Lecture 10

Mon. 9.24.2018

Hamiltonian vs. Lagrangian mechanics in Generalized Curvilinear Coordinates (GCC)

(Unit 1 Ch. 12, Unit 2 Ch. 2-7, Unit 3 Ch. 1-3)

Review of Lectures 8-9 procedures:

Lagrange prefers Covariant g_{mn} *with* Contravariant *velocity* $\dot{q}^m$

Hamilton prefers Contravariant g^{mn} *with* Covariant *momentum* p_m

- *Deriving Hamilton's equations from Lagrange's equations*
- *Expressing Hamiltonian* $H(p_m,q^n)$ *using* g^{mn} *and covariant momentum* p_m
  - *Polar-coordinate example of Hamilton's equations compared to Lagrange's*
    - *Hamilton's equations in Runga-Kutta (computer solution) form*

Examples of Hamiltonian mechanics in effective potentials

- *I*$I_{sotropic}$ $H_{armonic}$ $O_{scillator}$ *in polar coordinates and effective potential* *(Web Simulation: OscillatorPE - IHO)*
- *Coulomb orbits in polar coordinates and effective potential* *(Web Simulation: OscillatorPE - Coulomb)*

Examples of Hamiltonian mechanics in phase plots (Mostly for next Lecture 11)

- *1D Pendulum and phase plot* *(Web Simulations: Pendulum, Cycloidulum,)* *A running collection of links to course-relevant sites and articles*

2018 CMwBang! site

Class YouTube Channel

You-Tube site displays related videos world-wide

AIP publications

AJP article on superball dynamics

AAPT summer reading

These *are* hot off the presses. Out in MISC for quick reference.
https://modphys.hosted.uark.edu//ETC/MISC/Sorting_ultracold_atoms_in_a_three-dimensional_optical_lattice_in_a_realization_of_Maxwell%e2%80%99s_demon_-_Kumar-n-2018.pdf
https://modphys.hosted.uark.edu//ETC/MISC/Synthetic_three-dimensional_atomic_structures_assembled_atom_by_atom_-_Barredo-n-2018.pdf

Older ones:
https://modphys.hosted.uark.edu//ETC/MISC/Wave–particle_duality_of_C60_molecules_-_arndt-ltn-1999.pdf
https://modphys.hosted.uark.edu//ETC/MISC/Optical_Vortex_Knots_–_One_Photon__At_A_Time_-_Tempone-Wiltshire-Sr-2018.pdf

"Relawavity" and quantum basis of Lagrangian & Hamiltonian mechanics:
2-CW laser wave: https://modphys.hosted.uark.edu/markup/BohrItWeb.html?scenario=-30104&xPhasorFactor=0.5
Lagrangian vs Hamiltonian: https://modphys.hosted.uark.edu/markup/RelaWavityWeb.html?plotType=4,5&sigmaInd=0&swordLineWidth=3

Web Resources

Web Resources - front page

UAF Physics UTube channel

"Texts"

Classical Mechanics with a Bang!

Quantum Theory for the Computer Age

Principles of Symmetry, Dynamics, and Spectroscopy

Modern Physics and its Classical Foundations

Classes

2014 AMOP

2017 Group Theory for QM

2018 AMOP

2018 Adv Mechanics

AMOP Ch 0 Space-Time Symmetry - 2019
Seminar at Rochester Institute of Optics, Auxiliary slides, June 19, 2018

Quick Review of Lagrange Relations in Lectures 8-9

0^{th} and 1^{st} equations of Lagrange and Hamilton

p. 25 of Lecture 8

Starts out with simple demands for explicit-dependence, "loyalty" or "fealty to the colors"

Lagrangian *and* ***Estrangian***
have no explicit dependence
on ***momentum p***

$$\frac{\partial L}{\partial p_k} \equiv 0 \equiv \frac{\partial E}{\partial p_k}$$

Hamiltonian *and* ***Estrangian***
have no explicit dependence
on ***velocity v***

$$\frac{\partial H}{\partial v_k} \equiv 0 \equiv \frac{\partial E}{\partial v_k}$$

Lagrangian *and* ***Hamiltonian***
have no explicit dependence
on ***speedinum V***

$$\frac{\partial L}{\partial V_k} \equiv 0 \equiv \frac{\partial H}{\partial V_k}$$

Such non-dependencies hold in spite of "under-the-table" matrix and partial-differential connections

$$\nabla_v L = \frac{\partial L}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \frac{\mathbf{v} \bullet \mathbf{M} \bullet \mathbf{v}}{2}$$
$$= \mathbf{M} \bullet \mathbf{v} = \mathbf{p}$$

$$\begin{pmatrix} \frac{\partial L}{\partial v_1} \\ \frac{\partial L}{\partial v_2} \end{pmatrix} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

Lagrange's 1^{st} equation(s)

$$\frac{\partial L}{\partial v_k} = p_k \quad \text{or:} \quad \frac{\partial L}{\partial \mathbf{v}} = \mathbf{p}$$

$$\nabla_p H = \mathbf{v} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\partial}{\partial \mathbf{p}} \frac{\mathbf{p} \bullet \mathbf{M}^{-1} \bullet \mathbf{p}}{2}$$
$$= \mathbf{M}^{-1} \bullet \mathbf{p} = \mathbf{v}$$

$$\begin{pmatrix} \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \end{pmatrix} = \begin{pmatrix} m_1^{-1} & 0 \\ 0 & m_2^{-1} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Hamilton's 1^{st} equation(s)

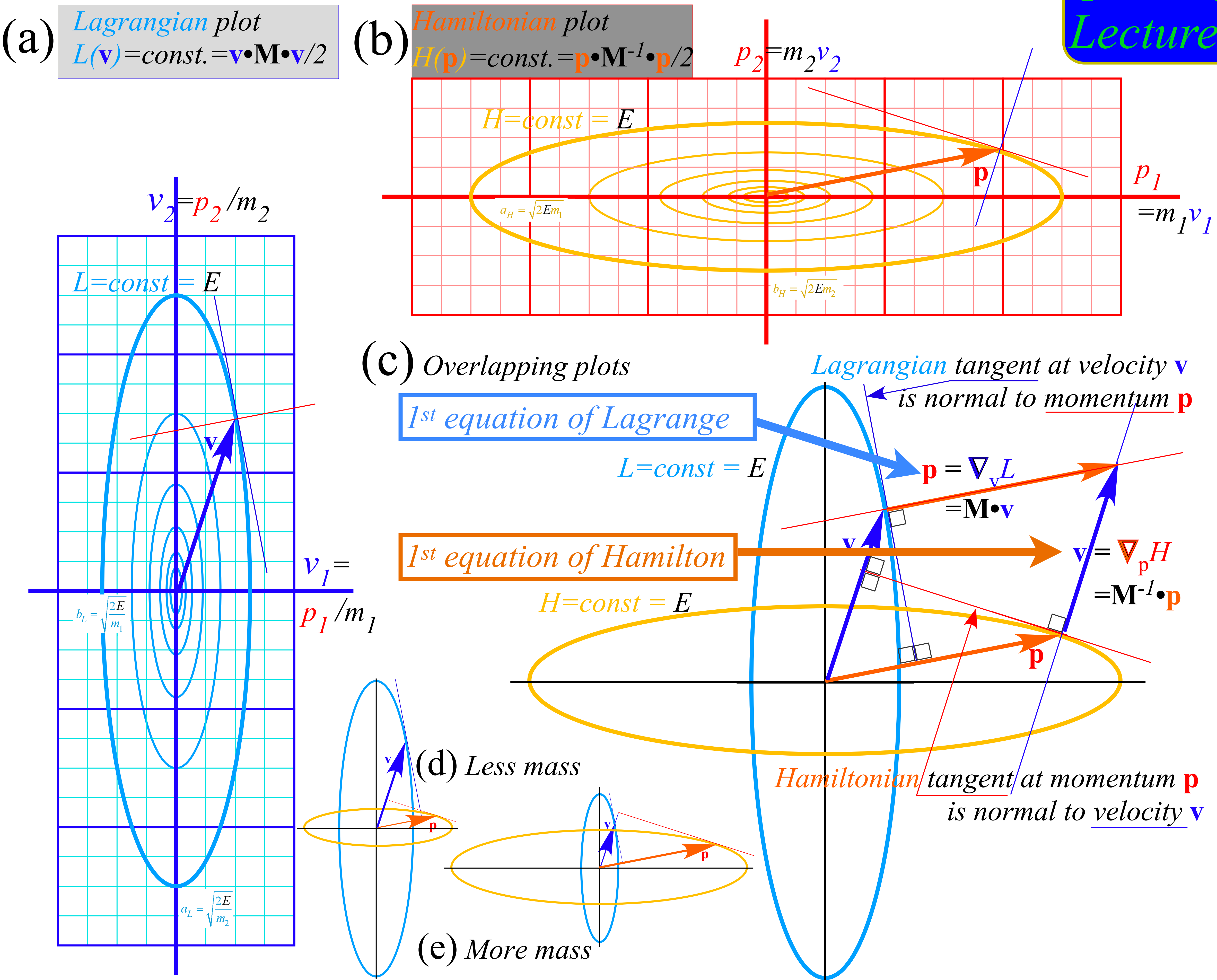
$$\frac{\partial H}{\partial p_k} = v_k \quad \text{or:} \quad \frac{\partial H}{\partial \mathbf{p}} = \mathbf{v}$$

Estrangian is neglected for now.
(It is related to dual ellipse geometry
in Lecture 8 p. 71-79 and 99-101)

†non-dependency due to
stationary-value effects
as shown on p. 28-31

Unit 1
Fig. 12.2

p. 25 of Lecture 8



(Review of Lecture 9)

Review of Lagrange Equations in Lecture 9

Lagrange prefers Covariant g_{mn} with Contravariant velocity $\dot{q}^m$

GCC Lagrangian definition

GCC "canonical" momentum p_m definition

→ *GCC "canonical" force F_m definition*

Coriolis "fictitious" forces (… and weather effects)

Lagrange prefers Covariant g_{mn} with Contravariant velocity

Lagrangian KE-U is supposed to be explicit function of velocity.

(Review of Lecture 9)

$$L(\mathbf{v})=\tfrac{1}{2}M\mathbf{v}\bullet\mathbf{v}-U=\tfrac{1}{2}M\dot{\mathbf{r}}\bullet\dot{\mathbf{r}}-U=\tfrac{1}{2}M(\mathbf{E}_m\dot{q}^m)\bullet(\mathbf{E}_n\dot{q}^n)-U=\tfrac{1}{2}M(g_{mn}\dot{q}^m\dot{q}^n)-U=L(\dot{q})$$

Use polar coordinate Covariant g_{mn} metric (1-page back)

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix}=\begin{pmatrix} \mathbf{E}_r\bullet\mathbf{E}_r & \mathbf{E}_r\bullet\mathbf{E}_\phi \\ \mathbf{E}_\phi\bullet\mathbf{E}_r & \mathbf{E}_\phi\bullet\mathbf{E}_\phi \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

This gives polar GCC form (Actually it's an OCC or Orthogonal Curvilinear Coordinate form)

$$L(\dot{r},\dot{\phi})=\tfrac{1}{2}M(g_{rr}\dot{r}^2+g_{\phi\phi}\dot{\phi}^2)-U(r,\phi)=\tfrac{1}{2}M(1\cdot\dot{r}^2+r^2\cdot\dot{\phi}^2)-U(r,\phi)$$

GCC Lagrange equations follow. 1st L-equation is momentum p_m definition for each coordinate q^m:

$$p_r=\frac{\partial L}{\partial\dot{r}}=M\,g_{rr}\dot{r}=M\,\dot{r}$$

Nothing too surprising; radial momentum p_r has the usual linear $M\cdot v$ form

$$p_\phi=\frac{\partial L}{\partial\dot{\phi}}=Mg_{\phi\phi}\dot{\phi}=Mr^2\dot{\phi}$$

Wow! $g_{\phi\phi}$ gives moment-of-inertia factor Mr^2 automatically for the angular momentum $p_\phi=Mr^2\omega$.

2nd L-equation involves total time derivative $\dot{p}_m$ for each momentum p_m:

$$\dot{p}_r=\frac{\partial L}{\partial r}=\frac{M}{2}\frac{\partial g_{\phi\phi}}{\partial r}\dot{\phi}^2-\frac{\partial U}{\partial r}=M\,r\,\dot{\phi}^2-\frac{\partial U}{\partial r}$$

Centrifugal force $Mr\omega^2$

$$\dot{p}_\phi=\frac{\partial L}{\partial\phi}=0-\frac{\partial U}{\partial\phi}$$

Angular momentum p_ϕ is conserved if potential U has no explicit ϕ-dependence

Find $\dot{p}_m$ directly from 1st L-equation: $\dot{p}_m\equiv\frac{dp_m}{dt}=\frac{d}{dt}M(g_{mn}\dot{q}^n)=M(\dot{g}_{mn}\dot{q}^n+g_{mn}\ddot{q}^n)$ Equate it to $\dot{p}_m$ in 2nd L-equation:

$$\dot{p}_r\equiv\frac{dp_r}{dt}=M\,\ddot{r}$$
$$=M\,r\,\dot{\phi}^2-\frac{\partial U}{\partial r}$$

Centrifugal (center-fleeing) force
equals total
Centripetal (center-pulling) force

$$\dot{p}_\phi\equiv\frac{dp_\phi}{dt}=2Mr\dot{r}\dot{\phi}+Mr^2\ddot{\phi}$$
$$=0-\frac{\partial U}{\partial\phi}$$

Torque relates to two distinct parts:
Coriolis and angular acceleration

Angular momentum p_ϕ is conserved if potential U has no explicit ϕ-dependence

(Review of Lecture 9)

Rewriting GCC Lagrange equations :

$$\dot{p}_r \equiv \frac{dp_r}{dt} = M\ddot{r}$$

$$= M r\dot{\phi}^2 - \frac{\partial U}{\partial r}$$

Centrifugal (center-fleeing) force
equals total
Centripetal (center-pulling) force

$$\dot{p}_\phi \equiv \frac{dp_\phi}{dt} = 2Mr\dot{r}\dot{\phi} + Mr^2\ddot{\phi}$$

$$= 0 - \frac{\partial U}{\partial \phi}$$

Torque relates to two distinct parts:
Coriolis and angular acceleration

Angular momentum p_ϕ is conserved if potential U has no explicit ϕ-dependence

Conventional forms

radial force: $M\ddot{r} = M r\dot{\phi}^2 - \frac{\partial U}{\partial r}$

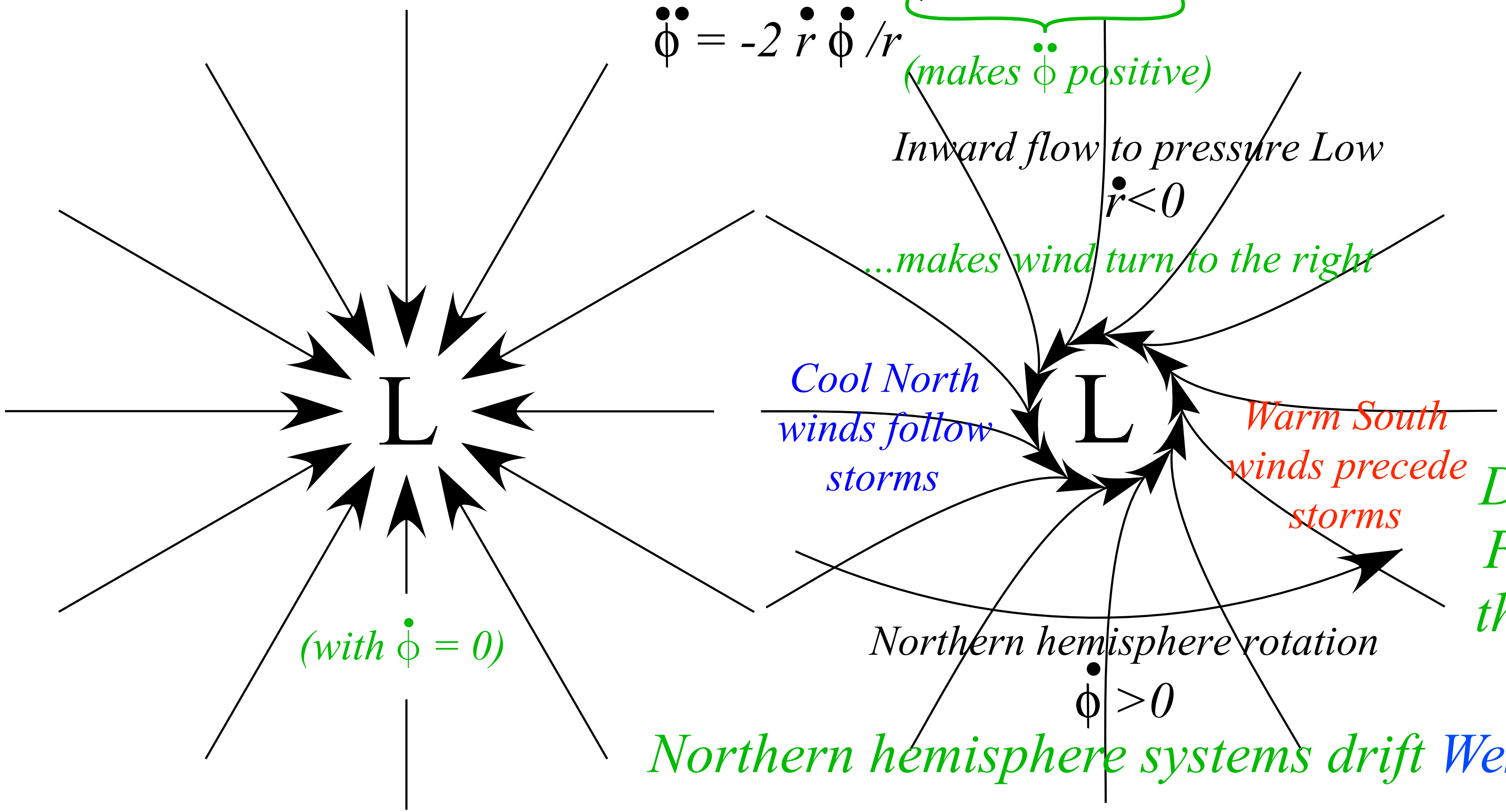
angular force or torque: $Mr^2\ddot{\phi} = -2Mr\dot{r}\dot{\phi} - \frac{\partial U}{\partial \phi}$

Field-free (U=0)

radial acceleration: $\ddot{r} = r\dot{\phi}^2$

angular acceleration: $\ddot{\phi} = -2\frac{\dot{r}\dot{\phi}}{r}$

Coriolis acceleration with $\underbrace{\dot{\phi}>0 \text{ and } \dot{r}<0}_{\text{(makes } \ddot{\phi} \text{ positive)}}$



$\ddot{\phi} = -2\,\dot{r}\,\dot{\phi}\,/r$

L

(with $\dot{\phi} = 0$)

Inward flow to pressure Low

$\dot{r}<0$

...makes wind turn to the right

Cool North winds follow storms

L

Warm South winds precede storms

Northern hemisphere rotation

$\dot{\phi}>0$

Northern hemisphere systems drift West to East

Effect on
Northern
Hemisphere
local weather

Cyclonic flow
around lows

Deep quantum rule:
Flow tries to mimic
the external rotation
(least relative v)

Hamilton prefers Contravariant g^{mn} *with* Covariant *momentum* p_m

→ *Deriving Hamilton's equations from Lagrange's equations*

Expressing Hamiltonian $H(p_m, q^n)$ *using* g^{mn} *and covariant momentum* p_m

Deriving Hamilton's equations from Lagrangian theory

Consider total time derivative of Lagrangian L=T-U
that is explicit function of coordinates and velocity $\dot{q}$ ...

$$\dot{L}\left(q,\dot{q},t\right)=\frac{dL}{dt}=\frac{\partial L}{\partial q^m}\frac{dq^m}{dt}+\frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt}$$

GCC velocity: $\dot{q}^m=\frac{dq^m}{dt}$

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...of coordinates and velocity and time, too. (You can safely drop last chain-rule factor [1=dt/dt])

$$\dot{L}(q,\dot{q},t)=\frac{dL}{dt}=\frac{\partial L}{\partial q^m}\frac{dq^m}{dt}+\frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt}+\frac{\partial L}{\partial t}\frac{dt}{dt}$$

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...of coordinates and velocity and <u>time</u>, too. (Imagine Mad Scientist turning U(t)-dial.)

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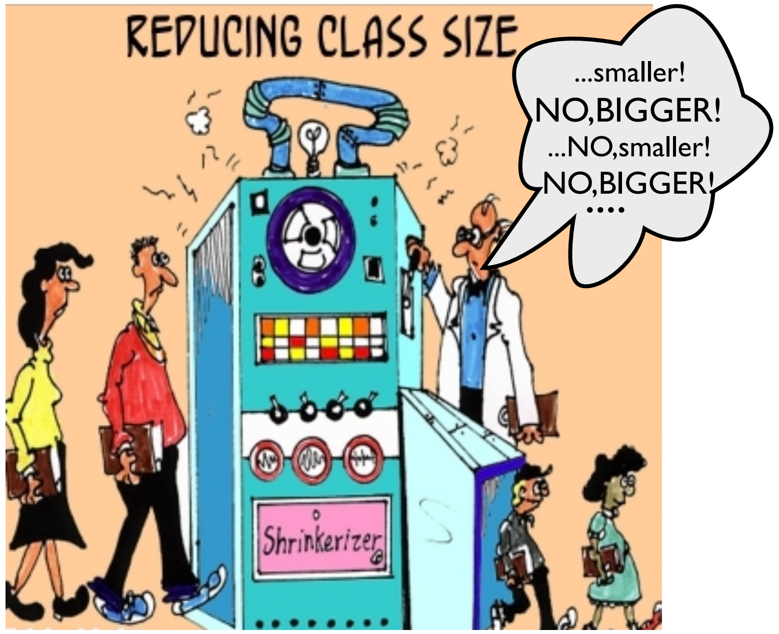

Cartoonish way to imagine explicit time dependence

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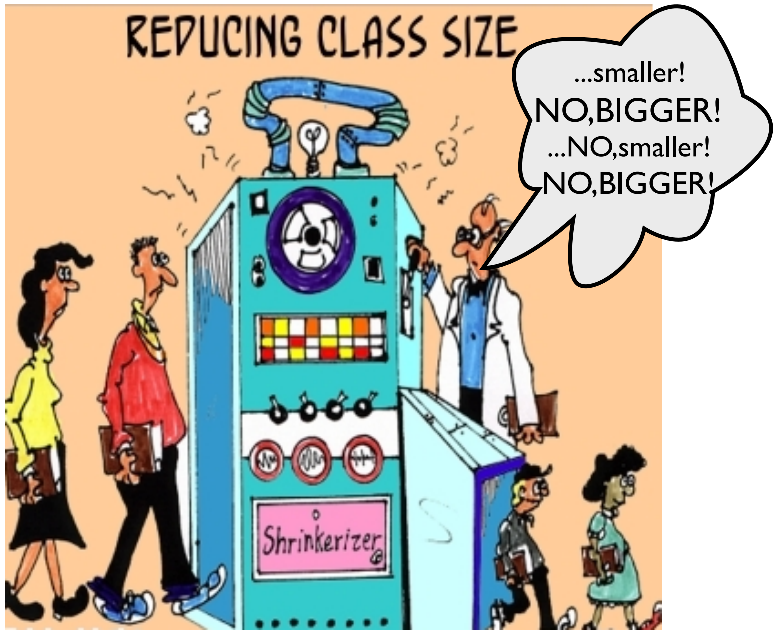
Recall Lagrange equations: $\boxed{\dot{p}_m=\frac{\partial L}{\partial q^m}}$ $\boxed{p_m=\frac{\partial L}{\partial \dot{q}^m}}$

$$\dot{L}(q,\dot{q},t)=\frac{dL}{dt}=\dot{p}_m\frac{dq^m}{dt}+p_m\frac{d\dot{q}^m}{dt}+\frac{\partial L}{\partial t}$$



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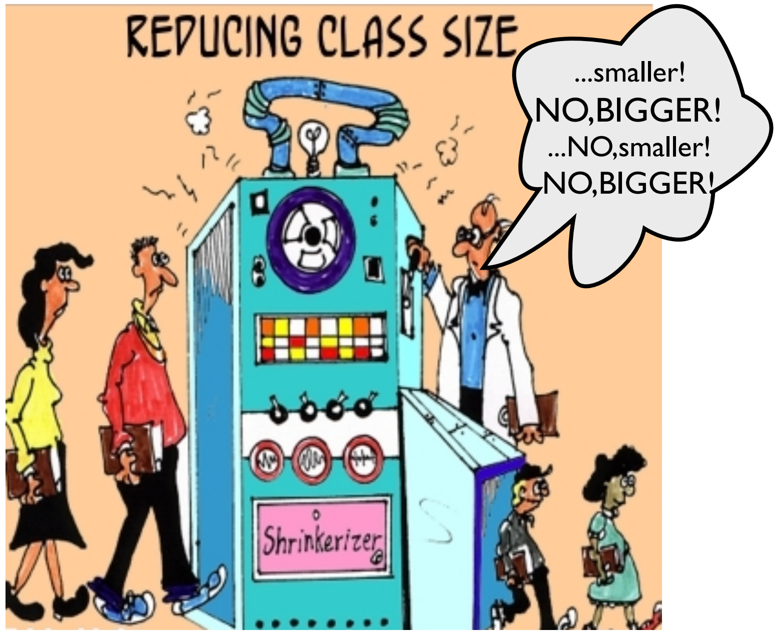
Use product rule:

$$\dot{u}\frac{dv}{dt}+u\frac{d\dot{v}}{dt}=\frac{d}{dt}(u\dot{v})$$

$$=\frac{dL}{dt}=\frac{d}{dt}\left(p_m\dot{q}^m\right)+\frac{\partial L}{\partial t}$$



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$$\dot L(q,\dot q, t) = \frac{dL}{dt} = \frac{\partial L}{\partial q^m}\frac{dq^m}{dt} + \frac{\partial L}{\partial \dot q^m}\frac{d\dot q^m}{dt}$$

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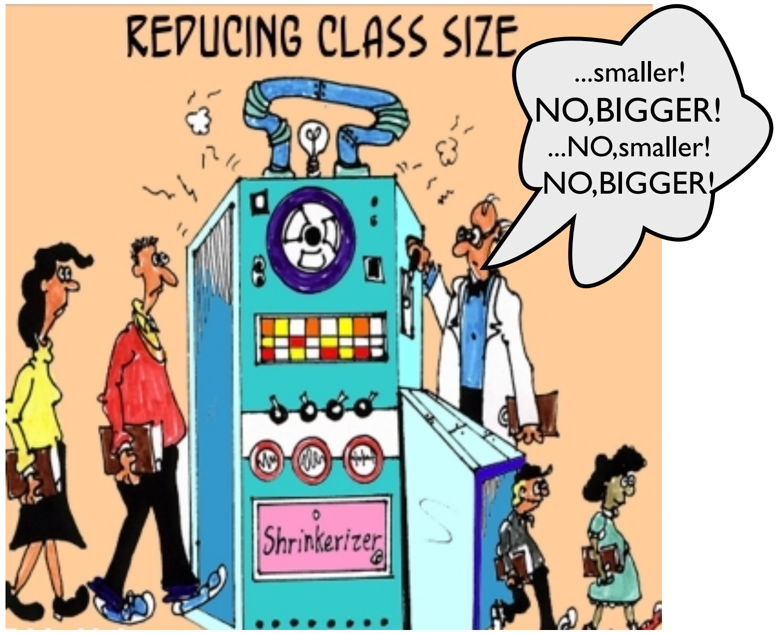
Use product rule:
$\dot u \frac{dv}{dt} + u \frac{d\dot v}{dt} = \frac{d}{dt}(u\dot v)$

$$= \frac{dL}{dt} = \frac{d}{dt}\left(p_m \dot q^m\right) + \frac{\partial L}{\partial t}$$

and switch the dL/dt and ∂L/∂t to define the Hamiltonian function $H(\mathbf{p}) = \mathbf{p}\bullet\mathbf{v} - L(\mathbf{v})$

$$\frac{d}{dt}\left(p_m \dot q^m - L\right) = -\frac{\partial L}{\partial t}$$

where: $H \equiv p_m \dot q^m - L$



Cartoonish way to imagine
explicit time dependence

Deriving Hamilton's equations from Lagrangian theory

Consider total time derivative of Lagrangian L=T-U
that is explicit function of coordinates and velocity $\dot q$...

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GCC velocity: $\dot q^m=\frac{dq^m}{dt}$

...of coordinates and velocity and <u>time</u>, too. (Imagine Mad Scientist turning U-dial.)

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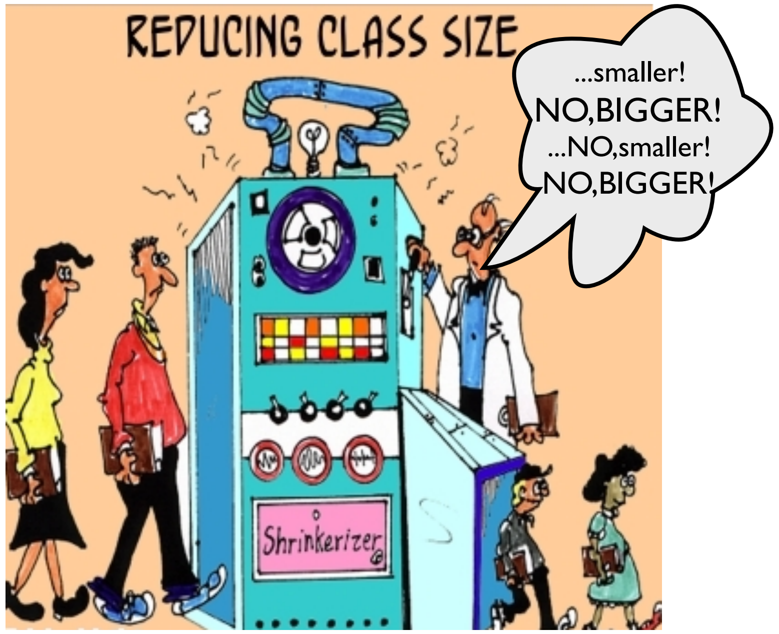

Cartoonish way to imagine
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and switch the dL/dt and ∂L/∂t to define the Hamiltonian function $H(\mathbf{p})=\mathbf{p}\bullet\mathbf{v}-L(\mathbf{v})$

$$\frac{d}{dt}\left(p_m\dot q^m-L\right)=-\frac{\partial L}{\partial t}\equiv\frac{dH}{dt}\qquad where: H\equiv p_m\dot q^m-L$$

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...of coordinates and velocity and time, too. (Imagine Mad Scientist turning U-dial.)

$$\dot{L}(q,\dot{q},t)=\frac{dL}{dt}=\frac{\partial L}{\partial q^m}\frac{dq^m}{dt}+\frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt}+\frac{\partial L}{\partial t}$$

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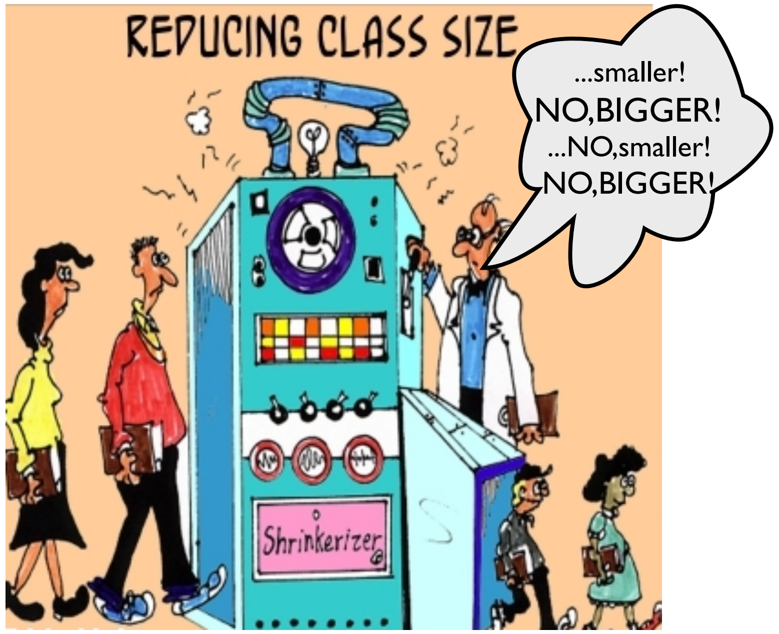

Define the Hamiltonian function $H(\mathbf{p})=\mathbf{p}\bullet\mathbf{v}-L(\mathbf{v})$

(That's the old Legendre transform)

$$\frac{d}{dt}\left(p_m\dot{q}^m-L\right)=-\frac{\partial L}{\partial t}\equiv\frac{dH}{dt}\quad \text{where}: H\equiv p_m\dot{q}^m-L$$

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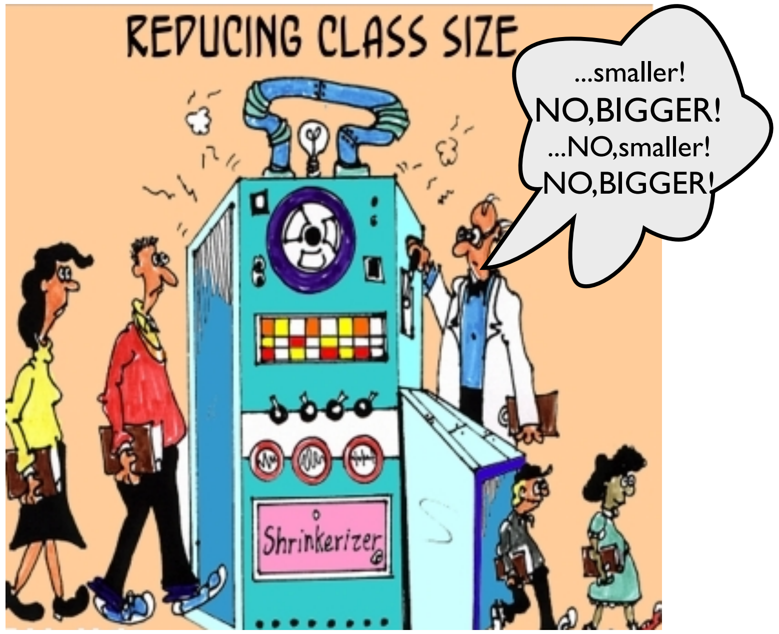

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Use product rule: $\dot{u}\frac{dv}{dt}+u\frac{d\dot{v}}{dt}=\frac{d}{dt}(u\dot{v})$

$$=\frac{dL}{dt}=\frac{d}{dt}\left(p_m\dot{q}^m\right)+\frac{\partial L}{\partial t}$$

Define the Hamiltonian function $H(\mathbf{p})=\mathbf{p}\bullet\mathbf{v}-L(\mathbf{v})$

(That's the old Legendre transform)

(Recall: $\frac{\partial L}{\partial p_m}\equiv 0$ *and:* $\frac{\partial H}{\partial \dot{q}^m}\equiv 0$ *)*

$$\frac{d}{dt}\left(p_m\dot{q}^m-L\right)=-\frac{\partial L}{\partial t}\equiv\frac{dH}{dt}\qquad where: H\equiv p_m\dot{q}^m-L$$

Deriving Hamilton's equations from Lagrangian theory

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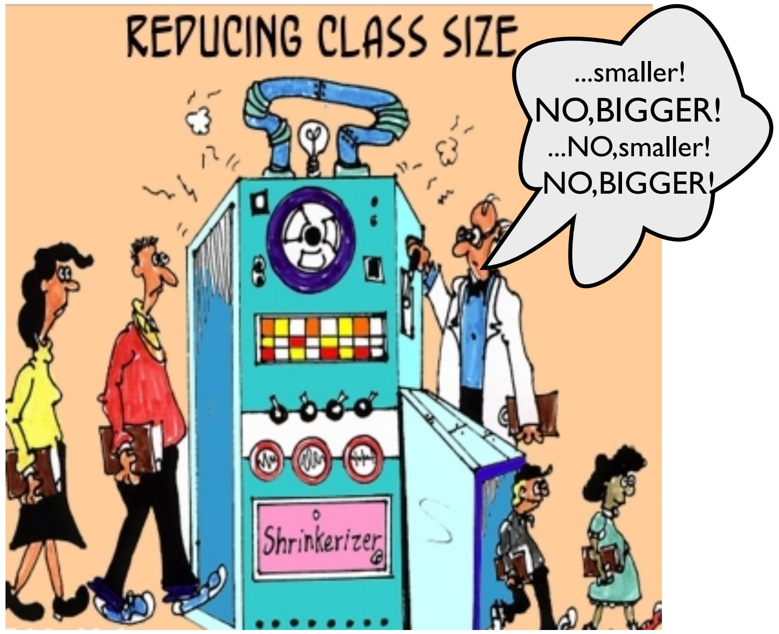
GCC velocity: $\dot{q}^m = \frac{dq^m}{dt}$

…of coordinates and velocity and time, too. (Imagine Mad Scientist turning U-dial.)



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Deriving Hamilton's equations from Lagrangian theory

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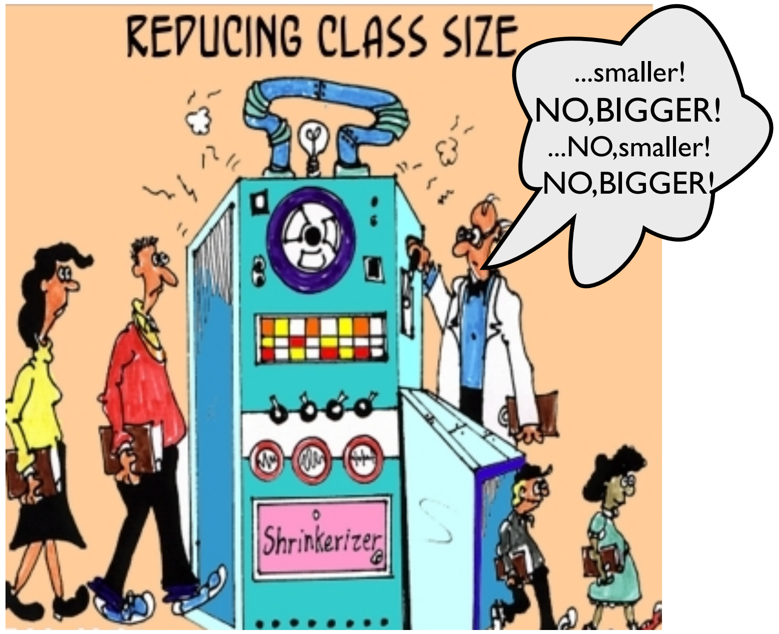

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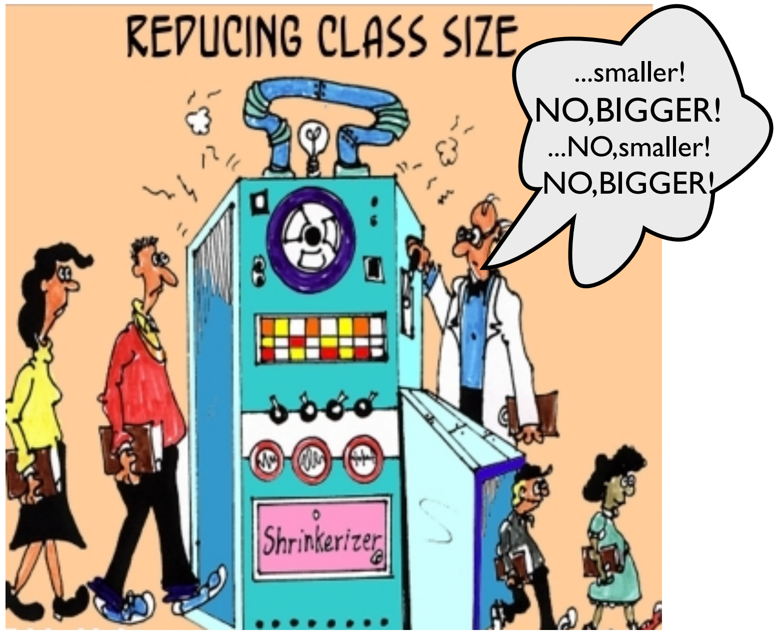

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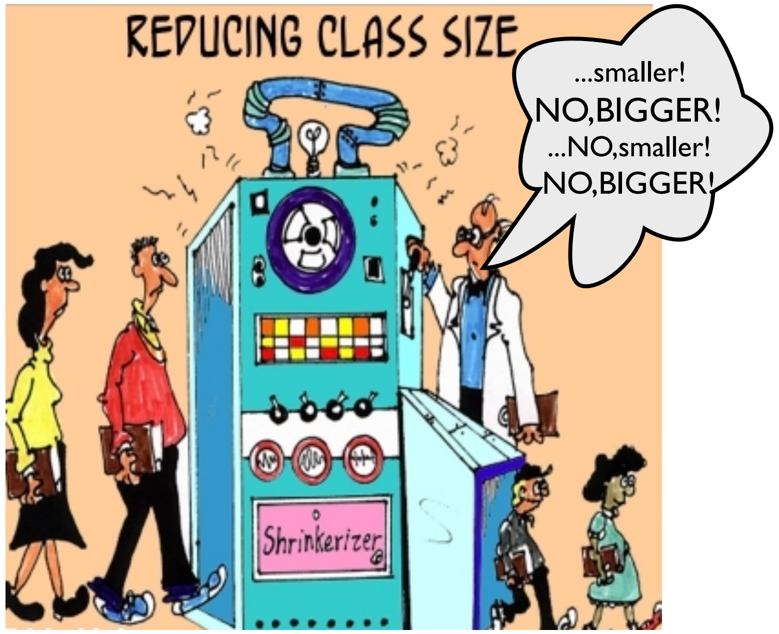

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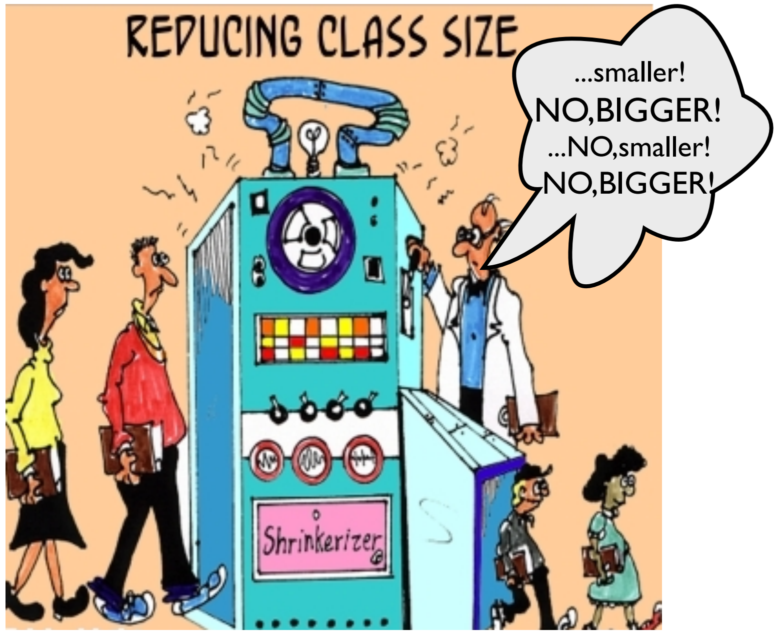

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Hamilton's 2nd GCC equation

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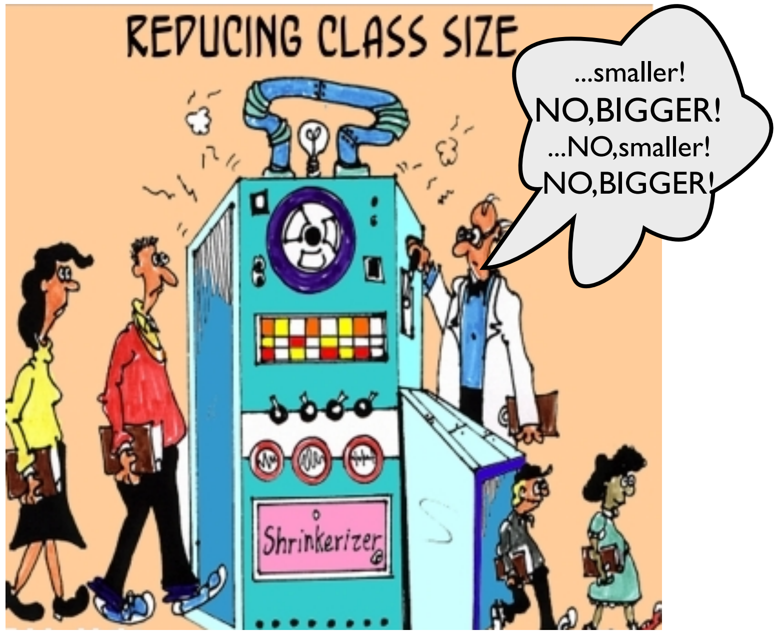

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a most **peculiar** relation
involving partial vs total

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Hamilton's 1st GCC equation

$$\frac{\partial H}{\partial p_m} = \dot{q}^m$$

Hamilton's 2nd GCC equation

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Hamilton prefers Contravariant g^{mn} with Covariant momentum p_m

- Deriving Hamilton's equations from Lagrange's equations
- → Expressing Hamiltonian $H(p_m,q^n)$ using g^{mn} and covariant momentum p_m
  - Polar-coordinate example of Hamilton's equations compared to Lagrange's
    - Hamilton's equations in Runga-Kutta (computer solution) form

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Using Legendre transform of Lagrangian $L=T\text{-}U$ with covariant metric definitions of L and p_m

We already have: $H = p_m\dot{q}^m - L$ *and:* $L(\dot{q}) = \frac{1}{2} M g_{mn}\dot{q}^m\dot{q}^n - U$ *and:* $p_m = \frac{\partial L}{\partial \dot{q}^m} = M g_{mn}\dot{q}^n$

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This gives an “illegal dependence” for the Hamiltonian (It musn’t be “explicit” in velocity $\dot{q}^m$.)

$$H = \frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n + U = T + U$$

(Numerically correct ONLY!)

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details on next pages

(Formally **and** Numerically correct)

Details of metric tensor algebra:

Given: $H = \frac{1}{2} M g_{mn} \dot{q}^{m} \dot{q}^{n} + U$

Let: $\dot{q}^{m} = \frac{1}{M} g^{mn'} p_{n'}$

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Metric tensor symmetry:

$$g_{mn} = g_{nm}$$

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(Always applies)

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(Numerically correct ONLY!)

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$$\boxed{H = \frac{1}{2M} g^{mn} p_m p_n + U = T + U \equiv E}$$

(Formally **and** Numerically correct)

Polar coordinate Lagrangian was given as:

$$L(\dot{r}, \dot{\phi}, r, \phi) = \tfrac{1}{2} M \left(g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2 \right) - U(r, \phi)$$

Hamilton prefers Contravariant g^{mn} with Covariant momentum p_m

Using Legendre transform of Lagrangian $L=T-U$ with covariant metric definitions of L and p_m

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Polar coordinate Lagrangian was given as:

See covariant polar metric $g_{\mu\upsilon}$ on next page (p35)

$$L(\dot{r},\dot{\phi},r,\phi)=\frac{1}{2}M(g_{rr}\dot{r}^2+g_{\phi\phi}\dot{\phi}^2)-U(r,\phi)=\frac{1}{2}M(\dot{r}^2+r^2\cdot\dot{\phi}^2)-U(r,\phi)$$

Covariant polar metric $g_{\mu\upsilon}$ *[from p53 of Lecture 9]* *Contravariant polar metric* $g^{\mu\upsilon}$

Covariant g_{mn} *vs.* Invariant $\delta_m{}^n$ *vs.* Contravariant g^{mn}

$$\mathbf{E}_m \bullet \mathbf{E}_n = \frac{\partial \mathbf{r}}{\partial q^m} \bullet \frac{\partial \mathbf{r}}{\partial q^n} \equiv g_{mn} \qquad \mathbf{E}_m \bullet \mathbf{E}^n = \frac{\partial \mathbf{r}}{\partial q^m} \bullet \frac{\partial q^n}{\partial \mathbf{r}} = \delta_m^n \qquad \mathbf{E}^m \bullet \mathbf{E}^n = \frac{\partial q^m}{\partial \mathbf{r}} \bullet \frac{\partial q^n}{\partial \mathbf{r}} \equiv g^{mn}$$

Covariant
metric tensor
g_{mn}

Invariant
Kroneker unit tensor

$$\delta_m^n \equiv \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

Contravariant
metric tensor
g^{mn}

Polar coordinate examples (again):

$$\langle J \rangle = \begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} = \cos\phi & \frac{\partial x}{\partial \phi} = -r\sin\phi \\ \frac{\partial y}{\partial r} = \sin\phi & \frac{\partial y}{\partial \phi} = r\cos\phi \end{pmatrix}$$

$$\uparrow \mathbf{E}_1 \quad \uparrow \mathbf{E}_2 \qquad\qquad \uparrow \mathbf{E}_r \qquad\qquad \uparrow \mathbf{E}_\phi$$

$$\langle K \rangle = \langle J^{-1} \rangle = \begin{pmatrix} \frac{\partial r}{\partial x} = \cos\phi & \frac{\partial r}{\partial y} = \sin\phi \\ \frac{\partial \phi}{\partial x} = \frac{-\sin\phi}{r} & \frac{\partial \phi}{\partial y} = \frac{\cos\phi}{r} \end{pmatrix} \begin{array}{l} \leftarrow \mathbf{E}^r = \mathbf{E}^1 \\ \leftarrow \mathbf{E}^\phi = \mathbf{E}^2 \end{array}$$

Covariant g_{mn}

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r \bullet \mathbf{E}_r & \mathbf{E}_r \bullet \mathbf{E}_\phi \\ \mathbf{E}_\phi \bullet \mathbf{E}_r & \mathbf{E}_\phi \bullet \mathbf{E}_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

Invariant δ_m^n

$$\begin{pmatrix} \delta_r^r & \delta_r^\phi \\ \delta_\phi^r & \delta_\phi^\phi \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r \bullet \mathbf{E}^r & \mathbf{E}_r \bullet \mathbf{E}^\phi \\ \mathbf{E}_\phi \bullet \mathbf{E}^r & \mathbf{E}_\phi \bullet \mathbf{E}^\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Contravariant g^{mn}

$$\begin{pmatrix} g^{rr} & g^{r\phi} \\ g^{\phi r} & g^{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}^r \bullet \mathbf{E}^r & \mathbf{E}^r \bullet \mathbf{E}^\phi \\ \mathbf{E}^\phi \bullet \mathbf{E}^r & \mathbf{E}^\phi \bullet \mathbf{E}^\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}$$

Hamilton prefers Contravariant g^{mn} with Covariant momentum p_m

Using Legendre transform of Lagrangian $L=T-U$ with covariant metric definitions of L and p_m

We already have: $H = p_m\dot{q}^m - L$ and: $L(\dot{q}) = \frac{1}{2} M g_{mn}\dot{q}^m \dot{q}^n - U$ and: $p_m = \frac{\partial L}{\partial \dot{q}^m} = M g_{mn}\dot{q}^n$

Now we combine all these:

$$H = p_m\dot{q}^m - L = \left(M g_{mn}\dot{q}^n\right)\dot{q}^m - \left(\frac{1}{2} M g_{mn}\dot{q}^m\dot{q}^n - U\right)$$

$$= M g_{mn}\dot{q}^m\dot{q}^n - \frac{1}{2} M g_{mn}\dot{q}^m \dot{q}^n + U$$

This gives an "illegal dependence" for the Hamiltonian (It musn't be "explicit" in velocity $\dot{q}^m$.)

$$H = \frac{1}{2} M g_{mn}\dot{q}^m\dot{q}^n + U = T + U$$

(Numerically correct ONLY!)

An inverse metric relation $\dot{q}^m = \frac{1}{M} g^{mn} p_n$ gives correct form that depends on momentum p_m.

$$\boxed{H = \frac{1}{2M} g^{mn} p_m p_n + U = T + U \equiv E}$$

(Formally **and** Numerically correct)

Polar coordinate Lagrangian was given as: See covariant polar metric $g_{\mu\upsilon}$ on p39

$$L(\dot{r},\dot{\phi},r,\phi) = \frac{1}{2} M(g_{rr}\dot{r}^2 + g_{\phi\phi}\dot{\phi}^2) - U(r,\phi) = \frac{1}{2} M(\dot{r}^2 + r^2\cdot\dot{\phi}^2) - U(r,\phi)$$

Polar coordinate Hamiltonian is given here:

$$H(p_r,p_\phi,r,\phi) = \frac{1}{2M}(g^{rr}p_r{}^2 + g^{\phi\phi}p_\phi{}^2) + U(r,\phi)$$

Hamilton prefers Contravariant g^{mn} with Covariant momentum p_m

Using Legendre transform of Lagrangian $L=T-U$ with covariant metric definitions of L and p_m

We already have: $H = p_m\dot{q}^m - L$ and: $L(\dot{q}) = \frac{1}{2}Mg_{mn}\dot{q}^m\dot{q}^n - U$ and: $p_m = \frac{\partial L}{\partial \dot{q}^m} = Mg_{mn}\dot{q}^n$

Now we combine all these:

$$H = p_m\dot{q}^m - L = \left(Mg_{mn}\dot{q}^n\right)\dot{q}^m - \left(\frac{1}{2}Mg_{mn}\dot{q}^m\dot{q}^n - U\right)$$

$$= Mg_{mn}\dot{q}^m\dot{q}^n - \frac{1}{2}Mg_{mn}\dot{q}^m\dot{q}^n + U$$

This gives an "illegal dependence" for the Hamiltonian (It musn't be "explicit" in velocity $\dot{q}^m$.)

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(Numerically correct ONLY!)

An inverse metric relation $\dot{q}^m = \frac{1}{M}g^{mn}p_n$ gives correct form that depends on momentum p_m.

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$$L(\dot{r},\dot{\phi},r,\phi) = \frac{1}{2}M(g_{rr}\dot{r}^2 + g_{\phi\phi}\dot{\phi}^2) - U(r,\phi) = \frac{1}{2}M(\dot{r}^2 + r^2\cdot\dot{\phi}^2) - U(r,\phi)$$

Polar coordinate Hamiltonian is given here: Contravariant polar metric $g^{\mu\upsilon}$ on p35

$$H(p_r,p_\phi,r,\phi) = \frac{1}{2M}(g^{rr}p_r^2 + g^{\phi\phi}p_\phi^2) + U(r,\phi) = \frac{1}{2M}(p_r^2 + \frac{1}{r^2}\cdot p_\phi^2) + U(r,\phi)$$

Hamilton prefers <u>Contra</u>*variant* g^{mn} *with* <u>Co</u>*variant* **momentum** p_m

Deriving Hamilton's equations from Lagrange's equations

Expressing Hamiltonian $H(p_m,q^n)$ *using* g^{mn} *and covariant momentum* p_m

→ *Polar-coordinate example of Hamilton's equations compared to Lagrange's*

Hamilton's equations in Runga-Kutta (computer solution) form

Polar coordinate example of Hamilton's equations

$$H(p_r,p_\phi,r,\phi)=\frac{1}{2M}(g^{rr}p_r^2+g^{\phi\phi}p_\phi^2)+U(r,\phi)=\frac{1}{2M}(p_r^2+\frac{1}{r^2}\cdot p_\phi^2)+U(r,\phi)$$

Hamiltonian $H(p_r,p_\phi,r,\phi)=\frac{1}{2M}(p_r^2+\frac{1}{r^2}\cdot p_\phi^2)+U(r,\phi)$ in 2D-polar coordinates satisfies:

Hamilton's 1st equations: $\frac{\partial H}{\partial p_m}=\dot{q}^m$ Hamilton's 2nd equations: $\frac{\partial H}{\partial q^m}=-\dot{p}_m$

Polar coordinate example of Hamilton's equations

$$H(p_r,p_\phi,r,\phi)=\frac{1}{2M}(g^{rr}p_r^2+g^{\phi\phi}p_\phi^2)+U(r,\phi)=\frac{1}{2M}(p_r^2+\frac{1}{r^2}\cdot p_\phi^2)+U(r,\phi)$$

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Hamilton's 2nd equations: $\frac{\partial H}{\partial q^m}=-\dot{p}_m$

$$\frac{\partial H}{\partial p_r}=\dot{r}$$

$$\frac{\partial H}{\partial p_\phi}=\dot{\phi}$$

Polar coordinate example of Hamilton's equations

$$H(p_r,p_\phi,r,\phi)=\frac{1}{2M}(g^{rr}p_r^2+g^{\phi\phi}p_\phi^2)+U(r,\phi)=\frac{1}{2M}(p_r^2+\frac{1}{r^2}\cdot p_\phi^2)+U(r,\phi)$$

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$$\frac{\partial H}{\partial p_\phi}=\dot{\phi}$$

$$\frac{\partial H}{\partial r}=-\dot{p}_r$$

$$\frac{\partial H}{\partial \phi}=-\dot{p}_\phi$$

Polar coordinate example of Hamilton's equations

$$H(p_r,p_\phi,r,\phi)=\frac{1}{2M}(g^{rr}p_r^2+g^{\phi\phi}p_\phi^2)+U(r,\phi)=\frac{1}{2M}(p_r^2+\frac{1}{r^2}\cdot p_\phi^2)+U(r,\phi)$$

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Hamilton's 1st equations: $\frac{\partial H}{\partial p_m}=\dot{q}^m$

Hamilton's 2nd equations: $\frac{\partial H}{\partial q^m}=-\dot{p}_m$

$$\frac{\partial H}{\partial p_r}=\dot{r}$$

$$\frac{\partial H}{\partial p_r}=\frac{p_r}{M}$$

$$\frac{\partial H}{\partial p_\phi}=\dot{\phi}$$

$$\frac{\partial H}{\partial r}=-\dot{p}_r$$

$$\frac{\partial H}{\partial \phi}=-\dot{p}_\phi$$

Polar coordinate example of Hamilton's equations

$$H(p_r,p_\phi,r,\phi)=\frac{1}{2M}(g^{rr}p_r^2+g^{\phi\phi}p_\phi^2)+U(r,\phi)=\frac{1}{2M}(p_r^2+\frac{1}{r^2}\cdot p_\phi^2)+U(r,\phi)$$

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Polar coordinate example of Hamilton's equations

$$H(p_r,p_\phi,r,\phi)=\frac{1}{2M}(g^{rr}p_r^2+g^{\phi\phi}p_\phi^2)+U(r,\phi)=\frac{1}{2M}(p_r^2+\frac{1}{r^2}\cdot p_\phi^2)+U(r,\phi)$$

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$$\frac{\partial H}{\partial p_\phi}=\frac{p_\phi}{Mr^2}$$

$$\frac{\partial H}{\partial r}=-\dot{p}_r$$

$$\frac{\partial H}{\partial r}=-\cancel{2}\frac{p_\phi^2}{\cancel{2}Mr^3}+\frac{\partial U(r,\phi)}{\partial r}$$

$$\frac{\partial H}{\partial \phi}=-\dot{p}_\phi$$

Polar coordinate example of Hamilton's equations

$$H(p_r,p_\phi,r,\phi)=\frac{1}{2M}(g^{rr}p_r^2+g^{\phi\phi}p_\phi^2)+U(r,\phi)=\frac{1}{2M}(p_r^2+\frac{1}{r^2}\cdot p_\phi^2)+U(r,\phi)$$

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$$\frac{\partial H}{\partial p_\phi}=\frac{p_\phi}{Mr^2}$$

$$\frac{\partial H}{\partial r}=\text{-}\dot{p}_r$$

$$\frac{\partial H}{\partial r}=-\cancel{2}\frac{p_\phi^2}{\cancel{2}Mr^3}+\frac{\partial U(r,\phi)}{\partial r}$$

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Polar coordinate example of Hamilton's equations

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Polar coordinate example of Hamilton's equations

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Polar coordinate example of Hamilton's equations

$$H(p_r,p_\phi,r,\phi)=\frac{1}{2M}(g^{rr}p_r^2+g^{\phi\phi}p_\phi^2)+U(r,\phi)=\frac{1}{2M}(p_r^2+\frac{1}{r^2}\cdot p_\phi^2)+U(r,\phi)$$

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Polar coordinate example of Hamilton's equations

$$H(p_r,p_\phi,r,\phi)=\frac{1}{2M}(g^{rr}p_r^2+g^{\phi\phi}p_\phi^2)+U(r,\phi)=\frac{1}{2M}(p_r^2+\frac{1}{r^2}\cdot p_\phi^2)+U(r,\phi)$$

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$$=Mr\dot{\phi}^2-\partial_r U(r,\phi)$$

$$\frac{\partial H}{\partial \phi}=-\dot{p}_\phi$$

$$\frac{\partial H}{\partial \phi}=\frac{\partial U(r,\phi)}{\partial \phi}$$

$$\dot{p}_\phi=-\frac{\partial U(r,\phi)}{\partial \phi}$$

Polar coordinate example of Hamilton's equations

$$H(p_r,p_\phi,r,\phi)=\frac{1}{2M}(g^{rr}p_r^2+g^{\phi\phi}p_\phi^2)+U(r,\phi)=\frac{1}{2M}(p_r^2+\frac{1}{r^2}\cdot p_\phi^2)+U(r,\phi)$$

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$$\dot{p}_r=M\ddot{r}=\frac{p_\phi^2}{Mr^3}-\frac{\partial U(r,\phi)}{\partial r}$$

$$=Mr\dot{\phi}^2-\partial_r U(r,\phi)$$

$$\frac{\partial H}{\partial \phi}=-\dot{p}_\phi$$

$$\frac{\partial H}{\partial \phi}=\frac{\partial U(r,\phi)}{\partial \phi}$$

$$\dot{p}_\phi=-\frac{\partial U(r,\phi)}{\partial \phi}$$

$$p_\phi=Mr^2\dot{\phi}$$

$$\dot{p}_\phi=2Mr\dot{r}\dot{\phi}+Mr^2\ddot{\phi}=-\partial_\phi U(r,\phi)$$

Compare these Hamilton's equations to Lagrange's on next page...

Lagrange prefers Covariant g_{mn} with Contravariant velocity

Lagrangian KE-U is supposed to be explicit function of velocity.

(Review of Lecture 9)

$$L(\mathbf{v}) = \tfrac{1}{2} M\mathbf{v}\bullet\mathbf{v} - U = \tfrac{1}{2} M\dot{\mathbf{r}}\bullet\dot{\mathbf{r}} - U = \tfrac{1}{2} M(\mathbf{E}_m \dot{q}^m)\bullet(\mathbf{E}_n \dot{q}^n) - U = \tfrac{1}{2} M(g_{mn}\dot{q}^m \dot{q}^n) - U = L(\dot{q})$$

Use polar coordinate Covariant g_{mn} metric (1-page back)

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r \bullet \mathbf{E}_r & \mathbf{E}_r \bullet \mathbf{E}_\phi \\ \mathbf{E}_\phi \bullet \mathbf{E}_r & \mathbf{E}_\phi \bullet \mathbf{E}_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

This gives polar GCC form (Actually it's an OCC or Orthogonal Curvilinear Coordinate form)

$$L(\dot{r},\dot{\phi}) = \tfrac{1}{2} M(g_{rr}\dot{r}^2 + g_{\phi\phi}\dot{\phi}^2) - U(r,\phi) = \tfrac{1}{2} M(1 \cdot \dot{r}^2 + r^2 \cdot \dot{\phi}^2) - U(r,\phi)$$

GCC Lagrange equations follow. 1st L-equation is momentum p_m definition for each coordinate q^m:

$$p_r = \frac{\partial L}{\partial \dot{r}} = M\, g_{rr}\dot{r} = M\,\dot{r}$$

Nothing too surprising; radial momentum p_r has the usual linear $M \cdot v$ form

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = M g_{\phi\phi}\dot{\phi} = M r^2 \dot{\phi}$$

Wow! $g_{\phi\phi}$ gives moment-of-inertia factor Mr^2 automatically for the angular momentum $p_\phi = Mr^2\omega$.

2nd L-equation involves total time derivative $\dot{p}_m$ for each momentum p_m:

$$\dot{p}_r = \frac{\partial L}{\partial r} = \frac{M}{2}\frac{\partial g_{\phi\phi}}{\partial r}\dot{\phi}^2 - \frac{\partial U}{\partial r} = M\, r\,\dot{\phi}^2 - \frac{\partial U}{\partial r}$$

Centrifugal force $M r\omega^2$

$$\dot{p}_\phi = \frac{\partial L}{\partial \phi} = 0 - \frac{\partial U}{\partial \phi}$$

Angular momentum p_ϕ is conserved if potential U has no explicit ϕ-dependence

Find $\dot{p}_m$ directly from 1st L-equation: $\dot{p}_m \equiv \frac{dp_m}{dt} = \frac{d}{dt} M(g_{mn}\dot{q}^n) = M(\dot{g}_{mn}\dot{q}^n + g_{mn}\ddot{q}^n)$ Equate it to $\dot{p}_m$ in 2nd L-equation:

$$\dot{p}_r \equiv \frac{dp_r}{dt} = M\,\ddot{r}$$

$$= M\, r\,\dot{\phi}^2 - \frac{\partial U}{\partial r}$$

Centrifugal (center-fleeing) force equals total Centripetal (center-pulling) force

$$\dot{p}_\phi \equiv \frac{dp_\phi}{dt} = 2Mr\dot{r}\dot{\phi} + Mr^2\ddot{\phi}$$

Torque relates to two distinct parts: Coriolis and angular acceleration

$$= 0 - \frac{\partial U}{\partial \phi}$$

Angular momentum p_ϕ is conserved if potential U has no explicit ϕ-dependence

Hamilton prefers Contravariant g^{mn} *with* Covariant ***momentum*** p_m

Deriving Hamilton's equations from Lagrange's equations

Expressing Hamiltonian $H(p_m,q^n)$ *using* g^{mn} *and covariant momentum* p_m

Polar-coordinate example of Hamilton's equations compared to Lagrange's

→ *Hamilton's equations in Runga-Kutta (computer solution) form*

Polar coordinate example: Hamilton's equations in Runga-Kutta form

$$p_r = M\dot{r}$$

$$\dot{p}_r = M\ddot{r} = \frac{p_\phi^2}{Mr^3} - \frac{\partial U(r,\phi)}{\partial r}$$

$$= Mr\dot{\phi}^2 - \partial_r U(r,\phi)$$

$$p_\phi = Mr^2\dot{\phi}$$

$$\dot{p}_\phi = 2Mr\dot{r}\dot{\phi} + Mr^2\ddot{\phi} = -\partial_\phi U(r,\phi)$$

Runga-Kutta form:

$$\dot{x}_1 = \dot{x}_1(x_1, x_2, x_3, \ldots)$$

$$\dot{x}_2 = \dot{x}_2(x_1, x_2, x_3, \ldots)$$

$$\dot{x}_3 = \dot{x}_3(x_1, x_2, x_3, \ldots)$$

$$\vdots$$

Polar coordinate example: Hamilton's equations in Runga-Kutta form

$$p_r = M\dot{r}$$

$$\dot{p}_r = M\ddot{r} = \frac{p_\phi^2}{Mr^3} - \frac{\partial U(r,\phi)}{\partial r}$$

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$$p_\phi = Mr^2\dot{\phi}$$

$$\dot{p}_\phi = 2Mr\dot{r}\dot{\phi} + Mr^2\ddot{\phi} = -\partial_\phi U(r,\phi)$$

Hamiltonian eqs. in Runga-Kutta form:

$$\dot{r} = \dot{r}(r, p_r, \phi, p_\phi) \qquad = \frac{p_r}{M}$$

$$\dot{p}_r = \dot{p}_r(r, p_r, \phi, p_\phi) \qquad = \frac{p_\phi^2}{Mr^3} - \partial_r U(r,\phi)$$

$$\dot{\phi} = \dot{\phi}(r, p_r, \phi, p_\phi) \qquad = \frac{p_\phi}{Mr^2}$$

$$\dot{p}_\phi = \dot{p}_\phi(r, p_r, \phi, p_\phi) \qquad = -\partial_\phi U(r,\phi)$$

Runga-Kutta form:

$$\dot{x}_1 = \dot{x}_1(x_1, x_2, x_3, \ldots)$$

$$\dot{x}_2 = \dot{x}_2(x_1, x_2, x_3, \ldots)$$

$$\dot{x}_3 = \dot{x}_3(x_1, x_2, x_3, \ldots)$$

$$\vdots$$

Examples of Hamiltonian mechanics in effective potentials

➡ Isotropic Harmonic Oscillator in polar coordinates and effective potential (Web Simulation: OscillatorPE - IHO)

Coulomb orbits in polar coordinates and effective potential (Web Simulation: OscillatorPE - Coulomb)

Effective potential analysis (Reducing 2D-problem to 1D-problem)

Polar coordinate Hamiltonian can take advantage of H-conservation and p_m-conservation

Consider polar coordinate Hamiltonian for $I_{sotropic}$ $H_{armonic}$ $O_{scillator}$ potential $U(r) = kr^2/2$:

$$H(p_r, p_\phi, r, \phi) = \frac{1}{2M}(g^{rr}p_r{}^2 + g^{\phi\phi}p_\phi{}^2) + k\cdot r^2/2 = \frac{1}{2M}(p_r{}^2 + \frac{1}{r^2}\cdot p_\phi{}^2) + \frac{k\cdot r^2}{2} = E = const.$$

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H is not explicit function of ϕ , and so Hamilton's 2nd says: $\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0$

Thus momentum p_ϕ is conserved constant: $p_\phi = \ell = const.$

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Same applies to any radial potential U(r)

$$E = \frac{p_r{}^2}{2M} + \overbrace{\underbrace{\frac{\ell^2}{2Mr^2}}_{\text{"centifugal-barrier" PE}} + \overbrace{U(r)}^{\text{"real" PE}}}^{\text{"effective" PE}}$$

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$$\frac{p_r^2}{2M}+\frac{p_\phi^2}{2Mr^2}+\frac{k\cdot r^2}{2}=\frac{p_r^2}{2M}+\frac{\ell^2}{2Mr^2}+\frac{k\cdot r^2}{2}=E=const.$$

Solving for momentum: $p_r^2=2ME-\frac{\ell^2}{r^2}-Mk\cdot r^2$

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$$p_r=M\dot{r}=\sqrt{2ME-\frac{\ell^2}{r^2}-Mk\cdot r^2}=\sqrt{2M}\sqrt{E-\frac{\ell^2}{2Mr^2}-\frac{k}{2}\cdot r^2}$$

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$$\frac{p_r^2}{2M} + \frac{p_\phi^2}{2Mr^2} + \frac{k\cdot r^2}{2} = \frac{p_r^2}{2M} + \frac{\ell^2}{2Mr^2} + \frac{k\cdot r^2}{2} = E = const.$$

Solving for momentum: $p_r^2 = 2ME - \frac{\ell^2}{r^2} - Mk\cdot r^2$

$$p_r = M\dot{r} = \sqrt{2ME - \frac{\ell^2}{r^2} - Mk\cdot r^2} = \sqrt{2M}\sqrt{E - \frac{\ell^2}{2Mr^2} - \frac{k}{2}\cdot r^2}$$

Radial KE is: $\frac{M\dot{r}^2}{2} = E - \frac{\ell^2}{2Mr^2} - \frac{k}{2}\cdot r^2$

Effective potential analysis (Reducing 2D-problem to 1D-problem)

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Radial velocity:

$$\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2E}{M} - \frac{\ell^2}{M^2r^2} - \frac{k}{M}\cdot r^2}$$

Effective potential analysis (Reducing 2D-problem to 1D-problem)

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Same applies to any radial potential $U(r)$

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Radial velocity:

$$\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2E}{M} - \frac{\ell^2}{M^2 r^2} - \frac{k}{M}\cdot r^2}$$

Time vs r: $t = \int_{r<}^{r>} \frac{dr}{\sqrt{\frac{2E}{M} - \frac{\ell^2}{M^2 r^2} - \frac{k}{M}\cdot r^2}}$

Effective potential analysis (Reducing 2D-problem to 1D-problem)

Polar coordinate Hamiltonian can take advantage of H-conservation and p_m-conservation

Consider polar coordinate Hamiltonian for I_{sotropic} H_{armonic} $O_{\text{scillator}}$ potential $U(r) = kr^2/2$:

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Same applies to any radial potential U(r)

$$E = \frac{p_r{}^2}{2M} + \overbrace{\underbrace{\frac{\ell^2}{2Mr^2}}_{\text{“centifugal-barrier” PE}} + \overbrace{U(r)}^{\text{“real” PE}}}^{\text{“effective” PE}}$$

Solving for momentum : $p_r^2 = 2ME - \frac{\ell^2}{r^2} - Mk\cdot r^2$

$$p_r = M\dot{r} = \sqrt{2ME - \frac{\ell^2}{r^2} - Mk\cdot r^2} = \sqrt{2M}\sqrt{E - \frac{\ell^2}{2Mr^2} - \frac{k}{2}\cdot r^2}$$

Called the “quadrature” or 1/4-cycle solution if $r_< = 0$ and $r_> =$ max amplitude

Radial KE is: $\frac{M\dot{r}^2}{2} = E - \frac{\ell^2}{2Mr^2} - \frac{k}{2}\cdot r^2$

Radial velocity:

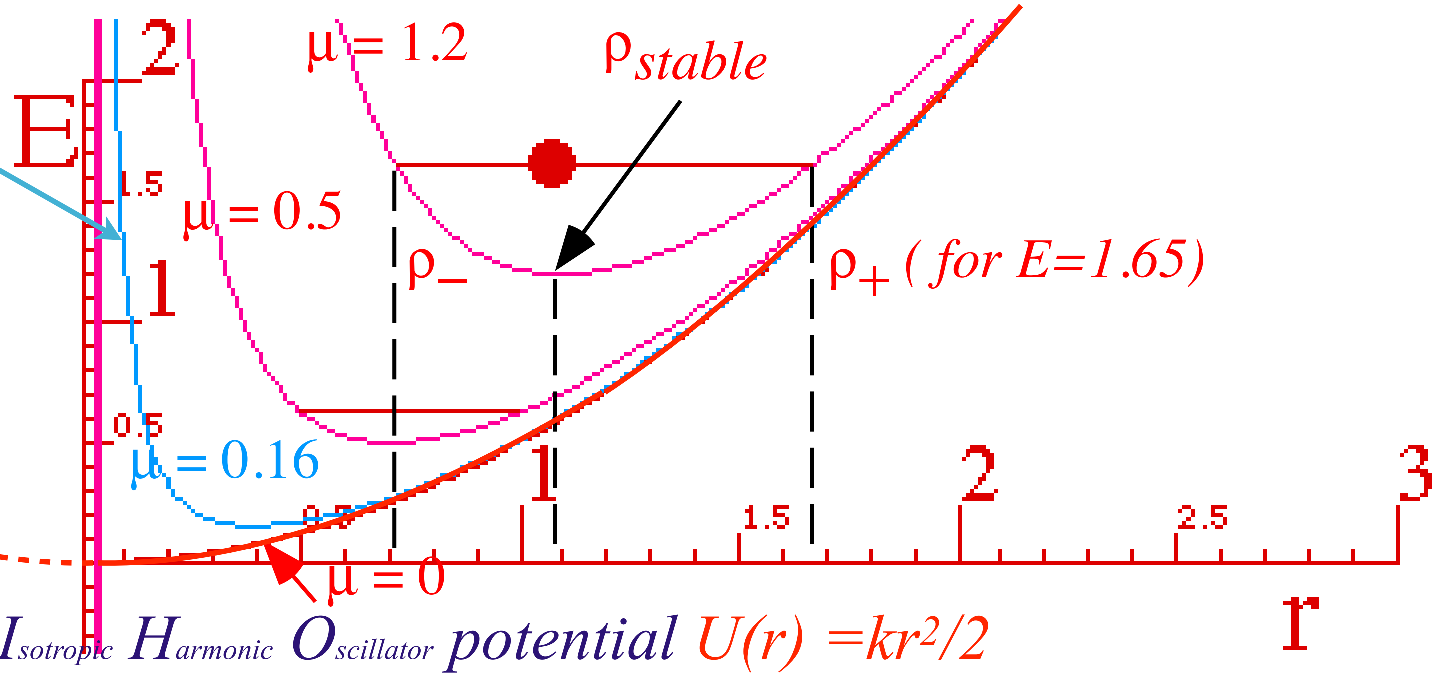
$$\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2E}{M} - \frac{\ell^2}{M^2r^2} - \frac{k}{M}\cdot r^2}$$

Time vs r: $t = \int_{r<}^{r>} \frac{dr}{\sqrt{\frac{2E}{M} - \frac{\ell^2}{M^2r^2} - \frac{k}{M}\cdot r^2}}$

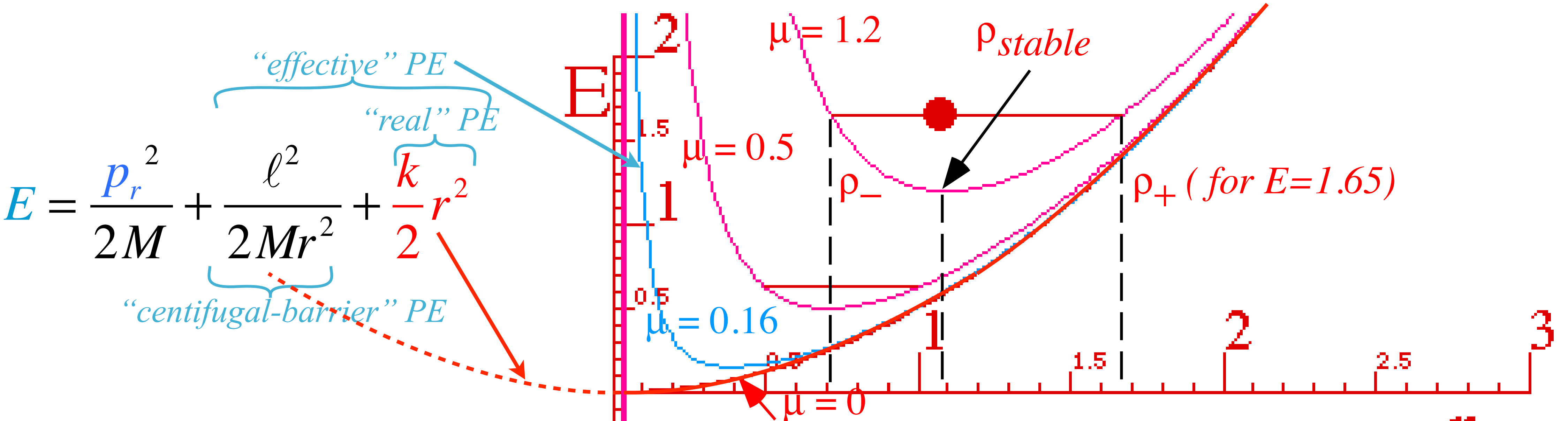
Time vs r for any radial U(r):

$$t = \int_{r<}^{r>} \frac{dr}{\sqrt{\frac{2E}{M} - \frac{\ell^2}{M^2r^2} - \frac{2U(r)}{M}}}$$

$$E=\frac{p_r^{\;2}}{2M}+\overbrace{\underbrace{\frac{\ell^2}{2Mr^2}}_{\text{“centifugal-barrier” PE}}+\overbrace{\frac{k}{2}r^2}^{\text{“real” PE}}}^{\text{“effective” PE}}$$



Hamiltonian dynamics for Isotropic Harmonic Oscillator potential $U(r) = kr^2/2$



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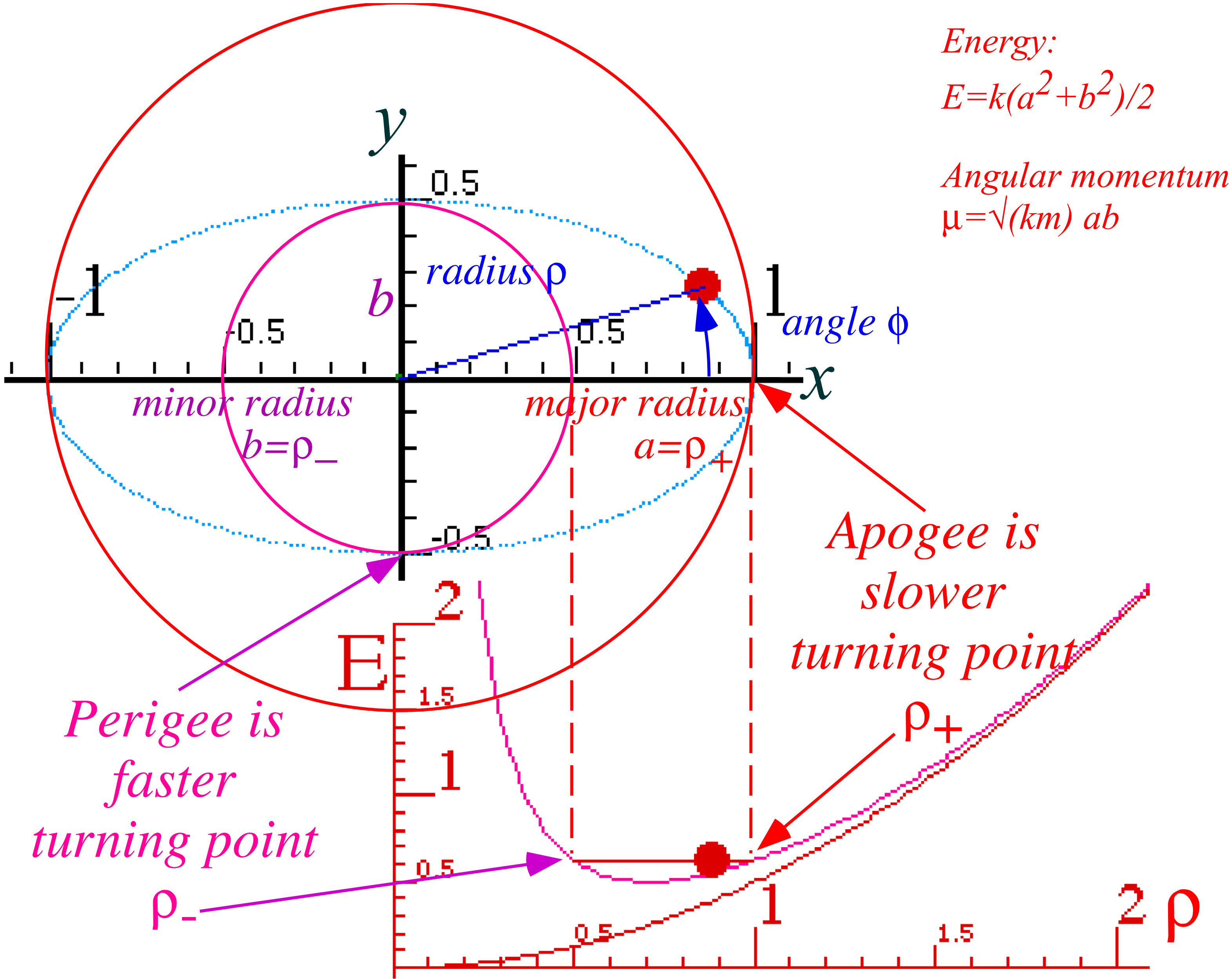
Hamiltonian dynamics for I_sotropic H_armonic O_scillator potential $U(r) = kr^2/2$

Energy:
$E=k(a^2+b^2)/2$

Angular momentum:
$\mu=\sqrt{(km)}\, ab$



Apogee is slower turning point

Perigee is faster turning point

I_sotropic H_armonic O_scillator in polar coordinates and effective potential (Web Simulation: OscillatorPE - IHO)

Examples of Hamiltonian mechanics in effective potentials

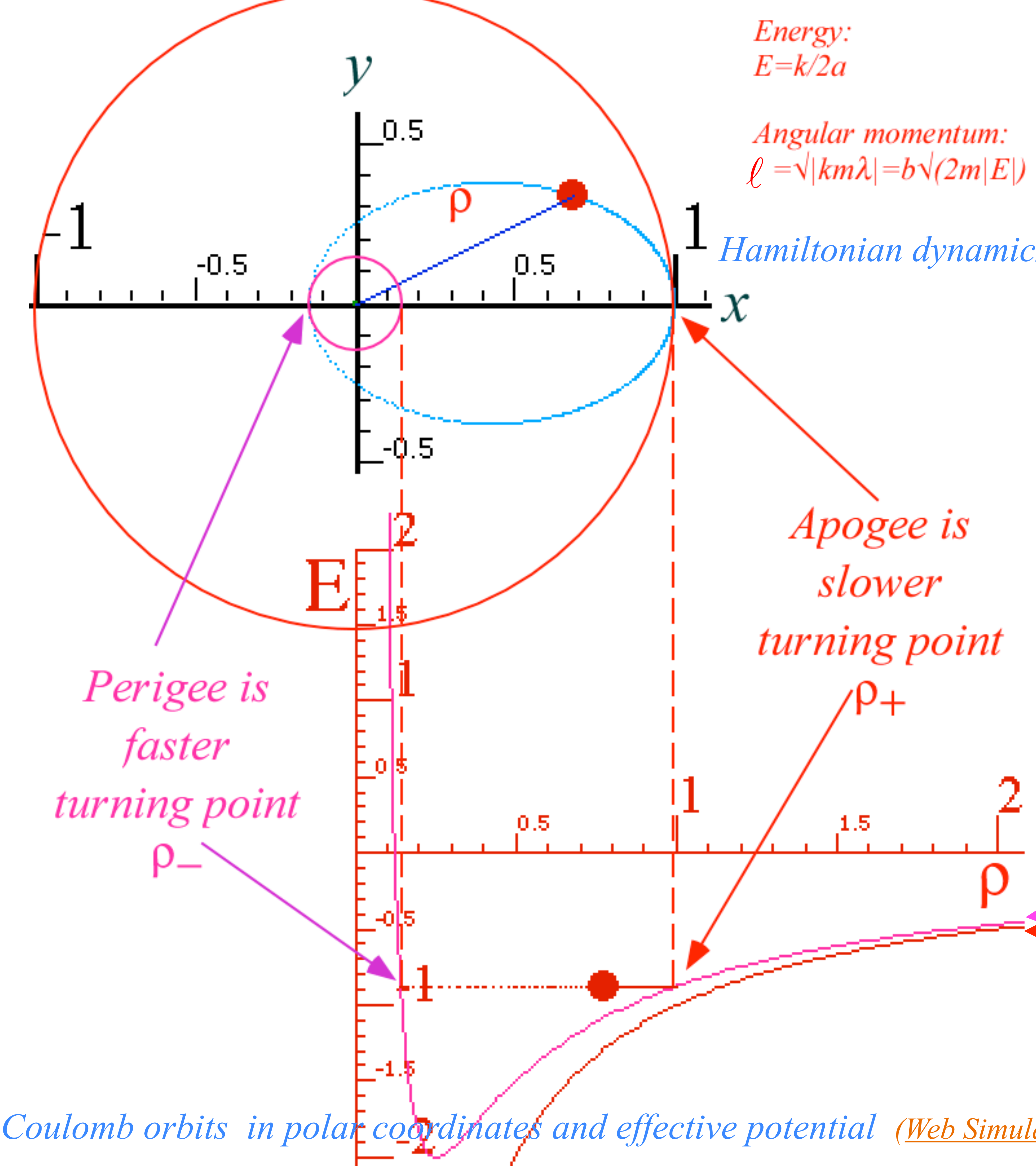
Isotropic Harmonic Oscillator in polar coordinates and effective potential (Web Simulation: OscillatorPE - IHO)

➡ Coulomb orbits in polar coordinates and effective potential (Web Simulation: OscillatorPE - Coulomb)

Hamiltonian dynamics for Coulomb potential $U(r) = -k/r$

Energy:
$E = k/2a$

Angular momentum:
$\ell = \sqrt{|km\lambda|} = b\sqrt{(2m|E|)}$



Apogee is slower turning point ρ_+

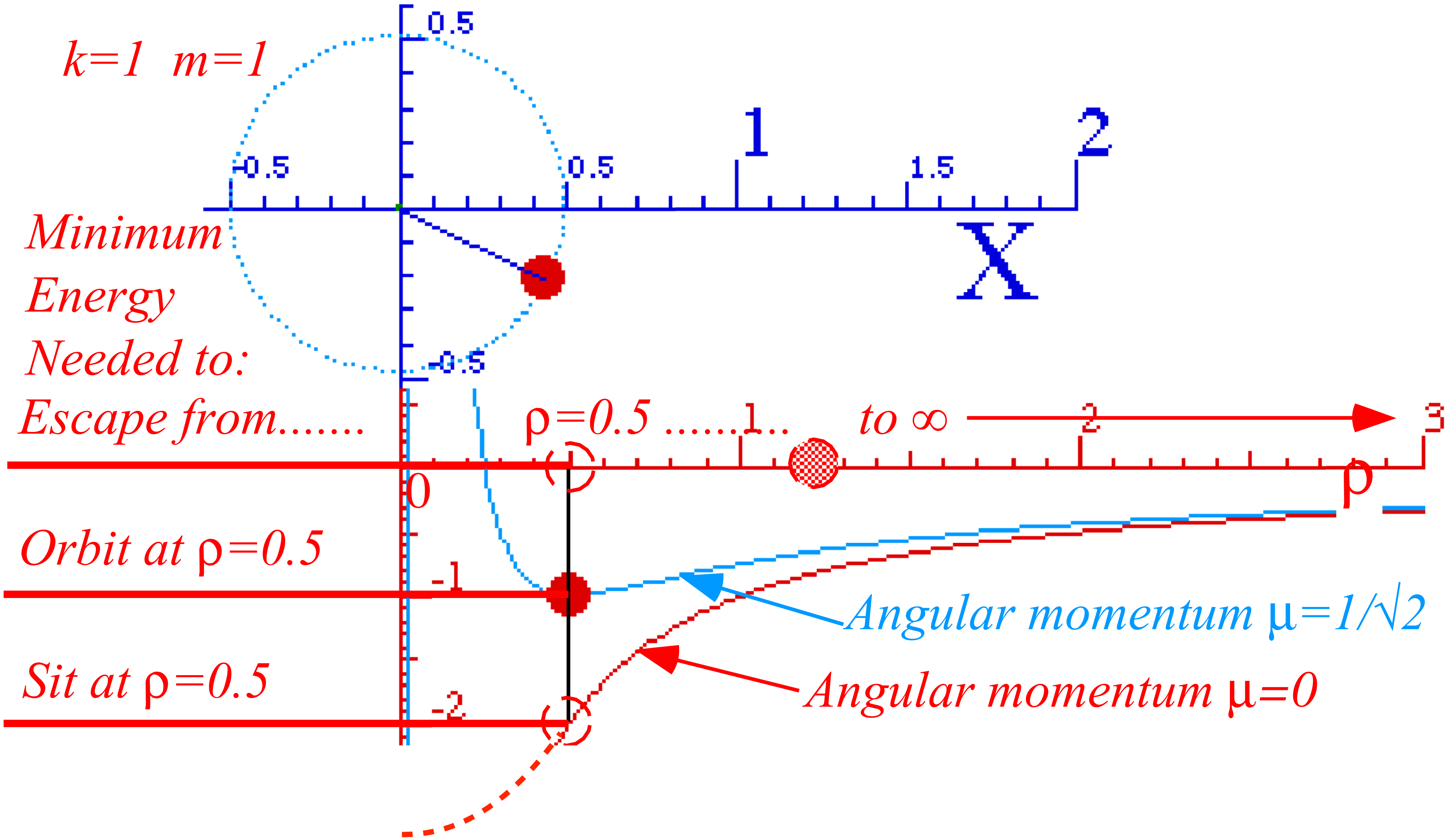
Perigee is faster turning point ρ_-

“effective” PE

“real” PE

$$E = \frac{p_r^2}{2M} + \overbrace{\underbrace{\frac{\ell^2}{2Mr^2}}_{\text{“centifugal-barrier” PE}} - \frac{\overbrace{k}^{\text{“real” PE}}}{r}}^{\text{“effective” PE}}$$

Coulomb orbits in polar coordinates and effective potential *(Web Simulation: OscillatorPE - Coulomb)*



k=1 m=1

Minimum Energy Needed to:

Escape from....... ρ=0.5 ......1... to ∞

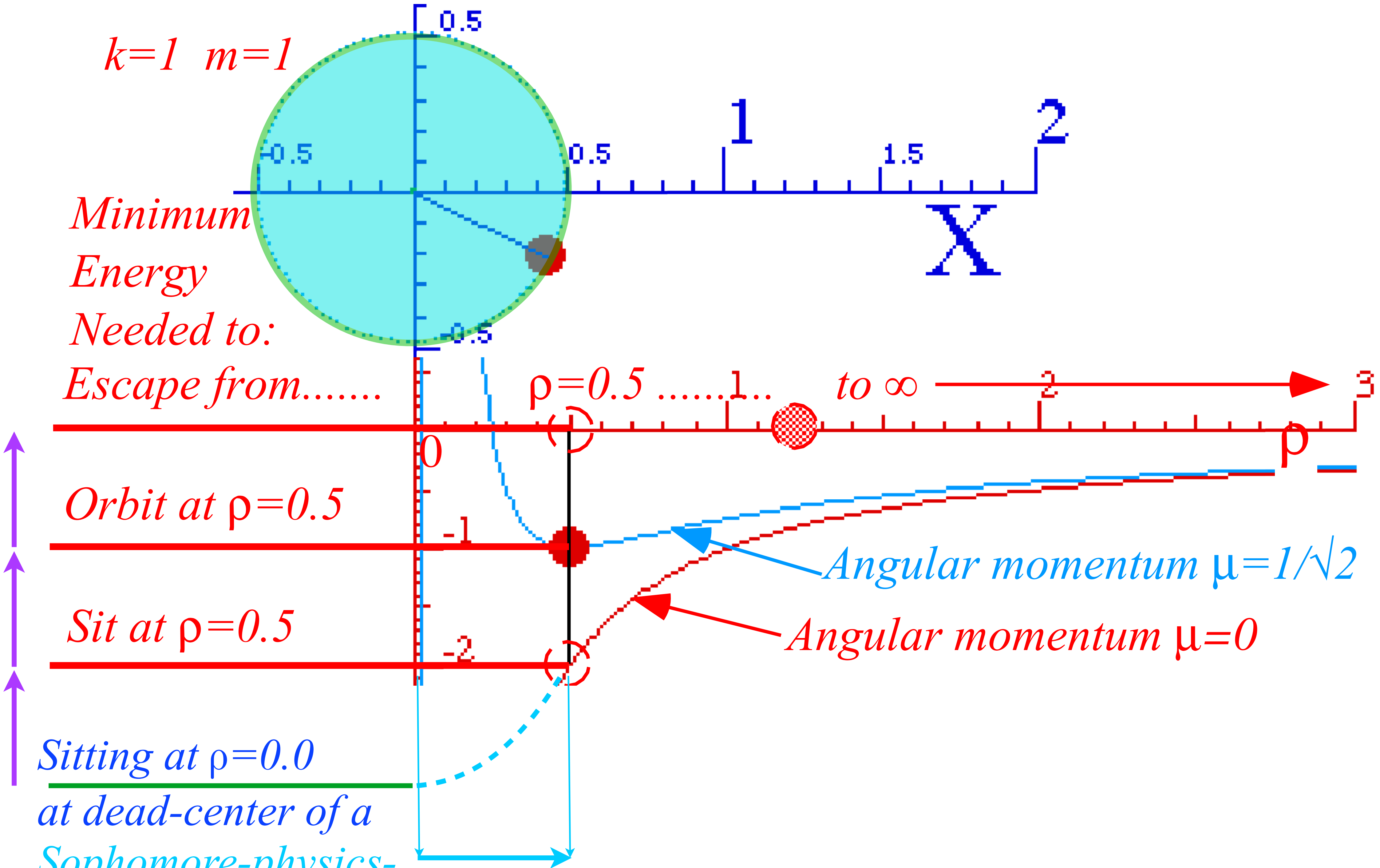
Orbit at ρ=0.5

Sit at ρ=0.5

Angular momentum μ=1/√2

Angular momentum μ=0

k=1 m=1



Minimum Energy Needed to:

Escape from....... ρ=0.5 ....1... to ∞

Orbit at ρ=0.5

Sit at ρ=0.5

The Three Equal Steps to ∞

Sitting at ρ=0.0
at dead-center of a
Sophomore-physics-
Earth of radius $R_{\oplus}$=0.5

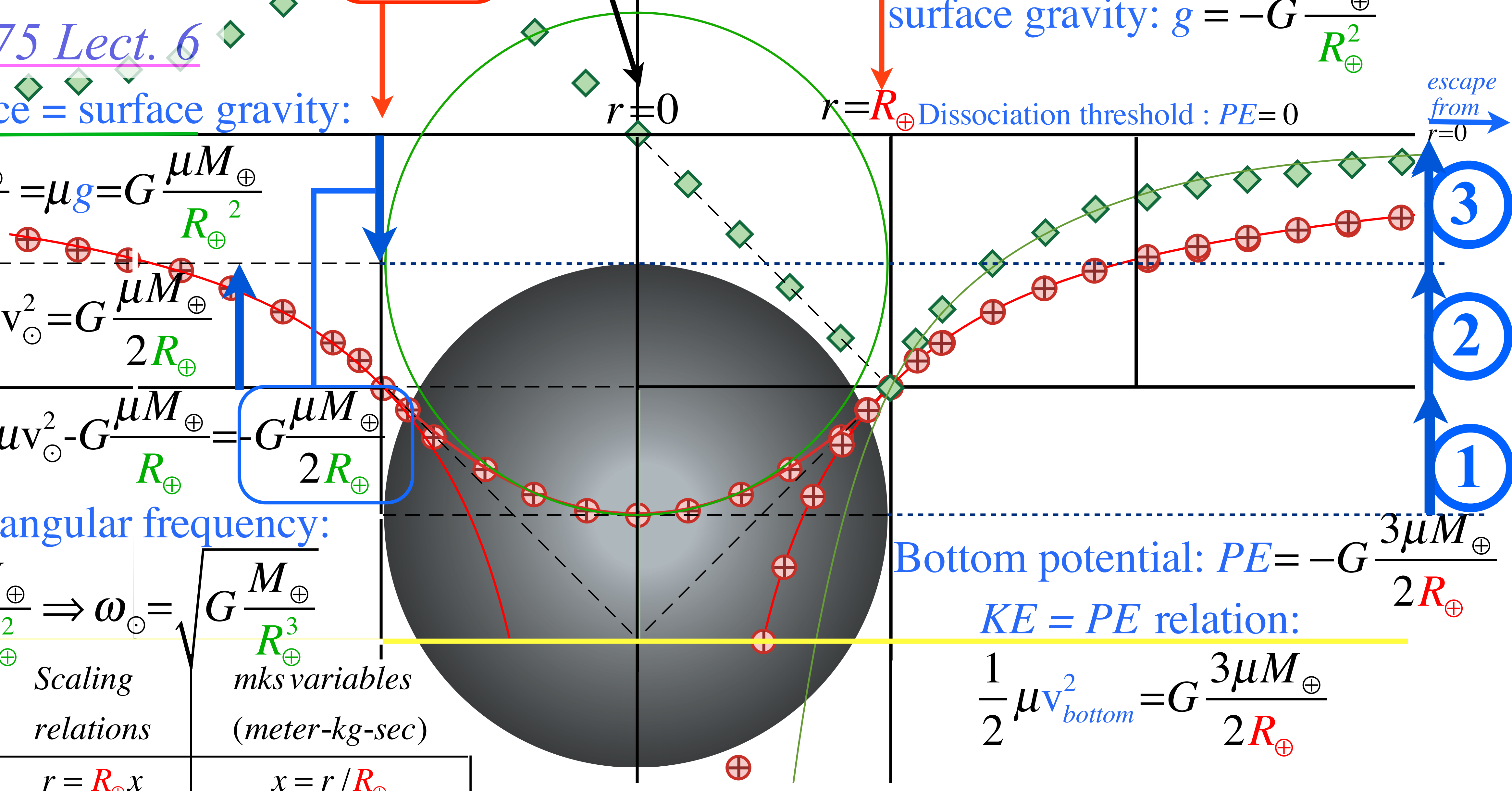
Angular momentum μ=1/√2

Angular momentum μ=0

From p. 74 Lect. 6, *on next page*

Sophomore-physics-Earth inside and out: "3-steps out of (or into) Hell" ...and surface orbit at $r=R_\oplus$

From p. 75 Lect. 6



surface gravity: $g=-G\frac{M_\oplus}{R_\oplus^2}$

Centifugal force = surface gravity:

$$\frac{\mu v_\odot^2}{R_\oplus}=\mu g=G\frac{\mu M_\oplus}{R_\oplus{}^2}$$

Orbit KE $=\frac{1}{2}\mu v_\odot^2=G\frac{\mu M_\oplus}{2R_\oplus}$

Orbit $E_\odot^{Total}=\frac{1}{2}\mu v_\odot^2-G\frac{\mu M_\oplus}{R_\oplus}=\boxed{-G\frac{\mu M_\oplus}{2R_\oplus}}$

$(r=R_\oplus)$-orbit angular frequency:

$$\omega_\odot^2 R_\oplus=G\frac{M_\oplus}{R_\oplus^2}\Rightarrow\omega_\odot=\sqrt{G\frac{M_\oplus}{R_\oplus^3}}$$

Bottom potential: $PE=-G\frac{3\mu M_\oplus}{2R_\oplus}$

$KE = PE$ relation:

$$\frac{1}{2}\mu v_{bottom}^2=G\frac{3\mu M_\oplus}{2R_\oplus}$$

Geometric (x,y) (Dimensionless)	Scaling relations	mks variables (meter-kg-sec)		
space coord.: x	$r=R_\oplus x$	$x=r/R_\oplus$		
PE for $\|x\|\geq 1$: $y^{PE}=\frac{-1}{x}$	$PE^{mks}(r)$ $=\frac{GM\mu}{R_\oplus}y^{PE}$	$PE^{mks}(r)=-\frac{GM\mu}{r}$ $=-\frac{GM\mu}{R_\oplus}\frac{1}{x}$	PE for $\|x\|<1$: $y^{PE}=\frac{x^2}{2}-\frac{3}{2}$	$PE^{mks}(r)=\frac{GM\mu}{R_\oplus}\left(\frac{r^2}{2R_\oplus^2}-\frac{3}{2}\right)$
$Force$ for $\|x\|\geq 1$: $y^{Force}=\frac{-1}{x^2}$	$F^{mks}(r)$ $=\frac{GM\mu}{R_\oplus^2}y^{Force}$	$F^{mks}(r)=-\frac{GM\mu}{r^2}$ $=-\frac{GM\mu}{R_\oplus^2}\frac{1}{x^2}$	$Force$ for $\|x\|<1$: $y^{Force}=-x$	$F^{mks}(r)=-\frac{GM\mu}{R_\oplus^3}r$

$(r=0)$-escape-velocity

$$v_{bottom}=\sqrt{3G\frac{M_\oplus}{R_\oplus}}$$

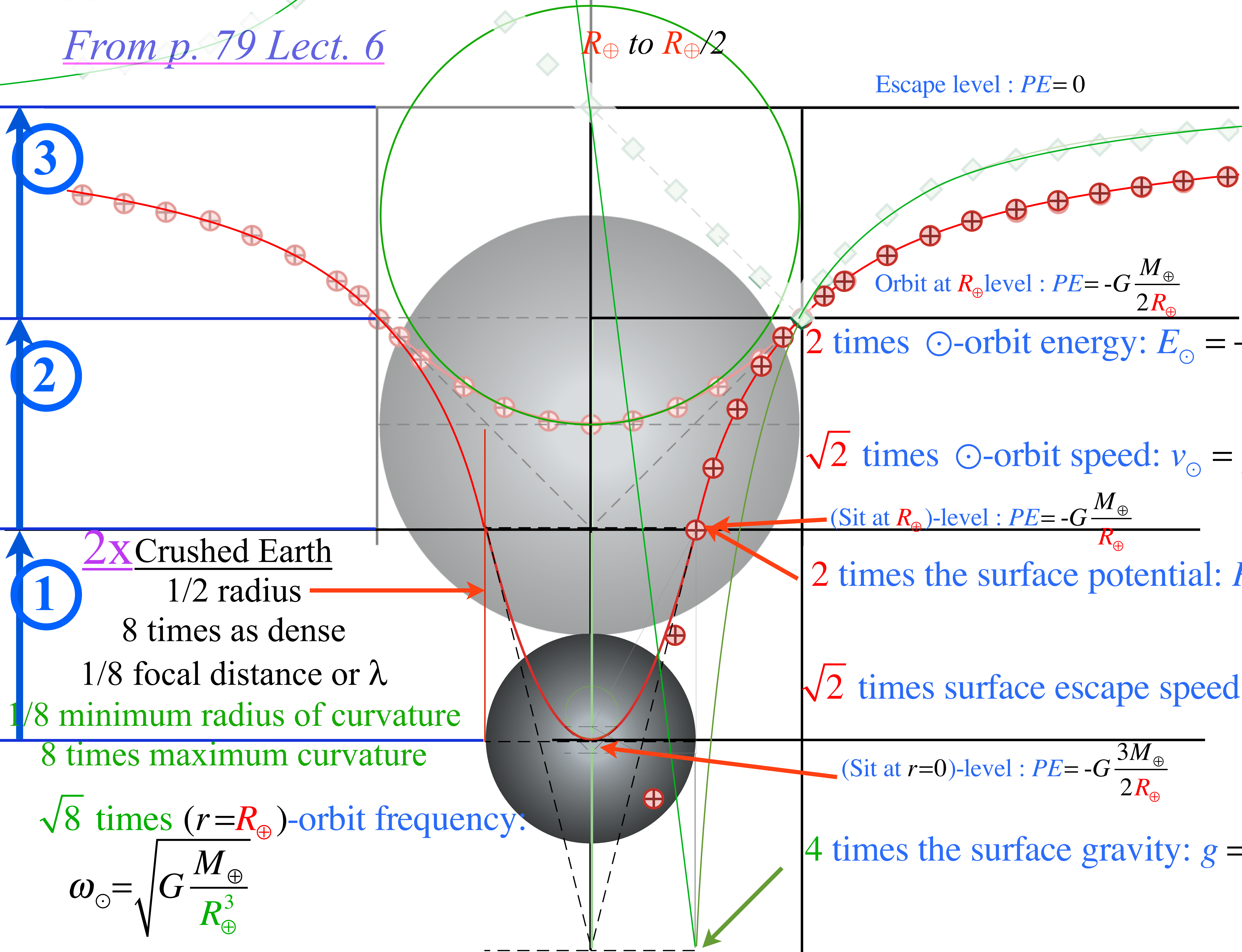
$(r=R_\oplus)$-escape velocity:

$$v_{escape}=\sqrt{2G\frac{M_\oplus}{R_\oplus}}$$

Sophomore-physics-Earth inside and out: "3-steps to Hell"

Suppose Earth radius crushed to 1/2: *($R_\oplus$=6.4·10^6m crushed to $R_\oplus$/2=3.2·10^6m)*

From p. 79 Lect. 6



$R_\oplus$ to $R_\oplus$/2

All formulas identical to ones derived on p.15 to 27. Imagine reducing $R_\oplus$ to $R_\oplus$/2

Escape level : $PE = 0$

Orbit at $R_\oplus$ level : $PE = -G\frac{M_\oplus}{2R_\oplus}$

2 times ⊙-orbit energy: $E_\odot = -G\frac{M_\oplus}{2R_\oplus}$

$\sqrt{2}$ times ⊙-orbit speed: $v_\odot = \sqrt{G\frac{M_\oplus}{R_\oplus}}$

(Sit at $R_\oplus$)-level : $PE = -G\frac{M_\oplus}{R_\oplus}$

2 times the surface potential: $PE = -G\frac{M_\oplus}{R_\oplus}$

$\sqrt{2}$ times surface escape speed: $v_e = \sqrt{G\frac{2M_\oplus}{R_\oplus}}$

(Sit at r=0)-level : $PE = -G\frac{3M_\oplus}{2R_\oplus}$

4 times the surface gravity: $g = -G\frac{M_\oplus}{R_\oplus^2}$

2x Crushed Earth

1/2 radius

8 times as dense

1/8 focal distance or λ

1/8 minimum radius of curvature

8 times maximum curvature

$\sqrt{8}$ times (r=$R_\oplus$)-orbit frequency:

$$\omega_\odot = \sqrt{G\frac{M_\oplus}{R_\oplus^3}}$$

Next Hamiltonian Lecture 11…

Examples of Hamiltonian mechanics in phase plots

1D Pendulum and phase plot (Web Simulations: Pendulum, Cycloidulum, JerkIt (Vertically Driven Pendulum))

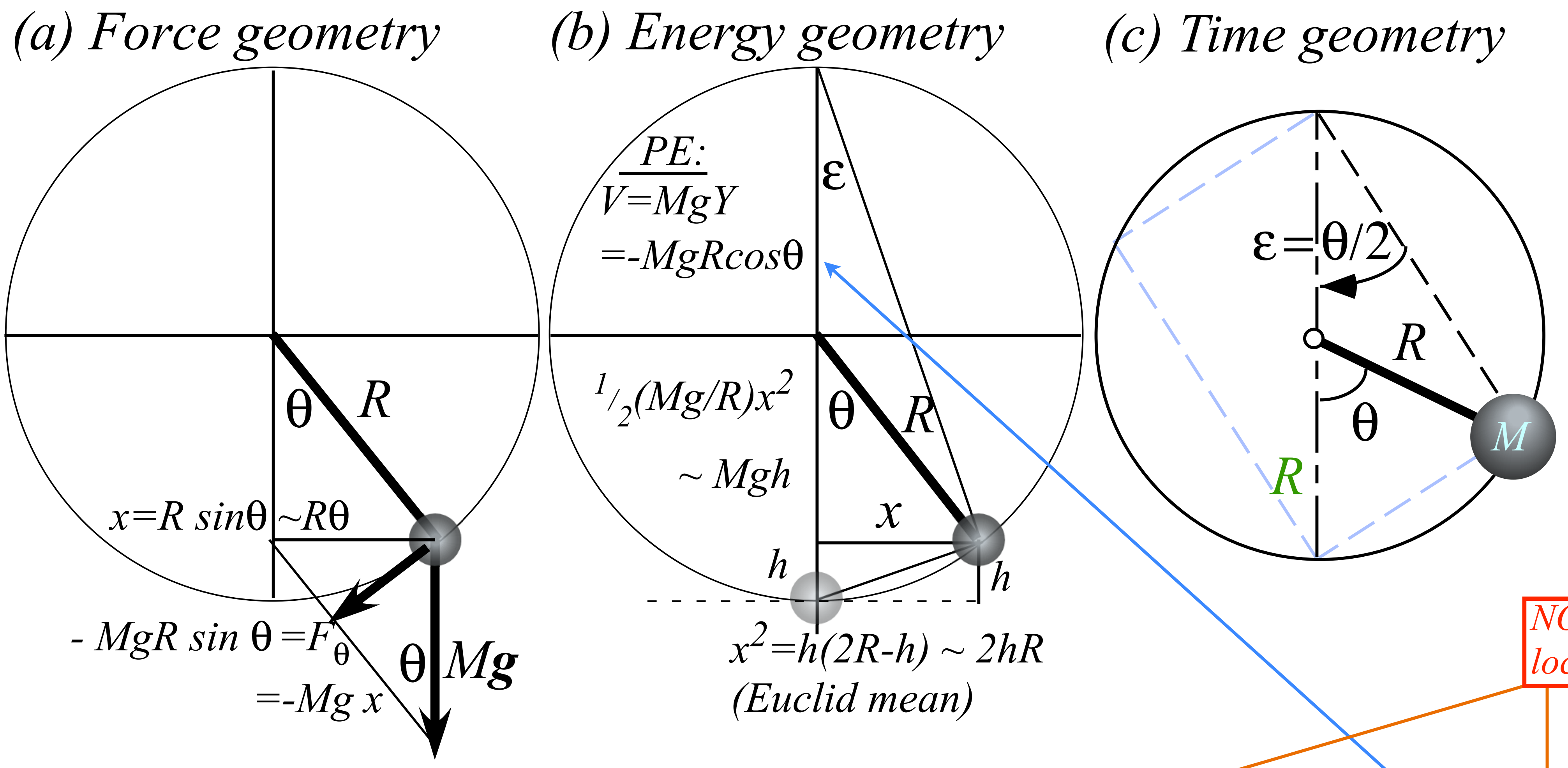
1D-HO phase-space control (Classic Simulation of "Catcher in the Eye", Web Simulation:JerkIt)

Web Simulation of atomic classical (or semi-classical) dynamics using varying phase control

Normally we'd stop here

Lecture 10 ends here
Mon. 9.24.2018

1D Pendulum and phase plot

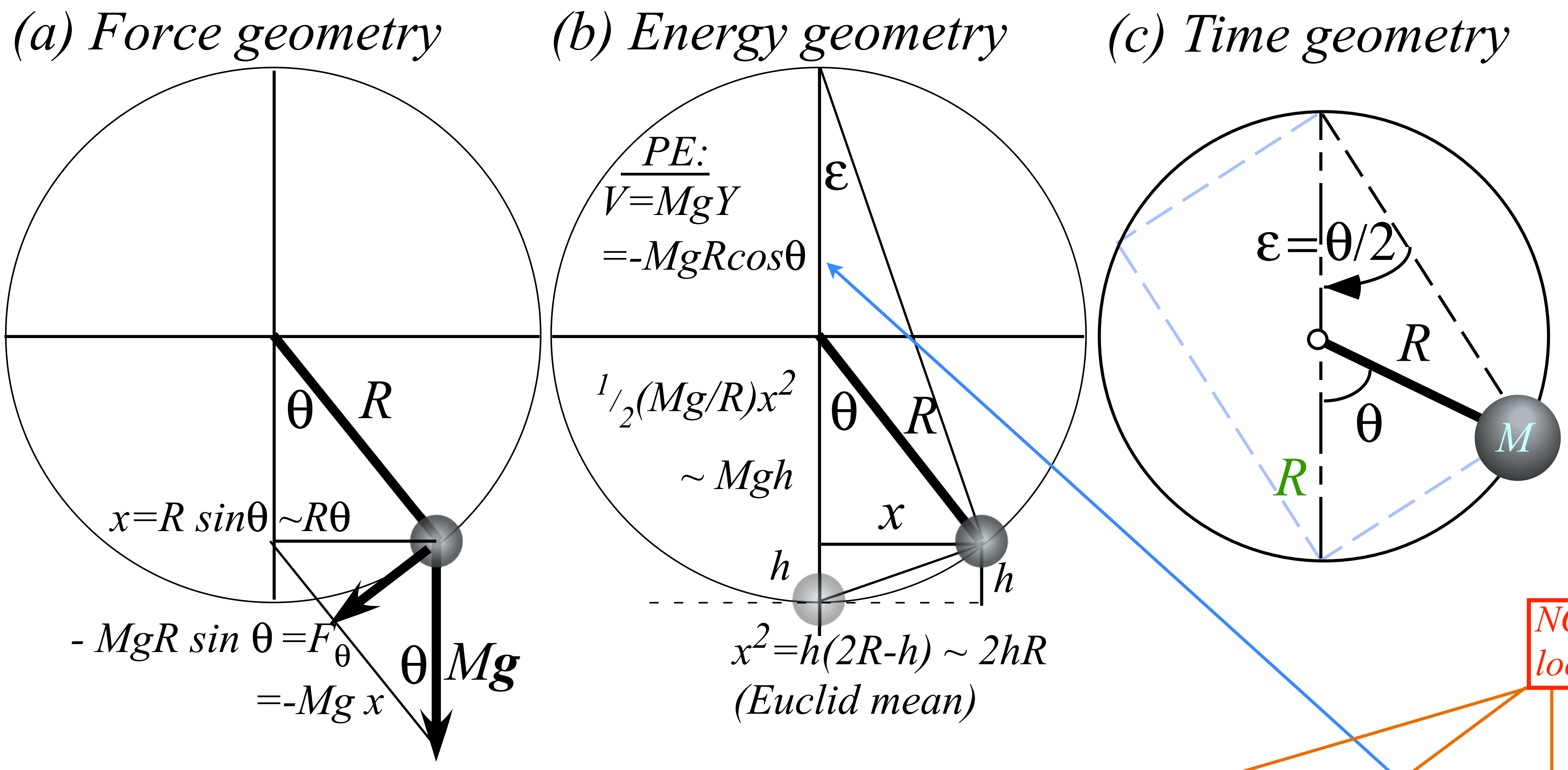


NOTE: Very common loci of ± sign blunders

Lagrangian function L= KE - PE = T - U where potential energy is U(θ) = $-MgR\cos\theta$

$$L(\dot{\theta},\theta)=\frac{1}{2}I\dot{\theta}^2-U(\theta)=\frac{1}{2}I\dot{\theta}^2+MgR\cos\theta$$

1D Pendulum and phase plot



NOTE: Very common loci of ± sign blunders

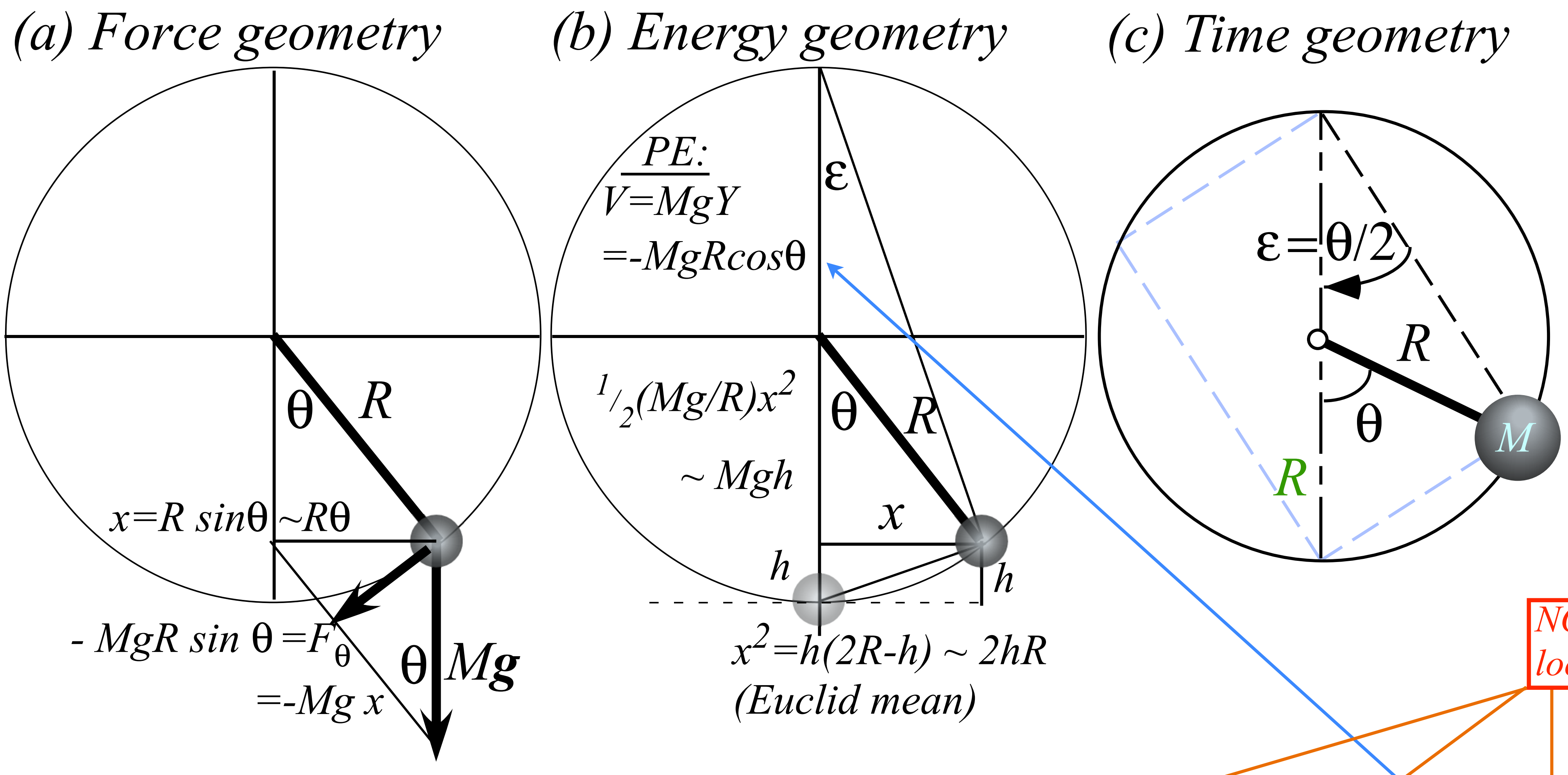
Lagrangian function L= KE - PE = T - U where potential energy is $U(\theta) = -MgR\cos\theta$

$$L(\dot{\theta},\theta) = \frac{1}{2}I\dot{\theta}^2 - U(\theta) = \frac{1}{2}I\dot{\theta}^2 + MgR\cos\theta$$

Hamiltonian function H= KE + PE = T +U where potential energy is $U(\theta) = -MgR\cos\theta$

$$H(p_\theta,\theta) = \frac{1}{2I}p_\theta{}^2 + U(\theta) = \frac{1}{2I}p_\theta{}^2 - MgR\cos\theta \ = E \quad =\text{const.}$$

1D Pendulum and phase plot



NOTE: Very common loci of ± sign blunders

Lagrangian function L= KE - PE = T - U where potential energy is U(θ) = $-MgR\cos\theta$

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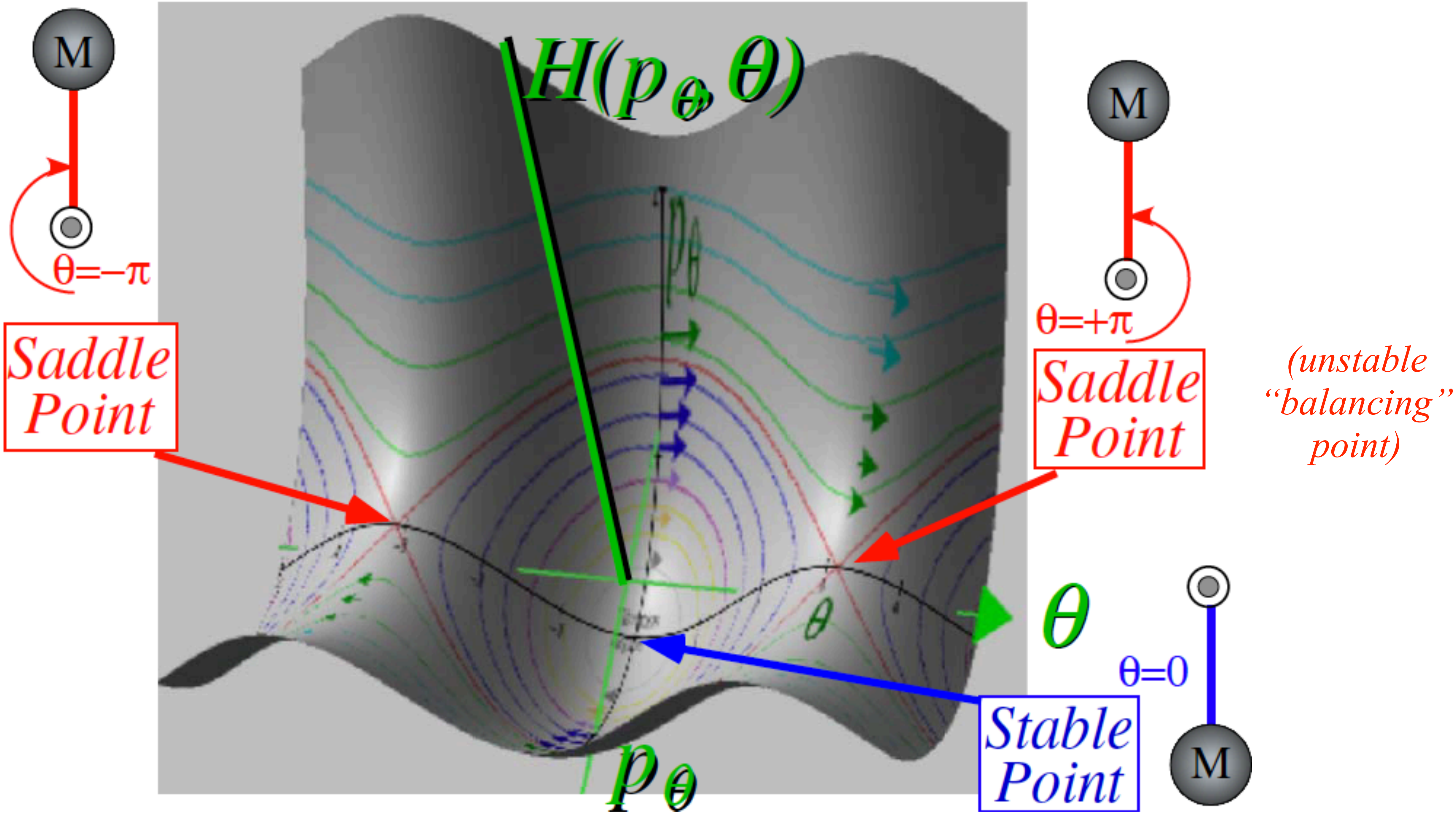
Hamiltonian function H= KE + PE = T +U where potential energy is U(θ) = $-MgR\cos\theta$

$$H(p_\theta,\theta)=\frac{1}{2I}p_\theta{}^2+U(\theta)=\frac{1}{2I}p_\theta{}^2-MgR\cos\theta \ = E \quad =\text{const.}$$

implies: $p_\theta=\sqrt{2I\left(E+MgR\cos\theta\right)}$



Saddle Point

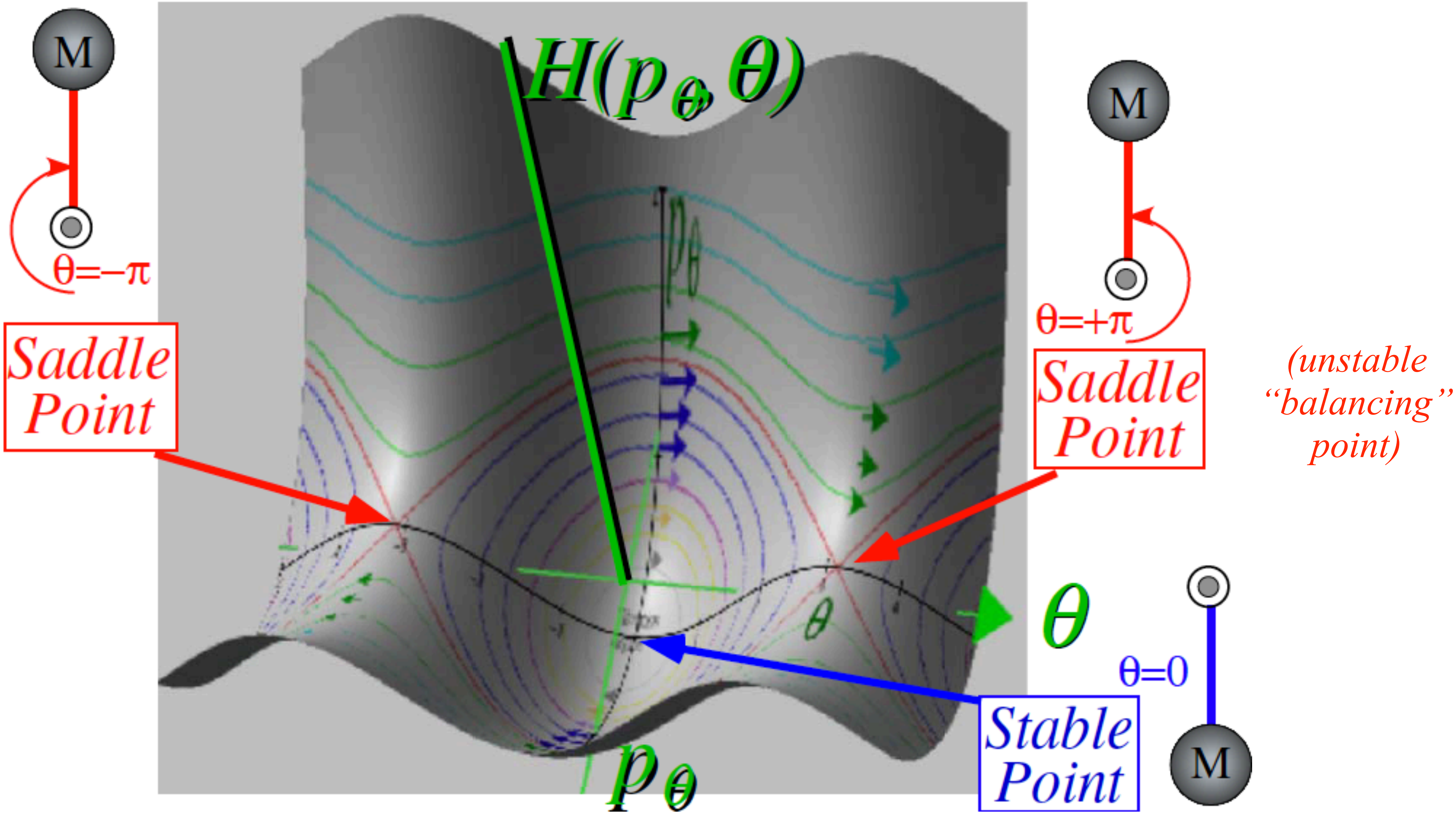
Saddle Point

(unstable "balancing" point)

Stable Point

Example of plot of Hamilton for 1D-solid pendulum in its Phase Space (θ,p_θ)

$$H(p_\theta,\theta)=E=\frac{1}{2I}p_\theta{}^2-MgR\cos\theta\ ,\quad \text{or:}\quad p_\theta=\sqrt{2I\left(E+MgR\cos\theta\right)}$$



Example of plot of Hamilton for 1D-solid pendulum in its Phase Space (θ,pθ)

$$H(p_\theta,\theta)=E=\frac{1}{2I}p_\theta{}^2-MgR\cos\theta\ ,\quad \text{or:}\quad p_\theta=\sqrt{2I\left(E+MgR\cos\theta\right)}$$

Funny way to look at Hamilton's equations:

$$\begin{pmatrix}\dot q\\ \dot p\end{pmatrix}=\begin{pmatrix}\partial_p H\\ -\partial_q H\end{pmatrix}=\mathbf{e_H}\times\left(-\nabla H\right)=\overrightarrow{\text{(H-axis)}}\times\overrightarrow{\text{(fall line)}},\text{ where:}\begin{cases}\overrightarrow{\text{(H-axis)}}=\mathbf{e_H}=\mathbf{e_q}\times\mathbf{e_p}\\ \overrightarrow{\text{(fall line)}}=-\nabla H\end{cases}$$

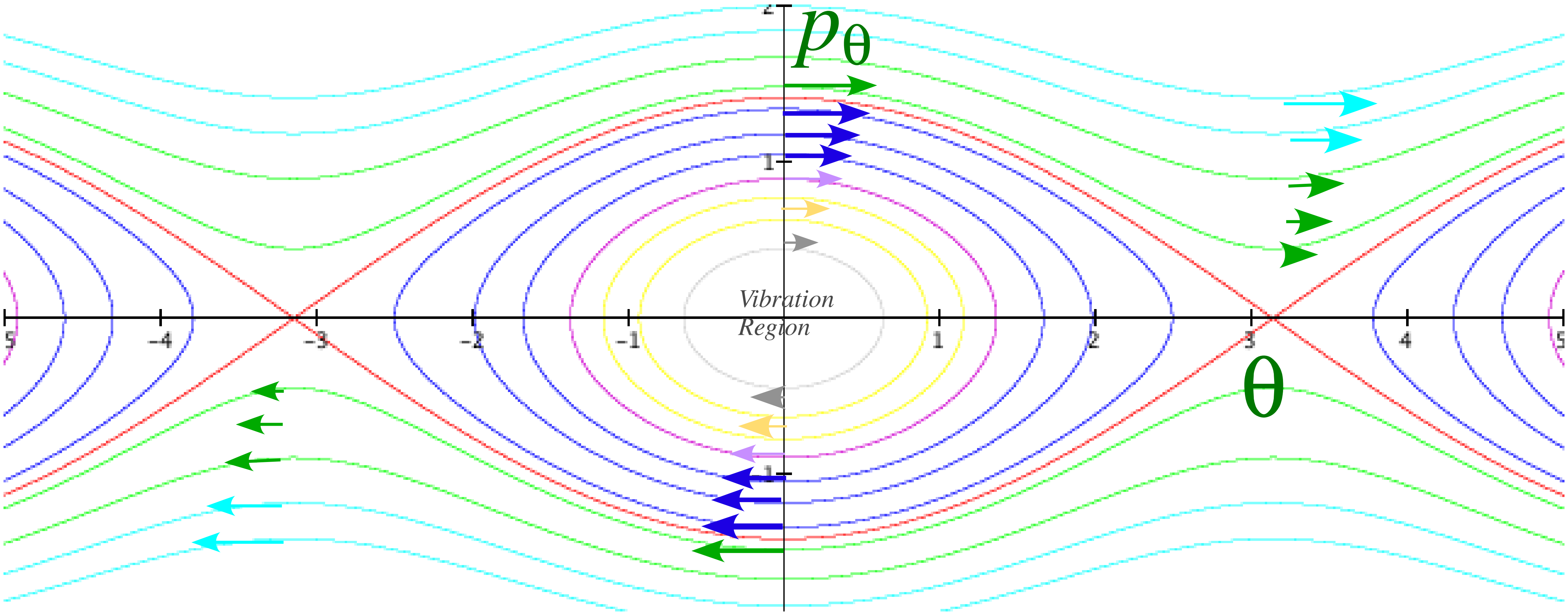
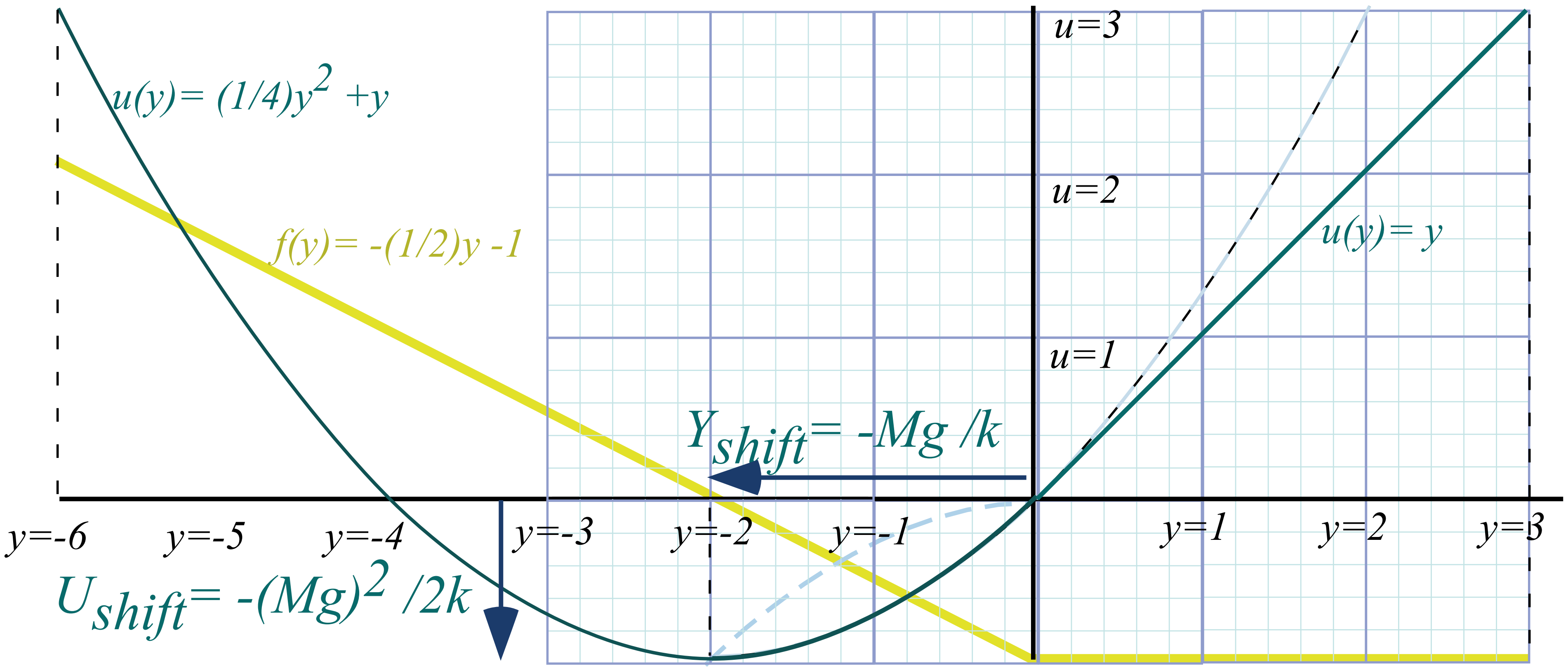


Fig. 2.7.2 Phase portrait or topography map for simple pendulum

(Unit 2 Chapter 7 Fig. 2)

$F(Y)=-kY-Mg$ $U(Y)=(1/2)kY^2+Mg\,Y$



Unit 1
Fig. 7.4

Web Simulation of atomic classical (or semi-classical) dynamics using varying phase control