

Lecture 1

Tue. 8.23.2016

1st axioms and theorems of classical mechanics

(Ch. 1 thru Ch. 3 of Unit 1)

Geometry of momentum conservation axiom (ala Occam's Razor)

Totally Inelastic "ka-runch" collisions (begin 4:1 graph project)*

*Perfectly Elastic "ka-bong" and Center Of Momentum (COM) symmetry**

+Intro to weighty-averages and vector notation

Comments on idealization in classical models

Geometry of Galilean translation symmetry

45° shift in (V_1, V_2) -space

Time reversal symmetry

...of COM collisions

Algebra, Geometry, and Physics of momentum conservation axiom

Vector algebra of collisions

Matrix or tensor algebra of collisions

Deriving Energy Conservation Theorem

Numerical details of collision tensor algebra

Note - Many of the underlined links throughout this lecture file link to the specific selected cases within those Web Simulators

**Launch Car Generic Collision Web Simulator*

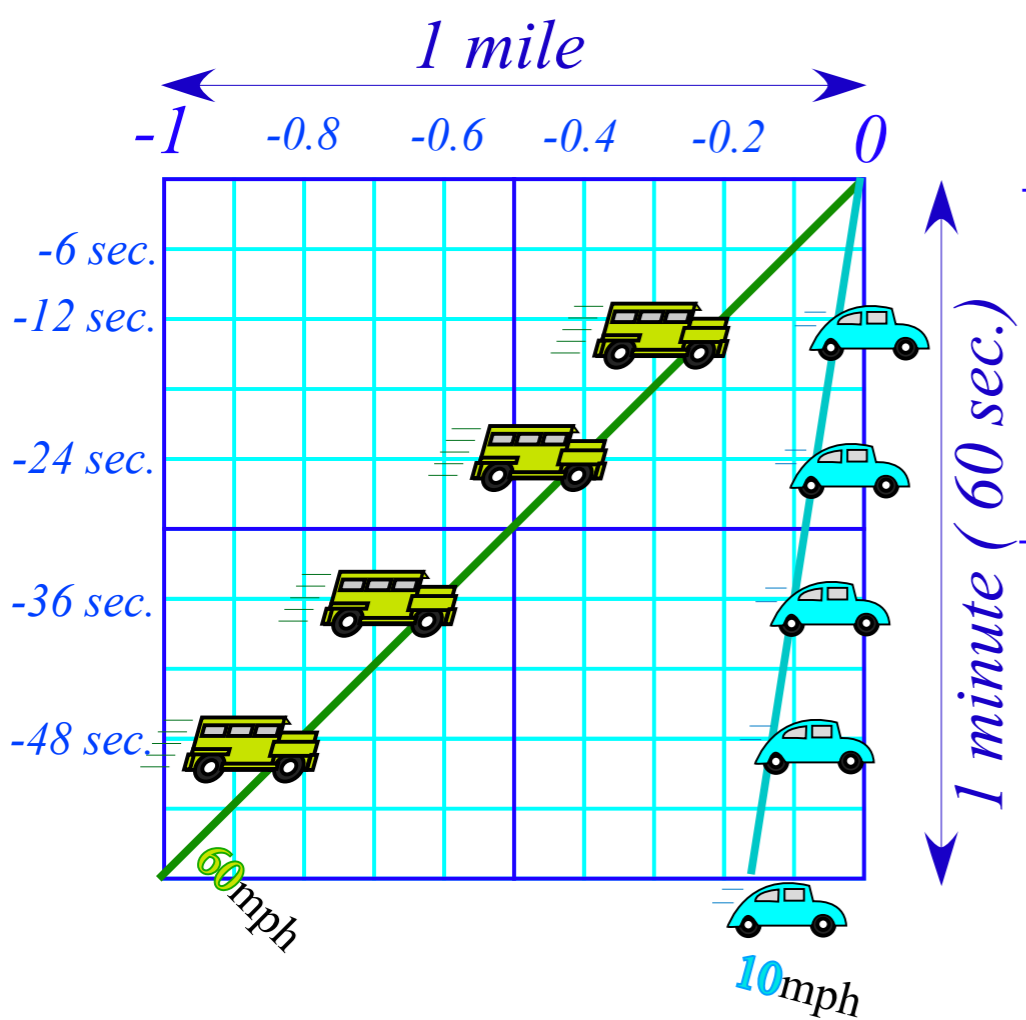
<http://www.uark.edu/ua/modphys/markup/CMMotionWeb.html>

**Launch Generic Superball Collision Web Simulator*

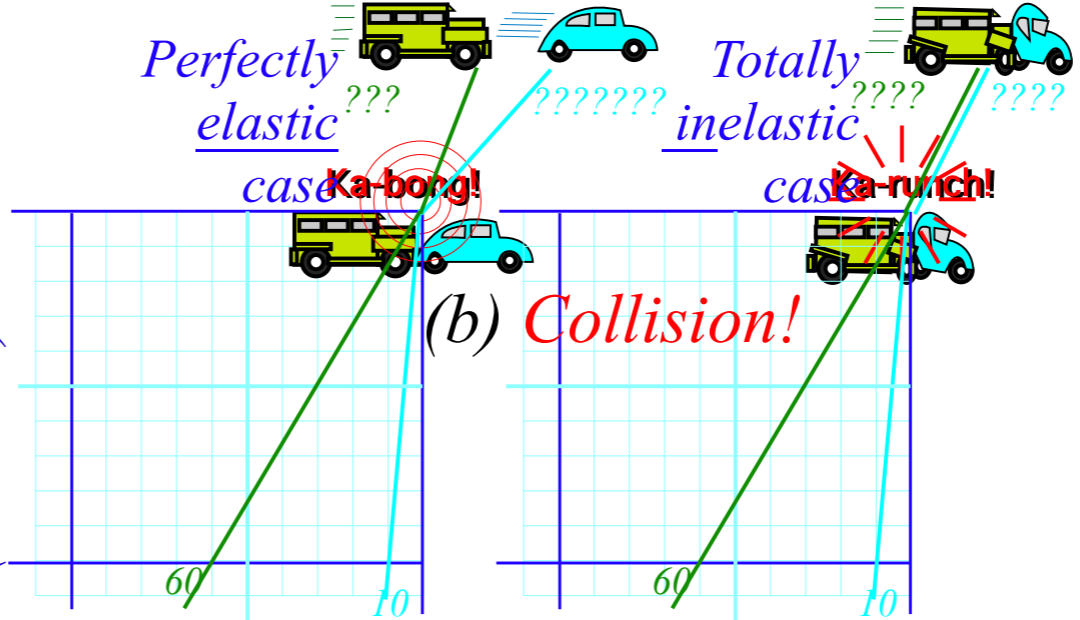
<http://www.uark.edu/ua/modphys/markup/BounceItWeb.html>

A problem in *space-time* : (60mph Cell-faxing 4ton SUV rear-ends 10mph 1ton VW)

Before collision.....



After collision...what velocities?



Car Simulator
Space vs Space
Elastic

Car Simulator
Space vs Space
Inelastic

Simulator
Elastic Collision
Dual Panel
Space vs Space
and
Space vs Time
(Newton)

Simulator
Inelastic Collision
Dual Panel
Space vs Space
and
Space vs Time
(Newton)

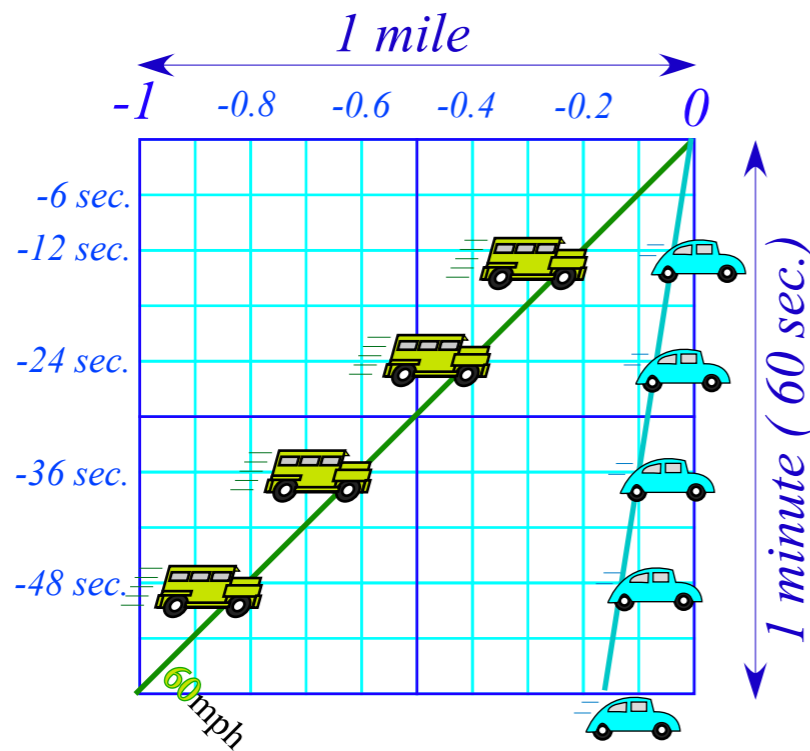
Simulator
Elastic Collision
Dual Panel
Space vs Space
and
Time vs. Space
(Minkowski)

Simulator
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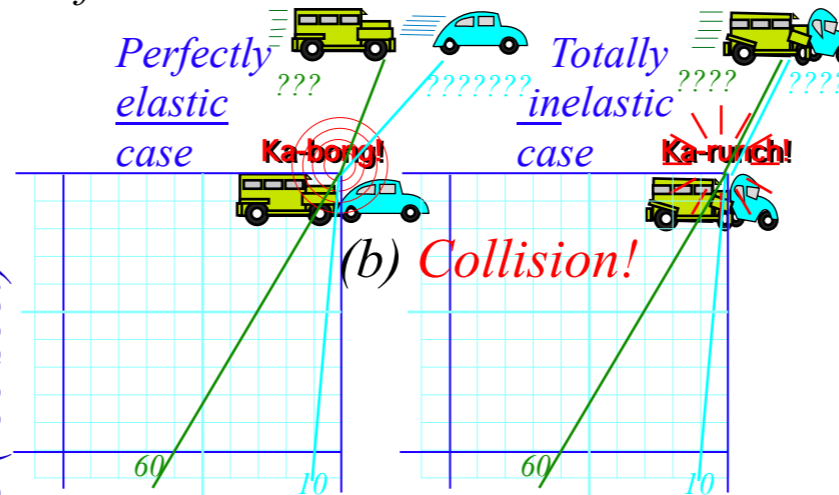
*Car Collision simulator <http://www.uark.edu/ua/modphys/markup/CMMotionWeb.html?scenario=20005>

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After collision...what velocities?



Conventional solution: Look up the usual momentum and energy formulas/axioms:
 $\sum_i m V_i(\text{initial}) = \sum_i m V_i(\text{final})$
 $\sum_i m V_i^2(\text{initial}) = \sum_i m V_i^2(\text{final})$
 and solve...

Let's see if we can solve this *easily* with just *one* (or one-and-a-half) axiom(s)

Axiom-1: All mass or masses keep their total momentum until it is changed by some outsider.

Note: Links in testing are only temporary.

Best to access the URLs that do not have "/testing.." nor "/devΩel/..." after the "/modphys".

*Car Collision simulator <http://www.uark.edu/ua/modphys/testing/markup/CMMotionWeb.html>

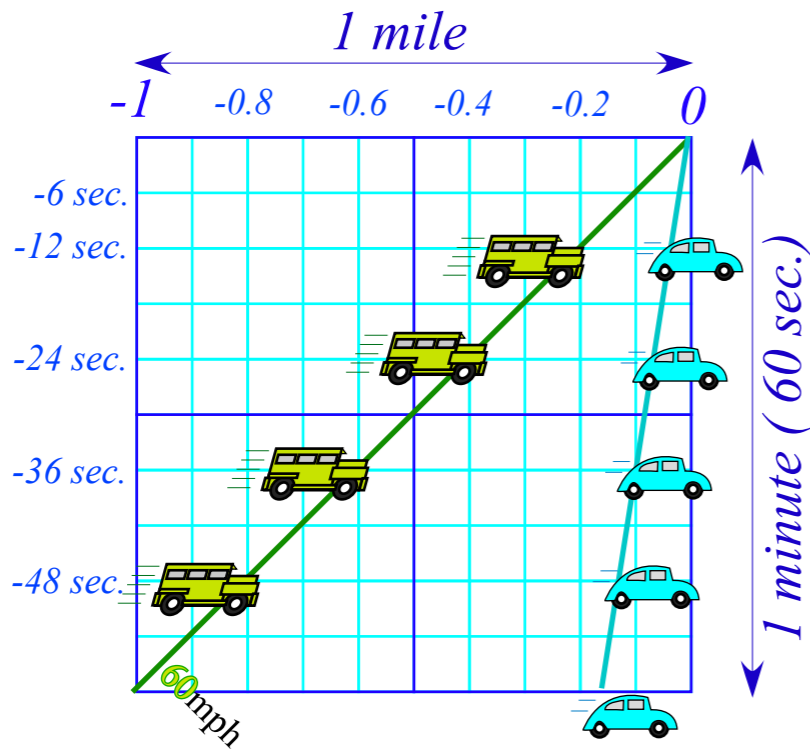
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Also, note the syntax needed to pass the scenario as a parameter

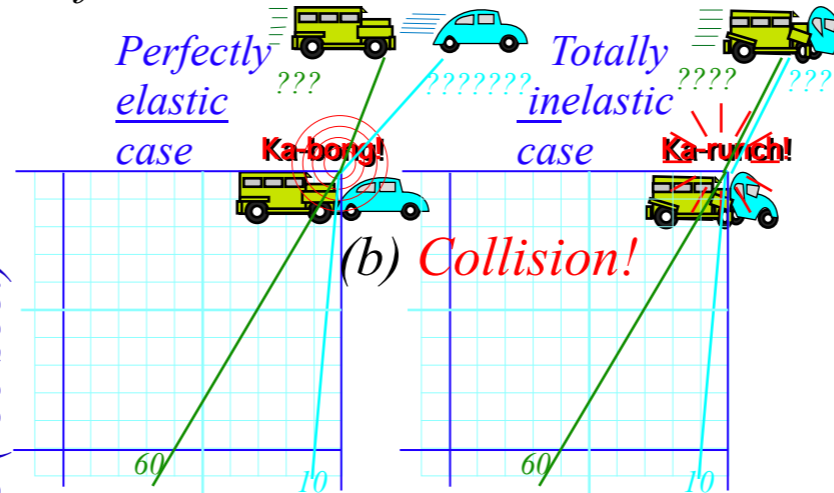
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 $P_{Total} = M_{SUV} V_{SUV} + M_{VW} V_{VW}$

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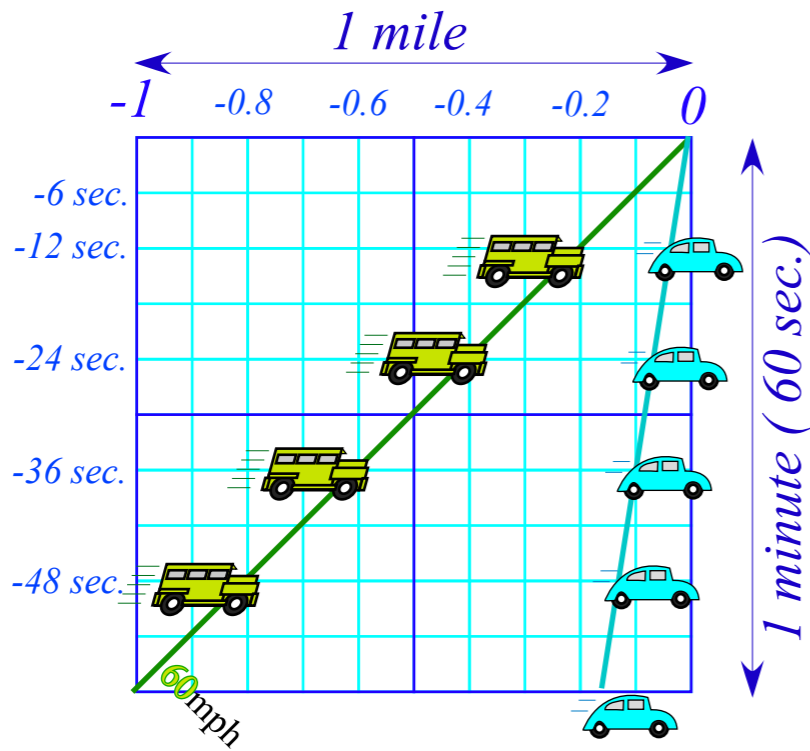
...But an UNconventional way is quicker and slicker....
 (Just have to draw 2 lines!)

Let's see if we can solve this *easily* with just **one** (or one-and-a-half) axiom(s)

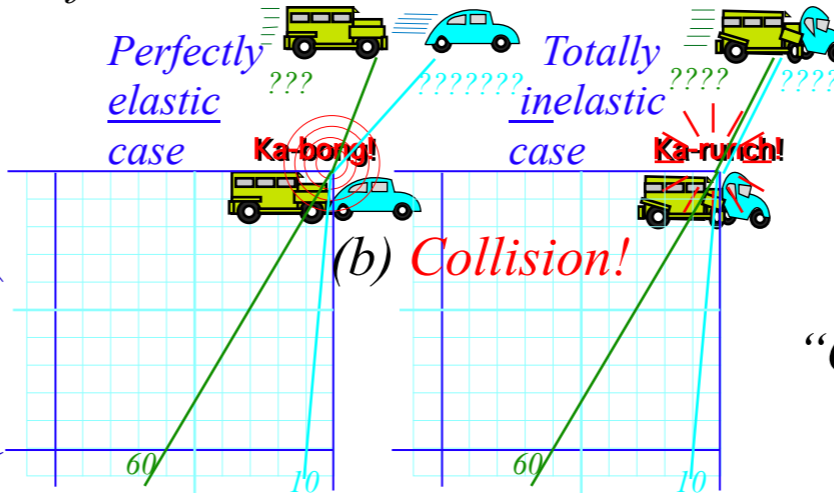
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Wielder of
 "Occam's Razor"



William of Ockham
 1285-1349

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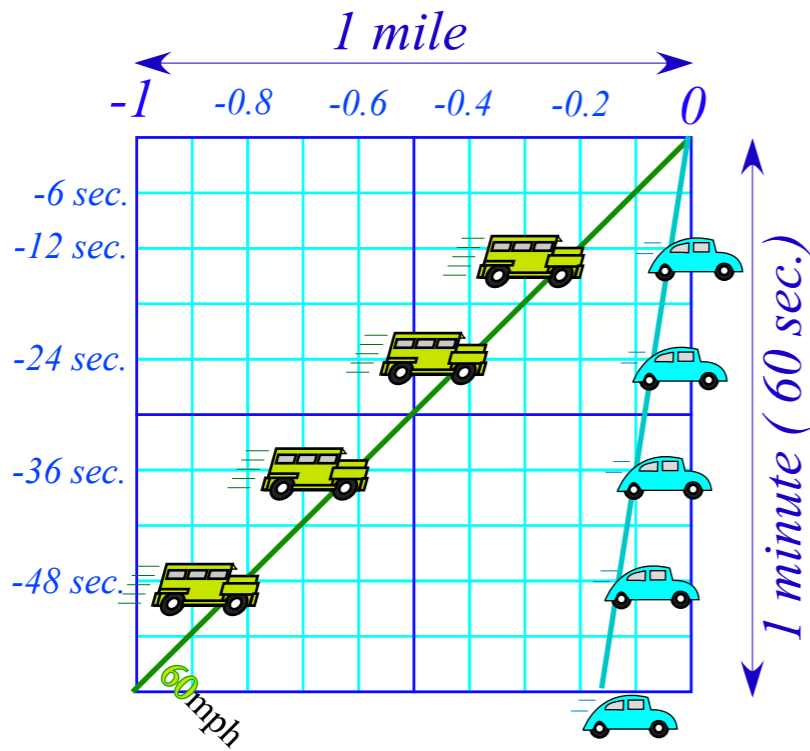
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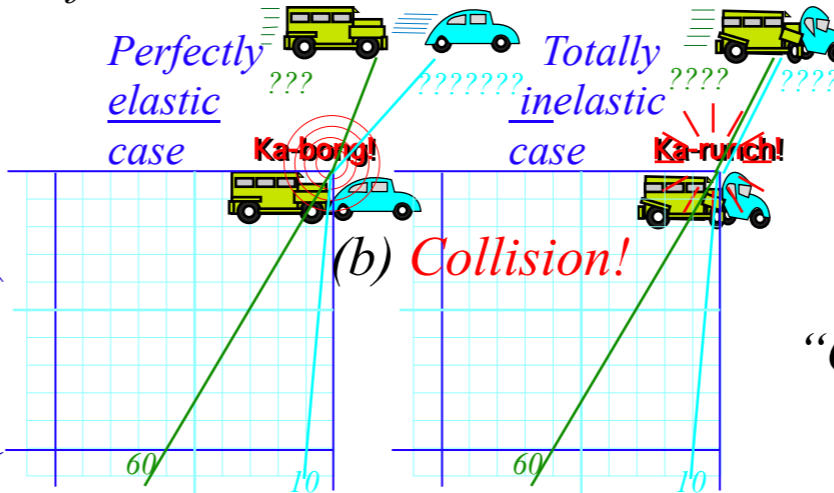
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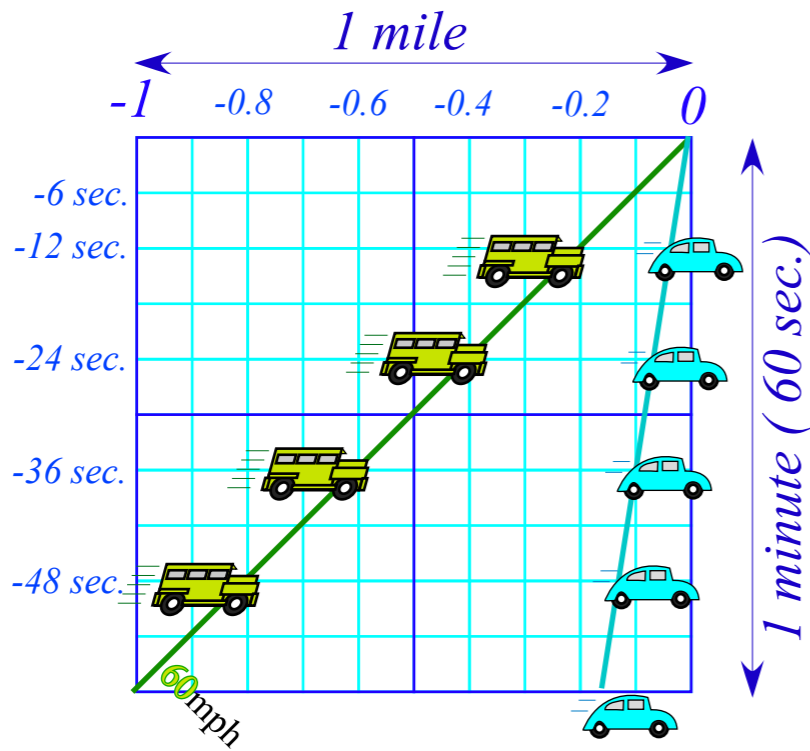
"*Pluralitas non set ponedata sine necessitate.*"

and has a number of interpretations:

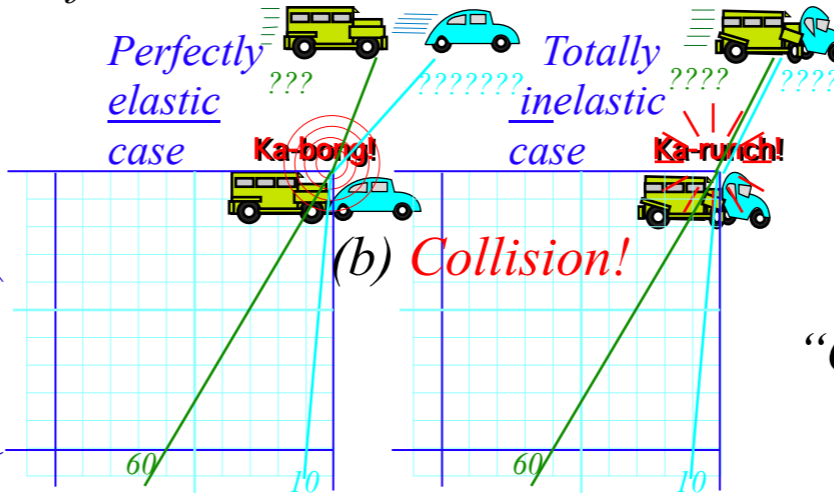
1. Literally: "Don't make pluralities of conjectures without necessity."
2. Logically: "Assume less to prove more."
3. Practical coding advice: "Keep it simple, make it powerful."

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GO! (INITIAL or IN)

STOP! (FINAL or FIN)

$$M_{SUV}V_{SUV}^{IN} + M_{VW}V_{VW}^{IN} = M_{SUV}V_{SUV}^{FIN} + M_{VW}V_{VW}^{FIN} = \text{constant}$$

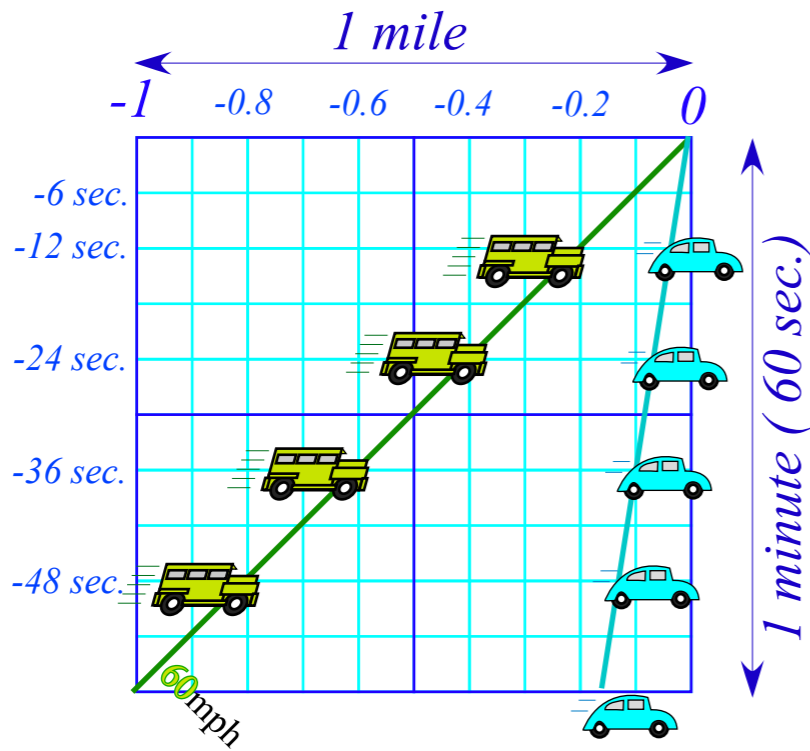
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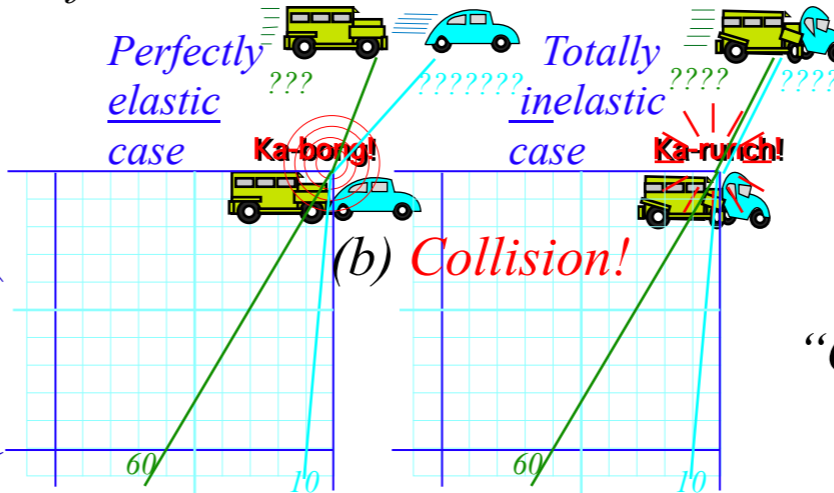
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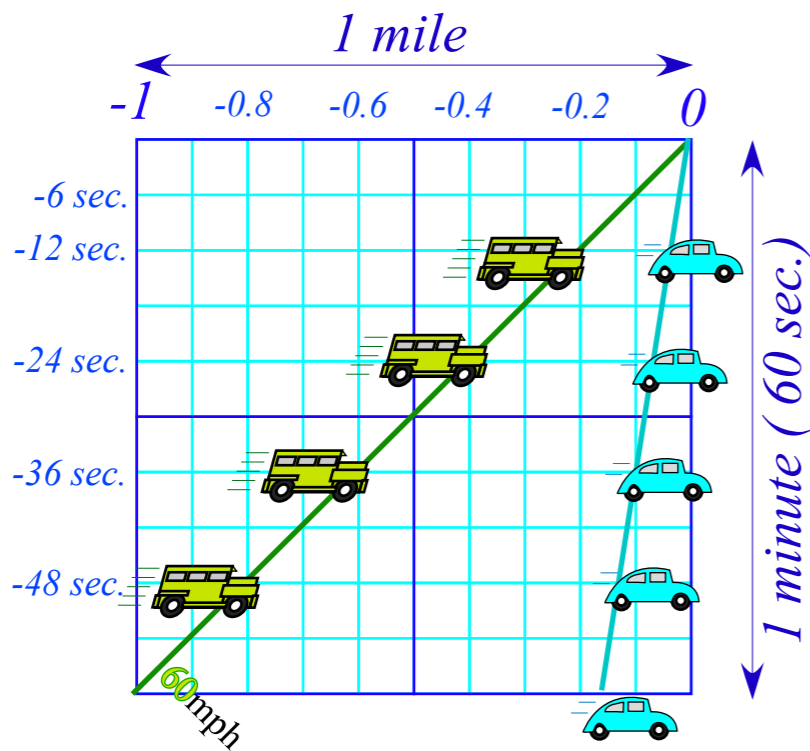
$$M_{SUV} V_{SUV}^{IN} + M_{VW} V_{VW}^{IN} = M_{SUV} V_{SUV}^{FIN} + M_{VW} V_{VW}^{FIN} = \text{constant}$$

$$4 \cdot 60 + 1 \cdot 10 = 4 \cdot ? + 1 \cdot ?? = 250$$

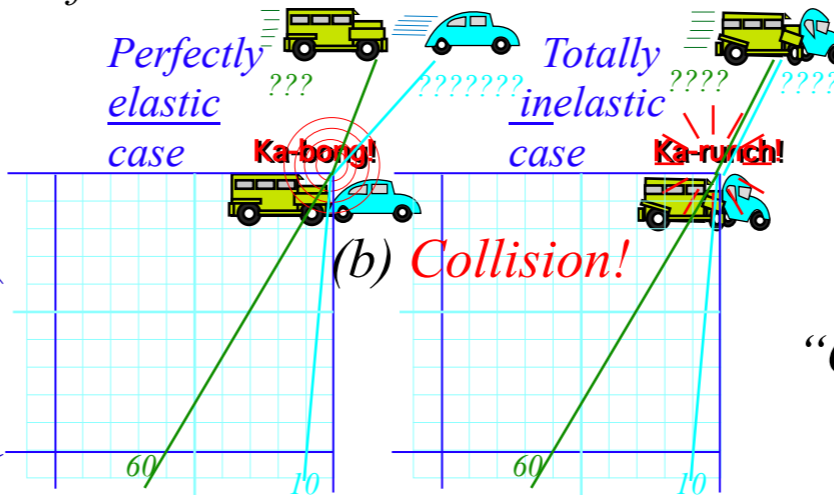
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It's a simple Cartesian equation

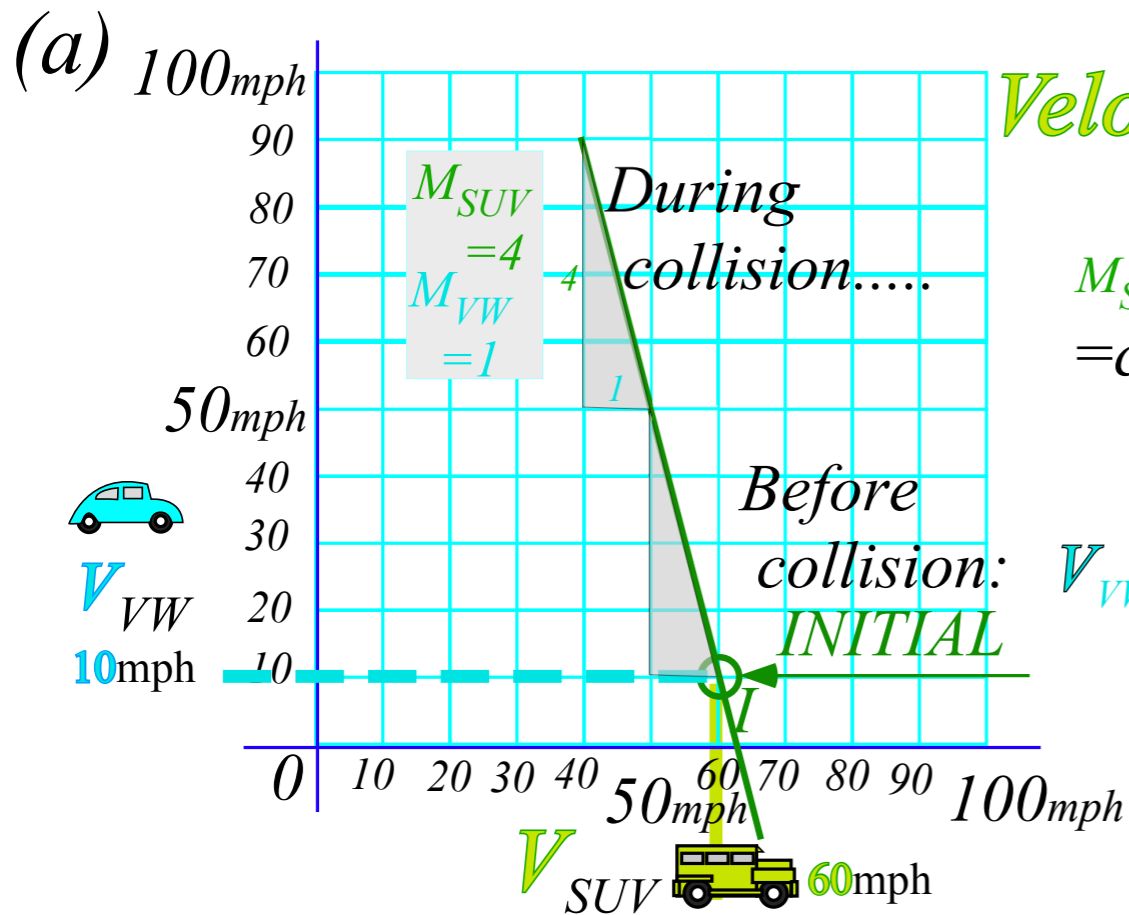
$$4 \cdot x + 1 \cdot y = 250$$



Rene Descartes
1596-1650

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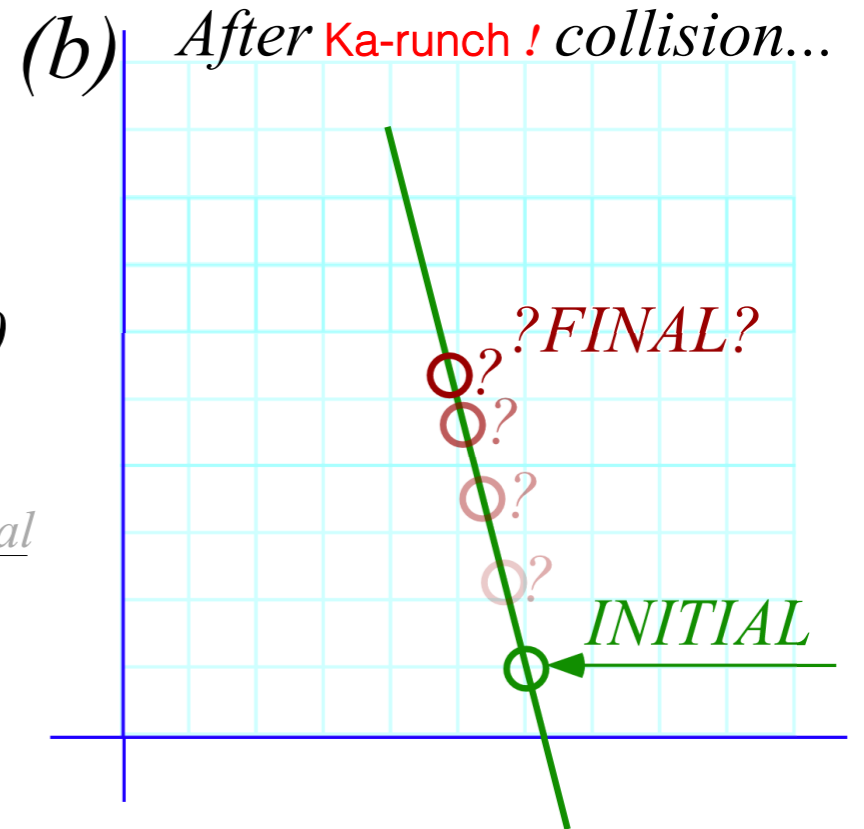
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Velocity-velocity plot of Axiom-1:

$$M_{SUV}V_{SUV} + M_{VW}V_{VW} = \text{constant} = P_{Total} = 250$$

$$V_{VW} = -\frac{M_{SUV}}{M_{VW}}V_{SUV} + \frac{P_{Total}}{M_{VW}} = -4V_{SUV} + 250$$



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It's a simple Cartesian equation

$$4 \cdot x + 1 \cdot y = 250$$



Rene Descartes
1596-1650

...with a simple Cartesian line-plot.

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Geometry of momentum conservation axiom

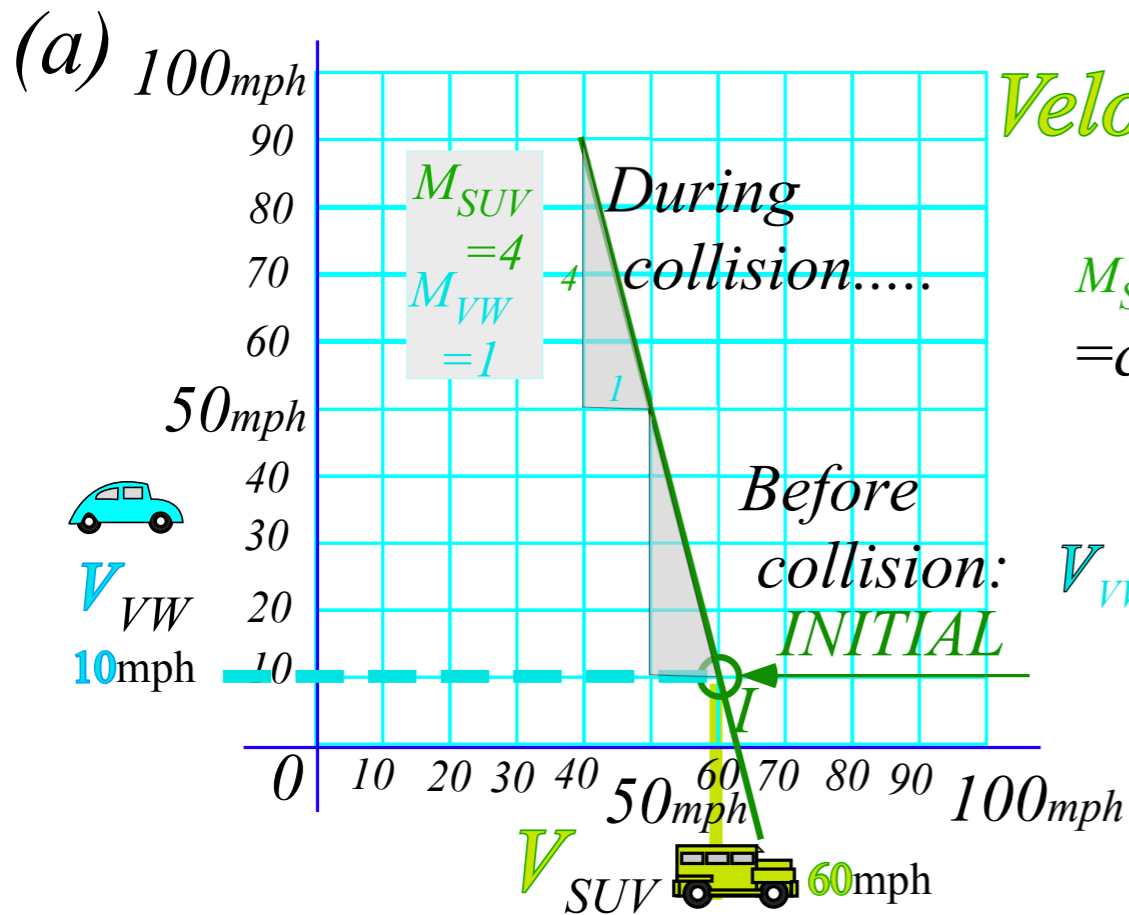


Totally Inelastic “ka-runch” collisions

Perfectly Elastic “ka-bong” and Center Of Momentum (COM) symmetry

+Intro to weighted-averages and vector notation

Comments on idealization in classical models

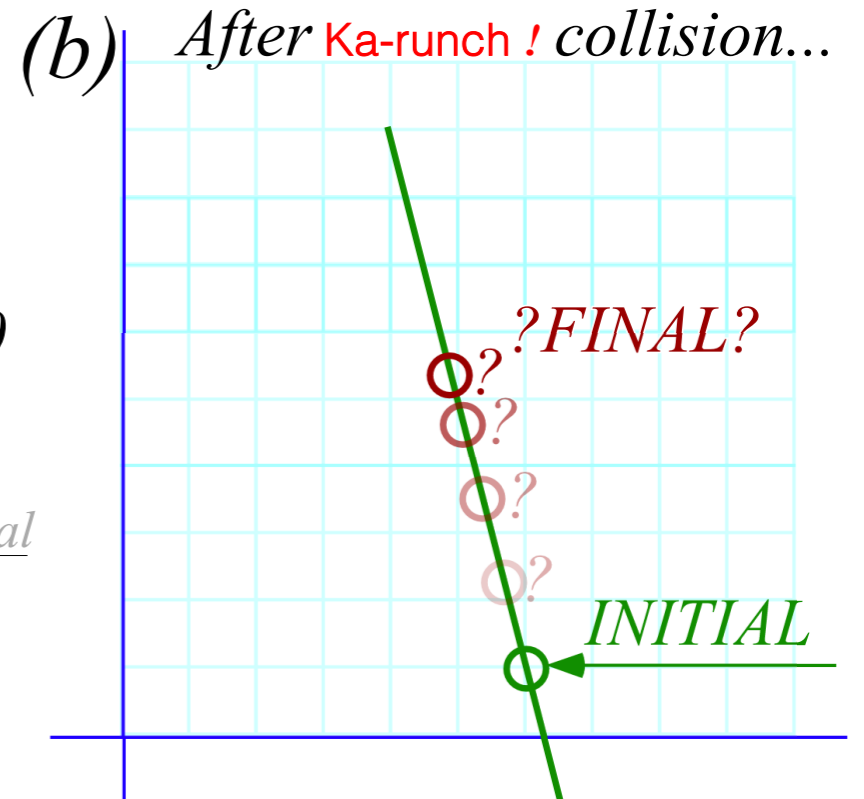


Velocity-velocity plot of Axiom-1:

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$$V_{VW} = -\frac{M_{SUV}}{M_{VW}} V_{SUV} + \frac{P_{Total}}{M_{VW}}$$

$$= -4 V_{SUV} + 250$$



It's a simple *Cartesian* equation...

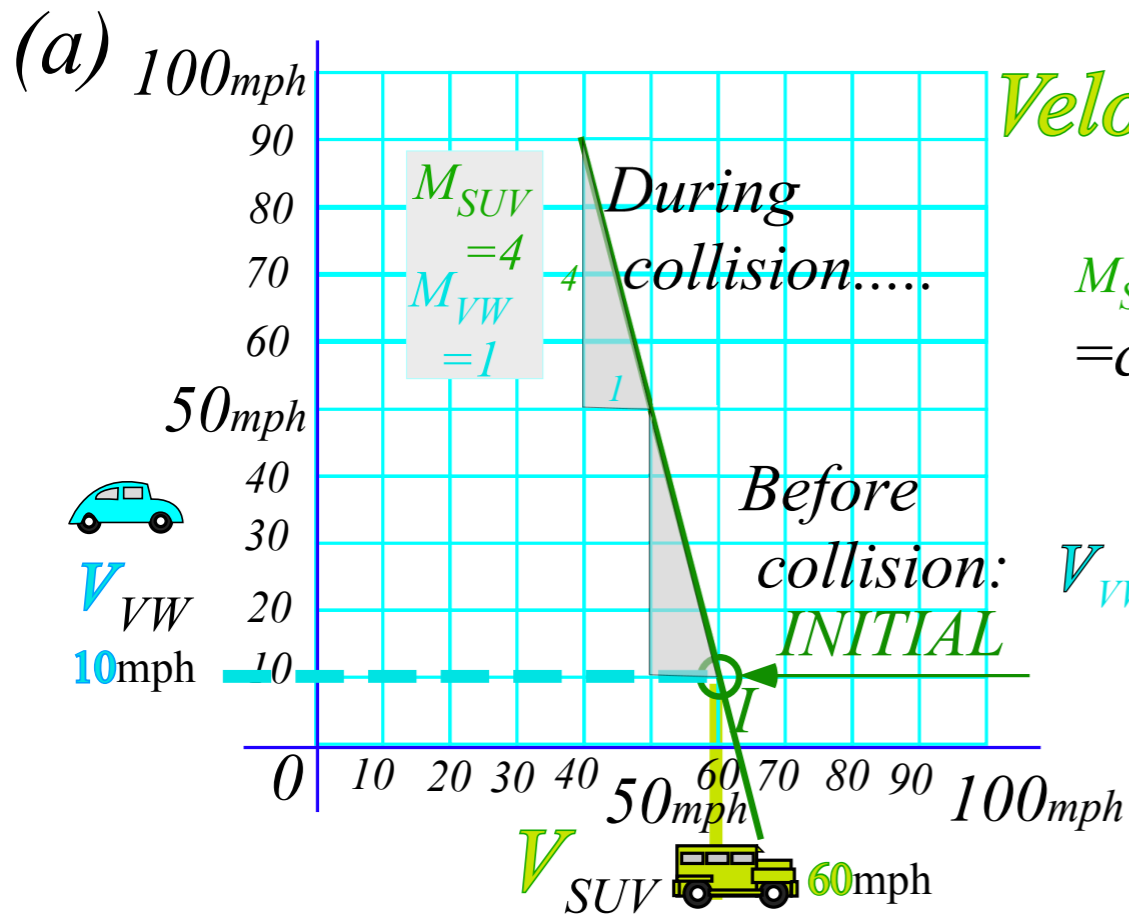
$$4 \cdot x + 1 \cdot y = 250$$

$$y = 250 - 4 \cdot x$$

...with a simple *Cartesian* line-plot.



Rene Descartes
1596-1650

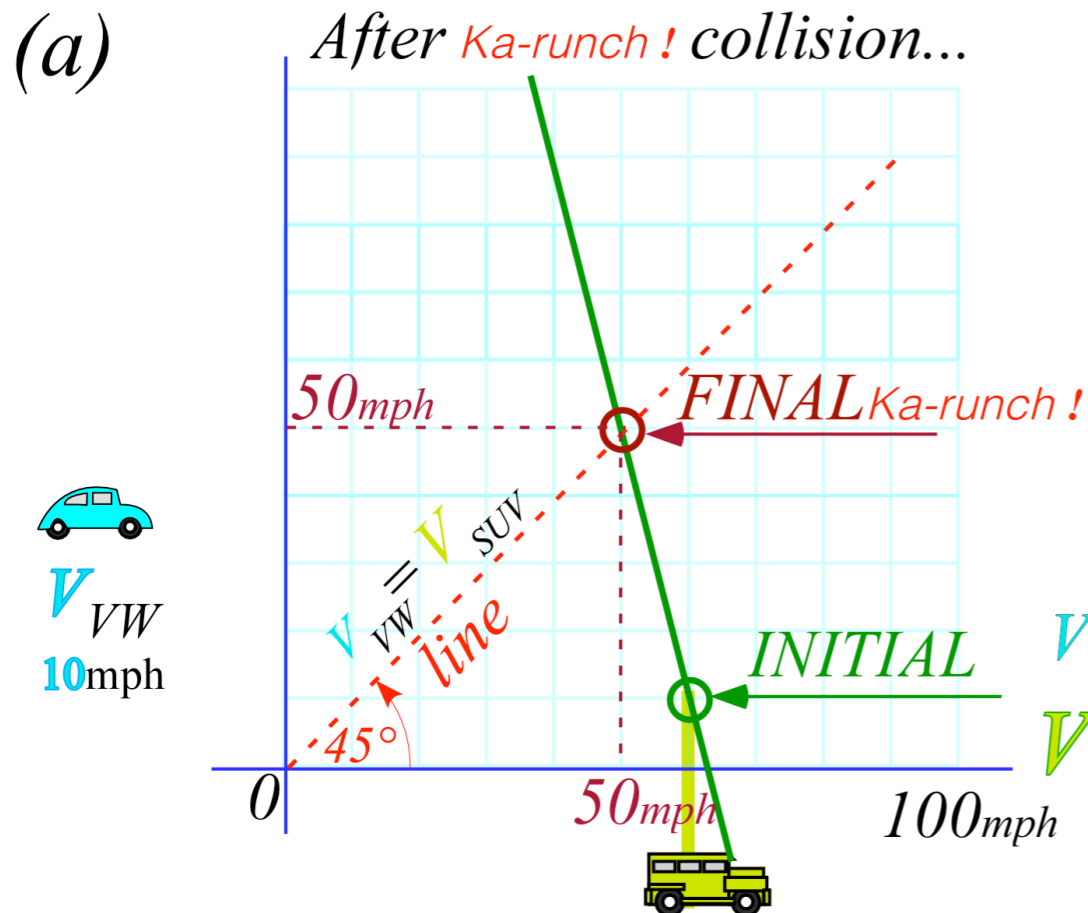
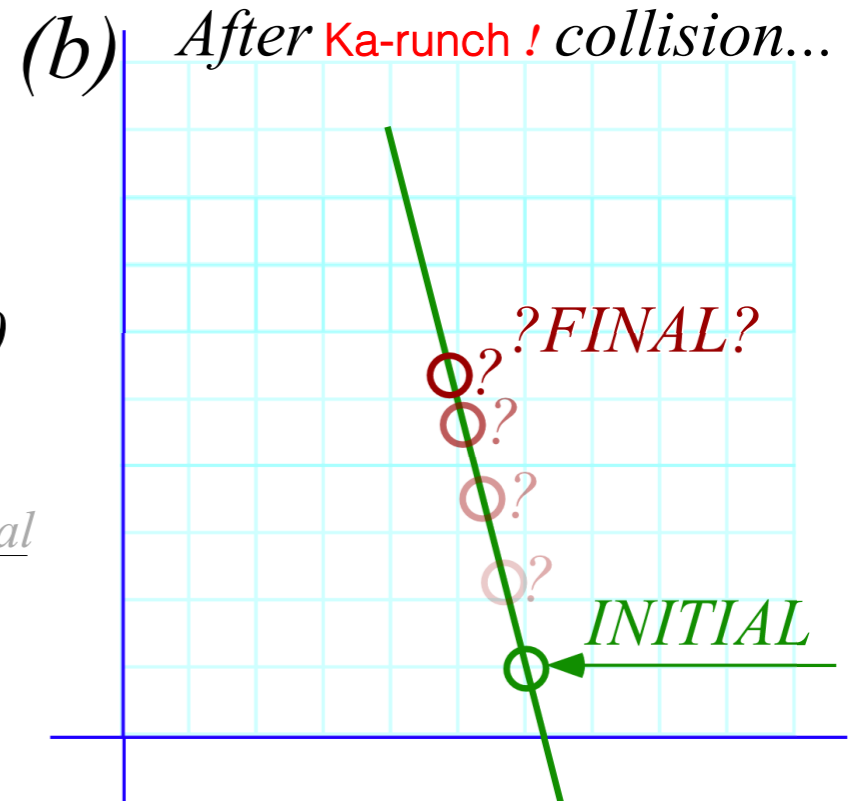


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$$= -4V_{SUV} + 250$$



It's a simple Cartesian equation...

$$4 \cdot x + 1 \cdot y = 250$$

$$y = 250 - 4 \cdot x$$

$$5 \cdot x = 250 \text{ ...with } y = x = 50$$

...with a simple Cartesian line-plot.



Rene Descartes
1596-1650

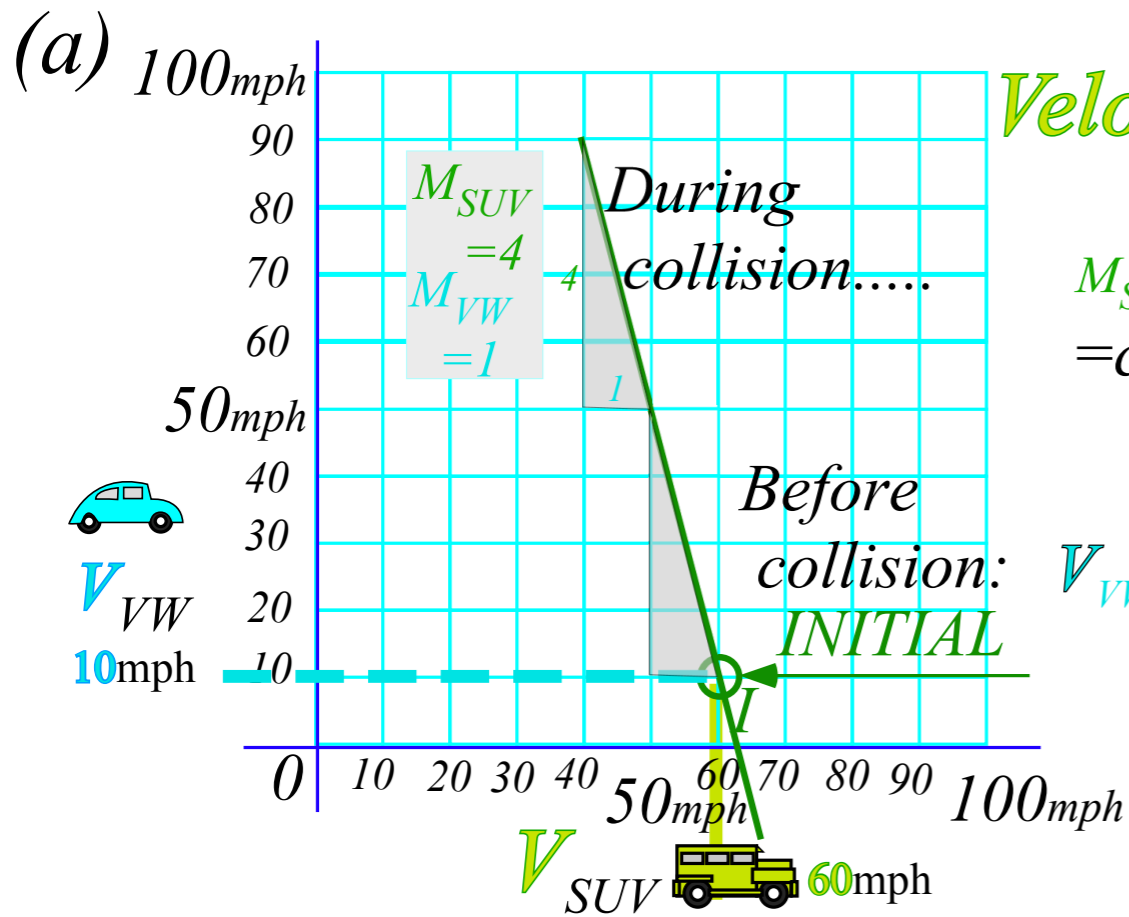
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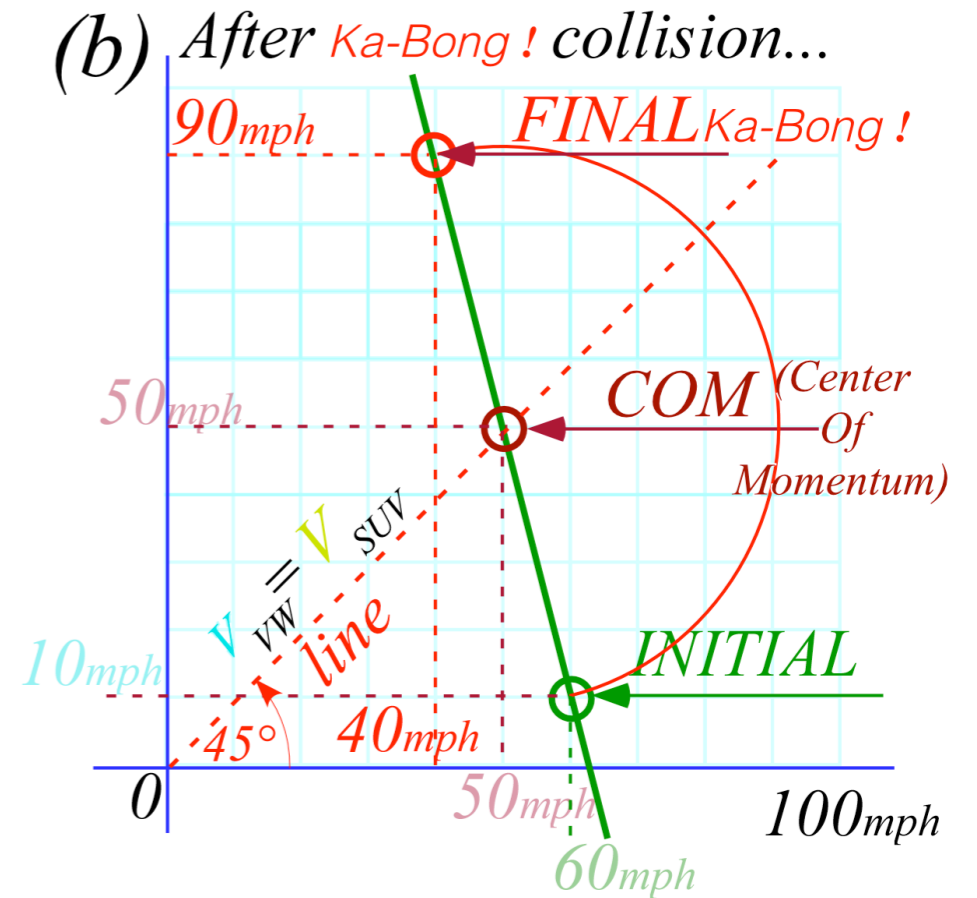
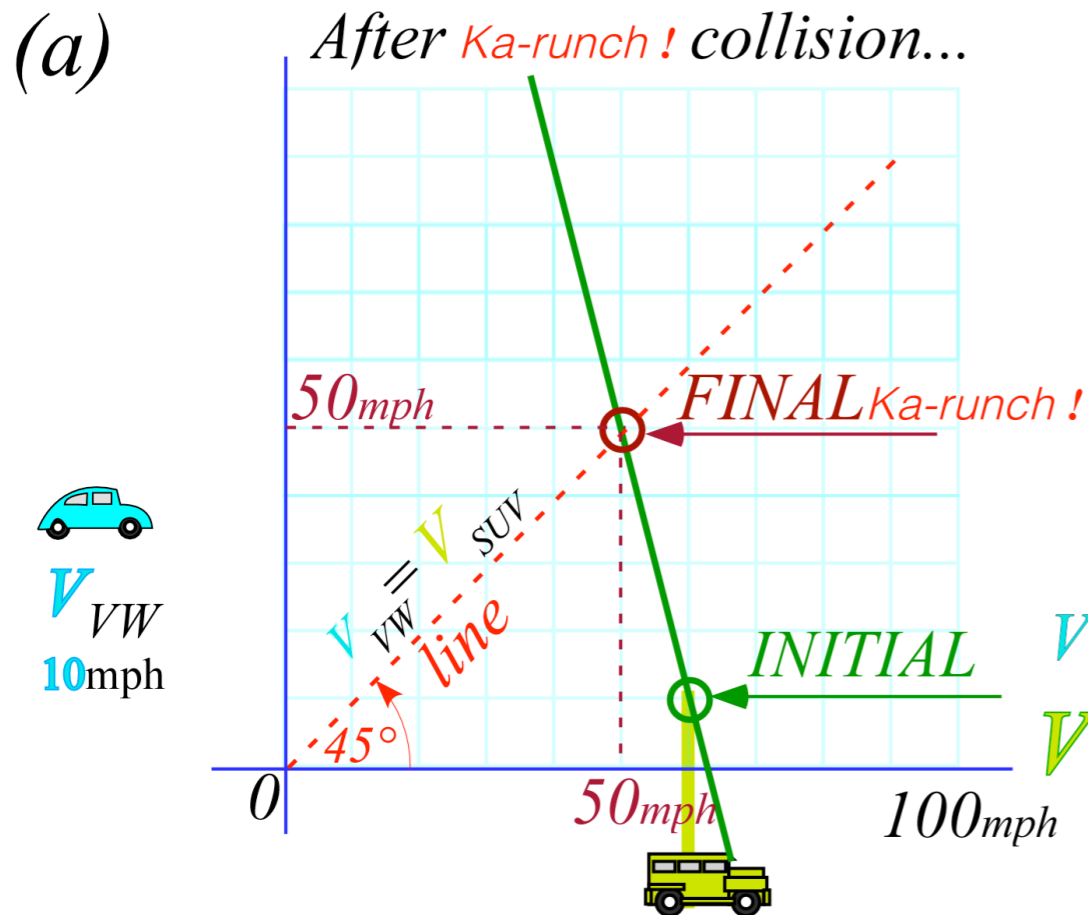
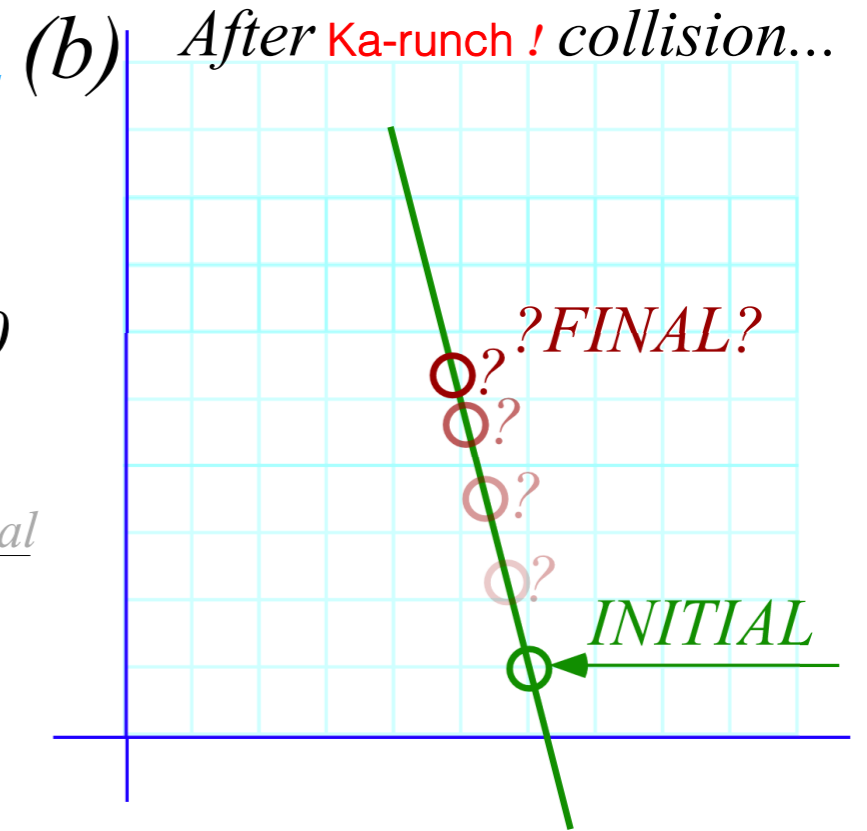


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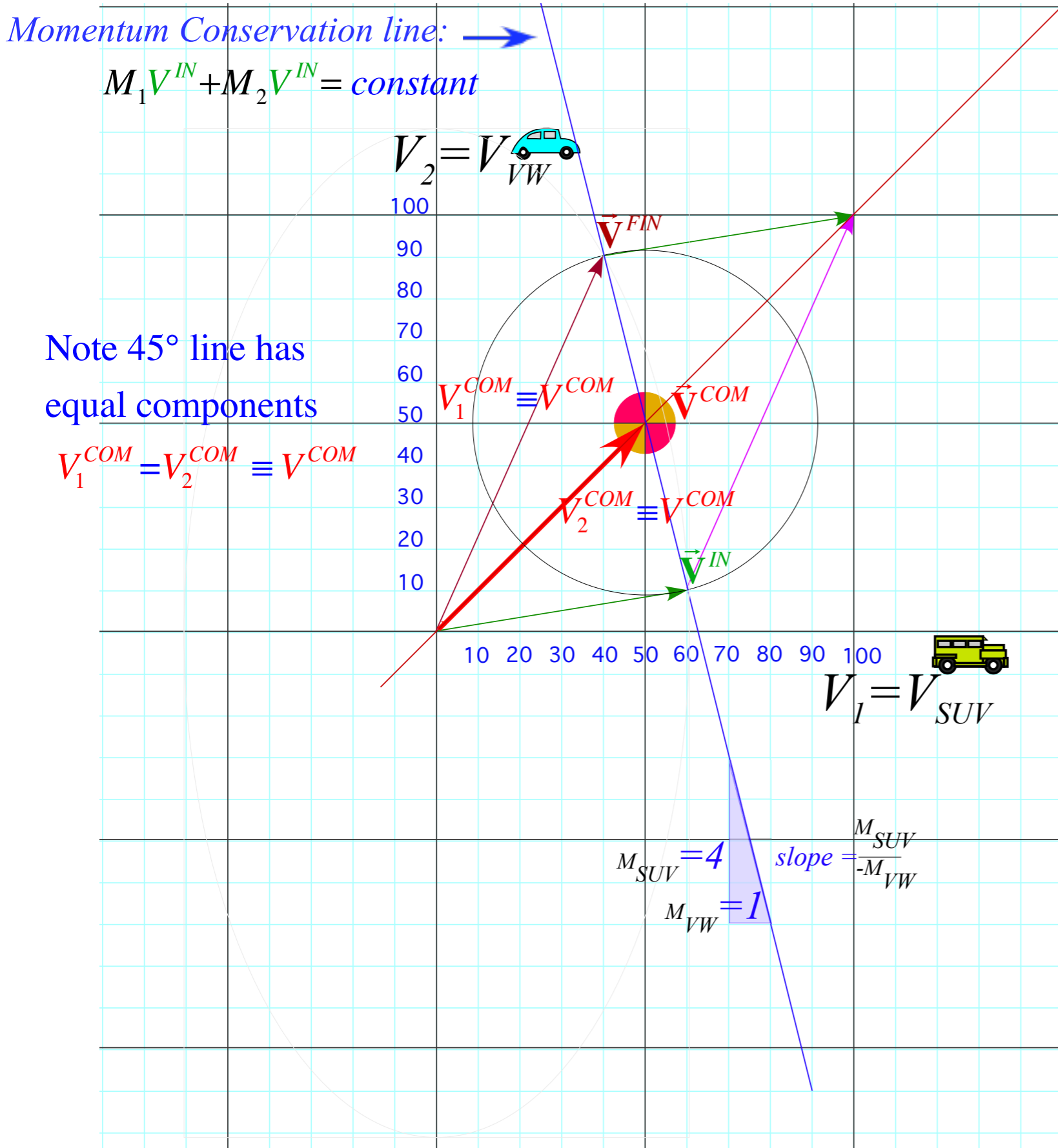
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Geometry of Momentum Conservation Axiom-1

$$(M_1+M_2)V^{COM} = M_1V_1^{IN} + M_2V_2^{IN} = M_1V_1^{COM} + M_2V_2^{COM} = M_1V_1^{FIN} + M_2V_2^{FIN} = M_{Total}V^{COM}$$



Geometry of Momentum Conservation Axiom-1

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Momentum Conservation line: \rightarrow

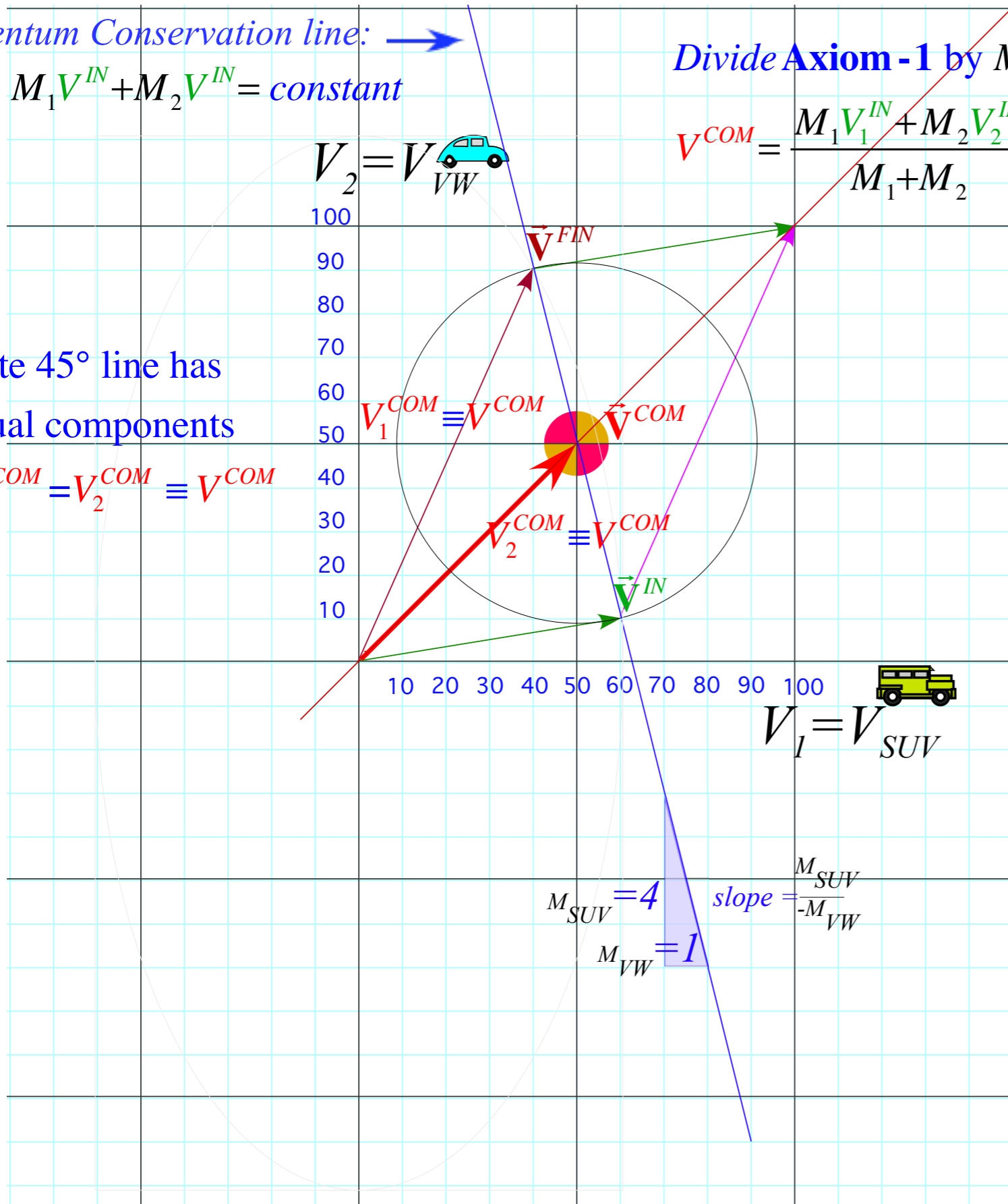
$$M_1V_1^{IN} + M_2V_2^{IN} = \text{constant}$$

Divide Axiom-1 by $M_{Total} = (M_1+M_2)$

$$V^{COM} = \frac{M_1V_1^{IN} + M_2V_2^{IN}}{M_1+M_2} = \frac{M_1V_1^{COM} + M_2V_2^{COM}}{M_1+M_2} = \frac{M_1V_1^{FIN} + M_2V_2^{FIN}}{M_1+M_2} = 50$$

Note 45° line has equal components

$$V_1^{COM} = V_2^{COM} \equiv V^{COM}$$



$$M_{SUV} = 4 \quad \text{slope} = \frac{M_{SUV}}{-M_{VW}}$$

$$M_{VW} = 1$$

Geometry of Momentum Conservation Axiom-1

$$(M_1+M_2)V^{COM} = M_1V_1^{IN} + M_2V_2^{IN} = M_1V_1^{COM} + M_2V_2^{COM} = M_1V_1^{FIN} + M_2V_2^{FIN} = M_{Total}V^{COM}$$

Momentum Conservation line: \rightarrow

$$M_1V_1^{IN} + M_2V_2^{IN} = \text{constant}$$

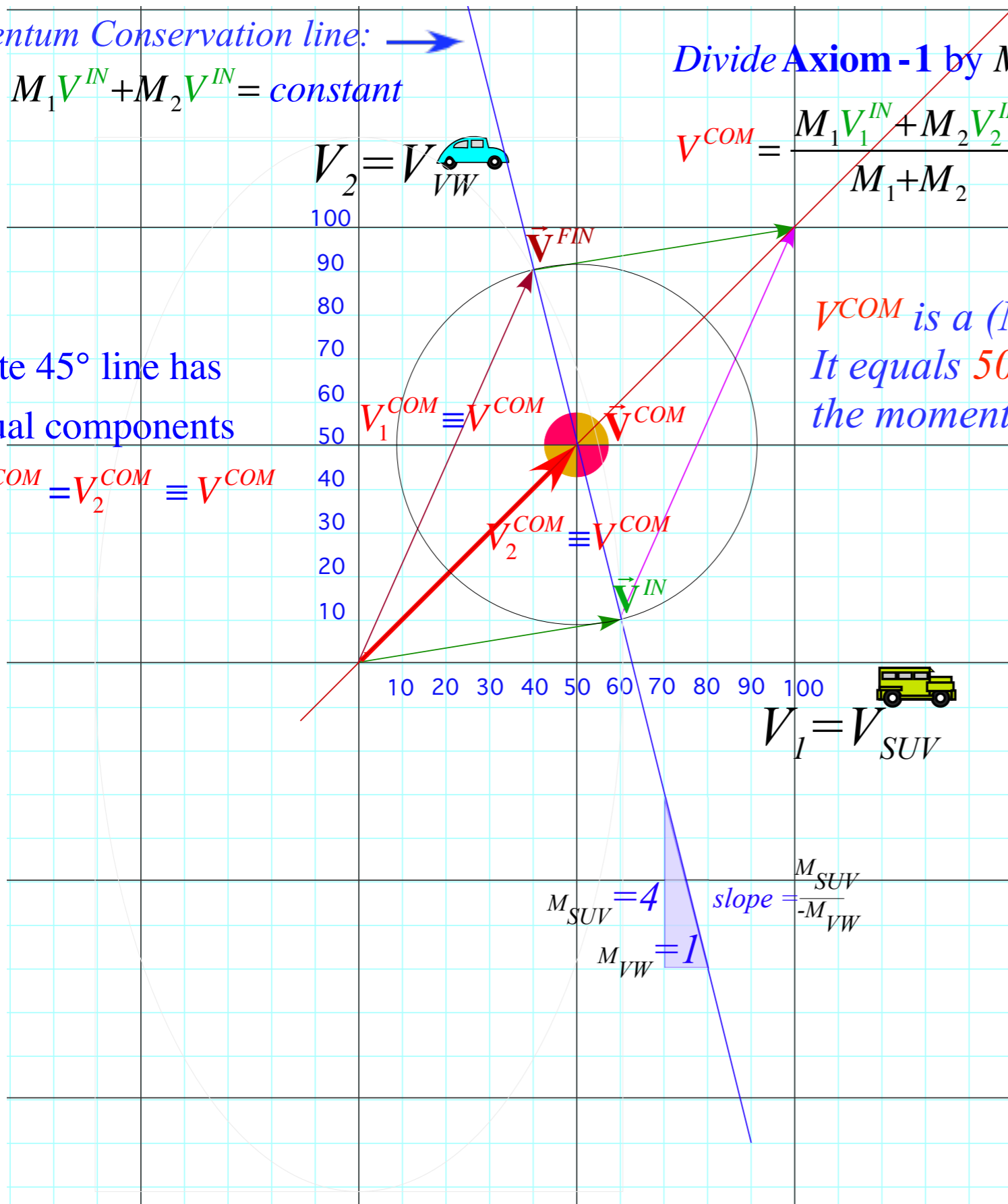
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Note 45° line has equal components

$$V_1^{COM} = V_2^{COM} \equiv V^{COM}$$

V^{COM} is a (M_1, M_2) Weighted Average of V_1 and V_2
It equals 50 for every point (V_1, V_2) on the momentum line



$$M_{SUV} = 4 \quad \text{slope} = \frac{M_{SUV}}{-M_{VW}}$$

$$M_{VW} = 1$$

Geometry of momentum conservation axiom

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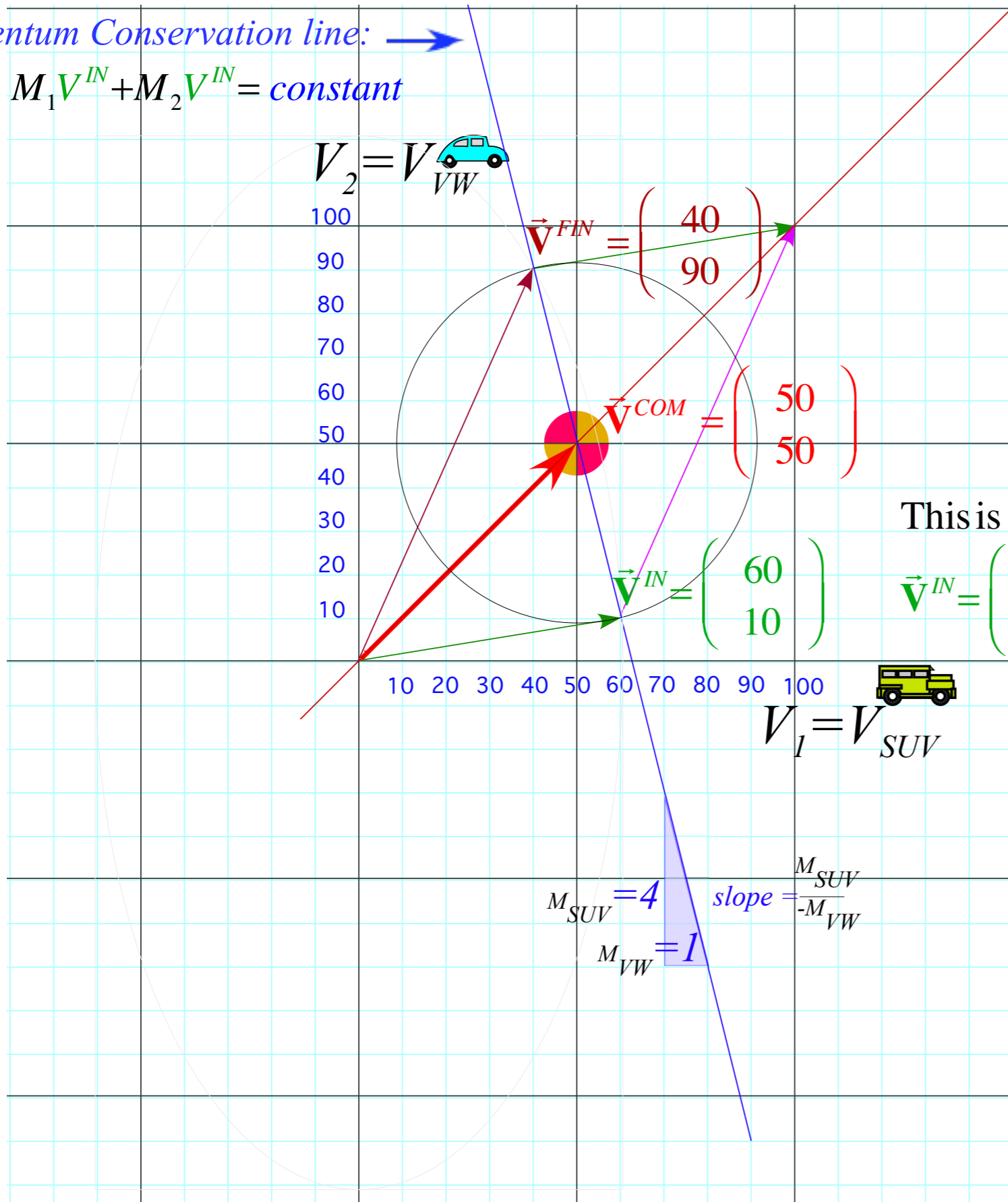


Geometry of Momentum Conservation Axiom - 1

$$(M_1 + M_2)V^{COM} = M_1V_1^{IN} + M_2V_2^{IN} = M_1V_1^{FIN} + M_2V_2^{FIN}$$

Momentum Conservation line: \rightarrow

$$M_1V_1^{IN} + M_2V_2^{IN} = \text{constant}$$



This is a *column-vector* (or *ket* $|IN\rangle$ in QM)

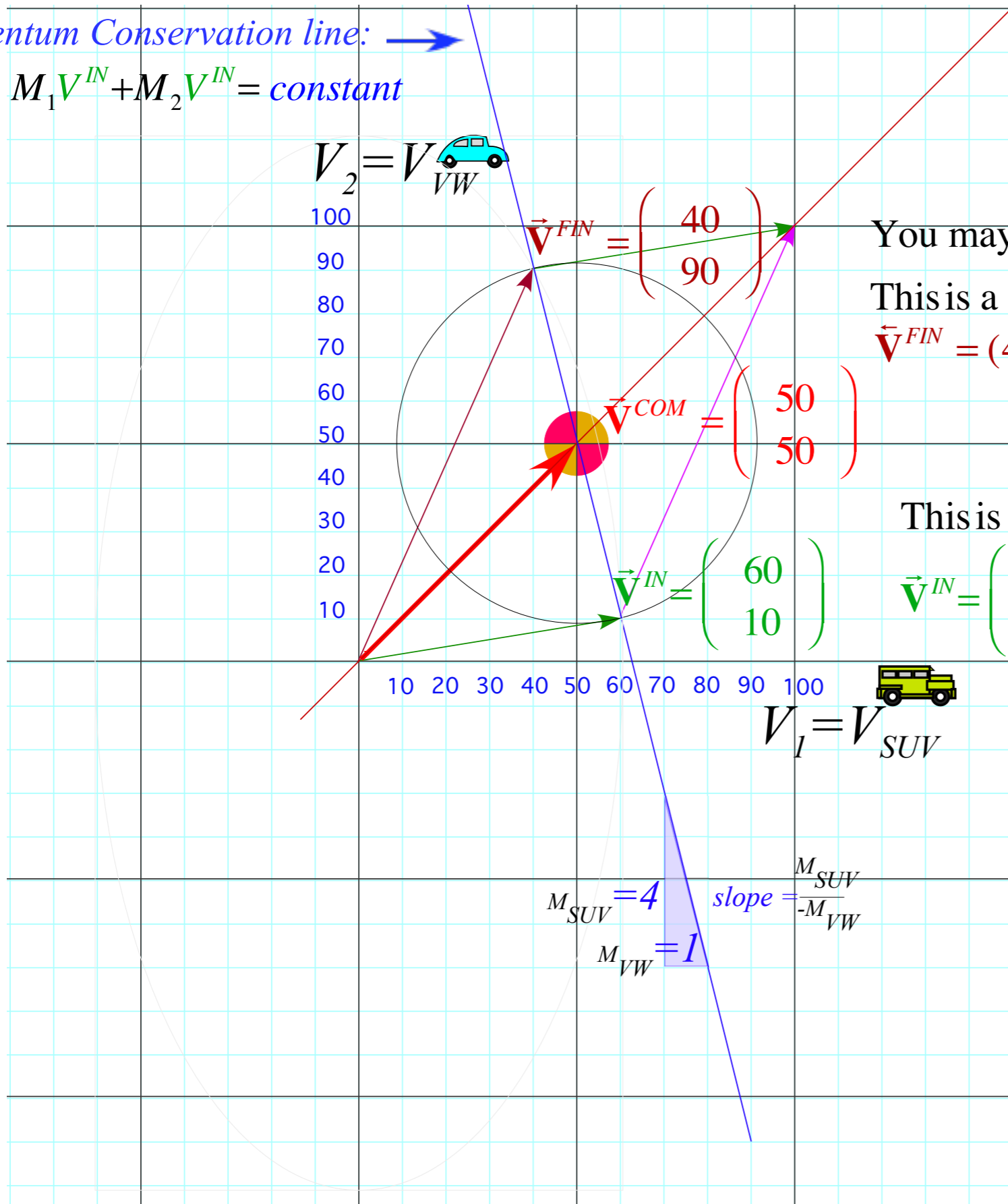
$$\vec{V}^{IN} = \begin{pmatrix} 60 \\ 10 \end{pmatrix}$$

Geometry of Momentum Conservation Axiom - 1

$$(M_1 + M_2)V^{COM} = M_1V_1^{IN} + M_2V_2^{IN} = M_1V_1^{FIN} + M_2V_2^{FIN}$$

Momentum Conservation line: \rightarrow

$$M_1V^{IN} + M_2V^{IN} = \text{constant}$$



You may be used to giving coordinates as (x,y).

This is a *row-vector* (or *bra* $\langle FIN|$ in QM)

$$\vec{V}^{FIN} = (40, 90)$$

This is a *column-vector* (or *ket* $|IN\rangle$ in QM)

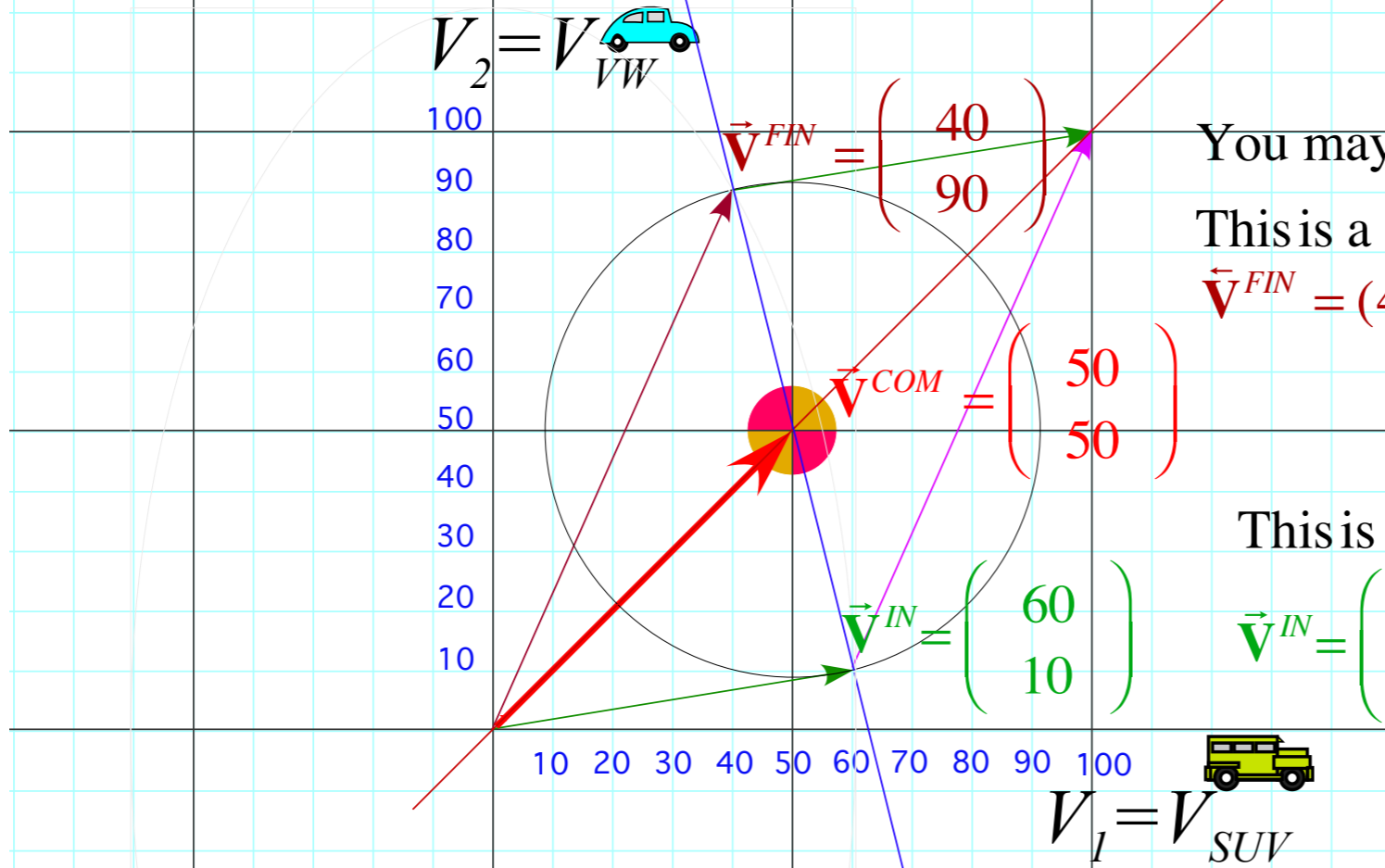
$$\vec{V}^{IN} = \begin{pmatrix} 60 \\ 10 \end{pmatrix}$$

Geometry of Momentum Conservation Axiom-1

$$(M_1+M_2)V^{COM} = M_1V_1^{IN} + M_2V_2^{IN} = M_1V_1^{FIN} + M_2V_2^{FIN}$$

Momentum Conservation line: \rightarrow

$$M_1V_1^{IN} + M_2V_2^{IN} = \text{constant}$$



You may be used to giving coordinates as (x,y).

This is a *row-vector* (or *bra* $\langle FIN|$ in QM)

$$\vec{V}^{FIN} = (40, 90)$$

This is a *column-vector* (or *ket* $|IN\rangle$ in QM)

$$\vec{V}^{IN} = \begin{pmatrix} 60 \\ 10 \end{pmatrix}$$

This is a *dot product* (or *scalar product*)

$$\vec{V}^{FIN} \bullet \vec{V}^{IN} = (40, 90) \bullet \begin{pmatrix} 60 \\ 10 \end{pmatrix} = \langle FIN|IN\rangle = 40 \cdot 60 + 90 \cdot 10 = 2400 + 900 = 3300$$

of a *row-vector* $\vec{V}^{FIN} = (40, 90)$ (or *bra* $\langle FIN|$)

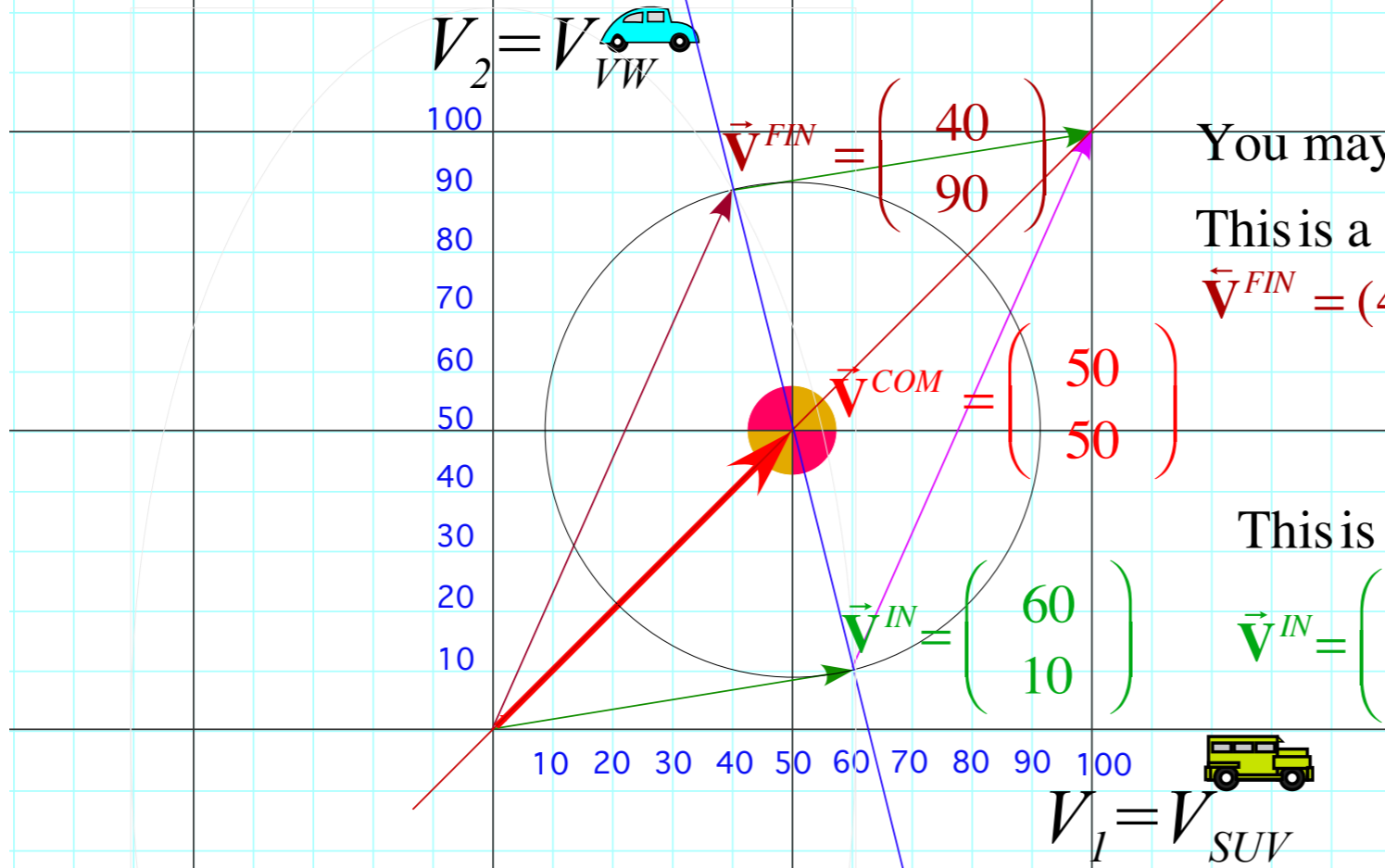
with *column-vector* $= \begin{pmatrix} 60 \\ 10 \end{pmatrix}$ (or *ket* $|IN\rangle$)

Geometry of Momentum Conservation Axiom-1

$$(M_1+M_2)V^{COM} = M_1V_1^{IN} + M_2V_2^{IN} = M_1V_1^{FIN} + M_2V_2^{FIN}$$

Momentum Conservation line: \rightarrow

$$M_1V_1^{IN} + M_2V_2^{IN} = \text{constant}$$



You may be used to giving coordinates as (x,y).

This is a *row-vector* (or *bra* $\langle FIN|$ in QM)

$$\vec{V}^{FIN} = (40, 90)$$

This is a *column-vector* (or *ket* $|IN\rangle$ in QM)

$$\vec{V}^{IN} = \begin{pmatrix} 60 \\ 10 \end{pmatrix}$$

This is a *outer product* (or *tensor product*)

$$\vec{V}^{IN} \otimes \vec{V}^{FIN} = \begin{pmatrix} 60 \\ 10 \end{pmatrix} \otimes (40, 90) = |IN\rangle\langle FIN| = \begin{pmatrix} 60 & 40 & 60 & 90 \\ 10 & 40 & 10 & 90 \end{pmatrix} = \begin{pmatrix} 2400 & 5400 \\ 400 & 900 \end{pmatrix}$$

of a *column-vector* $= \begin{pmatrix} 60 \\ 10 \end{pmatrix}$ (or *ket* $|IN\rangle$)

with a *row-vector* $\vec{V}^{FIN} = (40, 90)$ (or *bra* $\langle FIN|$)

Geometry of momentum conservation axiom

Totally Inelastic “ka-runch” collisions

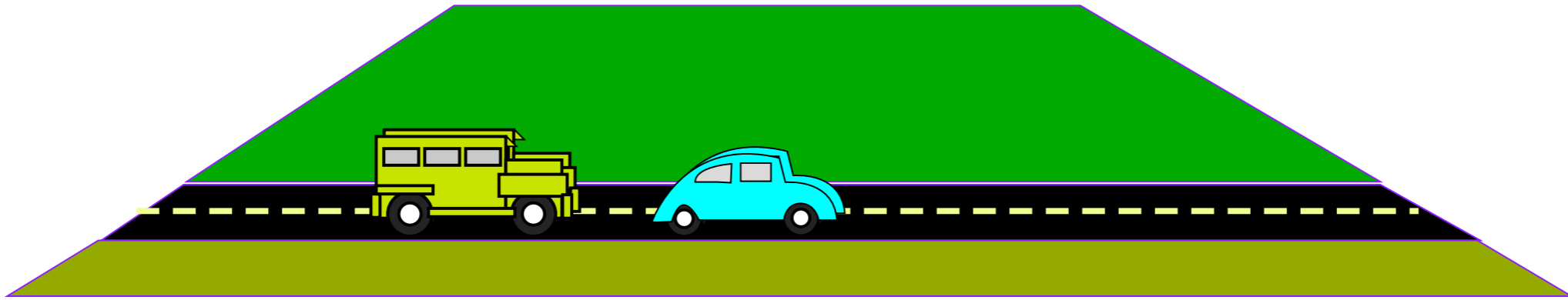
Perfectly Elastic “ka-bong” and Center Of Momentum (COM) symmetry

+Intro to weighty-averages and vector notation

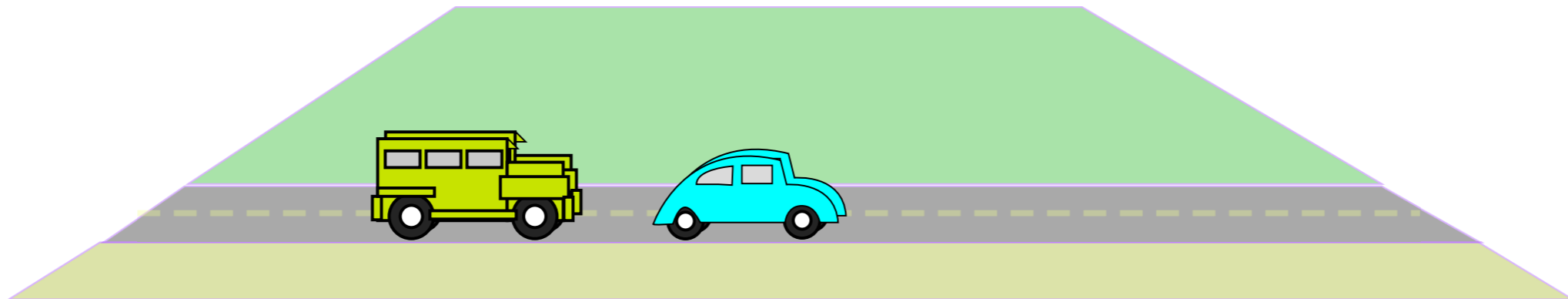
Comments on idealization in classical models



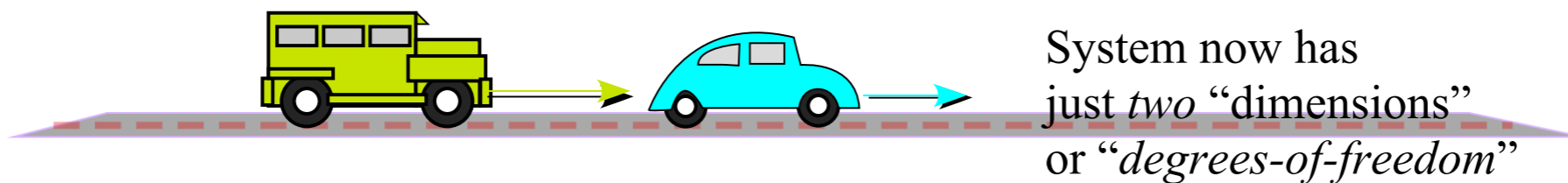
The SUV and VW *Idealized* thought experiments



Idealization 1. Ignore background.
(No rolling friction, air resistance, etc.)



Idealization 2. Make each 1-dimensional.
(Cars “constrained” to ride on frictionless rail)

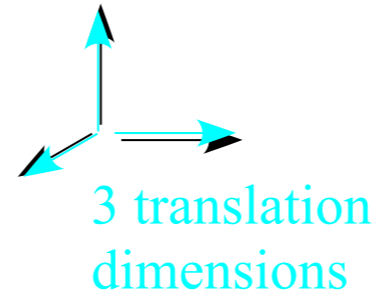
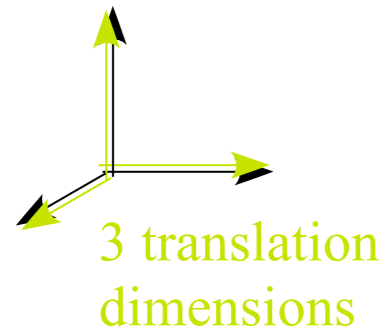


System now has
just *two* “dimensions”
or “degrees-of-freedom”

Landscape 1.1 Idealized model for collision model and thought experiments

Summary of Classical Mechanical Degrees of Freedom

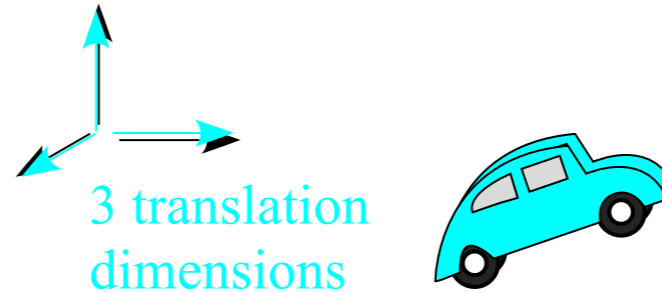
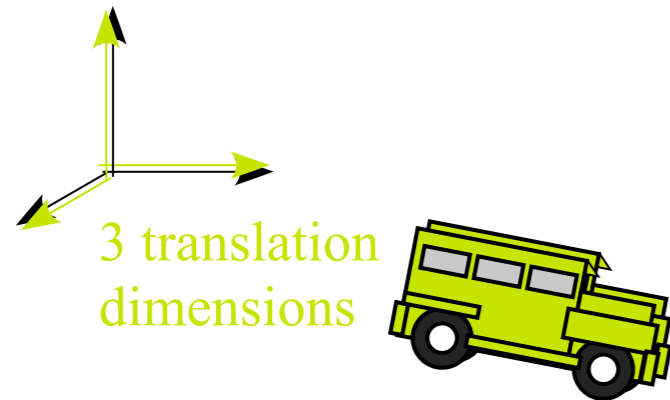
Translation (Each body has 3 translational degrees of freedom) (Introduced in Units 1 and 2)



6 translational degrees of freedom for SUV and VW.

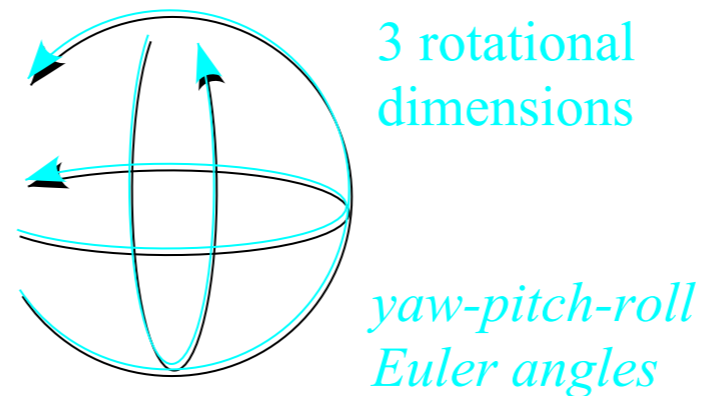
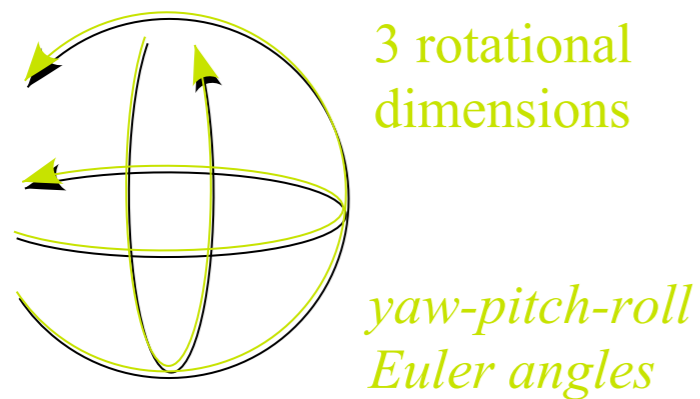
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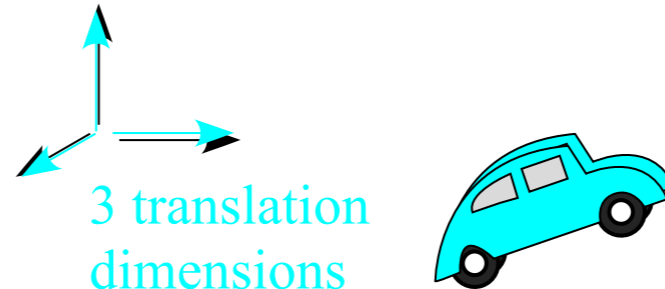
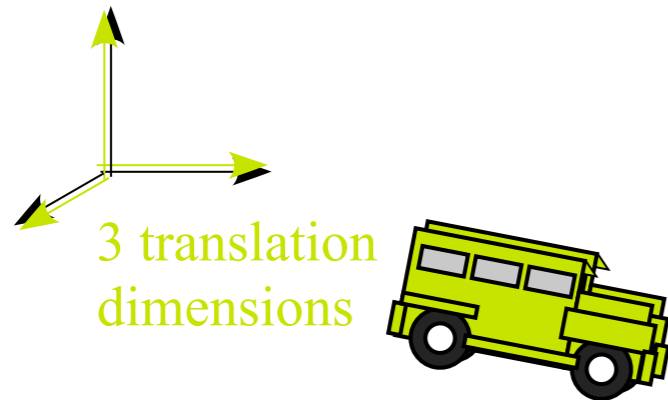
Rotation (Each body has 3 rotational degrees of freedom) (Introduced in Units 3 and 7)



6 rotational degrees of freedom for SUV and VW.

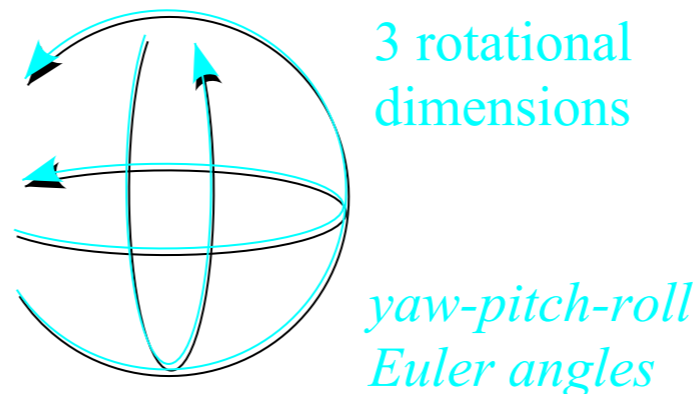
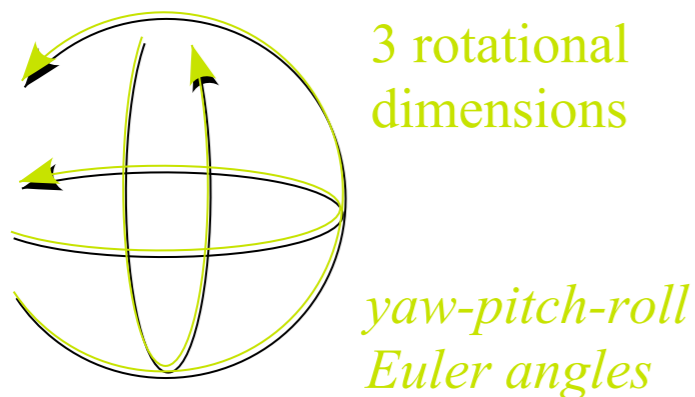
Summary of Classical Mechanical Degrees of Freedom

Translation (Each body has 3 translational degrees of freedom) (Introduced in Units 1 and 2)



6 translational degrees of freedom for SUV and VW.

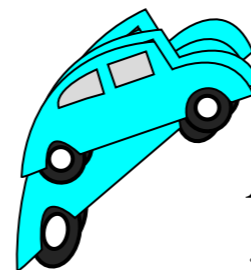
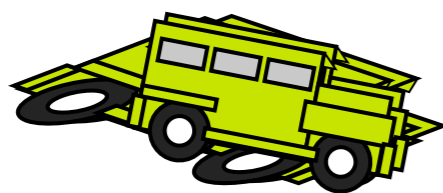
Rotation (Each body has 3 rotational degrees of freedom) (Introduced in Units 3 and 7)



6 rotational degrees of freedom for SUV and VW.

SUV and VW system involves 12 rigid-body degrees of freedom

Vibration (Each body has many vibrational degrees of freedom) (Introduced in Units 3-8)



Generalized Curvilinear Coordinates (GCC) introduced in Unit 1 Chapters 10-12

An N-atom molecule has $3N-6$ vibrational degrees of freedom

Geometry of Galilean translation symmetry



45° shift in (V_1, V_2) -space

Time reversal symmetry

...of COM collisions

A problem in *space-time* : (60mph Cell-faxing 4ton SUV rear-ends 10mph 1ton VW)

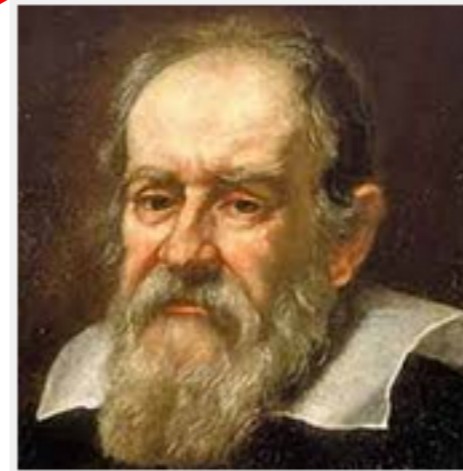
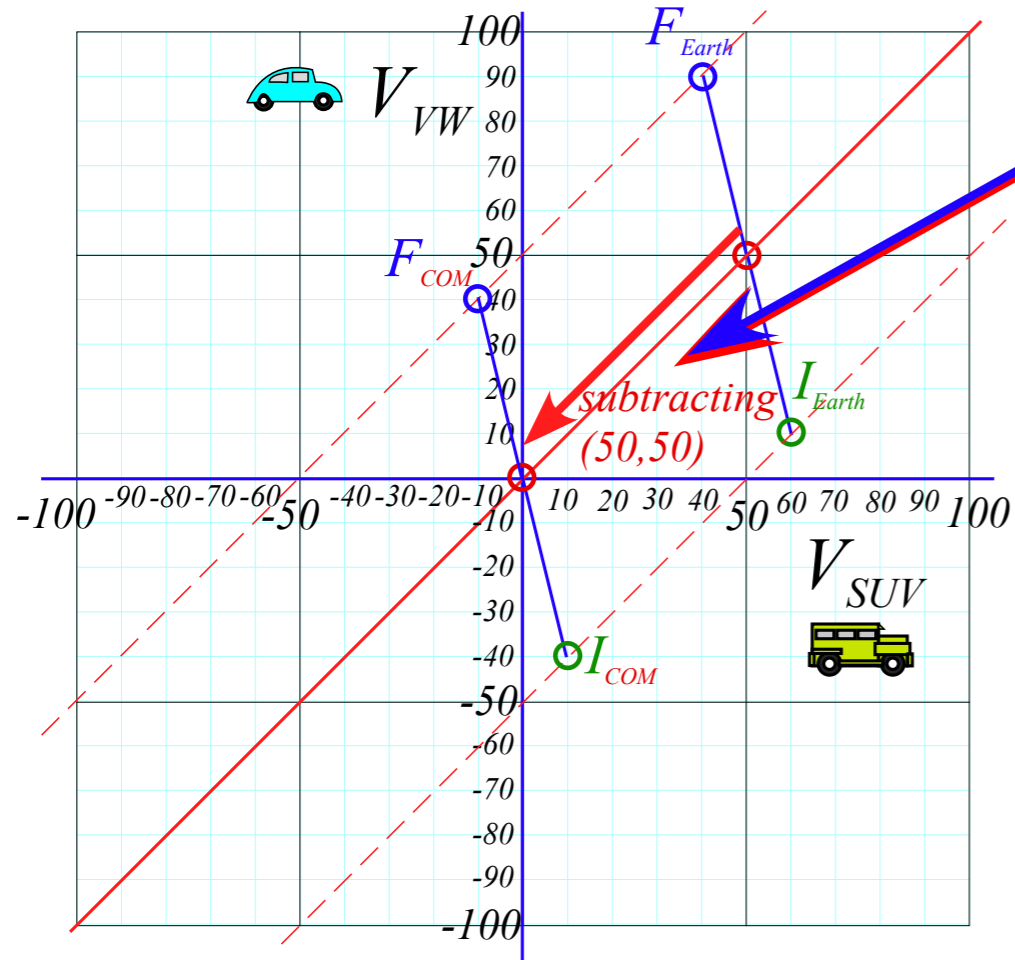
Geometry of Galilean translation (A *symmetry transformation*)

If you increase your velocity by 50 mph,...

(In some direction x, y , or z ...)

...the rest of the world appears to be 50 mph *slower* (In that direction...)

(a) Galileo transforms to *COM* frame



Galileo Galilei
1564-1642

Fig. 2.5a
in Unit 1

A problem in *space-time* : (60mph Cell-faxing 4ton SUV rear-ends 10mph 1ton VW)

Geometry of Galilean translation (A *symmetry transformation*)

If you increase your velocity by 50 mph,...

(In some direction $x, y, \text{ or } z \dots$)

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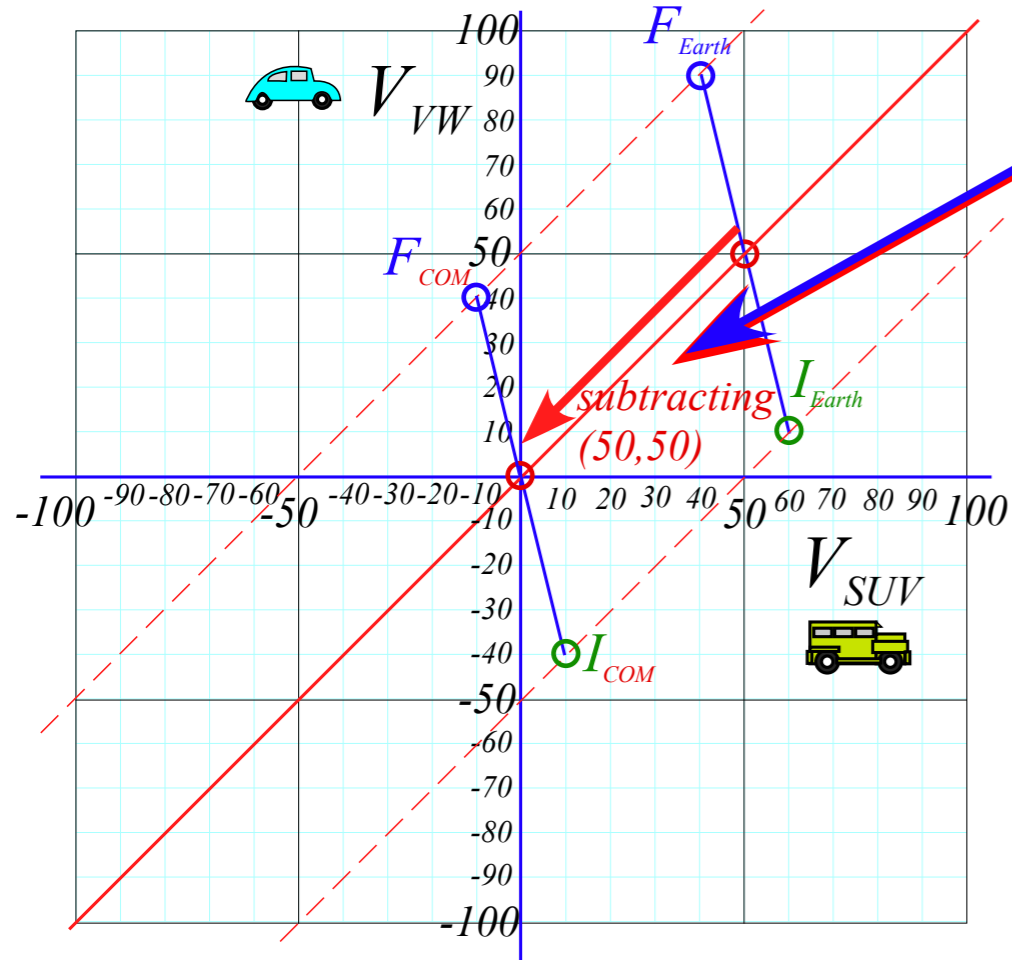


Fig. 2.5a
in Unit 1

(b) ... and to five or six other reference frames

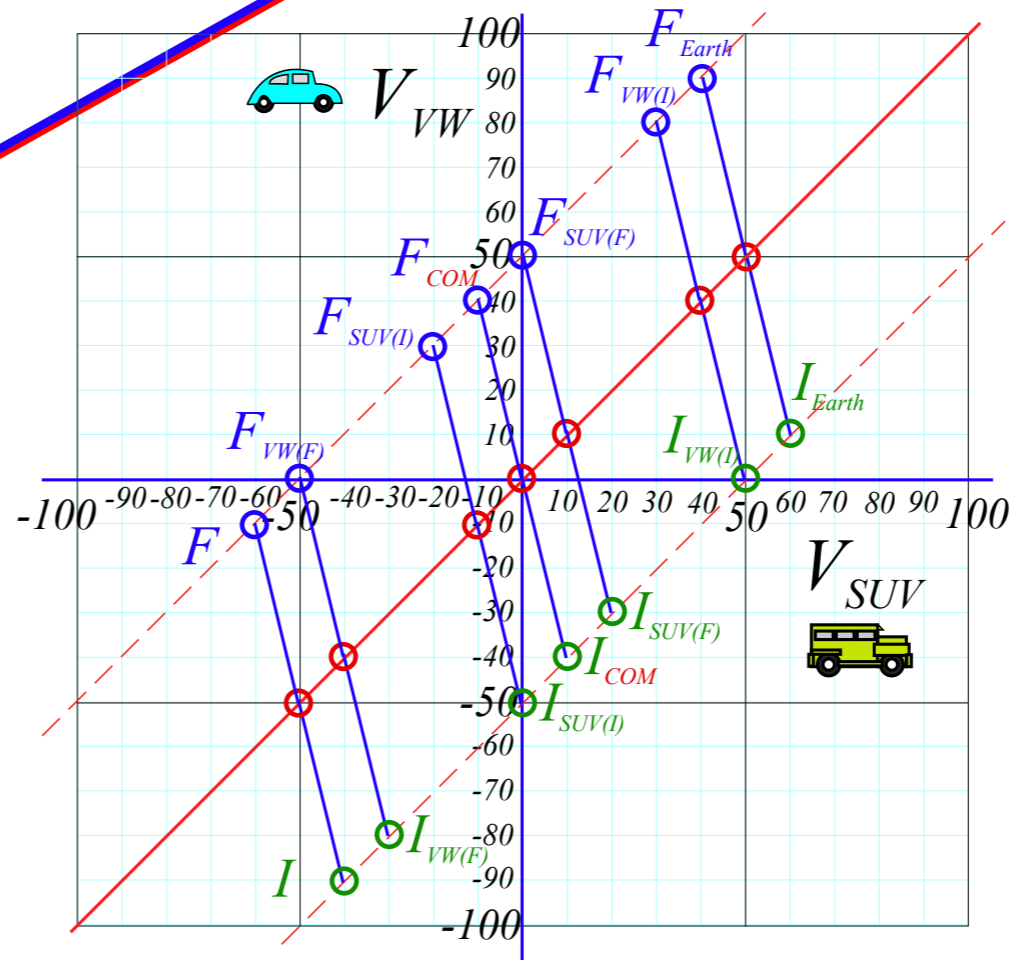
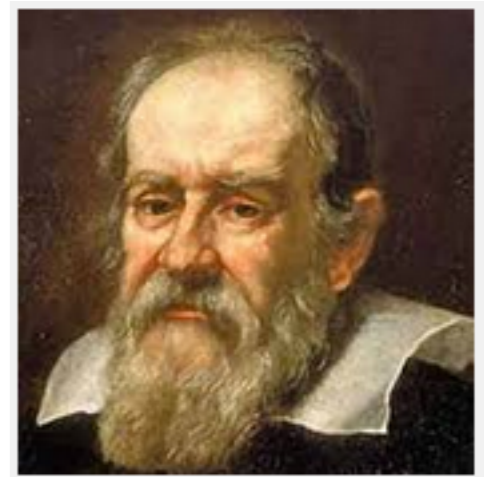


Fig. 2.5b
in Unit 1



Galileo Galilei
1564-1642

Geometry of Galilean translation symmetry

45° shift in (V_1, V_2) -space

Time reversal symmetry

...of COM collisions



A problem in *space-time* : (60mph Cell-faxing 4ton SUV rear-ends 10mph 1ton VW)

Final F and Initial I trade places ...

Geometry of Galilean translation (A symmetry transformation)

If you increase your velocity by 50 mph,...

...the rest of the world appears to be 50 mph slower

Time-reversal (F-I) symmetry pairs (Four examples)

(a) Galileo transforms to COM frame

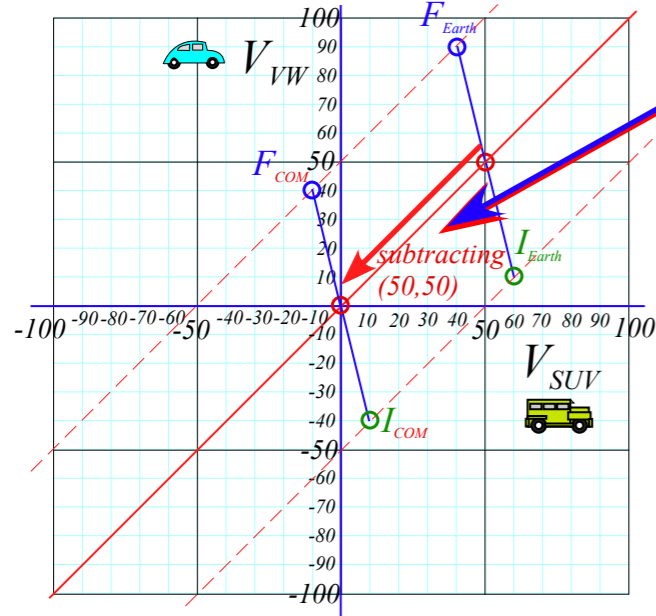


Fig. 2.5a
in Unit 1

(b) ... and to five or six other reference frames

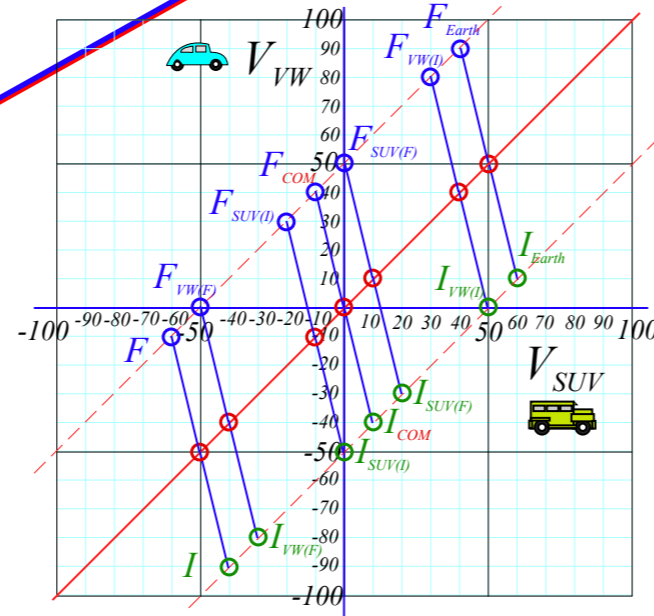
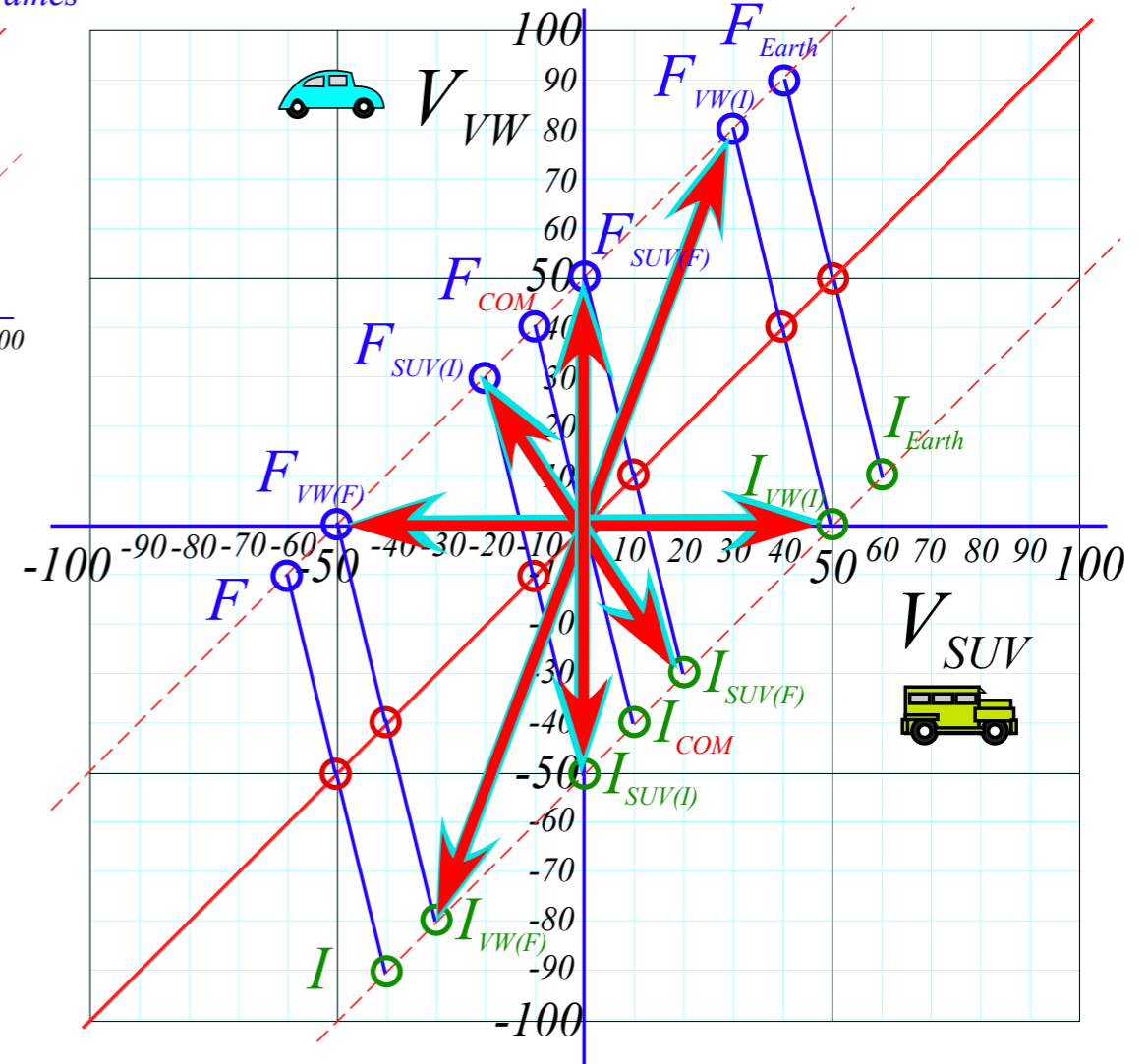


Fig. 2.5b
in Unit 1



*Time-reversal means flip t to $-t$...
(Run a movie backwards)*

A problem in *space-time* : (60mph Cell-faxing 4ton SUV rear-ends 10mph 1ton VW)

Final F and Initial I trade places ...

Geometry of Galilean translation (A symmetry transformation)

If you increase your velocity by 50 mph,...

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Time-reversal (F-I) symmetry pairs (Four examples)

(a) Galileo transforms to COM frame

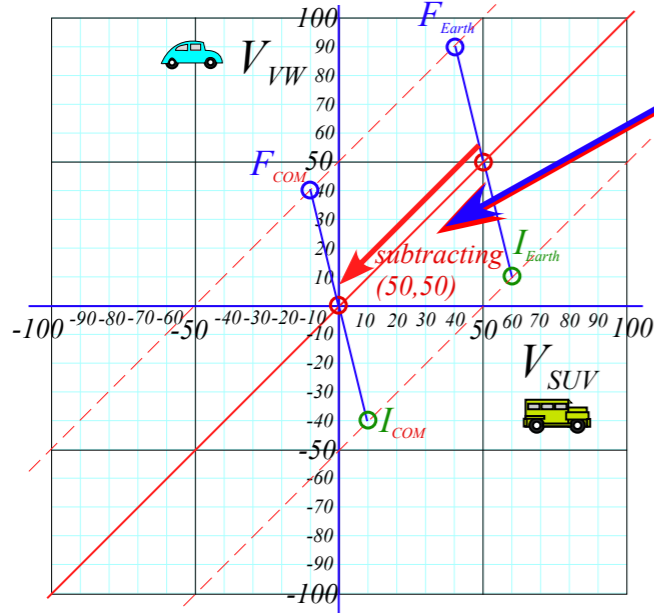


Fig. 2.5a
in Unit 1

(b) ... and to five or six other reference frames

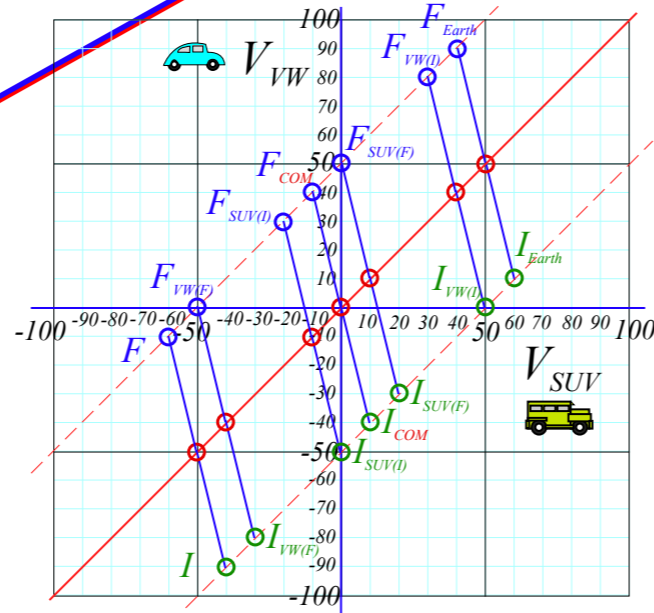
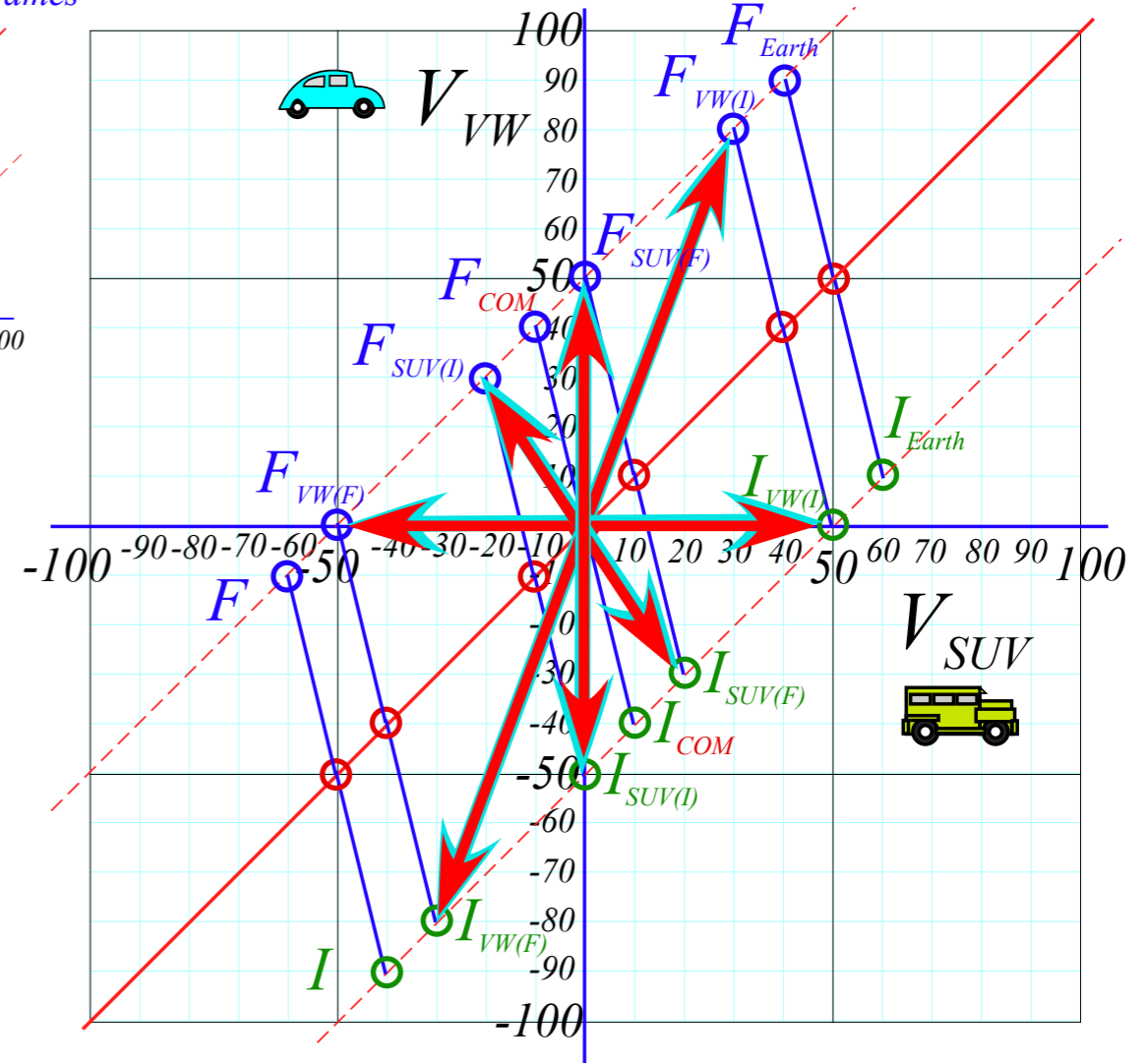


Fig. 2.5b
in Unit 1



*Time-reversal means flip t to $-t$...
(Run a movie backwards)*

*That means you flip Velocity V to $-V$...
(Everything goes backwards)*

Geometry of Galilean translation symmetry

45° shift in (V_1, V_2) -space

Time reversal symmetry

...of COM collisions



A problem in *space-time* : (60mph Cell-faxing 4ton SUV rear-ends 10mph 1ton VW)

Geometry of Galilean translation (A symmetry transformation)

If you increase your velocity by 50 mph,...

...the rest of the world appears to be 50 mph slower

THE
COM Time-reversal
symmetry pair
(Just 1 case)

(a) Galileo transforms to COM frame

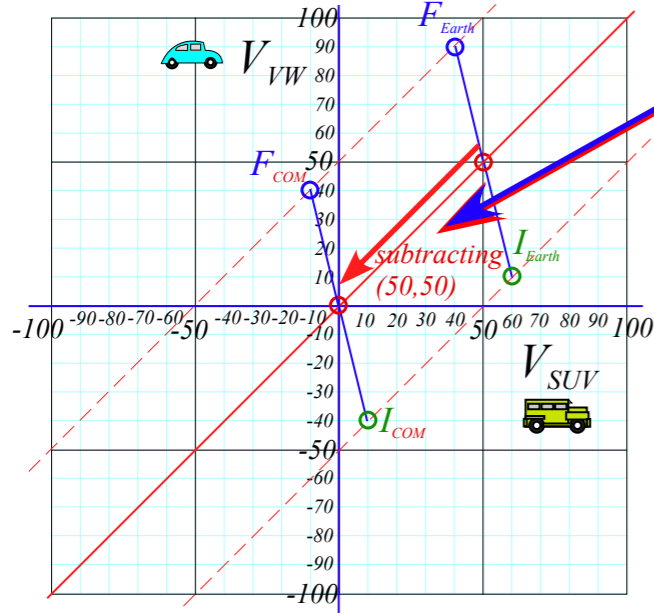


Fig. 2.5a
in Unit 1

(b) ... and to five or six other reference frames

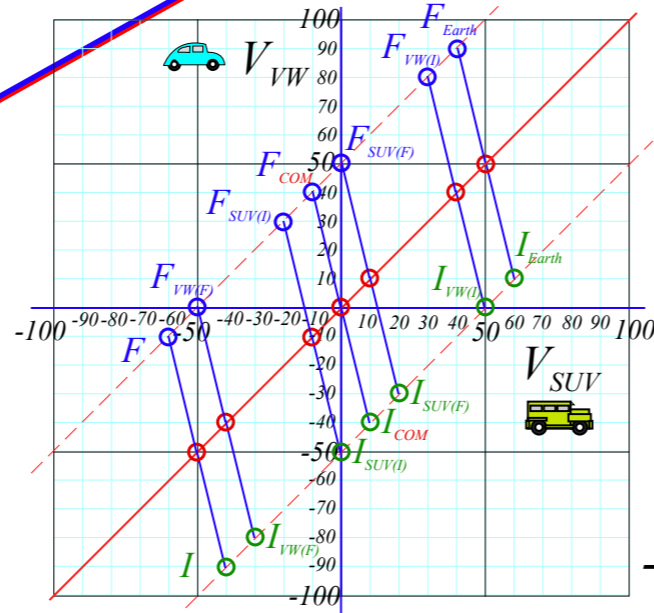
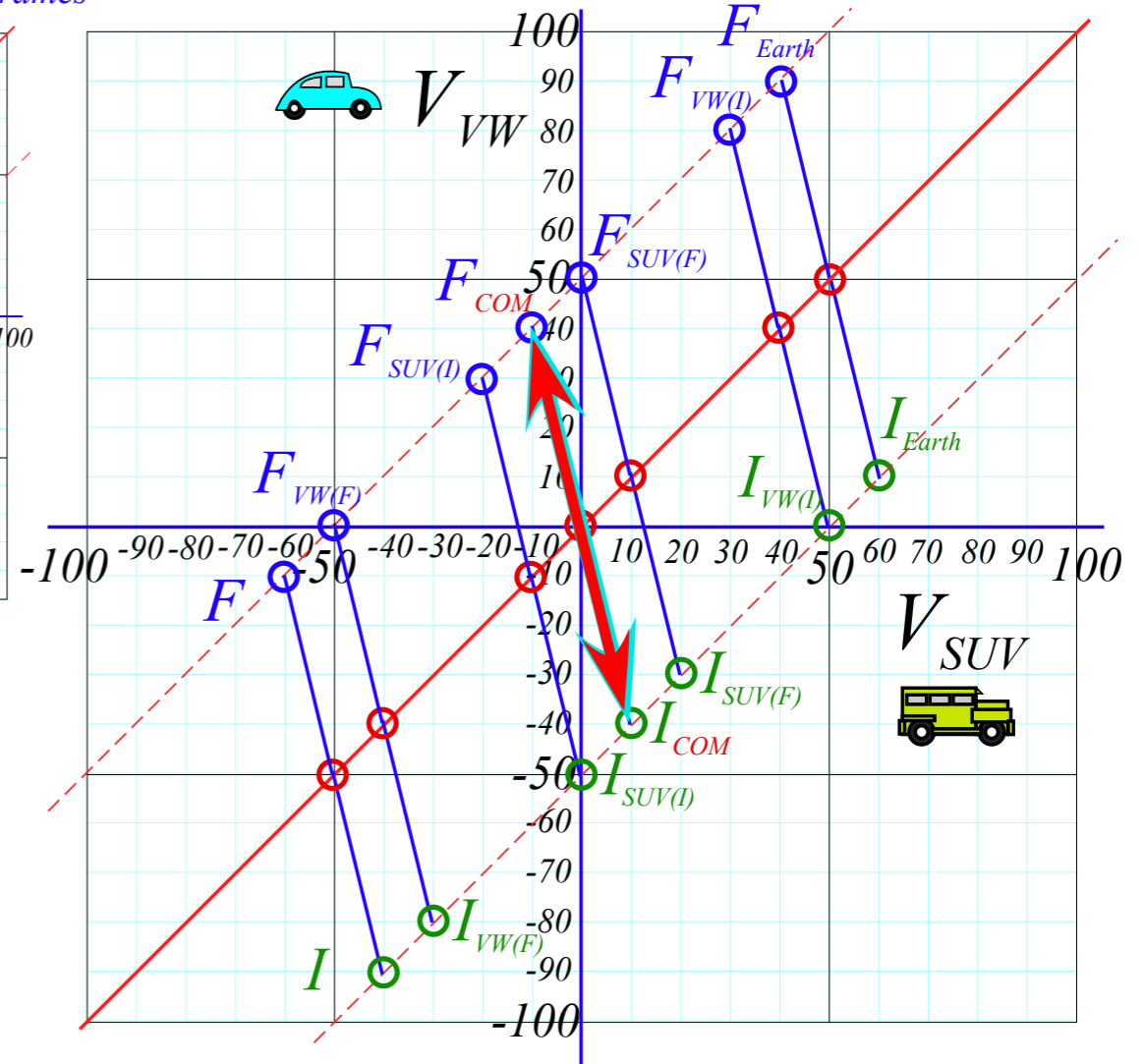


Fig. 2.5b
in Unit 1



*There is just one velocity frame
in which the time-reversed collision
looks just like the original collision*

*That is the
Center-of-Momentum
(COM)-frame*

*Time-reversal means flip t to $-t$...
(Run a movie backwards)*

*That means you flip Velocity V to $-V$...
(Everything goes backwards)*

Algebra, Geometry, and Physics of momentum conservation axiom

- *Vector algebra of collisions*
- Matrix or tensor algebra of collisions*
- Deriving Energy Conservation Theorem*
- Energy Ellipse geometry*

Quick lesson on

Gibb's notation for

dot (\bullet) product of matrix operator \mathbf{M} and column vector \mathbf{V}^{IN} :

$$\begin{aligned} & \vec{\mathbf{M}} \bullet \vec{\mathbf{V}}^{IN} \\ & \begin{pmatrix} A & B \\ C & D \end{pmatrix} \bullet \begin{pmatrix} x^{IN} \\ y^{IN} \end{pmatrix} \\ & = \begin{pmatrix} Ax^{IN} + By^{IN} \\ Cx^{IN} + Dy^{IN} \end{pmatrix} \end{aligned}$$

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Quick lesson on

Dirac notation is

much simpler:

$$\begin{aligned} & M |IN\rangle \\ & \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \langle x | IN \rangle \\ \langle y | IN \rangle \end{pmatrix} \end{aligned}$$

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$$\vec{\mathbf{M}} \bullet \vec{\mathbf{V}}^{IN}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \bullet \begin{pmatrix} x^{IN} \\ y^{IN} \end{pmatrix}$$

$$= \begin{pmatrix} Ax^{IN} + By^{IN} \\ Cx^{IN} + Dy^{IN} \end{pmatrix}$$

Quick lesson on

Dirac notation is

much simpler:

$$M|IN\rangle \quad (\dots \text{at first!})$$

$$\begin{pmatrix} \langle x|M|x\rangle & \langle x|M|y\rangle \\ \langle y|M|x\rangle & \langle y|M|y\rangle \end{pmatrix} \begin{pmatrix} \langle x|IN\rangle \\ \langle y|IN\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \langle x|M|x\rangle\langle x|IN\rangle + \langle x|M|y\rangle\langle y|IN\rangle \\ \langle y|M|x\rangle\langle x|IN\rangle + \langle y|M|y\rangle\langle y|IN\rangle \end{pmatrix}$$

Algebra, Geometry, and Physics of momentum conservation axiom

 *Vector algebra of collisions*

Matrix or tensor algebra of collisions

Deriving Energy Conservation Theorem

Energy Ellipse geometry

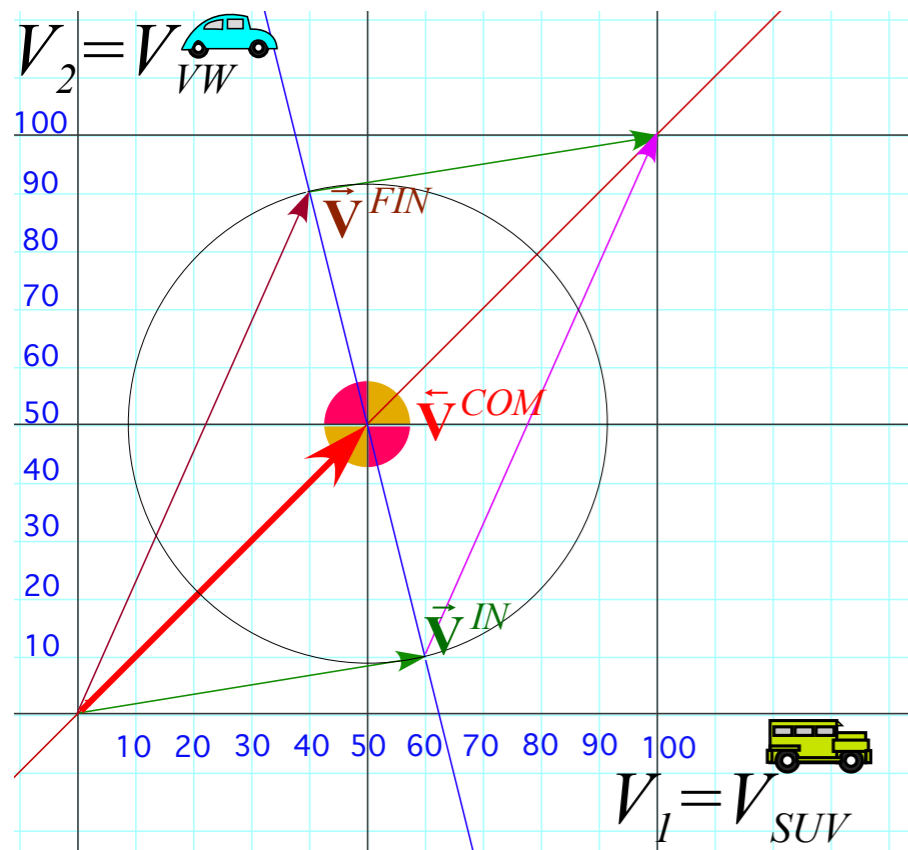
General Inertia Tensor \mathbf{M} or inertia matrix of 2 coefficients $M_{11}=M_1$ and $M_{22}=M_2$ for 2 dimensions

$$\left. \begin{aligned} P_1 &= M_1 V_1 + M_{12} V_2 \\ P_2 &= M_{21} V_1 + M_2 V_2 \end{aligned} \right\} \text{denoted : } \vec{\mathbf{P}} = \vec{\mathbf{M}} \cdot \vec{\mathbf{V}} \quad \text{or : } \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

Generalizing the definition of momentum...

With 45° diagonal \mathbf{V}^{COM} so: . $V_1^{COM} = V_2^{COM} \equiv V^{COM}$...and 3 ways to write *Axiom-1*

$$P_{Total} = M_1 V_1^{IN} + M_2 V_2^{IN} = M_1 V_1^{FIN} + M_2 V_2^{FIN} = M_1 V^{COM} + M_2 V^{COM} = (M_1 + M_2) V^{COM} = M_{Total} V^{COM}$$



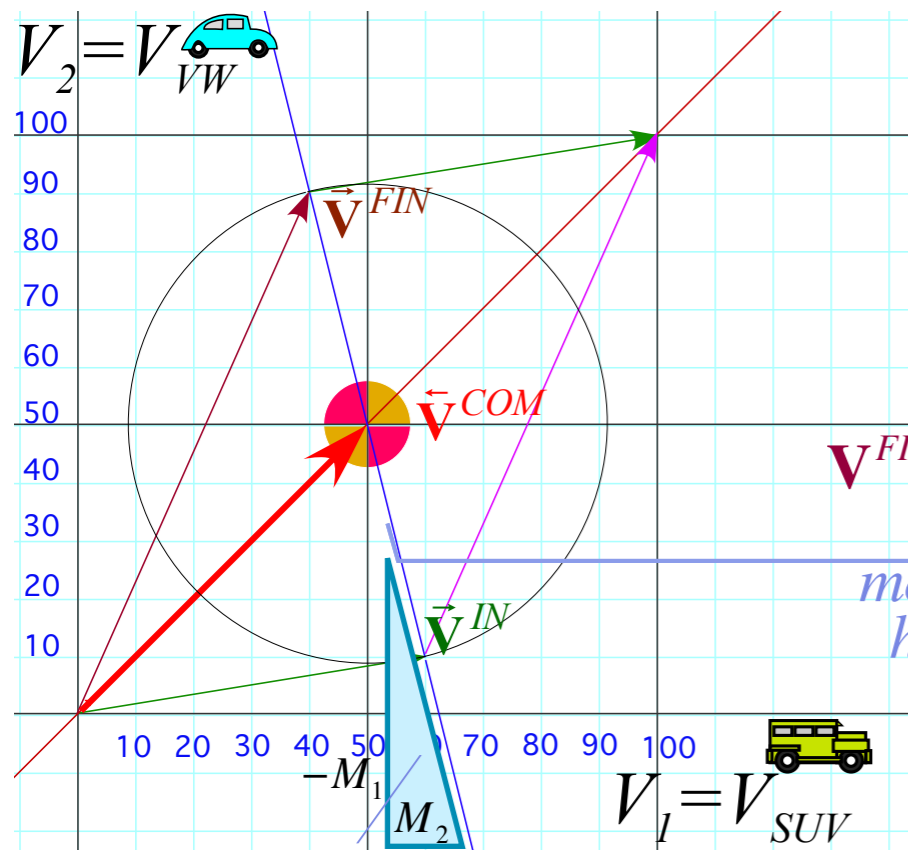
General Inertia Tensor \mathbf{M} or inertia matrix of 2 coefficients $M_{11}=M_1$ and $M_{22}=M_2$ for 2 dimensions

$$\left. \begin{aligned} P_1 &= M_1 V_1 + M_{12} V_2 \\ P_2 &= M_{21} V_1 + M_2 V_2 \end{aligned} \right\} \text{denoted : } \vec{\mathbf{P}} = \vec{\mathbf{M}} \cdot \vec{\mathbf{V}} \quad \text{or : } \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

Generalizing the definition of momentum...

With 45° diagonal \mathbf{V}^{COM} so: . $V_1^{COM} = V_2^{COM} \equiv V^{COM}$...and 3 ways to write *Axiom-1*

$$P_{Total} = M_1 V_1^{IN} + M_2 V_2^{IN} = M_1 V_1^{FIN} + M_2 V_2^{FIN} = M_1 V^{COM} + M_2 V^{COM} = (M_1 + M_2) V^{COM} = M_{Total} V^{COM}$$



Points of vectors \vec{V}^{FIN} and \vec{V}^{COM} and \vec{V}^{IN} all lie on momentum - conservation line having slope $-M_1 / M_2$

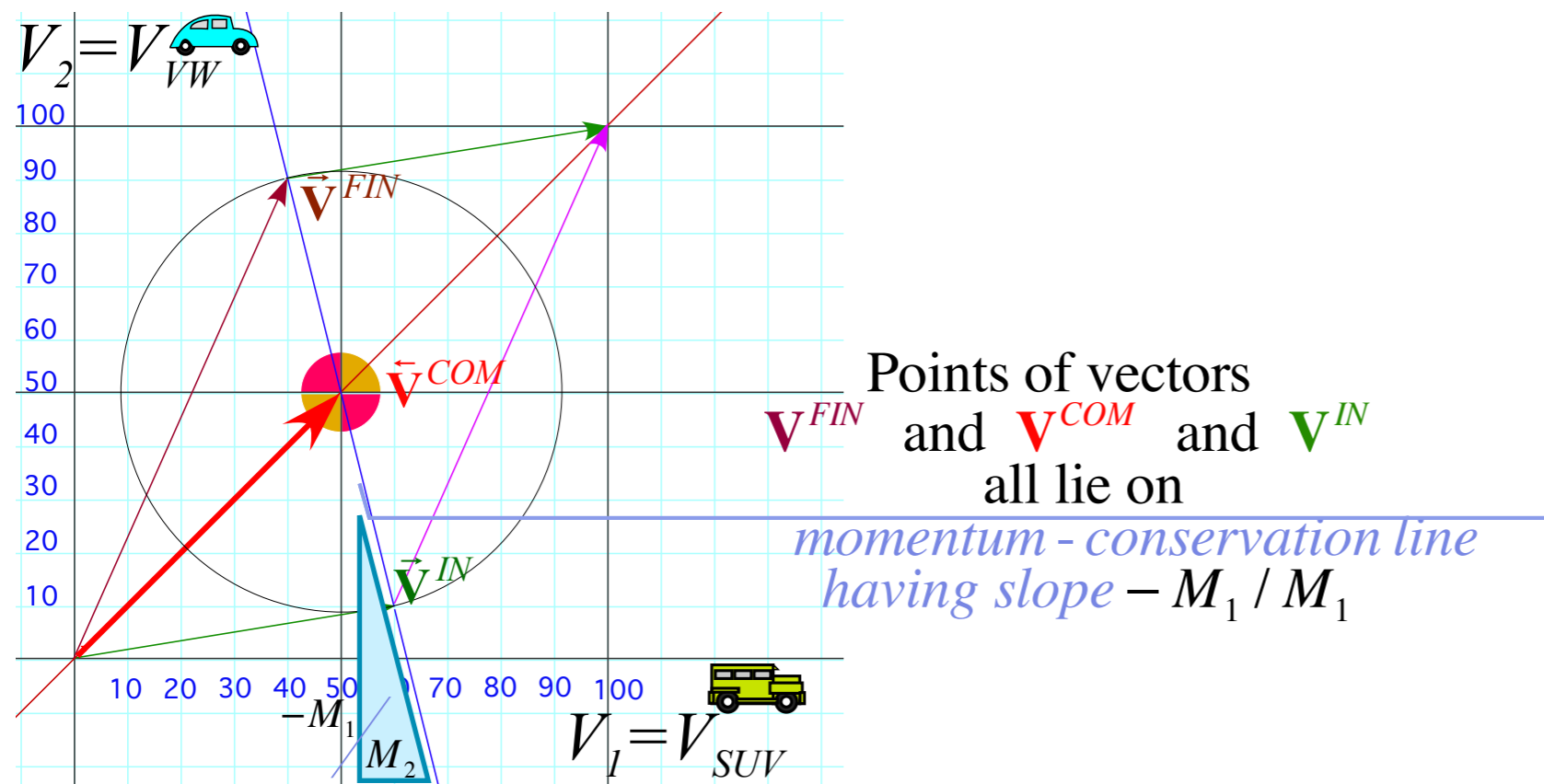
General Inertia Tensor \mathbf{M} or inertia matrix of 3 coefficients M_{11} , M_{22} and $M_{12}=M_{21}$ for 2 dimensions

$$\left. \begin{aligned} P_1 &= M_{11}V_1 + M_{12}V_2 \\ P_2 &= M_{21}V_1 + M_{22}V_2 \end{aligned} \right\} \text{denoted: } \vec{\mathbf{P}} = \vec{\mathbf{M}} \cdot \vec{\mathbf{V}} \text{ or: } \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

Generalizing the definition of momentum...some more...

With 45° diagonal \mathbf{V}^{COM} so: $V_1^{COM} = V_2^{COM} \equiv V^{COM}$...and 3 ways to write *Axiom-1*

$$P_{Total} = M_1 V_1^{IN} + M_2 V_2^{IN} = M_1 V_1^{FIN} + M_2 V_2^{FIN} = M_1 V^{COM} + M_2 V^{COM} = (M_1 + M_2) V^{COM} = M_{Total} V^{COM}$$

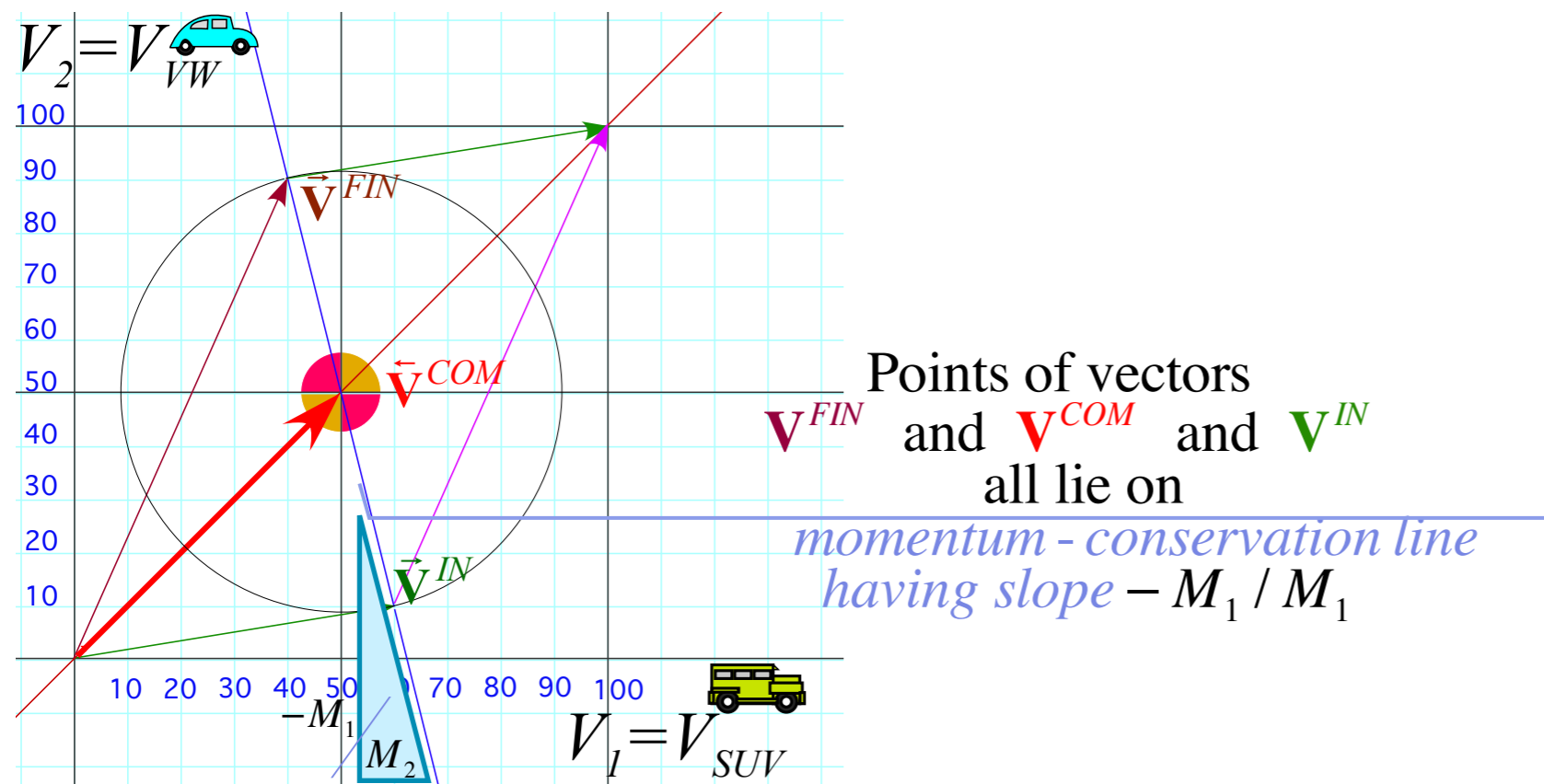


General Inertia Tensor \mathbf{M} or inertia matrix of $(n^2+n)/2$ coefficients $M_{jk} = M_{kj}$ for dimension $n=2, 3, \dots$

$$\left. \begin{aligned} P_1 &= M_{11}V_1 + M_{12}V_2 \dots \\ P_2 &= M_{21}V_1 + M_{22}V_2 \dots \\ \vdots &= \vdots \quad \vdots \quad \ddots \end{aligned} \right\} \begin{array}{l} \text{denoted : } \vec{\mathbf{P}} = \vec{\mathbf{M}} \cdot \vec{\mathbf{V}} \quad \text{or :} \\ \text{Generalizing the definition} \\ \text{of momentum...some more...and more} \end{array} \begin{pmatrix} P_1 \\ P_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & \dots \\ M_{21} & M_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ \vdots \end{pmatrix}$$

With 45° diagonal $\vec{\mathbf{V}}^{COM} = (\vec{\mathbf{V}}^{FIN} + \vec{\mathbf{V}}^{IN})/2$ so: $V_1^{COM} = V_2^{COM} \equiv V^{COM}$...and 3 ways to write *Axiom-1*

$$P_{Total} = M_1V_1^{IN} + M_2V_2^{IN} = M_1V_1^{FIN} + M_2V_2^{FIN} = M_1V^{COM} + M_2V^{COM} = (M_1+M_2)V^{COM} = M_{Total}V^{COM}$$



Algebra, Geometry, and Physics of momentum conservation axiom

Vector algebra of collisions

 *Matrix or tensor algebra of collisions*

Deriving Energy Conservation Theorem

Energy Ellipse geometry

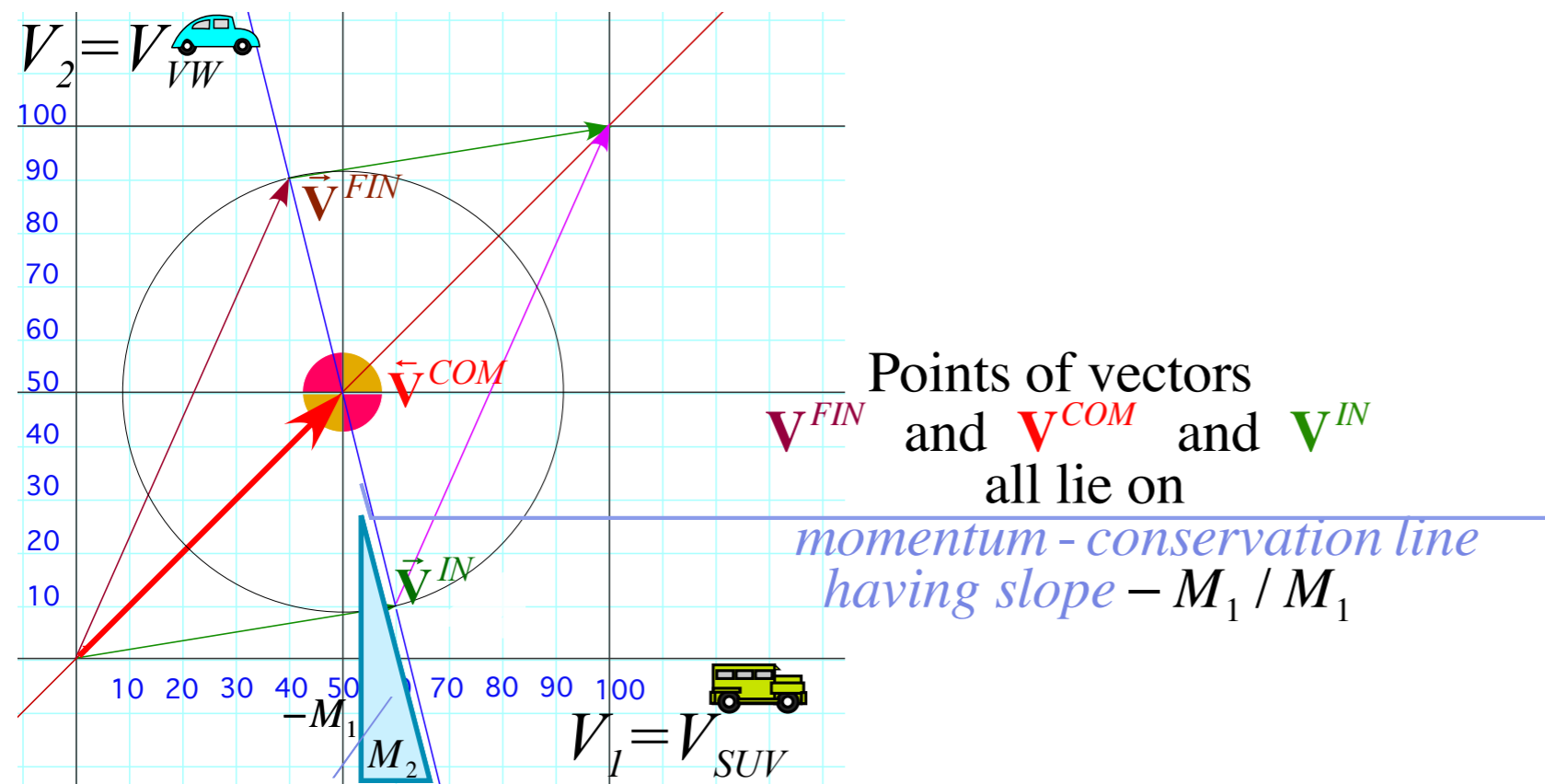
General Inertia Tensor \mathbf{M} or inertia matrix of 2 coefficients $M_{11}=M_1$ and $M_{22}=M_2$ for 2 dimensions

$$\left. \begin{aligned} P_1 &= M_1 V_1 + M_{12} V_2 \\ P_2 &= M_{21} V_1 + M_2 V_2 \end{aligned} \right\} \text{denoted : } \vec{\mathbf{P}} = \vec{\mathbf{M}} \cdot \vec{\mathbf{V}} \text{ or : } \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

Generalizing the definition of momentum...

With 45° diagonal $\vec{\mathbf{V}}^{COM} = (\vec{\mathbf{V}}^{FIN} + \vec{\mathbf{V}}^{IN})/2$ so: $V_1^{COM} = V_2^{COM} \equiv V^{COM}$...and 4 or 5 ways to write Axiom-1

$$P_{Total} = M_1 V_1^{IN} + M_2 V_2^{IN} = M_1 V_1^{FIN} + M_2 V_2^{FIN} = M_1 V^{COM} + M_2 V^{COM} = (M_1 + M_2) V^{COM} = M_{Total} V^{COM}$$



General Inertia Tensor \mathbf{M} or inertia matrix of 2 coefficients $M_{11}=M_1$ and $M_{22}=M_2$ for 2 dimensions

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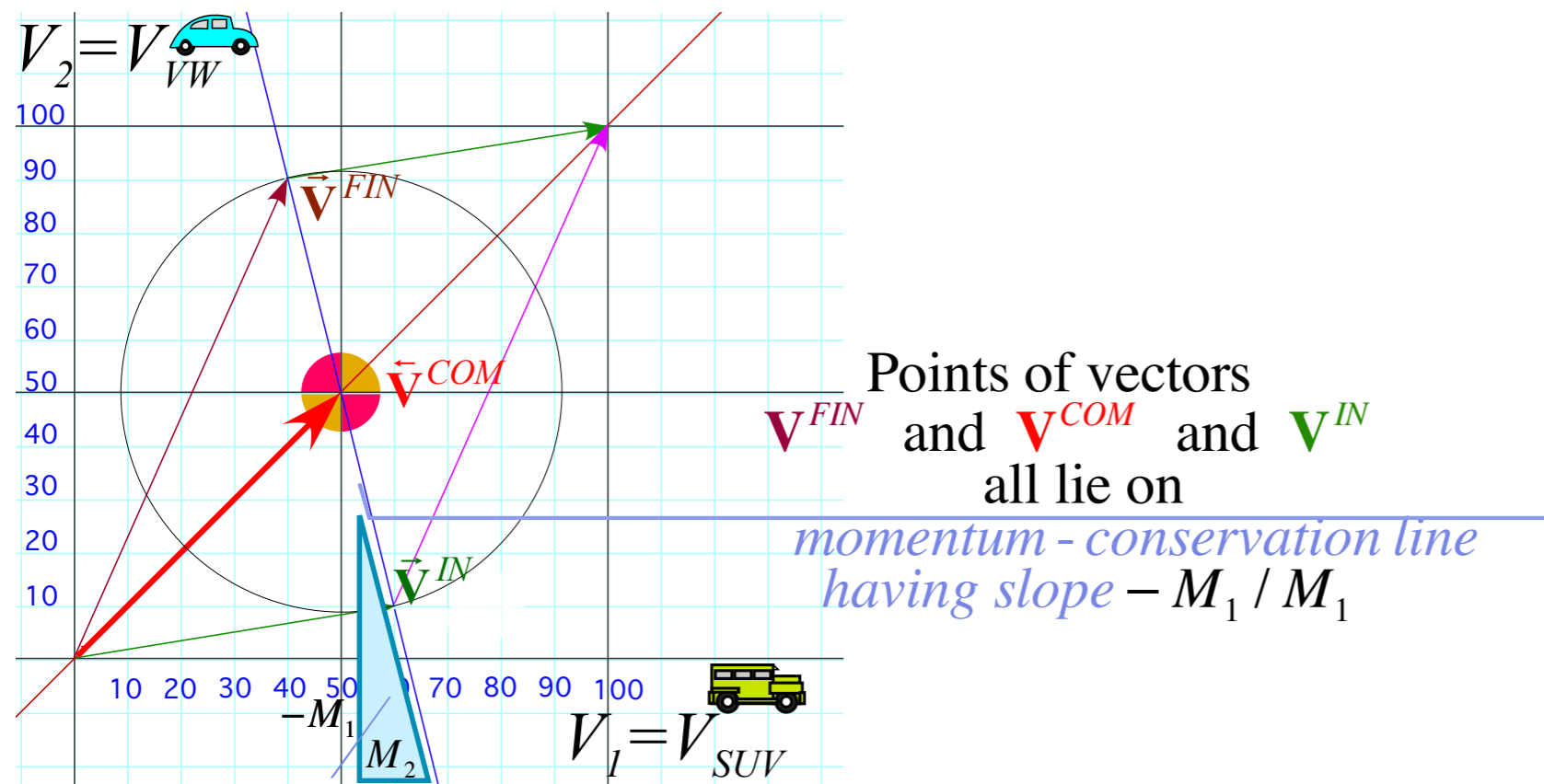
Generalizing the definition of momentum...

With 45° diagonal $\vec{\mathbf{V}}^{COM} = (\vec{\mathbf{V}}^{FIN} + \vec{\mathbf{V}}^{IN})/2$ so: $V_1^{COM} = V_2^{COM} \equiv V^{COM}$...and 4 or 5 ways to write Axiom-1

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Numerical details of collision tensor algebra

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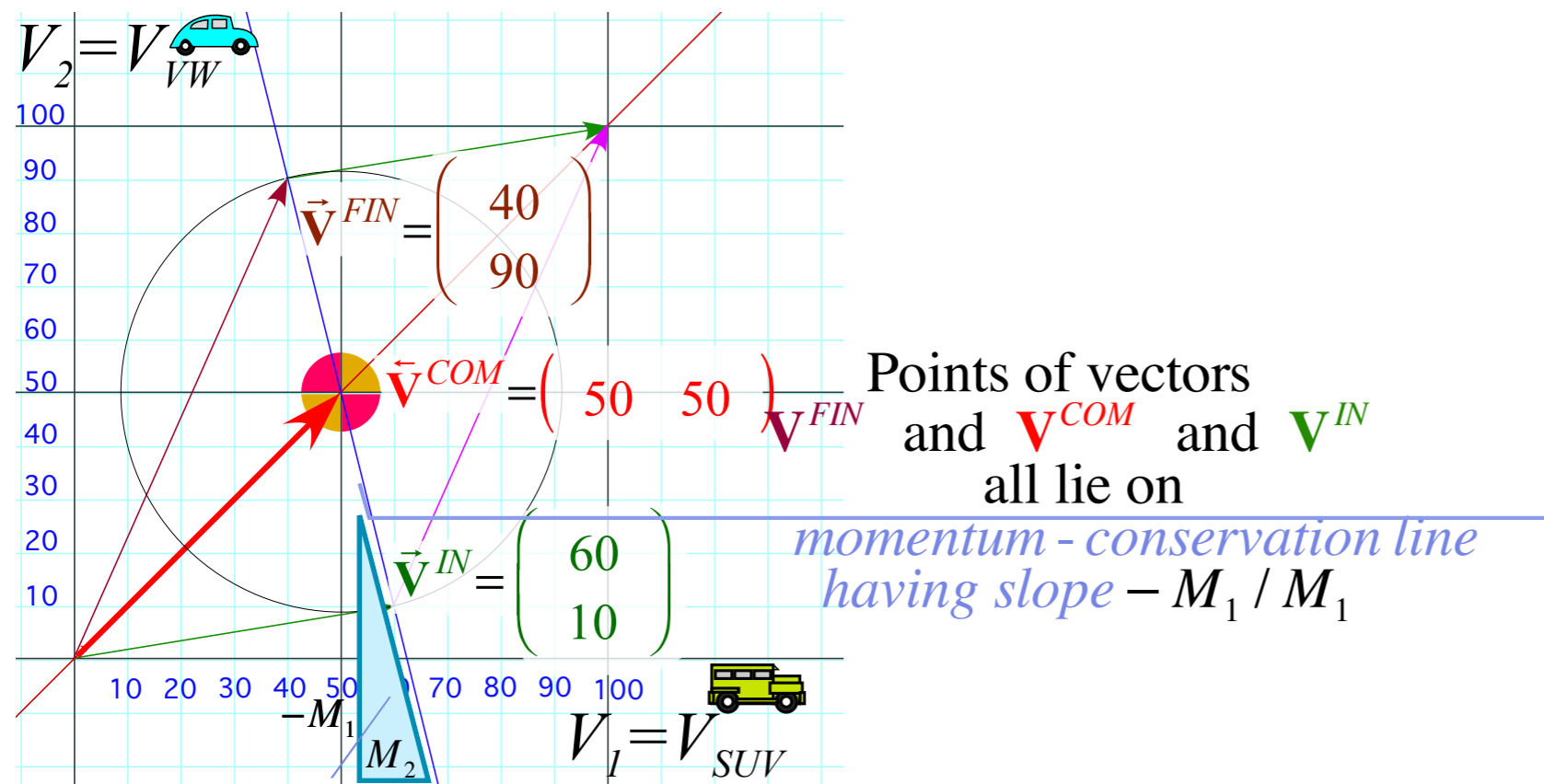
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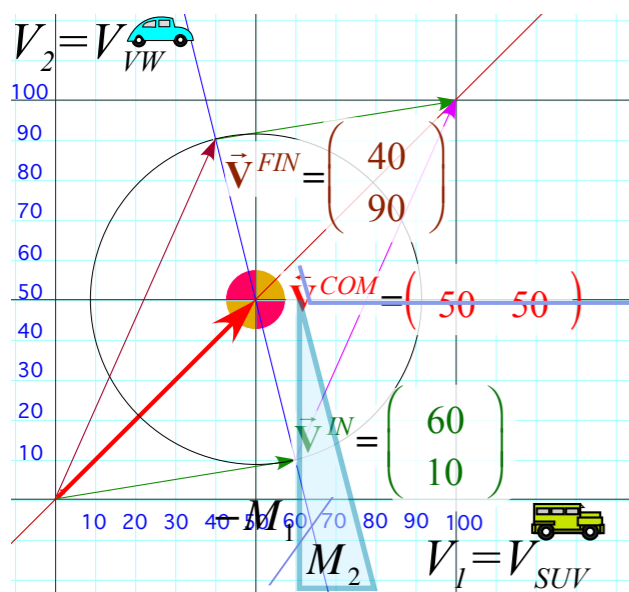
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Points of vectors \mathbf{V}^{FIN} and \mathbf{V}^{COM} and \mathbf{V}^{IN} all lie on momentum - conservation line having slope $-M_1 / M_1$

Algebra, Geometry, and Physics of momentum conservation axiom

Vector algebra of collisions

Matrix or tensor algebra of collisions

 *Deriving Energy Conservation Theorem*

Energy Ellipse geometry

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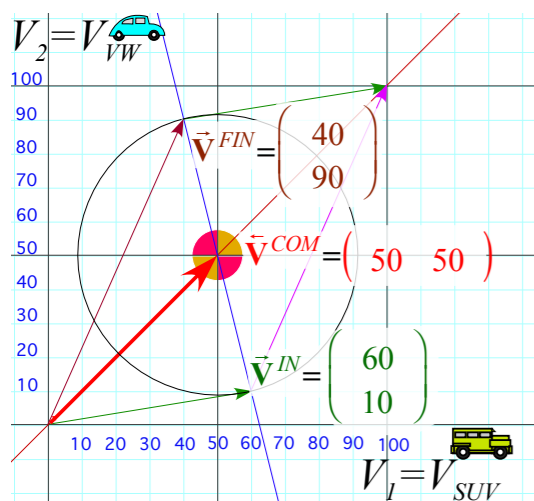
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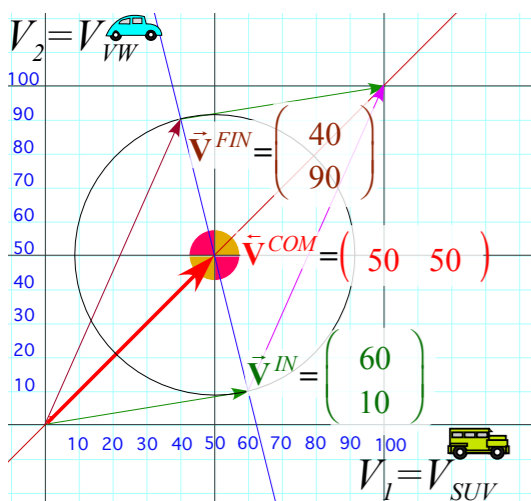
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Numerical details of collision tensor algebra

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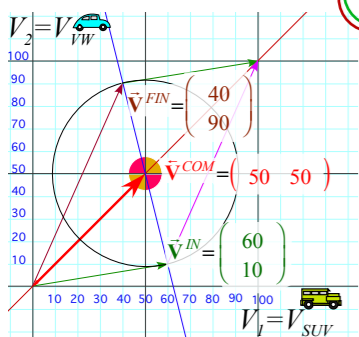
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Transpose symmetry ($M_{jk} = M_{kj}$) of \mathbf{M} -matrix makes 'lopsided' **FIN-IN**-terms equal:

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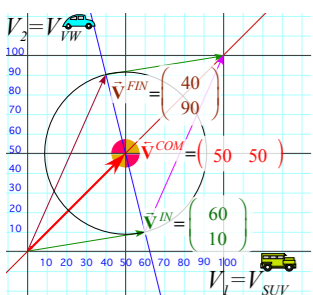
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FIN-IN-term is subtracted to give

Conservation of Kinetic Energy

$$KE = \frac{1}{2} \vec{V} \cdot \vec{M} \cdot \vec{V} = \frac{1}{2} M_1 V_1^2 + \frac{1}{2} M_2 V_2^2$$



$$\vec{V}^{FIN} \cdot \vec{M} \cdot \vec{V}^{IN} = \vec{V}^{IN} \cdot \vec{M} \cdot \vec{V}^{FIN}$$

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$P_{Total} = 250$ is the same at **IN**, **FIN**, and **COM**. Now use *T*-symmetry: $\vec{V}^{COM} = (\vec{V}^{FIN} + \vec{V}^{IN})/2$ (**Axiom-2**)

$$\begin{aligned} V^{COM} P_{Total} &= \frac{\vec{V}^{FIN} + \vec{V}^{IN}}{2} \cdot \vec{M} \cdot \vec{V}^{IN} = \frac{\vec{V}^{FIN} + \vec{V}^{IN}}{2} \cdot \vec{M} \cdot \vec{V}^{FIN} = \frac{\vec{V}^{FIN} + \vec{V}^{IN}}{2} \cdot \vec{M} \cdot \frac{\vec{V}^{FIN} + \vec{V}^{IN}}{2} \\ &= \frac{1}{2} \vec{V}^{FIN} \cdot \vec{M} \cdot \vec{V}^{IN} + \frac{1}{2} \vec{V}^{IN} \cdot \vec{M} \cdot \vec{V}^{IN} = \frac{1}{2} \vec{V}^{FIN} \cdot \vec{M} \cdot \vec{V}^{FIN} + \frac{1}{2} \vec{V}^{IN} \cdot \vec{M} \cdot \vec{V}^{FIN} \end{aligned}$$

$$V^{COM} P_{Total} - \frac{1}{2} \vec{V}^{FIN} \cdot \vec{M} \cdot \vec{V}^{IN} = \frac{1}{2} \vec{V}^{IN} \cdot \vec{M} \cdot \vec{V}^{IN} = \frac{1}{2} \vec{V}^{FIN} \cdot \vec{M} \cdot \vec{V}^{FIN}$$

$$50 \cdot 250 - \frac{1}{2} \cdot 10,500$$

$$\frac{1}{2} \begin{pmatrix} 60 & 10 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 60 \\ 10 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 40 & 90 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 40 \\ 90 \end{pmatrix}$$

FIN-IN-term is subtracted to give

Conservation of Kinetic Energy

$$KE = \frac{1}{2} \vec{V} \cdot \vec{M} \cdot \vec{V} = \frac{1}{2} M_1 V_1^2 + \frac{1}{2} M_2 V_2^2$$

General Inertia Tensor M or inertia matrix of 2 coefficients $M_{11}=M_1$ and $M_{22}=M_2$ for 2 dimensions

$$\left. \begin{aligned} P_1 &= M_1 V_1 + M_{12} V_2 \\ P_2 &= M_{21} V_1 + M_2 V_2 \end{aligned} \right\} \text{denoted: } \vec{P} = \vec{M} \cdot \vec{V} \text{ or: } \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

Generalizing the definition of momentum...

With 45° diagonal $\vec{V}^{COM} = (\vec{V}^{FIN} + \vec{V}^{IN})/2$ so: $V_1^{COM} = V_2^{COM} \equiv V^{COM}$...and 4 or 5 ways to write **Axiom-1**

$$P_{Total} = M_1 V_1^{IN} + M_2 V_2^{IN} = M_1 V_1^{FIN} + M_2 V_2^{FIN} = M_1 V^{COM} + M_2 V^{COM} = (M_1 + M_2) V^{COM} = M_{Total} V^{COM}$$

A product of total momentum P_{Total} and V^{COM} is expressed by *tensor quadratic forms* $\mathbf{v} \cdot \mathbf{M} \cdot \mathbf{u}$

$$V^{COM} P_{Total} = \vec{V}^{COM} \cdot \vec{M} \cdot \vec{V}^{IN} = \vec{V}^{COM} \cdot \vec{M} \cdot \vec{V}^{FIN} = \vec{V}^{COM} \cdot \vec{M} \cdot \vec{V}^{COM} = V^{COM} M_{Total} V^{COM}$$

Write this out with the numbers used in Fig. 1.3 where $V^{COM} = 50$.

$$50 P_{Total} = \begin{pmatrix} 50 & 50 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 60 \\ 10 \end{pmatrix} = \begin{pmatrix} 50 & 50 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 40 \\ 90 \end{pmatrix} = 50 M_{Total} 50 = 12,500$$

= $100 \cdot 125$ = $100 \cdot 125$ = $50 \cdot 250$

$P_{Total} = 250$ is the same at **IN**, **FIN**, and **COM**. Now use *T*-symmetry: $\vec{V}^{COM} = (\vec{V}^{FIN} + \vec{V}^{IN})/2$ (**Axiom-2**)

$$V^{COM} P_{Total} = \frac{\vec{V}^{FIN} + \vec{V}^{IN}}{2} \cdot \vec{M} \cdot \vec{V}^{IN} = \frac{\vec{V}^{FIN} + \vec{V}^{IN}}{2} \cdot \vec{M} \cdot \vec{V}^{FIN} = \frac{\vec{V}^{FIN} + \vec{V}^{IN}}{2} \cdot \vec{M} \cdot \frac{\vec{V}^{FIN} + \vec{V}^{IN}}{2}$$

$$= \frac{1}{2} \vec{V}^{FIN} \cdot \vec{M} \cdot \vec{V}^{IN} + \frac{1}{2} \vec{V}^{IN} \cdot \vec{M} \cdot \vec{V}^{IN} = \frac{1}{2} \vec{V}^{FIN} \cdot \vec{M} \cdot \vec{V}^{FIN} + \frac{1}{2} \vec{V}^{IN} \cdot \vec{M} \cdot \vec{V}^{FIN}$$

$$V^{COM} P_{Total} - \frac{1}{2} \vec{V}^{FIN} \cdot \vec{M} \cdot \vec{V}^{IN} = \frac{1}{2} \vec{V}^{IN} \cdot \vec{M} \cdot \vec{V}^{IN} = \frac{1}{2} \vec{V}^{FIN} \cdot \vec{M} \cdot \vec{V}^{FIN}$$

$$50 \cdot 250 - \frac{1}{2} \cdot 10,500$$

$$12,500 - 5,250$$

$$\frac{1}{2} \begin{pmatrix} 60 & 10 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 60 \\ 10 \end{pmatrix} = \frac{1}{2} (60 \cdot 4 \cdot 60 + 10 \cdot 1 \cdot 10)$$

$$\frac{1}{2} \begin{pmatrix} 40 & 90 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 40 \\ 90 \end{pmatrix} = \frac{1}{2} (40 \cdot 4 \cdot 40 + 90 \cdot 1 \cdot 90)$$

FIN-IN-term is subtracted to give **Conservation of Kinetic Energy**

$$KE = \frac{1}{2} \vec{V} \cdot \vec{M} \cdot \vec{V} = \frac{1}{2} M_1 V_1^2 + \frac{1}{2} M_2 V_2^2$$

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A product of total momentum P_{Total} and V^{COM} is expressed by *tensor quadratic forms* $\mathbf{v} \cdot \mathbf{M} \cdot \mathbf{u}$

$$V^{COM} P_{Total} = \vec{V}^{COM} \cdot \vec{M} \cdot \vec{V}^{IN} = \vec{V}^{COM} \cdot \vec{M} \cdot \vec{V}^{FIN} = \vec{V}^{COM} \cdot \vec{M} \cdot \vec{V}^{COM} = V^{COM} M_{Total} V^{COM}$$

Write this out with the numbers used in Fig. 1.3 where $V^{COM} = 50$.

$$50 P_{Total} = \begin{pmatrix} 50 & 50 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 60 \\ 10 \end{pmatrix} = \begin{pmatrix} 50 & 50 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 40 \\ 90 \end{pmatrix} = 50 M_{Total} 50 = 12,500$$

= 100 · 125 = 100 · 125 = 50 · 250

$P_{Total} = 250$ is the same at **IN**, **FIN**, and **COM**. Now use *T*-symmetry: $\vec{V}^{COM} = (\vec{V}^{FIN} + \vec{V}^{IN})/2$ (**Axiom-2**)

$$V^{COM} P_{Total} = \frac{\vec{V}^{FIN} + \vec{V}^{IN}}{2} \cdot \vec{M} \cdot \vec{V}^{IN} = \frac{\vec{V}^{FIN} + \vec{V}^{IN}}{2} \cdot \vec{M} \cdot \vec{V}^{FIN} = \frac{\vec{V}^{FIN} + \vec{V}^{IN}}{2} \cdot \vec{M} \cdot \frac{\vec{V}^{FIN} + \vec{V}^{IN}}{2}$$

$$= \frac{1}{2} \vec{V}^{FIN} \cdot \vec{M} \cdot \vec{V}^{IN} + \frac{1}{2} \vec{V}^{IN} \cdot \vec{M} \cdot \vec{V}^{IN} = \frac{1}{2} \vec{V}^{FIN} \cdot \vec{M} \cdot \vec{V}^{FIN} + \frac{1}{2} \vec{V}^{IN} \cdot \vec{M} \cdot \vec{V}^{FIN}$$

$$V^{COM} P_{Total} - \frac{1}{2} \vec{V}^{FIN} \cdot \vec{M} \cdot \vec{V}^{IN} = \frac{1}{2} \vec{V}^{IN} \cdot \vec{M} \cdot \vec{V}^{IN} = \frac{1}{2} \vec{V}^{FIN} \cdot \vec{M} \cdot \vec{V}^{FIN}$$

$$50 \cdot 250 - \frac{1}{2} \cdot 10,500$$

$$12,500 - 5,250 = 7,250$$

$$\frac{1}{2} \begin{pmatrix} 60 & 10 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 60 \\ 10 \end{pmatrix} = \frac{1}{2} (60 \cdot 4 \cdot 60 + 10 \cdot 1 \cdot 10)$$

$$= \frac{1}{2} (2 \cdot 3600 + 50) = 7250$$

$$\frac{1}{2} \begin{pmatrix} 40 & 90 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 40 \\ 90 \end{pmatrix} = \frac{1}{2} (40 \cdot 4 \cdot 40 + 90 \cdot 1 \cdot 90)$$

$$= \frac{1}{2} (2 \cdot 1600 + \frac{1}{2} \cdot 8100) = 7250$$

FIN-IN-term is subtracted to give **Conservation of Kinetic Energy**

$$KE = \frac{1}{2} \vec{V} \cdot \vec{M} \cdot \vec{V} = \frac{1}{2} M_1 V_1^2 + \frac{1}{2} M_2 V_2^2$$

Algebra, Geometry, and Physics of momentum conservation axiom

Vector algebra of collisions

Matrix or tensor algebra of collisions

Deriving Energy Conservation Theorem

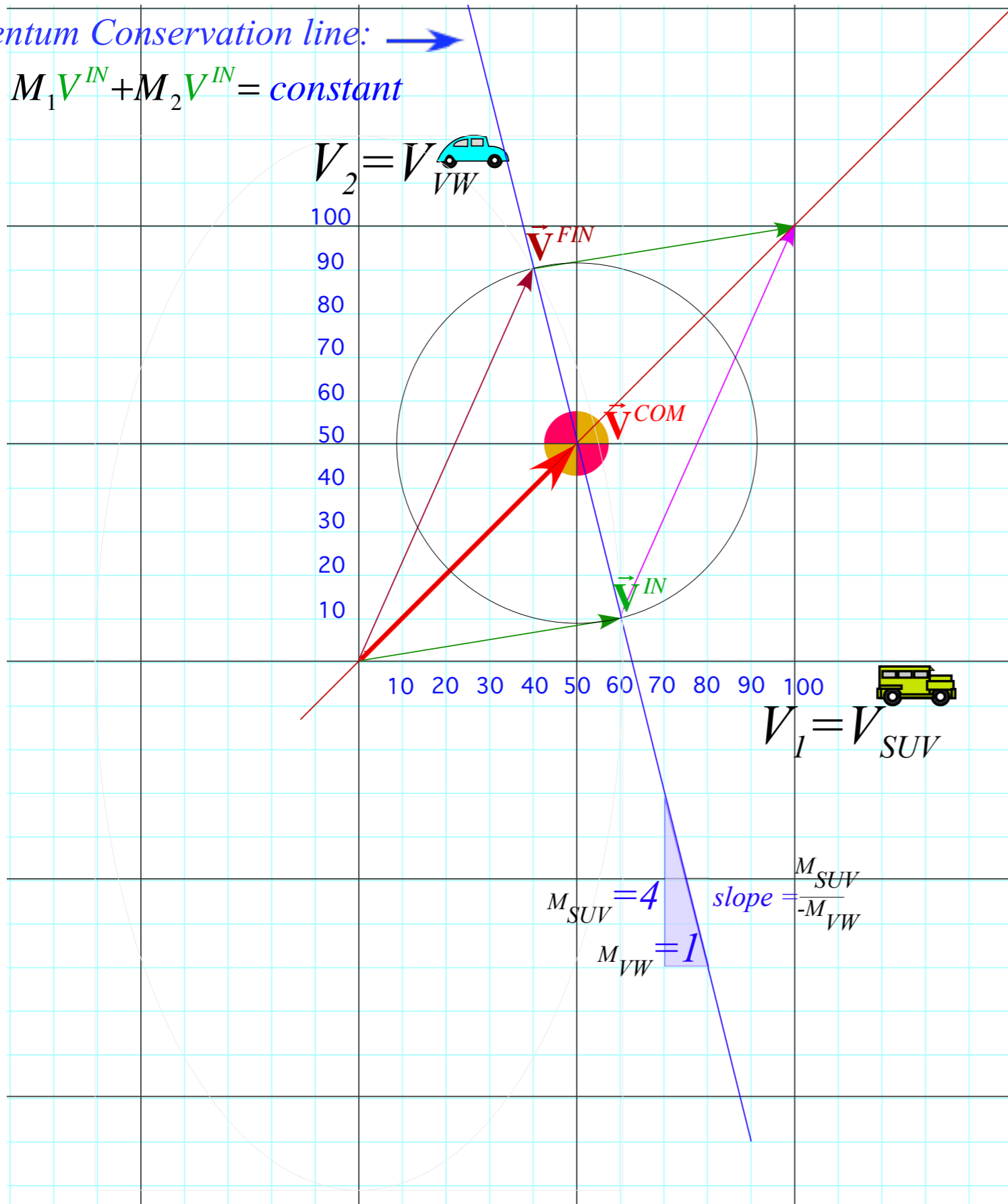
 *Energy Ellipse geometry*

Geometry of Momentum Conservation Axiom - 1

$$(M_1 + M_2)V^{COM} = M_1V_1^{IN} + M_2V_2^{IN} = M_1V_1^{FIN} + M_2V_2^{FIN}$$

Momentum Conservation line: →

$$M_1V_1^{IN} + M_2V_2^{IN} = \text{constant}$$



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[Basic elastic Collision](#)
[Dual Panel](#)
[Space vs Space](#)
 and
[V\(VW\) vs. V\(SUV\)](#)

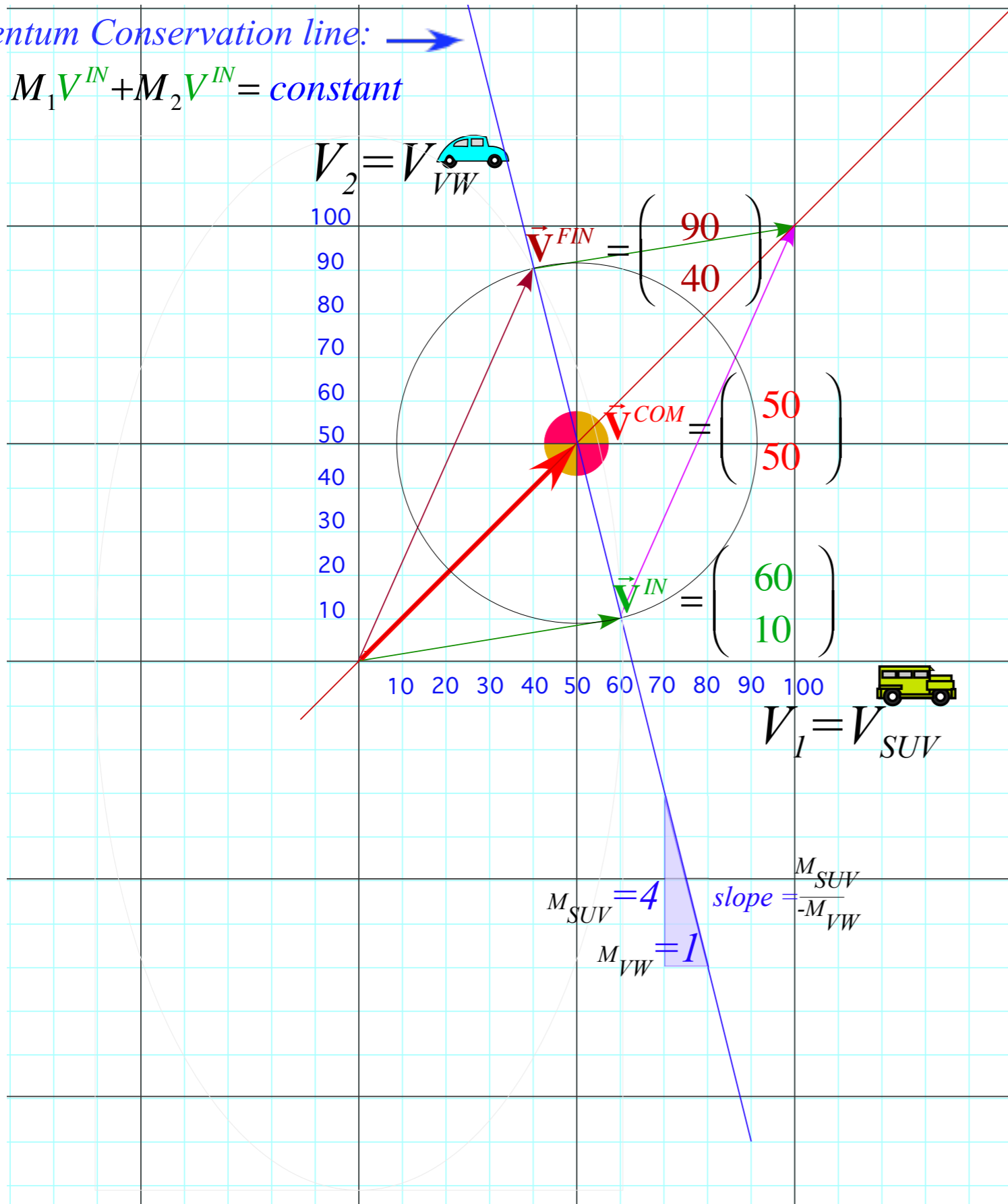
[BounceIt](#)
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 and
[V\(VW\) vs. V\(SUV\)](#)

Geometry of Momentum Conservation Axiom - 1

$$(M_1 + M_2)V^{COM} = M_1V_1^{IN} + M_2V_2^{IN} = M_1V_1^{FIN} + M_2V_2^{FIN}$$

Momentum Conservation line: \rightarrow

$$M_1V_1^{IN} + M_2V_2^{IN} = \text{constant}$$



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Basic elastic Collision
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 Space vs Space
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 V(VW) vs. V(SUV)

Geometry of Momentum Conservation Axiom-1

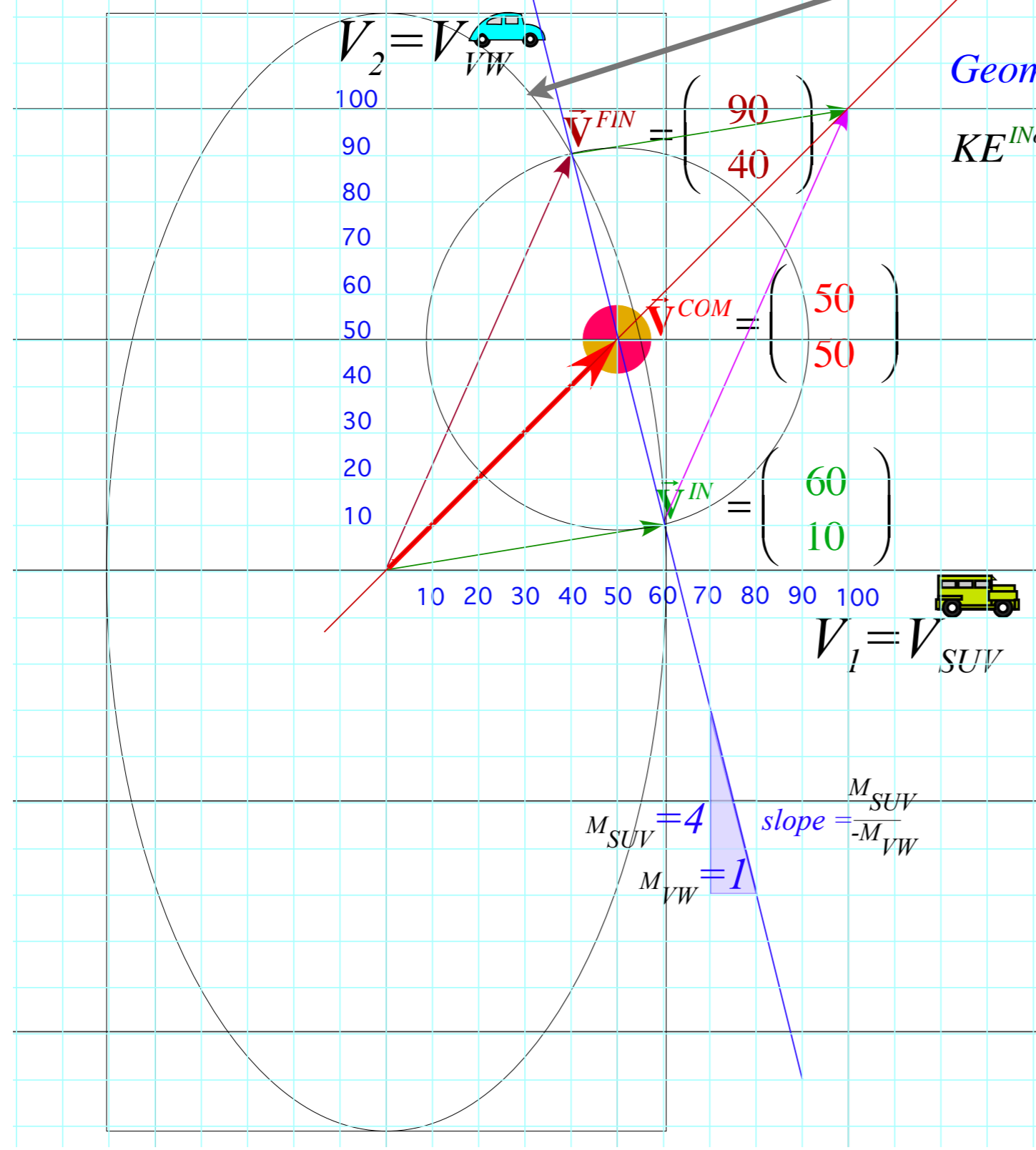
Geometry of Kinetic Energy Conservation Axiom-2/Theorem-1

$$(M_1+M_2)V^{COM} = M_1V_1^{IN} + M_2V_2^{IN} = M_1V_1^{FIN} + M_2V_2^{FIN} \quad KE^{IN} = \frac{1}{2}M_1(V_1^{IN})^2 + \frac{1}{2}M_2(V_2^{IN})^2 = \frac{1}{2}M_1(V_1^{FIN})^2 + \frac{1}{2}M_2(V_2^{FIN})^2 = KE^{FIN}$$

Momentum Conservation line: \rightarrow

$$M_1V_1^{IN} + M_2V_2^{IN} = \text{constant}$$

KE Conservation ellipse:



Geometry of KE Conservation Theorem -1

$$KE^{INorFIN} = \frac{1}{2}M_1(V_1)^2 + \frac{1}{2}M_2(V_2)^2 = \frac{1}{2}4(60)^2 + \frac{1}{2}1(10)^2 = 7,250$$

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Basic elastic Collision
 Dual Panel
 Space vs Space
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 V(VW) vs. V(SUV)

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 Superball Web Simulator
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 Dual Panel
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 V(VW) vs. V(SUV)

Geometry of Momentum Conservation Axiom-1

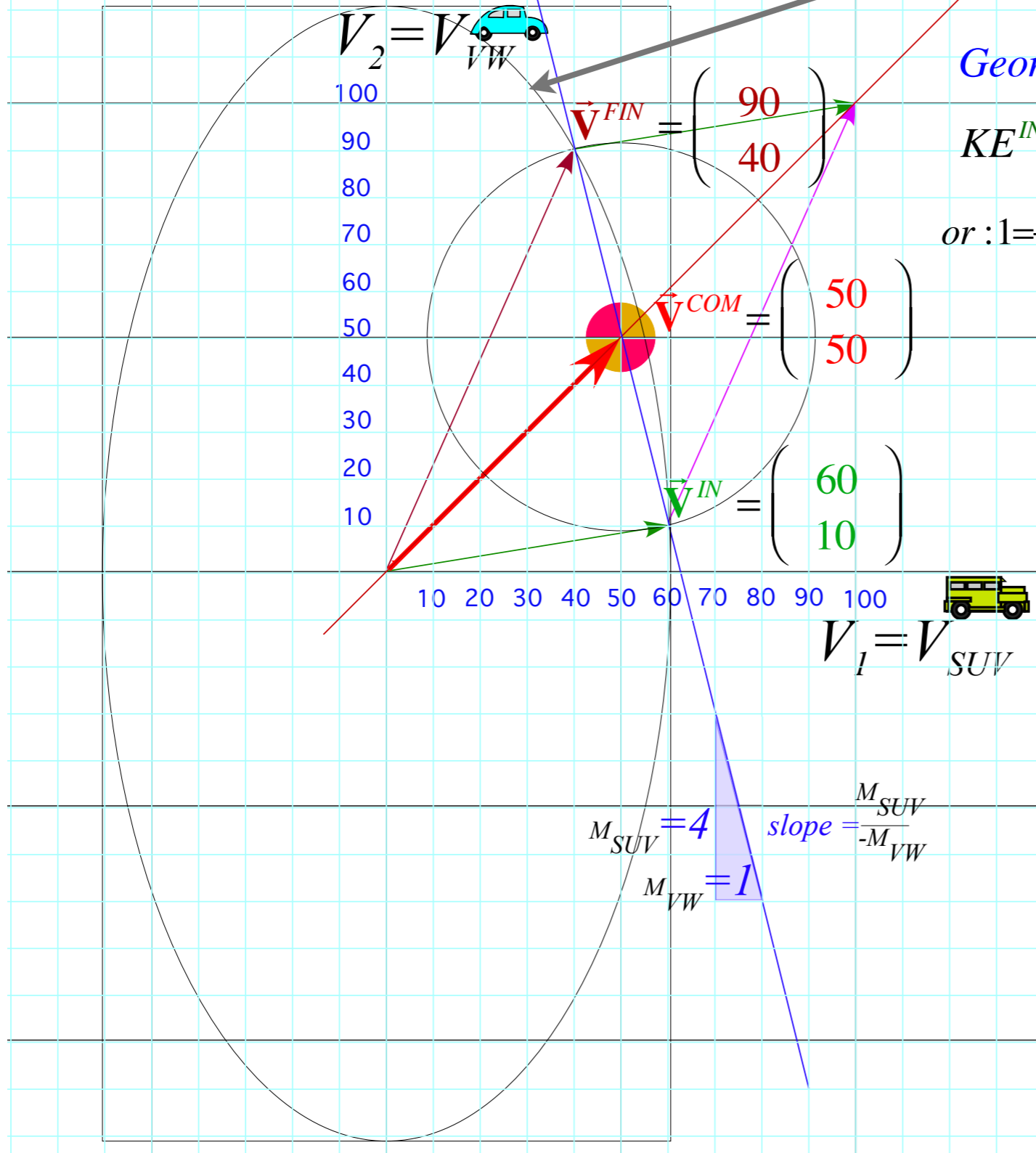
Geometry of Kinetic Energy Conservation Axiom-2/Theorem-1

$$(M_1+M_2)V^{COM} = M_1V_1^{IN} + M_2V_2^{IN} = M_1V_1^{FIN} + M_2V_2^{FIN} \quad KE^{IN} = \frac{1}{2}M_1(V_1^{IN})^2 + \frac{1}{2}M_2(V_2^{IN})^2 = \frac{1}{2}M_1(V_1^{FIN})^2 + \frac{1}{2}M_2(V_2^{FIN})^2 = KE^{FIN}$$

Momentum Conservation line: \rightarrow

$$M_1V_1^{IN} + M_2V_2^{IN} = \text{constant}$$

KE Conservation ellipse:



Geometry of KE Conservation Theorem -1

$$KE^{INorFIN} = \frac{1}{2}M_1(V_1)^2 + \frac{1}{2}M_2(V_2)^2 = \frac{1}{2}4(60)^2 + \frac{1}{2}1(10)^2 = 7,250$$

$$\text{or : } 1 = \frac{V_1^2}{\left(\frac{2 \cdot KE^{IFN}}{M_1}\right)} + \frac{V_2^2}{\left(\frac{2 \cdot KE^{IFN}}{M_2}\right)} = \frac{V_1^2}{\left(\sqrt{\frac{2 \cdot KE^{IFN}}{M_1}}\right)^2} + \frac{V_2^2}{\left(\sqrt{\frac{2 \cdot KE^{IFN}}{M_2}}\right)^2}$$

Collision Web Simulator
Basic elastic Collision
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 Space vs Space
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Geometry of Momentum Conservation Axiom-1

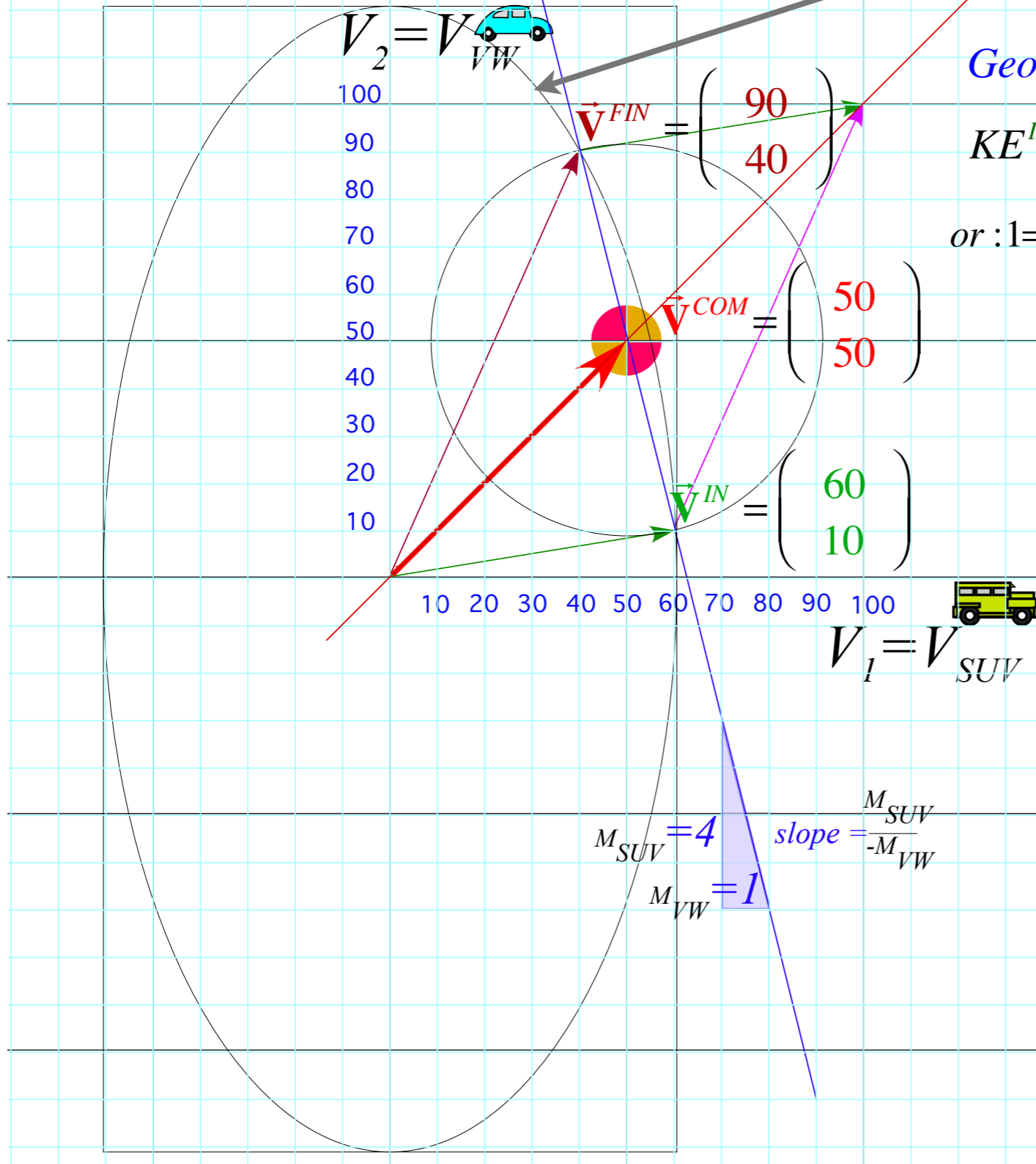
Geometry of Kinetic Energy Conservation Axiom-2/Theorem-1

$$(M_1+M_2)V^{COM} = M_1V_1^{IN} + M_2V_2^{IN} = M_1V_1^{FIN} + M_2V_2^{FIN} \quad KE^{IN} = \frac{1}{2}M_1(V_1^{IN})^2 + \frac{1}{2}M_2(V_2^{IN})^2 = \frac{1}{2}M_1(V_1^{FIN})^2 + \frac{1}{2}M_2(V_2^{FIN})^2 = KE^{FIN}$$

Momentum Conservation line: \rightarrow

$$M_1V_1^{IN} + M_2V_2^{IN} = \text{constant}$$

KE Conservation ellipse:



Geometry of KE Conservation Theorem -1

$$KE^{INorFIN} = \frac{1}{2}M_1(V_1)^2 + \frac{1}{2}M_2(V_2)^2 = \frac{1}{2}4(60)^2 + \frac{1}{2}1(10)^2 = 7,250$$

$$\text{or : } 1 = \frac{V_1^2}{\left(\frac{2 \cdot KE^{IFN}}{M_1}\right)} + \frac{V_2^2}{\left(\frac{2 \cdot KE^{IFN}}{M_2}\right)} = \frac{V_1^2}{\left(\sqrt{\frac{2 \cdot KE^{IFN}}{M_1}}\right)^2} + \frac{V_2^2}{\left(\sqrt{\frac{2 \cdot KE^{IFN}}{M_2}}\right)^2}$$

Has the algebraic form: $1 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

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Geometry of Momentum Conservation Axiom-1

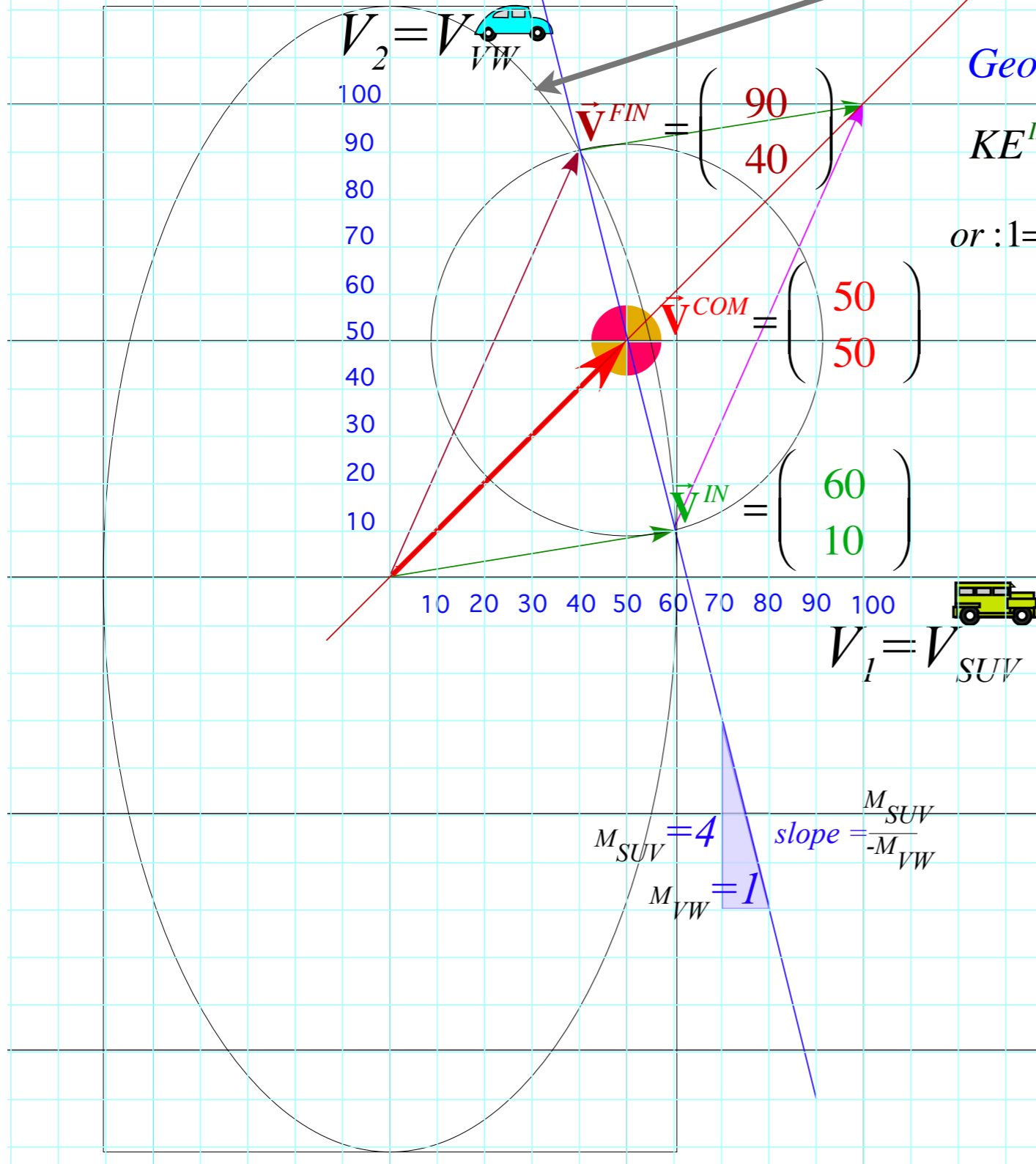
Geometry of Kinetic Energy Conservation Axiom-2/Theorem-1

$$(M_1+M_2)V^{COM} = M_1V_1^{IN} + M_2V_2^{IN} = M_1V_1^{FIN} + M_2V_2^{FIN} \quad KE^{IN} = \frac{1}{2}M_1(V_1^{IN})^2 + \frac{1}{2}M_2(V_2^{IN})^2 = \frac{1}{2}M_1(V_1^{FIN})^2 + \frac{1}{2}M_2(V_2^{FIN})^2 = KE^{FIN}$$

Momentum Conservation line: \rightarrow

$$M_1V_1^{IN} + M_2V_2^{IN} = \text{constant}$$

KE Conservation ellipse:



Geometry of KE Conservation Theorem -1

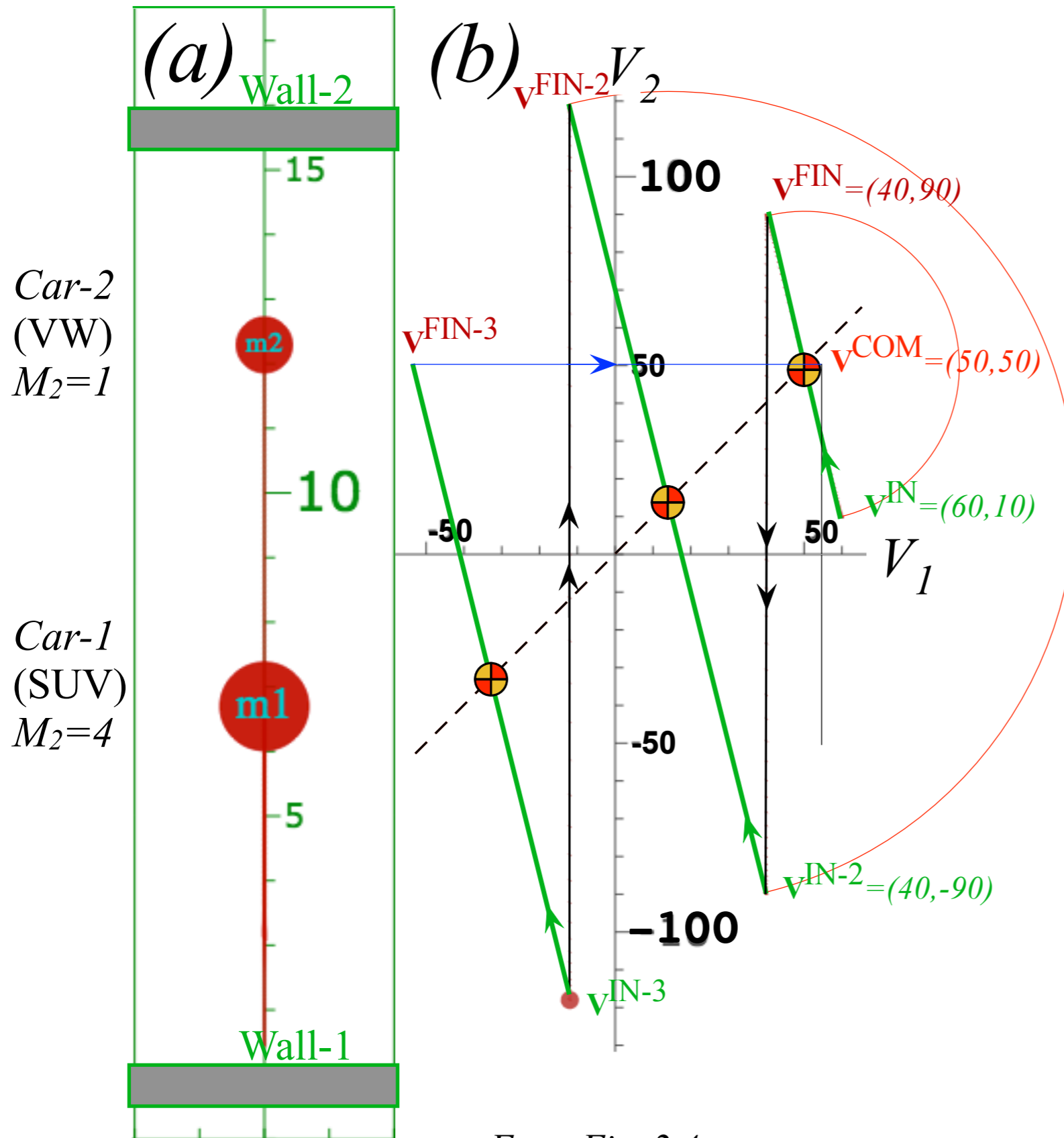
$$KE^{INorFIN} = \frac{1}{2}M_1(V_1)^2 + \frac{1}{2}M_2(V_2)^2 = \frac{1}{2}4(60)^2 + \frac{1}{2}1(10)^2 = 7,250$$

$$\text{or : } 1 = \frac{V_1^2}{\left(\frac{2 \cdot KE^{IFN}}{M_1}\right)} + \frac{V_2^2}{\left(\frac{2 \cdot KE^{IFN}}{M_2}\right)} = \frac{V_1^2}{\left(\sqrt{\frac{2 \cdot KE^{IFN}}{M_1}}\right)^2} + \frac{V_2^2}{\left(\sqrt{\frac{2 \cdot KE^{IFN}}{M_2}}\right)^2}$$

Has the algebraic form: $1 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

$$\begin{aligned} \text{elliptic radii : } a &= \sqrt{\frac{2KE^{INorFIN}}{M_1}} & b &= \sqrt{\frac{2KE^{INorFIN}}{M_2}} \\ &= \sqrt{\frac{2 \cdot 7,250}{4}} & &= \sqrt{\frac{2 \cdot 7,250}{1}} \\ &= 60.21 & &= 120.42 \end{aligned}$$

BounceIt Simulation: frictionless 1D-track with elastic bumper cars bouncing between walls



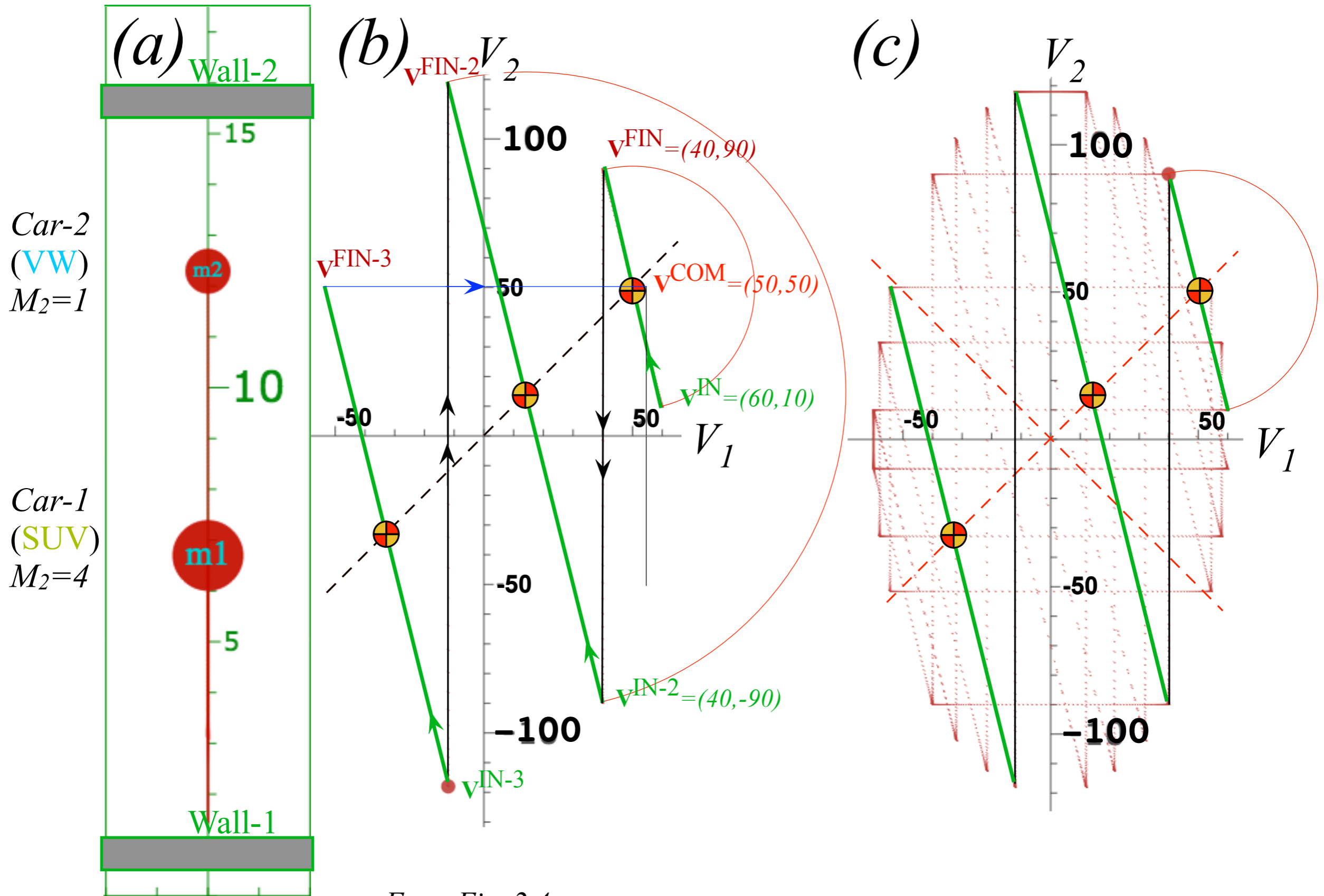
From Fig. 2.4

BounceIt
Superball Web Simulator
Repeated elastic Collisions
Dual Panel
Space vs Space
and
 $V(\text{VW})$ vs. $V(\text{SUV})$

Collision Web Simulator
Basic elastic Collision
Dual Panel
Space vs Space
and
 $V(\text{VW})$ vs. $V(\text{SUV})$

BounceIt
Superball Web Simulator
Basic elastic Collision
Dual Panel
Space vs Space
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 $V(\text{VW})$ vs. $V(\text{SUV})$

BounceIt Simulation: frictionless 1D-track with elastic bumper cars bouncing between walls



From Fig. 2.4

Transpose symmetry ($M_{jk} = M_{kj}$) of the \mathbf{M} -matrix implies: $\vec{V}^{FIN} \cdot \vec{M} \cdot \vec{V}^{IN} = \vec{V}^{IN} \cdot \vec{M} \cdot \vec{V}^{FIN}$

$$\begin{aligned} \begin{pmatrix} 40 & 90 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 60 \\ 10 \end{pmatrix} &= \begin{pmatrix} 60 & 10 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 40 \\ 90 \end{pmatrix} \\ &= 100 \cdot 105 = 100 \cdot 105 = 10,500 \end{aligned}$$

With ($M_{12} = 0 = M_{21}$) kinetic energy $KE^{INorFIN} = \frac{1}{2} \vec{V} \cdot \vec{M} \cdot \vec{V}$ is the same at $\mathbf{V} = \mathbf{V}^{IN}$ and $\mathbf{V} = \mathbf{V}^{FIN}$.

$$\begin{aligned} V^{COM} P_{Total} - \frac{\vec{V}^{FIN} \cdot \vec{M} \cdot \vec{V}^{IN}}{2} &= \frac{\vec{V}^{IN} \cdot \vec{M} \cdot \vec{V}^{IN}}{2} = \frac{\vec{V}^{FIN} \cdot \vec{M} \cdot \vec{V}^{FIN}}{2} = KE^{INorFIN} \\ 12,500 - \frac{10,500}{2} &= \frac{\begin{pmatrix} 60 & 10 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 60 \\ 10 \end{pmatrix}}{2} = \frac{\begin{pmatrix} 40 & 90 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 40 \\ 90 \end{pmatrix}}{2} = KE^{INorFIN} \\ 12,500 - 5,250 &= 7,250 = 7,250 \end{aligned}$$

Transpose symmetry ($M_{jk} = M_{kj}$) of the \mathbf{M} -matrix implies: $\vec{V}^{FIN} \cdot \vec{M} \cdot \vec{V}^{IN} = \vec{V}^{IN} \cdot \vec{M} \cdot \vec{V}^{FIN}$

$$\begin{aligned} \begin{pmatrix} 40 & 90 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 60 \\ 10 \end{pmatrix} &= \begin{pmatrix} 60 & 10 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 40 \\ 90 \end{pmatrix} \\ &= 100 \cdot 105 = 100 \cdot 105 = 10,500 \end{aligned}$$

With ($M_{12} = 0 = M_{21}$) kinetic energy $KE^{INorFIN} = \frac{1}{2} \vec{V} \cdot \vec{M} \cdot \vec{V}$ is the same at $\mathbf{V} = \mathbf{V}^{IN}$ and $\mathbf{V} = \mathbf{V}^{FIN}$.

$$\begin{aligned} V^{COM} P_{Total} - \frac{\vec{V}^{FIN} \cdot \vec{M} \cdot \vec{V}^{IN}}{2} &= \frac{\vec{V}^{IN} \cdot \vec{M} \cdot \vec{V}^{IN}}{2} = \frac{\vec{V}^{FIN} \cdot \vec{M} \cdot \vec{V}^{FIN}}{2} = KE^{INorFIN} \\ 12,500 - \frac{10,500}{2} &= \frac{\begin{pmatrix} 60 & 10 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 60 \\ 10 \end{pmatrix}}{2} = \frac{\begin{pmatrix} 40 & 90 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 40 \\ 90 \end{pmatrix}}{2} = KE^{INorFIN} \\ 12,500 - 5,250 &= 7,250 = 7,250 \end{aligned}$$

Consider kinetic energy KE^{COM} when $\mathbf{V} = \mathbf{V}^{COM}$. It is reduced by $1,000$ from $7,250$ to $6,250$.

$$\begin{aligned} KE^{COM} &= \frac{1}{2} \vec{V}^{COM} \cdot \vec{M} \cdot \vec{V}^{COM} = \frac{1}{2} \begin{pmatrix} 50 & 50 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 50 \\ 50 \end{pmatrix} \\ &= \frac{1}{2} (50 \cdot 4 \cdot 50 + 50 \cdot 1 \cdot 50) = 2 \cdot 2500 + \frac{1}{2} \cdot 2500 \\ &= 5000 + \frac{1}{2} \cdot 1250 = 6,250 \end{aligned}$$

Geometry of Momentum Conservation Axiom-1

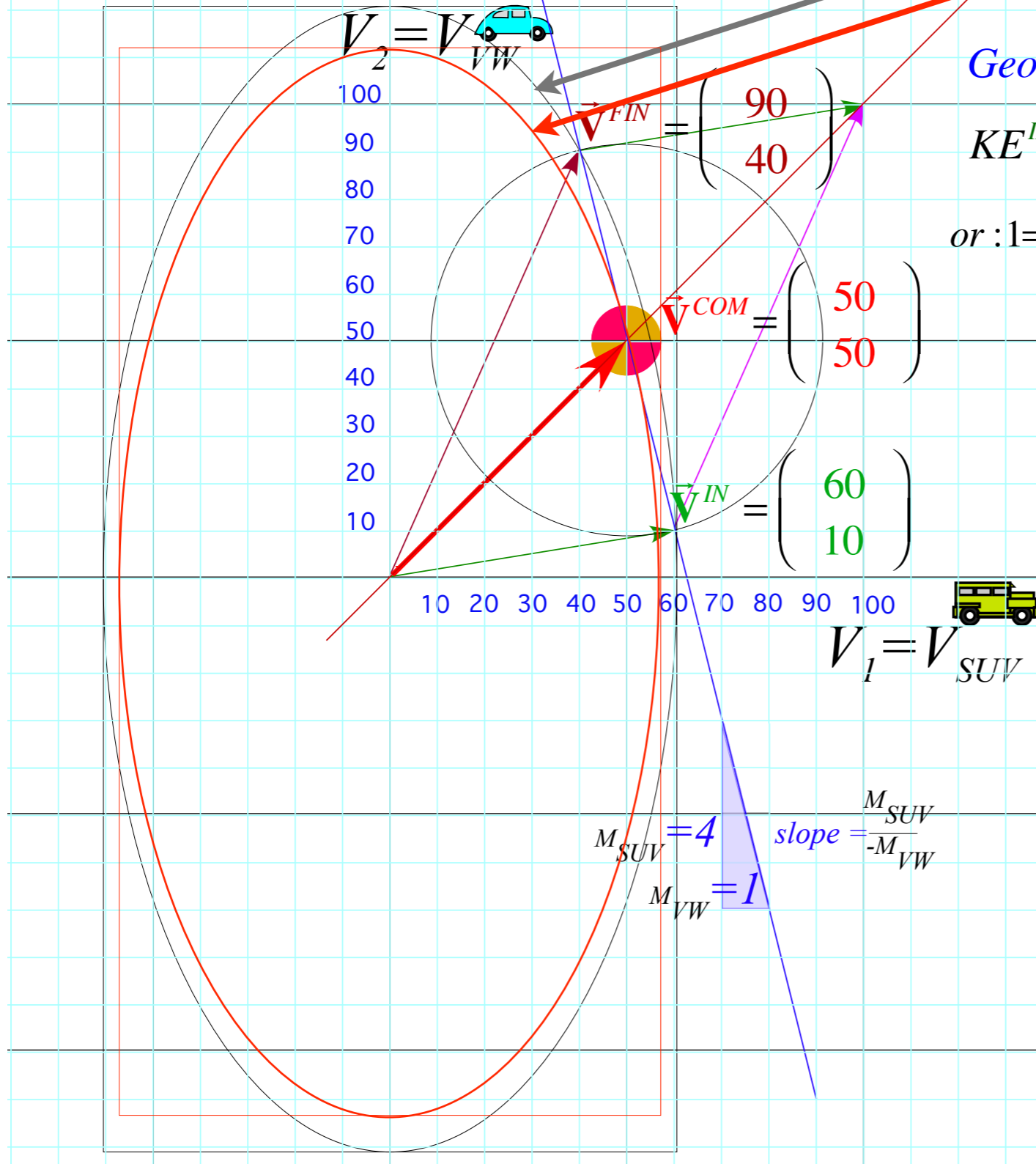
Geometry of Kinetic Energy Conservation Axiom-2/Theorem-1

$$(M_1+M_2)V^{COM} = M_1V_1^{IN} + M_2V_2^{IN} = M_1V_1^{FIN} + M_2V_2^{FIN} \quad KE^{IN} = \frac{1}{2}M_1(V_1^{IN})^2 + \frac{1}{2}M_2(V_2^{IN})^2 = \frac{1}{2}M_1(V_1^{FIN})^2 + \frac{1}{2}M_2(V_2^{FIN})^2 = KE^{FIN}$$

Momentum Conservation line: \rightarrow

$$M_1V_1^{IN} + M_2V_2^{IN} = \text{constant}$$

$KE^{INorFIN}$ Conservation ellipse:
 KE^{COM} Ka-runch ellipse:



Geometry of KE Conservation Theorem-1

$$KE^{INorFIN} = \frac{1}{2}M_1(V_1)^2 + \frac{1}{2}M_2(V_2)^2$$

$$\text{or : } 1 = \frac{V_1^2}{\left(\frac{2 \cdot KE^{IFN}}{M_1}\right)} + \frac{V_2^2}{\left(\frac{2 \cdot KE^{IFN}}{M_2}\right)} = \frac{V_1^2}{\left(\sqrt{\frac{2 \cdot KE^{IFN}}{M_1}}\right)^2} + \frac{V_2^2}{\left(\sqrt{\frac{2 \cdot KE^{IFN}}{M_2}}\right)^2}$$

Has the algebraic form: $1 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

Geometry of KE^{COM} at Center Of Momentum

$$KE^{COM} = \frac{1}{2}M_1(V_1^{COM})^2 + \frac{1}{2}M_2(V_2^{COM})^2$$

$$= \frac{1}{2}(M_1+M_2)(V^{COM})^2$$

elliptic radii: $a^{COM} = \sqrt{\frac{KE^{COM}}{M_1}} = \sqrt{\frac{2 \cdot 6,250}{4}} = 55.90$

$b^{COM} = \sqrt{\frac{KE^{COM}}{M_2}} = \sqrt{\frac{2 \cdot 6,250}{1}} = 111.80$

Transpose symmetry ($M_{jk} = M_{kj}$) of the \mathbf{M} -matrix implies: $\vec{V}^{FIN} \cdot \vec{M} \cdot \vec{V}^{IN} = \vec{V}^{IN} \cdot \vec{M} \cdot \vec{V}^{FIN}$

$$\begin{pmatrix} 40 & 90 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 60 \\ 10 \end{pmatrix} = \begin{pmatrix} 60 & 10 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 40 \\ 90 \end{pmatrix}$$

$$= 100 \cdot 105 = 100 \cdot 105 = 10,500$$

With ($M_{12} = 0 = M_{21}$) kinetic energy $KE^{INorFIN} = \frac{1}{2} \vec{V} \cdot \vec{M} \cdot \vec{V}$ is the same at $\mathbf{V} = \mathbf{V}^{IN}$ and $\mathbf{V} = \mathbf{V}^{FIN}$.

$$V^{COM} P_{Total} - \frac{\vec{V}^{FIN} \cdot \vec{M} \cdot \vec{V}^{IN}}{2} = \frac{\vec{V}^{IN} \cdot \vec{M} \cdot \vec{V}^{IN}}{2} = \frac{\vec{V}^{FIN} \cdot \vec{M} \cdot \vec{V}^{FIN}}{2} = KE^{INorFIN}$$

$$12,500 - \frac{10,500}{2} = \frac{\begin{pmatrix} 60 & 10 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 60 \\ 10 \end{pmatrix}}{2} = \frac{\begin{pmatrix} 40 & 90 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 40 \\ 90 \end{pmatrix}}{2} = KE^{INorFIN}$$

$$12,500 - 5,250 = 7,250 = 7,250 = KE^{INorFIN}$$

Consider kinetic energy KE^{COM} when $\mathbf{V} = \mathbf{V}^{COM}$. It is reduced by $1,000$ from $7,250$ to $6,250$.

$$KE^{COM} = V^{COM} P_{Total} - \frac{\vec{V}^{COM} \cdot \vec{M} \cdot \vec{V}^{COM}}{2} = \frac{1}{2} V^{COM} P_{Total} = \frac{\vec{V}^{COM} \cdot \vec{M} \cdot \vec{V}^{COM}}{2} = \frac{\vec{V}^{IN} \cdot \vec{M} \cdot \vec{V}^{IN}}{4} + \frac{\vec{V}^{FIN} \cdot \vec{M} \cdot \vec{V}^{FIN}}{4} = \frac{1}{2} KE^{INorFIN} + \frac{\vec{V}^{FIN} \cdot \vec{M} \cdot \vec{V}^{IN}}{4}$$

$$12,500 - \frac{12,500}{2} = 6,250 = 6,250 = 3,625 + 2,625 = 3,625 + 2,625$$

$$KE^{COM} = \frac{1}{2} \vec{V}^{COM} \cdot \vec{M} \cdot \vec{V}^{COM} = \frac{1}{2} \begin{pmatrix} 50 & 50 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 50 \\ 50 \end{pmatrix}$$

$$= \frac{1}{2} (50 \cdot 4 \cdot 50 + 50 \cdot 1 \cdot 50) = 2 \cdot 2500 + \frac{1}{2} \cdot 2500$$

$$= 5000 + \frac{1}{2} \cdot 1250 = 6,250$$

Transpose symmetry ($M_{jk} = M_{kj}$) of the \mathbf{M} -matrix implies: $\vec{V}^{FIN} \cdot \vec{M} \cdot \vec{V}^{IN} = \vec{V}^{IN} \cdot \vec{M} \cdot \vec{V}^{FIN}$

$$\begin{pmatrix} 40 & 90 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 60 \\ 10 \end{pmatrix} = \begin{pmatrix} 60 & 10 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 40 \\ 90 \end{pmatrix}$$

$$= 100 \cdot 105 = 100 \cdot 105 = 10,500$$

With ($M_{12} = 0 = M_{21}$) kinetic energy $KE^{INorFIN} = \frac{1}{2} \vec{V} \cdot \vec{M} \cdot \vec{V}$ is the same at $\mathbf{V} = \mathbf{V}^{IN}$ and $\mathbf{V} = \mathbf{V}^{FIN}$.

$$V^{COM} P_{Total} - \frac{\vec{V}^{FIN} \cdot \vec{M} \cdot \vec{V}^{IN}}{2} = \frac{\vec{V}^{IN} \cdot \vec{M} \cdot \vec{V}^{IN}}{2} = \frac{\vec{V}^{FIN} \cdot \vec{M} \cdot \vec{V}^{FIN}}{2} = KE^{INorFIN}$$

$$12,500 - \frac{10,500}{2} = \frac{\begin{pmatrix} 60 & 10 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 60 \\ 10 \end{pmatrix}}{2} = \frac{\begin{pmatrix} 40 & 90 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 40 \\ 90 \end{pmatrix}}{2} = KE^{INorFIN}$$

$$12,500 - 5,250 = 7,250 = 7,250$$

Consider kinetic energy KE^{COM} when $\mathbf{V} = \mathbf{V}^{COM}$. It is reduced by $1,000$ from $7,250$ to $6,250$.

$$KE^{COM} = V^{COM} P_{Total} - \frac{\vec{V}^{COM} \cdot \vec{M} \cdot \vec{V}^{COM}}{2} = \frac{1}{2} V^{COM} P_{Total} = \frac{\vec{V}^{COM} \cdot \vec{M} \cdot \vec{V}^{COM}}{2} = \frac{\vec{V}^{IN} \cdot \vec{M} \cdot \vec{V}^{IN}}{4} + \frac{\vec{V}^{FIN} \cdot \vec{M} \cdot \vec{V}^{FIN}}{4} = \frac{1}{2} KE^{INorFIN} + \frac{\vec{V}^{FIN} \cdot \vec{M} \cdot \vec{V}^{IN}}{4}$$

$$12,500 - \frac{12,500}{2} = 6,250 = 6,250 = 3,625 + 2,625 = 3,625 + 2,625$$

$$KE^{COM} = \frac{1}{2} \vec{V}^{COM} \cdot \vec{M} \cdot \vec{V}^{COM} = \frac{1}{2} \begin{pmatrix} 50 & 50 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 50 \\ 50 \end{pmatrix}$$

$$= \frac{1}{2} (50 \cdot 4 \cdot 50 + 50 \cdot 1 \cdot 50) = 2 \cdot 2500 + \frac{1}{2} \cdot 2500$$

$$= 5000 + \frac{1}{2} \cdot 1250 = 6,250$$

Introducing
Potential Energy = PE

Difference is inelastic “ka-Runch” $KE^{INorFIN} - KE^{COM}$. For elastic “ka-Bong” the $1,000$ is PE^{COM} of compression.

$$KE^{INorFIN} - KE^{COM} = \frac{\vec{V}^{IN} \cdot \vec{M} \cdot \vec{V}^{IN}}{4} - \frac{\vec{V}^{FIN} \cdot \vec{M} \cdot \vec{V}^{IN}}{4}$$

$$1,000 = 3,625 - 2,625$$

Transpose symmetry ($M_{jk} = M_{kj}$) of the \mathbf{M} -matrix implies: $\vec{V}^{FIN} \cdot \vec{M} \cdot \vec{V}^{IN} = \vec{V}^{IN} \cdot \vec{M} \cdot \vec{V}^{FIN}$

$$\begin{pmatrix} 40 & 90 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 60 \\ 10 \end{pmatrix} = \begin{pmatrix} 60 & 10 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 40 \\ 90 \end{pmatrix}$$

$$= 100 \cdot 105 = 100 \cdot 105 = 10,500$$

With ($M_{12} = 0 = M_{21}$) kinetic energy $KE^{INorFIN} = \frac{1}{2} \vec{V} \cdot \vec{M} \cdot \vec{V}$ is the same at $\mathbf{V} = \mathbf{V}^{IN}$ and $\mathbf{V} = \mathbf{V}^{FIN}$.

$$V^{COM} P_{Total} - \frac{\vec{V}^{FIN} \cdot \vec{M} \cdot \vec{V}^{IN}}{2} = \frac{\vec{V}^{IN} \cdot \vec{M} \cdot \vec{V}^{IN}}{2} = \frac{\vec{V}^{FIN} \cdot \vec{M} \cdot \vec{V}^{FIN}}{2} = KE^{INorFIN}$$

$$12,500 - \frac{10,500}{2} = \frac{\begin{pmatrix} 60 & 10 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 60 \\ 10 \end{pmatrix}}{2} = \frac{\begin{pmatrix} 40 & 90 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 40 \\ 90 \end{pmatrix}}{2} = KE^{INorFIN}$$

$$12,500 - 5,250 = 7,250 = 7,250 = KE^{INorFIN}$$

Consider kinetic energy KE^{COM} when $\mathbf{V} = \mathbf{V}^{COM}$. It is reduced by $1,000$ from $7,250$ to $6,250$.

$$KE^{COM} = V^{COM} P_{Total} - \frac{\vec{V}^{COM} \cdot \vec{M} \cdot \vec{V}^{COM}}{2} = \frac{1}{2} V^{COM} P_{Total} = \frac{\vec{V}^{COM} \cdot \vec{M} \cdot \vec{V}^{COM}}{2} = \frac{\vec{V}^{IN} \cdot \vec{M} \cdot \vec{V}^{IN}}{4} + \frac{\vec{V}^{FIN} \cdot \vec{M} \cdot \vec{V}^{FIN}}{4} = \frac{1}{2} KE^{INorFIN} + \frac{\vec{V}^{FIN} \cdot \vec{M} \cdot \vec{V}^{IN}}{4}$$

$$12,500 - \frac{12,500}{2} = 6,250 = 6,250 = 3,625 + 2,625 = 3,625 + 2,625$$

$$KE^{COM} = \frac{1}{2} \vec{V}^{COM} \cdot \vec{M} \cdot \vec{V}^{COM} = \frac{1}{2} \begin{pmatrix} 50 & 50 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 50 \\ 50 \end{pmatrix}$$

$$= \frac{1}{2} (50 \cdot 4 \cdot 50 + 50 \cdot 1 \cdot 50) = 2 \cdot 2500 + \frac{1}{2} \cdot 2500$$

$$= 5000 + \frac{1}{2} \cdot 1250 = 6,250$$

Introducing
Potential Energy = PE

Difference is inelastic “*ka-Runch*” $KE^{INorFIN} - KE^{COM}$. For elastic “*ka-Bong*” the $1,000$ is PE^{COM} of compression.

$$KE^{INorFIN} - KE^{COM} = \frac{\vec{V}^{IN} \cdot \vec{M} \cdot \vec{V}^{IN}}{4} - \frac{\vec{V}^{FIN} \cdot \vec{M} \cdot \vec{V}^{IN}}{4}$$

$$1,000 = 3,625 - 2,625$$

$$KE^{COM} = \frac{\vec{V}^{IN} \cdot \vec{M} \cdot \vec{V}^{IN}}{4} + \frac{\vec{V}^{FIN} \cdot \vec{M} \cdot \vec{V}^{IN}}{4}$$

$$6,250 = 3,625 + 2,625$$

Difference $KE^{INorFIN} - KE^{COM} = 1,000$ is the same in *all* frames including *COM*-frame where $\mathbf{V}^{COM} = \mathbf{0}$.

Geometry of Momentum Conservation Axiom-1

Geometry of Kinetic Energy Conservation Axiom-2/Theorem-1

$$(M_1+M_2)V^{COM} = M_1V_1^{IN} + M_2V_2^{IN} = M_1V_1^{FIN} + M_2V_2^{FIN} \quad KE^{IN} = \frac{1}{2}M_1(V_1^{IN})^2 + \frac{1}{2}M_2(V_2^{IN})^2 = \frac{1}{2}M_1(V_1^{FIN})^2 + \frac{1}{2}M_2(V_2^{FIN})^2 = KE^{FIN}$$

Momentum Conservation line: \rightarrow

$$M_1V^{IN} + M_2V^{IN} = \text{constant}$$

$KE^{INorFIN}$ Conservation ellipse:
 KE^{COM} Ka-runch ellipse:

Geometry of KE Conservation Theorem -1

$$KE^{INorFIN} = \frac{1}{2}M_1(V_1)^2 + \frac{1}{2}M_2(V_2)^2$$

$$\text{or : } 1 = \frac{V_1^2}{\left(\frac{2 \cdot KE^{IFN}}{M_1}\right)} + \frac{V_2^2}{\left(\frac{2 \cdot KE^{IFN}}{M_2}\right)} = \frac{V_1^2}{\left(\sqrt{\frac{2 \cdot KE^{IFN}}{M_1}}\right)^2} + \frac{V_2^2}{\left(\sqrt{\frac{2 \cdot KE^{IFN}}{M_2}}\right)^2}$$

Has the algebraic form: $1 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

Geometry of KE^{COM} at Center Of Momentum

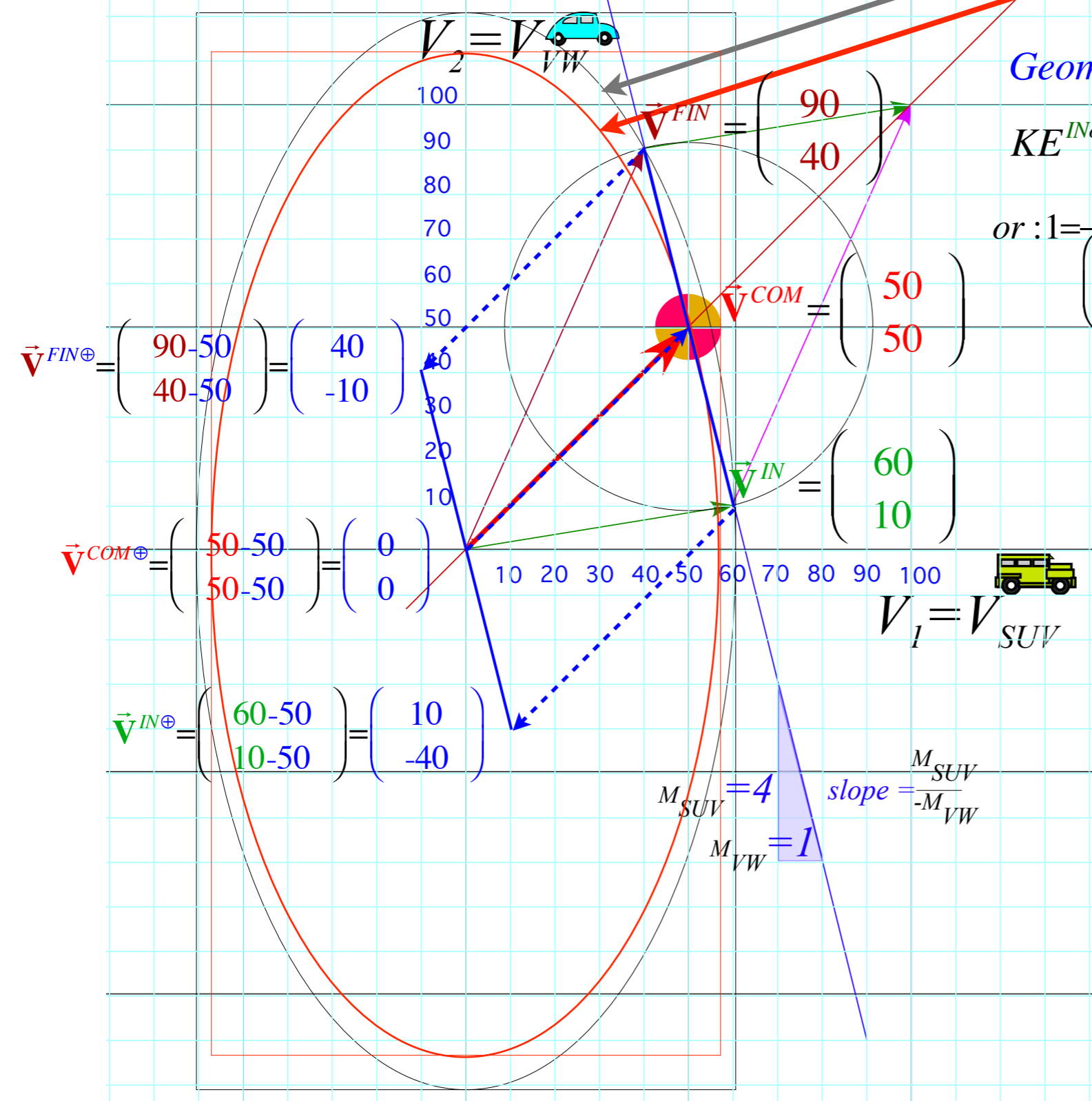
$$KE^{COM} = \frac{1}{2}M_1(V_1^{COM})^2 + \frac{1}{2}M_2(V_2^{COM})^2$$

$$= \frac{1}{2}(M_1+M_2)(V^{COM})^2$$

elliptic radii: $a^{COM} = \sqrt{\frac{KE^{COM}}{M_1}}$ $b^{COM} = \sqrt{\frac{KE^{COM}}{M_2}}$

$$= \sqrt{\frac{2 \cdot 6,250}{4}} \quad = \sqrt{\frac{2 \cdot 6,250}{1}}$$

$$= 55.90 \quad = 111.80$$



Geometry of Momentum Conservation Axiom-1

Geometry of Kinetic Energy Conservation Axiom-2/Theorem-1

$$(M_1+M_2)V^{COM} = M_1V_1^{IN} + M_2V_2^{IN} = M_1V_1^{FIN} + M_2V_2^{FIN} \quad KE^{IN} = \frac{1}{2}M_1(V_1^{IN})^2 + \frac{1}{2}M_2(V_2^{IN})^2 = \frac{1}{2}M_1(V_1^{FIN})^2 + \frac{1}{2}M_2(V_2^{FIN})^2 = KE^{FIN}$$

Momentum Conservation line: \rightarrow

$$M_1V^{IN} + M_2V^{IN} = \text{constant}$$

$KE^{INorFIN}$ Conservation ellipse:
 KE^{COM} Ka-runch ellipse:

Geometry of KE Conservation Theorem-1

$$KE^{INorFIN} = \frac{1}{2}M_1(V_1)^2 + \frac{1}{2}M_2(V_2)^2$$

$$\text{or : } 1 = \frac{V_1^2}{\left(\frac{2 \cdot KE^{IFN}}{M_1}\right)} + \frac{V_2^2}{\left(\frac{2 \cdot KE^{IFN}}{M_2}\right)} = \frac{V_1^2}{\left(\sqrt{\frac{2 \cdot KE^{IFN}}{M_1}}\right)^2} + \frac{V_2^2}{\left(\sqrt{\frac{2 \cdot KE^{IFN}}{M_2}}\right)^2}$$

Has the algebraic form: $1 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

Geometry of KE^{COM} at Center Of Momentum

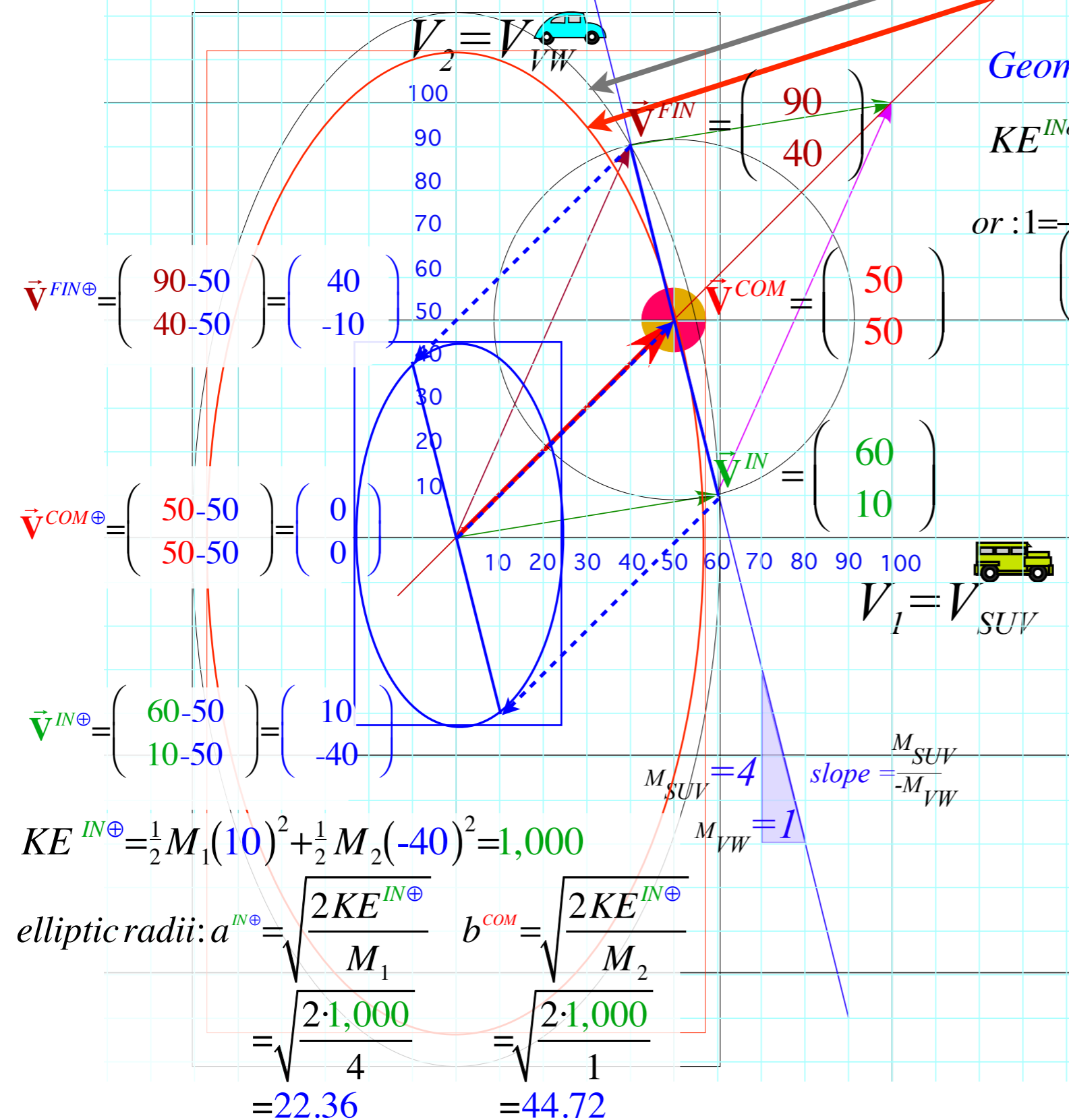
$$KE^{COM} = \frac{1}{2}M_1(V_1^{COM})^2 + \frac{1}{2}M_2(V_2^{COM})^2$$

$$= \frac{1}{2}(M_1+M_2)(V^{COM})^2$$

elliptic radii: $a^{COM} = \sqrt{\frac{KE^{COM}}{M_1}}$ $b^{COM} = \sqrt{\frac{KE^{COM}}{M_2}}$

$$= \sqrt{\frac{2 \cdot 6,250}{4}} \quad = \sqrt{\frac{2 \cdot 6,250}{1}}$$

$$= 55.90 \quad = 111.80$$



$$KE^{IN\oplus} = \frac{1}{2}M_1(10)^2 + \frac{1}{2}M_2(-40)^2 = 1,000$$

elliptic radii: $a^{IN\oplus} = \sqrt{\frac{2KE^{IN\oplus}}{M_1}}$ $b^{COM} = \sqrt{\frac{2KE^{IN\oplus}}{M_2}}$

$$= \sqrt{\frac{2 \cdot 1,000}{4}} \quad = \sqrt{\frac{2 \cdot 1,000}{1}}$$

$$= 22.36 \quad = 44.72$$

Developing
Conservation-of-Momentum
 The key axiom of mechanics
 leading to
Conservation-of-Energy Theorem

If and only if...
 there is **T-Symmetry**

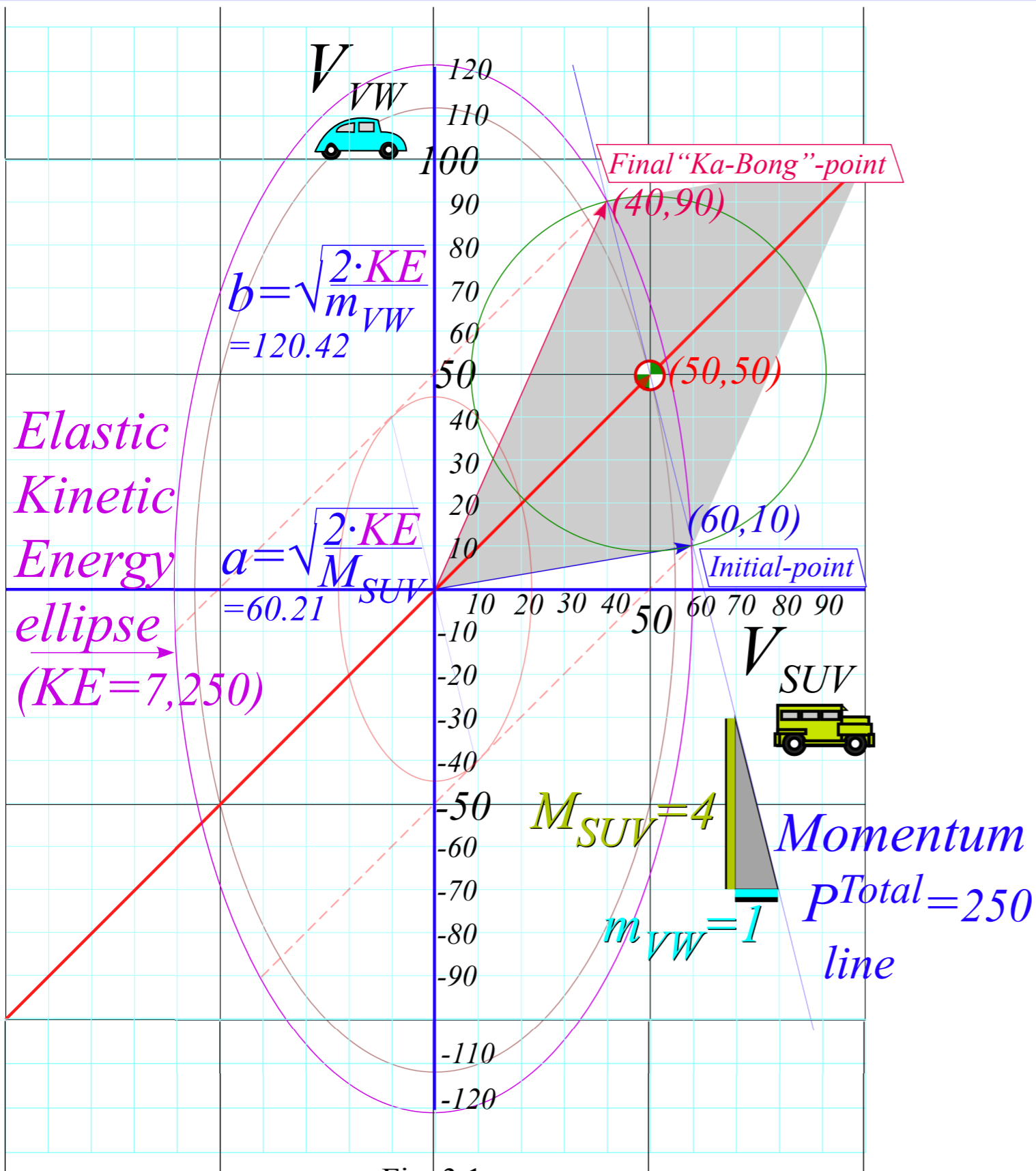


Fig. 3.1 a
 in Unit 1

Fig. 3.1

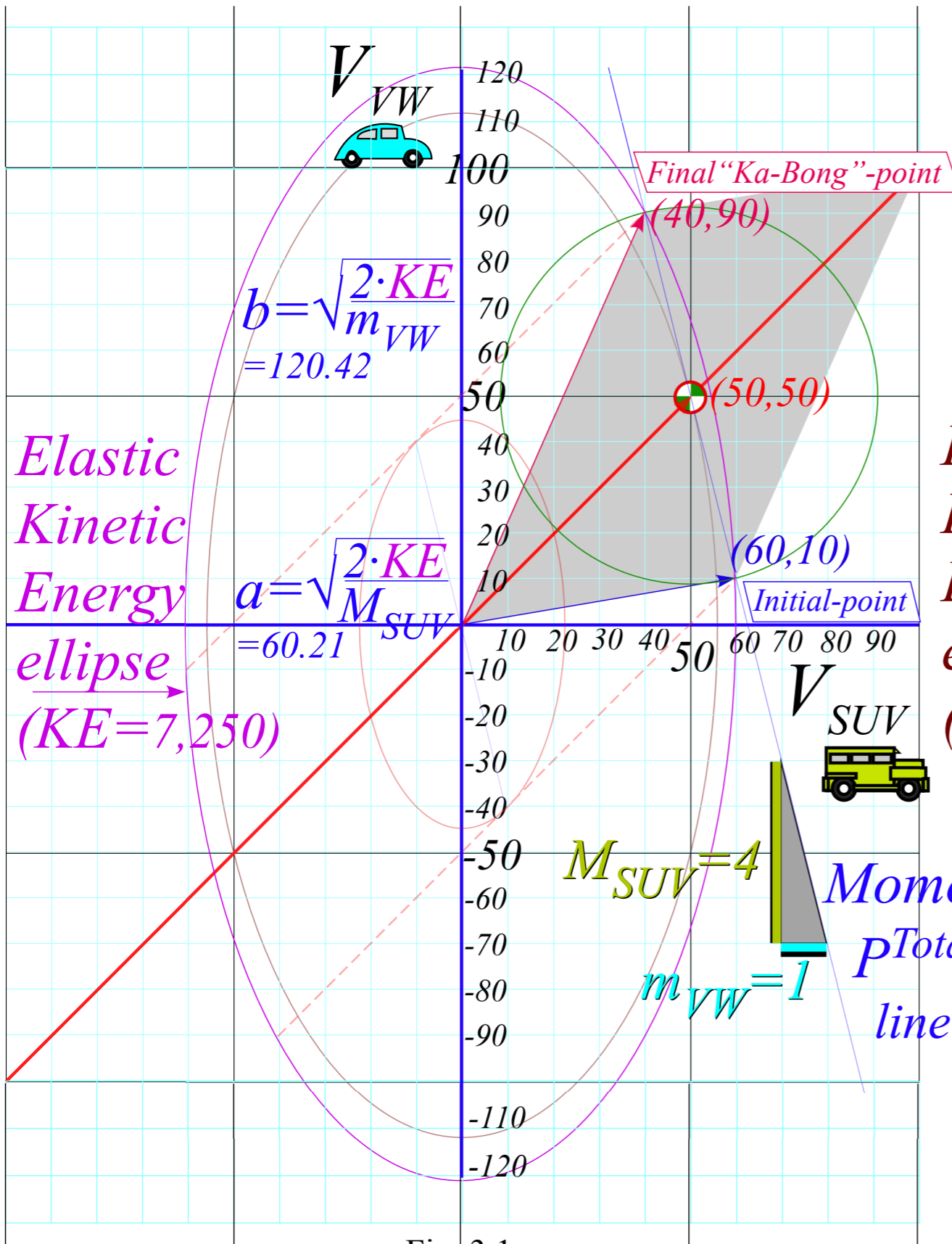


Fig. 3.1 a
in Unit 1

Inelastic Kinetic Energy ellipse ($IE = 6,250$)

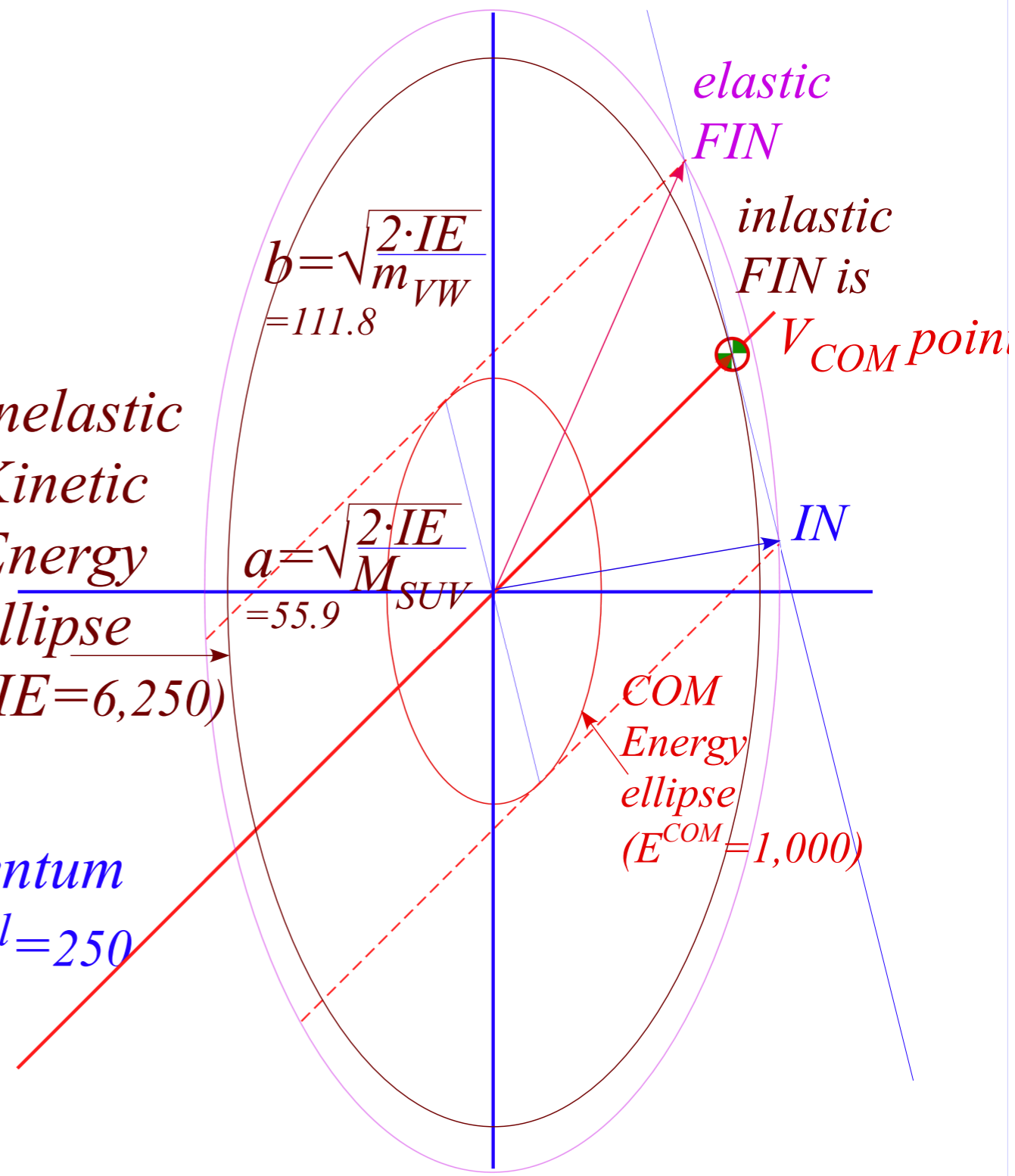


Fig. 3.1 b
in Unit 1

Fig. 3.1

As usual in physics, opposite extremes are easier to analyze than the generic “real(er) world” in between!

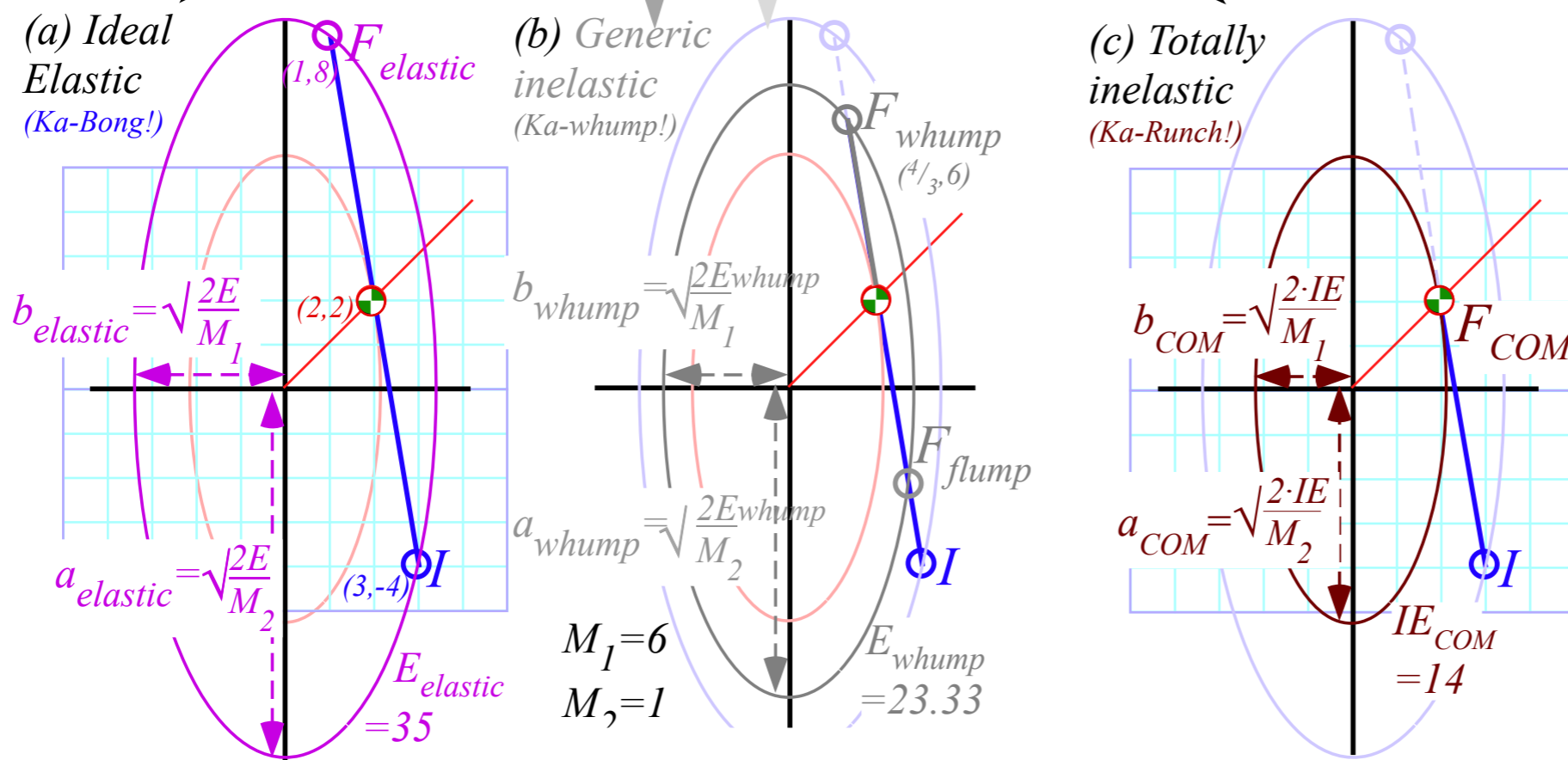


Fig. 2.3 (6-Ton SUV)

(During Bush II era an SUV with a mass of at least 6 tons allowed its owner to take a 100% write-off (up to \$100,000) on Federal Income Tax.)

Here T-Symmetry is best

Here T-Symmetry is less

Here T-Symmetry is least

Graph paper facilitates construction of energy ellipses given the two radii a and b in KE equation.

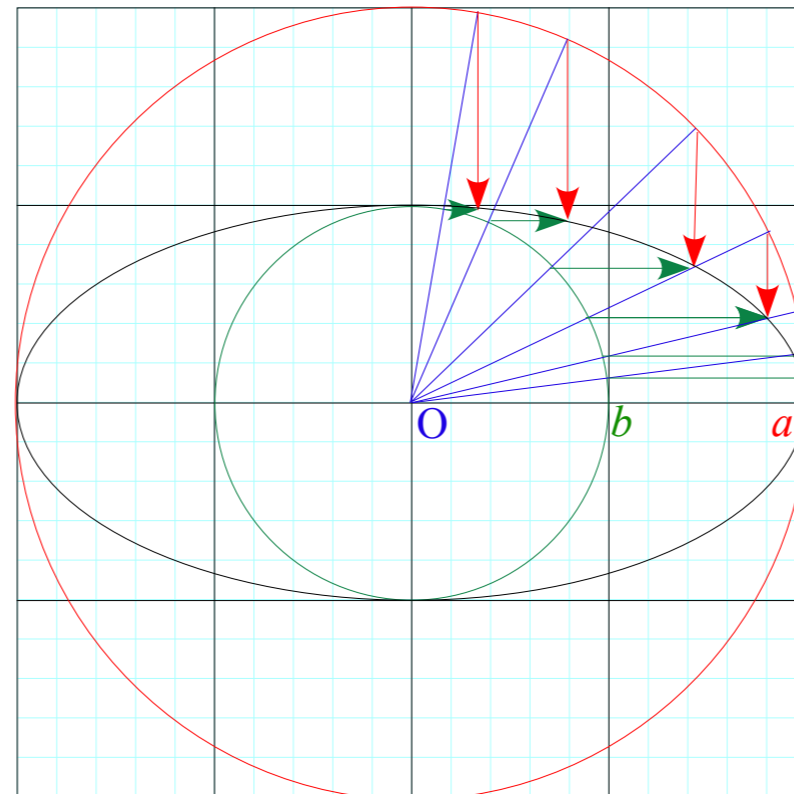
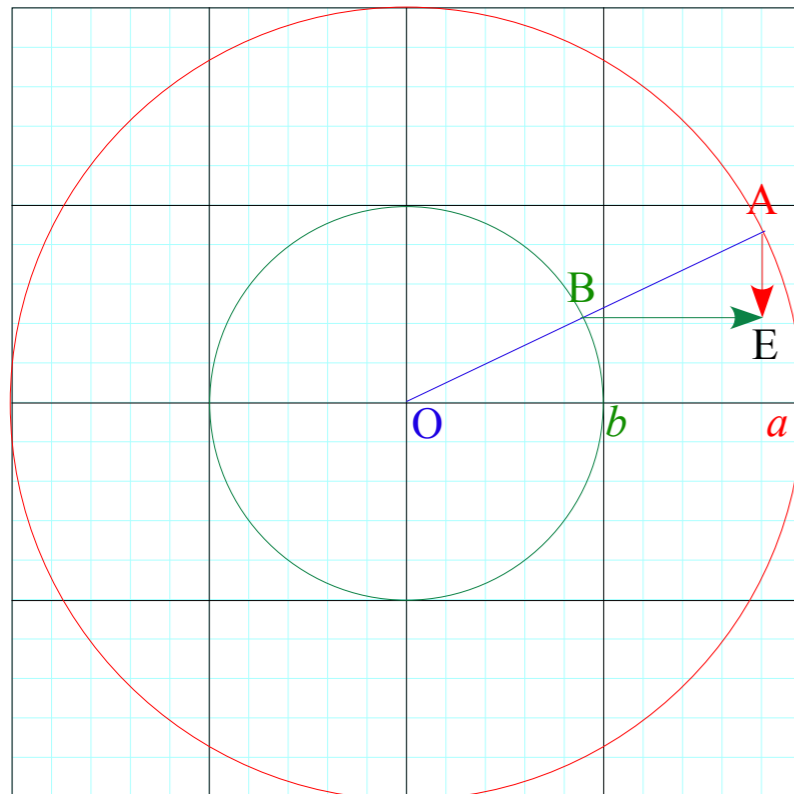
First step: draw concentric circles of radius a and b .

Then any radial line OBA “points” to point E on the ellipse.

Ellipse point E lies at the intersection of a vertical line AE thru radial intersection A with circle a and a horizontal line BE thru radial intersection B with circle b .

Graph grid helps locate E for a radius OBA , and usually there is no need to draw AE or BE .

You can pick x and find y or else *vice-versa*.

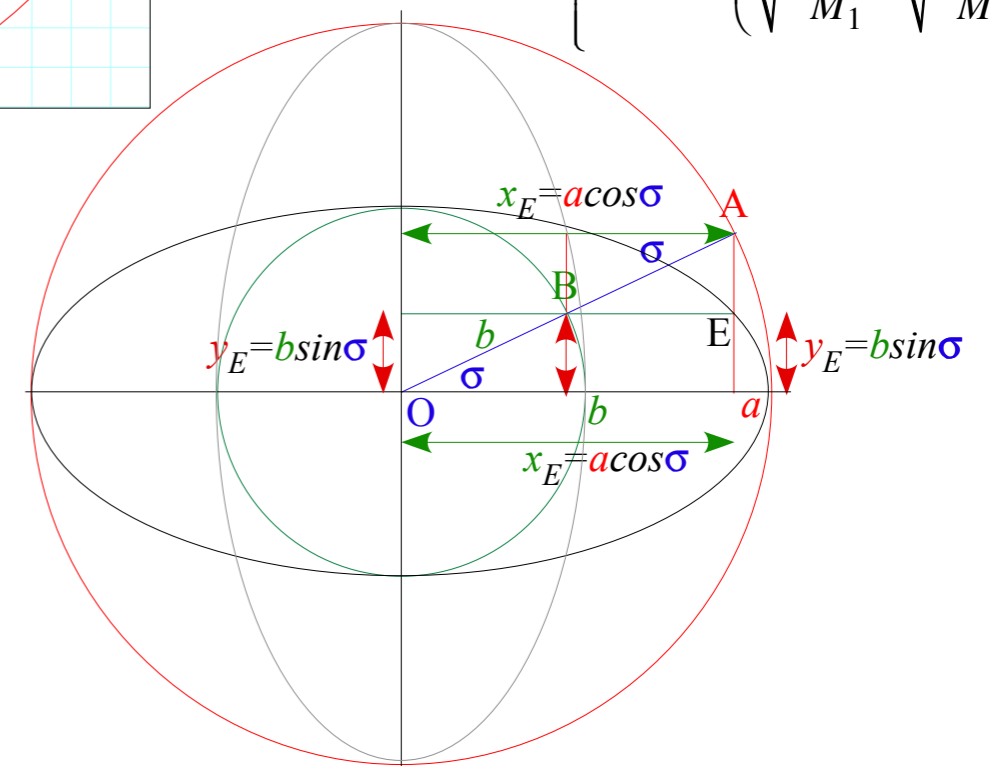


$$\frac{1}{2}M_1 \cdot V_1^2 + \frac{1}{2}M_2 \cdot V_2^2 = KE$$

$$\frac{V_1^2}{\left(\frac{2 \cdot KE}{M_1}\right)} + \frac{V_2^2}{\left(\frac{2 \cdot KE}{M_2}\right)} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\begin{cases} (x,y) = (V_1, V_2) \\ (a,b) = \left(\sqrt{\frac{2 \cdot KE}{M_1}}, \sqrt{\frac{2 \cdot KE}{M_2}} \right) \end{cases}$$

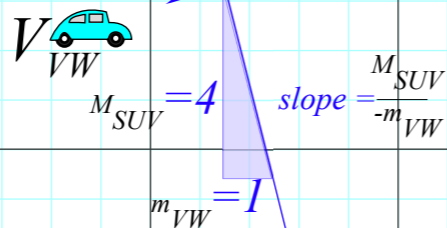


Ellipse coordinates $(x_E = a \cdot \cos \sigma, y_E = b \cdot \sin \sigma)$ are rescaled base and altitude

$(x_r = r \cdot \cos \sigma, y_r = r \cdot \sin \sigma)$ of Fig. 2.6.

Algebra, Geometry, and Physics of Momentum Conservation Axiom

Conservation of momentum line: \rightarrow (...one of ∞ -many...)



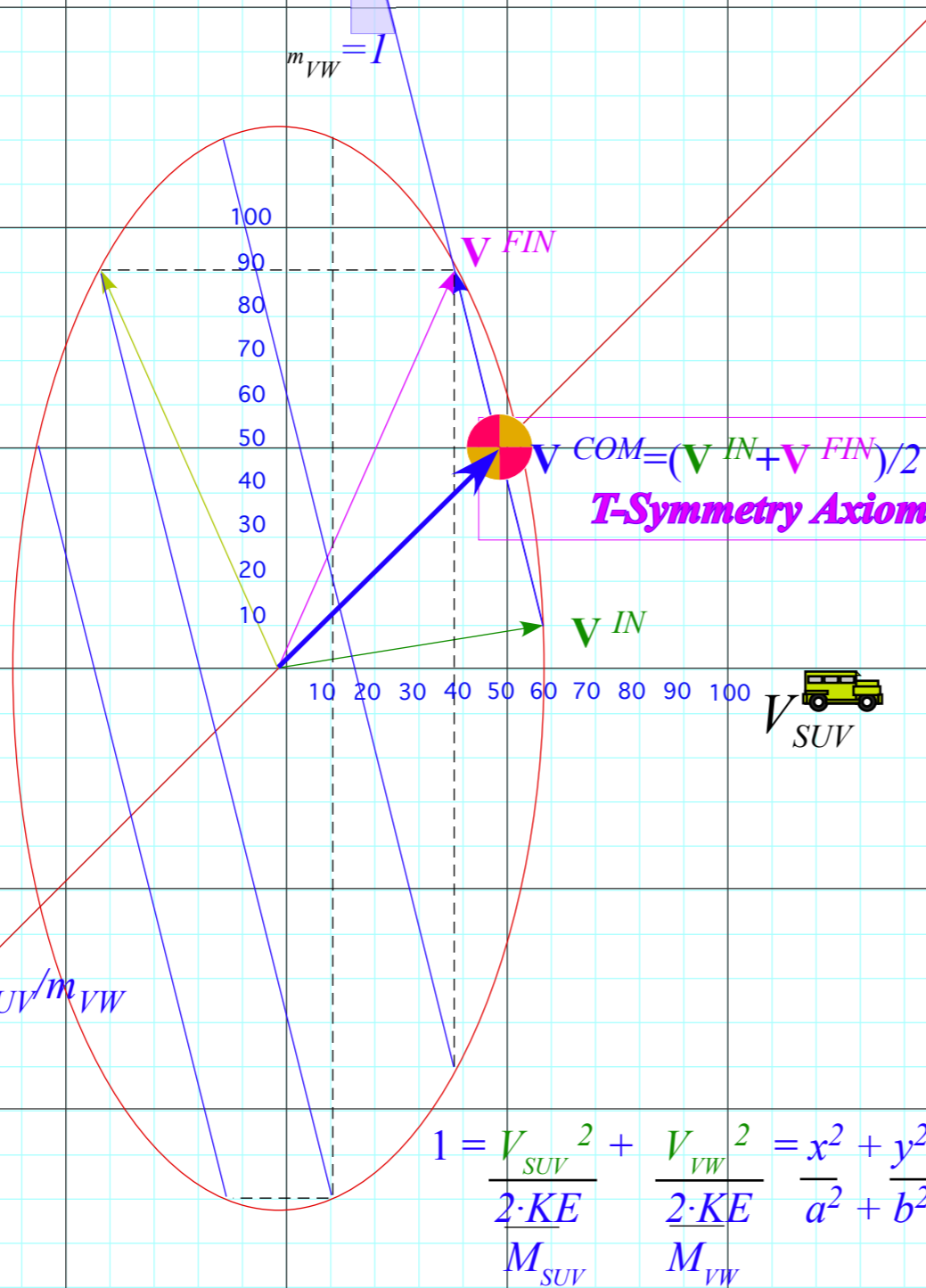
Momentum Conservation Axiom

plus

T-Symmetry Axiom
($M=M^T$ implied)

gives

Kinetic Energy Conservation Theorem



All lines of slope $-M_{SUV}/m_{VW}$
...are bisected by the
(slope=1)-COM line

$$1 = \frac{V_{SUV}^2}{\frac{2 \cdot KE}{M_{SUV}}} + \frac{V_{VW}^2}{\frac{2 \cdot KE}{M_{VW}}} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

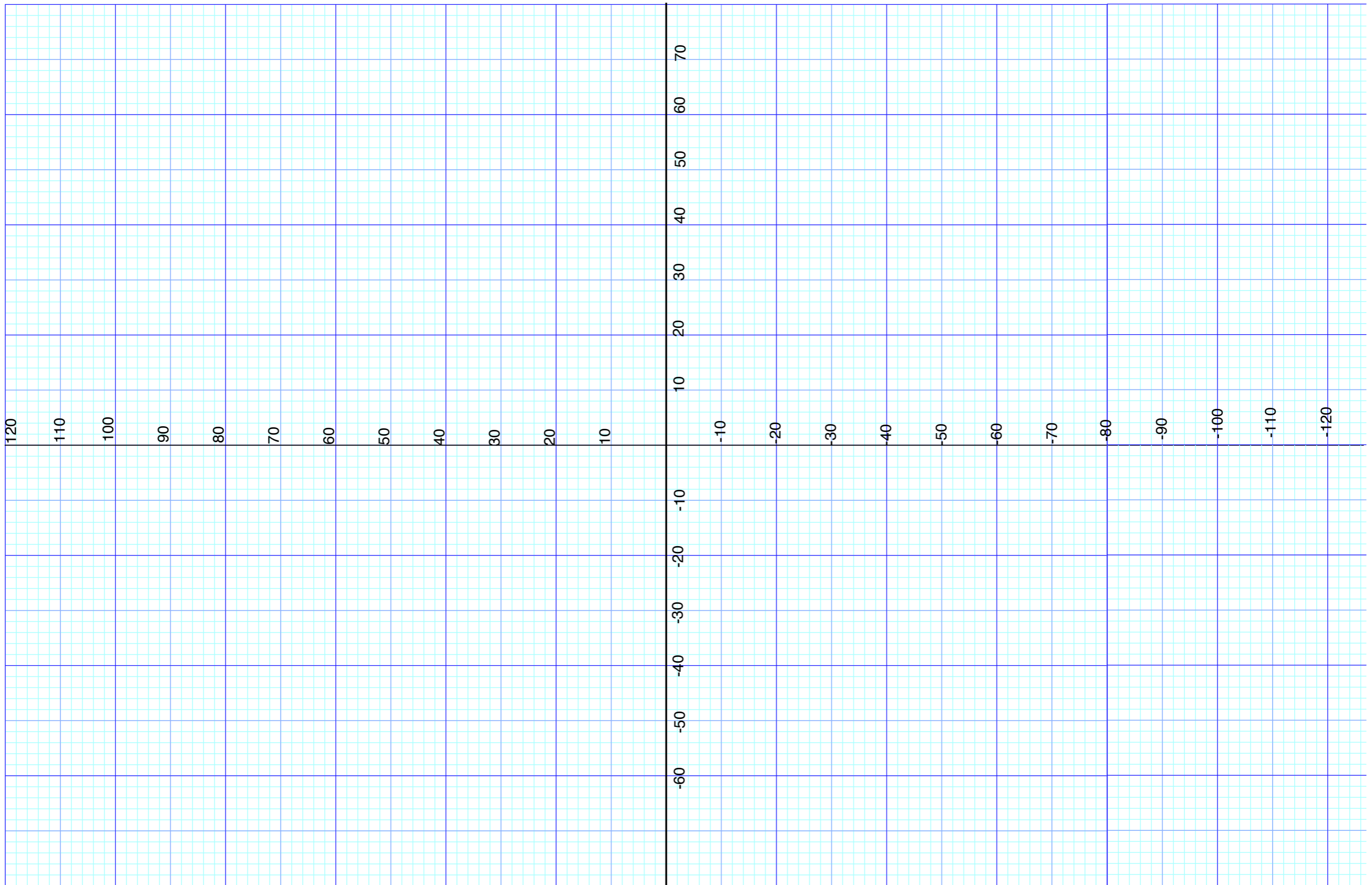
Developing
Conservation-of-Momentum
The key axiom of mechanics
leading to
Conservation-of-Energy Theorem

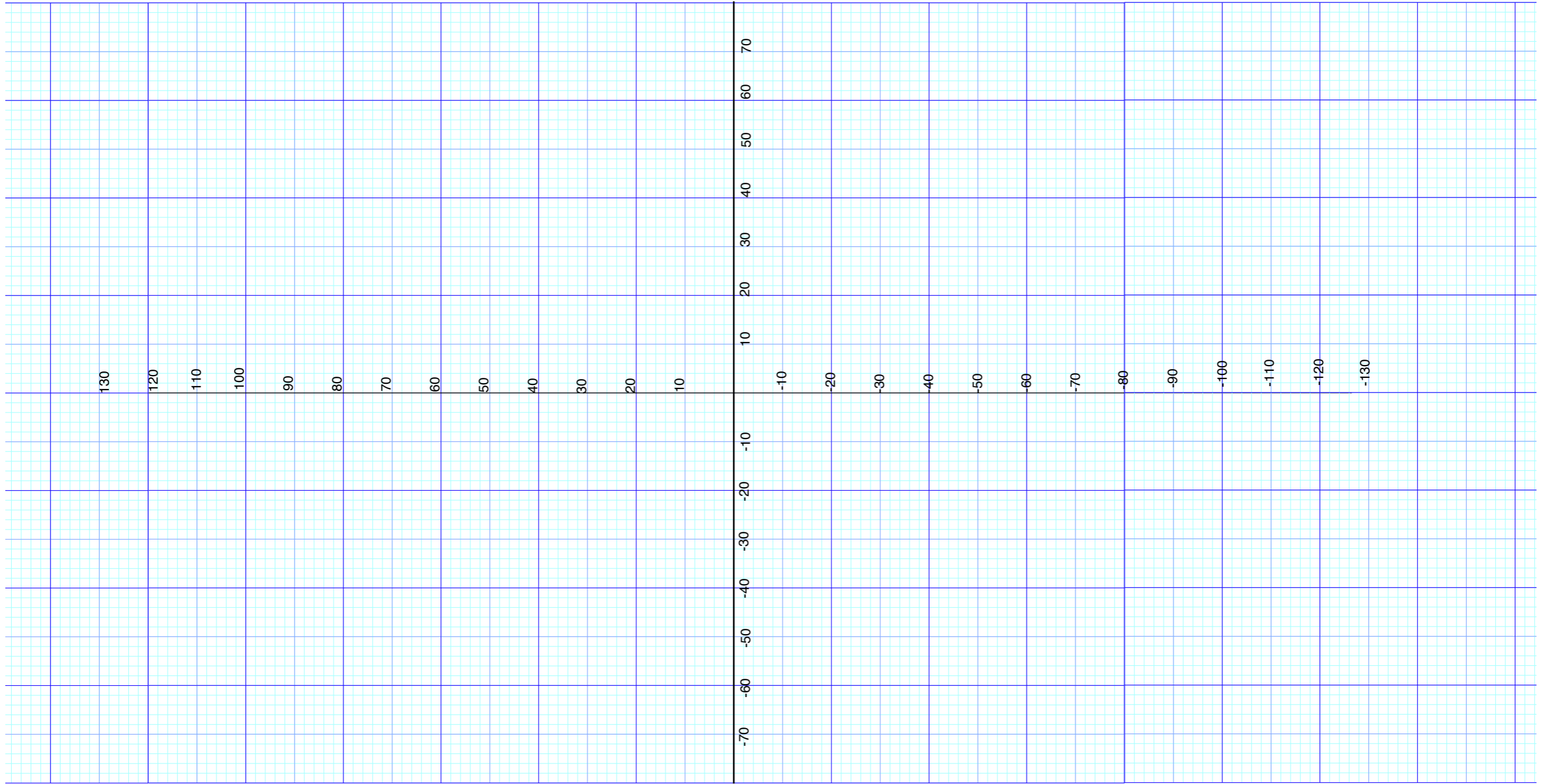
If and only if...
there is **T-Symmetry**

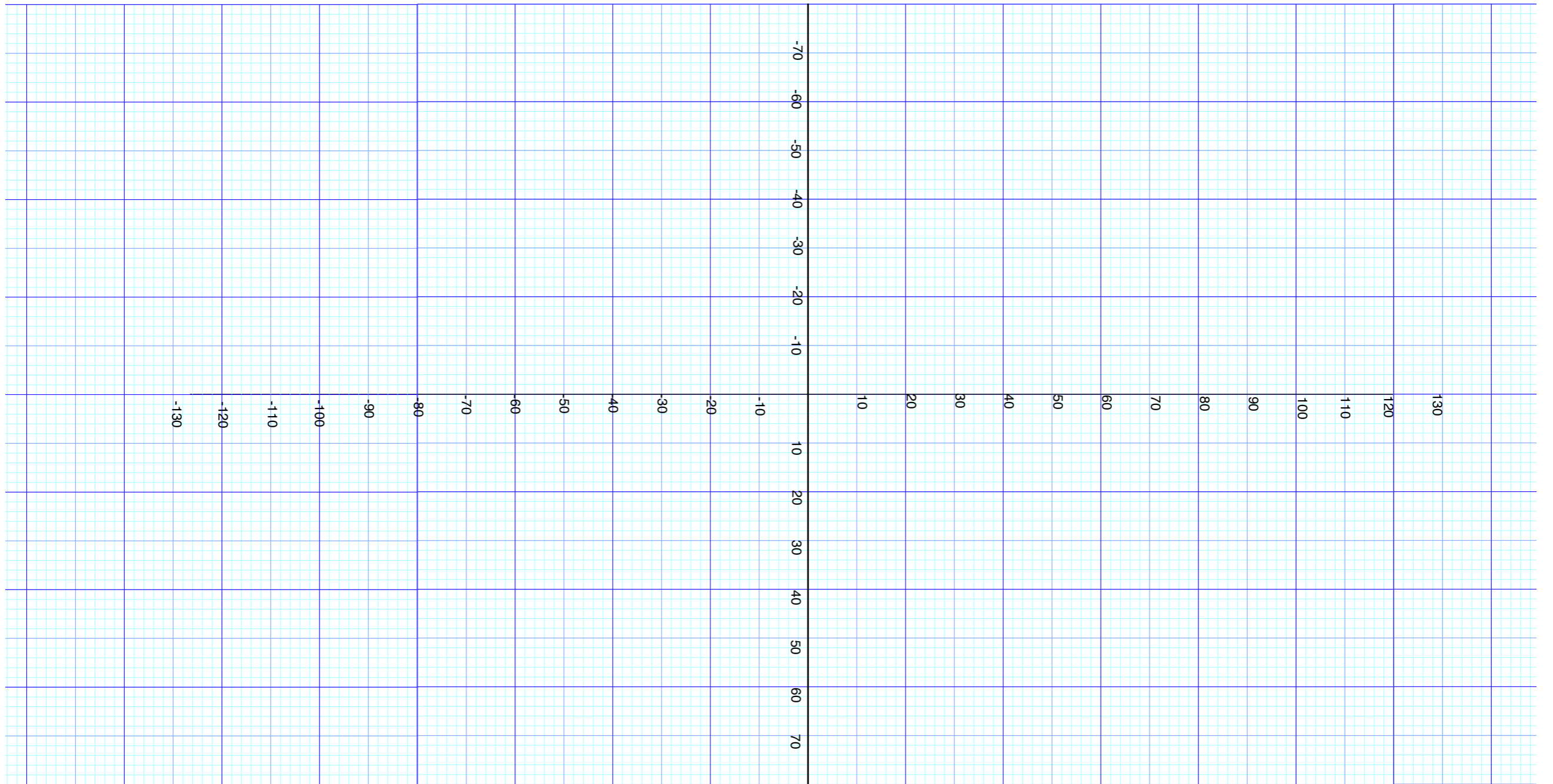
$$\mathbf{V}^{COM} \cdot \mathbf{M} \cdot \mathbf{V}^{COM} - 1/2 \mathbf{V}^{FIN} \cdot \mathbf{M} \cdot \mathbf{V}^{IN} = 1/2 \mathbf{V}^{IN} \cdot \mathbf{M} \cdot \mathbf{V}^{IN} = 1/2 \mathbf{V}^{FIN} \cdot \mathbf{M} \cdot \mathbf{V}^{FIN}$$

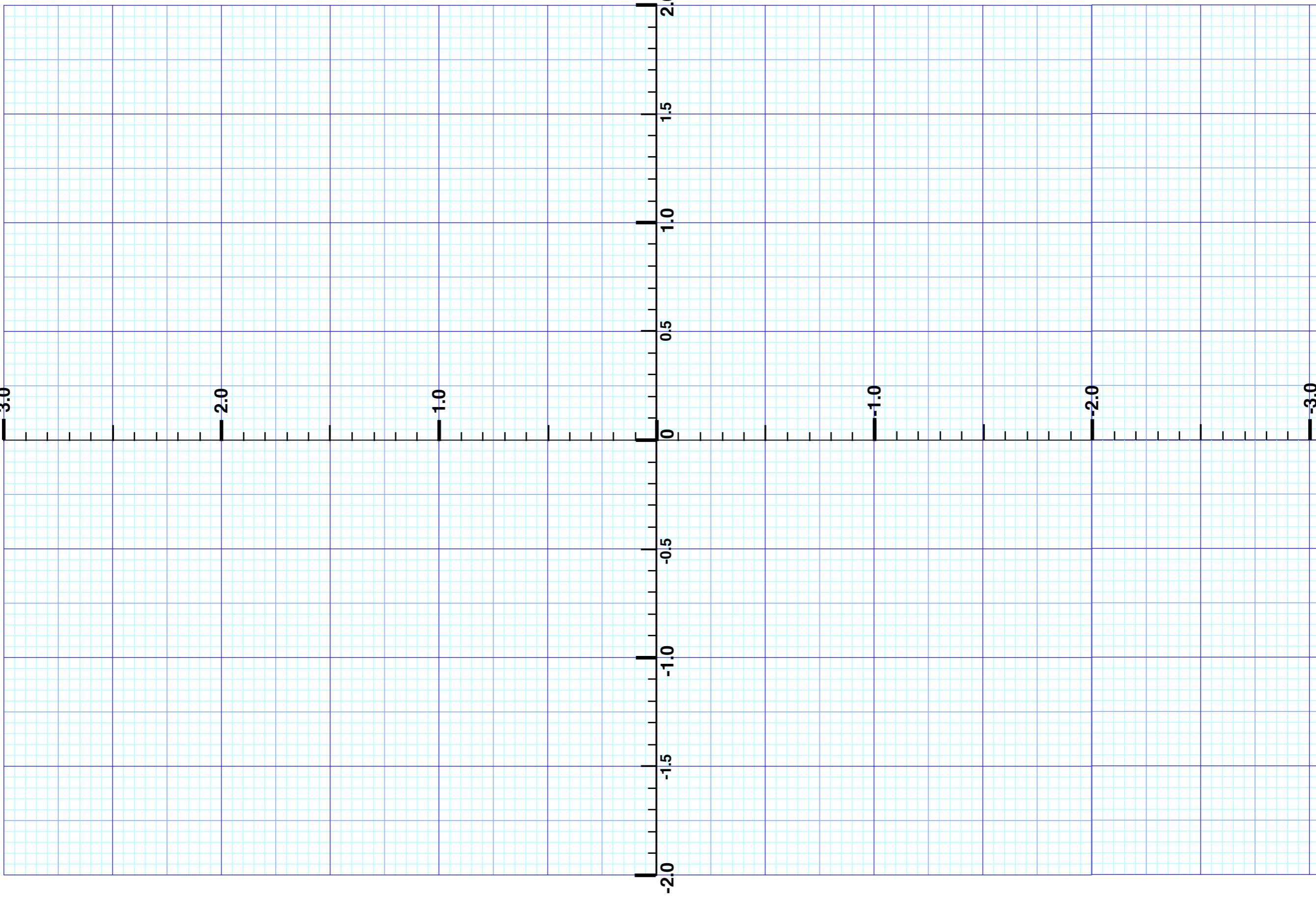
These are equations for energy conservation ellipse:

$$KE = 1/2 M_{SUV} V_{SUV}^2 + 1/2 M_{VW} V_{VW}^2$$









Note “crunch” energy $Elastic KE - inelastic IE = 0.21$ is the same in all frames including COM-frame.

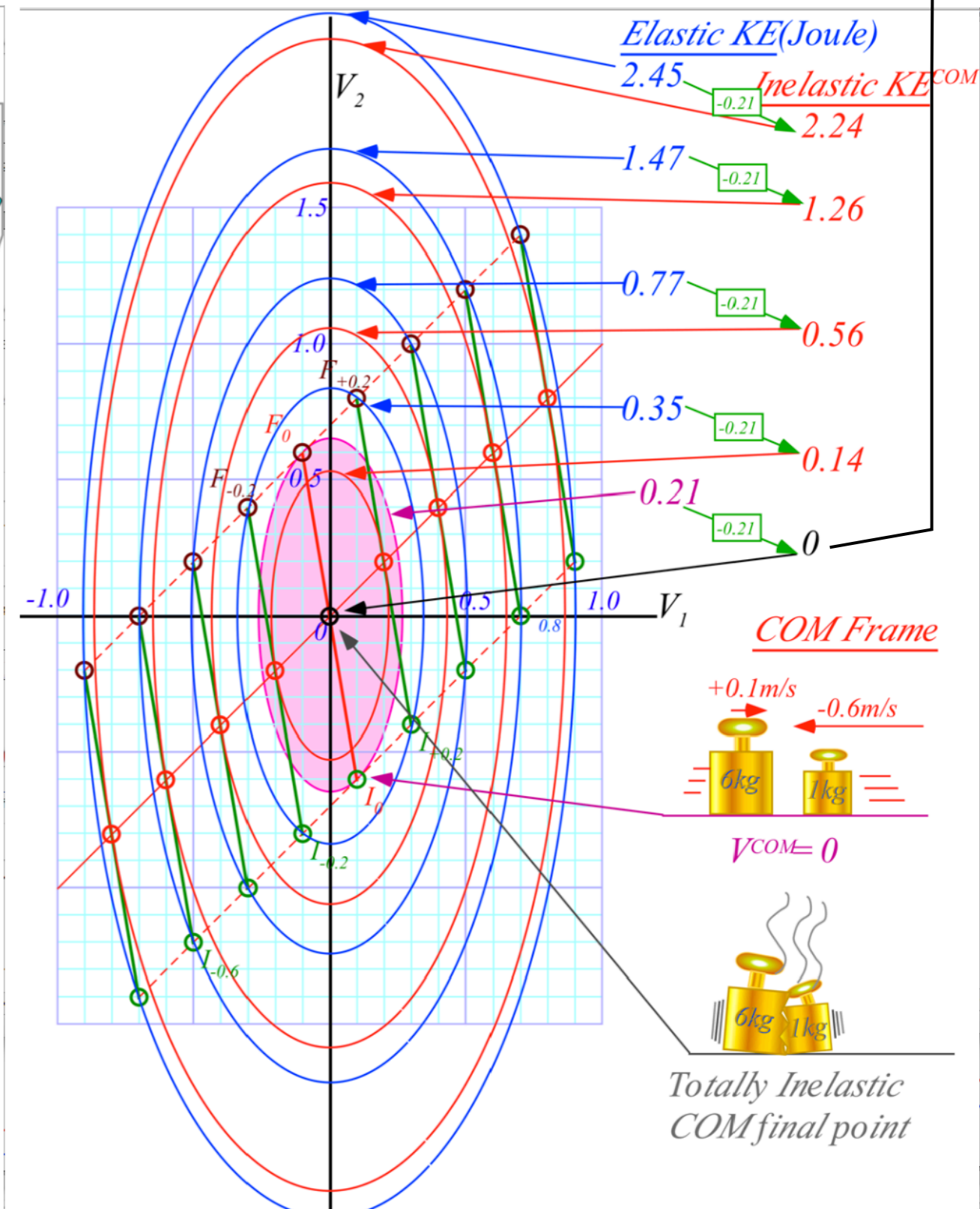
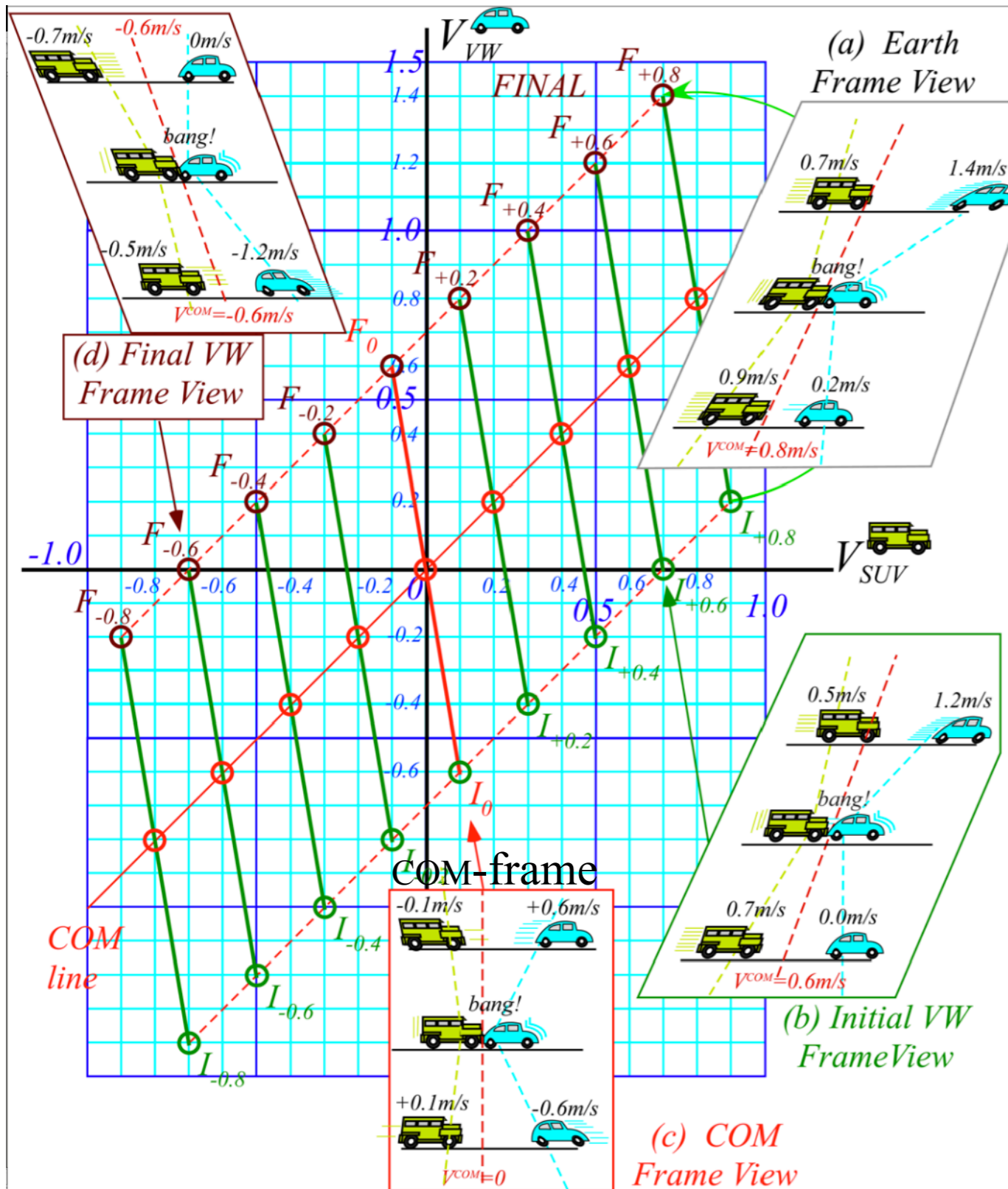


Fig. 3.4 Galilean Frame Views of collision like Fig. 2.5 or Fig. 3.1 with Bush (6:1) SUV.

- (a) Earth frame view
- (b) Initial VW frame (VW initially fixed)
- (c) COM frame view
- (d) Final VW frame (VW ends up fixed)

Fig. 3.5 Momentum ($P=const.$)-lines and energy ($KE=const.$)-ellipses appropriate for Fig. 3.4.