Lecture 13
Wed. 10.03.2018

Complex Variables, Series, and Field Coordinates II.

(Ch. 10 of Unit 1)

1. *The Story of e (A Tale of Great $Interest$)*
   - *How good are those power series?*
   - *Taylor-Maclaurin series, imaginary interest, and complex exponentials*
2. *What good are complex exponentials?*
   - *Easy trig*
   - *Easy 2D vector analysis*
   - *Easy oscillator phase analysis*
   - *Easy rotation and "dot" or "cross" products*
3. *Easy 2D vector calculus*
   - *Easy 2D vector derivatives*
   - *Easy 2D source-free field theory*
   - *Easy 2D vector field-potential theory*
4. *Riemann-Cauchy relations (What's analytic? What's not?)*
   - *Easy 2D curvilinear coordinate discovery*
   - *Easy 2D circulation and flux integrals*
   - *Easy 2D monopole, dipole, and 2^n-pole analysis*
   - *Easy 2^n-multipole field and potential expansion*
   - *Easy stereo-projection visualization*
   - *Cauchy integrals, Laurent-Maclaurin series*
5. *Mapping and Non-analytic 2D source field analysis*

Lecture 13 Wed. 10.03.18
Starts review here

Lect. 12
ended here

1. Complex numbers provide "automatic trigonometry"

2. Complex numbers add like vectors.

3. Complex exponentials $Ae^{-i\omega t}$ track position and velocity using Phasor Clock.

4. Complex products provide 2D rotation operations.

5. Complex products provide 2D "dot"(•) and "cross"(x) products.

(Review of topics in Lect.12)

6. Complex derivative contains "divergence"($\nabla\cdot\mathbf{F}$) and "curl"($\nabla\times\mathbf{F}$) of 2D vector field

7. Invent source-free 2D vector fields [$\nabla\cdot\mathbf{F}=0$ and $\nabla\times\mathbf{F}=0$]

8. Complex potential ϕ contains "scalar"($\mathbf{F}=\nabla\Phi$) and "vector"($\mathbf{F}=\nabla\times\mathbf{A}$) potentials

The half-n'-half results: (Riemann-Cauchy Derivative Relations)

9. Complex potentials define 2D Orthogonal Curvilinear Coordinates (OCC) of field

10. Complex integrals $\int f(z)dz$ count 2D "circulation"($\int\mathbf{F}\cdot d\mathbf{r}$) and "flux"($\int\mathbf{F}\times d\mathbf{r}$)

11. Complex integrals define 2D monopole fields and potentials

12. Complex derivatives give 2D dipole fields

13. More derivatives give 2D 2^N-pole fields…

14. ...and 2^N-pole multipole expansions of fields and potentials...

15. ...and Laurent Series...

16. ...and non-analytic source analysis.

17. …and mapping…

A running collection of links to course-relevant sites and articles

Physics Web Resources

Comprehensive Harter-Soft Resource Listing

UAF Physics YouTube channel

LearnIt Physics Web Applications

"Texts"

Classical Mechanics with a Bang!

Quantum Theory for the Computer Age

Principles of Symmetry, Dynamics, and Spectroscopy

Modern Physics and its Classical Foundations

Classes

2014 AMOP

2017 Group Theory for QM

2018 AMOP

2018 Adv Mechanics

Neat external material to start the class:
- AIP publications
- AJP article on superball dynamics
- AAPT summer reading

These *are* hot off the presses:
- Sorting ultracold atoms in a three-dimensional optical lattice in a realization of Maxwell's demon - Kumar-Nature-Letters-2018
- Synthetic three-dimensional atomic structures assembled atom by atom - Berredo-Nature-Letters-2018

Slightly Older ones:
- Wave–particle duality of C60 molecules
- Optical vortex knots – One Photon at a Time

"*Relawavity*" and quantum basis of *Lagrangian* & *Hamiltonian* mechanics:
- 2-CW laser wave - BohrIt Web App
- Lagrangian vs Hamiltonian - RelaWavity Web App

- AMOP Ch 0 Space-Time Symmetry - 2019
- Seminar at Rochester Institute of Optics, Auxiliary slides, June 19, 2018

6. Complex derivative contains "divergence"($\nabla\cdot\mathbf{F}$) and "curl"($\nabla\times\mathbf{F}$) of 2D vector field

Relation of (z,z^*) to $(x=\text{Re}\,z, y=\text{Im}\,z)$ defines a z-derivative $\frac{df}{dz}$ and "star" z^*-derivative. $\frac{df}{dz^*}$

$$z = x+iy \qquad x=\tfrac{1}{2}(z+z^*)$$

$$z^* = x-iy \qquad y=\tfrac{1}{2i}(z-z^*)$$

$$\frac{df}{dz} = \frac{\partial x}{\partial z}\frac{\partial f}{\partial x}+\frac{\partial y}{\partial z}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x}-\frac{i}{2}\frac{\partial f}{\partial y} \qquad \frac{d}{dz}=\frac{1}{2}\frac{\partial}{\partial x}-\frac{i}{2}\frac{\partial}{\partial y}$$

$$\frac{df}{dz^*} = \frac{\partial x}{\partial z^*}\frac{\partial f}{\partial x}+\frac{\partial y}{\partial z^*}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x}+\frac{i}{2}\frac{\partial f}{\partial y} \qquad \frac{d}{dz^*}=\frac{1}{2}\frac{\partial}{\partial x}+\frac{i}{2}\frac{\partial}{\partial y}$$

Derivative chain-rule shows real part of $\frac{df}{dz}$ has 2D divergence $\nabla\bullet\mathbf{f}$ and imaginary part has curl $\nabla\times\mathbf{f}$.

$$\frac{df}{dz}=\frac{d}{dz}(f_x+if_y)=\frac{1}{2}\left(\frac{\partial f}{\partial x}-i\frac{\partial f}{\partial y}\right)(f_x+if_y)=\frac{1}{2}\underbrace{\left(\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}\right)}+\frac{i}{2}\underbrace{\left(\frac{\partial f_y}{\partial x}-\frac{\partial f_x}{\partial y}\right)}=\frac{1}{2}\nabla\bullet\mathbf{f}+\frac{i}{2}|\nabla\times\mathbf{f}|_{Z\perp(x,y)}$$

7. Invent source-free 2D vector fields [$\nabla\cdot\mathbf{F}=0$ and $\nabla\times\mathbf{F}=0$]

We can invent *source-free 2D vector fields* that are both *zero-divergence* and *zero-curl*.
Take any function $f(z)$, conjugate it (change all i's to $-i$) to give $f^*(z^*)$ for which $\boxed{\frac{df^*}{dz}=0}$.

For example: if $f(z)=a\cdot z$ then $f^*(z^*)=a\cdot z^*=a(x-iy)$ is not function of z so it has *zero z-derivative*.

$\mathbf{F}=(F_x,F_y)=(f^*_x,f^*_y)=(a\cdot x,-a\cdot y)$ *has zero divergence:* $\nabla\bullet\mathbf{F}=0$ *and has zero curl:* $|\nabla\times\mathbf{F}|=0$.

$$\nabla\bullet\mathbf{F}=\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}=\frac{\partial(ax)}{\partial x}+\frac{\partial F(-ay)}{\partial y}=0 \qquad |\nabla\times\mathbf{F}|_{Z\perp(x,y)}=\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}=\frac{\partial(-ay)}{\partial x}-\frac{\partial F(ax)}{\partial y}=0$$

A *DFL* field $\mathbf{F}$ (*Divergence-Free-Laminar*)

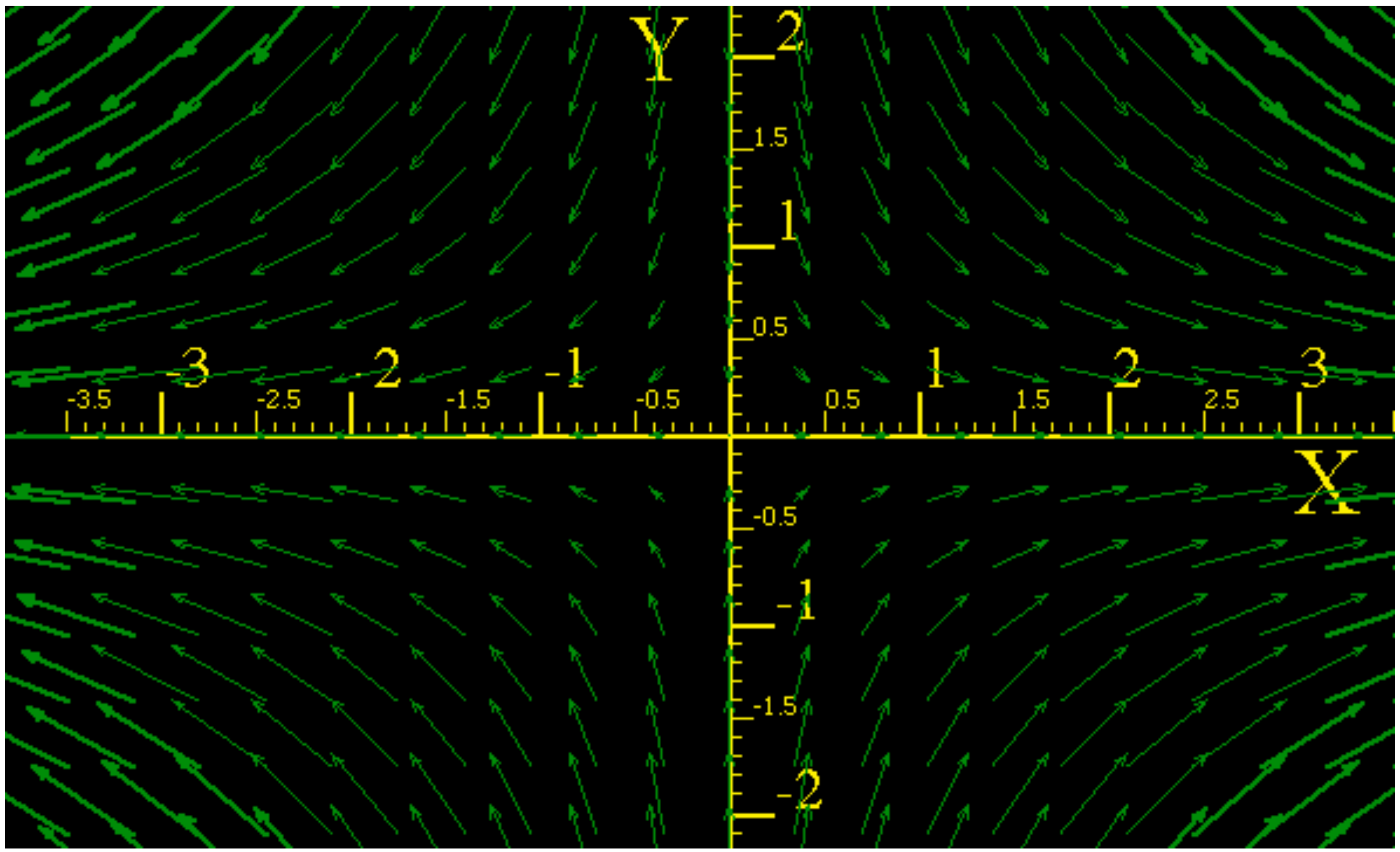
7. Invent source-free 2D vector fields [$\nabla\cdot\mathbf{F}=0$ and $\nabla\times\mathbf{F}=0$]

We can invent *source-free 2D vector fields* that are both *zero-divergence* and *zero-curl*.
Take any function *f(z)*, conjugate it (change all *i*'s to *–i*) to give *f*(z*)* for which $\boxed{\frac{df^*}{dz}=0}$.

For example: if *f(z)=a·z* then *f*(z*)=a·z*=a(x-iy)* is not function of *z* so it has *zero z-derivative*.

$\mathbf{F}=(F_x,F_y)=(f^*_x,f^*_y)=(a\cdot x,-a\cdot y)$ has *zero divergence*: $\nabla\bullet\mathbf{F}=0$ and has *zero curl*: $|\nabla\times\mathbf{F}|=0$.

$$\nabla\bullet\mathbf{F}=\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}=\frac{\partial(ax)}{\partial x}+\frac{\partial F(-ay)}{\partial y}=0 \qquad |\nabla\times\mathbf{F}|_{Z\perp(x,y)}=\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}=\frac{\partial(-ay)}{\partial x}-\frac{\partial F(ax)}{\partial y}=0$$



precursor to Unit 1 Fig. 10.7

$\mathbf{F}=(f^*_x,f^*_y)=(a\cdot x,-a\cdot y)$ is a *divergence-free laminar (DFL)* field.

What Good Are Complex Exponentials? (contd.) *(Review of topics in Lect.12)*

8. Complex potential ϕ contains "scalar"($\mathbf{F}=\nabla\Phi$) and "vector"($\mathbf{F}=\nabla \times \mathbf{A}$) potentials

Any *DFL* field $\mathbf{F}$ is a gradient of a *scalar potential field* Φ or a curl of a *vector potential field* $\mathbf{A}$.

$$\mathbf{F}=\nabla\Phi \qquad\qquad \mathbf{F}=\nabla\times\mathbf{A}$$

A *complex potential* $\phi(z)=\Phi(x,y)+i\mathbf{A}(x,y)$ exists whose z-derivative is $f(z)=d\phi/dz$.

Its complex conjugate $\phi^*(z^*)=\Phi(x,y)-i\mathbf{A}(x,y)$ has z^*-derivative $f^*(z^*)=d\phi^*/dz^*$ giving *DFL* field $\mathbf{F}$.

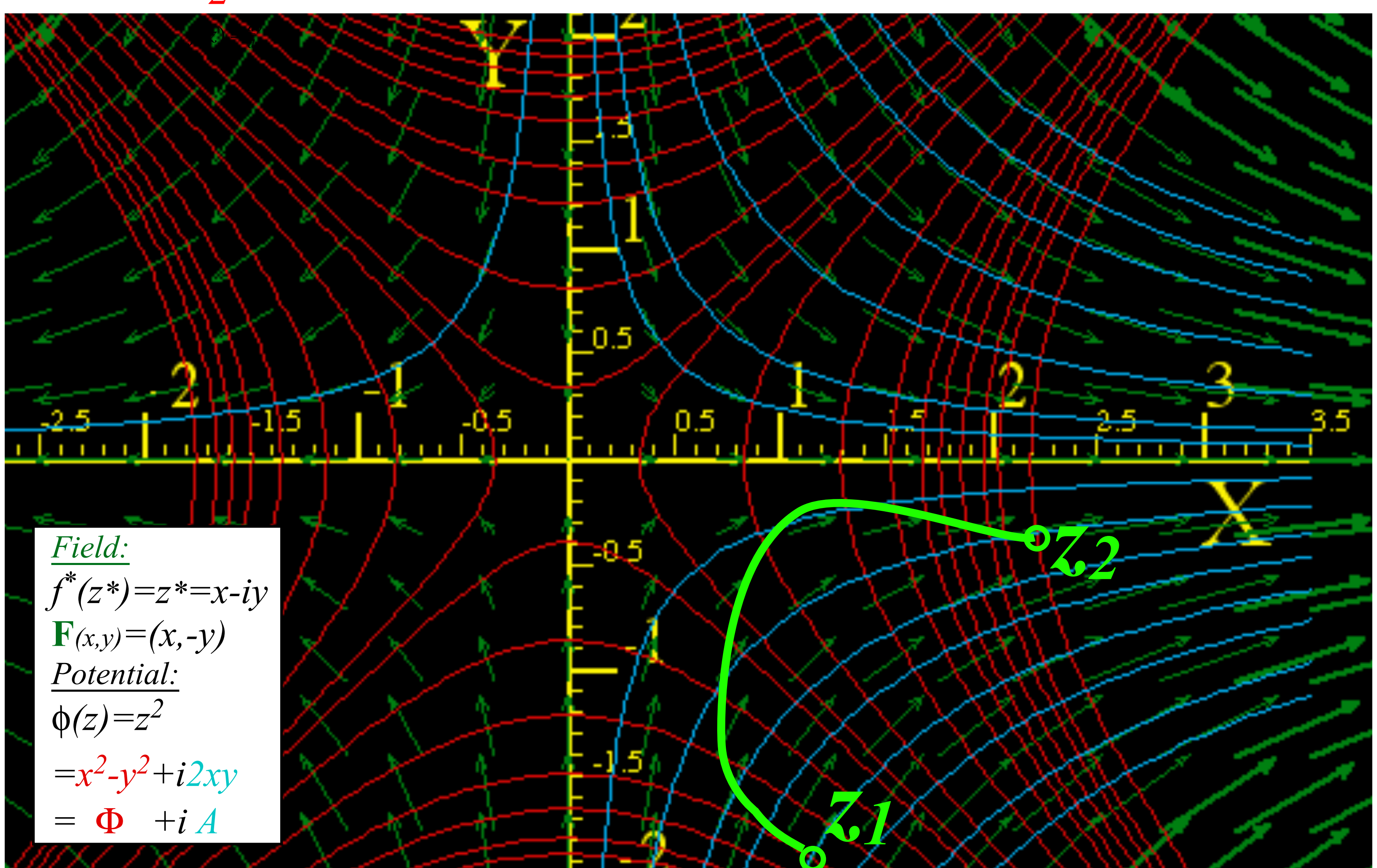
To find $\phi=\Phi+i\mathbf{A}$ integrate $f(z)=a\cdot z$ to get ϕ and isolate real (Re $\phi=\Phi$) and imaginary (Im $\phi=\mathbf{A}$) parts.

$$f(z)=\frac{d\phi}{dz} \quad\Rightarrow\quad \phi = \overbrace{\Phi}^{=\frac{1}{2}a(x^2-y^2)} + i\,\overbrace{\mathbf{A}}^{axy} = \int f\cdot dz = \int az\cdot dz = \frac{1}{2}az^2 = \frac{1}{2}a(x+iy)^2$$

$$=\frac{1}{2}a(x^2-y^2)+i\,axy$$

BONUS! Get a free coordinate system!

Unit 1 Fig. 10.7



The (Φ,$\mathbf{A}$) grid is a GCC coordinate system*:

$q^1=\Phi=(x^2-y^2)/2=$ *const.*

$q^2=\mathbf{A}=(xy)=$ *const.*

*Actually it's OCC.

8. (contd.) Complex potential ϕ contains "scalar"($\mathbf{F}=\nabla\Phi$) and "vector"($\mathbf{F}=\nabla\times\mathbf{A}$) potentials

...and either one (or half-n'-half!) works just as well.

Derivative $\frac{d\phi^*}{dz^*}$ has 2D gradient $\nabla\Phi=\begin{pmatrix}\frac{\partial\Phi}{\partial x}\\ \frac{\partial\Phi}{\partial y}\end{pmatrix}$ of scalar Φ and curl $\nabla\times\mathbf{A}=\begin{pmatrix}\frac{\partial A}{\partial y}\\ -\frac{\partial A}{\partial x}\end{pmatrix}$ of vector $\mathbf{A}$ (*and they're equal!*)

The half-n'-half result

$$\frac{d}{dz^*}\phi^* = \frac{d}{dz^*}(\Phi - iA) = \tfrac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(\Phi - iA) = \tfrac{1}{2}\left(\frac{\partial\Phi}{\partial x}+i\frac{\partial\Phi}{\partial y}\right)+\tfrac{1}{2}\left(\frac{\partial A}{\partial y}-i\frac{\partial A}{\partial x}\right) = \tfrac{1}{2}\nabla\Phi+\tfrac{1}{2}\nabla\times\mathbf{A}$$

Note, *mathematician definition* of force field $\mathbf{F}=+\nabla\Phi$ replaces usual physicist's definition $\mathbf{F}=-\nabla\Phi$

Given ϕ:

$$\boxed{\begin{aligned}\phi &= \Phi + i\,A\\ &= \tfrac{1}{2}a(x^2-y^2) + i\,axy\end{aligned}}$$

The half-n'-half result

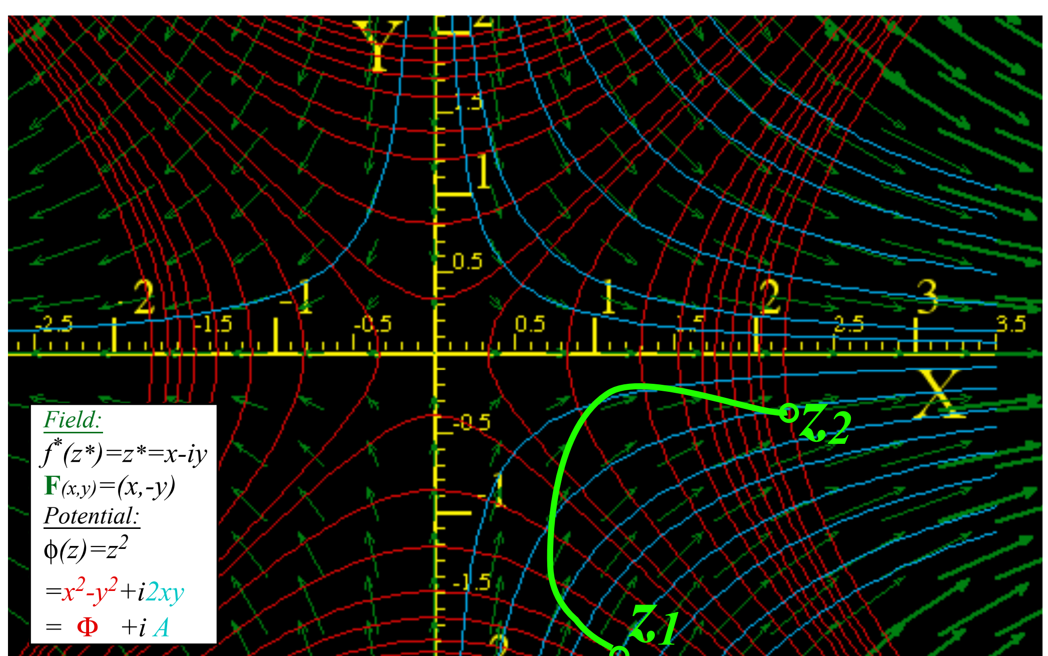
find:

$$\nabla\Phi=\begin{pmatrix}\frac{\partial\Phi}{\partial x}\\ \frac{\partial\Phi}{\partial y}\end{pmatrix}=\begin{pmatrix}\frac{\partial}{\partial x}\frac{a}{2}(x^2-y^2)\\ \frac{\partial}{\partial y}\frac{a}{2}(x^2-y^2)\end{pmatrix}=\begin{pmatrix}ax\\ -ay\end{pmatrix}=\mathbf{F}$$

or find:

$$\nabla\times\mathbf{A}=\begin{pmatrix}\frac{\partial A}{\partial y}\\ -\frac{\partial A}{\partial x}\end{pmatrix}=\begin{pmatrix}\frac{\partial}{\partial y}axy\\ -\frac{\partial}{\partial x}axy\end{pmatrix}=\begin{pmatrix}ax\\ -ay\end{pmatrix}=\mathbf{F}$$

Scalar *static potential lines* Φ=*const.* and vector *flux potential lines* $\mathbf{A}$=*const.* define *DFL field-net*.



The half-n'-half results are called Riemann-Cauchy Derivative Relations

$$\frac{\partial\Phi}{\partial x}=\frac{\partial A}{\partial y} \quad\text{is:}\quad \frac{\partial\,\mathrm{Re}f(z)}{\partial x}=\frac{\partial\,\mathrm{Im}f(z)}{\partial y}$$

$$\frac{\partial\Phi}{\partial y}=-\frac{\partial A}{\partial x} \quad\text{is:}\quad \frac{\partial\,\mathrm{Re}f(z)}{\partial y}=-\frac{\partial\,\mathrm{Im}f(z)}{\partial x}$$

Review (z,z) to (x,y) transformation relations* *(Review of topics in Lect.12)*

$$z = x + iy \qquad x = \frac{1}{2}(z + z^*)$$

$$z^* = x - iy \qquad y = \frac{1}{2i}(z - z^*)$$

$$\frac{df}{dz} = \frac{\partial x}{\partial z}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{1}{2i}\frac{\partial f}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)f$$

$$\frac{df}{dz^*} = \frac{\partial x}{\partial z^*}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z^*}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{1}{2i}\frac{\partial f}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)f$$

Criteria for a field function $f = f_x(x,y) + i f_y(x,y)$ to be an ***analytic function f(z)*** *of $z=x+iy$:*

First, f(z) must not be a function of $z^*=x-iy$, that is: $\frac{df}{dz^*} = 0$

This implies f(z) satisfies differential equations known as the ***Riemann-Cauchy conditions***

$$\frac{df}{dz^*} = 0 = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial f_x}{\partial x} - \frac{\partial f_y}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_y}{\partial x} + \frac{\partial f_x}{\partial y}\right) \text{ implies : } \boxed{\frac{\partial f_x}{\partial x} = \frac{\partial f_y}{\partial y} \quad \text{and :} \quad \frac{\partial f_y}{\partial x} = -\frac{\partial f_x}{\partial y}}$$

$$\frac{df}{dz} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right) = \frac{\partial f_x}{\partial x} + i\frac{\partial f_y}{\partial x} = \frac{\partial f_y}{\partial y} - i\frac{\partial f_x}{\partial y} = \frac{\partial}{\partial x}(f_x + i f_y) = \frac{\partial}{\partial iy}(f_x + i f_y)$$

Criteria for a field function $f = f_x(x,y) + i f_y(x,y)$ to be an ***analytic function f(z*)*** *of $z^*=x-iy$:*

First, f(z) must not be a function of $z=x+iy$, that is:* $\frac{df}{dz} = 0$

This implies f(z) satisfies differential equations we call* ***Anti-Riemann-Cauchy conditions***

$$\frac{df}{dz} = 0 = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right) = \text{ implies : } \boxed{\frac{\partial f_x}{\partial x} = -\frac{\partial f_y}{\partial y} \quad \text{and :} \quad \frac{\partial f_y}{\partial x} = \frac{\partial f_x}{\partial y}}$$

$$\frac{df}{dz^*} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial f_x}{\partial x} - \frac{\partial f_y}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_y}{\partial x} + \frac{\partial f_x}{\partial y}\right) = \frac{\partial f_x}{\partial x} + i\frac{\partial f_y}{\partial x} = -\frac{\partial f_y}{\partial y} + i\frac{\partial f_x}{\partial y} = \frac{\partial}{\partial x}(f_x + i f_y) = -\frac{\partial}{\partial iy}(f_x + i f_y)$$

What's analytic? (...and what's not?)

Example: Q: Is $f(x,y) = 2x + i4y$ an analytic function of $z=x+iy$?

Well, test it using definitions: $z = x + iy$ and: $z^* = x - iy$

or: $x = (z+z^*)/2$ and: $y =-i(z-z^*)/2$

$$f(x,y) = 2x + i4y = 2\,(z+z^*)/2 + i4(-i(z-z^*)/2)$$
$$= z+z^* + (2z-2z^*)$$
$$= 3z-z^*$$

A: *NO!* *It's a function of z and z* so not analytic for either.*

Example 2: Q: Is $r(x,y) = x^2 + y^2$ an analytic function of $z=x+iy$?

A: *NO!* $r(xy)=z^*z$ *is a function of z and z* so not analytic for either.*

Example 3: Q: Is $s(x,y) = x^2-y^2 + 2ixy$ an analytic function of $z=x+iy$?

A: *YES!* $s(xy)=(x+iy)^2 =z^2$ *is analytic function of z.* *(Yay!)*

4. Riemann-Cauchy conditions *What's analytic? (...and what's not?)*

Easy 2D circulation and flux integrals

Easy 2D curvilinear coordinate discovery

Easy 2D monopole, dipole, and 2^n-pole analysis

Easy 2^n-multipole field and potential expansion

Easy stereo-projection visualization

What Good Are Complex Exponentials? (contd.) *(Review of topics in Lect.12)*

*9. Complex integrals ∫ f(z)dz count 2D "circulation"(∫**F**•d**r**) and "flux"(∫**F**x d**r**)*

Integral of $f(z)$ between point z_1 and point z_2 is potential difference $\Delta\phi = \phi(z_2) - \phi(z_1)$

$$\Delta\phi = \phi(z_2) - \phi(z_1) = \int_{z_1}^{z_2} f(z)dz = \underbrace{\Phi(x_2,y_2) - \Phi(x_1,y_1)}_{\Delta\Phi} + i\,[\underbrace{\mathbf{A}(x_2,y_2) - \mathbf{A}(x_1,y_1)}_{\Delta\mathbf{A}}]$$

$$\Delta\phi = \Delta\Phi + i\,\Delta\mathbf{A}$$

In *DFL*-field **F**, $\Delta\phi$ is independent of the integration path $z(t)$ connecting z_1 and z_2.

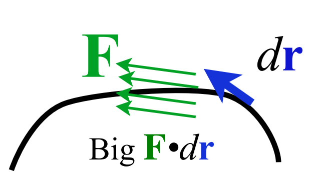
$$\int f(z)dz = \int \left(f^*(z^*)\right)^* dz = \int \left(f^*(z^*)\right)^* (dx + i\,dy) = \int \left(f_x^* + i f_y^*\right)^* (dx + i\,dy) = \int \left(f_x^* - i f_y^*\right)(dx + i\,dy)$$

$$= \int (f_x^*\,dx + f_y^*\,dy) + i\int (f_x^*\,dy - f_y^*\,dx)$$

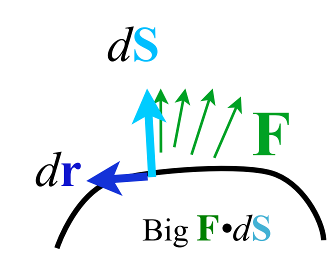
$$= \int \mathbf{F}\bullet d\mathbf{r} \quad + i\int \mathbf{F}\times d\mathbf{r}\bullet\hat{\mathbf{e}}_Z$$

$$= \int \mathbf{F}\bullet d\mathbf{r} \quad + i\int \mathbf{F}\bullet d\mathbf{r}\times\hat{\mathbf{e}}_Z$$

$$= \boxed{\int \mathbf{F}\bullet d\mathbf{r}} \quad \boxed{+ i\int \mathbf{F}\bullet d\mathbf{S}} \qquad \text{where:} \quad d\mathbf{S} = d\mathbf{r}\times\hat{\mathbf{e}}_Z$$



Real part $\int_1^2 \mathbf{F}\bullet d\mathbf{r} = \Delta\Phi$
sums **F** projections *along* path $d\mathbf{r}$ that is, *circulation* on path to get $\Delta\Phi = \Phi(x_2,y_2) - \Phi(x_1,y_1)$

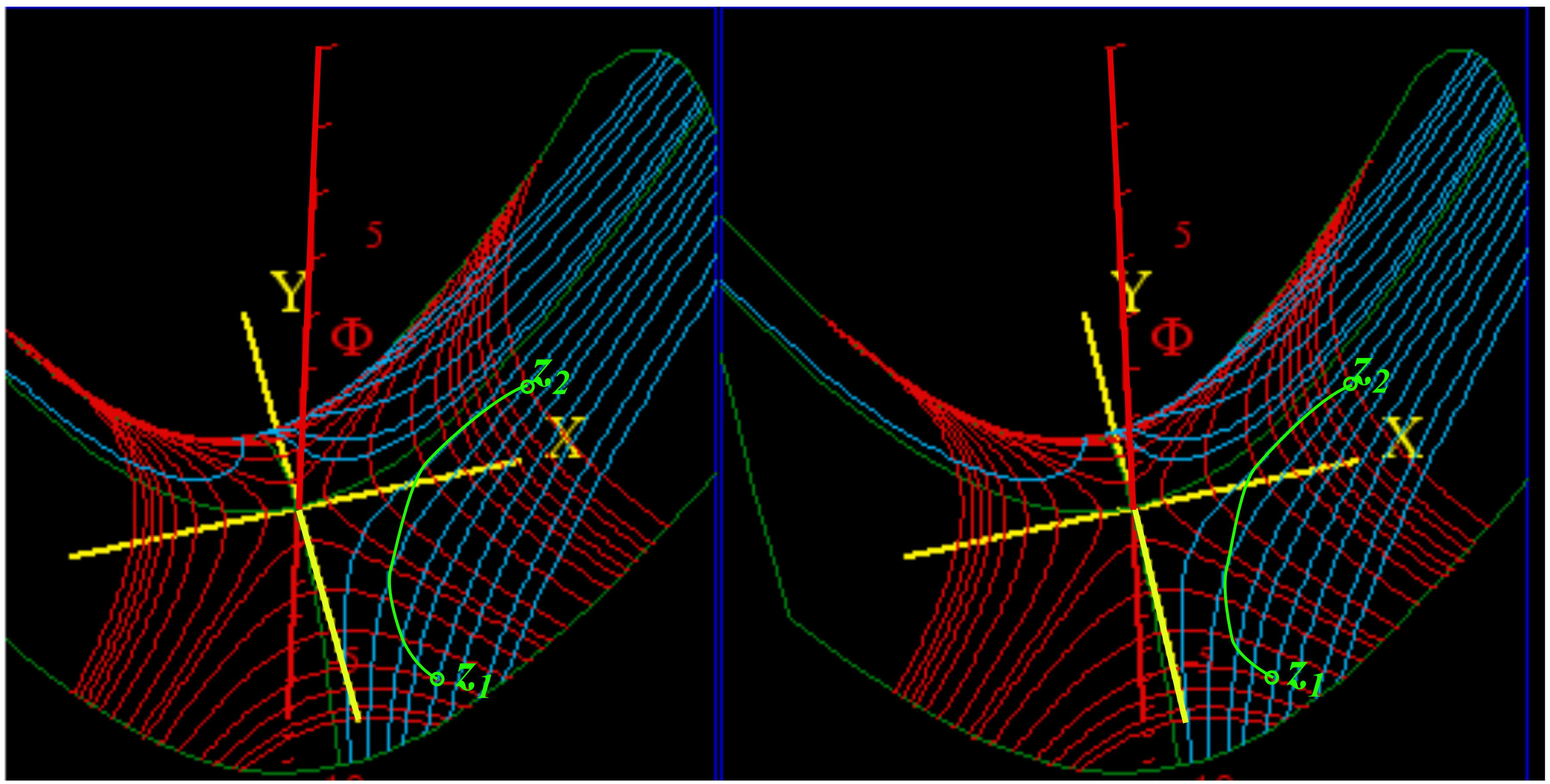


Imaginary part $\int_1^2 \mathbf{F}\bullet d\mathbf{S} = \Delta\mathbf{A}$
sums **F** projection *across* path $d\mathbf{r}$ that is, *flux* thru surface elements $d\mathbf{S} = d\mathbf{r}\times\mathbf{e}_Z$ normal to $d\mathbf{r}$ to get $\Delta\mathbf{A} = \mathbf{A}(x_2,y_2) - \mathbf{A}(x_1,y_1)$

Here the scalar potential $\Phi=(x^2-y^2)/2$ is stereo-plotted vs. (x,y)

The $\Phi=(x^2-y^2)/2=const.$ curves are topography lines

The $A=(xy)=const.$ curves are streamlines normal to topography lines



4. Riemann-Cauchy conditions *What's analytic? (...and what's <u>not</u>?)*

Easy 2D circulation and flux integrals

Easy 2D curvilinear coordinate discovery

Easy 2D monopole, dipole, and 2^n-pole analysis

Easy 2^n-multipole field and potential expansion

Easy stereo-projection visualization

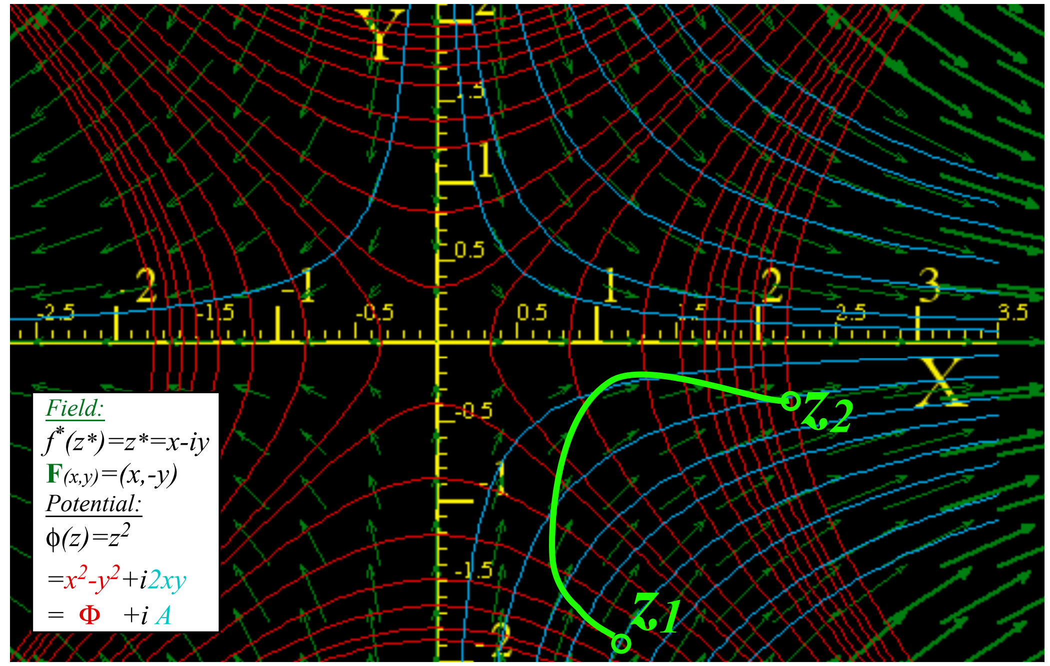
10. Complex potentials define 2D Orthogonal Curvilinear Coordinates (OCC) of field

The (Φ,A) grid is a GCC coordinate system*:

$q^1 = \Phi = (x^2-y^2)/2 = const.$

$q^2 = A = (xy) = const.$

*Actually it's OCC.



$$Kajobian = \begin{pmatrix} \frac{\partial q^1}{\partial x} & \frac{\partial q^1}{\partial y} \\ \frac{\partial q^2}{\partial x} & \frac{\partial q^2}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial \Phi}{\partial x} & \frac{\partial \Phi}{\partial y} \\ \frac{\partial A}{\partial x} & \frac{\partial A}{\partial y} \end{pmatrix} = \begin{array}{c} \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \end{array} \begin{array}{l} \leftarrow \mathbf{E}^{\Phi} \\ \leftarrow \mathbf{E}^{A} \end{array}$$

$$Jacobian = \begin{pmatrix} \frac{\partial x}{\partial q^1} & \frac{\partial x}{\partial q^2} \\ \frac{\partial y}{\partial q^1} & \frac{\partial y}{\partial q^2} \end{pmatrix} = \begin{array}{c} \begin{pmatrix} \frac{\partial x}{\partial \Phi} & \frac{\partial x}{\partial A} \\ \frac{\partial y}{\partial \Phi} & \frac{\partial y}{\partial A} \end{pmatrix} \\ \begin{array}{cc} \uparrow & \uparrow \\ \mathbf{E}_{\Phi} & \mathbf{E}_{A} \end{array} \end{array} = \frac{1}{r^2} \begin{array}{c} \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \\ \begin{array}{cc} \uparrow & \uparrow \\ \mathbf{E}_{\Phi} & \mathbf{E}_{A} \end{array} \end{array}$$

$$Metric\,tensor = \begin{pmatrix} g_{\Phi\Phi} & g_{\Phi A} \\ g_{A\Phi} & g_{AA} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_{\Phi}\bullet\mathbf{E}_{\Phi} & \mathbf{E}_{\Phi}\bullet\mathbf{E}_{A} \\ \mathbf{E}_{A}\bullet\mathbf{E}_{\Phi} & \mathbf{E}_{A}\bullet\mathbf{E}_{A} \end{pmatrix} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2 \end{pmatrix} \text{ where: } r^2 = x^2 + y^2$$

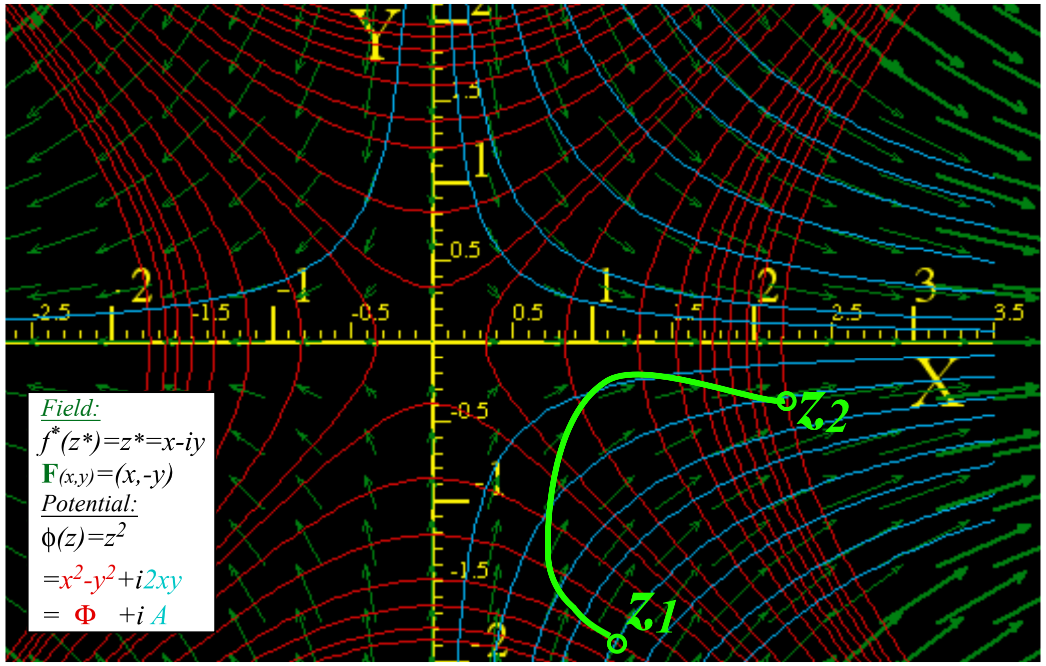
10. Complex potentials define 2D Orthogonal Curvilinear Coordinates (OCC) of field

The (Φ,A) grid is a GCC coordinate system*:

$q^1 = \Phi = (x^2 - y^2)/2 = const.$

$q^2 = A = (xy) = const.$

*Actually it's OCC.



$$Kajobian = \begin{pmatrix} \frac{\partial q^1}{\partial x} & \frac{\partial q^1}{\partial y} \\ \frac{\partial q^2}{\partial x} & \frac{\partial q^2}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial \Phi}{\partial x} & \frac{\partial \Phi}{\partial y} \\ \frac{\partial A}{\partial x} & \frac{\partial A}{\partial y} \end{pmatrix} = \begin{array}{cc} \begin{pmatrix} x & -y \\ y & x \end{pmatrix} & \begin{array}{l} \leftarrow \mathbf{E}^{\Phi} \\ \leftarrow \mathbf{E}^{A} \end{array} \end{array}$$

$$Jacobian = \begin{pmatrix} \frac{\partial x}{\partial q^1} & \frac{\partial x}{\partial q^2} \\ \frac{\partial y}{\partial q^1} & \frac{\partial y}{\partial q^2} \end{pmatrix} = \begin{array}{c} \begin{pmatrix} \frac{\partial x}{\partial \Phi} & \frac{\partial x}{\partial A} \\ \frac{\partial y}{\partial \Phi} & \frac{\partial y}{\partial A} \end{pmatrix} \\ \begin{array}{cc} \uparrow & \uparrow \\ \mathbf{E}_{\Phi} & \mathbf{E}_{A} \end{array} \end{array} = \frac{1}{r^2} \begin{array}{c} \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \\ \begin{array}{cc} \uparrow & \uparrow \\ \mathbf{E}_{\Phi} & \mathbf{E}_{A} \end{array} \end{array}$$

$$Metric\,tensor = \begin{pmatrix} g_{\Phi\Phi} & g_{\Phi A} \\ g_{A\Phi} & g_{AA} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_{\Phi} \bullet \mathbf{E}_{\Phi} & \mathbf{E}_{\Phi} \bullet \mathbf{E}_{A} \\ \mathbf{E}_{A} \bullet \mathbf{E}_{\Phi} & \mathbf{E}_{A} \bullet \mathbf{E}_{A} \end{pmatrix} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2 \end{pmatrix} \text{ where: } r^2 = x^2 + y^2$$

Riemann-Cauchy Derivative Relations make coordinates orthogonal

$$\nabla \Phi = \begin{pmatrix} \frac{\partial \Phi}{\partial x} \\ \frac{\partial \Phi}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} \frac{a}{2}(x^2 - y^2) \\ \frac{\partial}{\partial y} \frac{a}{2}(x^2 - y^2) \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = \mathbf{F}$$

The half-n'-half results assure

$$\mathbf{E}_{\Phi} \bullet \mathbf{E}_{A} = \frac{\partial \Phi}{\partial x}\frac{\partial A}{\partial x} + \frac{\partial \Phi}{\partial y}\frac{\partial A}{\partial y}$$

$$= -\frac{\partial \Phi}{\partial x}\frac{\partial \Phi}{\partial y} + \frac{\partial \Phi}{\partial y}\frac{\partial \Phi}{\partial x} = 0$$

$$\nabla \times \mathbf{A} = \begin{pmatrix} \frac{\partial \mathbf{A}}{\partial y} \\ -\frac{\partial \mathbf{A}}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial y} axy \\ -\frac{\partial}{\partial x} axy \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = \mathbf{F}$$

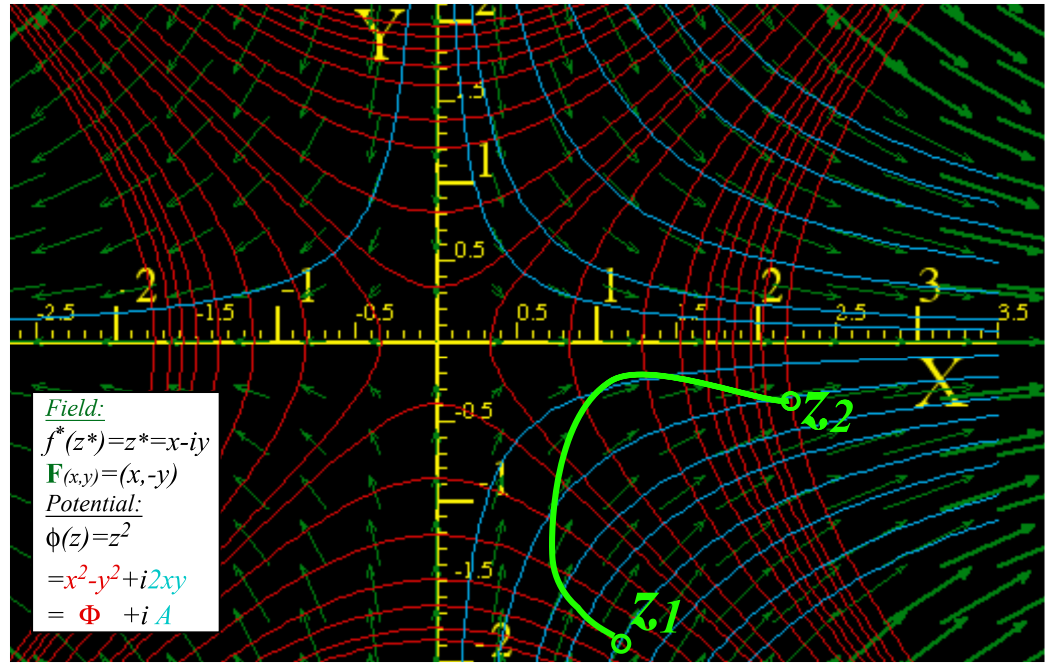
10. Complex potentials define 2D Orthogonal Curvilinear Coordinates (OCC) of field

The (Φ,A) grid is a GCC coordinate system*:

$q^1 = \Phi = (x^2-y^2)/2 = const.$

$q^2 = A = (xy) = const.$

*Actually it's OCC.



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or Riemann-Cauchy

Zero divergence requirement: $0 = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} = \frac{\partial}{\partial x}\frac{\partial \Phi}{\partial x} + \frac{\partial}{\partial y}\frac{\partial \Phi}{\partial y} = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$ *potential Φ obeys Laplace equation*

and so does $\mathbf{A}$

4. Riemann-Cauchy conditions *What's analytic? (...and what's not?)*

Easy 2D circulation and flux integrals

Easy 2D curvilinear coordinate discovery

Easy 2D monopole, dipole, and 2^n-pole analysis

Easy 2^n-multipole field and potential expansion

Easy stereo-projection visualization

11. Complex integrals define 2D monopole fields and potentials

Of all power-law fields $f(z)=az^n$ one lacks a power-law potential $\phi(z)=\frac{a}{n+1}z^{n+1}$. It is the $n=-1$ case.

Unit monopole field: $f(z)=\frac{1}{z}=z^{-1}$ $\qquad$ $f(z)=\frac{a}{z}=az^{-1}$ Source-a monopole

It has a *logarithmic potential* $\phi(z)=a\cdot\ln(z)=a\cdot\ln(x+iy)$.

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$$\phi(z) = \Phi + i\mathbf{A} = \int f(z)dz = \int \frac{a}{z}dz = a\ln(z)$$

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$$\phi(z)=\underbrace{\Phi}_{=a\ln(r)}+i\underbrace{\mathbf{A}}_{a\theta}=\int f(z)dz=\int \frac{a}{z}dz=a\ln(z)=a\ln(re^{i\theta})$$

$$=a\ln(r)+i\,a\theta$$

What Good Are Complex Exponentials? (contd.)

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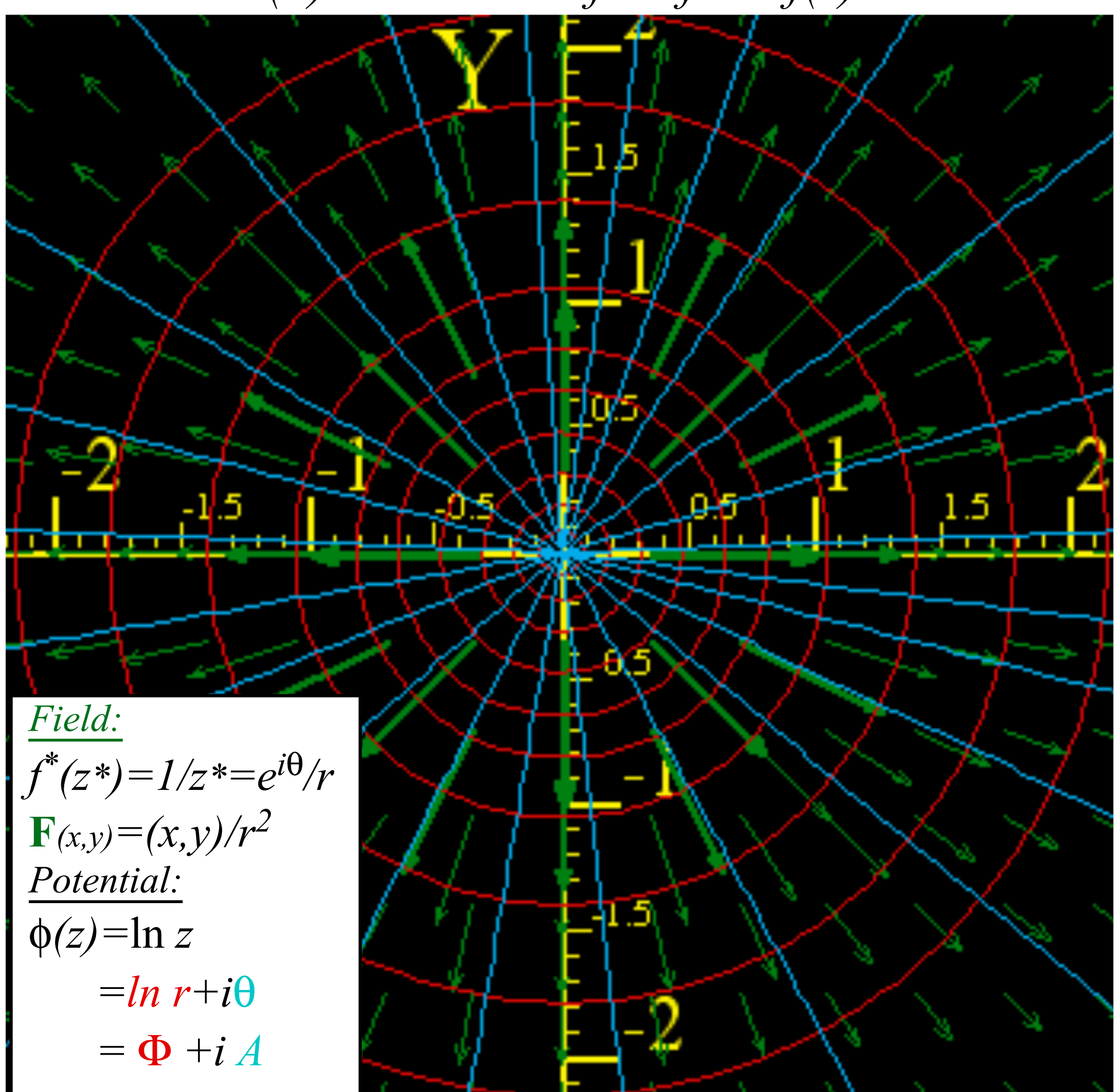
(a) Unit Z-line-flux field f(z)=1/z



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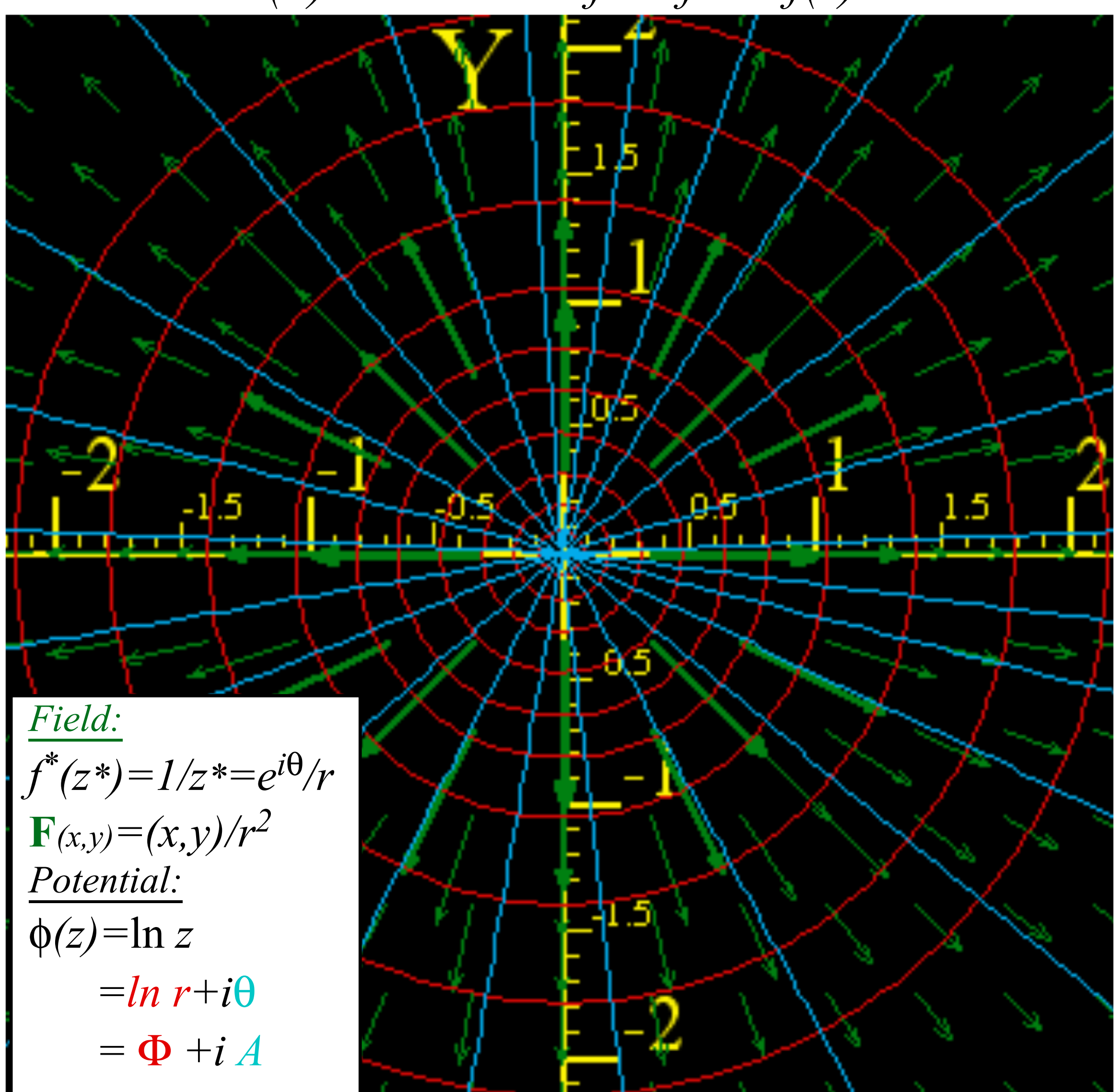
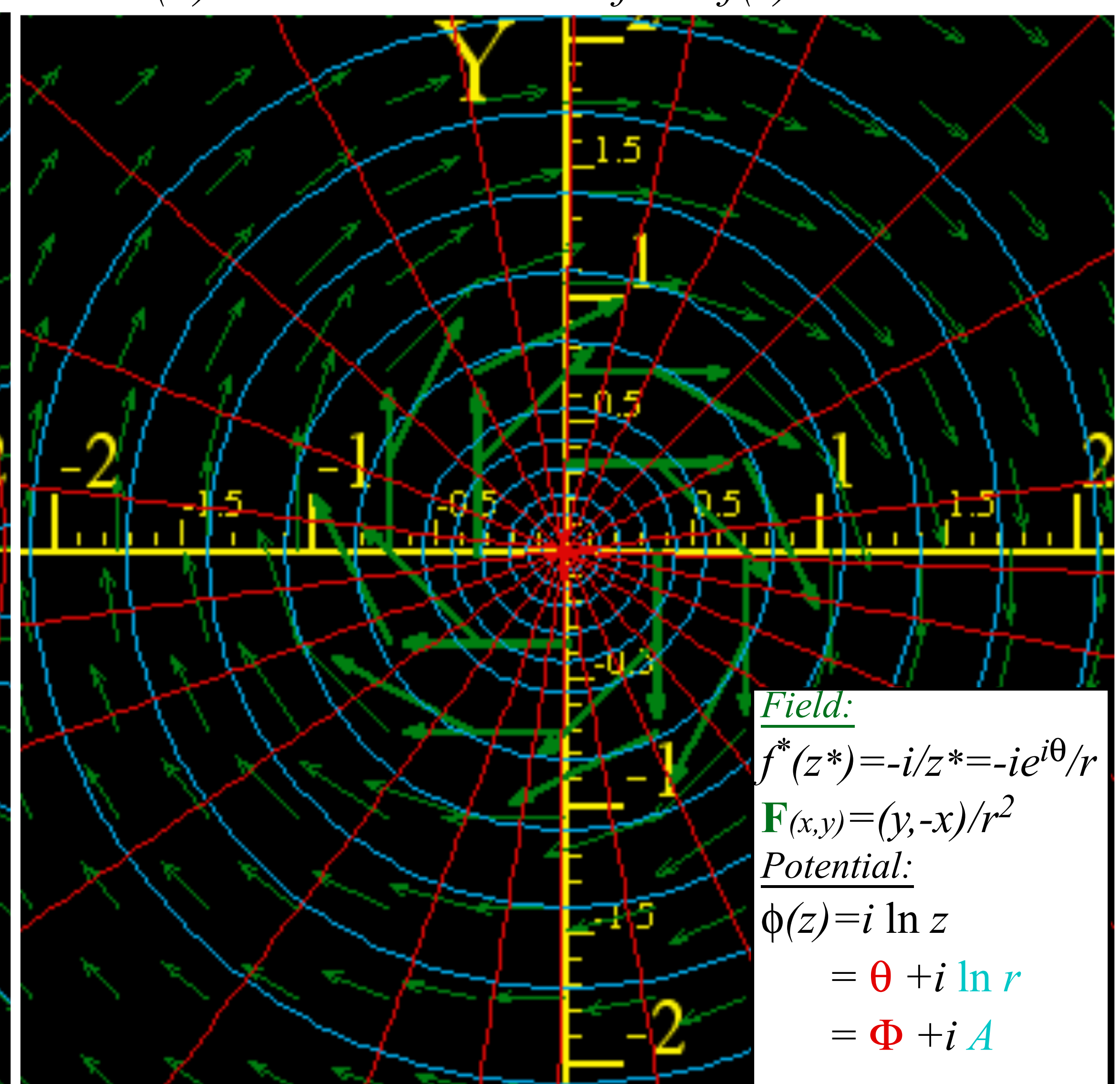
(a) Unit Z-line-flux field $f(z)=1/z$



(b) Unit Z-line-vortex field $f(z)=i/z$



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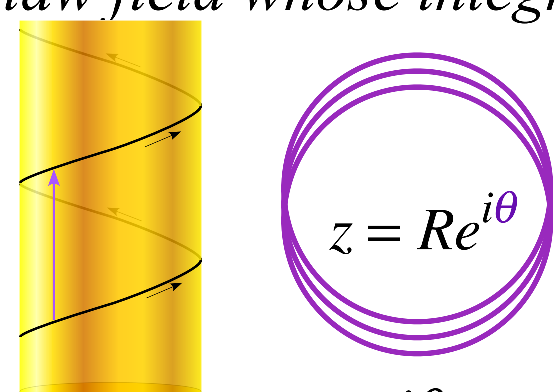
$$\phi(z)=\underbrace{\Phi}_{=a\ln(r)}+\underbrace{i\mathbf{A}}_{i\,a\theta}=\int f(z)dz=\int\frac{a}{z}dz=a\ln(z)=a\ln(re^{i\theta})$$

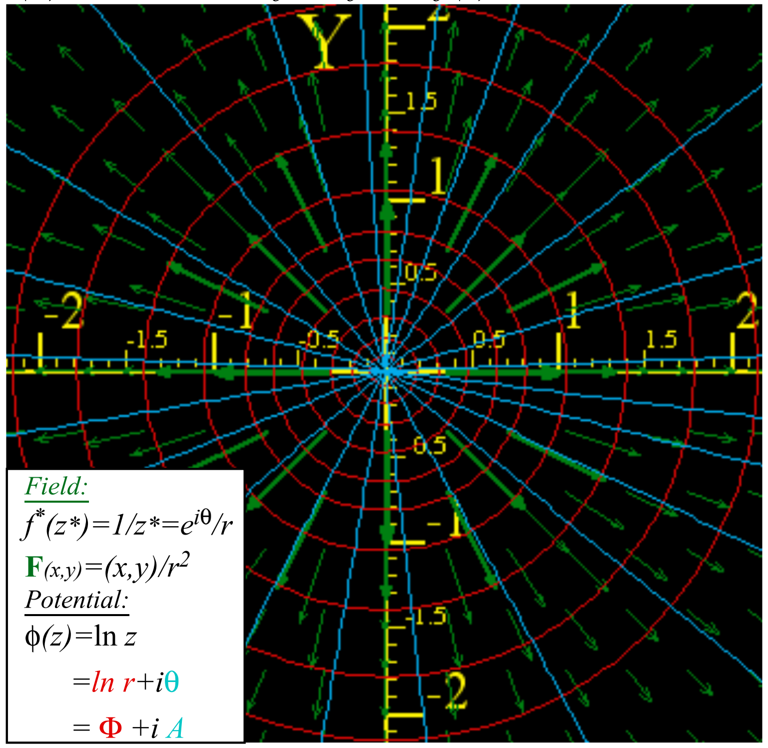
A monopole field is the only power-law field whose integral (potential) depends on path of integration.



path that goes N times around origin (r=0) at constant r = R.

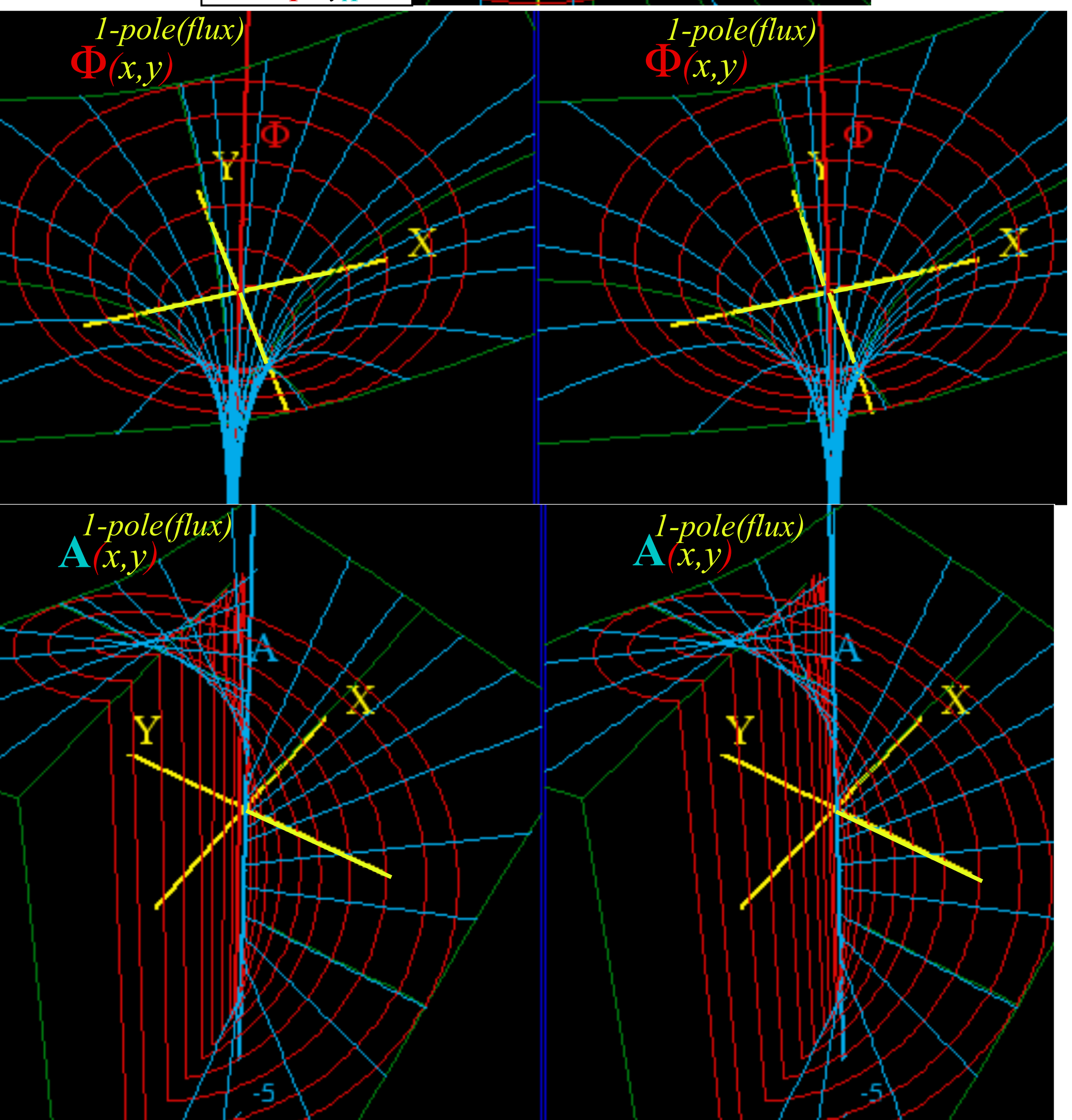
$$\Delta\phi=\oint f(z)dz=a\oint\frac{dz}{z}=a\int_{\theta=0}^{\theta=2\pi N}\frac{d(Re^{i\theta})}{Re^{i\theta}}=a\int_{\theta=0}^{\theta=2\pi N}i\,d\theta=ai\,\theta\Big|_0^{2\pi N}=2a\pi iN$$

(a) *Unit Z-line-flux field f(z)=1/z*



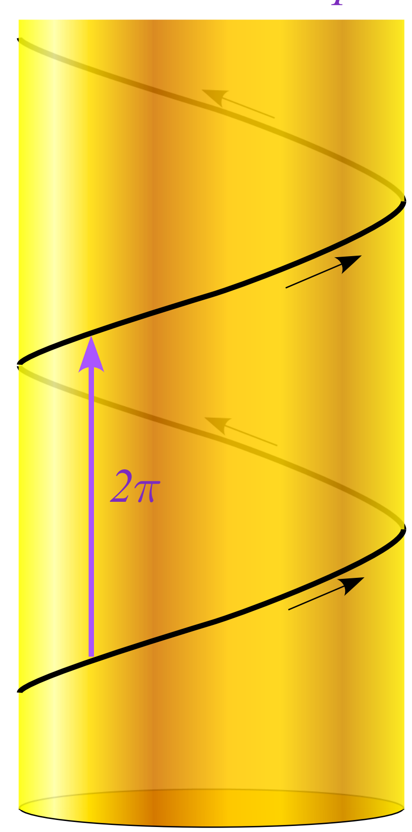
$$\phi(z) = \underbrace{\Phi}_{\ln(r)} + i\underbrace{\mathbf{A}}_{\theta} = \int f(z)dz = \int \frac{a}{z}dz = a\ln(re^{i\theta})$$

$$= \ln(r) + i\theta \qquad \textit{(For } a=1\textit{)}$$



Each turn around origin

*adds 2πi to vector potential i***A**



(For a=1)

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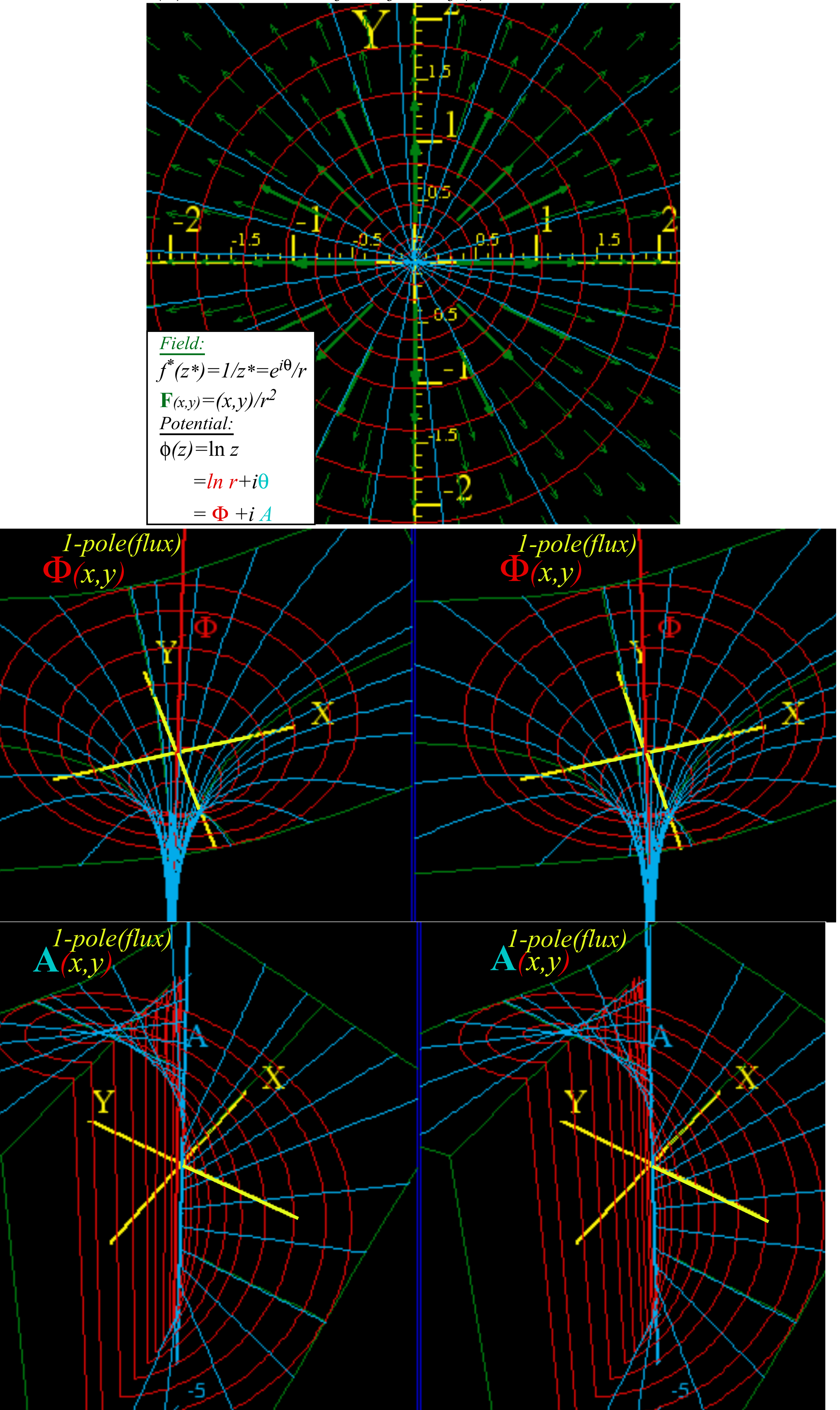
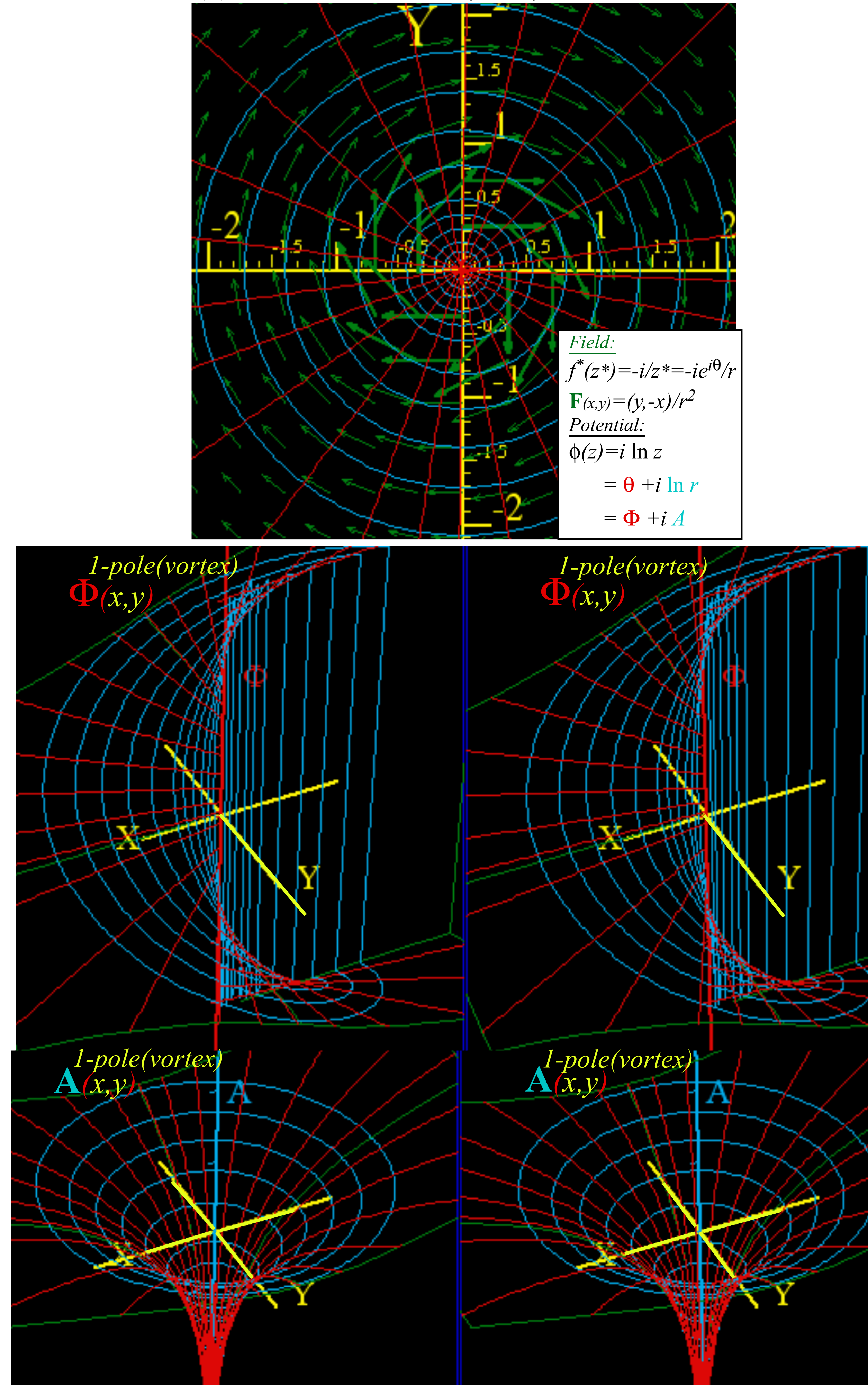


(b) Unit Z-line-vortex field $f(z)=i/z$

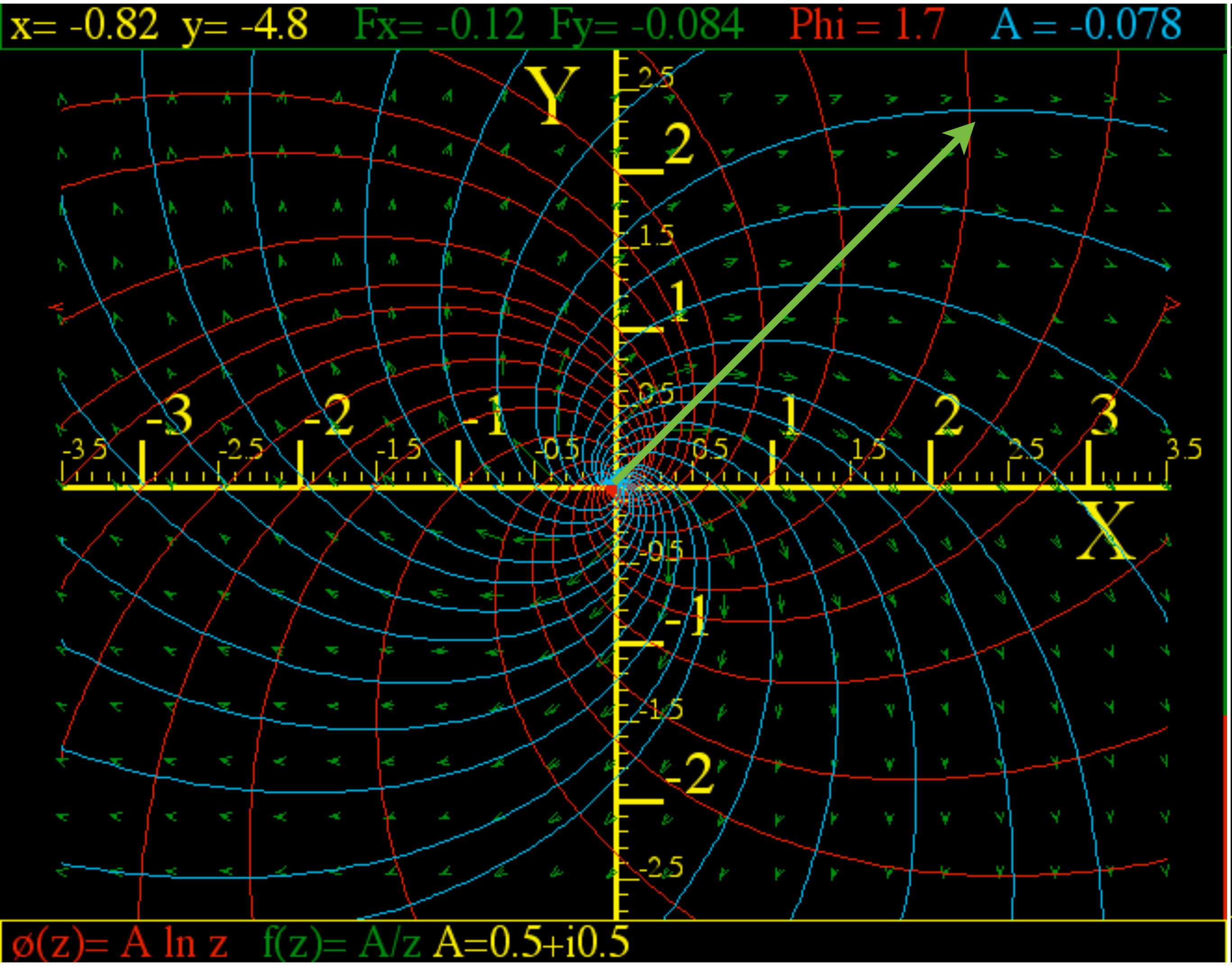


What Good Are Complex Exponentials? (contd.)

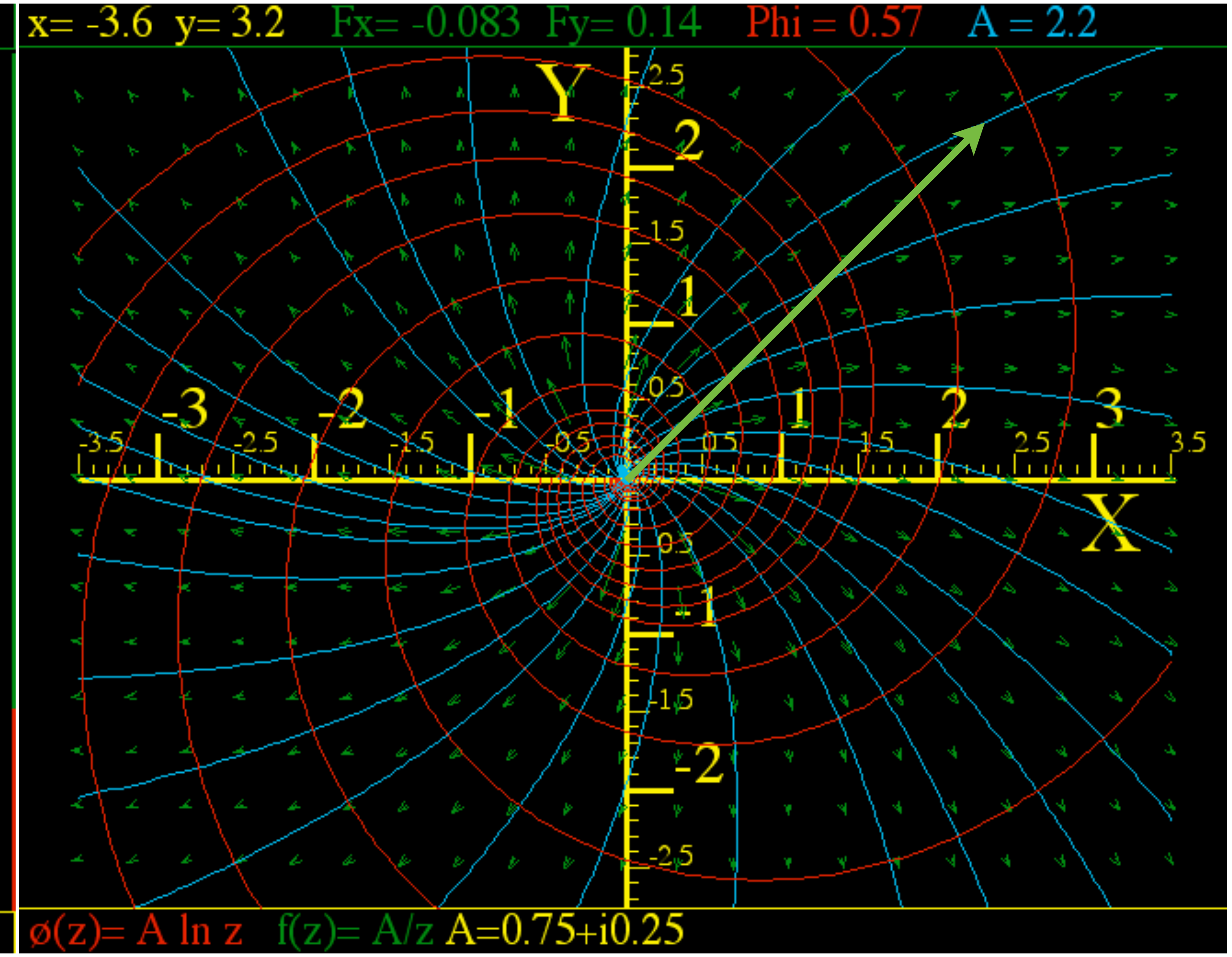
$f(z) = (0.5 + i0.5)/z = e^{i\pi/4}/z\sqrt{2}$

$f(z) = (0.75 + i0.25)/z = e^{i18^{\circ}}/z\sqrt{n}$

“Vortex”



“Hurricane”



4. Riemann-Cauchy conditions *What's analytic? (...and what's not?)*

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12. Complex derivatives give 2D dipole fields

Start with $f(z)=az^{-1}$: 2D line *monopole field* and is its *monopole potential* $\phi(z)=a\ln z$ of source strength a.

$$f^{1\text{-}pole}(z)=\frac{a}{z}=\frac{d\phi^{1\text{-}pole}}{dz} \qquad \phi^{1\text{-}pole}(z)=a\ln z$$

Now let these two line-sources of equal but opposite source constants $+a$ and $-a$ be located at $z=\pm\Delta/2$ separated by a small interval Δ. This sum (actually difference) of $f^{1\text{-}pole}$-fields is called a *dipole field*.

$$f^{dipole}(z)=\frac{a}{z+\frac{\Delta}{2}}-\frac{a}{z-\frac{\Delta}{2}}=\frac{-a\cdot\Delta}{z^2-\frac{\Delta^2}{4}} \qquad \phi^{dipole}(z)=a\ln(z-\tfrac{\Delta}{2})-a\ln(z+\tfrac{\Delta}{2})=a\ln\frac{z-\frac{\Delta}{2}}{z+\frac{\Delta}{2}}$$

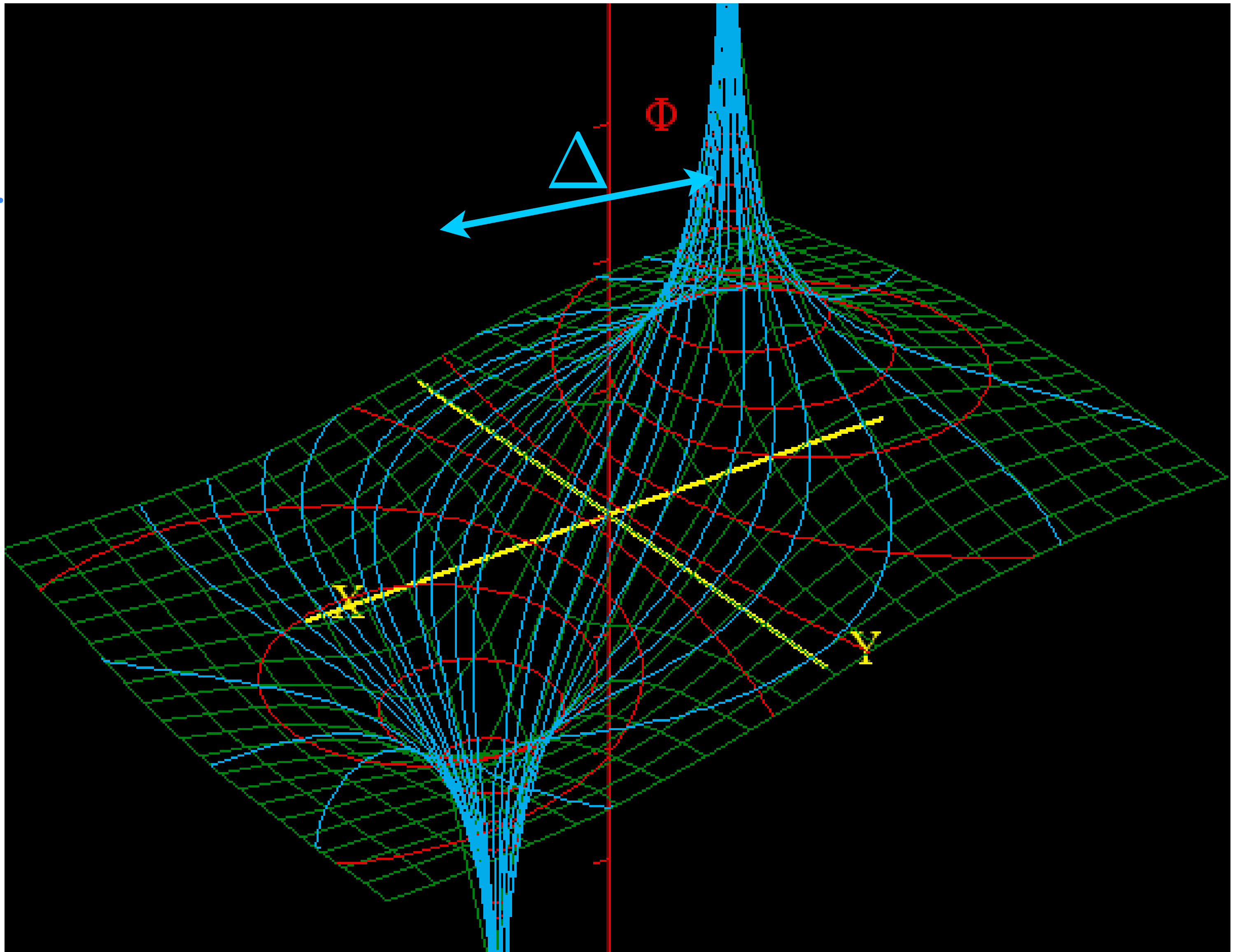
This is like the derivative definition:

$$\frac{df}{dz}=\frac{f(z+\Delta)-f(z)}{\Delta}$$

or:

$$\frac{df}{dz}=\frac{f(z+\frac{\Delta}{2})-f(z-\frac{\Delta}{2})}{\Delta}$$

if Δ is infinitesimal ($\Delta\to 0$)



So-called "physical dipole" has finite Δ

(+)(−) separation

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If interval Δ is *tiny* and is divided out we get a *point-dipole field* $f^{2\text{-}pole}$ that is the z-derivative of $f^{1\text{-}pole}$.

$$f^{2\text{-}pole}=\frac{-a}{z^2}=\frac{df^{1\text{-}pole}}{dz}=\frac{d\phi^{2\text{-}pole}}{dz} \qquad \phi^{2\text{-}pole}=\frac{a}{z}=\frac{d\phi^{1\text{-}pole}}{dz}$$

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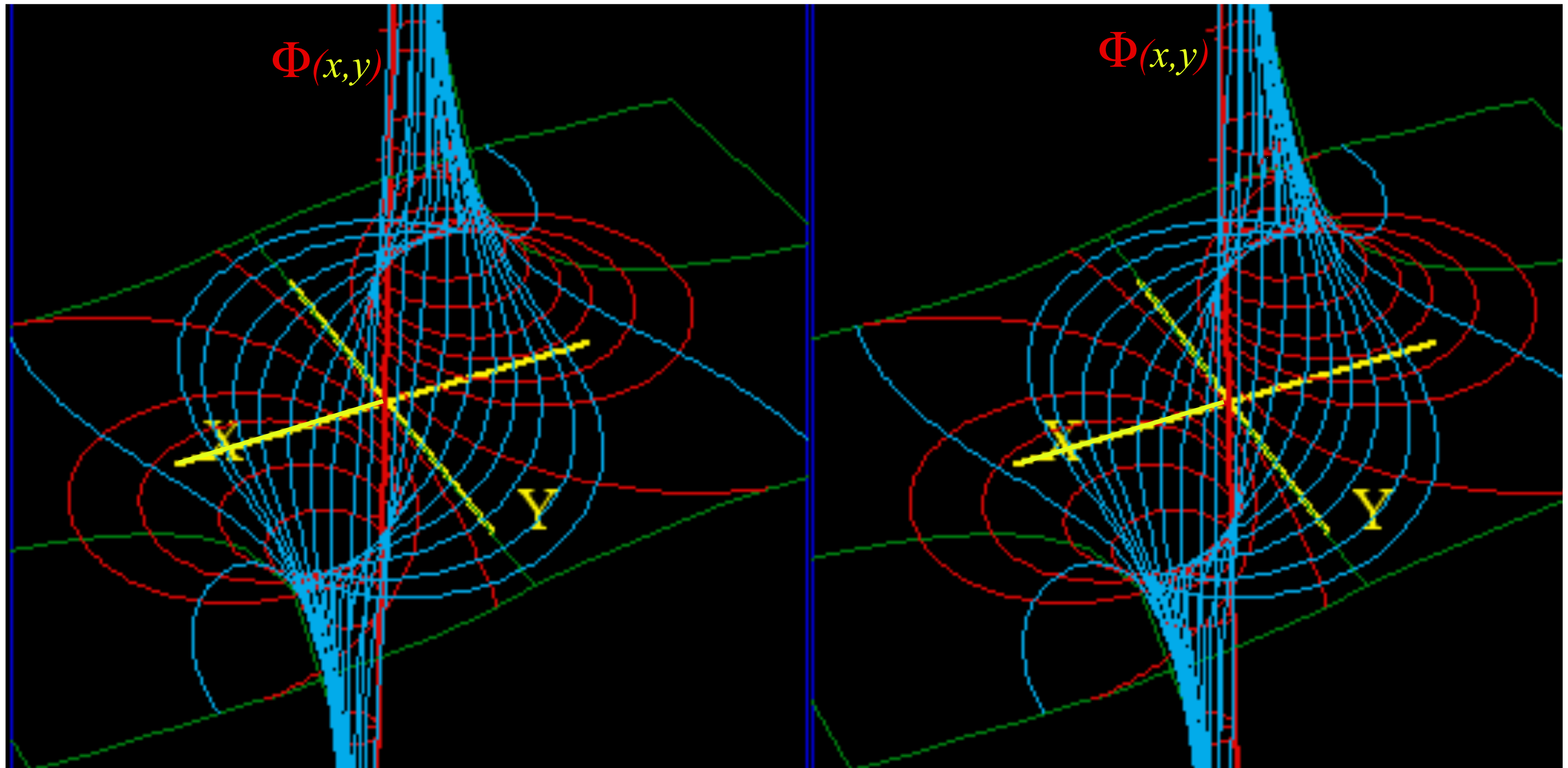

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A *point-dipole potential* $\phi^{2\text{-}pole}$ (whose z-derivative is $f^{2\text{-}pole}$) is a z-derivative of $\phi^{1\text{-}pole}$.

$$\phi^{2\text{-}pole}=\frac{a}{z}=\frac{a}{x+iy}=\frac{a}{x+iy}\frac{x-iy}{x-iy}=\frac{ax}{x^2+y^2}+i\frac{-ay}{x^2+y^2}=\frac{a}{r}\cos\theta-i\frac{a}{r}\sin\theta$$

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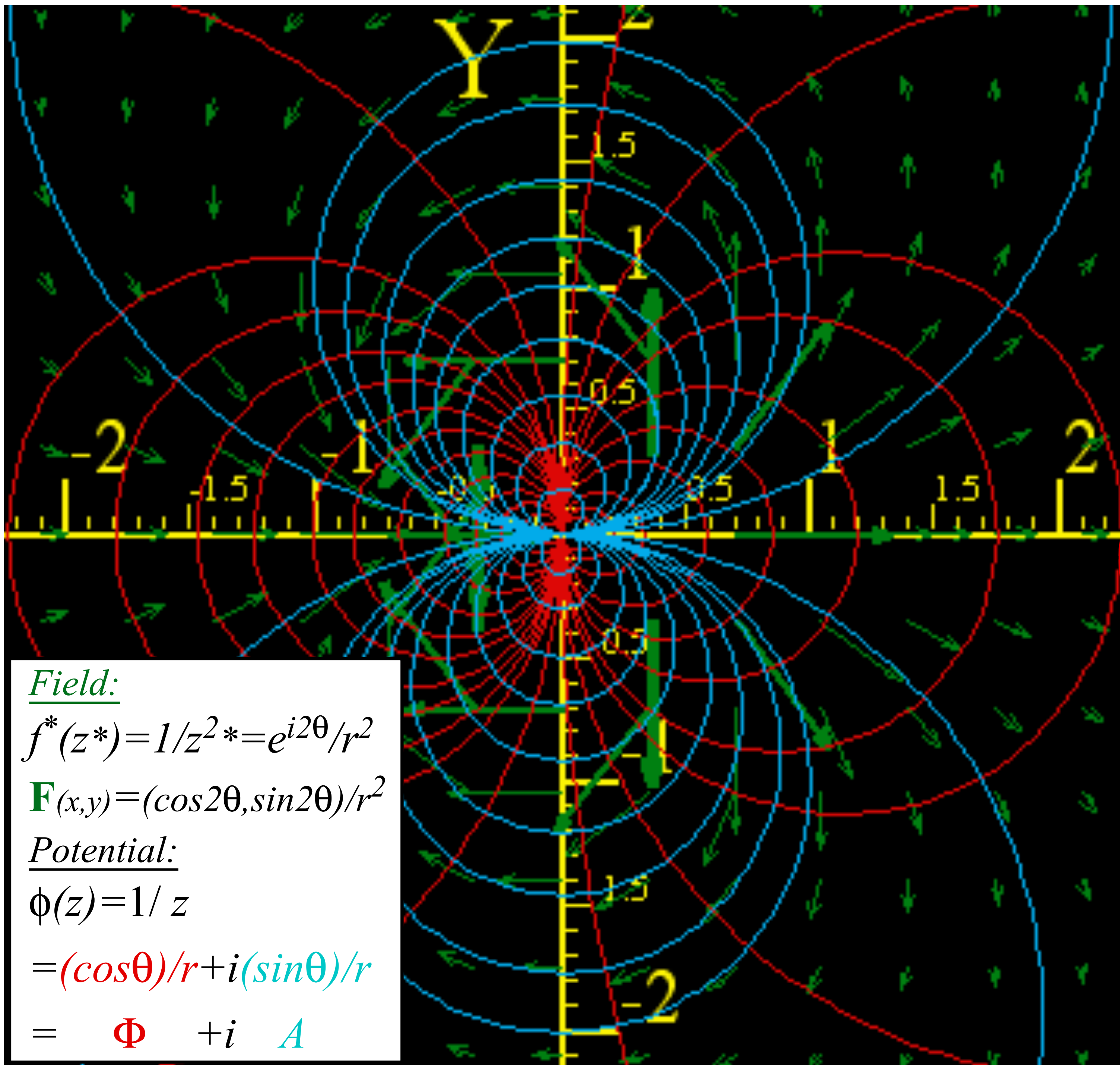

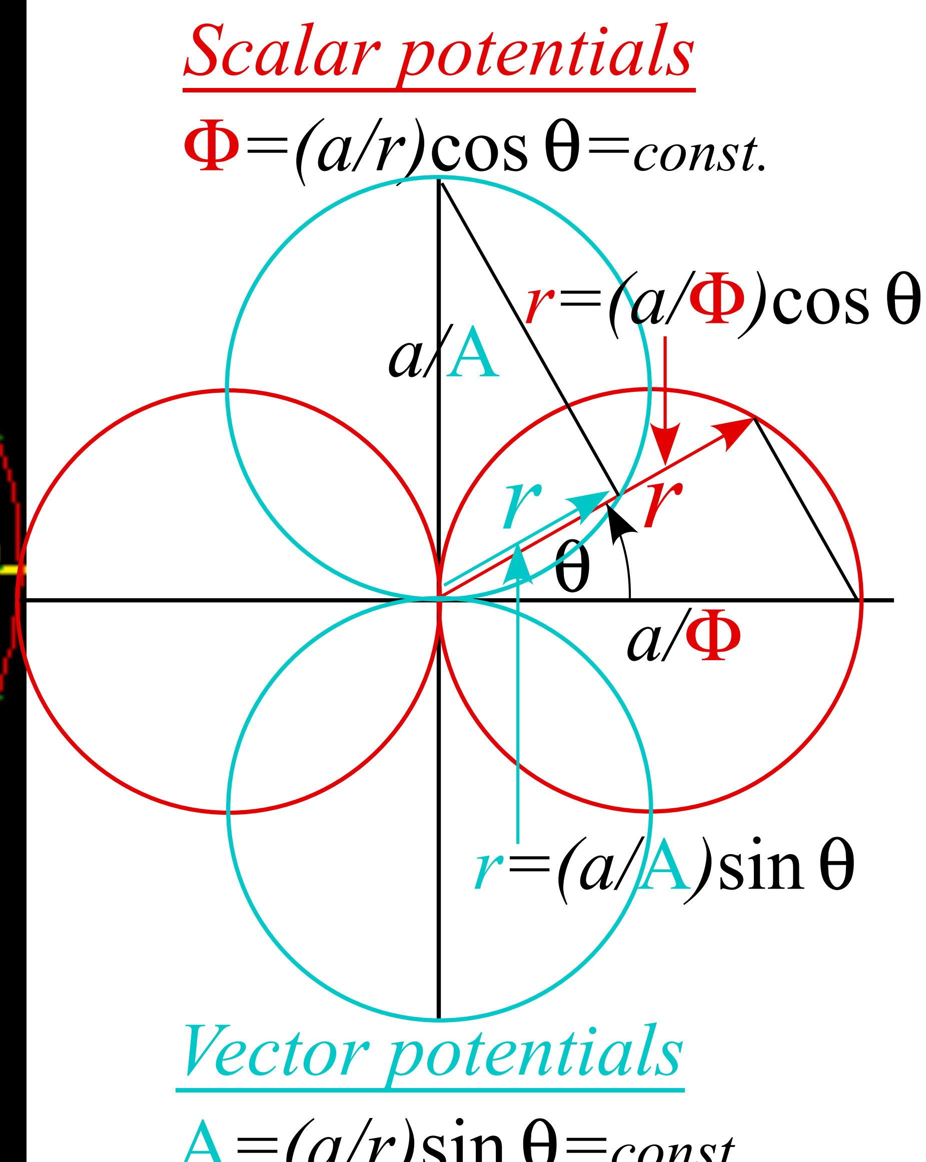

2^n-pole analysis *(quadrupole: 2^2=4-pole, octapole: 2^3=8-pole, …, pole dancer,*

What if we put a *(-)*copy of a *2-pole* near its original?

Well, the result is *4-pole* or *quadrupole* field $f^{\text{4-pole}}$ and potential $\phi^{\text{4-pole}}$.

Each a z-derivative of $f^{\text{2-pole}}$ and $\phi^{\text{2-pole}}$.

$$f^{4\text{-}pole} = \frac{a}{z^3} = \frac{1}{2}\frac{df^{2\text{-}pole}}{dz} = \frac{d\phi^{4\text{-}pole}}{dz}$$

$$\phi^{4\text{-}pole} = -\frac{a}{2z^2} = \frac{1}{2}\frac{d\phi^{2\text{-}pole}}{dz}$$

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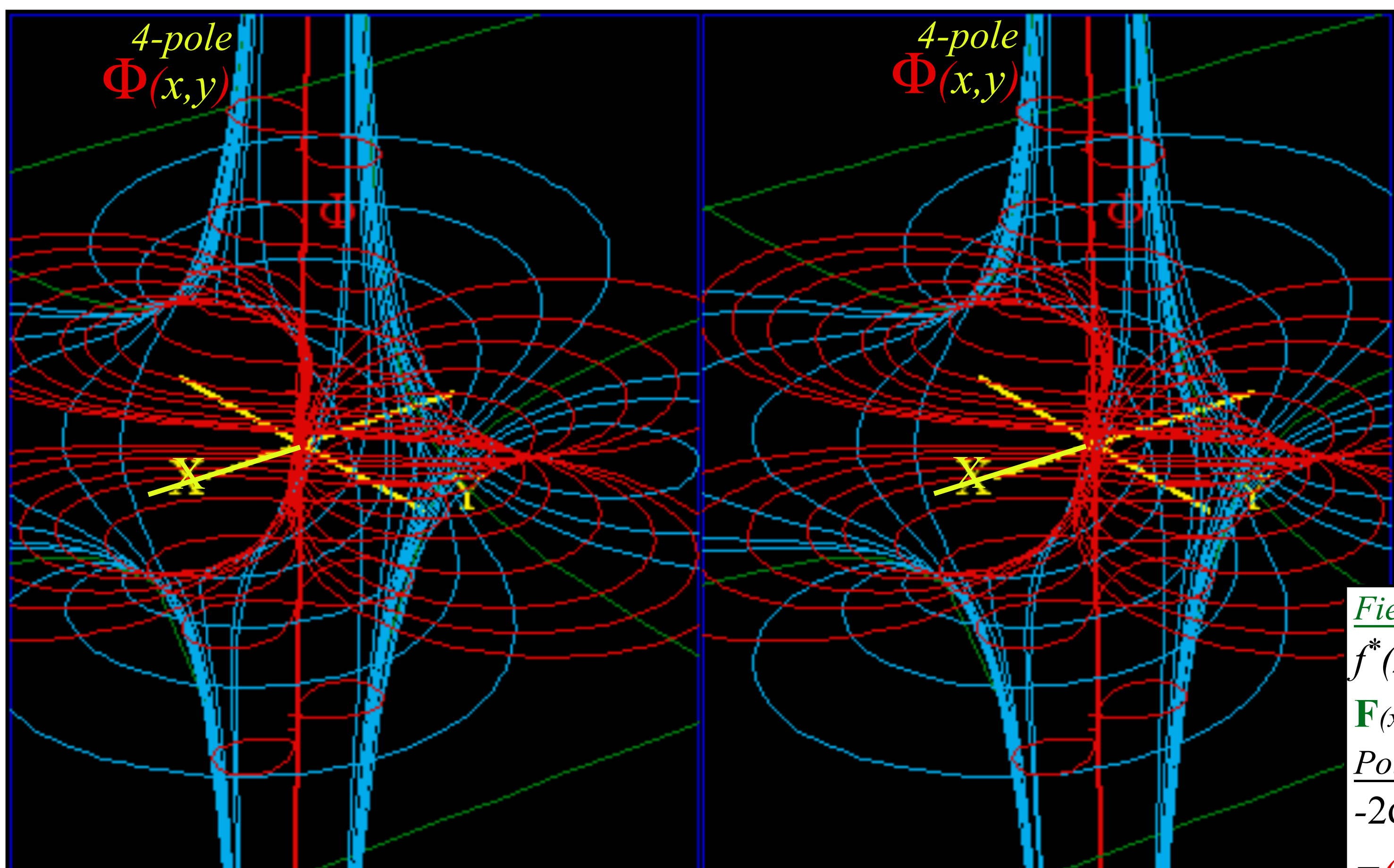
What if we put a *(-)*copy of a *2-pole* near its original?

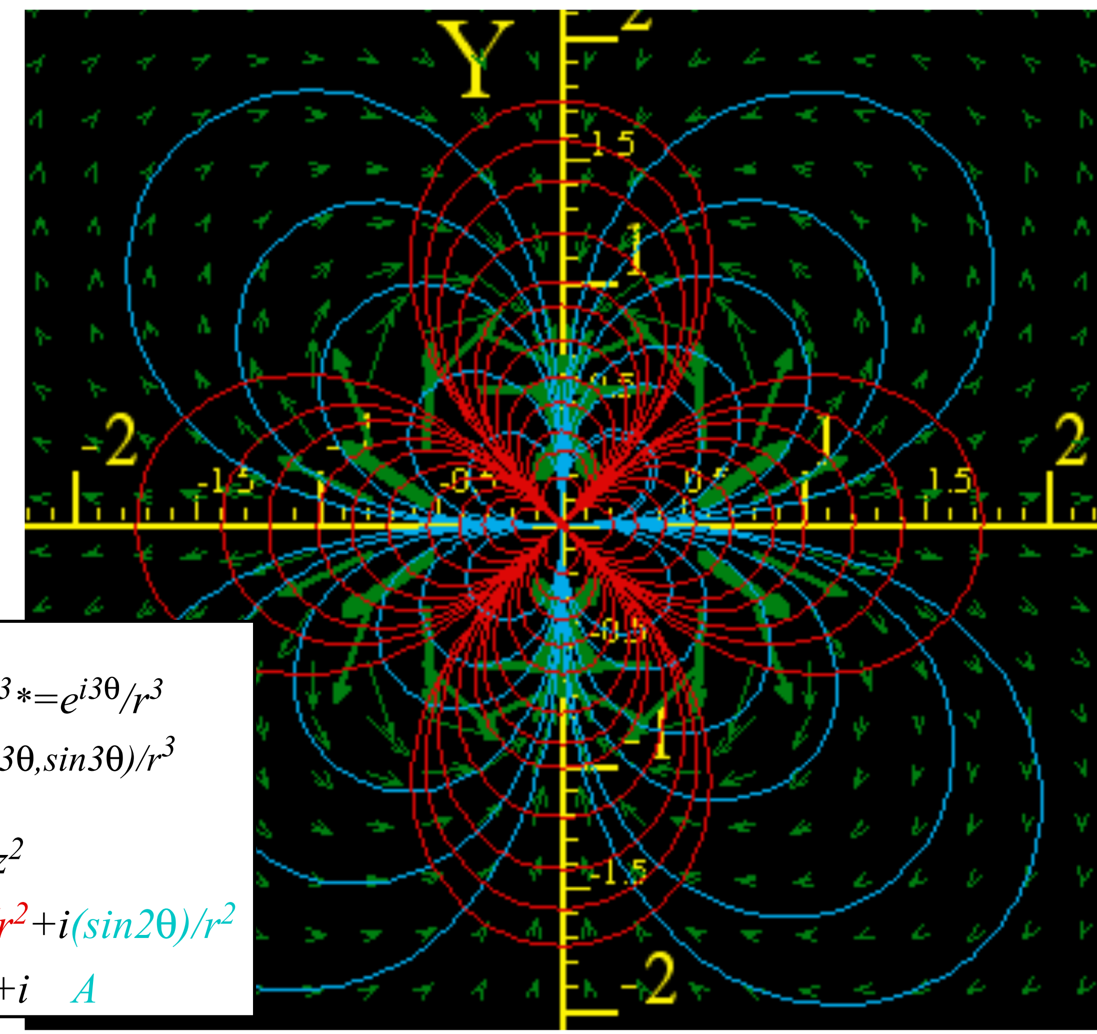
Well, the result is *4-pole* or *quadrupole* field $f^{4\text{-}pole}$ and potential $\phi^{4\text{-}pole}$.

Each a *z*-derivative of $f^{2\text{-}pole}$ and $\phi^{2\text{-}pole}$.

$$f^{4\text{-}pole} = \frac{a}{z^3} = \frac{1}{2}\frac{df^{2\text{-}pole}}{dz} = \frac{d\phi^{4\text{-}pole}}{dz}$$

$$\phi^{4\text{-}pole} = -\frac{a}{2z^2} = \frac{1}{2}\frac{d\phi^{2\text{-}pole}}{dz}$$





Field:
$f^*(z^*) = 1/z^{3*} = e^{i3\theta}/r^3$
$\mathbf{F}(x,y) = (\cos 3\theta, \sin 3\theta)/r^3$
Potential:
$-2\phi(z) = 1/z^2$
$= (\cos 2\theta)/r^2 + i(\sin 2\theta)/r^2$
$= \quad \Phi \quad + i \quad A$

4. Riemann-Cauchy conditions *What's analytic? (...and what's* ***not****?)*

Easy 2D circulation and flux integrals

Easy 2D curvilinear coordinate discovery

Easy 2D monopole, dipole, and 2^n-pole analysis

Easy 2^n-multipole field and potential expansion

Easy stereo-projection visualization

2^n-pole analysis: Laurent series *(Generalization of Maclaurin-Taylor series)*

Laurent series or *multipole expansion* of a given complex field function $f(z)$ around $z=0$.

$$\frac{d\phi}{dz} = f(z) = \dots a_{-3}z^{-3} + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + \dots$$

$\cdots$ 2^2-pole	2^1-pole	2^0-pole	2^1-pole	2^2-pole	2^3-pole	2^4-pole	2^5-pole	2^6-pole $\cdots$
(quadrupole)	*(dipole)*	*(monopole)*	*(dipole)*	*(quadrupole)*	*(octapole)*	*(hexadecapole)*		
at $z=0$	at $z=0$	at $z=0$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$

$$\int f\,dz = \phi(z) = \dots \frac{a_{-3}}{-2}z^{-2} + \frac{a_{-2}}{-1}z^{-1} + a_{-1}\ln z + a_0 z + \frac{a_1}{2}z^2 + \frac{a_2}{3}z^3 + \frac{a_3}{4}z^4 + \frac{a_4}{5}z^5 + \frac{a_5}{6}z^6 + \dots$$

All field terms $a_{m-1}z^{m-1}$ except *1-pole* $\frac{a_{-1}}{z}$ have potential term $a_{m-1}z^m/m$ of a *2^m-pole*.

These are located at $z=0$ for $m<0$ and at $z=\infty$ for $m>0$.

(octapole)$_0$	*(quadrupole)$_0$*	*(dipole)$_0$*	*(monopole)*	*(dipole)$_\infty$*	*(quadrupole)$_\infty$*	*(octapole)$_\infty$*

$$\phi(z) = \dots \frac{a_{-4}}{-3}z^{-3} + \frac{a_{-3}}{-2}z^{-2} + \frac{a_{-2}}{-1}z^{-1} + a_{-1}\ln z + a_0 z + \frac{a_1}{2}z^2 + \frac{a_2}{3}z^3 + \dots$$

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at *z=0*	at *z=0*	at *z=0*	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$

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(*with* $z=w^{-1}$)

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at $z=0$	at $z=0$	at $z=0$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$

$\int f\,dz=$

$$\phi(z)=\ldots\frac{a_{-3}}{-2}z^{-2}+\frac{a_{-2}}{-1}z^{-1}+a_{-1}\ln z+a_0z+\frac{a_1}{2}z^2+\frac{a_2}{3}z^3+\frac{a_3}{4}z^4+\frac{a_4}{5}z^5+\frac{a_5}{6}z^6+\ldots$$

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(with $z\rightarrow w$)

$$=\ldots\frac{a_2}{3}z^{-3}+\frac{a_1}{2}z^{-2}+a_0z^{-1}-a_{-1}\ln z+\frac{a_{-2}}{-1}z+\frac{a_{-3}}{-2}z^2+\frac{a_{-4}}{-3}z^3+\ldots$$

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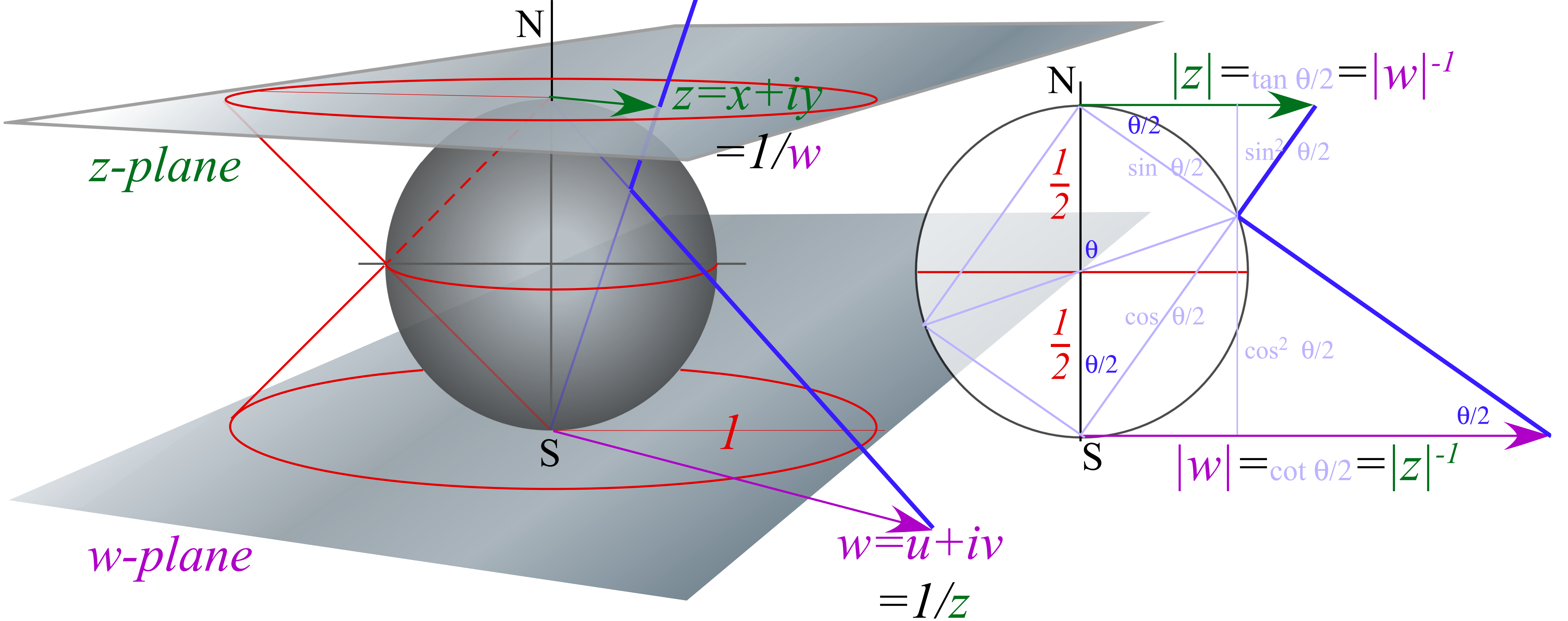

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(*with* $z \to w$)

$$= \ldots \frac{a_2}{3} z^{-2} + \frac{a_1}{2} z^{-2} + a_0 z^{-1} - a_{-1} \ln z + \frac{a_{-2}}{-1} z + \frac{a_{-3}}{-2} z^2 + \frac{a_{-4}}{-3} z^3 + \ldots$$

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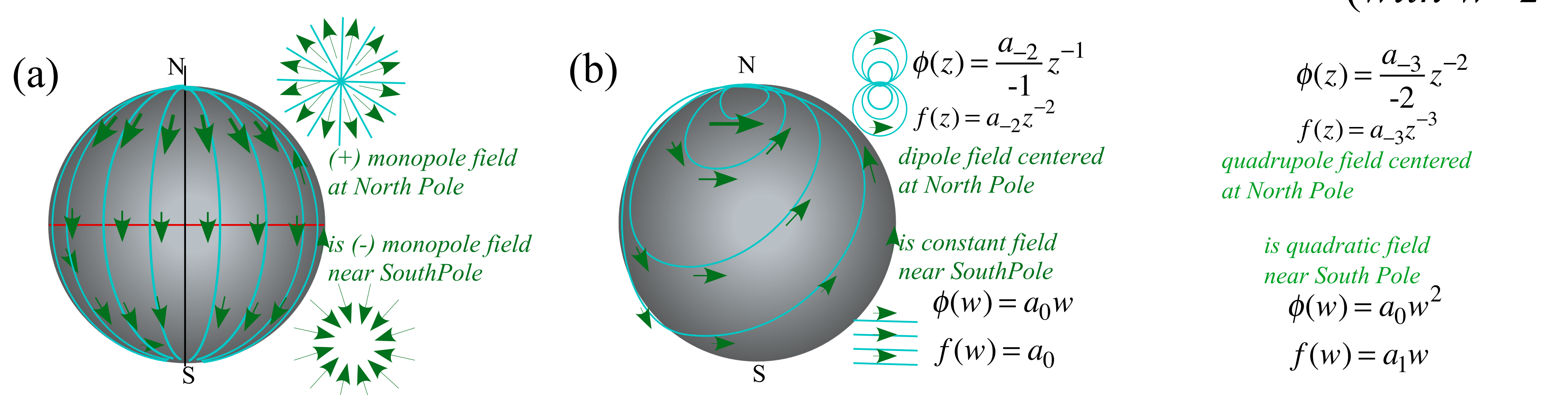

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$$f(z) = \ldots a_{-3}z^{-3} + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + \ldots$$

Of all *2^m-pole* field terms $a_{m-1}z^{m-1}$, only the $m=0$ monopole $a_{-1}z^{-1}$ has a non-zero loop integral (10.39).

$$\oint f(z)dz = \oint a_{-1}z^{-1}dz = 2\pi i a_{-1} \qquad a_{-1} = \frac{1}{2\pi i}\oint f(z)dz$$

This *$m=1$-pole* constant-a_{-1} formula is just the first in a series of *Laurent coefficient expressions*.

$$\cdots a_{-3} = \frac{1}{2\pi i}\oint z^2 f(z)dz \ , \ a_{-2} = \frac{1}{2\pi i}\oint z^1 f(z)dz \ , \ a_{-1} = \frac{1}{2\pi i}\oint f(z)dz \ , \ a_0 = \frac{1}{2\pi i}\oint \frac{f(z)}{z}dz \ , \ a_1 = \frac{1}{2\pi i}\oint \frac{f(z)}{z^2}dz \ ,\cdots$$

Source analysis starts with 1-pole loop integrals $\oint z^{-1}dz = 2\pi i$ or, with origin shifted $\oint (z-a)^{-1}dz = 2\pi i$.

They hold for any loop about point-a. Function *f(z)* is just *f(a)* on a *tiny* circle around point-a.

(assume *tiny* circle around $z=a$)

$$\oint \frac{f(z)}{z-a}dz = \oint \frac{f(a)}{z-a}dz = f(a)\oint \frac{1}{z-a}dz = 2\pi i f(a)$$

(but any contour that doesn't "touch a gives same answer)

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$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n \qquad where:\; a_n = \frac{1}{2\pi i}\oint \frac{f(z)}{(z-a)^{n+1}}dz \left(= \frac{1}{n!}\frac{d^n f(a)}{da^n} \quad for:\; n \geq 0 \right)$$

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(quadrupole)₀ *(dipole)₀* *(monopole)* *(dipole)∞* *(quadrupole)∞* *(octapole)∞* *(hexadecapole)∞* ...

$$f(z) = \ldots a_{-3}z^{-3} + \underbrace{a_{-2}z^{-2}}_{\text{dipole moment}} + \underbrace{a_{-1}z^{-1}}_{\text{monopole moment}} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + \ldots$$

5. Mapping and Non-analytic 2D source field analysis

The half-n'-half results

are called

Riemann-Cauchy

Derivative Relations

$$\boxed{\begin{aligned}\frac{\partial \Phi}{\partial x} &= \frac{\partial \mathbf{A}}{\partial y}\\ \frac{\partial \Phi}{\partial y} &= -\frac{\partial \mathbf{A}}{\partial x}\end{aligned}} \begin{aligned}\text{is:}\\ \text{is:}\end{aligned} \boxed{\begin{aligned}\frac{\partial \mathrm{Re}\,\phi(z)}{\partial x} &= \frac{\partial \mathrm{Im}\,\phi(z)}{\partial y}\\ \frac{\partial \mathrm{Re}\,\phi(z)}{\partial y} &= -\frac{\partial \mathrm{Im}\,\phi(z)}{\partial x}\end{aligned}} \begin{aligned}\text{or:}\\ \text{or:}\end{aligned} \boxed{\begin{aligned}\frac{\partial \mathrm{Re}\,f(z)}{\partial x} &= \frac{\partial \mathrm{Im}\,f(z)}{\partial y}\\ \frac{\partial \mathrm{Re}\,f(z)}{\partial y} &= -\frac{\partial \mathrm{Im}\,f(z)}{\partial x}\end{aligned}} \begin{aligned}\text{is:}\\ \text{is:}\end{aligned} \boxed{\begin{aligned}\frac{\partial f_x(z)}{\partial x} &= \frac{\partial f_y(z)}{\partial y}\\ \frac{\partial f_x(z)}{\partial y} &= -\frac{\partial f_y(z)}{\partial x}\end{aligned}}$$

RC applies to analytic potential $\phi(z) = \Phi + i\mathbf{A}$ *and analytic field* $f(z) = f_x + if_y$ *and any analytic function*

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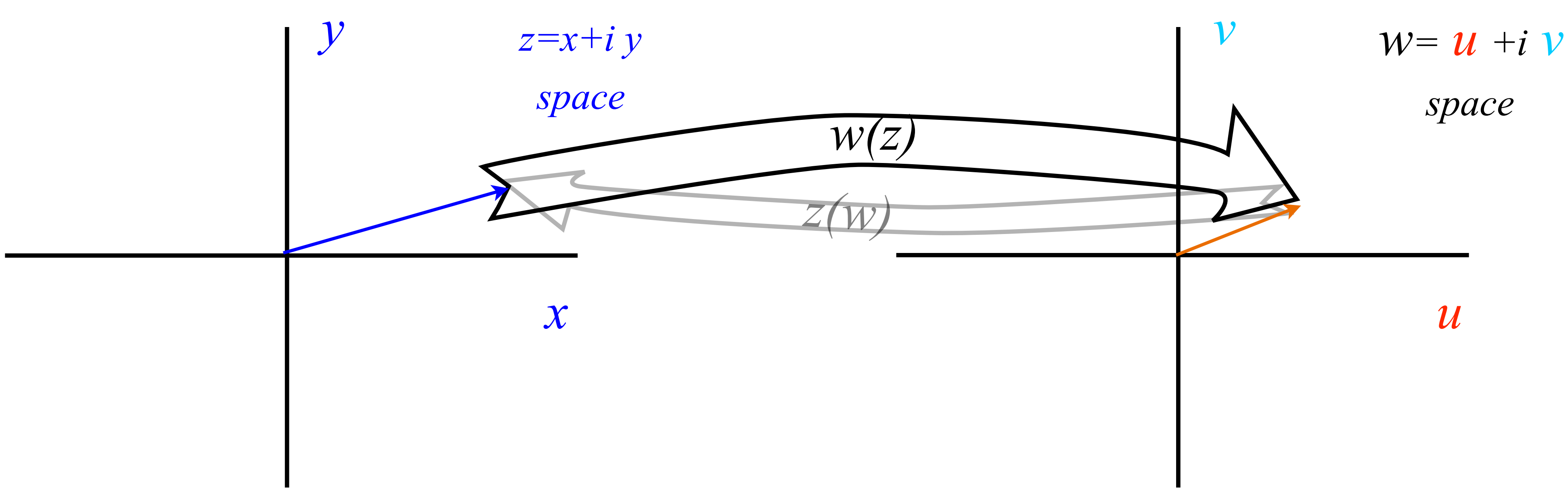
Common notation for mapping: $w(z)=u+iv$



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$$\boxed{\begin{aligned}\frac{\partial \Phi}{\partial x}&=\frac{\partial A}{\partial y}\\ \frac{\partial \Phi}{\partial y}&=-\frac{\partial A}{\partial x}\end{aligned}} \text{ is: } \boxed{\begin{aligned}\frac{\partial \mathrm{Re}\,\phi(z)}{\partial x}&=\frac{\partial \mathrm{Im}\,\phi(z)}{\partial y}\\ \frac{\partial \mathrm{Re}\,\phi(z)}{\partial y}&=-\frac{\partial \mathrm{Im}\,\phi(z)}{\partial x}\end{aligned}} \text{ or: } \boxed{\begin{aligned}\frac{\partial \mathrm{Re}\,f(z)}{\partial x}&=\frac{\partial \mathrm{Im}\,f(z)}{\partial y}\\ \frac{\partial \mathrm{Re}\,f(z)}{\partial y}&=-\frac{\partial \mathrm{Im}\,f(z)}{\partial x}\end{aligned}} \text{ is: } \boxed{\begin{aligned}\frac{\partial f_x(z)}{\partial x}&=\frac{\partial f_y(z)}{\partial y}\\ \frac{\partial f_x(z)}{\partial y}&=-\frac{\partial f_y(z)}{\partial x}\end{aligned}}$$

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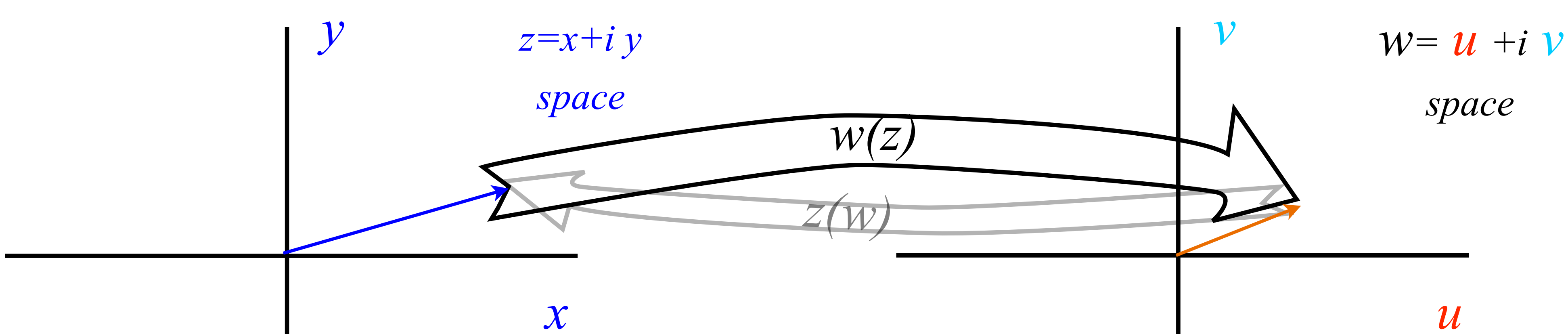

Jacobian for mapping:

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$$

$$dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy$$

$$\begin{pmatrix} du \\ dv \end{pmatrix}=\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix}$$

Complex derivative for mapping:

$$\frac{dw}{dz}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)(u+iv)=\frac{1}{2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{i}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

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Derivative Relations

$\frac{\partial \Phi}{\partial x} = \frac{\partial A}{\partial y}$	is:	$\frac{\partial \mathrm{Re}\,\phi(z)}{\partial x} = \frac{\partial \mathrm{Im}\,\phi(z)}{\partial y}$	or:	$\frac{\partial \mathrm{Re}\,f(z)}{\partial x} = \frac{\partial \mathrm{Im}\,f(z)}{\partial y}$	is:	$\frac{\partial f_x(z)}{\partial x} = \frac{\partial f_y(z)}{\partial y}$
$\frac{\partial \Phi}{\partial y} = -\frac{\partial A}{\partial x}$	is:	$\frac{\partial \mathrm{Re}\,\phi(z)}{\partial y} = -\frac{\partial \mathrm{Im}\,\phi(z)}{\partial x}$	or:	$\frac{\partial \mathrm{Re}\,f(z)}{\partial y} = -\frac{\partial \mathrm{Im}\,f(z)}{\partial x}$	is:	$\frac{\partial f_x(z)}{\partial y} = -\frac{\partial f_y(z)}{\partial x}$

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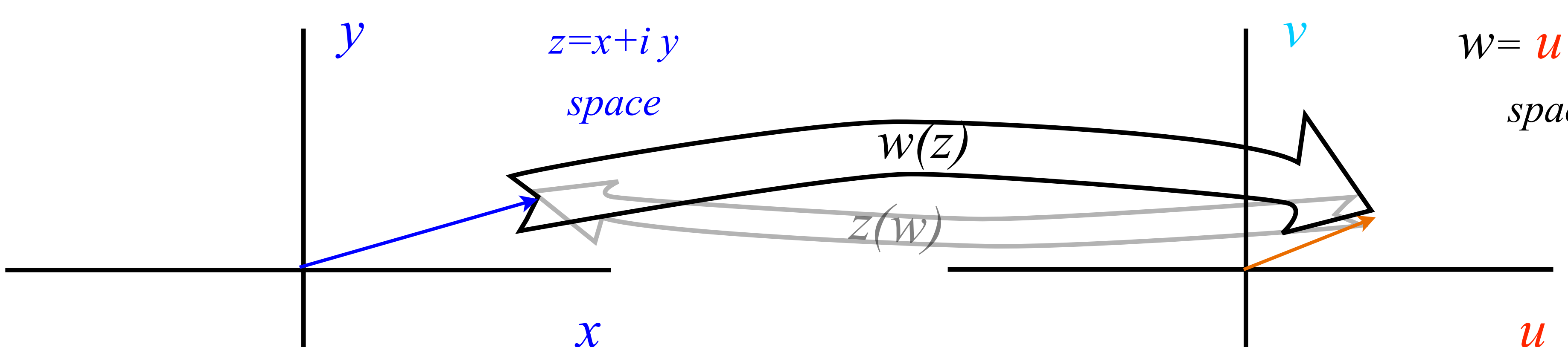

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$
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Jacobian for mapping:

$$\begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$$
$$= \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial v}{\partial y} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

Complex derivative for mapping:

$$\frac{dw}{dz} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(u + iv) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$
$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

Complex derivative abs-square:

$$\left|\frac{dw}{dz}\right|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$$

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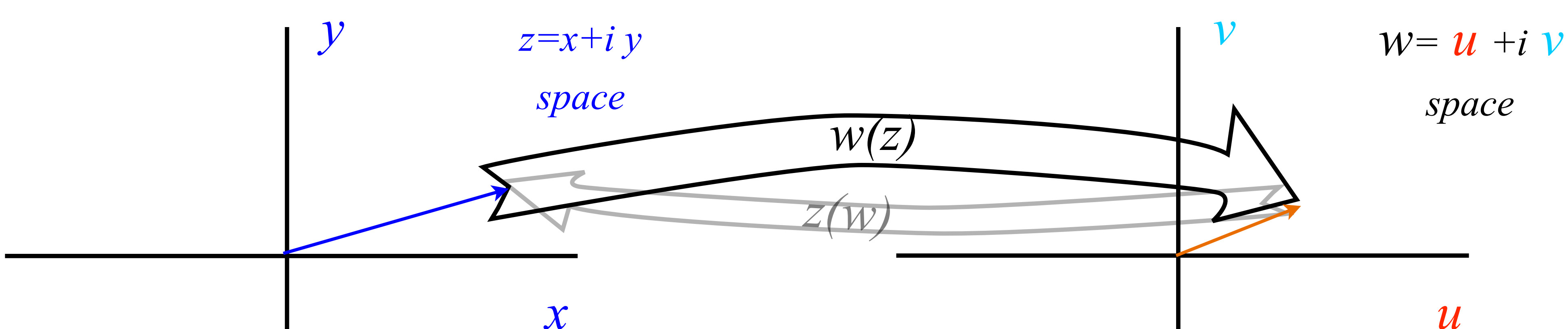

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$$=\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x}\end{pmatrix}\begin{pmatrix}dx\\ dy\end{pmatrix}=\begin{pmatrix}\frac{\partial v}{\partial y} & -\frac{\partial v}{\partial x}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix}\begin{pmatrix}dx\\ dy\end{pmatrix}$$

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$$\frac{dw}{dz}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)(u+iv)=\frac{1}{2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{i}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$
$$=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}\qquad=\frac{\partial v}{\partial y}+i\frac{\partial v}{\partial x}$$

Complex derivative abs-square:

$$\left|\frac{dw}{dz}\right|^2=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2=\det|J|$$

...equals Jacobian Determinant

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Derivative Relations

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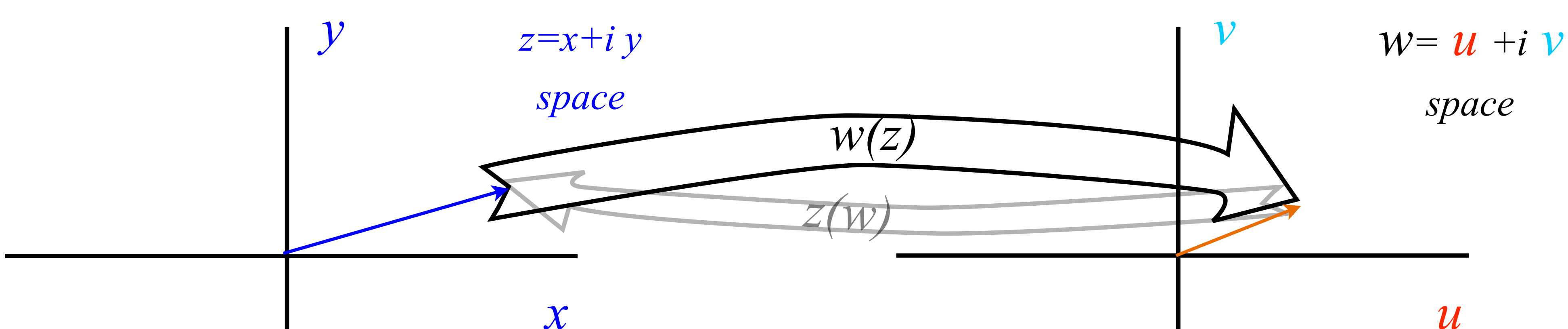


Important result:

$$dw=\sqrt{J}\cdot e^{i\theta}\cdot dz$$

is scaled rotation of dz.

Jacobian for mapping is scaled rotation:

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$$
$$dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy$$

$$\begin{pmatrix} du \\ dv \end{pmatrix}=\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix}=\sqrt{\det J}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix}$$

$$=\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix}=\begin{pmatrix} \frac{\partial v}{\partial y} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix}$$

Complex derivative for mapping:

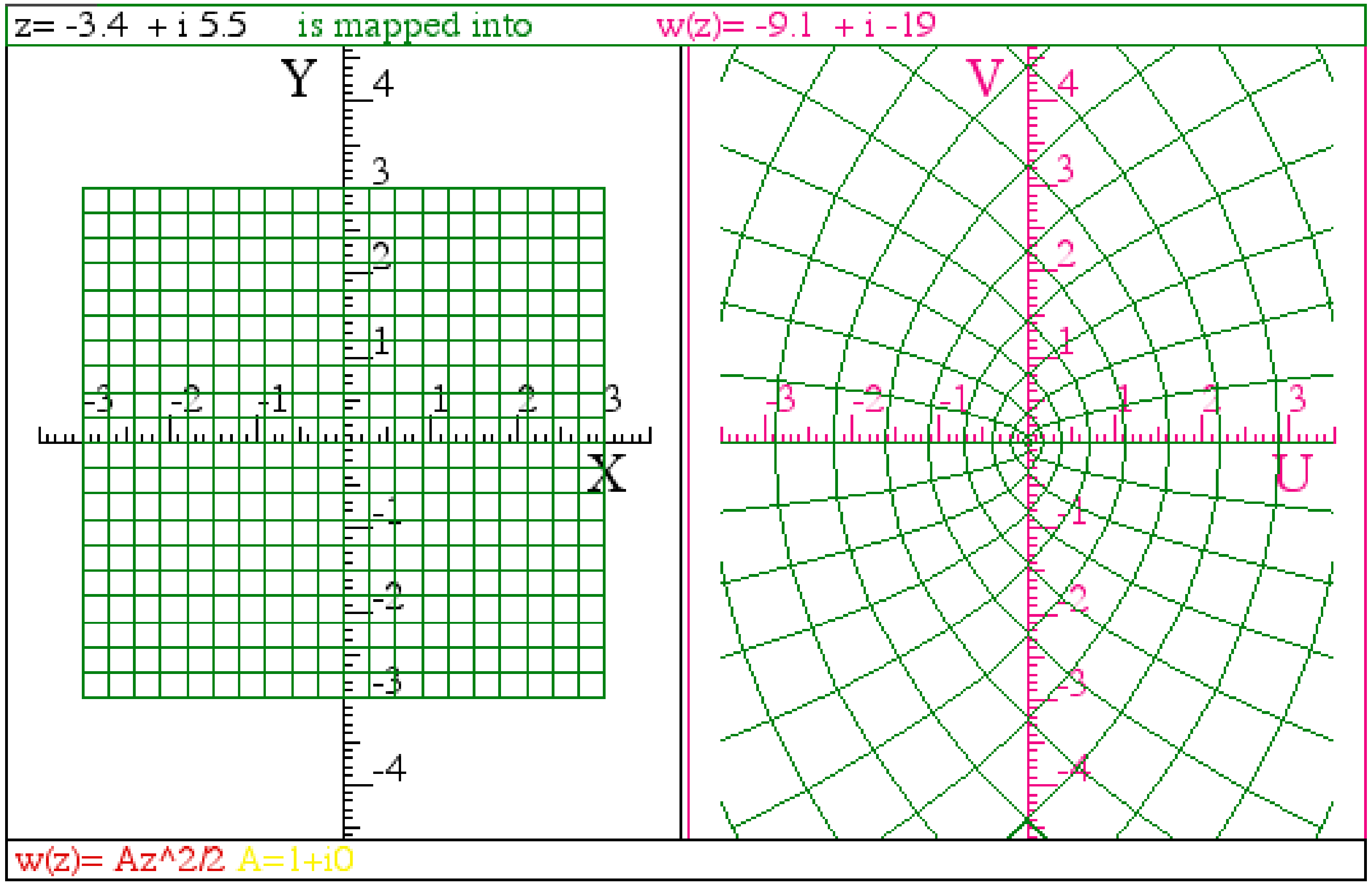
$$\frac{dw}{dz}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)(u+iv)=\frac{1}{2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{i}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$
$$=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}\qquad=\frac{\partial v}{\partial y}+i\frac{\partial v}{\partial x}$$

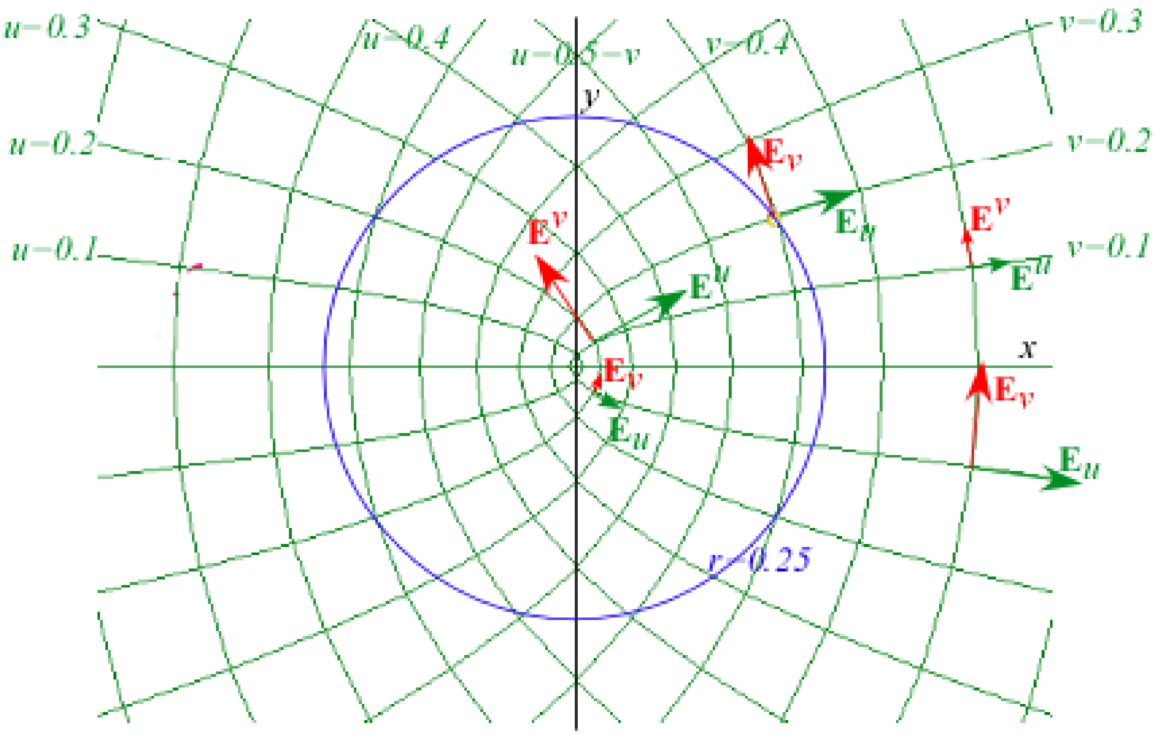
Complex derivative abs-square:

$$\left|\frac{dw}{dz}\right|^2=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2=\det|J|$$

...equals Jacobian Determinant

$w(z) = z^2$ gives parabolic OCC





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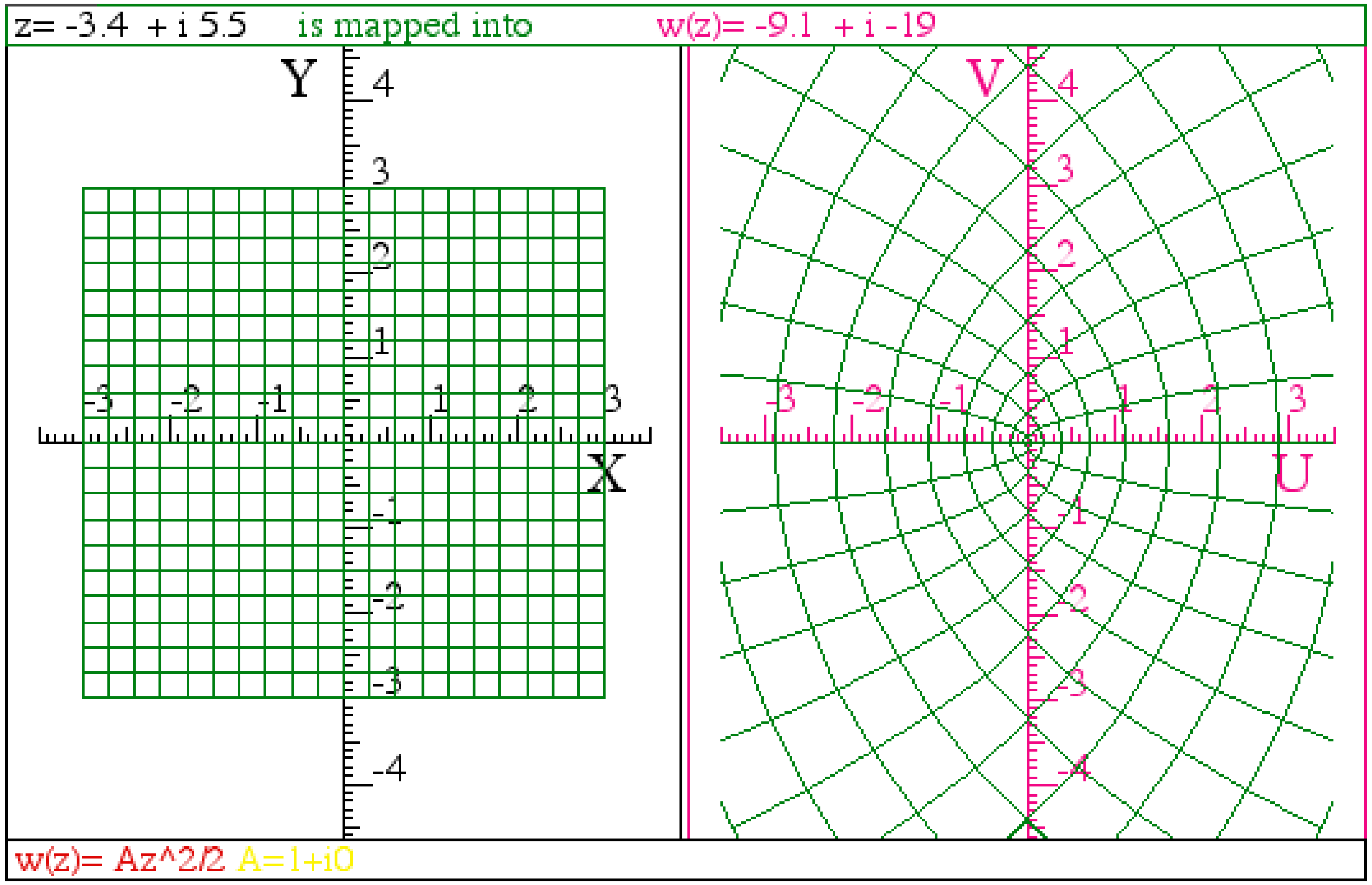
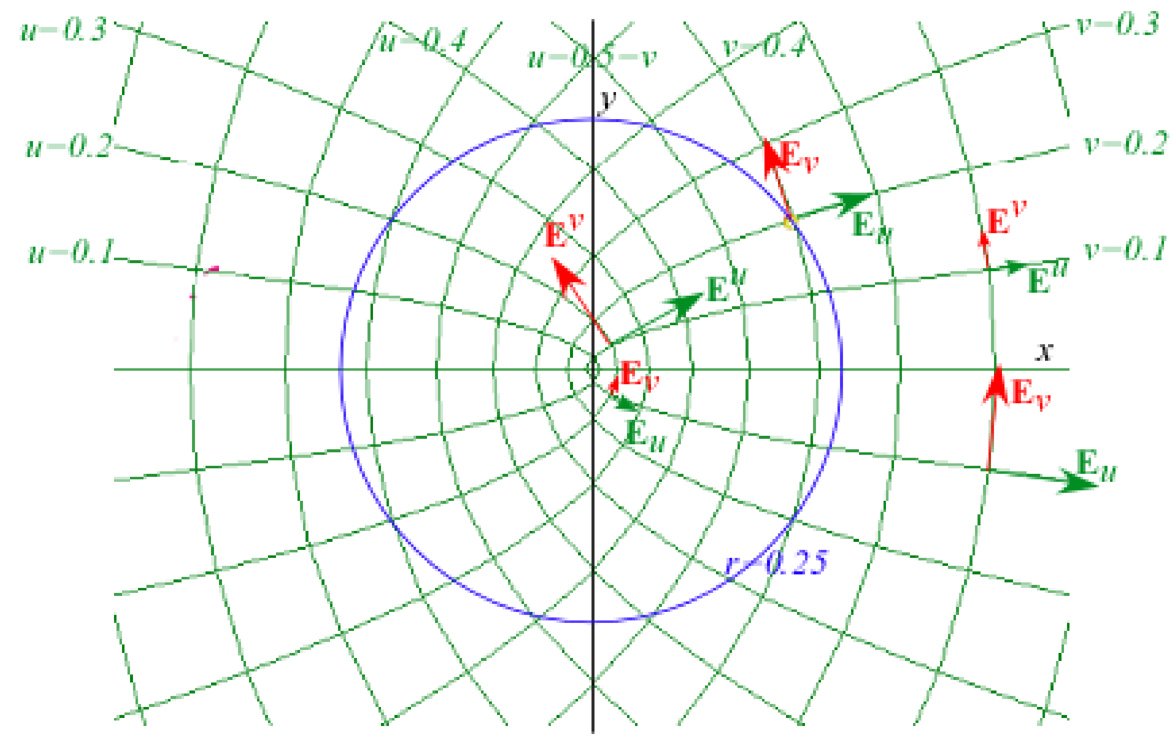
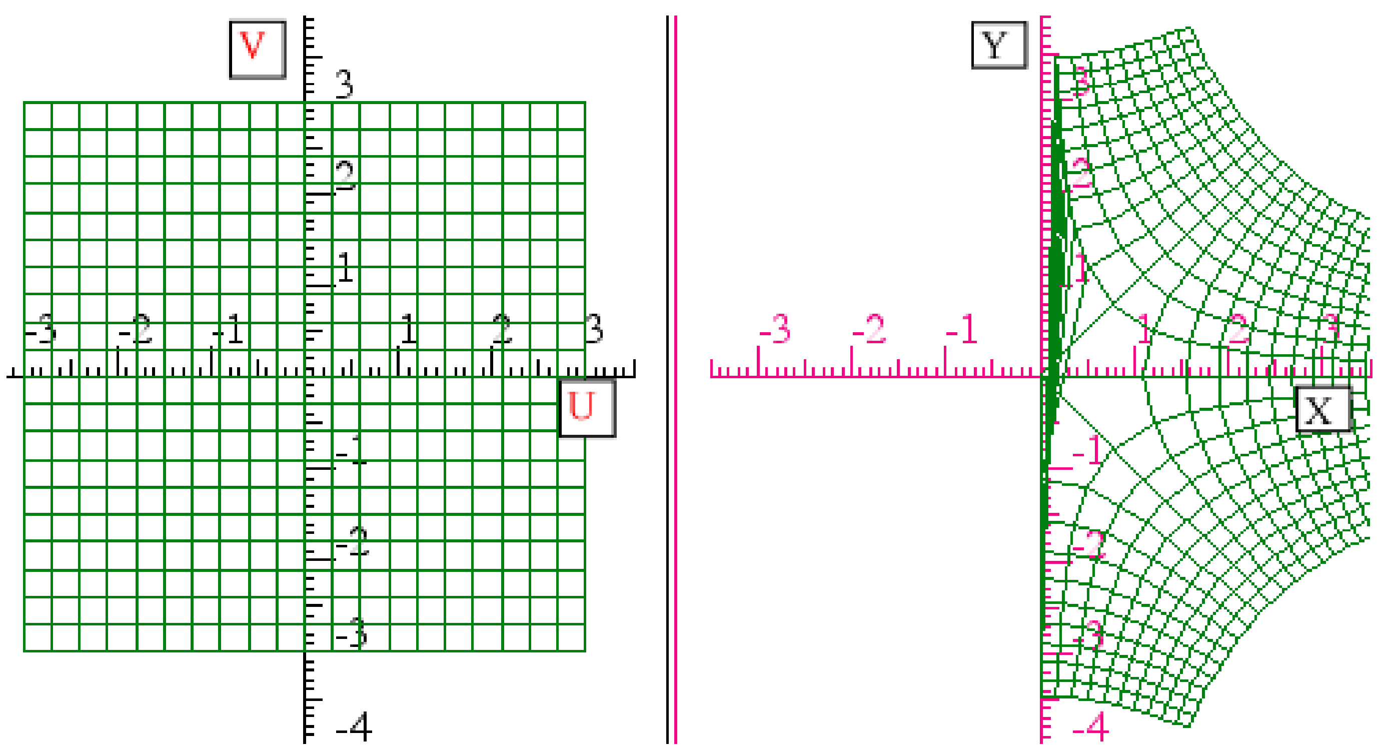




Inverse: $z(w) = w^{1/2}$ gives hyperbolic OCC



5. Mapping and Non-analytic 2D source field analysis

Non-analytic potential, force, and source field functions (Excerpts of Unit 1-Ch.10 and AnalyIt)

A general 2D complex field may have:

1. non-analytic *potential field function* $\phi(z,z^*)=\Phi(x,y)+iA(x,y)$,
2. non-analytic *force field function* $f(z,z^*) = fx(x,y) + ify(x,y)$,
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$$2\frac{df^*}{dz}=s^*(z,z^*) \qquad 2\frac{df}{dz^*}=s(z,z^*)$$

Field-*f*-from-potential-ϕ equations are like the older ($f(z)=\frac{d\phi}{dz}$ or $f^*(z*)=\frac{d\phi*}{dz*}$) but with an extra factor of *2*.

$$2\frac{d\phi}{dz}=f(z,z^*) \qquad 2\frac{d\phi^*}{dz^*}=f^*(z,z^*)$$

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$$2\frac{d\phi}{dz}=f(z,z^*) \qquad\qquad 2\frac{d\phi^*}{dz^*}=f^*(z,z^*)$$

The new source equations expand into a real and imaginary parts that are divergence and curl terms, respectively.

$$s^*(z,z^*)=2\frac{df^*}{dz}=\left[\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right]\left[f_x^*(x,y)+if_y^*(x,y)\right] \quad =\rho-i\,I\ , \quad \text{where: } f_x^*=f_x\ , \ \text{ and: } f_y^*=-f_y$$

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Real part: *Poisson scalar source equation* (*charge density* ρ). Imaginary part: *Biot-Savart vector source equation*(*current density I*)

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Field-*vs*-potential has Re and Im parts that are x and y components of grad Φ and curlA_Z from potential $\phi=\Phi+iA$ or $\phi^*=\Phi-iA$.

$$f^*(z,z^*)=2\frac{d\phi^*}{dz^*}=\left[\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right]\left(\Phi-iA\right)=f_x^*+if_y^*$$

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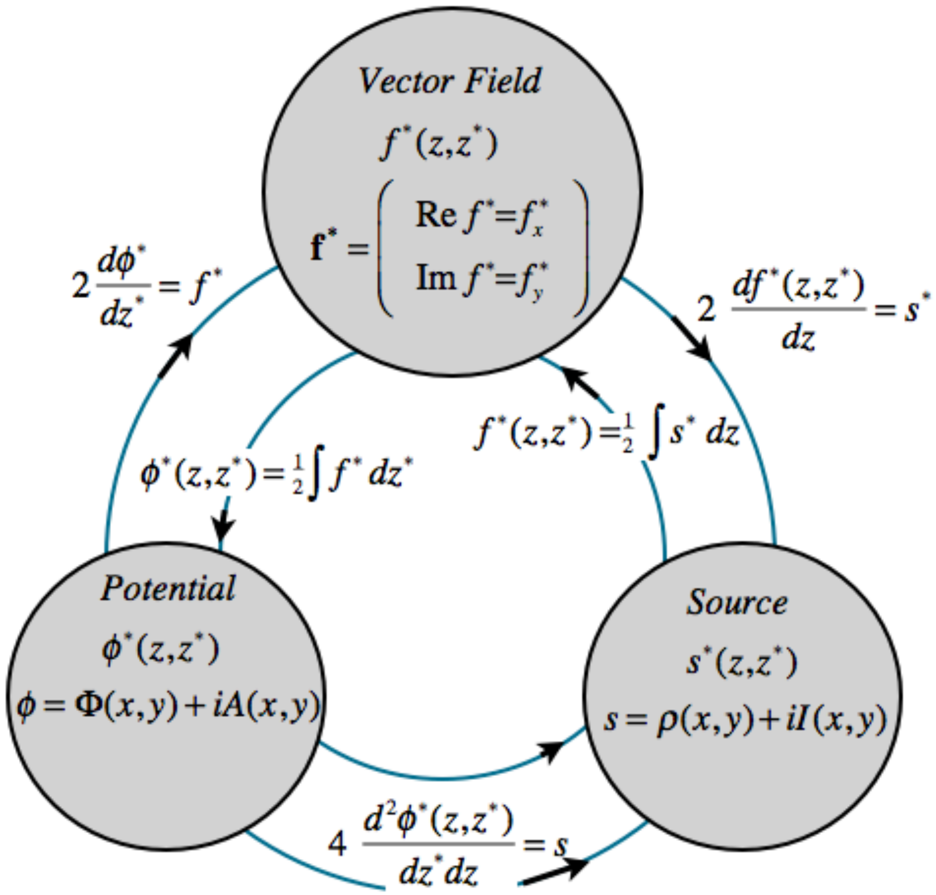
(For source-free analytic functions these two fields are identical.)

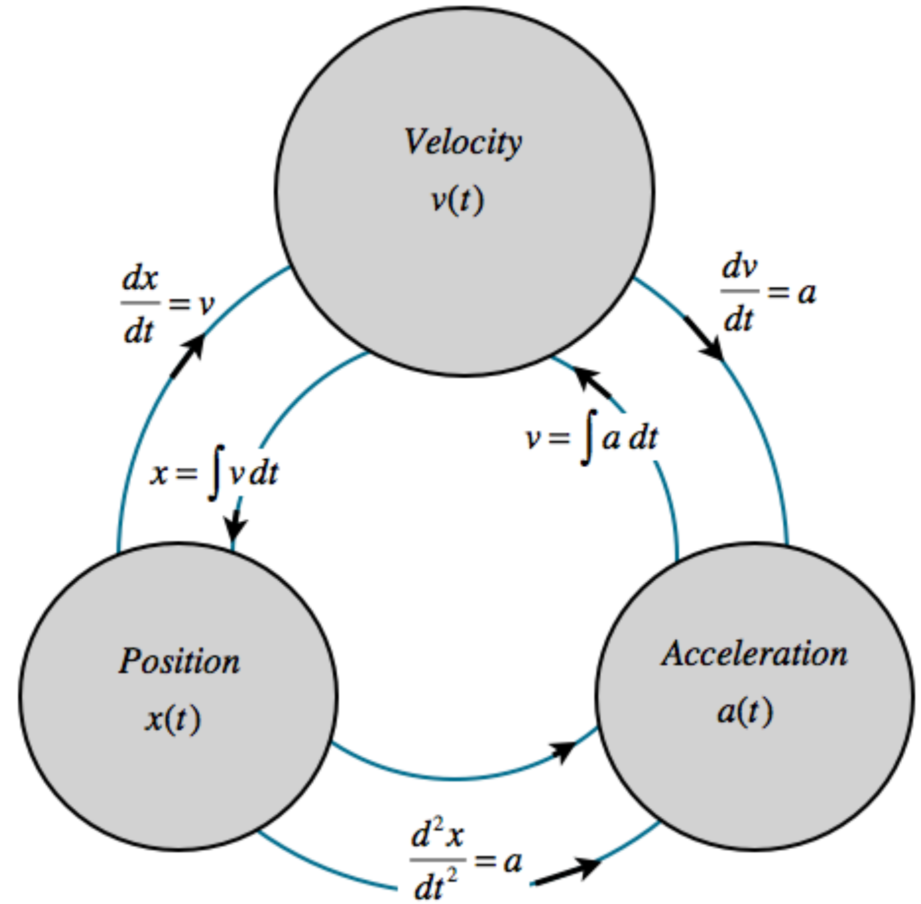
Gradient of scalar potential is the *longitudinal field* $\mathbf{f}_\mathbf{L}^*$ and curl of a vector potential is the *transverse field* $\mathbf{f}_\mathbf{T}^*$.

Total field is: $\mathbf{f}^*=\mathbf{f}_\mathbf{L}^*+\mathbf{f}_\mathbf{T}^*$ $\qquad \mathbf{f}_\mathbf{L}^*=\nabla\Phi$ $\qquad \mathbf{f}_\mathbf{T}^*=\nabla\times\mathbf{A}$

Potential, force, and source field equations *vs.* *position, velocity, and acceleration equations*

Field equations *Newton equations*





Potential and source field theory reduced to sophomore mechanics of 1D-motion!

Example 1 Consider a non-analytic field $f(z) = (z^*)^2$ or $f^*(z) = z^2$.

Non-analytic source s^* is derivative of field f^*

$$s^*(z,z^*) = 2\frac{df^*}{dz} = 4z = 4x + i4y,$$

$$or: \quad \rho = 4x, \quad and: \quad I = -4y.$$

Non-analytic potential ϕ is integral of field f^*

$$\phi(z,z^*) = \frac{1}{2}\int f(z)\,dz = \frac{1}{2}\int (z^*)^2\,dz = \frac{z(z^*)^2}{2} = \frac{(x+iy)(x^2-y^2-i2xy)}{2},$$

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The longitudinal field $\mathbf{f_L}^*$ is quite different from the transverse field $\mathbf{f_T}^*$

$$\mathbf{f}_{\mathbf{L}}^* = \nabla\Phi = \nabla\left(\frac{x^3+xy^2}{2}\right) = \begin{pmatrix} \frac{\partial\Phi}{\partial x} \\ \frac{\partial\Phi}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{3x^2+y^2}{2} \\ xy \end{pmatrix},$$

$$\mathbf{f}_{\mathbf{T}}^* = \nabla\times\mathbf{A} = \nabla\times\left(\frac{-y^3-yx^2}{2}\mathbf{e_z}\right) = \begin{pmatrix} \frac{\partial A}{\partial y} \\ -\frac{\partial A}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{-3y^2-x^2}{2} \\ xy \end{pmatrix}.$$

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Longitudinal field $\mathbf{f_L}^*$ has no curl and the transverse field $\mathbf{f_T}^*$ has no divergence. Sum field $\mathbf{f}$ has both.

$$\mathbf{f}^* = \mathbf{f}_\mathbf{L}^* + \mathbf{f}_\mathbf{T}^* = \begin{pmatrix} \frac{3x^2+y^2}{2} \\ xy \end{pmatrix} + \begin{pmatrix} \frac{-3y^2-x^2}{2} \\ xy \end{pmatrix} = \begin{pmatrix} x^2-y^2 \\ 2xy \end{pmatrix}, \quad \nabla\bullet\mathbf{f}^* = \nabla\bullet\mathbf{f}_\mathbf{L}^* = 4x = \rho, \quad \nabla\times\mathbf{f}^* = \nabla\times\mathbf{f}_\mathbf{T}^* = 4y = -I.$$

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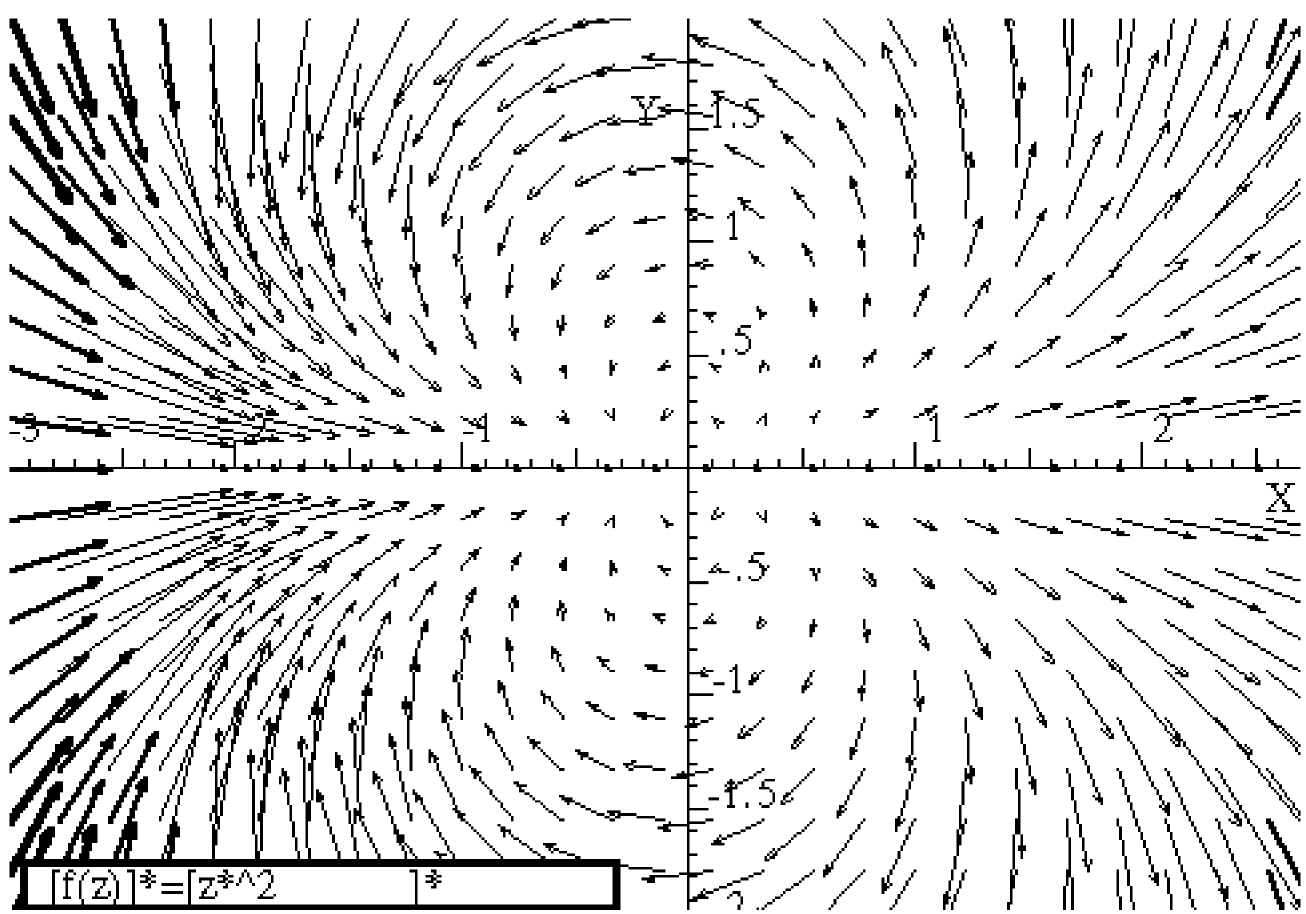
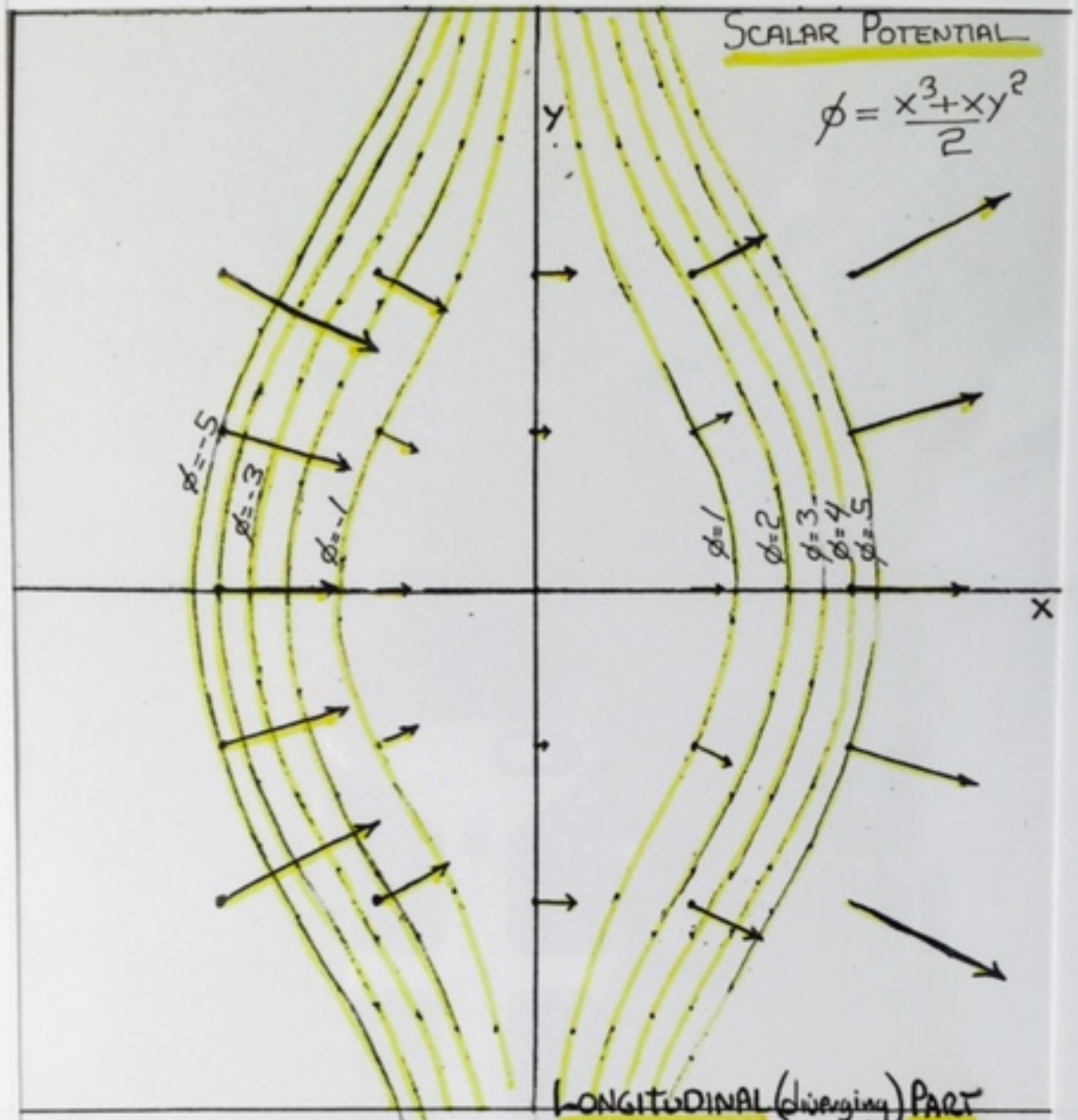


$\mathbf{f}_\mathbf{L}^* + \mathbf{f}_\mathbf{T}^*$

Fig.10.17 Force field vectors for non-analytic function $f(z) = (z^*)^2$



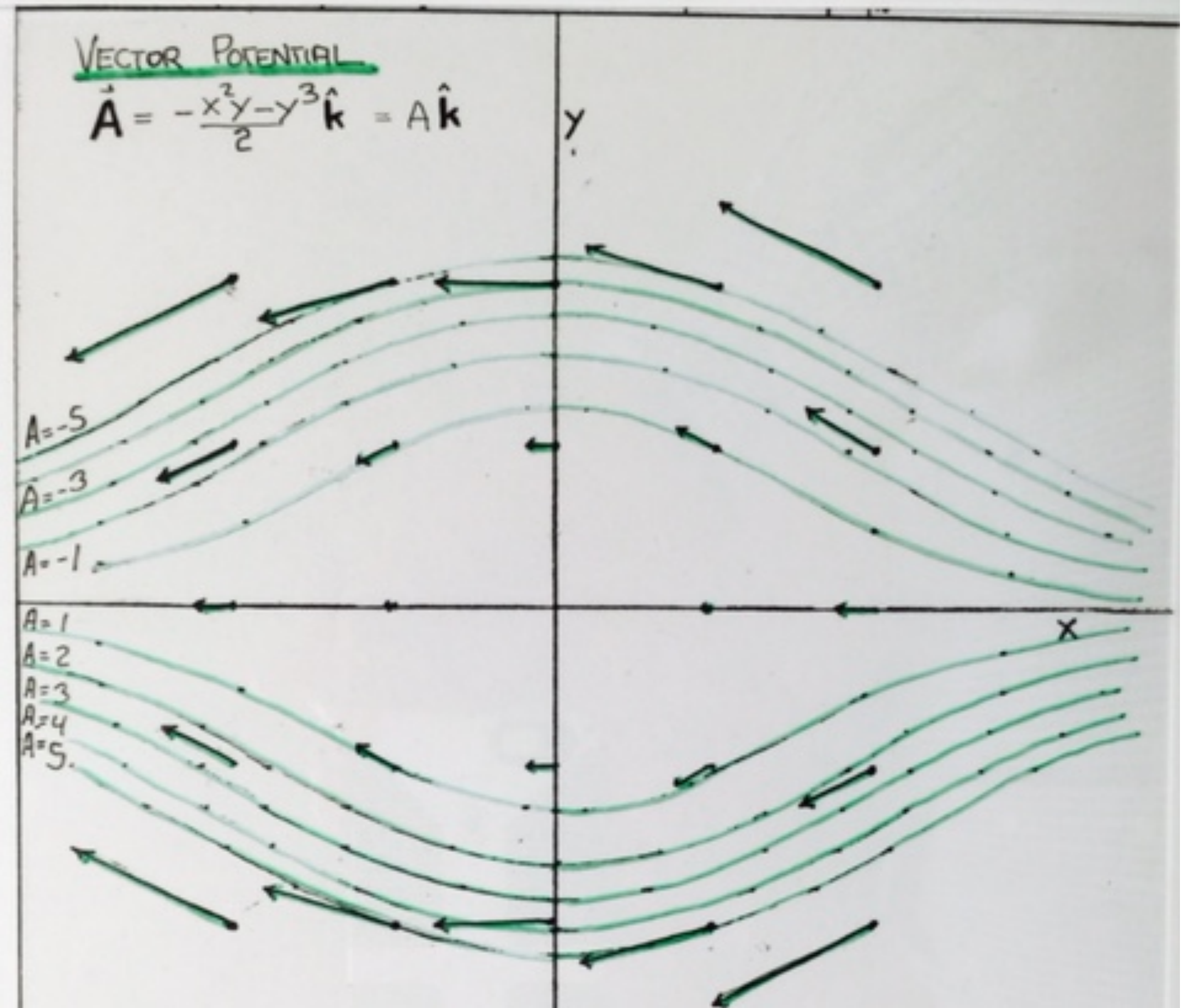
LONGITUDINAL (diverging) PART

$$F_L = \begin{pmatrix} \frac{3x^2+y^2}{2} \\ xy \end{pmatrix}$$

$$= \nabla\phi$$

$$= \nabla\left(\frac{x^3+xy^2}{2}\right)$$

$$\nabla\times F_L = 0$$



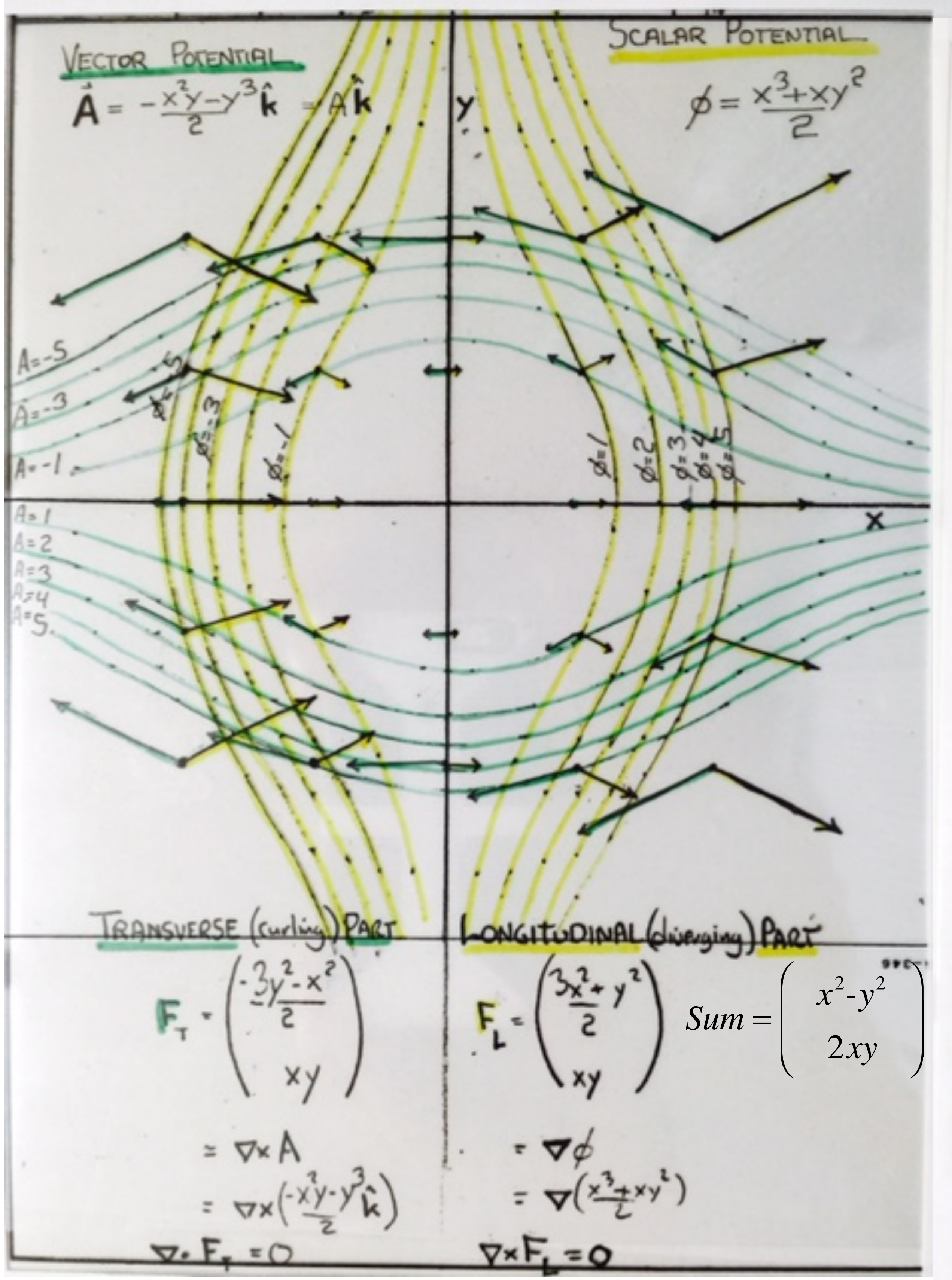
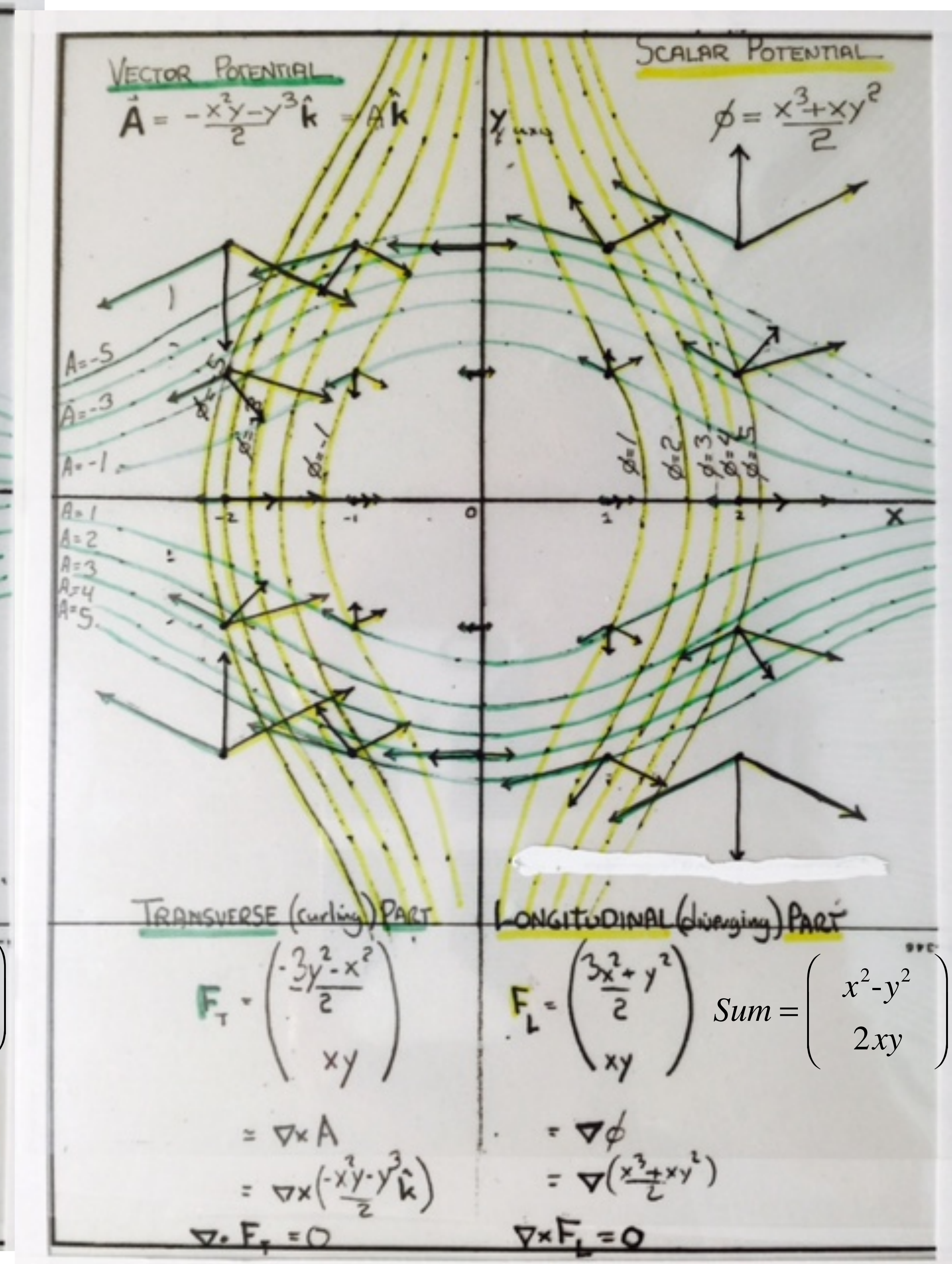
TRANSVERSE (curling) PART

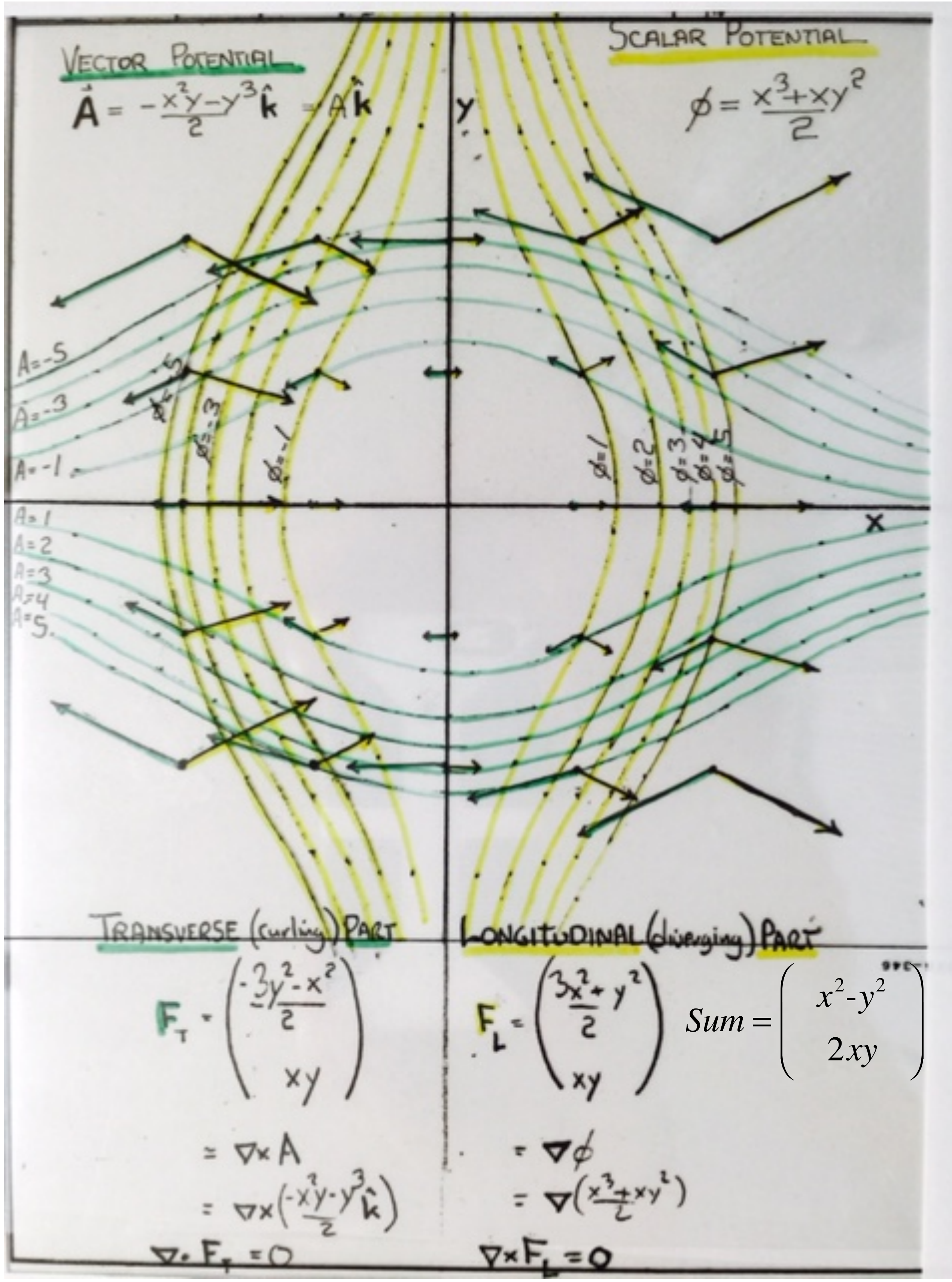
$$F_T = \begin{pmatrix} \frac{3y^2-x^2}{2} \\ xy \end{pmatrix}$$

$$= \nabla\times A$$

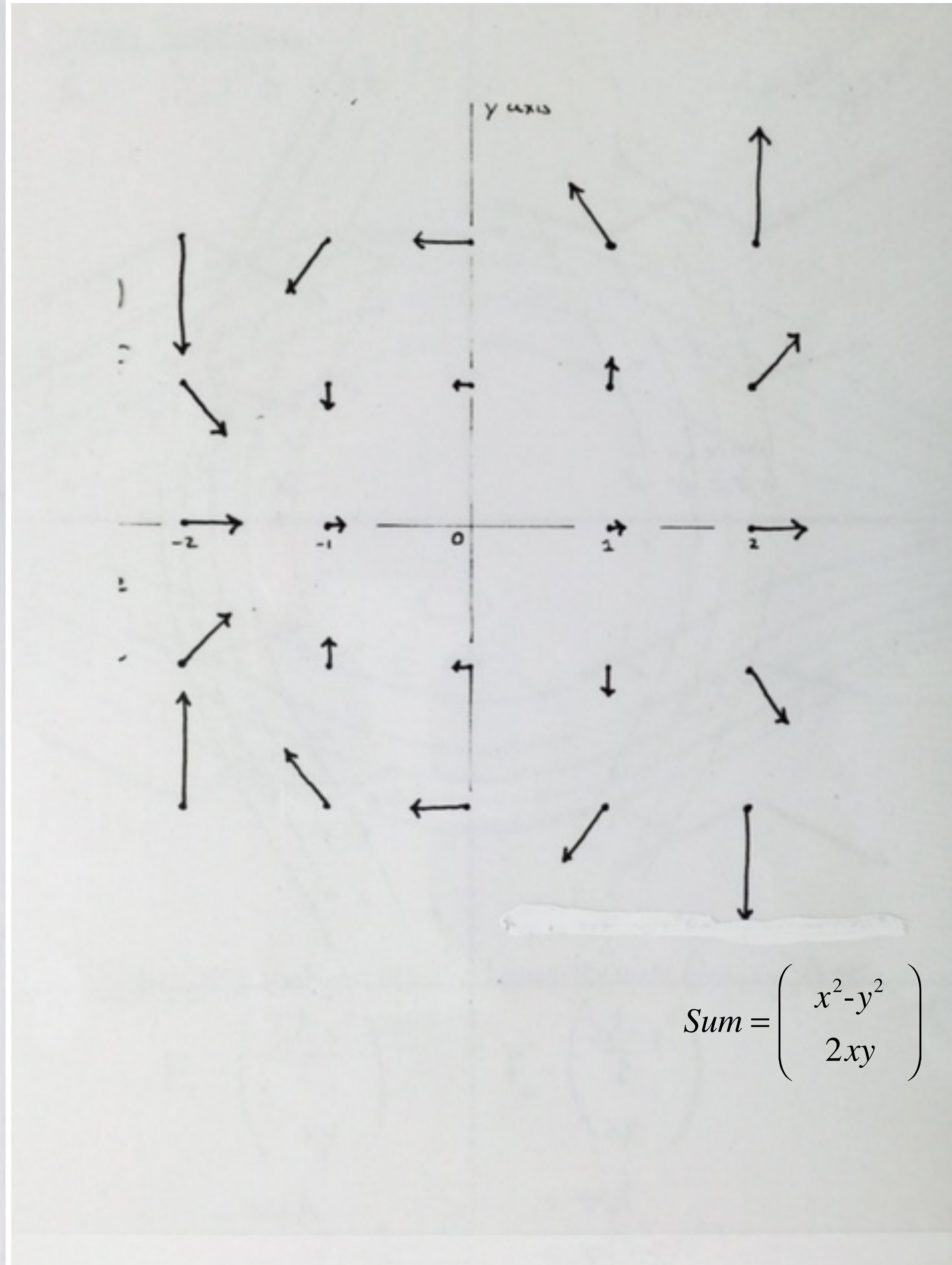
$$= \nabla\times\left(\frac{-x^2y-y^3}{2}\hat{k}\right)$$

$$\nabla\cdot F_T = 0$$

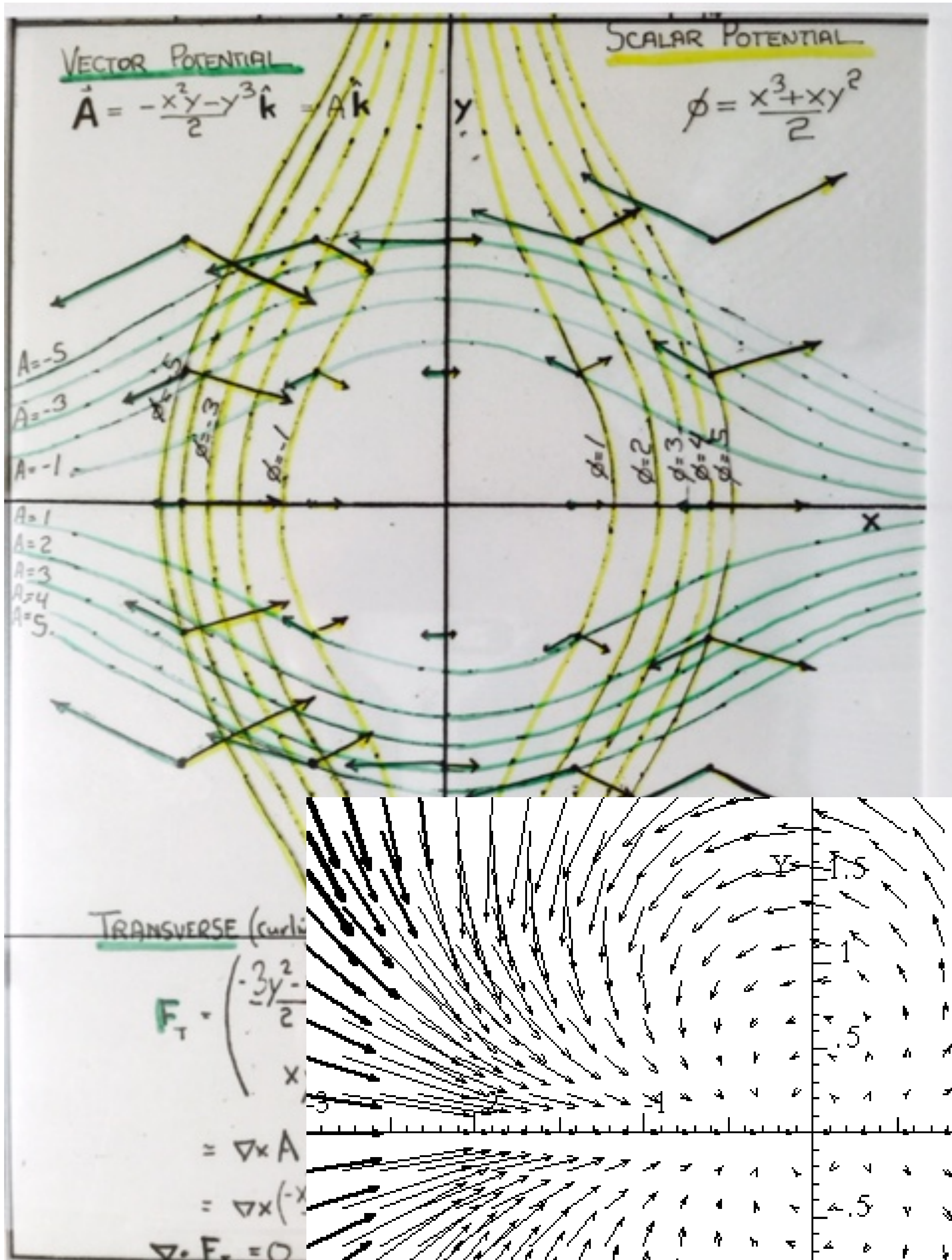

VECTOR POTENTIAL
$\vec{A} = -\frac{x^2y - y^3}{2}\hat{k} = A\hat{k}$
SCALAR POTENTIAL
$\phi = \frac{x^3 + xy^2}{2}$
y
x
A=-5
A=-3
A=-1
A=1
A=2
A=3
A=4
A=5
$\phi=-3$
$\phi=-1$
$\phi=1$
$\phi=2$
$\phi=3$
TRANSVERSE (curling) PART
LONGITUDINAL (diverging) PART
$F_T = \begin{pmatrix} \frac{3y^2 - x^2}{2} \\ xy \end{pmatrix}$
$F_L = \begin{pmatrix} \frac{3x^2 + y^2}{2} \\ xy \end{pmatrix}$
$Sum = \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix}$
$= \nabla \times A$
$= \nabla \times \left(\frac{-x^2y - y^3}{2}\hat{k}\right)$
$= \nabla\phi$
$= \nabla\left(\frac{x^3 + xy^2}{2}\right)$
$\nabla \cdot F_T = 0$
$\nabla \times F_L = 0$

VECTOR POTENTIAL
$\vec{A} = -\frac{x^2y - y^3}{2}\hat{k} = A\hat{k}$
SCALAR POTENTIAL
$\phi = \frac{x^3 + xy^2}{2}$
y
x
A=-5
A=-3
A=-1
A=1
A=2
A=3
A=4
A=5
-2
-1
0
1
2
$\phi=-1$
$\phi=1$
$\phi=2$
$\phi=3$
TRANSVERSE (curling) PART
LONGITUDINAL (diverging) PART
$F_T = \begin{pmatrix} \frac{3y^2 - x^2}{2} \\ xy \end{pmatrix}$
$F_L = \begin{pmatrix} \frac{3x^2 + y^2}{2} \\ xy \end{pmatrix}$
$Sum = \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix}$
$= \nabla \times A$
$= \nabla \times \left(\frac{-x^2y - y^3}{2}\hat{k}\right)$
$= \nabla\phi$
$= \nabla\left(\frac{x^3 + xy^2}{2}\right)$
$\nabla \cdot F_T = 0$
$\nabla \times F_L = 0$

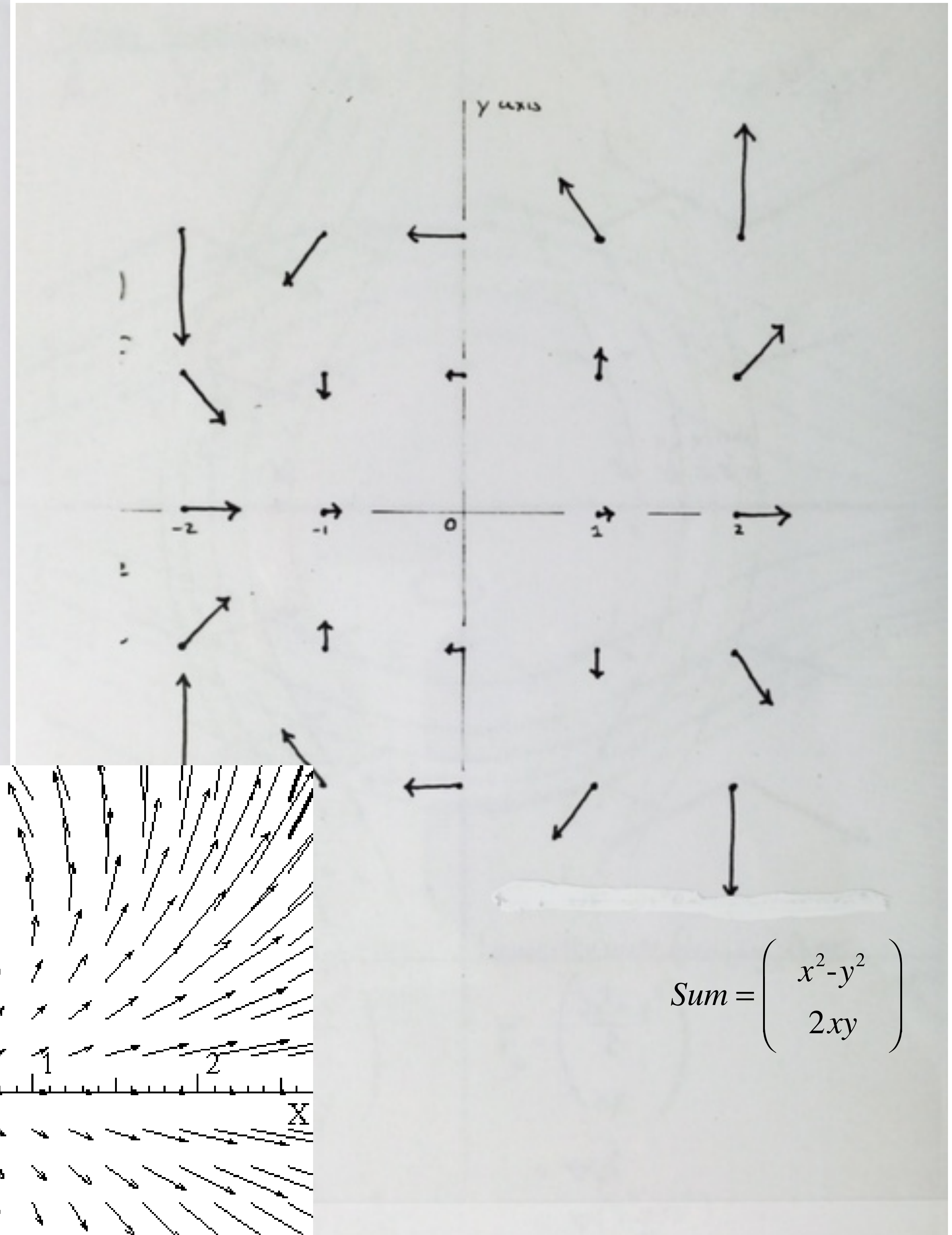


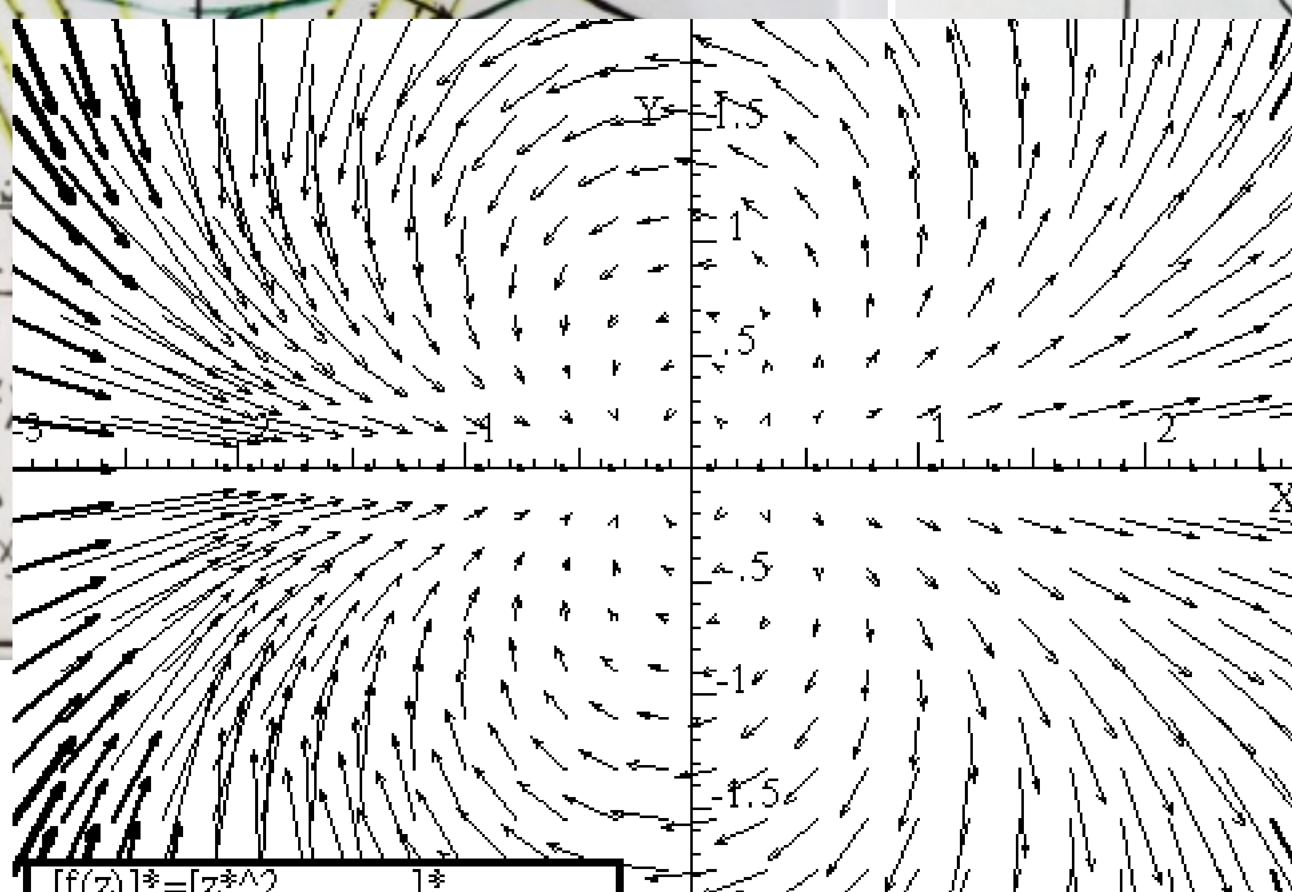
$$Sum = \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix}$$



$$Sum = \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix}$$


VECTOR POTENTIAL
$\vec{A} = -\frac{x^2y - y^3}{2}\hat{k} = A\hat{k}$
SCALAR POTENTIAL
$\phi = \frac{x^3 + xy^2}{2}$
y
x
A=-5
A=-3
A=-1
A=1
A=2
A=3
A=4
A=5
TRANSVERSE


y axis
-2
-1
0
1
2
$Sum = \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix}$


Y
1.5
1
.5
-1
1
2
X
-.5
-1
-1.5
[f(z)]*=[z*^2]*