Lecture 12
Mon. 10.01.2018

Complex Variables, Series, and Field Coordinates I.

(Ch. 10 of Unit 1)

1. The Story of e (A Tale of Great $Interest$)
  How good are those power series?
  Taylor-Maclaurin series, imaginary interest, and complex exponentials
2. What good are complex exponentials?
  Easy trig
  Easy 2D vector analysis
  Easy oscillator phase analysis
  Easy rotation and "dot" or "cross" products
3. Easy 2D vector calculus
  Easy 2D vector derivatives
  Easy 2D source-free field theory
  Easy 2D vector field-potential theory
4. Riemann-Cauchy relations (What's analytic? What's not?)
  Easy 2D curvilinear coordinate discovery
  Easy 2D circulation and flux integrals
  Easy 2D monopole, dipole, and 2^n-pole analysis
  Easy 2^n-multipole field and potential expansion
  Easy stereo-projection visualization
  Cauchy integrals, Laurent-Maclaurin series

Lect. 12
ends here

Lecture 14 Tue. 10.15
starts here

What good are complex quantities?

1. Complex numbers provide "automatic trigonometry"

2. Complex numbers add like vectors.

3. Complex exponentials $Ae^{-i\omega t}$ track position and velocity using Phasor Clock.

4. Complex products provide 2D rotation operations.

5. Complex products provide 2D "dot"(•) and "cross"(x) products.

6. Complex derivative contains "divergence"($\nabla\cdot\mathbf{F}$) and "curl"($\nabla\times\mathbf{F}$) of 2D vector field

7. Invent source-free 2D vector fields [$\nabla\cdot\mathbf{F}=0$ and $\nabla\times\mathbf{F}=0$]

8. Complex potential ϕ contains "scalar"($\mathbf{F}=\nabla\Phi$) and "vector"($\mathbf{F}=\nabla\times\mathbf{A}$) potentials
The half-n'-half results: (Riemann-Cauchy Derivative Relations)

9. Complex potentials define 2D Orthogonal Curvilinear Coordinates (OCC) of field

10. Complex integrals $\int f(z)dz$ count 2D "circulation"($\int\mathbf{F}\cdot d\mathbf{r}$) and "flux"($\int\mathbf{F}\times d\mathbf{r}$)

11. Complex integrals define 2D monopole fields and potentials

12. Complex derivatives give 2D dipole fields

Lecture 15 Thur. 10.17
starts here

13. More derivatives give 2D 2^N-pole fields…

14. …and 2^N-pole multipole expansions of fields and potentials...

15. ...and Laurent Series...

16. Mapping and non-analytic source analysis.

A running collection of links to course-relevant sites and articles

Physics Web Resources

Comprehensive Harter-Soft Resource Listing

UAF Physics YouTube channel

LearnIt Physics Web Applications

"Texts"

Classical Mechanics with a Bang!

Quantum Theory for the Computer Age

Principles of Symmetry, Dynamics, and Spectroscopy

Modern Physics and its Classical Foundations

Classes

2014 AMOP

2017 Group Theory for QM

2018 AMOP

2018 Adv Mechanics

Neat external material to start the class:
- AIP publications
- AJP article on superball dynamics
- AAPT summer reading

These *are* hot off the presses:
- Sorting ultracold atoms in a three-dimensional optical lattice in a realization of Maxwell's demon - Kumar-Nature-Letters-2018
- Synthetic three-dimensional atomic structures assembled atom by atom - Berredo-Nature-Letters-2018

Slightly Older ones:
- Wave–particle duality of C60 molecules
- Optical vortex knots – One Photon at a Time

"*Relawavity*" and quantum basis of *Lagrangian* & *Hamiltonian* mechanics:
- 2-CW laser wave - BohrIt Web App
- Lagrangian vs Hamiltonian - RelaWavity Web App

- AMOP Ch 0 Space-Time Symmetry - 2019
- Seminar at Rochester Institute of Optics, Auxiliary slides, June 19, 2018

The Story of e (A Tale of Great $Interest$)

British spelling: *intrest*

Simple *interest* at some rate r based on a 1 year period.

You gave a principal $p(0)$ to the bank and some time t later they would pay you $p(t)=(1+r\cdot t)p(0)$.

\$1.00 at rate $r=1$ (like Israel and Brazil that once had 100% interest.) gives \$2.00 at $t=1$ year.

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$$p^{\frac{1}{2}}(t)=(1+r\cdot\tfrac{t}{2})p(\tfrac{t}{2})=(1+r\cdot\tfrac{t}{2})\cdot(1+r\cdot\tfrac{t}{2})p(0)=\tfrac{3}{2}\cdot\tfrac{3}{2}\cdot 1=\tfrac{9}{4}=2.25$$

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Trimester compounded interest gives $p(\tfrac{t}{3})=(1+r\cdot\tfrac{t}{3})p(0)$ at the *1/3*rd-period $\tfrac{t}{3}$ or 1st trimester and then use that to figure the 2nd trimester and so on. Now \$1.00 at rate $r=1$ earns \$2.37.

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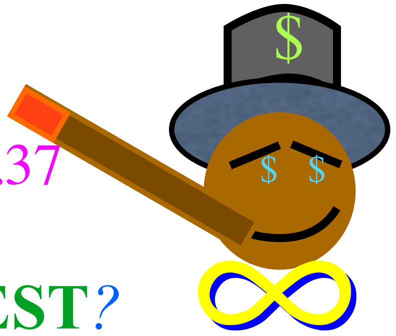

So if you compound interest more and more frequently, do you approach **INFININTEREST***?*

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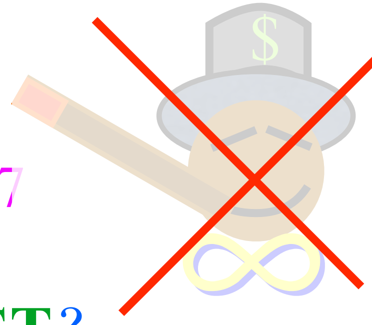

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$$p^{\frac{1}{1}}(t)=(1+r\cdot\frac{t}{1})^{1}p(0)=\left(\frac{2}{1}\right)^{1}\cdot 1=\frac{2}{1}=2.00$$

+25¢

$$p^{\frac{1}{2}}(t)=(1+r\cdot\frac{t}{2})^{2}p(0)=\left(\frac{3}{2}\right)^{2}\cdot 1=\frac{9}{4}=2.25$$

+12¢

$$p^{\frac{1}{3}}(t)=(1+r\cdot\frac{t}{3})^{3}p(0)=\left(\frac{4}{3}\right)^{3}\cdot 1=\frac{64}{27}=2.37$$

+7¢

$$p^{\frac{1}{4}}(t)=(1+r\cdot\frac{t}{4})^{4}p(0)=\left(\frac{5}{4}\right)^{4}\cdot 1=\frac{625}{256}=2.44$$

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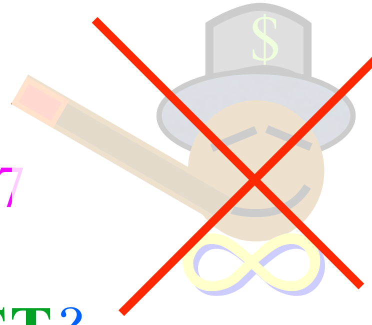

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$$p^{\frac{1}{4}}(t) = (1+r\cdot\tfrac{t}{4})^4 p(0) = \left(\tfrac{5}{4}\right)^4\cdot 1 = \tfrac{625}{256} = 2.44$$

+17¢

Monthly: $p^{\frac{1}{12}}(t) = (1+r\cdot\tfrac{t}{12})^{12} p(0) = \left(\tfrac{13}{12}\right)^{12}\cdot 1 = 2.613$

+8¢

Weekly: $p^{\frac{1}{52}}(t) = (1+r\cdot\tfrac{t}{52})^{52} p(0) = \left(\tfrac{53}{52}\right)^{52}\cdot 1 = 2.693$

+2¢

Daily: $p^{\frac{1}{365}}(t) = (1+r\cdot\tfrac{t}{365})^{365} p(0) = \left(\tfrac{366}{365}\right)^{365}\cdot 1 = 2.7145$

+zip¢

Hrly: $p^{\frac{1}{8760}}(t) = (1+r\cdot\tfrac{t}{8760})^{8760} p(0) = \left(\tfrac{8761}{8760}\right)^{8760}\cdot 1 = 2.7181$

Interest product formula is really inefficient: 10^6 products for 6-figures! .. .10^9 products for 9 ...

$$p^{1/m}(1) = (1+\tfrac{1}{m})^m \xrightarrow[m\to\infty]{} \mathbf{2.718281828459..} = e$$

Let: $m\cdot r\cdot t=n$

or: $1/m= r\cdot t/n$

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$p^{1/m}(1) = \mathbf{2.718}1459268$ for $m = 10{,}000$

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$p^{1/m}(1) = \mathbf{2.71828}04693$ for $m = 1{,}000{,}000$

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Can improve computational efficiency using binomial theorem:

$$(x+y)^n = x^n + n\cdot x^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \ldots + n\cdot xy^{n-1} + y^n$$

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Define: *Factorials*(!):

$0! = 1 = 1!$, $2! = 1\cdot 2$, $3! = 1\cdot 2\cdot 3, \ldots$

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As $n\to\infty$ let :

$$n(n-1)\to n^2,$$

$$n(n-1)(n-2)\to n^3, etc.$$

$$\boxed{e^{r\cdot t} = 1 + r\cdot t + \frac{1}{2!}(r\cdot t)^2 + \frac{1}{3!}(r\cdot t)^3 + \ldots = \sum_{p=0}^{o}\frac{(r\cdot t)^p}{p!}}$$

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Precision order:

(o=1)-e-series = **2**.00000 =1+1

(o=2)-e-series = **2**.50000 =1+1+1/2

(o=3)-e-series = **2**.66667 =1+1+1/2+1/6

(o=4)-e-series = **2.7**0833 =1+1+1/2+1/6+1/24

(o=5)-e-series = **2.71**667 =1+1+1/2+1/6+1/24+1/120

(o=6)-e-series = **2.718**05 =1+1+1/2+1/6+1/24+1/120+1/720

(o=7)-e-series = **2.7182**5

(o=8)-e-series = **2.71828**

About 12 summed quotients
for 6-figure precision (A lot better!)

Power Series Good! Need general power series development

Start with a general power series with constant coefficients c_0, c_1, *etc.*

Set $t=0$ to get $c_0 = x(0)$.

$$x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 + \ldots + c_n t^n +$$

Power Series Good! *Need general power series development*

Start with a general power series with constant coefficients c_0, c_1, *etc.*

Set $t=0$ to get $c_0 = x(0)$.

$$x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 + \ldots + c_n t^n +$$

Rate of change of position $x(t)$ is *velocity* $v(t)$.

Set $t=0$ to get $c_1 = v(0)$.

$$v(t) = \frac{d}{dt} x(t) = 0 + c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + 5c_5 t^4 + \ldots + nc_n t^{n-1} +$$

Power Series Good! Need general power series development

Start with a general power series with constant coefficients *c_0, c_1, etc.*

Set $t=0$ to get $c_0 = x(0)$.

$$x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 + \ldots + c_n t^n +$$

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Change of velocity $v(t)$ is *acceleration $a(t)$*.

Set $t=0$ to get $c_2 = \frac{1}{2} a(0)$.

$$a(t) = \frac{d}{dt} v(t) = 0 + 2c_2 + 2{\cdot}3c_3 t + 3{\cdot}4c_4 t^2 + 4{\cdot}5c_5 t^3 + \ldots + n(n-1)c_n t^{n-2} +$$

Power Series Good! *Need general power series development*

Start with a general power series with constant coefficients c_0, c_1, *etc.*

Set $t=0$ to get $c_0 = x(0)$.

$$x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 + \ldots + c_n t^n +$$

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$$v(t) = \frac{d}{dt} x(t) = 0 + c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + 5c_5 t^4 + \ldots + nc_n t^{n-1} +$$

Change of velocity *v(t)* is *acceleration a(t)*.

Set $t=0$ to get $c_2 = \frac{1}{2} a(0)$.

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Change of acceleration *a(t)* is *jerk j(t)*. (*Jerk* is NASA term.)

Set $t=0$ to get $c_3 = \frac{1}{3!} j(0)$.

$$j(t) = \frac{d}{dt} a(t) = 0 + 2\cdot 3c_3 + 2\cdot 3\cdot 4c_4 t + 3\cdot 4\cdot 5c_5 t^2 + \ldots + n(n-1)(n-2)c_n t^{n-3} +$$

Power Series Good! *Need general power series development*

Start with a general power series with constant coefficients c_0, c_1, *etc.*

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Set $t=0$ to get $c_3 = \frac{1}{3!} j(0)$.

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Change of jerk *j(t)* is *inauguration i(t)*. (Be silly like NASA!)

Set $t=0$ to get $c_4 = \frac{1}{4!} i(0)$.

$$i(t) = \frac{d}{dt} j(t) = 0 + 2\cdot3\cdot4c_4 + 2\cdot3\cdot4\cdot5c_5 t + \ldots + n(n-1)(n-2)(n-3)c_n t^{n-4} +$$

Power Series Good! *Need general power series development*

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$$x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 + \ldots + c_n t^n +$$

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$$v(t) = \frac{d}{dt} x(t) = 0 + c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + 5c_5 t^4 + \ldots + nc_n t^{n-1} +$$

Change of velocity $v(t)$ is *acceleration $a(t)$*. Set $t=0$ to get $c_2 = \frac{1}{2} a(0)$.

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Gives Maclaurin (or Taylor) power series

$$\boxed{x(t) = x(0) + v(0)t + \frac{1}{2!} a(0)t^2 + \frac{1}{3!} j(0)t^3 + \frac{1}{4!} i(0)t^4 + \frac{1}{5!} r(0)t^5 + \ldots + \frac{1}{n!} x^{(n)} t^n +}$$

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Start with a general power series with constant coefficients c_0, c_1, *etc.*

Set $t=0$ to get $c_0 = x(0)$.

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Set $t=0$ to get $c_1 = v(0)$.

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Set $t=0$ to get $c_2 = \tfrac{1}{2} a(0)$.

$$a(t) = \frac{d}{dt} v(t) = 0 + 2c_2 + 2\cdot 3c_3 t + 3\cdot 4c_4 t^2 + 4\cdot 5c_5 t^3 + \ldots + n(n-1)c_n t^{n-2} +$$

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Change of jerk $j(t)$ is *inauguration $i(t)$*. (Be silly like NASA!)

Set $t=0$ to get $c_4 = \tfrac{1}{4!} i(0)$.

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Gives Maclaurin (or Taylor) power series

$$\boxed{\boxed{x(t) = x(0) + v(0)t + \tfrac{1}{2!} a(0)t^2} + \tfrac{1}{3!} j(0)t^3 + \tfrac{1}{4!} i(0)t^4 + \tfrac{1}{5!} r(0)t^5 + \ldots + \tfrac{1}{n!} x^{(n)} t^n +}$$

Good old UP I formula!

Power Series Good! *Need general power series development*

Start with a general power series with constant coefficients c_0, c_1, *etc.*

Set $t=0$ to get $c_0 = x(0)$.

$$x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 + \ldots + c_n t^n +$$

Rate of change of position $x(t)$ is *velocity $v(t)$*.

Set $t=0$ to get $c_1 = v(0)$.

$$v(t) = \frac{d}{dt} x(t) = 0 + c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + 5c_5 t^4 + \ldots + nc_n t^{n-1} +$$

Change of velocity $v(t)$ is *acceleration $a(t)$*.

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$$a(t) = \frac{d}{dt} v(t) = 0 + 2c_2 + 2\cdot 3c_3 t + 3\cdot 4c_4 t^2 + 4\cdot 5c_5 t^3 + \ldots + n(n-1)c_n t^{n-2} +$$

Change of acceleration $a(t)$ is *jerk $j(t)$*. (*Jerk* is NASA term.)

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Change of jerk $j(t)$ is *inauguration $i(t)$*. (Be silly like NASA!)

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Gives Maclaurin (or Taylor) power series

$$\boxed{\boxed{x(t) = x(0) + v(0)t + \frac{1}{2!} a(0)t^2} + \frac{1}{3!} j(0)t^3 + \frac{1}{4!} i(0)t^4 + \frac{1}{5!} r(0)t^5 + \ldots + \frac{1}{n!} x^{(n)} t^n +}$$

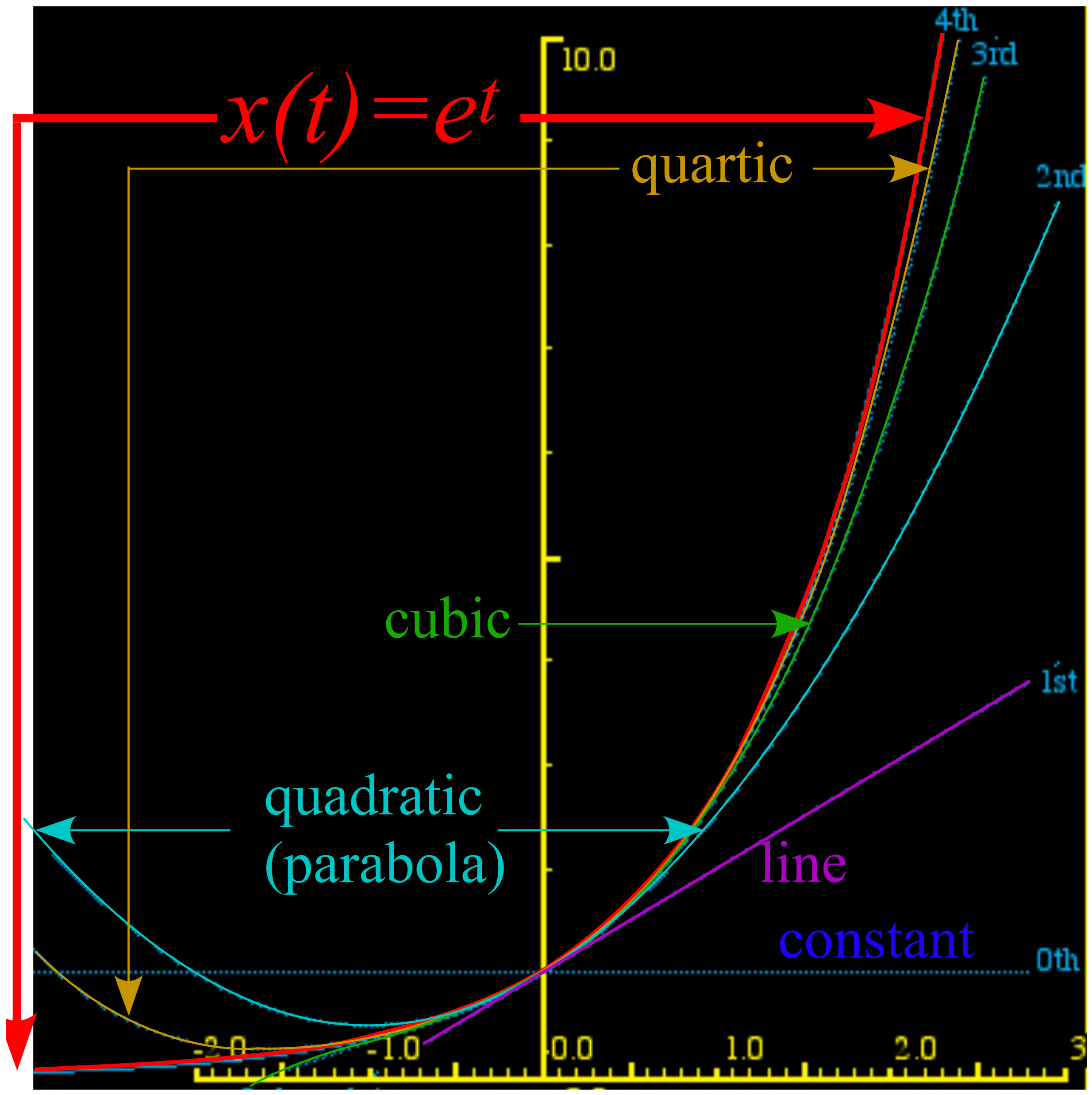
Setting all initial values to $1 = x(0) = v(0) = a(0) = j(0) = i(0) = \ldots.$

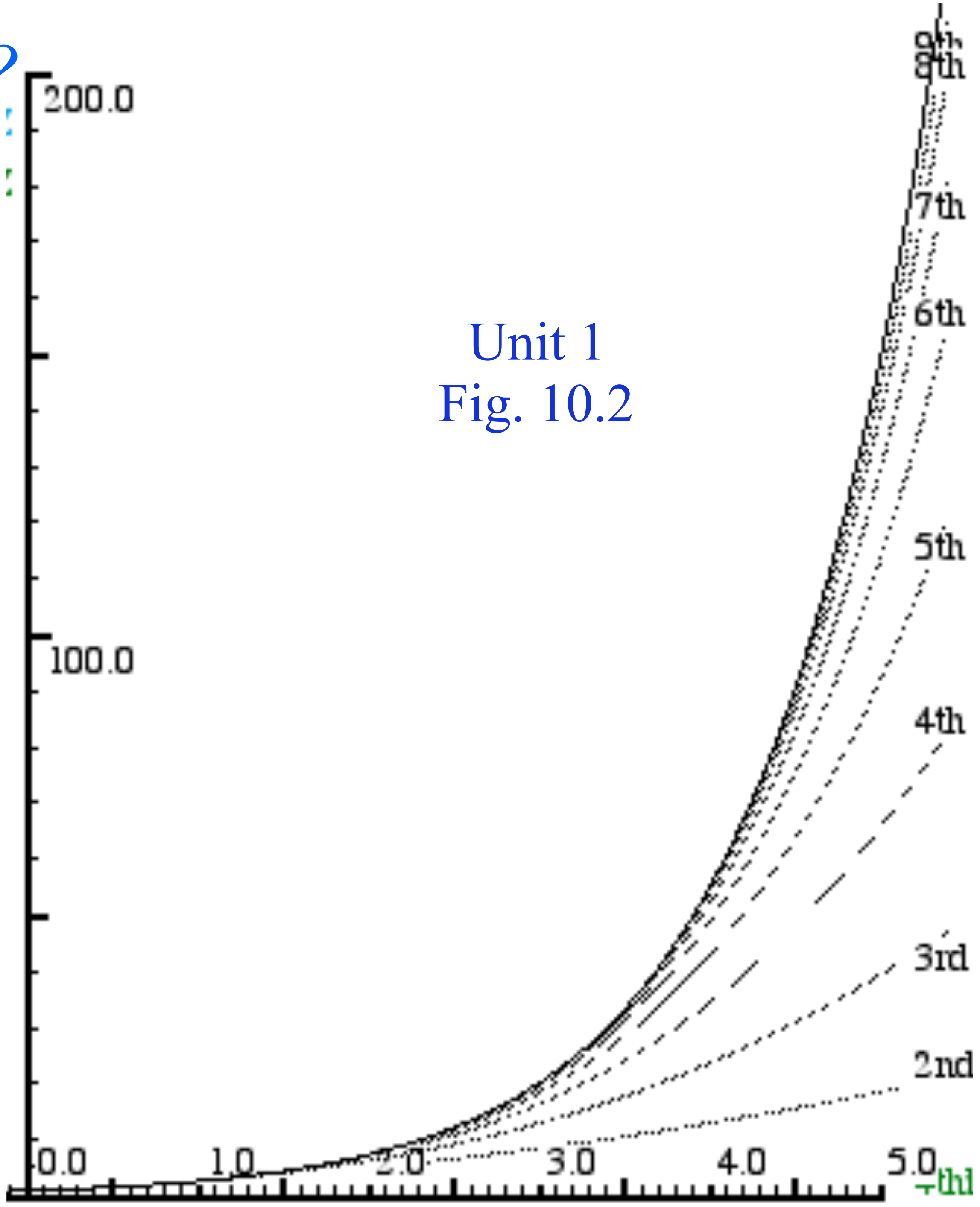
Good old UP I formula!

gives *exponential:*

$$e^t = 1 + t + \frac{1}{2!} t^2 + \frac{1}{3!} t^3 + \frac{1}{4!} t^4 + \frac{1}{5!} t^5 + \ldots + \frac{1}{n!} t^n +$$

But, how good are power series?





Unit 1
Fig. 10.2

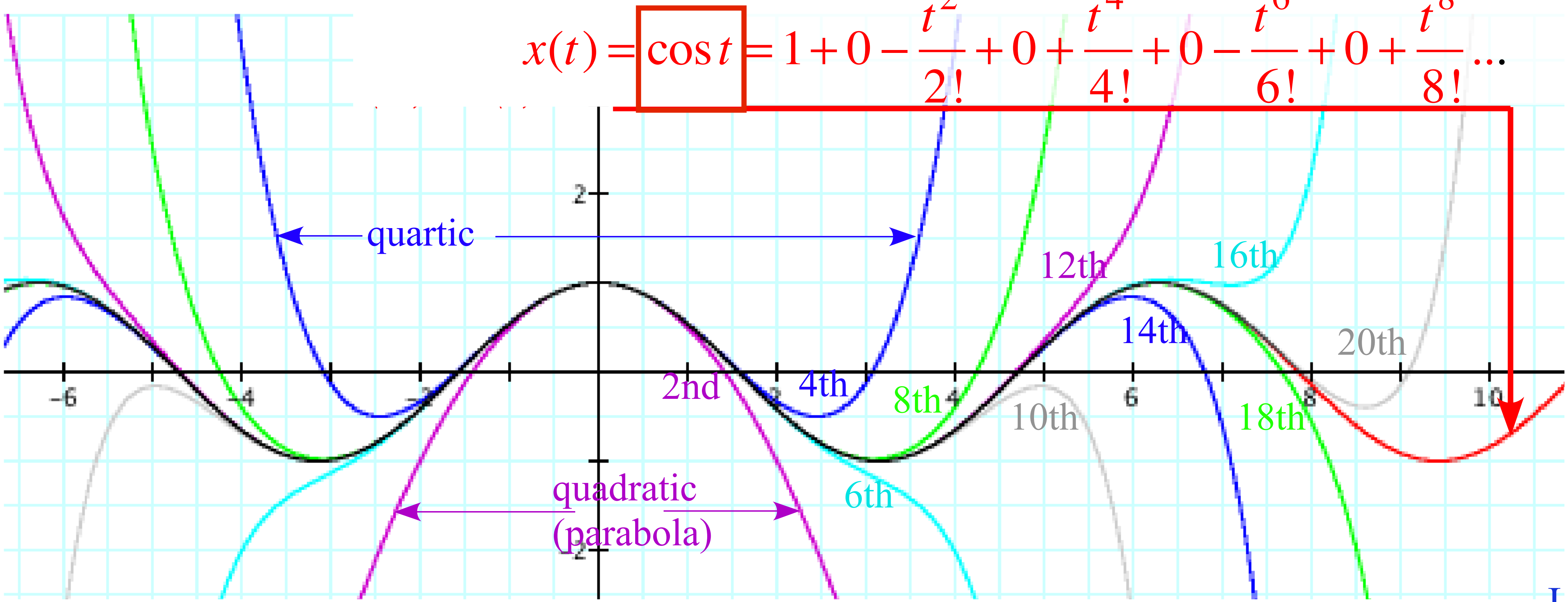
Gives Maclaurin (or Taylor) power series

$$x(t)=x(0)+v(0)t+\frac{1}{2!}a(0)t^2+\frac{1}{3!}j(0)t^3+\frac{1}{4!}i(0)t^4+\frac{1}{5!}r(0)t^5+\ldots+\frac{1}{n!}x^{(n)}t^n+$$

Setting all initial values to $1=x(0)=v(0)=a(0)=j(0)=i(0)=\ldots.$

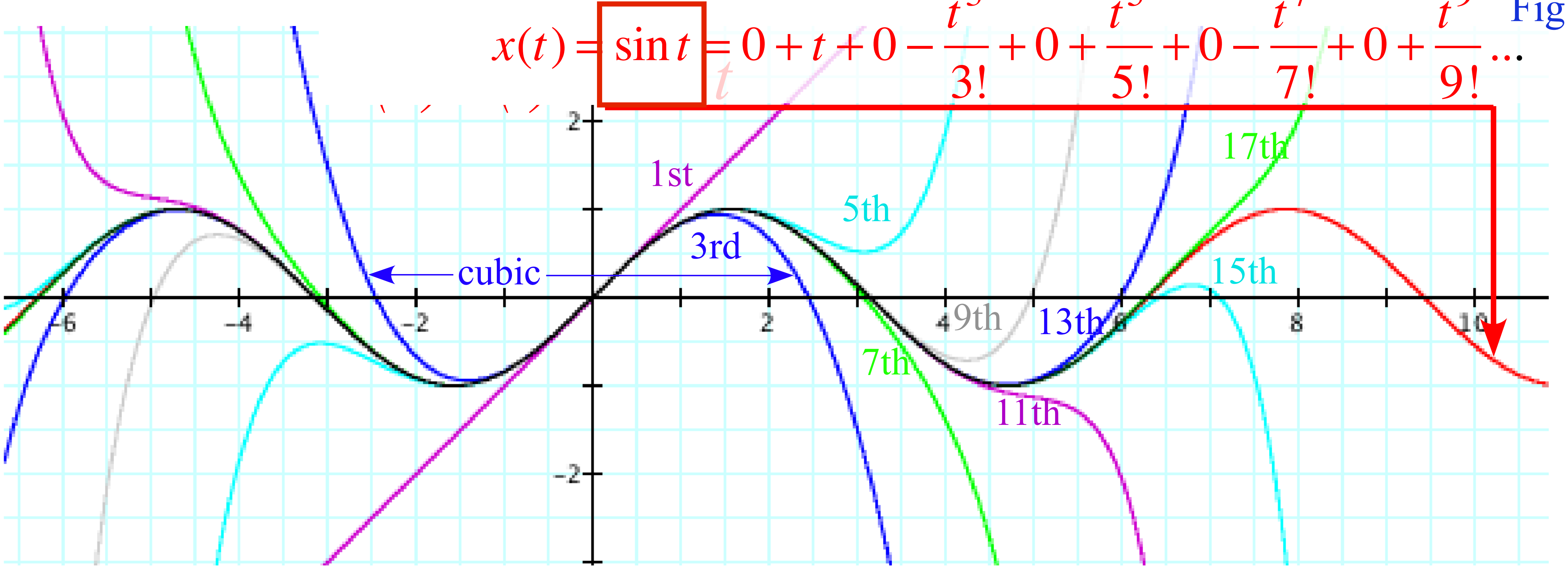
gives *exponential:* $$e^t=1+t+\frac{1}{2!}t^2+\frac{1}{3!}t^3+\frac{1}{4!}t^4+\frac{1}{5!}t^5+\ldots+\frac{1}{n!}t^n+$$

How good are power series? Depends...



$$x(t) = \boxed{\cos t} = 1 + 0 - \frac{t^2}{2!} + 0 + \frac{t^4}{4!} + 0 - \frac{t^6}{6!} + 0 + \frac{t^8}{8!} \ldots$$

Unit 1
Fig. 10.3



$$x(t) = \boxed{\sin t} = 0 + t + 0 - \frac{t^3}{3!} + 0 + \frac{t^5}{5!} + 0 - \frac{t^7}{7!} + 0 + \frac{t^9}{9!} \ldots$$

1. The Story of e (A Tale of Great $Interest$)

How good are those power series?

Taylor-Maclaurin series,

→ *imaginary interest, and complex exponentials*

Suppose the fancy bankers really went bonkers and made interest rate r an *imaginary number* $r=i\theta$.

Imaginary number $i=\sqrt{-1}$ powers have *repeat-after-4-pattern:* $i^0=1$, $i^1=i$, $i^2=-1$, $i^3=-i$, $i^4=1$, *etc...*

$$e^{i\theta}=1+i\theta+\frac{(i\theta)^2}{2!}+\frac{(i\theta)^3}{3!}+\frac{(i\theta)^4}{4!}+\frac{(i\theta)^5}{5!}+\ldots \qquad \text{(From exponential series)}$$

$$=1+i\theta-\frac{\theta^2}{2!}-i\frac{\theta^3}{3!}+\frac{\theta^4}{4!}+i\frac{\theta^5}{5!}-\ldots \qquad (i=\sqrt{-1}\text{ imples: } i^1=i,\ i^2=-1,\ i^3=-i,\ i^4=+1,\ i^5=i,\ldots)$$

$$=\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\ldots\right)+\left(i\theta-i\frac{\theta^3}{3!}+i\frac{\theta^5}{5!}-\ldots\right)$$

Suppose the fancy bankers really went bonkers and made interest rate r an *imaginary number* $r=i\theta$.

Imaginary number $i=\sqrt{-1}$ powers have *repeat-after-4-pattern:* $i^0=1$, $i^1=i$, $i^2=-1$, $i^3=-i$, $i^4=1$, *etc...*

$$e^{i\theta}=1+i\theta+\frac{(i\theta)^2}{2!}+\frac{(i\theta)^3}{3!}+\frac{(i\theta)^4}{4!}+\frac{(i\theta)^5}{5!}+\dots$$

(From exponential series)

$$=1+i\theta-\frac{\theta^2}{2!}-i\frac{\theta^3}{3!}+\frac{\theta^4}{4!}+i\frac{\theta^5}{5!}-\dots$$

($i=\sqrt{-1}$ imples: $i^1=i$, $i^2=-1$, $i^3=-i$, $i^4=+1$, $i^5=i$,...)

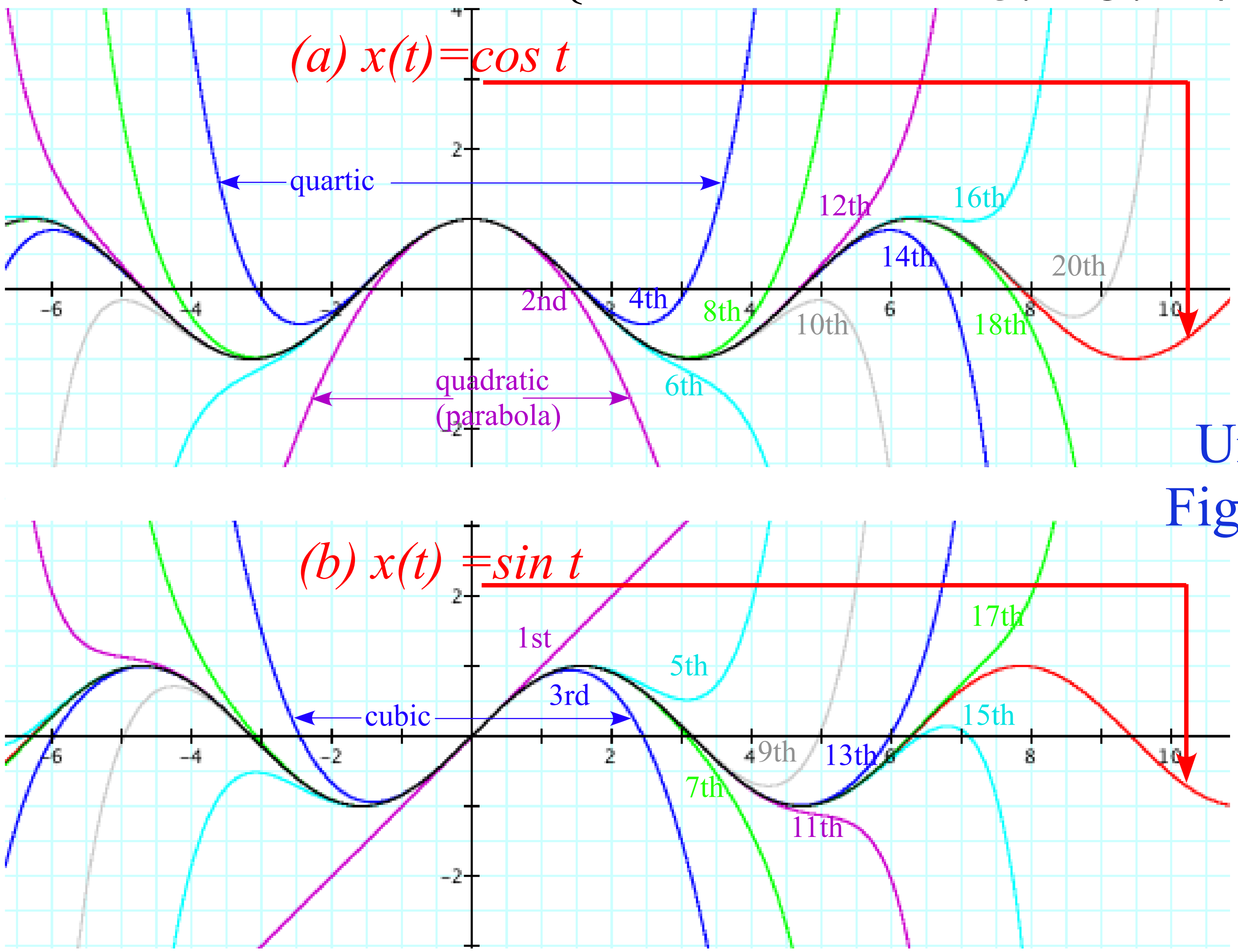
$$=\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\dots\right)+\left(i\theta-i\frac{\theta^3}{3!}+i\frac{\theta^5}{5!}-\dots\right)$$

To match series for

$$\begin{cases} \textit{cosine}: \cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots \\ \textit{sine}: \sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots \end{cases}$$

$$\boxed{e^{i\theta}=\cos\theta+i\sin\theta}$$

Euler-DeMoivre Theorem



Unit 1
Fig. 10.3

Suppose the fancy bankers really went bonkers and made interest rate *r* an *imaginary number* $r=i\theta$.

Imaginary number $i=\sqrt{-1}$ powers have *repeat-after-4-pattern:* $i^0=1,\ i^1=i,\ i^2=-1,\ i^3=-i,\ i^4=1, etc...$

$$e^{i\theta}=1+i\theta+\frac{(i\theta)^2}{2!}+\frac{(i\theta)^3}{3!}+\frac{(i\theta)^4}{4!}+\frac{(i\theta)^5}{5!}+\ldots$$

(From exponential series)

$$=1+i\theta-\frac{\theta^2}{2!}\quad -i\frac{\theta^3}{3!}\quad +\frac{\theta^4}{4!}\quad +i\frac{\theta^5}{5!}\quad -\ldots$$

($i=\sqrt{-1}$ imples: $i^1=i,\ i^2=-1,\ i^3=-i,\ i^4=+1,\ i^5=i,\ldots$)

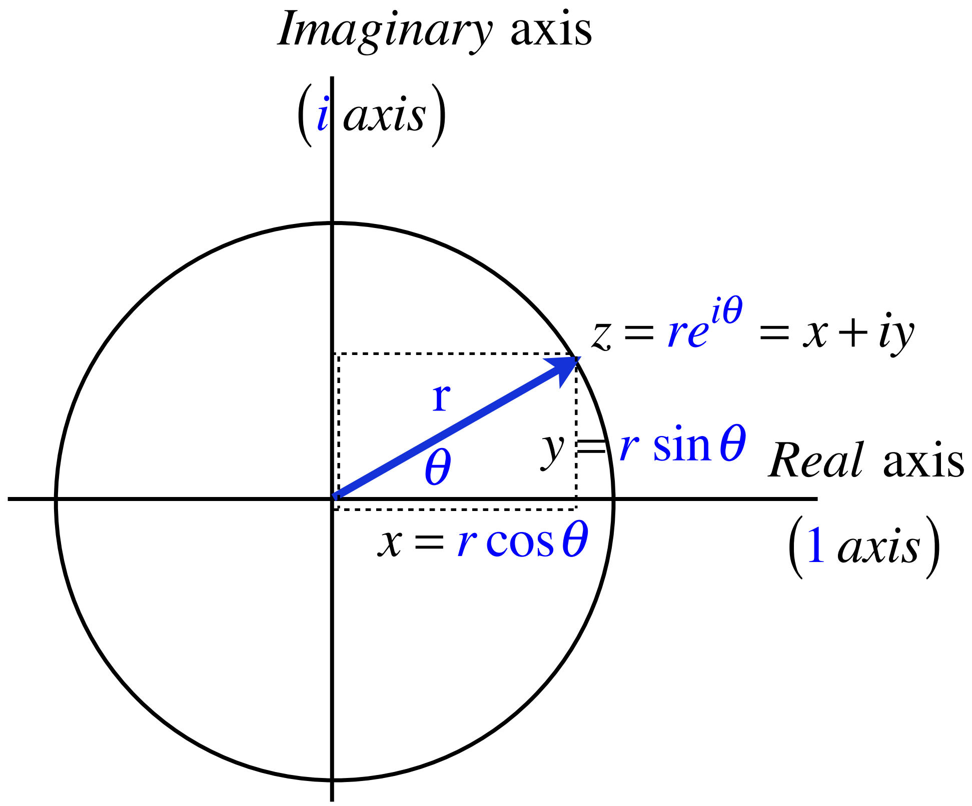
$$=\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\ldots\right)+\left(i\theta-i\frac{\theta^3}{3!}+i\frac{\theta^5}{5!}-\ldots\right)$$

To match series for

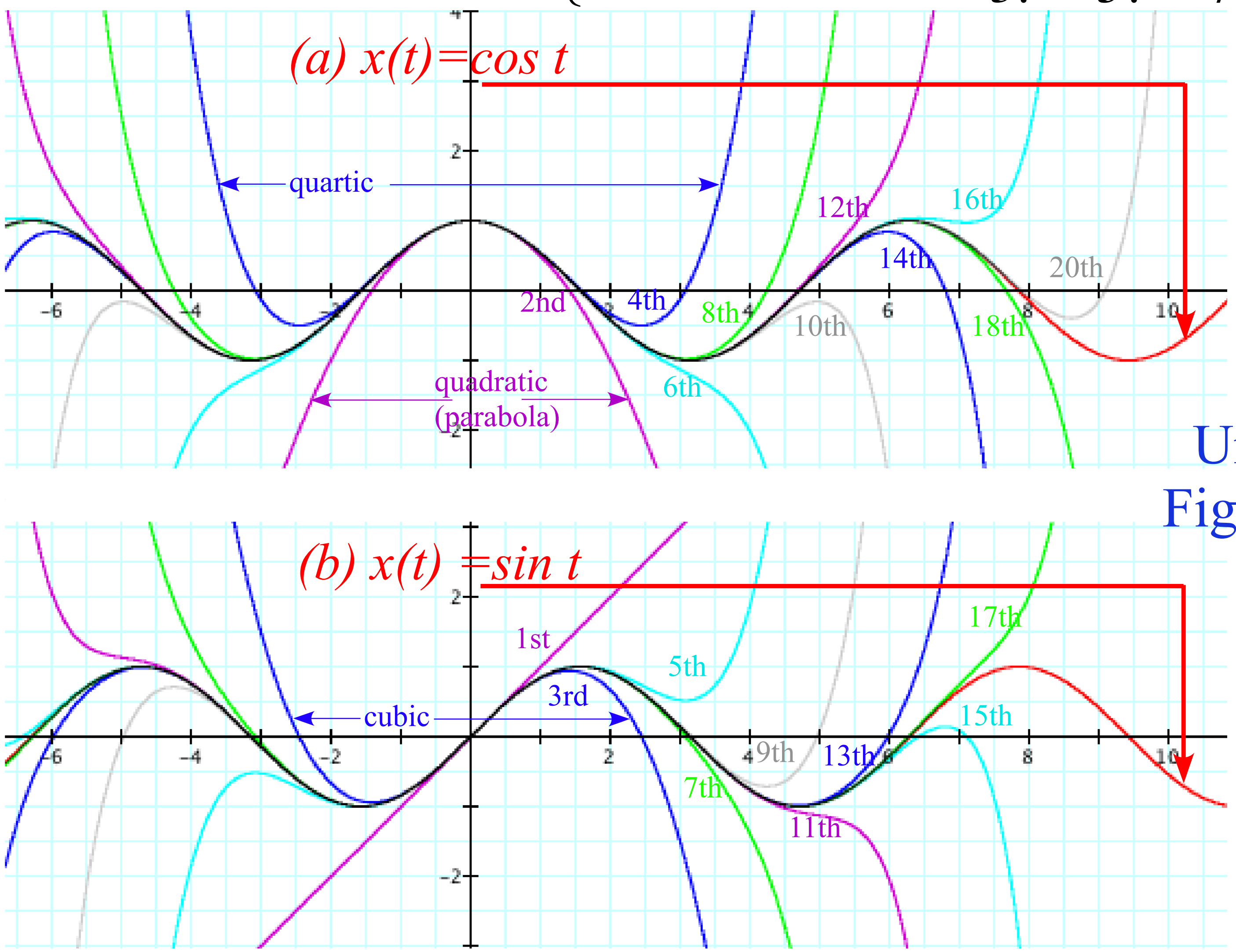
$$\begin{cases} \textit{cosine}: \cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots \\ \textit{sine}: \sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots \end{cases}$$

$$e^{i\theta}\ =\ \cos\theta\ +\ i\sin\theta$$

Euler-DeMoivre Theorem



$$re^{i\theta}\ =\ r\cos\theta+\ i\sin\theta$$



Unit 1 Fig. 10.3

2. What Good Are Complex Exponentials?

Easy trig

Easy 2D vector analysis

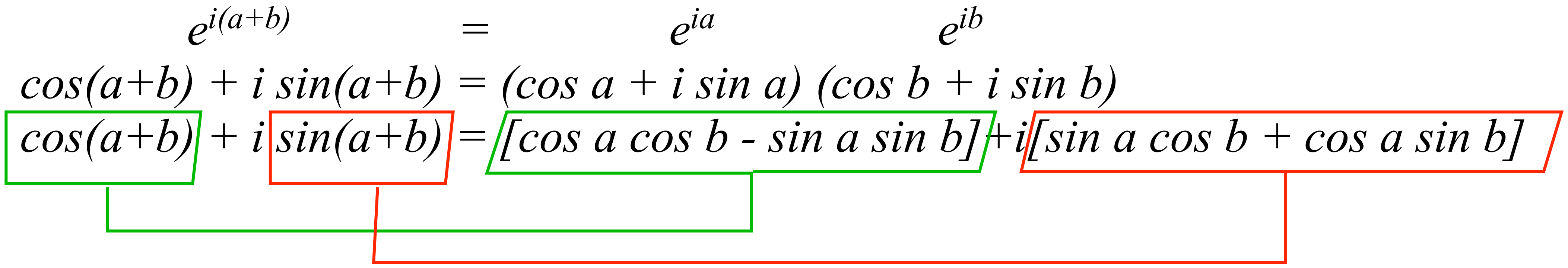
Easy oscillator phase analysis

Easy rotation and “dot” or “cross” products

What Good Are Complex Exponentials?

1. Complex numbers provide "automatic trigonometry"

Can't remember is cos$(a+b)$ or sin$(a+b)$? Just factor $e^{i(a+b)} = e^{ia}e^{ib}$...



$$e^{i(a+b)} = e^{ia}\, e^{ib}$$

$$\cos(a+b) + i \sin(a+b) = (\cos a + i \sin a)\,(\cos b + i \sin b)$$

$$\boxed{\cos(a+b)} + i\,\boxed{\sin(a+b)} = \boxed{[\cos a \cos b - \sin a \sin b]} + i\,\boxed{[\sin a \cos b + \cos a \sin b]}$$

What Good Are Complex Exponentials?

1. Complex numbers provide "automatic trigonometry"

Can't remember is cos$(a+b)$ or sin$(a+b)$? Just factor $e^{i(a+b)} = e^{ia}e^{ib}$...

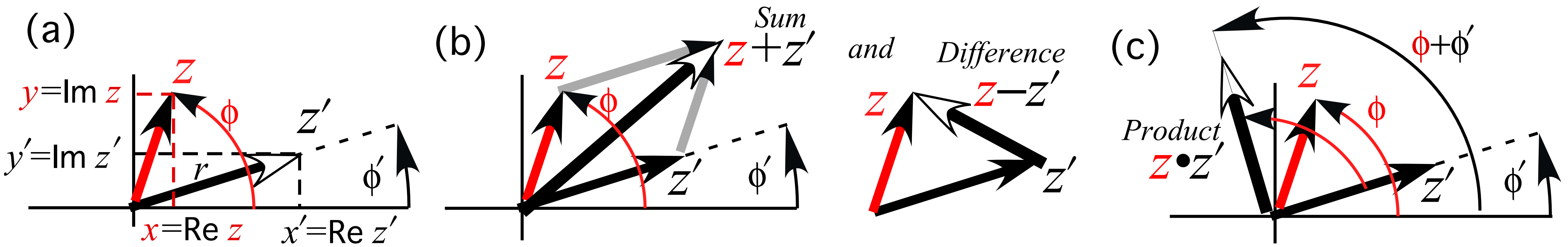
$$\begin{array}{ccccc} e^{i(a+b)} & = & e^{ia} & & e^{ib} \\ \cos(a+b) + i\sin(a+b) & = & (\cos a + i\sin a) & & (\cos b + i\sin b) \end{array}$$

$$\boxed{\cos(a+b)} + i\,\boxed{\sin(a+b)} = \boxed{[\cos a\cos b - \sin a\sin b]} + i\,\boxed{[\sin a\cos b + \cos a\sin b]}$$

2. Complex numbers add like vectors.

$$z_{sum} = z + z' = (x + iy) + (x' + iy') = (x + x') + i(y + y')$$

$$z_{diff} = z - z' = (x + iy) - (x' + iy') = (x - x') + i(y - y')$$



Unit 1
Fig. 10.6

$$|z_{SUM}| = \sqrt{(z+z')^*(z+z')} = \sqrt{\left(re^{i\phi} + r'e^{i\phi'}\right)^*\left(re^{i\phi} + r'e^{i\phi'}\right)} = \sqrt{\left(re^{-i\phi} + r'e^{-i\phi'}\right)\left(re^{i\phi} + r'e^{i\phi'}\right)}$$

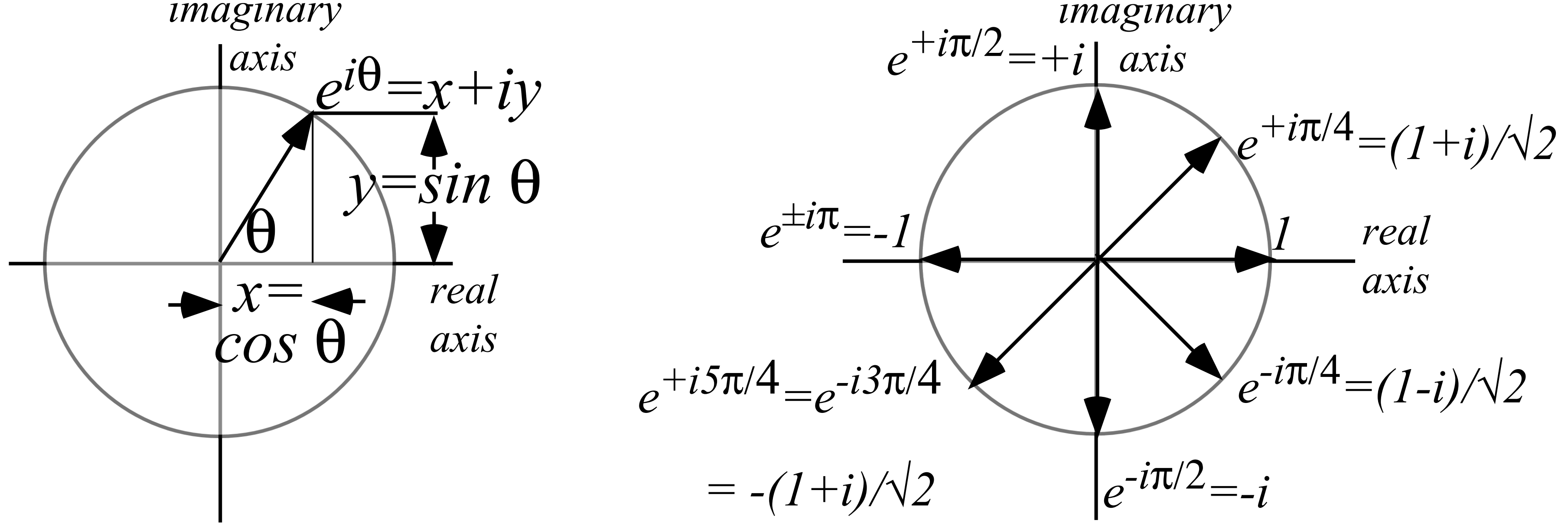
$$= \sqrt{r^2 + r'^2 + rr'\left(e^{i(\phi-\phi')} + e^{-i(\phi-\phi')}\right)} = \sqrt{r^2 + r'^2 + 2rr'\cos(\phi-\phi')}$$

(quick derivation of Cosine Law)

What Good Are Complex Exponentials? (contd.)

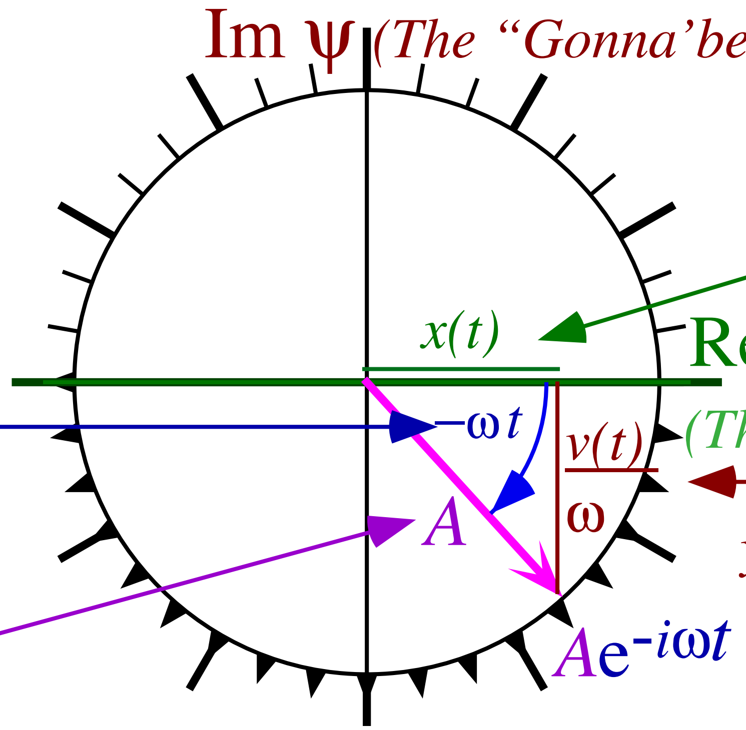
3. Complex exponentials $Ae^{-i\omega t}$ track position <u>and</u> velocity using Phasor Clock.

(a) Complex plane and unit vectors



(b) Quantum Phasor Clock $\psi = Ae^{-i\omega t} = A\cos\omega t - i\,A\sin\omega t = x+iy$

Unit 1
Fig. 10.5



Re ψ
$x(t) = A\cos\omega t$
<u>CARTESIAN</u>
<u>COMPONENTS</u>

(The "Is")

Im ψ
$y(t)=v(t)/\omega= -A\sin\omega t$

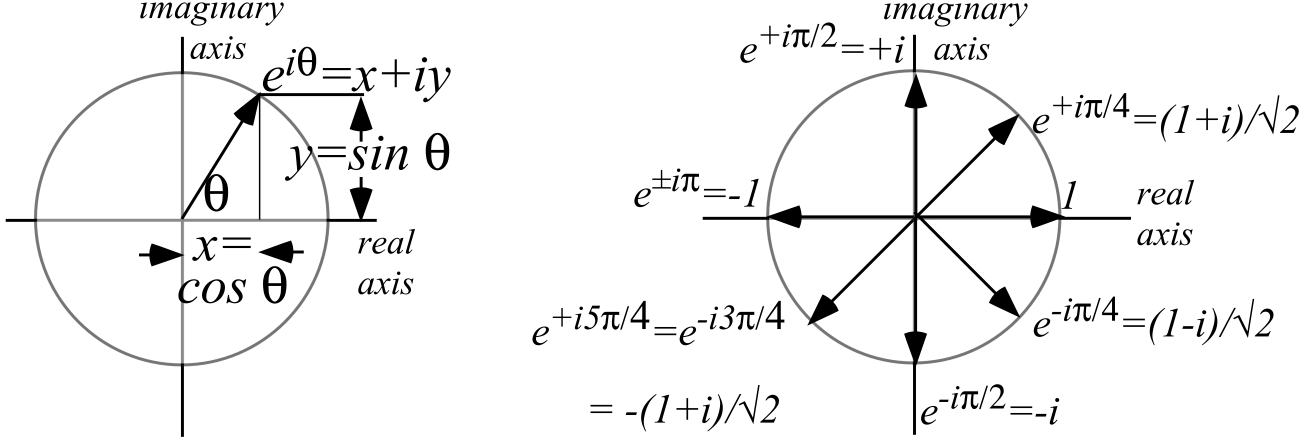
Phase angle or *Argument*
$\theta=-\omega t = \mathrm{ATAN}[v(t)/\omega x(t)]$
<u>POLAR</u>
<u>COMPONENTS</u>

Magnitude or *Modulus*
$A = |\psi| = \sqrt{\psi^*\psi}$

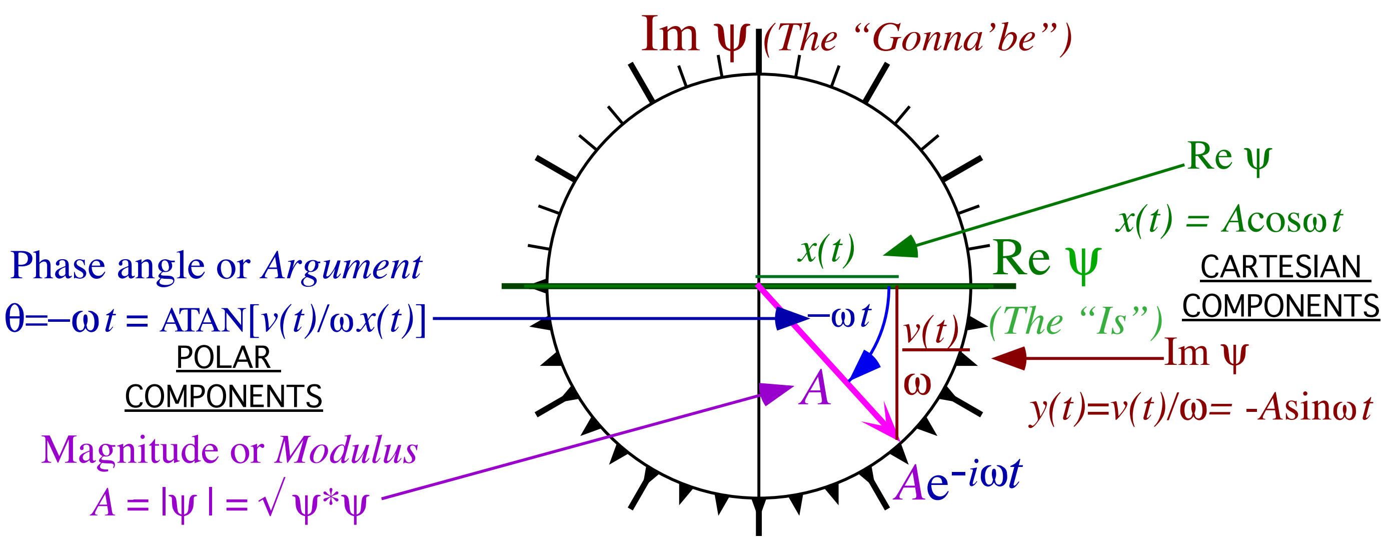
What Good Are Complex Exponentials? (contd.)

3.Complex exponentials $Ae^{-i\omega t}$ track position <u>and</u> velocity using Phasor Clock.

(a) Complex plane and unit vectors



(b) Quantum Phasor Clock $\psi = Ae^{-i\omega t} = A\cos\omega t - i\,A\sin\omega t = x+iy$



Unit 1
Fig. 10.5

Some Rect-vs-Polar *relations worth remembering*

$$\text{Cartesian }(x,y)\text{ form}\begin{cases}\psi_x = \mathrm{Re}\,\psi(t) = x(t) = A\cos\omega t = \dfrac{\psi+\psi^*}{2}\\[2ex] \psi_y = \mathrm{Im}\,\psi(t) = \dfrac{v(t)}{\omega} = -A\sin\omega t = \dfrac{\psi-\psi^*}{2i}\end{cases}$$

$$\psi = re^{+i\theta} = re^{-i\omega t} = r(\cos\omega t - i\sin\omega t)$$

$$\psi^* = re^{-i\theta} = re^{+i\omega t} = r(\cos\omega t + i\sin\omega t)$$

$$\text{Polar }(r,\theta)\text{ form}\begin{cases} r = A = |\psi| = \sqrt{\psi_x^{\,2}+\psi_y^{\,2}} = \sqrt{\psi^*\psi}\\[1ex] \theta = -\omega t = \arctan(\psi_y/\psi_x)\end{cases}$$

$$\cos\theta = \tfrac{1}{2}(e^{+i\theta}+e^{-i\theta}) \qquad \mathrm{Re}\,\psi = \frac{\psi+\psi^*}{2}$$

$$\sin\theta = \tfrac{1}{2i}(e^{+i\theta}-e^{-i\theta}) \qquad \mathrm{Im}\,\psi = \frac{\psi-\psi^*}{2i}$$

2. What Good Are Complex Exponentials?

Easy trig

Easy 2D vector analysis

Easy oscillator phase analysis

→ *Easy rotation and “dot” or “cross” products*

What Good Are Complex Exponentials? (contd.)

4. Complex products provide 2D rotation operations.

$$e^{i\phi}\cdot z = (\cos\phi + i\sin\phi)\cdot(x + iy) = x\cos\phi - y\sin\phi \qquad + i \quad (x\sin\phi + y\cos\phi)$$

$$\mathbf{R}_{+\phi}\bullet\mathbf{r} \;= (x\cos\phi - y\sin\phi)\hat{\mathbf{e}}_x + (x\sin\phi + y\cos\phi)\hat{\mathbf{e}}_y$$

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\bullet\begin{pmatrix} x \\ y \end{pmatrix} = \qquad \begin{pmatrix} x\cos\phi - y\sin\phi \\ x\sin\phi + y\cos\phi \end{pmatrix}$$

What Good Are Complex Exponentials? (contd.)

4. Complex products provide 2D rotation operations.

$$e^{i\phi}\cdot z = (\cos\phi + i\,\sin\phi)\cdot(x + iy) = \;x\cos\phi - y\sin\phi \qquad + i\;\;(x\sin\phi + y\cos\phi\,)$$

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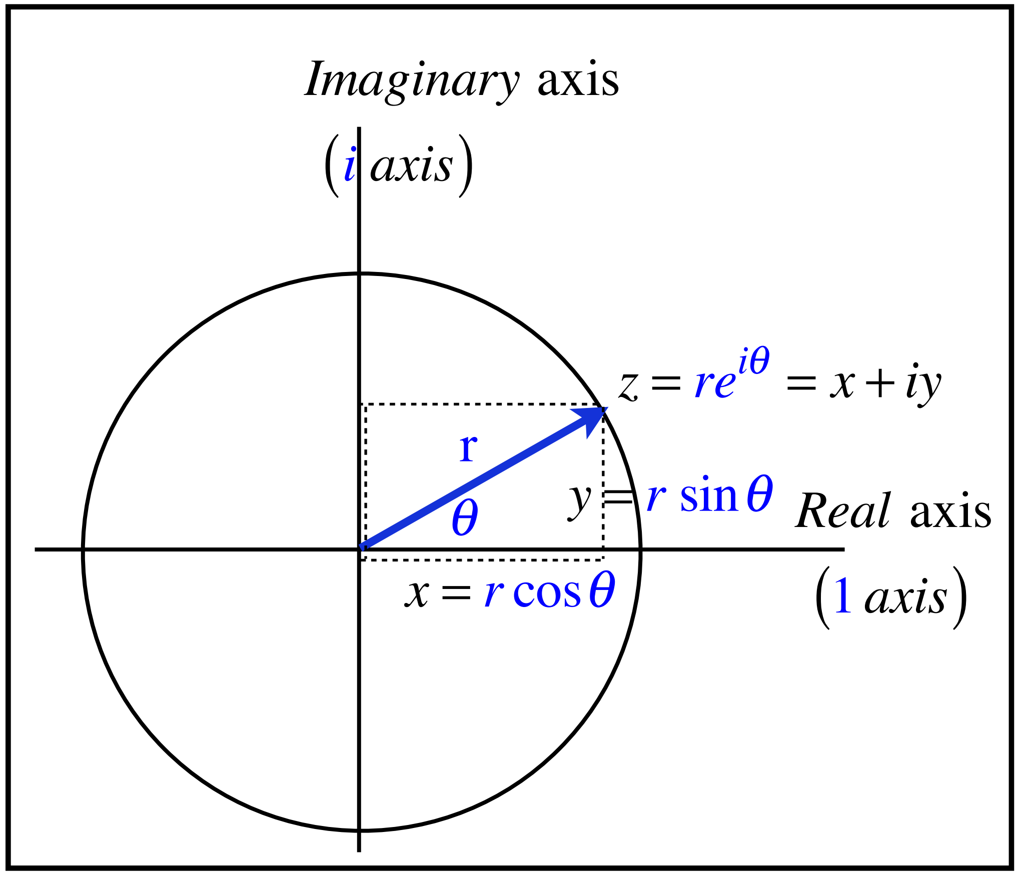
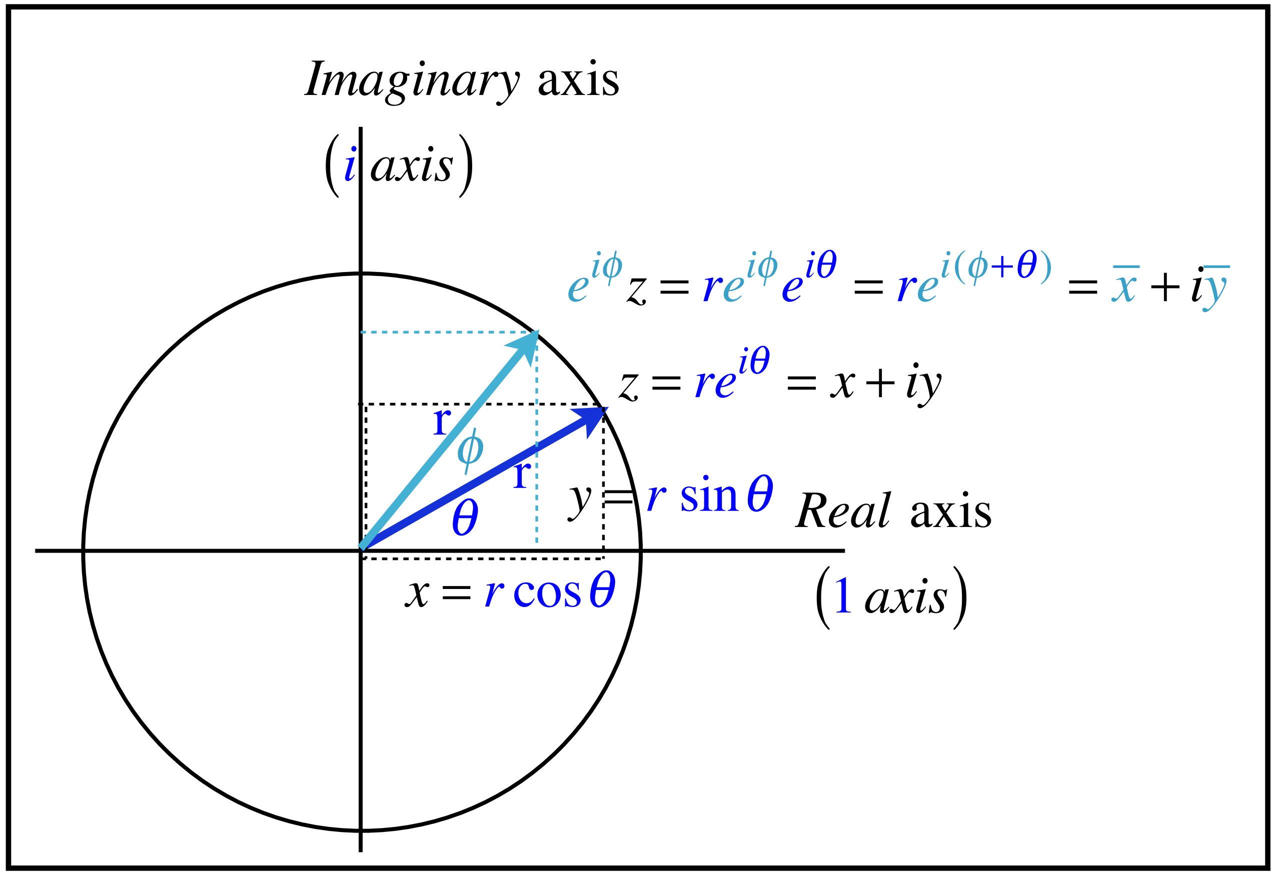
$$\begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}\bullet\begin{pmatrix}x\\ y\end{pmatrix} = \qquad \begin{pmatrix}x\cos\phi - y\sin\phi\\ x\sin\phi + y\cos\phi\end{pmatrix}$$

$e^{i\phi}$ acts on this: $z = re^{i\theta}$



to give this: $e^{i\phi}\; e^{i\phi}z = re^{i\phi}e^{i\theta}$



What Good Are Complex Exponentials? (contd.)

4. Complex products provide 2D rotation operations.

$$e^{i\phi}\cdot z = (\cos\phi + i\sin\phi)\cdot(x + iy) = x\cos\phi - y\sin\phi \quad + i \quad (x\sin\phi + y\cos\phi)$$

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5. Complex products provide 2D "dot"(•) and "cross"(x) products.

Two complex numbers $A=A_x+iA_y$ and $B=B_x+iB_y$ and their "star" (*)-product A^*B.

$$A^*B = (A_x + iA_y)^*(B_x + iB_y) = (A_x - iA_y)(B_x + iB_y)$$

$$= (A_xB_x + A_yB_y) + i(A_xB_y - A_yB_x) = \mathbf{A}\bullet\mathbf{B} + i\,|\,\mathbf{A}\times\mathbf{B}\,|_{Z\perp(x,y)}$$

Real part is scalar or "dot"(•) product **A•B**.

Imaginary part is vector or "cross"(×) product, but just the Z-component *normal* to xy-plane.

Rewrite A^*B in polar form.

$$A^*B = (|A|e^{i\theta_A})^*(|B|e^{i\theta_B}) = |A|e^{-i\theta_A}\,|B|e^{i\theta_B} = |A||B|e^{i(\theta_B-\theta_A)}$$

$$= |A||B|\cos(\theta_B - \theta_A) + i|A||B|\sin(\theta_B - \theta_A) = \mathbf{A}\bullet\mathbf{B} + i\,|\,\mathbf{A}\times\mathbf{B}\,|_{Z\perp(x,y)}$$

What Good Are Complex Exponentials? (contd.)

4. Complex products provide 2D rotation operations.

$$e^{i\phi}\cdot z = (\cos\phi + i\sin\phi)\cdot(x+iy) = x\cos\phi - y\sin\phi \qquad + i\ \ (x\sin\phi + y\cos\phi)$$

$$\mathbf{R}_{+\phi}\bullet\mathbf{r} = (x\cos\phi - y\sin\phi)\hat{\mathbf{e}}_x + (x\sin\phi + y\cos\phi)\hat{\mathbf{e}}_y$$

$$\begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}\bullet\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}x\cos\phi - y\sin\phi\\ x\sin\phi + y\cos\phi\end{pmatrix}$$

5. Complex products provide 2D “dot”(•) and “cross”(x) products.

Two complex numbers $A=A_x+iA_y$ and $B=B_x+iB_y$ and their “star” (*)-product $A*B$.

$$A*B = (A_x+iA_y)^*(B_x+iB_y) = (A_x-iA_y)(B_x+iB_y)$$

$$= (A_xB_x + A_yB_y) + i(A_xB_y - A_yB_x) = \mathbf{A}\bullet\mathbf{B} + i\,|\mathbf{A}\times\mathbf{B}|_{Z\perp(x,y)}$$

Real part is scalar or “dot”(•) product **A•B**.

Imaginary part is vector or “cross”(×) product, but just the *Z*-component *normal* to *xy*-plane.

Rewrite $A*B$ in polar form.

$$A*B = \left(|A|e^{i\theta_A}\right)^*\left(|B|e^{i\theta_B}\right) = |A|e^{-i\theta_A}\,|B|e^{i\theta_B} = |A||B|e^{i(\theta_B-\theta_A)}$$

$$= |A||B|\cos(\theta_B-\theta_A) + i|A||B|\sin(\theta_B-\theta_A) = \mathbf{A}\bullet\mathbf{B} + i\,|\mathbf{A}\times\mathbf{B}|_{Z\perp(x,y)}$$

$$\mathbf{A}\bullet\mathbf{B} = |A||B|\cos(\theta_B-\theta_A)$$
$$= |A|\cos\theta_A|B|\cos\theta_B + |A|\sin\theta_A|B|\sin\theta_B$$
$$= \qquad A_xB_x \qquad + \qquad A_yB_y$$

$$|\mathbf{A}\times\mathbf{B}| = |A||B|\sin(\theta_B-\theta_A)$$
$$= |A|\cos\theta_A|B|\sin\theta_B - |A|\sin\theta_A|B|\cos\theta_B$$
$$= \qquad A_xB_y \qquad - \qquad A_yB_x$$

What Good are complex variables?

- *Easy 2D vector calculus*
  - *Easy 2D vector derivatives*
    - *Easy 2D source-free field theory*
      - *Easy 2D vector field-potential theory*

What Good Are Complex Exponentials? (contd.)

6. Complex derivative contains “divergence”(∇•F) and “curl”(∇xF) of 2D vector field

Relation of *(z,z*)* to *(x*=Re*z,y*=Im*z)* defines a *z*-derivative $\frac{df}{dz}$ and “star” *z**-derivative. $\frac{df}{dz*}$

$$z = x + iy \qquad x = \tfrac{1}{2}(z + z*)$$

$$z^* = x - iy \qquad y = \tfrac{1}{2i}(z - z*)$$

Applying chain-rule

$$\frac{df}{dz} = \frac{\partial x}{\partial z}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{i}{2}\frac{\partial f}{\partial y} \qquad \frac{d}{dz} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{i}{2}\frac{\partial}{\partial y}$$

$$\frac{df}{dz*} = \frac{\partial x}{\partial z*}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z*}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{i}{2}\frac{\partial f}{\partial y} \qquad \frac{d}{dz*} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{i}{2}\frac{\partial}{\partial y}$$

6. Complex derivative contains "divergence"($\nabla\cdot\mathbf{F}$) and "curl"($\nabla\times\mathbf{F}$) of 2D vector field

Relation of (z,z^*) to $(x=\mathrm{Re}\,z, y=\mathrm{Im}\,z)$ defines a z-derivative $\frac{df}{dz}$ and "star" z^*-derivative. $\frac{df}{dz^*}$

$$z = x + iy \qquad x = \tfrac{1}{2}(z + z^*)$$

$$z^* = x - iy \qquad y = \tfrac{1}{2i}(z - z^*)$$

$$\frac{df}{dz} = \frac{\partial x}{\partial z}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{i}{2}\frac{\partial f}{\partial y}$$

$$\frac{d}{dz} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{i}{2}\frac{\partial}{\partial y}$$

$$\frac{df}{dz^*} = \frac{\partial x}{\partial z^*}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z^*}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{i}{2}\frac{\partial f}{\partial y}$$

$$\frac{d}{dz^*} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{i}{2}\frac{\partial}{\partial y}$$

$$\frac{d}{dz} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{i}{2}\frac{\partial}{\partial y}$$

Derivative chain-rule shows real part of $\frac{df}{dz}$ has 2D divergence $\nabla\bullet\mathbf{f}$ and imaginary part has curl $\nabla\times\mathbf{f}$.

$$\frac{df}{dz} = \frac{d}{dz}(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\overbrace{\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right)} + \frac{i}{2}\overbrace{\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right)} = \frac{1}{2}\nabla\bullet\mathbf{f} + \frac{i}{2}|\nabla\times\mathbf{f}|_{Z\perp(x,y)}$$

What Good Are Complex Exponentials? (contd.)

6. Complex derivative contains "divergence"($\nabla\cdot\mathbf{F}$) and "curl"($\nabla\times\mathbf{F}$) of 2D vector field

Relation of *(z,z*)* to *(x=*Re*z,y=*Im*z)* defines a *z*-derivative $\frac{df}{dz}$ and "star" *z**-derivative. $\frac{df}{dz^*}$

$$z \;= x+iy \qquad x=\tfrac{1}{2}\,(z\;+z^*)$$

$$z^* = x-iy \qquad y=\tfrac{1}{2i}\,(z\;-z^*)$$

$$\frac{df}{dz}=\frac{\partial x}{\partial z}\frac{\partial f}{\partial x}+\frac{\partial y}{\partial z}\frac{\partial f}{\partial y}=\frac{1}{2}\frac{\partial f}{\partial x}-\frac{i}{2}\frac{\partial f}{\partial y} \qquad \frac{d}{dz}=\frac{1}{2}\frac{\partial}{\partial x}-\frac{i}{2}\frac{\partial}{\partial y}$$

$$\frac{df}{dz^*}=\frac{\partial x}{\partial z^*}\frac{\partial f}{\partial x}+\frac{\partial y}{\partial z^*}\frac{\partial f}{\partial y}=\frac{1}{2}\frac{\partial f}{\partial x}+\frac{i}{2}\frac{\partial f}{\partial y} \qquad \frac{d}{dz^*}=\frac{1}{2}\frac{\partial}{\partial x}+\frac{i}{2}\frac{\partial}{\partial y}$$

$$\frac{d}{dz}=\frac{1}{2}\frac{\partial}{\partial x}-\frac{i}{2}\frac{\partial}{\partial y}$$

Derivative chain-rule shows real part of $\frac{df}{dz}$ has 2D divergence $\nabla\bullet\mathbf{f}$ and imaginary part has curl $\nabla\times\mathbf{f}$.

$$\frac{df}{dz}=\frac{d}{dz}\,(f_x+i\,f_y)=\tfrac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)(f_x+i\,f_y)=\tfrac{1}{2}\left(\overbrace{\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}}\right)+\tfrac{i}{2}\left(\overbrace{\frac{\partial f_y}{\partial x}-\frac{\partial f_x}{\partial y}}\right)=\tfrac{1}{2}\,\nabla\bullet\mathbf{f}+\tfrac{i}{2}\,|\nabla\times\mathbf{f}|_{Z\perp(x,y)}$$

7. Invent source-free 2D vector fields [$\nabla\cdot\mathbf{F}=0$ and $\nabla\times\mathbf{F}=0$]

We can invent *source-free 2D vector fields* that are both *zero-divergence* and *zero-curl*.
Take any function *f(z)*, conjugate it (change all *i*'s to –*i*) to give *f*(z*)* for which $\boxed{\frac{df^*}{dz}=0}$.

6. Complex derivative contains "divergence"($\nabla\cdot\mathbf{F}$) and "curl"($\nabla\times\mathbf{F}$) of 2D vector field

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Derivative chain-rule shows real part of $\frac{df}{dz}$ has 2D divergence $\nabla\bullet\mathbf{f}$ and imaginary part has curl $\nabla\times\mathbf{f}$.

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Take any function *f(z)*, conjugate it (change all *i*'s to –*i*) to give *f*(z*)* for which $\boxed{\frac{df^*}{dz} = 0}$

For example: if *f(z)=a·z* then *f*(z*)=a·z*=a(x-iy)* is not function of *z* so it has *zero z-derivative*.

$\mathbf{F}=(F_x,F_y)=(f^*_x,f^*_y)=(a\cdot x,-a\cdot y)$ *has zero divergence:* $\nabla\bullet\mathbf{F}=0$ *and has zero curl:* $|\nabla\times\mathbf{F}|=0$.

$$\nabla\bullet\mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} = \frac{\partial(ax)}{\partial x} + \frac{\partial F(-ay)}{\partial y} = 0 \qquad |\nabla\times\mathbf{F}|_{Z\perp(x,y)} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = \frac{\partial(-ay)}{\partial x} - \frac{\partial F(ax)}{\partial y} = 0$$

A *DFL* field **F** (*Divergence-Free-Laminar*)

What Good Are Complex Exponentials? (contd.)

7. Invent source-free 2D vector fields [$\nabla\bullet\mathbf{F}=0$ and $\nabla\times\mathbf{F}=0$]

We can invent *source-free 2D vector fields* that are both *zero-divergence* and *zero-curl*. Take any function *f(z)*, conjugate it (change all *i*'s to *–i*) to give *f*(z*)* for which $\boxed{\frac{df^*}{dz}=0}$.

For example: if $f(z)=a\cdot z$ then $f^*(z^*)=a\cdot z^*=a(x-iy)$ is not function of *z* so it has *zero z-derivative*.

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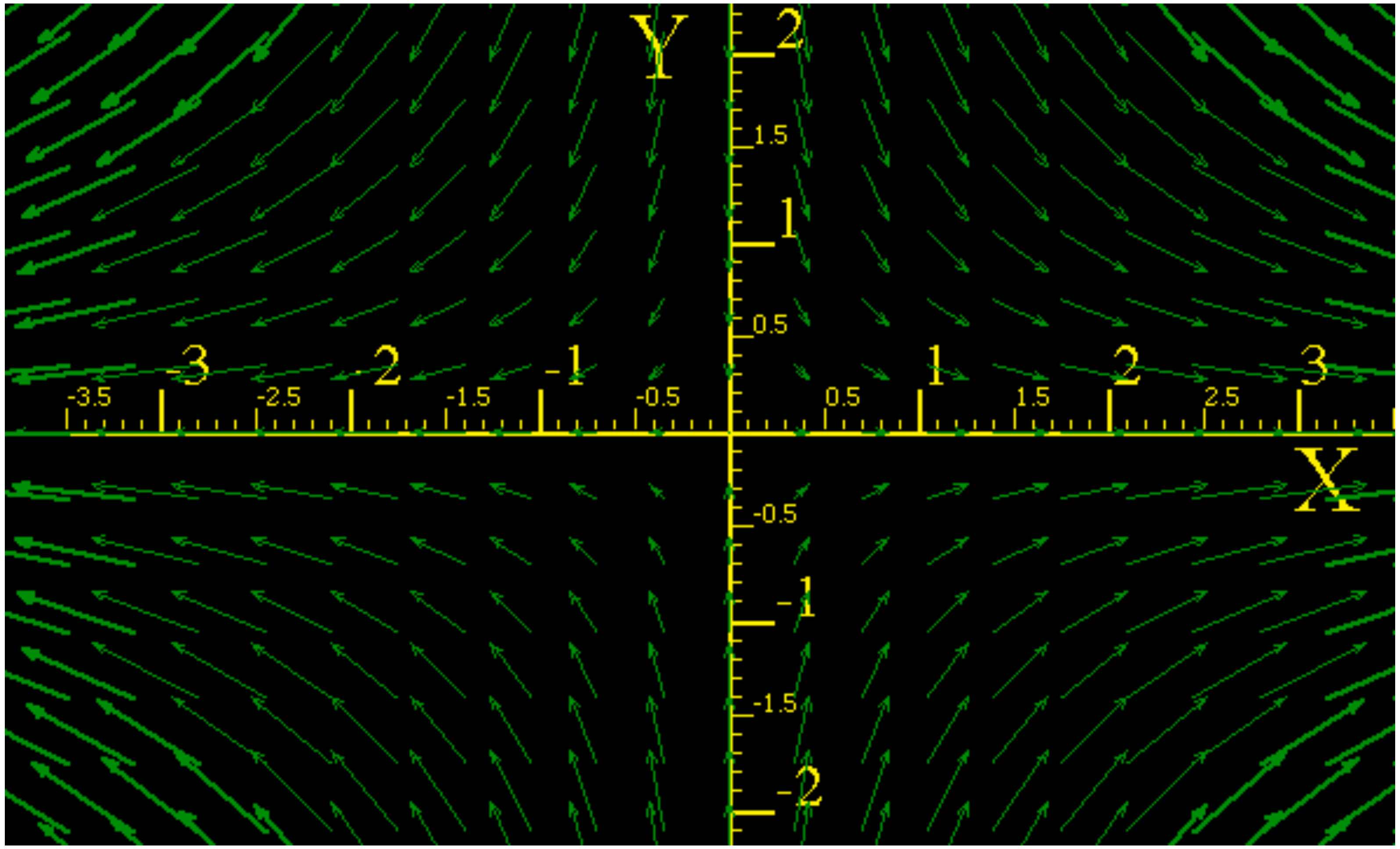

precursor to
Unit 1
Fig. 10.7

$\mathbf{F}=(f^*_x,f^*_y)=(a\cdot x,-a\cdot y)$ is a *divergence-free laminar (DFL)* field.

What Good are complex variables?

Easy 2D vector calculus
Easy 2D vector derivatives
Easy 2D source-free field theory
→ *Easy 2D vector field-potential theory*

What Good Are Complex Exponentials? (contd.)

8. Complex potential ϕ contains “scalar”($\mathbf{F}=\nabla\mathbf{\Phi}$) and “vector”($\mathbf{F}=\nabla x\mathbf{A}$) potentials

Any *DFL* field **F** is a gradient of a *scalar potential field* **Φ** or a curl of a *vector potential field* **A**.

$$\mathbf{F}=\nabla\mathbf{\Phi} \qquad\qquad \mathbf{F}=\nabla\times\mathbf{A}$$

A *complex potential* $\phi(z)=\mathbf{\Phi}(x,y)+i\mathbf{A}(x,y)$ exists whose z-derivative is $f(z)=d\phi/dz$.

Its complex conjugate $\phi^*(z^*)=\mathbf{\Phi}(x,y)-i\mathbf{A}(x,y)$ has z^*-derivative $f^*(z^*)=d\phi^*/dz^*$ giving *DFL* field **F**.

What Good Are Complex Exponentials? (contd.)

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To find $\phi=\mathbf{\Phi}+i\mathbf{A}$ integrate $f(z)=a\cdot z$ to get ϕ and isolate real ($\mathrm{Re}\,\phi=\mathbf{\Phi}$) and imaginary ($\mathrm{Im}\,\phi=\mathbf{A}$) parts.

What Good Are Complex Exponentials? (contd.)

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$$f(z)=\tfrac{d\phi}{dz} \quad\Rightarrow\quad \phi = \quad \mathbf{\Phi} \quad + i\ \ \mathbf{A}=\int f\cdot dz=\int az\cdot dz=\tfrac{1}{2}\,az^2$$

What Good Are Complex Exponentials? (contd.)

*8. Complex potential ϕ contains "scalar"(**F=**$\nabla\Phi$) and "vector"(**F=**∇x**A**) potentials*

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$$f(z)=\tfrac{d\phi}{dz} \quad\Rightarrow\quad \phi=\overbrace{\tfrac{1}{2}a(x^2-y^2)}^{\Phi}+i\overbrace{axy}^{\mathbf{A}}=\int f\cdot dz=\int az\cdot dz=\tfrac{1}{2}az^2=\tfrac{1}{2}a(x+iy)^2$$

What Good Are Complex Exponentials? (contd.)

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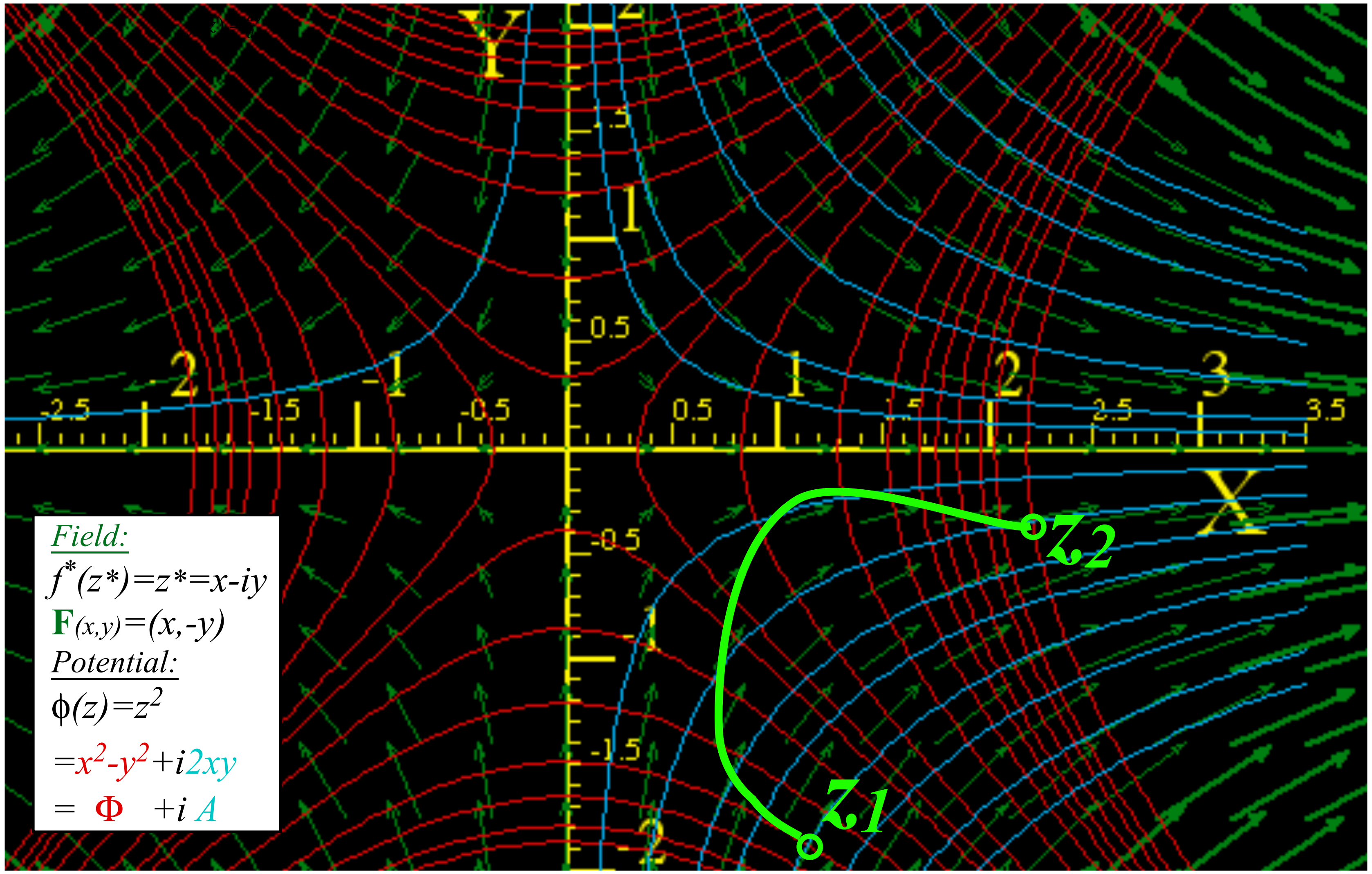

Unit 1
Fig. 10.7

What Good Are Complex Exponentials? (contd.)

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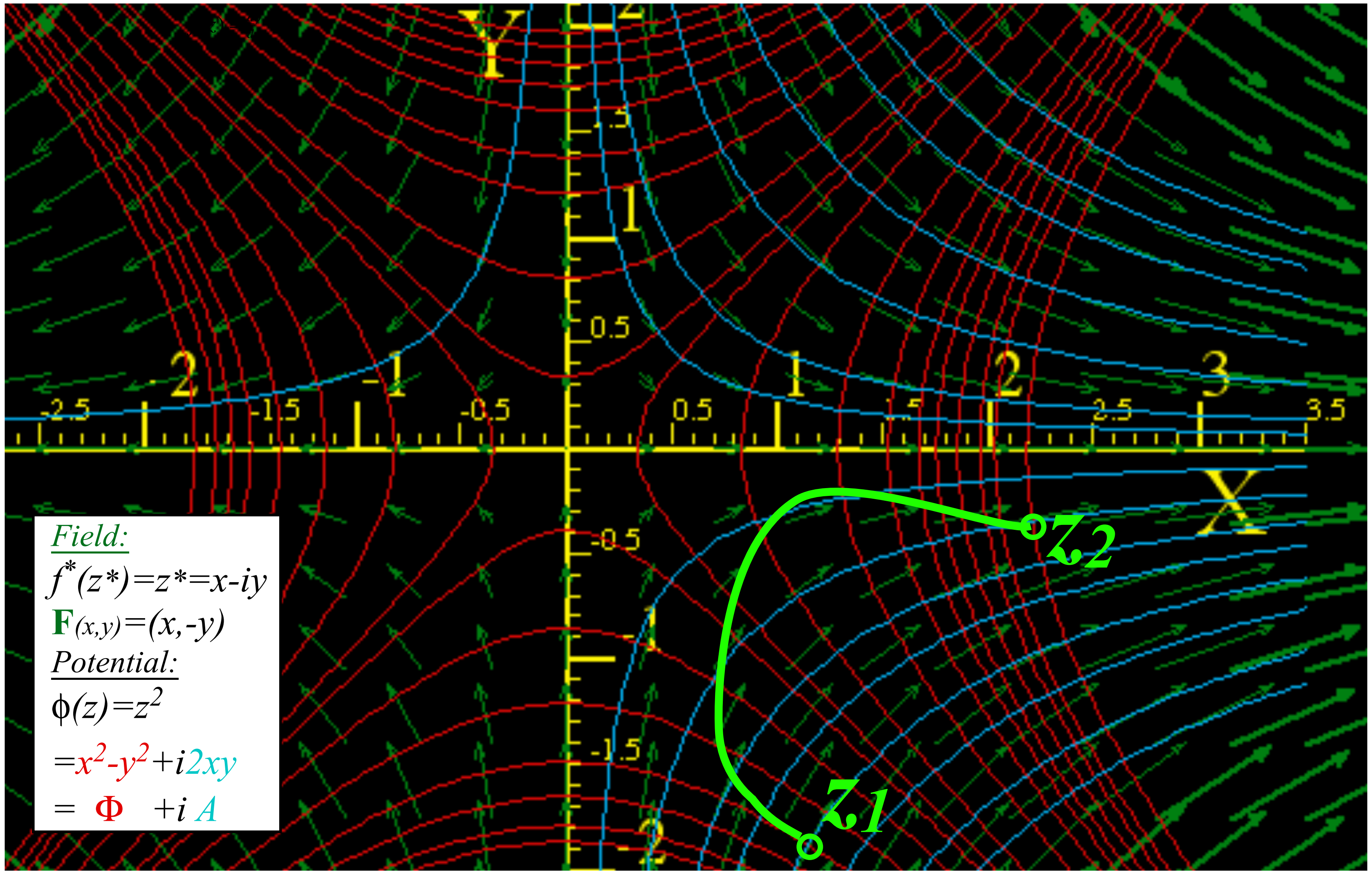
coordinate
system!



Unit 1
Fig. 10.7

The (**Φ**,**A**) grid is a GCC coordinate system*:

$q^1=\Phi=(x^2-y^2)/2=\text{const.}$

$q^2=A=(xy)=\text{const.}$

*Actually it's OCC.

What Good are complex variables?

Easy 2D vector calculus

Easy 2D vector derivatives

Easy 2D source-free field theory

Easy 2D vector field-potential theory

The half-n'-half results: (Riemann-Cauchy Derivative Relations)

What Good Are Complex Exponentials? (contd.)

*8. (contd.) Complex potential ϕ contains "scalar"(**F=**$\nabla\Phi$) and "vector"(**F=**∇x**A**) potentials*

...and either one (or half-n'-half!) works just as well.

Derivative $\frac{d\phi*}{dz*}$ has 2D gradient $\nabla\Phi=\begin{pmatrix}\frac{\partial\Phi}{\partial x}\\ \frac{\partial\Phi}{\partial y}\end{pmatrix}$ of scalar $\mathbf{\Phi}$ and curl $\nabla\times\mathbf{A}=\begin{pmatrix}\frac{\partial\mathbf{A}}{\partial y}\\ -\frac{\partial\mathbf{A}}{\partial x}\end{pmatrix}$ of vector **A** (*and they're equal!*)

$f(z)=\frac{d\phi}{dz} \quad \Rightarrow$

$$\frac{d}{dz^*}\phi^*=\frac{d}{dz^*}(\mathbf{\Phi}-i\mathbf{A})=\frac{1}{2}(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y})(\mathbf{\Phi}-i\mathbf{A})=\frac{1}{2}\overbrace{(\frac{\partial\mathbf{\Phi}}{\partial x}+i\frac{\partial\mathbf{\Phi}}{\partial y})}+\frac{1}{2}\overbrace{(\frac{\partial\mathbf{A}}{\partial y}-i\frac{\partial\mathbf{A}}{\partial x})}=\frac{1}{2}\nabla\mathbf{\Phi}+\frac{1}{2}\nabla\times\mathbf{A}$$

$$\frac{d}{dz}=\frac{1}{2}\frac{\partial}{\partial x}-\frac{i}{2}\frac{\partial}{\partial y}$$

$$\frac{d}{dz*}=\frac{1}{2}\frac{\partial}{\partial x}+\frac{i}{2}\frac{\partial}{\partial y}$$

What Good Are Complex Exponentials? (contd.)

*8. (contd.) Complex potential ϕ contains "scalar"(**F=**$\nabla\mathbf{\Phi}$) and "vector"(**F=**∇x**A**) potentials*

...and either one (or half-n'-half!) works just as well.

Derivative $\frac{d\phi*}{dz*}$ has 2D gradient $\nabla\mathbf{\Phi}=\begin{pmatrix}\frac{\partial\mathbf{\Phi}}{\partial x}\\ \frac{\partial\mathbf{\Phi}}{\partial y}\end{pmatrix}$ of scalar **Φ** and curl $\nabla\times\mathbf{A}=\begin{pmatrix}\frac{\partial\mathbf{A}}{\partial y}\\ -\frac{\partial\mathbf{A}}{\partial x}\end{pmatrix}$ of vector **A** (*and they're equal!*)

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Note, *mathematician definition* of force field **F**=+∇**Φ** replaces usual physicist's definition **F**=-∇**Φ**

What Good Are Complex Exponentials? (contd.)

8. (contd.) Complex potential ϕ contains "scalar"($\mathbf{F}=\nabla\mathbf{\Phi}$) and "vector"($\mathbf{F}=\nabla x\mathbf{A}$) potentials

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Note, *mathematician definition* of force field $\mathbf{F}=+\nabla\mathbf{\Phi}$ replaces usual physicist's definition $\mathbf{F}=-\nabla\mathbf{\Phi}$

Given ϕ:

$$\boxed{\begin{aligned}\phi&=\quad\mathbf{\Phi}\quad+i\ \ \mathbf{A}\\ &=\tfrac{1}{2}\,a(x^2-y^2)\ +i\ \ axy\end{aligned}}$$

The half-n'-half result

find:

$$\nabla\mathbf{\Phi}=\begin{pmatrix}\frac{\partial\mathbf{\Phi}}{\partial x}\\ \frac{\partial\mathbf{\Phi}}{\partial y}\end{pmatrix}=\begin{pmatrix}\frac{\partial}{\partial x}\frac{a}{2}(x^2-y^2)\\ \frac{\partial}{\partial y}\frac{a}{2}(x^2-y^2)\end{pmatrix}=\begin{pmatrix}ax\\ -ay\end{pmatrix}=\mathbf{F}$$

or find:

$$\nabla\times\mathbf{A}=\begin{pmatrix}\frac{\partial\mathbf{A}}{\partial y}\\ -\frac{\partial\mathbf{A}}{\partial x}\end{pmatrix}=\begin{pmatrix}\frac{\partial}{\partial y}axy\\ -\frac{\partial}{\partial x}axy\end{pmatrix}=\begin{pmatrix}ax\\ -ay\end{pmatrix}=\mathbf{F}$$

What Good Are Complex Exponentials? (contd.)

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$$\frac{d}{dz^*}\phi^*=\frac{d}{dz^*}(\mathbf{\Phi}-i\mathbf{A})=\tfrac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(\mathbf{\Phi}-i\mathbf{A})=\tfrac{1}{2}\overbrace{\left(\frac{\partial\mathbf{\Phi}}{\partial x}+i\frac{\partial\mathbf{\Phi}}{\partial y}\right)}+\tfrac{1}{2}\overbrace{\left(\frac{\partial\mathbf{A}}{\partial y}-i\frac{\partial\mathbf{A}}{\partial x}\right)}=\tfrac{1}{2}\nabla\mathbf{\Phi}+\tfrac{1}{2}\nabla\times\mathbf{A}$$

Note, *mathematician definition* of force field **F**=+$\nabla\Phi$ replaces usual physicist’s definition **F**=-$\nabla\Phi$

Given ϕ: $$\boxed{\begin{aligned}\phi &= \quad \mathbf{\Phi} \quad + i \ \mathbf{A}\\ &=\tfrac{1}{2}a(x^2-y^2)+i\ axy\end{aligned}}$$

The half-n’-half result

find:

$$\nabla\mathbf{\Phi}=\begin{pmatrix}\frac{\partial\mathbf{\Phi}}{\partial x}\\ \frac{\partial\mathbf{\Phi}}{\partial y}\end{pmatrix}=\begin{pmatrix}\frac{\partial}{\partial x}\frac{a}{2}(x^2-y^2)\\ \frac{\partial}{\partial y}\frac{a}{2}(x^2-y^2)\end{pmatrix}=\begin{pmatrix}ax\\ -ay\end{pmatrix}=\mathbf{F}$$

or find:

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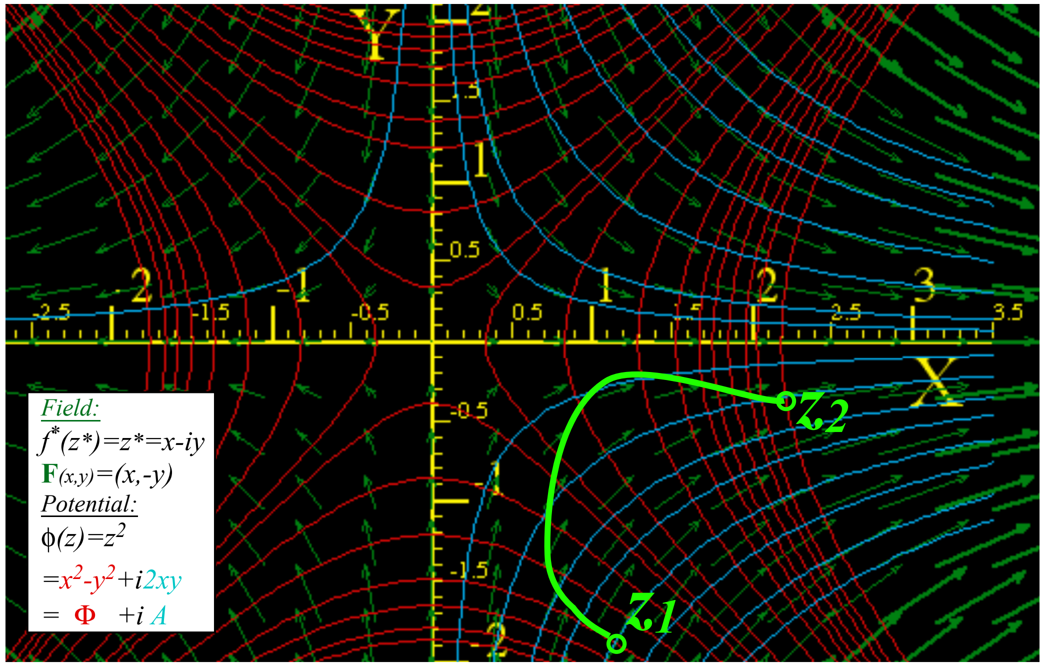
Scalar *static potential lines* **Φ**=*const.* and vector *flux potential lines* **A**=*const.* define *DFL field-net*.



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Derivative $\frac{d\phi^*}{dz^*}$ has 2D gradient $\nabla\Phi = \begin{pmatrix}\frac{\partial\Phi}{\partial x}\\ \frac{\partial\Phi}{\partial y}\end{pmatrix}$ of scalar **Φ** and curl $\nabla\times\mathbf{A} = \begin{pmatrix}\frac{\partial\mathbf{A}}{\partial y}\\ -\frac{\partial\mathbf{A}}{\partial x}\end{pmatrix}$ of vector **A** (*and they're equal!*)

The half-n'-half result

$$\frac{d}{dz^*}\phi^* = \frac{d}{dz^*}(\Phi - i\mathbf{A}) = \tfrac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(\Phi - i\mathbf{A}) = \tfrac{1}{2}\overbrace{\left(\frac{\partial\Phi}{\partial x}+i\frac{\partial\Phi}{\partial y}\right)} + \tfrac{1}{2}\overbrace{\left(\frac{\partial\mathbf{A}}{\partial y}-i\frac{\partial\mathbf{A}}{\partial x}\right)} = \tfrac{1}{2}\nabla\Phi + \tfrac{1}{2}\nabla\times\mathbf{A}$$

Note, *mathematician definition* of force field **F**=+∇**Φ** replaces usual physicist's definition **F**=-∇**Φ**

Given ϕ:

$$\boxed{\begin{aligned}\phi &= \Phi + i\,\mathbf{A}\\ &= \tfrac{1}{2}a(x^2-y^2) + i\,axy\end{aligned}}$$

The half-n'-half result

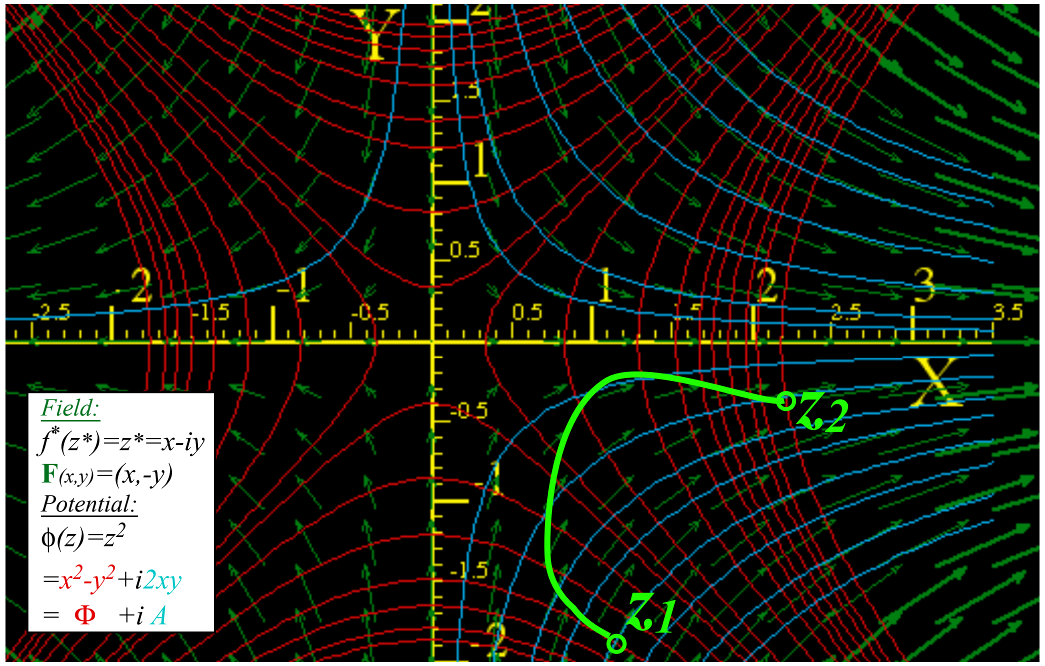
find:

$$\nabla\Phi = \begin{pmatrix}\frac{\partial\Phi}{\partial x}\\ \frac{\partial\Phi}{\partial y}\end{pmatrix} = \begin{pmatrix}\frac{\partial}{\partial x}\frac{a}{2}(x^2-y^2)\\ \frac{\partial}{\partial y}\frac{a}{2}(x^2-y^2)\end{pmatrix} = \begin{pmatrix}ax\\ -ay\end{pmatrix} = \mathbf{F}$$

or find:

$$\nabla\times\mathbf{A} = \begin{pmatrix}\frac{\partial\mathbf{A}}{\partial y}\\ -\frac{\partial\mathbf{A}}{\partial x}\end{pmatrix} = \begin{pmatrix}\frac{\partial}{\partial y}axy\\ -\frac{\partial}{\partial x}axy\end{pmatrix} = \begin{pmatrix}ax\\ -ay\end{pmatrix} = \mathbf{F}$$

Scalar *static potential lines* **Φ**=*const.* and vector *flux potential lines* **A**=*const.* define *DFL field-net*.



Field:
$f^*(z^*)=z^*=x-iy$
F$(x,y)=(x,-y)$
Potential:
$\phi(z)=z^2$
$=x^2-y^2+i2xy$
$=\ \Phi\ +i\ A$

The half-n'-half results

are called

Riemann-Cauchy

Derivative Relations

$$\frac{\partial\Phi}{\partial x}=\frac{\partial\mathbf{A}}{\partial y}\quad\text{is:}\quad\frac{\partial\,\mathrm{Re}f(z)}{\partial x}=\frac{\partial\,\mathrm{Im}f(z)}{\partial y}$$

$$\frac{\partial\Phi}{\partial y}=-\frac{\partial\mathbf{A}}{\partial x}\quad\text{is:}\quad\frac{\partial\,\mathrm{Re}f(z)}{\partial y}=-\frac{\partial\,\mathrm{Im}f(z)}{\partial x}$$

→ *4. Riemann-Cauchy conditions* *What's analytic? (...and what's <u>not</u>?)*

Review (z,z*) to (x,y) transformation relations

$$z = x + iy \qquad x = \tfrac{1}{2}(z + z^*)$$

$$z^* = x - iy \qquad y = \tfrac{1}{2i}(z - z^*)$$

$$\frac{df}{dz} = \frac{\partial x}{\partial z}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{1}{2i}\frac{\partial f}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)f$$

$$\frac{df}{dz^*} = \frac{\partial x}{\partial z^*}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z^*}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{1}{2i}\frac{\partial f}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)f$$

Criteria for a field function $f = f_x(x,y) + i\, f_y(x,y)$ to be an ***analytic function f(z)*** of $z=x+iy$:

First, f(z) must not be a function of $z^*=x-iy$, that is: $\frac{df}{dz^*} = 0$

This implies f(z) satisfies differential equations known as the **Riemann-Cauchy conditions**

$$\frac{df}{dz^*} = 0 = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial f_x}{\partial x} - \frac{\partial f_y}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_y}{\partial x} + \frac{\partial f_x}{\partial y}\right) \text{ implies: } \boxed{\frac{\partial f_x}{\partial x} = \frac{\partial f_y}{\partial y} \quad \text{and: } \quad \frac{\partial f_y}{\partial x} = -\frac{\partial f_x}{\partial y}}$$

$$\frac{df}{dz} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right) = \frac{\partial f_x}{\partial x} + i\frac{\partial f_y}{\partial x} = \frac{\partial f_y}{\partial y} - i\frac{\partial f_x}{\partial y} = \frac{\partial}{\partial x}(f_x + i f_y) = \frac{\partial}{\partial iy}(f_x + i f_y)$$

Review (z,z) to (x,y) transformation relations*

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Criteria for a field function $f = f_x(x,y) + i\, f_y(x,y)$ to be an ***analytic function f(z*)*** *of $z^*=x-iy$:*

First, f(z) must not be a function of $z=x+iy$, that is:* $\frac{df}{dz} = 0$

This implies f(z) satisfies differential equations we call* ***Anti-Riemann-Cauchy conditions***

$$\frac{df}{dz} = 0 = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right) = \text{implies: } \frac{\partial f_x}{\partial x} = -\frac{\partial f_y}{\partial y} \quad \text{and: } \frac{\partial f_y}{\partial x} = \frac{\partial f_x}{\partial y}$$

$$\frac{df}{dz^*} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial f_x}{\partial x} - \frac{\partial f_y}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_y}{\partial x} + \frac{\partial f_x}{\partial y}\right) = \frac{\partial f_x}{\partial x} + i\frac{\partial f_y}{\partial x} = -\frac{\partial f_y}{\partial y} + i\frac{\partial f_x}{\partial y} = \frac{\partial}{\partial x}(f_x + i f_y) = -\frac{\partial}{\partial iy}(f_x + i f_y)$$

What's analytic? (...and what's not?)

Example: Is $f(x,y) = 2x + iy$ an analytic function of $z=x+iy$?

What's analytic? (...and what's not?)

Example: Q: Is $f(x,y) = 2x + i4y$ an analytic function of $z=x+iy$?

Well, test it using definitions: $z = x + iy$ and: $z^* = x - iy$

or: $x = (z+z^*)/2$ and: $y = -i(z-z^*)/2$

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$$= z+z^* + (2z-2z^*)$$

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$$\begin{aligned} f(x,y) = 2x + i4y &= 2\,(z+z^*)/2 + i4(-i(z-z^*)/2) \\ &= z+z^* + (2z-2z^*) \\ &= 3z-z^* \end{aligned}$$

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$$

A: *NO!* *It's a function of z and z* so not analytic for either.*

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A: *NO!* *It's a function of z <u>and</u> z* so not analytic for <u>either</u>.*

Example 2: Q: Is $r(x,y) = x^2 + y^2$ an analytic function of $z=x+iy$?

A: *NO!* *r(xy)=z*z is a function of z <u>and</u> z* so not analytic for <u>either</u>.*

What's analytic? (...and what's not?)

Example: Q: Is $f(x,y) = 2x + i4y$ an analytic function of $z=z+iy$?

Well, test it using definitions: $z = x + iy$ and: $z^* = x - iy$
or: $x = (z+z^*)/2$ and: $y =-i(z-z^*)/2$

$$
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f(x,y) = 2x + i4y &= 2\,(z+z^*)/2 + i4(-i(z-z^*)/2) \\
&= z+z^* \quad + \quad (2z-2z^*) \\
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\end{aligned}
$$

A: *NO!* *It's a function of* z *and* z^* *so not analytic for either.*

Example 2: Q: Is $r(x,y) = x^2 + y^2$ an analytic function of $z=x+iy$?

A: *NO!* $r(xy)=z^*z$ *is a function of* z *and* z^* *so not analytic for either.*

Example 3: Q: Is $s(x,y) = x^2-y^2 + 2ixy$ an analytic function of $z=x+iy$?

A: *YES!* $s(xy)=(x+iy)^2 =z^2$ *is analytic function of* z*.* *(Yay!)*

4. *Riemann-Cauchy conditions* *What’s analytic? (...and what’s not?)*

→ *Easy 2D circulation and flux integrals*
- *Easy 2D curvilinear coordinate discovery*
  - *Easy 2D monopole, dipole, and 2^n-pole analysis*
    - *Easy 2^n-multipole field and potential expansion*
      - *Easy stereo-projection visualization*

What Good Are Complex Exponentials? (contd.)

*9. Complex integrals ∫ f(z)dz count 2D "circulation"(∫**F•dr**) and "flux"(∫**Fxdr**)*

Integral of *f(z)* between point z_1 and point z_2 is potential difference $\Delta\phi = \phi(z_2) - \phi(z_1)$

$$\Delta\phi = \phi(z_2) - \phi(z_1) = \int_{z_1}^{z_2} f(z)dz = \underbrace{\mathbf{\Phi}(x_2,y_2) - \mathbf{\Phi}(x_1,y_1)}_{\Delta\mathbf{\Phi}} + i\underbrace{[\mathbf{A}(x_2,y_2) - \mathbf{A}(x_1,y_1)]}_{\Delta\mathbf{A}}$$

$$\Delta\phi = \Delta\mathbf{\Phi} + i\,\Delta\mathbf{A}$$

In *DFL*-field **F**, $\Delta\phi$ is independent of the integration path $z(t)$ connecting z_1 and z_2.

What Good Are Complex Exponentials? (contd.)

*9. Complex integrals ∫ f(z)dz count 2D "circulation"(∫**F•dr**) and "flux"(∫**F**x**dr**)*

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$$\Delta\phi = \phi(z_2) - \phi(z_1) = \int_{z_1}^{z_2} f(z)dz = \underbrace{\mathbf{\Phi}(x_2,y_2) - \mathbf{\Phi}(x_1,y_1)}_{\Delta\mathbf{\Phi}} + i\,[\underbrace{\mathbf{A}(x_2,y_2) - \mathbf{A}(x_1,y_1)}_{\Delta\mathbf{A}}]$$

$$\Delta\phi = \qquad \Delta\mathbf{\Phi} \qquad + i \qquad \Delta\mathbf{A}$$

In *DFL*-field **F**, $\Delta\phi$ is independent of the integration path $z(t)$ connecting z_1 and z_2.

$$\int f(z)\,dz = \int\left(f^*(z^*)\right)^* dz = \int\left(f^*(z^*)\right)^*\left(dx + i\,dy\right) = \int\left(f_x^* + i\,f_y^*\right)^*\left(dx + i\,dy\right) = \int\left(f_x^* - i\,f_y^*\right)\left(dx + i\,dy\right)$$

$$= \int(f_x^*\,dx + f_y^*\,dy) + i\int(f_x^*\,dy - f_y^*\,dx)$$

$$= \int \mathbf{F}\bullet d\mathbf{r} \quad + i\int \mathbf{F}\times d\mathbf{r}\bullet\hat{\mathbf{e}}_Z$$

$$= \int \mathbf{F}\bullet d\mathbf{r} \quad + i\int \mathbf{F}\bullet d\mathbf{r}\times\hat{\mathbf{e}}_Z$$

$$= \int \mathbf{F}\bullet d\mathbf{r} \quad + i\int \mathbf{F}\bullet d\mathbf{S} \qquad \text{where:} \quad d\mathbf{S} = d\mathbf{r}\times\hat{\mathbf{e}}_Z$$

What Good Are Complex Exponentials? (contd.)

9. Complex integrals ∫ f(z)dz count 2D "circulation"(∫F•dr) and "flux"(∫Fxdr)

Integral of $f(z)$ between point z_1 and point z_2 is potential difference $\Delta\phi = \phi(z_2) - \phi(z_1)$

$$\Delta\phi = \phi(z_2) - \phi(z_1) = \int_{z_1}^{z_2} f(z)dz = \underbrace{\Phi(x_2,y_2) - \Phi(x_1,y_1)} + i[\underbrace{A(x_2,y_2) - A(x_1,y_1)}]$$

$$\Delta\phi = \qquad \Delta\Phi \qquad + i \qquad \Delta A$$

In *DFL*-field **F**, $\Delta\phi$ is independent of the integration path $z(t)$ connecting z_1 and z_2.

$$\int f(z)dz = \int \left(f^*(z^*)\right)^* dz = \int \left(f^*(z^*)\right)^* (dx + i\,dy) = \int \left(f_x^* + i\,f_y^*\right)^* (dx + i\,dy) = \int \left(f_x^* - i\,f_y^*\right)(dx + i\,dy)$$

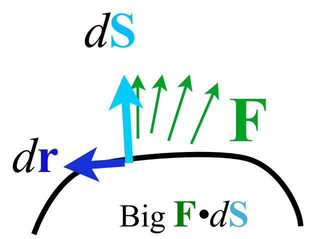
$$= \int (f_x^*\,dx + f_y^*\,dy) + i \int (f_x^*\,dy - f_y^*\,dx)$$

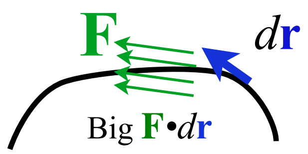
$$= \qquad \int \mathbf{F} \bullet d\mathbf{r} \qquad + i \int \mathbf{F} \times d\mathbf{r} \bullet \hat{\mathbf{e}}_Z$$

$$= \qquad \int \mathbf{F} \bullet d\mathbf{r} \qquad + i \int \mathbf{F} \bullet d\mathbf{r} \times \hat{\mathbf{e}}_Z$$

$$= \qquad \boxed{\int \mathbf{F} \bullet d\mathbf{r}} \qquad \boxed{+ i \int \mathbf{F} \bullet d\mathbf{S}}$$

where: $d\mathbf{S} = d\mathbf{r} \times \hat{\mathbf{e}}_Z$





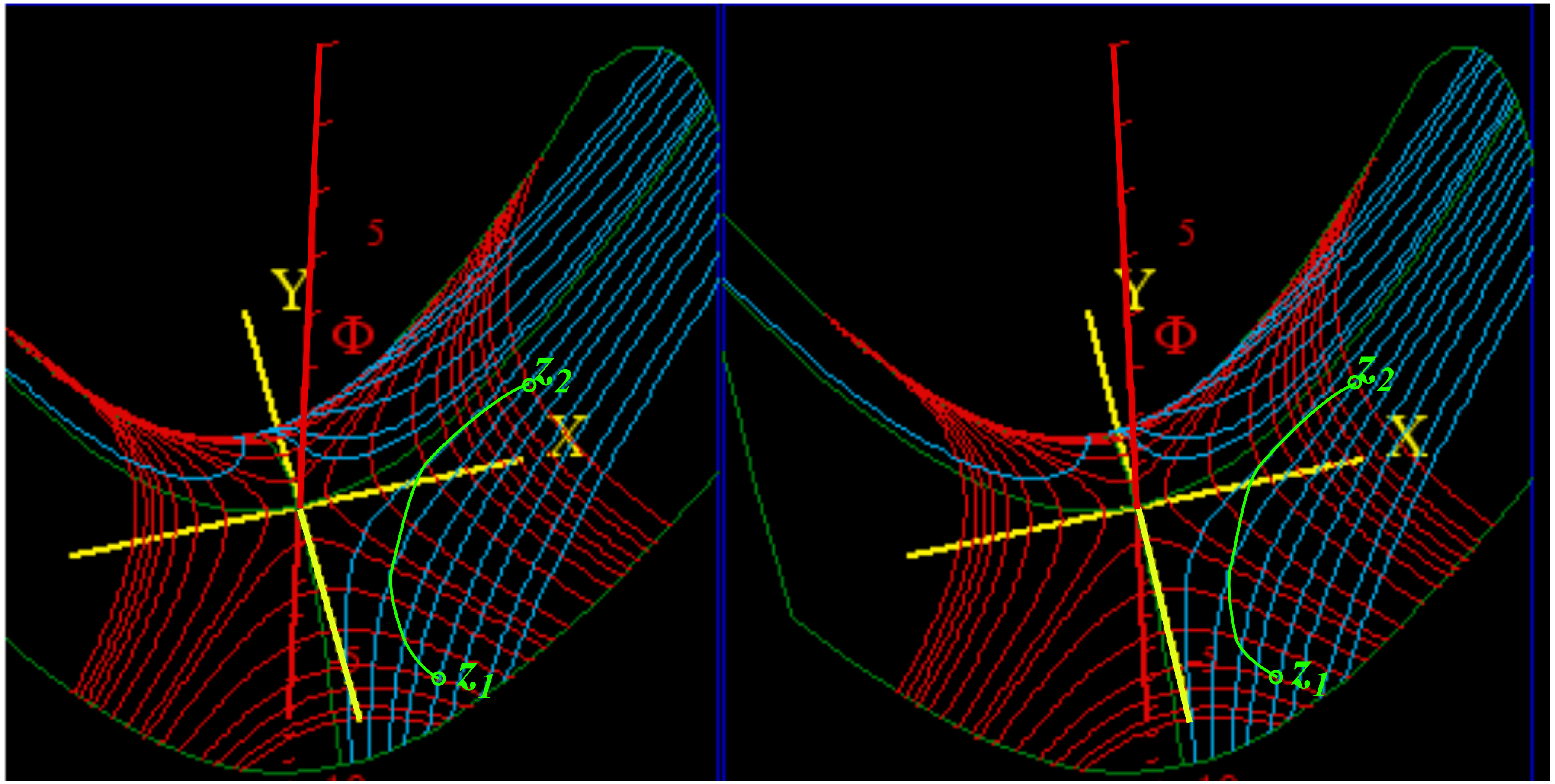
Real part $\int_1^2 \mathbf{F} \bullet d\mathbf{r} = \Delta\Phi$
sums **F** projections *along* path
*d***r** that is, *circulation* on path
to get $\Delta\Phi$.

Imaginary part $\int_1^2 \mathbf{F} \bullet d\mathbf{S} = \Delta A$
sums **F** projection *across* path *d***r**
that is, *flux* thru surface
elements $d\mathbf{S} = d\mathbf{r} \times \mathbf{e}_Z$ normal to *d***r**
to get ΔA.

Here the scalar potential $\Phi=(x^2-y^2)/2$ *is stereo-plotted vs.* (x,y)

The $\Phi=(x^2-y^2)/2=const.$ *curves are topography lines*

The $A=(xy)=const.$ *curves are streamlines normal to topography lines*



4. Riemann-Cauchy conditions *What's analytic? (...and what's not?)*

Easy 2D circulation and flux integrals

→ *Easy 2D curvilinear coordinate discovery*

Easy 2D monopole, dipole, and 2^n-pole analysis

Easy 2^n-multipole field and potential expansion

Easy stereo-projection visualization

What Good Are Complex Exponentials? (contd.)

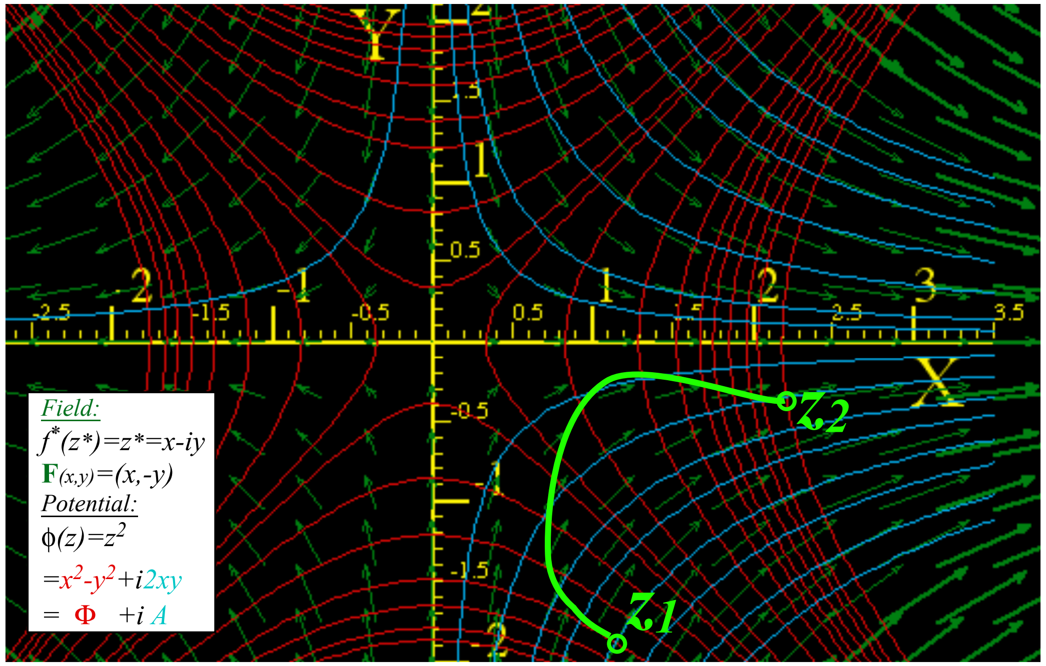
10. Complex potentials define 2D Orthogonal Curvilinear Coordinates (OCC) of field

The (Φ,A) grid is a GCC coordinate system*:

$q^1 = \Phi = (x^2-y^2)/2 = const.$

$q^2 = A = (xy) = const.$

*Actually it's OCC.



$$Kajobian = \begin{pmatrix} \frac{\partial q^1}{\partial x} & \frac{\partial q^1}{\partial y} \\ \frac{\partial q^2}{\partial x} & \frac{\partial q^2}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial \Phi}{\partial x} & \frac{\partial \Phi}{\partial y} \\ \frac{\partial A}{\partial x} & \frac{\partial A}{\partial y} \end{pmatrix} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{array}{l} \leftarrow \mathbf{E}^{\Phi} \\ \leftarrow \mathbf{E}^{A} \end{array}$$

$$Jacobian = \begin{pmatrix} \frac{\partial x}{\partial q^1} & \frac{\partial x}{\partial q^2} \\ \frac{\partial y}{\partial q^1} & \frac{\partial y}{\partial q^2} \end{pmatrix} = \begin{array}{cc} \begin{pmatrix} \frac{\partial x}{\partial \Phi} & \frac{\partial x}{\partial A} \\ \frac{\partial y}{\partial \Phi} & \frac{\partial y}{\partial A} \end{pmatrix} \\ \begin{array}{cc} \uparrow & \uparrow \\ \mathbf{E}_{\Phi} & \mathbf{E}_{A} \end{array} \end{array} = \frac{1}{r^2} \begin{array}{c} \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \\ \begin{array}{cc} \uparrow & \uparrow \\ \mathbf{E}_{\Phi} & \mathbf{E}_{A} \end{array} \end{array}$$

$$Metric\,tensor = \begin{pmatrix} g_{\Phi\Phi} & g_{\Phi A} \\ g_{A\Phi} & g_{AA} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_{\Phi}\bullet\mathbf{E}_{\Phi} & \mathbf{E}_{\Phi}\bullet\mathbf{E}_{A} \\ \mathbf{E}_{A}\bullet\mathbf{E}_{\Phi} & \mathbf{E}_{A}\bullet\mathbf{E}_{A} \end{pmatrix} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2 \end{pmatrix} \text{ where: } r^2 = x^2 + y^2$$

What Good Are Complex Exponentials? (contd.)

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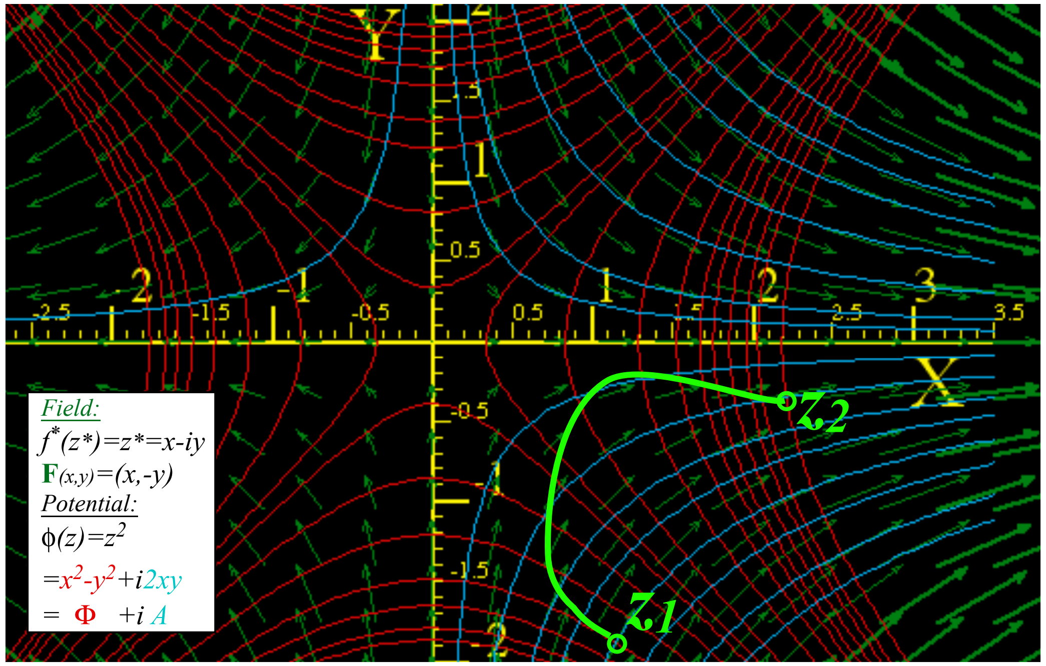

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Riemann-Cauchy Derivative Relations make coordinates orthogonal

$$\nabla\Phi = \begin{pmatrix} \frac{\partial \Phi}{\partial x} \\ \frac{\partial \Phi}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x}\frac{a}{2}(x^2 - y^2) \\ \frac{\partial}{\partial y}\frac{a}{2}(x^2 - y^2) \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = \mathbf{F}$$

The half-n'-half results assure

$$\mathbf{E}_{\Phi}\bullet\mathbf{E}_{A} = \frac{\partial \Phi}{\partial x}\frac{\partial A}{\partial x} + \frac{\partial \Phi}{\partial y}\frac{\partial A}{\partial y}$$
$$= -\frac{\partial \Phi}{\partial x}\frac{\partial \Phi}{\partial y} + \frac{\partial \Phi}{\partial y}\frac{\partial \Phi}{\partial x} = 0$$

$$\nabla\times\mathbf{A} = \begin{pmatrix} \frac{\partial \mathbf{A}}{\partial y} \\ -\frac{\partial \mathbf{A}}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial y} axy \\ -\frac{\partial}{\partial x} axy \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = \mathbf{F}$$

What Good Are Complex Exponentials? (contd.)

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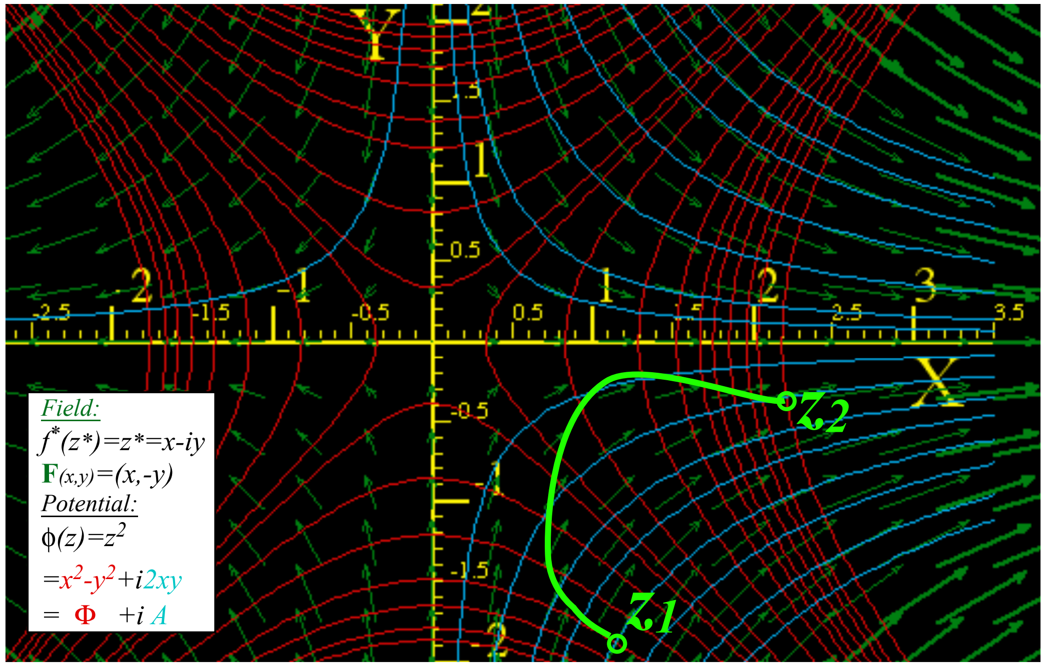

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The half-n'-half results assure

$$\mathbf{E}_{\Phi}\bullet\mathbf{E}_{A} = \frac{\partial \Phi}{\partial x}\frac{\partial A}{\partial x} + \frac{\partial \Phi}{\partial y}\frac{\partial A}{\partial y} = -\frac{\partial \Phi}{\partial x}\frac{\partial \Phi}{\partial y} + \frac{\partial \Phi}{\partial y}\frac{\partial \Phi}{\partial x} = 0$$

$$\nabla\times\mathbf{A} = \begin{pmatrix} \frac{\partial \mathbf{A}}{\partial y} \\ -\frac{\partial \mathbf{A}}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial y}axy \\ -\frac{\partial}{\partial x}axy \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = \mathbf{F}$$

Zero divergence requirement: $0 = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} = \frac{\partial}{\partial x}\frac{\partial \Phi}{\partial x} + \frac{\partial}{\partial y}\frac{\partial \Phi}{\partial y} = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$ *potential Φ obeys Laplace equation*

What Good Are Complex Exponentials? (contd.)

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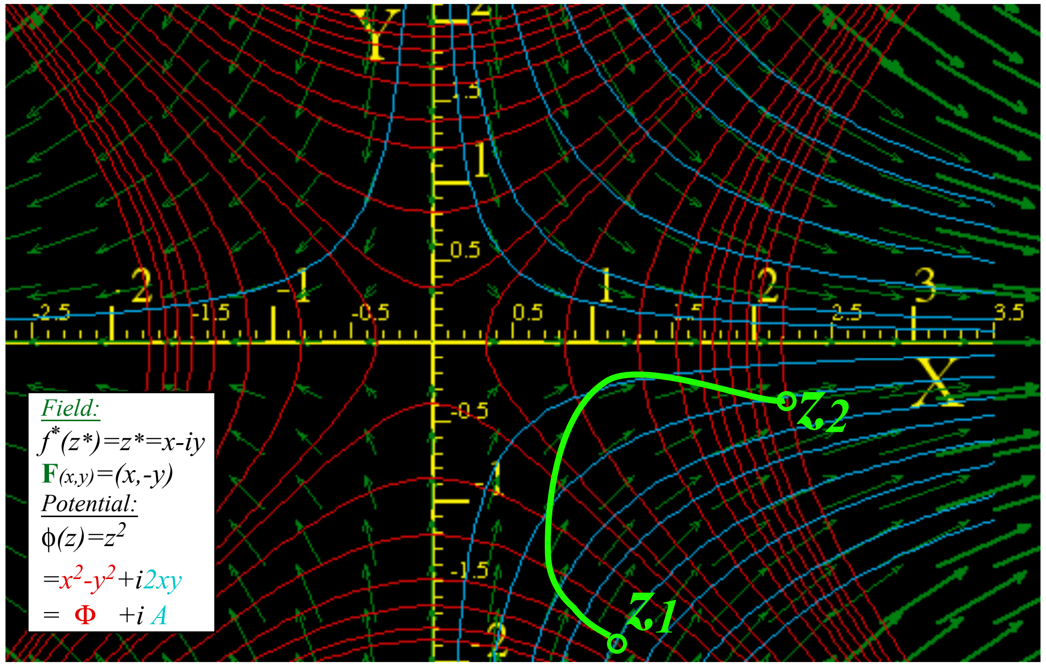

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$$\begin{matrix} \uparrow & \uparrow \\ \mathbf{E}_{\Phi} & \mathbf{E}_{A} \end{matrix} \qquad \begin{matrix} \uparrow & \uparrow \\ \mathbf{E}_{\Phi} & \mathbf{E}_{A} \end{matrix}$$

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$$= -\frac{\partial \Phi}{\partial x}\frac{\partial \Phi}{\partial y} + \frac{\partial \Phi}{\partial y}\frac{\partial \Phi}{\partial x} = 0$$

$$\nabla\times A = \begin{pmatrix} \frac{\partial A}{\partial y} \\ -\frac{\partial A}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial y}axy \\ -\frac{\partial}{\partial x}axy \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = \mathbf{F}$$

or Riemann-Cauchy

Zero divergence requirement:

$$0 = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} = \frac{\partial}{\partial x}\frac{\partial \Phi}{\partial x} + \frac{\partial}{\partial y}\frac{\partial \Phi}{\partial y} = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$

and so does A

potential Φ obeys Laplace equation

4. Riemann-Cauchy conditions *What's analytic? (...and what's not?)*

- *Easy 2D circulation and flux integrals*
  - *Easy 2D curvilinear coordinate discovery*
    - → *Easy 2D monopole, dipole, and 2^n-pole analysis*
      - *Easy 2^n-multipole field and potential expansion*
        - *Easy stereo-projection visualization*

11. Complex integrals define 2D monopole fields and potentials

Of all power-law fields $f(z)=az^n$ one lacks a power-law potential $\phi(z)=\frac{a}{n+1}z^{n+1}$. It is the $n = -1$ case.

Unit monopole field: $f(z)=\frac{1}{z}=z^{-1}$

$f(z)=\frac{a}{z}=az^{-1}$ Source-a monopole

It has a *logarithmic potential* $\phi(z)=a\cdot\ln(z)=a\cdot\ln(x+iy)$.

What Good Are Complex Exponentials? (contd.)

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$$\phi(z)=\quad\Phi\quad+\quad i\mathbf{A}=\int f(z)dz=\int\frac{a}{z}dz=a\ln(z)$$

What Good Are Complex Exponentials? (contd.)

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$$\phi(z) = \Phi + i\mathbf{A} = \int f(z)dz = \int \frac{a}{z}dz = a\ln(z) = a\ln(re^{i\theta})$$

$$= \underbrace{a\ln(r)}_{} + i\,\underbrace{a\theta}_{}$$

What Good Are Complex Exponentials? (contd.)

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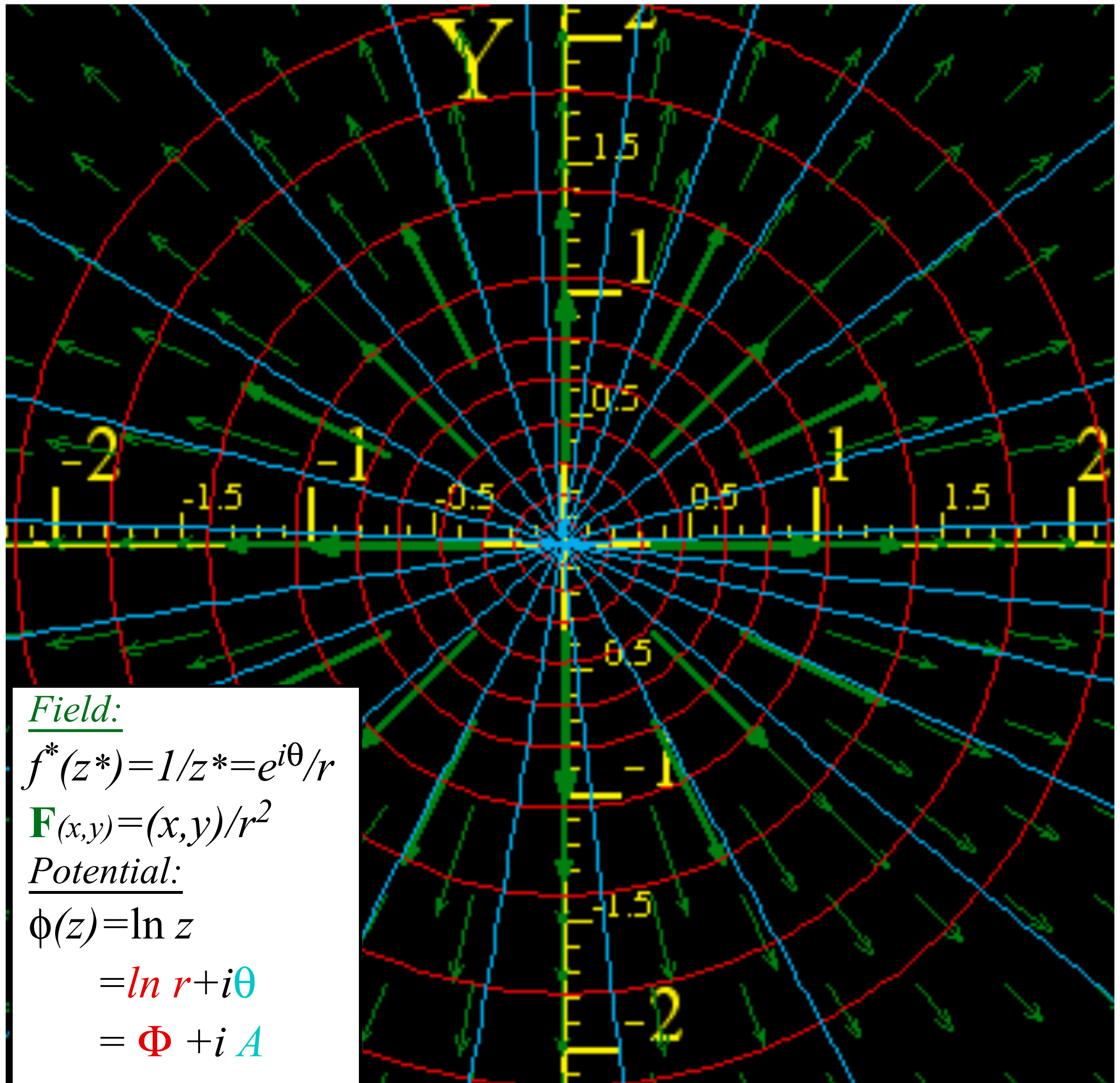
$$= \overbrace{a\ln(r)} + i\,\overbrace{a\theta}$$

(a) Unit Z-line-flux field $f(z)=1/z$



Field:

$f^*(z^*)=1/z^*=e^{i\theta}/r$

$\mathbf{F}(x,y)=(x,y)/r^2$

Potential:

$\phi(z)=\ln z$

$=\ln r+i\theta$

$=\Phi+iA$

Lecture 12 Mon. 10.01
May end here

What Good Are Complex Exponentials? (contd.)

11. Complex integrals define 2D monopole fields and potentials

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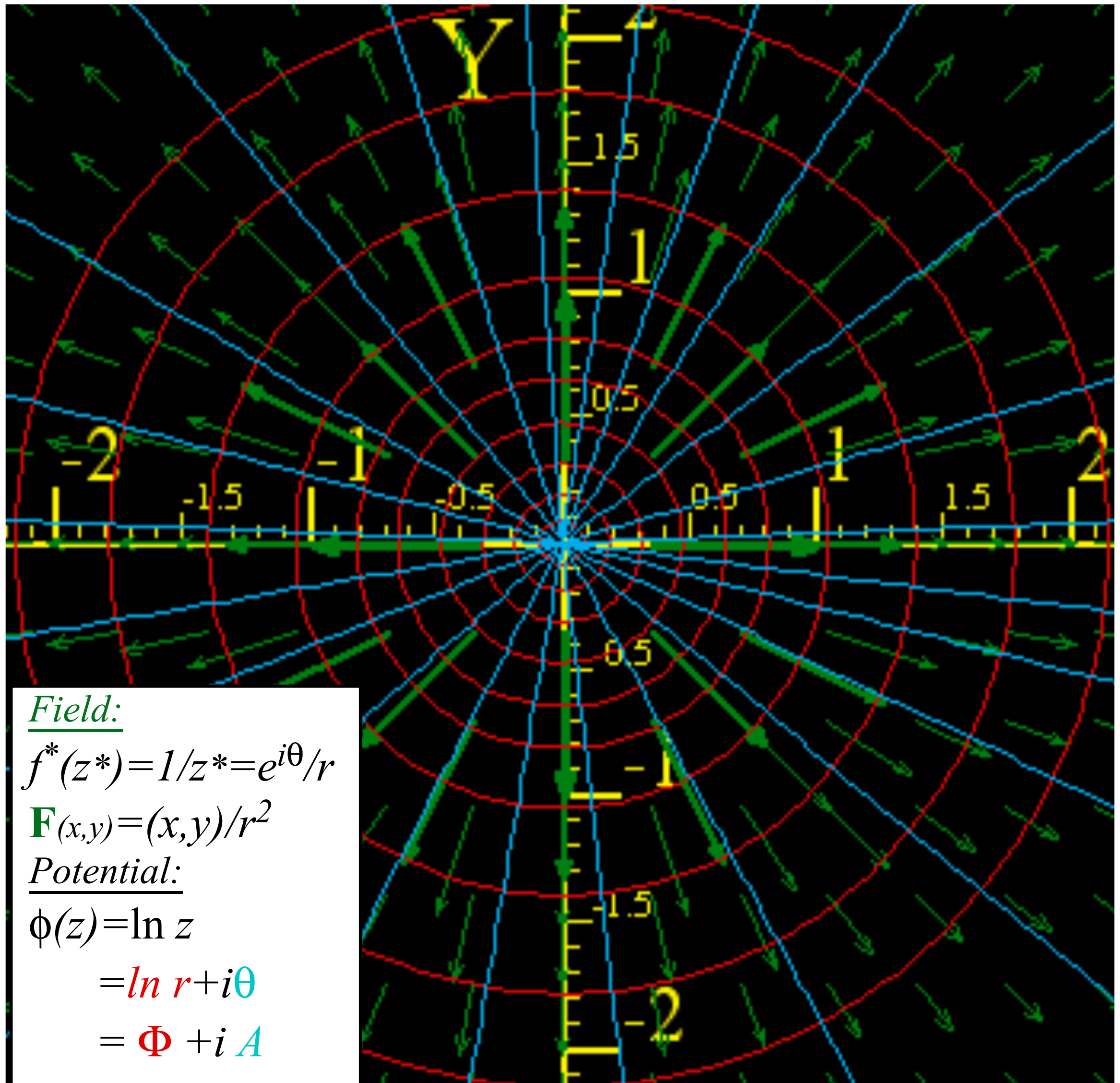
$$\phi(z)=\underbrace{\Phi}_{=a\ln(r)} + i\underbrace{\mathbf{A}}_{a\theta} = \int f(z)dz = \int \frac{a}{z}dz = a\ln(z) = a\ln(re^{i\theta})$$

(a) Unit Z-line-flux field f(z)=1/z



Field:

$f^*(z^*)=1/z^*=e^{i\theta}/r$

$\mathbf{F}_{(x,y)}=(x,y)/r^2$

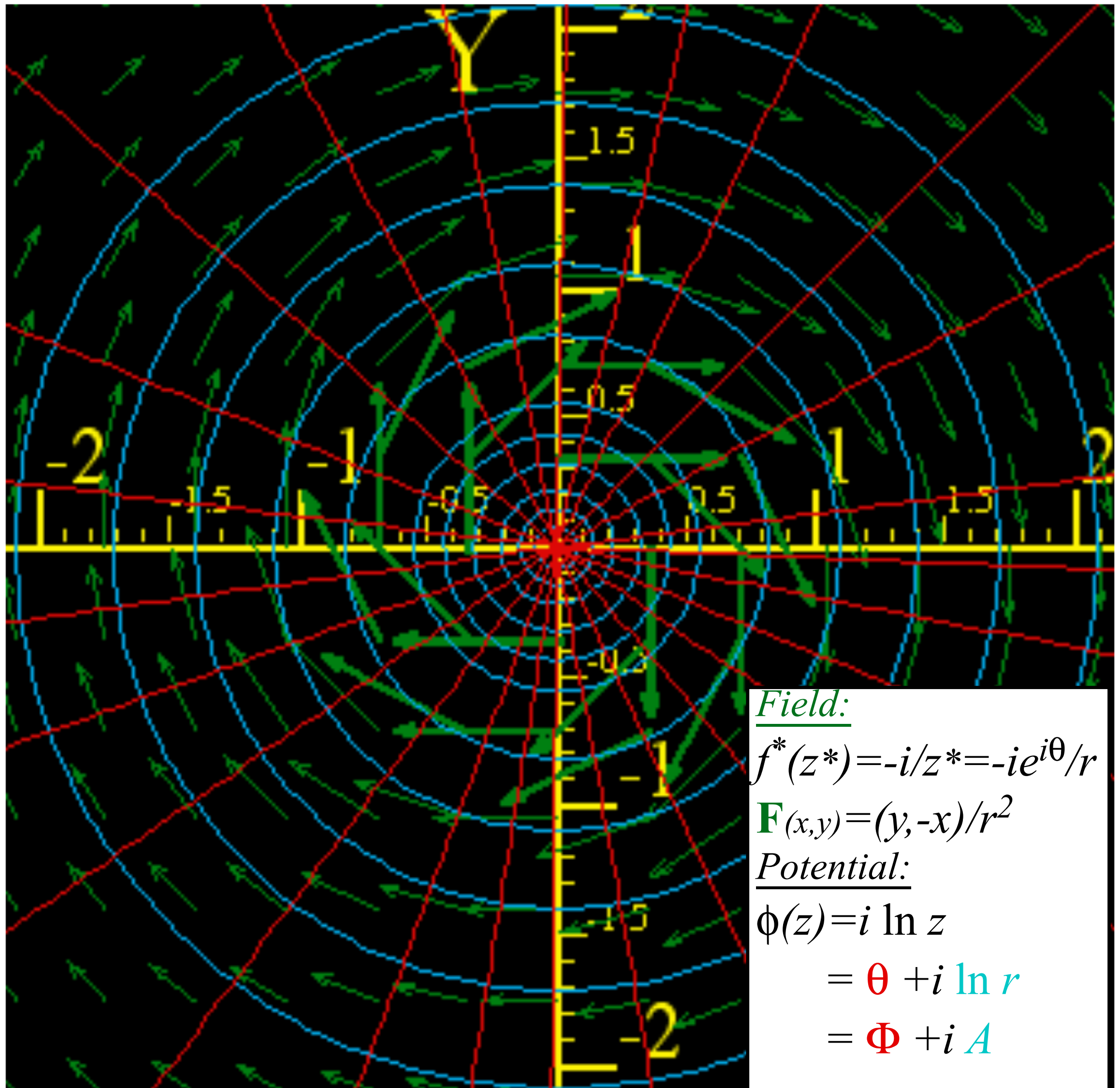
Potential:

$\phi(z)=\ln z$

$=\ln r+i\theta$

$=\Phi+iA$

(b) Unit Z-line-vortex field f(z)=i/z



Field:

$f^*(z^*)=-i/z^*=-ie^{i\theta}/r$

$\mathbf{F}_{(x,y)}=(y,-x)/r^2$

Potential:

$\phi(z)=i\ln z$

$=\theta+i\ln r$

$=\Phi+iA$

What Good Are Complex Exponentials? (contd.)

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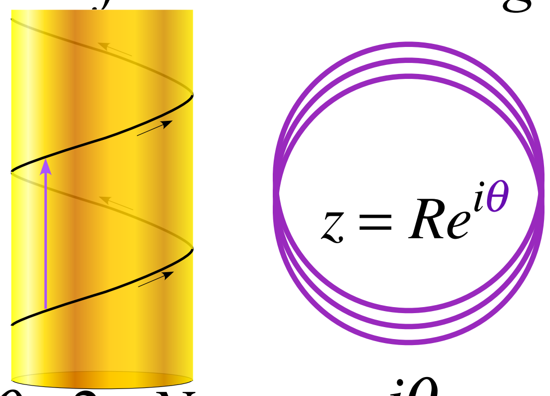
$$\phi(z)=\underbrace{\Phi}_{=a\ln(r)}+i\underbrace{\mathbf{A}}_{a\theta}=\int f(z)dz=\int\frac{a}{z}dz=a\ln(z)=a\ln(re^{i\theta})$$

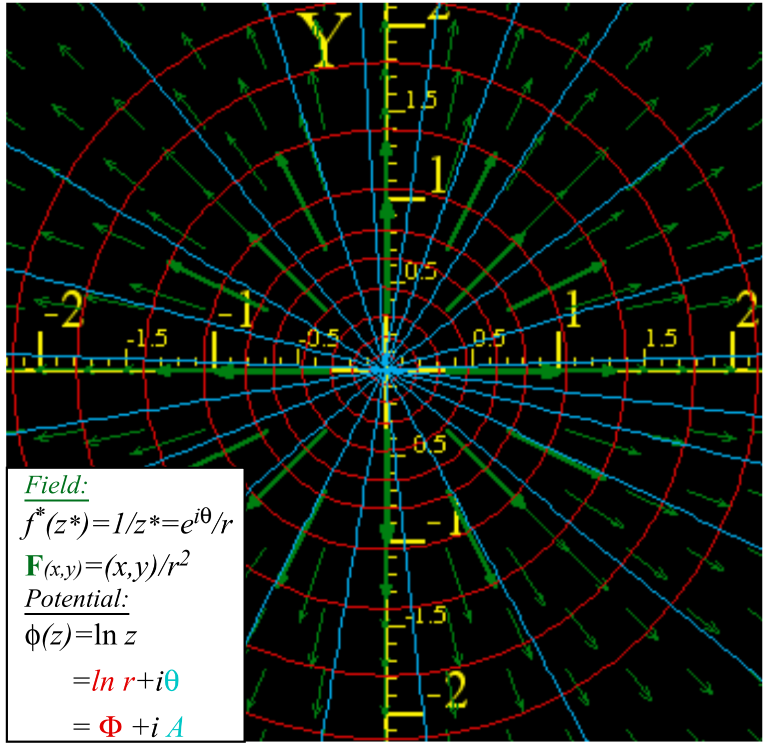
A monopole field is the only power-law field whose integral (potential) depends on path of integration.



path that goes N times around origin (r=0) at constant r = R.

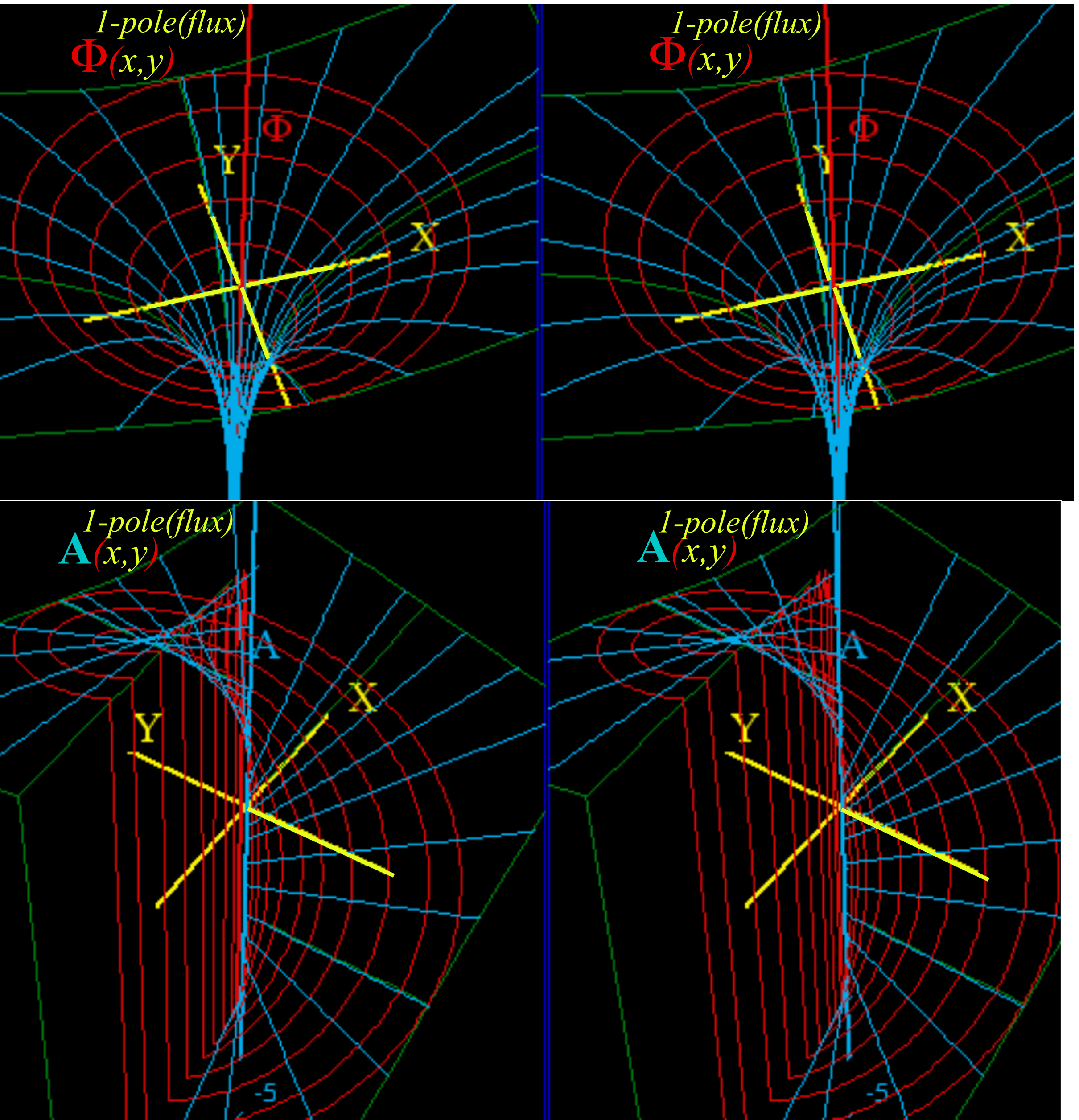
$$\Delta\phi=\oint f(z)dz=a\oint\frac{dz}{z}=a\int_{\theta=0}^{\theta=2\pi N}\frac{d(Re^{i\theta})}{Re^{i\theta}}=a\int_{\theta=0}^{\theta=2\pi N}id\theta=ai\,\theta\Big|_0^{2\pi N}=2a\pi iN$$

(a) Unit Z-line-flux field f(z)=1/z



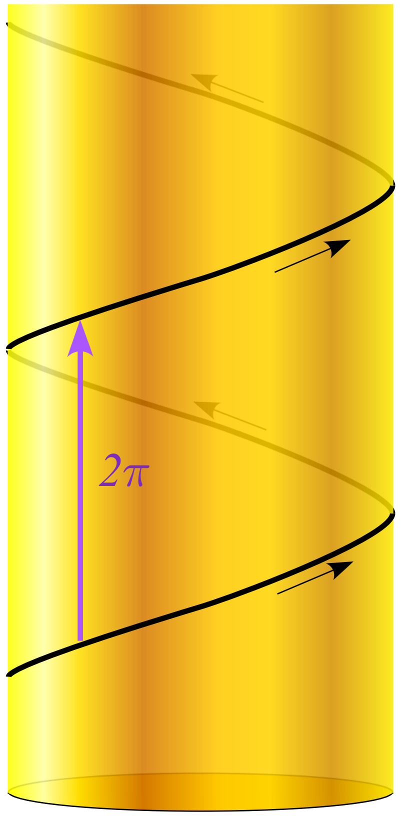
$$\begin{aligned}\phi(z) &= \underbrace{\Phi} + i\underbrace{\mathbf{A}} = \int f(z)dz = \int \frac{a}{z}dz = a\ln(re^{i\theta}) \\ &= \ln(r) + i\theta\end{aligned}$$

(For a=1)



Each turn around origin

*adds 2πi to vector potential i**A***



(For a=1)

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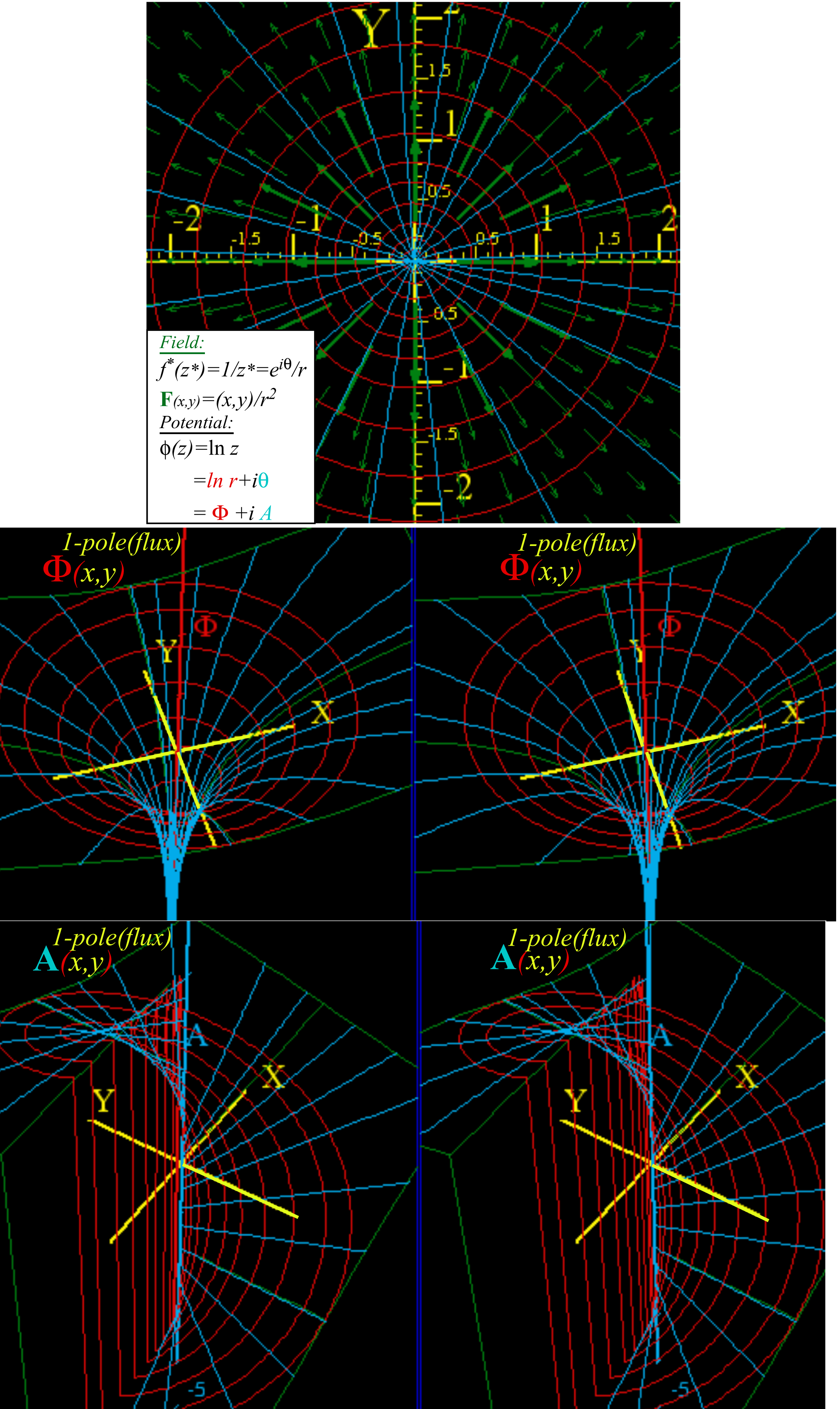
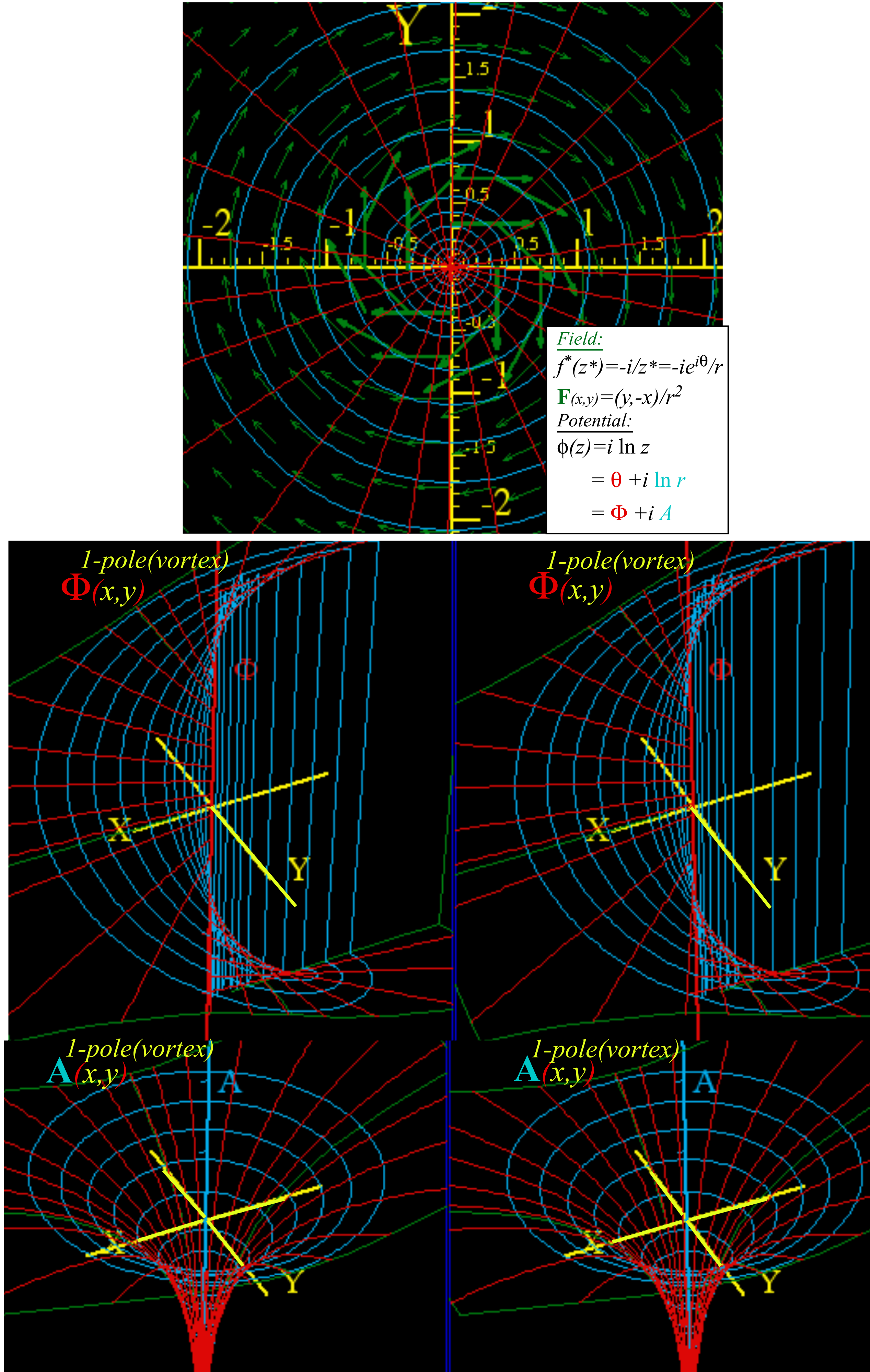


(b) Unit Z-line-vortex field f(z)=i/z

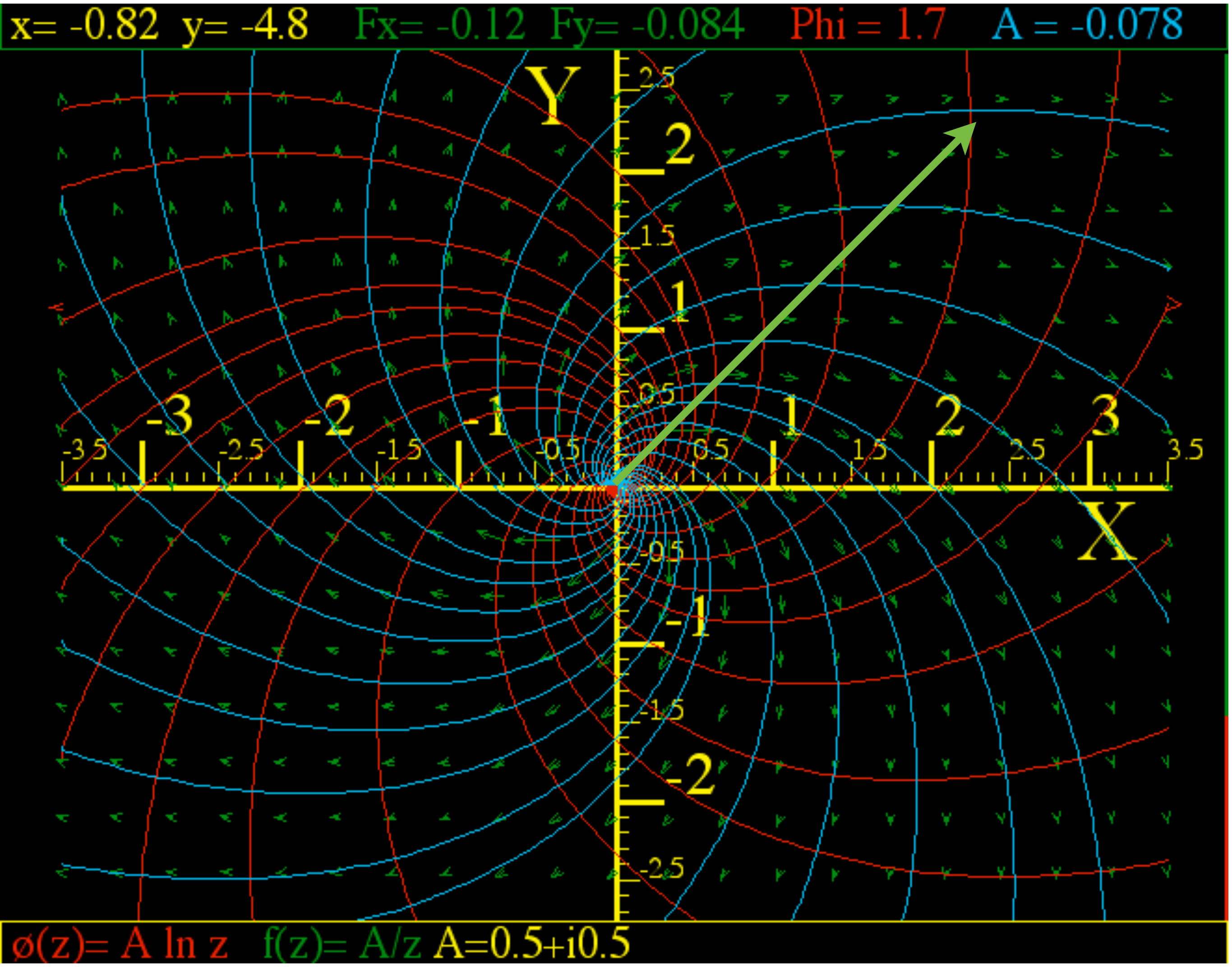


What Good Are Complex Exponentials? (contd.)

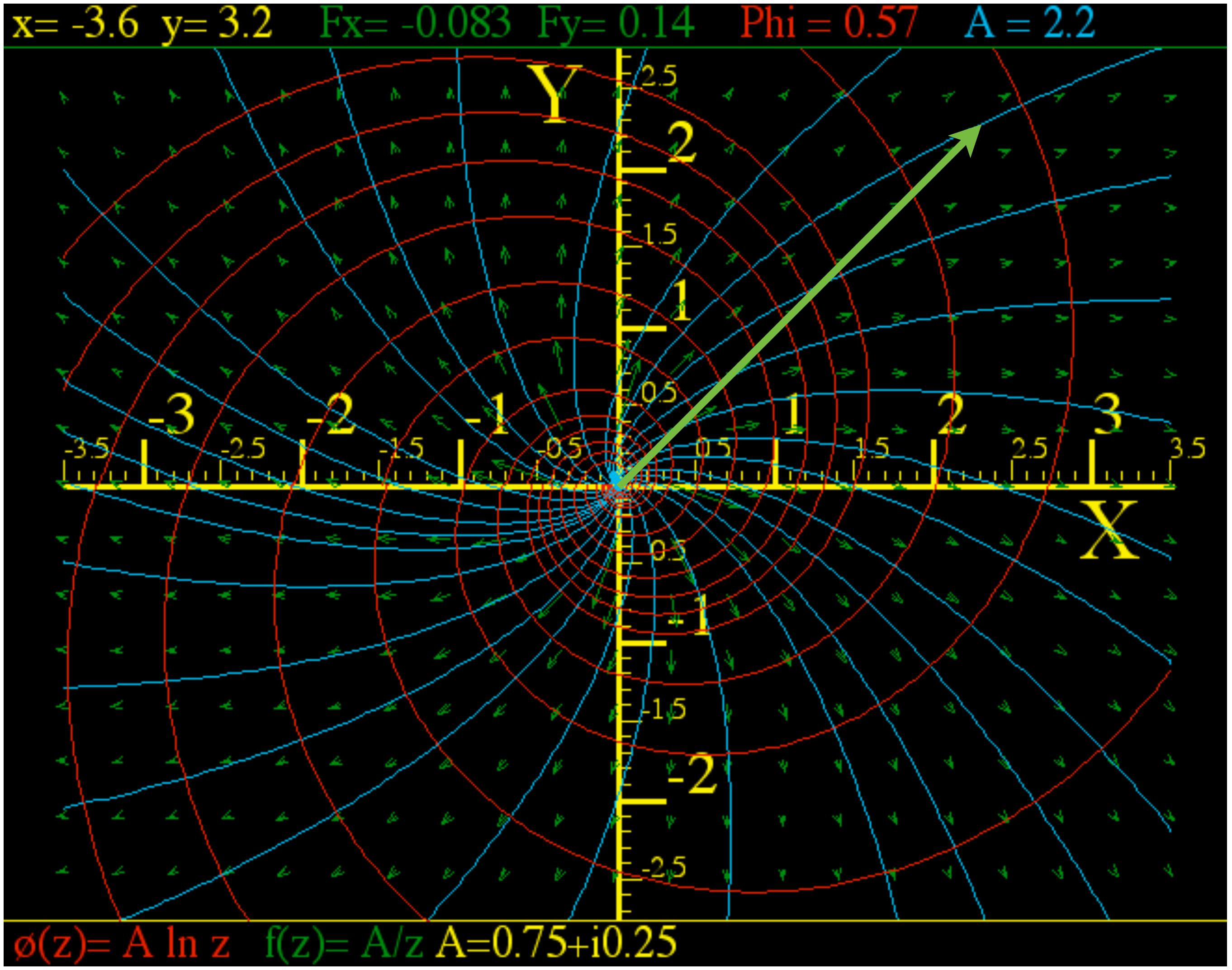
$f(z) = (0.5 + i0.5)/z = e^{i\pi/4}/z\sqrt{2}$

$f(z) = (0.75 + i0.25)/z = e^{i18^\circ}/z\sqrt{n}$

“Vortex”



“Hurricane”



4. Riemann-Cauchy conditions *What's analytic? (...and what's not?)*

- *Easy 2D circulation and flux integrals*
  - *Easy 2D curvilinear coordinate discovery*
    - → *Easy 2D monopole, dipole, and 2^n-pole analysis*
      - *Easy 2^n-multipole field and potential expansion*
        - *Easy stereo-projection visualization*

12. Complex derivatives give 2D dipole fields

Start with $f(z)=az^{-1}$: 2D line *monopole field* and is its *monopole potential* $\phi(z)=a\ln z$ of source strength a.

$$f^{1\text{-}pole}(z)=\frac{a}{z}=\frac{d\phi^{1\text{-}pole}}{dz} \qquad \phi^{1\text{-}pole}(z)=a\ln z$$

Now let these two line-sources of equal but opposite source constants $+a$ and $-a$ be located at $z=\pm\Delta/2$ separated by a small interval Δ. This sum (actually difference) of $f^{1\text{-}pole}$-fields is called a *dipole field*.

$$f^{dipole}(z)=\frac{a}{z+\frac{\Delta}{2}}-\frac{a}{z-\frac{\Delta}{2}}=\frac{-a\cdot\Delta}{z^2-\frac{\Delta^2}{4}} \qquad \phi^{dipole}(z)=a\ln(z-\tfrac{\Delta}{2})-a\ln(z+\tfrac{\Delta}{2})=a\ln\frac{z-\frac{\Delta}{2}}{z+\frac{\Delta}{2}}$$

This is like the derivative definition:

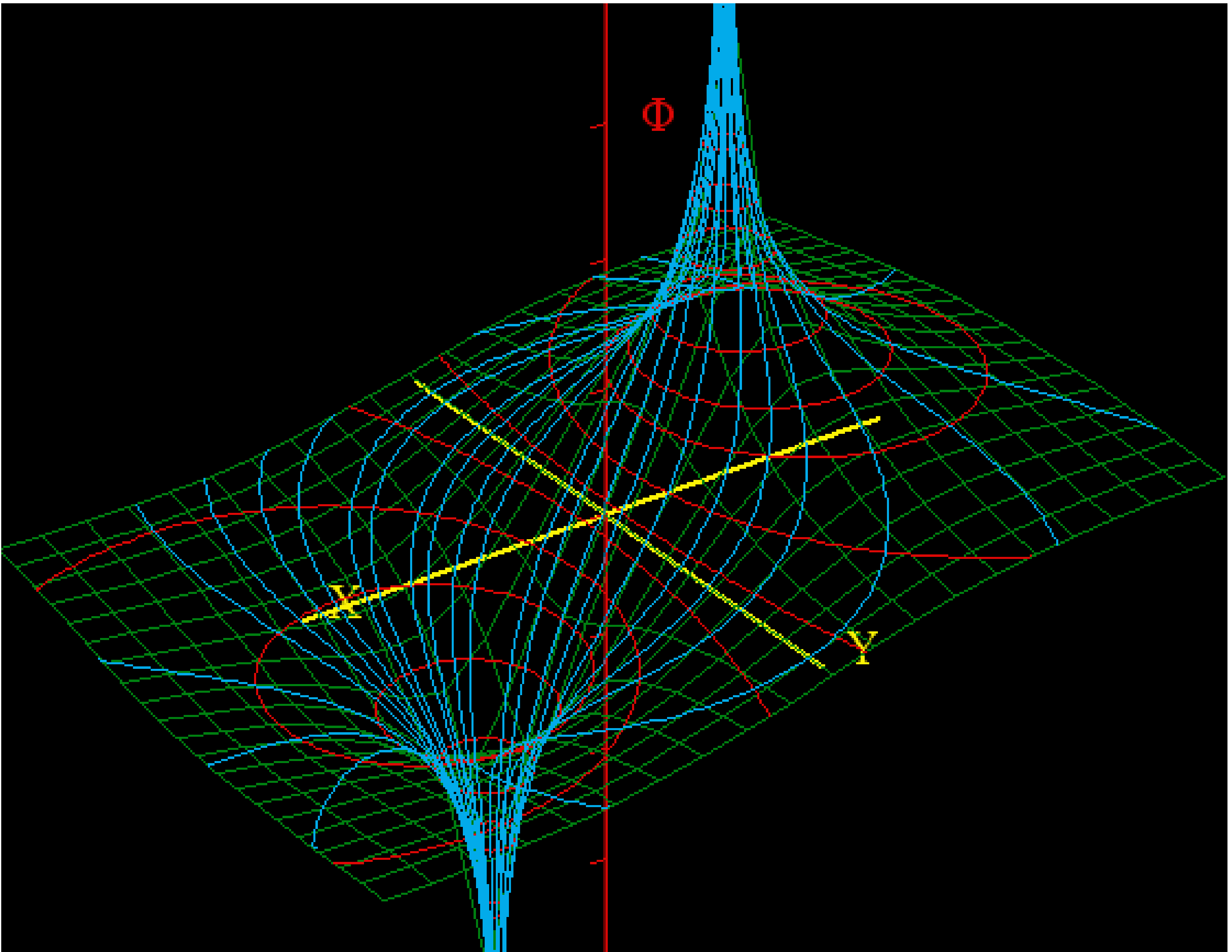
$$\frac{df}{dz}=\frac{f(z+\Delta)-f(z)}{\Delta}$$

or:

$$\frac{df}{dz}=\frac{f(z+\frac{\Delta}{2})-f(z-\frac{\Delta}{2})}{\Delta}$$

if Δ is infinitesimal

(Δ→0)



So-called
“physical dipole”
has finite Δ

(+)(−) separation

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If interval Δ is *tiny* and is divided out we get a *point-dipole field* $f^{2\text{-}pole}$ that is the z-derivative of $f^{1\text{-}pole}$.

$$f^{2\text{-}pole}=\frac{-a}{z^2}=\frac{df^{1\text{-}pole}}{dz}=\frac{d\phi^{2\text{-}pole}}{dz} \qquad \phi^{2\text{-}pole}=\frac{a}{z}=\frac{d\phi^{1\text{-}pole}}{dz}$$

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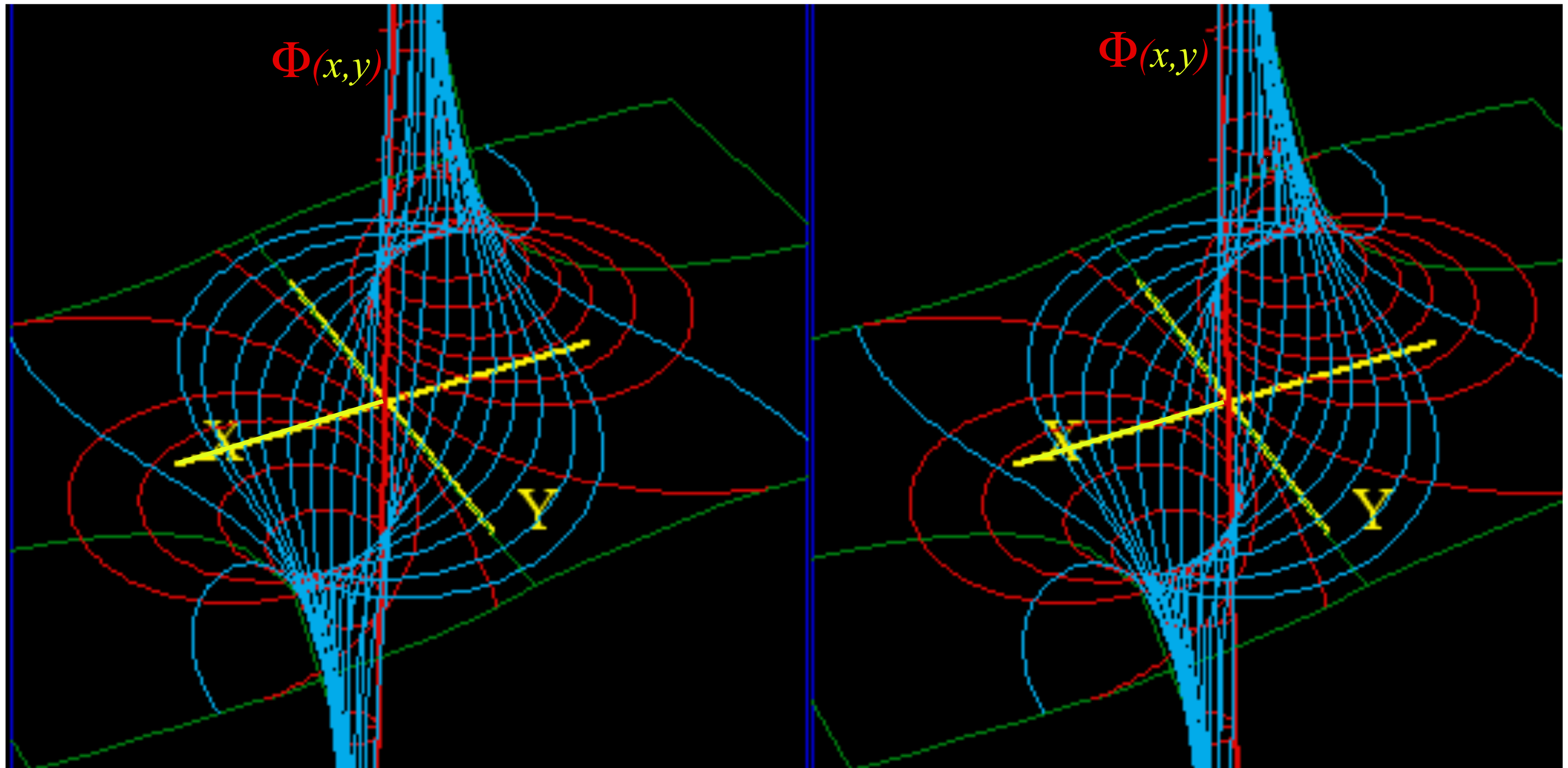

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$$f^{2\text{-}pole}=\frac{-a}{z^2}=\frac{df^{1\text{-}pole}}{dz}=\frac{d\phi^{2\text{-}pole}}{dz} \qquad \phi^{2\text{-}pole}=\frac{a}{z}=\frac{d\phi^{1\text{-}pole}}{dz}$$

A *point-dipole potential* $\phi^{2\text{-}pole}$ (whose *z*-derivative is $f^{2\text{-}pole}$) is a *z*-derivative of $\phi^{1\text{-}pole}$.

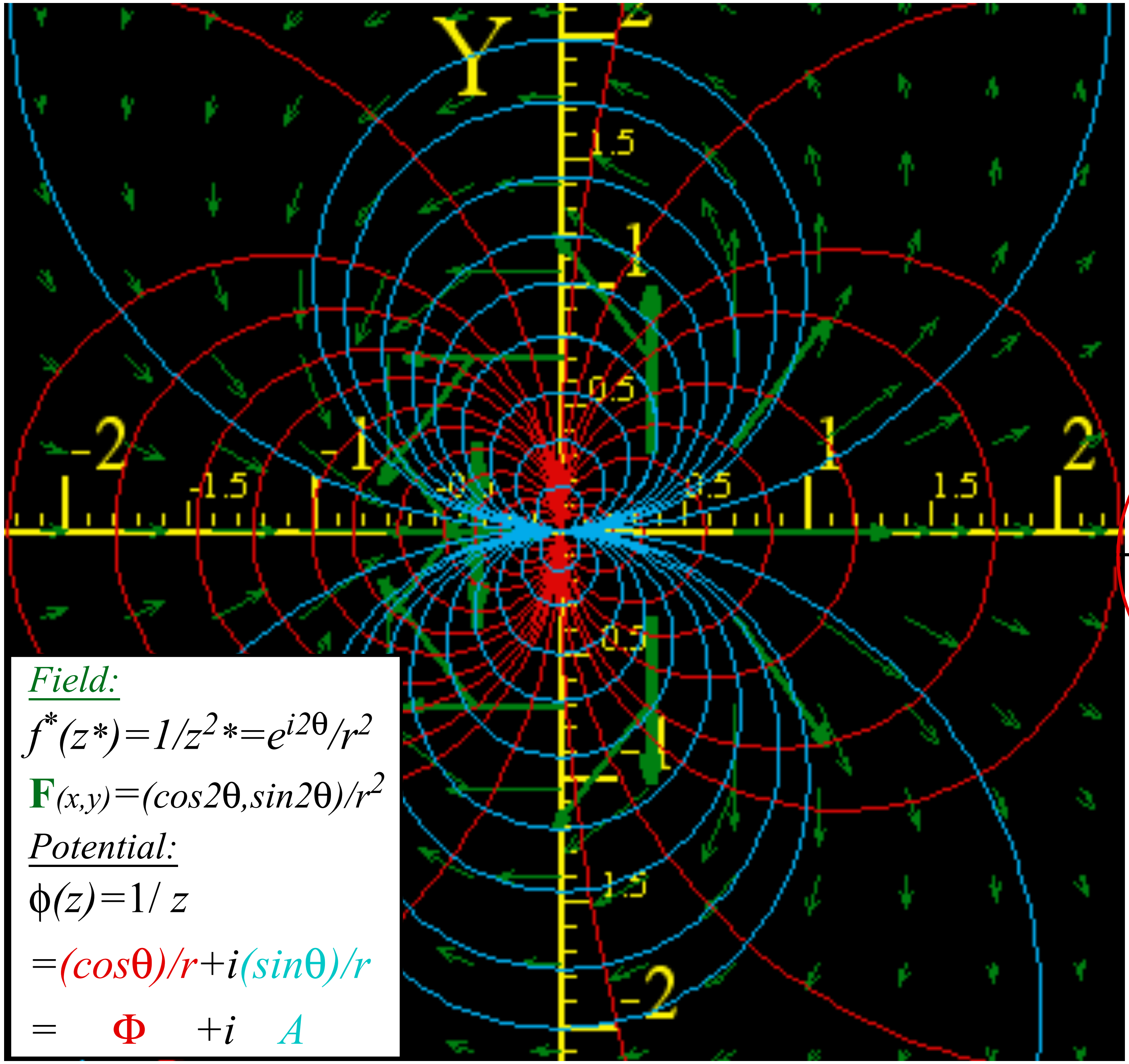
$$\phi^{2\text{-}pole}=\frac{a}{z}=\frac{a}{x+iy}=\frac{a}{x+iy}\frac{x-iy}{x-iy}=\frac{ax}{x^2+y^2}+i\frac{-ay}{x^2+y^2}=\frac{a}{r}\cos\theta-i\frac{a}{r}\sin\theta$$

$$=\Phi^{2\text{-}pole}+i\,\mathbf{A}^{2\text{-}pole}$$

A *point-dipole potential* $\phi^{2\text{-}pole}$ (whose *z*-derivative is $f^{2\text{-}pole}$) is a *z*-derivative of $\phi^{1\text{-}pole}$.

$$\phi^{2\text{-}pole} = \frac{a}{z} = \frac{a}{x+iy} = \frac{a}{x+iy}\frac{x-iy}{x-iy} = \frac{ax}{x^2+y^2} + i\frac{-ay}{x^2+y^2} = \frac{a}{r}\cos\theta - i\frac{a}{r}\sin\theta$$

$$= \Phi^{2\text{-}pole} + i\, A^{2\text{-}pole}$$



Field:

$f^*(z^*)=1/z^{2*}=e^{i2\theta}/r^2$

$\mathbf{F}(x,y)=(\cos 2\theta,\sin 2\theta)/r^2$

Potential:

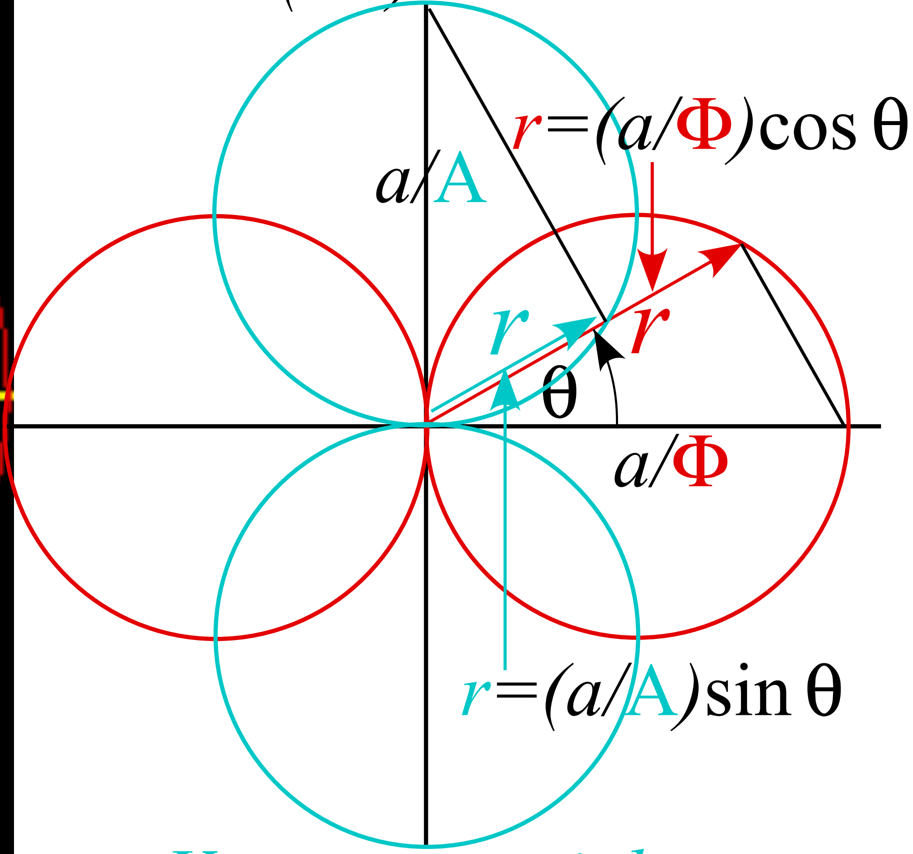
$\phi(z)=1/z$

$=(\cos\theta)/r+i(\sin\theta)/r$

$= \Phi \quad +i \quad A$

Scalar potentials

$\Phi=(a/r)\cos\theta=const.$



Vector potentials

$A=(a/r)\sin\theta=const.$

2^n-*pole analysis* *(quadrupole:2^2=4-pole, octapole:2^3=8-pole,…, pole dancer,*

What if we put a *(-)*copy of a *2-pole* near its original?

Well, the result is *4-pole* or *quadrupole* field $f^{\text{4-pole}}$ and potential $\phi^{\text{4-pole}}$.

Each a z-derivative of $f^{\text{2-pole}}$ and $\phi^{\text{2-pole}}$.

$$f^{4\text{-}pole} = \frac{a}{z^3} = \frac{1}{2}\frac{df^{2\text{-}pole}}{dz} = \frac{d\phi^{4\text{-}pole}}{dz} \qquad \phi^{4\text{-}pole} = -\frac{a}{2z^2} = \frac{1}{2}\frac{d\phi^{2\text{-}pole}}{dz}$$

2^n-pole analysis *(quadrupole: 2^2=4-pole, octapole: 2^3=8-pole, …, pole dancer,*

What if we put a *(-)*copy of a *2-pole* near its original?

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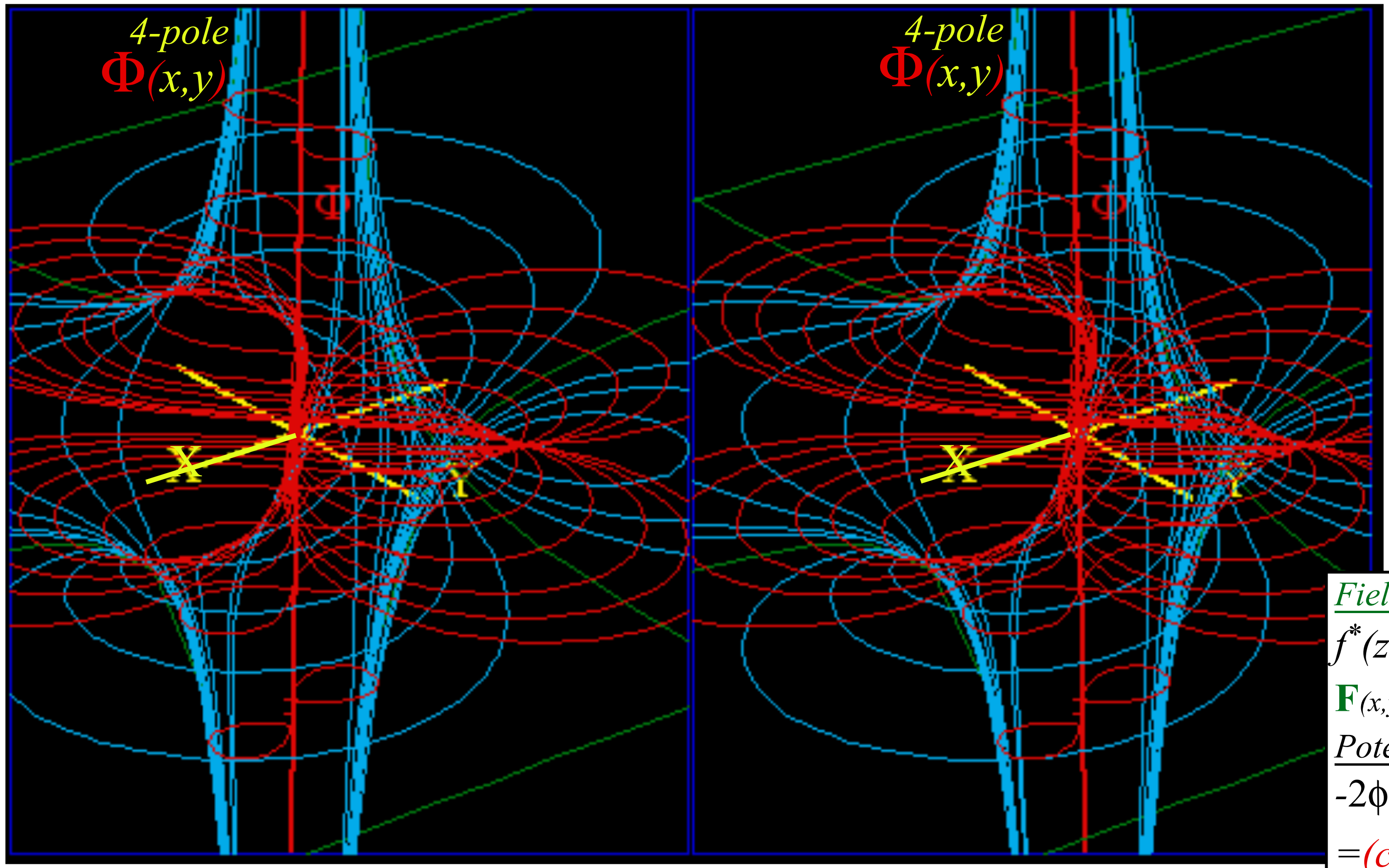

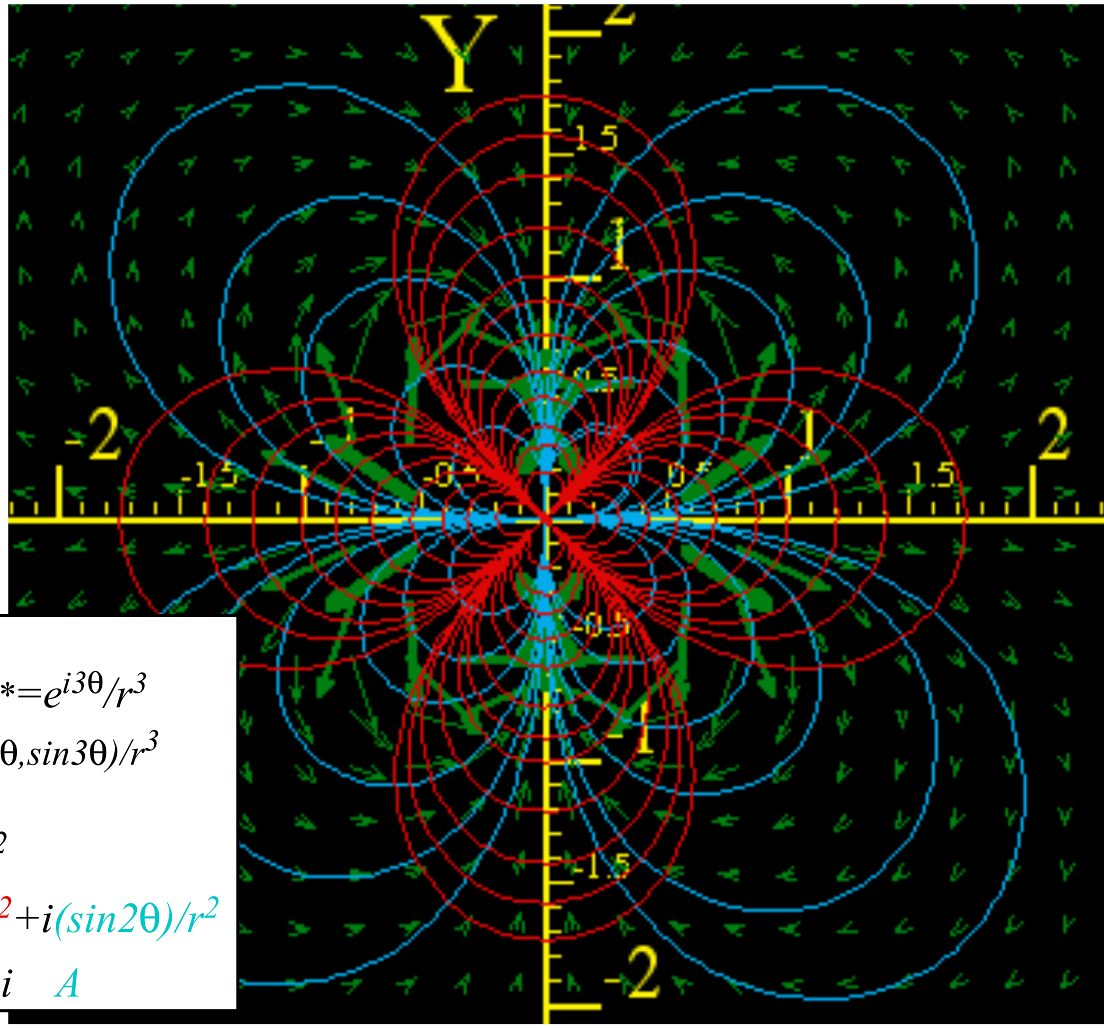

Field:

$f^*(z^*)=1/z^{3*}=e^{i3\theta}/r^3$

$\mathbf{F}(x,y)=(\cos 3\theta,\sin 3\theta)/r^3$

Potential:

$-2\phi(z)=1/z^2$

$=(\cos 2\theta)/r^2+i(\sin 2\theta)/r^2$

$= \Phi + i \; A$

4. Riemann-Cauchy conditions *What's analytic? (...and what's not?)*

Easy 2D circulation and flux integrals

Easy 2D curvilinear coordinate discovery

Easy 2D monopole, dipole, and 2^n*-pole analysis*

→ *Easy* 2^n*-multipole field and potential expansion*

Easy stereo-projection visualization

2^n-pole analysis: Laurent series *(Generalization of Maclaurin-Taylor series)*

Laurent series or *multipole expansion* of a given complex field function *f(z)* around *z=0*.

$$\frac{d\phi}{dz} = f(z) = \ldots a_{-3}z^{-3} + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + \ldots$$

⋯ 2^2-pole	2^1-pole	2^0-pole	2^1-pole	2^2-pole	2^3-pole	2^4-pole	2^5-pole	2^6-pole ⋯
(quadrupole)	*(dipole)*	*(monopole)*	*(dipole)*	*(quadrupole)*	*(octapole)*	*(hexadecapole)*		
at *z=0*	at *z=0*	at *z=0*	at *z=∞*	at *z=∞*	at *z=∞*	at *z=∞*	at *z=∞*	at *z=∞*

$$\int f\, dz = \phi(z) = \ldots \frac{a_{-3}}{-2}z^{-2} + \frac{a_{-2}}{-1}z^{-1} + a_{-1}\ln z + a_0 z + \frac{a_1}{2}z^2 + \frac{a_2}{3}z^3 + \frac{a_3}{4}z^4 + \frac{a_4}{5}z^5 + \frac{a_5}{6}z^6 + \ldots$$

All field terms $a_{m-1}z^{m-1}$ except *1-pole* $\frac{a_{-1}}{z}$ have potential term $a_{m-1}z^m/m$ of a *2^m-pole.*

These are located at $z=0$ for $m<0$ and at $z=\infty$ for $m>0$.

	(octapole)$_0$	*(quadrupole)*$_0$	*(dipole)*$_0$	*(monopole)*	*(dipole)*$_\infty$	*(quadrupole)*$_\infty$	*(octapole)*$_\infty$	
$\phi(z) = \ldots$	$\frac{a_{-4}}{-3}z^{-3}$	$+\frac{a_{-3}}{-2}z^{-2}$	$+\frac{a_{-2}}{-1}z^{-1}$	$+ a_{-1}\ln z$	$+ a_0 z$	$+\frac{a_1}{2}z^2$	$+\frac{a_2}{3}z^3$	$+\ldots$

2^n-pole analysis: Laurent series *(Generalization of Maclaurin-Taylor series)*

Laurent series or *multipole expansion* of a given complex field function *f(z)* around *z=0*.

$$\frac{d\phi}{dz}=f(z)=\ldots a_{-3}z^{-3} + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + \ldots$$

$\cdots 2^2$-pole	2^1-pole	2^0-pole	2^1-pole	2^2-pole	2^3-pole	2^4-pole	2^5-pole	2^6-pole $\cdots$
(quadrupole)	*(dipole)*	*(monopole)*	*(dipole)*	*(quadrupole)*	*(octapole)*	*(hexadecapole)*		
at *z=0*	at *z=0*	at *z=0*	at *z=∞*	at *z=∞*	at *z=∞*	at *z=∞*	at *z=∞*	at *z=∞*

$$\int f\,dz =$$

$$\phi(z)=\ldots\frac{a_{-3}}{-2}z^{-2}+\frac{a_{-2}}{-1}z^{-1}+a_{-1}\ln z+a_0 z+\frac{a_1}{2}z^2+\frac{a_2}{3}z^3+\frac{a_3}{4}z^4+\frac{a_4}{5}z^5+\frac{a_5}{6}z^6+\ldots$$

All field terms $a_{m-1}z^{m-1}$ except *1-pole* $\frac{a_{-1}}{z}$ have potential term $a_{m-1}z^m/m$ of a *2^m-pole.*

These are located at $z=0$ for $m<0$ and at $z=\infty$ for $m>0$.

(octapole)$_0$ *(quadrupole)$_0$* *(dipole)$_0$* *(monopole)* *(dipole)$_\infty$* *(quadrupole)$_\infty$* *(octapole)$_\infty$*

$$\phi(z)=\ldots\frac{a_{-3}}{-2}z^{-2}+\frac{a_{-3}}{-2}z^{-2}+\frac{a_{-2}}{-1}z^{-1}+a_{-1}\ln z+a_0 z+\frac{a_1}{2}z^2+\frac{a_2}{3}z^3+\ldots$$

$$\phi(w)=\ldots\frac{a_{-4}}{-3}w^{-3}+\frac{a_{-3}}{-2}w^{-2}+\frac{a_{-2}}{-1}w^{-1}+a_{-1}\ln w+a_0 w+\frac{a_1}{2}w^2+\frac{a_2}{3}w^3+\ldots$$

(*with $z=w^{-1}$*)

2^n-pole analysis: *Laurent series* (Generalization of Maclaurin-Taylor series)

Laurent series or *multipole expansion* of a given complex field function *f(z)* around *z=0*.

$$\frac{d\phi}{dz}=f(z)=\ldots a_{-3}z^{-3}+a_{-2}z^{-2}+a_{-1}z^{-1}+a_0+a_1z+a_2z^2+a_3z^3+a_4z^4+a_5z^5+\ldots$$

$a_{-3}z^{-3}$	$a_{-2}z^{-2}$	$a_{-1}z^{-1}$	a_0	a_1z	a_2z^2	a_3z^3	a_4z^4	a_5z^5
⋯ 2^2-pole	2^1-pole	2^0-pole	2^1-pole	2^2-pole	2^3-pole	2^4-pole	2^5-pole	2^6-pole ⋯
(quadrupole)	*(dipole)*	*(monopole)*	*(dipole)*	*(quadrupole)*	*(octapole)*	*(hexadecapole)*		
at *z=0*	at *z=0*	at *z=0*	at *z=∞*	at *z=∞*	at *z=∞*	at *z=∞*	at *z=∞*	at *z=∞*

$$\int f\,dz=\phi(z)=\ldots\frac{a_{-3}}{-2}z^{-2}+\frac{a_{-2}}{-1}z^{-1}+a_{-1}\ln z+a_0z+\frac{a_1}{2}z^2+\frac{a_2}{3}z^3+\frac{a_3}{4}z^4+\frac{a_4}{5}z^5+\frac{a_5}{6}z^6+\ldots$$

All field terms $a_{m-1}z^{m-1}$ except *1-pole* $\frac{a_{-1}}{z}$ have potential term $a_{m-1}z^m/m$ of a *2^m-pole*.

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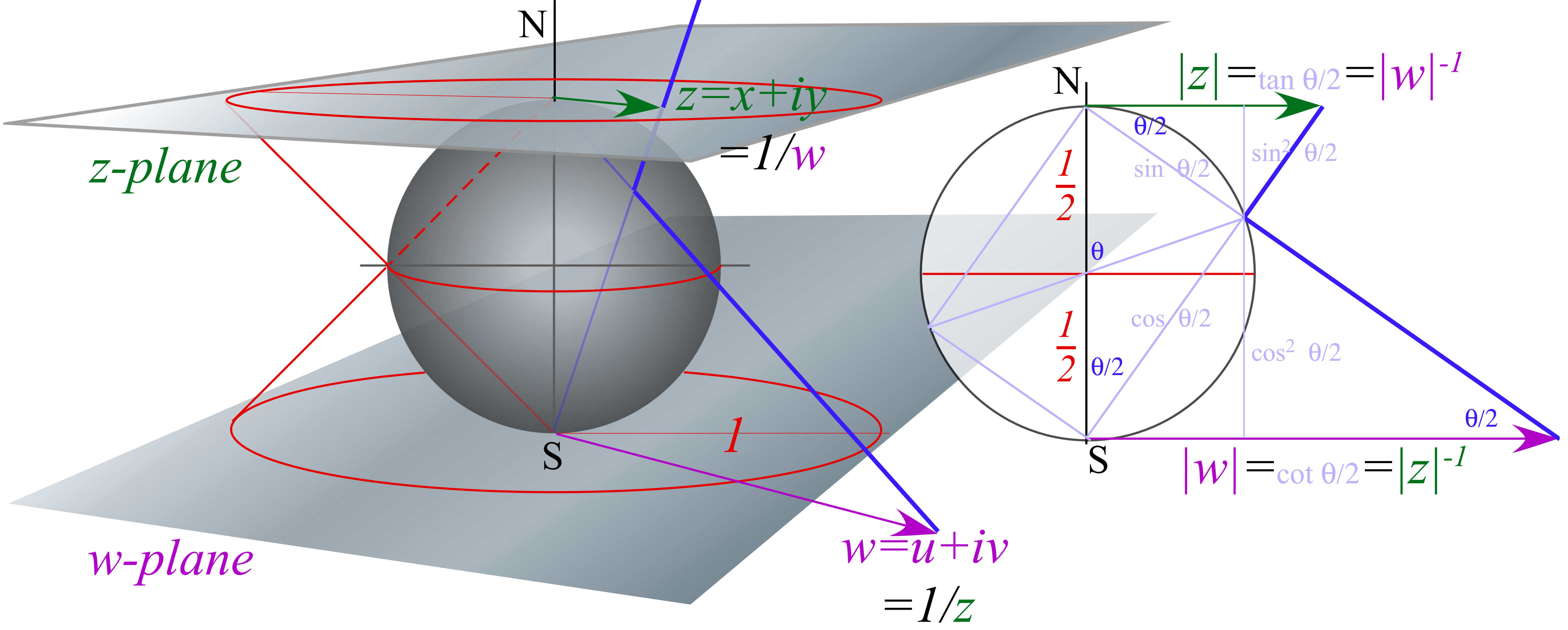
$$\phi(z)=\ldots\overset{(\text{octapole})_0}{\frac{a_{-4}}{-3}z^{-3}}+\overset{(\text{quadrupole})_0}{\frac{a_{-3}}{-2}z^{-2}}+\overset{(\text{dipole})_0}{\frac{a_{-2}}{-1}z^{-1}}+\overset{(\text{monopole})}{a_{-1}\ln z}+\overset{(\text{dipole})_\infty}{a_0z}+\overset{(\text{quadrupole})_\infty}{\frac{a_1}{2}z^2}+\overset{(\text{octapole})_\infty}{\frac{a_2}{3}z^3}+\ldots$$

$$\phi(w)=\ldots\frac{a_{-4}}{-3}w^{-3}+\frac{a_{-3}}{-2}w^{-2}+\frac{a_{-2}}{-1}w^{-1}+a_{-1}\ln w+a_0w+\frac{a_1}{2}w^2+\frac{a_2}{3}w^3+\ldots$$

(*with z⟶w*)

$$=\ldots\frac{a_2}{3}z^{-3}+\frac{a_1}{2}z^{-2}+a_0z^{-1}-a_{-1}\ln z+\frac{a_{-2}}{-1}z+\frac{a_{-3}}{-2}z^2+\frac{a_{-4}}{-3}z^3+\ldots$$

(*with $w=z^{-1}$*)



$$\phi(z) = \dots \frac{a_{-4}}{-3} z^{-3} + \frac{a_{-3}}{-2} z^{-2} + \frac{a_{-2}}{-1} z^{-1} + a_{-1}\ln z + a_0 z + \frac{a_1}{2} z^2 + \frac{a_2}{3} z^3 + \dots$$

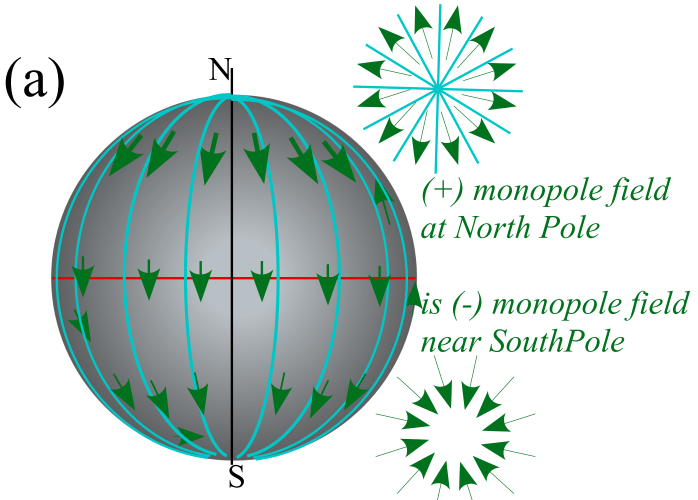
(octapole)$_0$ (quadrupole)$_0$ (dipole)$_0$ (monopole) (dipole)$_\infty$ (quadrupole)$_\infty$ (octapole)$_\infty$

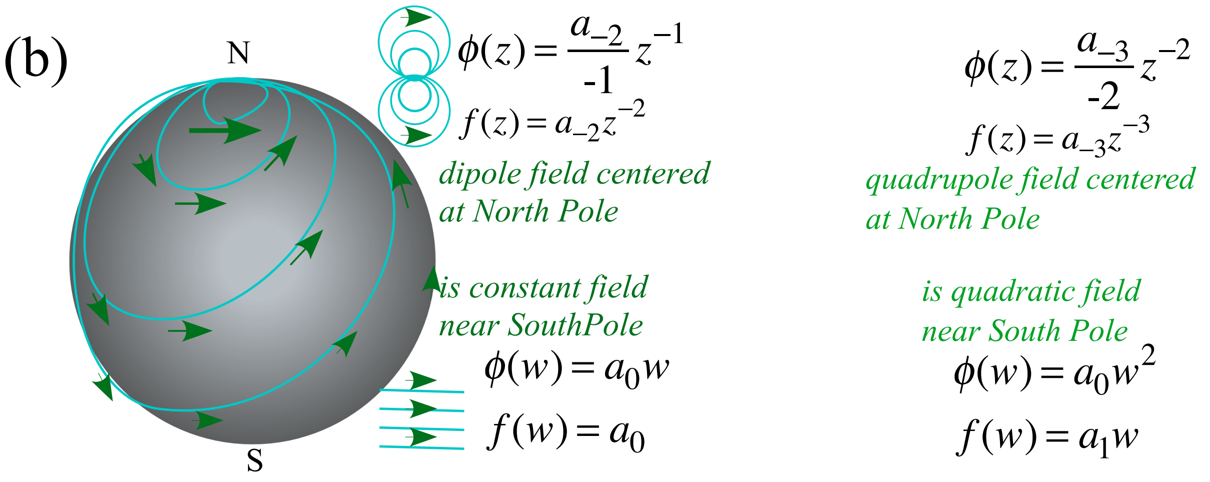
$$\phi(w) = \dots \frac{a_{-4}}{-3} w^{-3} + \frac{a_{-3}}{-2} w^{-2} + \frac{a_{-2}}{-1} w^{-1} + a_{-1}\ln w + a_0 w + \frac{a_1}{2} w^2 + \frac{a_2}{3} w^3 + \dots$$

(with z⟶w)

$$= \dots \frac{a_2}{3} z^{-2} + \frac{a_1}{2} z^{-2} + a_0 z^{-1} - a_{-1}\ln z + \frac{a_{-2}}{-1} z + \frac{a_{-3}}{-2} z^2 + \frac{a_{-4}}{-3} z^3 + \dots$$

(with $w=z^{-1}$)





$$f(z) = \ldots a_{-3}z^{-3} + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + \ldots$$

Of all 2^m*-pole* field terms $a_{m-1}z^{m-1}$, only the $m=0$ monopole $a_{-1}z^{-1}$ has a non-zero loop integral (10.39).

$$\oint f(z)dz = \oint a_{-1}z^{-1}dz = 2\pi i a_{-1} \qquad\qquad a_{-1} = \tfrac{1}{2\pi i} \oint f(z)dz$$

$$f(z) = \dots a_{-3}z^{-3} + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + \dots$$

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$$\oint f(z)dz = \oint a_{-1}z^{-1}dz = 2\pi i a_{-1} \qquad a_{-1} = \tfrac{1}{2\pi i}\oint f(z)dz$$

This $m=1$-*pole* constant-a_{-1} formula is just the first in a series of *Laurent coefficient expressions*.

$$\cdots a_{-3} = \tfrac{1}{2\pi i}\oint z^2 f(z)dz \;,\; a_{-2} = \tfrac{1}{2\pi i}\oint z^1 f(z)dz \;,\; a_{-1} = \tfrac{1}{2\pi i}\oint f(z)dz \;,\; a_0 = \tfrac{1}{2\pi i}\oint \frac{f(z)}{z}dz \;,\; a_1 = \tfrac{1}{2\pi i}\oint \frac{f(z)}{z^2}dz \;,\cdots$$

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(quadrupole)$_0$ (dipole)$_0$ (monopole) (dipole)$_\infty$ (quadrupole)$_\infty$ (octapole)$_\infty$ (hexadecapole)$_\infty$ ...

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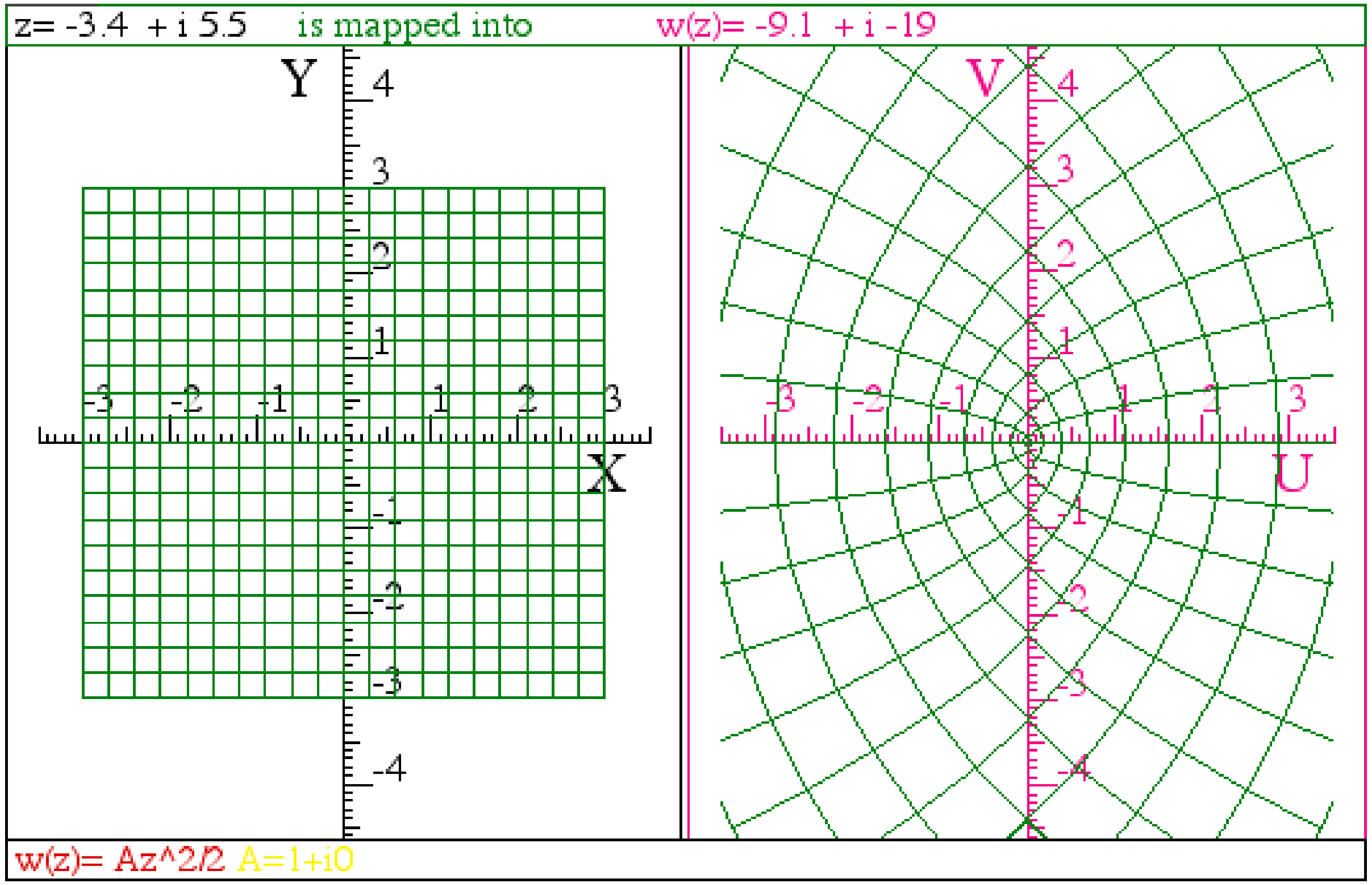
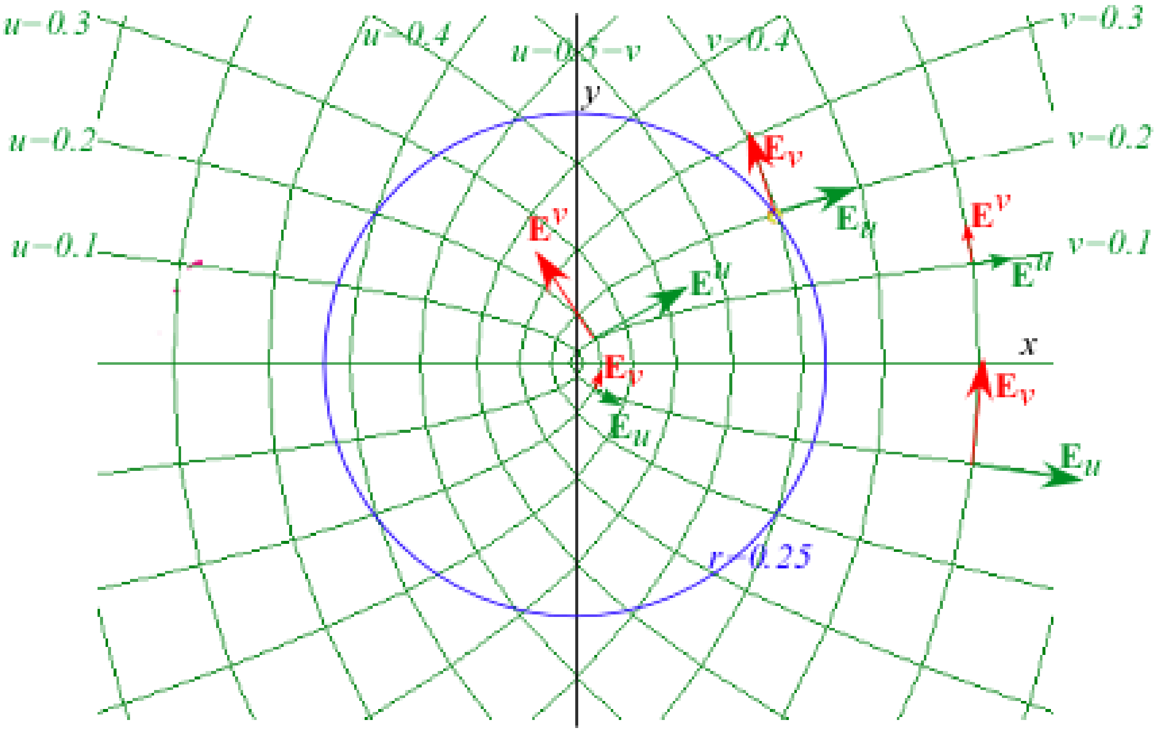
$w(z) = z^2$ gives parabolic OCC

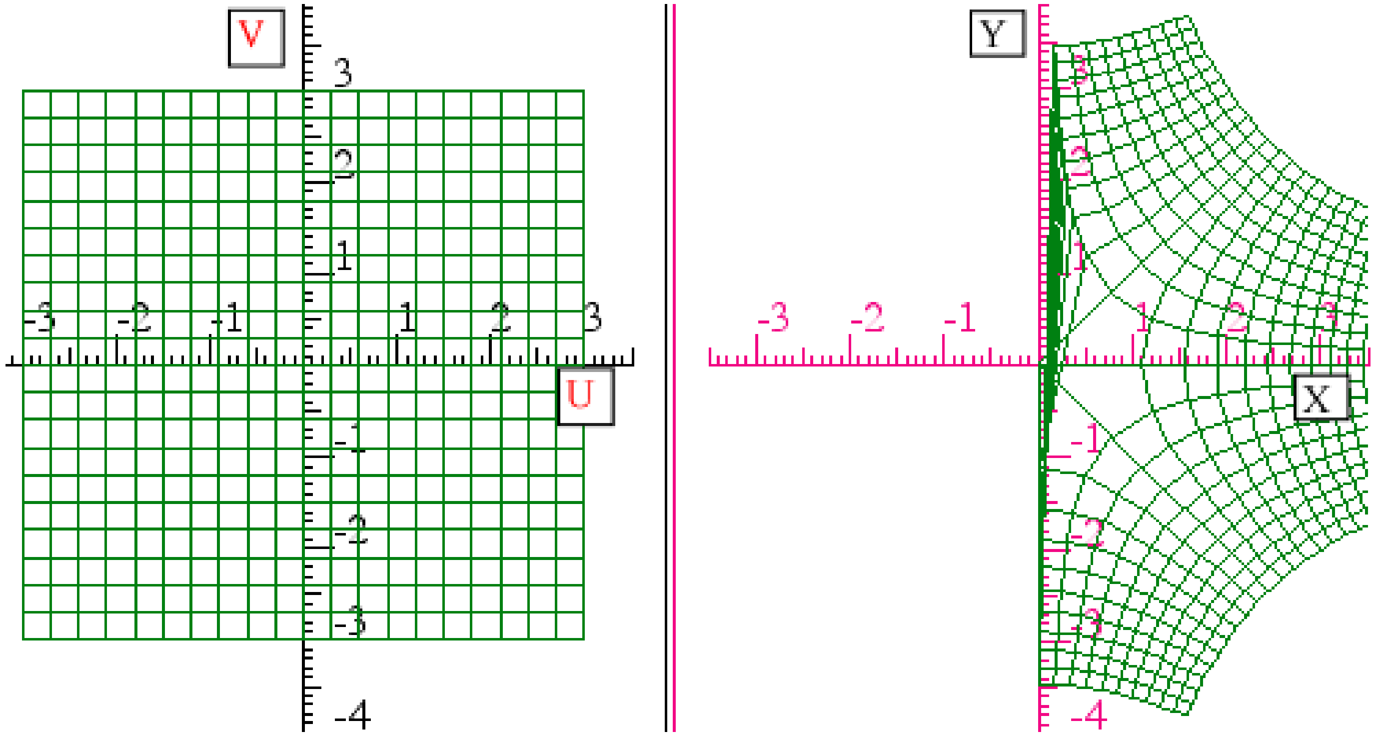


z= -3.4 + i 5.5 is mapped into w(z)= -9.1 + i -19

w(z)= Az^2/2 A=1+i0



Inverse: $z(w) = w^{1/2}$ gives hyperbolic OCC



$w = (u + iv) = z^2 = (x + iy)^2$ is analytic function of z and w

Expansion: $u = x^2 - y^2$ and $v = 2xy$ may be solved using $|w| = |z^2| = |z|^2$

Expansion: $|w| = \sqrt{u^2 + v^2} = x^2 + y^2 = |z|^2$

Solution: $x^2 = \dfrac{u + \sqrt{u^2 + v^2}}{2}$ $\quad y^2 = \dfrac{-u + \sqrt{u^2 + v^2}}{2}$

$$\begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{E}}^u \\ \bar{\mathbf{E}}^v \end{pmatrix} = \begin{pmatrix} 2x & -2y \\ +2y & 2x \end{pmatrix}$$

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