Lecture 28
Thur. 12.01.2016

Multi-particle and Rotational Dynamics

(Ch. 2-7 of Unit 6 12.01.16)

2-Particle orbits
- *Ptolemetric or LAB view and reduced mass*
- *Copernican or COM view and reduced coupling*

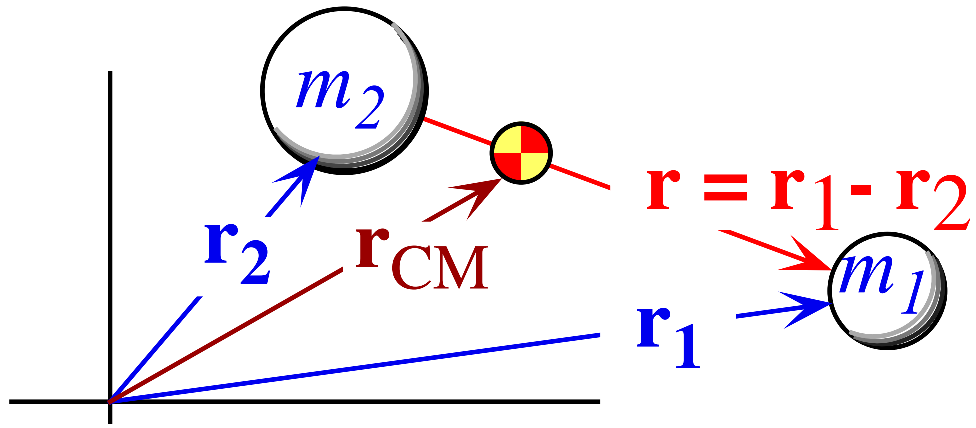
2-Particle orbits and scattering: LAB-vs.-COM frame views
- *Ruler & compass construction (or not)*

Rotational equivalent of Newton's **F**=*d***p**/*dt equations:* **N**=*d***L**/*dt*
- *How to make my boomerang come back*
- *The gyrocompass and mechanical spin analogy*

Rotational momentum and velocity tensor relations
- *Quadratic form geometry and duality (again)*
  - *angular velocity* $\boldsymbol{\omega}$*-ellipsoid vs. angular momentum* **L***-ellipsoid*
  - *Lagrangian* $\boldsymbol{\omega}$*-equations vs. Hamiltonian momentum* **L***-equation*

Rotational Energy Surfaces (RES) and Constant Energy Surfaces (CES)
- *Symmetric, asymmetric, and spherical-top dynamics (Constant* **L***)*
  - *BOD-frame cone rolling on LAB frame cone*
- *Deformable spherical rotor RES and semi-classical rotational states and spectra*
  - *Cycloidal geometry of flying levers*
    - *Practical poolhall application*

2-Particle orbits and center-of-mass (CM) coordinate frame



$$\mathbf{r}_{CM} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$

Defining *relative coordinate vector*

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$$

and *mass-weighted-average* or *center-of-mass coordinate vector* $\mathbf{r}_{CM}$

$$\bar{\mathbf{r}} = \mathbf{r}_{CM} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$

The inverse coordinate transformation.

$$\mathbf{r}_1 = \mathbf{r}_{CM} + \frac{m_2\mathbf{r}}{m_1 + m_2}, \qquad \mathbf{r}_2 = \mathbf{r}_{CM} - \frac{m_1\mathbf{r}}{m_1 + m_2}$$

2-Particle orbits

➔ *Ptolemetric or LAB view and reduced mass*

Copernican or COM view and reduced coupling

Reduced mass: Ptolemetric views

Radial inter-particle force $\mathbf{F}_{12}$ is on m_1 due to m_2 and $\mathbf{F}_{21} = -\mathbf{F}_{12}$ is on m_2 due to m_1

$$\mathbf{F}_{12} = \mathbf{F}(r)\mathbf{e_r} = -\mathbf{F}_{21} \quad = F(r)\hat{\mathbf{r}} = \quad F(r)\frac{\mathbf{r}}{r} = \quad \frac{F(r)}{r}\left(\mathbf{r_1} - \mathbf{r_2}\right)$$

$\mathbf{F}_{12}$ acts along relative coordinate vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

Depends only upon the relative distance $r = |\, \mathbf{r}_1 - \mathbf{r}_2 \,|$

$$\mathbf{F}_{12} = m_1\ddot{\mathbf{r}}_\mathbf{1} = \quad F(r)\hat{\mathbf{r}} = \quad F(r)\frac{\mathbf{r}}{r} = \quad \frac{F(r)}{r}\left(\mathbf{r_1} - \mathbf{r_2}\right)$$

$$\mathbf{F}_{21} = m_2\ddot{\mathbf{r}}_\mathbf{2} = -F(r)\hat{\mathbf{r}} = -F(r)\frac{\mathbf{r}}{r} = -\frac{F(r)}{r}\left(\mathbf{r_1} - \mathbf{r_2}\right)$$

Re-scaled force: A Copernican view

$$\mathbf{r_1} = \frac{m_2\mathbf{r}}{m_1 + m_2} = \frac{\mu}{m_1}\mathbf{r}\ , \qquad \mathbf{r}_2 = \frac{-m_1\mathbf{r}}{m_1 + m_2} = \frac{-\mu}{m_2}\mathbf{r}$$

relative radius vector

$$\frac{m_1}{\mu}\mathbf{r_1} = \mathbf{r} = \frac{-m_2}{\mu}\mathbf{r}_2$$

Reduced mass: Ptolemetric views

Radial inter-particle force $\mathbf{F}_{12}$ is on m_1 due to m_2 and $\mathbf{F}_{21} = -\mathbf{F}_{12}$ is on m_2 due to m_1

$$\mathbf{F}_{12} = \mathbf{F}(r)\mathbf{e_r} = -\mathbf{F}_{21} \quad = F(r)\hat{\mathbf{r}} = \quad F(r)\frac{\mathbf{r}}{r} = \quad \frac{F(r)}{r}\left(\mathbf{r_1} - \mathbf{r_2}\right)$$

$\mathbf{F}_{12}$ acts along relative coordinate vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

Depends only upon the relative distance $r = |\mathbf{r}_1 - \mathbf{r}_2|$

$$\mathbf{F}_{12} = m_1\ddot{\mathbf{r}}_\mathbf{1} = \quad F(r)\hat{\mathbf{r}} = \quad F(r)\frac{\mathbf{r}}{r} = \quad \frac{F(r)}{r}\left(\mathbf{r_1} - \mathbf{r_2}\right)$$

$$\mathbf{F}_{21} = m_2\ddot{\mathbf{r}}_\mathbf{2} = -F(r)\hat{\mathbf{r}} = -F(r)\frac{\mathbf{r}}{r} = -\frac{F(r)}{r}\left(\mathbf{r_1} - \mathbf{r_2}\right)$$

Sum $\mathbf{F}_{12}+\mathbf{F}_{21}$ yields zero because of Newton's 3^{rd} -law action-reaction cancellation.

$$(m_1 + m_2)\ddot{\mathbf{r}}_\mathbf{CM} = m_1\ddot{\mathbf{r}}_\mathbf{1} + m_2\ddot{\mathbf{r}}_\mathbf{2} = \mathbf{0}$$

Re-scaled force: A Copernican view

$$\mathbf{r_1} = \frac{m_2\mathbf{r}}{m_1 + m_2} = \frac{\mu}{m_1}\mathbf{r}\ , \qquad \mathbf{r}_2 = \frac{-m_1\mathbf{r}}{m_1 + m_2} = \frac{-\mu}{m_2}\mathbf{r}$$

relative radius vector

$$\frac{m_1}{\mu}\mathbf{r_1} = \mathbf{r} = \frac{-m_2}{\mu}\mathbf{r}_2$$

Reduced mass: Ptolemetric views

Radial inter-particle force $\mathbf{F}_{12}$ is on m_1 due to m_2 and $\mathbf{F}_{21} = -\mathbf{F}_{12}$ is on m_2 due to m_1

$$\mathbf{F}_{12} = \mathbf{F}(r)\mathbf{e_r} = -\mathbf{F}_{21} \quad = F(r)\hat{\mathbf{r}} = \quad F(r)\frac{\mathbf{r}}{r} = \quad \frac{F(r)}{r}\left(\mathbf{r_1} - \mathbf{r_2}\right)$$

$\mathbf{F}_{12}$ acts along relative coordinate vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

Depends only upon the relative distance $r = |\mathbf{r}_1 - \mathbf{r}_2|$

$$\mathbf{F}_{12} = m_1\ddot{\mathbf{r}}_\mathbf{1} = \quad F(r)\hat{\mathbf{r}} = \quad F(r)\frac{\mathbf{r}}{r} = \quad \frac{F(r)}{r}\left(\mathbf{r_1} - \mathbf{r_2}\right)$$

$$\mathbf{F}_{21} = m_2\ddot{\mathbf{r}}_\mathbf{2} = -F(r)\hat{\mathbf{r}} = -F(r)\frac{\mathbf{r}}{r} = -\frac{F(r)}{r}\left(\mathbf{r_1} - \mathbf{r_2}\right)$$

Sum $\mathbf{F}_{12}+\mathbf{F}_{21}$ yields zero because of Newton's 3rd -law action-reaction cancellation.

$$(m_1 + m_2)\ddot{\mathbf{r}}_\mathbf{CM} = m_1\ddot{\mathbf{r}}_\mathbf{1} + m_2\ddot{\mathbf{r}}_\mathbf{2} = \mathbf{0}$$

Difference $\mathbf{F}_{12}$-$\mathbf{F}_{21}$ reduces to $\mu\ddot{\mathbf{r}} = \mathbf{F}(r)$ using *reduced mass*: $\mu = \dfrac{m_2 m_1}{m_1 + m_2}$ $\quad \ddot{\mathbf{r}}_\mathbf{CM} = \mathbf{0}$

Re-scaled force: A Copernican view

$$\mathbf{r_1} = \frac{m_2\mathbf{r}}{m_1 + m_2} = \frac{\mu}{m_1}\mathbf{r}\ , \qquad \mathbf{r}_2 = \frac{-m_1\mathbf{r}}{m_1 + m_2} = \frac{-\mu}{m_2}\mathbf{r}$$

relative radius vector

$$\frac{m_1}{\mu}\mathbf{r_1} = \mathbf{r} = \frac{-m_2}{\mu}\mathbf{r}_2$$

Reduced mass: Ptolemetric views

Radial inter-particle force $\mathbf{F}_{12}$ is on m_1 due to m_2 and $\mathbf{F}_{21} = -\mathbf{F}_{12}$ is on m_2 due to m_1

$$\mathbf{F}_{12} = \mathbf{F}(r)\mathbf{e_r} = -\mathbf{F}_{21} = F(r)\hat{\mathbf{r}} = F(r)\frac{\mathbf{r}}{r} = \frac{F(r)}{r}(\mathbf{r_1}-\mathbf{r_2})$$

$\mathbf{F}_{12}$ acts along relative coordinate vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

Depends only upon the relative distance $r = |\mathbf{r}_1 - \mathbf{r}_2|$

$$\mathbf{F}_{12} = m_1\ddot{\mathbf{r}}_\mathbf{1} = F(r)\hat{\mathbf{r}} = F(r)\frac{\mathbf{r}}{r} = \frac{F(r)}{r}(\mathbf{r_1}-\mathbf{r_2})$$

$$\mathbf{F}_{21} = m_2\ddot{\mathbf{r}}_\mathbf{2} = -F(r)\hat{\mathbf{r}} = -F(r)\frac{\mathbf{r}}{r} = -\frac{F(r)}{r}(\mathbf{r_1}-\mathbf{r_2})$$

Sum $\mathbf{F}_{12}+\mathbf{F}_{21}$ yields zero because of Newton's 3rd -law action-reaction cancellation.

$$(m_1+m_2)\ddot{\mathbf{r}}_\mathbf{CM} = m_1\ddot{\mathbf{r}}_\mathbf{1} + m_2\ddot{\mathbf{r}}_\mathbf{2} = \mathbf{0}$$

Difference $\mathbf{F}_{12}-\mathbf{F}_{21}$ reduces to $\mu\ddot{\mathbf{r}} = \mathbf{F}(r)$ using *reduced mass*: $\boxed{\mu = \dfrac{m_2 m_1}{m_1+m_2}}$ $\quad \ddot{\mathbf{r}}_\mathbf{CM} = \mathbf{0}$

$$[\quad m_1\ddot{\mathbf{r}}_\mathbf{1}\quad] - [\quad m_2\ddot{\mathbf{r}}_\mathbf{2}\quad] = \frac{2F(r)}{r}(\mathbf{r_1}-\mathbf{r_2})$$

$$\boxed{\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} = \frac{m_1+m_2}{m_1 m_2}}$$

$$\left[m_1\ddot{\mathbf{r}}_\mathbf{CM} + \frac{m_1 m_2\ddot{\mathbf{r}}}{m_1+m_2}\right] - \left[m_2\ddot{\mathbf{r}}_\mathbf{CM} + \frac{m_2 m_1\ddot{\mathbf{r}}}{m_1+m_2}\right] = \frac{2F(r)}{r}(\mathbf{r_1}-\mathbf{r_2})$$

$$\mu\ddot{\mathbf{r}} = F(r)\hat{\mathbf{r}} = F(r)\mathbf{e_r} = \mathbf{F}(r)$$

Re-scaled force: A Copernican view

$$\mathbf{r_1} = \frac{m_2\mathbf{r}}{m_1+m_2} = \frac{\mu}{m_1}\mathbf{r}, \qquad \mathbf{r}_2 = \frac{-m_1\mathbf{r}}{m_1+m_2} = \frac{-\mu}{m_2}\mathbf{r}$$

relative radius vector

$$\frac{m_1}{\mu}\mathbf{r_1} = \mathbf{r} = \frac{-m_2}{\mu}\mathbf{r}_2$$

Reduced mass: Ptolemetric views

Radial inter-particle force $\mathbf{F}_{12}$ is on m_1 due to m_2 and $\mathbf{F}_{21} = -\mathbf{F}_{12}$ is on m_2 due to m_1

$$\mathbf{F}_{12} = \mathbf{F}(r)\mathbf{e_r} = -\mathbf{F}_{21} = F(r)\hat{\mathbf{r}} = F(r)\frac{\mathbf{r}}{r} = \frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$\mathbf{F}_{12}$ acts along relative coordinate vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

Depends only upon the relative distance $r = |\mathbf{r}_1 - \mathbf{r}_2|$

$$\mathbf{F}_{12} = m_1\ddot{\mathbf{r}}_1 = F(r)\hat{\mathbf{r}} = F(r)\frac{\mathbf{r}}{r} = \frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$$\mathbf{F}_{21} = m_2\ddot{\mathbf{r}}_2 = -F(r)\hat{\mathbf{r}} = -F(r)\frac{\mathbf{r}}{r} = -\frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

Sum $\mathbf{F}_{12}+\mathbf{F}_{21}$ yields zero because of Newton's 3rd -law action-reaction cancellation.

$$(m_1 + m_2)\ddot{\mathbf{r}}_{\mathbf{CM}} = m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 = \mathbf{0}$$

Difference $\mathbf{F}_{12}-\mathbf{F}_{21}$ reduces to $\mu\ddot{\mathbf{r}} = \mathbf{F}(r)$ using *reduced mass*: $\boxed{\mu = \frac{m_2 m_1}{m_1 + m_2}}$ $\quad \ddot{\mathbf{r}}_{\mathbf{CM}} = \mathbf{0}$

$$[\ m_1\ddot{\mathbf{r}}_1\ ] - [\ m_2\ddot{\mathbf{r}}_2\ ] = \frac{2F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$$\left[m_1\ddot{\mathbf{r}}_{\mathbf{CM}} + \frac{m_1 m_2\ddot{\mathbf{r}}}{m_1 + m_2}\right] - \left[m_2\ddot{\mathbf{r}}_{\mathbf{CM}} + \frac{m_2 m_1\ddot{\mathbf{r}}}{m_1 + m_2}\right] = \frac{2F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$$\boxed{\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} = \frac{m_1 + m_2}{m_1 m_2}}$$

$$\mu = \frac{m_2}{1 + \frac{m_2}{m_1}} = m_2\left(1 - \frac{m_2}{m_1}\ldots\right) \quad (m_1 >> m_2)$$

$$\mu = \frac{m_1}{1 + \frac{m_1}{m_2}} = m_1\left(1 - \frac{m_1}{m_2}\ldots\right) \quad (m_2 >> m_1)$$

$$\mu\ddot{\mathbf{r}} = F(r)\hat{\mathbf{r}} = F(r)\mathbf{e_r} = \mathbf{F}(r)$$

Re-scaled force: A Copernican view

$$\mathbf{r_1} = \frac{m_2\mathbf{r}}{m_1 + m_2} = \frac{\mu}{m_1}\mathbf{r}\ , \qquad \mathbf{r}_2 = \frac{-m_1\mathbf{r}}{m_1 + m_2} = \frac{-\mu}{m_2}\mathbf{r}$$

relative radius vector

$$\frac{m_1}{\mu}\mathbf{r_1} = \mathbf{r} = \frac{-m_2}{\mu}\mathbf{r}_2$$

Reduced mass: Ptolemetric views

Radial inter-particle force $\mathbf{F}_{12}$ is on m_1 due to m_2 and $\mathbf{F}_{21} = -\mathbf{F}_{12}$ is on m_2 due to m_1

$$\mathbf{F}_{12} = \mathbf{F}(r)\mathbf{e_r} = -\mathbf{F}_{21} \quad = F(r)\hat{\mathbf{r}} = \quad F(r)\frac{\mathbf{r}}{r} = \quad \frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$\mathbf{F}_{12}$ acts along relative coordinate vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

Depends only upon the relative distance $r = |\mathbf{r}_1 - \mathbf{r}_2|$

$$\mathbf{F}_{12} = m_1\ddot{\mathbf{r}}_\mathbf{1} = \quad F(r)\hat{\mathbf{r}} = \quad F(r)\frac{\mathbf{r}}{r} = \quad \frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$$\mathbf{F}_{21} = m_2\ddot{\mathbf{r}}_\mathbf{2} = -F(r)\hat{\mathbf{r}} = -F(r)\frac{\mathbf{r}}{r} = -\frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

Sum $\mathbf{F}_{12}+\mathbf{F}_{21}$ yields zero because of Newton's 3^{rd} -law action-reaction cancellation.

$$(m_1 + m_2)\ddot{\mathbf{r}}_\mathbf{CM} = m_1\ddot{\mathbf{r}}_\mathbf{1} + m_2\ddot{\mathbf{r}}_\mathbf{2} = \mathbf{0}$$

Difference $\mathbf{F}_{12}-\mathbf{F}_{21}$ reduces to $\mu\ddot{\mathbf{r}} = \mathbf{F}(r)$ using *reduced mass*: $\boxed{\mu = \frac{m_2 m_1}{m_1 + m_2}}$ $\quad \ddot{\mathbf{r}}_\mathbf{CM} = \mathbf{0}$

$$[\quad m_1\ddot{\mathbf{r}}_\mathbf{1} \quad] - [\quad m_2\ddot{\mathbf{r}}_\mathbf{2} \quad] = \frac{2F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$$\boxed{\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} = \frac{m_1 + m_2}{m_1 m_2}}$$

$$\left[m_1\ddot{\mathbf{r}}_\mathbf{CM} + \frac{m_1 m_2 \ddot{\mathbf{r}}}{m_1 + m_2}\right] - \left[m_2\ddot{\mathbf{r}}_\mathbf{CM} + \frac{m_2 m_1 \ddot{\mathbf{r}}}{m_1 + m_2}\right] = \frac{2F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$$\mu = \frac{m_2}{1 + \frac{m_2}{m_1}} = m_2\left(1 - \frac{m_2}{m_1}\ldots\right) \quad (m_1 >> m_2)$$

(Why it's reduced)

$$\mu = \frac{m_1}{1 + \frac{m_1}{m_2}} = m_1\left(1 - \frac{m_1}{m_2}\ldots\right) \quad (m_2 >> m_1)$$

$$\mu\ddot{\mathbf{r}} = F(r)\hat{\mathbf{r}} = F(r)\mathbf{e_r} = \mathbf{F}(r)$$

Re-scaled force: A Copernican view

$$\mathbf{r_1} = \frac{m_2\mathbf{r}}{m_1 + m_2} = \frac{\mu}{m_1}\mathbf{r}\,, \qquad \mathbf{r}_2 = \frac{-m_1\mathbf{r}}{m_1 + m_2} = \frac{-\mu}{m_2}\mathbf{r}$$

relative radius vector

$$\frac{m_1}{\mu}\mathbf{r_1} = \mathbf{r} = \frac{-m_2}{\mu}\mathbf{r}_2$$

2-Particle orbits

Ptolemetric view and reduced mass

➜ *Copernican view and reduced coupling*

Reduced mass: Ptolemetric views

Radial inter-particle force $\mathbf{F}_{12}$ is on m_1 due to m_2 and $\mathbf{F}_{21} = -\mathbf{F}_{12}$ is on m_2 due to m_1

$$\mathbf{F}_{12} = \mathbf{F}(r)\mathbf{e_r} = -\mathbf{F}_{21} \quad = F(r)\hat{\mathbf{r}} = \quad F(r)\frac{\mathbf{r}}{r} = \quad \frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$\mathbf{F}_{12}$ acts along relative coordinate vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

Depends only upon the relative distance $r = |\mathbf{r}_1 - \mathbf{r}_2|$

$$\mathbf{F}_{12} = m_1\ddot{\mathbf{r}}_1 = \quad F(r)\hat{\mathbf{r}} = \quad F(r)\frac{\mathbf{r}}{r} = \quad \frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$$\mathbf{F}_{21} = m_2\ddot{\mathbf{r}}_2 = -F(r)\hat{\mathbf{r}} = -F(r)\frac{\mathbf{r}}{r} = -\frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

Sum $\mathbf{F}_{12}+\mathbf{F}_{21}$ yields zero because of Newton's 3rd -law action-reaction cancellation.

$$(m_1 + m_2)\ddot{\mathbf{r}}_{\mathbf{CM}} = m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 = \mathbf{0}$$

Difference $\mathbf{F}_{12}-\mathbf{F}_{21}$ reduces to $\mu\ddot{\mathbf{r}} = \mathbf{F}(r)$ using *reduced mass*: $\boxed{\mu = \frac{m_2 m_1}{m_1 + m_2}}$ $\quad \ddot{\mathbf{r}}_{\mathbf{CM}} = \mathbf{0}$

$$[\quad m_1\ddot{\mathbf{r}}_1 \quad] - [\quad m_2\ddot{\mathbf{r}}_2 \quad] = \frac{2F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$$\left[m_1\ddot{\mathbf{r}}_{\mathbf{CM}} + \frac{m_1 m_2 \ddot{\mathbf{r}}}{m_1 + m_2}\right] - \left[m_2\ddot{\mathbf{r}}_{\mathbf{CM}} + \frac{m_2 m_1 \ddot{\mathbf{r}}}{m_1 + m_2}\right] = \frac{2F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$$\boxed{\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} = \frac{m_1 + m_2}{m_1 m_2}}$$

$$\mu = \frac{m_2}{1 + \frac{m_2}{m_1}} = m_2\left(1 - \frac{m_2}{m_1}\ldots\right) \quad (m_1 >> m_2)$$

(Why it's reduced)

$$\mu = \frac{m_1}{1 + \frac{m_1}{m_2}} = m_1\left(1 - \frac{m_1}{m_2}\ldots\right) \quad (m_2 >> m_1)$$

$$\mu\ddot{\mathbf{r}} = F(r)\hat{\mathbf{r}} = F(r)\mathbf{e_r} = \mathbf{F}(r)$$

Re-scaled force: A Copernican view

$$\mathbf{r_1} = \frac{m_2\mathbf{r}}{m_1 + m_2} = \frac{\mu}{m_1}\mathbf{r}\,, \qquad \mathbf{r}_2 = \frac{-m_1\mathbf{r}}{m_1 + m_2} = \frac{-\mu}{m_2}\mathbf{r}$$

relative radius vector

$$\frac{m_1}{\mu}\mathbf{r_1} = \mathbf{r} = \frac{-m_2}{\mu}\mathbf{r}_2$$

(Here we get "reduced" coupling constants)

each particle keeps it original mass m_1 or m_2, but feels *coordinate-re-scaled force field $F(m_1 r_1/\mu)$* or *$F(m_2 r_2/\mu)$* field

$$\mathbf{F}_{12} = m_1\ddot{\mathbf{r}}_1 = \quad F(\frac{m_1}{\mu}r_1)\hat{\mathbf{r}}_1 = -\mathbf{F}_{21}$$

$$\mathbf{F}_{21} = m_2\ddot{\mathbf{r}}_2 = F(\frac{m_2}{\mu}r_2)\hat{\mathbf{r}}_2 = -\mathbf{F}_{12}$$

Reduced mass: Ptolemetric views

Radial inter-particle force $\mathbf{F}_{12}$ is on m_1 due to m_2 and $\mathbf{F}_{21} = -\mathbf{F}_{12}$ is on m_2 due to m_1

$$\mathbf{F}_{12} = \mathbf{F}(r)\mathbf{e_r} = -\mathbf{F}_{21} \quad = F(r)\hat{\mathbf{r}} = \quad F(r)\frac{\mathbf{r}}{r} = \quad \frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$\mathbf{F}_{12}$ acts along relative coordinate vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

Depends only upon the relative distance $r = |\mathbf{r}_1 - \mathbf{r}_2|$

$$\mathbf{F}_{12} = m_1\ddot{\mathbf{r}}_\mathbf{1} = \quad F(r)\hat{\mathbf{r}} = \quad F(r)\frac{\mathbf{r}}{r} = \quad \frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$$\mathbf{F}_{21} = m_2\ddot{\mathbf{r}}_\mathbf{2} = -F(r)\hat{\mathbf{r}} = -F(r)\frac{\mathbf{r}}{r} = -\frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

Sum $\mathbf{F}_{12}+\mathbf{F}_{21}$ yields zero because of Newton's 3rd -law action-reaction cancellation.

$$(m_1 + m_2)\ddot{\mathbf{r}}_\mathbf{CM} = m_1\ddot{\mathbf{r}}_\mathbf{1} + m_2\ddot{\mathbf{r}}_\mathbf{2} = \mathbf{0}$$

Difference $\mathbf{F}_{12}-\mathbf{F}_{21}$ reduces to $\mu\ddot{\mathbf{r}} = \mathbf{F}(r)$ using $\boxed{\textit{reduced mass}: \quad \mu = \frac{m_2 m_1}{m_1 + m_2}}$ $\quad \ddot{\mathbf{r}}_\mathbf{CM} = \mathbf{0}$

$$[\quad m_1\ddot{\mathbf{r}}_\mathbf{1} \quad] - [\quad m_2\ddot{\mathbf{r}}_\mathbf{2} \quad] = \frac{2F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$$\left[m_1\ddot{\mathbf{r}}_\mathbf{CM} + \frac{m_1 m_2 \ddot{\mathbf{r}}}{m_1 + m_2}\right] - \left[m_2\ddot{\mathbf{r}}_\mathbf{CM} + \frac{m_2 m_1 \ddot{\mathbf{r}}}{m_1 + m_2}\right] = \frac{2F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$$\boxed{\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} = \frac{m_1 + m_2}{m_1 m_2}}$$

$$\mu = \frac{m_2}{1 + \frac{m_2}{m_1}} = m_2\left(1 - \frac{m_2}{m_1}\ldots\right) \quad (m_1 >> m_2)$$

(Why it's reduced)

$$\mu = \frac{m_1}{1 + \frac{m_1}{m_2}} = m_1\left(1 - \frac{m_1}{m_2}\ldots\right) \quad (m_2 >> m_1)$$

$$\mu\ddot{\mathbf{r}} = F(r)\hat{\mathbf{r}} = F(r)\mathbf{e_r} = \mathbf{F}(r)$$

Re-scaled force: A Copernican view

$$\mathbf{r_1} = \frac{m_2 \mathbf{r}}{m_1 + m_2} = \frac{\mu}{m_1}\mathbf{r}, \qquad \mathbf{r}_2 = \frac{-m_1 \mathbf{r}}{m_1 + m_2} = \frac{-\mu}{m_2}\mathbf{r}$$

relative radius vector

$$\frac{m_1}{\mu}\mathbf{r_1} = \mathbf{r} = \frac{-m_2}{\mu}\mathbf{r}_2$$

(Here we get "reduced" coupling constants)

each particle keeps it original mass m_1 or m_2, but feels

coordinate-re-scaled force field $F(m_1 r_1/\mu)$ or $F(m_2 r_2/\mu)$ field

$$\mathbf{F}_{12} = m_1\ddot{\mathbf{r}}_\mathbf{1} = \; F(\frac{m_1}{\mu}r_1)\hat{\mathbf{r}}_1 = -\mathbf{F}_{21}$$

$$\mathbf{F}_{21} = m_2\ddot{\mathbf{r}}_\mathbf{2} = F(\frac{m_2}{\mu}r_2)\hat{\mathbf{r}}_2 = -\mathbf{F}_{12}$$

$$F(r) = \frac{k}{r^2} \text{ becomes: } F(\frac{m_1}{\mu}r_1) = \frac{\mu^2}{m_1^2}\frac{k}{r_1^2}$$

(Coulomb)

$$k \to k_1 = k\,\mu^2 / m_1^2, \qquad k \to k_2 = k\,\mu^2 / m_2^2$$

Reduced mass: Ptolemetric views

Radial inter-particle force $\mathbf{F}_{12}$ is on m_1 due to m_2 and $\mathbf{F}_{21} = -\mathbf{F}_{12}$ is on m_2 due to m_1

$$\mathbf{F}_{12} = \mathbf{F}(r)\mathbf{e_r} = -\mathbf{F}_{21} \quad = F(r)\hat{\mathbf{r}} = F(r)\frac{\mathbf{r}}{r} = \frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$\mathbf{F}_{12}$ acts along relative coordinate vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

Depends only upon the relative distance $r = |\mathbf{r}_1 - \mathbf{r}_2|$

$$\mathbf{F}_{12} = m_1\ddot{\mathbf{r}}_1 = F(r)\hat{\mathbf{r}} = F(r)\frac{\mathbf{r}}{r} = \frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$$\mathbf{F}_{21} = m_2\ddot{\mathbf{r}}_2 = -F(r)\hat{\mathbf{r}} = -F(r)\frac{\mathbf{r}}{r} = -\frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

Sum $\mathbf{F}_{12}+\mathbf{F}_{21}$ yields zero because of Newton's 3rd -law action-reaction cancellation.

$$(m_1 + m_2)\ddot{\mathbf{r}}_{\mathbf{CM}} = m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 = \mathbf{0}$$

Difference $\mathbf{F}_{12}-\mathbf{F}_{21}$ reduces to $\mu\ddot{\mathbf{r}} = \mathbf{F}(r)$ using *reduced mass*: $\boxed{\mu = \frac{m_2 m_1}{m_1 + m_2}}$ $\qquad \ddot{\mathbf{r}}_{\mathbf{CM}} = \mathbf{0}$

$$[\quad m_1\ddot{\mathbf{r}}_1 \quad] - [\quad m_2\ddot{\mathbf{r}}_2 \quad] = \frac{2F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$$\left[m_1\ddot{\mathbf{r}}_{\mathbf{CM}} + \frac{m_1 m_2 \ddot{\mathbf{r}}}{m_1 + m_2}\right] - \left[m_2\ddot{\mathbf{r}}_{\mathbf{CM}} + \frac{m_2 m_1 \ddot{\mathbf{r}}}{m_1 + m_2}\right] = \frac{2F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$$\boxed{\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} = \frac{m_1 + m_2}{m_1 m_2}}$$

$$\mu = \frac{m_2}{1 + \frac{m_2}{m_1}} = m_2\left(1 - \frac{m_2}{m_1}\ldots\right) \quad (m_1 >> m_2)$$

(Why it's reduced)

$$\mu = \frac{m_1}{1 + \frac{m_1}{m_2}} = m_1\left(1 - \frac{m_1}{m_2}\ldots\right) \quad (m_2 >> m_1)$$

$$\mu\ddot{\mathbf{r}} = F(r)\hat{\mathbf{r}} = F(r)\mathbf{e_r} = \mathbf{F}(r)$$

Re-scaled force: A Copernican view

$$\mathbf{r}_1 = \frac{m_2\mathbf{r}}{m_1 + m_2} = \frac{\mu}{m_1}\mathbf{r}, \qquad \mathbf{r}_2 = \frac{-m_1\mathbf{r}}{m_1 + m_2} = \frac{-\mu}{m_2}\mathbf{r}$$

relative radius vector $\quad \frac{m_1}{\mu}\mathbf{r}_1 = \mathbf{r} = \frac{-m_2}{\mu}\mathbf{r}_2$

(Here we get "reduced" coupling constants)

each particle keeps it original mass m_1 or m_2, but feels
coordinate-re-scaled force field $F(m_1 r_1/\mu)$ or $F(m_2 r_2/\mu)$ field

$$\mathbf{F}_{12} = m_1\ddot{\mathbf{r}}_1 = F(\frac{m_1}{\mu}r_1)\hat{\mathbf{r}}_1 = -\mathbf{F}_{21}$$

$$\mathbf{F}_{21} = m_2\ddot{\mathbf{r}}_2 = F(\frac{m_2}{\mu}r_2)\hat{\mathbf{r}}_2 = -\mathbf{F}_{12}$$

$F(r) = \frac{k}{r^2}$ becomes: $F(\frac{m_1}{\mu}r_1) = \frac{\mu^2}{m_1^2}\frac{k}{r_1^2}$

(Coulomb)

$k \to k_1 = k\mu^2/m_1^2, \quad k \to k_2 = k\mu^2/m_2^2$

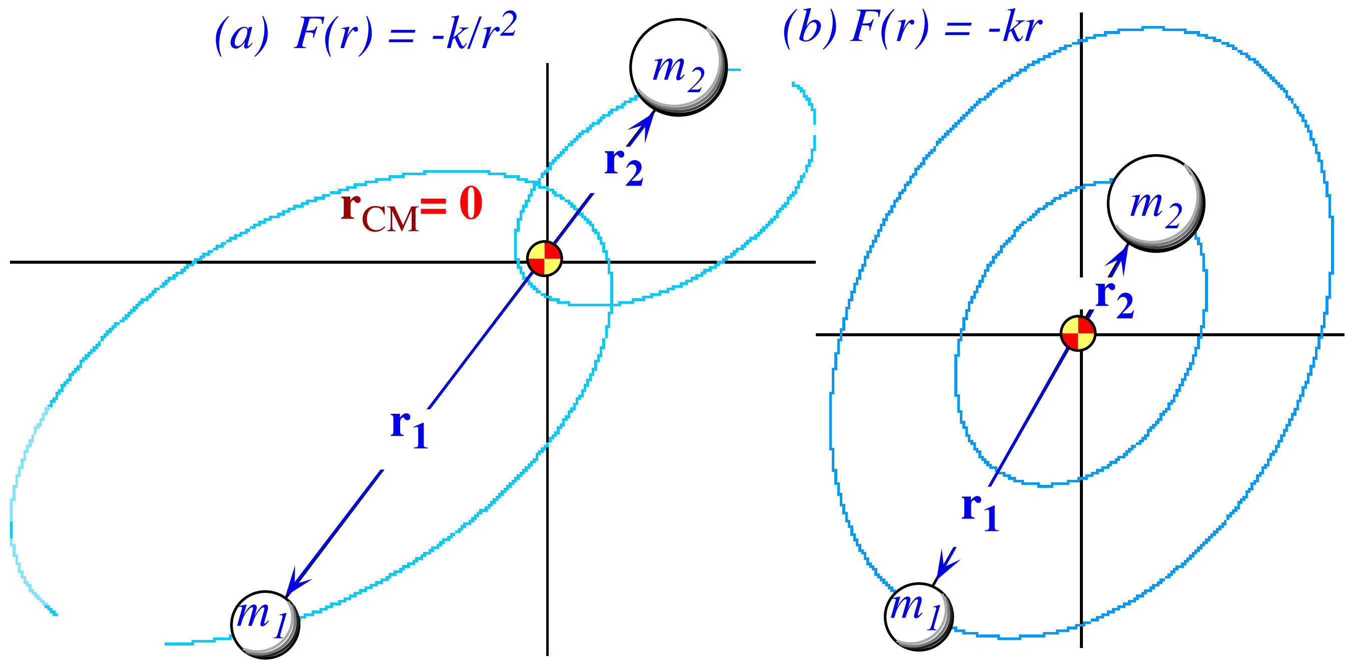
$F(r) = -kr$ becomes: $F(\frac{m_1}{\mu}r_1) = -\frac{m_1}{\mu}k\,r_1$

(Harmonic Oscillator)

$k \to k_1 = k m_1/\mu, \quad k \to k_2 = k m_2/\mu$

2-Particle orbits and scattering: LAB-vs.-COM frame views
Ruler & compass construction (or not)

Examples of Coulomb and harmonic oscillator 2-particle “Copernican” orbits in CM system.



CoulIt Web Simulations
Coulombic Orbit (CM Frame)

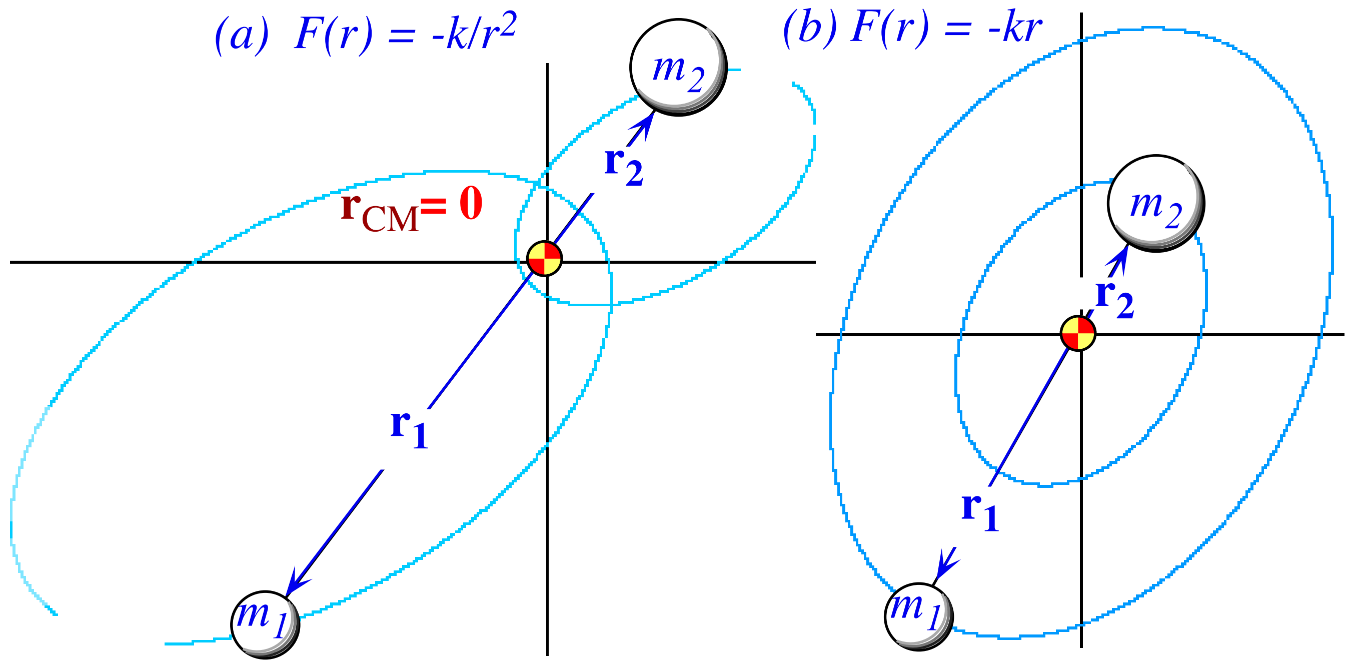
Coulombic Orbit (Lab Frame)

CoulIt Web Simulations
Hooke Orbit (CM Frame)

Hooke Orbit (Lab Frame)

Two particles are in synchronous motion around fixed CM origin.
Orbit periods are identical to each other.
Orbits are mass-scaled copies with equal aspect ratio (*a/b*), eccentricity, and orientation.

Examples of Coulomb and harmonic oscillator 2-particle "Copernican" orbits in CM system.



CoulIt Web Simulations
Coulombic Orbit (CM Frame)

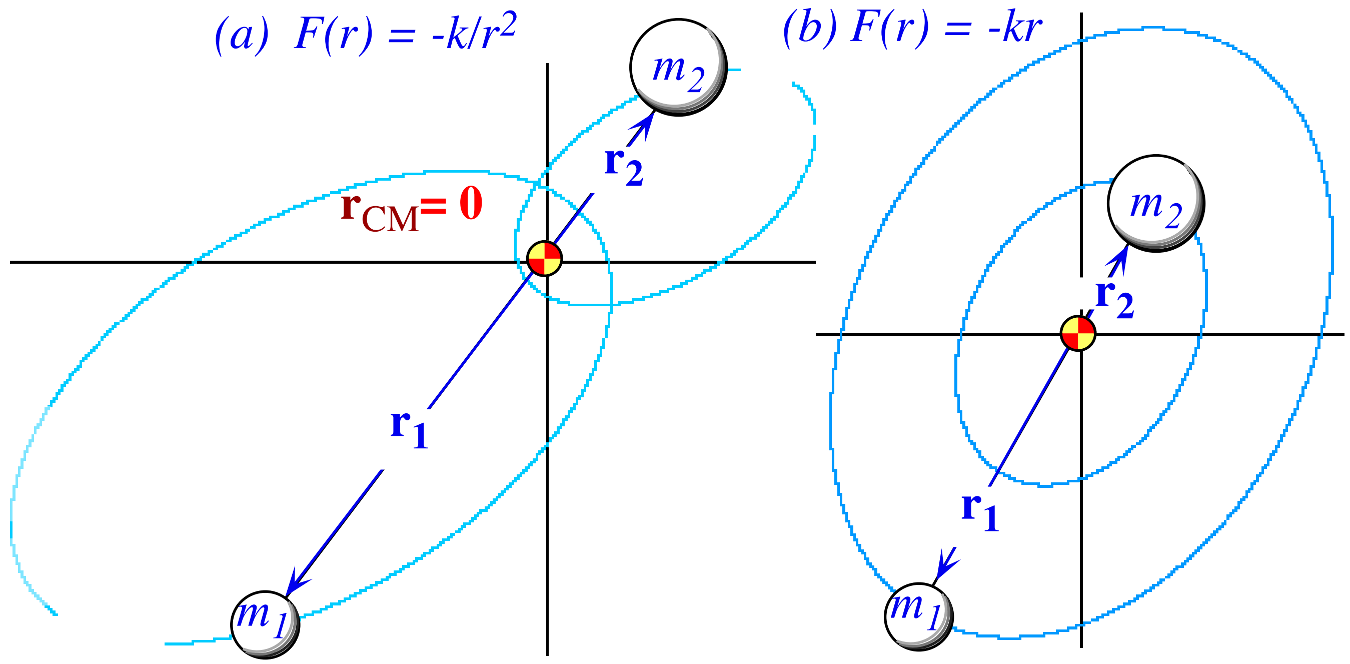
Coulombic Orbit (Lab Frame)

CoulIt Web Simulations
Hooke Orbit (CM Frame)

Hooke Orbit (Lab Frame)

Two particles are in synchronous motion around fixed CM origin.
Orbit periods are identical to each other.
Orbits are mass-scaled copies with equal aspect ratio (a/b), eccentricity, and orientation.
Orbits differ in size of axes (a_1 , b_1) and (a_2 , b_2)
Orbits differ in placement of center (for the Coulomb case) or foci (for the oscillator).

Examples of Coulomb and harmonic oscillator 2-particle "Copernican" orbits in CM system.



CoulIt Web Simulations
Coulombic Orbit (CM Frame)

Coulombic Orbit (Lab Frame)

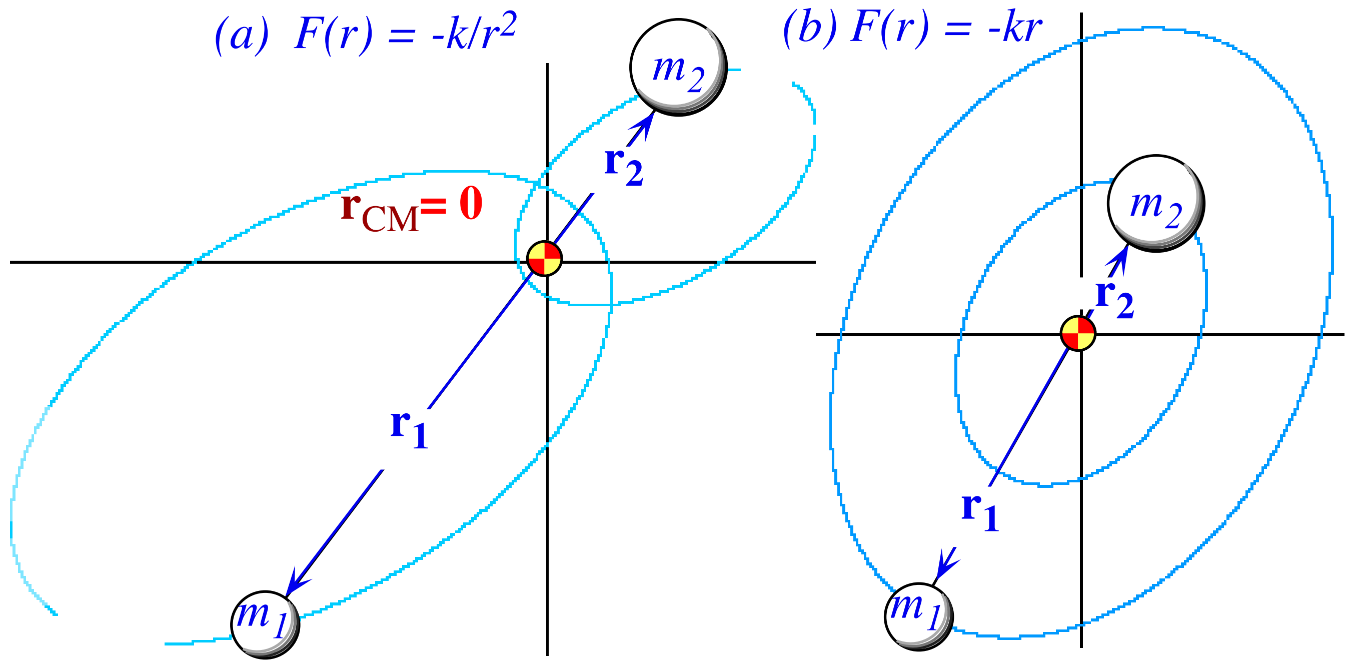
CoulIt Web Simulations
Hooke Orbit (CM Frame)

Hooke Orbit (Lab Frame)

Two particles are in synchronous motion around fixed CM origin.
Orbit periods are identical to each other.
Orbits are mass-scaled copies with equal aspect ratio (a/b), eccentricity, and orientation.
Orbits differ in size of axes (a_1 , b_1) and (a_2 , b_2)
Orbits differ in placement of center (for the Coulomb case) or foci (for the oscillator).
Orbit axial dimensions (a_k , b_k) and λ_k are in inverse proportion to mass values.

$$a_1 m_1 = a_2 m_2 = a\mu \ , \qquad b_1 m_1 = b_2 m_2 = b\mu \qquad \lambda_1 m_1 = \lambda_2 m_2 = \lambda\mu$$

Examples of Coulomb and harmonic oscillator 2-particle "Copernican" orbits in CM system.



CoulIt Web Simulations
Coulombic Orbit (CM Frame)

Coulombic Orbit (Lab Frame)

CoulIt Web Simulations
Hooke Orbit (CM Frame)

Hooke Orbit (Lab Frame)

Two particles are in synchronous motion around fixed CM origin.
Orbit periods are identical to each other.
Orbits are mass-scaled copies with equal aspect ratio (a/b), eccentricity, and orientation.
Orbits differ in size of axes (a_1 , b_1) and (a_2 , b_2)
Orbits differ in placement of center (for the Coulomb case) or foci (for the oscillator).
Orbit axial dimensions (a_k , b_k) and λ_k are in inverse proportion to mass values.

$$a_1 m_1 = a_2 m_2 = a\mu \ , \qquad b_1 m_1 = b_2 m_2 = b\mu \qquad \lambda_1 m_1 = \lambda_2 m_2 = \lambda\mu$$

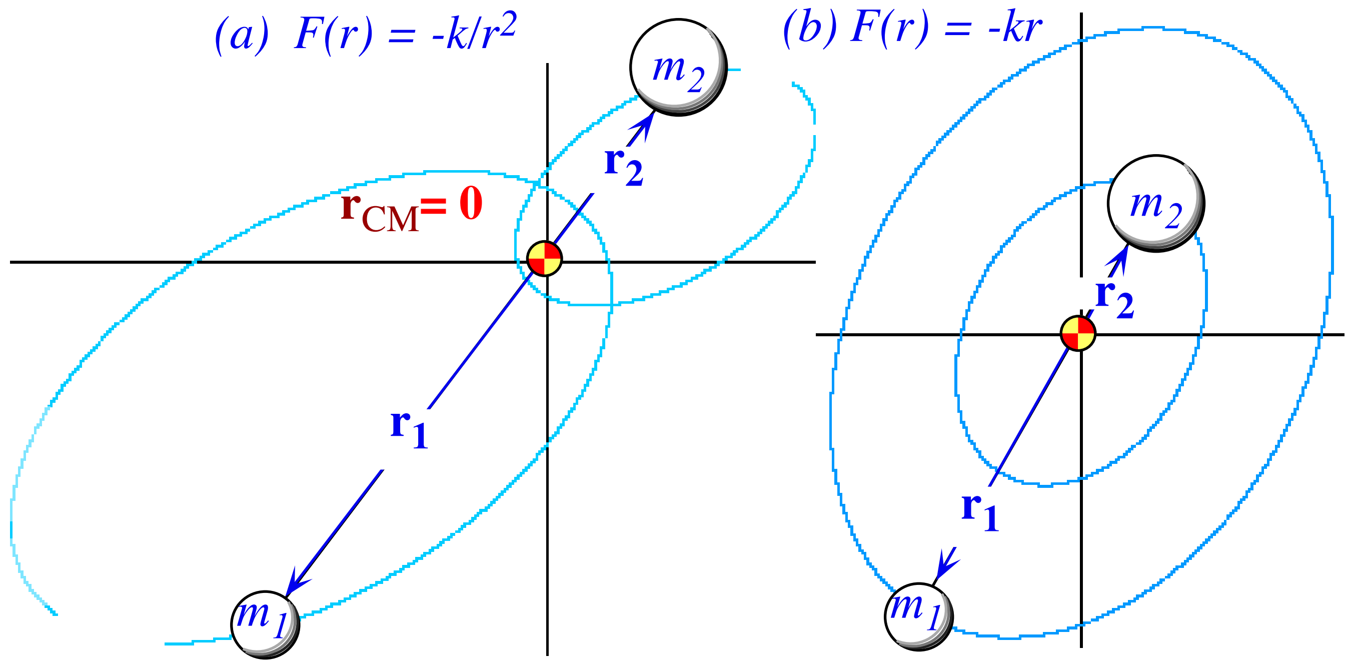
Harmonic oscillator periods and Coulomb orbit periods and eccentricity must match

$$T_{IHO} = 2\pi\sqrt{\frac{\mu}{k}} = 2\pi\sqrt{\frac{m_1}{k_1}} = 2\pi\sqrt{\frac{m_2}{k_2}} \qquad T_{Coul} = 2\pi\sqrt{\frac{\mu\, a^3}{k}} = 2\pi\sqrt{\frac{m_1 a_1^{\ 3}}{k_1}} = 2\pi\sqrt{\frac{m_2 a_2^{\ 3}}{k_2}} \qquad \varepsilon_1 = \varepsilon_2 = \varepsilon$$

Examples of Coulomb and harmonic oscillator 2-particle "Copernican" orbits in CM system.

[CoulIt Web Simulations](#)
[Coulombic Orbit (CM Frame)](#)

[Coulombic Orbit (Lab Frame)](#)



[CoulIt Web Simulations](#)
[Hooke Orbit (CM Frame)](#)

[Hooke Orbit (Lab Frame)](#)

Two particles are in synchronous motion around fixed CM origin.
Orbit periods are identical to each other.
Orbits are mass-scaled copies with equal aspect ratio (a/b), eccentricity, and orientation.
Orbits differ in size of axes (a_1 , b_1) and (a_2 , b_2)
Orbits differ in placement of center (for the Coulomb case) or foci (for the oscillator).
Orbit axial dimensions (a_k , b_k) and λ_k are in inverse proportion to mass values.

$$a_1 m_1 = a_2 m_2 = a\mu \ , \qquad b_1 m_1 = b_2 m_2 = b\mu \qquad \lambda_1 m_1 = \lambda_2 m_2 = \lambda\mu$$

Harmonic oscillator periods and Coulomb orbit periods and eccentricity must match

$$T_{IHO} = 2\pi\sqrt{\frac{\mu}{k}} = 2\pi\sqrt{\frac{m_1}{k_1}} = 2\pi\sqrt{\frac{m_2}{k_2}} \qquad T_{Coul} = 2\pi\sqrt{\frac{\mu\, a^3}{k}} = 2\pi\sqrt{\frac{m_1 a_1{}^3}{k_1}} = 2\pi\sqrt{\frac{m_2 a_2{}^3}{k_2}} \qquad \varepsilon_1 = \varepsilon_2 = \varepsilon$$

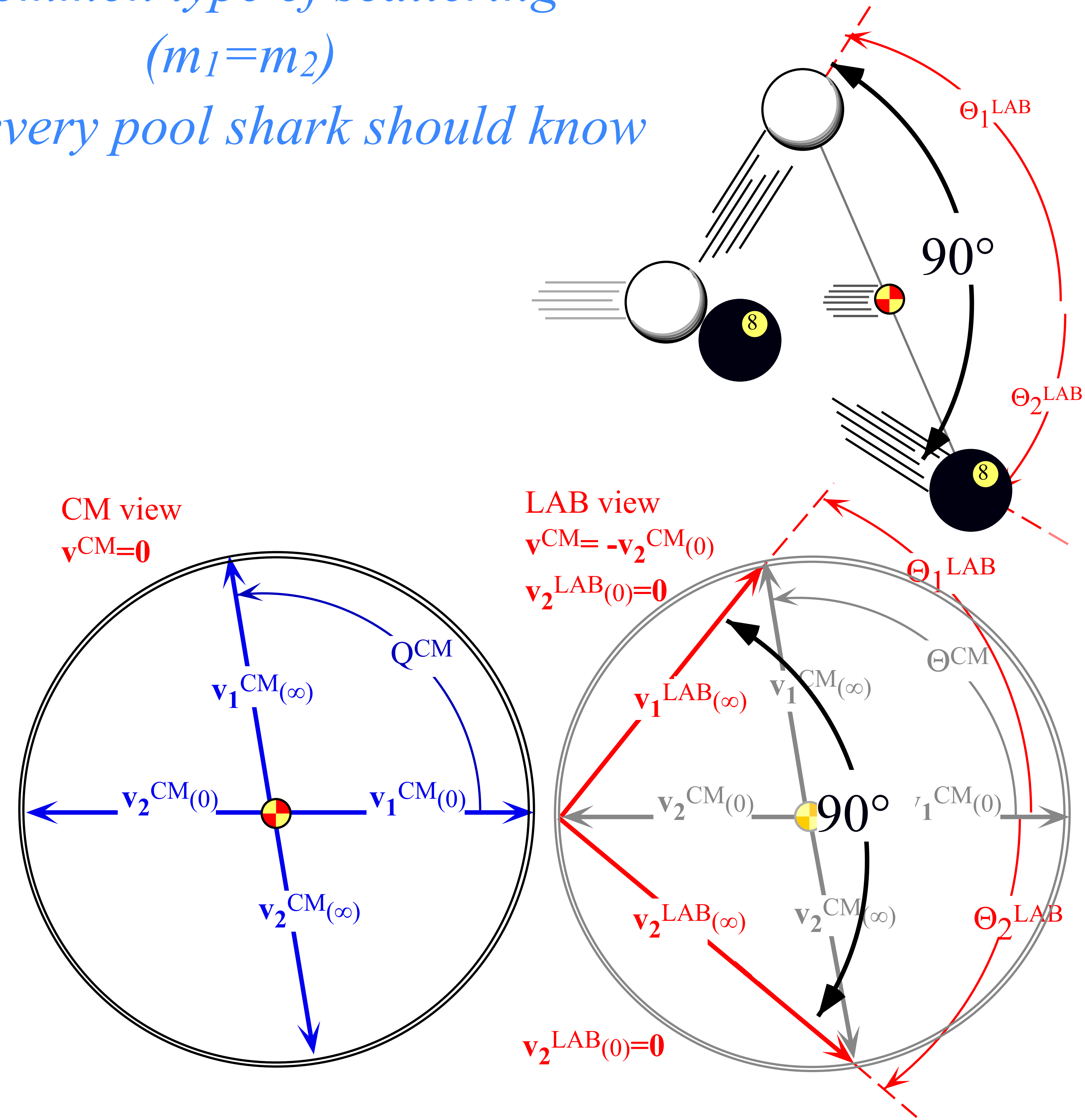
Three Coulomb orbit energy values satisfy the same proportion relation as their axes

$$E_1 m_1 = E_2 m_2 = E\mu \ , \quad \text{where:} \quad |E_1| = \frac{|k_1|}{2a_1} \ , \quad |E_2| = \frac{|k_2|}{2a_2} \ , \quad |E| = \frac{|k|}{2a} \ .$$

Energy values and axes satisfy similar sum relations

$$E_1 + E_2 = \frac{m_1}{\mu}E + \frac{m_2}{\mu}E = E \ , \quad \text{and:} \quad a_1 + a_2 = \frac{m_1}{\mu}a + \frac{m_2}{\mu}a = a$$

A common type of scattering
($m_1=m_2$)
...that every pool shark should know



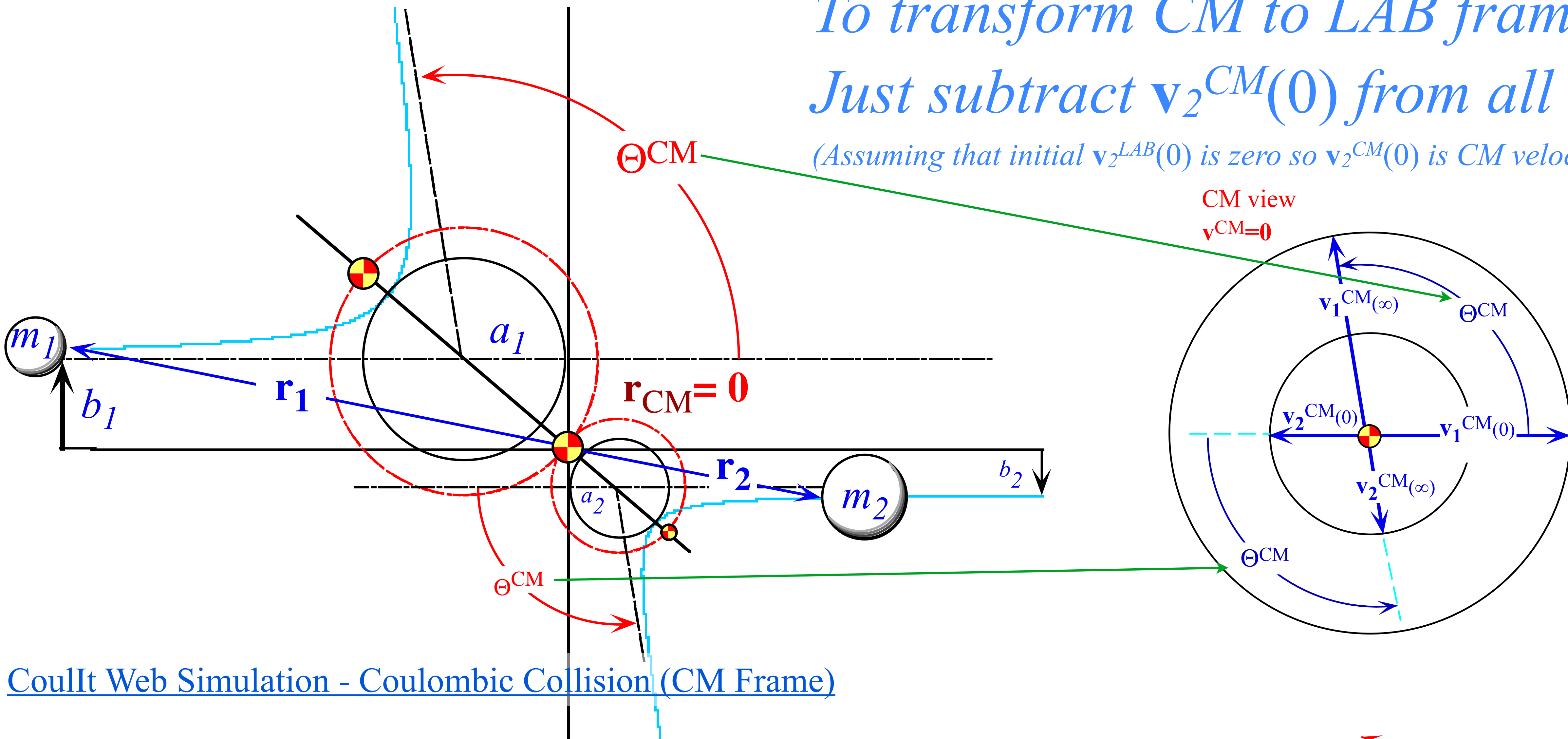
[BounceIt Web Simulations](#)
[Hard Collision (CM Frame)](#)

[Hard Collision (Lab Frame)](#)

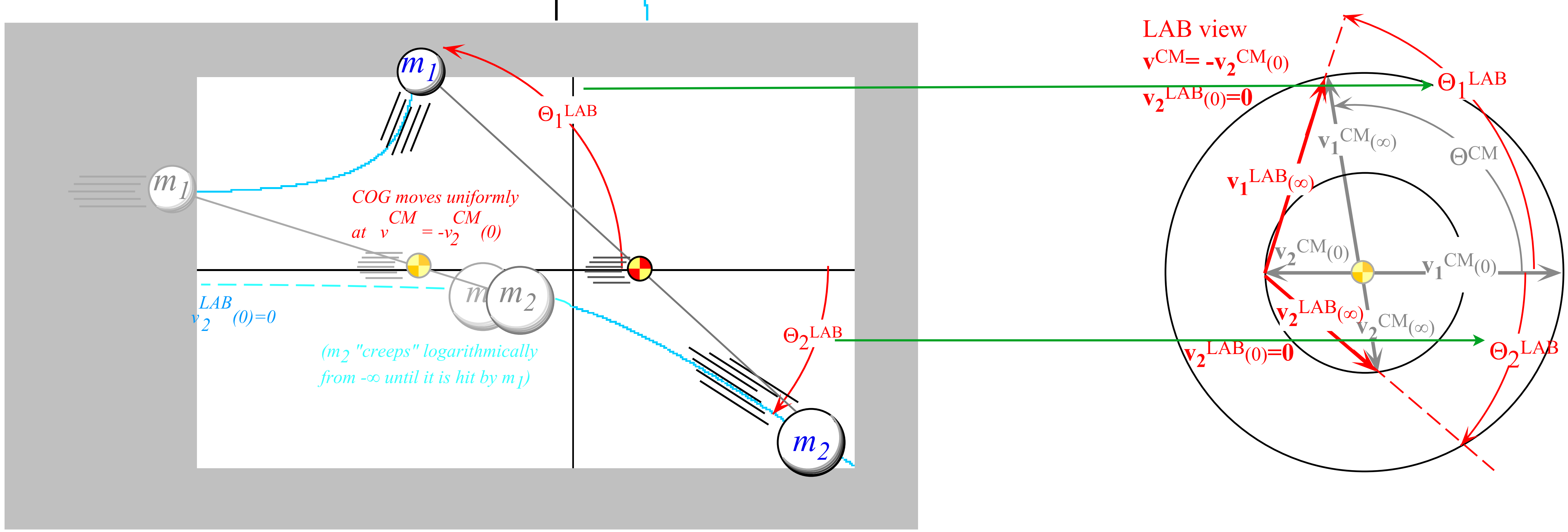
To transform CM to LAB frame

Just subtract $\mathbf{v}_2^{CM}(0)$ from all

(Assuming that initial $\mathbf{v}_2^{LAB}(0)$ is zero so $\mathbf{v}_2^{CM}(0)$ is CM velocity in LAB)



[CoulIt Web Simulation - Coulombic Collision (CM Frame)]



[CoulIt Web Simulation - Coulombic Collision (LAB Frame)]

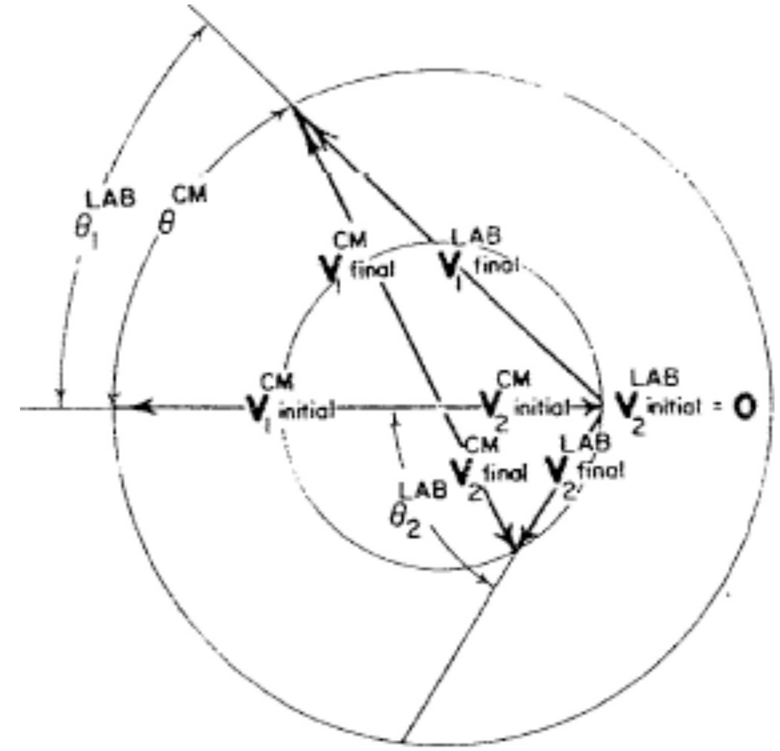
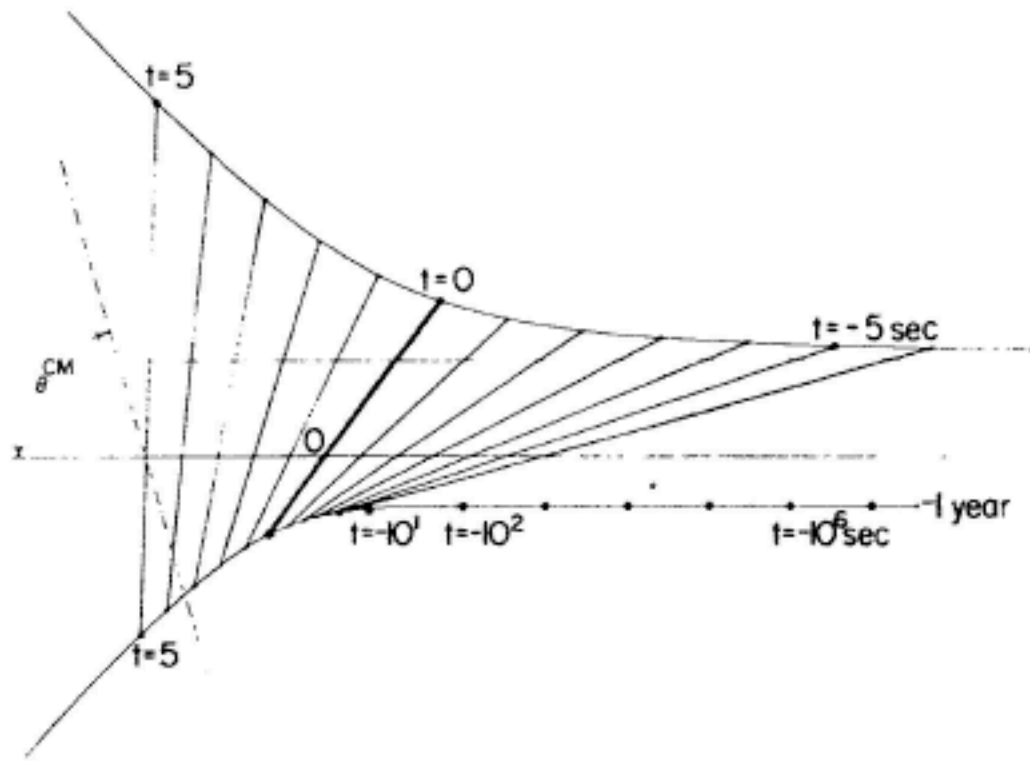
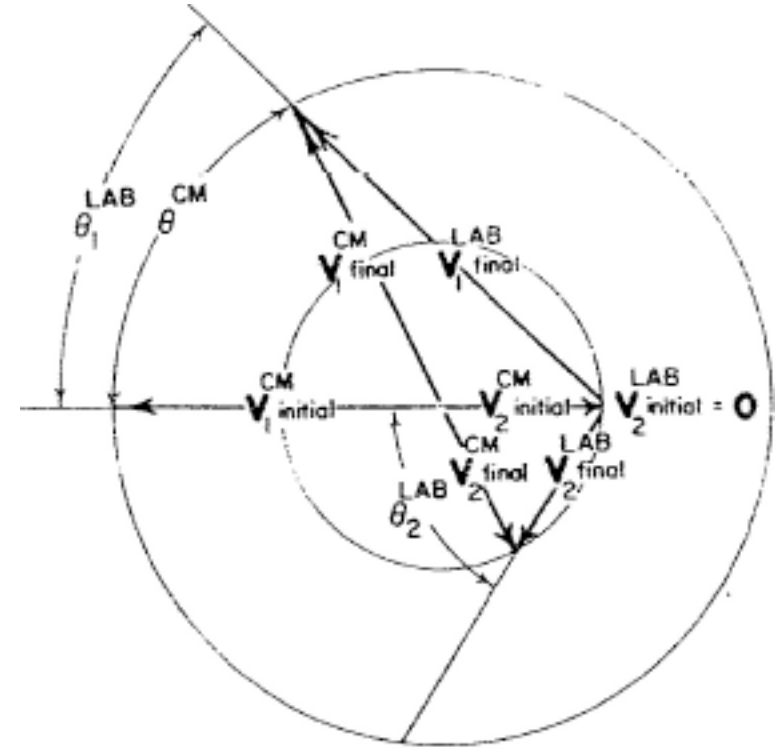


FIG. 4. Given the center of mass scattering angle θ^{CM} (from Fig. 3) and the mass ratio (2:1 in this case) a vector addition construction produces angles $\theta_1{}^{LAB}$ and $\theta_2{}^{LAB}$ shown here.

22

Geometrical Aspects of Classical Coulomb Scattering





The trouble with the Coulomb field is...

$$\int t^{-1}dt = \ln t + C$$

FIG. 5. The laboratory picture of Fig. 3. The scattering begins with both particles infinitely far to the right. The heavier particle is at rest and the lighter particle is moving left about 0.3 mile per day in the scale of this drawing. When the heavier particle first appears on this picture, one or two years before the "collision," it is creeping extremely slowly leftward, while the lighter particle is still over a hundred miles off to the right. The heavier particle continues creeping until finally the lighter particle arrives in the picture and moves through in about 12 sec. Most of the momentum is transferred in 3 or 4 sec.

$$v_2^{\mathbf{LAB}}(t) = \int (|F|/m_2)dt$$

$$\cong \int k\,dt/m_2[v_1^{\mathrm{CM}}(\mathrm{initial})t]^2$$

$$\cong [-k/m_2 v_1^{\mathrm{CM}}(\mathrm{initial})^2]t^{-1}$$

From: [Geometric aspects of classical Coulomb scattering](#)
American Journal of Physics 40,1852-1856 (1972)
Class project when I taught Jr. CM at Georgia Tech
(Just 5 students)

1856 / December 1972

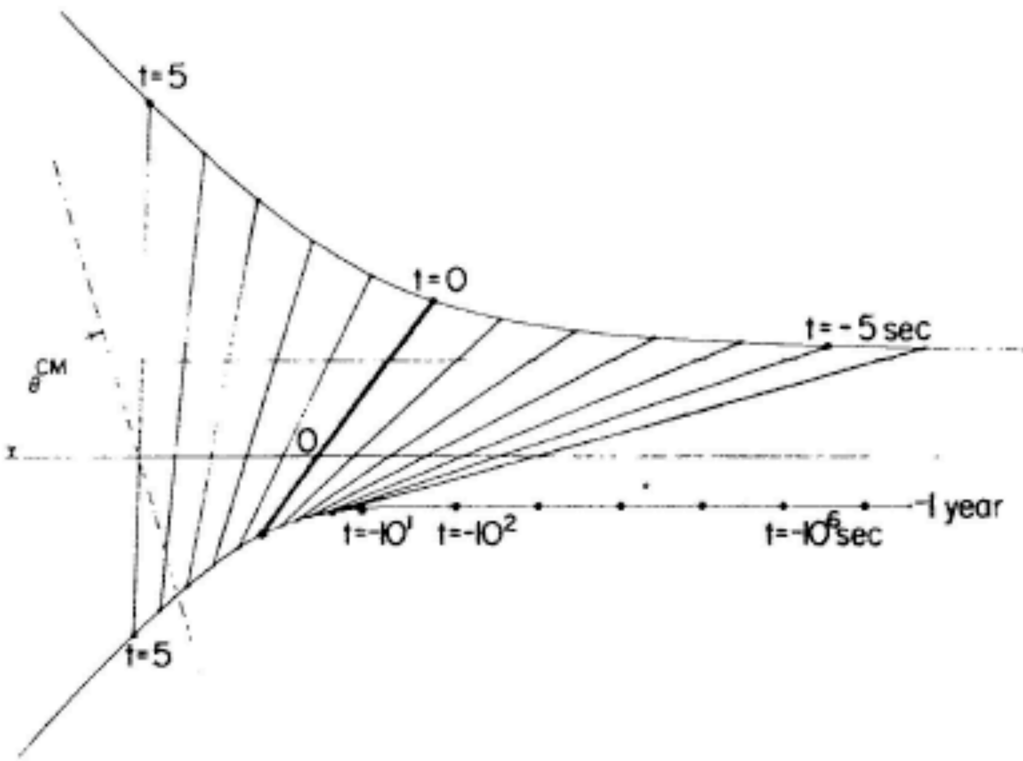


FIG. 5. The laboratory picture of Fig. 3. The scattering begins with both particles infinitely far to the right. The heavier particle is at rest and the lighter particle is moving left about 0.3 mile per day in the scale of this drawing. When the heavier particle first appears on this picture, one or two years before the "collision," it is creeping extremely slowly leftward, while the lighter particle is still over a hundred miles off to the right. The heavier particle continues creeping until finally the lighter particle arrives in the picture and moves through in about 12 sec. Most of the momentum is transferred in 3 or 4 sec.

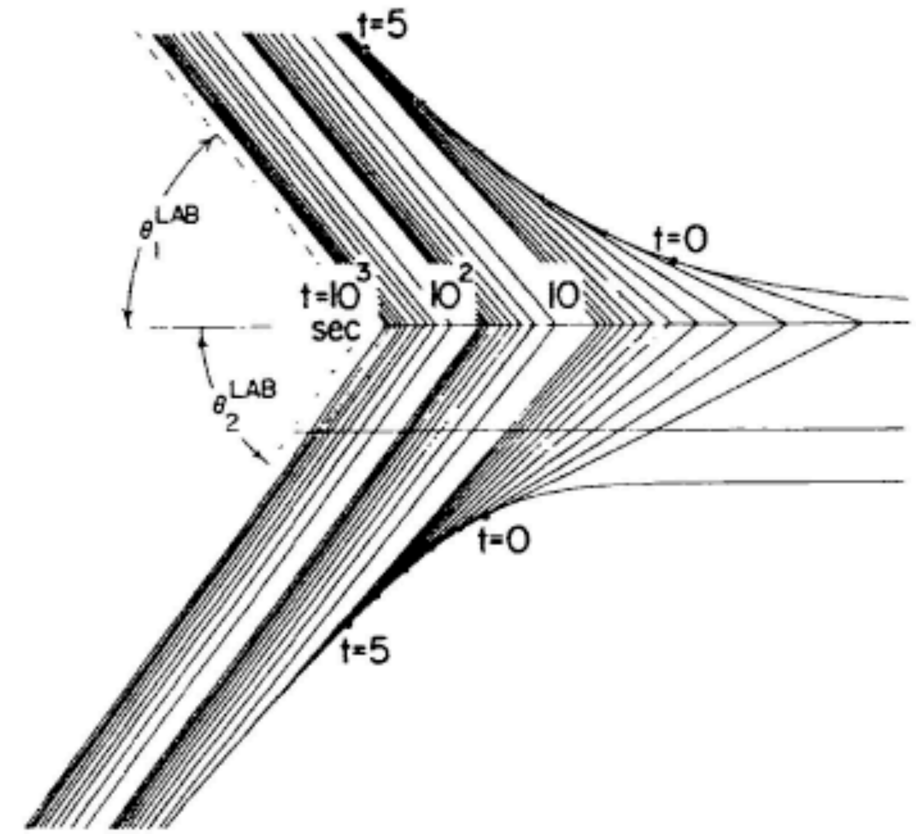


FIG. 6. Logarithmic recession of tangents demonstrates the nonexistence of asymptotes, for pure Coulomb scattering in laboratory system. At $t=10^3$ the slopes of the tangents are shy of θ_1^{LAB} and θ_2^{LAB} by only 0.02° and 0.04°, respectively.

From: Geometric aspects of classical Coulomb scattering
American Journal of Physics 40,1852-1856 (1972)
Class project when I taught Jr. CM at Georgia Tech
(Just 5 students)

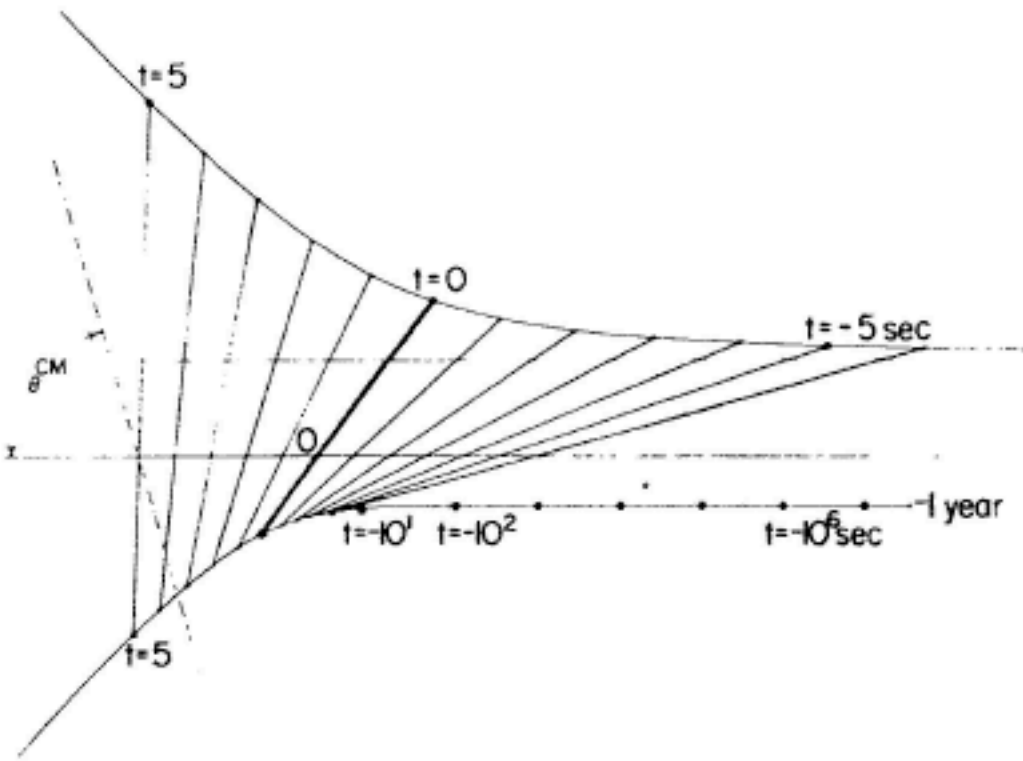


FIG. 5. The laboratory picture of Fig. 3. The scattering begins with both particles infinitely far to the right. The heavier particle is at rest and the lighter particle is moving left about 0.3 mile per day in the scale of this drawing. When the heavier particle first appears on this picture, one or two years before the "collision," it is creeping extremely slowly leftward, while the lighter particle is still over a hundred miles off to the right. The heavier particle continues creeping until finally the lighter particle arrives in the picture and moves through in about 12 sec. Most of the momentum is transferred in 3 or 4 sec.

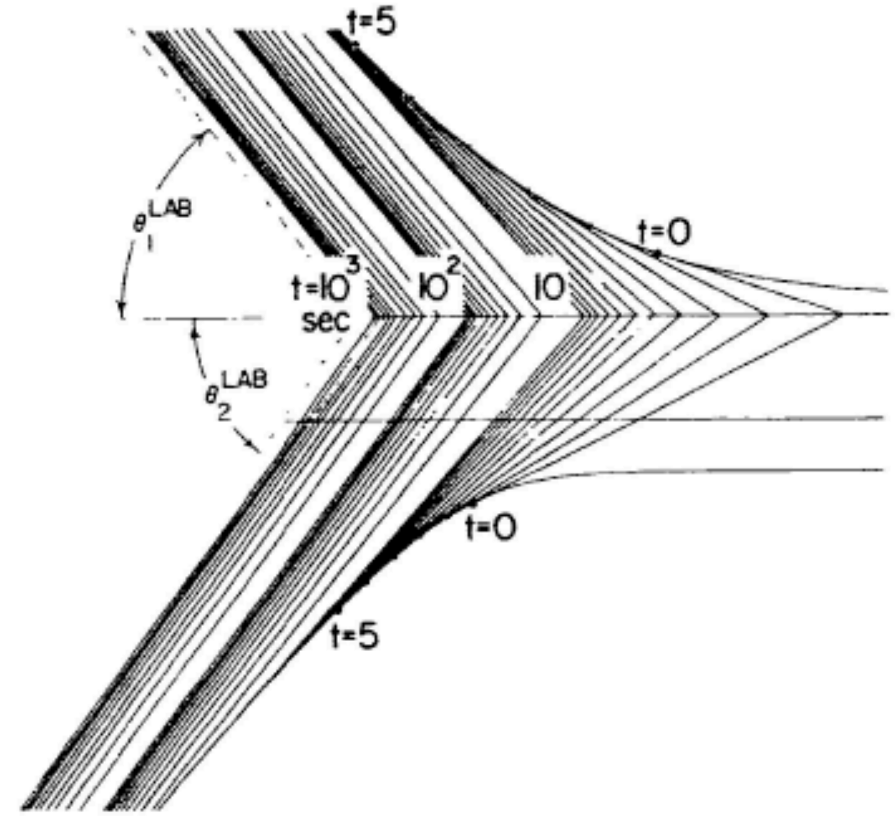


FIG. 6. Logarithmic recession of tangents demonstrates the nonexistence of asymptotes, for pure Coulomb scattering in laboratory system. At $t = 10^3$ the slopes of the tangents are shy of θ_1^{LAB} and θ_2^{LAB} by only 0.02° and 0.04°, respectively.

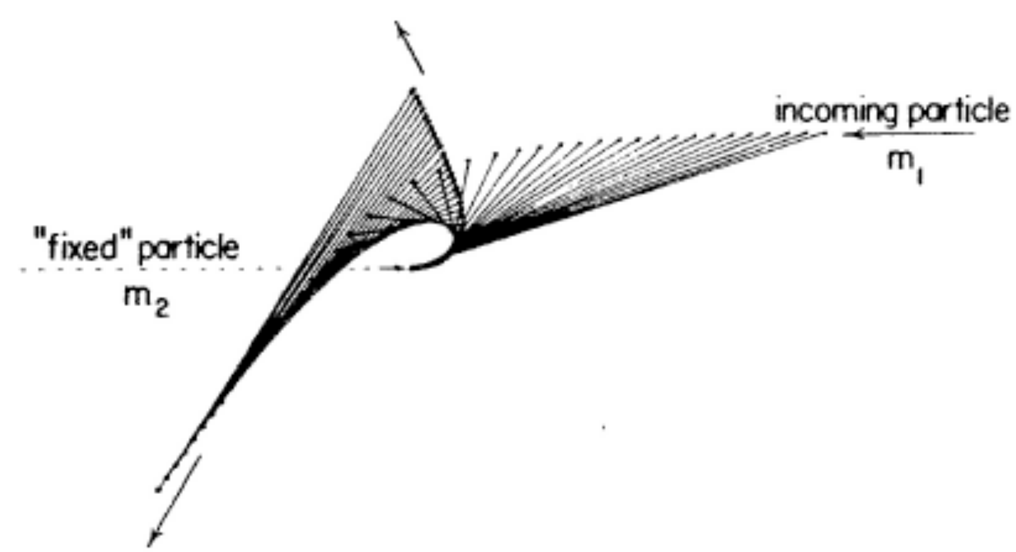


FIG. 7. Attractive Coulomb scattering in laboratory system. This has the same "anomalies" as the repulsive case.

From: [Geometric aspects of classical Coulomb scattering](#)
American Journal of Physics 40,1852-1856 (1972)
Class project when I taught Jr. CM at Georgia Tech
(Just 5 students)

Rotational equivalent of Newton's $\mathbf{F}=d\mathbf{p}/dt$ *equations:* $\mathbf{N}=d\mathbf{L}/dt$

How to make my boomerang come back

The gyrocompass and mechanical spin analogy

Rotational equivalent of Newton's **F**=*d***p**/*dt equations:* **N**=*d***L**/*dt*

Angular momentum vector $\mathbf{L}_j$ of a mass m_j is its linear momentum $\mathbf{p}_j$ times its lever arm as given by the *angular momentum cross-product relation* $\mathbf{L}_j = \mathbf{r}_j \times m_j \dot{\mathbf{r}}_j \equiv \mathbf{r}_j \times \mathbf{p}_j$

Rotational equivalent of Newton's **F**=*d***p**/*dt equations:* **N**=*d***L**/*dt*

Angular momentum vector $\mathbf{L}_j$ of a mass m_j is its linear momentum $\mathbf{p}_j$ times its lever arm as given by the *angular momentum cross-product relation* $\mathbf{L}_j = \mathbf{r}_j \times m_j \dot{\mathbf{r}}_j \equiv \mathbf{r}_j \times \mathbf{p}_j$

The sum-total angular momentum is $\mathbf{L} = \mathbf{L}^{total} = \sum_{j=1}^{3} \mathbf{L}_j = \sum_{j=1}^{3} \mathbf{r}_j \times m_j \dot{\mathbf{r}}_j$

Rotational equivalent of Newton's **F**=d**p**/dt *equations:* **N**=d**L**/dt

Angular momentum vector $\mathbf{L}_j$ of a mass m_j is its linear momentum $\mathbf{p}_j$ times its lever arm as given by the *angular momentum cross-product relation* $\mathbf{L}_j = \mathbf{r}_j \times m_j \dot{\mathbf{r}}_j \equiv \mathbf{r}_j \times \mathbf{p}_j$

The sum-total angular momentum is $\mathbf{L} = \mathbf{L}^{total} = \sum_{j=1}^{3} \mathbf{L}_j = \sum_{j=1}^{3} \mathbf{r}_j \times m_j \dot{\mathbf{r}}_j$

d**L** /dt gives a rotor Newton equation relating rotor momentum **r**×**p** to rotor force or *torque* **r**×**F**.

$$\frac{d\mathbf{L}}{dt} = \sum_{j=1}^{3} \mathbf{r}_j \times m_j \ddot{\mathbf{r}}_j = \sum_{j=1}^{3} \mathbf{r}_j \times \mathbf{F}_j^{total}$$

$$= \sum_{j=1}^{3} \mathbf{r}_j \times \mathbf{F}_j^{applied} + \sum_{j=1}^{3} \mathbf{r}_j \times \left(\sum_{k=1(k\neq j)}^{3} \mathbf{F}_{jk}^{constraint} \right)$$

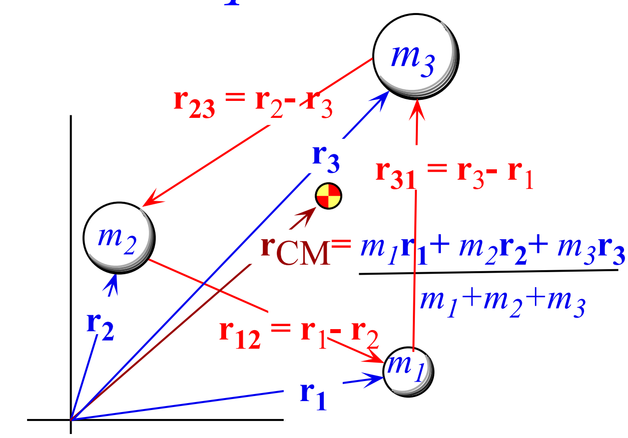


Fig. 6.4.1 Three-particle coordinate vectors

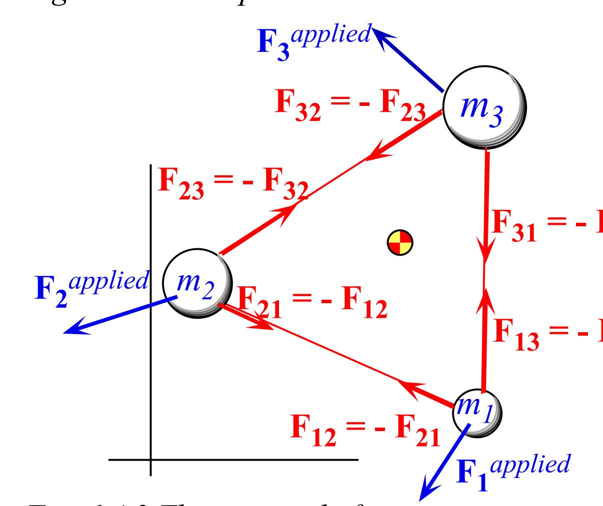


Fig. 6.4.2 Three-particle force vectors

Angular momentum vector $\mathbf{L}_j$ of a mass m_j is its linear momentum $\mathbf{p}_j$ times its lever arm as given by the *angular momentum cross-product relation* $\mathbf{L}_j = \mathbf{r}_j \times m_j \dot{\mathbf{r}}_j \equiv \mathbf{r}_j \times \mathbf{p}_j$

The sum-total angular momentum is $\mathbf{L} = \mathbf{L}^{total} = \sum_{j=1}^{3} \mathbf{L}_j = \sum_{j=1}^{3} \mathbf{r}_j \times m_j \dot{\mathbf{r}}_j$

d**L** /dt gives a rotor Newton equation relating rotor momentum **r**×**p** to rotor force or *torque* **r**×**F**.

$$\frac{d\mathbf{L}}{dt} = \sum_{j=1}^{3} \mathbf{r}_j \times m_j \ddot{\mathbf{r}}_j = \sum_{j=1}^{3} \mathbf{r}_j \times \mathbf{F}_j^{total}$$

$$= \sum_{j=1}^{3} \mathbf{r}_j \times \mathbf{F}_j^{applied} + \sum_{j=1}^{3} \mathbf{r}_j \times \left(\sum_{k=1(k\neq j)}^{3} \mathbf{F}_{jk}^{constraint} \right)$$

Internal constraint or coupling force terms appear at first to be a nuisance.

$$\sum_{j=1}^{3} \sum_{k=1(k\neq j)}^{3} \mathbf{r}_j \times \mathbf{F}_{jk}^{constraint} = \mathbf{r}_1 \times \left(\mathbf{F}_{12} + \mathbf{F}_{13}^{constraint}\right) + \mathbf{r}_2 \times \left(\mathbf{F}_{21} + \mathbf{F}_{23}^{constraint}\right) + \mathbf{r}_3 \times \left(\mathbf{F}_{31} + \mathbf{F}_{32}^{constraint}\right)$$

$$= \left(\mathbf{r}_1 - \mathbf{r}_2\right) \times \mathbf{F}_{12}^{constraint} + \left(\mathbf{r}_1 - \mathbf{r}_3\right) \times \mathbf{F}_{13}^{constraint} + \left(\mathbf{r}_2 - \mathbf{r}_3\right) \times \mathbf{F}_{23}^{constraint} = \mathbf{0}$$

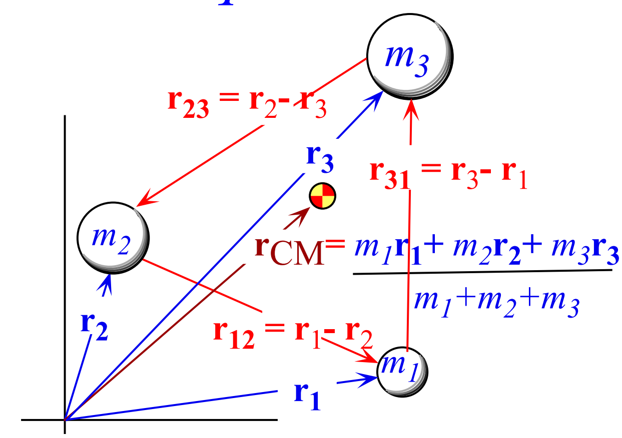


Fig. 6.4.1 Three-particle coordinate vectors

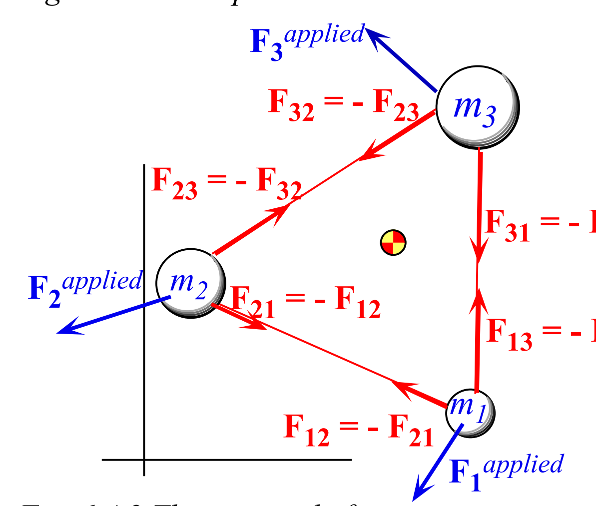


Fig. 6.4.2 Three-particle force vectors

Rotational equivalent of Newton's **F**=*d***p**/*dt equations:* **N**=*d***L**/*dt*

Angular momentum vector $\mathbf{L}_j$ of a mass m_j is its linear momentum $\mathbf{p}_j$ times its lever arm as given by the *angular momentum cross-product relation* $\mathbf{L}_j = \mathbf{r}_j \times m_j \dot{\mathbf{r}}_j \equiv \mathbf{r}_j \times \mathbf{p}_j$

The sum-total angular momentum is $\mathbf{L} = \mathbf{L}^{total} = \sum_{j=1}^{3} \mathbf{L}_j = \sum_{j=1}^{3} \mathbf{r}_j \times m_j \dot{\mathbf{r}}_j$

d**L** /dt gives a rotor Newton equation relating rotor momentum **r**×**p** to rotor force or *torque* **r**×**F**.

$$\frac{d\mathbf{L}}{dt} = \sum_{j=1}^{3} \mathbf{r}_j \times m_j \ddot{\mathbf{r}}_j = \sum_{j=1}^{3} \mathbf{r}_j \times \mathbf{F}_j^{total}$$

$$= \sum_{j=1}^{3} \mathbf{r}_j \times \mathbf{F}_j^{applied} + \sum_{j=1}^{3} \mathbf{r}_j \times \left(\sum_{k=1(k\neq j)}^{3} \mathbf{F}_{jk}^{constraint} \right)$$

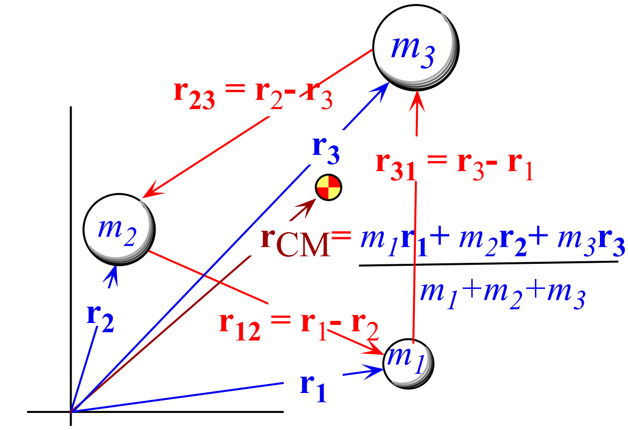


Fig. 6.4.1 Three-particle coordinate vectors

Internal constraint or coupling force terms appear at first to be a nuisance.

$$\sum_{j=1}^{3} \sum_{k=1(k\neq j)}^{3} \mathbf{r}_j \times \mathbf{F}_{jk}^{constraint} = \mathbf{r}_1 \times \left(\mathbf{F}_{12} + \mathbf{F}_{13}^{constraint} \right) + \mathbf{r}_2 \times \left(\mathbf{F}_{21} + \mathbf{F}_{23}^{constraint} \right) + \mathbf{r}_3 \times \left(\mathbf{F}_{31} + \mathbf{F}_{32}^{constraint} \right)$$

$$= \left(\mathbf{r}_1 - \mathbf{r}_2 \right) \times \mathbf{F}_{12}^{constraint} + \left(\mathbf{r}_1 - \mathbf{r}_3 \right) \times \mathbf{F}_{13}^{constraint} + \left(\mathbf{r}_2 - \mathbf{r}_3 \right) \times \mathbf{F}_{23}^{constraint} = \mathbf{0}$$

However, they vanish if coupling forces act along lines connecting the masses.

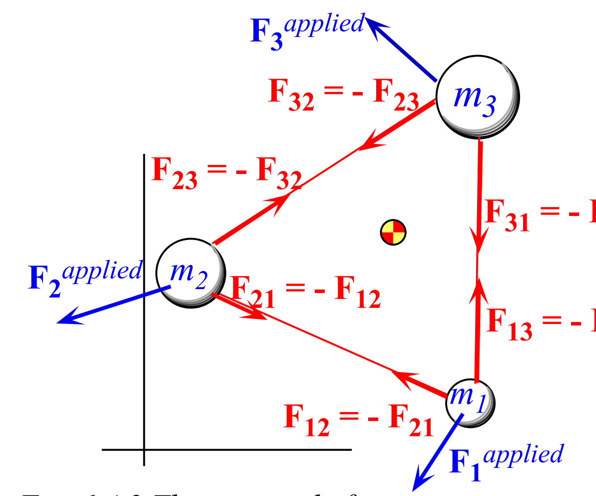


Fig. 6.4.2 Three-particle force vectors

Rotational equivalent of Newton's $\mathbf{F}=d\mathbf{p}/dt$ *equations:* $\mathbf{N}=d\mathbf{L}/dt$

Angular momentum vector $\mathbf{L}_j$ of a mass m_j is its linear momentum $\mathbf{p}_j$ times its lever arm as given by the *angular momentum cross-product relation* $\mathbf{L}_j = \mathbf{r}_j \times m_j \dot{\mathbf{r}}_j \equiv \mathbf{r}_j \times \mathbf{p}_j$

The sum-total angular momentum is $\mathbf{L} = \mathbf{L}^{total} = \sum_{j=1}^{3} \mathbf{L}_j = \sum_{j=1}^{3} \mathbf{r}_j \times m_j \dot{\mathbf{r}}_j$

d**L** /dt gives a rotor Newton equation relating rotor momentum **r**×**p** to rotor force or *torque* **r**×**F**.

$$\frac{d\mathbf{L}}{dt} = \sum_{j=1}^{3} \mathbf{r}_j \times m_j \ddot{\mathbf{r}}_j = \sum_{j=1}^{3} \mathbf{r}_j \times \mathbf{F}_j^{total}$$

$$= \sum_{j=1}^{3} \mathbf{r}_j \times \mathbf{F}_j^{applied} + \sum_{j=1}^{3} \mathbf{r}_j \times \left(\sum_{k=1(k\neq j)}^{3} \mathbf{F}_{jk}^{constraint} \right)$$

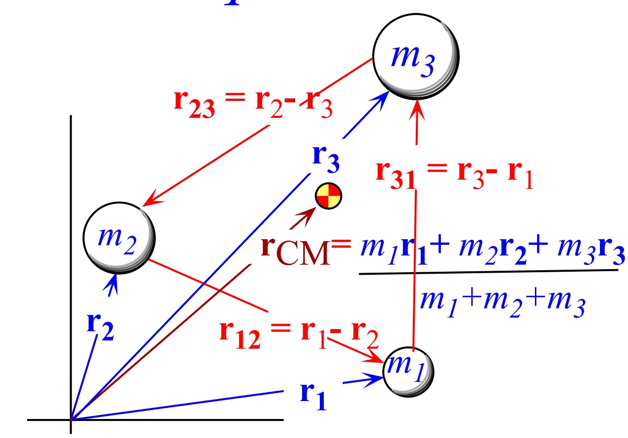


Fig. 6.4.1 Three-particle coordinate vectors

Internal constraint or coupling force terms appear at first to be a nuisance.

$$\sum_{j=1}^{3} \sum_{k=1(k\neq j)}^{3} \mathbf{r}_j \times \mathbf{F}_{jk}^{constraint} = \mathbf{r}_1 \times \left(\mathbf{F}_{12} + \mathbf{F}_{13}^{constraint}\right) + \mathbf{r}_2 \times \left(\mathbf{F}_{21} + \mathbf{F}_{23}^{constraint}\right) + \mathbf{r}_3 \times \left(\mathbf{F}_{31} + \mathbf{F}_{32}^{constraint}\right)$$

$$= \left(\mathbf{r}_1 - \mathbf{r}_2\right) \times \mathbf{F}_{12}^{constraint} + \left(\mathbf{r}_1 - \mathbf{r}_3\right) \times \mathbf{F}_{13}^{constraint} + \left(\mathbf{r}_2 - \mathbf{r}_3\right) \times \mathbf{F}_{23}^{constraint} = \mathbf{0}$$

However, they vanish if coupling forces act along lines connecting the masses.

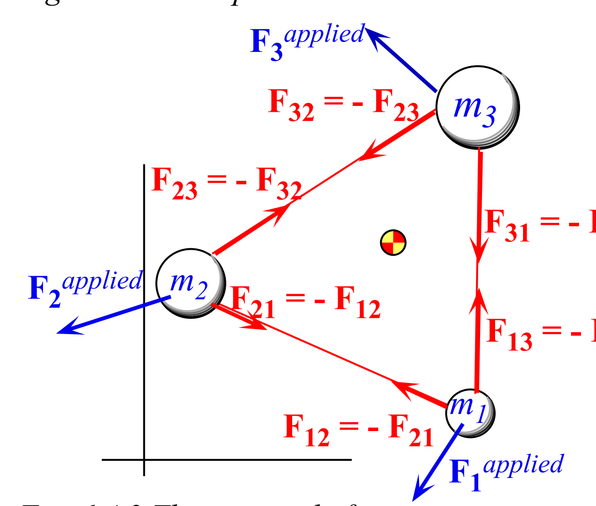


Fig. 6.4.2 Three-particle force vectors

The results are the *rotational Newton's equation*.

$$\frac{d\mathbf{L}}{dt} = \mathbf{N} \text{, where: } \mathbf{N} = \sum_{j=1}^{3} \mathbf{N}_j \quad \text{and:} \quad \mathbf{N}_j = \sum_{j=1}^{3} \mathbf{r}_j \times \mathbf{F}_j^{applied}$$

Angular momentum vector $\mathbf{L}_j$ of a mass m_j is its linear momentum $\mathbf{p}_j$ times its lever arm as given by the *angular momentum cross-product relation* $\mathbf{L}_j = \mathbf{r}_j \times m_j \dot{\mathbf{r}}_j \equiv \mathbf{r}_j \times \mathbf{p}_j$

The sum-total angular momentum is $\mathbf{L} = \mathbf{L}^{total} = \sum_{j=1}^{3} \mathbf{L}_j = \sum_{j=1}^{3} \mathbf{r}_j \times m_j \dot{\mathbf{r}}_j$

d**L** /dt gives a rotor Newton equation relating rotor momentum **r**x**p** to rotor force or *torque* **r**x**F**.

$$\frac{d\mathbf{L}}{dt} = \sum_{j=1}^{3} \mathbf{r}_j \times m_j \ddot{\mathbf{r}}_j = \sum_{j=1}^{3} \mathbf{r}_j \times \mathbf{F}_j^{total}$$

$$= \sum_{j=1}^{3} \mathbf{r}_j \times \mathbf{F}_j^{applied} + \sum_{j=1}^{3} \mathbf{r}_j \times \left(\sum_{k=1(k\neq j)}^{3} \mathbf{F}_{jk}^{constraint} \right)$$

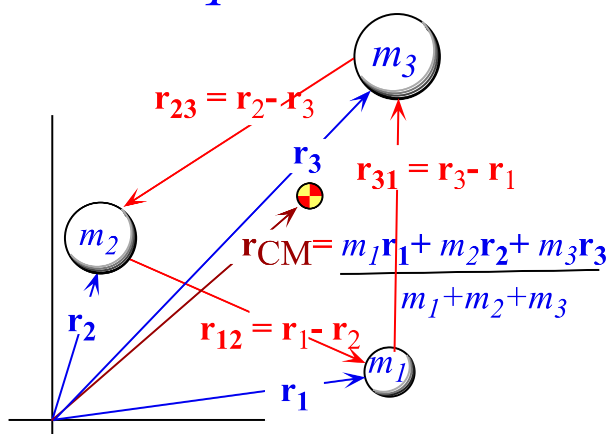


Fig. 6.4.1 Three-particle coordinate vectors

Internal constraint or coupling force terms appear at first to be a nuisance.

$$\sum_{j=1}^{3} \sum_{k=1(k\neq j)}^{3} \mathbf{r}_j \times \mathbf{F}_{jk}^{constraint} = \mathbf{r}_1 \times \left(\mathbf{F}_{12} + \mathbf{F}_{13}^{constraint}\right) + \mathbf{r}_2 \times \left(\mathbf{F}_{21} + \mathbf{F}_{23}^{constraint}\right) + \mathbf{r}_3 \times \left(\mathbf{F}_{31} + \mathbf{F}_{32}^{constraint}\right)$$

$$= \left(\mathbf{r}_1 - \mathbf{r}_2\right) \times \mathbf{F}_{12}^{constraint} + \left(\mathbf{r}_1 - \mathbf{r}_3\right) \times \mathbf{F}_{13}^{constraint} + \left(\mathbf{r}_2 - \mathbf{r}_3\right) \times \mathbf{F}_{23}^{constraint} = \mathbf{0}$$

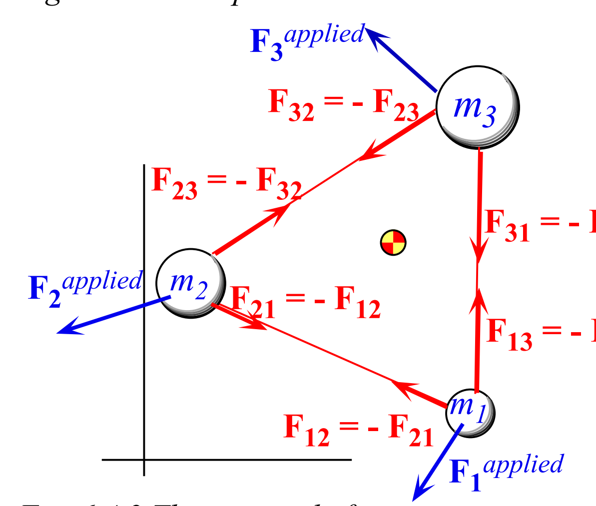


Fig. 6.4.2 Three-particle force vectors

However, they vanish if coupling forces act along lines connecting the masses.

The results are the *rotational Newton's equation*.

$$\frac{d\mathbf{L}}{dt} = \mathbf{N} \text{ , where: } \mathbf{N} = \sum_{j=1}^{3} \mathbf{N}_j \quad \text{and:} \quad \mathbf{N}_j = \sum_{j=1}^{3} \mathbf{r}_j \times \mathbf{F}_j^{applied}$$

Taken together with *translational Newton's equation* the six equations describe rigid body mechanics.

$$\frac{d\mathbf{P}}{dt} = \mathbf{F} \text{ , where: } \mathbf{F} = \sum_{j=1}^{3} \mathbf{F}_j^{applied}$$

Rotational equivalent of Newton's **F**=*d***p**/*dt equations:* **N**=*d***L**/*dt*

Angular momentum vector $\mathbf{L}_j$ of a mass m_j is its linear momentum $\mathbf{p}_j$ times its lever arm as given by the *angular momentum cross-product relation* $\mathbf{L}_j = \mathbf{r}_j \times m_j \dot{\mathbf{r}}_j \equiv \mathbf{r}_j \times \mathbf{p}_j$

The sum-total angular momentum is $\mathbf{L} = \mathbf{L}^{total} = \sum_{j=1}^{3} \mathbf{L}_j = \sum_{j=1}^{3} \mathbf{r}_j \times m_j \dot{\mathbf{r}}_j$

$d\mathbf{L}/dt$ gives a rotor Newton equation relating rotor momentum $\mathbf{r}\times\mathbf{p}$ to rotor force or *torque* $\mathbf{r}\times\mathbf{F}$.

$$\frac{d\mathbf{L}}{dt} = \sum_{j=1}^{3} \mathbf{r}_j \times m_j \ddot{\mathbf{r}}_j = \sum_{j=1}^{3} \mathbf{r}_j \times \mathbf{F}_j^{total}$$

$$= \sum_{j=1}^{3} \mathbf{r}_j \times \mathbf{F}_j^{applied} + \sum_{j=1}^{3} \mathbf{r}_j \times \left(\sum_{k=1(k\neq j)}^{3} \mathbf{F}_{jk}^{constraint} \right)$$

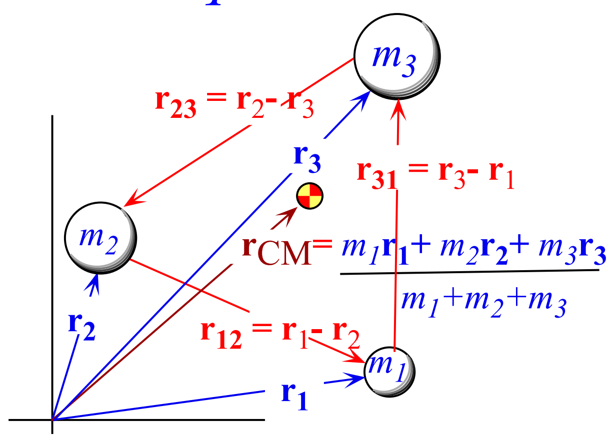


Fig. 6.4.1 Three-particle coordinate vectors

Internal constraint or coupling force terms appear at first to be a nuisance.

$$\sum_{j=1}^{3} \sum_{k=1(k\neq j)}^{3} \mathbf{r}_j \times \mathbf{F}_{jk}^{constraint} = \mathbf{r}_1 \times \left(\mathbf{F}_{12} + \mathbf{F}_{13}^{constraint} \right) + \mathbf{r}_2 \times \left(\mathbf{F}_{21} + \mathbf{F}_{23}^{constraint} \right) + \mathbf{r}_3 \times \left(\mathbf{F}_{31} + \mathbf{F}_{32}^{constraint} \right)$$

$$= \left(\mathbf{r}_1 - \mathbf{r}_2 \right) \times \mathbf{F}_{12}^{constraint} + \left(\mathbf{r}_1 - \mathbf{r}_3 \right) \times \mathbf{F}_{13}^{constraint} + \left(\mathbf{r}_2 - \mathbf{r}_3 \right) \times \mathbf{F}_{23}^{constraint} = \mathbf{0}$$

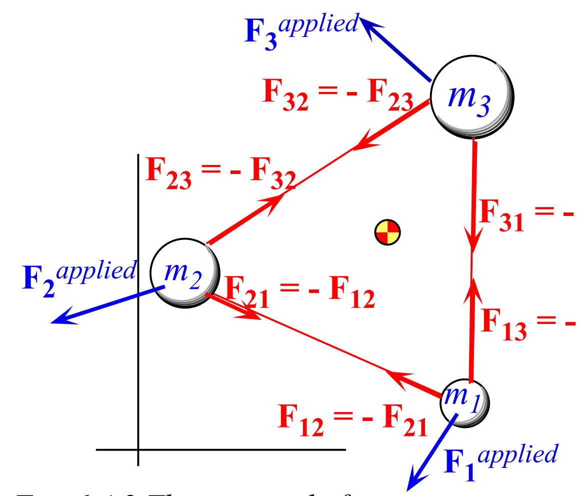


Fig. 6.4.2 Three-particle force vectors

However, they vanish if coupling forces act along lines connecting the masses.

The results are the *rotational Newton's equation*.

$$\frac{d\mathbf{L}}{dt} = \mathbf{N} \text{ , where: } \mathbf{N} = \sum_{j=1}^{3} \mathbf{N}_j \quad \text{and:} \quad \mathbf{N}_j = \sum_{j=1}^{3} \mathbf{r}_j \times \mathbf{F}_j^{applied}$$

Taken together with *translational Newton's equation* the six equations describe rigid body mechanics.

$$\frac{d\mathbf{P}}{dt} = \mathbf{F} \text{ , where: } \mathbf{F} = \sum_{j=1}^{3} \mathbf{F}_j^{applied}$$

Remaining 3N-6 equations consist of normal mode or GCC equations of some kind.

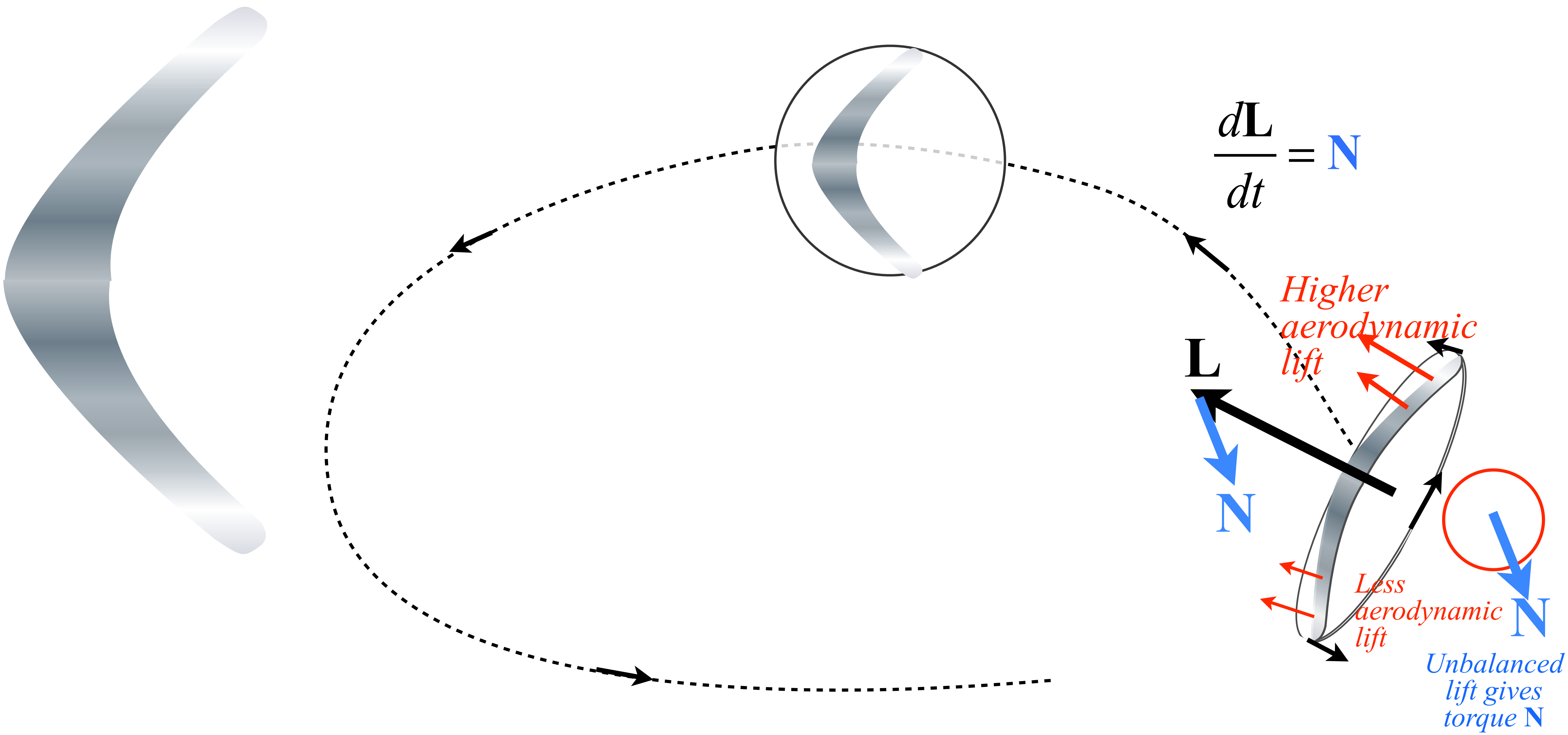
Rotational equivalent of Newton's $\mathbf{F}=d\mathbf{p}/dt$ *equations:* $\mathbf{N}=d\mathbf{L}/dt$

➔ *How to make my boomerang come back*

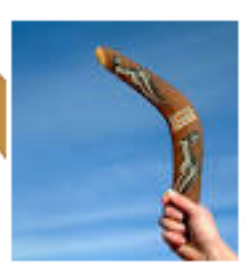
The gyrocompass and mechanical spin analogy

The Australian Boomerang (that comes back!)

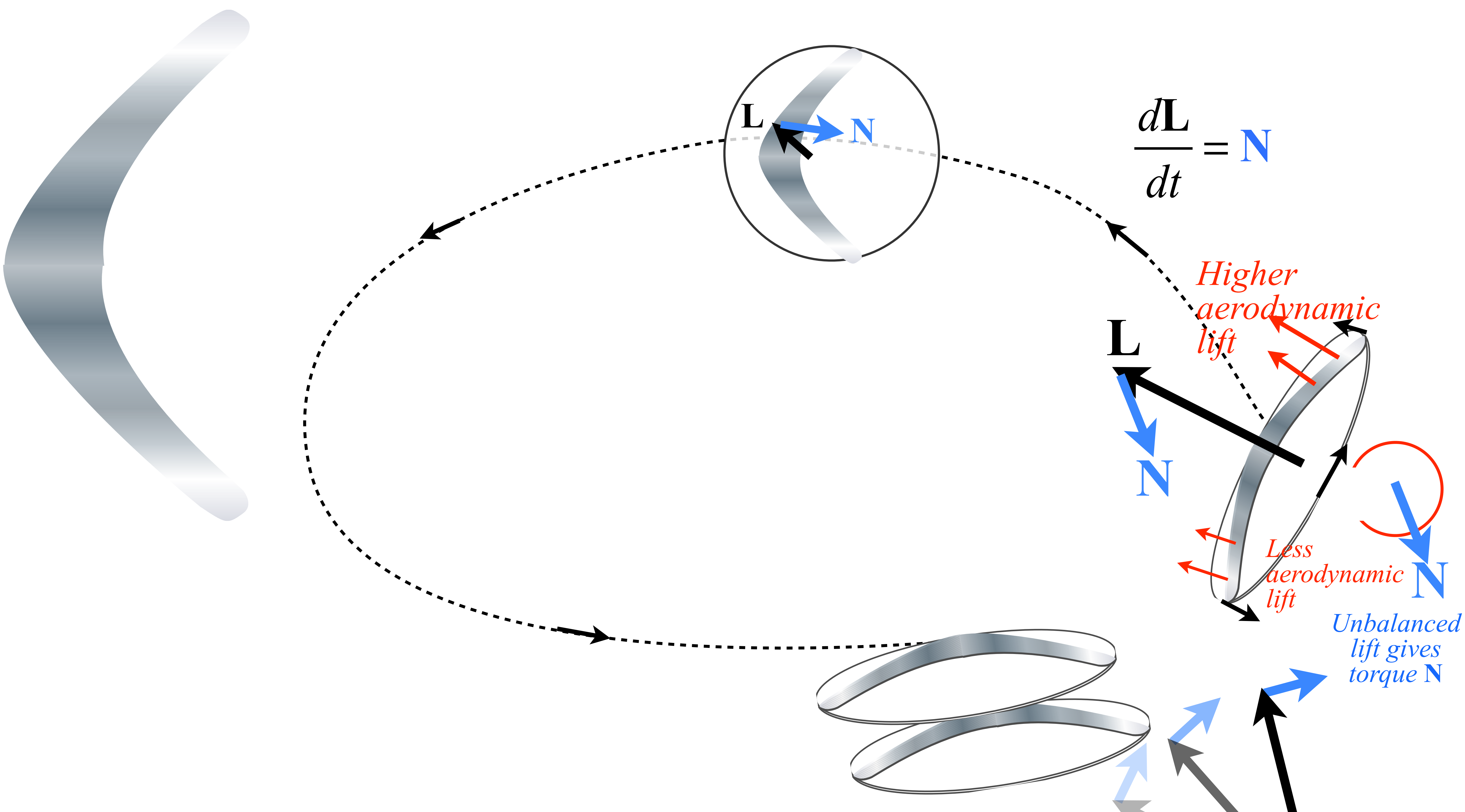
$$\frac{d\mathbf{L}}{dt} = \mathbf{N}$$







The Australian Boomerang (that comes back and hovers down!)



Small lifting torque due to "bad-air" of leading blade hitting trailing one left-to-right may cause boomerang to level and hover. Stronger effect in 3-blade boomers causes figure-8 paths.

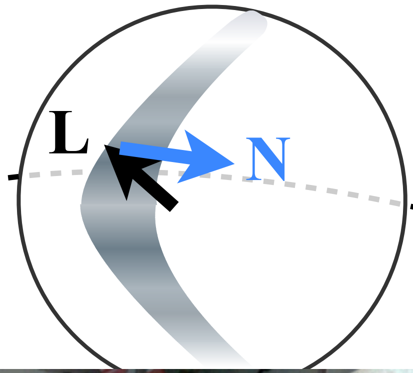
The Australian Boomerang (that comes back and hovers down!)

Charlie Drake's famous 1961 song:

My boomerang won't come back!

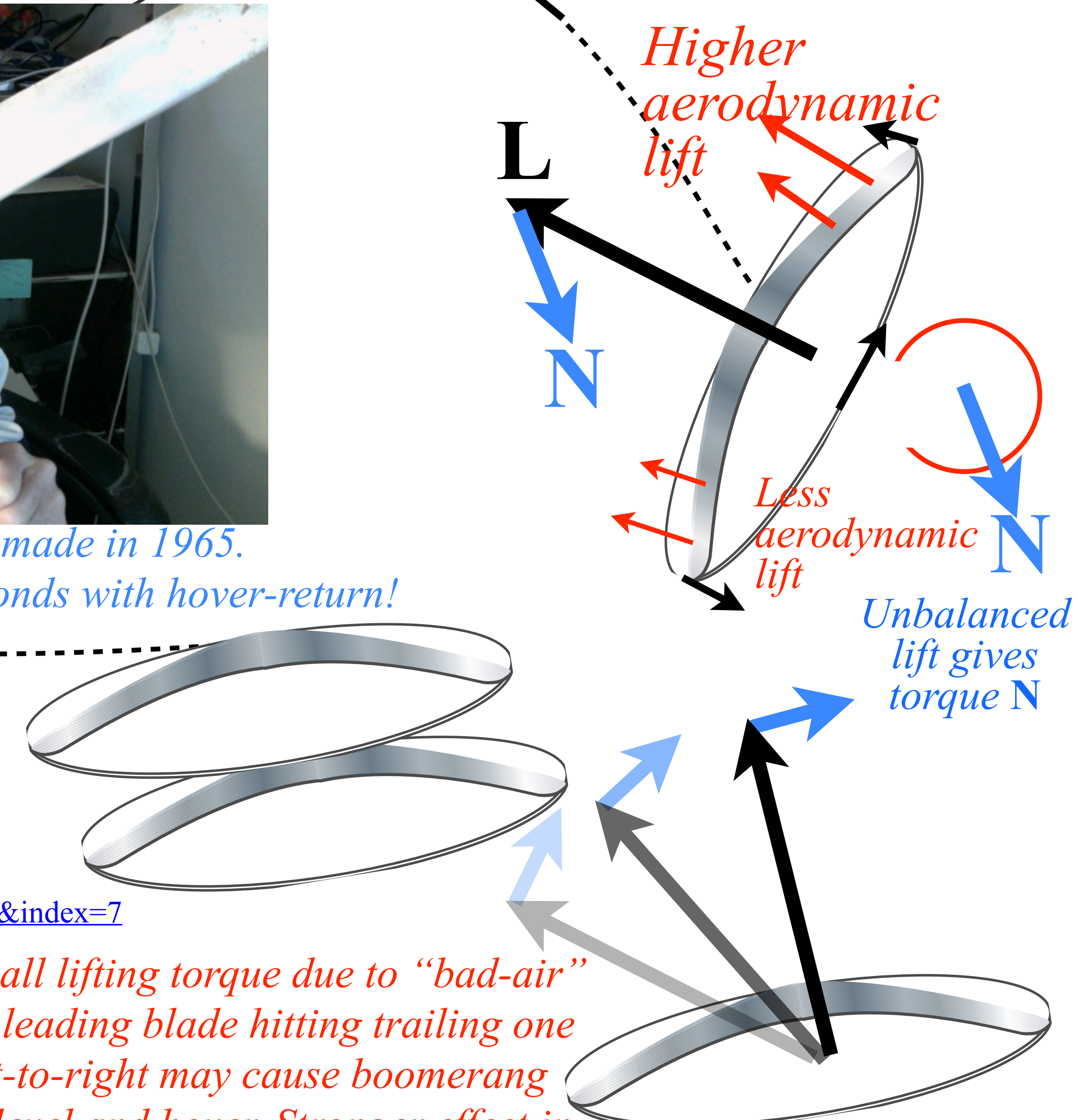
My boomerang won't come back!
My boomerang won't come back!
I've waved the thing all over the place
Practiced til' I was blue in the face*
I'm a big disgrace
to the Aborigine Race
My boomerang won't come back!

**blue later replaced black*

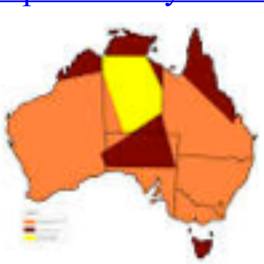


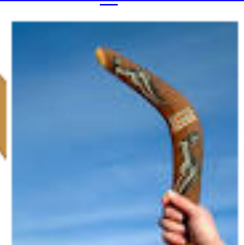
$$\frac{d\mathbf{L}}{dt} = \mathbf{N}$$

Aluminum boomerang I made in 1965.
It once flew over 18 seconds with hover-return!



https://www.youtube.com/watch?v=EXJR5NWM_xI&list=PLGwmGldCxzLxbPlFVG8Z89WZIBuT4m0Ii&index=7





Small lifting torque due to "bad-air" of leading blade hitting trailing one left-to-right may cause boomerang to level and hover. Stronger effect in 3-blade boomers causes figure-8 paths.

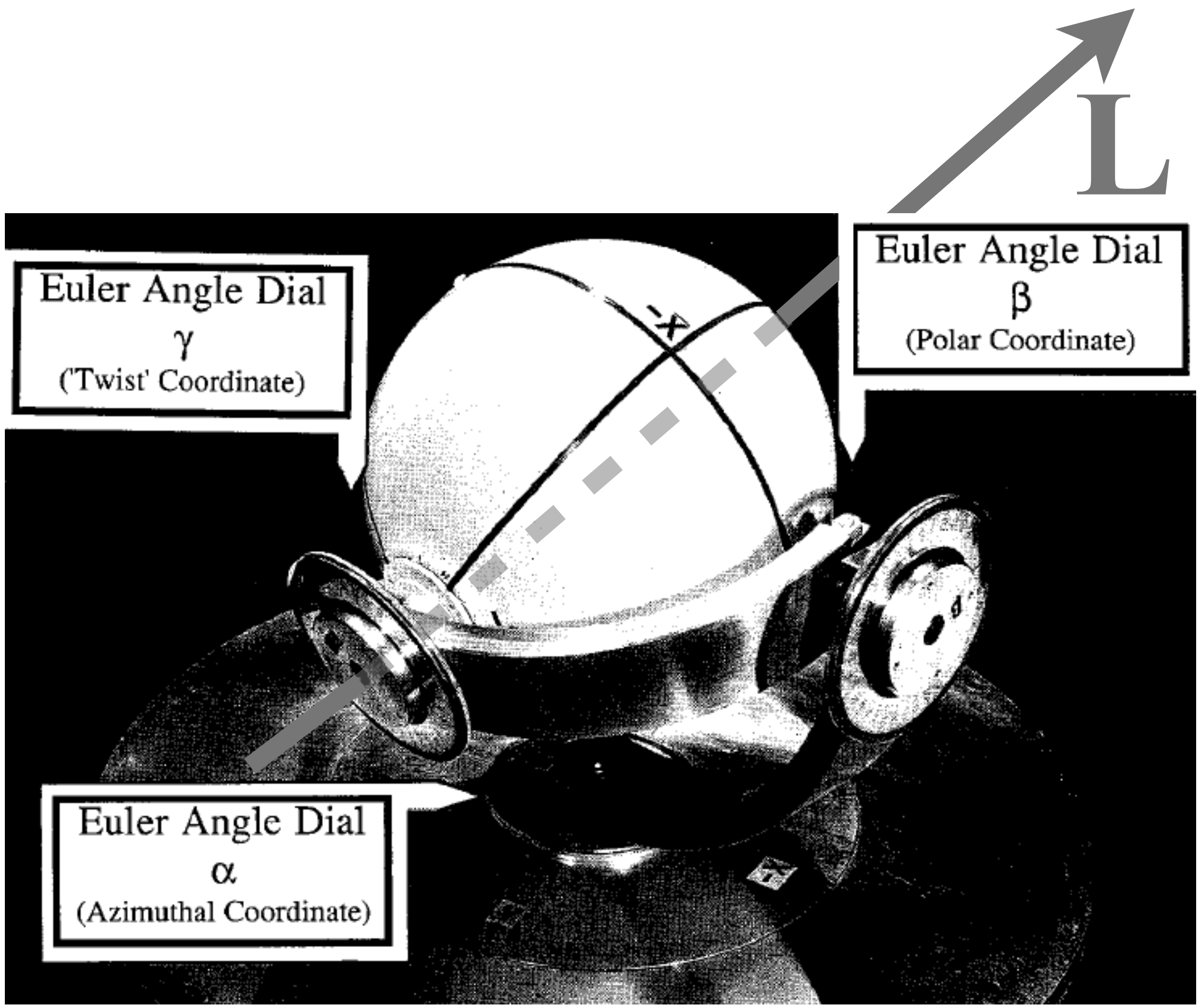
Rotational equivalent of Newton's $\mathbf{F}=d\mathbf{p}/dt$ *equations:* $\mathbf{N}=d\mathbf{L}/dt$

How to make my boomerang come back

➡ *The gyrocompass and mechanical spin analogy*

The gyrocompass and mechanical spin analogy

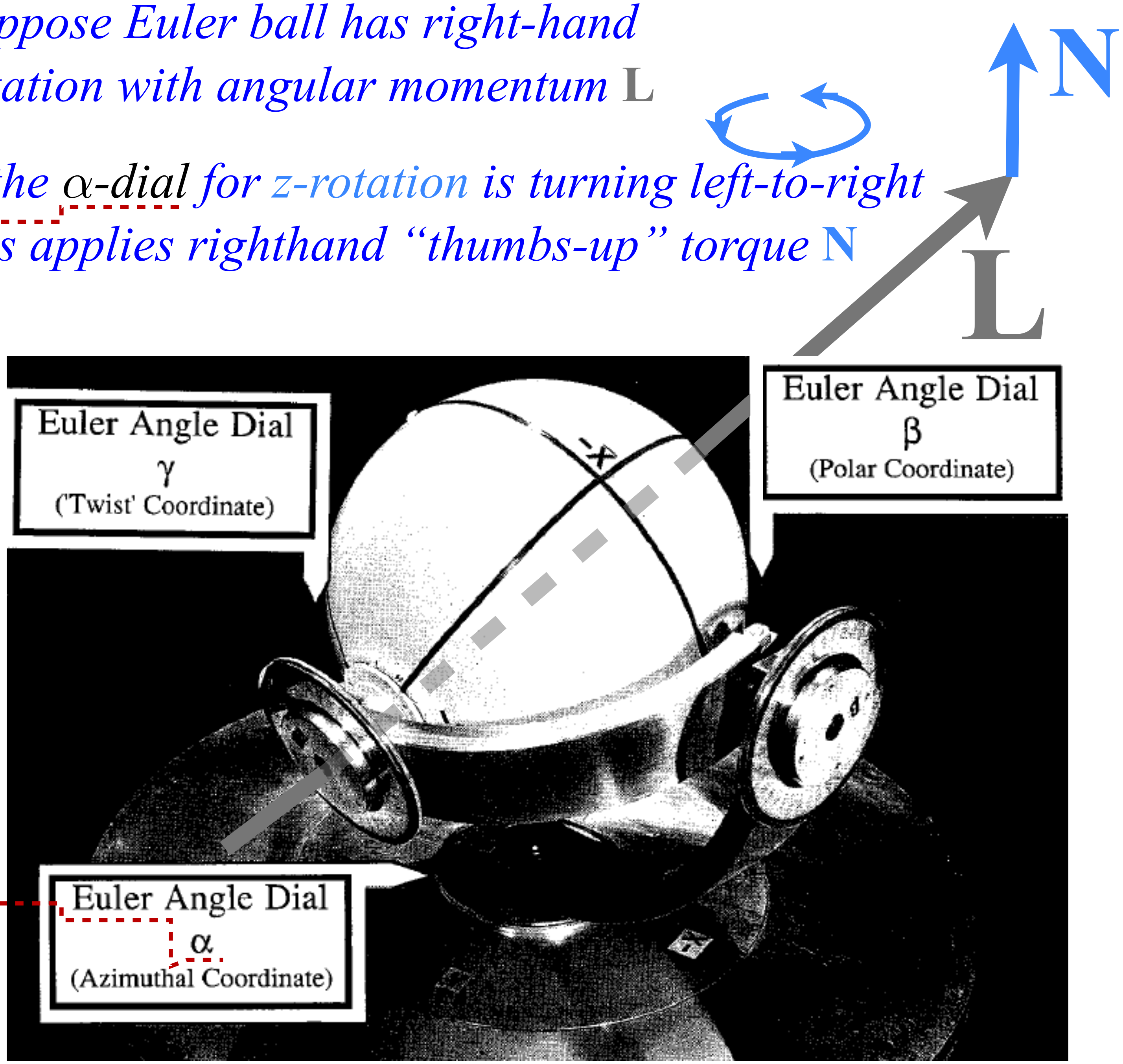
Suppose Euler ball has right-hand rotation with angular momentum **L**



The gyrocompass and mechanical spin analogy

Suppose Euler ball has right-hand rotation with angular momentum **L**

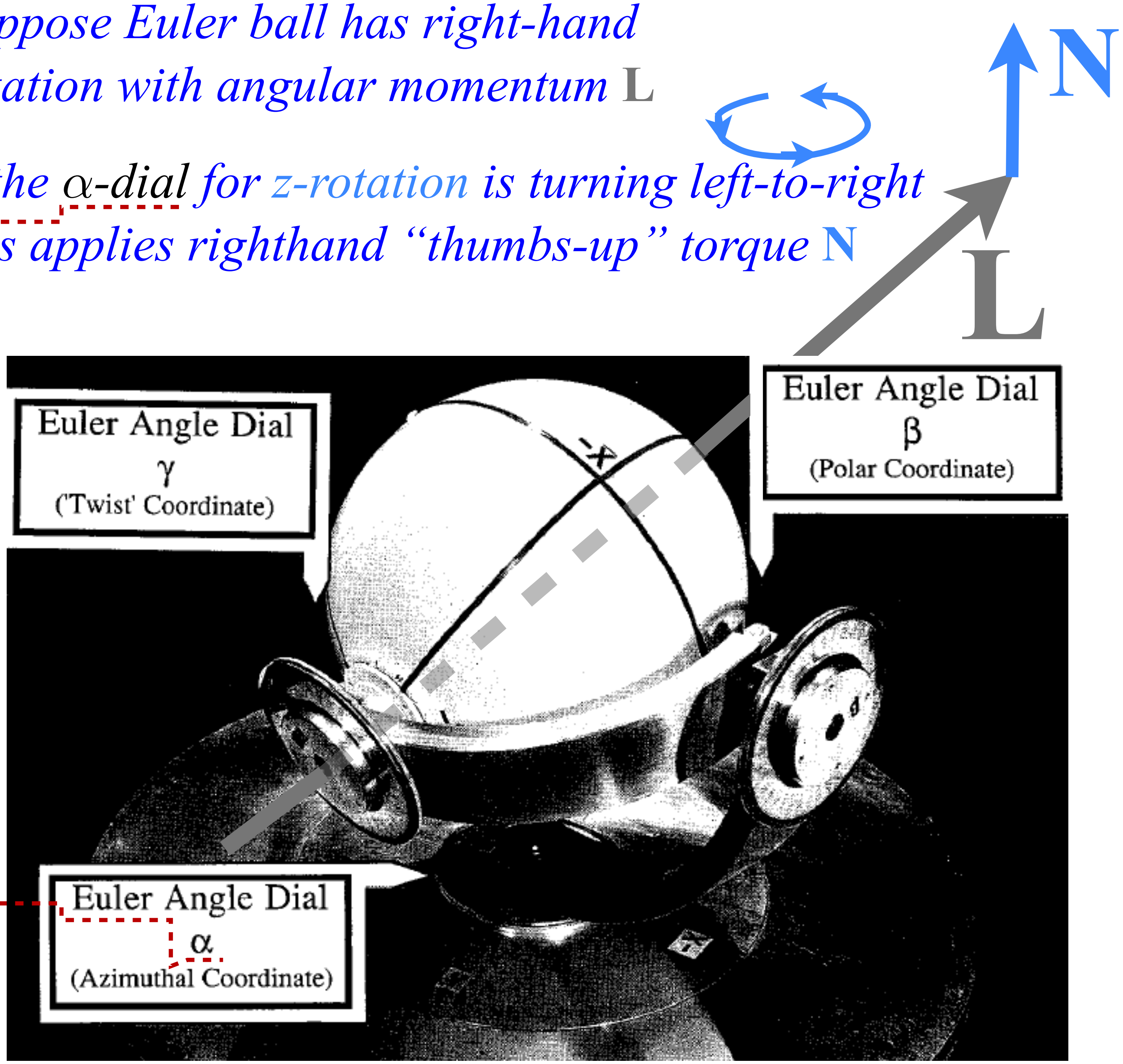
If the α-dial for z-rotation is turning left-to-right this applies righthand "thumbs-up" torque **N**

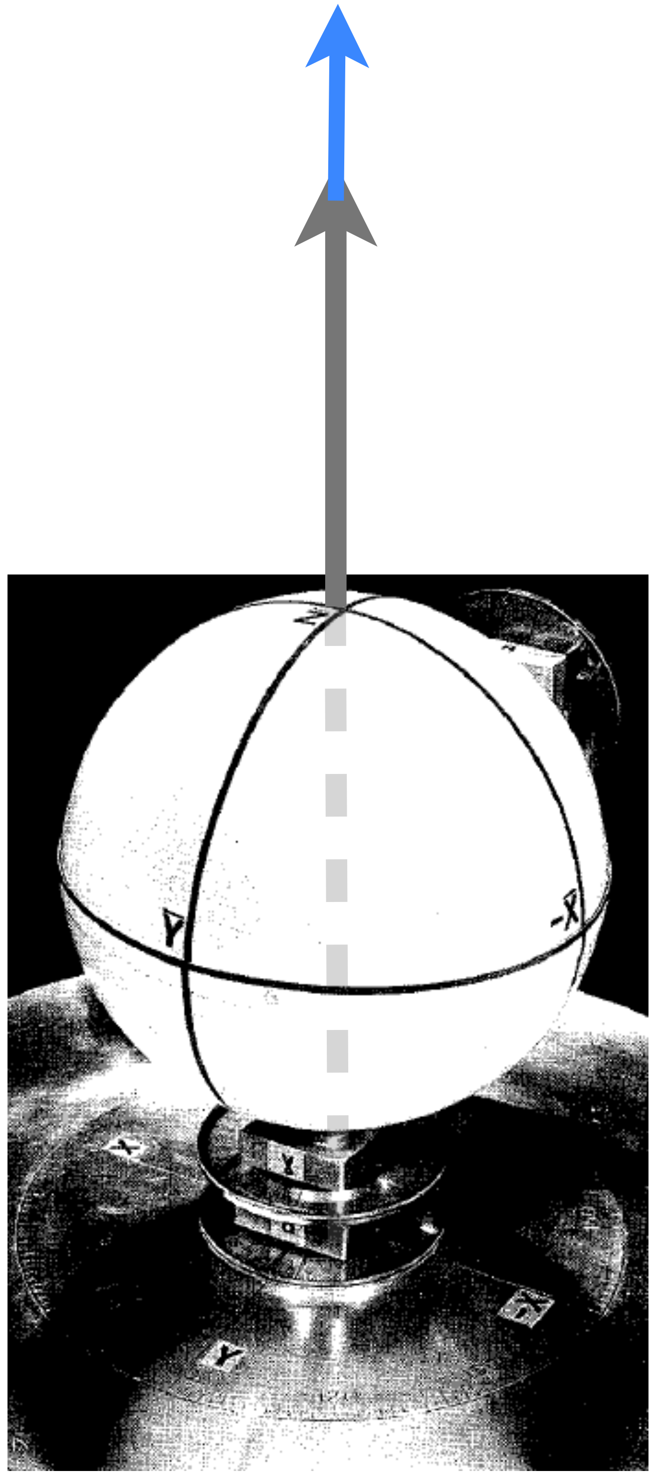


The gyrocompass and mechanical spin analogy

Suppose Euler ball has right-hand rotation with angular momentum **L**

If the α-dial for z-rotation is turning left-to-right this applies righthand "thumbs-up" torque **N**



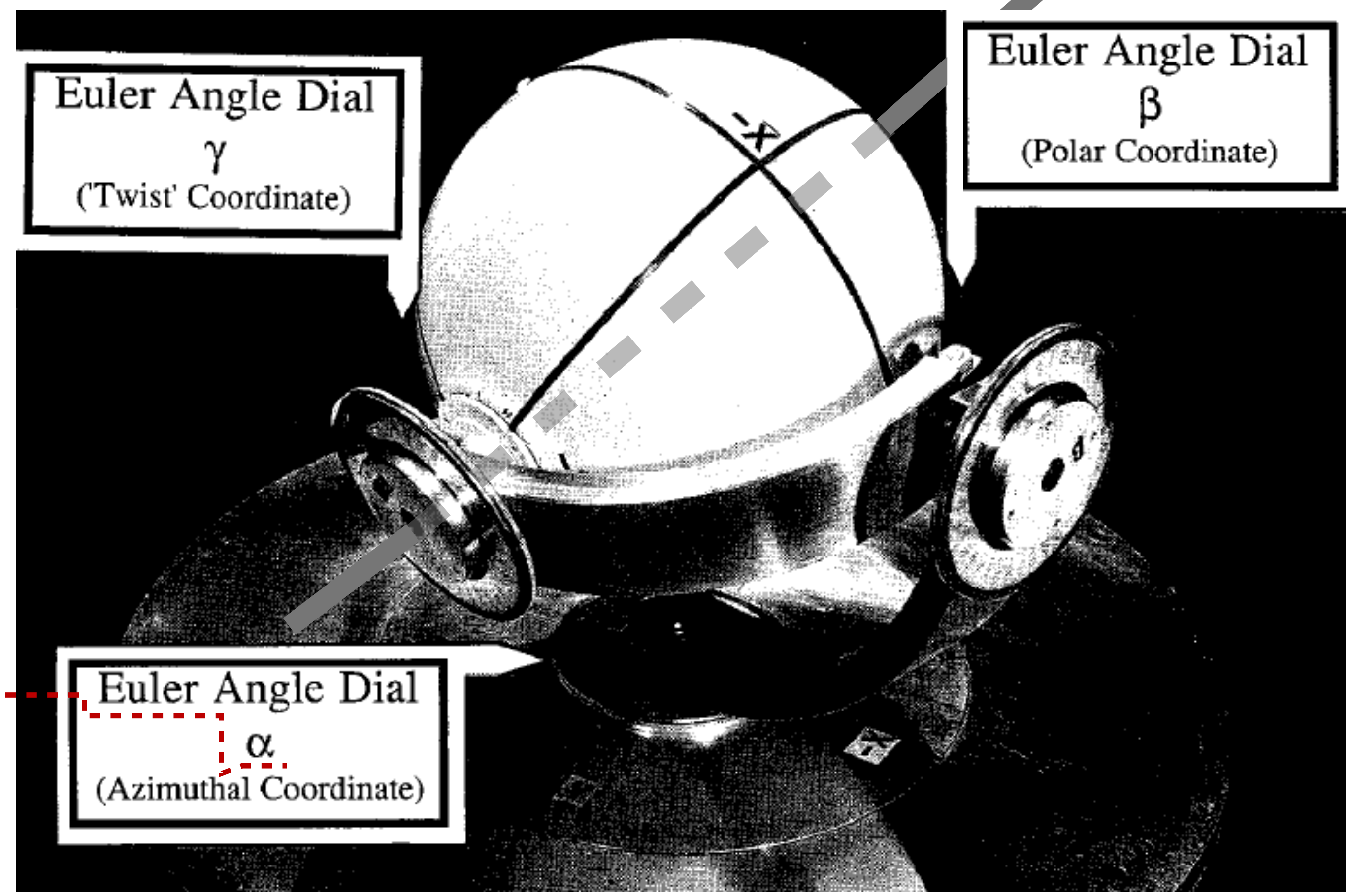


Then the ball tends to line-up with z-axis (and may go past z, then come back, etc. in a precessional or "hunting" motion)

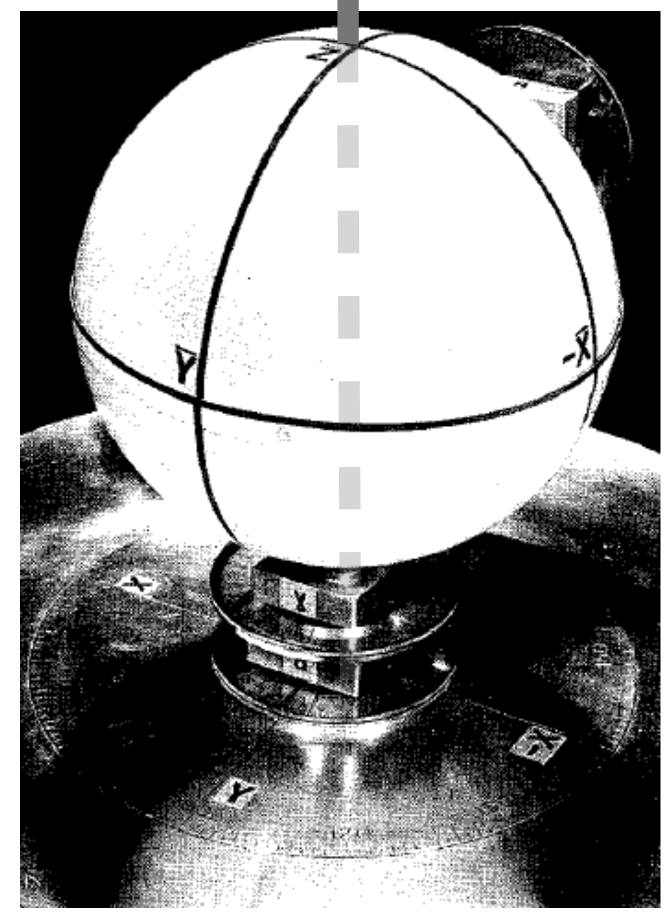
The gyrocompass and mechanical spin analogy

Suppose Euler ball has right-hand rotation with angular momentum **L**

If the α-dial *for z-rotation is turning left-to-right this applies righthand "thumbs-up" torque* **N**



A very high speed ball in a gyro-compass will similarly seek true North due to Earth rotation.



Then the ball tends to line-up with z-axis (and may go past z, then come back, etc. in a precessional or "hunting" motion)

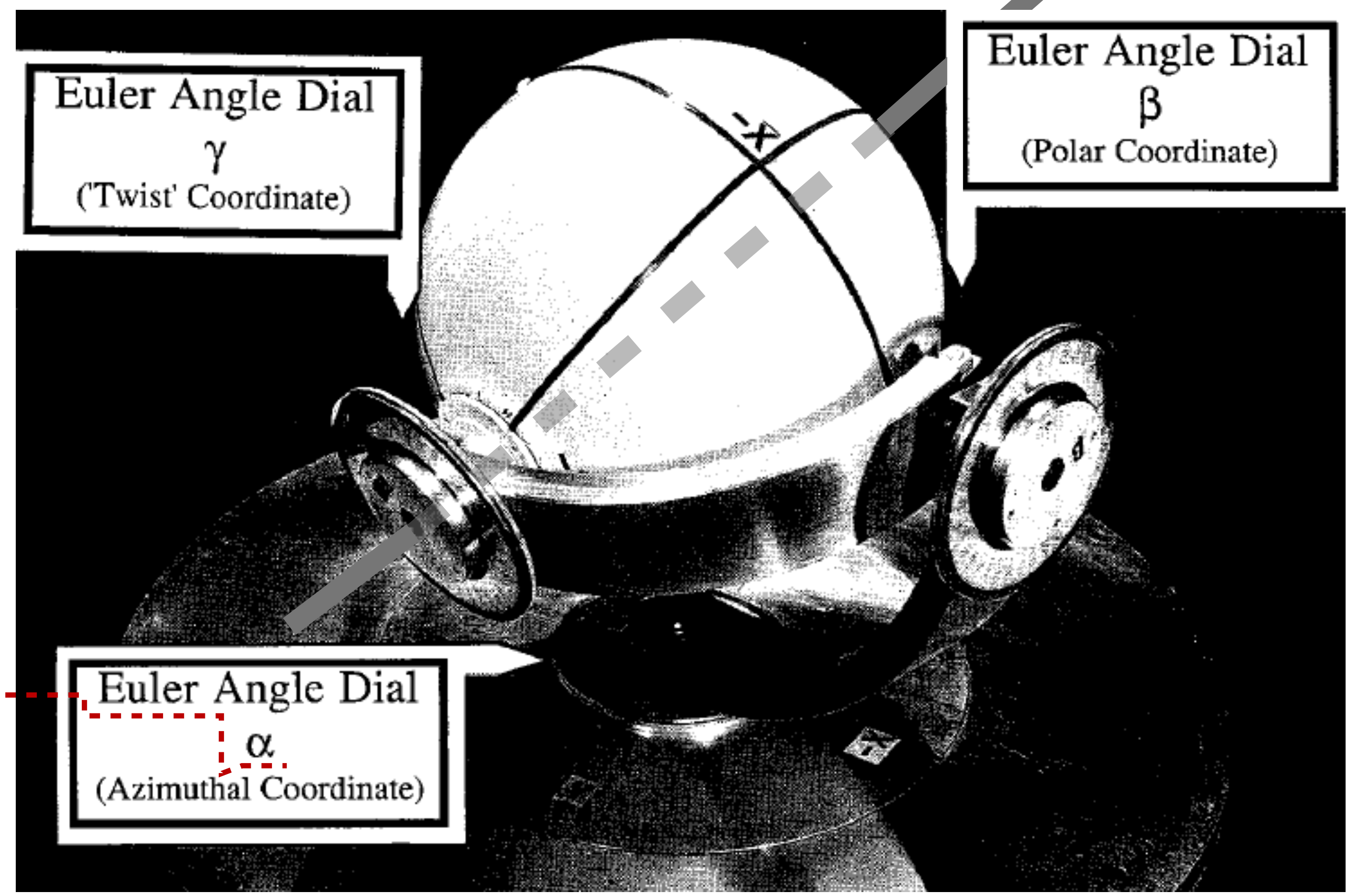
The gyrocompass and mechanical spin analogy

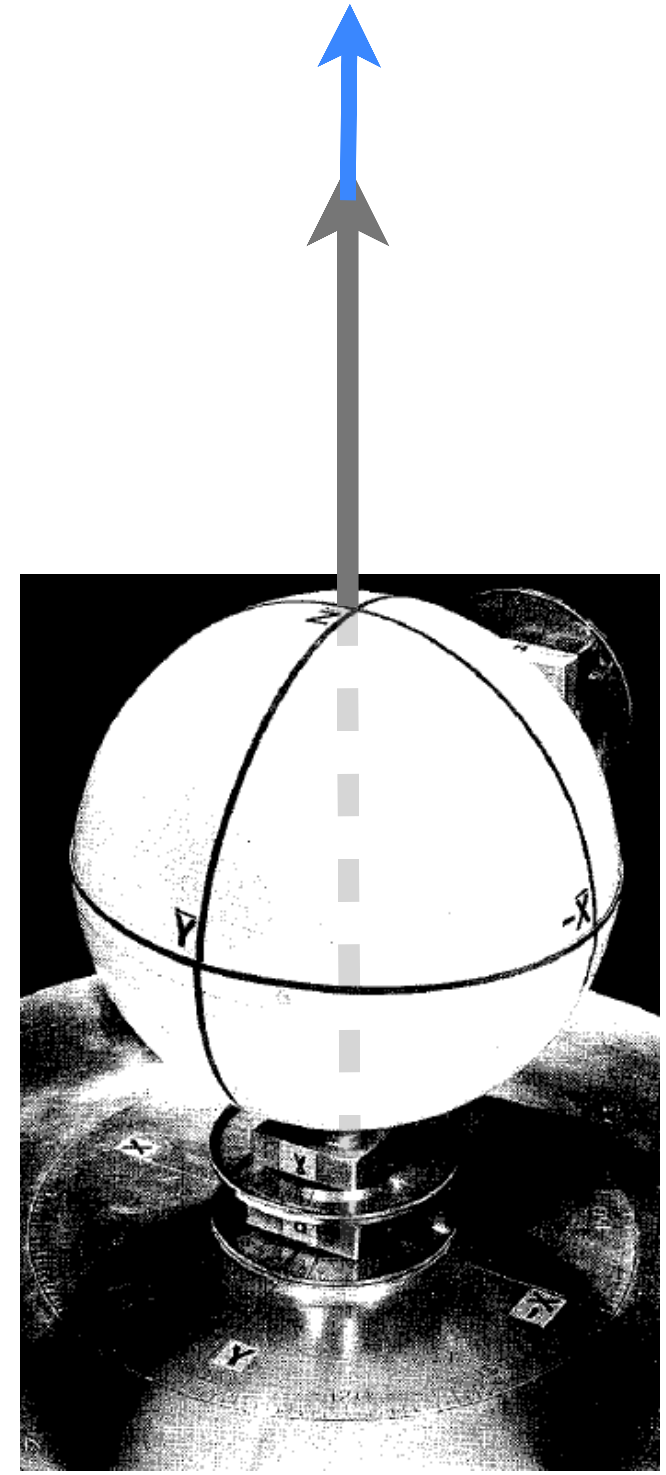
Suppose Euler ball has right-hand rotation with angular momentum **L**

If the α*-dial for z-rotation is turning left-to-right this applies righthand "thumbs-up" torque* **N**

N

L





A very high speed ball in a gyro-compass will similarly seek true North due to Earth rotation.

Then the ball tends to line-up with z-axis (and may go past z, then come back, etc. in a precessional or "hunting" motion)

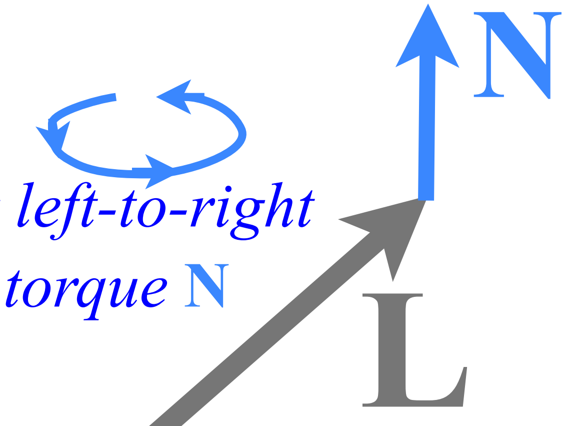
This is analogous to the tendency for spin magnetic moments to allign (or precess about) the B-direction of a magnetic field

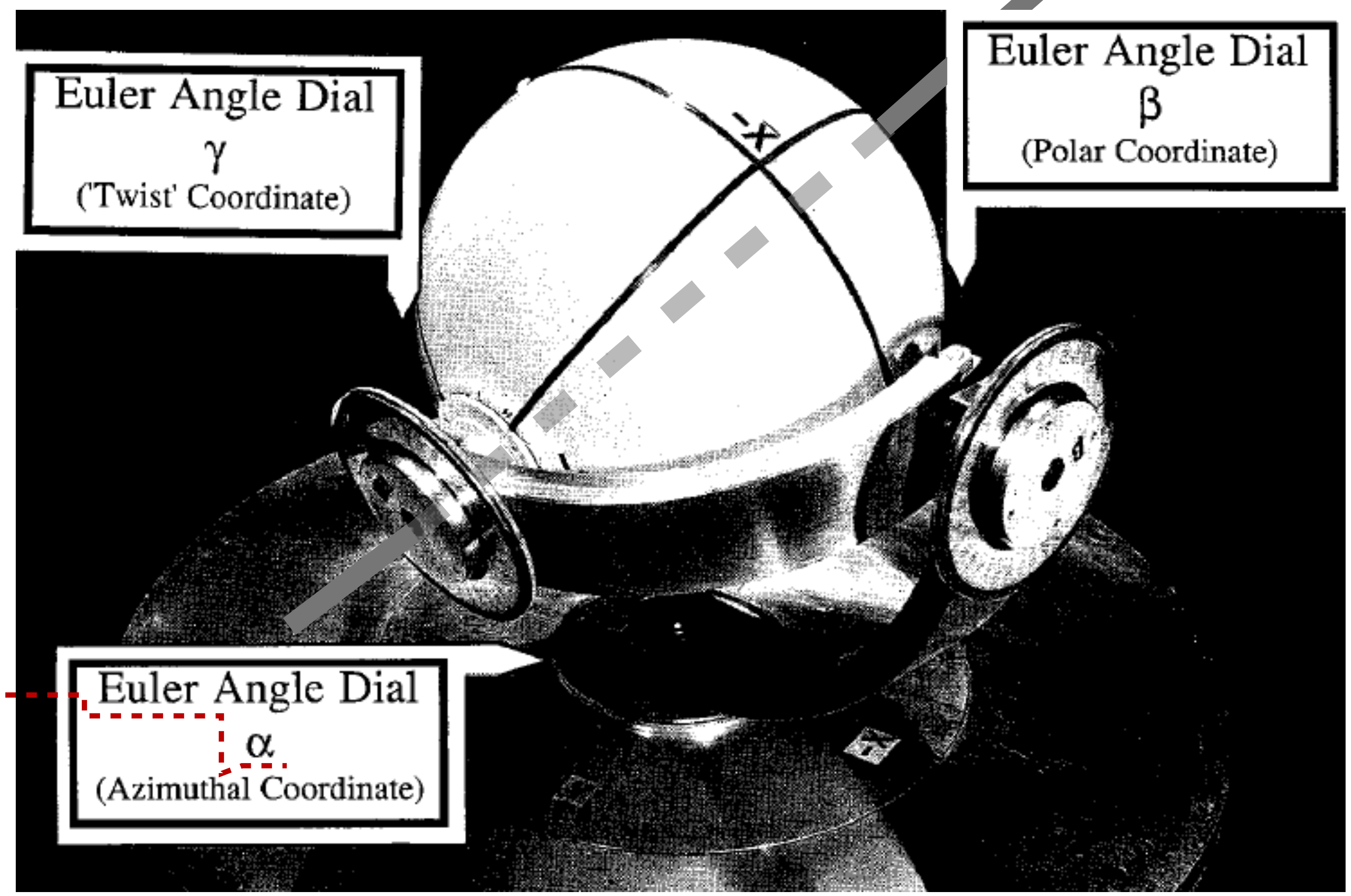
Recall **S***-precession discussion in CMwB Unit 4 Ch.4 and Lect.26*

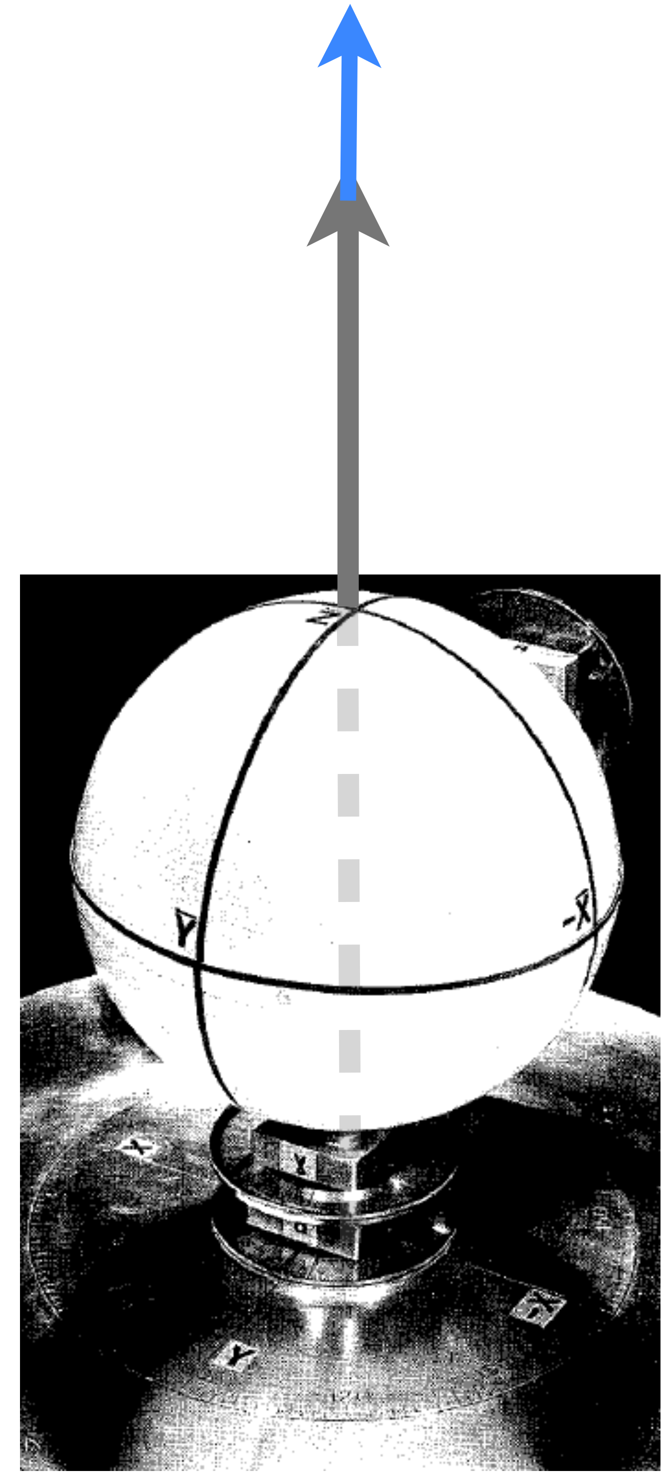
The gyrocompass and mechanical spin analogy

Suppose Euler ball has right-hand rotation with angular momentum **L**

If the α-dial for z-rotation is turning left-to-right this applies righthand "thumbs-up" torque **N**







A very high speed ball in a gyro-compass will similarly seek true North due to Earth rotation.

Then the ball tends to line-up with z-axis (and may go past z, then come back, etc. in a precessional or "hunting" motion)

General Rule: Gyros tend to "line-up" so they are rotating with whatever is most closely coupled to them.

This is analogous to the tendency for spin magnetic moments to allign (or precess about) the B-direction of a magnetic field

Recall **S***-precession discussion in CMwB Unit 4 Ch.4 and Lect.26*

Rotational momentum and velocity tensor relations

Quadratic form geometry and duality (again)

angular velocity $\boldsymbol{\omega}$*-ellipsoid vs. angular momentum* $\mathbf{L}$*-ellipsoid*

Lagrangian $\boldsymbol{\omega}$*-equations vs. Hamiltonian momentum* $\mathbf{L}$*-equation*

Inertia tensors

Consider N-body *angular velocity* $\boldsymbol{\omega}$ and *angular momentum* $\mathbf{L}$ relations with Levi-Civita analysis

$$\dot{\mathbf{r}}_j = \boldsymbol{\omega} \times \mathbf{r}_j \quad \text{and} \quad \mathbf{L} = \sum_{j=1}^{N} \mathbf{r}_j \times m_j \dot{\mathbf{r}}_j = \sum_{j=1}^{N} m_j \mathbf{r}_j \times \left(\boldsymbol{\omega} \times \mathbf{r}_j\right) \quad \text{with} \quad \mathbf{A} \times \left(\mathbf{B} \times \mathbf{C}\right) = \left(\mathbf{A} \bullet \mathbf{C}\right)\mathbf{B} - \left(\mathbf{A} \bullet \mathbf{B}\right)\mathbf{C}$$

Consider mass m instantaneously at $\mathbf{r}_m = (x_m, y_m, z_m) = r(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$ on a bent axle rotating in a fixed bearing:

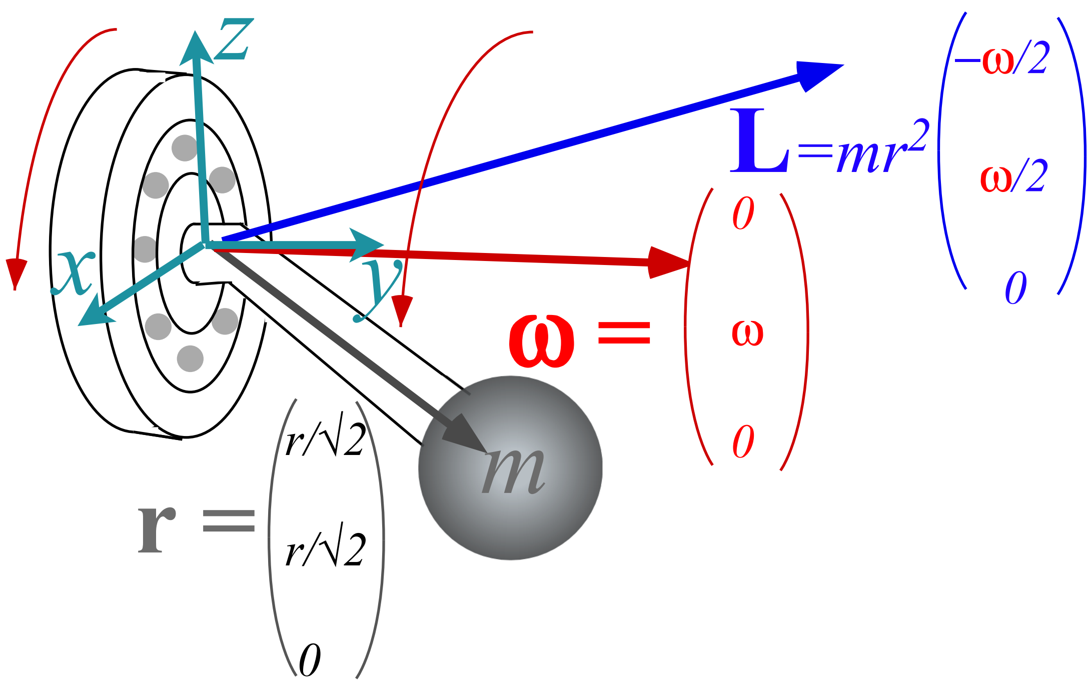


Fig. 6.5.1 Angular momentum for mass rotating on bent axle.

Inertia tensors

Consider N-body *angular velocity* $\boldsymbol{\omega}$ and *angular momentum* $\mathbf{L}$ relations with Levi-Civita analysis

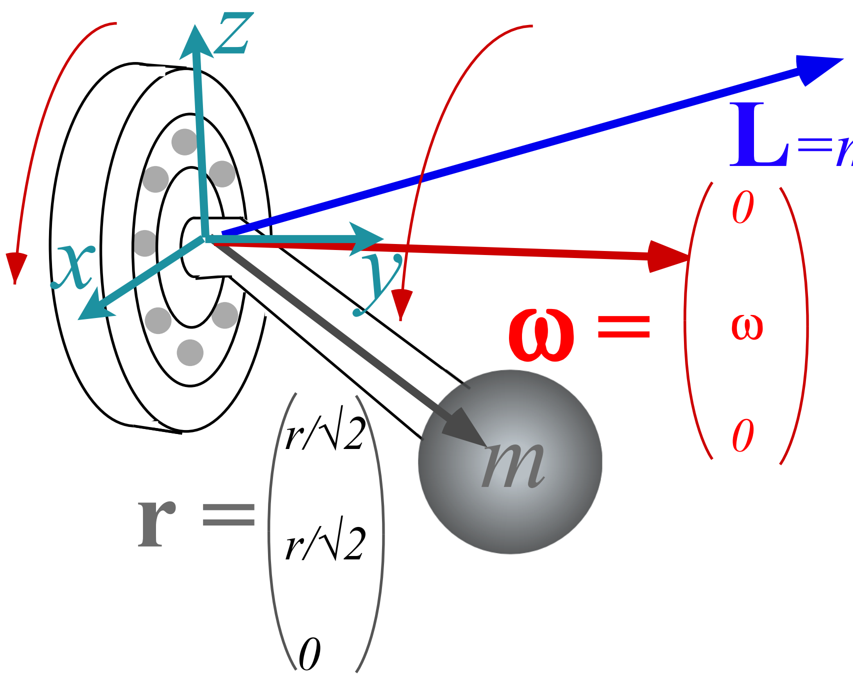
$$\dot{\mathbf{r}}_j = \boldsymbol{\omega} \times \mathbf{r}_j \qquad \text{and} \qquad \mathbf{L} = \sum_{j=1}^{N} \mathbf{r}_j \times m_j \dot{\mathbf{r}}_j = \sum_{j=1}^{N} m_j \mathbf{r}_j \times \left(\boldsymbol{\omega} \times \mathbf{r}_j \right) \qquad \text{with} \qquad \mathbf{A} \times \left(\mathbf{B} \times \mathbf{C} \right) = \left(\mathbf{A} \bullet \mathbf{C} \right) \mathbf{B} - \left(\mathbf{A} \bullet \mathbf{B} \right) \mathbf{C}$$

This produces the *rotational inertia tensor* $\mathbf{I}$: $\qquad \overleftrightarrow{\mathbf{I}} = \sum_{j=1}^{N} \overleftrightarrow{\mathbf{I}}_j = \sum_{j=1}^{N} m_j \left[\left(\mathbf{r}_j \bullet \mathbf{r}_j \right) \mathbf{1} - \mathbf{r}_j \mathbf{r}_j \right]$

in the ω-to-$\mathbf{L}$ relation:

$$\mathbf{L} = \sum_{j=1}^{N} m_j \left[\left(\mathbf{r}_j \bullet \mathbf{r}_j \right) \boldsymbol{\omega} - \left(\mathbf{r}_j \bullet \boldsymbol{\omega} \right) \mathbf{r}_j \right] = \sum_{j=1}^{N} m_j \left[\left(\mathbf{r}_j \bullet \mathbf{r}_j \right) \mathbf{1} - \mathbf{r}_j \mathbf{r}_j \right] \bullet \boldsymbol{\omega} = \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega}$$

Consider mass m instantaneously at $\mathbf{r}_m = (x_m, y_m, z_m) = r(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$ on a bent axle rotating in a fixed bearing:



$$\mathbf{L} = mr^2 \begin{pmatrix} -\omega/2 \\ \omega/2 \\ 0 \end{pmatrix} \qquad \boldsymbol{\omega} = \begin{pmatrix} 0 \\ \omega \\ 0 \end{pmatrix} \qquad \mathbf{r} = \begin{pmatrix} r/\sqrt{2} \\ r/\sqrt{2} \\ 0 \end{pmatrix}$$

Fig. 6.5.1 Angular momentum for mass rotating on bent axle.

Inertia tensors

Consider N-body *angular velocity* $\boldsymbol{\omega}$ and *angular momentum* $\mathbf{L}$ relations with Levi-Civita analysis

$$\dot{\mathbf{r}}_j = \boldsymbol{\omega}\times\mathbf{r}_j \qquad \text{and} \qquad \mathbf{L}=\sum_{j=1}^{N}\mathbf{r}_j\times m_j\dot{\mathbf{r}}_j=\sum_{j=1}^{N}m_j\mathbf{r}_j\times\left(\boldsymbol{\omega}\times\mathbf{r}_j\right) \qquad \text{with} \quad \mathbf{A}\times(\mathbf{B}\times\mathbf{C})=(\mathbf{A}\bullet\mathbf{C})\mathbf{B}-(\mathbf{A}\bullet\mathbf{B})\mathbf{C}$$

This produces the *rotational inertia tensor* $\mathbf{I}$:

$$\overleftrightarrow{\mathbf{I}}=\sum_{j=1}^{N}\overleftrightarrow{\mathbf{I}}_j=\sum_{j=1}^{N}m_j\left[\left(\mathbf{r}_j\bullet\mathbf{r}_j\right)\mathbf{1}-\mathbf{r}_j\mathbf{r}_j\right]$$

in the ω-to-$\mathbf{L}$ relation:

$$\mathbf{L}=\sum_{j=1}^{N}m_j\left[\left(\mathbf{r}_j\bullet\mathbf{r}_j\right)\boldsymbol{\omega}-\left(\mathbf{r}_j\bullet\boldsymbol{\omega}\right)\mathbf{r}_j\right]=\sum_{j=1}^{N}m_j\left[\left(\mathbf{r}_j\bullet\mathbf{r}_j\right)\mathbf{1}-\mathbf{r}_j\mathbf{r}_j\right]\bullet\boldsymbol{\omega}=\overleftrightarrow{\mathbf{I}}\bullet\boldsymbol{\omega}$$

Matrix form of the ω-to-$\mathbf{L}$ relation using the *inertia matrix* $\langle\mathbf{I}\rangle$

$$\begin{pmatrix}L_x\\L_y\\L_z\end{pmatrix}=\sum_{j=1}^{N}m_j\begin{pmatrix}y_j^2+z_j^2 & -x_jy_j & -x_jz_j\\ -y_jx_j & x_j^2+z_j^2 & -y_jz_j\\ -z_jx_j & -z_jy_j & x_j^2+y_j^2\end{pmatrix}\begin{pmatrix}\omega_x\\\omega_y\\\omega_z\end{pmatrix} \qquad \left\langle\overleftrightarrow{\mathbf{I}}\right\rangle=\sum_{j=1}^{N}\left\langle\overleftrightarrow{\mathbf{I}}_j\right\rangle=\sum_{j=1}^{N}m_j\begin{pmatrix}y_j^2+z_j^2 & -x_jy_j & -x_jz_j\\ -y_jx_j & x_j^2+z_j^2 & -y_jz_j\\ -z_jx_j & -z_jy_j & x_j^2+y_j^2\end{pmatrix}$$

Consider mass m instantaneously at $\mathbf{r}_m=(x_m,y_m,z_m)=r(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0)$ on a bent axle rotating in a fixed bearing:

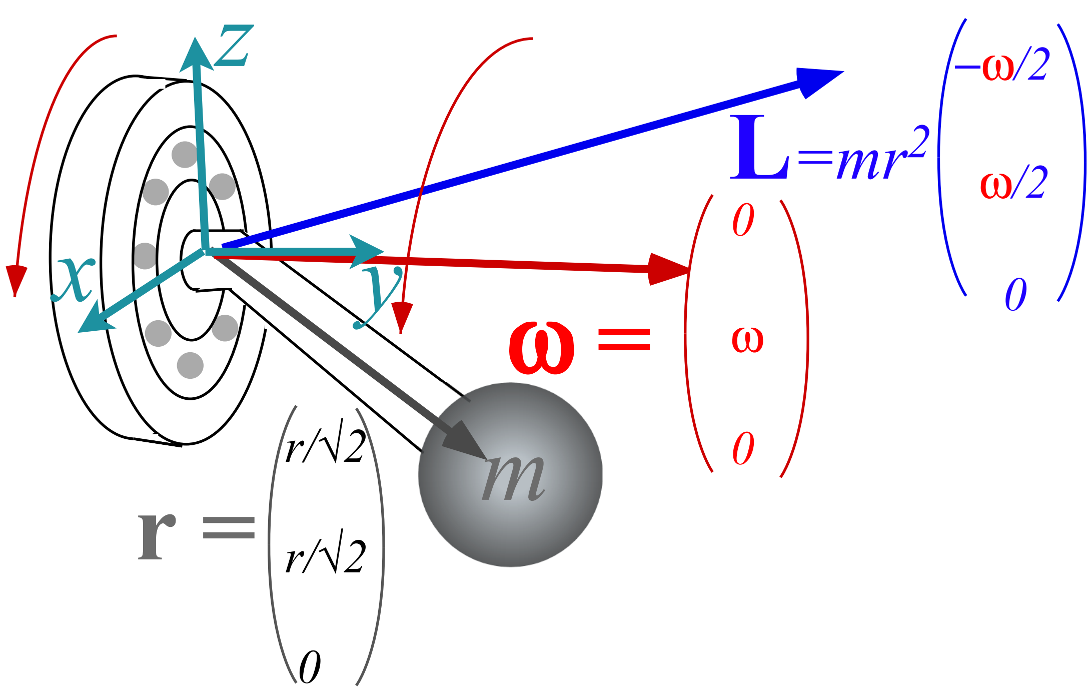


Fig. 6.5.1 Angular momentum for mass rotating on bent axle.

Inertia tensors

Consider N-body *angular velocity* $\boldsymbol{\omega}$ and *angular momentum* $\mathbf{L}$ relations with Levi-Civita analysis

$$\dot{\mathbf{r}}_j = \boldsymbol{\omega}\times\mathbf{r}_j \quad \text{and} \quad \mathbf{L}=\sum_{j=1}^{N}\mathbf{r}_j\times m_j\dot{\mathbf{r}}_j=\sum_{j=1}^{N}m_j\mathbf{r}_j\times\left(\boldsymbol{\omega}\times\mathbf{r}_j\right) \quad \text{with} \quad \mathbf{A}\times\left(\mathbf{B}\times\mathbf{C}\right)=\left(\mathbf{A}\bullet\mathbf{C}\right)\mathbf{B}-\left(\mathbf{A}\bullet\mathbf{B}\right)\mathbf{C}$$

This produces the *rotational inertia tensor* $\mathbf{I}$:

$$\overleftrightarrow{\mathbf{I}}=\sum_{j=1}^{N}\overleftrightarrow{\mathbf{I}}_j=\sum_{j=1}^{N}m_j\left[\left(\mathbf{r}_j\bullet\mathbf{r}_j\right)\mathbf{1}-\mathbf{r}_j\mathbf{r}_j\right]$$

in the $\boldsymbol{\omega}$-to-$\mathbf{L}$ relation:

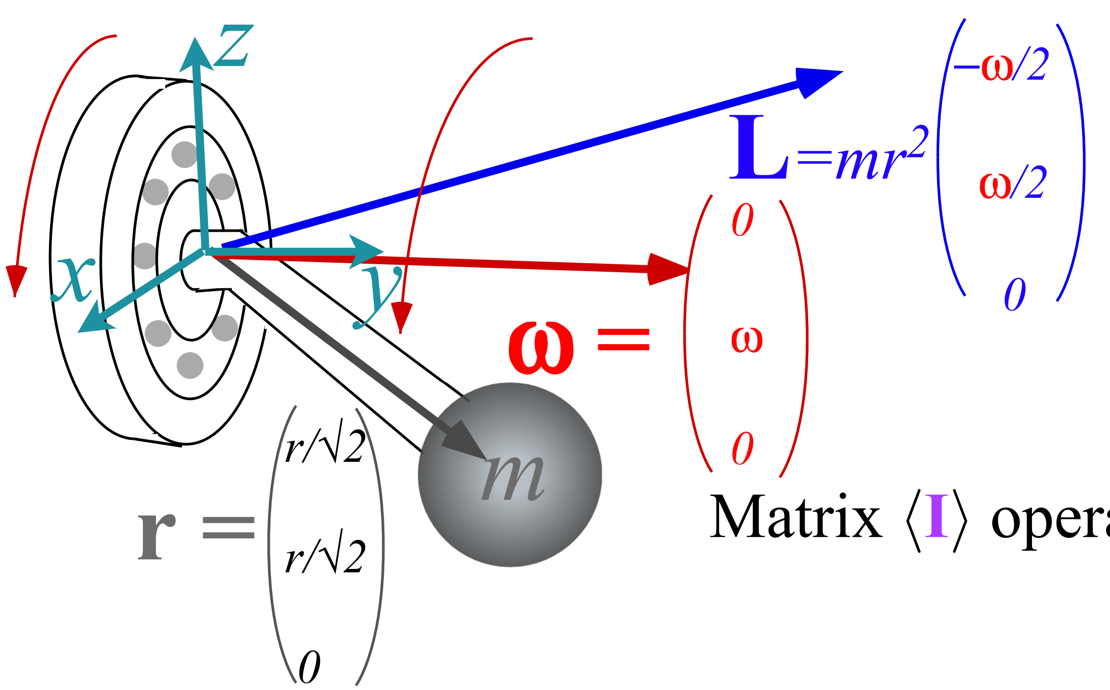
$$\mathbf{L}=\sum_{j=1}^{N}m_j\left[\left(\mathbf{r}_j\bullet\mathbf{r}_j\right)\boldsymbol{\omega}-\left(\mathbf{r}_j\bullet\boldsymbol{\omega}\right)\mathbf{r}_j\right]=\sum_{j=1}^{N}m_j\left[\left(\mathbf{r}_j\bullet\mathbf{r}_j\right)\mathbf{1}-\mathbf{r}_j\mathbf{r}_j\right]\bullet\boldsymbol{\omega}=\overleftrightarrow{\mathbf{I}}\bullet\boldsymbol{\omega}$$

Matrix form of the $\boldsymbol{\omega}$-to-$\mathbf{L}$ relation using the *inertia matrix* $\langle\mathbf{I}\rangle$

$$\begin{pmatrix}L_x\\L_y\\L_z\end{pmatrix}=\sum_{j=1}^{N}m_j\begin{pmatrix}y_j^2+z_j^2 & -x_jy_j & -x_jz_j\\ -y_jx_j & x_j^2+z_j^2 & -y_jz_j\\ -z_jx_j & -z_jy_j & x_j^2+y_j^2\end{pmatrix}\begin{pmatrix}\omega_x\\\omega_y\\\omega_z\end{pmatrix}$$

$$\left\langle\overleftrightarrow{\mathbf{I}}\right\rangle=\sum_{j=1}^{N}\left\langle\overleftrightarrow{\mathbf{I}}_j\right\rangle=\sum_{j=1}^{N}m_j\begin{pmatrix}y_j^2+z_j^2 & -x_jy_j & -x_jz_j\\ -y_jx_j & x_j^2+z_j^2 & -y_jz_j\\ -z_jx_j & -z_jy_j & x_j^2+y_j^2\end{pmatrix}$$

Consider mass m instantaneously at $\mathbf{r}_m=(x_m,y_m,z_m)=r(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0)$ on a bent axle rotating in a fixed bearing:



Instantaneous matrix $\langle\mathbf{I}\rangle$ of inertia is:

$$\left\langle\overleftrightarrow{\mathbf{I}}\right\rangle=mr^2\begin{pmatrix}\left(1/\sqrt{2}\right)^2+0 & -\left(1/\sqrt{2}\right)\left(1/\sqrt{2}\right) & -\left(1/\sqrt{2}\right)0\\ -\left(1/\sqrt{2}\right)\left(1/\sqrt{2}\right) & \left(1/\sqrt{2}\right)^2+0 & -\left(1/\sqrt{2}\right)0\\ -0\left(1/\sqrt{2}\right) & -0\left(1/\sqrt{2}\right) & \left(1/\sqrt{2}\right)^2+\left(1/\sqrt{2}\right)^2\end{pmatrix}=mr^2\begin{pmatrix}1/2 & -1/2 & 0\\ -1/2 & 1/2 & 0\\ 0 & 0 & 1\end{pmatrix}$$

Matrix $\langle\mathbf{I}\rangle$ operates on angular velocity $\boldsymbol{\omega}$ to give angular momentum $\mathbf{L}$

Fig. 6.5.1 Angular momentum for mass rotating on bent axle.

Inertia tensors

Consider N-body *angular velocity* $\boldsymbol{\omega}$ and *angular momentum* $\mathbf{L}$ relations with Levi-Civita analysis

$$\dot{\mathbf{r}}_j = \boldsymbol{\omega}\times\mathbf{r}_j \quad \text{and} \quad \mathbf{L} = \sum_{j=1}^{N} \mathbf{r}_j \times m_j \dot{\mathbf{r}}_j = \sum_{j=1}^{N} m_j \mathbf{r}_j \times \left(\boldsymbol{\omega}\times\mathbf{r}_j\right) \quad \text{with} \quad \mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\bullet\mathbf{C})\mathbf{B} - (\mathbf{A}\bullet\mathbf{B})\mathbf{C}$$

This produces the *rotational inertia tensor* $\mathbf{I}$:
$$\overleftrightarrow{\mathbf{I}} = \sum_{j=1}^{N} \overleftrightarrow{\mathbf{I}}_j = \sum_{j=1}^{N} m_j \left[\left(\mathbf{r}_j \bullet \mathbf{r}_j\right)\mathbf{1} - \mathbf{r}_j\mathbf{r}_j \right]$$

in the ω-to-$\mathbf{L}$ relation:
$$\mathbf{L} = \sum_{j=1}^{N} m_j \left[\left(\mathbf{r}_j \bullet \mathbf{r}_j\right)\boldsymbol{\omega} - \left(\mathbf{r}_j \bullet \boldsymbol{\omega}\right)\mathbf{r}_j \right] = \sum_{j=1}^{N} m_j \left[\left(\mathbf{r}_j \bullet \mathbf{r}_j\right)\mathbf{1} - \mathbf{r}_j\mathbf{r}_j \right] \bullet \boldsymbol{\omega} = \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega}$$

Matrix form of the ω-to-$\mathbf{L}$ relation using the *inertia matrix* $\langle \mathbf{I} \rangle$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \sum_{j=1}^{N} m_j \begin{pmatrix} y_j^2+z_j^2 & -x_j y_j & -x_j z_j \\ -y_j x_j & x_j^2+z_j^2 & -y_j z_j \\ -z_j x_j & -z_j y_j & x_j^2+y_j^2 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$$\left\langle \overleftrightarrow{\mathbf{I}} \right\rangle = \sum_{j=1}^{N} \left\langle \overleftrightarrow{\mathbf{I}}_j \right\rangle = \sum_{j=1}^{N} m_j \begin{pmatrix} y_j^2+z_j^2 & -x_j y_j & -x_j z_j \\ -y_j x_j & x_j^2+z_j^2 & -y_j z_j \\ -z_j x_j & -z_j y_j & x_j^2+y_j^2 \end{pmatrix}$$

Consider mass m instantaneously at $\mathbf{r}_m = (x_m, y_m, z_m) = r(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$ on a bent axle rotating in a fixed bearing:

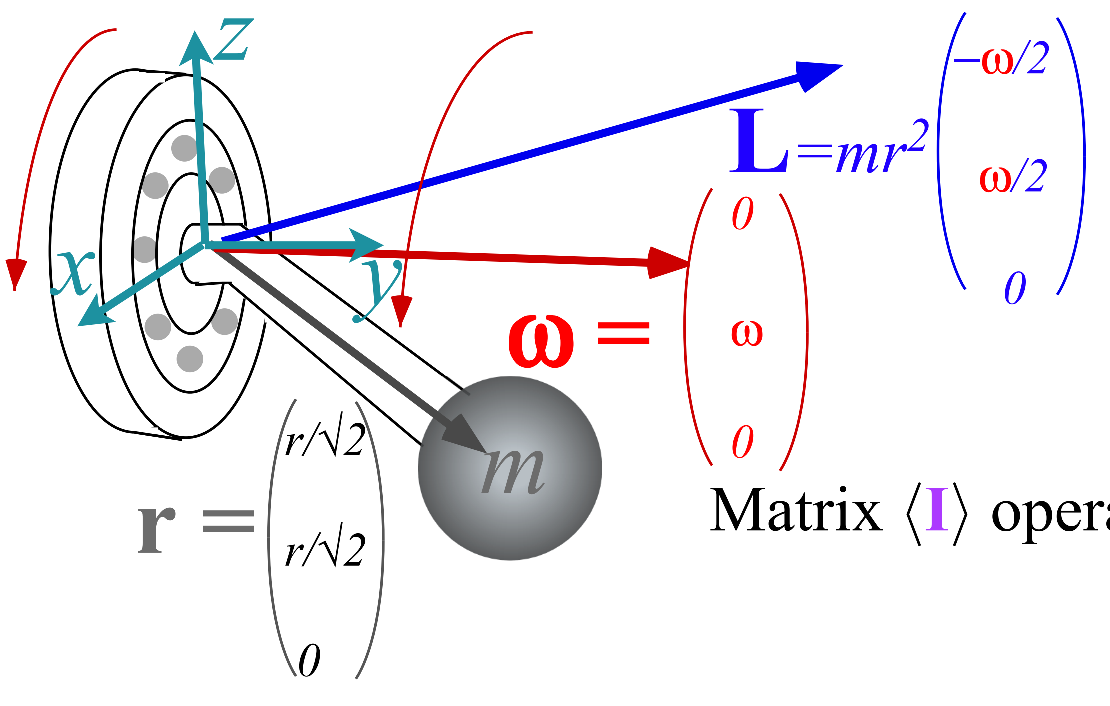


Fig. 6.5.1 Angular momentum for mass rotating on bent axle.

Instantaneous matrix $\langle \mathbf{I} \rangle$ of inertia is:

$$\left\langle \overleftrightarrow{\mathbf{I}} \right\rangle = mr^2 \begin{pmatrix} \left(1/\sqrt{2}\right)^2+0 & -\left(1/\sqrt{2}\right)\left(1/\sqrt{2}\right) & -\left(1/\sqrt{2}\right)0 \\ -\left(1/\sqrt{2}\right)\left(1/\sqrt{2}\right) & \left(1/\sqrt{2}\right)^2+0 & -\left(1/\sqrt{2}\right)0 \\ -0\left(1/\sqrt{2}\right) & -0\left(1/\sqrt{2}\right) & \left(1/\sqrt{2}\right)^2+\left(1/\sqrt{2}\right)^2 \end{pmatrix} = mr^2 \begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Matrix $\langle \mathbf{I} \rangle$ operates on angular velocity $\boldsymbol{\omega}$ to give angular momentum $\mathbf{L}$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = mr^2 \begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \omega \\ 0 \end{pmatrix} = mr^2 \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \end{pmatrix} \omega$$

Inertia tensors

Consider N-body *angular velocity* $\boldsymbol{\omega}$ and *angular momentum* $\mathbf{L}$ relations with Levi-Civita analysis

$$\dot{\mathbf{r}}_j = \boldsymbol{\omega}\times\mathbf{r}_j \qquad \text{and} \qquad \mathbf{L} = \sum_{j=1}^{N} \mathbf{r}_j \times m_j \dot{\mathbf{r}}_j = \sum_{j=1}^{N} m_j \mathbf{r}_j \times \left(\boldsymbol{\omega}\times\mathbf{r}_j\right) \qquad \text{with} \qquad \mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\bullet\mathbf{C})\mathbf{B} - (\mathbf{A}\bullet\mathbf{B})\mathbf{C}$$

This produces the *rotational inertia tensor* $\mathbf{I}$:

$$\overleftrightarrow{\mathbf{I}} = \sum_{j=1}^{N} \overleftrightarrow{\mathbf{I}}_j = \sum_{j=1}^{N} m_j \left[\left(\mathbf{r}_j\bullet\mathbf{r}_j\right)\mathbf{1} - \mathbf{r}_j\mathbf{r}_j\right]$$

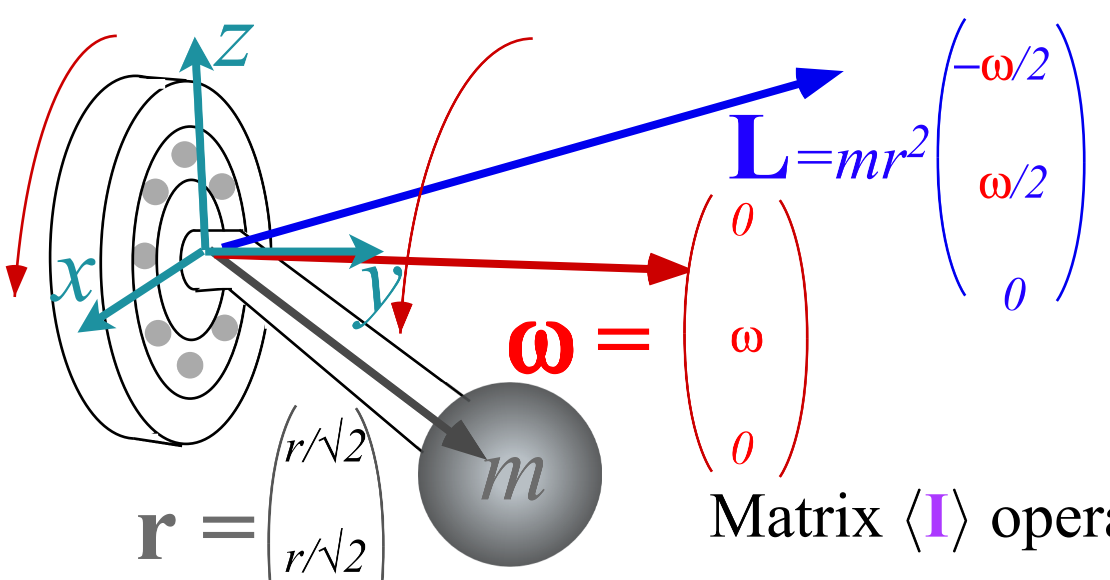
in the $\boldsymbol{\omega}$-to-$\mathbf{L}$ relation:

$$\mathbf{L} = \sum_{j=1}^{N} m_j \left[\left(\mathbf{r}_j\bullet\mathbf{r}_j\right)\boldsymbol{\omega} - \left(\mathbf{r}_j\bullet\boldsymbol{\omega}\right)\mathbf{r}_j\right] = \sum_{j=1}^{N} m_j \left[\left(\mathbf{r}_j\bullet\mathbf{r}_j\right)\mathbf{1} - \mathbf{r}_j\mathbf{r}_j\right]\bullet\boldsymbol{\omega} = \overleftrightarrow{\mathbf{I}}\bullet\boldsymbol{\omega}$$

Matrix form of the $\boldsymbol{\omega}$-to-$\mathbf{L}$ relation using the *inertia matrix* $\langle\mathbf{I}\rangle$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \sum_{j=1}^{N} m_j \begin{pmatrix} y_j^2+z_j^2 & -x_jy_j & -x_jz_j \\ -y_jx_j & x_j^2+z_j^2 & -y_jz_j \\ -z_jx_j & -z_jy_j & x_j^2+y_j^2 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \qquad \left\langle\overleftrightarrow{\mathbf{I}}\right\rangle = \sum_{j=1}^{N} \left\langle\overleftrightarrow{\mathbf{I}}_j\right\rangle = \sum_{j=1}^{N} m_j \begin{pmatrix} y_j^2+z_j^2 & -x_jy_j & -x_jz_j \\ -y_jx_j & x_j^2+z_j^2 & -y_jz_j \\ -z_jx_j & -z_jy_j & x_j^2+y_j^2 \end{pmatrix}$$

Consider mass m instantaneously at $\mathbf{r}_m = (x_m, y_m, z_m) = r(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$ on a bent axle rotating in a fixed bearing:



Instantaneous matrix $\langle\mathbf{I}\rangle$ of inertia is:

$$\left\langle\overleftrightarrow{\mathbf{I}}\right\rangle = mr^2 \begin{pmatrix} \left(1/\sqrt{2}\right)^2+0 & -\left(1/\sqrt{2}\right)\left(1/\sqrt{2}\right) & -\left(1/\sqrt{2}\right)0 \\ -\left(1/\sqrt{2}\right)\left(1/\sqrt{2}\right) & \left(1/\sqrt{2}\right)^2+0 & -\left(1/\sqrt{2}\right)0 \\ -0\left(1/\sqrt{2}\right) & -0\left(1/\sqrt{2}\right) & \left(1/\sqrt{2}\right)^2+\left(1/\sqrt{2}\right)^2 \end{pmatrix} = mr^2 \begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Matrix $\langle\mathbf{I}\rangle$ operates on angular velocity $\boldsymbol{\omega}$ to give angular momentum $\mathbf{L}$

Bearing torque is: $\mathbf{N} = \dfrac{d\mathbf{L}}{dt} = \boldsymbol{\omega}\times\mathbf{L}$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = mr^2 \begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \omega \\ 0 \end{pmatrix} = mr^2 \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \end{pmatrix} \omega$$

Fig. 6.5.1 Angular momentum for mass rotating on bent axle.

Kinetic energy in terms of velocity ω and rotational Lagrangian

Kinetic energy T of a rotating rigid body can be expressed in terms of the inertia matrix $\mathbf{I}$

$$T=\frac{1}{2}\sum_{j=1}^{3} m_j\dot{\mathbf{r}}_j\bullet\dot{\mathbf{r}}_j=\frac{1}{2}\sum_{j=1}^{3} m_j(\boldsymbol{\omega}\times\mathbf{r}_j)\bullet(\boldsymbol{\omega}\times\mathbf{r}_j)$$

Levi-Civita identity

$$(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D})=(\mathbf{A}\bullet\mathbf{C})(\mathbf{B}\bullet\mathbf{D})-(\mathbf{A}\bullet\mathbf{D})(\mathbf{B}\bullet\mathbf{C})$$

$$T=\frac{1}{2}\sum_{j=1}^{3} m_j\left[(\boldsymbol{\omega}\bullet\boldsymbol{\omega})(\mathbf{r}_j\bullet\mathbf{r}_j)-(\boldsymbol{\omega}\bullet\mathbf{r}_j)(\mathbf{r}_j\bullet\boldsymbol{\omega})\right]$$

$$=\frac{1}{2}\boldsymbol{\omega}\bullet\sum_{j=1}^{3} m_j\left[(\mathbf{r}_j\bullet\mathbf{r}_j)\mathbf{1}-(\mathbf{r}_j)(\mathbf{r}_j)\right]\bullet\boldsymbol{\omega}$$

$$=\frac{1}{2}\boldsymbol{\omega}\bullet\overleftrightarrow{\mathbf{I}}\bullet\boldsymbol{\omega}$$

Kinetic energy is a *quadratic form*

$$T=\frac{1}{2}\begin{pmatrix}\omega_x & \omega_y & \omega_y\end{pmatrix}\begin{pmatrix}I_{xx} & I_{xy} & I_{xz}\\ I_{yx} & I_{yy} & I_{yz}\\ I_{zx} & I_{zy} & I_{zz}\end{pmatrix}\begin{pmatrix}\omega_x\\ \omega_y\\ \omega_z\end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix}\langle\omega|x\rangle & \langle\omega|y\rangle & \langle\omega|z\rangle\end{pmatrix}\begin{pmatrix}\langle x|\mathbf{I}|x\rangle & \langle x|\mathbf{I}|y\rangle & \langle x|\mathbf{I}|z\rangle\\ \langle y|\mathbf{I}|x\rangle & \langle y|\mathbf{I}|y\rangle & \langle y|\mathbf{I}|z\rangle\\ \langle z|\mathbf{I}|x\rangle & \langle z|\mathbf{I}|y\rangle & \langle z|\mathbf{I}|z\rangle\end{pmatrix}\begin{pmatrix}\langle x|\omega\rangle\\ \langle y|\omega\rangle\\ \langle z|\omega\rangle\end{pmatrix}\quad\text{(Dirac notation)}$$

$$=\frac{1}{2}\begin{pmatrix}\omega_x & \omega_y & \omega_y\end{pmatrix}\sum_{j=1}^{3} m_j\begin{pmatrix}y_j^2+z_j^2 & -x_jy_j & -x_jz_j\\ -y_jx_j & x_j^2+z_j^2 & -y_jz_j\\ -z_jx_j & -z_jy_j & x_j^2+y_j^2\end{pmatrix}\begin{pmatrix}\omega_x\\ \omega_y\\ \omega_z\end{pmatrix}$$

Simplifies in *principle inertial axes {X,Y,Z}* or *body eigen-axes*

$$T=\frac{1}{2}\begin{pmatrix}\omega_X & \omega_Y & \omega_Z\end{pmatrix}\begin{pmatrix}I_{XX} & I_{XY} & I_{XZ}\\ I_{YX} & I_{YY} & I_{YZ}\\ I_{ZX} & I_{ZY} & I_{ZZ}\end{pmatrix}\begin{pmatrix}\omega_X\\ \omega_Y\\ \omega_Z\end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix}\omega_X & \omega_Y & \omega_Z\end{pmatrix}\begin{pmatrix}I_{XX} & 0 & 0\\ 0 & I_{YY} & 0\\ 0 & 0 & I_{ZZ}\end{pmatrix}\begin{pmatrix}\omega_X\\ \omega_Y\\ \omega_Z\end{pmatrix}=\frac{I_{XX}\omega_X^2}{2}+\frac{I_{YY}\omega_Y^2}{2}+\frac{I_{ZZ}\omega_Z^2}{2}$$

Kinetic energy in terms of momentum L and rotational Hamiltonian

$\mathbf{L} = \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega}$, generally implies: $\boldsymbol{\omega} = \overleftrightarrow{\mathbf{I}}^{-1} \bullet \mathbf{L}$

Express kinetic energy T in terms of angular velocity ω , momentum $\mathbf{L}$, or both at once. once

$$T = \frac{1}{2}\boldsymbol{\omega} \bullet \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega} \bullet \mathbf{L} = \frac{1}{2}\mathbf{L} \bullet \boldsymbol{\omega} = \frac{1}{2}\mathbf{L} \bullet \overleftrightarrow{\mathbf{I}}^{-1} \bullet \mathbf{L}$$

$$T = \frac{1}{2}\begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix}\begin{pmatrix} I_{XX} & I_{XY} & I_{XZ} \\ I_{YX} & I_{YY} & I_{YZ} \\ I_{ZX} & I_{ZY} & I_{ZZ} \end{pmatrix}^{-1}\begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix}\begin{pmatrix} 1/I_{XX} & 0 & 0 \\ 0 & 1/I_{YY} & 0 \\ 0 & 0 & 1/I_{ZZ} \end{pmatrix}\begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix} = \frac{L_X^2}{2I_{XX}} + \frac{L_Y^2}{2I_{YY}} + \frac{L_Z^2}{2I_{ZZ}}$$

Kinetic energy in terms of momentum L and rotational Hamiltonian

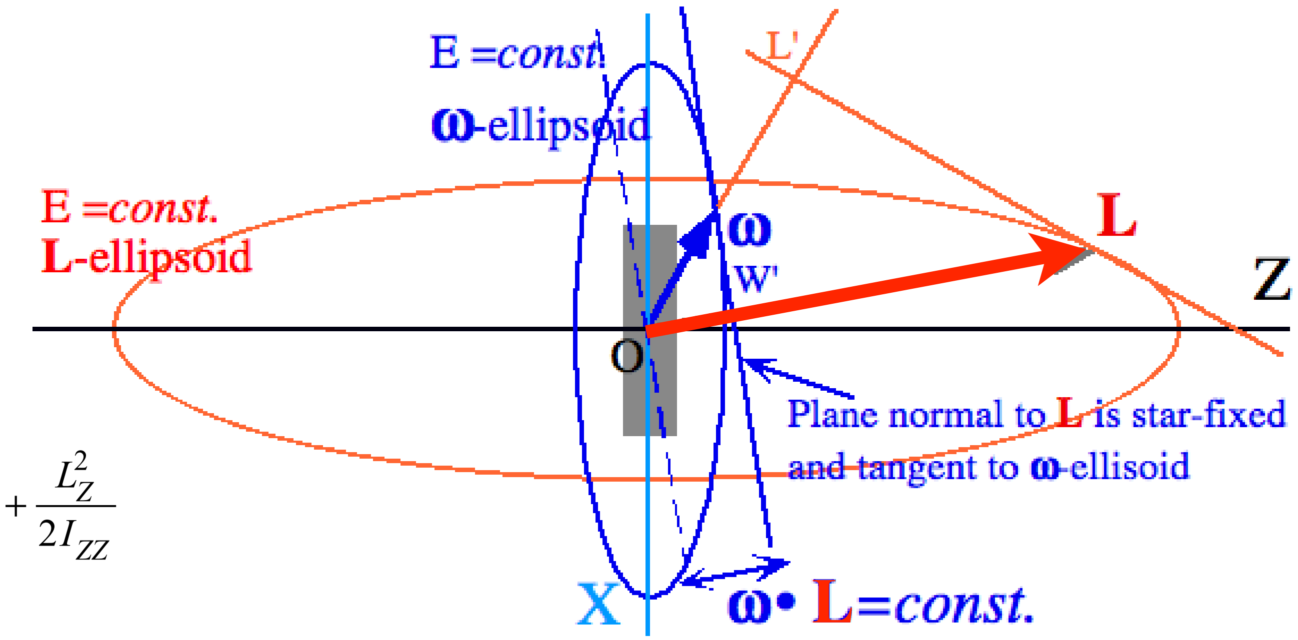
$\mathbf{L} = \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega}$, generally implies: $\boldsymbol{\omega} = \overleftrightarrow{\mathbf{I}}^{-1} \bullet \mathbf{L}$

Express kinetic energy T in terms of angular velocity ω , momentum **L**, or both at once. once

$$T = \frac{1}{2}\boldsymbol{\omega} \bullet \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega} \bullet \mathbf{L} = \frac{1}{2}\mathbf{L} \bullet \boldsymbol{\omega} = \frac{1}{2}\mathbf{L} \bullet \overleftrightarrow{\mathbf{I}}^{-1} \bullet \mathbf{L}$$

$$T = \frac{1}{2}\begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix}\begin{pmatrix} I_{XX} & I_{XY} & I_{XZ} \\ I_{YX} & I_{YY} & I_{YZ} \\ I_{ZX} & I_{ZY} & I_{ZZ} \end{pmatrix}^{-1}\begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix}\begin{pmatrix} 1/I_{XX} & 0 & 0 \\ 0 & 1/I_{YY} & 0 \\ 0 & 0 & 1/I_{ZZ} \end{pmatrix}\begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix} = \frac{L_X^2}{2I_{XX}} + \frac{L_Y^2}{2I_{YY}} + \frac{L_Z^2}{2I_{ZZ}}$$



Hamiltonian form is the equation of the *angular momentum or* ***L**-ellipsoid*

Lagrangian form is the equation of the *angular velocity or ω-ellipsoid*

$\frac{1}{2}\omega\bullet$

Kinetic energy in terms of momentum L and rotational Hamiltonian

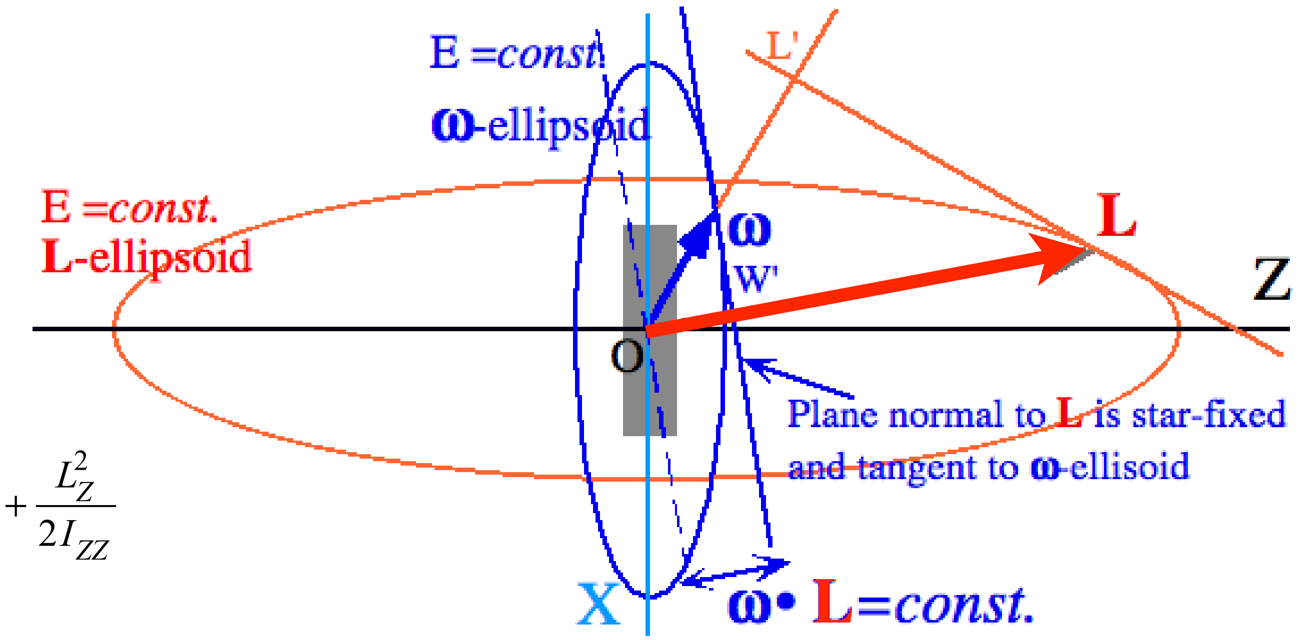
$\mathbf{L} = \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega}$, generally implies: $\boldsymbol{\omega} = \overleftrightarrow{\mathbf{I}}^{-1} \bullet \mathbf{L}$

Express kinetic energy T in terms of angular velocity ω , momentum **L**, or both at once. once

$$T = \frac{1}{2}\boldsymbol{\omega} \bullet \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega} \bullet \mathbf{L} = \frac{1}{2}\mathbf{L} \bullet \boldsymbol{\omega} = \frac{1}{2}\mathbf{L} \bullet \overleftrightarrow{\mathbf{I}}^{-1} \bullet \mathbf{L}$$

$$T = \frac{1}{2}\begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix} \begin{pmatrix} I_{XX} & I_{XY} & I_{XZ} \\ I_{YX} & I_{YY} & I_{YZ} \\ I_{ZX} & I_{ZY} & I_{ZZ} \end{pmatrix}^{-1} \begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix} \begin{pmatrix} 1/I_{XX} & 0 & 0 \\ 0 & 1/I_{YY} & 0 \\ 0 & 0 & 1/I_{ZZ} \end{pmatrix} \begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix} = \frac{L_X^2}{2I_{XX}} + \frac{L_Y^2}{2I_{YY}} + \frac{L_Z^2}{2I_{ZZ}}$$



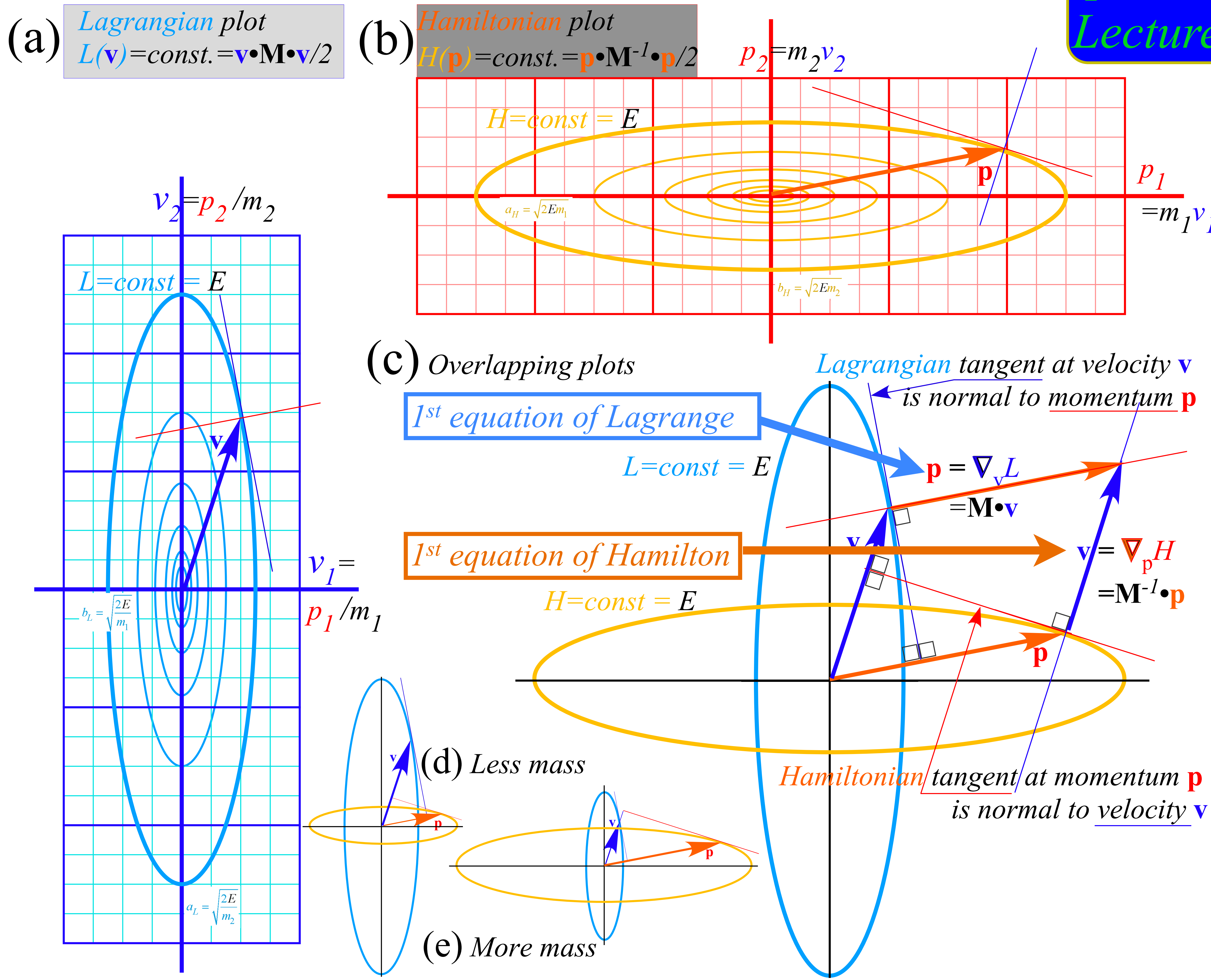
Hamiltonian form is the equation of the *angular momentum or* ***L**-ellipsoid*

Lagrangian form is the equation of the *angular velocity or ω-ellipsoid*

$\frac{1}{2}\omega\bullet$

Recall quadratic forms for Lagrangian and Hamiltonian in Lecture 10 unit 1?

Unit 1
Fig. 12.2



Kinetic energy in terms of momentum L and rotational Hamiltonian

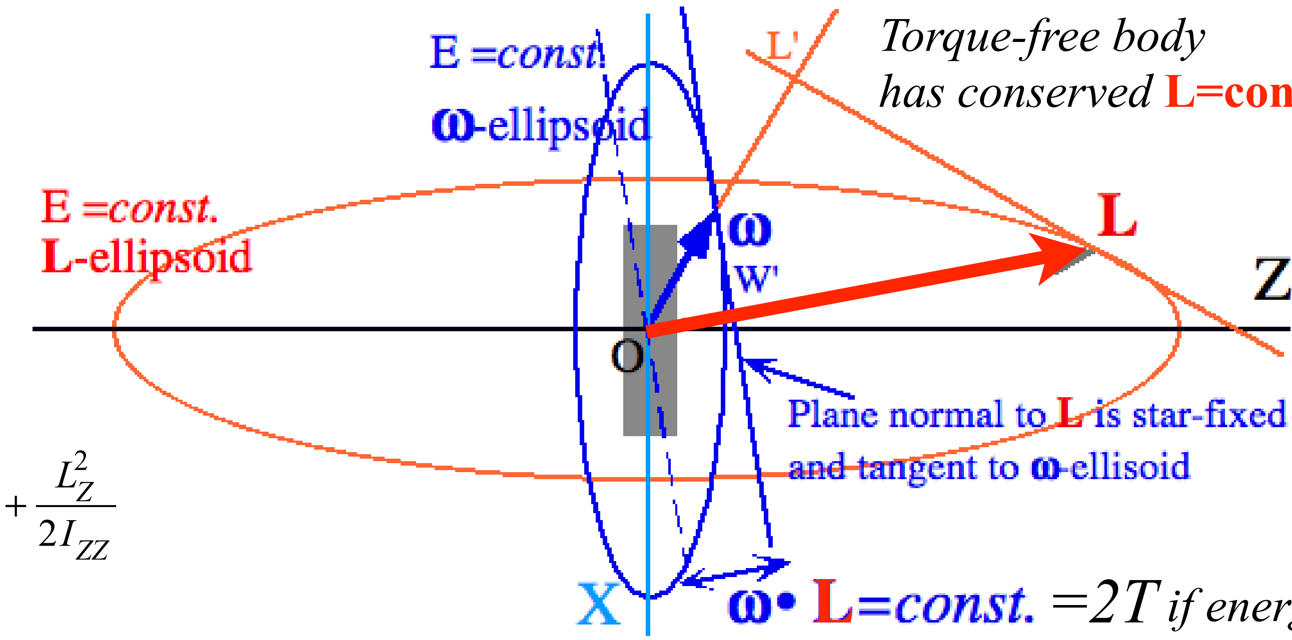
$$\mathbf{L} = \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega}\ , \quad \text{generally implies:} \quad \boldsymbol{\omega} = \overleftrightarrow{\mathbf{I}}^{-1} \bullet \mathbf{L}$$

Express kinetic energy T in terms of angular velocity ω , momentum **L**, or both at once. once

$$T = \frac{1}{2}\boldsymbol{\omega} \bullet \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega} \bullet \mathbf{L} = \frac{1}{2}\mathbf{L} \bullet \boldsymbol{\omega} = \frac{1}{2}\mathbf{L} \bullet \overleftrightarrow{\mathbf{I}}^{-1} \bullet \mathbf{L}$$

$$T = \frac{1}{2}\begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix}\begin{pmatrix} I_{XX} & I_{XY} & I_{XZ} \\ I_{YX} & I_{YY} & I_{YZ} \\ I_{ZX} & I_{ZY} & I_{ZZ} \end{pmatrix}^{-1}\begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix}\begin{pmatrix} 1/I_{XX} & 0 & 0 \\ 0 & 1/I_{YY} & 0 \\ 0 & 0 & 1/I_{ZZ} \end{pmatrix}\begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix} = \frac{L_X^2}{2I_{XX}} + \frac{L_Y^2}{2I_{YY}} + \frac{L_Z^2}{2I_{ZZ}}$$



Torque-free body has conserved **L=const**.

$\boldsymbol{\omega}\bullet$ **L**$=const. = 2T$ *if energy is not dissipated internally*

Hamiltonian form is the equation of the *angular momentum or* ***L**-ellipsoid*

Lagrangian form is the equation of the *angular velocity or ω-ellipsoid*

ω is generally not conserved unless it is aligned to **L** *or body has symmetry*

Kinetic energy in terms of momentum L and rotational Hamiltonian

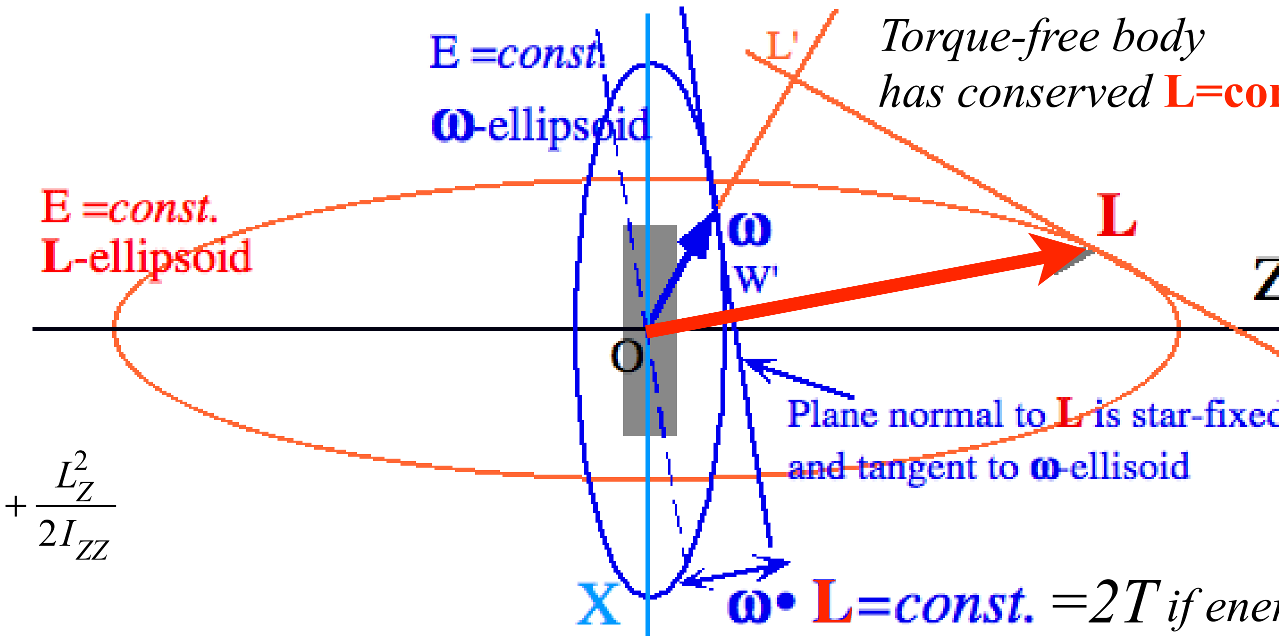
$\mathbf{L} = \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega}$, generally implies: $\boldsymbol{\omega} = \overleftrightarrow{\mathbf{I}}^{-1} \bullet \mathbf{L}$

Express kinetic energy T in terms of angular velocity ω , momentum **L**, or both at once. once

$$T = \frac{1}{2}\boldsymbol{\omega} \bullet \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega} \bullet \mathbf{L} = \frac{1}{2}\mathbf{L} \bullet \boldsymbol{\omega} = \frac{1}{2}\mathbf{L} \bullet \overleftrightarrow{\mathbf{I}}^{-1} \bullet \mathbf{L}$$

$$T = \frac{1}{2}\begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix}\begin{pmatrix} I_{XX} & I_{XY} & I_{XZ} \\ I_{YX} & I_{YY} & I_{YZ} \\ I_{ZX} & I_{ZY} & I_{ZZ} \end{pmatrix}^{-1}\begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix}\begin{pmatrix} 1/I_{XX} & 0 & 0 \\ 0 & 1/I_{YY} & 0 \\ 0 & 0 & 1/I_{ZZ} \end{pmatrix}\begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix} = \frac{L_X^2}{2I_{XX}} + \frac{L_Y^2}{2I_{YY}} + \frac{L_Z^2}{2I_{ZZ}}$$



Torque-free body has conserved **L=const**.

$\boldsymbol{\omega}\bullet$ **L**=*const.* =2*T* *if energy is not dissipated internally*

Hamiltonian form is the equation of the *angular momentum or* ***L****-ellipsoid*

Lagrangian form is the equation of the *angular velocity or ω-ellipsoid*

ω is generally not conserved unless it is aligned to **L** *or body has symmetry*

$$\text{Canonical momentum:} \quad p_\mu = \frac{\partial L}{\partial \dot{q}^\mu} \quad (\text{where: } L = T)$$

$$\mathbf{L} = \frac{\partial T}{\partial \boldsymbol{\omega}} = \nabla_{\boldsymbol{\omega}} T = \frac{\partial}{\partial \boldsymbol{\omega}} \frac{\boldsymbol{\omega} \bullet \mathbf{I} \bullet \boldsymbol{\omega}}{2} = \mathbf{I} \bullet \boldsymbol{\omega}$$

Kinetic energy in terms of momentum L and rotational Hamiltonian

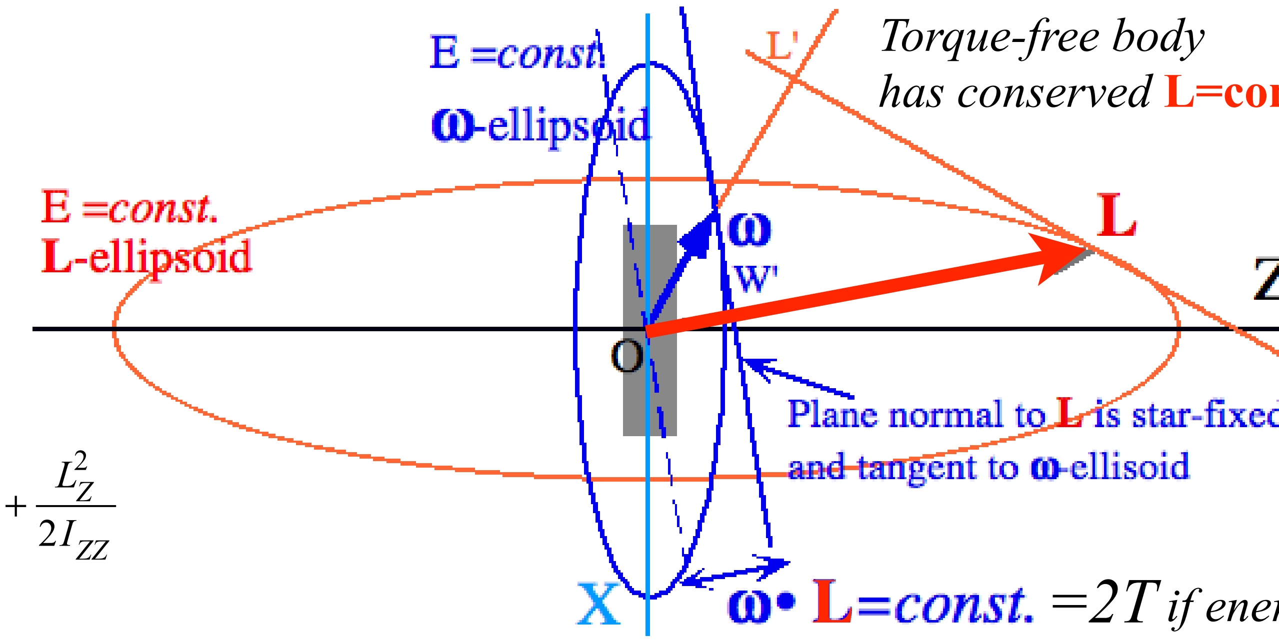
$\mathbf{L} = \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega}$, generally implies: $\boldsymbol{\omega} = \overleftrightarrow{\mathbf{I}}^{-1} \bullet \mathbf{L}$

Express kinetic energy T in terms of angular velocity ω , momentum **L**, or both at once. once

$$T = \frac{1}{2}\boldsymbol{\omega} \bullet \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega} \bullet \mathbf{L} = \frac{1}{2}\mathbf{L} \bullet \boldsymbol{\omega} = \frac{1}{2}\mathbf{L} \bullet \overleftrightarrow{\mathbf{I}}^{-1} \bullet \mathbf{L}$$

$$T = \frac{1}{2}\begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix}\begin{pmatrix} I_{XX} & I_{XY} & I_{XZ} \\ I_{YX} & I_{YY} & I_{YZ} \\ I_{ZX} & I_{ZY} & I_{ZZ} \end{pmatrix}^{-1}\begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix}\begin{pmatrix} 1/I_{XX} & 0 & 0 \\ 0 & 1/I_{YY} & 0 \\ 0 & 0 & 1/I_{ZZ} \end{pmatrix}\begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix} = \frac{L_X^2}{2I_{XX}} + \frac{L_Y^2}{2I_{YY}} + \frac{L_Z^2}{2I_{ZZ}}$$



Torque-free body has conserved **L=const**.

ω• L=*const.* =2*T if energy is not dissipated internally*

Hamiltonian form is the equation of the *angular momentum or* **L***-ellipsoid*

Lagrangian form is the equation of the *angular velocity or ω-ellipsoid*

ω is generally not conserved unless it is aligned to **L** *or body has symmetry*

Canonical momentum: $p_\mu = \frac{\partial L}{\partial \dot{q}^\mu}$ (where: $L = T$)

$$\mathbf{L} = \frac{\partial T}{\partial \boldsymbol{\omega}} = \nabla_{\boldsymbol{\omega}} T = \frac{\partial}{\partial \boldsymbol{\omega}}\frac{\boldsymbol{\omega} \bullet \mathbf{I} \bullet \boldsymbol{\omega}}{2} = \mathbf{I} \bullet \boldsymbol{\omega}$$

Hamilton's 1st equations : $\dot{q}^\mu = \frac{\partial H}{\partial p_\mu}$ (where: $H = T$)

$$\boldsymbol{\omega} = \frac{\partial H}{\partial \mathbf{L}} = \nabla_{\mathbf{L}} H = \frac{\partial}{\partial \mathbf{L}}\frac{\mathbf{L} \bullet \mathbf{I}^{-1} \bullet \mathbf{L}}{2} = \mathbf{I}^{-1} \bullet \mathbf{L}$$

Kinetic energy in terms of momentum L and rotational Hamiltonian

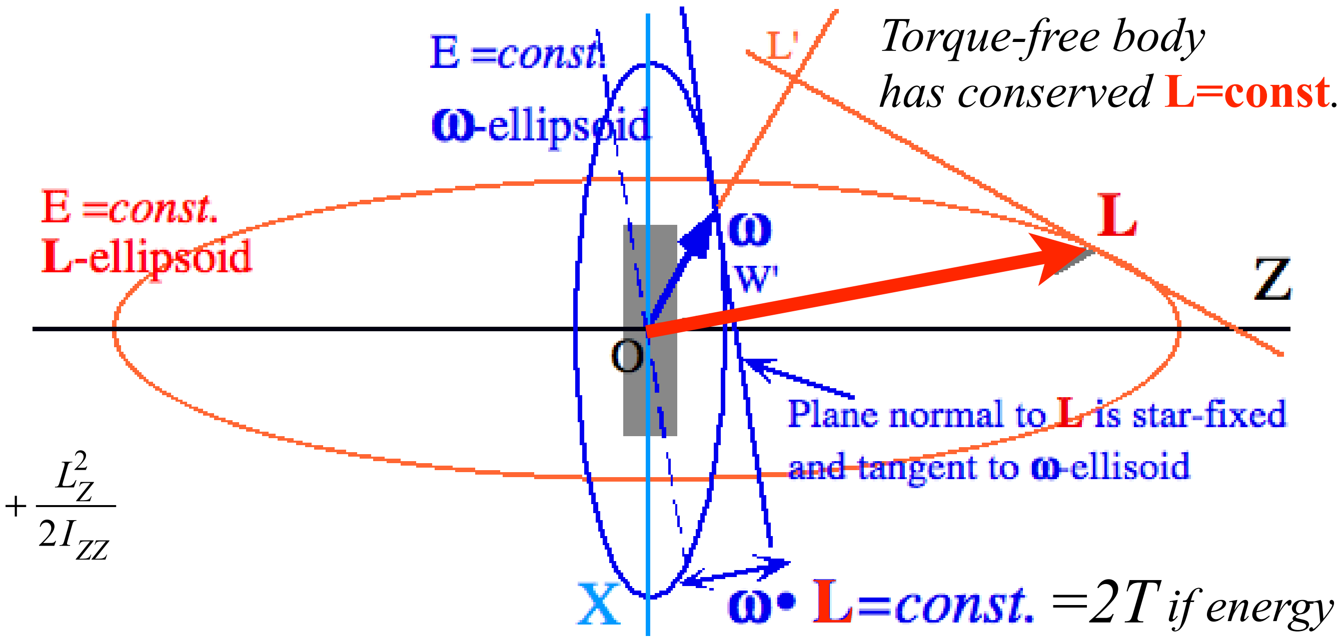
$\mathbf{L} = \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega}$, generally implies: $\boldsymbol{\omega} = \overleftrightarrow{\mathbf{I}}^{-1} \bullet \mathbf{L}$

Express kinetic energy T in terms of angular velocity ω , momentum $\mathbf{L}$, or both at once. once

$$T = \frac{1}{2}\boldsymbol{\omega} \bullet \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega} \bullet \mathbf{L} = \frac{1}{2}\mathbf{L} \bullet \boldsymbol{\omega} = \frac{1}{2}\mathbf{L} \bullet \overleftrightarrow{\mathbf{I}}^{-1} \bullet \mathbf{L}$$

$$T = \frac{1}{2}\begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix} \begin{pmatrix} I_{XX} & I_{XY} & I_{XZ} \\ I_{YX} & I_{YY} & I_{YZ} \\ I_{ZX} & I_{ZY} & I_{ZZ} \end{pmatrix}^{-1} \begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix}$$

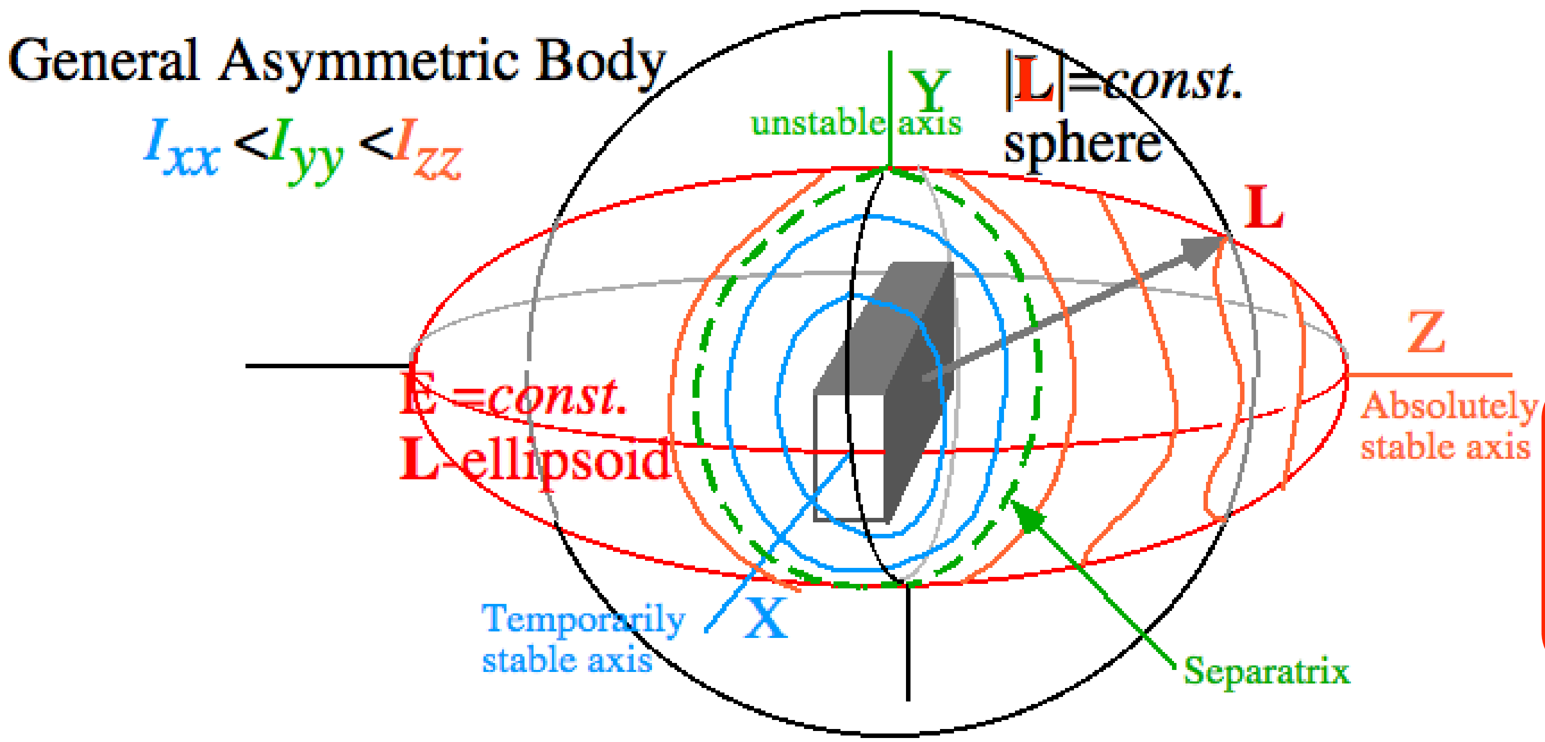
$$= \frac{1}{2}\begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix} \begin{pmatrix} 1/I_{XX} & 0 & 0 \\ 0 & 1/I_{YY} & 0 \\ 0 & 0 & 1/I_{ZZ} \end{pmatrix} \begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix} = \frac{L_X^2}{2I_{XX}} + \frac{L_Y^2}{2I_{YY}} + \frac{L_Z^2}{2I_{ZZ}}$$



Hamiltonian form is the equation of the *angular momentum or* ***L****-ellipsoid*

Lagrangian form is the equation of the *angular velocity or ω-ellipsoid*

ω is generally not conserved unless it is aligned to ***L*** *or body has symmetry*



Canonical momentum: $p_\mu = \dfrac{\partial L}{\partial \dot{q}^\mu}$ (where: $L = T$)

$$\mathbf{L} = \frac{\partial T}{\partial \boldsymbol{\omega}} = \nabla_{\boldsymbol{\omega}} T = \frac{\partial}{\partial \boldsymbol{\omega}} \frac{\boldsymbol{\omega} \bullet \mathbf{I} \bullet \boldsymbol{\omega}}{2} = \mathbf{I} \bullet \boldsymbol{\omega}$$

Hamilton's 1st equations : $\dot{q}^\mu = \dfrac{\partial H}{\partial p_\mu}$ (where: $H = T$)

$$\boldsymbol{\omega} = \frac{\partial H}{\partial \mathbf{L}} = \nabla_{\mathbf{L}} H = \frac{\partial}{\partial \mathbf{L}} \frac{\mathbf{L} \bullet \mathbf{I}^{-1} \bullet \mathbf{L}}{2} = \mathbf{I}^{-1} \bullet \mathbf{L}$$

Kinetic energy in terms of momentum L and rotational Hamiltonian

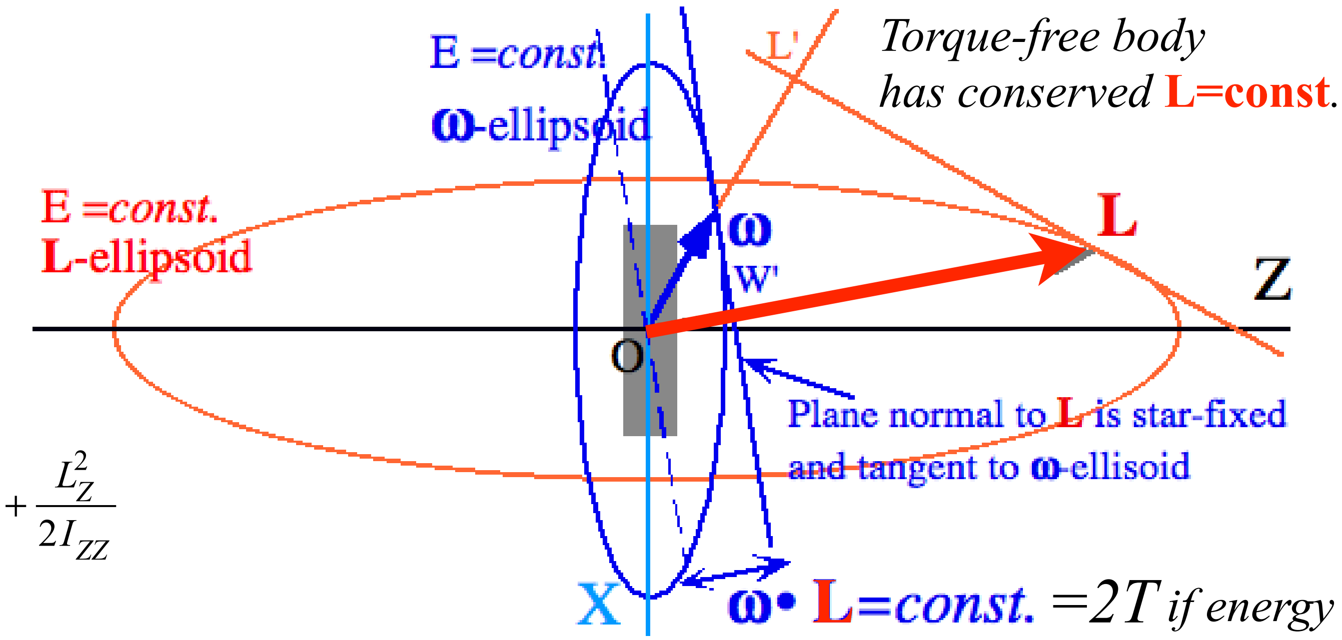
$\mathbf{L} = \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega}$, generally implies: $\boldsymbol{\omega} = \overleftrightarrow{\mathbf{I}}^{-1} \bullet \mathbf{L}$

Express kinetic energy T in terms of angular velocity ω , momentum $\mathbf{L}$, or both at once. once

$$T = \frac{1}{2}\boldsymbol{\omega} \bullet \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega} \bullet \mathbf{L} = \frac{1}{2}\mathbf{L} \bullet \boldsymbol{\omega} = \frac{1}{2}\mathbf{L} \bullet \overleftrightarrow{\mathbf{I}}^{-1} \bullet \mathbf{L}$$

$$T = \frac{1}{2}\begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix}\begin{pmatrix} I_{XX} & I_{XY} & I_{XZ} \\ I_{YX} & I_{YY} & I_{YZ} \\ I_{ZX} & I_{ZY} & I_{ZZ} \end{pmatrix}^{-1}\begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix}$$

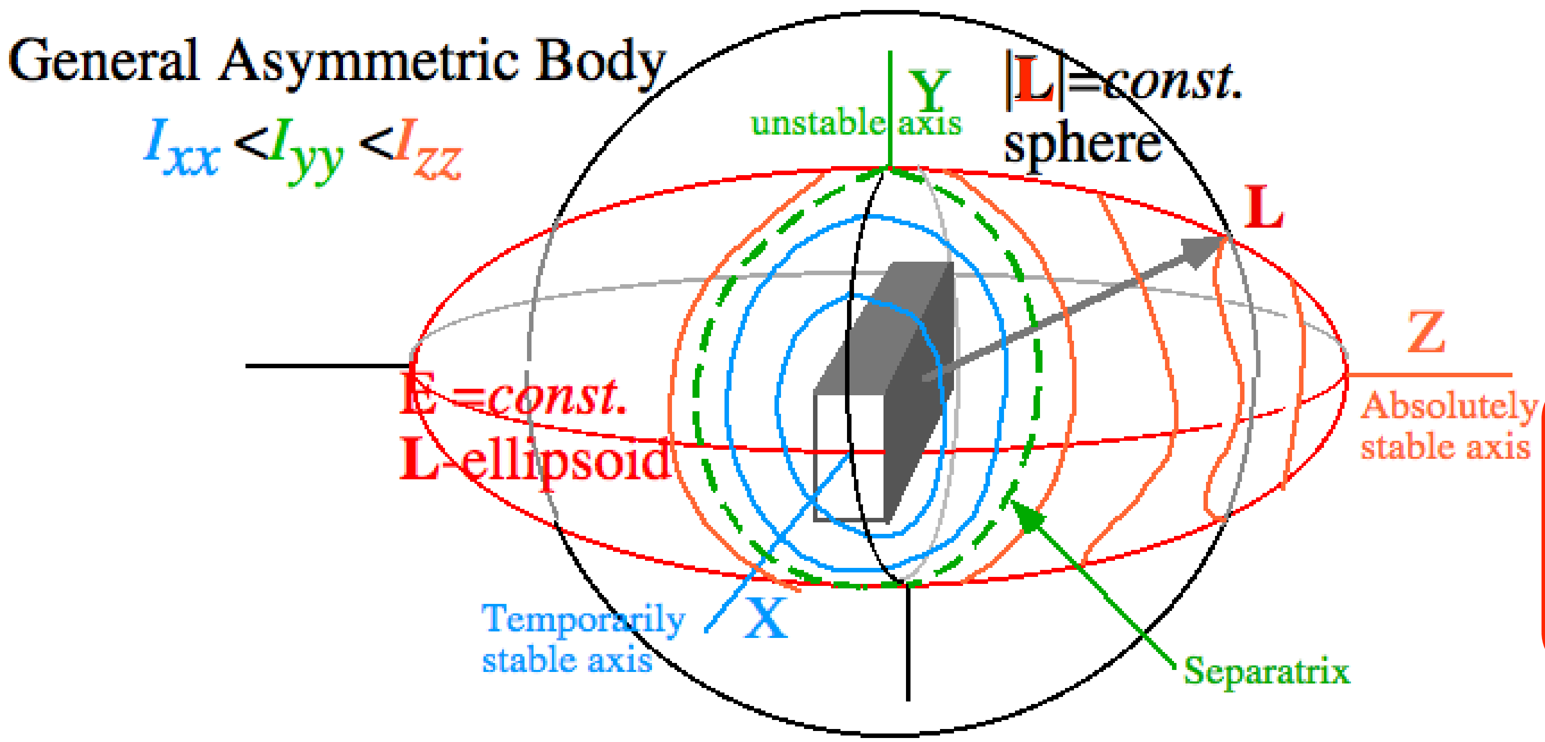
$$= \frac{1}{2}\begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix}\begin{pmatrix} 1/I_{XX} & 0 & 0 \\ 0 & 1/I_{YY} & 0 \\ 0 & 0 & 1/I_{ZZ} \end{pmatrix}\begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix} = \frac{L_X^2}{2I_{XX}} + \frac{L_Y^2}{2I_{YY}} + \frac{L_Z^2}{2I_{ZZ}}$$



Hamiltonian form is the equation of the *angular momentum or* **L***-ellipsoid*

Lagrangian form is the equation of the *angular velocity or ω-ellipsoid*

ω is generally not conserved unless it is aligned to **L** *or body has symmetry*



Canonical momentum: $p_\mu = \dfrac{\partial L}{\partial \dot{q}^\mu}$ (where: $L = T$)

$$\mathbf{L} = \frac{\partial T}{\partial \boldsymbol{\omega}} = \nabla_{\boldsymbol{\omega}} T = \frac{\partial}{\partial \boldsymbol{\omega}} \frac{\boldsymbol{\omega} \bullet \mathbf{I} \bullet \boldsymbol{\omega}}{2} = \mathbf{I} \bullet \boldsymbol{\omega}$$

Hamilton's 1st equations : $\dot{q}^\mu = \dfrac{\partial H}{\partial p_\mu}$ (where: $H = T$)

$$\boldsymbol{\omega} = \frac{\partial H}{\partial \mathbf{L}} = \nabla_{\mathbf{L}} H = \frac{\partial}{\partial \mathbf{L}} \frac{\mathbf{L} \bullet \mathbf{I}^{-1} \bullet \mathbf{L}}{2} = \mathbf{I}^{-1} \bullet \mathbf{L}$$

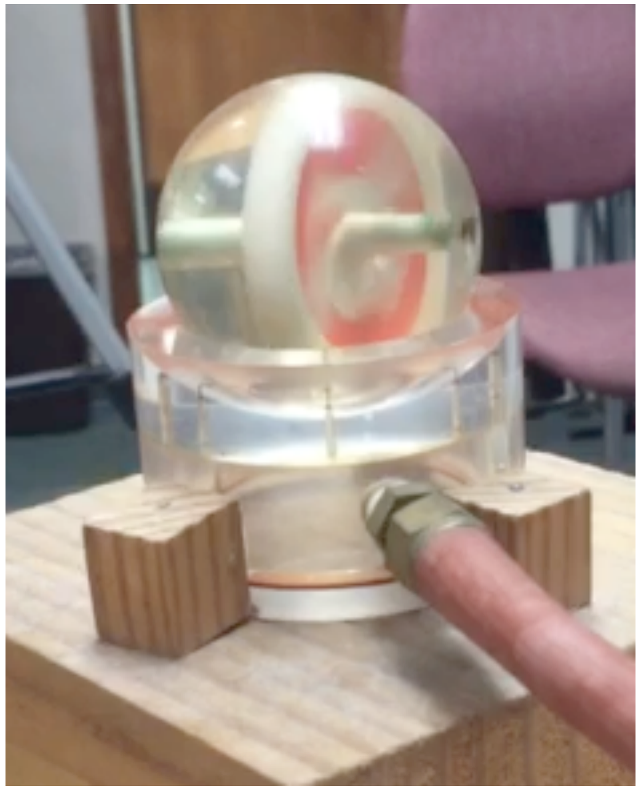
In body frame momentum **L** *moves along intersection of* **L***-ellipsoid and* **L***-sphere (Length |***L***| is constant in any classical frame.)*

Asymmetric Top Demo video

Rotational Energy Surfaces (RES)

→ *Symmetric, asymmetric, and spherical-top dynamics (Constant* **L***)*

BOD-frame cone rolling on LAB frame cone



Singular Motion of Asymetric Rotators
AJP (44) p1080

Asymmetric-top dynamics (Constant **L***)*

1. NASA Space station video

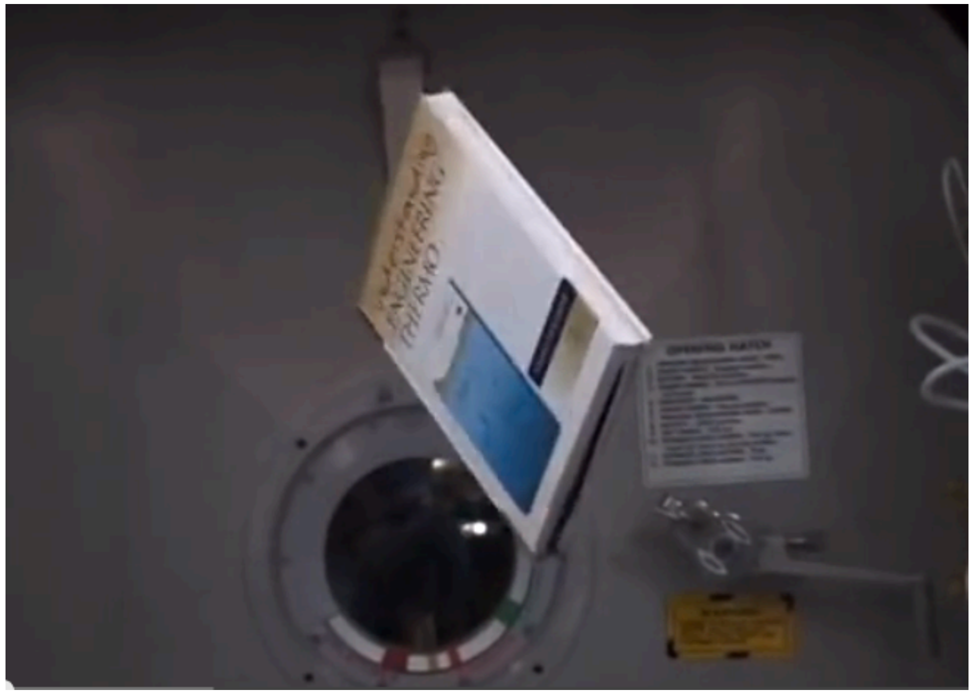
https://youtu.be/1n-HMSCDYtM

For those physist who are brave of heart, make note the video's comments

2. UAF lab air-supported asymmetric top video

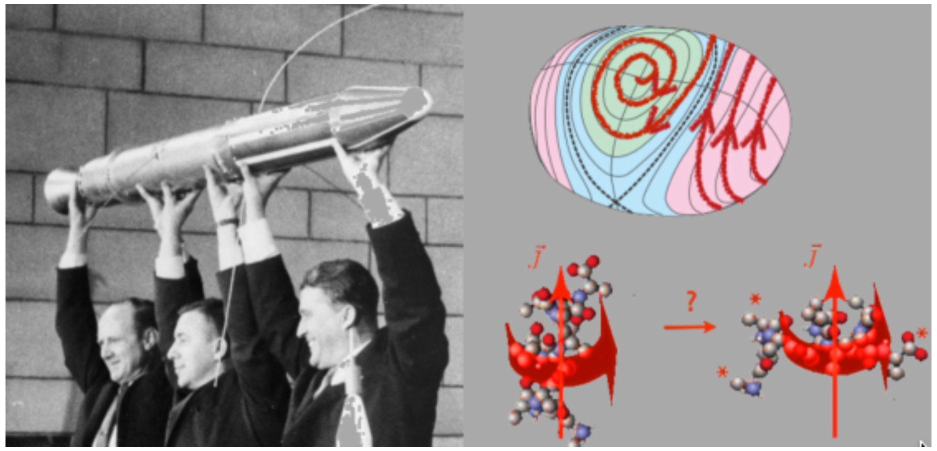
https://youtu.be/HWjGvCaqx5g

3. NASA-Rotating Solid Bodies in Microgravity (2008)



https://www.youtube.com/watch?v=BPMjcN-sBJ4

4. Early NASA-JPL satellite blunder (1958)



To be Continued ⇒several pages ahead

Comments following Space Lab video of asymmetric rotation show that it is not a widely understood phenomenon

Bagnon DuJour • 3 months ago

As the handles spins out it dips down a bit before becoming detached and that linear momentum travels through the angular momentum until the equilibrium requires the flip to maintain the path of least resistance. If they could spin it perfectly without the dip, it would not turn like that.

Reply • Share ›

Bill Aldridge → Bagnon DuJour • 3 months ago

So you are saying, when they put their hands on the tip, i dip, you dip, we dip.

Reply • Share ›

EVERYONE is born an atheist → Bagnon DuJour • 3 months ago

Exactly. Not sure why this was even posted. Maybe it was just going to b used as a basic physics example for schools.

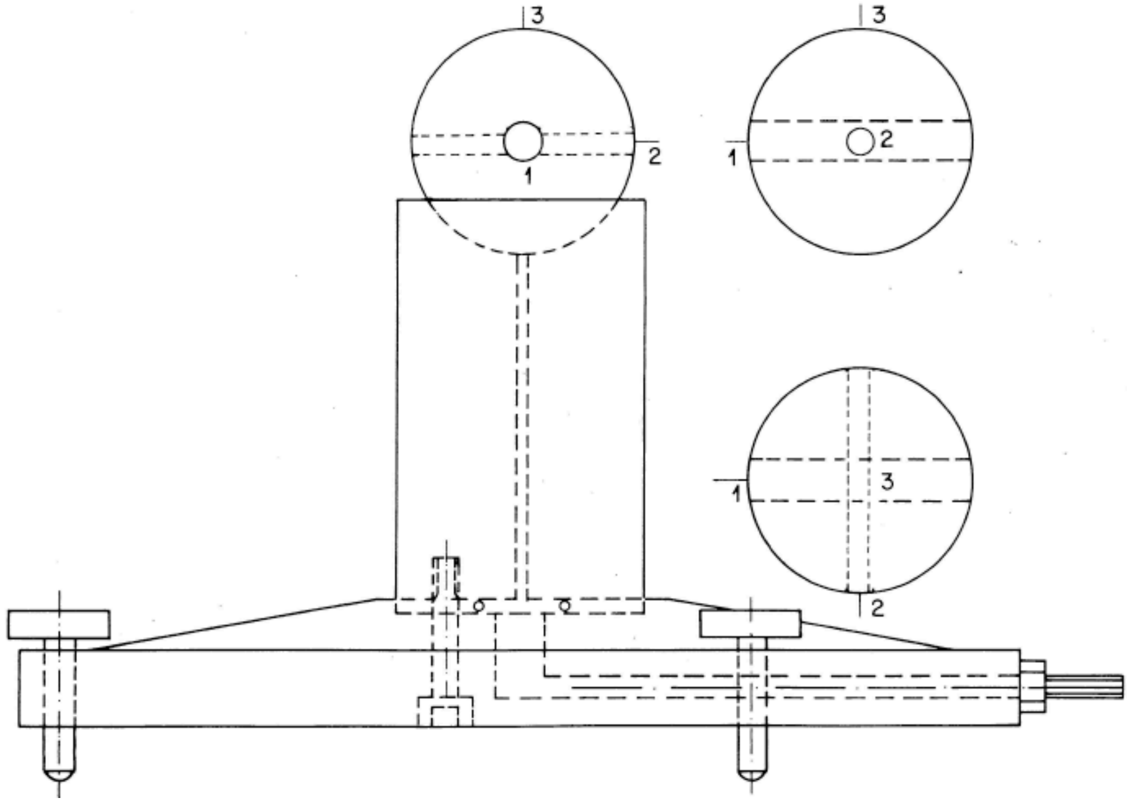
Reply • Share ›

Tim Johnson → Bagnon DuJour • 3 months ago

It sounds like you have a handle on what's going on here.

1 • Reply • Share ›

Bocce-Ball Asymmetric Top we built at USC *(donated to Cal. Museum of Science & Industry)*



$$I_1 = (2M/5 + m_2/3)R^2 + m_1 r_1{}^2/2 + m_2 r_2{}^2/4,$$
$$I_2 = (2M/5 + m_1/3)R^2 + m_2 r_2{}^2/2 + m_1 r_1{}^2/4, \quad (8)$$
$$I_3 = (2M/5 + m_1/3 + m_2/3)R^2 + m_1 r_1{}^2/4 + m_2 r_2{}^2/4,$$

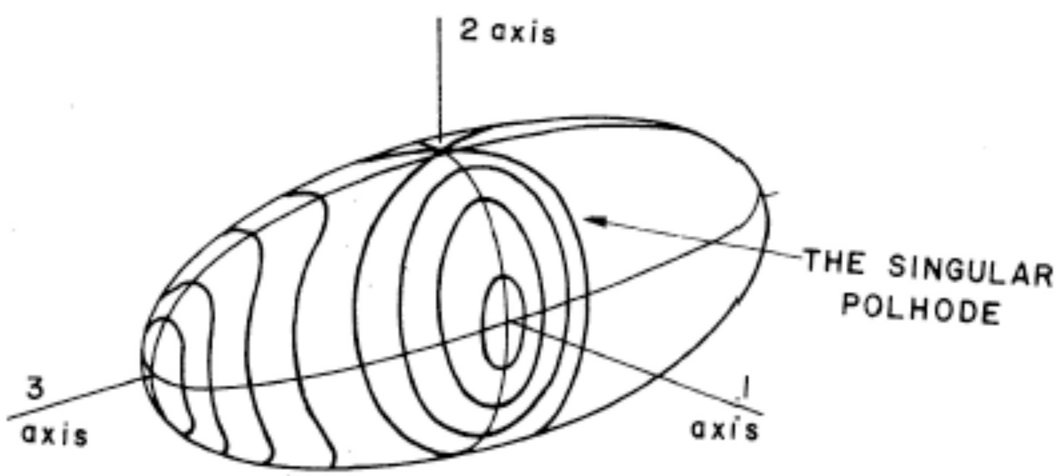


Fig. 3. Polhodes. A family of constraint curves for the vector ω in the body system, or "polhodes," are separated into two distinct groups by a curve called the singular polhode.

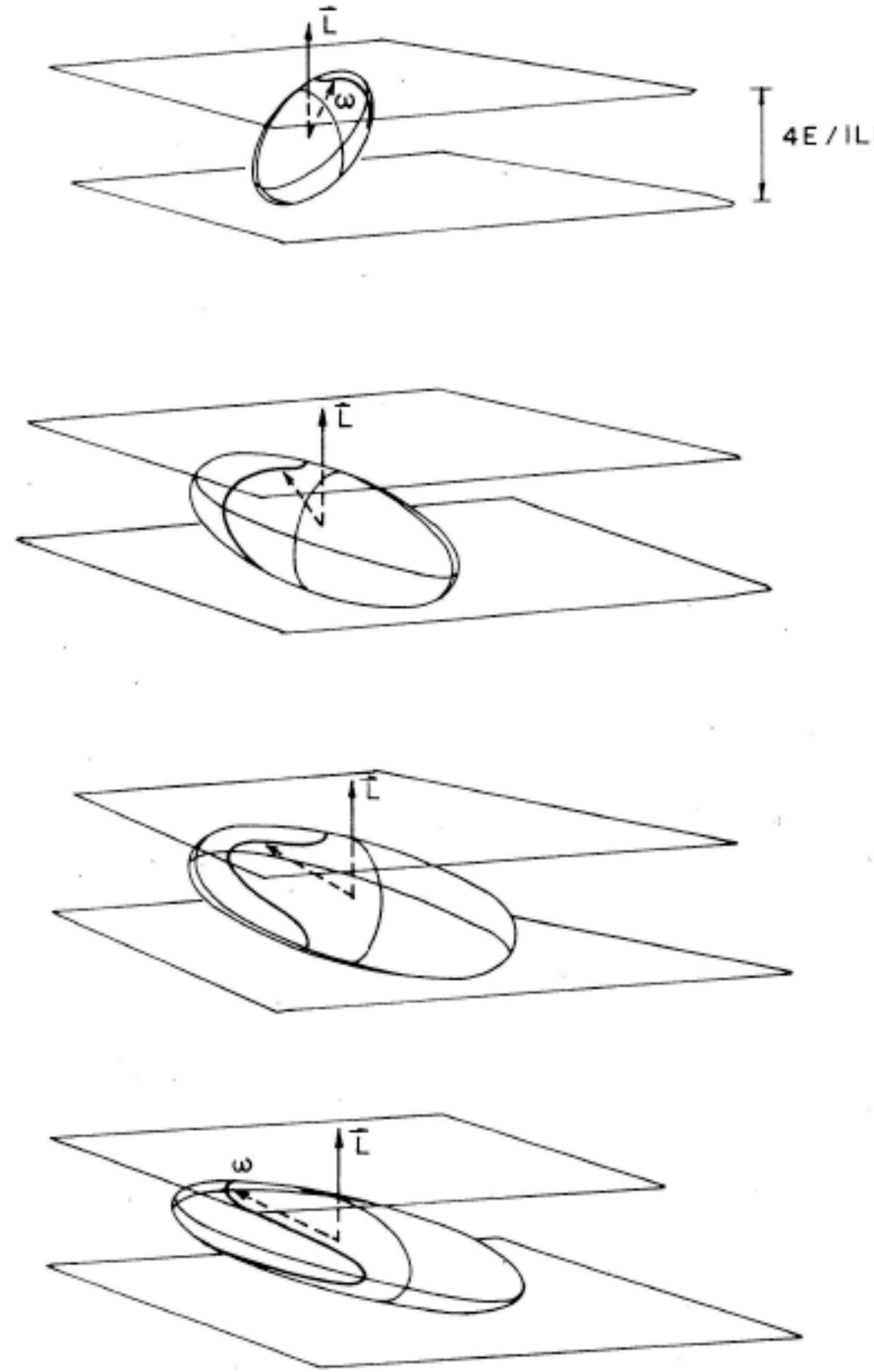


Fig. 4. Model of rotational motion near the singular polhode.

Bocce-Ball Asymmetric Top Motion solved by Euler's equation and elliptic integrals

$$\dot{\mathbf{L}} = \boldsymbol{\omega} \times \mathbf{L}, \tag{9}$$

which takes the following form for the 2 component:

$$\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3)/I_2 = 0. \tag{10}$$

Solving Eq. (10) for $\omega \equiv \omega_2$ using Eqs. (5) and (6), we obtain the following:

$$\dot{\omega} = (a - b\omega^2)^{1/2}(c - d\omega^2)^{1/2}/I_2(I_1 I_3)^{1/2}, \tag{11}$$

where the constants a–d [Eq. (12)] depend on initial conditions and the inertial moments as follows:

$$a = 2EI_3 - L^2, \quad b = I_2(I_3 - I_2),$$
$$c = L^2 - 2EI_1, \quad d = I_2(I_2 - I_1),$$
$$a = I_2(I_3 - I_2)W^2 \cos^2 \epsilon,$$
$$c = [I_2(I_2 - I_1) \cos^2 \epsilon + I_3(I_3 - I_1) \sin^2 \epsilon)W^2, \tag{12}$$

where we have assumed initial conditions

$$\omega_1(0) = 0, \quad \omega_2(0) = W \cos \epsilon, \quad \omega_3(0) = W \sin \epsilon. \tag{13}$$

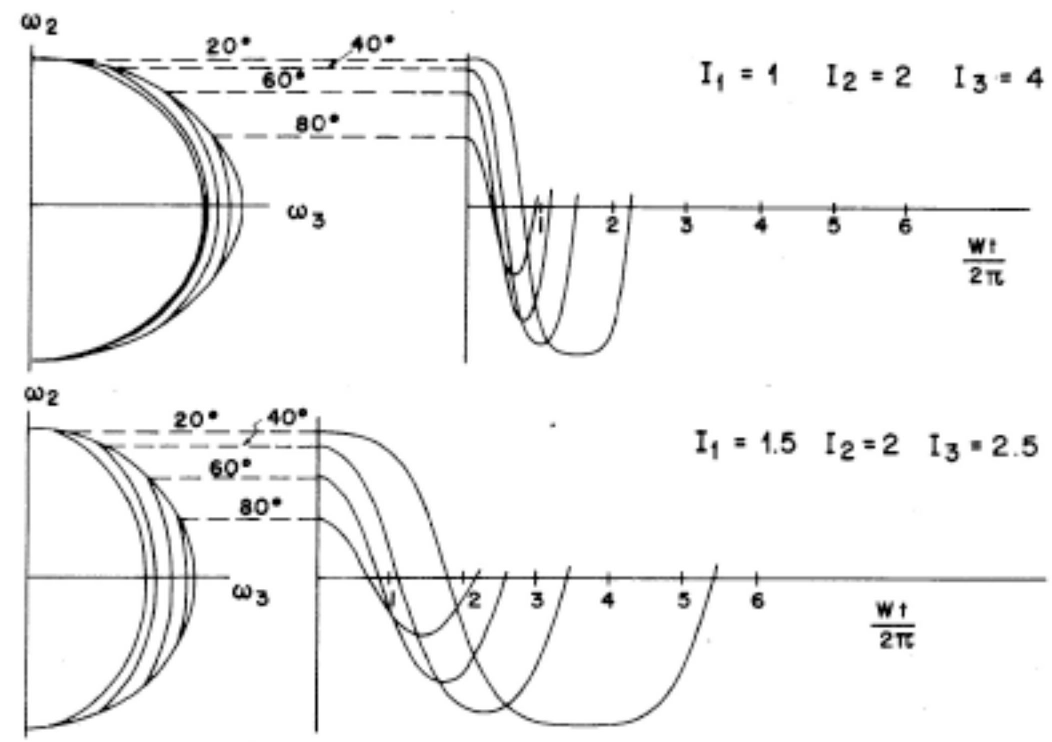


Fig. 6. Exact solutions. The motion of the ω vector for an asymmetric and a not-so-asymmetric body are compared. Various polhodes are shown on the left-hand side while the corresponding time behavior is plotted on the right-hand side.

$$\dot{\mathbf{L}} = \boldsymbol{\omega} \times \mathbf{L}, \tag{9}$$

which takes the following form for the 2 component:

$$\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3)/I_2 = 0. \tag{10}$$

Solving Eq. (10) for $\omega \equiv \omega_2$ using Eqs. (5) and (6), we obtain the following:

$$\dot{\omega} = (a - b\omega^2)^{1/2}(c - d\omega^2)^{1/2}/I_2(I_1I_3)^{1/2}, \tag{11}$$

where the constants a–d [Eq. (12)] depend on initial conditions and the inertial moments as follows:

$$\begin{gathered} a = 2EI_3 - L^2, \quad b = I_2(I_3 - I_2), \\ c = L^2 - 2EI_1, \quad d = I_2(I_2 - I_1), \\ a = I_2(I_3 - I_2)W^2 \cos^2 \epsilon, \\ c = [I_2(I_2 - I_1) \cos^2 \epsilon + I_3(I_3 - I_1) \sin^2 \epsilon)W^2, \end{gathered} \tag{12}$$

where we have assumed initial conditions

$$\omega_1(0) = 0, \quad \omega_2(0) = W \cos \epsilon, \quad \omega_3(0) = W \sin \epsilon. \tag{13}$$

$$t = \left(\frac{I_1I_2I_3}{(I_3 - I_2)(L^2 - 2EI_1)}\right)^{1/2} \times \int_0^{\Omega'} \frac{d\Omega}{(1 - \Omega^2)^{1/2}(1 - k^2\Omega^2)^{1/2}}, \tag{14}$$

where the following substitutions were made:

$$k = (ad/bc)^{1/2}, \quad \omega = (a/b)^{1/2}\Omega = \Omega W \cos \epsilon. \tag{15}$$

A further substitution $\Omega = \sin\phi$ reduces the integral

$$\int_0^{\Omega'} \frac{d\Omega}{(1 - \Omega^2)^{1/2}(1 - k^2\Omega^2)^{1/2}} = \int_0^{\phi'} \frac{d\phi}{(1 - k^2 \sin^2 \phi)^{1/2}} \equiv \mathrm{sn}^{-1}(\phi', k). \tag{16}$$

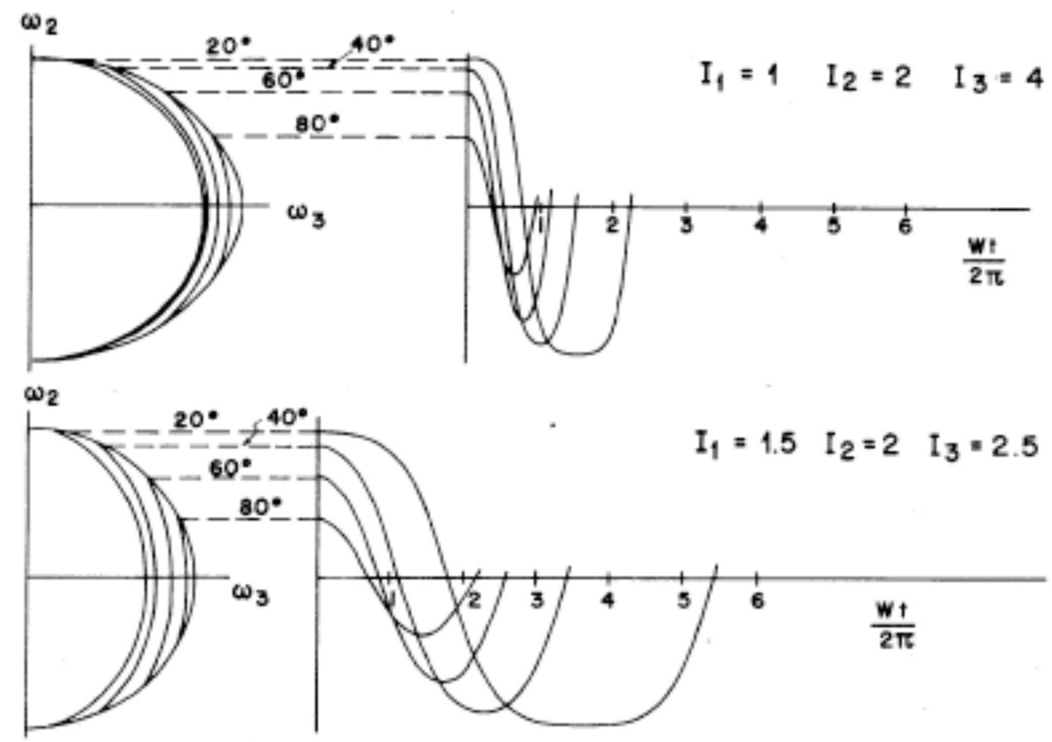


Fig. 6. Exact solutions. The motion of the ω vector for an asymmetric and a not-so-asymmetric body are compared. Various polhodes are shown on the left-hand side while the corresponding time behavior is plotted on the right-hand side.

$$\dot{\mathbf{L}} = \boldsymbol{\omega} \times \mathbf{L}, \quad (9)$$

which takes the following form for the 2 component:

$$\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3)/I_2 = 0. \quad (10)$$

Solving Eq. (10) for $\omega \equiv \omega_2$ using Eqs. (5) and (6), we obtain the following:

$$\dot{\omega} = (a - b\omega^2)^{1/2}(c - d\omega^2)^{1/2}/I_2(I_1I_3)^{1/2}, \quad (11)$$

where the constants a–d [Eq. (12)] depend on initial conditions and the inertial moments as follows:

$$\begin{gathered} a = 2EI_3 - L^2, \quad b = I_2(I_3 - I_2), \\ c = L^2 - 2EI_1, \quad d = I_2(I_2 - I_1), \\ a = I_2(I_3 - I_2)W^2\cos^2\epsilon, \\ c = [I_2(I_2 - I_1)\cos^2\epsilon + I_3(I_3 - I_1)\sin^2\epsilon)W^2, \quad (12) \end{gathered}$$

where we have assumed initial conditions

$$\omega_1(0) = 0, \quad \omega_2(0) = W\cos\epsilon, \quad \omega_3(0) = W\sin\epsilon. \quad (13)$$

$$t = \left(\frac{I_1I_2I_3}{(I_3 - I_2)(L^2 - 2EI_1)}\right)^{1/2} \times \int_0^{\Omega'} \frac{d\Omega}{(1 - \Omega^2)^{1/2}(1 - k^2\Omega^2)^{1/2}}, \quad (14)$$

where the following substitutions were made:

$$k = (ad/bc)^{1/2}, \quad \omega = (a/b)^{1/2}\Omega = \Omega W\cos\epsilon. \quad (15)$$

A further substitution $\Omega = \sin\phi$ reduces the integral

$$\int_0^{\Omega'} \frac{d\Omega}{(1 - \Omega^2)^{1/2}(1 - k^2\Omega^2)^{1/2}} = \int_0^{\phi'} \frac{d\phi}{(1 - k^2\sin^2\phi)^{1/2}} \equiv \mathrm{sn}^{-1}(\phi', k). \quad (16)$$

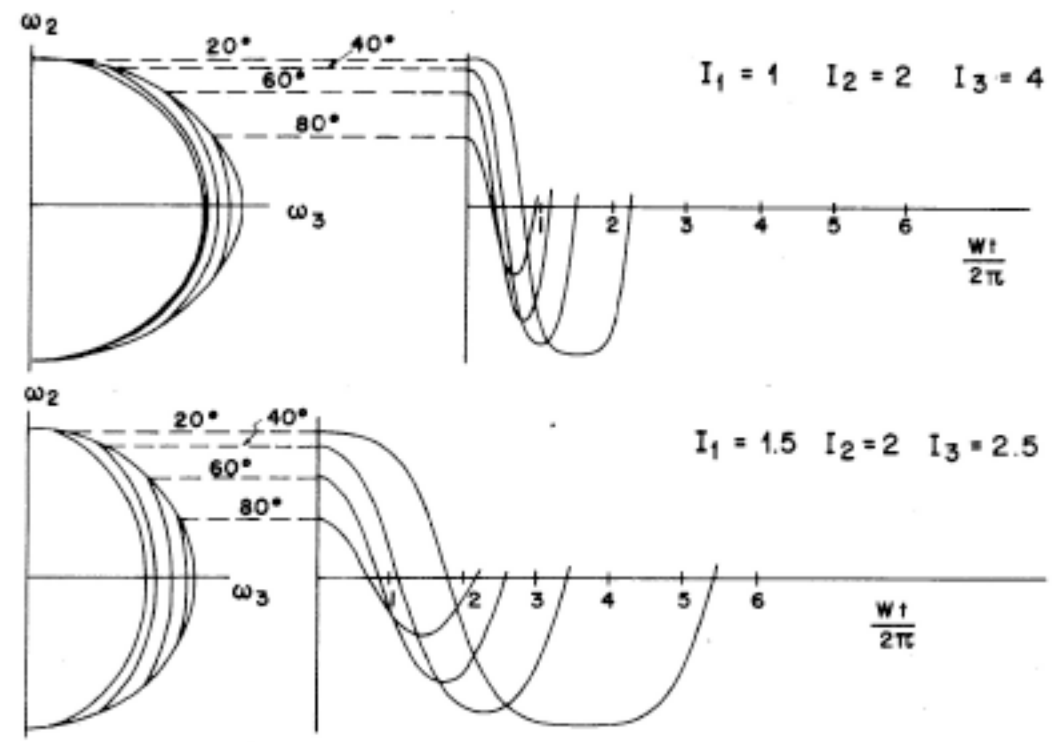


Fig. 6. Exact solutions. The motion of the ω vector for an asymmetric and a not-so-asymmetric body are compared. Various polhodes are shown on the left-hand side while the corresponding time behavior is plotted on the right-hand side.

$$t = \frac{2}{W} \times \left(\frac{I_1I_2I_3}{(I_3 - I_2)[I_2(I_2 - I_1)\cos^2\epsilon + I_3(I_3 - I_1)\sin^2\epsilon]}\right)^{1/2} \times \mathrm{sn}^{-1}\left(\frac{\pi}{2}, k\right), \quad (17a)$$

$$t \to \frac{2}{W}\left(\frac{I_1I_2}{(I_3 - I_2)(I_2 - I_1)}\right)^{1/2} \mathrm{sn}^{-1}\left(\frac{\pi}{2}, k\right), \quad (17b)$$

where

$$k = \left(\frac{I_2(I_2 - I_1)}{I_2(I_2 - I_1)\cos^2\epsilon + I_3(I_3 - I_1)\sin^2\epsilon}\right)^{1/2}\cos\epsilon, \quad (18a)$$

$$k \to 1 - (I_3/I_2)[(I_3 - I_1)/(I_2 - I_1)](\epsilon^2/2). \quad (18b)$$

Bocce-Ball Asymmetric Top Motion solved by Euler's equation and elliptic integrals

$$\dot{\mathbf{L}} = \boldsymbol{\omega} \times \mathbf{L}, \qquad (9)$$

which takes the following form for the 2 component:

$$\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3)/I_2 = 0. \qquad (10)$$

Solving Eq. (10) for $\omega \equiv \omega_2$ using Eqs. (5) and (6), we obtain the following:

$$\dot{\omega} = (a - b\omega^2)^{1/2}(c - d\omega^2)^{1/2}/I_2(I_1 I_3)^{1/2}, \qquad (11)$$

where the constants a–d [Eq. (12)] depend on initial conditions and the inertial moments as follows:

$$\begin{aligned} a &= 2EI_3 - L^2, \quad b = I_2(I_3 - I_2), \\ c &= L^2 - 2EI_1, \quad d = I_2(I_2 - I_1), \\ a &= I_2(I_3 - I_2)W^2 \cos^2 \epsilon, \\ c &= [I_2(I_2 - I_1)\cos^2 \epsilon + I_3(I_3 - I_1)\sin^2 \epsilon)W^2, \end{aligned} \qquad (12)$$

where we have assumed initial conditions

$$\omega_1(0) = 0, \quad \omega_2(0) = W \cos \epsilon, \quad \omega_3(0) = W \sin \epsilon. \qquad (13)$$

$$t = \left(\frac{I_1 I_2 I_3}{(I_3 - I_2)(L^2 - 2EI_1)}\right)^{1/2} \times \int_0^{\Omega'} \frac{d\Omega}{(1 - \Omega^2)^{1/2}(1 - k^2\Omega^2)^{1/2}}, \qquad (14)$$

where the following substitutions were made:

$$k = (ad/bc)^{1/2}, \quad \omega = (a/b)^{1/2}\Omega = \Omega W \cos \epsilon. \qquad (15)$$

A further substitution $\Omega = \sin\phi$ reduces the integral

$$\int_0^{\Omega'} \frac{d\Omega}{(1 - \Omega^2)^{1/2}(1 - k^2\Omega^2)^{1/2}} = \int_0^{\phi'} \frac{d\phi}{(1 - k^2 \sin^2 \phi)^{1/2}} \equiv \mathrm{sn}^{-1}(\phi', k). \qquad (16)$$

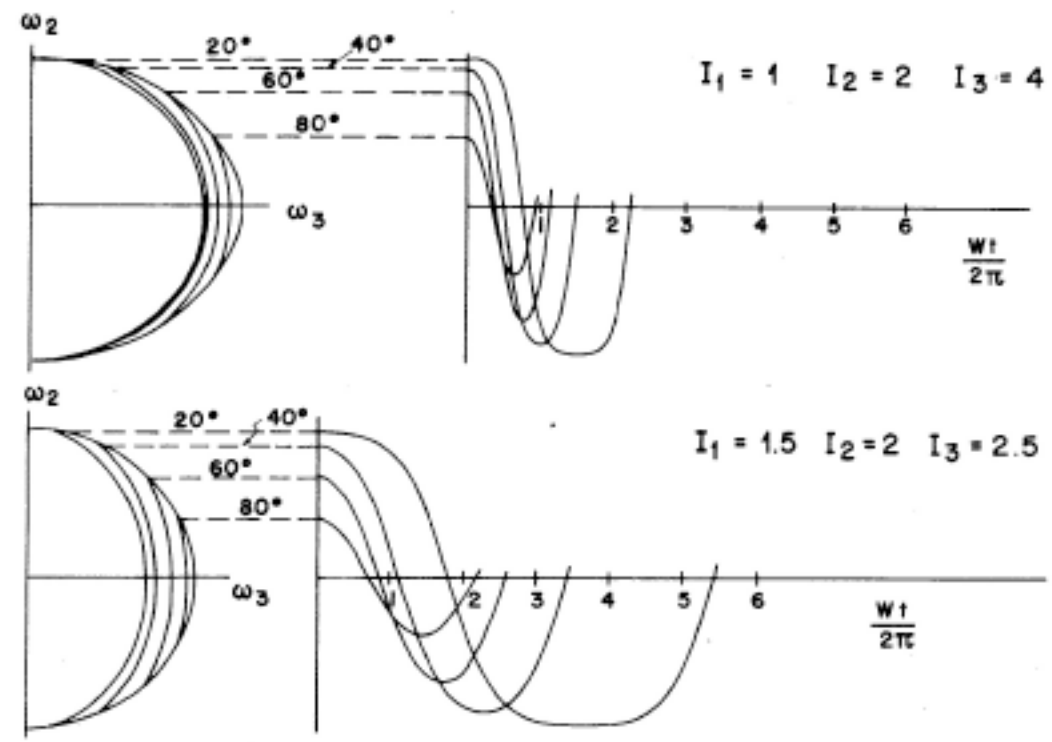
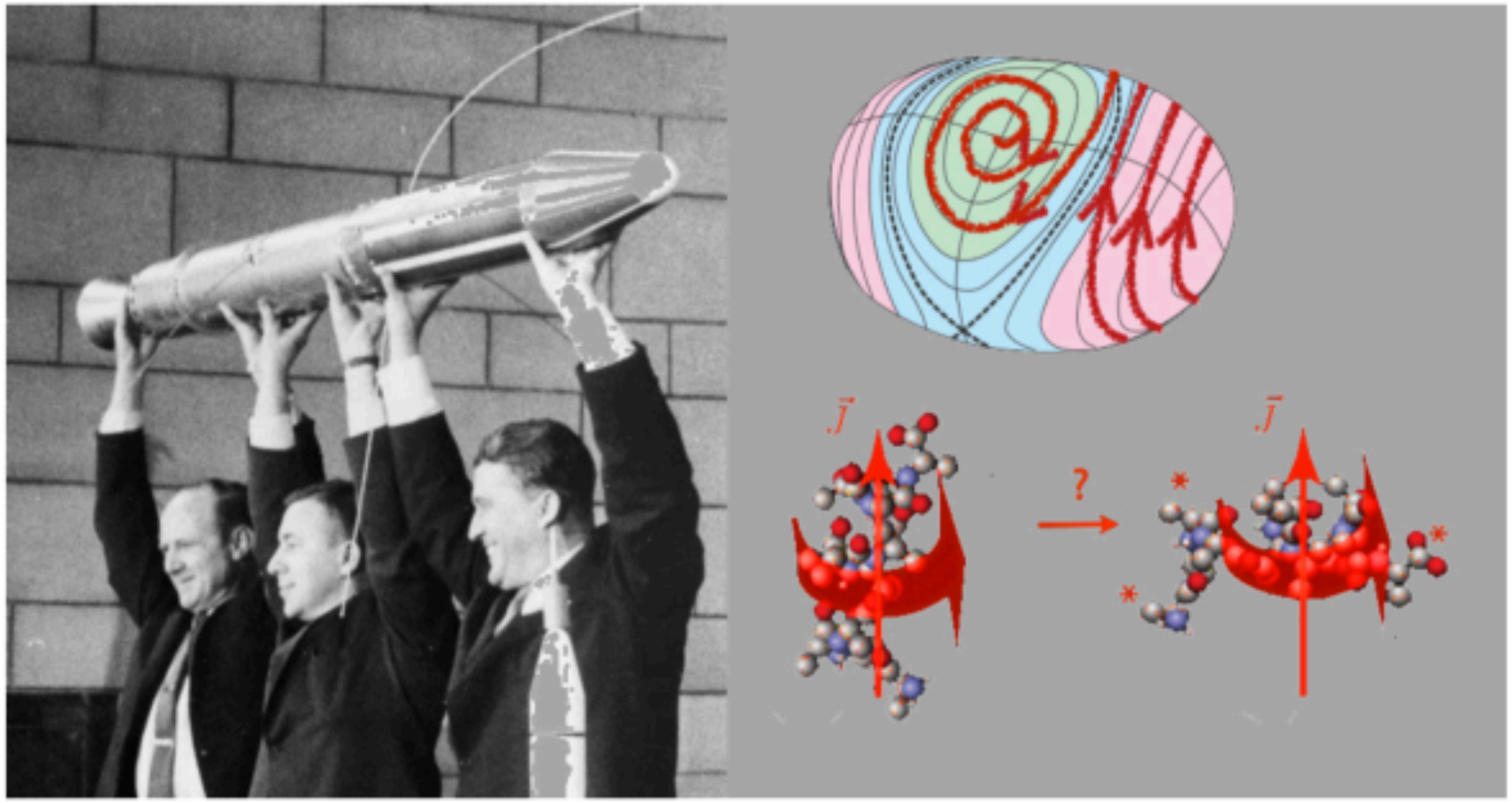


Fig. 6. Exact solutions. The motion of the ω vector for an asymmetric and a not-so-asymmetric body are compared. Various polhodes are shown on the left-hand side while the corresponding time behavior is plotted on the right-hand side.

The limiting forms [Eqs. (17) and (18b)] become good approximations for $\epsilon < 10°$. The approximate number of revolutions accomplished by a body before it overturns is given by the product of $W/2\pi$, the number of revolutions per second, and the right-hand side of Eq. (17b). Exact solutions for various I_j and ϵ are displayed in Fig. 6.

If one desires to increase the reversal time, it should be done through the first factor in Eq. (17b). The integral in the second factor is usually only as large as 7 or 8 in our experiments ($\epsilon = 10°$ gives 3.1, $\epsilon = 1°$ gives 5.4, and $\epsilon = 0°$ $1'$ gives 9.5). This is a good demonstration of the behavior of an elliptic function near its singularity.

4. Early NASA-JPL satellite blunder (1958)



From text in preparation by Rick Heller on semi-classical dynamics of polyatomic molecules

Figure 10.3: NASA-JPL early blunder. Rockets are not rigid bodies, especially with floppy whip antennas attached. The Explorer 1 satellite was the first one launched successfully by the United States. Seen in the left panel are James van Allen (center), William Pickering (left), and Werner von Braun, with a full-size model of the satellite, just after it was successfully orbited in 1958. As this press conference took place, the satellite was busily tumbling out of control. Van Allen soon realized that the intermittent signal from the satellite was due to tumbling. Fortunately, enough antennas were bristling from the satellite that it still gave much useful data, resulting in discovery of the van Allen radiation belts. The tumbling took place because friction due to slight wobbling is converted to heat, lowering the rotational energy, but not changing the angular momentum. The only way for this to happen is for the satellite to start rotating around a lower energy axis, until it bottoms out in end and over and tumbling at the lowest possible rotational energy for the given angular momentum. The author thanks Prof. William Harter for pointing out the existence and the physics of this story.

Rotational Energy Surface (RES) is quadratic multipole function plotted radially

$$E=\frac{\mathbf{J}_x^2}{2I_x}+\frac{\mathbf{J}_y^2}{2I_y}+\frac{\mathbf{J}_z^2}{2I_z}\quad \text{with } J=const.$$

$$=J^2\left(\frac{\sin^2\theta\cos^2\phi}{2I_x}+\frac{\sin^2\theta\sin^2\phi}{2I_y}+\frac{\cos^2\theta}{2I_z}\right)$$

Constant Energy Surface (CES) is asymmetric ellipsoid of constant E

$$E=\frac{\mathbf{J}_x^2}{2I_x}+\frac{\mathbf{J}_y^2}{2I_y}+\frac{\mathbf{J}_z^2}{2I_z}=const.$$

$$or:\quad \frac{\mathbf{J}_x^2}{2EI_x}+\frac{\mathbf{J}_y^2}{2EI_y}+\frac{\mathbf{J}_z^2}{2EI_z}=1$$

Here notation L or $\mathbf{L}$ for angular momentum is replaced by J or $\mathbf{J}$

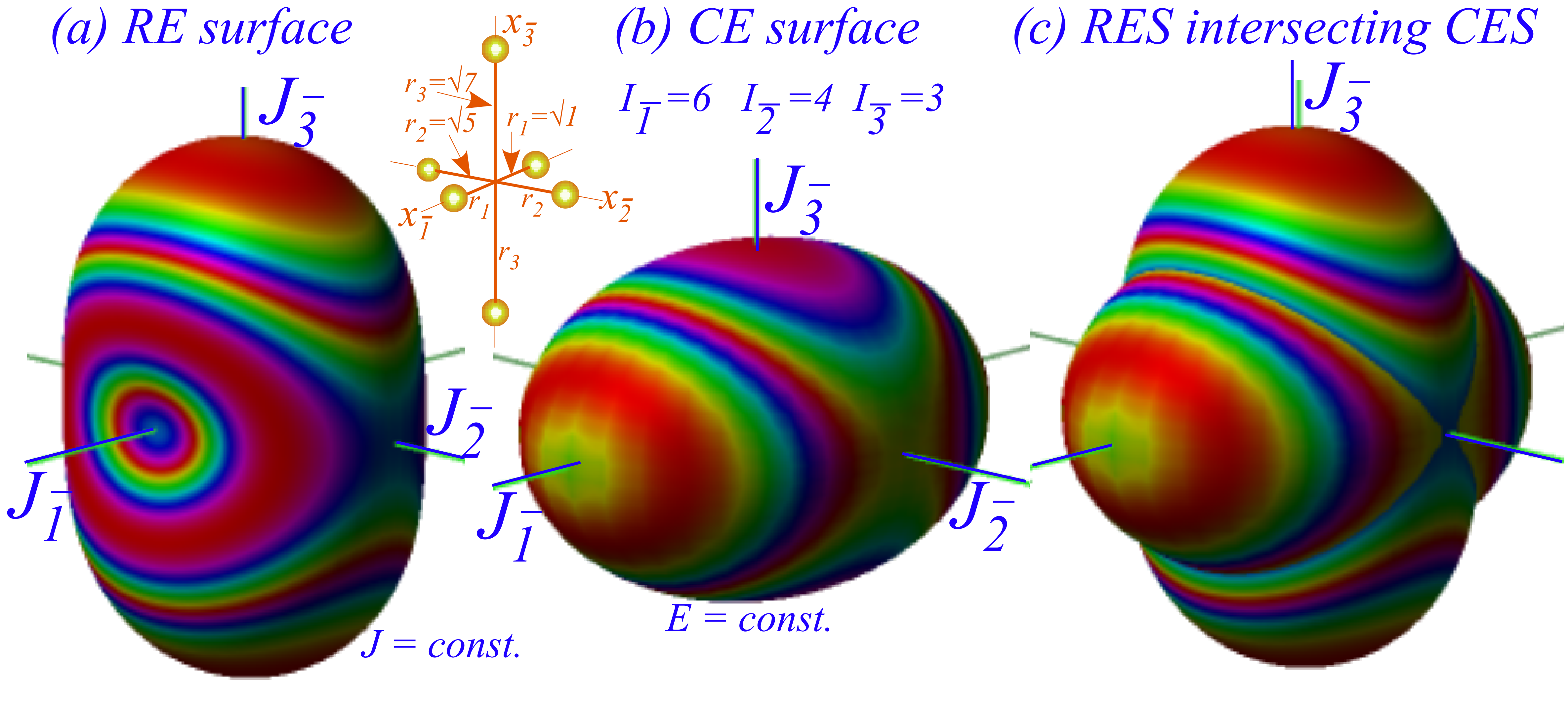


Fig. 6.8.1 Rigid rotor surfaces (a) RES polynomial, (b) CES ellipsoid, and (c) RES and CES intersected.

RES and CES for nearly-symmetric prolate rotors and nearly-symmetric oblate rotors

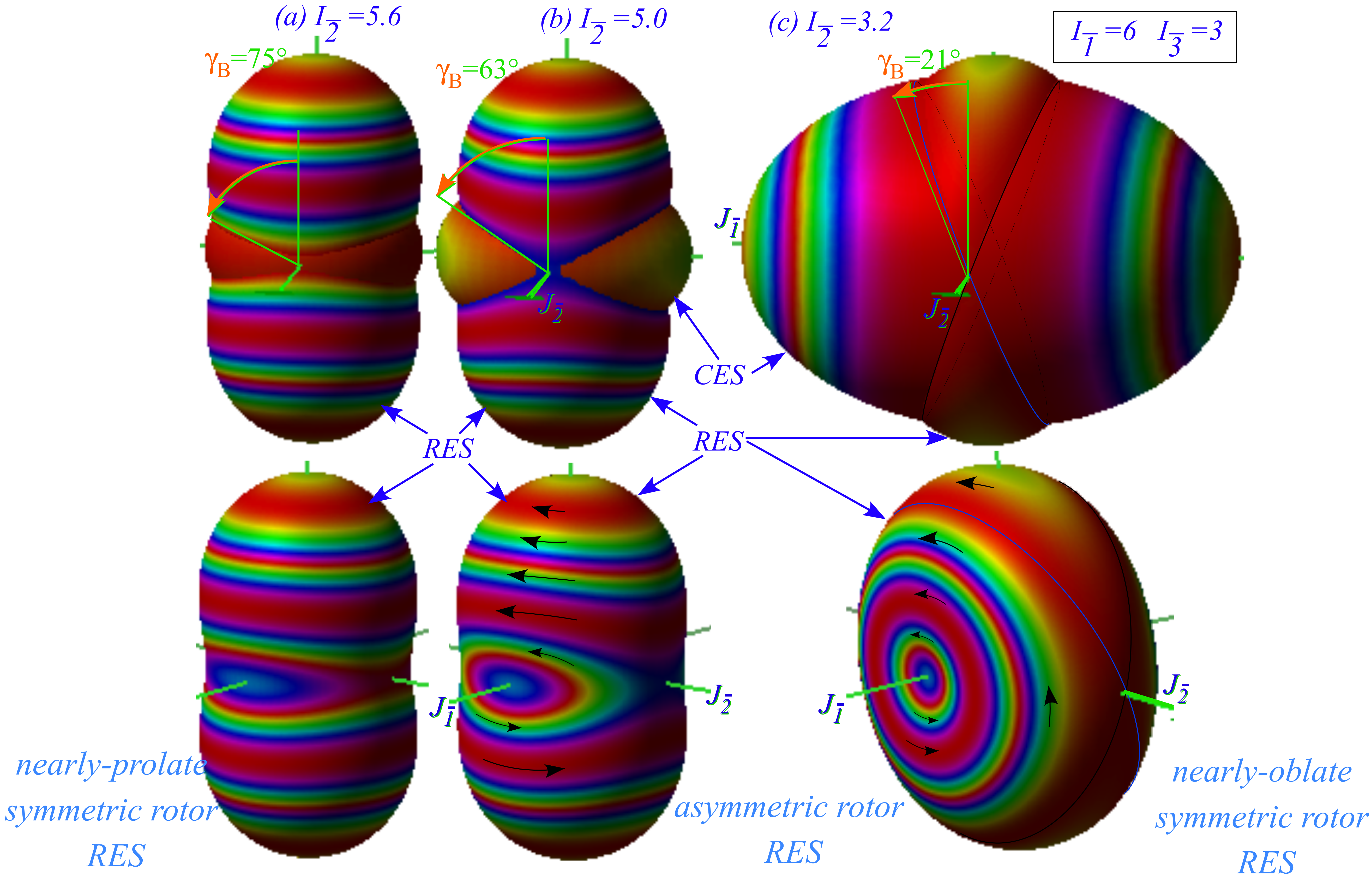
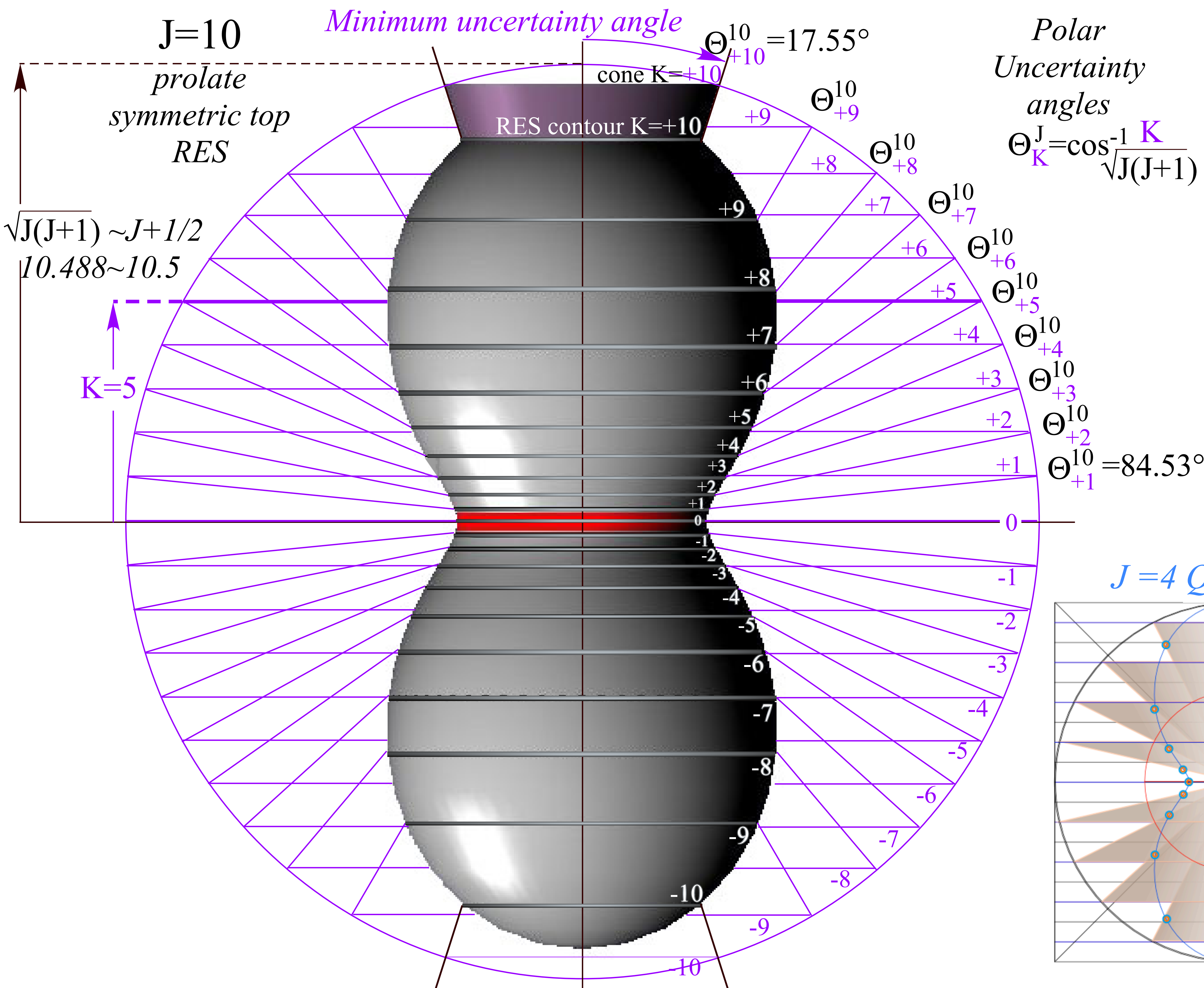
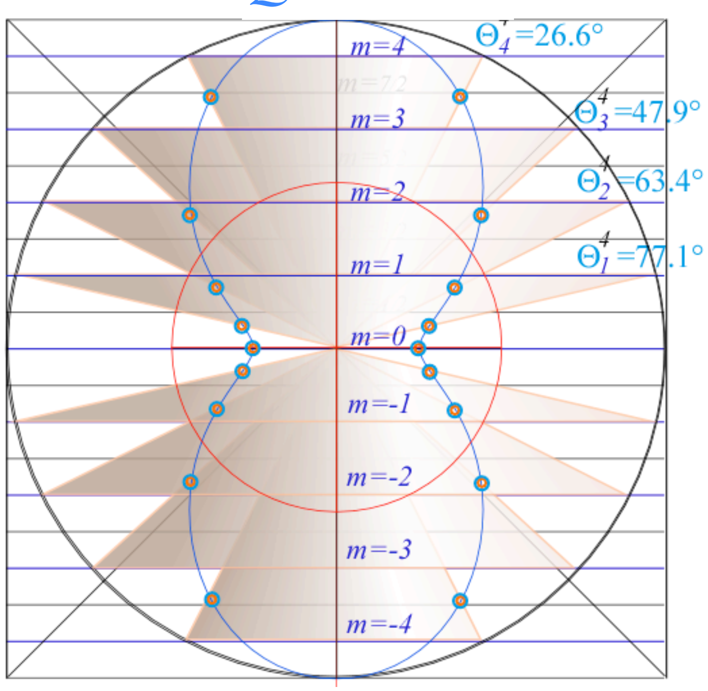


Fig. 6.8.2 Fixed-J- RES with CES at separatrix $E=J^2/2I_{\bar{2}}$ as $I_{\bar{2}}$ varies. (a) $I_{\bar{2}}=5.6$ and $\gamma_B=75.5°$ (Nearly prolate low-E CES), (b) $I_{\bar{2}}=5.0$ and $\gamma_B=63.4°$, (c) $I_{\bar{2}}=3.2$ and $\gamma_B=20.7°$ (Nearly oblate high-E CES).

RES for symmetric prolate rotor locates $J=10$ quantum $(-J<K<J)$ levels (at RES-quantum cone intersections)

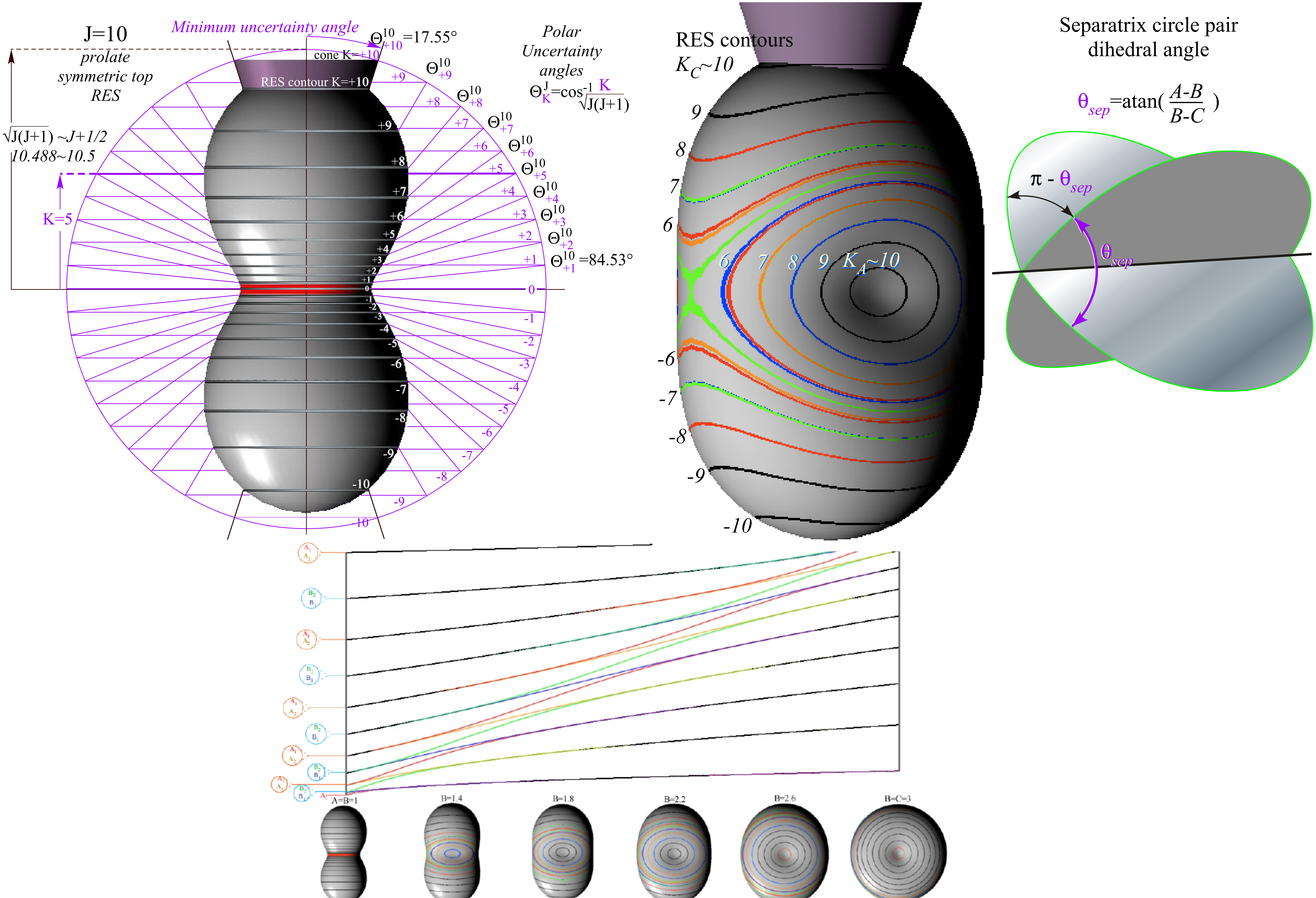


$J=4$ Quantum cones



Link to pdf of: W. G. Harter and J C. Mitchell ,International Journal of Molecular Science, 14, 714-806 (2013) Fig. 1-5 p.730

RES for symmetric and asymmetric rotor approximates J =10 (-J<K<J) levels *(near RES-quantum cone levels)*



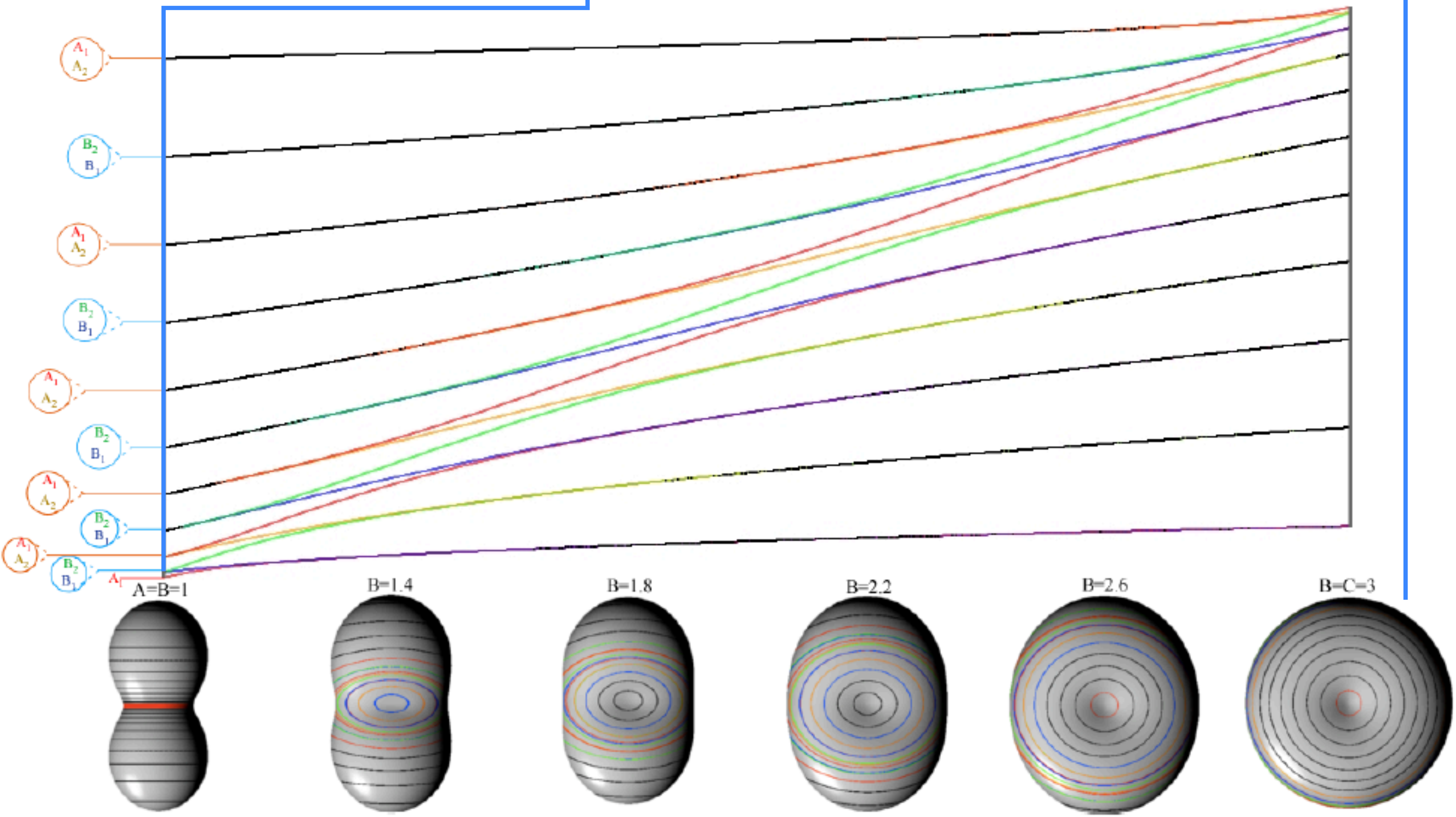
75

Link to pdf of: [W. G. Harter and J C. Mitchell ,International Journal of Molecular Science, 14, 714-806 (2013) Fig. 1-5 p.730]

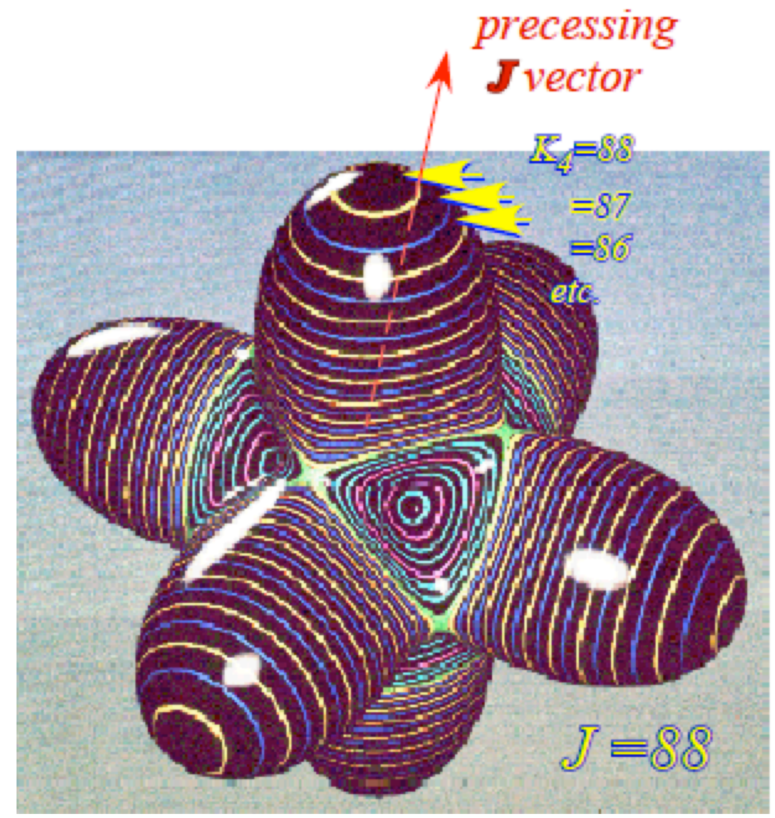
RES for symmetric prolate rotor locates $J=10$ quantum ($-J<K<J$) levels (at RES-quantum cone intersections)

$$E = A\mathbf{J}_x^2 + B\mathbf{J}_y^2 + C\mathbf{J}_z^2 \quad \text{with } J = const.$$

Spectra varies as symmetric prolate RES changes through a range of asymmetric RES to oblate RES



Link to pdf of: W. G. Harter and J C. Mitchell ,International Journal of Molecular Science, 14, 714-806 (2013) Fig. 1-5 p.730



Rotational Energy Surfaces (RES)

→ *Symmetric, asymmetric, and spherical-top dynamics (Constant* **J***)*

BOD-frame cone rolling on LAB frame cone

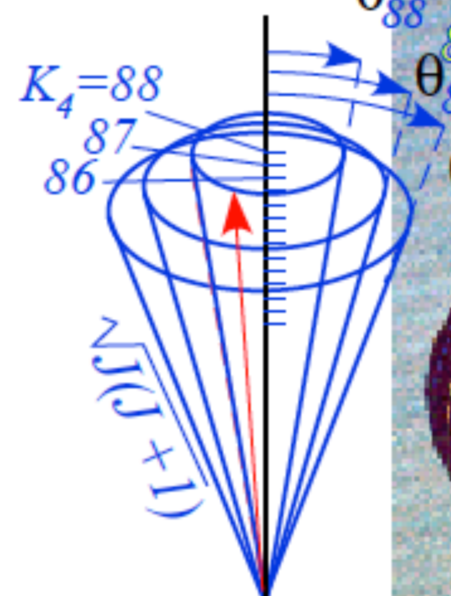
RES for spherical rotor approximates J =88 (-J<K<J) levels of SF_6

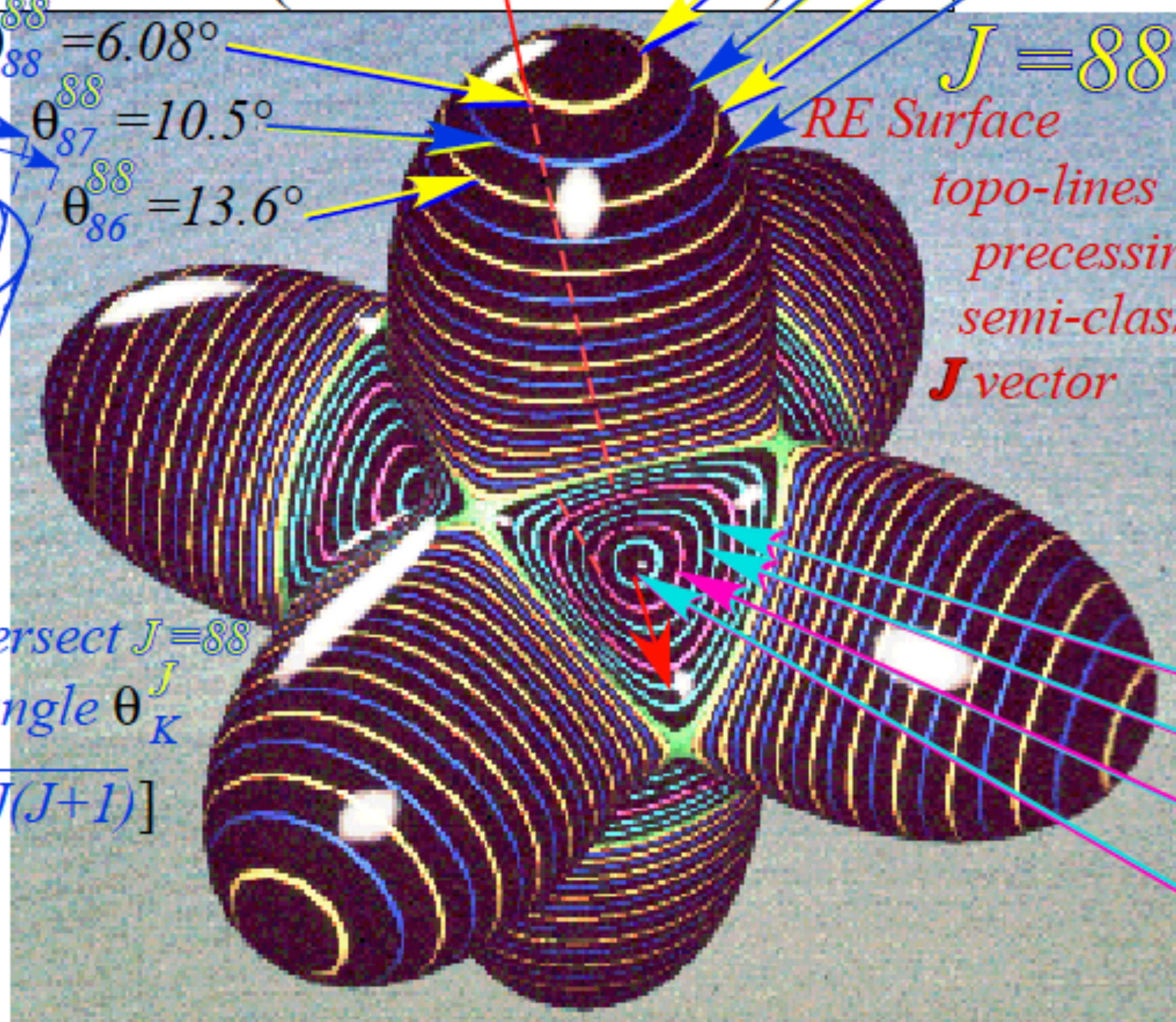
$$\langle H\rangle \sim \nu_{vib} + BJ(J+1) + \langle H^{Scalar\ Coriolis}\rangle + \langle H^{Tensor\ Centrifugal}\rangle + \langle H^{Tensor\ Coriolis}\rangle + \langle H^{Nuclear\ Spin}\rangle + \ldots$$

O_h or T_d Spherical Top: *(Hecht CH_4 Hamiltonian 1960)*

$$\mathbf{H} = B\left(\mathbf{J}_x^2 + \mathbf{J}_y^2 + \mathbf{J}_z^2\right) + t_{440}\left(\mathbf{J}_x^4 + \mathbf{J}_y^4 + \mathbf{J}_z^4 - \frac{3}{5}J^4\right) + \cdots$$

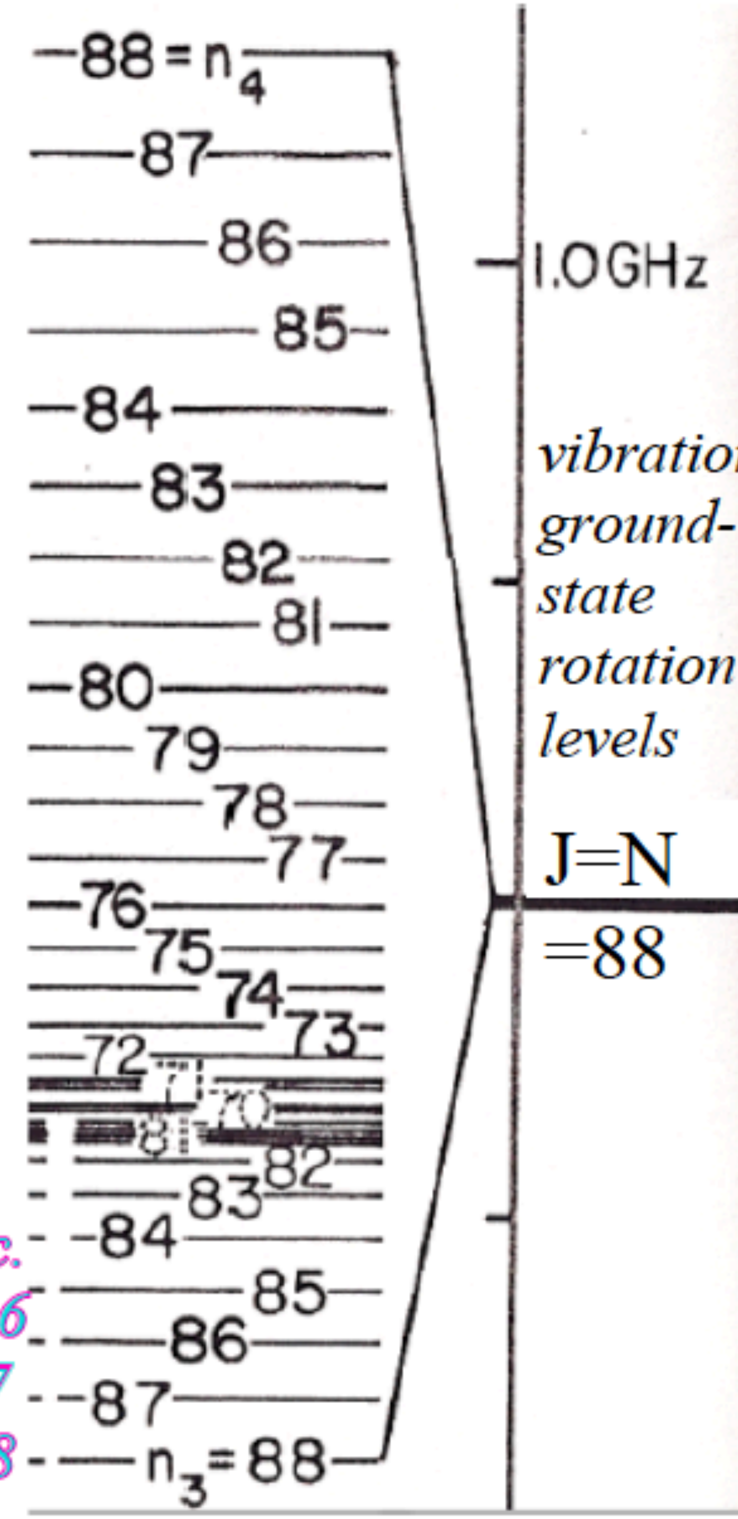
$$= \quad B\mathbf{J}^2 \quad + t_{440}\left(\mathbf{T}_0^4 + \sqrt{\frac{5}{14}}\left[\mathbf{T}_4^4 + \mathbf{T}_{-4}^4\right]\right) + \cdots$$





(J,K) cones intersect J=88 RE surface at angle θ^J_K

$$\theta^J_K = \text{acos}[K/\sqrt{J(J+1)}]$$



(next page shows slice)

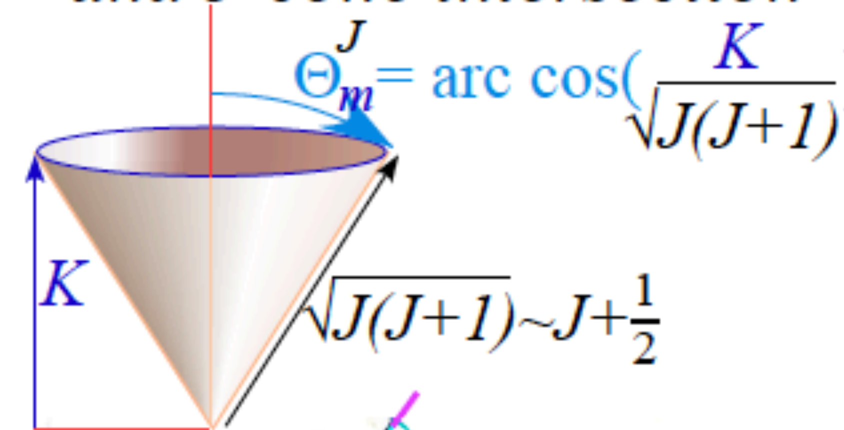
Link to pdf of: *W. G. Harter and J C. Mitchell ,International Symposium on Molecular Spectroscopy, OSU Columbus (2009)*

SF_6 Spectra of O_h Ro-vibronic Hamiltonian described by RE Tensor Topography

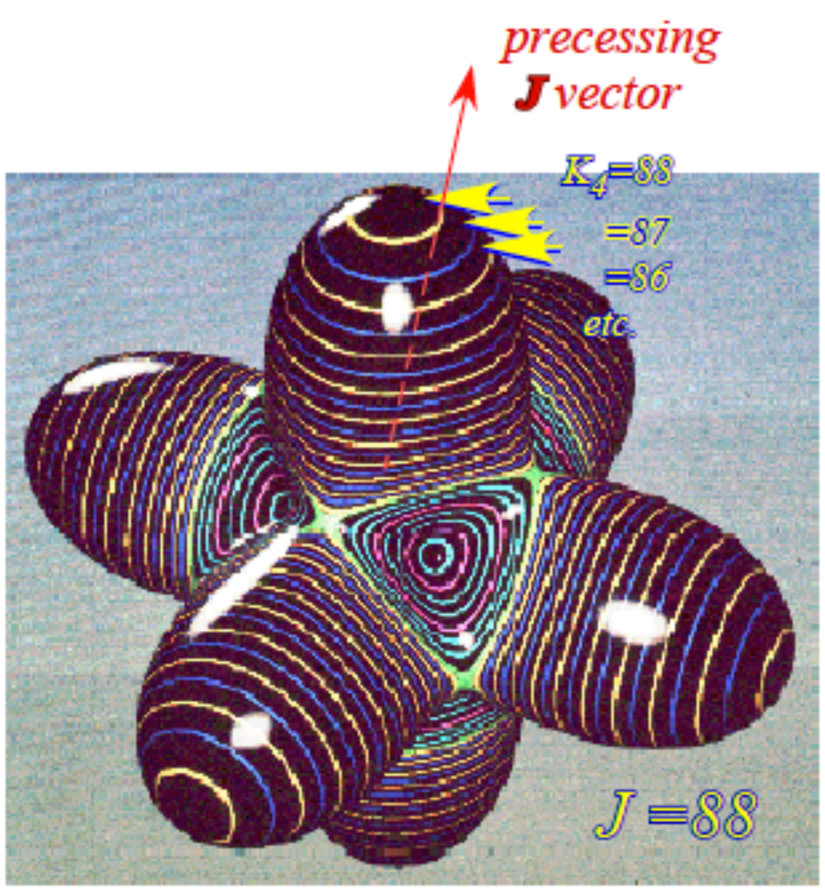
and J-cone intersection

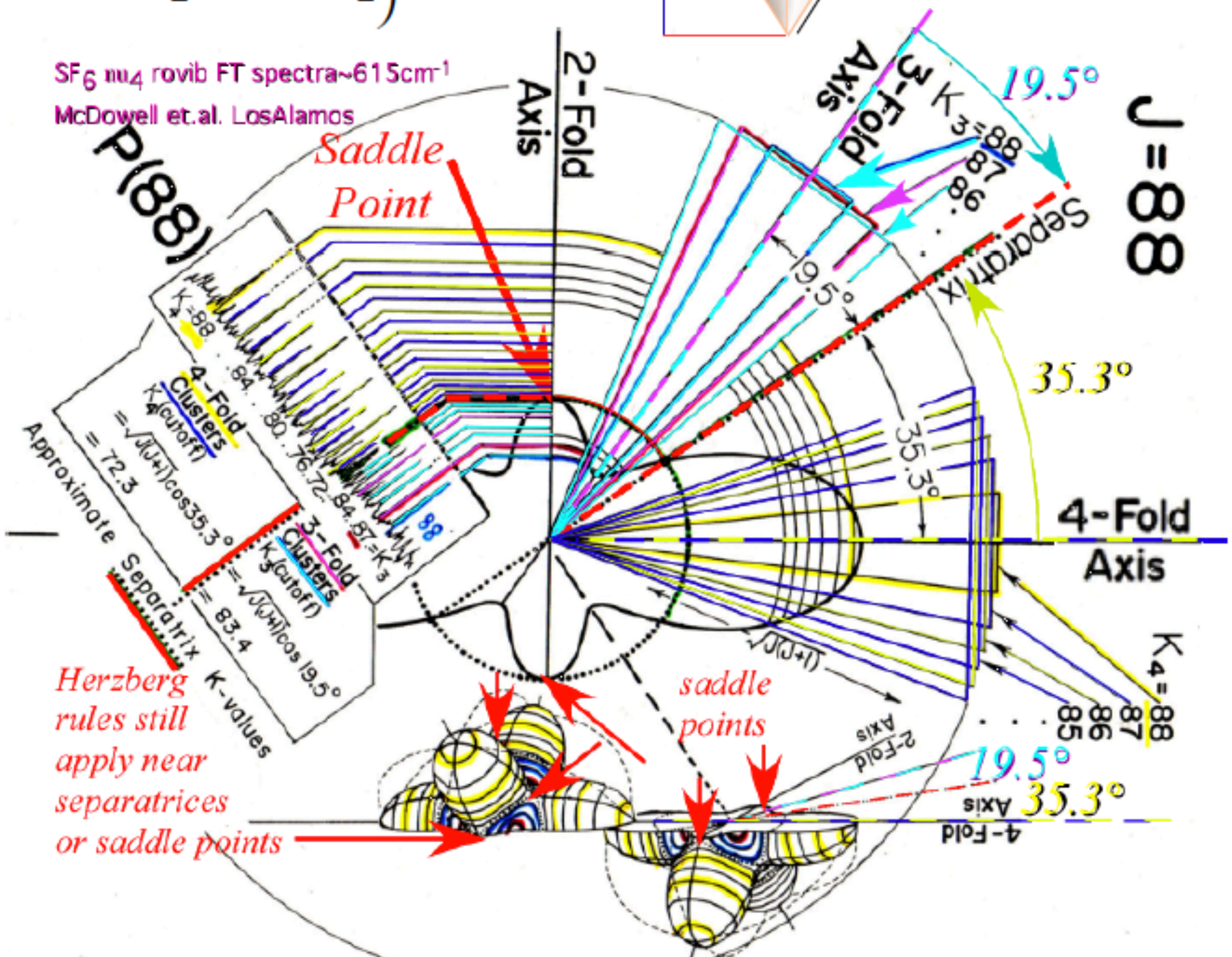
$$\mathbf{H} = B\left(\mathbf{J}_x^2 + \mathbf{J}_y^2 + \mathbf{J}_z^2\right) + t_{440}\left(\mathbf{J}_x^4 + \mathbf{J}_y^4 + \mathbf{J}_z^4 - \frac{3}{5}J^4\right) + \cdots$$

$$= \quad B\mathbf{J}^2 \quad + t_{440}\left(\mathbf{T}_0^4 + \sqrt{\frac{5}{14}}\left[\mathbf{T}_4^4 + \mathbf{T}_{-4}^4\right]\right) + \cdots$$



Rovibronic Energy (RE) Tensor Surface



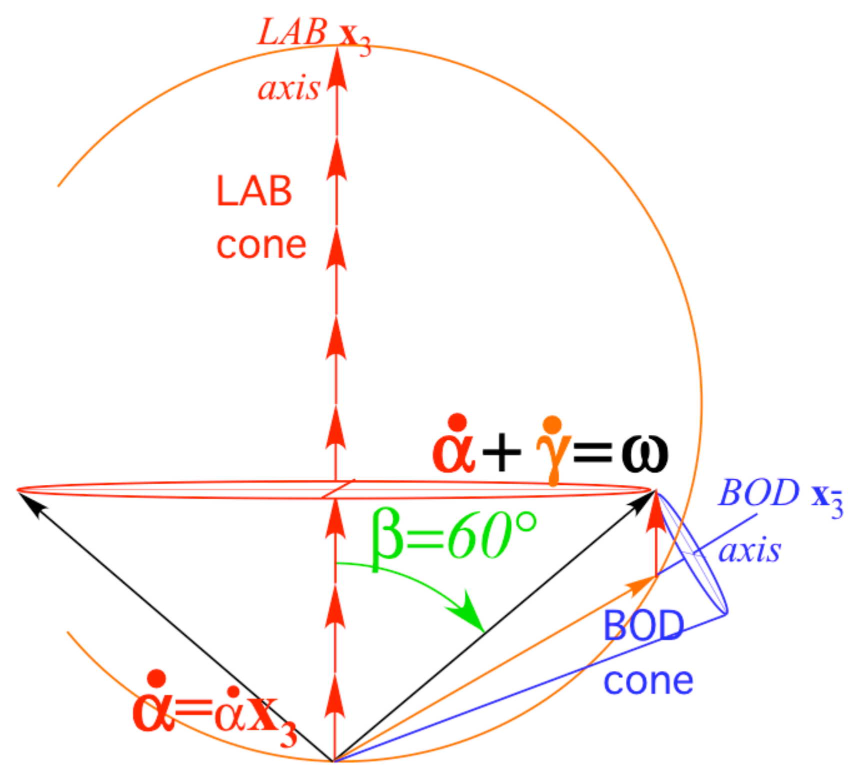


Link to pdf of: W. G. Harter and J C. Mitchell ,International Symposium on Molecular Spectroscopy, OSU Columbus (2009)

Rotational Energy Surfaces (RES)

Symmetric, asymmetric, and spherical-top dynamics (Constant **J***)*

BOD-frame cone rolling on LAB frame cone



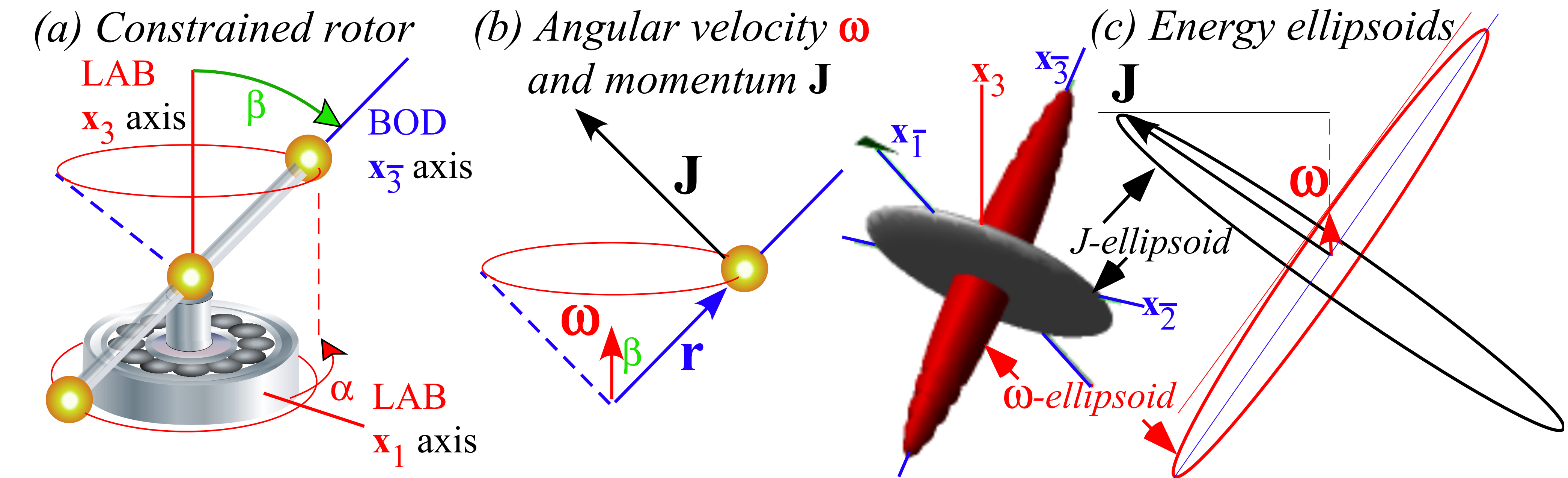


Fig. 6.7.1 Elementary ω-constrained rotor and angular velocity-momentum geometry.

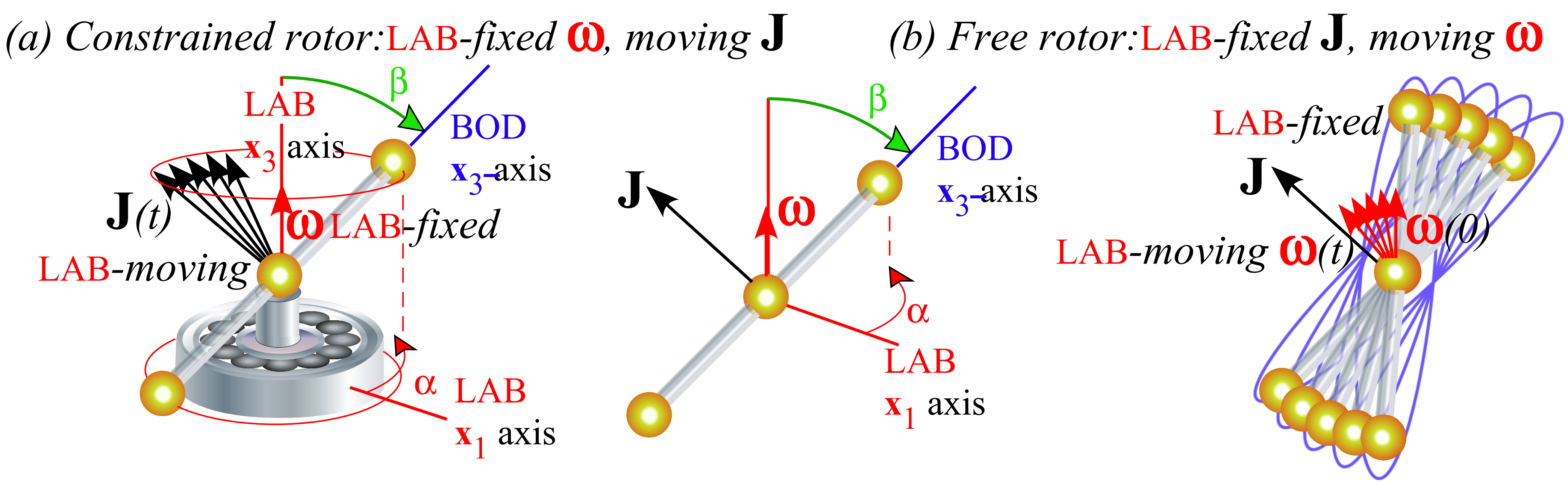
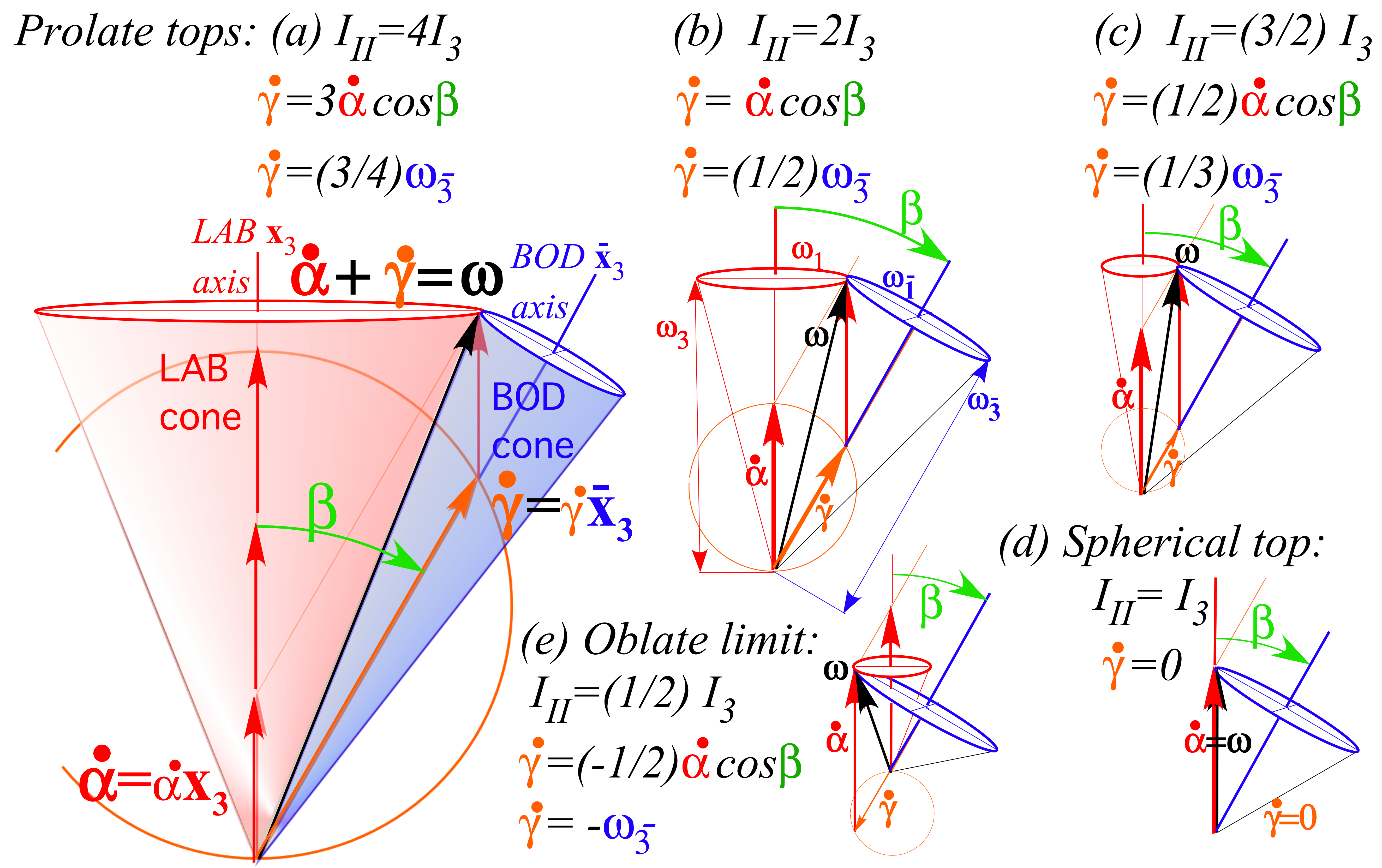


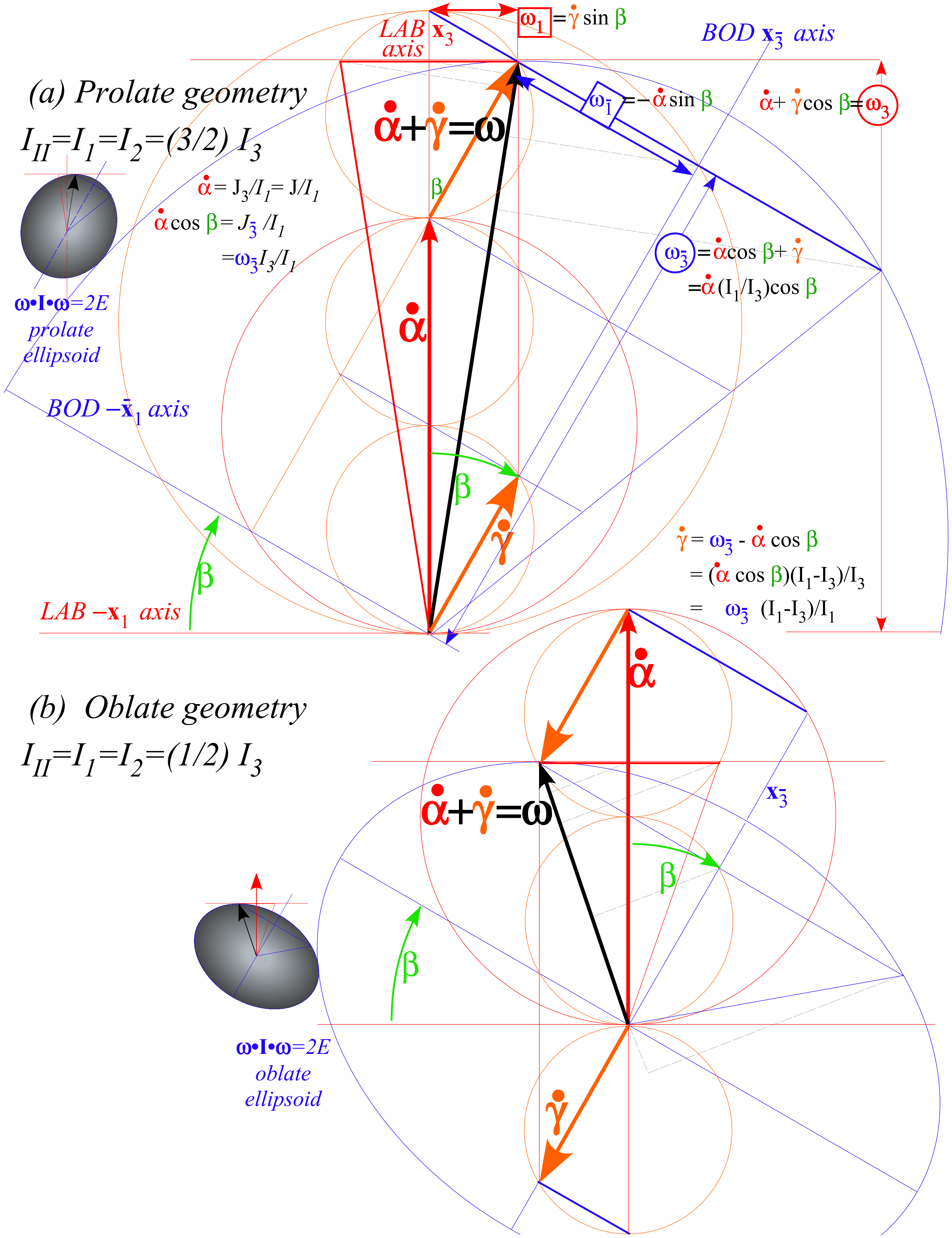
Fig. 6.7.2 Free rotor cut loose from LAB-constraining ω-axis changes dynamics accordingly.

..this was the kind of dynamics that started me dropping superballs...



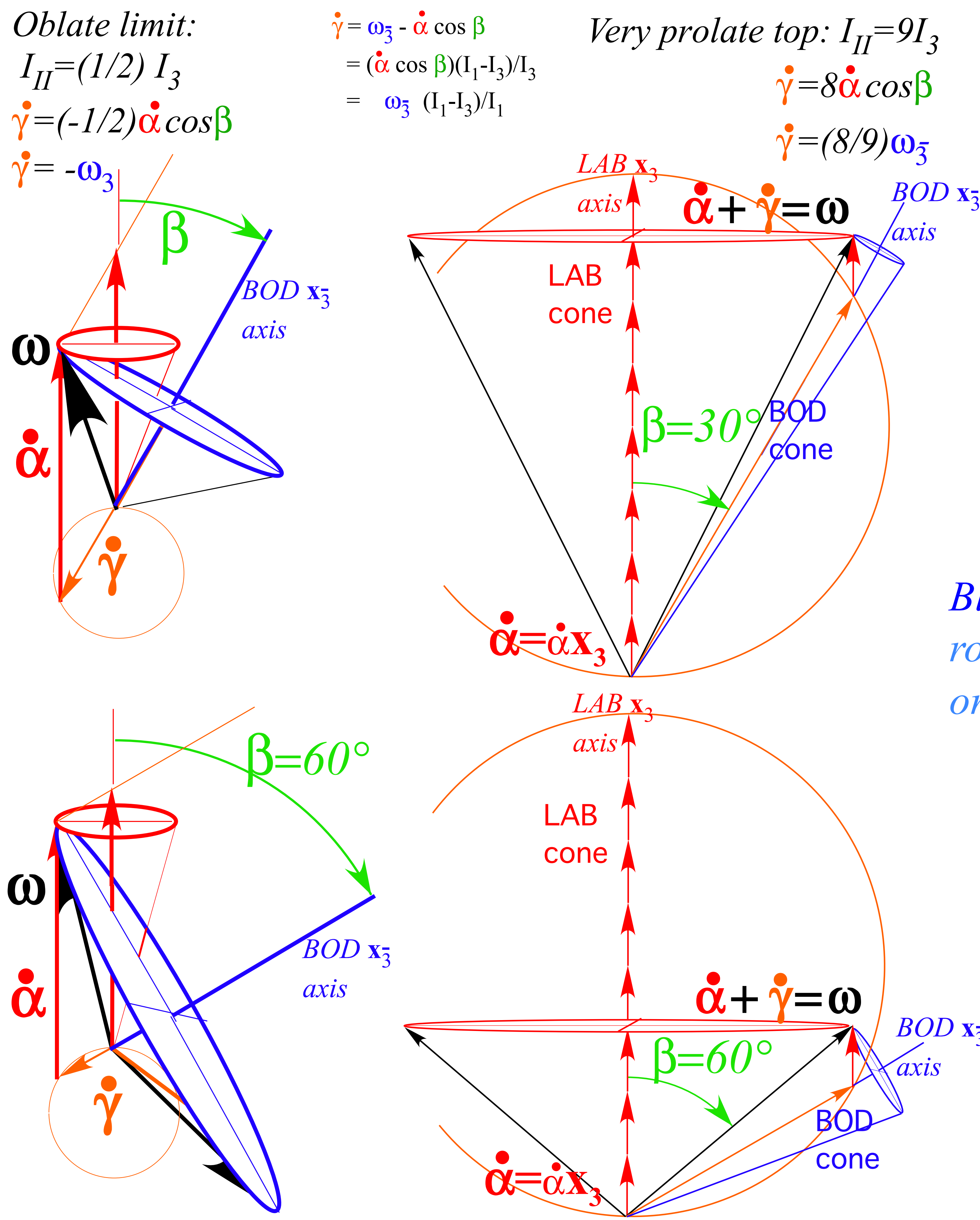
Blue BOD-frame cones roll (around ω-sticking axis)without slipping on red LAB-frame cone

Fig. 6.7.3 Symmetric top ω-cones for $\beta=30°$ *and inertial ratios: (a)* $\frac{I_{II}-I_3}{I_3}=3$*, (b) 1, (c)* $\frac{1}{2}$ *,(d) 0, (e)* $-\frac{1}{2}$.



Blue BOD-frame cones roll without slipping on red LAB-frame cone

Fig. 6.7.4 Detailed geometry of symmetric top kinetics. (a) Prolate case. (b) Most-oblate case



Blue BOD-frame cones roll without slipping on red LAB-frame cone

Fig. 6.7.5 Extreme cases (Oblate vs. Prolate) of symmetric-top geometry.

→ *Cycloidal geometry of flying levers*
Practical poolhall application

If you hammer a stick at a point h meters from its center
you give it some linear momentum Π
and some angular momentum $\Lambda = h \cdot \Pi$

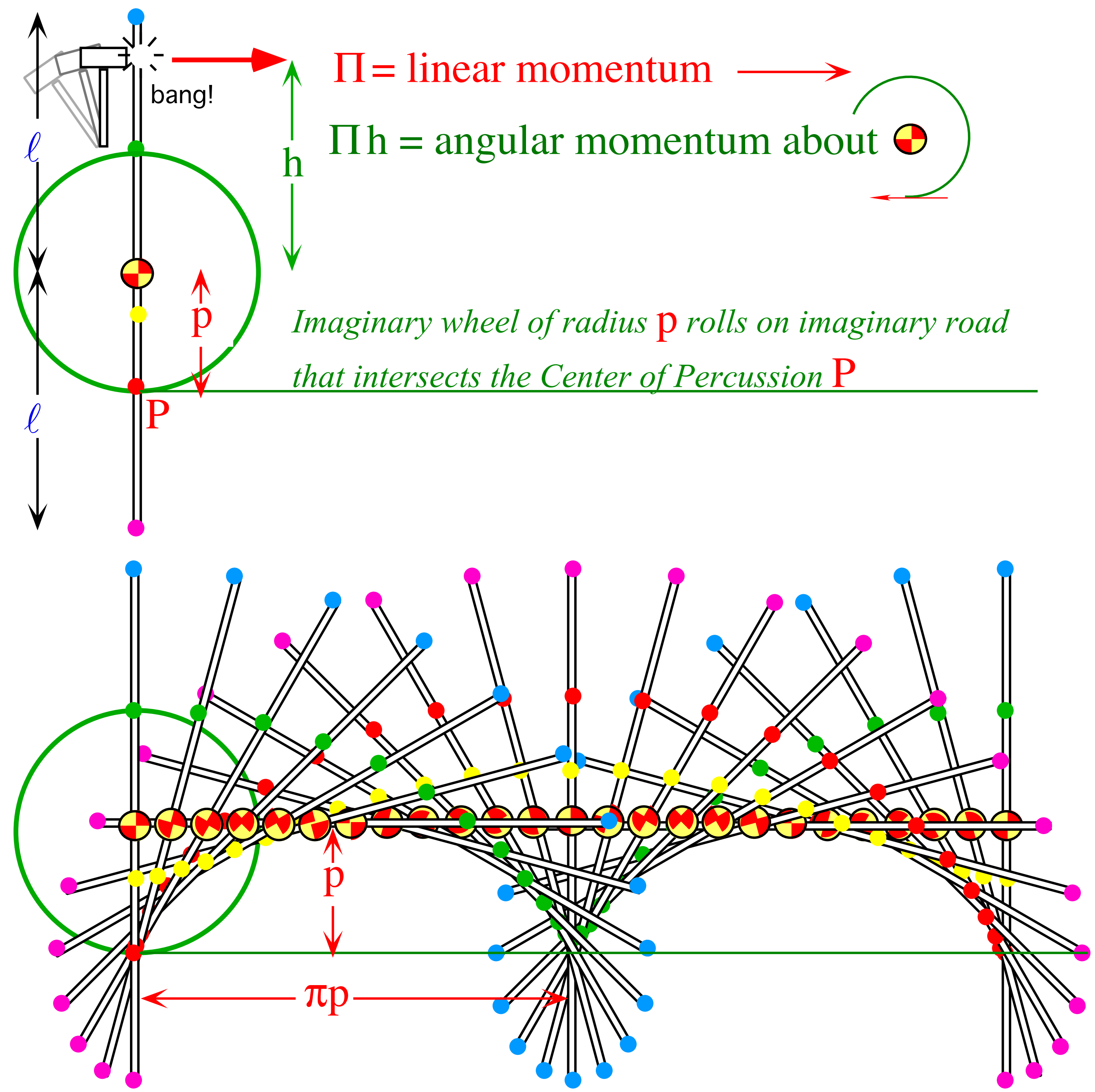


Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

If you hammer a stick at a point h meters from its center
you give it some linear momentum Π
and some angular momentum $\Lambda = h \cdot \Pi$

Resulting angular velocity ω about the center
is angular momentum Λ divided by
moment of inertia $I = M\,\ell^2/3$ of the stick.

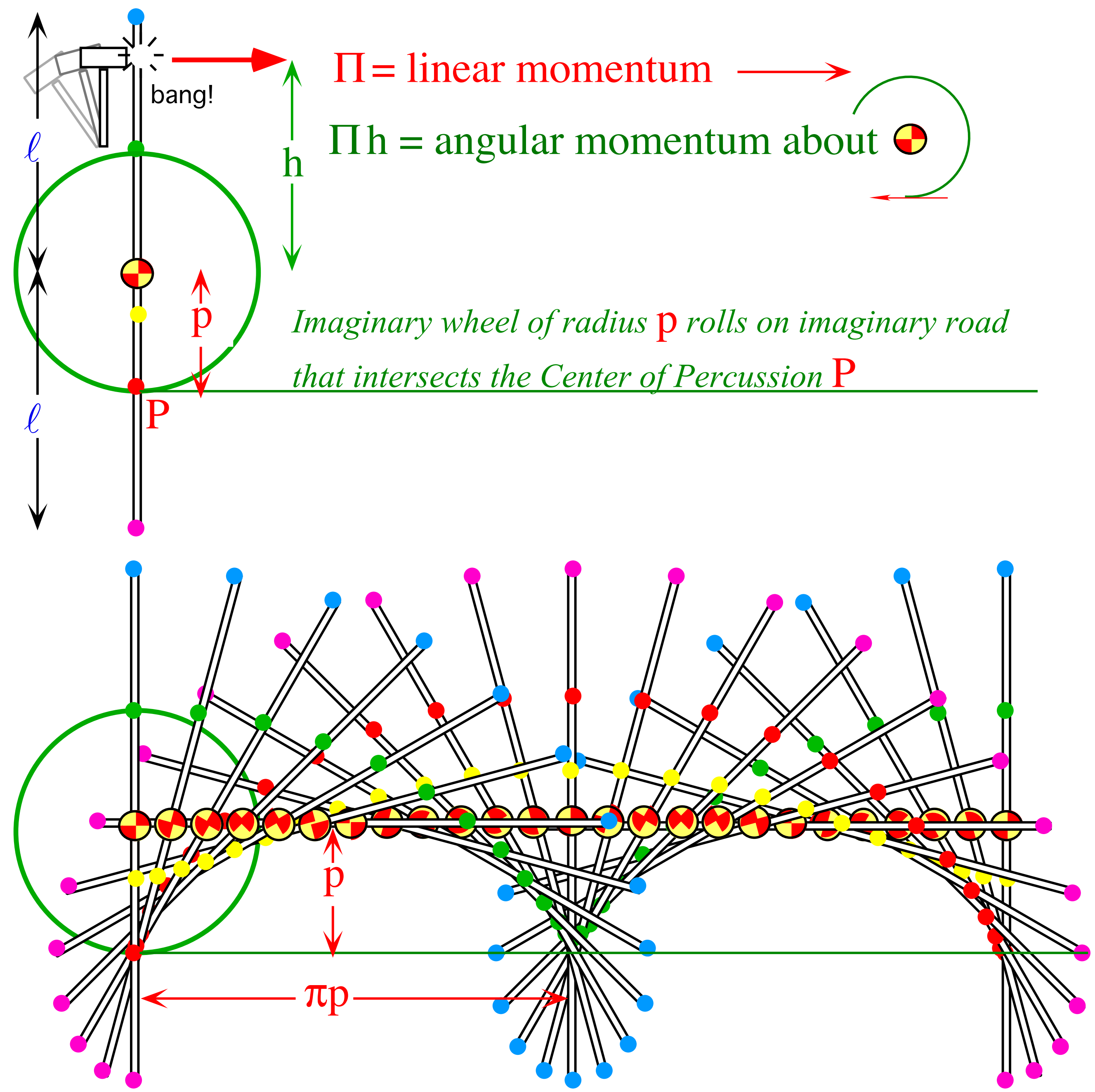


Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

If you hammer a stick at a point h meters from its center
you give it some linear momentum Π
and some angular momentum $\Lambda = h \cdot \Pi$

Resulting angular velocity ω about the center
is angular momentum Λ divided by
moment of inertia $I = M\,\ell^2/3$ of the stick.

$\omega = \Lambda / I \qquad (=3\Lambda/(M\,\ell^2)$ for stick)

$\quad = h\Pi / I \quad (=3h\Pi/(M\,\ell^2)$ for stick)

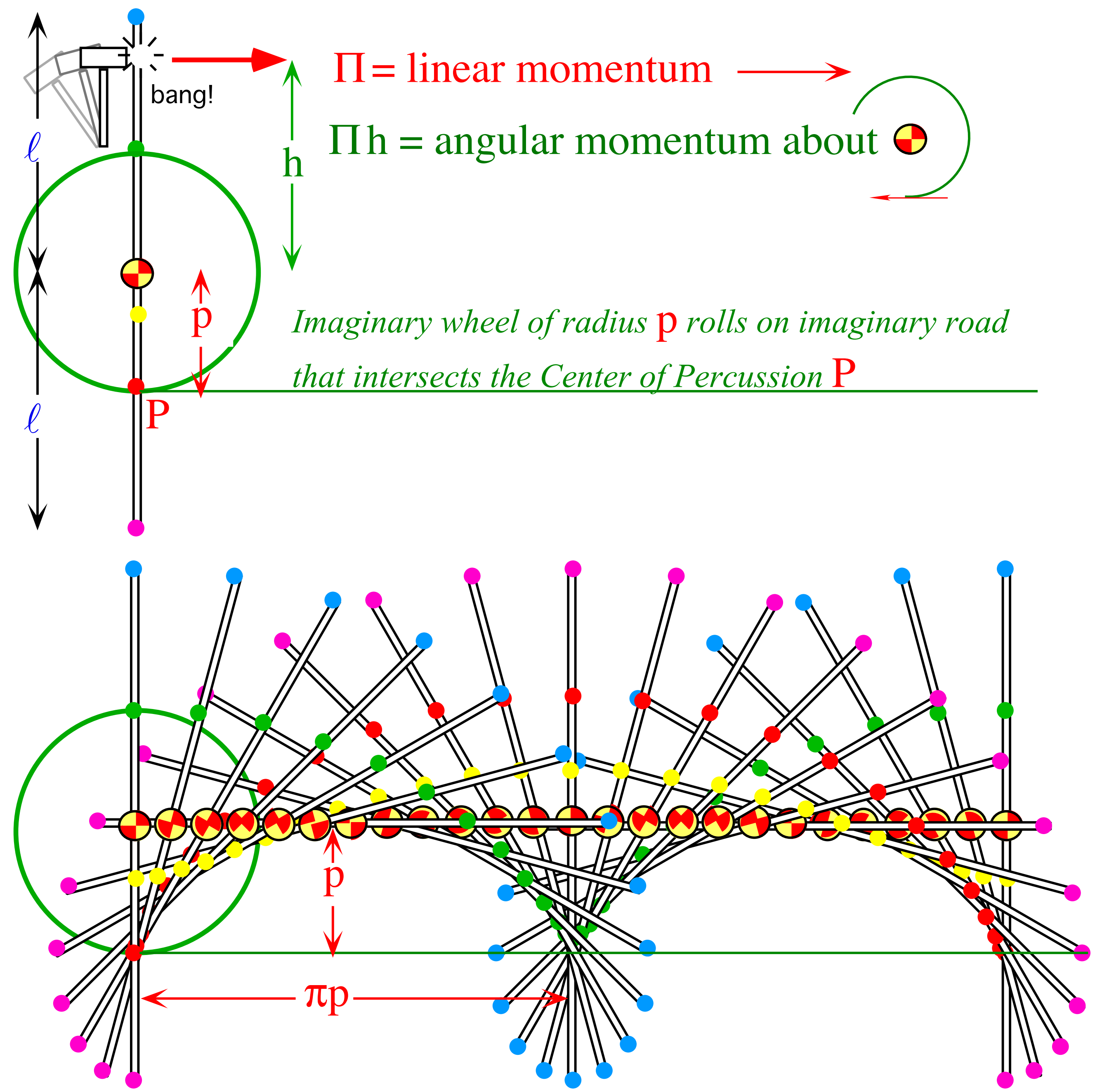


Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

If you hammer a stick at a point h meters from its center
you give it some linear momentum Π
and some angular momentum $\Lambda = h{\cdot}\Pi$

Resulting angular velocity ω about the center
is angular momentum Λ divided by
moment of inertia $I = M\,\ell^2/3$ of the stick.

$$\omega = \Lambda\,/\,I \qquad (=3\Lambda\,/(M\,\ell^2)\ \text{ for stick})$$
$$= h\Pi\,/\,I \quad (=3h\Pi/(M\,\ell^2)\text{for stick})$$

One point P, or *center of percussion* (CoP), is
on the wheel where speed $p\omega$ due to rotation
just cancels translational speed V_{Center} of stick.

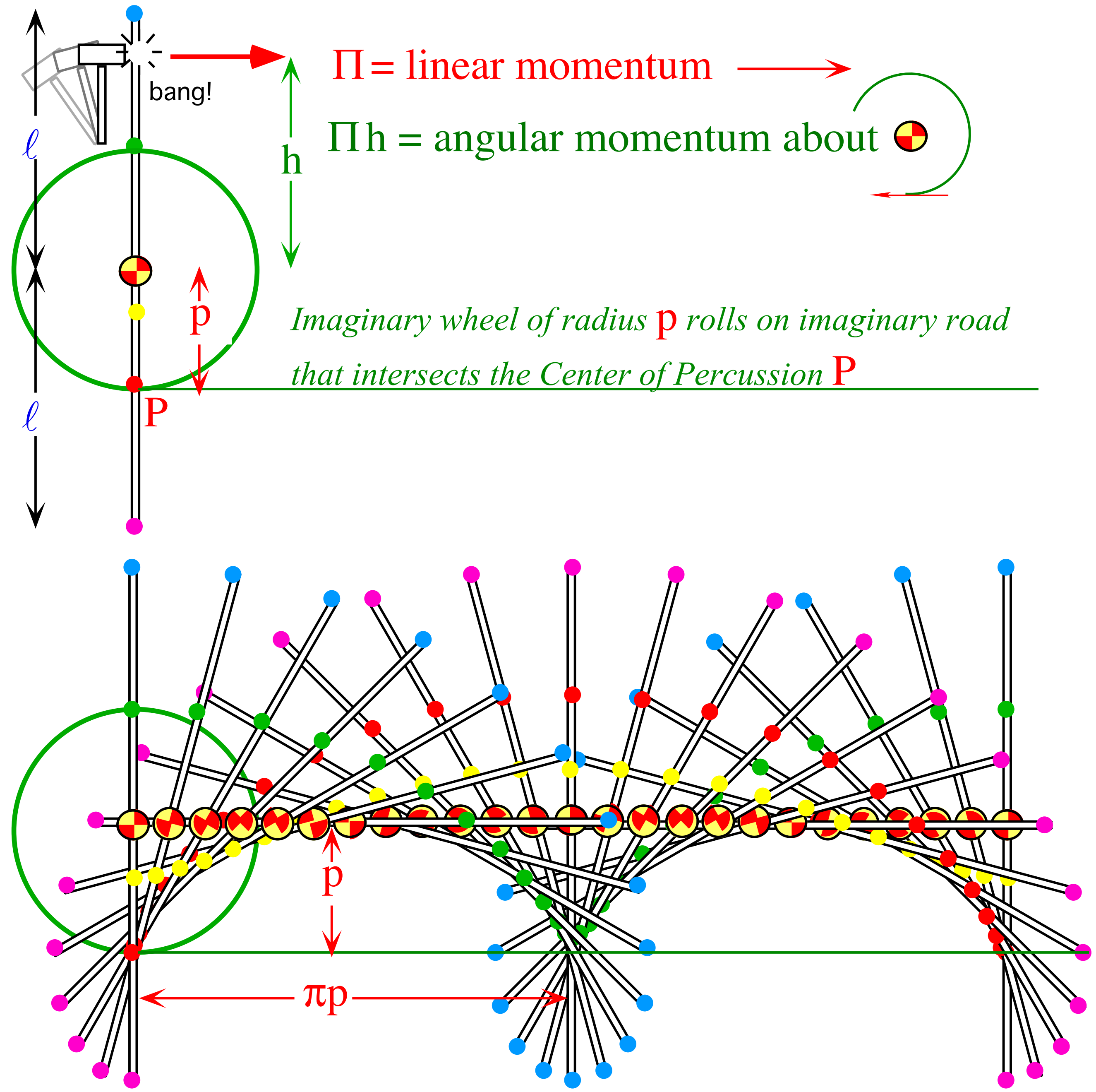
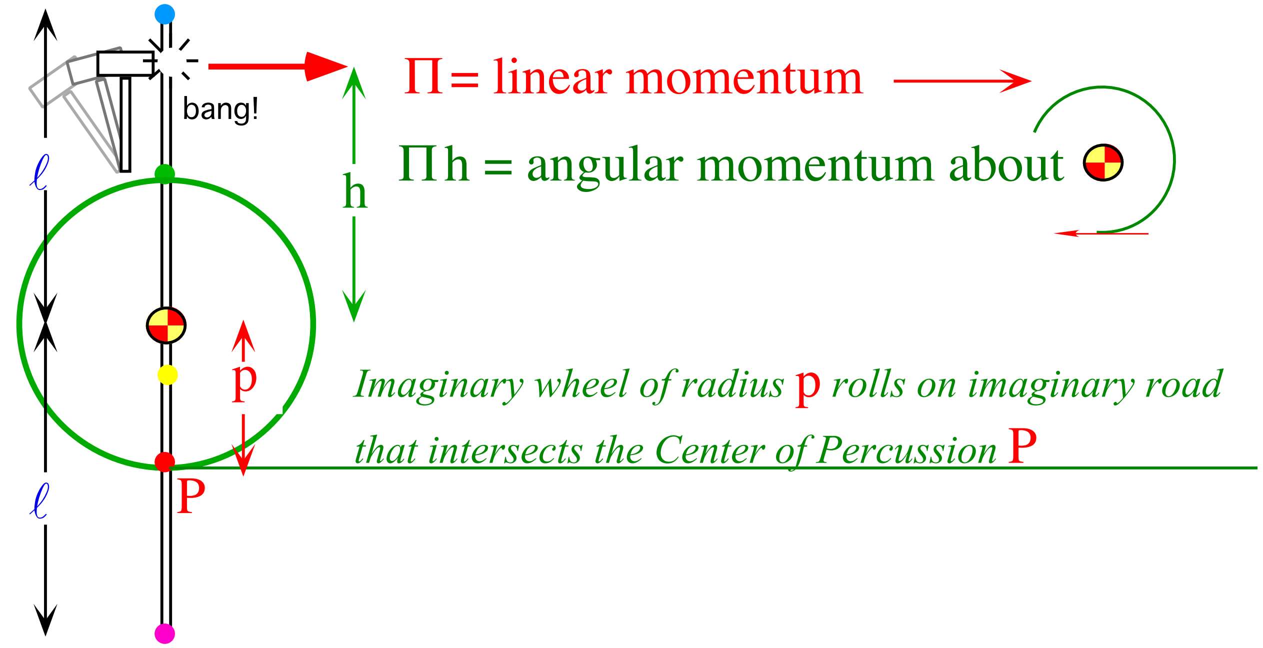


Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

If you hammer a stick at a point h meters from its center
you give it some linear momentum Π
and some angular momentum $\Lambda = h \cdot \Pi$



Resulting angular velocity ω about the center
is angular momentum Λ divided by
moment of inertia $I = M\,\ell^2/3$ of the stick.

$$\omega = \Lambda / I \qquad (=3\Lambda/(M\,\ell^2)\ \text{for stick})$$
$$= h\Pi / I \quad (=3h\Pi/(M\,\ell^2)\ \text{for stick})$$

One point P, or *center of percussion* (CoP), is
on the wheel where speed $p\omega$ due to rotation
just cancels translational speed V_{Center} of stick.

$$\Pi / M = V_{Center} = |p\omega| = p \cdot h\Pi / I$$

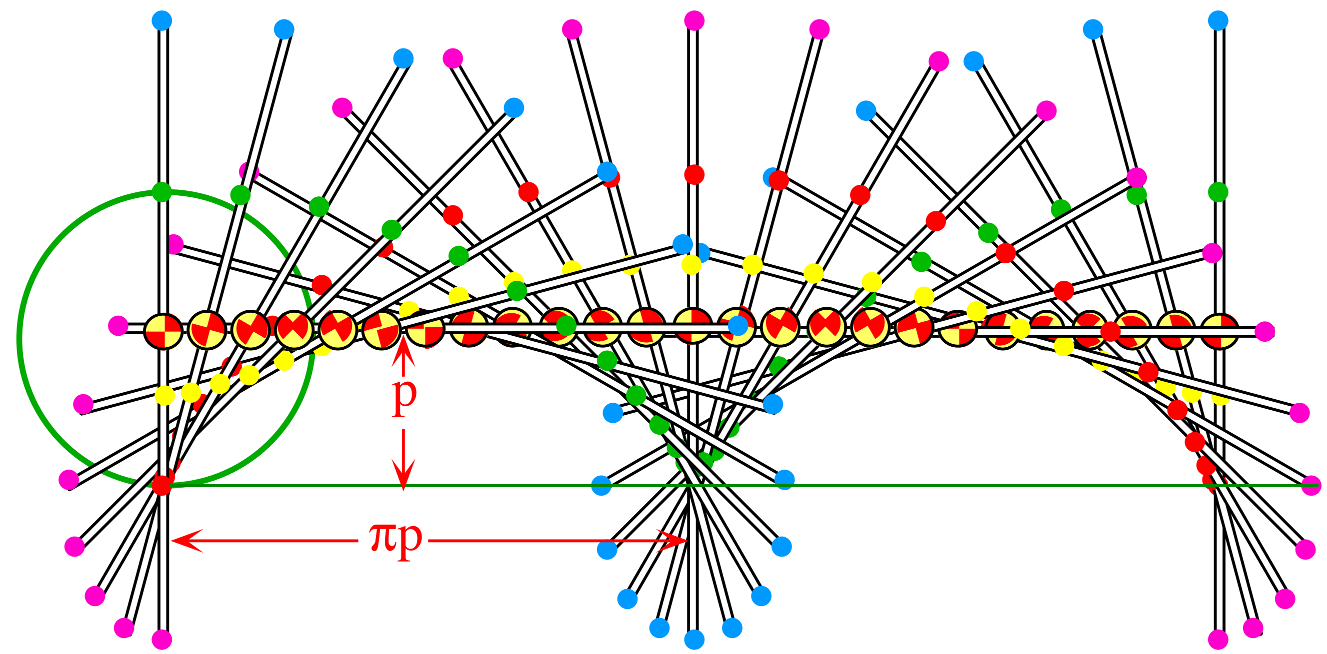


Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

If you hammer a stick at a point h meters from its center
you give it some linear momentum Π
and some angular momentum $\Lambda = h\cdot\Pi$

Resulting angular velocity ω about the center
is angular momentum Λ divided by
moment of inertia $I = M\,\ell^2/3$ of the stick.

$$\omega = \Lambda / I \qquad (=3\Lambda/(M\,\ell^2) \text{ for stick})$$
$$= h\Pi / I \qquad (=3h\Pi/(M\,\ell^2) \text{ for stick})$$

One point P, or *center of percussion* (CoP), is
on the wheel where speed $p\omega$ due to rotation
just cancels translational speed V_{Center} of stick.

$$\Pi/M = V_{Center} = |p\omega| = p\cdot h\Pi/I$$
$$I/M = \qquad = \qquad = p\cdot h$$

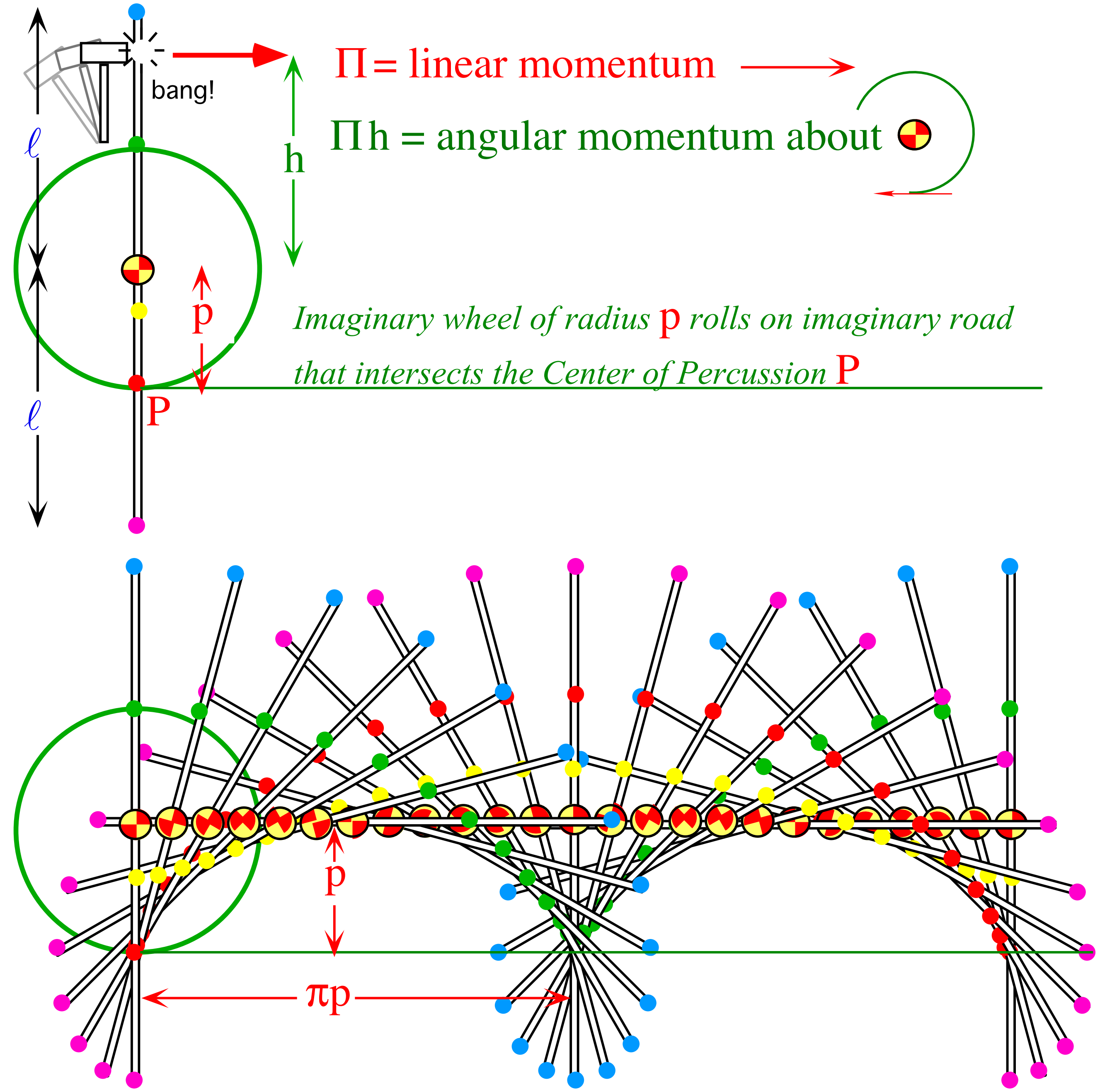
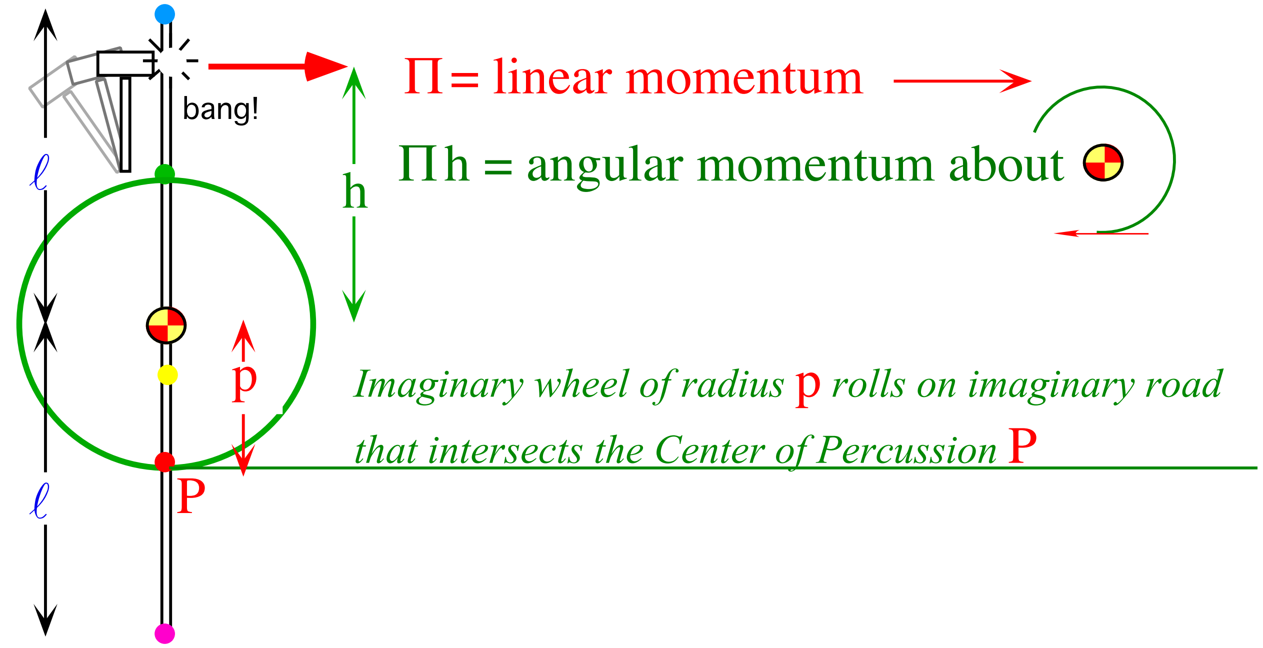


Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

If you hammer a stick at a point h meters from its center
you give it some linear momentum Π
and some angular momentum $\Lambda = h\cdot\Pi$



Resulting angular velocity ω about the center
is angular momentum Λ divided by
moment of inertia $I = M\,\ell^2/3$ of the stick.

$$\omega = \Lambda / I \qquad (=3\Lambda/(M\,\ell^2)\ \text{for stick})$$
$$= h\Pi / I \quad (=3h\Pi/(M\,\ell^2)\ \text{for stick})$$

One point P, or *center of percussion* (CoP), is
on the wheel where speed $p\omega$ due to rotation
just cancels translational speed V_{Center} of stick.

$$\Pi/M = V_{Center} = |p\omega| = p\cdot h\Pi/I$$
$$I/M = \quad = \quad = p\cdot h \qquad \textit{or: } p = I/(Mh)$$

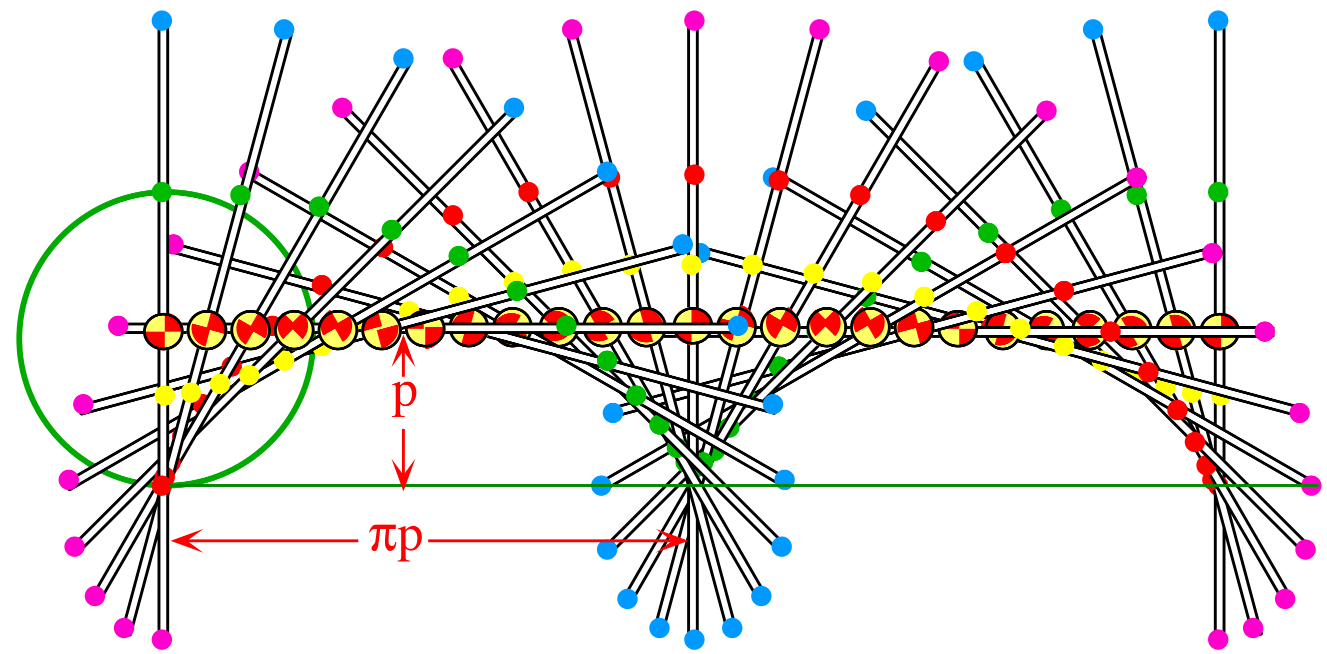
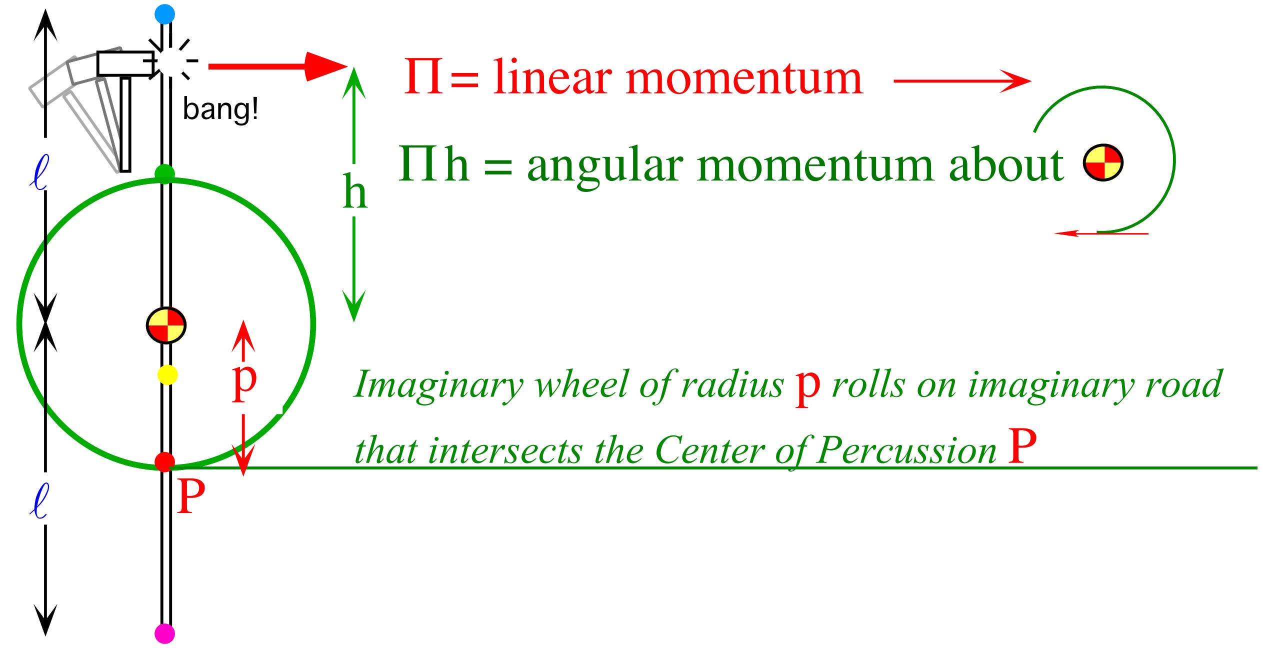


Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

If you hammer a stick at a point h meters from its center
you give it some linear momentum Π
and some angular momentum $\Lambda = h\cdot\Pi$



Resulting angular velocity ω about the center
is angular momentum Λ divided by
moment of inertia $I = M\,\ell^2/3$ of the stick.

$$\omega = \Lambda / I \qquad (=3\Lambda/(M\,\ell^2)\ \text{for stick})$$
$$= h\Pi / I \qquad (=3h\Pi/(M\,\ell^2)\text{for stick})$$

One point P, or *center of percussion* (CoP), is
on the wheel where speed $p\omega$ due to rotation
just cancels translational speed V_{Center} of stick.

$$\Pi/M = V_{Center} = |p\omega| = p\cdot h\Pi/I$$

$$I/M = \qquad = \qquad = p\cdot h \qquad \textit{or: } p = I/(Mh)$$

P follows a normal cycloid made by a circle
of radius $p=I/(Mh)$ rolling on an imaginary road
thru point P in direction of Π.

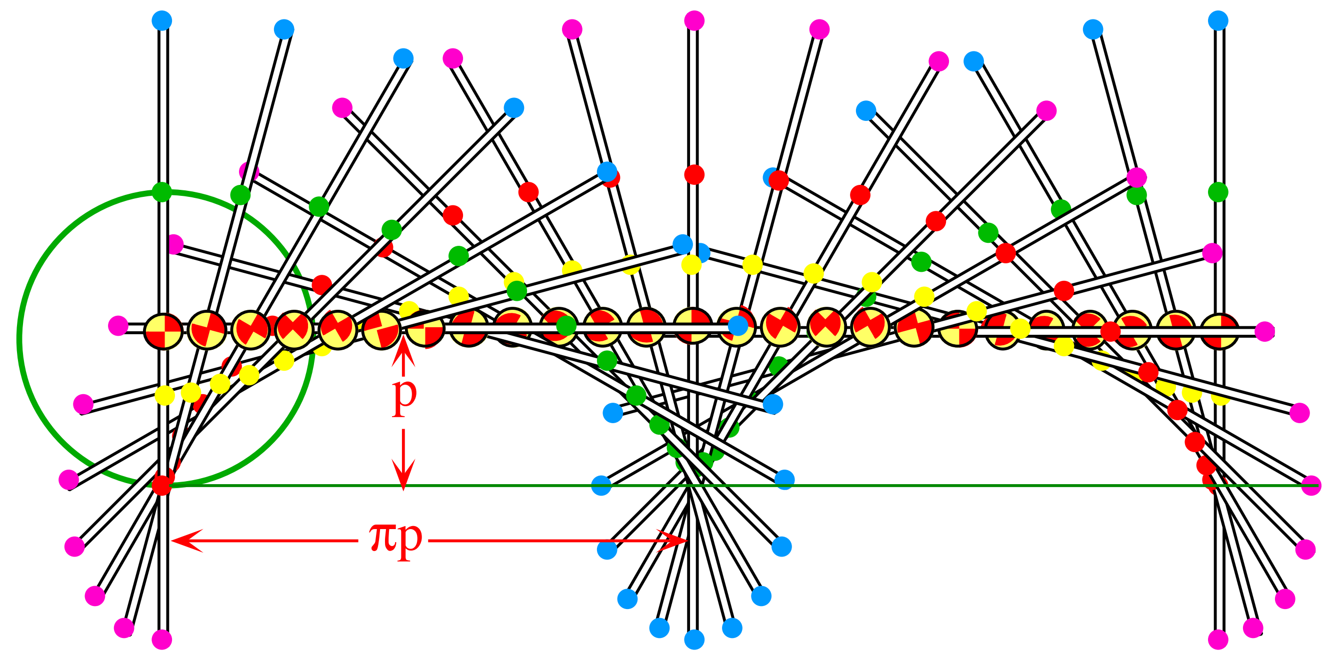


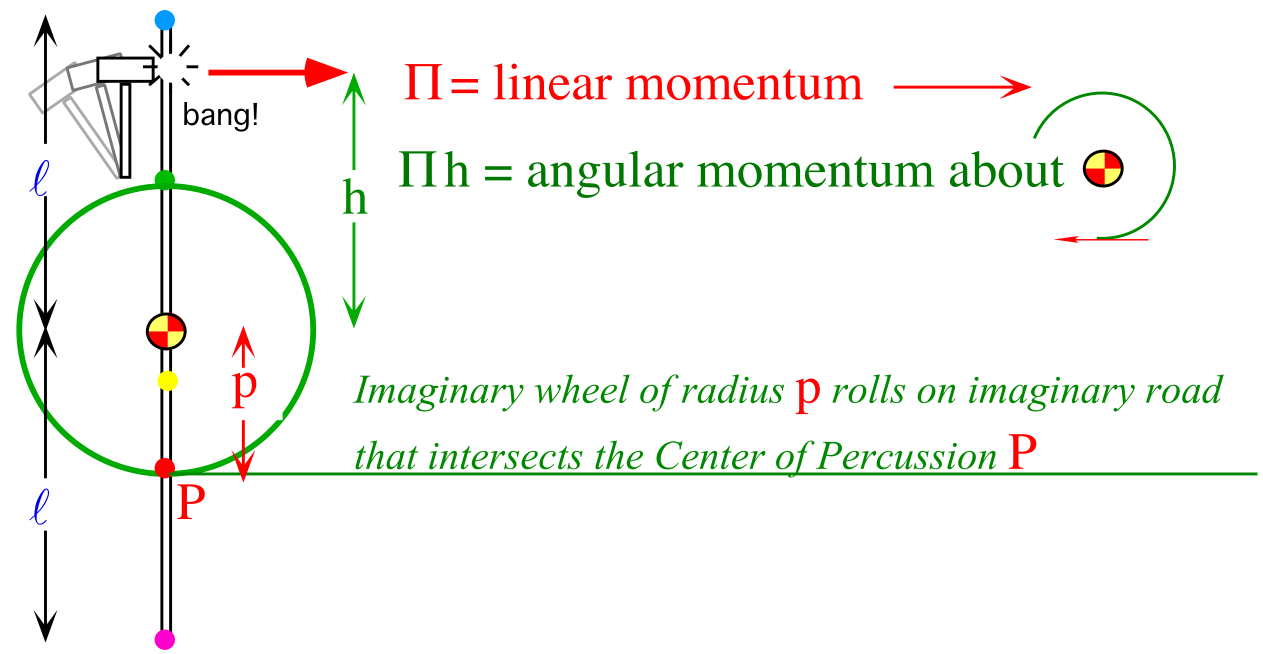
Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

If you hammer a stick at a point h meters from its center you give it some linear momentum Π and some angular momentum $\Lambda = h \cdot \Pi$

Resulting angular velocity ω about the center is angular momentum Λ divided by moment of inertia $I = M\,\ell^2/3$ of the stick.

$\omega = \Lambda / I \qquad (=3\Lambda/(M\,\ell^2)$ for stick)

$\quad = h\Pi / I \quad (=3h\Pi/(M\,\ell^2)$ for stick)



One point P, or *center of percussion* (CoP), is on the wheel where speed $p\omega$ due to rotation just cancels translational speed V_{Center} of stick.

$\Pi/M = V_{Center} = |p\omega| = p \cdot h\Pi/I$

$I/M = \qquad = \qquad = p \cdot h \qquad$ *or:* $p = I/(Mh)$

P follows a normal cycloid made by a circle of radius $p=I/(Mh)$ rolling on an imaginary road thru point P in direction of Π.

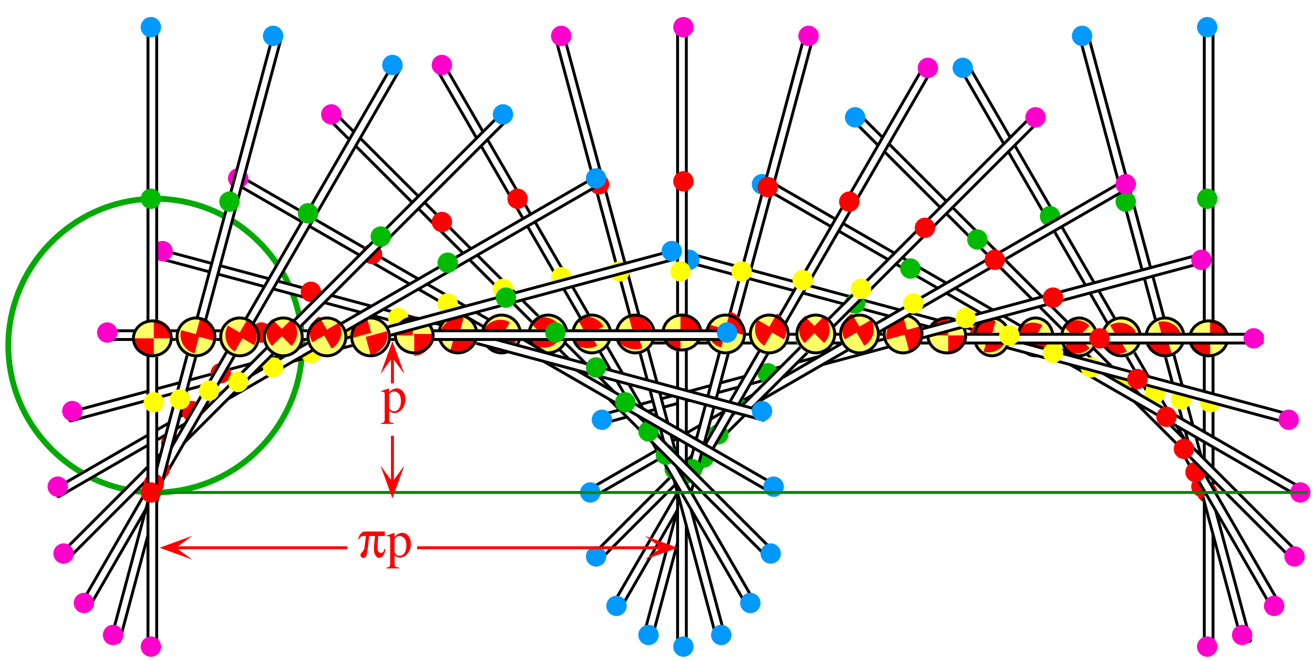


Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

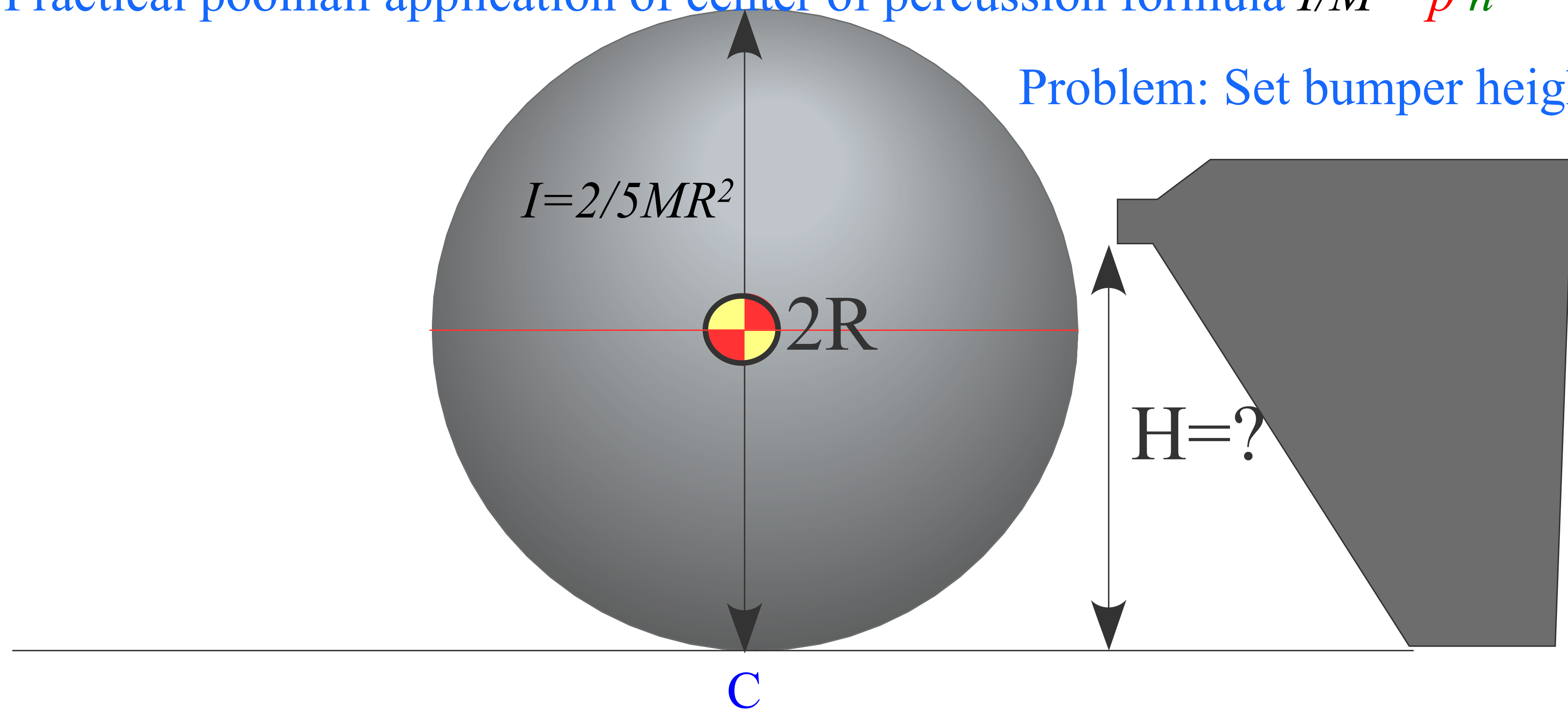
The *percussion radius* $p = \ell^2/3h$ is of the CoP point that has no velocity just after hammer hits at h.

Cycloidal geometry of flying levers

→ *Practical poolhall application*

Practical poolhall application of center of percussion formula $I/M = p \cdot h$

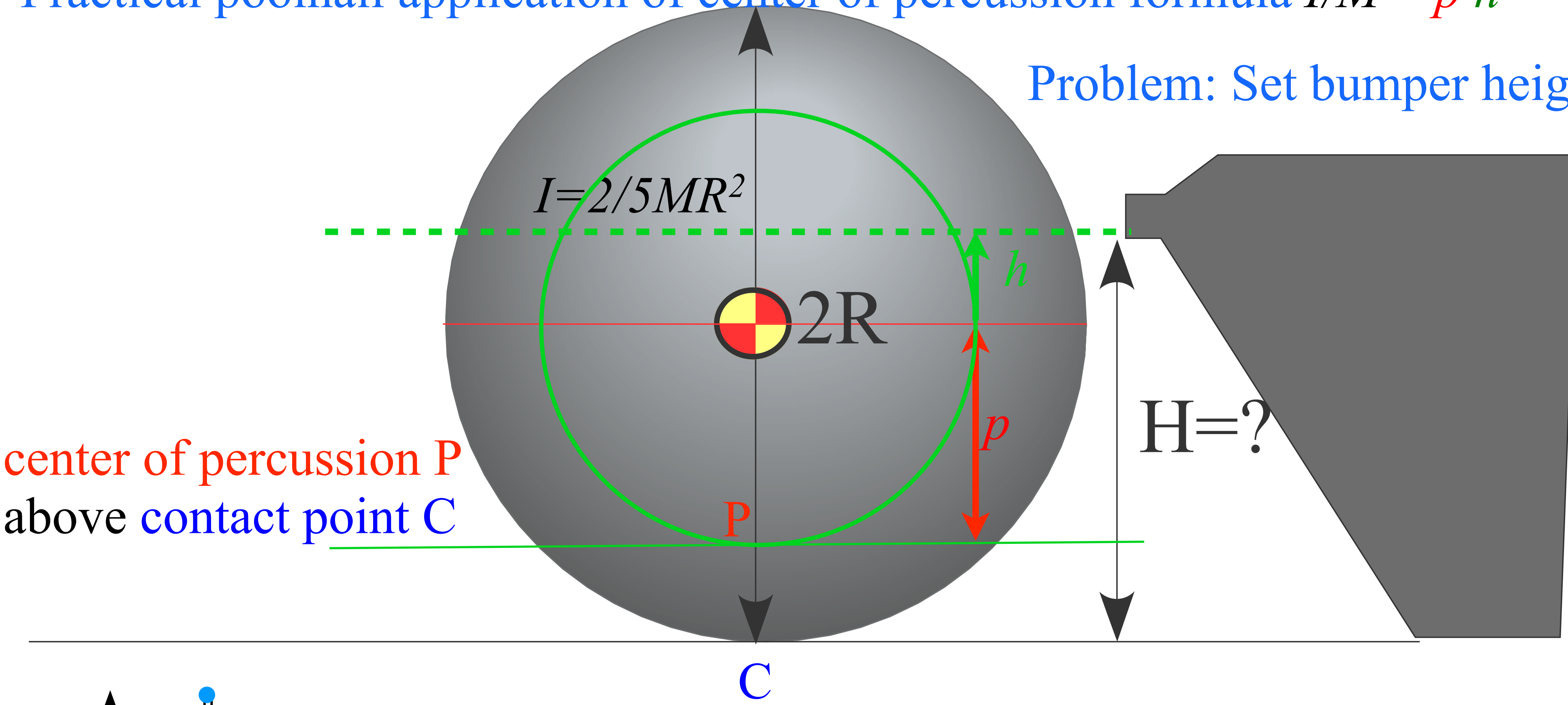
Problem: Set bumper height H so ball does not skid.



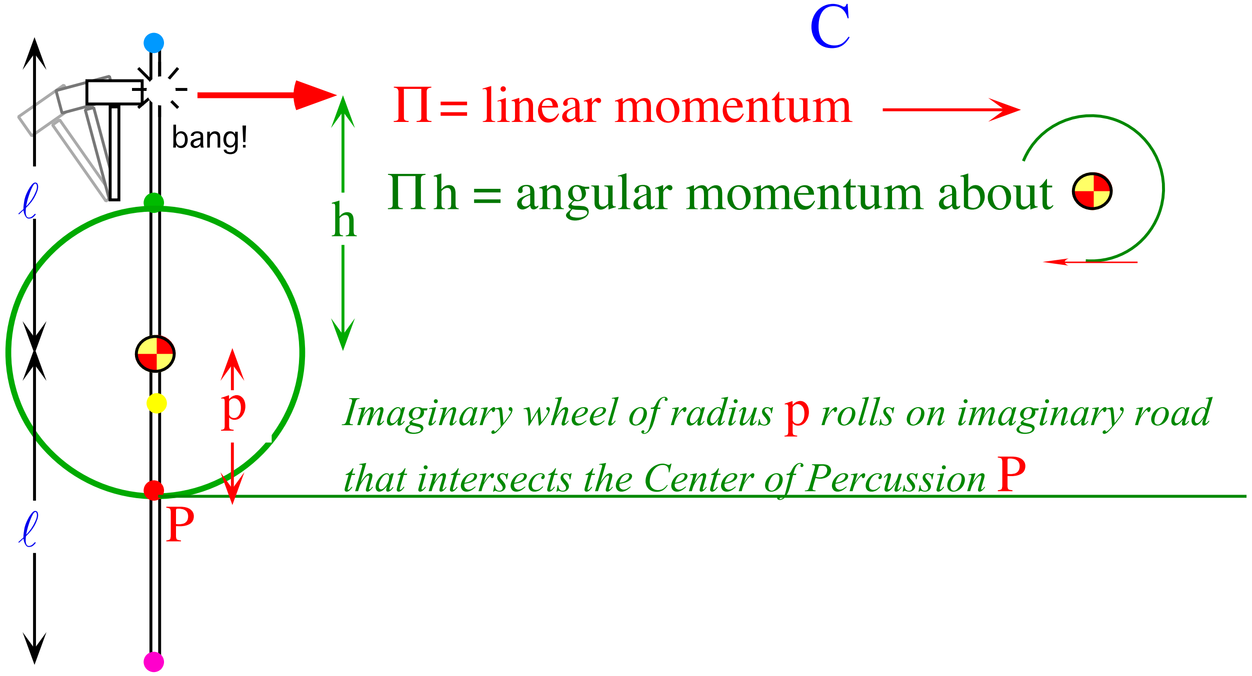
Practical poolhall application of center of percussion formula $I/M = p \cdot h$

Problem: Set bumper height H so ball does not skid.

Where should bumper height H be set to make ball contact point C at the center of percussion P?

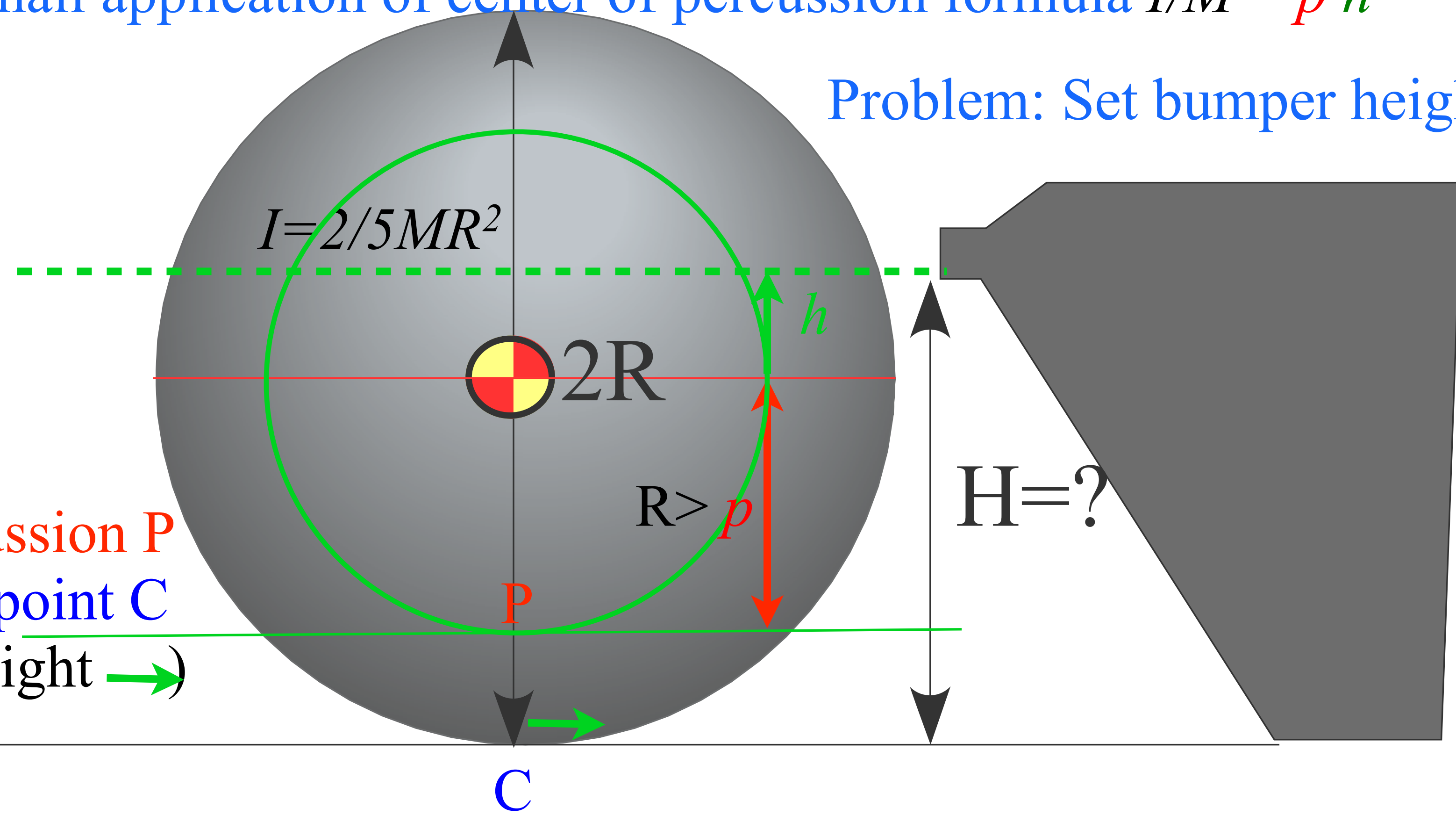


$I/M = p \cdot h$



Practical poolhall application of center of percussion formula $I/M = p \cdot h$

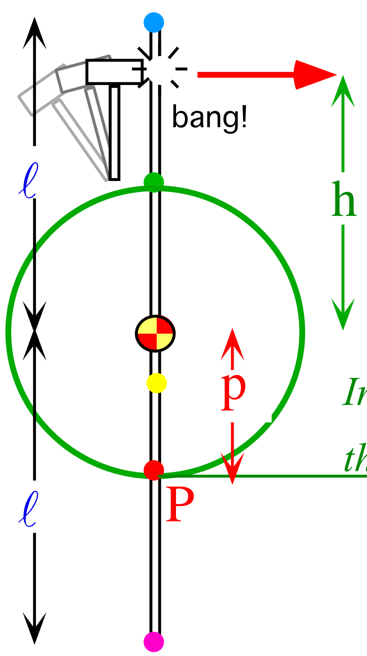
Problem: Set bumper height H so ball does not skid.



Where should bumper height H be set to make ball contact point C at the center of percussion P?

center of percussion P above contact point C (Ball skids to right →)

$I/M = p \cdot h$



Π = linear momentum →

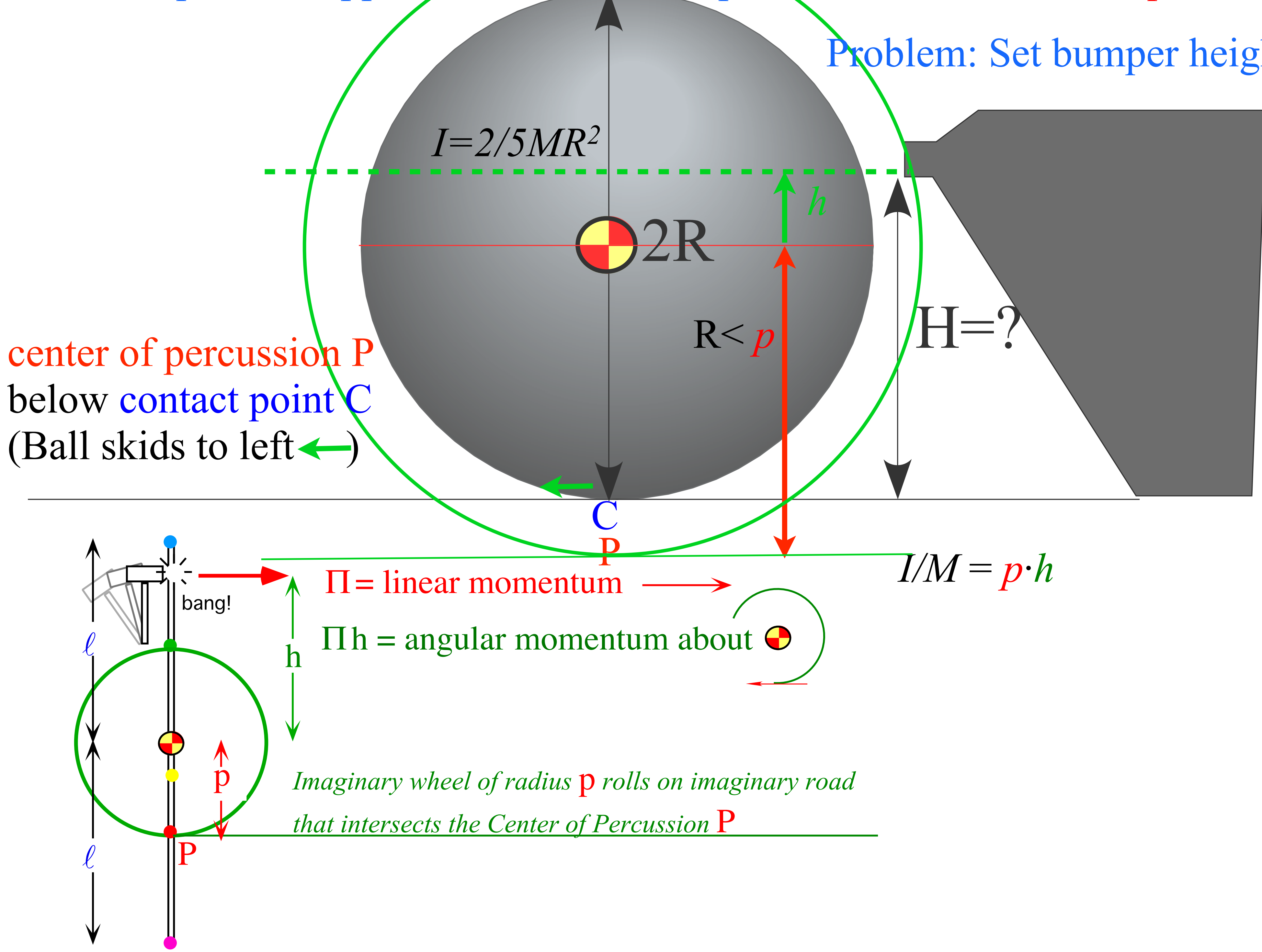
Πh = angular momentum about

Imaginary wheel of radius p *rolls on imaginary road that intersects the Center of Percussion* P

Practical poolhall application of center of percussion formula $I/M = p \cdot h$

Problem: Set bumper height H so ball does not skid.

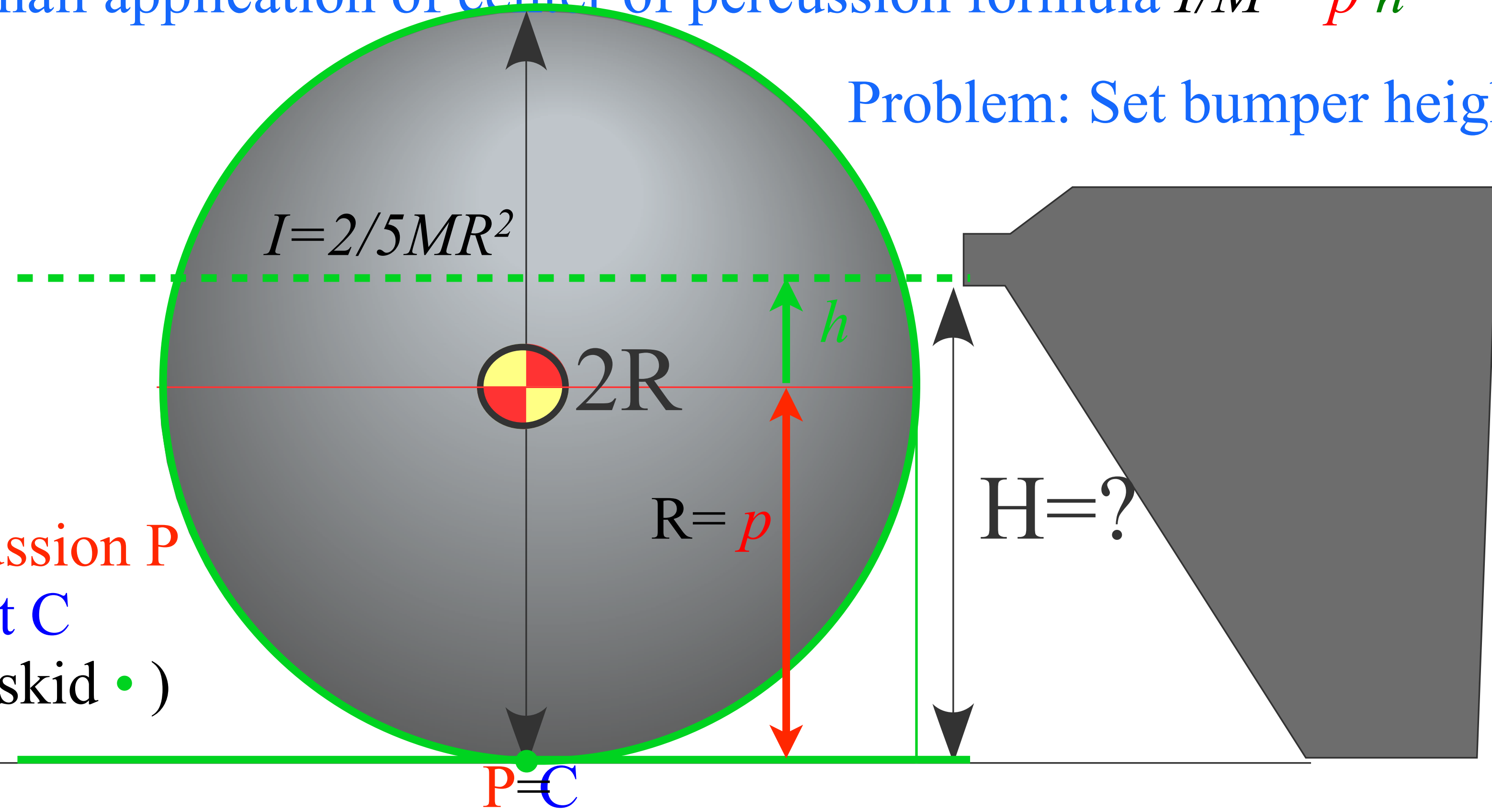
Where should bumper height H be set to make ball contact point C at the center of percussion P?



99

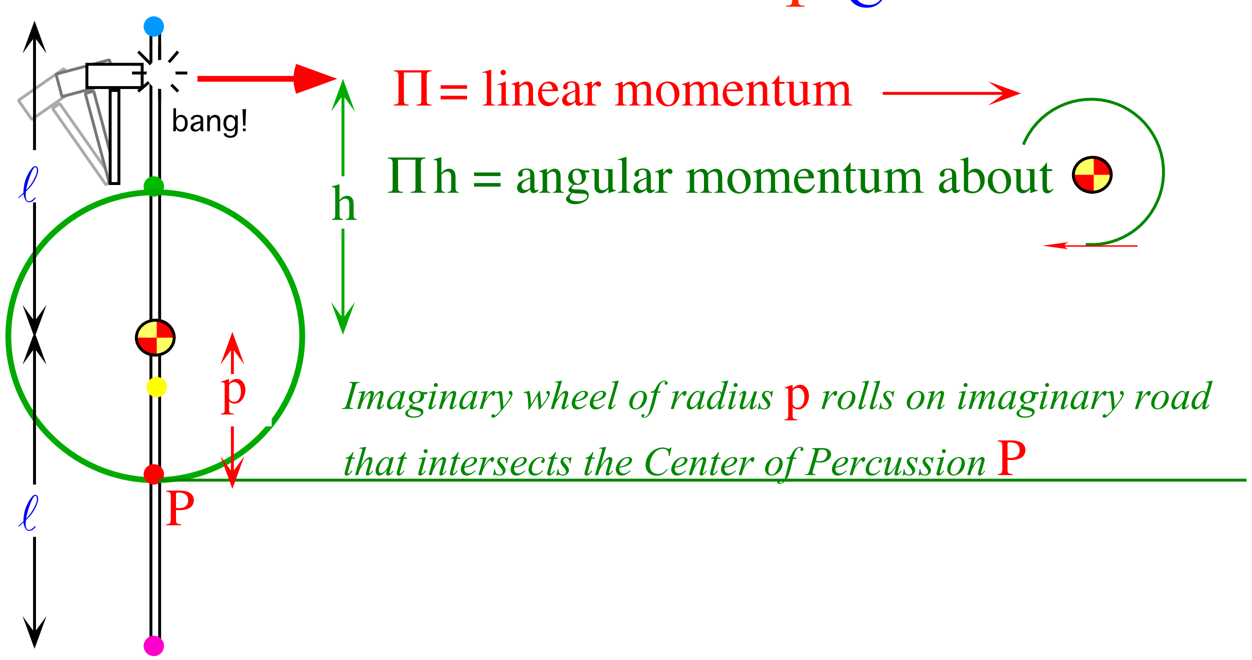
Practical poolhall application of center of percussion formula $I/M = p \cdot h$

Problem: Set bumper height H so ball does not skid.



Where should bumper height H be set to make ball contact point C at the center of percussion P?

center of percussion P
at contact point C
(Ball does not skid •)



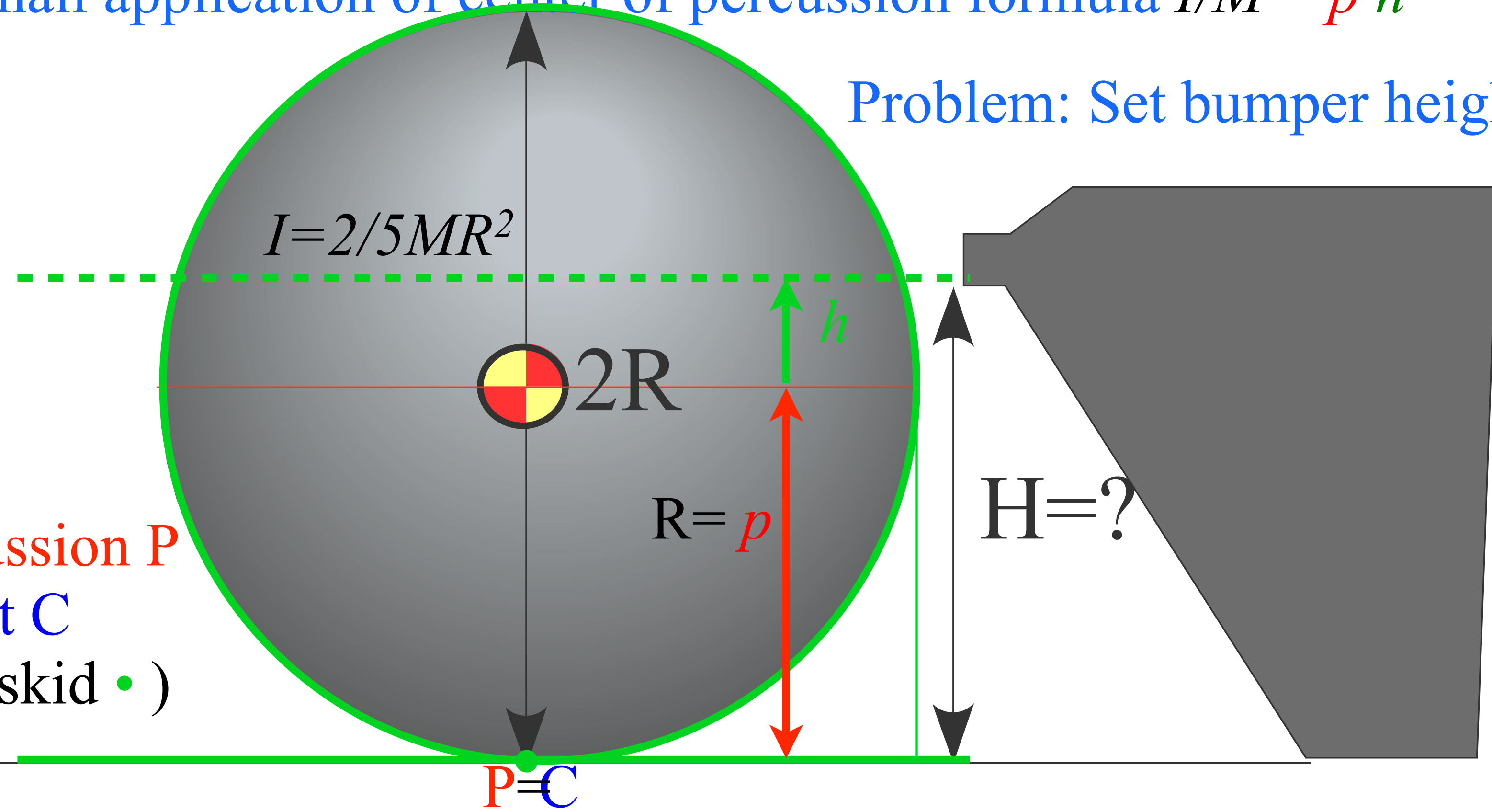
$I/M = p \cdot h$

$h = I/Mp = I/MR$ (For R= p)

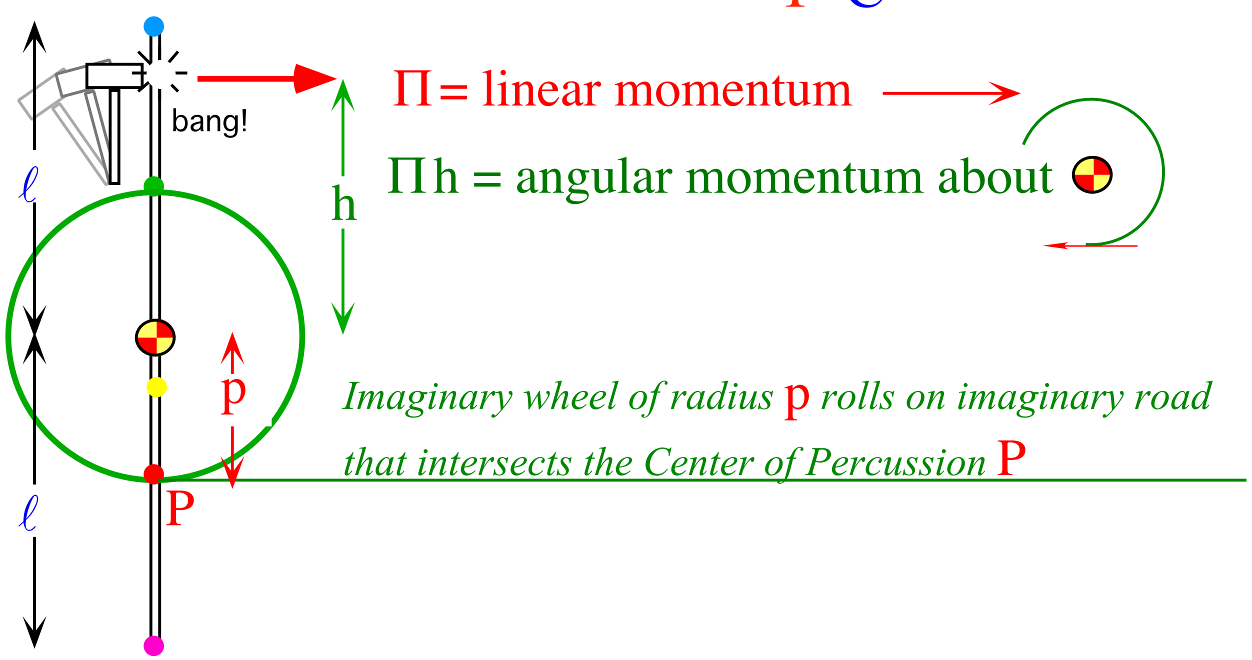
Practical poolhall application of center of percussion formula $I/M = p \cdot h$

Problem: Set bumper height H so ball does not skid.

Where should bumper height H be set to make ball contact point C at the center of percussion P?



center of percussion P
at contact point C
(Ball does not skid •)



$I/M = p \cdot h$

$h = I/Mp = I/MR$ (For R= p)

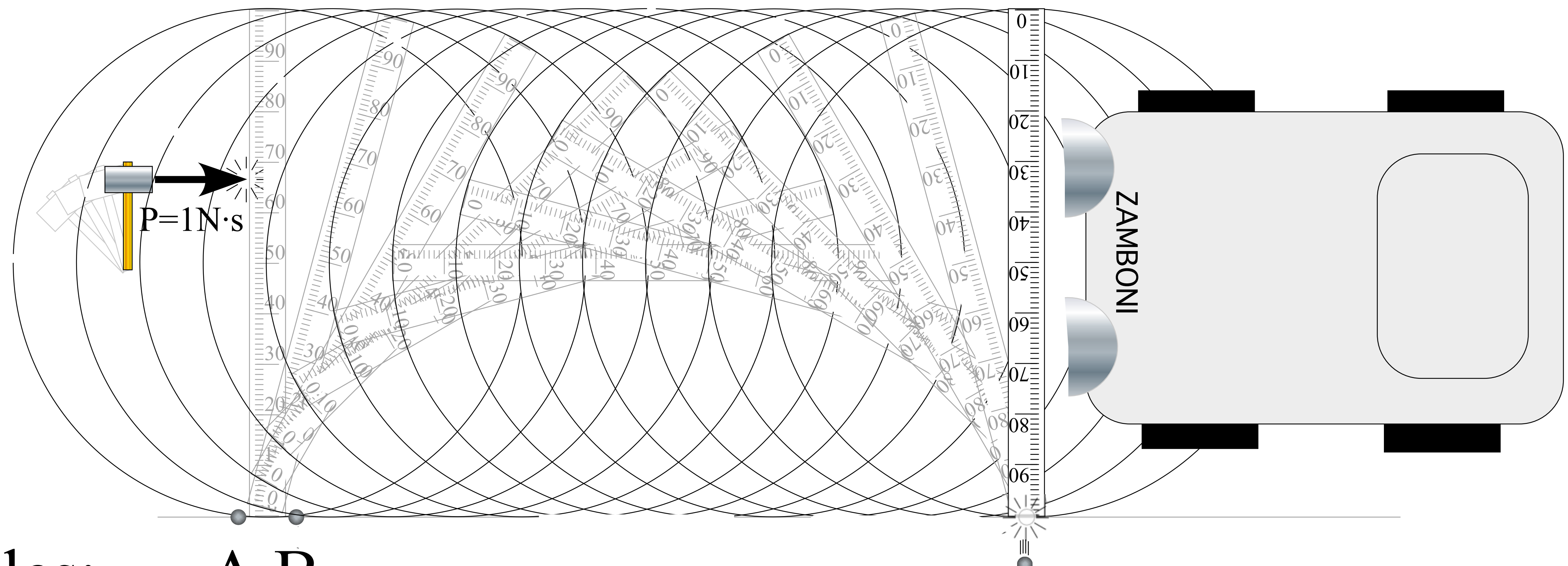
$= 2/5MR^2/MR$

$= 2/5R$

For: H= $R+h = 7/10(2R)$ ball does not skid.

The Zamboni-Ice-Shot problem

(Assumes frictionless ice rink)



Marbles: A B

C

Where on a meter-stick do you hit it so as to <u>not</u> disturb marbles A or B and...

...knock marble C down as shown.