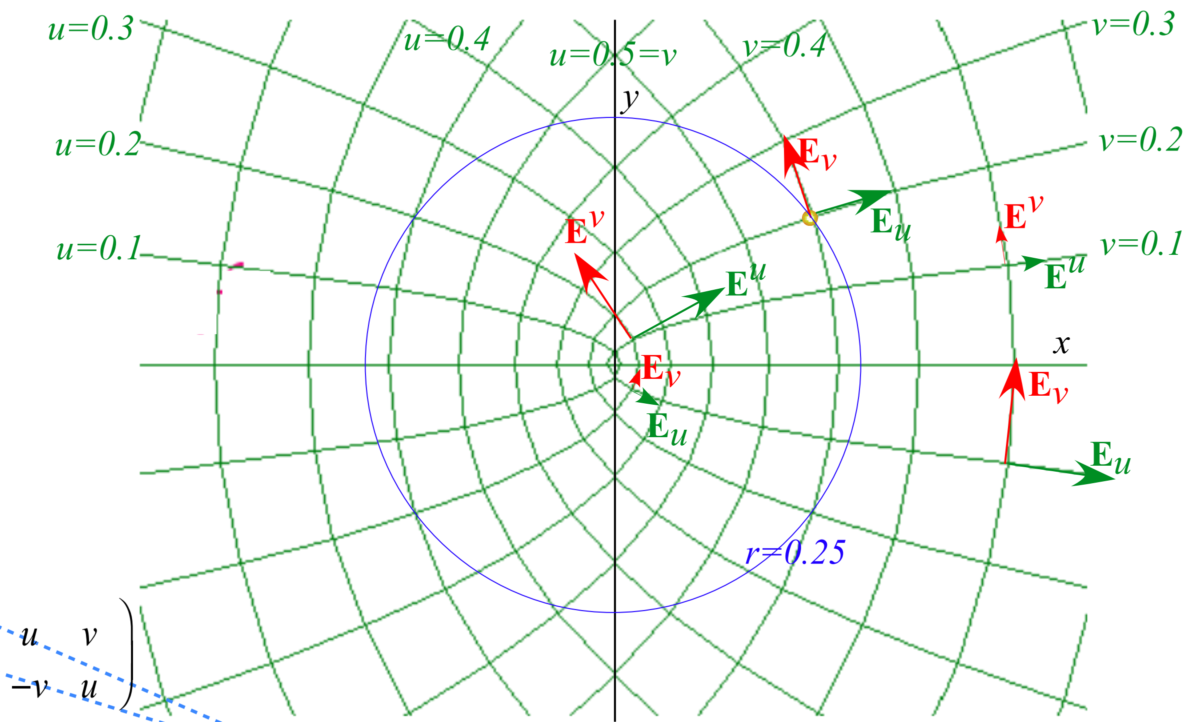

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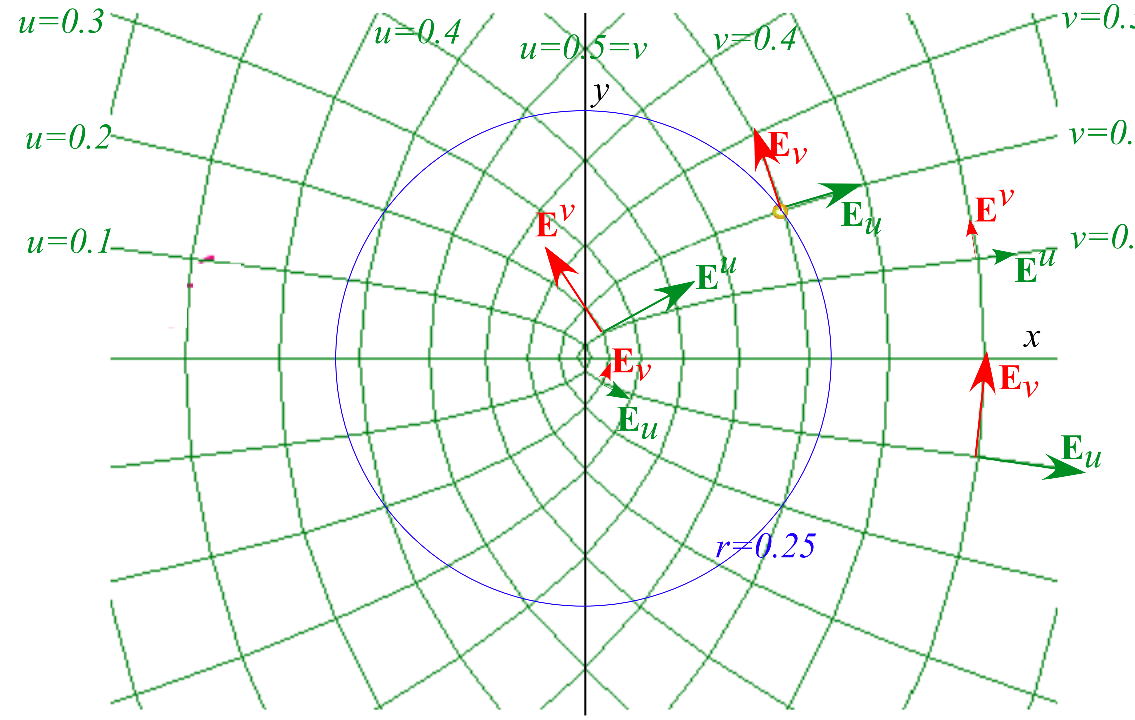

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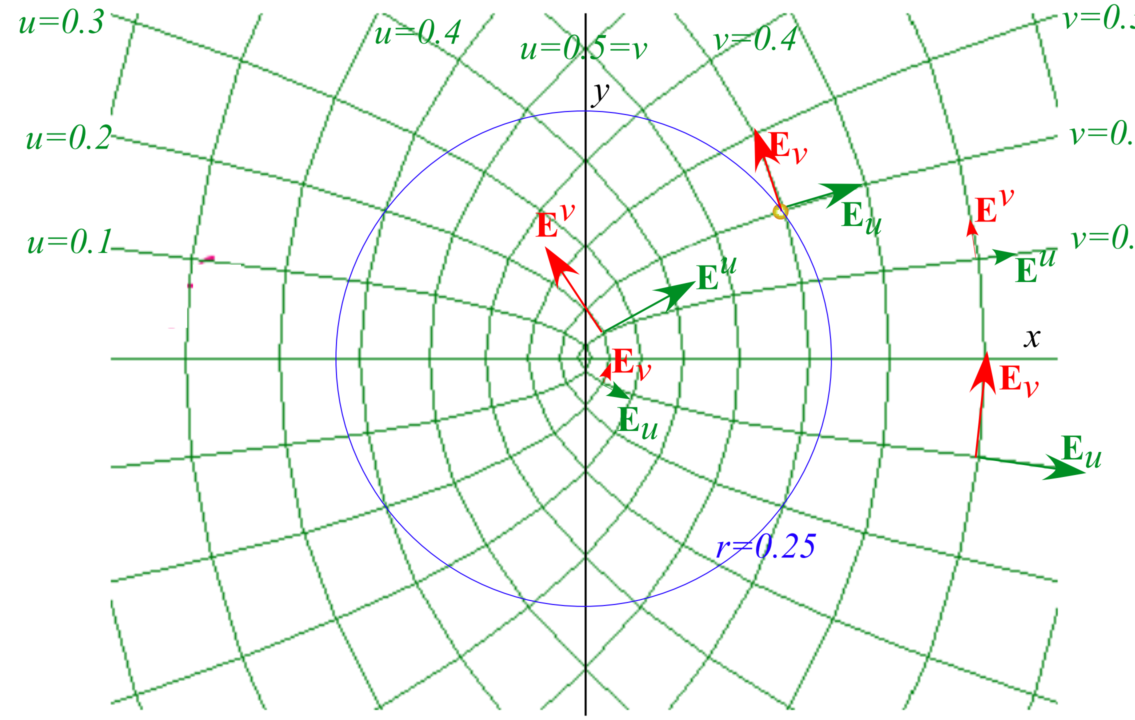

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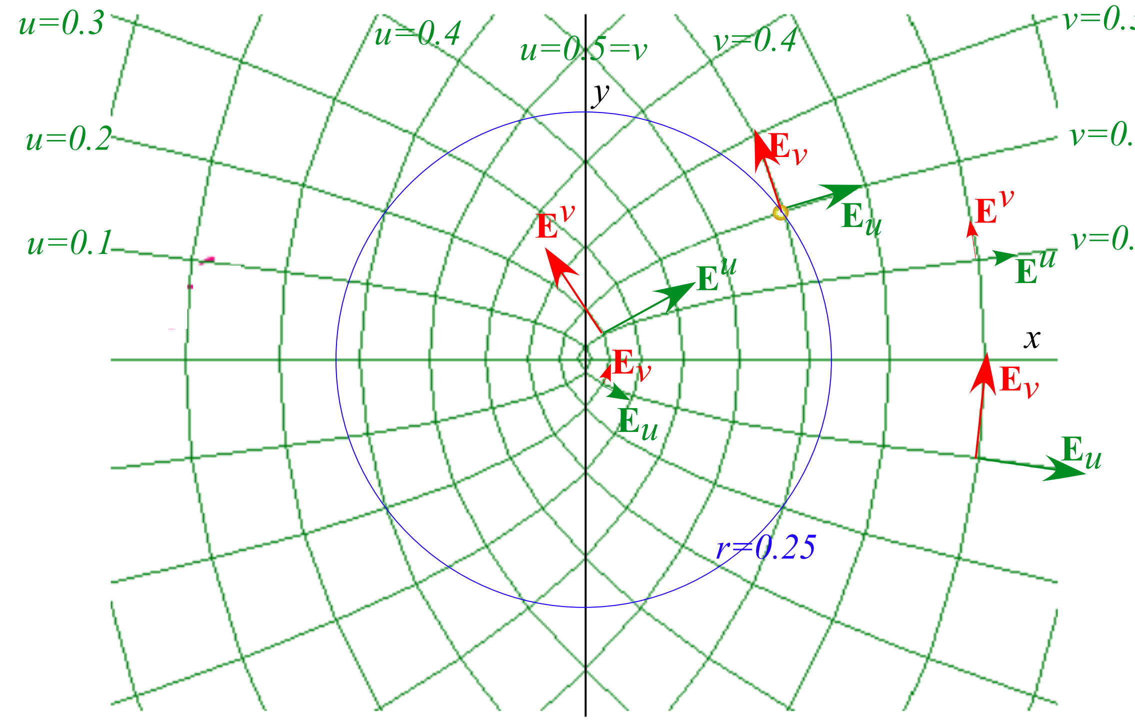

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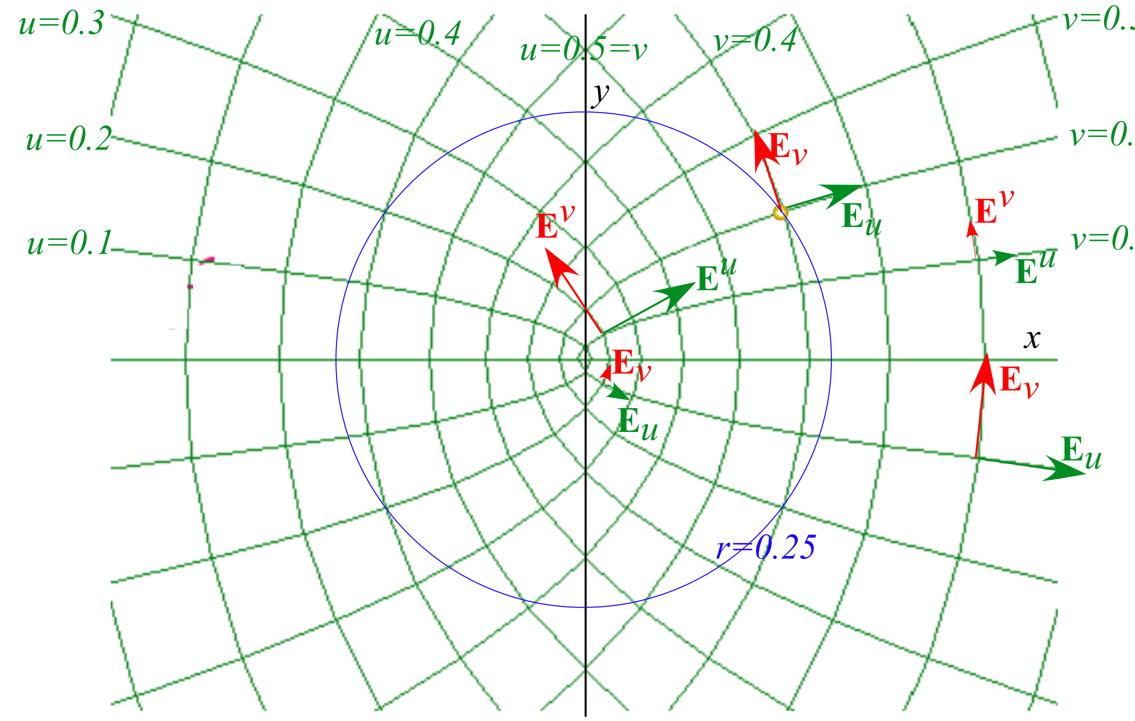

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$$H=\tfrac{1}{2m}(g^{ab}p_a p_b)+V=\tfrac{1}{2m}(g^{uu}p_u^2+g^{vv}p_v^2)+V=\frac{p_u^2+p_v^2}{8m(u^2+v^2)}+V$$

$$V=\varepsilon x+k/r$$

Stark-Coulomb potential

For a *Stark-Coulomb potential* Hamiltonian *(H=E)* is constant and *separable* into u and v parts.

$$4(u^2+v^2)E=\tfrac{1}{2m}(p_u^2+p_v^2)+4(u^4-v^4)\varepsilon+4k \quad \text{for: } H=E \text{ and: } V=\varepsilon x+\frac{k}{r}=\varepsilon(u^2-v^2)+\frac{k}{u^2+v^2}$$

Each sub-Hamiltonian part h_u and h_v is a constant. Together they sum to zero total energy $0=h_u+h_v$.

$$0=\tfrac{1}{2m}p_u^2-4Eu^2+4\varepsilon u^4 \quad + \quad \tfrac{1}{2m}p_v^2-4Ev^2-4\varepsilon v^4+4k \quad = \quad h_u \quad + \quad h_v$$

Zero Stark-field *(ε=0)* gives h_u or h_v harmonic oscillation if $E<0$. It's unstable or anharmonic otherwise.

$$\dot{p}_u=-\frac{\partial h_u}{\partial u}=-8Eu+16\varepsilon u^3 \qquad \dot{u}=\frac{\partial h_u}{\partial p_u}=p_u/m \qquad \dot{p}_v=-\frac{\partial h_v}{\partial v}=-8Ev-16\varepsilon v^3 \qquad \dot{v}=\frac{\partial h_v}{\partial p_v}=p_v/m$$

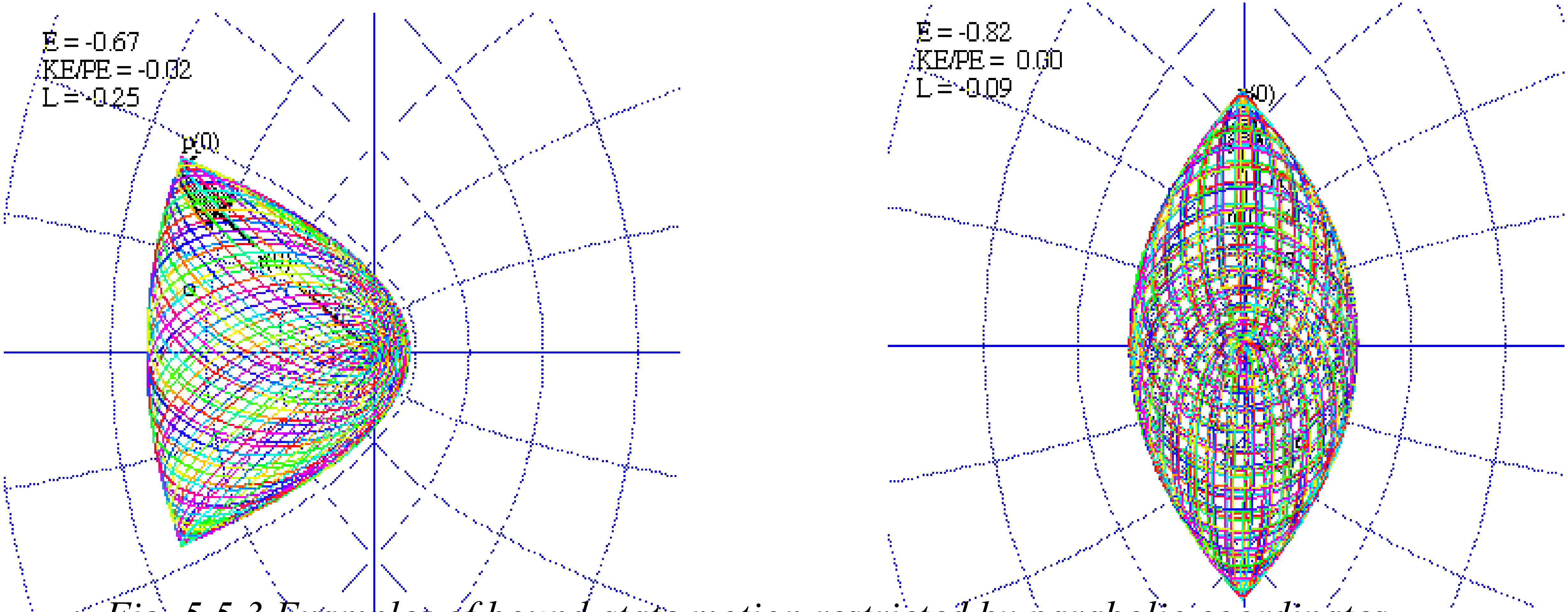


Fig. 5.5.3 Examples of bound-state motion restricted by parabolic coordinates

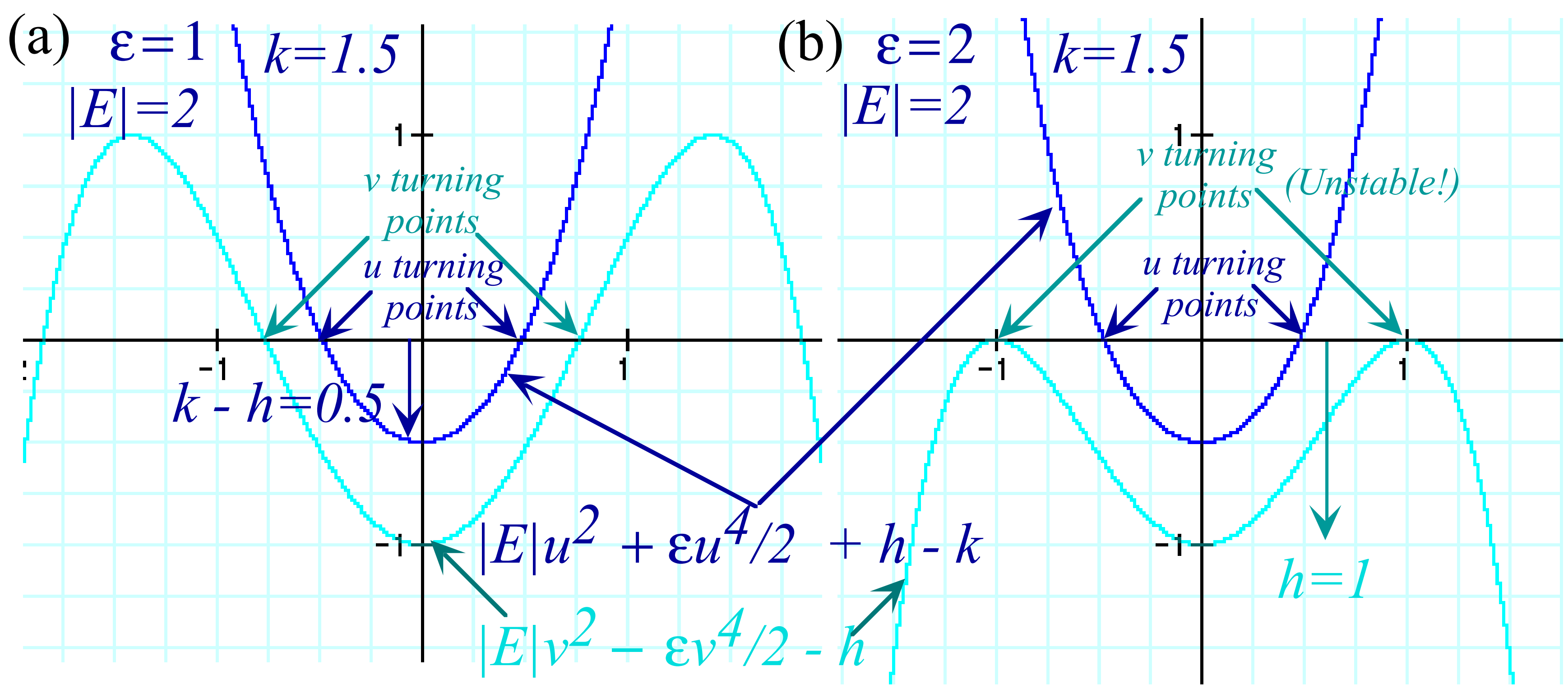
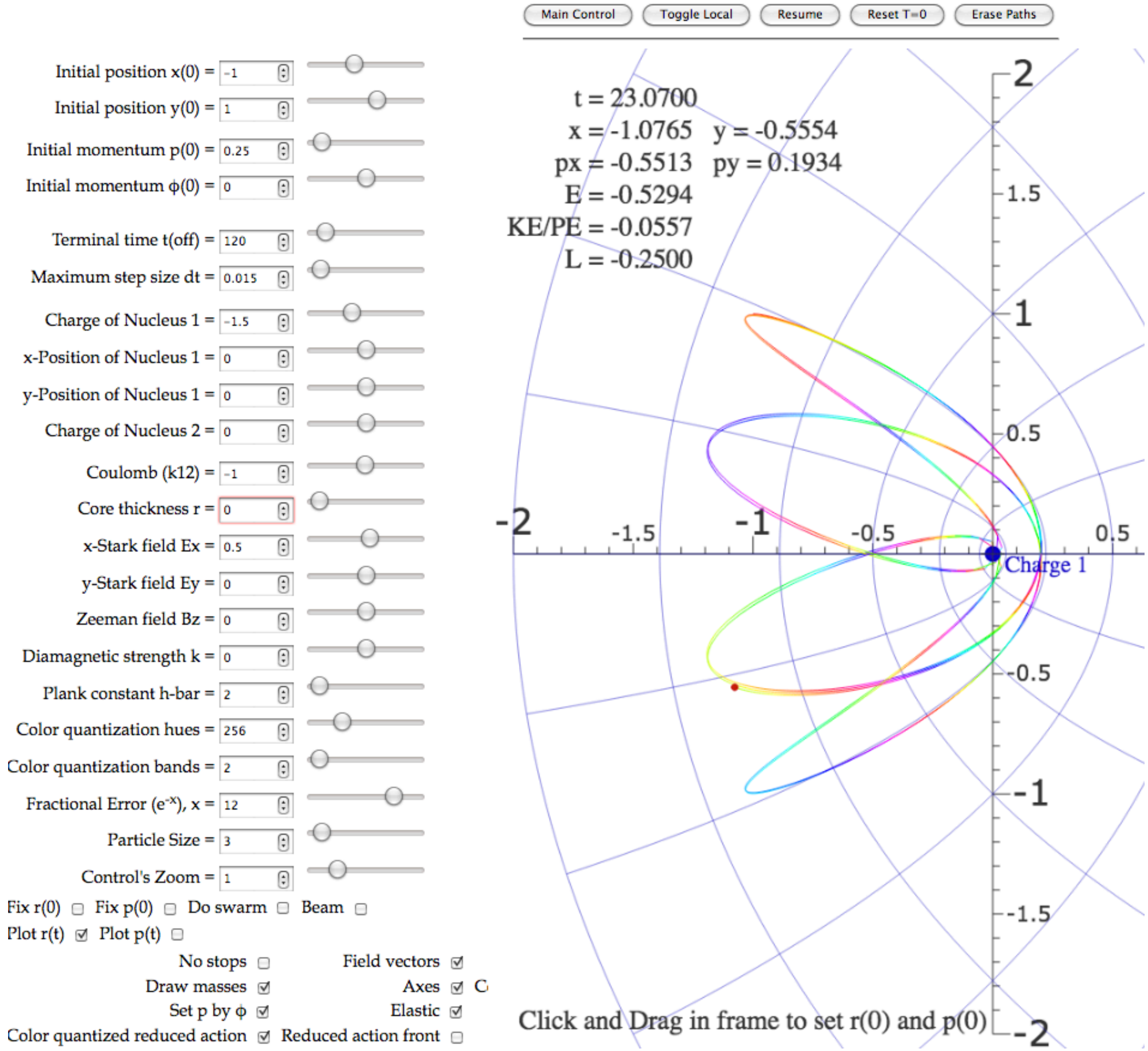


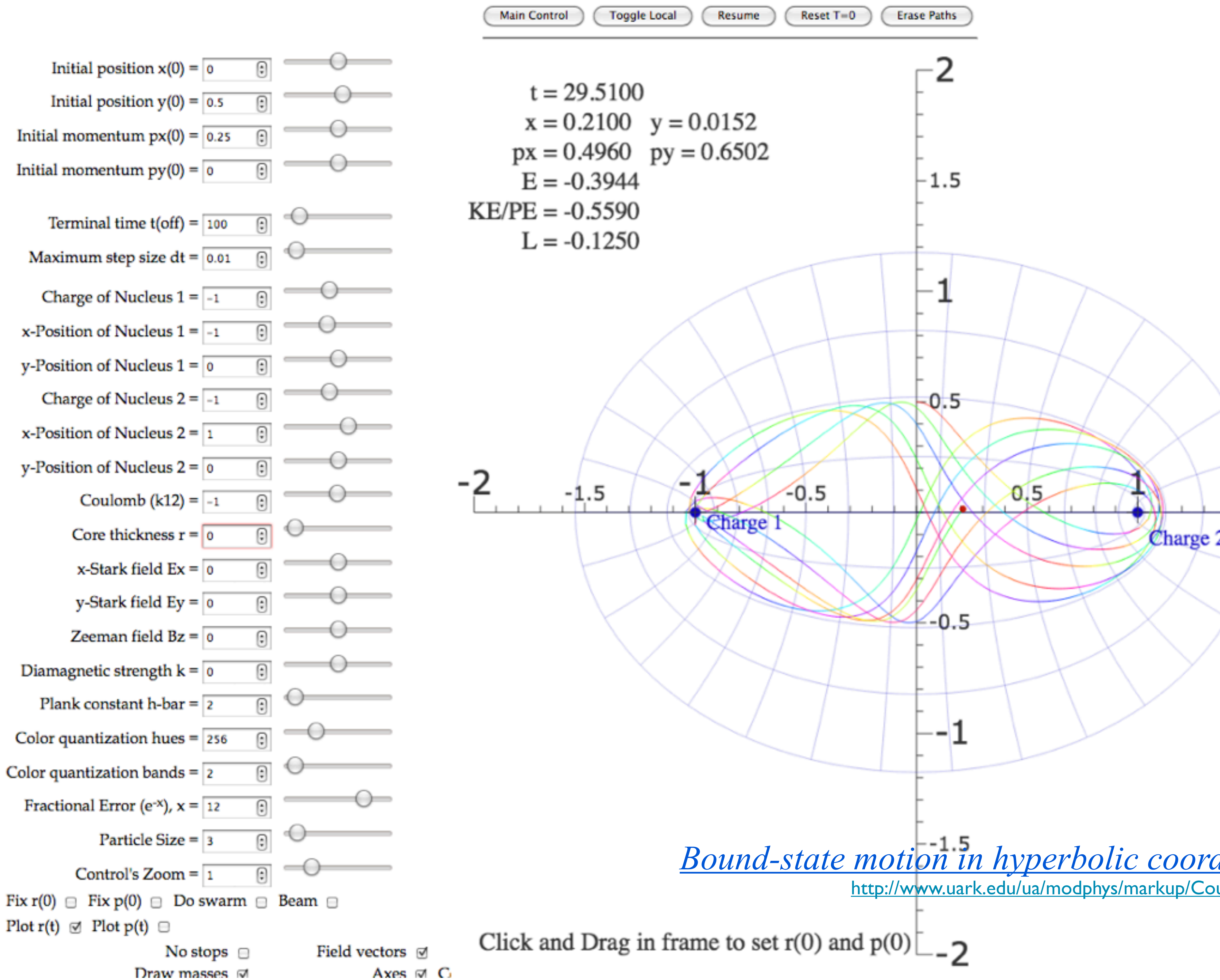
Fig. 5.5.2 Effective potentials for parabolic coordinates

Examples of bound-state motion restricted by parabolic coordinates
(H classical electronic Stark-field orbits with color-quantization)



[*Bound-state motion in parabolic coordinates*](http://www.uark.edu/ua/modphys/markup/CoulItWeb.html) http://www.uark.edu/ua/modphys/markup/CoulItWeb.html

Examples of bound-state motion restricted by hyperbolic-elliptic coordinates (H2+-ion classical electronic orbits with color-quantization)



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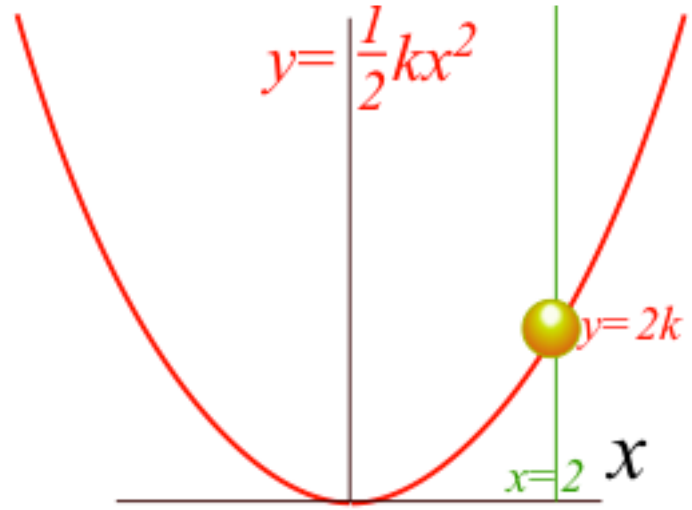
Simple constrained problem...



...and a variety of solutions

Other Ways to do constraint analysis

Way 3. OCC constraint webs

Preview of atomic-Stark orbits

Classical Hamiltonian separability

→ *Way 4. Lagrange multipliers*

Lagrange multiplier as eigenvalues

Multiple multipliers

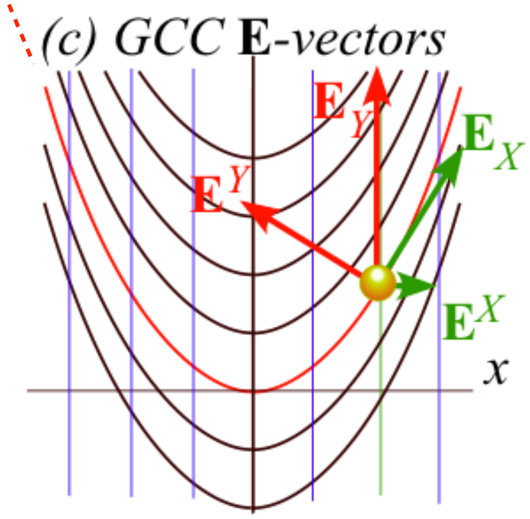
"Non-Holonomic" multipliers

Lagrange multiplier approaches

Lagrange multiplier or λ-method. The constraining parabola $y=1/2kx^2$ is defined as follows.

$$c^1 = \frac{1}{2}kx^2 - y = 0$$

(Back to "Stupid-Parabolic" GCC)



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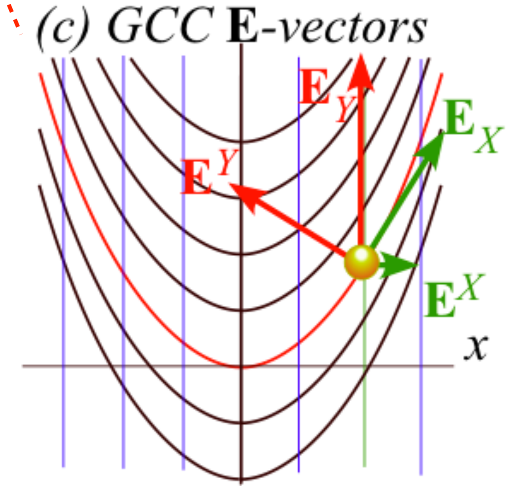

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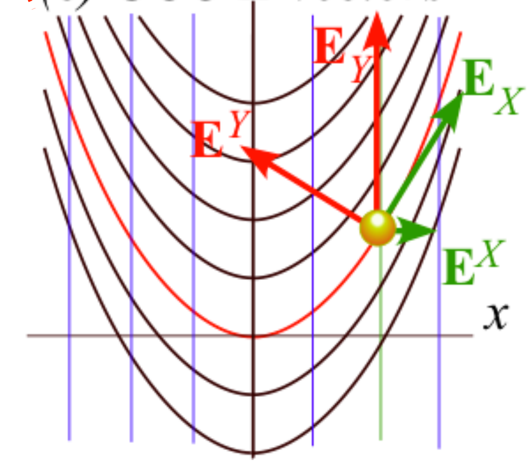
(c) GCC **E**-*vectors*



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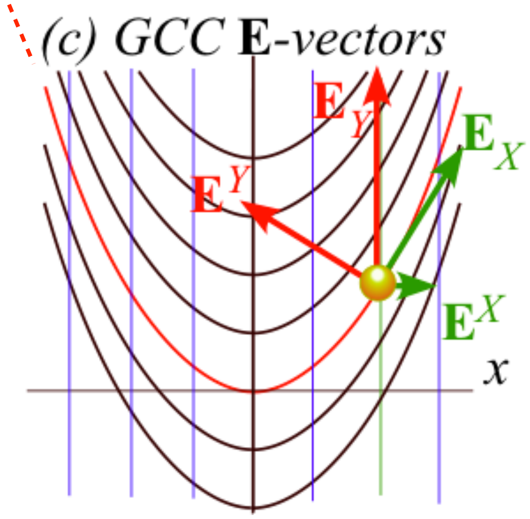

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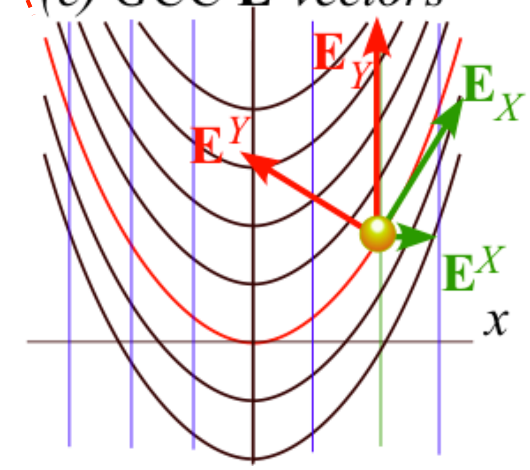

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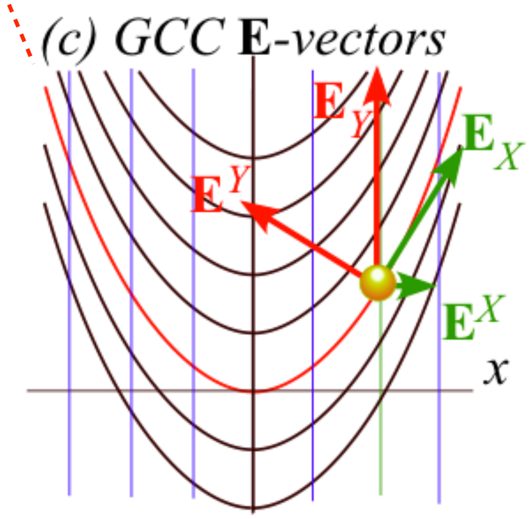

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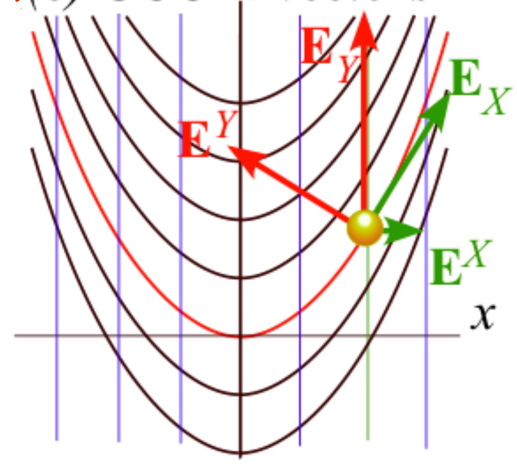

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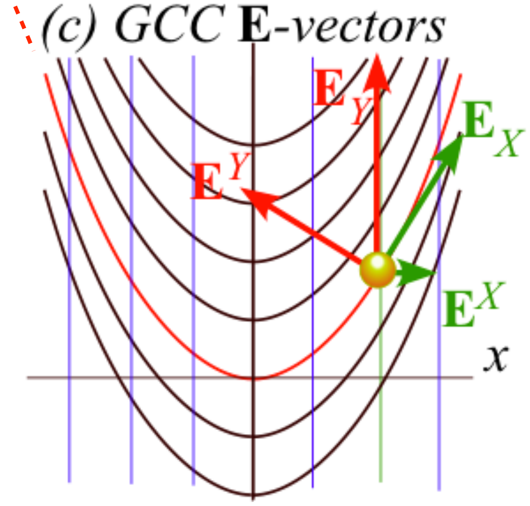

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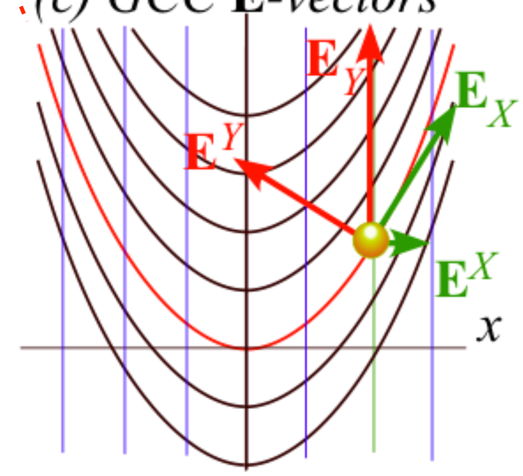

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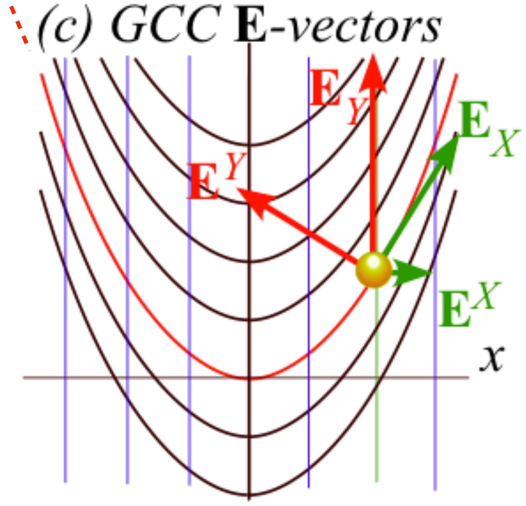

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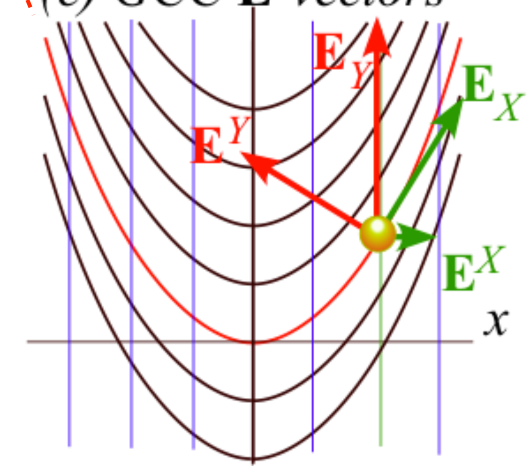

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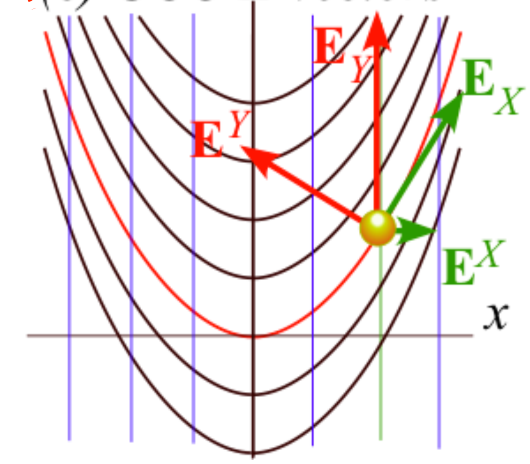
(c) GCC ***E****-vectors*



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$$\lambda = -m(k\dot{x}^2 + kx\ddot{x} + g)$$

Then the λ function gives the new constrained *x*-equation of motion.

$$m\ddot{x} = \lambda kx = -m(k\dot{x}^2 + kx\ddot{x} + g)kx$$

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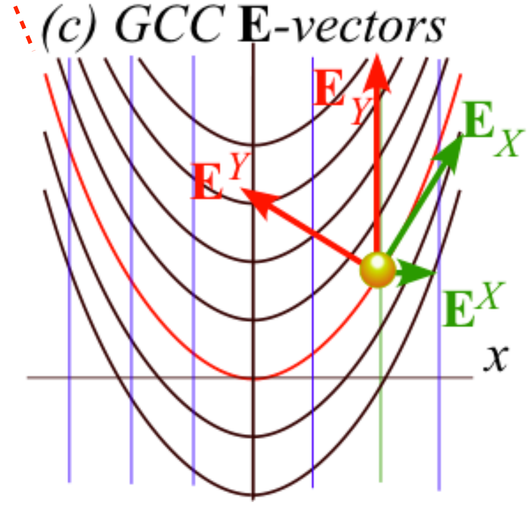

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$$\lambda = -m(k\dot{x}^2 + kx\ddot{x} + g)$$

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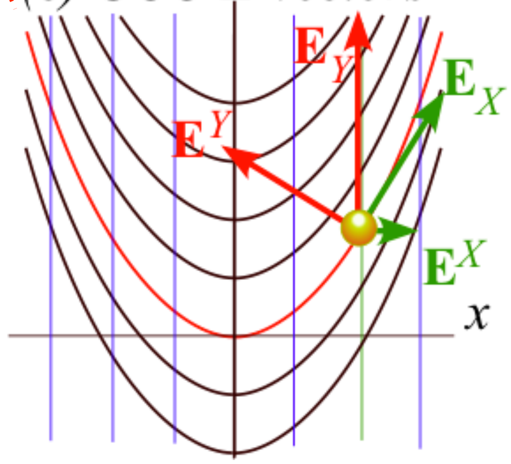
Lagrange multiplier approaches

Lagrange multiplier or λ-method. The constraining parabola $y=1/2kx^2$ is defined as follows.

$$c^1 = \tfrac{1}{2} kx^2 - y = 0$$

(Back to "Stupid-Parabolic" GCC)

Imagine this is a coordinate line. Its normal constraining force **F** is along its c^1-gradient . $(\mathbf{F} \propto \nabla c^1)$



$$\mathbf{F} = \lambda \nabla c^1 = \lambda \nabla(\tfrac{1}{2}kx^2 - y) = \lambda \begin{pmatrix} \frac{\partial c^1}{\partial x} \\ \frac{\partial c^1}{\partial y} \end{pmatrix} = \lambda \begin{pmatrix} kx \\ -1 \end{pmatrix}$$

Proportionality factor $\lambda = F_1^c$ is a *Lagrange multiplier*.

It is like a covariant constraint component F_1^c of a contravariant vector $\mathbf{E}^1 = \nabla c^1$ that arises if $c^1(x,y) = const.$ was a coordinate line causing a constraint force $\mathbf{F} = F_1^c \nabla c^1$.

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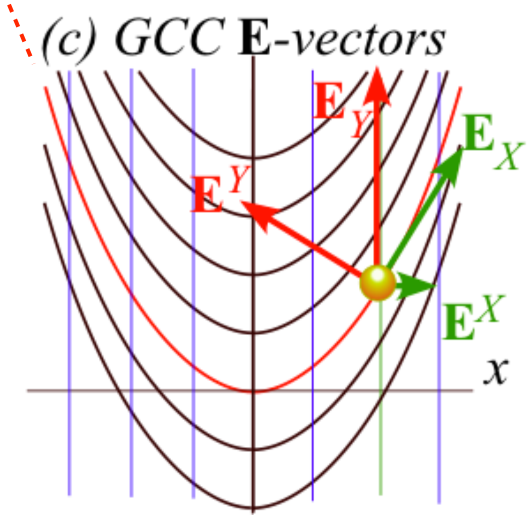

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(Same equation as on p.12)

$$\ddot{x} = \frac{-k\dot{x}^2 - g}{1 + k^2x^2}kx$$

Other Ways to do constraint analysis

Way 3. OCC constraint webs

Preview of atomic-Stark orbits

Classical Hamiltonian separability

Way 4. Lagrange multipliers

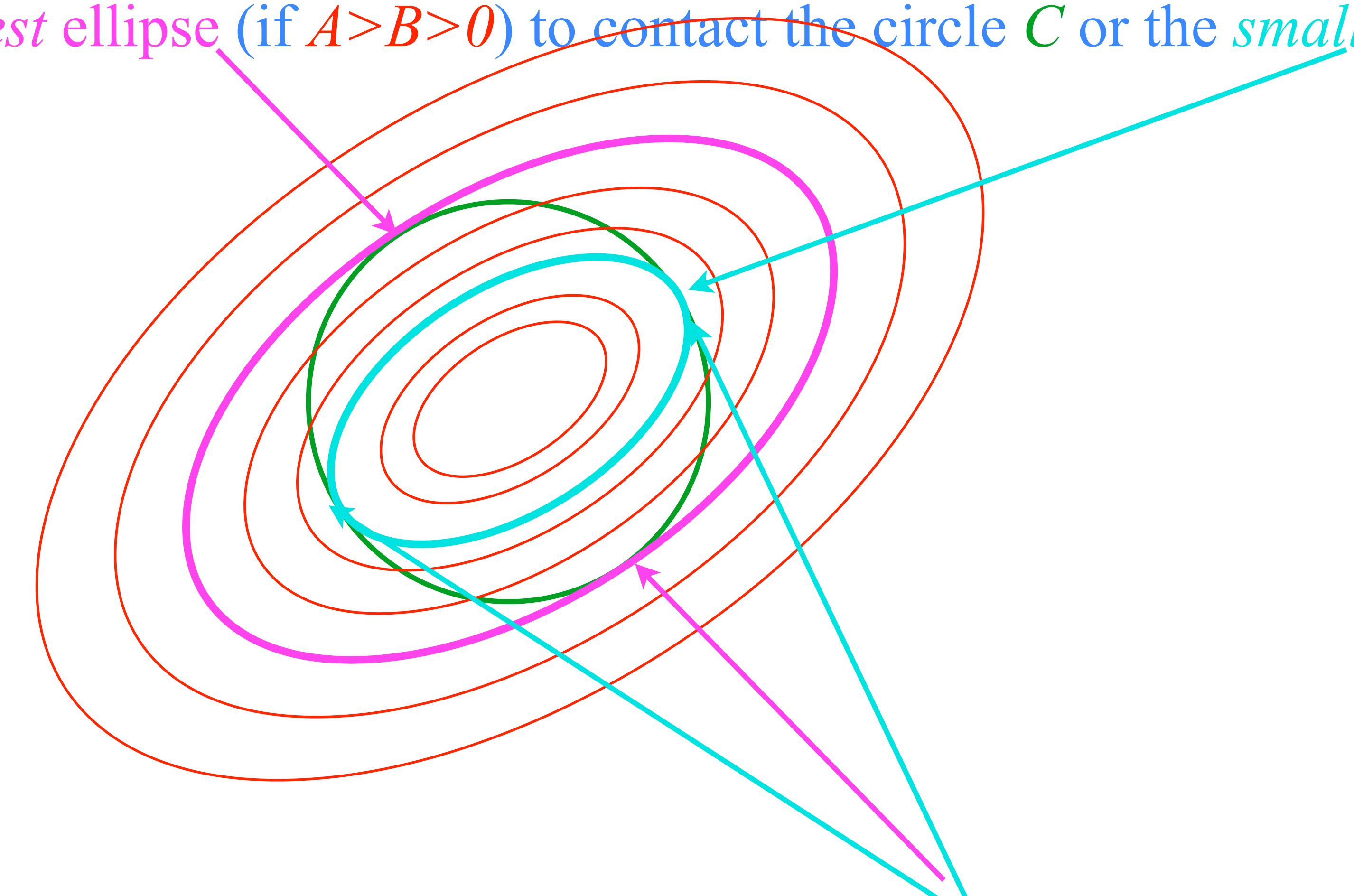
→ *Lagrange multiplier as eigenvalues*

Multiple multipliers

"Non-Holonomic" multipliers

Lagrange multiplier basics

Suppose you need to find maximum of $H=(Ax^2+Bxy+Ay^2)/2$ subject to constraint: $C=(x^2+y^2)/2=const.$
By geometry you are finding the *largest* ellipse (if $A>B>0$) to contact the circle C or the *smallest*.



Extreme cases occur only at *contact points*

The contact points satisfy
gradient proportionality equations:

$$\nabla H = \lambda \cdot \nabla C$$

$$\begin{pmatrix} \partial_x H \\ \partial_y H \end{pmatrix} = \lambda \cdot \begin{pmatrix} \partial_x C \\ \partial_y C \end{pmatrix}$$

$$\begin{pmatrix} Ax + By \\ Bx + Dy \end{pmatrix} = \lambda \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

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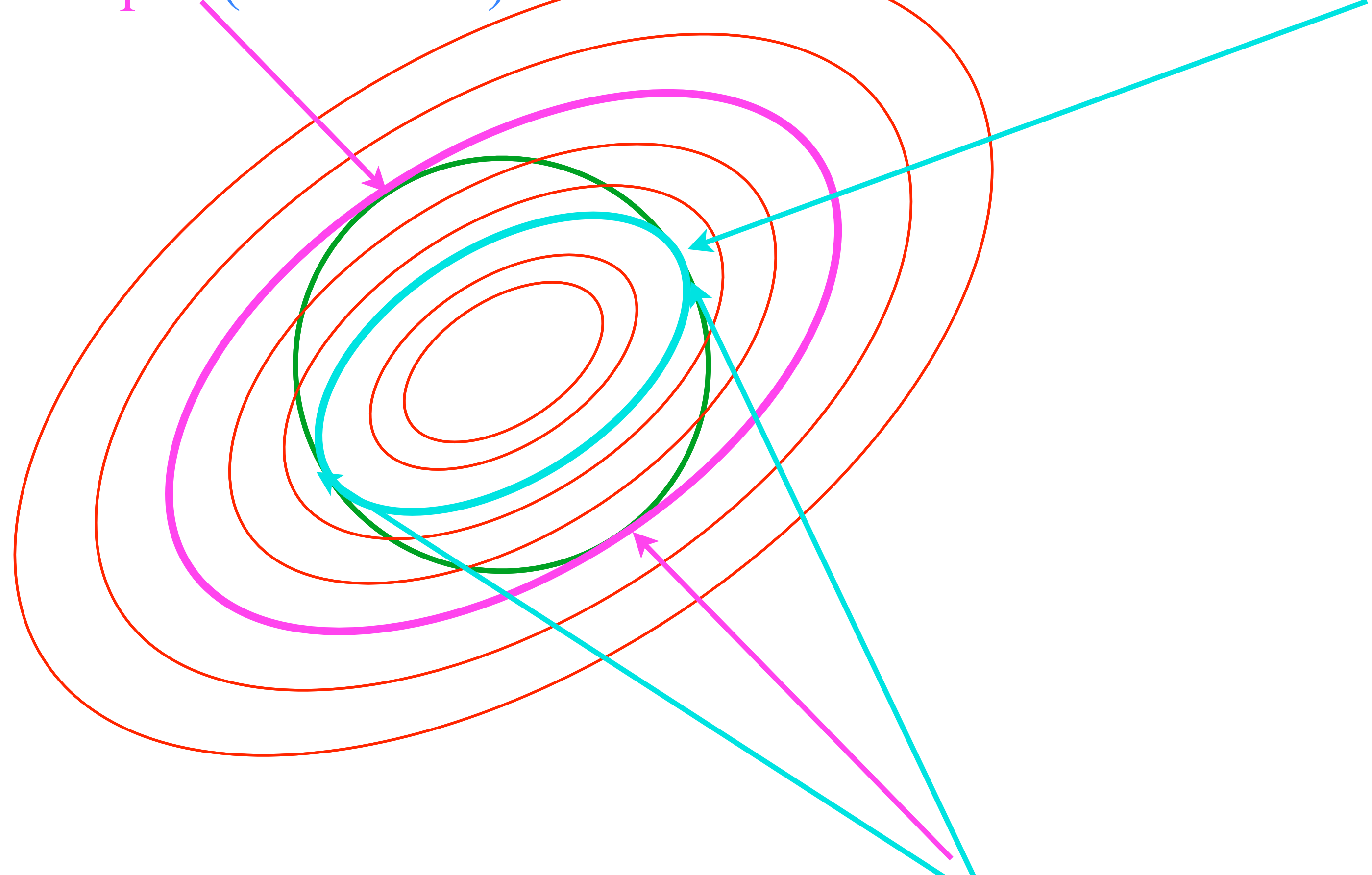

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This amounts to a *λ-eigenvalue-eigenvector* equation

$$\begin{pmatrix} A & B \\ B & D \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

(More about this in Units 4-6)

(Perhaps, this is why we often label eigenvalues λ with a Greek "L")

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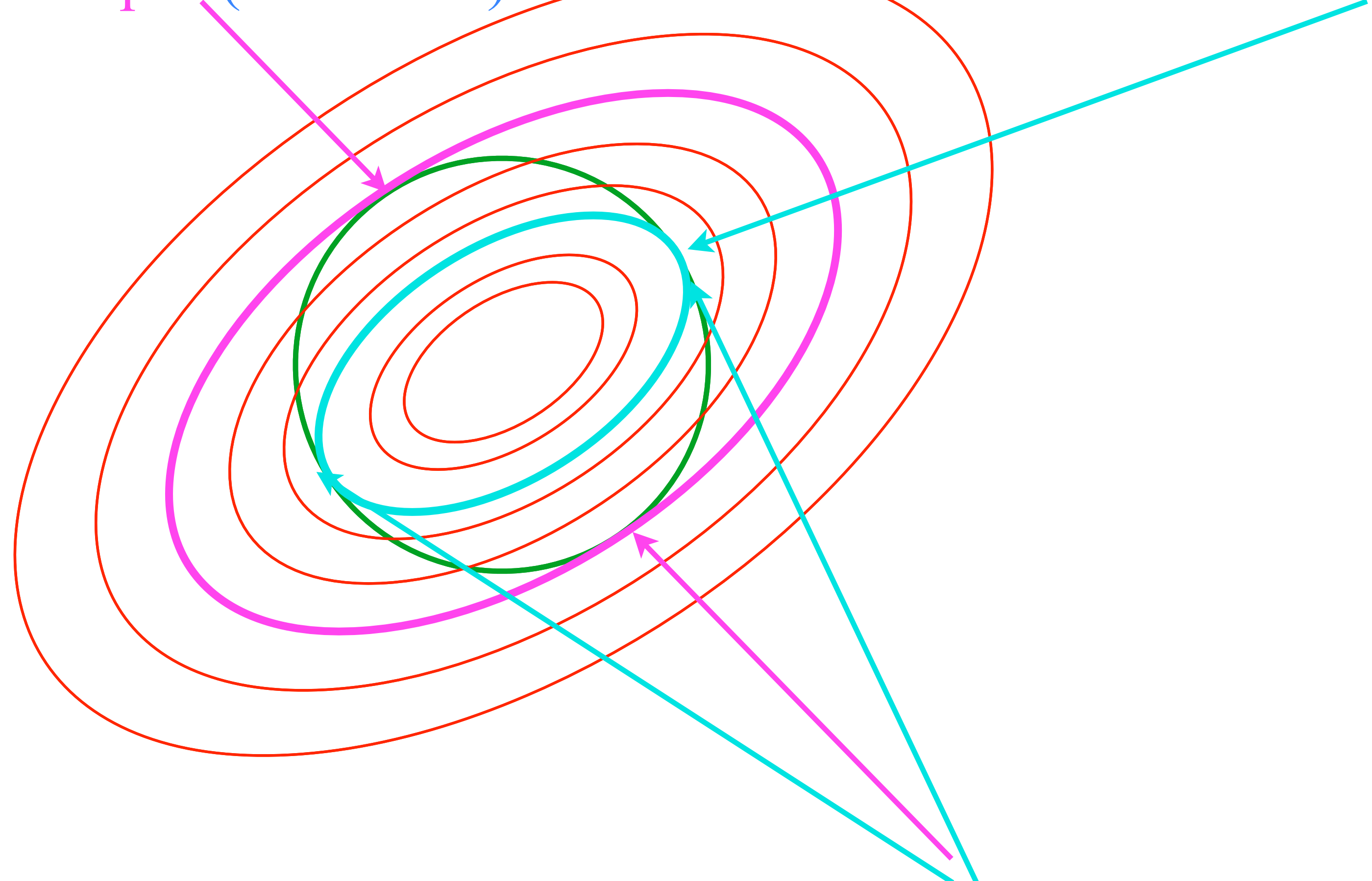

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Eigenvalues λ are *extreme* matrix "own"-values $\langle\psi|M|\psi\rangle$ subject *Norm-constraint* $\langle\psi|\psi\rangle=1$

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→ *Multiple multipliers*

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Lagrange multipliers also work for constraints $c(q^k) = const.$ that cut across GCC lines.
It is only necessary to express the gradient of $c(q^k)$ in terms of the GCC using chainsaw sum rule.

$$\nabla c = \frac{\partial c}{\partial x^j}\hat{\mathbf{e}}^j = \frac{\partial c}{\partial q^k}\mathbf{E}^k \qquad \frac{\partial c}{\partial q^k} = \frac{\quad}{\partial q^k}\frac{\partial c}{\quad} = \frac{\partial x^j}{\partial q^k}\frac{\partial c}{\partial x^j} = \frac{\partial \mathbf{r}}{\partial q^k}\bullet\frac{\partial c}{\partial \mathbf{r}} = \mathbf{E}_k\bullet\nabla c$$

Then the Lagrange equations for each GCC q^k will share a λ-multiplier on its c-gradient component.

$$\begin{pmatrix} \dot{p}_1 - \frac{\partial L}{\partial q^1} \\ \dot{p}_2 - \frac{\partial L}{\partial q^2} \\ \vdots \end{pmatrix} = \begin{pmatrix} \lambda\frac{\partial c}{\partial q^1} \\ \lambda\frac{\partial c}{\partial q^2} \\ \vdots \end{pmatrix} \qquad \dot{p}_k - \frac{\partial L}{\partial q^k} = \lambda\frac{\partial c}{\partial q^k}$$

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Two or more constraints $c^1(q^k) = const.,\ c^2(q^k) = const.,\cdots$ add two or more λ_γ terms to the equations.

$$\begin{pmatrix} \dot{p}_1 - \frac{\partial L}{\partial q^1} \\ \dot{p}_2 - \frac{\partial L}{\partial q^2} \\ \vdots \end{pmatrix} = \begin{pmatrix} \lambda_1\frac{\partial c^1}{\partial q^1} \\ \lambda_1\frac{\partial c^1}{\partial q^2} \\ \vdots \end{pmatrix} + \begin{pmatrix} \lambda_2\frac{\partial c^2}{\partial q^1} \\ \lambda_2\frac{\partial c^2}{\partial q^2} \\ \vdots \end{pmatrix} + \ldots \qquad \dot{p}_k - \frac{\partial L}{\partial q^k} = \lambda\gamma\frac{\partial c^\gamma}{\partial q^k}$$

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→ *"Non-Holonomic" multipliers*

Constraints may be determined by differential relations that are not integrable.
Lagrange methods use differentials and do not need integral c^{γ} surface functions.

Integral constraint differentials

$$0 = dc^1 = \frac{\partial c^1}{\partial q^1} dq^1 + \frac{\partial c^1}{\partial q^2} dq^2 + \ldots$$

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$$\vdots \qquad \vdots$$

General differential constraint relations

$$0 = C^1_1 dq^1 + C^1_2 dq^2 + \ldots$$

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I guess that means that integrable ones are *holonomic*. (But why do we need the bigger words?)

A requirement for integrability (or "holonomicty") is that double differentials are symmetric.

$$\frac{\partial^2 c^\gamma}{\partial q^j \partial q^k} = \frac{\partial^2 c^\gamma}{\partial q^k \partial q^j}$$

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General differential constraint relations

$$\frac{\partial C^{\gamma}_k}{\partial q^j} \quad \textit{may or may not be} \quad \frac{\partial C^{\gamma}_j}{\partial q^k}$$