

Lecture 14
Mon. 10.08.2018

**treb-yew-shay*

Introducing GCC Lagrangian `a la Trebuchet Dynamics*

Ch. 1-3 of Unit 2 and Unit 3 (Mostly Unit 2.)

The trebuchet (or ingenium) and its cultural relevancy (3000 BCE to 21st See Sci. Am. 273, 66 (July 1995))

The medieval ingenium (9th to 14th century) and modern re-enactments

Human kinesthetics and sports kinesiology

Review of Lagrangian equation derivation from Lecture 10 (Now with trebuchet model)

Coordinate geometry, Jacobian, velocity, kinetic energy, and dynamic metric tensor γ_{mn}

Structure of dynamic metric tensor γ_{mn}

Basic force, work, and acceleration

Lagrangian force equation

Canonical momentum and γ_{mn} tensor

Summary of Lagrange equations and force analysis (Mostly from Unit 2.)

Forces: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)

Multivalued functionality and connections

Covariant and contravariant relations

Tangent space vs. Normal space

Metric g_{mn} tensor geometric relations to length, area, and volume

A running collection of links to course-relevant sites and articles

Physics Web Resources

[Comprehensive Harter-Soft Resource Listing](#)

[UAF Physics YouTube channel](#)

[LearnIt Physics Web Applications](#)

Neat external material to start the class:

[AIP publications](#)

[AJP article on superball dynamics](#)

[AAPT summer reading](#)

These *are* hot off the presses:

[Sorting ultracold atoms in a three-dimensional optical lattice in a realization of Maxwell's demon - Kumar-Nature-Letters-2018](#)

[Synthetic three-dimensional atomic structures assembled atom by atom - Berredo-Nature-Letters-2018](#)

Slightly Older ones:

[Wave-particle duality of C60 molecules](#)

[Optical vortex knots – One Photon at a Time](#)

“*Relawavity*” and quantum basis of *Lagrangian & Hamiltonian* mechanics:

[2-CW laser wave - BohrIt Web App](#)

[Lagrangian vs Hamiltonian - RelaWavity Web App](#)

[AMOP Ch 0 Space-Time Symmetry - 2019](#)

[Seminar at Rochester Institute of Optics, Auxiliary slides, June 19, 2018](#)

New *Analyt Web Application now under development* in out testing area:

<https://modphys.hosted.uark.edu/testing/markup/AnalyItBJS.html>

“Texts”

[Classical Mechanics with a Bang!](#)

[Quantum Theory for the Computer Age](#)

[Principles of Symmetry, Dynamics, and Spectroscopy](#)

[Modern Physics and its Classical Foundations](#)

Classes

[2014 AMOP](#)

[2017 Group Theory for QM](#)

[2018 AMOP](#)

[2018 Adv Mechanics](#)

Unique sorted Links for Lecture 14

<http://thearmchaircritic.blogspot.com/2011/11/punkin-chunkin.html>

<http://www.sussexcountyonline.com/news/photos/punkinchunkin.html>

<http://www.twcenter.net/forums/showthread.php?358315-Shooting-range-for-medieval-siege-weapons-Anybody-knows>

<https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html>

<https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html?scenario=MontezumasRevenge>

<https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html?scenario=SeigeOfKenilworth>

https://modphys.hosted.uark.edu/pdfs/Journal_Pdfs/Trebuchet-SciAm_273_66_July_1995_chevedden1.pdf

Chapter 1. The Trebuchet: A dream problem for Galileo?

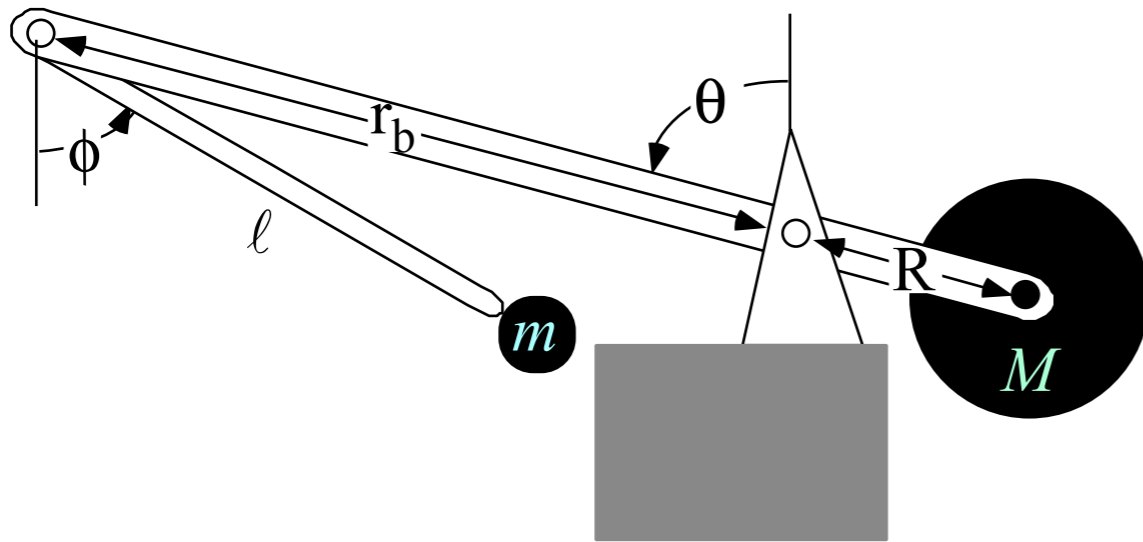
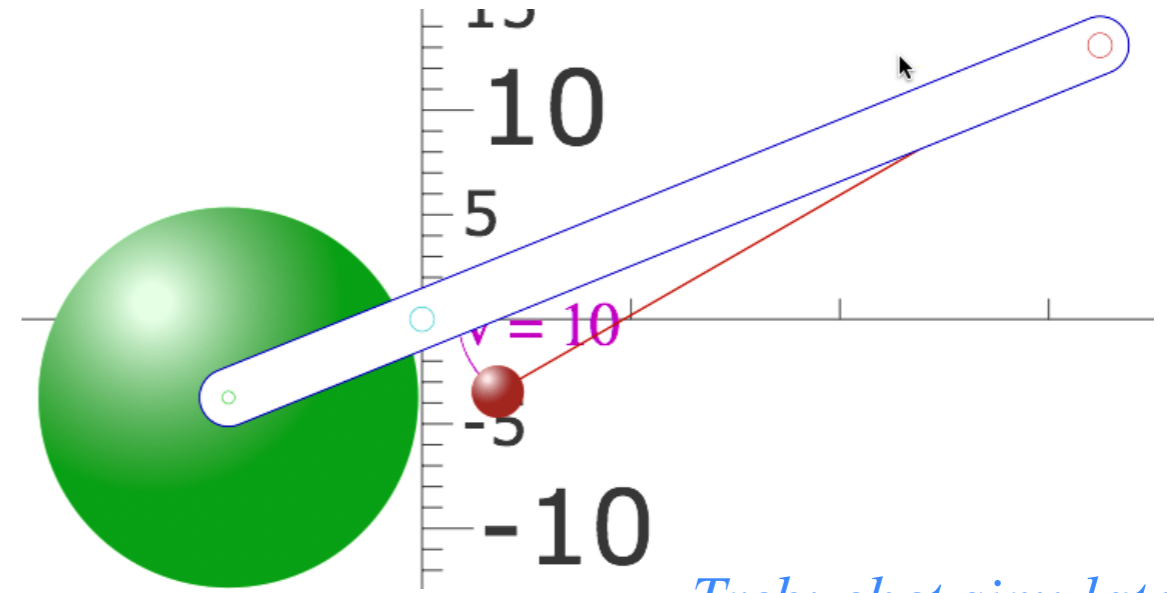


Fig. 2.1.1 An elementary ground-fixed trebuchet

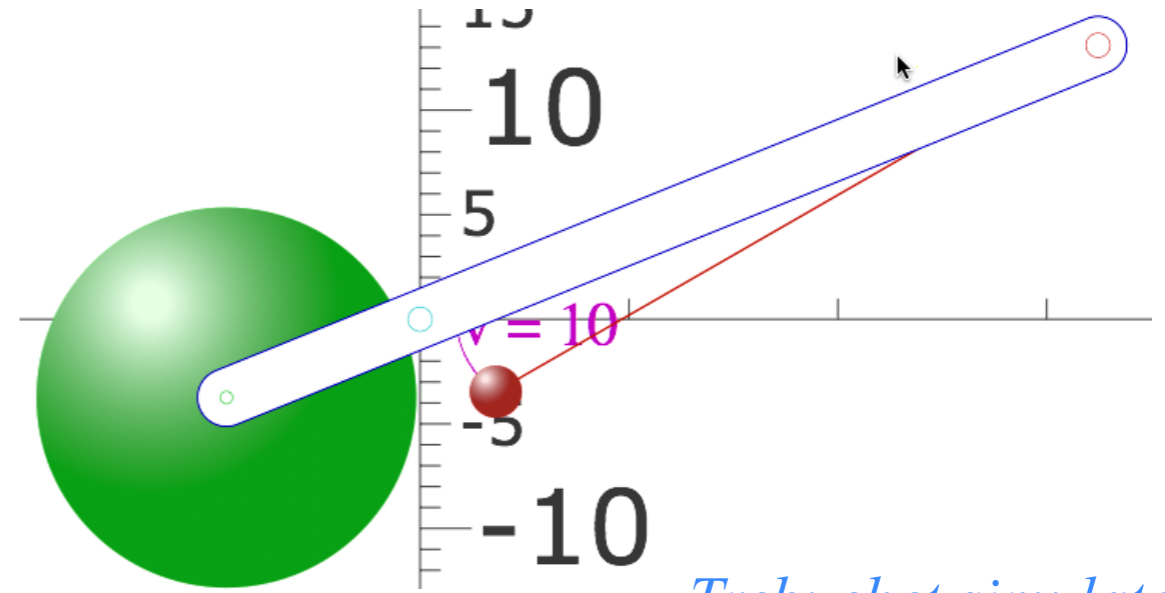
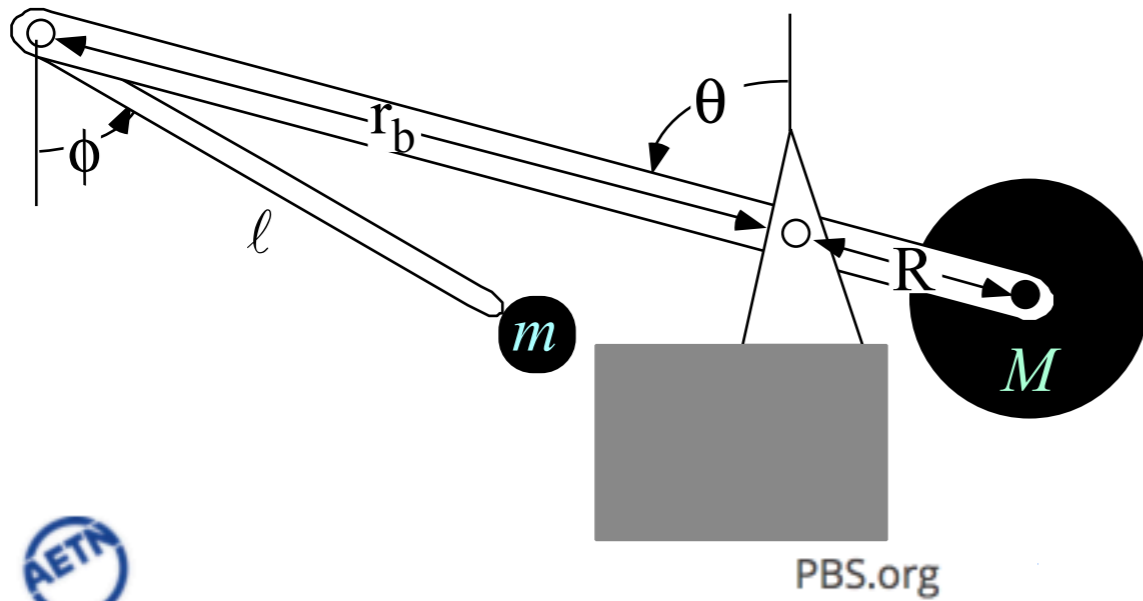


Trebuchet simulator

<https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html>

<https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html?scenario=SeigeOfKenilworth>

Chapter 1. The Trebuchet: A dream problem for Galileo?



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Siege of Kenilworth 1264-1267

<https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html?scenario=MontezumasRevenge>

Trebuchet Web App, use a URL with the string after the equal sign replaced with the desired scenario.

- PlotPosSpaceCourse
- PlotPosSpaceFine
- AnimateFlinger
- AnimateTrebuchet
- MontezumasRevenge
- SeigeOfKenilworth

<https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html?scenario=XXX> =>

<https://www.pbs.org/wgbh/nova/lostempires/trebuchet/builds.html>



The plan: NOVA and a team of master builders from England, Germany, France and the United States will reconstruct one of the most destructive of medieval weapons ever made: a giant trebuchet. They will raise the weapon in the shadows of Castle Urquhart, located on the shores of Loch Ness in the Scottish Highlands.

Chapter 1. The Trebuchet: A dream problem for Galileo?

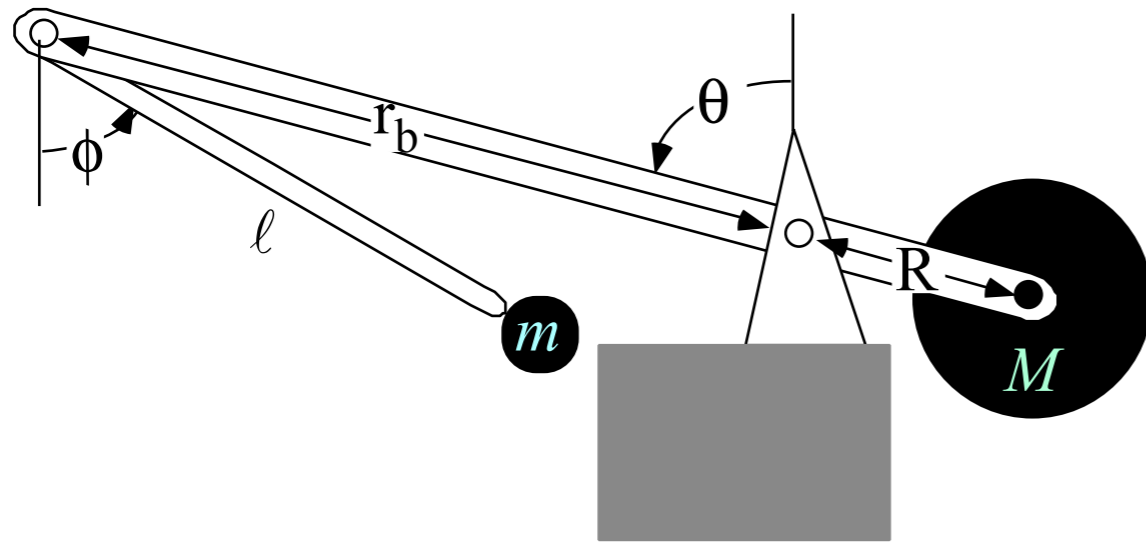
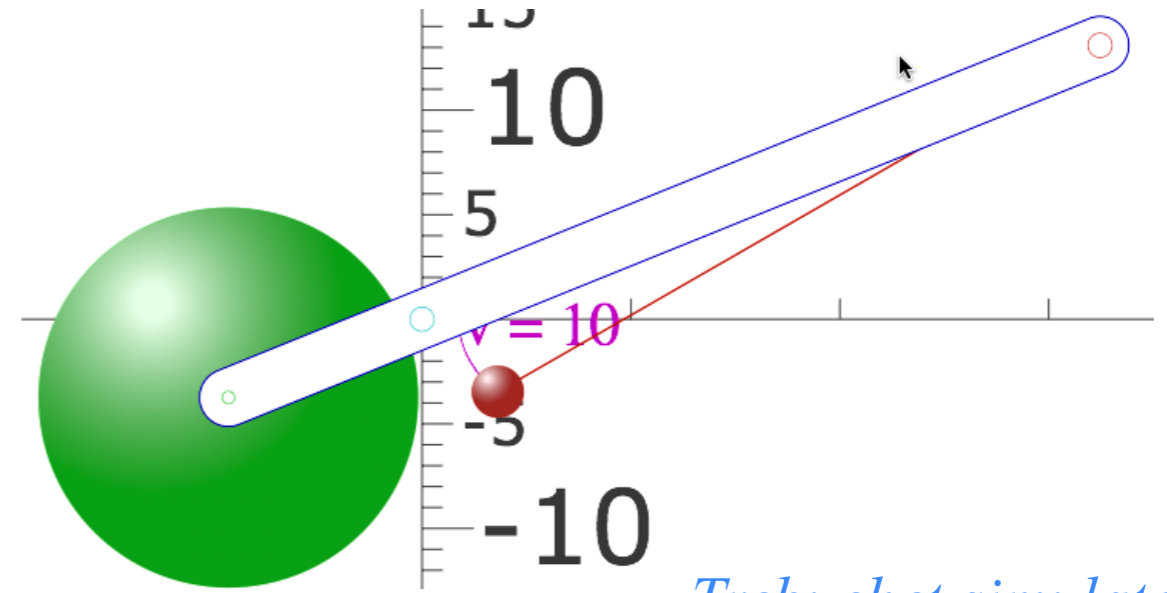


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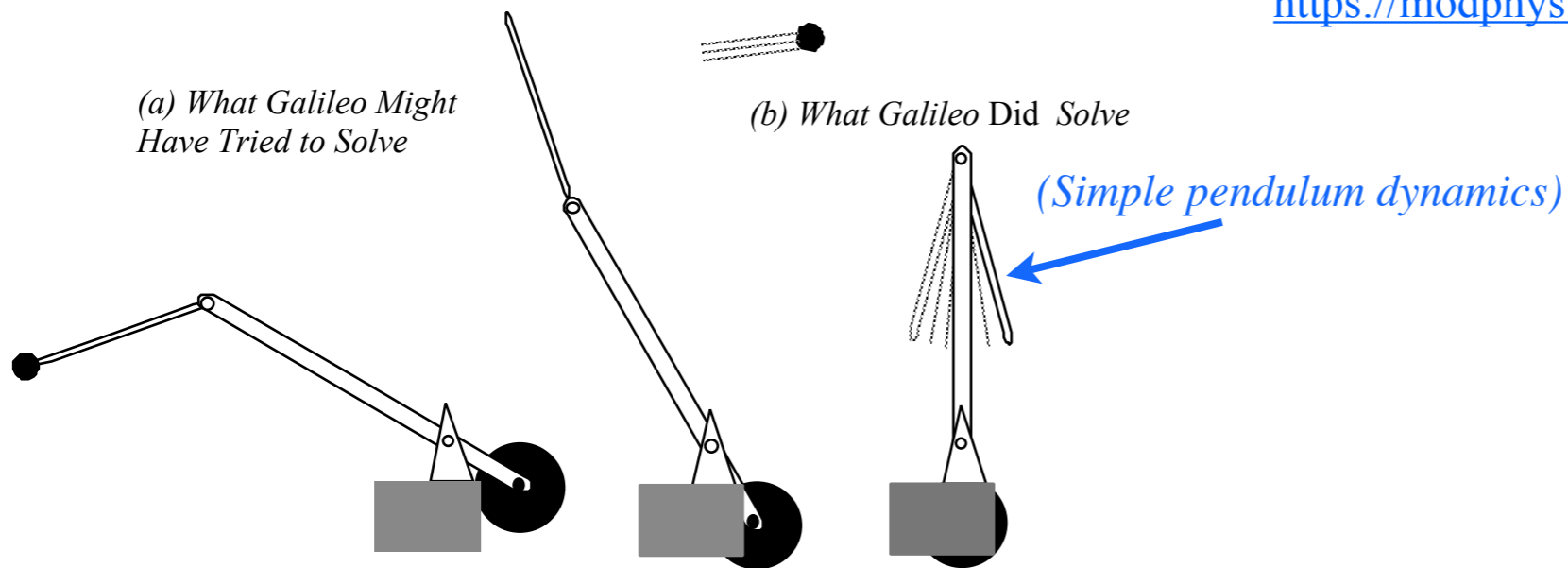


Fig. 2.1.2 Galileo's (supposed fictitious) problem

Chapter 1. The Trebuchet: A dream problem for Galileo?

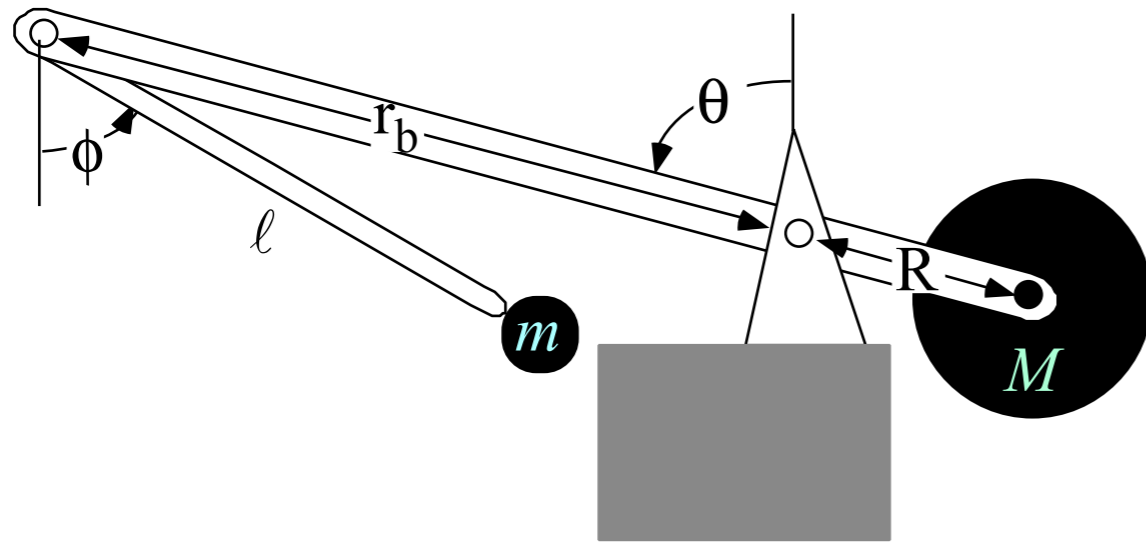
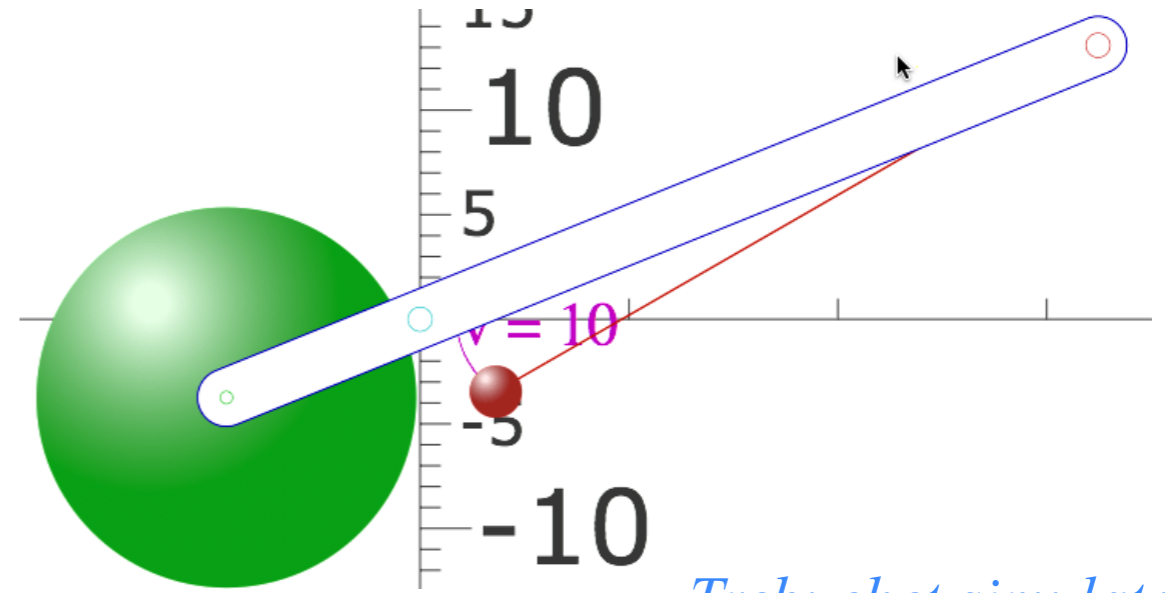


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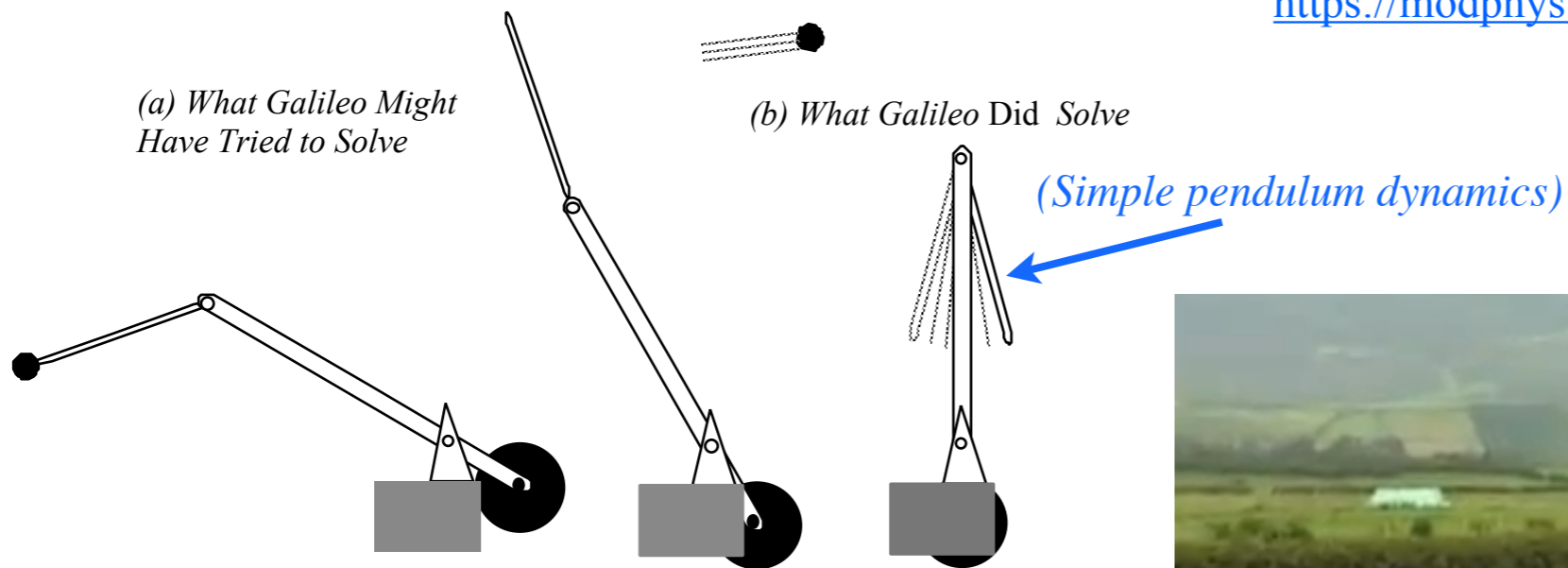


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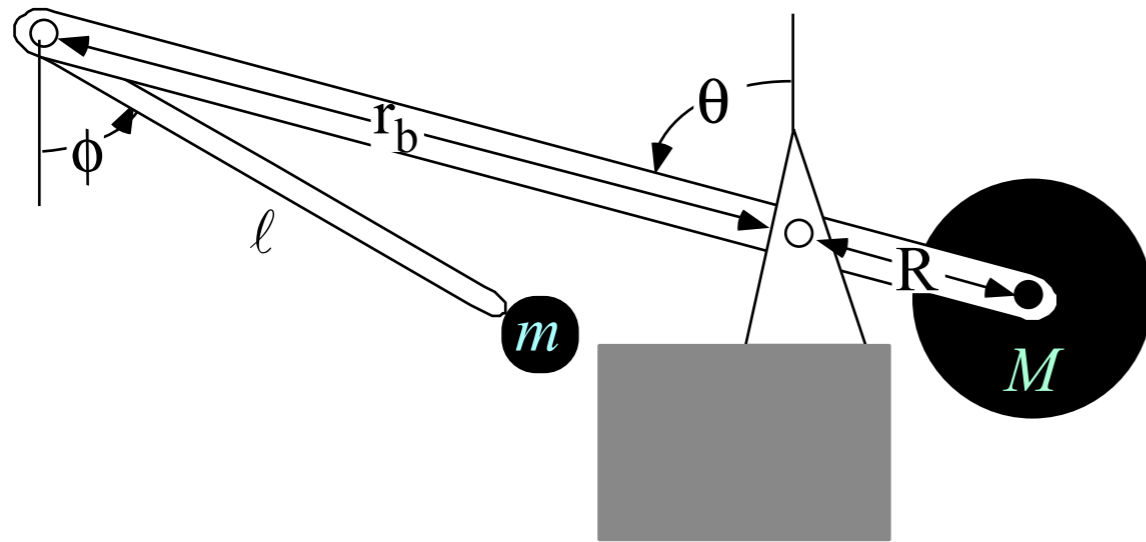
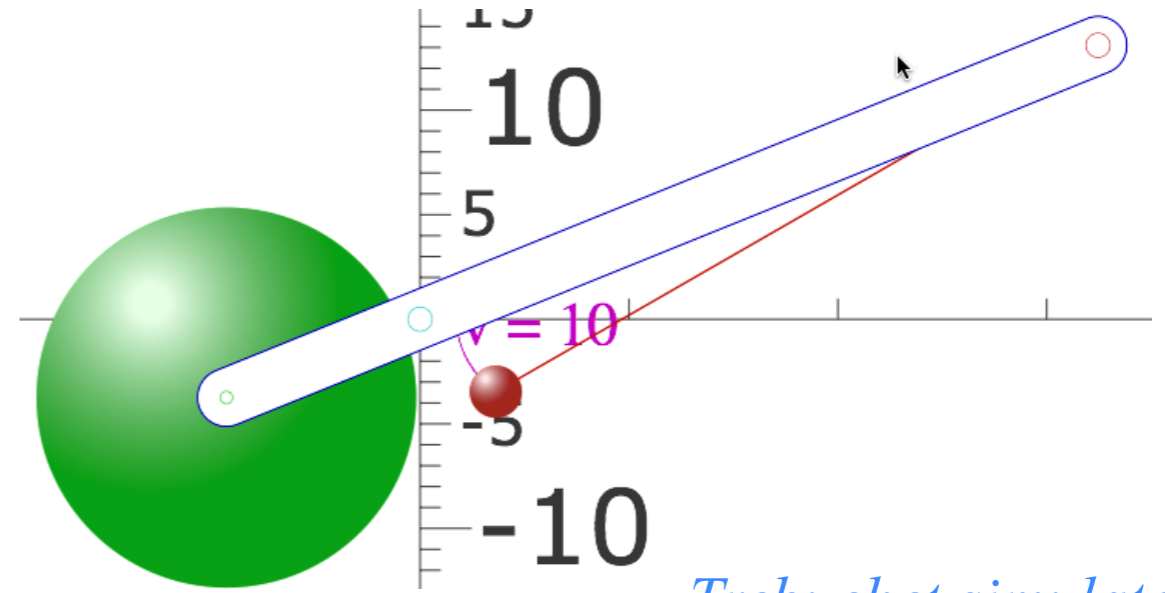


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Trebuchet simulator

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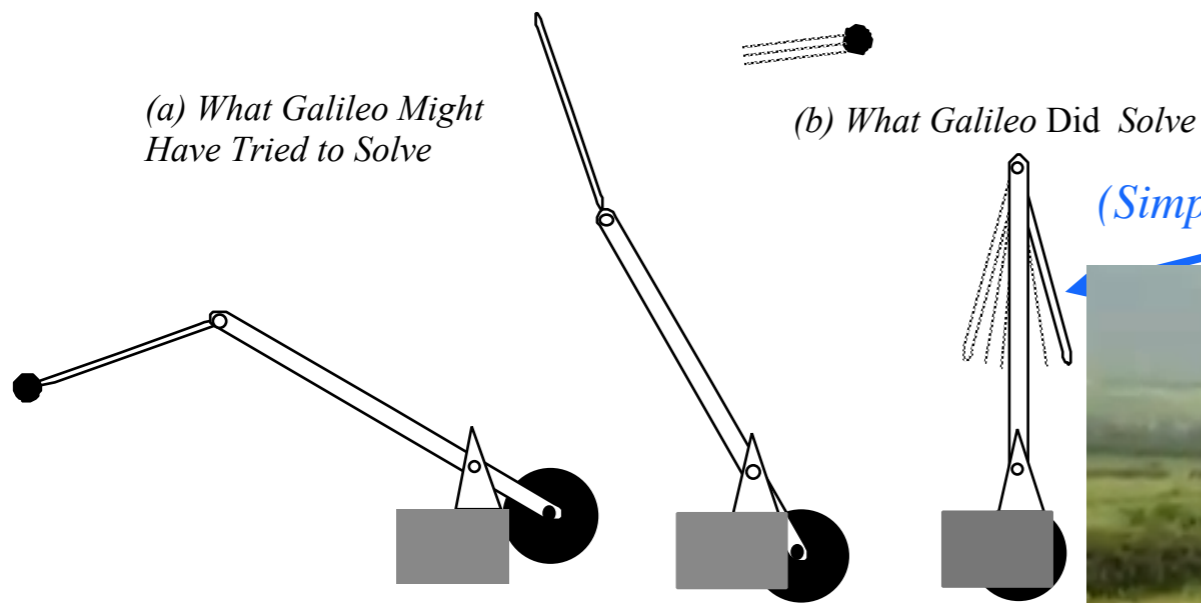


Fig. 2.1.2 Galileo's (supposed) problem



It's Halloween!...and time for Punkin' Chunkin' Trebuchets



<http://thearmchaircritic.blogspot.com/2011/11/punkin-chunkin.html>

As happened in history...Trebuchet is replaced by higher-tech (or lower tech)

Giant cannons can chunk-a-punkin over 4,000 ft. Trebuchet range max ~1,200ft.

<http://www.twcenter.net/forums/showthread.php?358315-Shooting-range-for-medieval-siege-weapons-Anybody-knows>



<http://www.sussexcountyonline.com/news/photos/punkinchunkin.html>

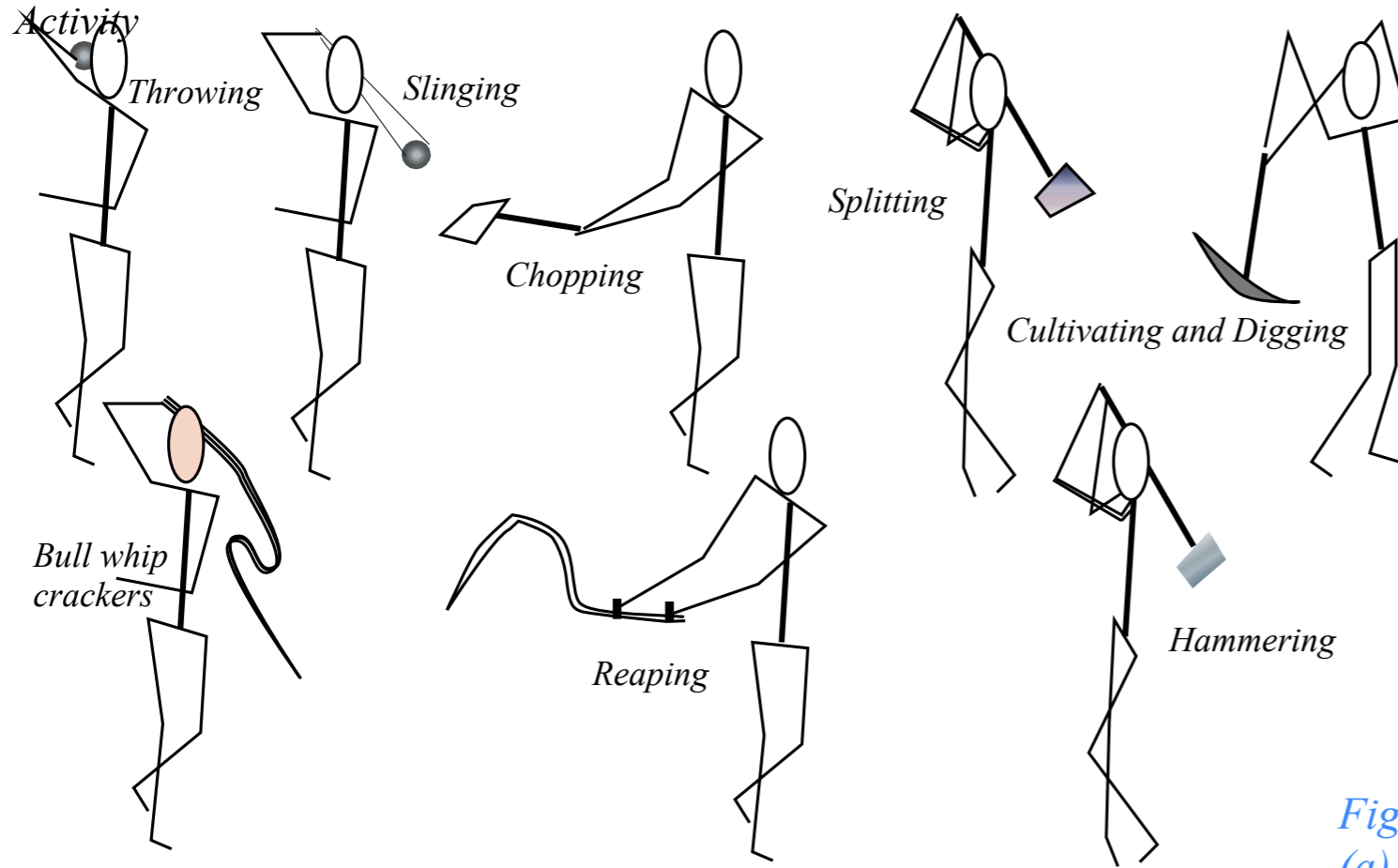


The trebuchet (or ingenium) and its cultural relevancy (3000 BCE to 21st See Sci. Am. 273, 66 (July 1995))

The medieval ingenium (9th to 14th century) and modern re-enactments

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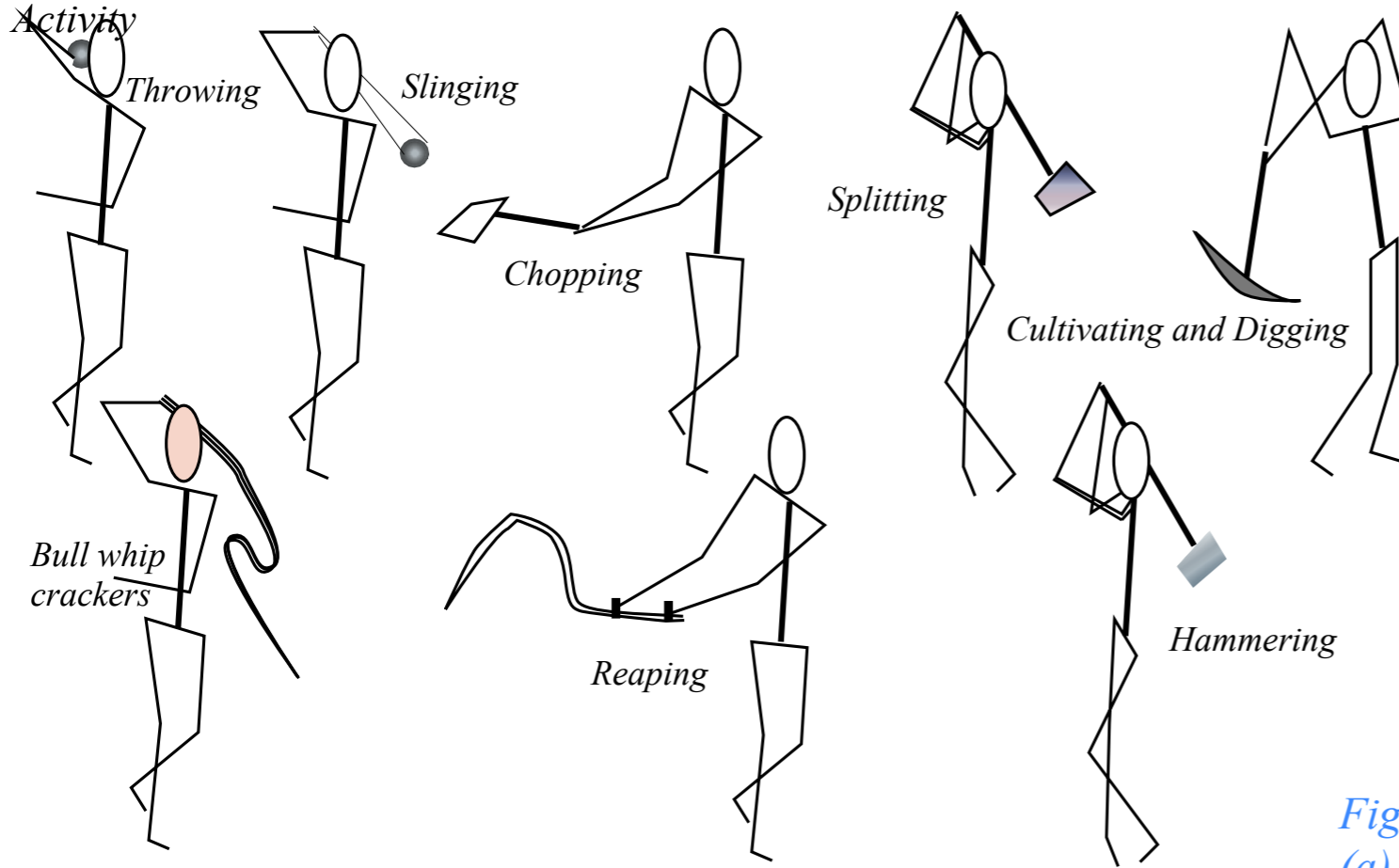
(a) Early Human Agriculture and Infrastructure Building



*Some technique required!
KE achieved by non-linear whip action
Must avoid injury*

*Fig. 2.1.3 Trebuchet-like motion of humans.
(a) Early work.*

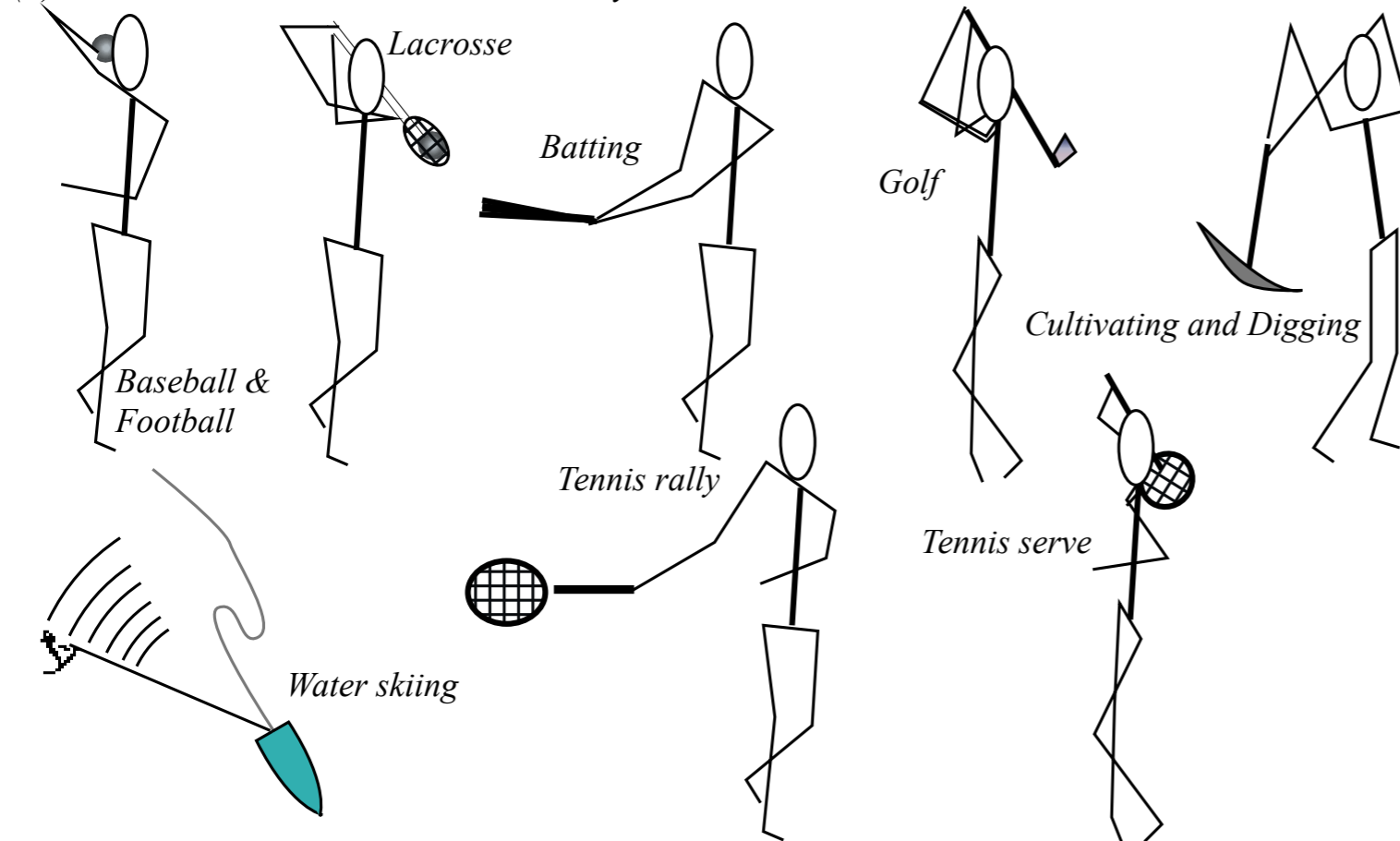
(a) Early Human Agriculture and Infrastructure Building



*Some technique required!
KE achieved by non-linear whip action
Must avoid injury*

*Fig. 2.1.3 Trebuchet-like motion of humans.
(a) Early work. (b) Later recreational kinesthetics.*

(b) Later Human Recreational Activity



*Some technique required!
KE achieved by non-linear whip action
Must avoid injury*

Review of Lagrangian equation derivation from Lecture 10 (Now with trebuchet model)

→ *Coordinate geometry, Jacobian, velocity, kinetic energy, and dynamic metric tensor γ_{mn}*

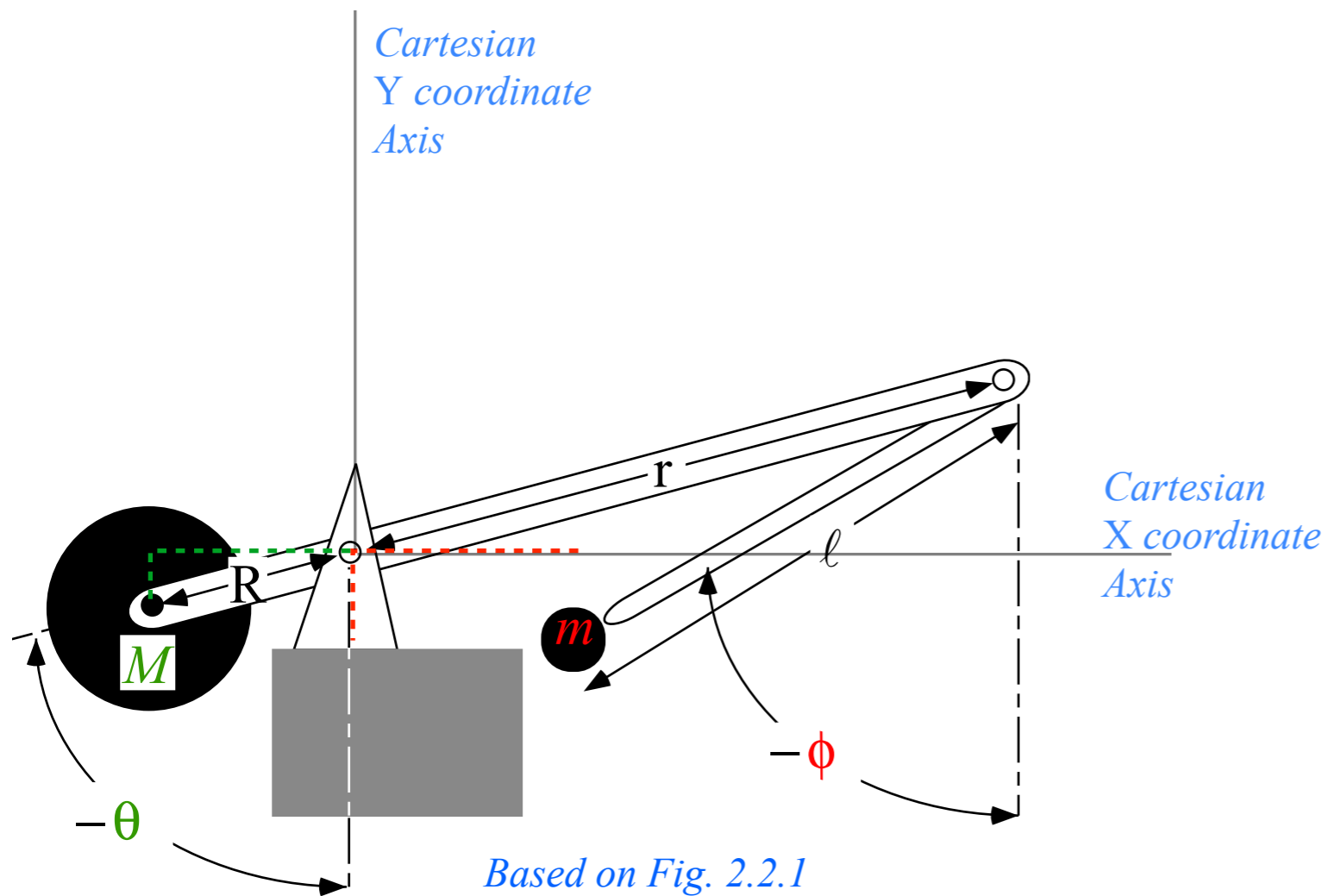
Structure of dynamic metric tensor γ_{mn}

Basic force, work, and acceleration

Lagrangian force equation

Canonical momentum and γ_{mn} tensor

geometry of trebuchet



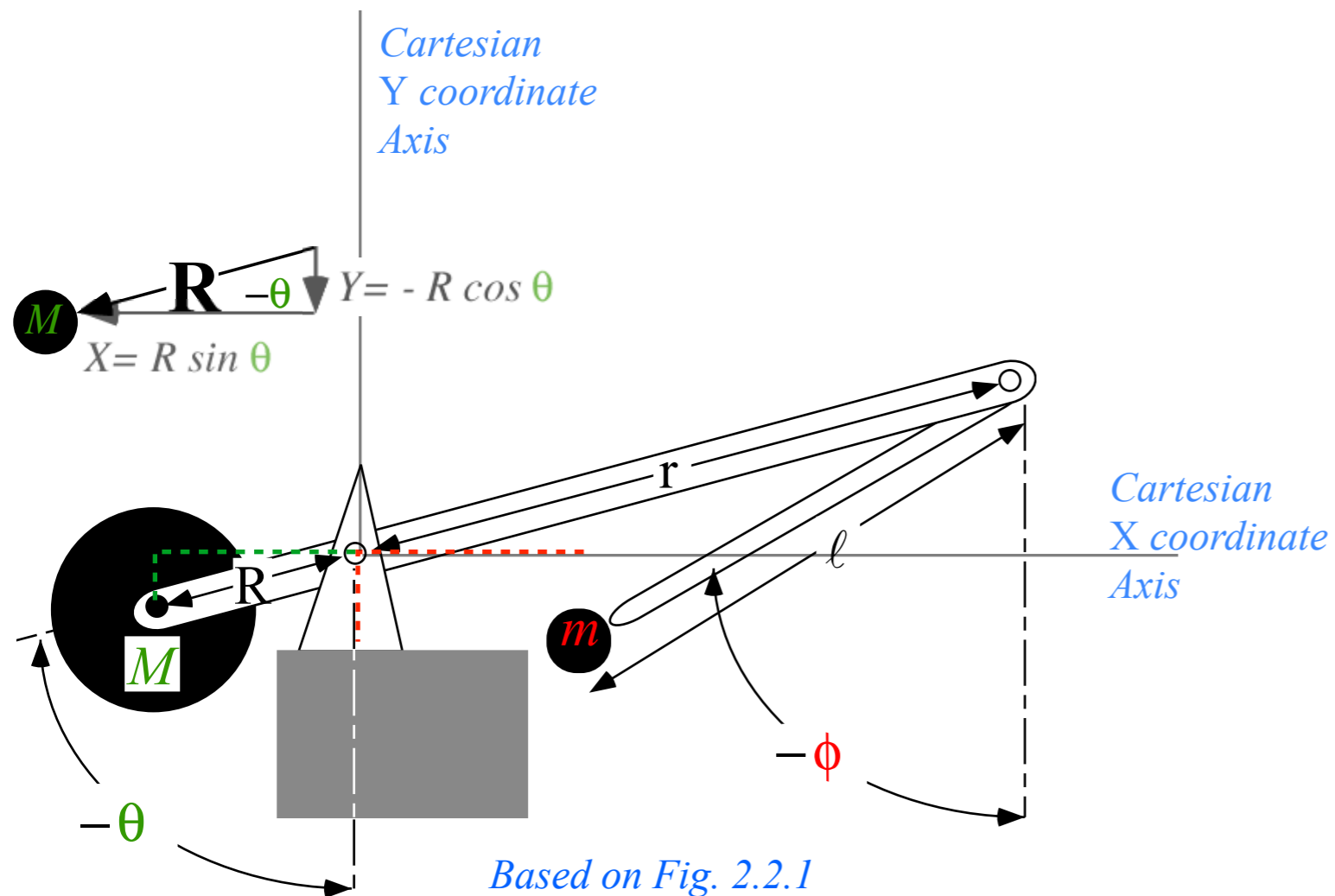
Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

Coordinates of M
(Driving weight Mg):

$$X = R \sin \theta$$

$$Y = -R \cos \theta$$

geometry of trebuchet



Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

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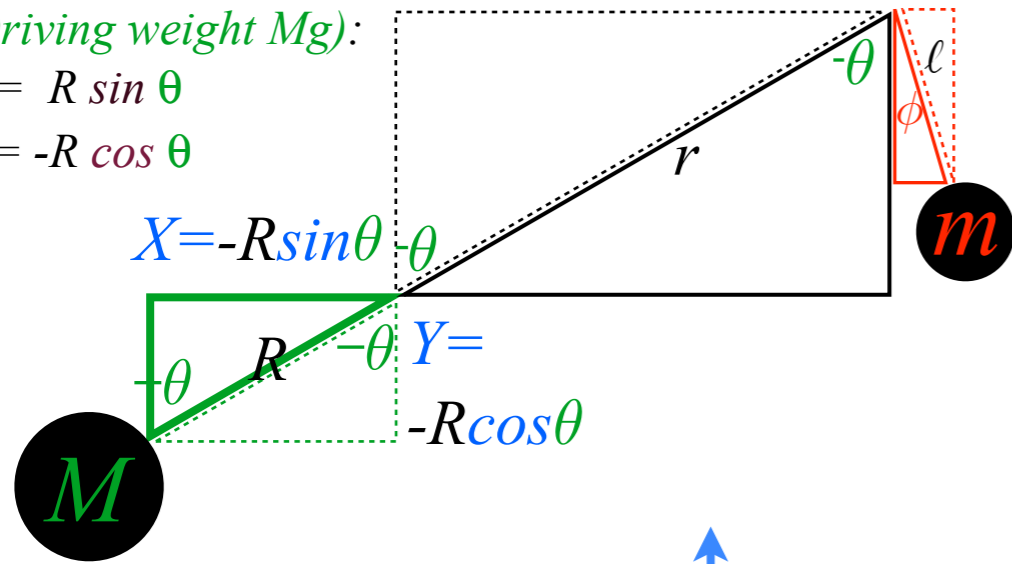
$$Y = -R \cos \theta$$

Coordinates of mass m

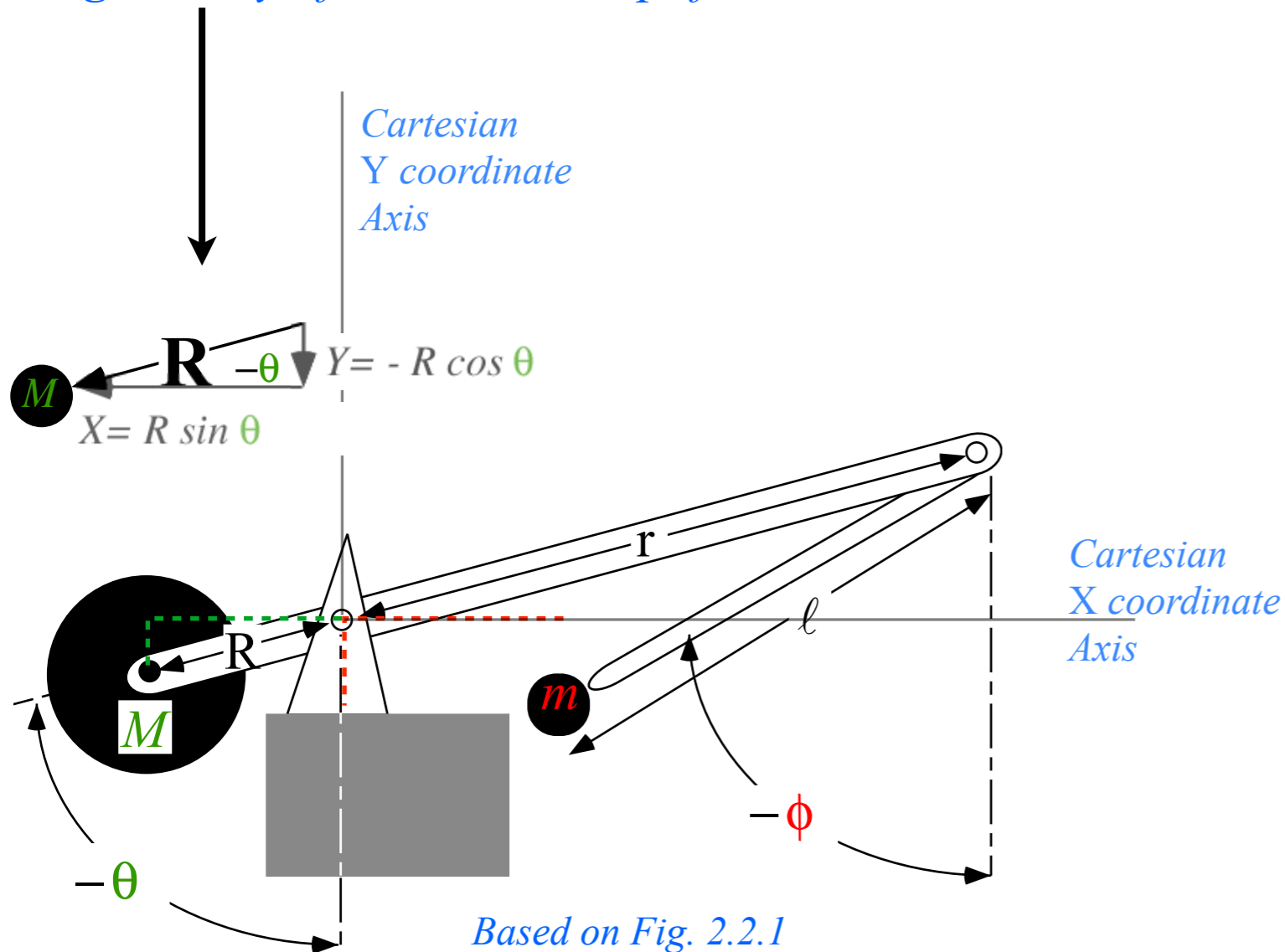
(Payload or projectile):

$$x = x_r + x_\ell = -r \sin \theta + \ell \sin \phi$$

$$y = y_r + y_\ell = r \cos \theta - \ell \cos \phi$$



geometry of trebuchet simplified somewhat...



Based on Fig. 2.2.1

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

Coordinates of M

(Driving weight Mg):

$$X = R \sin \theta$$

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$$x_r = r \sin(-\theta)$$

$$y_r = r \cos(-\theta)$$

$$X = -R \sin \theta$$

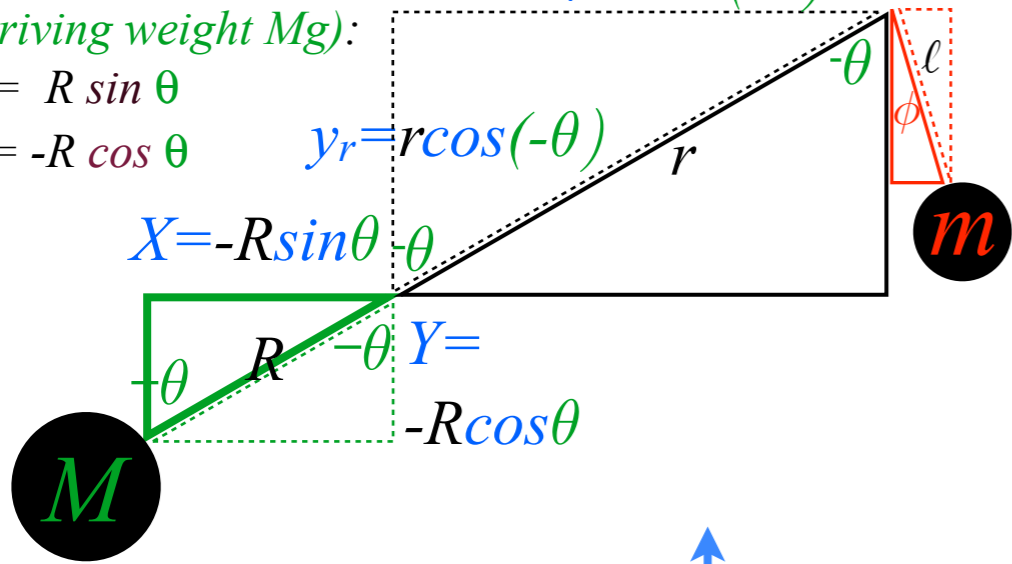
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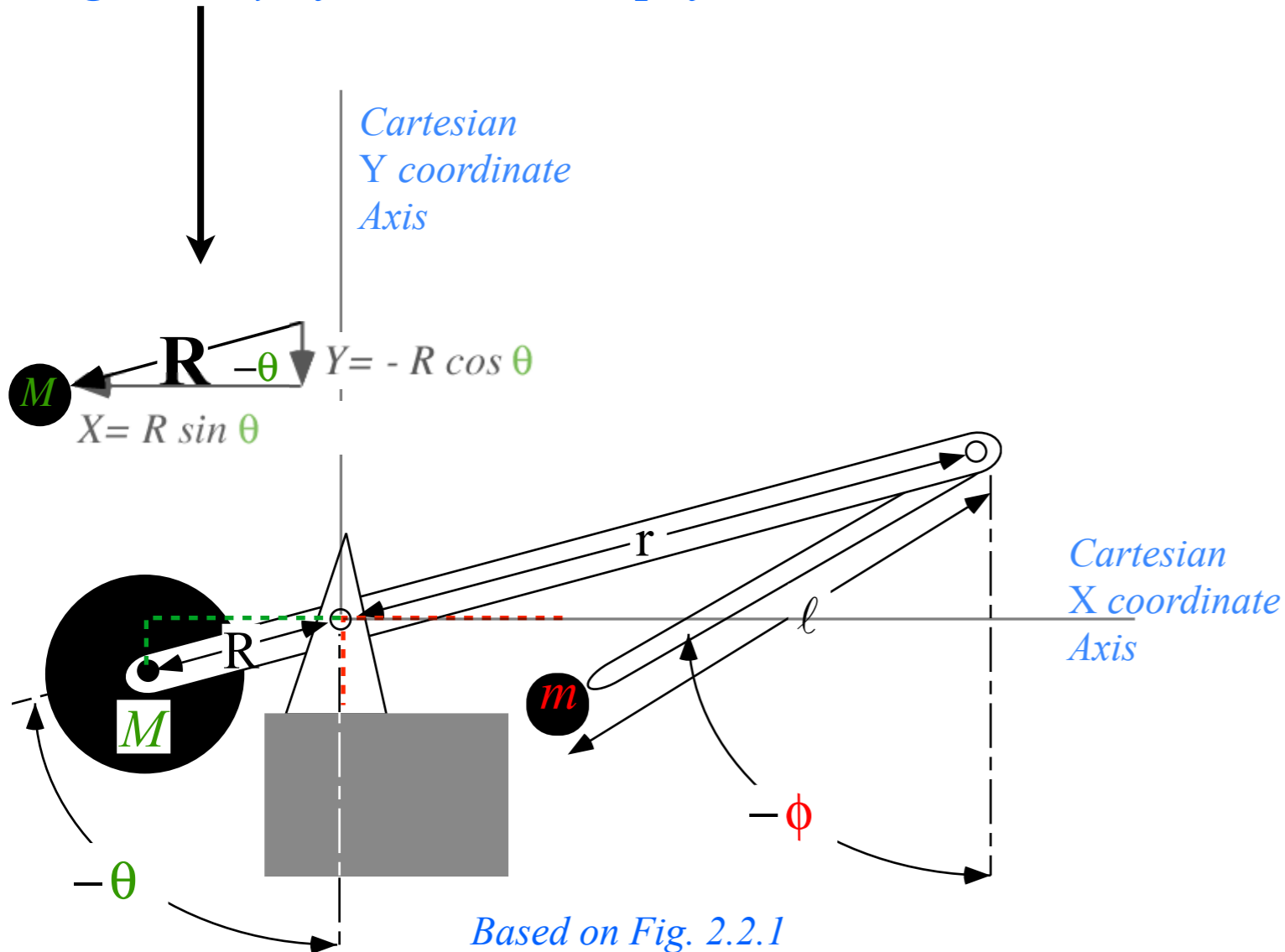
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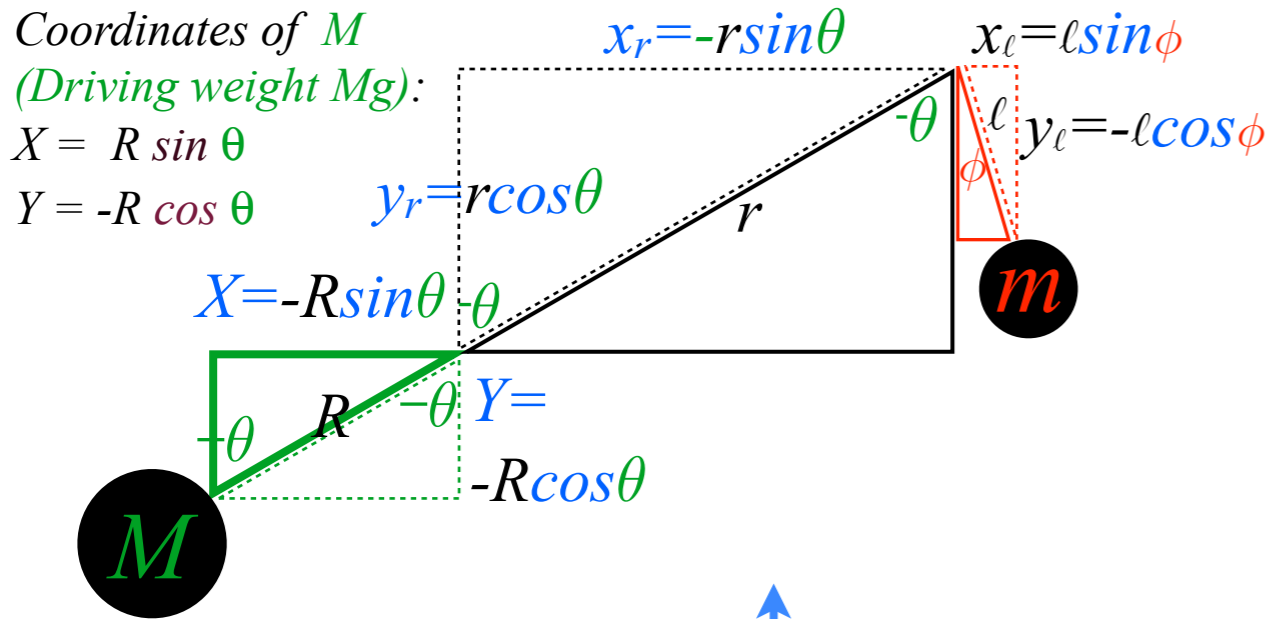
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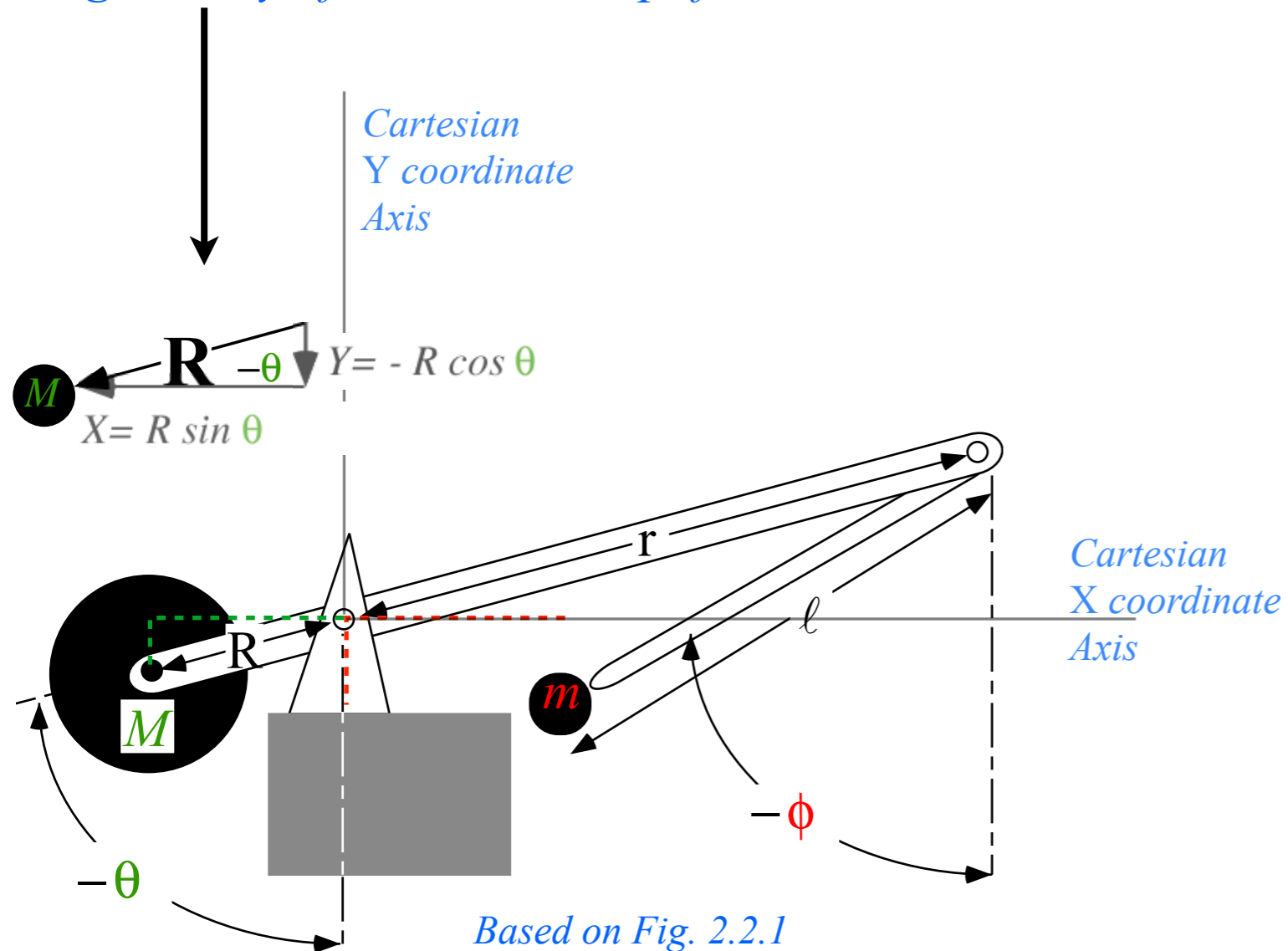
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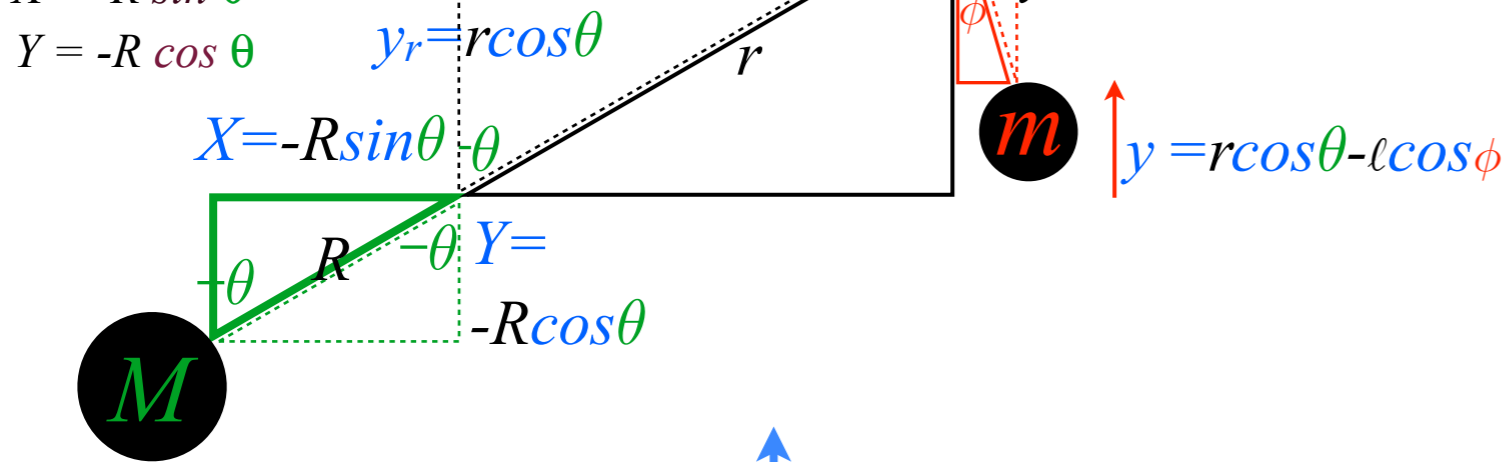


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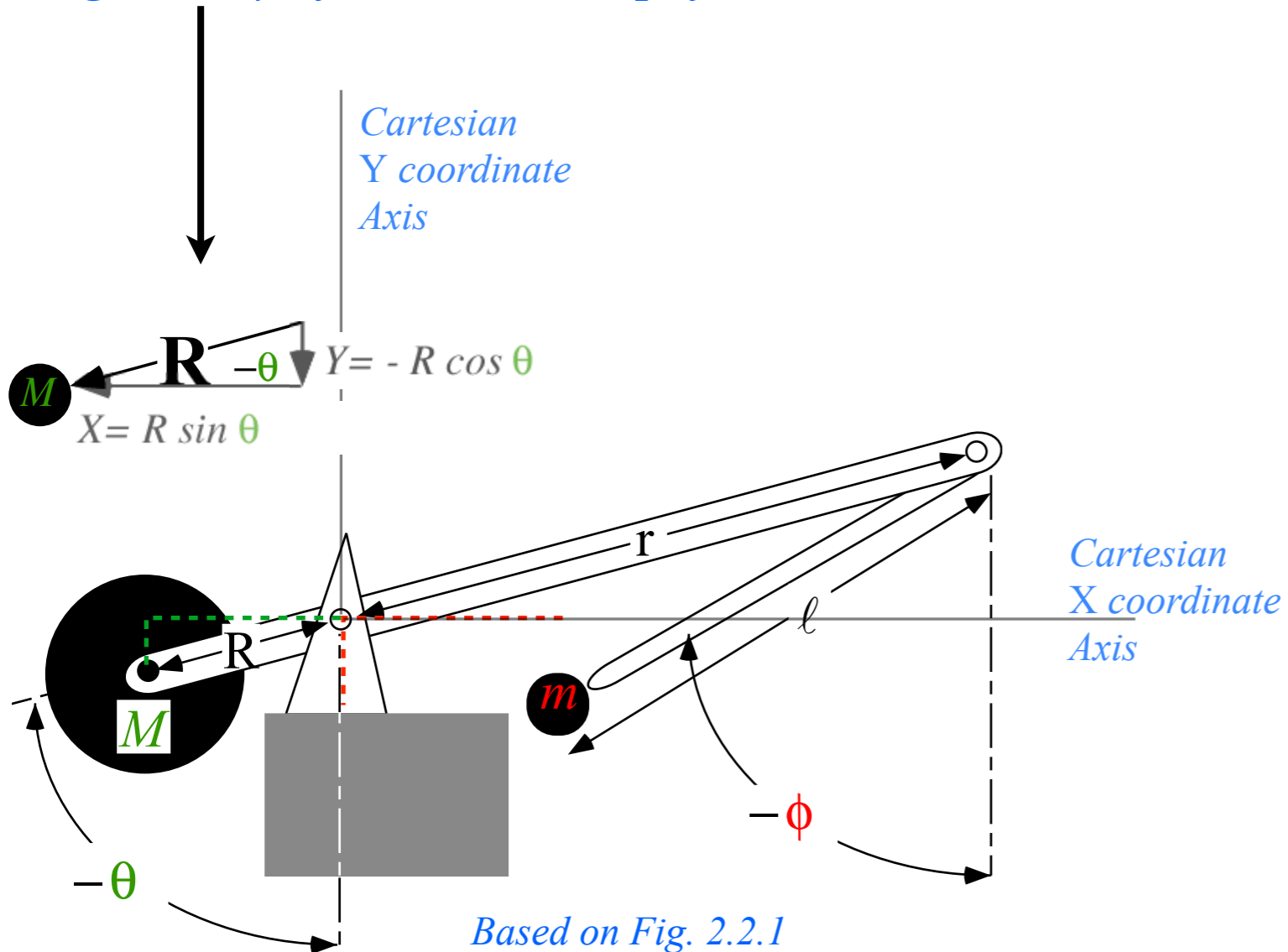
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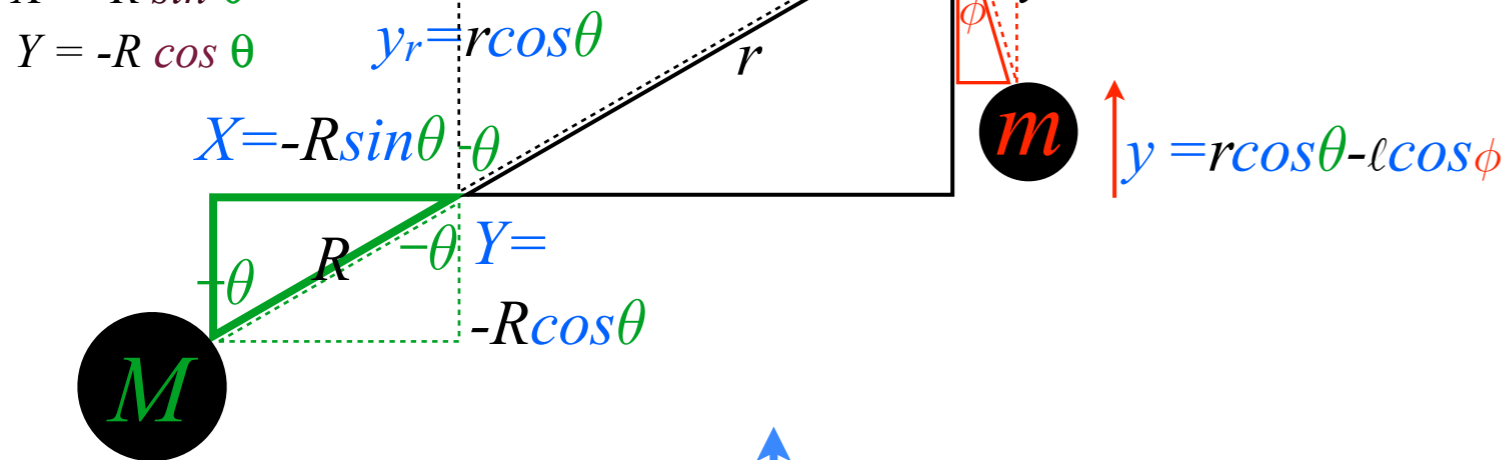


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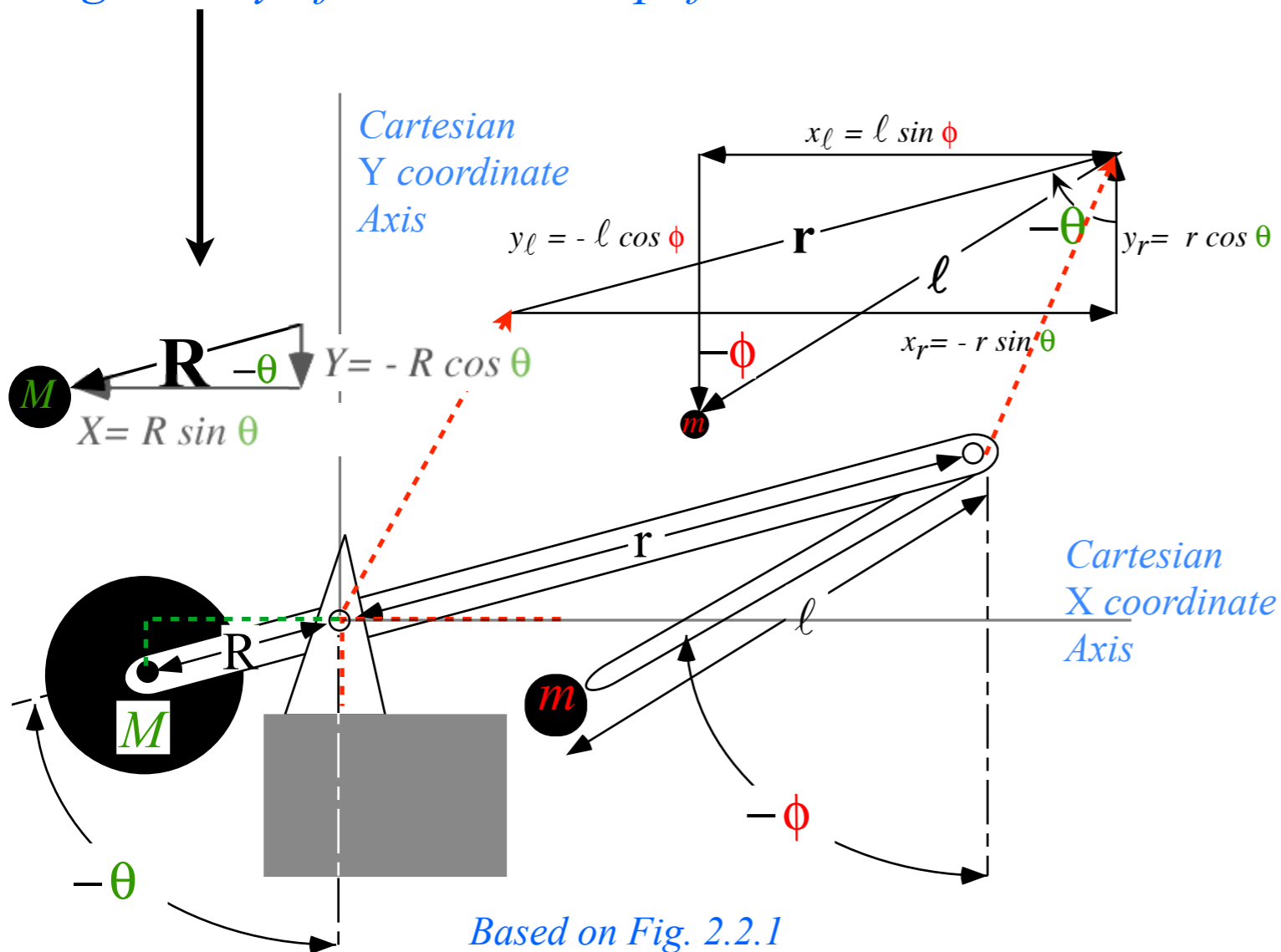
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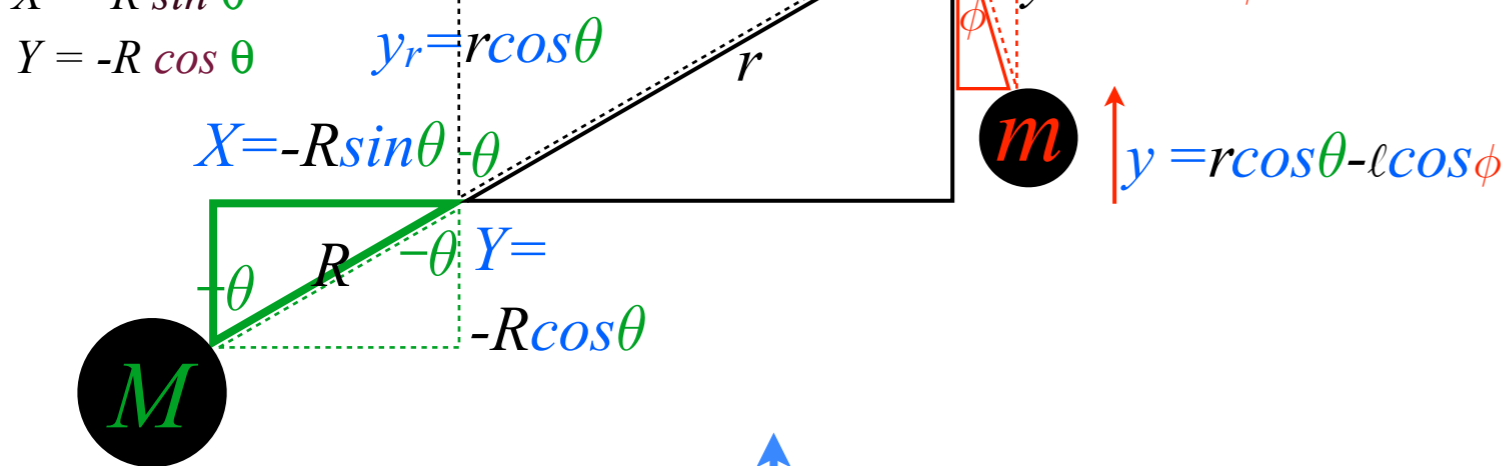
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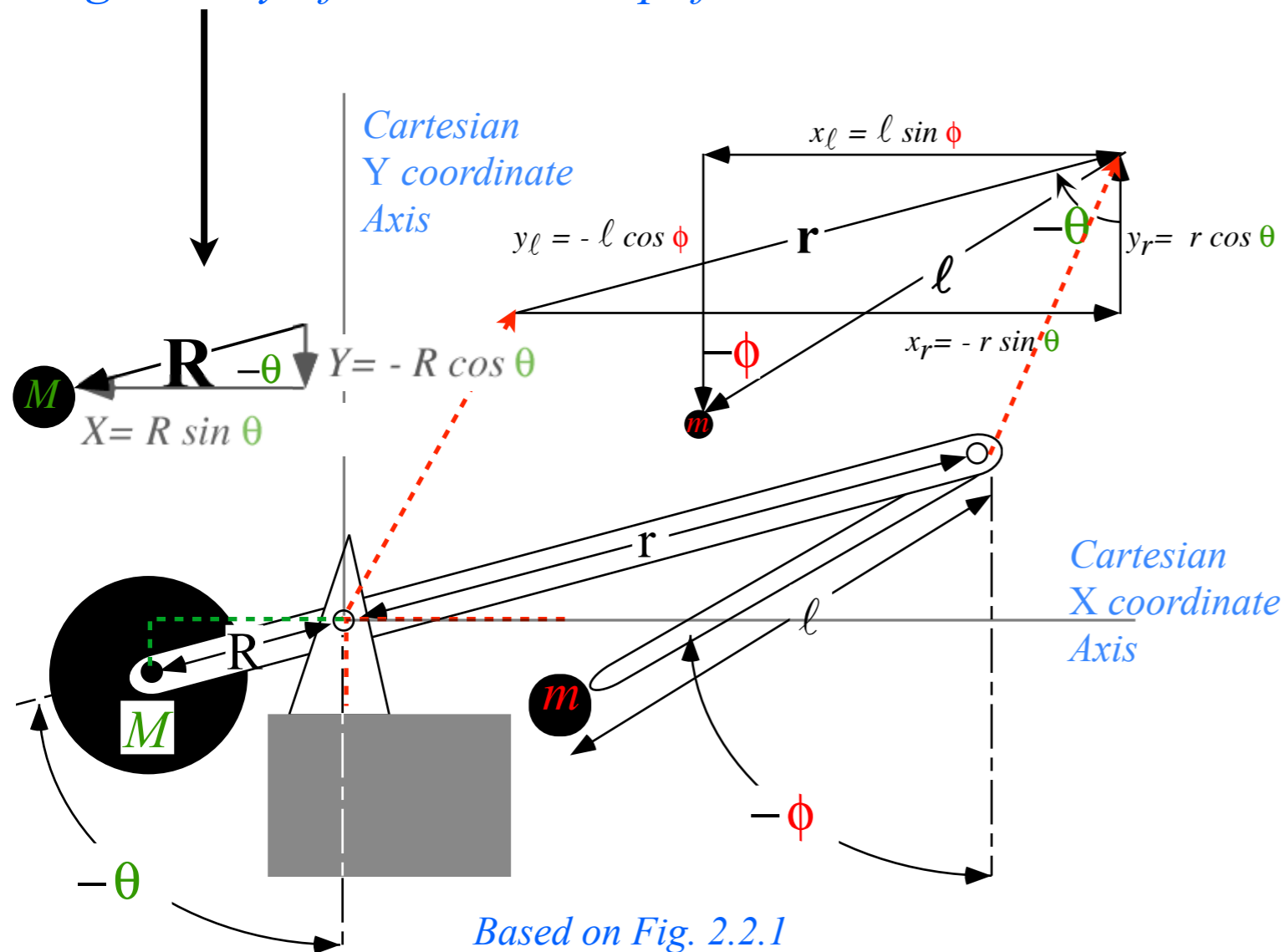
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1st differential and Jacobian relations:

$$dX = \frac{\partial X}{\partial \theta} d\theta + \frac{\partial X}{\partial \phi} d\phi, \quad dx = \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial \phi} d\phi,$$

$$dY = \frac{\partial Y}{\partial \theta} d\theta + \frac{\partial Y}{\partial \phi} d\phi, \quad dy = \frac{\partial y}{\partial \theta} d\theta + \frac{\partial y}{\partial \phi} d\phi.$$

geometry of trebuchet simplified somewhat...



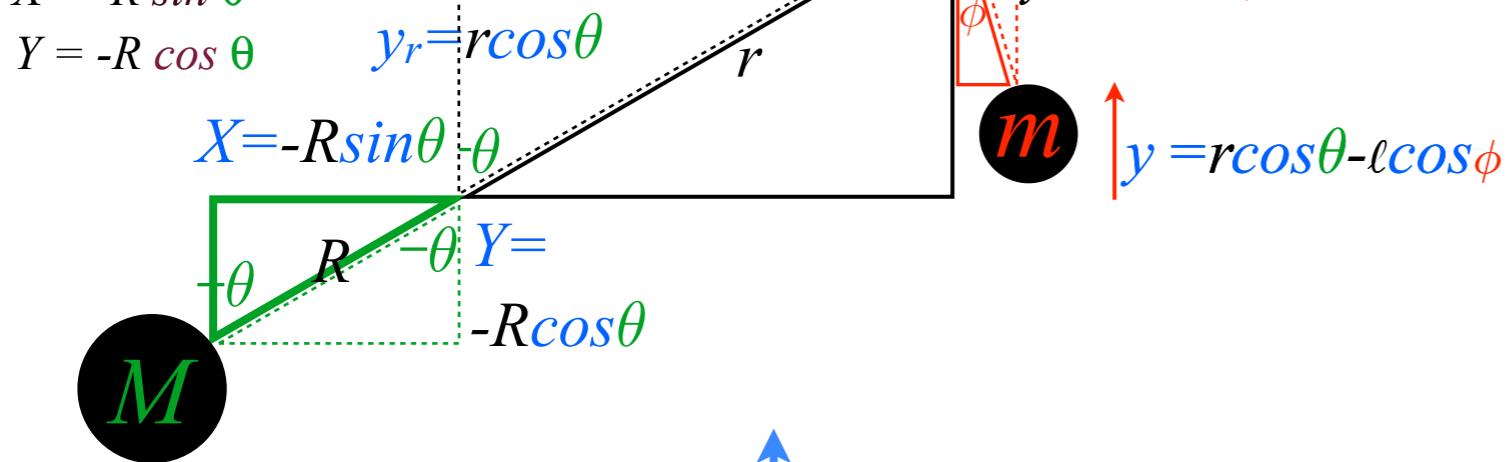
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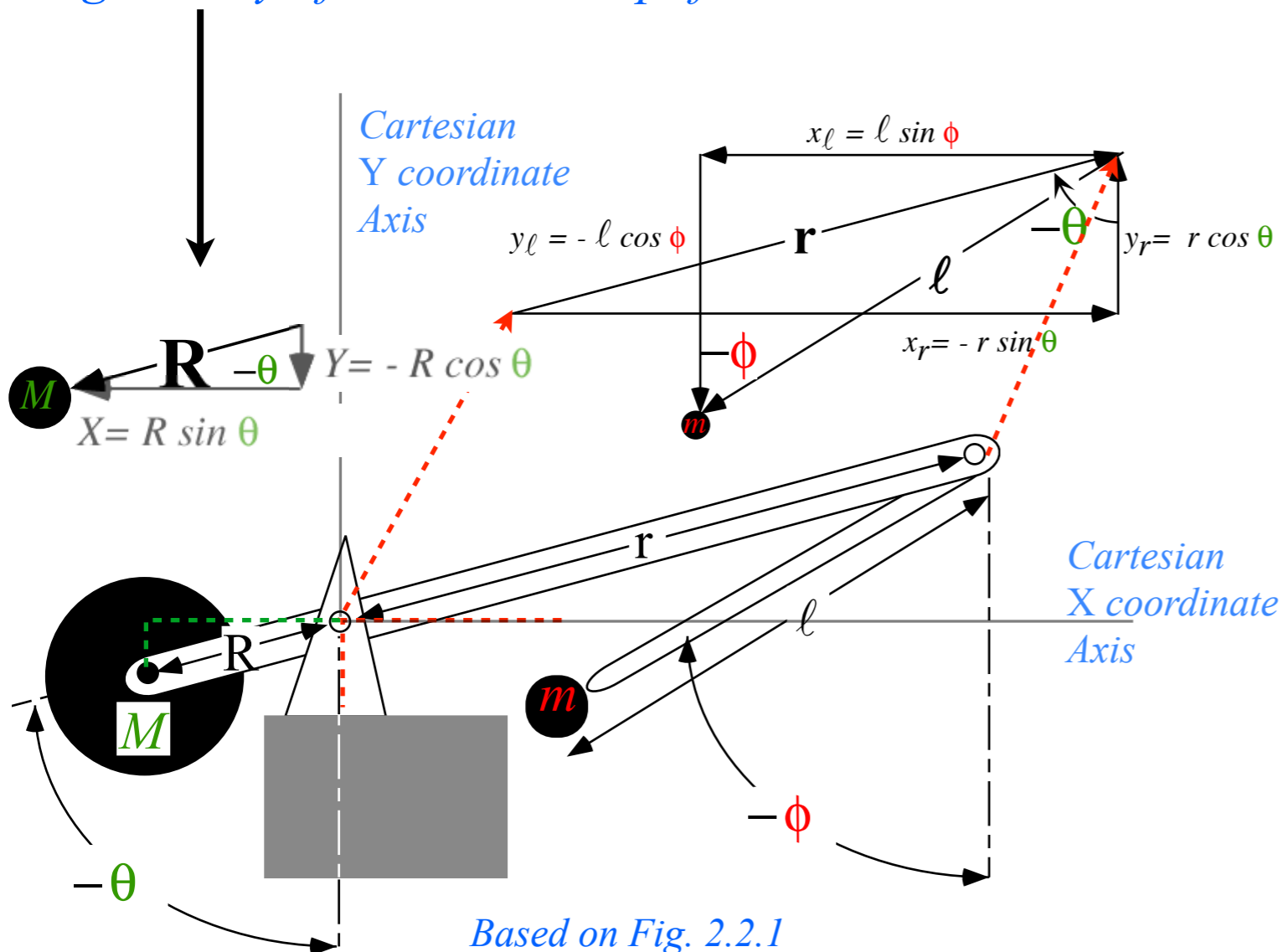
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$$dX = R \cos \theta d\theta + 0, \quad dx = -r \cos \theta d\theta + l \cos \phi d\phi,$$

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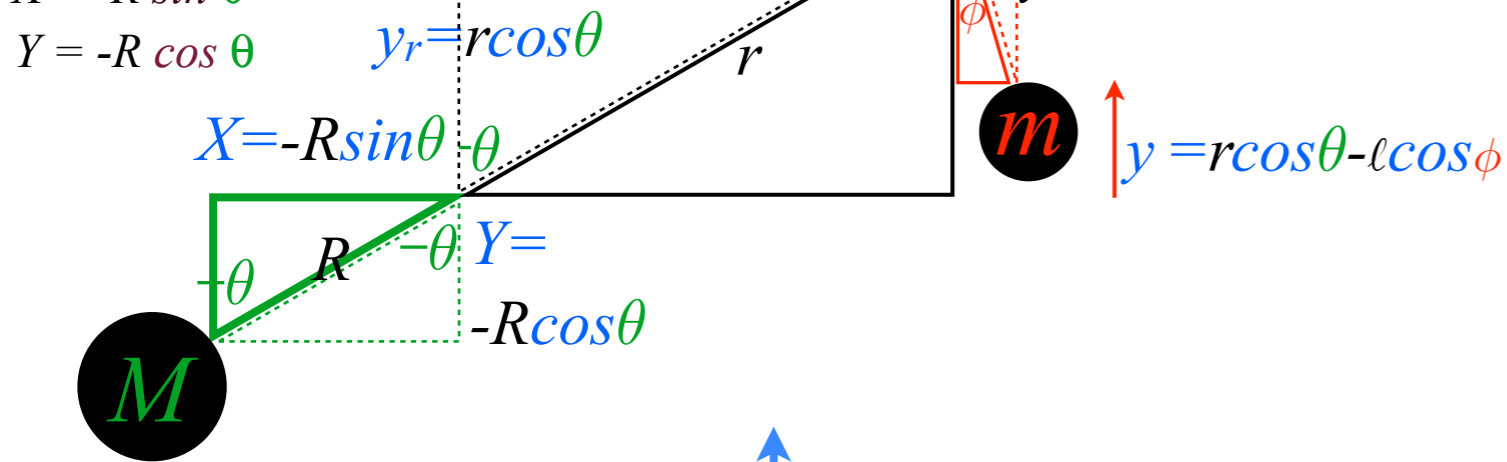


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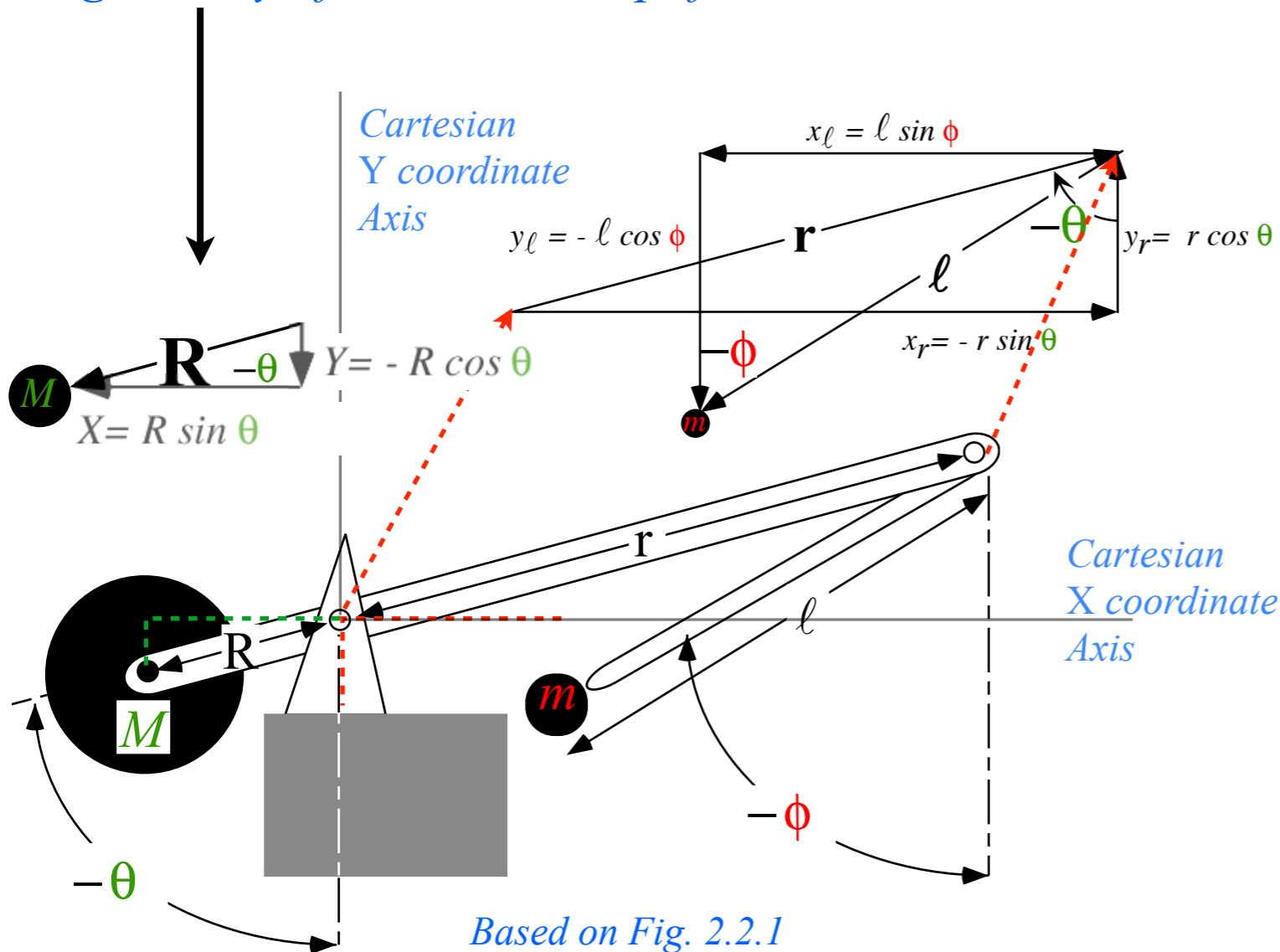
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Constraint relations:

$$c_R(X, Y) = X^2 + Y^2 = R^2 = \text{const.}$$

$$c_\ell(x_\ell, y_\ell) = x_\ell^2 + y_\ell^2 = \ell^2 = \text{const.}$$

$$c_r(x_r, y_r) = x_r^2 + y_r^2 = r^2 = \text{const.}$$

'Raw' Jacobian form

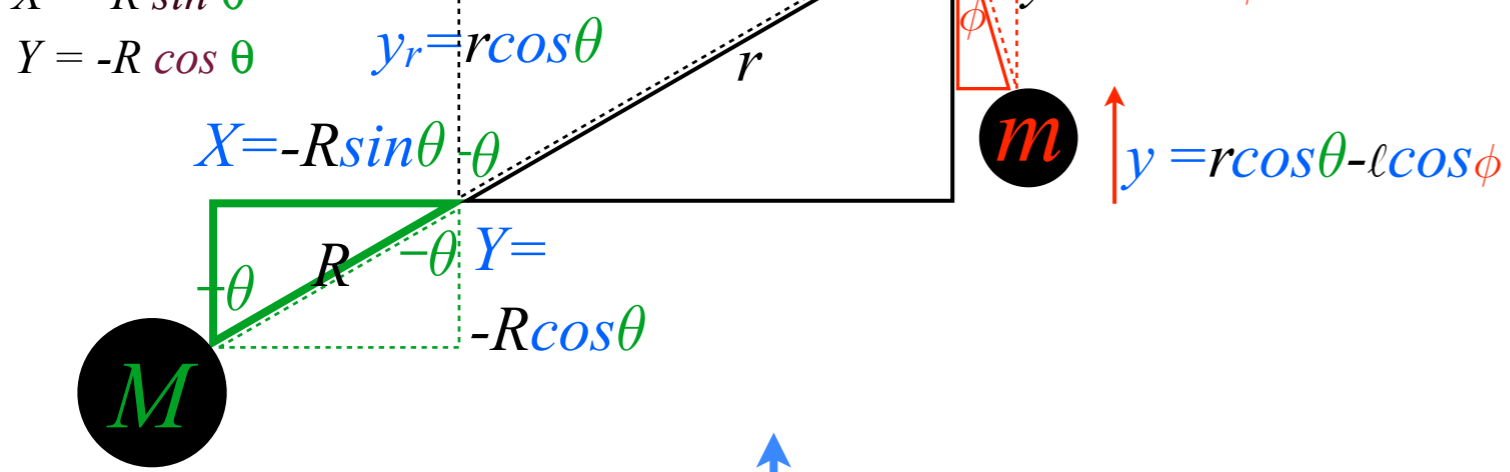
$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R \cos \theta & 0 \\ R \sin \theta & 0 \\ -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

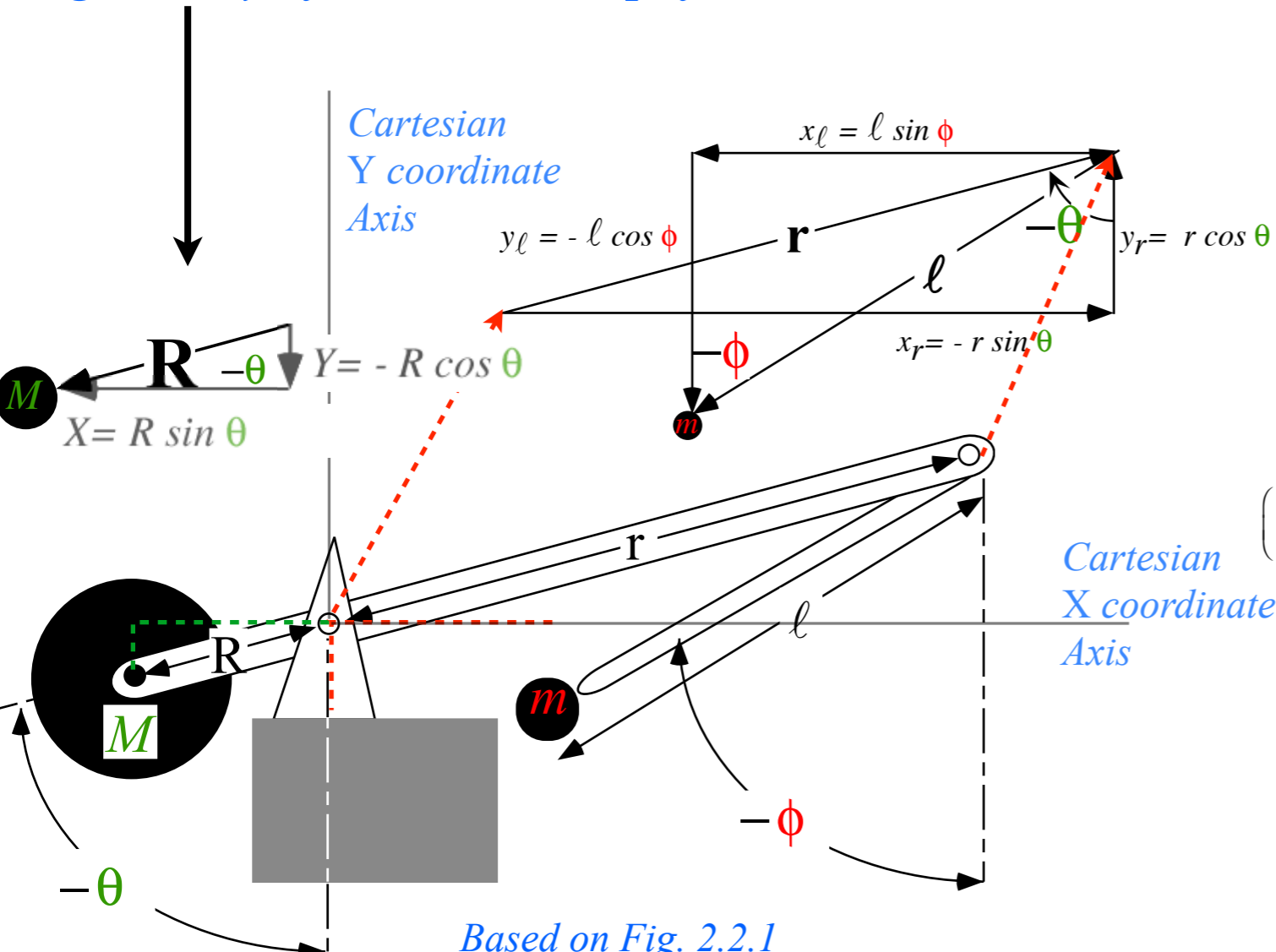
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$$c_r(x_r, y_r) = x_r^2 + y_r^2 = r^2 = \text{const.}$$

'Raw' Jacobian form

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R \cos \theta & 0 \\ R \sin \theta & 0 \\ -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

Finding a reduced Jacobian form

$$\begin{pmatrix} dX \\ dY \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R \cos \theta & 0 \\ R \sin \theta & 0 \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} \text{ FAILS since: } \det \begin{vmatrix} R \cos \theta & 0 \\ R \sin \theta & 0 \end{vmatrix} = 0$$

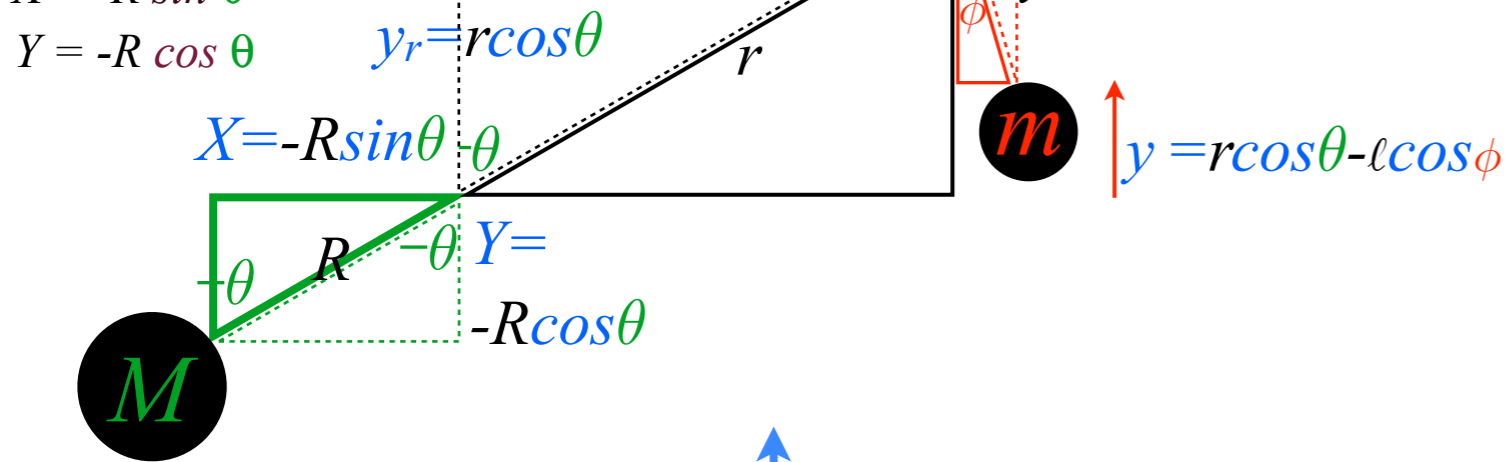
FAILS! (Always singular)

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

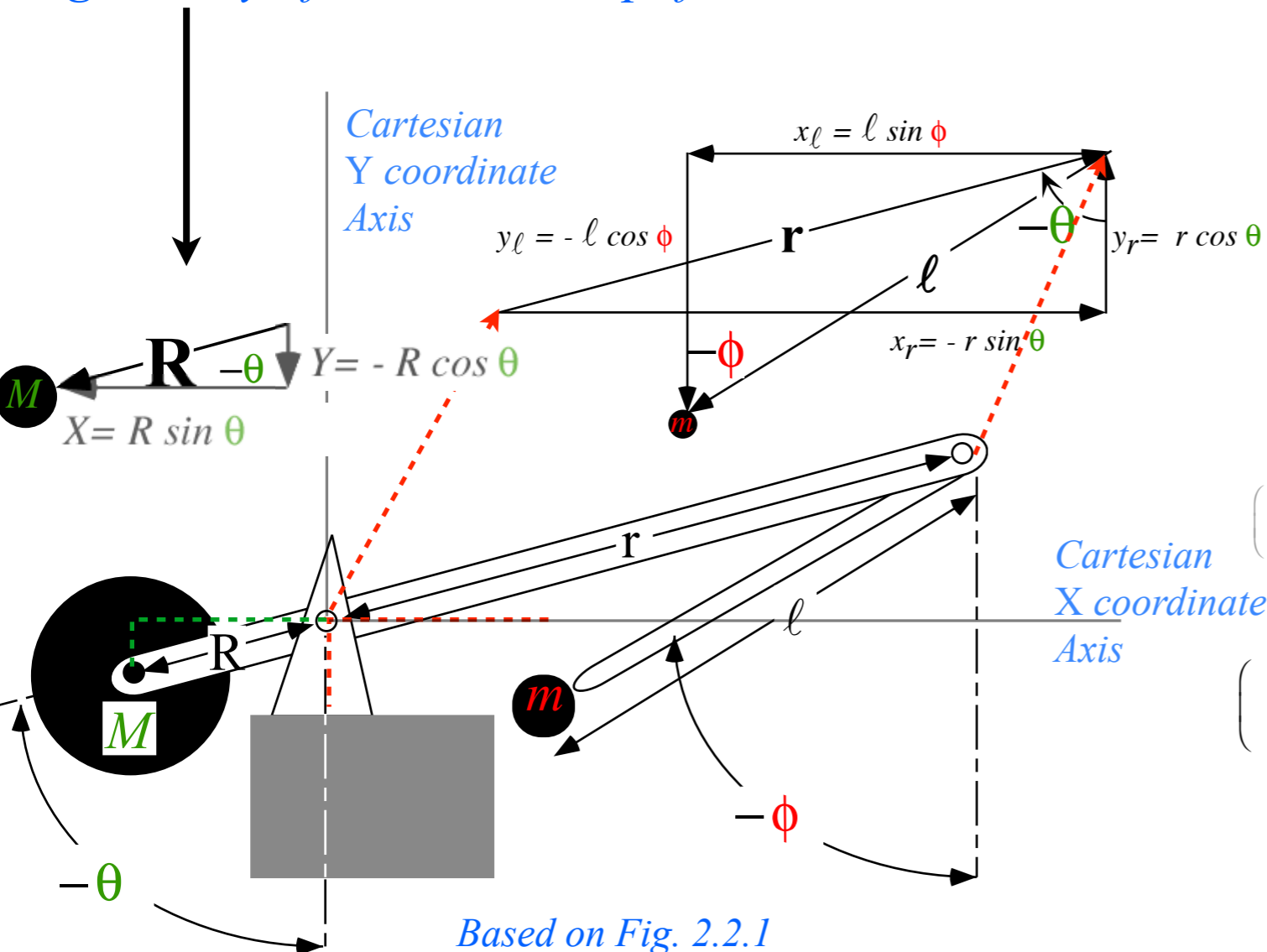
Coordinates of M
(Driving weight Mg):

$$X = R \sin \theta$$

$$Y = -R \cos \theta$$



geometry of trebuchet simplified somewhat...



Based on Fig. 2.2.1

Coordinates of mass m

(Payload or projectile):

$$x = x_r + x_l = -r \sin \theta + l \sin \phi$$

$$y = y_r + y_l = r \cos \theta - l \cos \phi$$

1st differential and Jacobian relations:

$$dX = \frac{\partial X}{\partial \theta} d\theta + \frac{\partial X}{\partial \phi} d\phi, \quad dx = \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial \phi} d\phi,$$

$$dY = \frac{\partial Y}{\partial \theta} d\theta + \frac{\partial Y}{\partial \phi} d\phi, \quad dy = \frac{\partial y}{\partial \theta} d\theta + \frac{\partial y}{\partial \phi} d\phi.$$

$$dX = R \cos \theta d\theta + 0, \quad dx = -r \cos \theta d\theta + l \cos \phi d\phi,$$

$$dY = R \sin \theta d\theta + 0, \quad dy = -r \sin \theta d\theta + l \sin \phi d\phi$$

Constraint relations:

$$c_R(X, Y) = X^2 + Y^2 = R^2 = \text{const.}$$

$$c_l(x_l, y_l) = x_l^2 + y_l^2 = l^2 = \text{const.}$$

$$c_r(x_r, y_r) = x_r^2 + y_r^2 = r^2 = \text{const.}$$

'Raw' Jacobian form

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R \cos \theta & 0 \\ R \sin \theta & 0 \\ -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

Finding a reduced Jacobian form

$$\begin{pmatrix} dX \\ dY \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R \cos \theta & 0 \\ R \sin \theta & 0 \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} \text{ FAILS since: } \det \begin{vmatrix} R \cos \theta & 0 \\ R \sin \theta & 0 \end{vmatrix} = 0$$

FAILS! (Always singular)

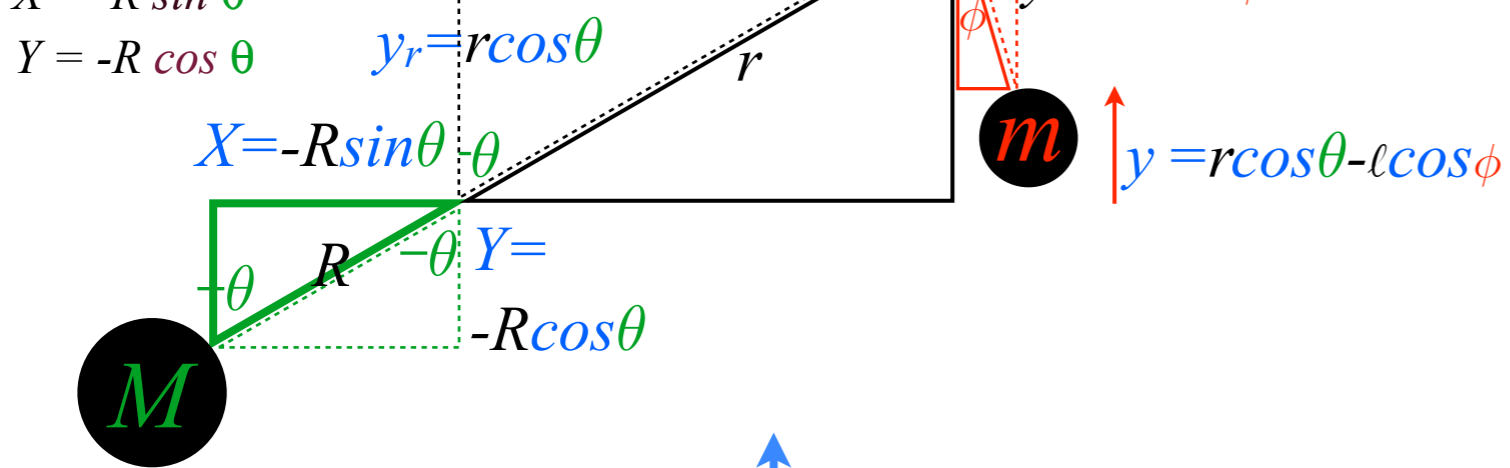
$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

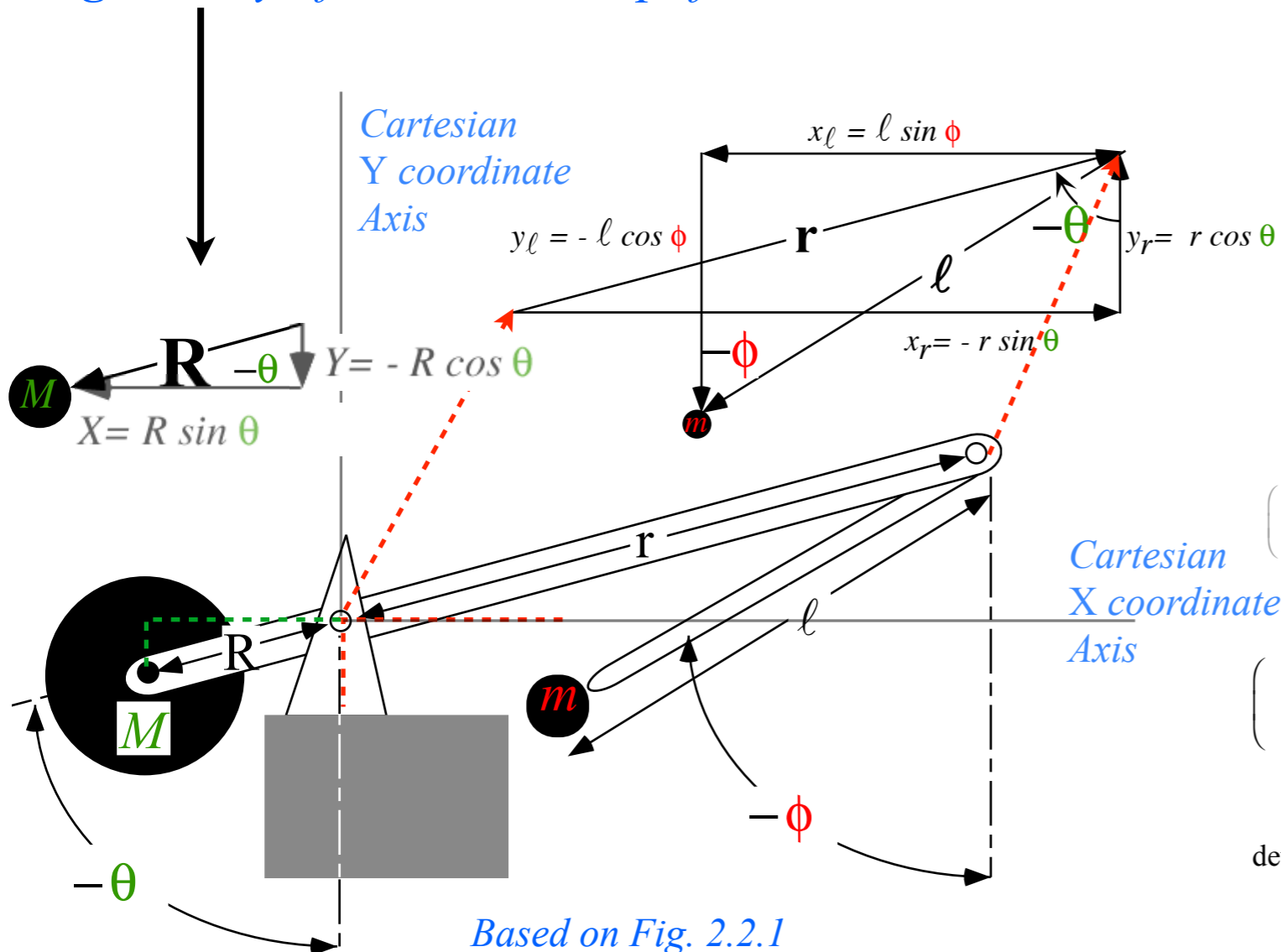
Coordinates of M
(Driving weight Mg):

$$X = R \sin \theta$$

$$Y = -R \cos \theta$$



geometry of trebuchet simplified somewhat...



Based on Fig. 2.2.1

Coordinates of mass m

(Payload or projectile):

$$x = x_r + x_l = -r \sin \theta + l \sin \phi$$

$$y = y_r + y_l = r \cos \theta - l \cos \phi$$

1st differential and Jacobian relations:

$$dX = \frac{\partial X}{\partial \theta} d\theta + \frac{\partial X}{\partial \phi} d\phi, \quad dx = \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial \phi} d\phi,$$

$$dY = \frac{\partial Y}{\partial \theta} d\theta + \frac{\partial Y}{\partial \phi} d\phi, \quad dy = \frac{\partial y}{\partial \theta} d\theta + \frac{\partial y}{\partial \phi} d\phi.$$

$$dX = R \cos \theta d\theta + 0, \quad dx = -r \cos \theta d\theta + l \cos \phi d\phi,$$

$$dY = R \sin \theta d\theta + 0, \quad dy = -r \sin \theta d\theta + l \sin \phi d\phi.$$

Constraint relations:

$$c_R(X, Y) = X^2 + Y^2 = R^2 = \text{const.}$$

$$c_l(x_l, y_l) = x_l^2 + y_l^2 = l^2 = \text{const.}$$

$$c_r(x_r, y_r) = x_r^2 + y_r^2 = r^2 = \text{const.}$$

'Raw' Jacobian form

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R \cos \theta & 0 \\ R \sin \theta & 0 \\ -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

Finding a reduced Jacobian form

$$\begin{pmatrix} dX \\ dY \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R \cos \theta & 0 \\ R \sin \theta & 0 \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} \text{ FAILS since: } \det \begin{vmatrix} R \cos \theta & 0 \\ R \sin \theta & 0 \end{vmatrix} = 0$$

FAILS! (Always singular)

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

$$\det \begin{vmatrix} -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{vmatrix} = -rl \cos \theta \sin \phi + rl \sin \theta \cos \phi = rl \sin(\theta - \phi)$$

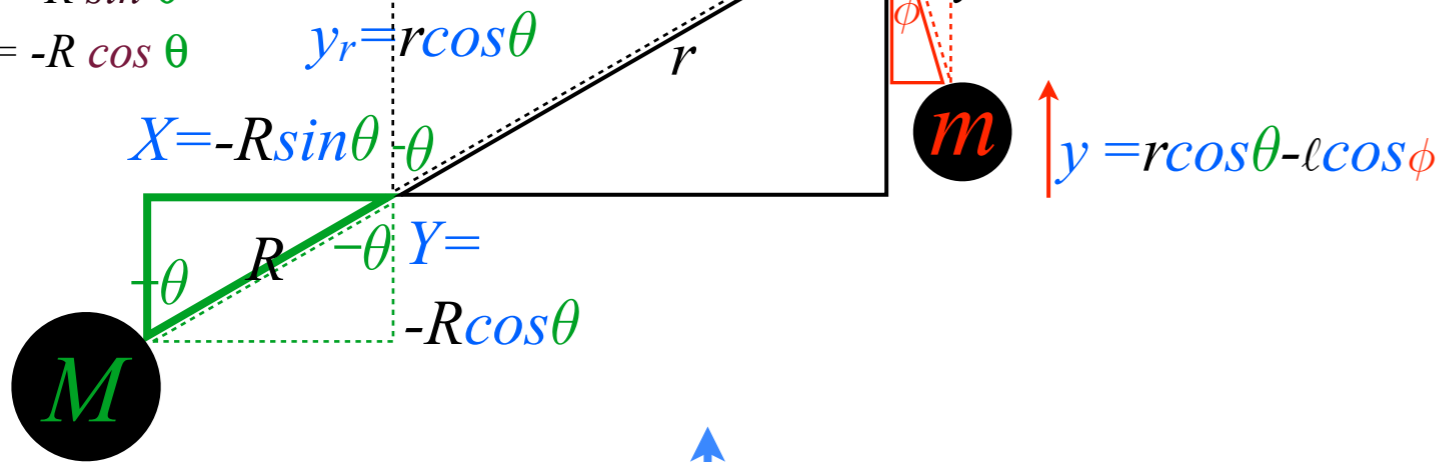
SUCCESS! (Usually non-singular)

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

Coordinates of M
(Driving weight Mg):

$$X = R \sin \theta$$

$$Y = -R \cos \theta$$



geometry of trebuchet simplified somewhat...

Coordinates of mass m

(Payload or projectile):

$$x = x_r + x_\ell = -r \sin \theta + \ell \sin \phi$$

$$y = y_r + y_\ell = r \cos \theta - \ell \cos \phi$$

1st differential and Jacobian relations:

$$dX = \frac{\partial X}{\partial \theta} d\theta + \frac{\partial X}{\partial \phi} d\phi, \quad dx = \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial \phi} d\phi,$$

$$dY = \frac{\partial Y}{\partial \theta} d\theta + \frac{\partial Y}{\partial \phi} d\phi, \quad dy = \frac{\partial y}{\partial \theta} d\theta + \frac{\partial y}{\partial \phi} d\phi.$$

$$dX = R \cos \theta d\theta + 0, \quad dx = -r \cos \theta d\theta + \ell \cos \phi d\phi,$$

$$dY = R \sin \theta d\theta + 0, \quad dy = -r \sin \theta d\theta + \ell \sin \phi d\phi.$$

Constraint relations:

$$c_R(X, Y) = X^2 + Y^2 = R^2 = \text{const.}$$

$$c_\ell(x_\ell, y_\ell) = x_\ell^2 + y_\ell^2 = \ell^2 = \text{const.}$$

$$c_r(x_r, y_r) = x_r^2 + y_r^2 = r^2 = \text{const.}$$

'Raw' Jacobian form

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R \cos \theta & 0 \\ R \sin \theta & 0 \\ -r \cos \theta & \ell \cos \phi \\ -r \sin \theta & \ell \sin \phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

Finding a reduced Jacobian form

$$\begin{pmatrix} dX \\ dY \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R \cos \theta & 0 \\ R \sin \theta & 0 \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} \text{ FAILS since: } \det \begin{vmatrix} R \cos \theta & 0 \\ R \sin \theta & 0 \end{vmatrix} = 0$$

Jacobian FAILS! (Always singular)

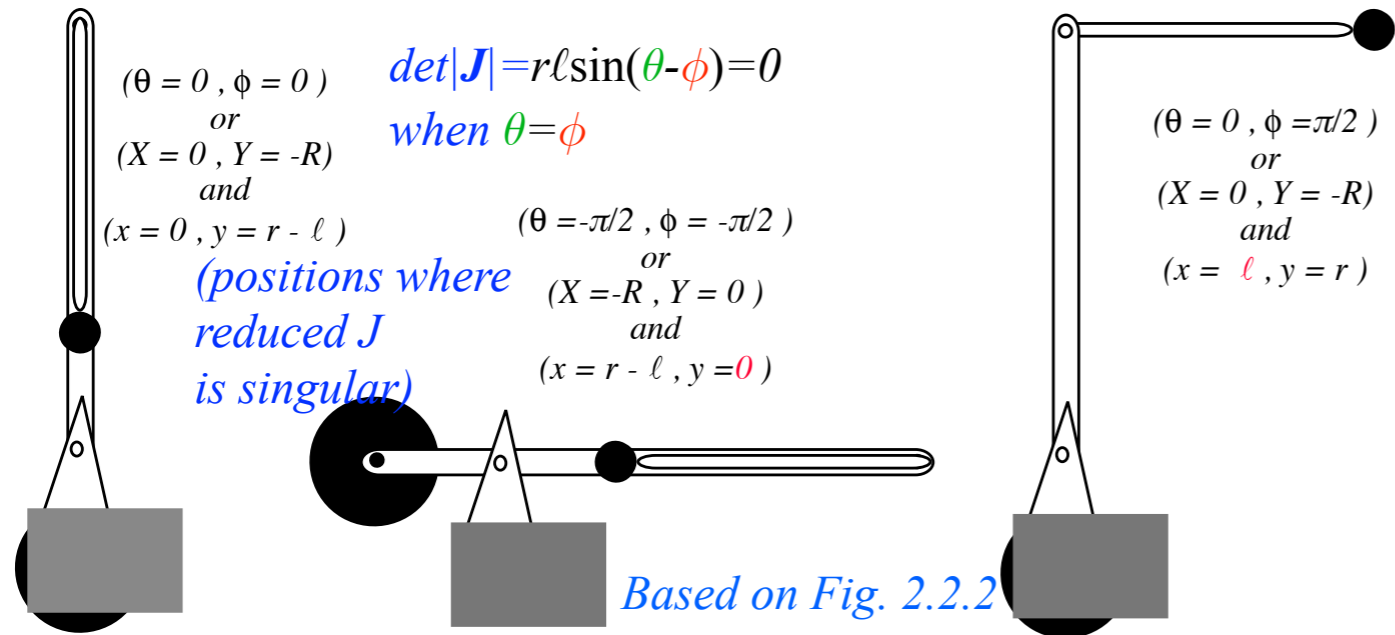
J-matrix

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} -r \cos \theta & \ell \cos \phi \\ -r \sin \theta & \ell \sin \phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

$$\det \begin{vmatrix} -r \cos \theta & \ell \cos \phi \\ -r \sin \theta & \ell \sin \phi \end{vmatrix} = -r\ell \cos \theta \sin \phi + r\ell \sin \theta \cos \phi = r\ell \sin(\theta - \phi)$$

SUCCESS! (Usually non-singular)

Fig. 2.2.2 Singular positions of the trebuchet



$$(\theta = 0, \phi = 0) \quad \det |J| = r\ell \sin(\theta - \phi) = 0$$

or
(X = 0, Y = -R)
and
(x = 0, y = r - l)

(positions where reduced J is singular)

$$(\theta = -\pi/2, \phi = -\pi/2)$$

or
(X = -R, Y = 0)
and
(x = r - l, y = 0)

$$(\theta = 0, \phi = \pi/2)$$

or
(X = 0, Y = -R)
and
(x = l, y = r)

Based on Fig. 2.2.2

Review of Lagrangian equation derivation from Lecture 10 (Now with trebuchet model)

→ *Coordinate geometry, Jacobian, velocity, kinetic energy, and dynamic metric tensor γ_{mn}*

Structure of dynamic metric tensor γ_{mn}

Basic force, work, and acceleration

Lagrangian force equation

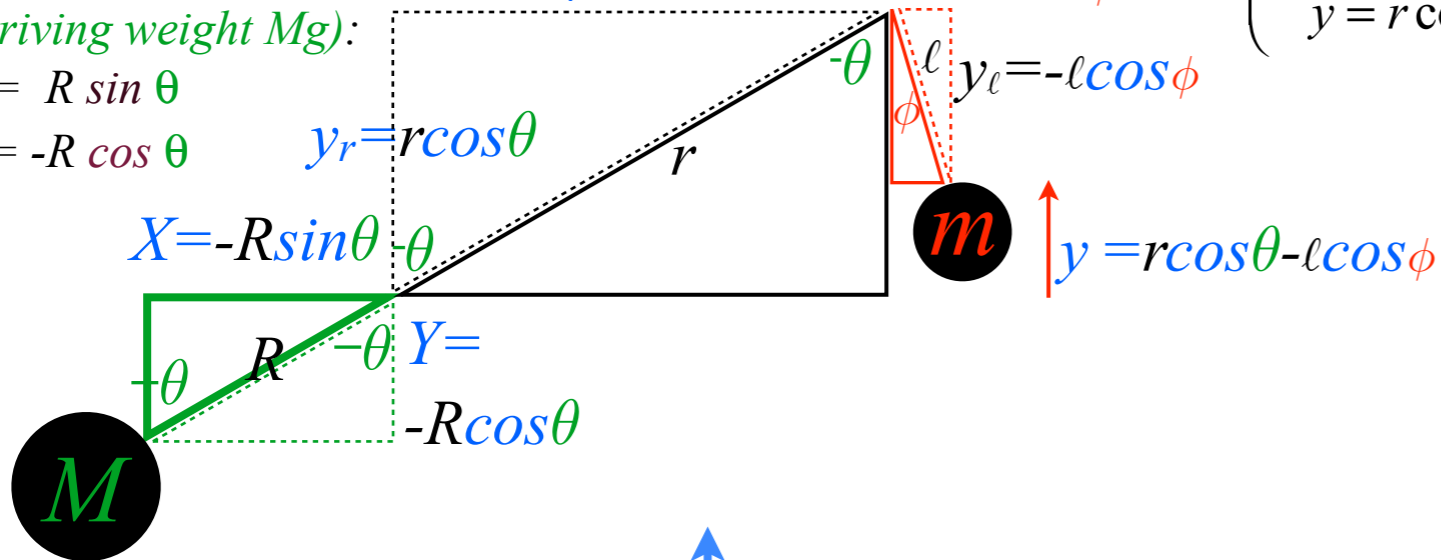
Canonical momentum and γ_{mn} tensor

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

Coordinates of M
(Driving weight Mg):

$$X = R \sin \theta$$

$$Y = -R \cos \theta$$



geometry of trebuchet simplified somewhat...

Coordinates of mass m

(Payload or projectile):

$$x = x_r + x_l = -r \sin \theta + l \sin \phi$$

$$y = y_r + y_l = r \cos \theta - l \cos \phi$$

1st differential and Jacobian relations:

$$dX = \frac{\partial X}{\partial \theta} d\theta + \frac{\partial X}{\partial \phi} d\phi, \quad dx = \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial \phi} d\phi,$$

$$dY = \frac{\partial Y}{\partial \theta} d\theta + \frac{\partial Y}{\partial \phi} d\phi, \quad dy = \frac{\partial y}{\partial \theta} d\theta + \frac{\partial y}{\partial \phi} d\phi.$$

$$dX = R \cos \theta d\theta + 0, \quad dx = -r \cos \theta d\theta + l \cos \phi d\phi$$

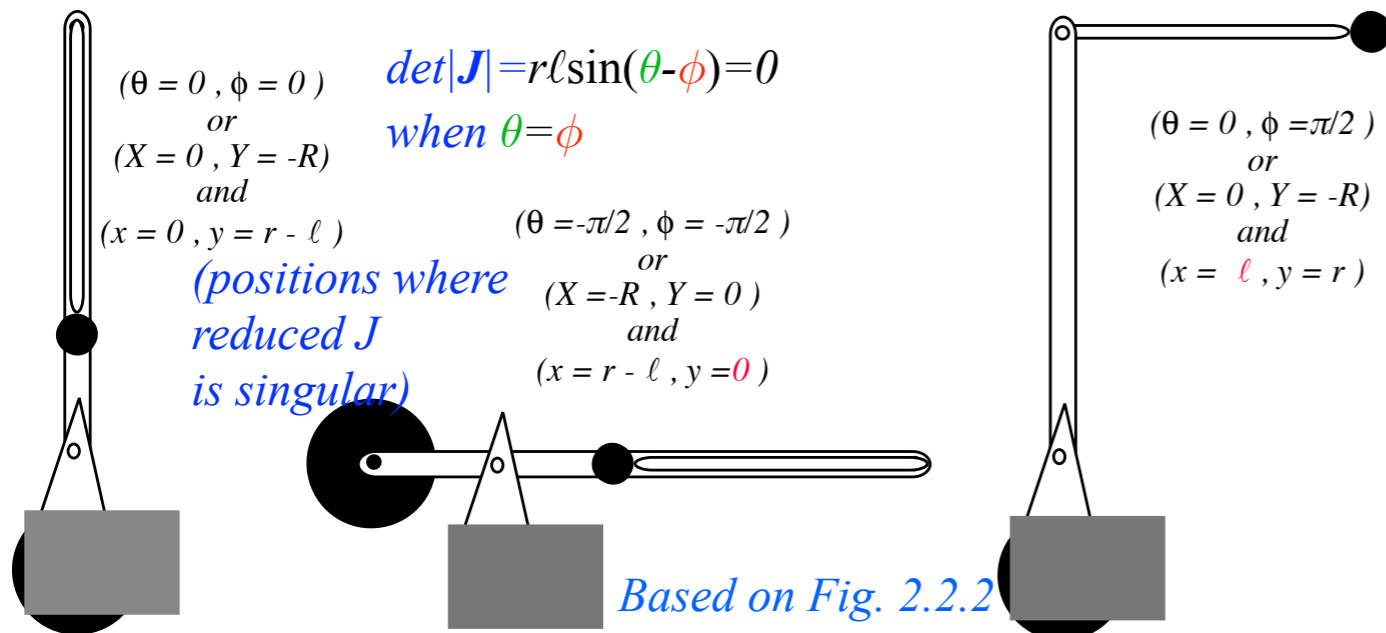
$$dY = R \sin \theta d\theta + 0, \quad dy = -r \sin \theta d\theta + l \sin \phi d\phi$$

GCC Velocity relations:

$$\dot{X} = R \cos \theta \dot{\theta} + 0, \quad \dot{x} = -r \cos \theta \dot{\theta} + l \cos \phi \dot{\phi}$$

$$\dot{Y} = R \sin \theta \dot{\theta} + 0, \quad \dot{y} = -r \sin \theta \dot{\theta} + l \sin \phi \dot{\phi}$$

Fig. 2.2.2 Singular positions of the trebuchet



$(\theta = 0, \phi = 0)$
or
 $(X = 0, Y = -R)$
and
 $(x = 0, y = r - l)$

$\det |J| = r l \sin(\theta - \phi) = 0$
when $\theta = \phi$

$(\theta = -\pi/2, \phi = -\pi/2)$
or
 $(X = -R, Y = 0)$
and
 $(x = r - l, y = 0)$

(positions where reduced J is singular)

$(\theta = 0, \phi = \pi/2)$
or
 $(X = 0, Y = -R)$
and
 $(x = l, y = r)$

Based on Fig. 2.2.2

Jacobian
J-matrix

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

$$\det \begin{vmatrix} -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{vmatrix} = -rl \cos \theta \sin \phi + rl \sin \theta \cos \phi = rl \sin(\theta - \phi)$$

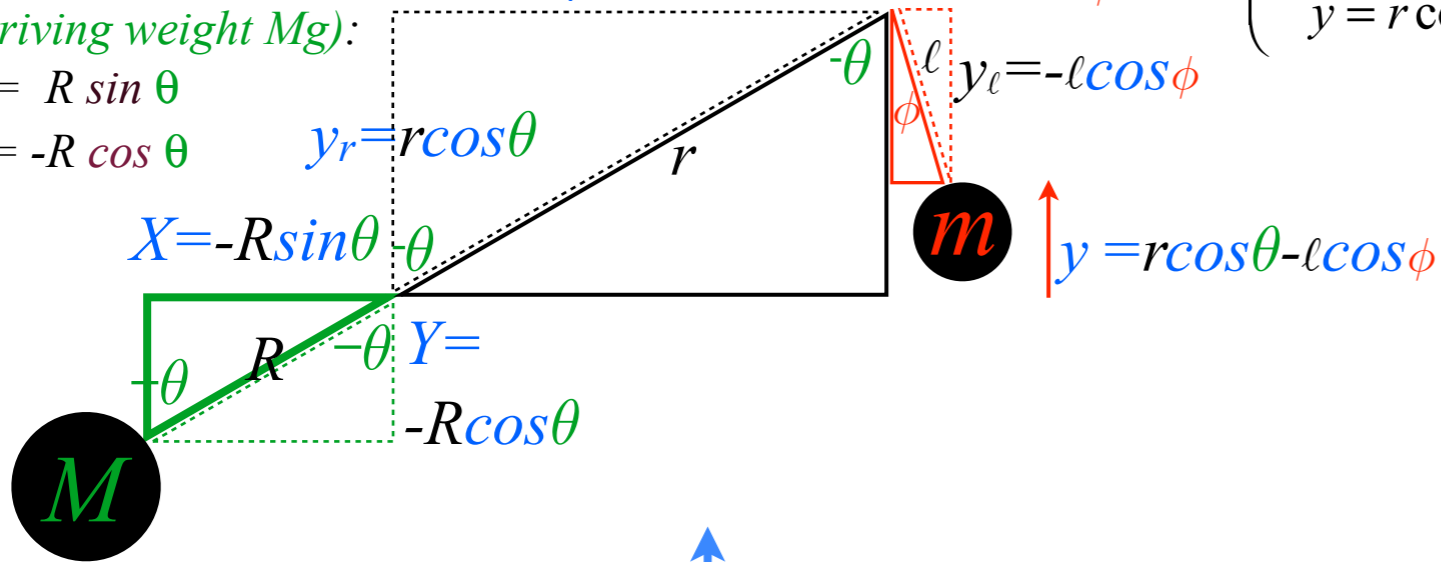
SUCCESS! (Usually non-singular)

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

Coordinates of M
(Driving weight Mg):

$$X = R \sin \theta$$

$$Y = -R \cos \theta$$



geometry of trebuchet simplified somewhat...

Coordinates of mass m

(Payload or projectile):

$$x = x_r + x_l = -r \sin \theta + l \sin \phi$$

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$$dX = R \cos \theta d\theta + 0, \quad dx = -r \cos \theta d\theta + l \cos \phi d\phi$$

$$dY = R \sin \theta d\theta + 0, \quad dy = -r \sin \theta d\theta + l \sin \phi d\phi$$

GCC Velocity relations:

$$\dot{X} = R \cos \theta \dot{\theta} + 0, \quad \dot{x} = -r \cos \theta \dot{\theta} + l \cos \phi \dot{\phi}$$

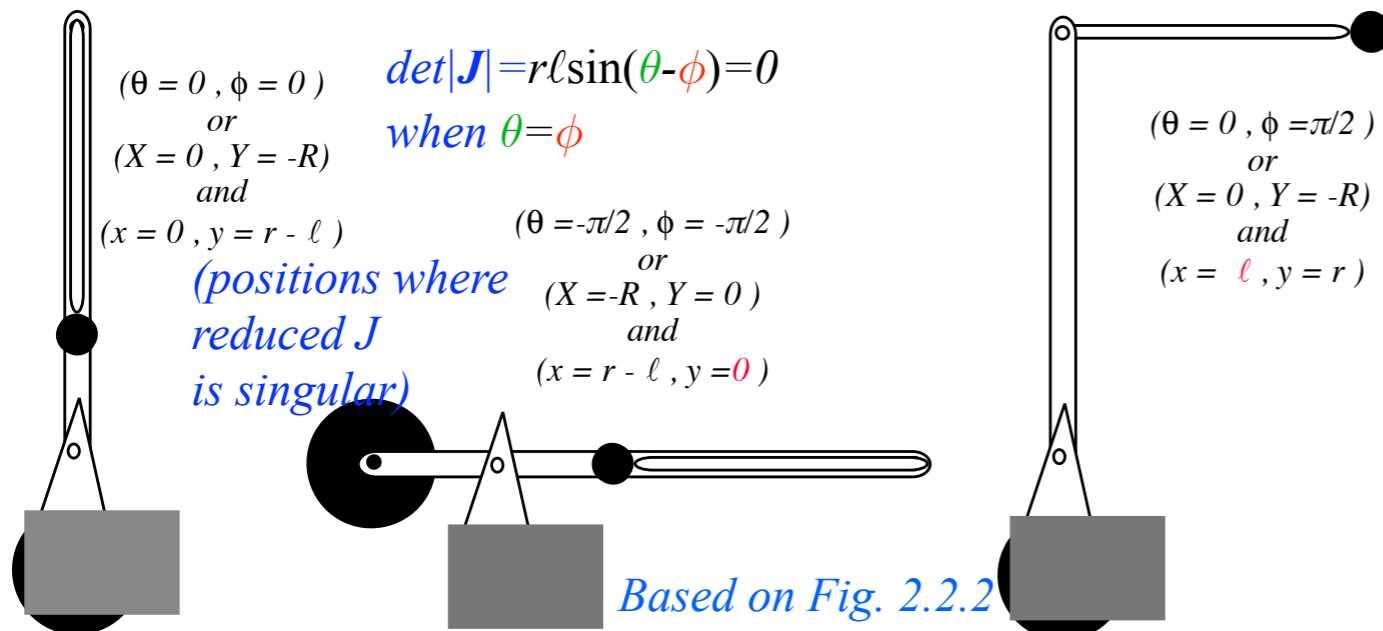
$$\dot{Y} = R \sin \theta \dot{\theta} + 0, \quad \dot{y} = -r \sin \theta \dot{\theta} + l \sin \phi \dot{\phi}$$

Jacobian \mathbf{J} -matrix velocity relations:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

\mathbf{J} -matrix

Fig. 2.2.2 Singular positions of the trebuchet



Jacobian \mathbf{J} -matrix

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

$$\det \begin{vmatrix} -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{vmatrix} = -rl \cos \theta \sin \phi + rl \sin \theta \cos \phi = rl \sin(\theta - \phi)$$

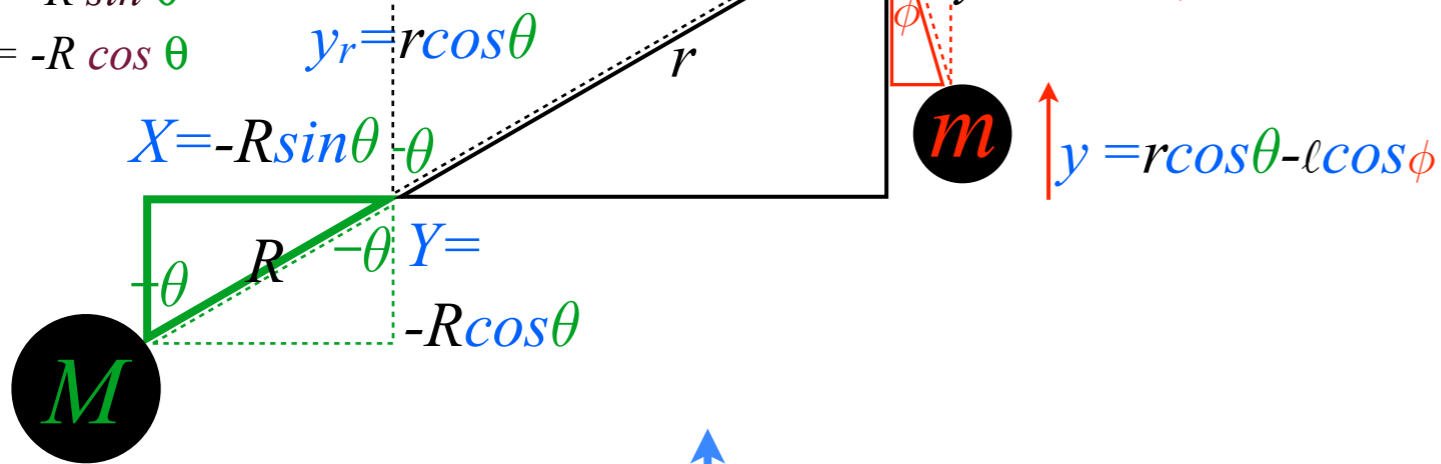
SUCCESS! (Usually non-singular)

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

Coordinates of M
(Driving weight Mg):

$$X = R \sin \theta$$

$$Y = -R \cos \theta$$



geometry of trebuchet simplified somewhat...

Coordinates of mass m

(Payload or projectile):

$$x = x_r + x_l = -r \sin \theta + l \sin \phi$$

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1st differential and Jacobian relations:

$$dX = \frac{\partial X}{\partial \theta} d\theta + \frac{\partial X}{\partial \phi} d\phi, \quad dx = \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial \phi} d\phi,$$

$$dY = \frac{\partial Y}{\partial \theta} d\theta + \frac{\partial Y}{\partial \phi} d\phi, \quad dy = \frac{\partial y}{\partial \theta} d\theta + \frac{\partial y}{\partial \phi} d\phi.$$

$$dX = R \cos \theta d\theta + 0, \quad dx = -r \cos \theta d\theta + l \cos \phi d\phi$$

$$dY = R \sin \theta d\theta + 0, \quad dy = -r \sin \theta d\theta + l \sin \phi d\phi$$

GCC Velocity relations:

$$\dot{X} = R \cos \theta \dot{\theta} + 0, \quad \dot{x} = -r \cos \theta \dot{\theta} + l \cos \phi \dot{\phi}$$

$$\dot{Y} = R \sin \theta \dot{\theta} + 0, \quad \dot{y} = -r \sin \theta \dot{\theta} + l \sin \phi \dot{\phi}$$

Jacobian J -matrix velocity relations:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Transpose
Jacobian J^T -matrix velocity relations:

$$\begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix} = \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} -r \cos \theta & -r \sin \theta \\ l \cos \phi & l \sin \phi \end{pmatrix}$$

J^T -matrix

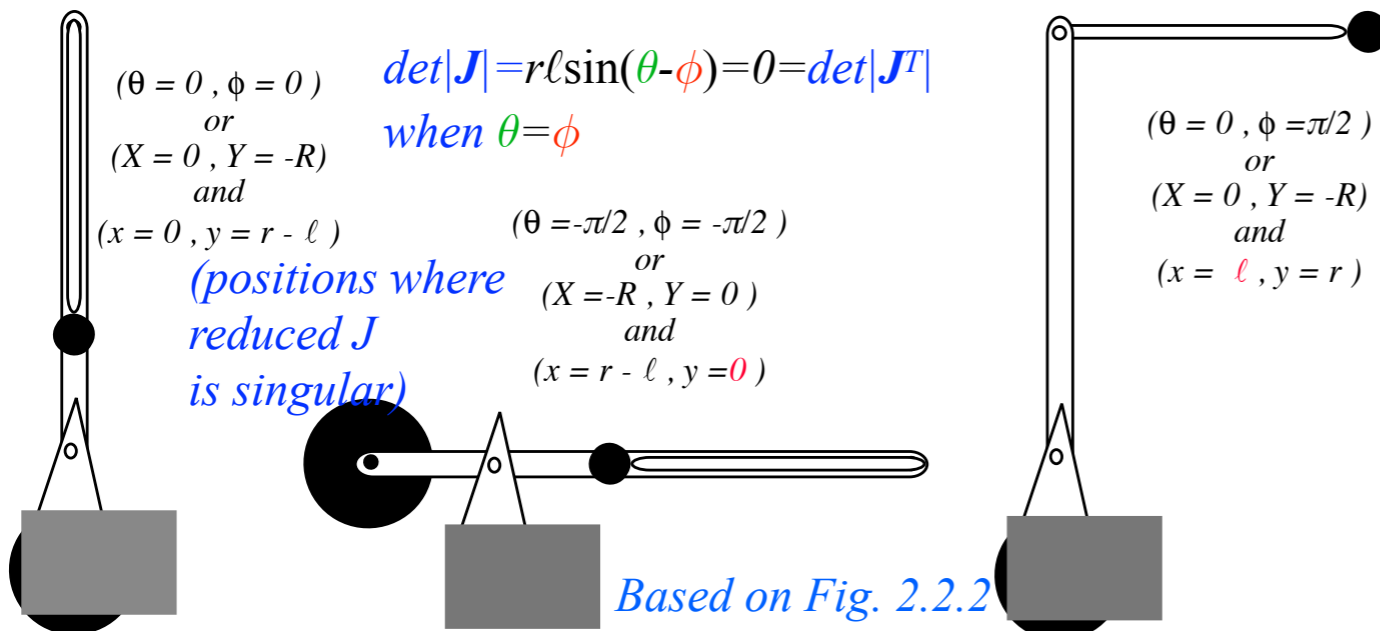
Jacobian J -matrix

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

$$\det \begin{vmatrix} -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{vmatrix} = -rl \cos \theta \sin \phi + rl \sin \theta \cos \phi = rl \sin(\theta - \phi)$$

SUCCESS! (Usually non-singular)

Fig. 2.2.2 Singular positions of the trebuchet



$$\det |J| = rl \sin(\theta - \phi) = 0 = \det |J^T|$$

when $\theta = \phi$

(positions where reduced J is singular)

Based on Fig. 2.2.2

Review of Lagrangian equation derivation from Lecture 10 (Now with trebuchet model)

→ *Coordinate geometry, Jacobian, velocity, kinetic energy, and dynamic metric tensor γ_{mn}*

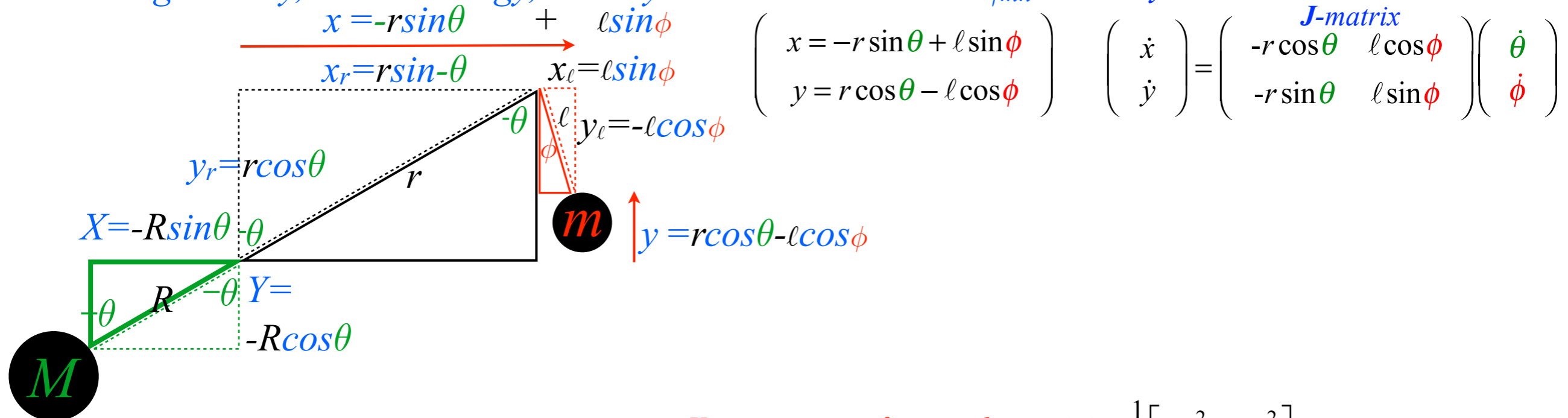
Structure of dynamic metric tensor γ_{mn}

Basic force, work, and acceleration

Lagrangian force equation

Canonical momentum and γ_{mn} tensor

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn} Velocity, Jacobian, and KE

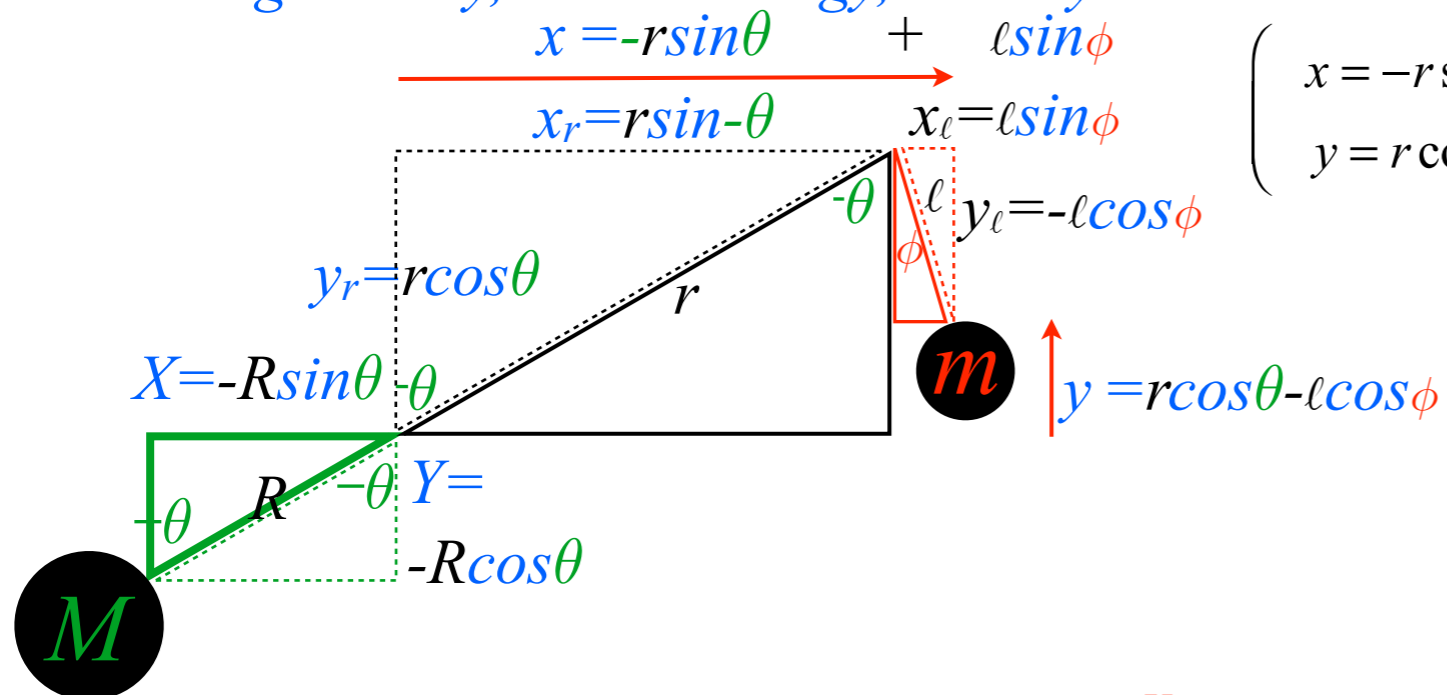


Kinetic energy of driver M

$$T(M) = \frac{1}{2} M \dot{X}^2 + \frac{1}{2} M \dot{Y}^2$$

Kinetic energy of projectile m $T(m) = \frac{1}{2} [m \dot{x}^2 + m \dot{y}^2]$

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn} Velocity, Jacobian, and KE



$$\begin{pmatrix} x = -r \sin \theta + l \sin \phi \\ y = r \cos \theta - l \cos \phi \end{pmatrix} \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

J-matrix

$$\begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix} = \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} -r \cos \theta & -r \sin \theta \\ l \cos \phi & l \sin \phi \end{pmatrix}$$

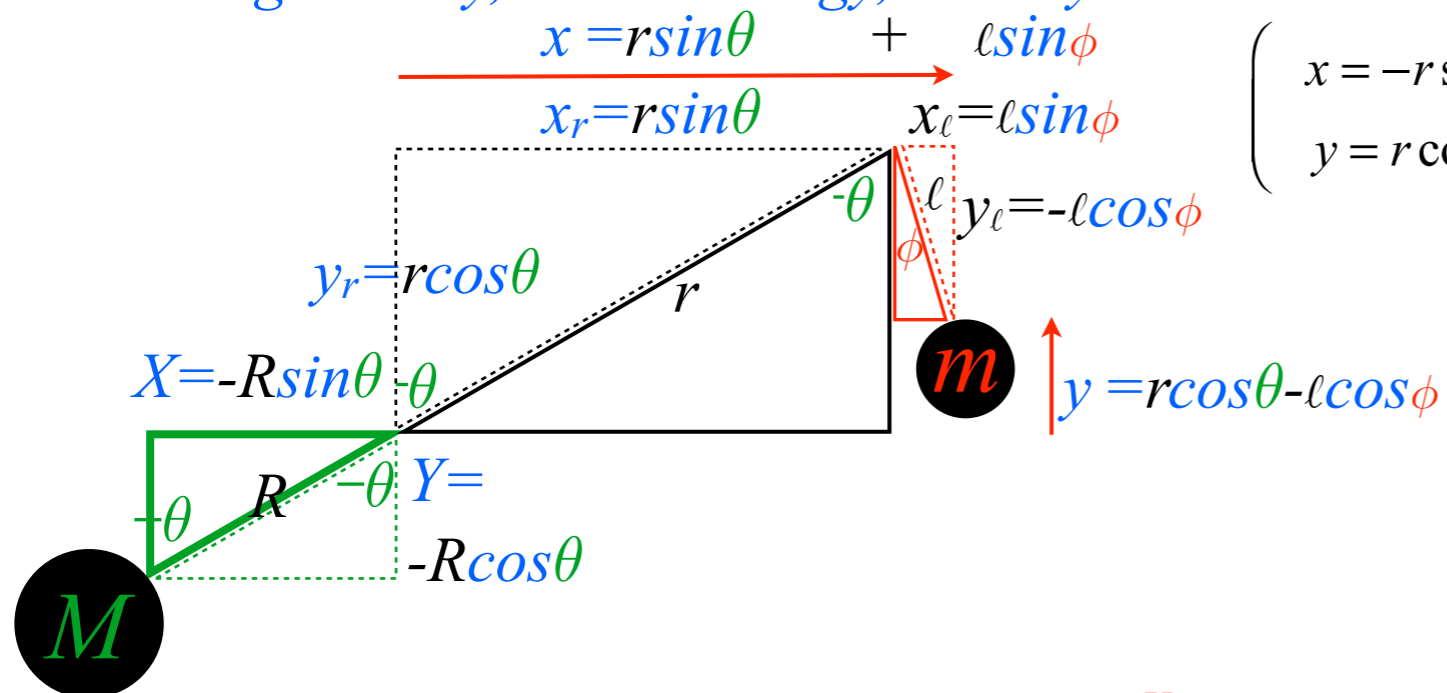
J^T-matrix

Kinetic energy of driver M

$$T(M) = \frac{1}{2} M \dot{X}^2 + \frac{1}{2} M \dot{Y}^2$$

Kinetic energy of projectile m $T(m) = \frac{1}{2} [m \dot{x}^2 + m \dot{y}^2] = \frac{1}{2} m \begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn} Velocity, Jacobian, and KE



$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -r \sin \theta + l \sin \phi \\ r \cos \theta - l \cos \phi \end{pmatrix} \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

J-matrix

$$\begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix} = \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} -r \cos \theta & -r \sin \theta \\ l \cos \phi & l \sin \phi \end{pmatrix}$$

J^T-matrix

Kinetic energy of driver M

$$T(M) = \frac{1}{2} M \dot{X}^2 + \frac{1}{2} M \dot{Y}^2$$

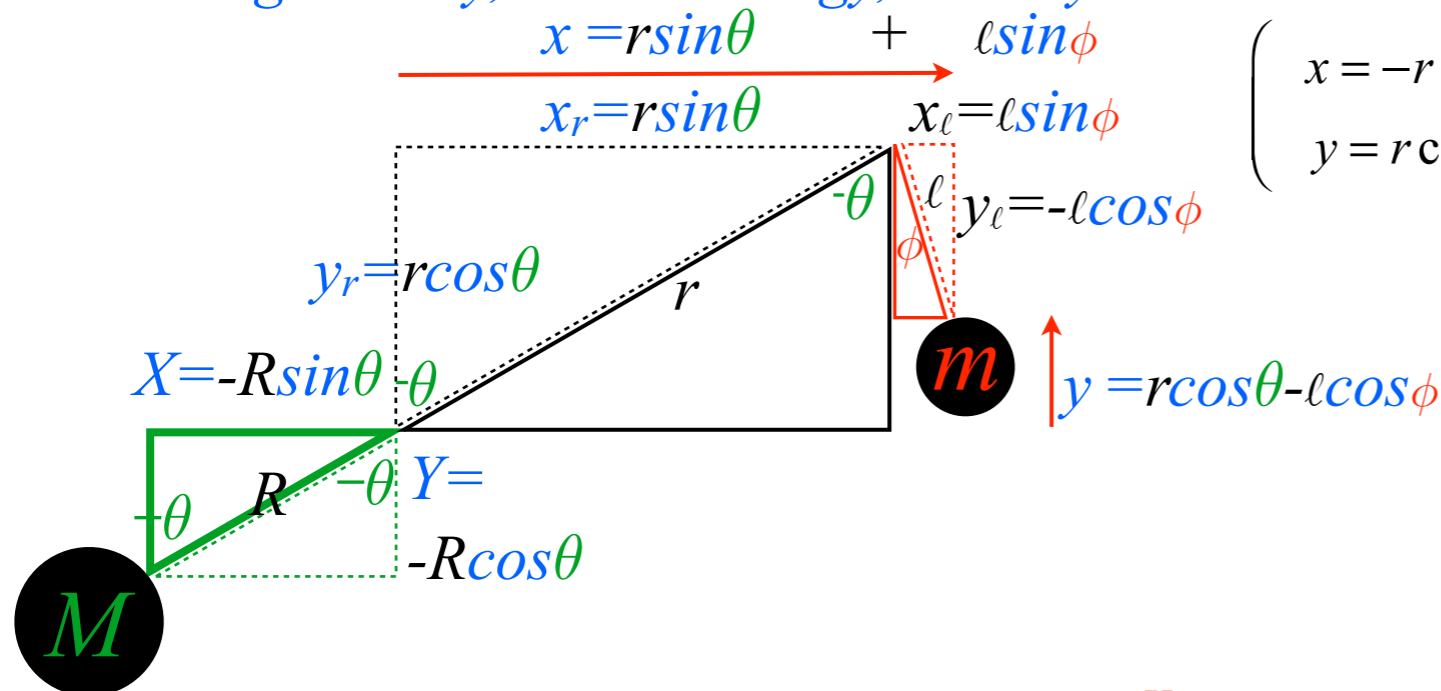
Kinetic energy of projectile m

$$T(m) = \frac{1}{2} [m \dot{x}^2 + m \dot{y}^2] = \frac{1}{2} m \begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$

$$T(m) = \frac{1}{2} m \begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix} \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} -r \cos \theta & -r \sin \theta \\ l \cos \phi & l \sin \phi \end{pmatrix} \begin{pmatrix} -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

J^T-matrix *J-matrix*

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn} Velocity, Jacobian, and KE



$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

J-matrix

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} -r \cos \theta & -r \sin \theta \\ l \cos \phi & l \sin \phi \end{pmatrix}$$

J^T-matrix

Kinetic energy of driver M

$$T(M) = \frac{1}{2} M \dot{X}^2 + \frac{1}{2} M \dot{Y}^2$$

$$= \frac{1}{2} M R^2 \dot{\theta}^2$$

Kinetic energy of projectile m

$$T(m) = \frac{1}{2} [m \dot{x}^2 + m \dot{y}^2] = \frac{1}{2} m \begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$

$$T(m) = \frac{1}{2} m \begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{2} m \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} -r \cos \theta & -r \sin \theta \\ l \cos \phi & l \sin \phi \end{pmatrix} \begin{pmatrix} -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

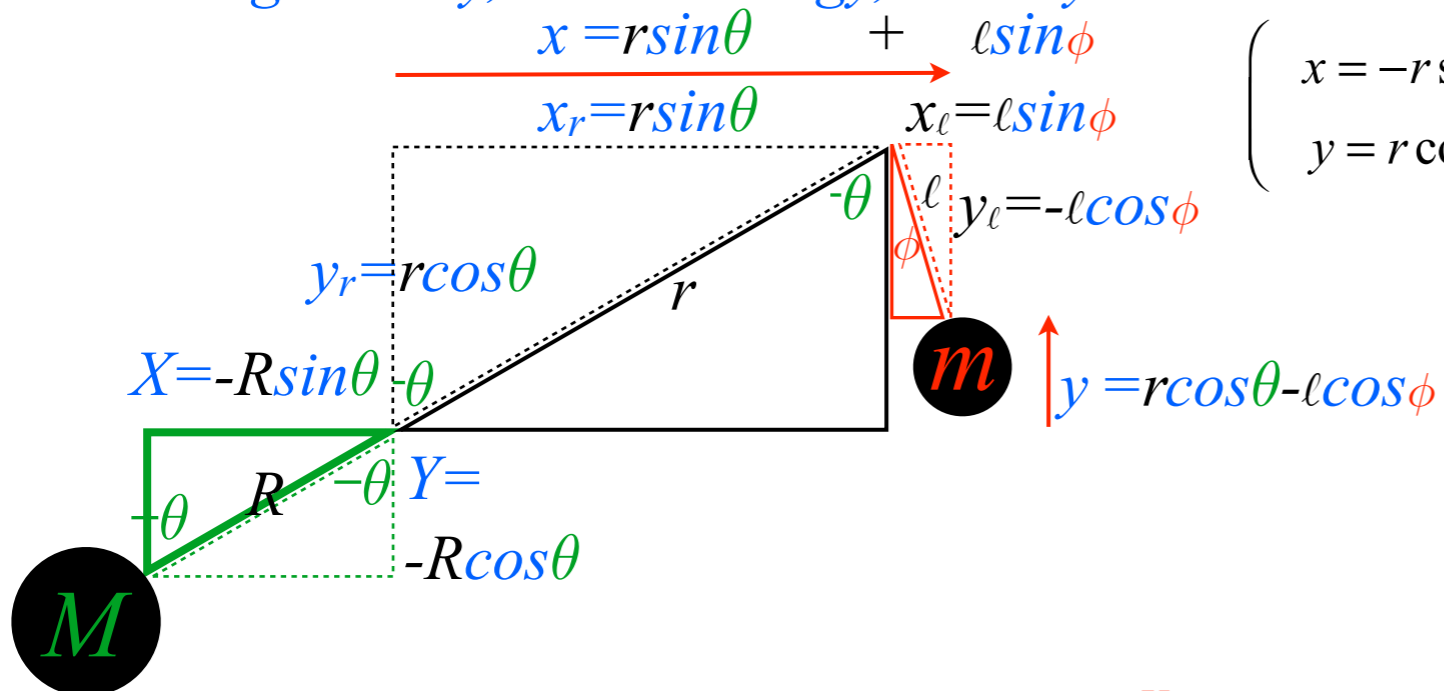
$$\text{Total KE} = T = T(M) + T(m) = \frac{1}{2} \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} M R^2 + m r^2 & -m r l \cos(\theta - \phi) \\ -m r l \cos(\theta - \phi) & m l^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

$$\text{Total KE} = T = \frac{1}{2} [(M R^2 + m r^2) \dot{\theta}^2 - 2 m r l \cos(\theta - \phi) \dot{\theta} \dot{\phi} + m l^2 \dot{\phi}^2]$$

$$T = \frac{1}{2} [M \dot{X}^2 + M \dot{Y}^2 + m \dot{x}^2 + m \dot{y}^2]$$

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

Velocity, Jacobian, and KE



$$\begin{pmatrix} x = -r \sin \theta + l \sin \phi \\ y = r \cos \theta - l \cos \phi \end{pmatrix}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

'Raw' Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R \cos \theta & 0 \\ R \sin \theta & 0 \\ -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

Kinetic energy of driver M

Kinetic energy of projectile m

$$T(M) = \frac{1}{2} M \dot{X}^2 + \frac{1}{2} M \dot{Y}^2$$

$$T(m) = \frac{1}{2} m \begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{2} m \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} -r \cos \theta & -r \sin \theta \\ l \cos \phi & l \sin \phi \end{pmatrix} \begin{pmatrix} -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

$$= \frac{1}{2} M R^2 \dot{\theta}^2$$

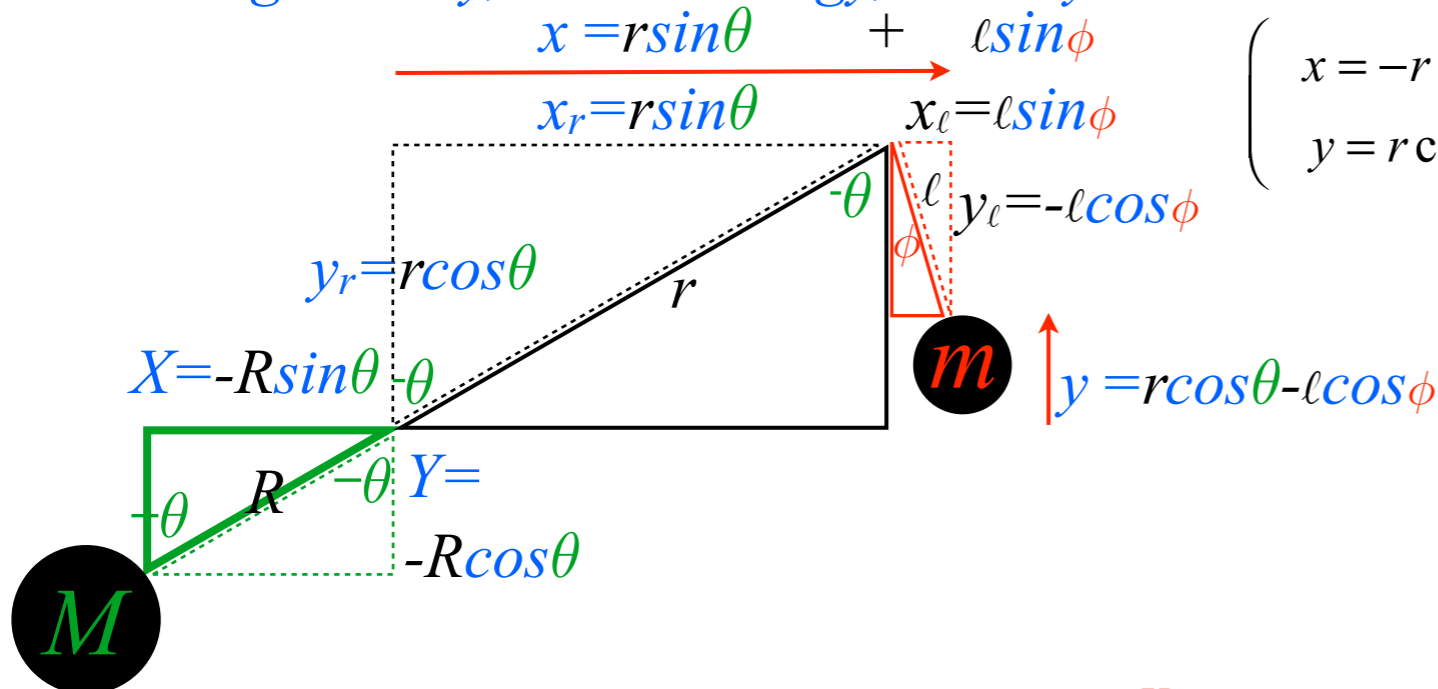
$$\text{Total KE} = T = T(M) + T(m) = \frac{1}{2} \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} M R^2 + m r^2 & -m r l \cos(\theta - \phi) \\ -m r l \cos(\theta - \phi) & m l^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

$$\text{Total KE} = T = \frac{1}{2} \left[(M R^2 + m r^2) \dot{\theta}^2 - 2 m r l \cos(\theta - \phi) \dot{\theta} \dot{\phi} + m l^2 \dot{\phi}^2 \right]$$

$$T = \frac{1}{2} \left[M \dot{X}^2 + M \dot{Y}^2 + m \dot{x}^2 + m \dot{y}^2 \right]$$

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

Velocity, Jacobian, and KE



$$\begin{pmatrix} x = -r \sin \theta + l \sin \phi \\ y = r \cos \theta - l \cos \phi \end{pmatrix}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Raw Jacobian

$$\begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R \cos \theta & 0 \\ R \sin \theta & 0 \\ -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

(x,y) to (theta, phi)

Jacobian

Kinetic energy of driver M

Kinetic energy of projectile m $T(m) = \frac{1}{2} [m\dot{x}^2 + m\dot{y}^2]$

$$T(M) = \frac{1}{2} M\dot{X}^2 + \frac{1}{2} M\dot{Y}^2$$

$$T(m) = \frac{1}{2} m \begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{2} m \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} -r \cos \theta & -r \sin \theta \\ l \cos \phi & l \sin \phi \end{pmatrix} \begin{pmatrix} -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

$$= \frac{1}{2} MR^2 \dot{\theta}^2$$

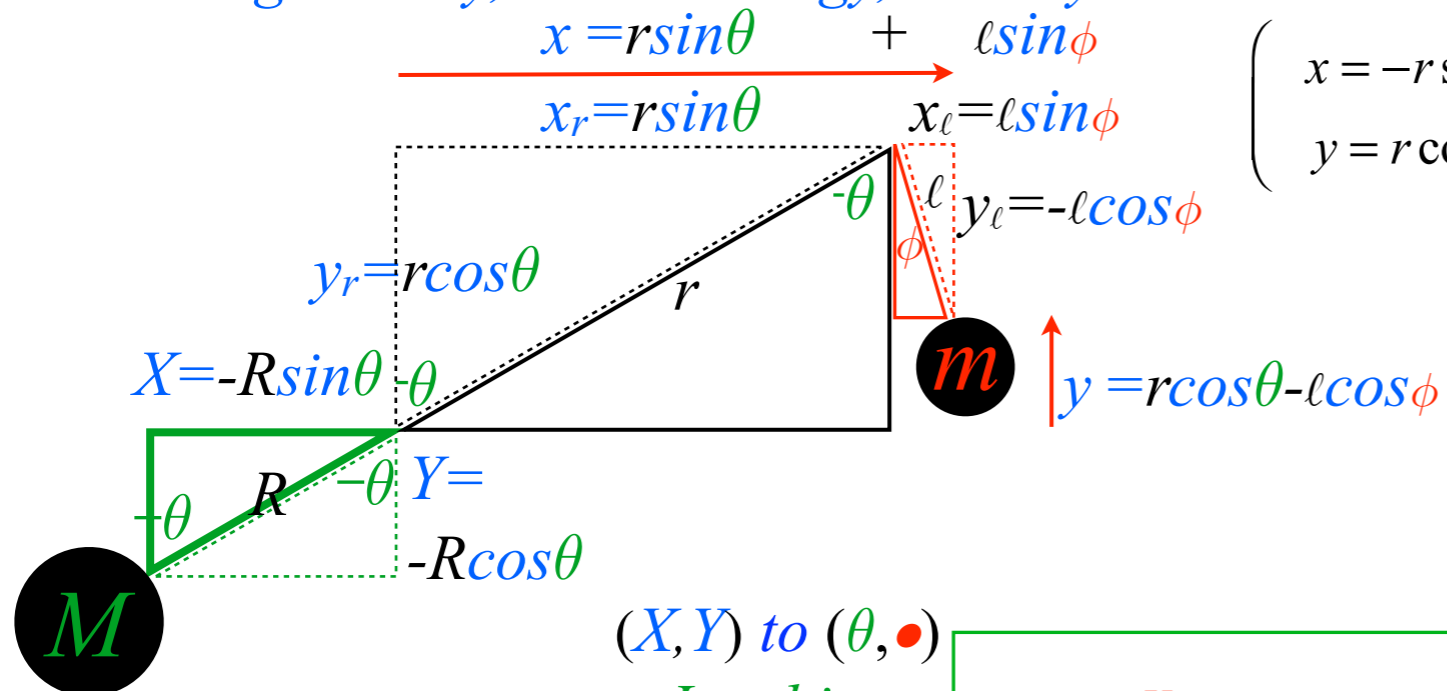
$$\text{Total KE} = T = T(M) + T(m) = \frac{1}{2} \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} MR^2 + mr^2 & -mr l \cos(\theta - \phi) \\ -mr l \cos(\theta - \phi) & ml^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

$$\text{Total KE} = T = \frac{1}{2} [(MR^2 + mr^2) \dot{\theta}^2 - 2mr l \cos(\theta - \phi) \dot{\theta} \dot{\phi} + ml^2 \dot{\phi}^2]$$

$$T = \frac{1}{2} [M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2]$$

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

Velocity, Jacobian, and KE



$$\begin{pmatrix} x = -r \sin \theta + l \sin \phi \\ y = r \cos \theta - l \cos \phi \end{pmatrix}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R \cos \theta & 0 \\ R \sin \theta & 0 \\ -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

(x,y) to (theta, phi)

(X,Y) to (theta, phi) Jacobian

Kinetic energy of driver M

Kinetic energy of projectile m $T(m) = \frac{1}{2} [m\dot{x}^2 + m\dot{y}^2]$

$$T(M) = \frac{1}{2} M \dot{X}^2 + \frac{1}{2} M \dot{Y}^2 = \frac{1}{2} M R^2 \dot{\theta}^2$$

$$T(m) = \frac{1}{2} m \begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{2} m \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} -r \cos \theta & -r \sin \theta \\ l \cos \phi & l \sin \phi \end{pmatrix} \begin{pmatrix} -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

$$\text{Total KE} = T = T(M) + T(m) = \frac{1}{2} \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} M R^2 + m r^2 & -m r l \cos(\theta - \phi) \\ -m r l \cos(\theta - \phi) & m l^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

$$\text{Total KE} = T = \frac{1}{2} [(M R^2 + m r^2) \dot{\theta}^2 - 2 m r l \cos(\theta - \phi) \dot{\theta} \dot{\phi} + m l^2 \dot{\phi}^2]$$

$$T = \frac{1}{2} [M \dot{X}^2 + M \dot{Y}^2 + m \dot{x}^2 + m \dot{y}^2]$$

Review of Lagrangian equation derivation from Lecture 10 (Now with trebuchet model)

→ *Coordinate geometry, Jacobian, velocity, kinetic energy, and dynamic metric tensor γ_{mn}*

Structure of dynamic metric tensor γ_{mn}

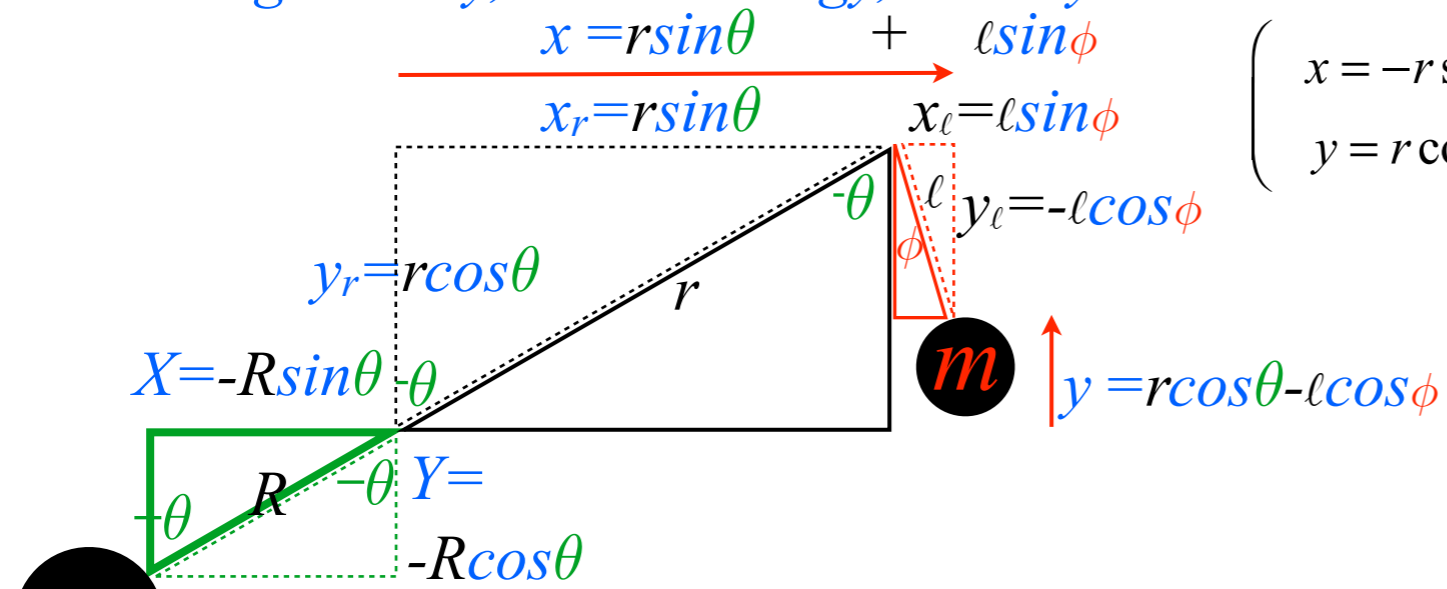
Basic force, work, and acceleration

Lagrangian force equation

Canonical momentum and γ_{mn} tensor

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

Velocity, Jacobian, and KE



$$\begin{pmatrix} x = -r \sin \theta + l \sin \phi \\ y = r \cos \theta - l \cos \phi \end{pmatrix}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Raw Jacobian

$$\begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R \cos \theta & 0 \\ R \sin \theta & 0 \\ -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

(x,y) to (theta, phi)

(X,Y) to (theta, phi) Jacobian

Kinetic energy of driver M

Kinetic energy of projectile m $T(m) = \frac{1}{2} [m\dot{x}^2 + m\dot{y}^2]$

$$T(M) = \frac{1}{2} M \dot{X}^2 + \frac{1}{2} M \dot{Y}^2 = \frac{1}{2} M R^2 \dot{\theta}^2$$

JT-matrix

$$T(m) = \frac{1}{2} m \begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{2} m \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} -r \cos \theta & -r \sin \theta \\ l \cos \phi & l \sin \phi \end{pmatrix} \begin{pmatrix} -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

J-matrix

Total KE = T = T(M) + T(m)

$$= \frac{1}{2} \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} M R^2 + m r^2 & -m r l \cos(\theta - \phi) \\ -m r l \cos(\theta - \phi) & m l^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

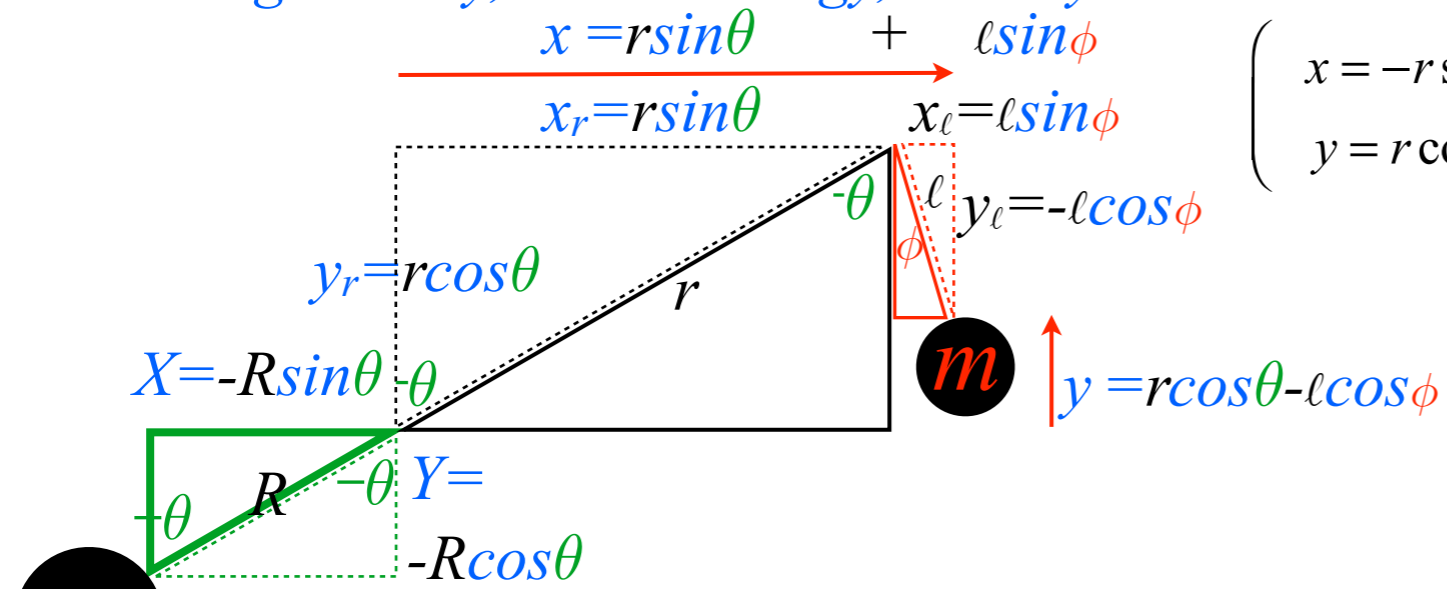
$$\begin{pmatrix} \gamma_{X,X} & & & \\ & \gamma_{Y,Y} & & \\ & & \gamma_{x,x} & \\ & & & \gamma_{y,y} \end{pmatrix} = \begin{pmatrix} M & & & \\ & M & & \\ & & m & \\ & & & m \end{pmatrix}$$

$$Total\ KE = T = \frac{1}{2} [M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2]$$

Dynamic metric tensor γ_{mn} in raw Cartesian X,Y and x,y

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

Velocity, Jacobian, and KE



$$\begin{pmatrix} x = -r \sin \theta + l \sin \phi \\ y = r \cos \theta - l \cos \phi \end{pmatrix}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Raw Jacobian

$$\begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R \cos \theta & 0 \\ R \sin \theta & 0 \\ -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

(x,y) to (theta, phi)

(X,Y) to (theta, phi) Jacobian

Kinetic energy of driver M

Kinetic energy of projectile m $T(m) = \frac{1}{2} [m\dot{x}^2 + m\dot{y}^2]$

$$T(M) = \frac{1}{2} M \dot{X}^2 + \frac{1}{2} M \dot{Y}^2 = \frac{1}{2} M R^2 \dot{\theta}^2$$

JT-matrix

$$T(m) = \frac{1}{2} m \begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{2} m \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} -r \cos \theta & -r \sin \theta \\ l \cos \phi & l \sin \phi \end{pmatrix} \begin{pmatrix} -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

J-matrix

Total KE = T = T(M) + T(m)

$$= \frac{1}{2} \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} MR^2 + mr^2 & -mrl \cos(\theta - \phi) \\ -mrl \cos(\theta - \phi) & ml^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

$$\begin{pmatrix} \gamma_{X,X} & & & \\ & \gamma_{Y,Y} & & \\ & & \gamma_{x,x} & \\ & & & \gamma_{y,y} \end{pmatrix} = \begin{pmatrix} M & & & \\ & M & & \\ & & m & \\ & & & m \end{pmatrix}$$

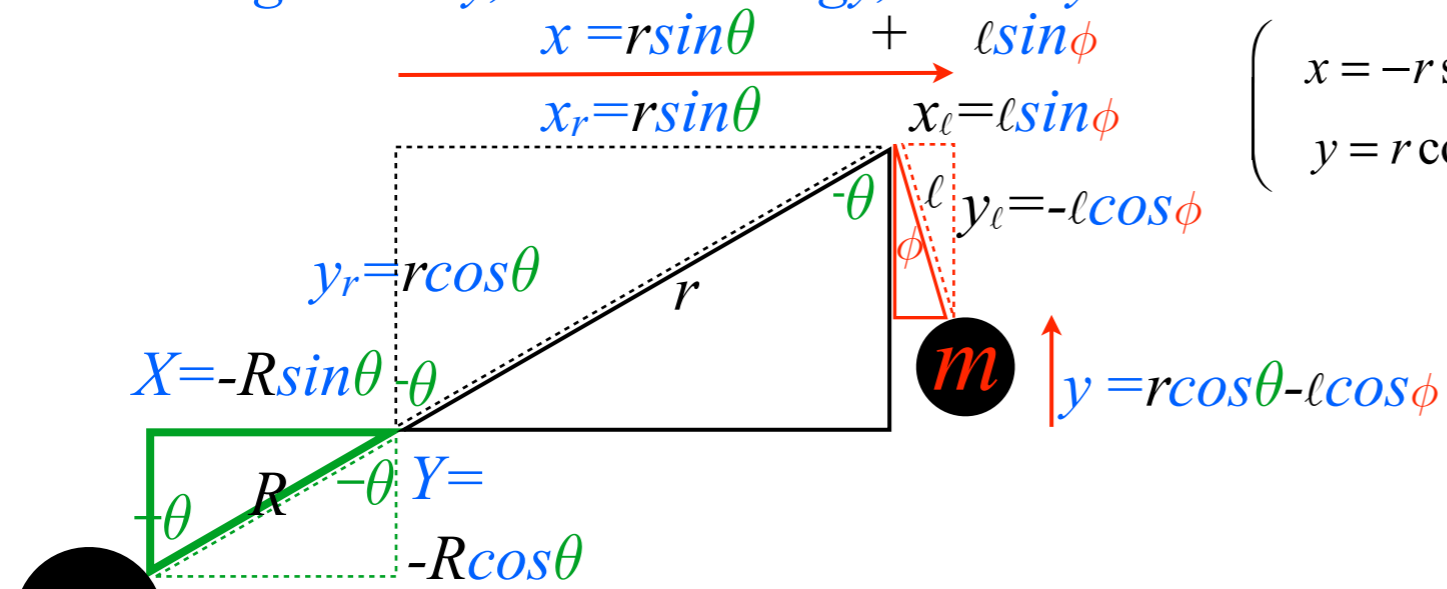
$$Total\ KE = T = \frac{1}{2} [M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2]$$

$$T = \frac{1}{2} [(MR^2 + mr^2)\dot{\theta}^2 - 2mrl \cos(\theta - \phi) \dot{\theta} \dot{\phi} + ml^2 \dot{\phi}^2]$$

Dynamic metric tensor γ_{mn} in raw Cartesian X,Y and x,y

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

Velocity, Jacobian, and KE



$$\begin{pmatrix} x = -r \sin \theta + l \sin \phi \\ y = r \cos \theta - l \cos \phi \end{pmatrix}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R \cos \theta & 0 \\ R \sin \theta & 0 \\ -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

(x,y) to (theta, phi)

M
Kinetic energy of driver M

(X,Y) to (theta, phi)
Jacobian

Kinetic energy of projectile m $T(m) = \frac{1}{2} [m\dot{x}^2 + m\dot{y}^2]$

$$T(M) = \frac{1}{2} M\dot{X}^2 + \frac{1}{2} M\dot{Y}^2$$

$$T(m) = \frac{1}{2} m \begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{2} m \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} -r \cos \theta & -r \sin \theta \\ l \cos \phi & l \sin \phi \end{pmatrix} \begin{pmatrix} -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

$$T(M) = \frac{1}{2} MR^2 \dot{\theta}^2$$

$$\text{Total KE} = T = T(M) + T(m) = \frac{1}{2} \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} MR^2 + mr^2 & -mrl \cos(\theta - \phi) \\ -mrl \cos(\theta - \phi) & ml^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

$$\begin{pmatrix} \gamma_{X,X} & & & \\ & \gamma_{Y,Y} & & \\ & & \gamma_{x,x} & \\ & & & \gamma_{y,y} \end{pmatrix} = \begin{pmatrix} M & & & \\ & M & & \\ & & m & \\ & & & m \end{pmatrix}$$

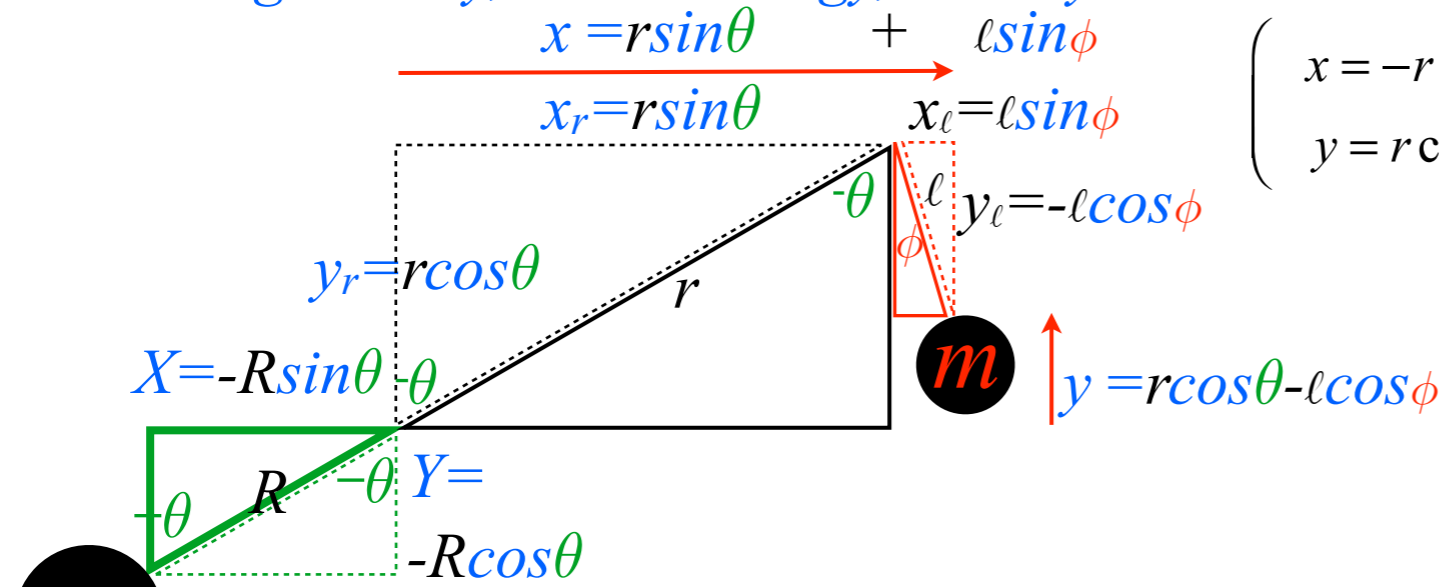
$$\text{Total KE} = T = \frac{1}{2} [M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2]$$

$$T = \frac{1}{2} [(MR^2 + mr^2)\dot{\theta}^2 - 2mrl \cos(\theta - \phi) \dot{\theta} \dot{\phi} + ml^2 \dot{\phi}^2] = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

Dynamic metric tensor γ_{mn}
in raw Cartesian X,Y and x,y

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

Velocity, Jacobian, and KE



$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -r \sin \theta + l \sin \phi \\ r \cos \theta - l \cos \phi \end{pmatrix}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Raw Jacobian

$$\begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R \cos \theta & 0 \\ R \sin \theta & 0 \\ -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

(x,y) to (theta, phi)

(X,Y) to (theta, phi)
Jacobian

Kinetic energy of driver M

$$T(M) = \frac{1}{2} M \dot{X}^2 + \frac{1}{2} M \dot{Y}^2 = \frac{1}{2} M R^2 \dot{\theta}^2$$

Kinetic energy of projectile m $T(m) = \frac{1}{2} [m\dot{x}^2 + m\dot{y}^2]$

$$T(m) = \frac{1}{2} m \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{2} m \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} -r \cos \theta & -r \sin \theta \\ l \cos \phi & l \sin \phi \end{pmatrix} \begin{pmatrix} -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

$$\text{Total KE} = T = T(M) + T(m) = \frac{1}{2} \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} MR^2 + mr^2 & -mrl \cos(\theta - \phi) \\ -mrl \cos(\theta - \phi) & ml^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

$$\begin{pmatrix} \gamma_{X,X} & & & \\ & \gamma_{Y,Y} & & \\ & & \gamma_{x,x} & \\ & & & \gamma_{y,y} \end{pmatrix} = \begin{pmatrix} M & & & \\ & M & & \\ & & m & \\ & & & m \end{pmatrix}$$

$$\text{Total KE} = T = \frac{1}{2} [M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2]$$

$$T = \frac{1}{2} [(MR^2 + mr^2)\dot{\theta}^2 - 2mrl \cos(\theta - \phi) \dot{\theta} \dot{\phi} + ml^2 \dot{\phi}^2] = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

Dynamic metric tensor γ_{mn}
in raw Cartesian X,Y and x,y

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mrl \cos(\theta - \phi) \\ -mrl \cos(\theta - \phi) & ml^2 \end{pmatrix} = \begin{pmatrix} MR^2 & 0 \\ 0 & 0 \end{pmatrix} + m \begin{pmatrix} -r \cos \theta & -r \sin \theta \\ l \cos \phi & l \sin \phi \end{pmatrix} \begin{pmatrix} -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix}$$

Dynamic metric tensor γ_{mn} in GCC θ and ϕ

Review of Lagrangian equation derivation from Lecture 10 (Now with trebuchet model)

Coordinate geometry, Jacobian, velocity, kinetic energy, and dynamic metric tensor γ_{mn}

→ *Structure of dynamic metric tensor γ_{mn}*

Basic force, work, and acceleration

Lagrangian force equation

Canonical momentum and γ_{mn} tensor

Kinetic energy of driver M

$$T(M) = \frac{1}{2} M \dot{X}^2 + \frac{1}{2} M \dot{Y}^2$$

$$= \frac{1}{2} M R^2 \dot{\theta}^2$$

Kinetic energy of projectile m

$$T(m) = \frac{1}{2} m \begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{2} m \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix}^T \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

$$= \frac{1}{2} m \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} -r \cos \theta & -r \sin \theta \\ l \cos \phi & l \sin \phi \end{pmatrix} \begin{pmatrix} -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

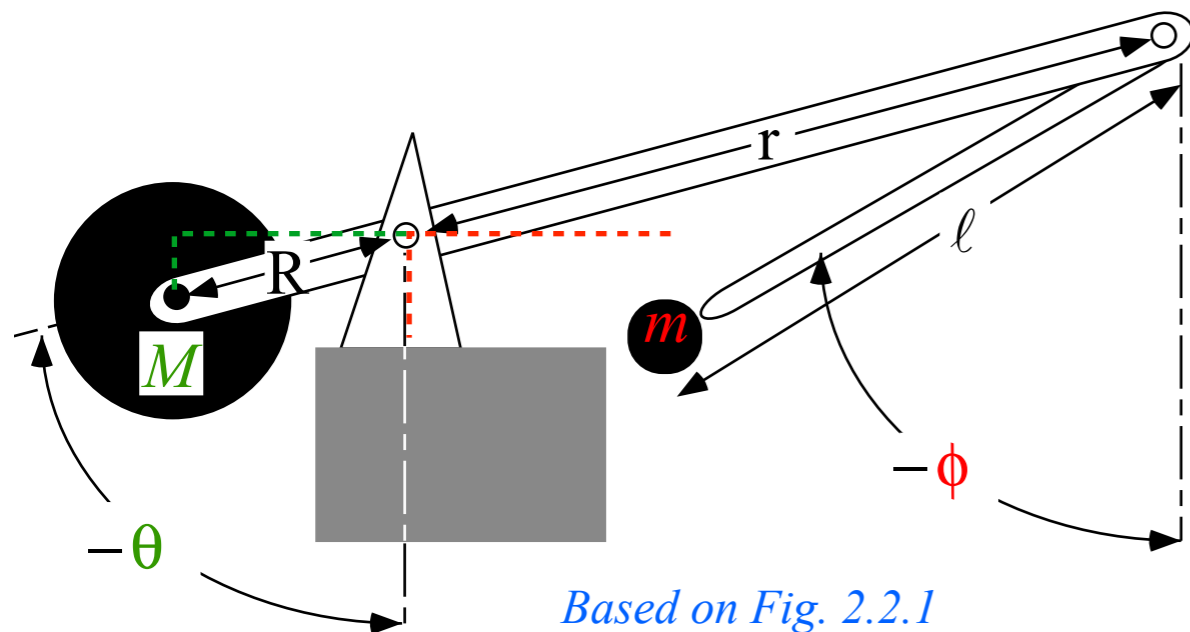
$$= \frac{1}{2} m \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} r^2 \cos^2 \theta + r^2 \sin^2 \theta & -rl \cos \theta \cos \phi - rl \sin \theta \sin \phi \\ -lr \cos \phi \cos \theta - rl \sin \theta \sin \phi & l^2 \cos^2 \phi + l^2 \sin^2 \phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Total kinetic energy of M and m

$$\text{Total KE} = T = T(M) + T(m) = \frac{1}{2} \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} MR^2 + mr^2 & -mrl \cos(\theta - \phi) \\ -mrl \cos(\theta - \phi) & ml^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2} \left[(MR^2 + mr^2) \dot{\theta}^2 - 2mrl \cos(\theta - \phi) \dot{\theta} \dot{\phi} + ml^2 \dot{\phi}^2 \right]$$

Dynamic metric tensor γ_{mn}

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}$$



Based on Fig. 2.2.1

Kinetic energy of driver M

$$T(M) = \frac{1}{2} M \dot{X}^2 + \frac{1}{2} M \dot{Y}^2$$

$$= \frac{1}{2} M R^2 \dot{\theta}^2$$

Kinetic energy of projectile m

$$T(m) = \frac{1}{2} m \begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{2} m \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix}^T \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

$$= \frac{1}{2} m \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} -r \cos \theta & -r \sin \theta \\ l \cos \phi & l \sin \phi \end{pmatrix} \begin{pmatrix} -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

$$= \frac{1}{2} m \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} r^2 \cos^2 \theta + r^2 \sin^2 \theta & -rl \cos \theta \cos \phi - rl \sin \theta \sin \phi \\ -lr \cos \phi \cos \theta - rl \sin \theta \sin \phi & l^2 \cos^2 \phi + l^2 \sin^2 \phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Total kinetic energy of M and m

$$\text{Total KE} = T = T(M) + T(m) = \frac{1}{2} \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} MR^2 + mr^2 & -mrl \cos(\theta - \phi) \\ -mrl \cos(\theta - \phi) & ml^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2} \left[(MR^2 + mr^2) \dot{\theta}^2 - 2mrl \cos(\theta - \phi) \dot{\theta} \dot{\phi} + ml^2 \dot{\phi}^2 \right]$$

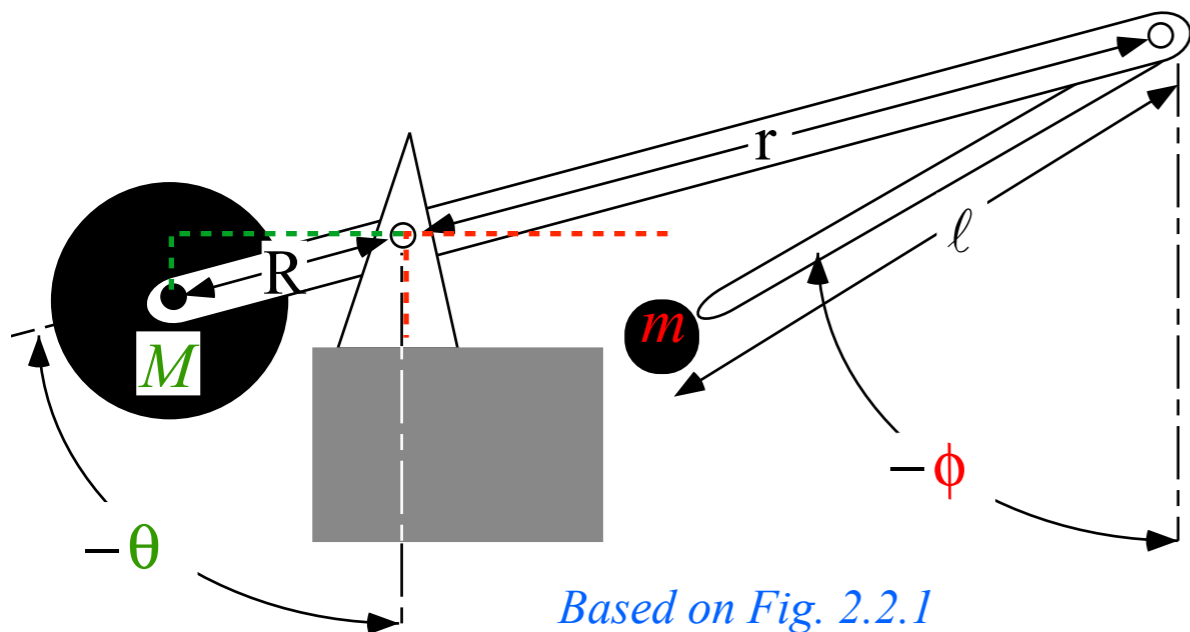
Dynamic metric tensor $\gamma_{mn} = \sum_{\text{mass } \mu} m(\mu) \frac{\partial x^j(\mu)}{\partial q^m} \frac{\partial x^j(\mu)}{\partial q^n}$

$$\begin{pmatrix} \gamma_{\theta, \theta} & \gamma_{\theta, \phi} \\ \gamma_{\phi, \theta} & \gamma_{\phi, \phi} \end{pmatrix} = \sum_{\text{mass } \mu} m(\mu) \frac{\partial \mathbf{r}(\mu)}{\partial q^m} \cdot \frac{\partial \mathbf{r}(\mu)}{\partial q^n}$$

$$= \sum_{\text{mass } \mu} m(\mu) \mathbf{E}_m(\mu) \cdot \mathbf{E}_n(\mu)$$

$$KE = \sum_{\text{mass } \mu} \frac{1}{2} m(\mu) \dot{x}^j(\mu) \dot{x}^j(\mu) = \sum_{\text{mass } \mu} \frac{1}{2} m(\mu) \frac{\partial x^j(\mu)}{\partial q^m} \frac{\partial x^j(\mu)}{\partial q^n} \dot{q}^m \dot{q}^n$$

$$= \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$



Kinetic energy of driver M

$$T(M) = \frac{1}{2} M \dot{X}^2 + \frac{1}{2} M \dot{Y}^2$$

$$= \frac{1}{2} MR^2 \dot{\theta}^2$$

Kinetic energy of projectile m

$$T(m) = \frac{1}{2} m \begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{2} m \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix}^T \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

$$= \frac{1}{2} m \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} -r \cos \theta & -r \sin \theta \\ l \cos \phi & l \sin \phi \end{pmatrix} \begin{pmatrix} -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

$$= \frac{1}{2} m \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} r^2 \cos^2 \theta + r^2 \sin^2 \theta & -rl \cos \theta \cos \phi - rl \sin \theta \sin \phi \\ -lr \cos \phi \cos \theta - rl \sin \theta \sin \phi & l^2 \cos^2 \phi + l^2 \sin^2 \phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Total kinetic energy of M and m

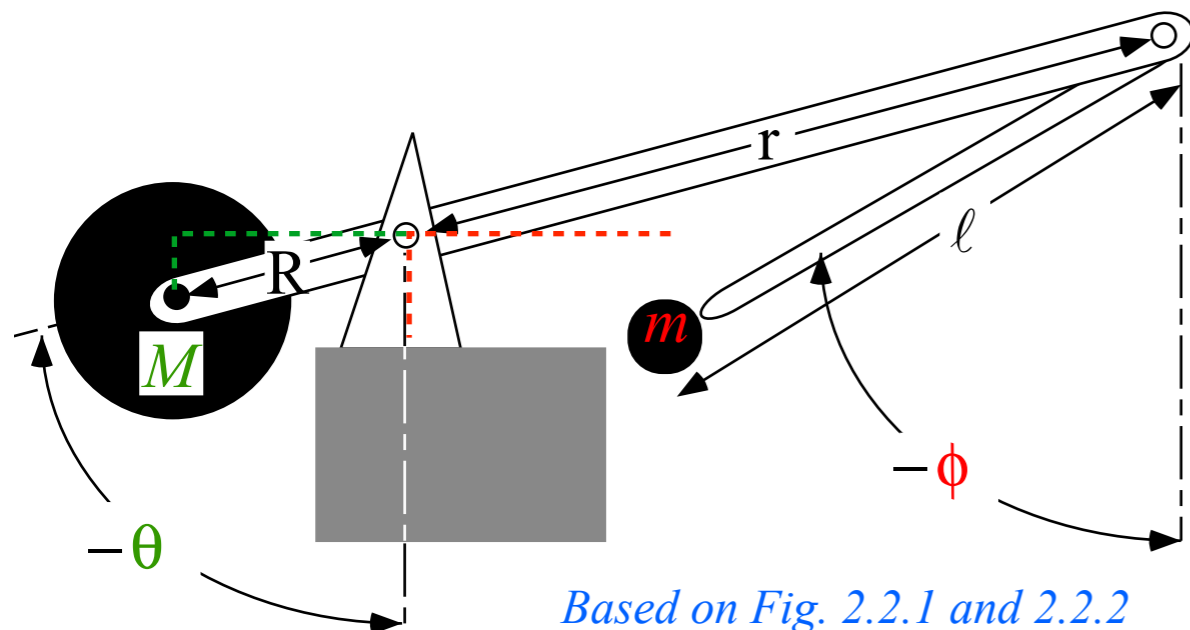
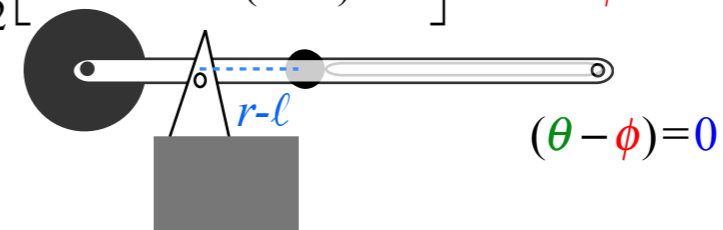
$$\text{Total KE} = T = T(M) + T(m) = \frac{1}{2} \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} MR^2 + mr^2 & -mrl \cos(\theta - \phi) \\ -mrl \cos(\theta - \phi) & ml^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2} \left[(MR^2 + mr^2) \dot{\theta}^2 - 2mrl \cos(\theta - \phi) \dot{\theta} \dot{\phi} + ml^2 \dot{\phi}^2 \right]$$

Dynamic metric tensor γ_{mn}

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}$$

Special cases (rigid rotation)

$$T = \frac{1}{2} \left[MR^2 \omega^2 + m(r-l)^2 \omega^2 \right] \text{ for: } \dot{\theta} = \dot{\phi} = \omega \text{ and } (\theta - \phi) = 0$$



Based on Fig. 2.2.1 and 2.2.2

Kinetic energy of driver M

$$T(M) = \frac{1}{2} M \dot{X}^2 + \frac{1}{2} M \dot{Y}^2$$

$$= \frac{1}{2} MR^2 \dot{\theta}^2$$

Kinetic energy of projectile m

$$T(m) = \frac{1}{2} m \begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{2} m \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix}^T \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

$$= \frac{1}{2} m \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} -r \cos \theta & -r \sin \theta \\ l \cos \phi & l \sin \phi \end{pmatrix} \begin{pmatrix} -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

$$= \frac{1}{2} m \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} r^2 \cos^2 \theta + r^2 \sin^2 \theta & -rl \cos \theta \cos \phi - rl \sin \theta \sin \phi \\ -lr \cos \phi \cos \theta - rl \sin \theta \sin \phi & l^2 \cos^2 \phi + l^2 \sin^2 \phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Total kinetic energy of M and m

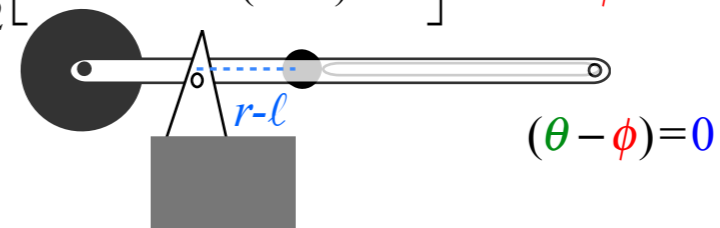
$$\text{Total KE} = T = T(M) + T(m) = \frac{1}{2} \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} MR^2 + mr^2 & -mrl \cos(\theta - \phi) \\ -mrl \cos(\theta - \phi) & ml^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2} \left[(MR^2 + mr^2) \dot{\theta}^2 - 2mrl \cos(\theta - \phi) \dot{\theta} \dot{\phi} + ml^2 \dot{\phi}^2 \right]$$

Dynamic metric tensor γ_{mn}

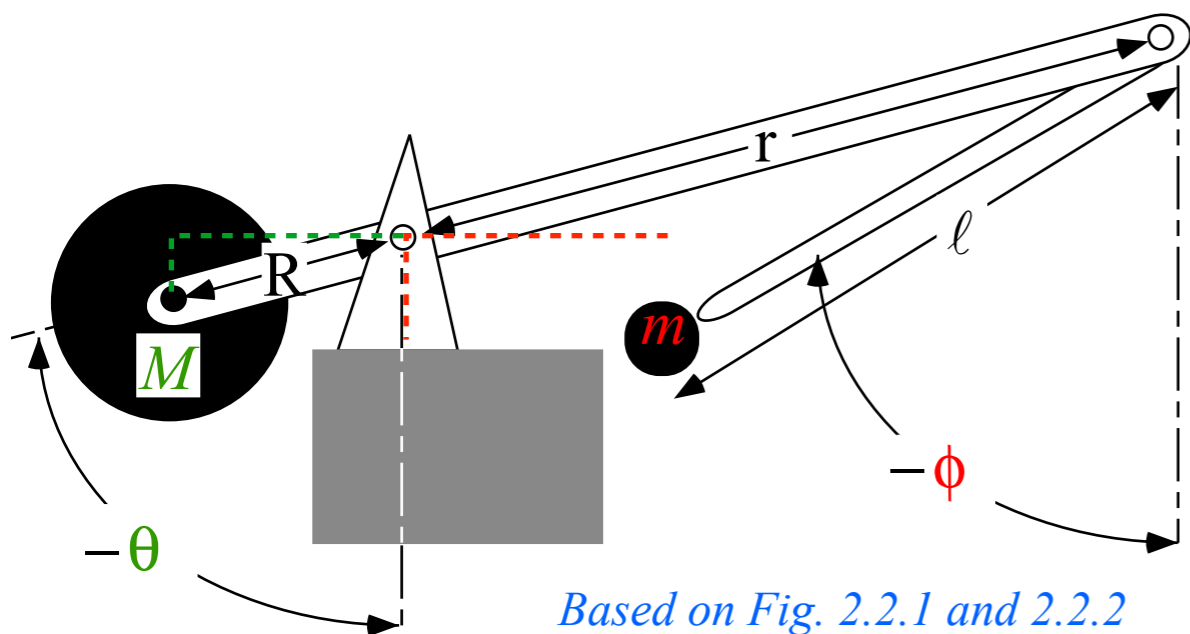
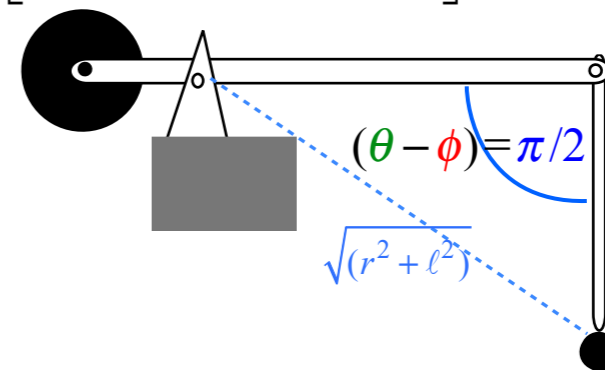
$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}$$

Special cases (rigid rotation)

$$T = \frac{1}{2} \left[MR^2 \omega^2 + m(r-l)^2 \omega^2 \right] \text{ for: } \dot{\theta} = \dot{\phi} = \omega \text{ and } (\theta - \phi) = 0$$



$$T = \frac{1}{2} \left[MR^2 \omega^2 + m(r^2 + l^2) \omega^2 \right] \text{ for: } \dot{\theta} = \dot{\phi} = \omega \text{ and } (\theta - \phi) = \pi/2$$



Based on Fig. 2.2.1 and 2.2.2

Kinetic energy of driver M

$$T(M) = \frac{1}{2} M \dot{X}^2 + \frac{1}{2} M \dot{Y}^2$$

$$= \frac{1}{2} M R^2 \dot{\theta}^2$$

Kinetic energy of projectile m

$$T(m) = \frac{1}{2} m \begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{2} m \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix}^T \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

$$= \frac{1}{2} m \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} -r \cos \theta & -r \sin \theta \\ l \cos \phi & l \sin \phi \end{pmatrix} \begin{pmatrix} -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

$$= \frac{1}{2} m \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} r^2 \cos^2 \theta + r^2 \sin^2 \theta & -rl \cos \theta \cos \phi - rl \sin \theta \sin \phi \\ -lr \cos \phi \cos \theta - rl \sin \theta \sin \phi & l^2 \cos^2 \phi + l^2 \sin^2 \phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Total kinetic energy of M and m

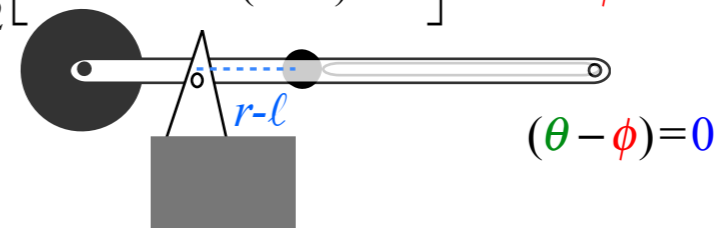
$$\text{Total KE} = T = T(M) + T(m) = \frac{1}{2} \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} MR^2 + mr^2 & -mrl \cos(\theta - \phi) \\ -mrl \cos(\theta - \phi) & ml^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2} \left[(MR^2 + mr^2) \dot{\theta}^2 - 2mrl \cos(\theta - \phi) \dot{\theta} \dot{\phi} + ml^2 \dot{\phi}^2 \right]$$

Dynamic metric tensor γ_{mn}

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}$$

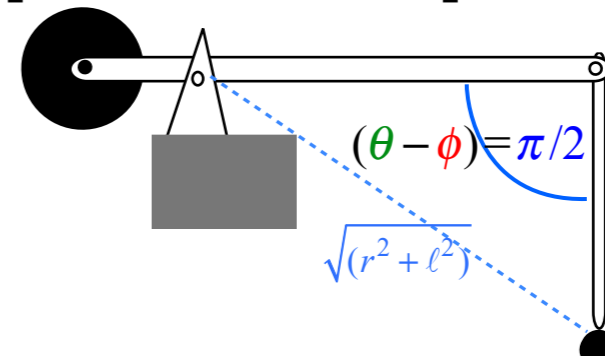
Special cases (rigid rotation)

$$T = \frac{1}{2} \left[MR^2 \omega^2 + m(r-l)^2 \omega^2 \right] \text{ for: } \dot{\theta} = \dot{\phi} = \omega \text{ and } (\theta - \phi) = 0$$



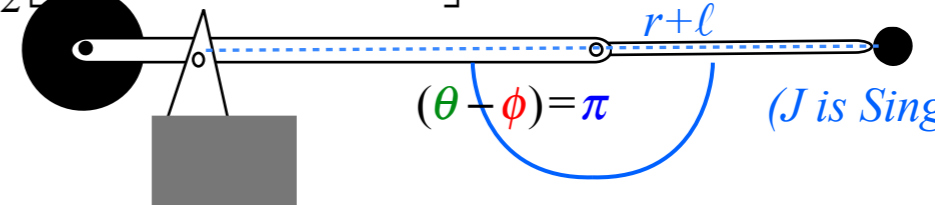
(J is Singular)

$$T = \frac{1}{2} \left[MR^2 \omega^2 + m(r^2 + l^2) \omega^2 \right] \text{ for: } \dot{\theta} = \dot{\phi} = \omega \text{ and } (\theta - \phi) = \pi/2$$

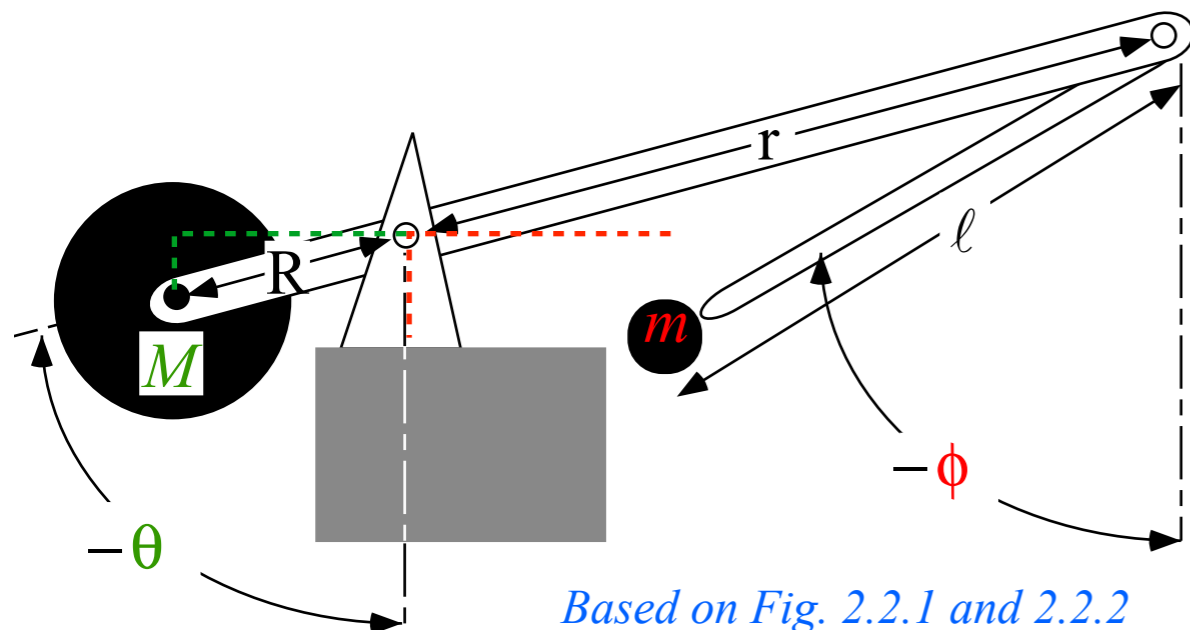


(J is Orthogonal)

$$T = \frac{1}{2} \left[MR^2 \omega^2 + m(r+l)^2 \omega^2 \right] \text{ for: } \dot{\theta} = \dot{\phi} = \omega \text{ and } (\theta - \phi) = \pi$$



(J is Singular)



Based on Fig. 2.2.1 and 2.2.2

Kinetic energy of driver M

$$T(M) = \frac{1}{2} M \dot{X}^2 + \frac{1}{2} M \dot{Y}^2$$

$$= \frac{1}{2} MR^2 \dot{\theta}^2$$

Kinetic energy of projectile m

$$T(m) = \frac{1}{2} m \begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{2} m \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix}^T \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

$$= \frac{1}{2} m \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} -r \cos \theta & -r \sin \theta \\ l \cos \phi & l \sin \phi \end{pmatrix} \begin{pmatrix} -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

$$= \frac{1}{2} m \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} r^2 \cos^2 \theta + r^2 \sin^2 \theta & -rl \cos \theta \cos \phi - rl \sin \theta \sin \phi \\ -lr \cos \phi \cos \theta - rl \sin \theta \sin \phi & l^2 \cos^2 \phi + l^2 \sin^2 \phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Total kinetic energy of M and m

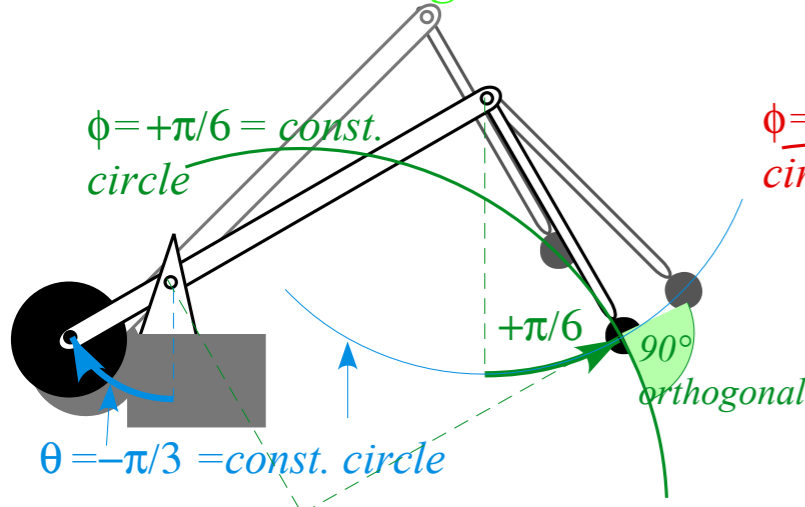
$$\text{Total KE} = T = T(M) + T(m) = \frac{1}{2} \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} MR^2 + mr^2 & -mrl \cos(\theta - \phi) \\ -mrl \cos(\theta - \phi) & ml^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2} \left[(MR^2 + mr^2) \dot{\theta}^2 - 2mrl \cos(\theta - \phi) \dot{\theta} \dot{\phi} + ml^2 \dot{\phi}^2 \right]$$

Dynamic metric tensor γ_{mn}

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}$$

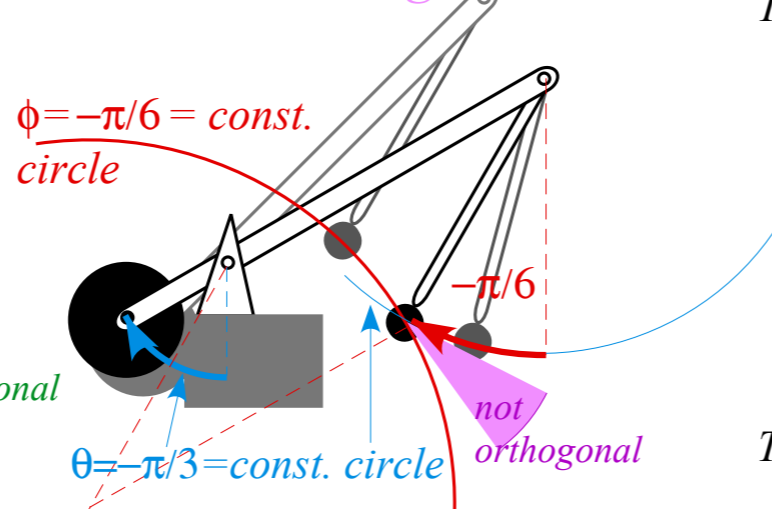
SPECIAL CASE

(a) When (θ, ϕ) coordinates are **orthogonal**



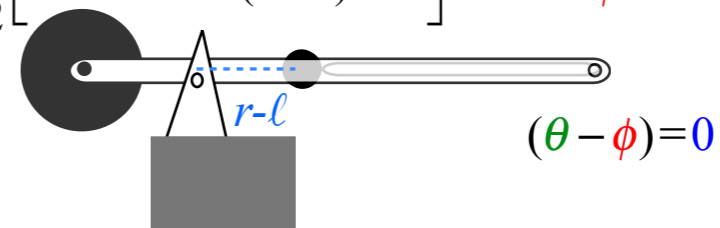
USUAL CASE

(b) When (θ, ϕ) coordinates are **not orthogonal**



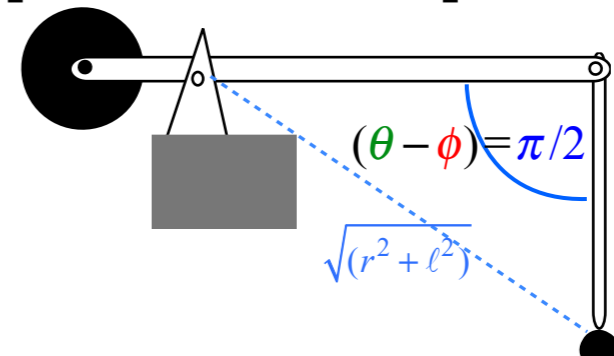
Special cases (rigid rotation)

$$T = \frac{1}{2} \left[MR^2 \omega^2 + m(r-l)^2 \omega^2 \right] \text{ for: } \dot{\theta} = \dot{\phi} = \omega \text{ and } (\theta - \phi) = 0$$



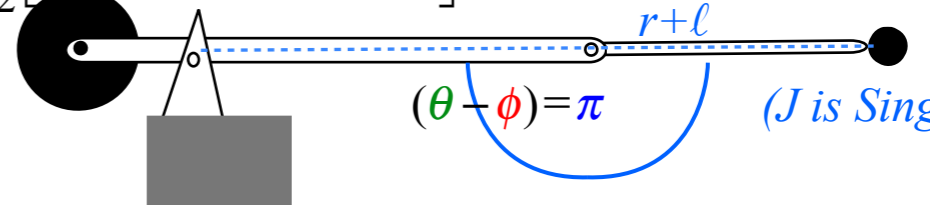
(J is Singular)

$$T = \frac{1}{2} \left[MR^2 \omega^2 + m(r^2 + l^2) \omega^2 \right] \text{ for: } \dot{\theta} = \dot{\phi} = \omega \text{ and } (\theta - \phi) = \pi/2$$



(J is Orthogonal)

$$T = \frac{1}{2} \left[MR^2 \omega^2 + m(r+l)^2 \omega^2 \right] \text{ for: } \dot{\theta} = \dot{\phi} = \omega \text{ and } (\theta - \phi) = \pi$$



(J is Singular)

Fig. 2.3.1 Examples of (θ, ϕ) intersections (a) orthogonal (special case), (b) non-orthogonal (typical).

Based on Fig. 2.3.1 and 2.2.2

Review of Lagrangian equation derivation from Lecture 10 (Now with trebuchet model)

Coordinate geometry, Jacobian, velocity, kinetic energy, and dynamic metric tensor γ_{mn}

Structure of dynamic metric tensor γ_{mn}

 *Basic force, work, and acceleration*

Lagrangian force equation

Canonical momentum and γ_{mn} tensor

Force, Work, and Acceleration

$$dW = F_X dX + F_Y dY + F_x dx + F_y dy$$

$$= M\ddot{X}dX + M\ddot{Y}dY + m\ddot{x}dx + m\ddot{y}dy$$

Write work-sums in columns:

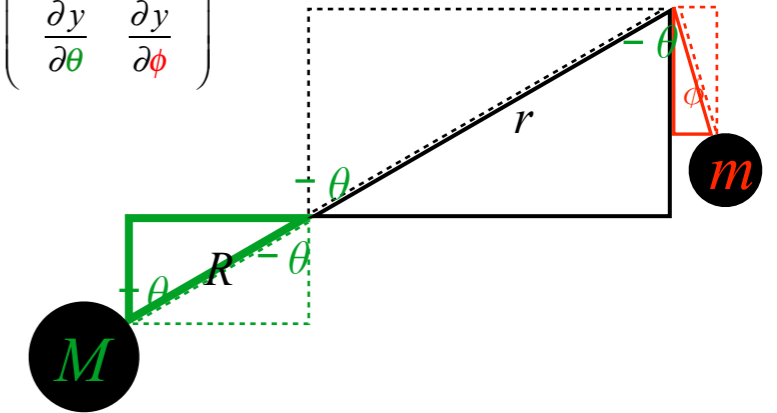
$$dW = F_X dX = M\ddot{X}dX$$

$$+ F_Y dY = M\ddot{Y}dY$$

$$+ F_x dx = m\ddot{x}dx$$

$$+ F_y dy = m\ddot{y}dy$$

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R \cos \theta & 0 \\ R \sin \theta & 0 \\ -r \cos \theta & \ell \cos \phi \\ -r \sin \theta & \ell \sin \phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Force, Work, and Acceleration

$$dW = F_X dX + F_Y dY + F_x dx + F_x dy$$

$$= M\ddot{X}dX + M\ddot{Y}dY + m\ddot{x}dx + m\ddot{y}dy$$

Write work-sums in columns: (Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$\begin{aligned}
 dW = F_X dX &= M\ddot{X}dX = F_X \frac{\partial X}{\partial \theta} d\theta + F_X \frac{\partial X}{\partial \phi} d\phi = M\ddot{X} \frac{\partial X}{\partial \theta} d\theta + M\ddot{X} \frac{\partial X}{\partial \phi} d\phi \\
 + F_Y dY &+ M\ddot{Y}dY + F_Y \frac{\partial Y}{\partial \theta} d\theta + F_Y \frac{\partial Y}{\partial \phi} d\phi + M\ddot{Y} \frac{\partial Y}{\partial \theta} d\theta + M\ddot{Y} \frac{\partial Y}{\partial \phi} d\phi \\
 + F_x dx &+ m\ddot{x}dx + F_x \frac{\partial x}{\partial \theta} d\theta + F_x \frac{\partial x}{\partial \phi} d\phi + m\ddot{x} \frac{\partial x}{\partial \theta} d\theta + m\ddot{x} \frac{\partial x}{\partial \phi} d\phi \\
 + F_y dy &+ m\ddot{y}dy + F_y \frac{\partial y}{\partial \theta} d\theta + F_y \frac{\partial y}{\partial \phi} d\phi + m\ddot{y} \frac{\partial y}{\partial \theta} d\theta + m\ddot{y} \frac{\partial y}{\partial \phi} d\phi
 \end{aligned}$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R \cos \theta & 0 \\ R \sin \theta & 0 \\ -r \cos \theta & \ell \cos \phi \\ -r \sin \theta & \ell \sin \phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

Force, Work, and Acceleration

$$dW = F_X dX + F_Y dY + F_x dx + F_y dy$$

$$= M\ddot{X}dX + M\ddot{Y}dY + m\ddot{x}dx + m\ddot{y}dy$$

Write work-sums in columns: (Using GCC $d\theta$ and $d\phi$ in Jacobian)

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 + F_Y dY &+ M\ddot{Y}dY + F_Y \frac{\partial Y}{\partial \theta} d\theta + F_Y \frac{\partial Y}{\partial \phi} d\phi + M\ddot{Y} \frac{\partial Y}{\partial \theta} d\theta + M\ddot{Y} \frac{\partial Y}{\partial \phi} d\phi \\
 + F_x dx &+ m\ddot{x}dx + F_x \frac{\partial x}{\partial \theta} d\theta + F_x \frac{\partial x}{\partial \phi} d\phi + m\ddot{x} \frac{\partial x}{\partial \theta} d\theta + m\ddot{x} \frac{\partial x}{\partial \phi} d\phi \\
 + F_y dy &+ m\ddot{y}dy + F_y \frac{\partial y}{\partial \theta} d\theta + F_y \frac{\partial y}{\partial \phi} d\phi + m\ddot{y} \frac{\partial y}{\partial \theta} d\theta + m\ddot{y} \frac{\partial y}{\partial \phi} d\phi
 \end{aligned}$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R \cos \theta & 0 \\ R \sin \theta & 0 \\ -r \cos \theta & \ell \cos \phi \\ -r \sin \theta & \ell \sin \phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

Assuming variables θ and ϕ are independent...

Force, Work, and Acceleration

$$dW = F_X dX + F_Y dY + F_x dx + F_y dy$$

$$= M\ddot{X}dX + M\ddot{Y}dY + m\ddot{x}dx + m\ddot{y}dy$$

Write work-sums in columns: (Using GCC $d\theta$ and $d\phi$ in Jacobian)

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 + F_Y dY &+ M\ddot{Y}dY + F_y \frac{\partial Y}{\partial \theta} d\theta + F_y \frac{\partial Y}{\partial \phi} d\phi + M\ddot{Y} \frac{\partial Y}{\partial \theta} d\theta + M\ddot{Y} \frac{\partial Y}{\partial \phi} d\phi \\
 + F_x dx &+ m\ddot{x}dx + F_x \frac{\partial x}{\partial \theta} d\theta + F_x \frac{\partial x}{\partial \phi} d\phi + m\ddot{x} \frac{\partial x}{\partial \theta} d\theta + m\ddot{x} \frac{\partial x}{\partial \phi} d\phi \\
 + F_y dy &+ m\ddot{y}dy + F_y \frac{\partial y}{\partial \theta} d\theta + F_y \frac{\partial y}{\partial \phi} d\phi + m\ddot{y} \frac{\partial y}{\partial \theta} d\theta + m\ddot{y} \frac{\partial y}{\partial \phi} d\phi
 \end{aligned}$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R \cos \theta & 0 \\ R \sin \theta & 0 \\ -r \cos \theta & \ell \cos \phi \\ -r \sin \theta & \ell \sin \phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

Assuming variables θ and ϕ are independent...

Set: $d\theta=1$ $d\phi=0$

$$\begin{aligned}
 F_x \frac{\partial X}{\partial \theta} &= M\ddot{X} \frac{\partial X}{\partial \theta} \\
 + F_Y \frac{\partial Y}{\partial \theta} &+ M\ddot{Y} \frac{\partial Y}{\partial \theta} \\
 + F_x \frac{\partial x}{\partial \theta} &+ m\ddot{x} \frac{\partial x}{\partial \theta} \\
 + F_y \frac{\partial y}{\partial \theta} &+ m\ddot{y} \frac{\partial y}{\partial \theta}
 \end{aligned}$$

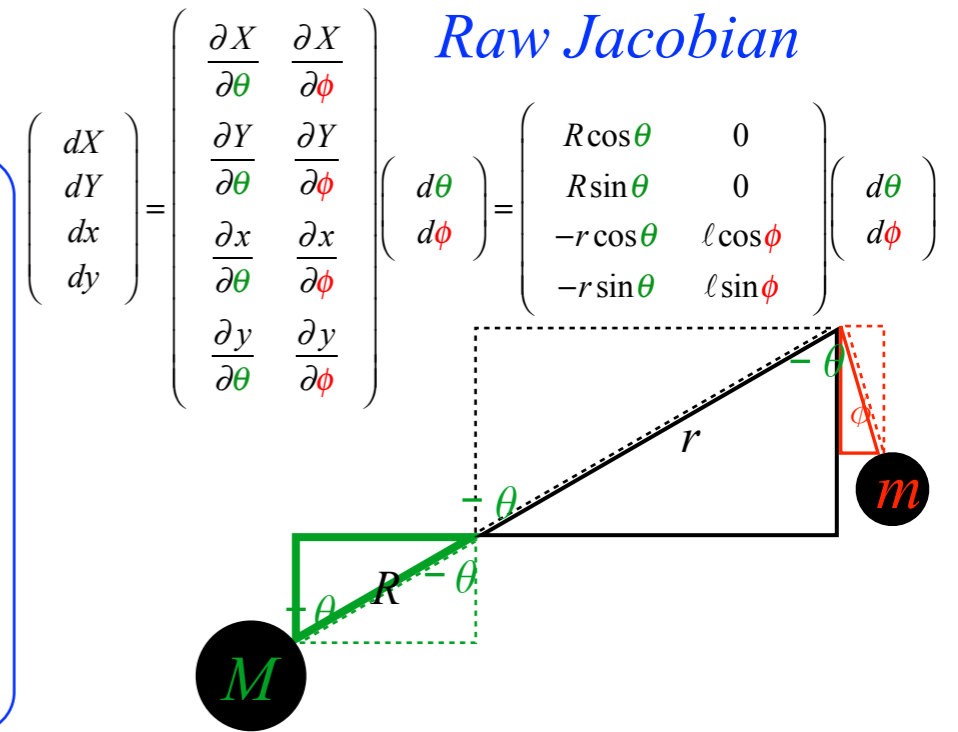
Force, Work, and Acceleration

$$dW = F_X dX + F_Y dY + F_x dx + F_y dy$$

$$= M\ddot{X}dX + M\ddot{Y}dY + m\ddot{x}dx + m\ddot{y}dy$$

Write work-sums in columns: (Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$\begin{aligned}
 dW = F_X dX &= M\ddot{X}dX = F_X \frac{\partial X}{\partial \theta} d\theta + F_X \frac{\partial X}{\partial \phi} d\phi = M\ddot{X} \frac{\partial X}{\partial \theta} d\theta + M\ddot{X} \frac{\partial X}{\partial \phi} d\phi \\
 + F_Y dY &+ M\ddot{Y}dY + F_Y \frac{\partial Y}{\partial \theta} d\theta + F_Y \frac{\partial Y}{\partial \phi} d\phi + M\ddot{Y} \frac{\partial Y}{\partial \theta} d\theta + M\ddot{Y} \frac{\partial Y}{\partial \phi} d\phi \\
 + F_x dx &+ m\ddot{x}dx + F_x \frac{\partial x}{\partial \theta} d\theta + F_x \frac{\partial x}{\partial \phi} d\phi + m\ddot{x} \frac{\partial x}{\partial \theta} d\theta + m\ddot{x} \frac{\partial x}{\partial \phi} d\phi \\
 + F_y dy &+ m\ddot{y}dy + F_y \frac{\partial y}{\partial \theta} d\theta + F_y \frac{\partial y}{\partial \phi} d\phi + m\ddot{y} \frac{\partial y}{\partial \theta} d\theta + m\ddot{y} \frac{\partial y}{\partial \phi} d\phi
 \end{aligned}$$



Assuming variables θ and ϕ are independent...

Set: $d\theta=1$ $d\phi=0$

$$\begin{aligned}
 F_X \frac{\partial X}{\partial \theta} &= M\ddot{X} \frac{\partial X}{\partial \theta} \\
 + F_Y \frac{\partial Y}{\partial \theta} &+ M\ddot{Y} \frac{\partial Y}{\partial \theta} \\
 + F_x \frac{\partial x}{\partial \theta} &+ m\ddot{x} \frac{\partial x}{\partial \theta} \\
 + F_y \frac{\partial y}{\partial \theta} &+ m\ddot{y} \frac{\partial y}{\partial \theta}
 \end{aligned}$$

Set: $d\theta=0$ $d\phi=1$

$$\begin{aligned}
 F_X \frac{\partial X}{\partial \phi} &= M\ddot{X} \frac{\partial X}{\partial \phi} \\
 + F_Y \frac{\partial Y}{\partial \phi} &+ M\ddot{Y} \frac{\partial Y}{\partial \phi} \\
 + F_x \frac{\partial x}{\partial \phi} &+ m\ddot{x} \frac{\partial x}{\partial \phi} \\
 + F_y \frac{\partial y}{\partial \phi} &+ m\ddot{y} \frac{\partial y}{\partial \phi}
 \end{aligned}$$

Force, Work, and Acceleration

$$dW = F_X dX + F_Y dY + F_x dx + F_y dy$$

$$= M\ddot{X}dX + M\ddot{Y}dY + m\ddot{x}dx + m\ddot{y}dy$$

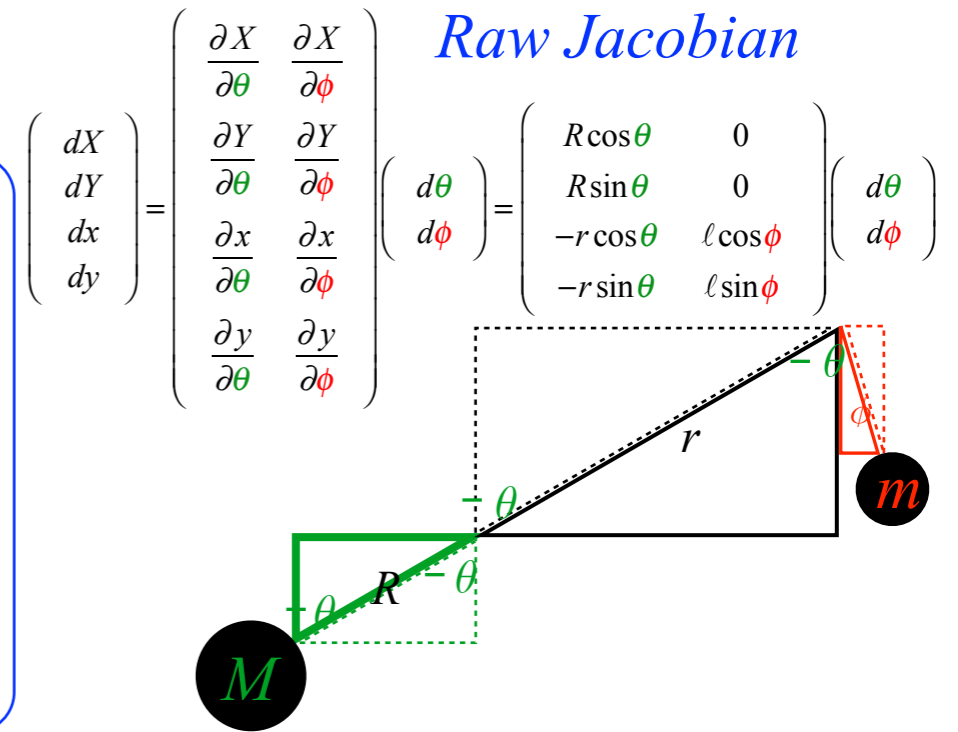
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$$dW = F_X dX = M\ddot{X}dX = F_X \frac{\partial X}{\partial \theta} d\theta + F_X \frac{\partial X}{\partial \phi} d\phi = M\ddot{X} \frac{\partial X}{\partial \theta} d\theta + M\ddot{X} \frac{\partial X}{\partial \phi} d\phi$$

$$+ F_Y dY + M\ddot{Y}dY + F_Y \frac{\partial Y}{\partial \theta} d\theta + F_Y \frac{\partial Y}{\partial \phi} d\phi + M\ddot{Y} \frac{\partial Y}{\partial \theta} d\theta + M\ddot{Y} \frac{\partial Y}{\partial \phi} d\phi$$

$$+ F_x dx + m\ddot{x}dx + F_x \frac{\partial x}{\partial \theta} d\theta + F_x \frac{\partial x}{\partial \phi} d\phi + m\ddot{x} \frac{\partial x}{\partial \theta} d\theta + m\ddot{x} \frac{\partial x}{\partial \phi} d\phi$$

$$+ F_y dy + m\ddot{y}dy + F_y \frac{\partial y}{\partial \theta} d\theta + F_y \frac{\partial y}{\partial \phi} d\phi + m\ddot{y} \frac{\partial y}{\partial \theta} d\theta + m\ddot{y} \frac{\partial y}{\partial \phi} d\phi$$



Lagrange trickery:

$$\ddot{X} \frac{\partial X}{\partial \theta} = \frac{d}{dt} \left(\dot{X} \frac{\partial X}{\partial \theta} \right) - \dot{X} \frac{d}{dt} \frac{\partial X}{\partial \theta}$$

(using $\frac{d}{dt}(\dot{X}U) = \ddot{X}U + \dot{X}\dot{U}$)

STEP
A

Set: $d\theta=1$ $d\phi=0$

$$F_X \frac{\partial X}{\partial \theta} = M\ddot{X} \frac{\partial X}{\partial \theta}$$

$$+ F_Y \frac{\partial Y}{\partial \theta} + M\ddot{Y} \frac{\partial Y}{\partial \theta}$$

$$+ F_x \frac{\partial x}{\partial \theta} + m\ddot{x} \frac{\partial x}{\partial \theta}$$

$$+ F_y \frac{\partial y}{\partial \theta} + m\ddot{y} \frac{\partial y}{\partial \theta}$$

Set: $d\theta=0$ $d\phi=1$

$$F_X \frac{\partial X}{\partial \phi} = M\ddot{X} \frac{\partial X}{\partial \phi}$$

$$+ F_Y \frac{\partial Y}{\partial \phi} + M\ddot{Y} \frac{\partial Y}{\partial \phi}$$

$$+ F_x \frac{\partial x}{\partial \phi} + m\ddot{x} \frac{\partial x}{\partial \phi}$$

$$+ F_y \frac{\partial y}{\partial \phi} + m\ddot{y} \frac{\partial y}{\partial \phi}$$

Force, Work, and Acceleration

$$dW = F_X dX + F_Y dY + F_x dx + F_y dy$$

$$= M\ddot{X}dX + M\ddot{Y}dY + m\ddot{x}dx + m\ddot{y}dy$$

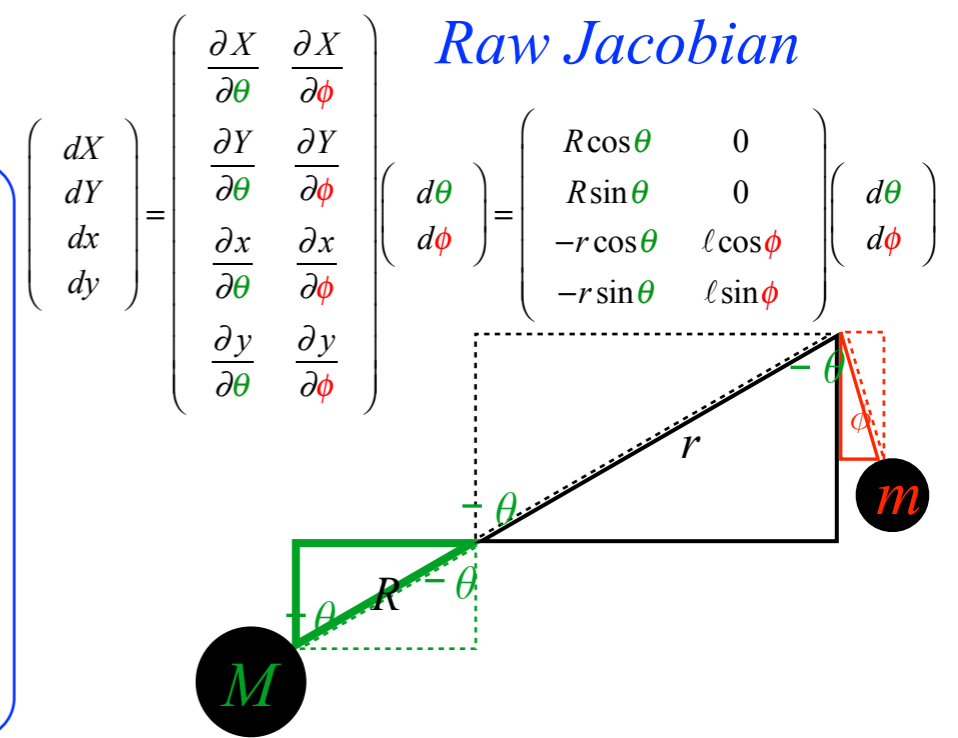
Write work-sums in columns: (Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$dW = F_X dX = M\ddot{X}dX = F_X \frac{\partial X}{\partial \theta} d\theta + F_X \frac{\partial X}{\partial \phi} d\phi = M\ddot{X} \frac{\partial X}{\partial \theta} d\theta + M\ddot{X} \frac{\partial X}{\partial \phi} d\phi$$

$$+ F_Y dY + M\ddot{Y} dY + F_Y \frac{\partial Y}{\partial \theta} d\theta + F_Y \frac{\partial Y}{\partial \phi} d\phi + M\ddot{Y} \frac{\partial Y}{\partial \theta} d\theta + M\ddot{Y} \frac{\partial Y}{\partial \phi} d\phi$$

$$+ F_x dx + m\ddot{x} dx + F_x \frac{\partial x}{\partial \theta} d\theta + F_x \frac{\partial x}{\partial \phi} d\phi + m\ddot{x} \frac{\partial x}{\partial \theta} d\theta + m\ddot{x} \frac{\partial x}{\partial \phi} d\phi$$

$$+ F_y dy + m\ddot{y} dy + F_y \frac{\partial y}{\partial \theta} d\theta + F_y \frac{\partial y}{\partial \phi} d\phi + m\ddot{y} \frac{\partial y}{\partial \theta} d\theta + m\ddot{y} \frac{\partial y}{\partial \phi} d\phi$$



Lagrange trickery:

$$\ddot{X} \frac{\partial X}{\partial \theta} = \frac{d}{dt} \left(\dot{X} \frac{\partial X}{\partial \theta} \right) - \dot{X} \frac{d}{dt} \frac{\partial X}{\partial \theta}$$

(using $\frac{d}{dt}(\dot{X}U) = \ddot{X}U + \dot{X}\dot{U}$)

STEP A

$$= \frac{d}{dt} \left(\dot{X} \frac{\partial X}{\partial \dot{\theta}} \right) - \dot{X} \frac{\partial \dot{X}}{\partial \theta}$$

by lemma 1: $\frac{\partial \dot{X}}{\partial \dot{q}} = \frac{\partial X}{\partial q}$

STEP B and lemma 2: $\frac{\partial \dot{X}}{\partial q} = \frac{d}{dt} \frac{\partial X}{\partial q}$

Lemmas from Lect.9

lemma 1: p.9.13

lemma 2: p.9.24

Set: $d\theta=1$ $d\phi=0$

$$F_X \frac{\partial X}{\partial \theta} = M\ddot{X} \frac{\partial X}{\partial \theta}$$

$$+ F_Y \frac{\partial Y}{\partial \theta} + M\ddot{Y} \frac{\partial Y}{\partial \theta}$$

$$+ F_x \frac{\partial x}{\partial \theta} + m\ddot{x} \frac{\partial x}{\partial \theta}$$

$$+ F_y \frac{\partial y}{\partial \theta} + m\ddot{y} \frac{\partial y}{\partial \theta}$$

Set: $d\theta=0$ $d\phi=1$

$$F_X \frac{\partial X}{\partial \phi} = M\ddot{X} \frac{\partial X}{\partial \phi}$$

$$+ F_Y \frac{\partial Y}{\partial \phi} + M\ddot{Y} \frac{\partial Y}{\partial \phi}$$

$$+ F_x \frac{\partial x}{\partial \phi} + m\ddot{x} \frac{\partial x}{\partial \phi}$$

$$+ F_y \frac{\partial y}{\partial \phi} + m\ddot{y} \frac{\partial y}{\partial \phi}$$

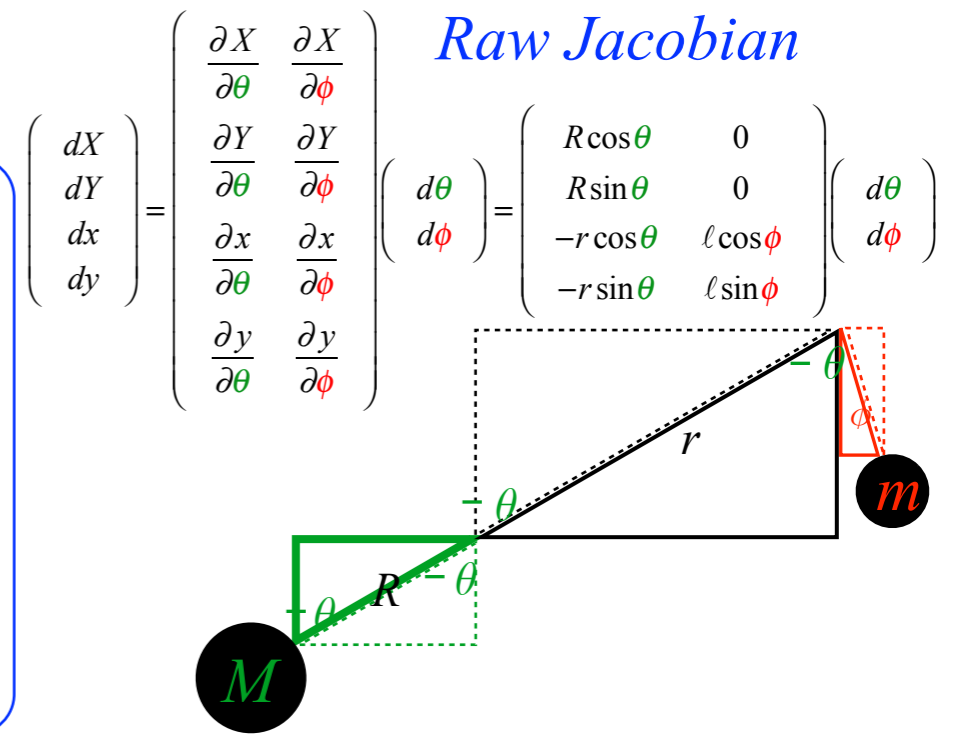
Force, Work, and Acceleration

$$dW = F_X dX + F_Y dY + F_x dx + F_y dy$$

$$= M\ddot{X}dX + M\ddot{Y}dY + m\ddot{x}dx + m\ddot{y}dy$$

Write work-sums in columns: (Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$\begin{aligned} dW = F_X dX &= M\ddot{X}dX = F_X \frac{\partial X}{\partial \theta} d\theta + F_X \frac{\partial X}{\partial \phi} d\phi = M\ddot{X} \frac{\partial X}{\partial \theta} d\theta + M\ddot{X} \frac{\partial X}{\partial \phi} d\phi \\ &+ F_Y dY = M\ddot{Y}dY = F_Y \frac{\partial Y}{\partial \theta} d\theta + F_Y \frac{\partial Y}{\partial \phi} d\phi = M\ddot{Y} \frac{\partial Y}{\partial \theta} d\theta + M\ddot{Y} \frac{\partial Y}{\partial \phi} d\phi \\ &+ F_x dx = m\ddot{x}dx = F_x \frac{\partial x}{\partial \theta} d\theta + F_x \frac{\partial x}{\partial \phi} d\phi = m\ddot{x} \frac{\partial x}{\partial \theta} d\theta + m\ddot{x} \frac{\partial x}{\partial \phi} d\phi \\ &+ F_y dy = m\ddot{y}dy = F_y \frac{\partial y}{\partial \theta} d\theta + F_y \frac{\partial y}{\partial \phi} d\phi = m\ddot{y} \frac{\partial y}{\partial \theta} d\theta + m\ddot{y} \frac{\partial y}{\partial \phi} d\phi \end{aligned}$$



Lagrange trickery:

$$\ddot{X} \frac{\partial X}{\partial \theta} = \frac{d}{dt} \left(\dot{X} \frac{\partial X}{\partial \theta} \right) - \dot{X} \frac{d}{dt} \frac{\partial X}{\partial \theta}$$

(using $\frac{d}{dt}(\dot{X}U) = \ddot{X}U + \dot{X}\dot{U}$)

STEP
A

$$= \frac{d}{dt} \left(\dot{X} \frac{\partial \dot{X}}{\partial \dot{\theta}} \right) - \dot{X} \frac{\partial \dot{X}}{\partial \dot{\theta}}$$

by lemma 1: $\frac{\partial X}{\partial q} = \frac{\partial \dot{X}}{\partial \dot{q}}$

STEP
B

and lemma 2: $\frac{\partial \dot{X}}{\partial \dot{q}} = \frac{d}{dt} \frac{\partial X}{\partial q}$

STEP
C

$$= \frac{d}{dt} \left(\frac{\partial (\dot{X}^2 / 2)}{\partial \dot{\theta}} \right) - \frac{\partial (\dot{X}^2 / 2)}{\partial \dot{\theta}}$$

(using $\frac{\partial (U^2 / 2)}{\partial q} = U \frac{\partial U}{\partial q}$)

Set: $d\theta=1$ $d\phi=0$

$$\begin{aligned} F_X \frac{\partial X}{\partial \theta} &= M\ddot{X} \frac{\partial X}{\partial \theta} \\ + F_Y \frac{\partial Y}{\partial \theta} &+ M\ddot{Y} \frac{\partial Y}{\partial \theta} \\ + F_x \frac{\partial x}{\partial \theta} &+ m\ddot{x} \frac{\partial x}{\partial \theta} \\ + F_y \frac{\partial y}{\partial \theta} &+ m\ddot{y} \frac{\partial y}{\partial \theta} \end{aligned}$$

Set: $d\theta=0$ $d\phi=1$

$$\begin{aligned} F_X \frac{\partial X}{\partial \phi} &= M\ddot{X} \frac{\partial X}{\partial \phi} \\ + F_Y \frac{\partial Y}{\partial \phi} &+ M\ddot{Y} \frac{\partial Y}{\partial \phi} \\ + F_x \frac{\partial x}{\partial \phi} &+ m\ddot{x} \frac{\partial x}{\partial \phi} \\ + F_y \frac{\partial y}{\partial \phi} &+ m\ddot{y} \frac{\partial y}{\partial \phi} \end{aligned}$$

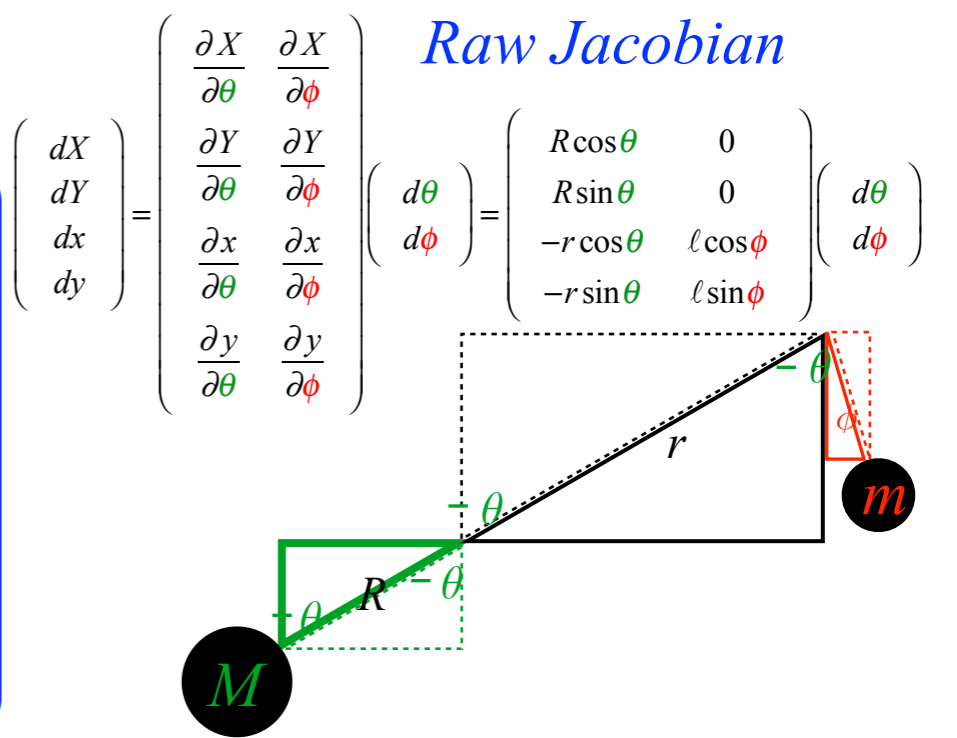
Force, Work, and Acceleration

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$$= M\ddot{X}dX + M\ddot{Y}dY + m\ddot{x}dx + m\ddot{y}dy$$

Write work-sums in columns: (Using GCC $d\theta$ and $d\phi$ in Jacobian)

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 + F_Y dY &+ M\ddot{Y}dY + F_Y \frac{\partial Y}{\partial \theta} d\theta + F_Y \frac{\partial Y}{\partial \phi} d\phi + M\ddot{Y} \frac{\partial Y}{\partial \theta} d\theta + M\ddot{Y} \frac{\partial Y}{\partial \phi} d\phi \\
 + F_x dx &+ m\ddot{x}dx + F_x \frac{\partial x}{\partial \theta} d\theta + F_x \frac{\partial x}{\partial \phi} d\phi + m\ddot{x} \frac{\partial x}{\partial \theta} d\theta + m\ddot{x} \frac{\partial x}{\partial \phi} d\phi \\
 + F_y dy &+ m\ddot{y}dy + F_y \frac{\partial y}{\partial \theta} d\theta + F_y \frac{\partial y}{\partial \phi} d\phi + m\ddot{y} \frac{\partial y}{\partial \theta} d\theta + m\ddot{y} \frac{\partial y}{\partial \phi} d\phi
 \end{aligned}$$



Lagrange trickery:

$$\begin{aligned}
 \ddot{X} \frac{\partial X}{\partial \theta} &= \frac{d}{dt} \left(\dot{X} \frac{\partial X}{\partial \theta} \right) - \dot{X} \frac{d}{dt} \frac{\partial X}{\partial \theta} \\
 &= \frac{d}{dt} \left(\dot{X} \frac{\partial \dot{X}}{\partial \dot{\theta}} \right) - \dot{X} \frac{\partial \dot{X}}{\partial \dot{\theta}} \\
 &= \frac{d}{dt} \left(\frac{\partial (\dot{X}^2 / 2)}{\partial \dot{\theta}} \right) - \frac{\partial (\dot{X}^2 / 2)}{\partial \theta}
 \end{aligned}$$

(using $\frac{d}{dt}(\dot{X}U) = \ddot{X}U + \dot{X}\dot{U}$)

STEP A **STEP B** and lemma 2: $\frac{\partial \dot{X}}{\partial q} = \frac{d}{dt} \frac{\partial X}{\partial q}$ **STEP C** (using $\frac{\partial (U^2 / 2)}{\partial q} = U \frac{\partial U}{\partial q}$)

Set: $d\theta=1$ $d\phi=0$

$$\begin{aligned}
 F_X \frac{\partial X}{\partial \theta} &= M\ddot{X} \frac{\partial X}{\partial \theta} = \frac{d}{dt} \frac{\partial \frac{M\dot{X}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{X}^2}{2}}{\partial \theta} \\
 + F_Y \frac{\partial Y}{\partial \theta} &+ M\ddot{Y} \frac{\partial Y}{\partial \theta} + \frac{d}{dt} \frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \theta} \\
 + F_x \frac{\partial x}{\partial \theta} &+ m\ddot{x} \frac{\partial x}{\partial \theta} + \frac{d}{dt} \frac{\partial \frac{M\dot{x}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{x}^2}{2}}{\partial \theta} \\
 + F_y \frac{\partial y}{\partial \theta} &+ m\ddot{y} \frac{\partial y}{\partial \theta} + \frac{d}{dt} \frac{\partial \frac{M\dot{y}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{y}^2}{2}}{\partial \theta}
 \end{aligned}$$

Set: $d\theta=0$ $d\phi=1$

$$\begin{aligned}
 F_X \frac{\partial X}{\partial \phi} &= M\ddot{X} \frac{\partial X}{\partial \phi} = \frac{d}{dt} \frac{\partial \frac{M\dot{X}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{X}^2}{2}}{\partial \phi} \\
 + F_Y \frac{\partial Y}{\partial \phi} &+ M\ddot{Y} \frac{\partial Y}{\partial \phi} + \frac{d}{dt} \frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \phi} \\
 + F_x \frac{\partial x}{\partial \phi} &+ m\ddot{x} \frac{\partial x}{\partial \phi} + \frac{d}{dt} \frac{\partial \frac{M\dot{x}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{x}^2}{2}}{\partial \phi} \\
 + F_y \frac{\partial y}{\partial \phi} &+ m\ddot{y} \frac{\partial y}{\partial \phi} + \frac{d}{dt} \frac{\partial \frac{M\dot{y}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{y}^2}{2}}{\partial \phi}
 \end{aligned}$$

Force, Work, and Acceleration

$$dW = F_X dX + F_Y dY + F_x dx + F_y dy$$

$$= M\ddot{X} dX + M\ddot{Y} dY + m\ddot{x} dx + m\ddot{y} dy$$

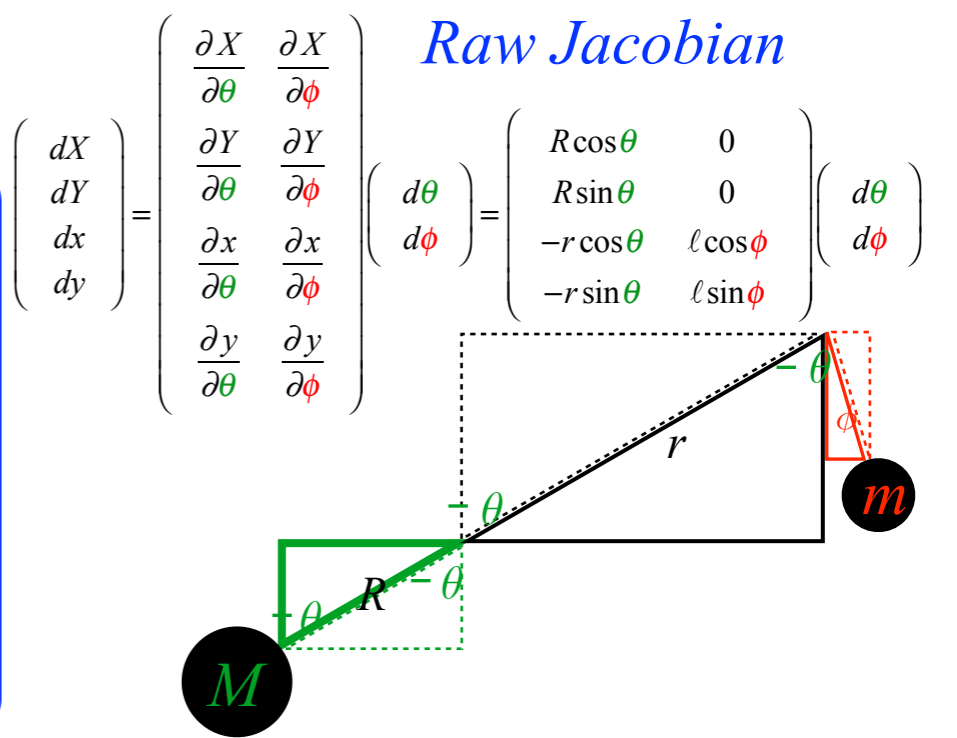
Write work-sums in columns: (Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$dW = F_X dX = M\ddot{X} dX = F_X \frac{\partial X}{\partial \theta} d\theta + F_X \frac{\partial X}{\partial \phi} d\phi = M\ddot{X} \frac{\partial X}{\partial \theta} d\theta + M\ddot{X} \frac{\partial X}{\partial \phi} d\phi$$

$$+ F_Y dY + M\ddot{Y} dY + F_Y \frac{\partial Y}{\partial \theta} d\theta + F_Y \frac{\partial Y}{\partial \phi} d\phi + M\ddot{Y} \frac{\partial Y}{\partial \theta} d\theta + M\ddot{Y} \frac{\partial Y}{\partial \phi} d\phi$$

$$+ F_x dx + m\ddot{x} dx + F_x \frac{\partial x}{\partial \theta} d\theta + F_x \frac{\partial x}{\partial \phi} d\phi + m\ddot{x} \frac{\partial x}{\partial \theta} d\theta + m\ddot{x} \frac{\partial x}{\partial \phi} d\phi$$

$$+ F_y dy + m\ddot{y} dy + F_y \frac{\partial y}{\partial \theta} d\theta + F_y \frac{\partial y}{\partial \phi} d\phi + m\ddot{y} \frac{\partial y}{\partial \theta} d\theta + m\ddot{y} \frac{\partial y}{\partial \phi} d\phi$$



Lagrange trickery:

STEP D Add up first and last columns for each variable θ and ϕ

Set: $d\theta=1$ $d\phi=0$

$$F_X \frac{\partial X}{\partial \theta} = M\ddot{X} \frac{\partial X}{\partial \theta} = \frac{d}{dt} \frac{\partial \frac{M\dot{X}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{X}^2}{2}}{\partial \theta}$$

$$+ F_Y \frac{\partial Y}{\partial \theta} + M\ddot{Y} \frac{\partial Y}{\partial \theta} + \frac{d}{dt} \frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \theta}$$

$$+ F_x \frac{\partial x}{\partial \theta} + m\ddot{x} \frac{\partial x}{\partial \theta} + \frac{d}{dt} \frac{\partial \frac{M\dot{x}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{x}^2}{2}}{\partial \theta}$$

$$+ F_y \frac{\partial y}{\partial \theta} + m\ddot{y} \frac{\partial y}{\partial \theta} + \frac{d}{dt} \frac{\partial \frac{M\dot{y}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{y}^2}{2}}{\partial \theta}$$

Set: $d\theta=0$ $d\phi=1$

$$F_X \frac{\partial X}{\partial \phi} = M\ddot{X} \frac{\partial X}{\partial \phi} = \frac{d}{dt} \frac{\partial \frac{M\dot{X}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{X}^2}{2}}{\partial \phi}$$

$$+ F_Y \frac{\partial Y}{\partial \phi} + M\ddot{Y} \frac{\partial Y}{\partial \phi} + \frac{d}{dt} \frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \phi}$$

$$+ F_x \frac{\partial x}{\partial \phi} + m\ddot{x} \frac{\partial x}{\partial \phi} + \frac{d}{dt} \frac{\partial \frac{M\dot{x}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{x}^2}{2}}{\partial \phi}$$

$$+ F_y \frac{\partial y}{\partial \phi} + m\ddot{y} \frac{\partial y}{\partial \phi} + \frac{d}{dt} \frac{\partial \frac{M\dot{y}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{y}^2}{2}}{\partial \phi}$$

Force, Work, and Acceleration

$$dW = F_X dX + F_Y dY + F_x dx + F_y dy$$

$$= M\ddot{X}dX + M\ddot{Y}dY + m\ddot{x}dx + m\ddot{y}dy$$

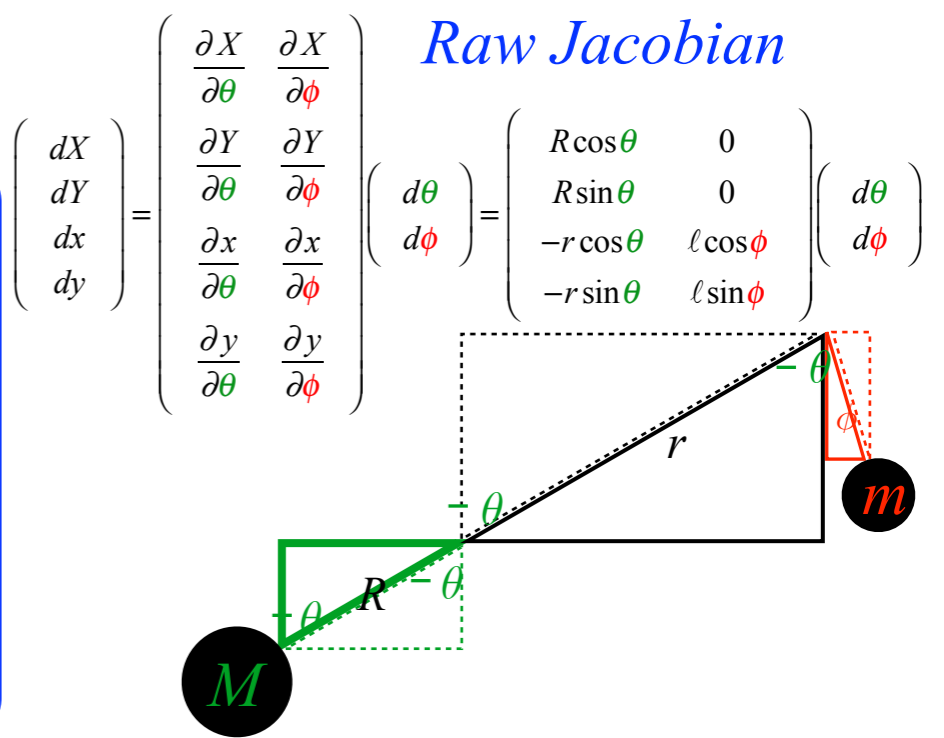
Write work-sums in columns: (Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$dW = F_X dX = M\ddot{X}dX = F_X \frac{\partial X}{\partial \theta} d\theta + F_X \frac{\partial X}{\partial \phi} d\phi = M\ddot{X} \frac{\partial X}{\partial \theta} d\theta + M\ddot{X} \frac{\partial X}{\partial \phi} d\phi$$

$$+ F_Y dY + M\ddot{Y}dY + F_Y \frac{\partial Y}{\partial \theta} d\theta + F_Y \frac{\partial Y}{\partial \phi} d\phi + M\ddot{Y} \frac{\partial Y}{\partial \theta} d\theta + M\ddot{Y} \frac{\partial Y}{\partial \phi} d\phi$$

$$+ F_x dx + m\ddot{x}dx + F_x \frac{\partial x}{\partial \theta} d\theta + F_x \frac{\partial x}{\partial \phi} d\phi + m\ddot{x} \frac{\partial x}{\partial \theta} d\theta + m\ddot{x} \frac{\partial x}{\partial \phi} d\phi$$

$$+ F_y dy + m\ddot{y}dy + F_y \frac{\partial y}{\partial \theta} d\theta + F_y \frac{\partial y}{\partial \phi} d\phi + m\ddot{y} \frac{\partial y}{\partial \theta} d\theta + m\ddot{y} \frac{\partial y}{\partial \phi} d\phi$$



Lagrange trickery:

STEP D Add up first and last columns for each variable θ and ϕ for: $T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$

Let: $F_X \frac{\partial X}{\partial \theta} + F_Y \frac{\partial Y}{\partial \theta} + F_x \frac{\partial x}{\partial \theta} + F_y \frac{\partial y}{\partial \theta} \equiv F_\theta$ Defines F_θ

$$\equiv F_\theta = \frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta}$$

Set: $d\theta=1$ $d\phi=0$

$$F_X \frac{\partial X}{\partial \theta} = M\ddot{X} \frac{\partial X}{\partial \theta} = \frac{d}{dt} \frac{\partial \frac{M\dot{X}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{X}^2}{2}}{\partial \theta}$$

$$+ F_Y \frac{\partial Y}{\partial \theta} + M\ddot{Y} \frac{\partial Y}{\partial \theta} + \frac{d}{dt} \frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \theta}$$

$$+ F_x \frac{\partial x}{\partial \theta} + m\ddot{x} \frac{\partial x}{\partial \theta} + \frac{d}{dt} \frac{\partial \frac{M\dot{x}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{x}^2}{2}}{\partial \theta}$$

$$+ F_y \frac{\partial y}{\partial \theta} + m\ddot{y} \frac{\partial y}{\partial \theta} + \frac{d}{dt} \frac{\partial \frac{M\dot{y}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{y}^2}{2}}{\partial \theta}$$

Set: $d\theta=0$ $d\phi=1$

$$F_X \frac{\partial X}{\partial \phi} = M\ddot{X} \frac{\partial X}{\partial \phi} = \frac{d}{dt} \frac{\partial \frac{M\dot{X}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{X}^2}{2}}{\partial \phi}$$

$$+ F_Y \frac{\partial Y}{\partial \phi} + M\ddot{Y} \frac{\partial Y}{\partial \phi} + \frac{d}{dt} \frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \phi}$$

$$+ F_x \frac{\partial x}{\partial \phi} + m\ddot{x} \frac{\partial x}{\partial \phi} + \frac{d}{dt} \frac{\partial \frac{M\dot{x}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{x}^2}{2}}{\partial \phi}$$

$$+ F_y \frac{\partial y}{\partial \phi} + m\ddot{y} \frac{\partial y}{\partial \phi} + \frac{d}{dt} \frac{\partial \frac{M\dot{y}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{y}^2}{2}}{\partial \phi}$$

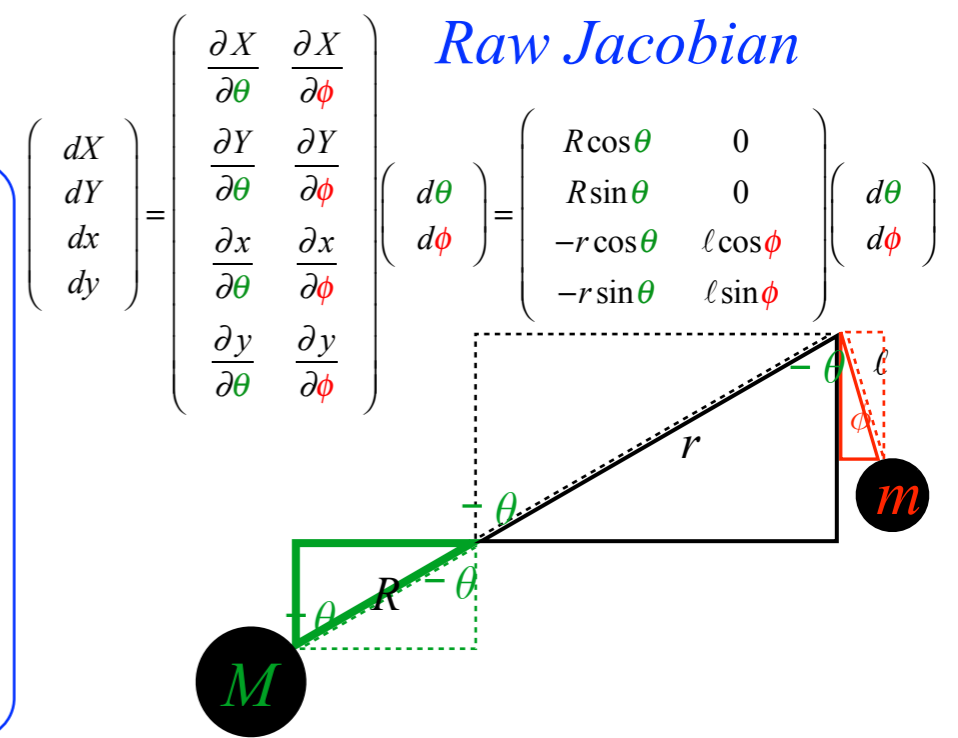
Force, Work, and Acceleration

$$dW = F_X dX + F_Y dY + F_x dx + F_y dy$$

$$= M\ddot{X}dX + M\ddot{Y}dY + m\ddot{x}dx + m\ddot{y}dy$$

Write work-sums in columns: (Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$\begin{aligned}
 dW = F_X dX &= M\ddot{X}dX = F_X \frac{\partial X}{\partial \theta} d\theta + F_X \frac{\partial X}{\partial \phi} d\phi = M\ddot{X} \frac{\partial X}{\partial \theta} d\theta + M\ddot{X} \frac{\partial X}{\partial \phi} d\phi \\
 + F_Y dY &+ M\ddot{Y}dY + F_Y \frac{\partial Y}{\partial \theta} d\theta + F_Y \frac{\partial Y}{\partial \phi} d\phi + M\ddot{Y} \frac{\partial Y}{\partial \theta} d\theta + M\ddot{Y} \frac{\partial Y}{\partial \phi} d\phi \\
 + F_x dx &+ m\ddot{x}dx + F_x \frac{\partial x}{\partial \theta} d\theta + F_x \frac{\partial x}{\partial \phi} d\phi + m\ddot{x} \frac{\partial x}{\partial \theta} d\theta + m\ddot{x} \frac{\partial x}{\partial \phi} d\phi \\
 + F_y dy &+ m\ddot{y}dy + F_y \frac{\partial y}{\partial \theta} d\theta + F_y \frac{\partial y}{\partial \phi} d\phi + m\ddot{y} \frac{\partial y}{\partial \theta} d\theta + m\ddot{y} \frac{\partial y}{\partial \phi} d\phi
 \end{aligned}$$



Lagrange trickery:

STEP D

Add up first and last columns for each variable θ and ϕ for: $T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$

Let: $F_X \frac{\partial X}{\partial \theta} + F_Y \frac{\partial Y}{\partial \theta} + F_x \frac{\partial x}{\partial \theta} + F_y \frac{\partial y}{\partial \theta} \equiv F_\theta$ Defines F_θ

$$\equiv F_\theta = \frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta}$$

Let: $F_X \frac{\partial X}{\partial \phi} + F_Y \frac{\partial Y}{\partial \phi} + F_x \frac{\partial x}{\partial \phi} + F_y \frac{\partial y}{\partial \phi} \equiv F_\phi$ Defines F_ϕ

$$\equiv F_\phi = \frac{d}{dt} \frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi}$$

Set: $d\theta=1$ $d\phi=0$

$$\begin{aligned}
 F_X \frac{\partial X}{\partial \theta} &= M\ddot{X} \frac{\partial X}{\partial \theta} = \frac{d}{dt} \frac{\partial \frac{M\dot{X}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{X}^2}{2}}{\partial \theta} \\
 + F_Y \frac{\partial Y}{\partial \theta} &+ M\ddot{Y} \frac{\partial Y}{\partial \theta} + \frac{d}{dt} \frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \theta} \\
 + F_x \frac{\partial x}{\partial \theta} &+ m\ddot{x} \frac{\partial x}{\partial \theta} + \frac{d}{dt} \frac{\partial \frac{M\dot{x}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{x}^2}{2}}{\partial \theta} \\
 + F_y \frac{\partial y}{\partial \theta} &+ m\ddot{y} \frac{\partial y}{\partial \theta} + \frac{d}{dt} \frac{\partial \frac{M\dot{y}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{y}^2}{2}}{\partial \theta}
 \end{aligned}$$

Set: $d\theta=0$ $d\phi=1$

$$\begin{aligned}
 F_X \frac{\partial X}{\partial \phi} &= M\ddot{X} \frac{\partial X}{\partial \phi} = \frac{d}{dt} \frac{\partial \frac{M\dot{X}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{X}^2}{2}}{\partial \phi} \\
 + F_Y \frac{\partial Y}{\partial \phi} &+ M\ddot{Y} \frac{\partial Y}{\partial \phi} + \frac{d}{dt} \frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \phi} \\
 + F_x \frac{\partial x}{\partial \phi} &+ m\ddot{x} \frac{\partial x}{\partial \phi} + \frac{d}{dt} \frac{\partial \frac{M\dot{x}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{x}^2}{2}}{\partial \phi} \\
 + F_y \frac{\partial y}{\partial \phi} &+ m\ddot{y} \frac{\partial y}{\partial \phi} + \frac{d}{dt} \frac{\partial \frac{M\dot{y}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{y}^2}{2}}{\partial \phi}
 \end{aligned}$$

Review of Lagrangian equation derivation from Lecture 10 (Now with trebuchet model)

Coordinate geometry, Jacobian, velocity, kinetic energy, and dynamic metric tensor γ_{mn}

Structure of dynamic metric tensor γ_{mn}

Basic force, work, and acceleration

 *Lagrangian force equation*

Canonical momentum and γ_{mn} tensor

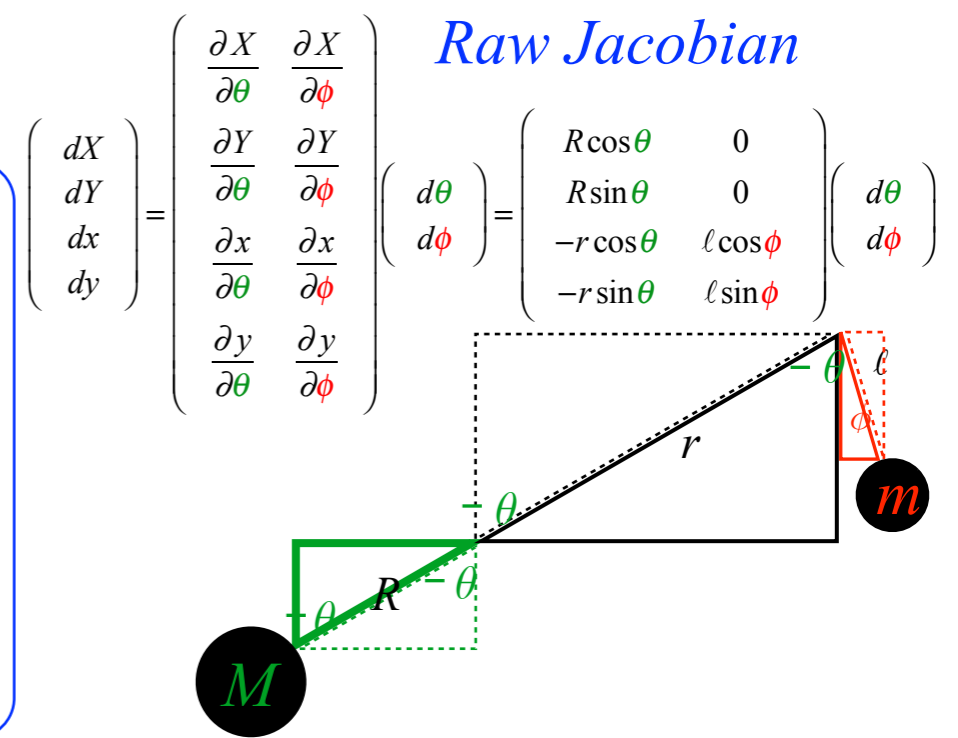
Force, Work, and Acceleration

$$dW = F_X dX + F_Y dY + F_x dx + F_y dy$$

$$= M\ddot{X}dX + M\ddot{Y}dY + m\ddot{x}dx + m\ddot{y}dy$$

Write work-sums in columns: (Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$\begin{aligned}
 dW = F_X dX &= M\ddot{X}dX = F_X \frac{\partial X}{\partial \theta} d\theta + F_X \frac{\partial X}{\partial \phi} d\phi = M\ddot{X} \frac{\partial X}{\partial \theta} d\theta + M\ddot{X} \frac{\partial X}{\partial \phi} d\phi \\
 + F_Y dY &+ M\ddot{Y}dY + F_Y \frac{\partial Y}{\partial \theta} d\theta + F_Y \frac{\partial Y}{\partial \phi} d\phi + M\ddot{Y} \frac{\partial Y}{\partial \theta} d\theta + M\ddot{Y} \frac{\partial Y}{\partial \phi} d\phi \\
 + F_x dx &+ m\ddot{x}dx + F_x \frac{\partial x}{\partial \theta} d\theta + F_x \frac{\partial x}{\partial \phi} d\phi + m\ddot{x} \frac{\partial x}{\partial \theta} d\theta + m\ddot{x} \frac{\partial x}{\partial \phi} d\phi \\
 + F_y dy &+ m\ddot{y}dy + F_y \frac{\partial y}{\partial \theta} d\theta + F_y \frac{\partial y}{\partial \phi} d\phi + m\ddot{y} \frac{\partial y}{\partial \theta} d\theta + m\ddot{y} \frac{\partial y}{\partial \phi} d\phi
 \end{aligned}$$



Lagrange trickery:

STEP D

Add up first and last columns for each variable θ and ϕ for:

$$T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$$

$$\begin{aligned}
 \text{Let: } &F_X \frac{\partial X}{\partial \theta} + F_Y \frac{\partial Y}{\partial \theta} + F_x \frac{\partial x}{\partial \theta} + F_y \frac{\partial y}{\partial \theta} \\
 \equiv F_\theta &= \frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta}
 \end{aligned}$$

$$\begin{aligned}
 \text{Let: } &F_X \frac{\partial X}{\partial \phi} + F_Y \frac{\partial Y}{\partial \phi} + F_x \frac{\partial x}{\partial \phi} + F_y \frac{\partial y}{\partial \phi} \\
 \equiv F_\phi &= \frac{d}{dt} \frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi}
 \end{aligned}$$

Completes derivation of Lagrange covariant-force equation for each GCC variable θ and ϕ .

$$\begin{aligned}
 &F_X R \cos \theta + F_Y R \sin \theta - F_x r \cos \theta - F_y r \sin \theta \\
 \equiv F_\theta &= \frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta}
 \end{aligned}$$

$$\begin{aligned}
 &F_X \cdot 0 + F_Y \cdot 0 + F_x \ell \cos \phi + F_y \ell \sin \phi \\
 \equiv F_\phi &= \frac{d}{dt} \frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi}
 \end{aligned}$$

Force, Work, and Acceleration

$$dW = F_X dX + F_Y dY + F_x dx + F_y dy$$

$$= M\ddot{X} dX + M\ddot{Y} dY + m\ddot{x} dx + m\ddot{y} dy$$

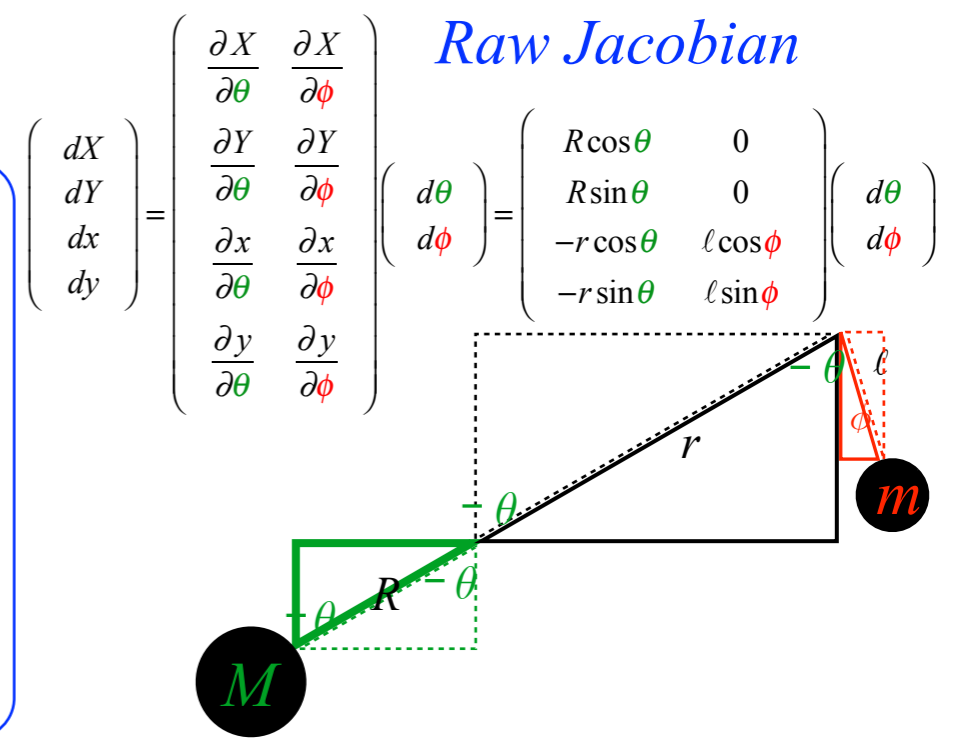
Write work-sums in columns: (Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$dW = F_X dX = M\ddot{X} dX = F_X \frac{\partial X}{\partial \theta} d\theta + F_X \frac{\partial X}{\partial \phi} d\phi = M\ddot{X} \frac{\partial X}{\partial \theta} d\theta + M\ddot{X} \frac{\partial X}{\partial \phi} d\phi$$

$$+ F_Y dY + M\ddot{Y} dY + F_Y \frac{\partial Y}{\partial \theta} d\theta + F_Y \frac{\partial Y}{\partial \phi} d\phi + M\ddot{Y} \frac{\partial Y}{\partial \theta} d\theta + M\ddot{Y} \frac{\partial Y}{\partial \phi} d\phi$$

$$+ F_x dx + m\ddot{x} dx + F_x \frac{\partial x}{\partial \theta} d\theta + F_x \frac{\partial x}{\partial \phi} d\phi + m\ddot{x} \frac{\partial x}{\partial \theta} d\theta + m\ddot{x} \frac{\partial x}{\partial \phi} d\phi$$

$$+ F_y dy + m\ddot{y} dy + F_y \frac{\partial y}{\partial \theta} d\theta + F_y \frac{\partial y}{\partial \phi} d\phi + m\ddot{y} \frac{\partial y}{\partial \theta} d\theta + m\ddot{y} \frac{\partial y}{\partial \phi} d\phi$$



Lagrange trickery:

STEP D Add up first and last columns for each variable θ and ϕ for:

$$T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$$

Let: $F_X \frac{\partial X}{\partial \theta} + F_Y \frac{\partial Y}{\partial \theta} + F_x \frac{\partial x}{\partial \theta} + F_y \frac{\partial y}{\partial \theta} \equiv F_\theta$ (Defines F_θ)

$$\equiv F_\theta = \frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta}$$

Let: $F_X \frac{\partial X}{\partial \phi} + F_Y \frac{\partial Y}{\partial \phi} + F_x \frac{\partial x}{\partial \phi} + F_y \frac{\partial y}{\partial \phi} \equiv F_\phi$ (Defines F_ϕ)

$$\equiv F_\phi = \frac{d}{dt} \frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi}$$

Completes derivation of Lagrange covariant-force equation for each GCC variable θ and ϕ .

$$F_X R \cos \theta + F_Y R \sin \theta - F_x r \cos \theta - F_y r \sin \theta$$

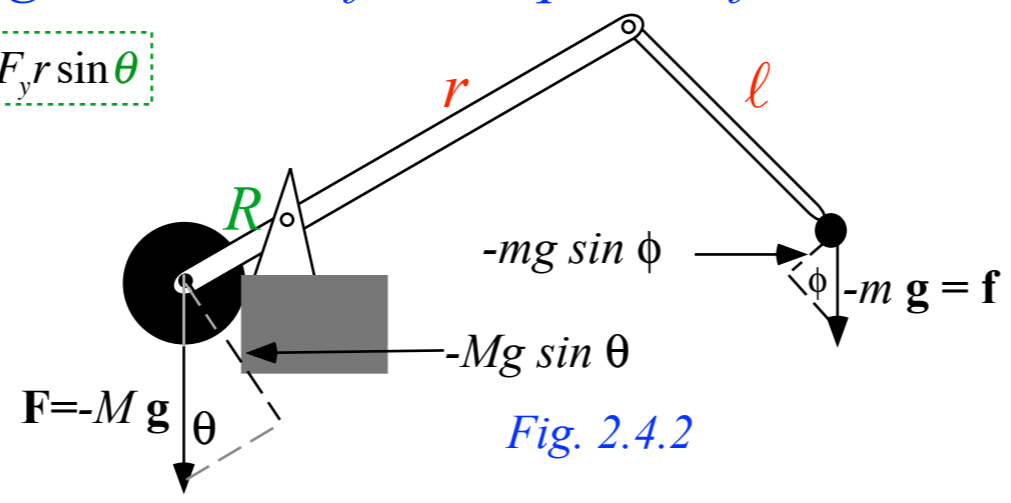
$$\equiv F_\theta = \frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta}$$

Add F_θ gravity given

$(F_X=0, F_Y=-Mg)$

$(F_x=0, F_y=-mg)$

$$F_\theta = \frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta} = -MgR \sin \theta + mgr \sin \theta$$



$$F_X \cdot 0 + F_Y \cdot 0 + F_x \ell \cos \phi + F_y \ell \sin \phi$$

$$\equiv F_\phi = \frac{d}{dt} \frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi}$$

These are competing torques on main beam R

Force, Work, and Acceleration

$$dW = F_X dX + F_Y dY + F_x dx + F_y dy$$

$$= M\ddot{X} dX + M\ddot{Y} dY + m\ddot{x} dx + m\ddot{y} dy$$

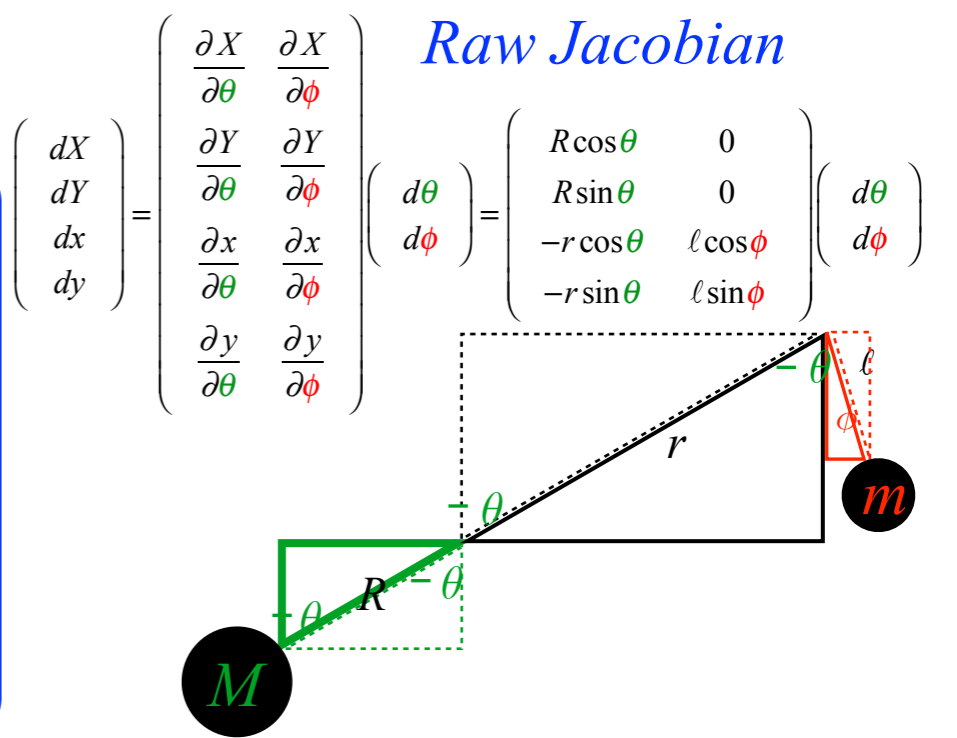
Write work-sums in columns: (Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$dW = F_X dX = M\ddot{X} dX = F_X \frac{\partial X}{\partial \theta} d\theta + F_X \frac{\partial X}{\partial \phi} d\phi = M\ddot{X} \frac{\partial X}{\partial \theta} d\theta + M\ddot{X} \frac{\partial X}{\partial \phi} d\phi$$

$$+ F_Y dY + M\ddot{Y} dY + F_Y \frac{\partial Y}{\partial \theta} d\theta + F_Y \frac{\partial Y}{\partial \phi} d\phi + M\ddot{Y} \frac{\partial Y}{\partial \theta} d\theta + M\ddot{Y} \frac{\partial Y}{\partial \phi} d\phi$$

$$+ F_x dx + m\ddot{x} dx + F_x \frac{\partial x}{\partial \theta} d\theta + F_x \frac{\partial x}{\partial \phi} d\phi + m\ddot{x} \frac{\partial x}{\partial \theta} d\theta + m\ddot{x} \frac{\partial x}{\partial \phi} d\phi$$

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Lagrange trickery:

STEP D Add up first and last columns for each variable θ and ϕ for:

$$T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$$

Let: $F_X \frac{\partial X}{\partial \theta} + F_Y \frac{\partial Y}{\partial \theta} + F_x \frac{\partial x}{\partial \theta} + F_y \frac{\partial y}{\partial \theta} \equiv F_\theta$ (Defines F_θ)

$$\equiv F_\theta = \frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta}$$

Let: $F_X \frac{\partial X}{\partial \phi} + F_Y \frac{\partial Y}{\partial \phi} + F_x \frac{\partial x}{\partial \phi} + F_y \frac{\partial y}{\partial \phi} \equiv F_\phi$ (Defines F_ϕ)

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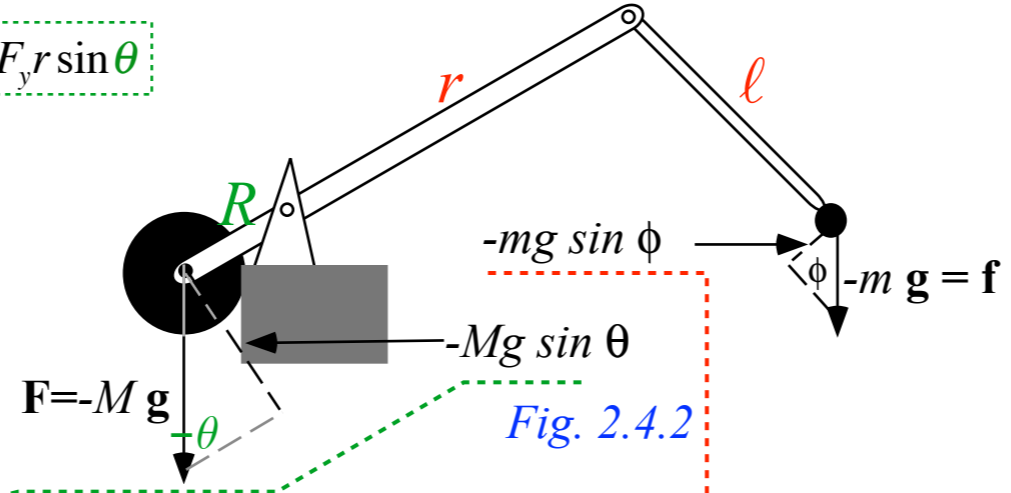
Completes derivation of Lagrange covariant-force equation for each GCC variable θ and ϕ .

$$F_X R \cos \theta + F_Y R \sin \theta - F_x r \cos \theta - F_y r \sin \theta$$

$$\equiv F_\theta = \frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta}$$

Add F_θ gravity given
 $(F_X = 0, F_Y = -Mg)$
 $(F_x = 0, F_y = -mg)$

$$F_\theta = \frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta} = -MgR \sin \theta + mgr \sin \theta$$



$$F_X \cdot 0 + F_Y \cdot 0 + F_x \ell \cos \phi + F_y \ell \sin \phi$$

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These are competing torques on main beam R...

... and a torque on throwing lever l

Force, Work, and Acceleration

$$dW = F_X dX + F_Y dY + F_x dx + F_y dy$$

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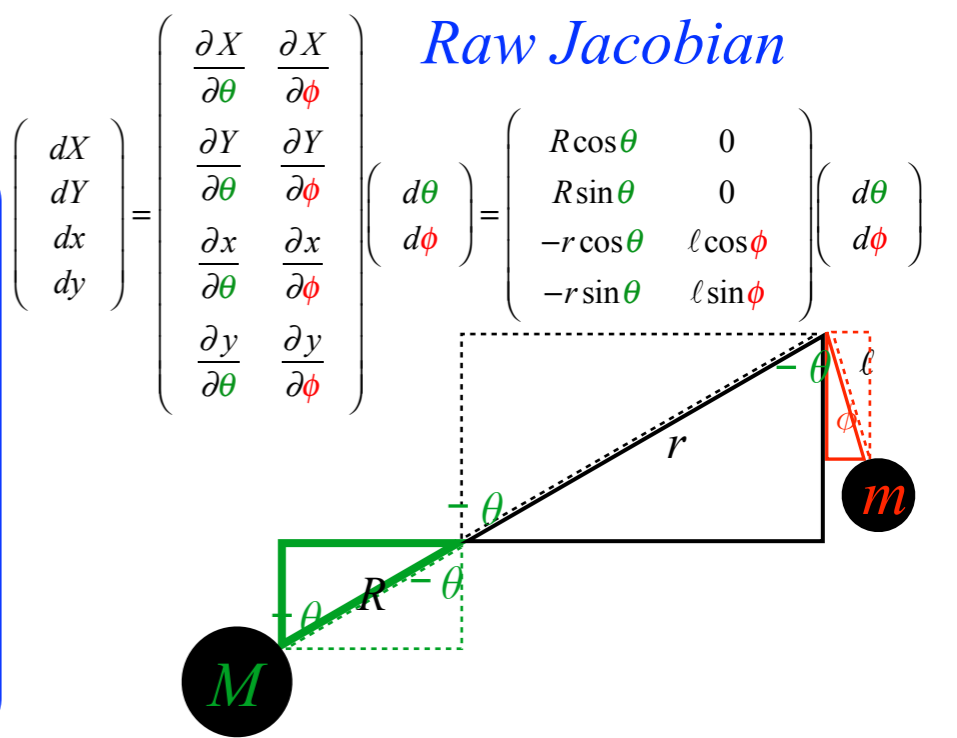
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$$+ F_Y dY + M\ddot{Y}dY + F_Y \frac{\partial Y}{\partial \theta} d\theta + F_Y \frac{\partial Y}{\partial \phi} d\phi + M\ddot{Y} \frac{\partial Y}{\partial \theta} d\theta + M\ddot{Y} \frac{\partial Y}{\partial \phi} d\phi$$

$$+ F_x dx + m\ddot{x}dx + F_x \frac{\partial x}{\partial \theta} d\theta + F_x \frac{\partial x}{\partial \phi} d\phi + m\ddot{x} \frac{\partial x}{\partial \theta} d\theta + m\ddot{x} \frac{\partial x}{\partial \phi} d\phi$$

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Lagrange trickery:

STEP D Add up first and last columns for each variable θ and ϕ for:

$$T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$$

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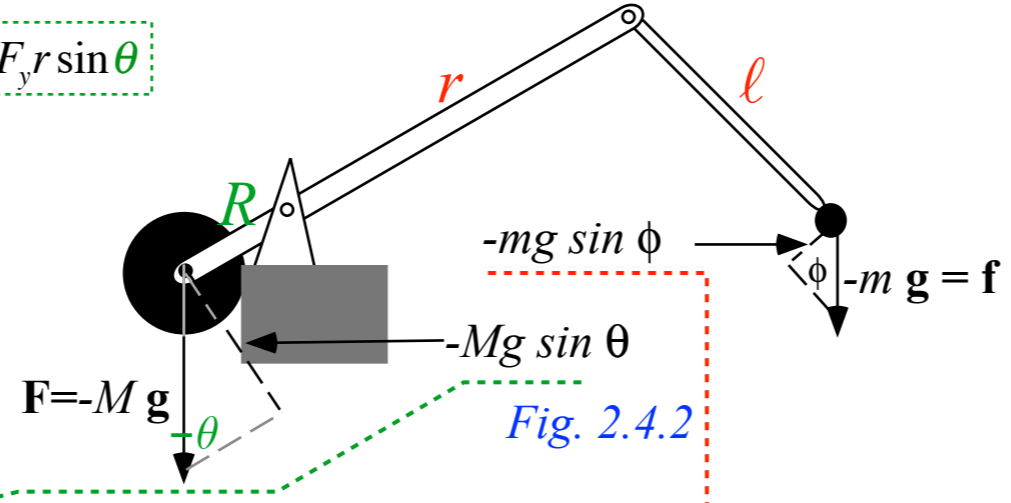
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 $(F_X = 0, F_Y = -Mg)$
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$$F_\theta = \frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta} = -MgR \sin \theta + mgr \sin \theta$$



Q: Are there \pm sign errors here?

$$F_X \cdot 0 + F_Y \cdot 0 + F_x \ell \cos \phi + F_y \ell \sin \phi$$

$$\equiv F_\phi = \frac{d}{dt} \frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi}$$

Add F_ϕ gravity given
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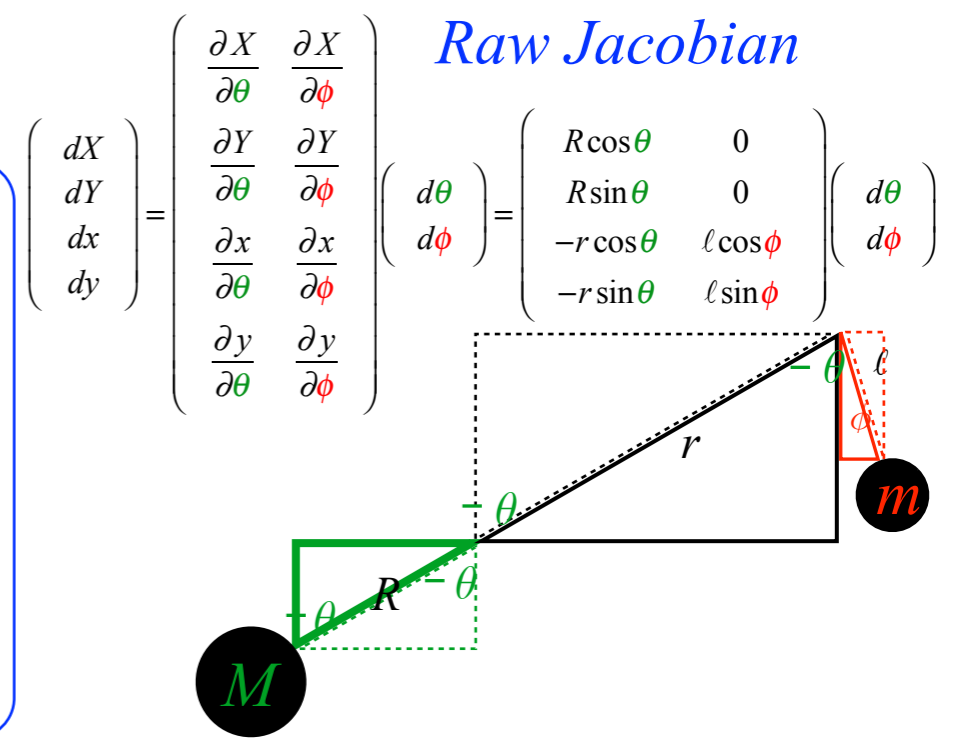
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$$dW = F_X dX = M\ddot{X}dX = F_X \frac{\partial X}{\partial \theta} d\theta + F_X \frac{\partial X}{\partial \phi} d\phi = M\ddot{X} \frac{\partial X}{\partial \theta} d\theta + M\ddot{X} \frac{\partial X}{\partial \phi} d\phi$$

$$+ F_Y dY + M\ddot{Y}dY + F_Y \frac{\partial Y}{\partial \theta} d\theta + F_Y \frac{\partial Y}{\partial \phi} d\phi + M\ddot{Y} \frac{\partial Y}{\partial \theta} d\theta + M\ddot{Y} \frac{\partial Y}{\partial \phi} d\phi$$

$$+ F_x dx + m\ddot{x}dx + F_x \frac{\partial x}{\partial \theta} d\theta + F_x \frac{\partial x}{\partial \phi} d\phi + m\ddot{x} \frac{\partial x}{\partial \theta} d\theta + m\ddot{x} \frac{\partial x}{\partial \phi} d\phi$$

$$+ F_y dy + m\ddot{y}dy + F_y \frac{\partial y}{\partial \theta} d\theta + F_y \frac{\partial y}{\partial \phi} d\phi + m\ddot{y} \frac{\partial y}{\partial \theta} d\theta + m\ddot{y} \frac{\partial y}{\partial \phi} d\phi$$



Lagrange trickery:

STEP D

Add up first and last columns for each variable θ and ϕ for:

$$T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$$

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$$\equiv F_\theta = \frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta}$$

Let: $F_X \frac{\partial X}{\partial \phi} + F_Y \frac{\partial Y}{\partial \phi} + F_x \frac{\partial x}{\partial \phi} + F_y \frac{\partial y}{\partial \phi} \equiv F_\phi$ (Defines F_ϕ)

$$\equiv F_\phi = \frac{d}{dt} \frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi}$$

Completes derivation of Lagrange covariant-force equation for each GCC variable θ and ϕ .

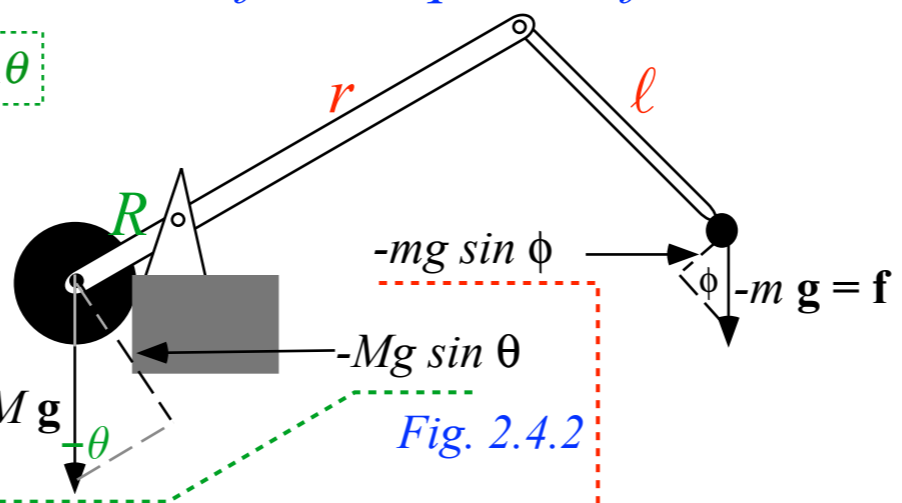
$$F_X R \cos \theta + F_Y R \sin \theta - F_x r \cos \theta - F_y r \sin \theta$$

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Add F_θ gravity given
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These are competing torques on main beam R...



Q: Are there \pm sign errors here?
 A: No. Beam in $-\theta$ position.

$$F_X \cdot 0 + F_Y \cdot 0 + F_x \ell \cos \phi + F_y \ell \sin \phi$$

$$\equiv F_\phi = \frac{d}{dt} \frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi}$$

Add F_ϕ gravity given
 $(F_X=0, F_Y=-Mg)$
 $(F_x=0, F_y=-mg)$

$$F_\phi = \frac{d}{dt} \frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi} = -mgl \sin \phi$$

... and a torque on throwing lever l

Review of Lagrangian equation derivation from Lecture 10 (Now with trebuchet model)

Coordinate geometry, Jacobian, velocity, kinetic energy, and dynamic metric tensor γ_{mn}

Structure of dynamic metric tensor γ_{mn}

Basic force, work, and acceleration

Lagrangian force equation

 *Canonical momentum and γ_{mn} tensor*

Canonical momentum and γ_{mn} tensor

Standard formulation of $p_m = \frac{\partial T}{\partial \dot{q}^m}$

Total KE = $T = T(M) + T(m)$

$$= \frac{1}{2} \left[(MR^2 + mr^2) \dot{\theta}^2 - 2mr\ell \cos(\theta - \phi) \dot{\theta} \dot{\phi} + m\ell^2 \dot{\phi}^2 \right]$$

The γ_{mn} tensor/matrix formulation

Total KE = $T = T(M) + T(m)$

$$= \frac{1}{2} \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

where:

$$\gamma_{mn} \text{ tensor is } \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell \cos(\theta - \phi) \\ -mr\ell \cos(\theta - \phi) & m\ell^2 \end{pmatrix}$$

Canonical momentum and γ_{mn} tensor

Standard formulation of $p_m = \frac{\partial T}{\partial \dot{q}^m}$

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$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}} \left(\frac{1}{2} (MR^2 + mr^2) \dot{\theta}^2 - mr\ell \cos(\theta - \phi) \dot{\theta} \dot{\phi} + \frac{1}{2} m\ell^2 \dot{\phi}^2 \right)$$

$$= (MR^2 + mr^2) \dot{\theta} - mr\ell \dot{\phi} \cos(\theta - \phi)$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}} \left(\frac{1}{2} (MR^2 + mr^2) \dot{\theta}^2 - mr\ell \cos(\theta - \phi) \dot{\theta} \dot{\phi} + \frac{1}{2} m\ell^2 \dot{\phi}^2 \right)$$

$$= m\ell^2 \dot{\phi} - mr\ell \dot{\theta} \cos(\theta - \phi)$$

The γ_{mn} tensor/matrix formulation

$$\text{Total KE} = T = T(M) + T(m)$$

$$= \frac{1}{2} \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

$$\text{where: } \gamma_{mn} \text{ tensor is } \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell \cos(\theta - \phi) \\ -mr\ell \cos(\theta - \phi) & m\ell^2 \end{pmatrix}$$

Canonical momentum and γ_{mn} tensor

Standard formulation of $p_m = \frac{\partial T}{\partial \dot{q}^m}$

$$\text{Total KE} = T = T(\mathbf{M}) + T(\mathbf{m})$$

$$= \frac{1}{2} \left[(MR^2 + mr^2) \dot{\theta}^2 - 2mr\ell \cos(\theta - \phi) \dot{\theta} \dot{\phi} + ml^2 \dot{\phi}^2 \right]$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}} \left(\frac{1}{2} (MR^2 + mr^2) \dot{\theta}^2 - mr\ell \cos(\theta - \phi) \dot{\theta} \dot{\phi} + \frac{1}{2} ml^2 \dot{\phi}^2 \right)$$

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The γ_{mn} tensor/matrix formulation

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Momentum γ_{mn} -matrix theorem: (matrix-proof on page 78)

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial T}{\partial \dot{\theta}} \\ \frac{\partial T}{\partial \dot{\phi}} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \text{ if: } \gamma_{\phi,\theta} = \gamma_{\theta,\phi} \text{ (symmetry)}$$

$$= \begin{pmatrix} MR^2 + mr^2 & -mr\ell \cos(\theta - \phi) \\ -mr\ell \cos(\theta - \phi) & ml^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

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$$= \frac{1}{2} \left[(MR^2 + mr^2) \dot{\theta}^2 - 2mrl \cos(\theta - \phi) \dot{\theta} \dot{\phi} + ml^2 \dot{\phi}^2 \right]$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}} \left(\frac{1}{2} (MR^2 + mr^2) \dot{\theta}^2 - mrl \cos(\theta - \phi) \dot{\theta} \dot{\phi} + \frac{1}{2} ml^2 \dot{\phi}^2 \right)$$

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$$= ml^2 \dot{\phi} - mrl \dot{\theta} \cos(\theta - \phi)$$

The γ_{mn} tensor/matrix formulation

$$\text{Total KE} = T = T(\mathbf{M}) + T(\mathbf{m})$$

$$= \frac{1}{2} \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

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$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mrl \cos(\theta - \phi) \\ -mrl \cos(\theta - \phi) & ml^2 \end{pmatrix}$$

Momentum γ_{mn} -matrix theorem: (matrix-proof on page 43)

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial T}{\partial \dot{\theta}} \\ \frac{\partial T}{\partial \dot{\phi}} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \text{ if: } \gamma_{\phi,\theta} = \gamma_{\theta,\phi} \text{ (symmetry)}$$

$$= \begin{pmatrix} MR^2 + mr^2 & -mrl \cos(\theta - \phi) \\ -mrl \cos(\theta - \phi) & ml^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Momentum γ_{mn} -tensor theorem: (proof here)

$$p_m = \gamma_{mn} \dot{q}^n$$

proof: Given: $p_m = \frac{\partial T}{\partial \dot{q}^m}$ where: $T = \frac{1}{2} \gamma_{jk} \dot{q}^j \dot{q}^k$

Then: $p_m = \frac{\partial}{\partial \dot{q}^m} \frac{1}{2} \gamma_{jk} \dot{q}^j \dot{q}^k = \frac{1}{2} \gamma_{jk} \frac{\partial \dot{q}^j}{\partial \dot{q}^m} \dot{q}^k + \frac{1}{2} \gamma_{jk} \dot{q}^j \frac{\partial \dot{q}^k}{\partial \dot{q}^m}$

Canonical momentum and γ_{mn} tensor

Standard formulation of $p_m = \frac{\partial T}{\partial \dot{q}^m}$

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$$= \frac{1}{2} \left[(MR^2 + mr^2) \dot{\theta}^2 - 2mrl \cos(\theta - \phi) \dot{\theta} \dot{\phi} + ml^2 \dot{\phi}^2 \right]$$

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$$= (MR^2 + mr^2) \dot{\theta} - mrl \dot{\phi} \cos(\theta - \phi)$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}} \left(\frac{1}{2} (MR^2 + mr^2) \dot{\theta}^2 - mrl \cos(\theta - \phi) \dot{\theta} \dot{\phi} + \frac{1}{2} ml^2 \dot{\phi}^2 \right)$$

$$= ml^2 \dot{\phi} - mrl \dot{\theta} \cos(\theta - \phi)$$

The γ_{mn} tensor/matrix formulation

Total KE = $T = T(M) + T(m)$

$$= \frac{1}{2} \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

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$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial T}{\partial \dot{\theta}} \\ \frac{\partial T}{\partial \dot{\phi}} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \text{ if: } \gamma_{\phi,\theta} = \gamma_{\theta,\phi} \text{ (symmetry)}$$

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$$p_m = \gamma_{mn} \dot{q}^n$$

proof: Given: $p_m = \frac{\partial T}{\partial \dot{q}^m}$ where: $T = \frac{1}{2} \gamma_{jk} \dot{q}^j \dot{q}^k$

$$\text{Then: } p_m = \frac{\partial}{\partial \dot{q}^m} \frac{1}{2} \gamma_{jk} \dot{q}^j \dot{q}^k = \frac{1}{2} \gamma_{jk} \frac{\partial \dot{q}^j}{\partial \dot{q}^m} \dot{q}^k + \frac{1}{2} \gamma_{jk} \dot{q}^j \frac{\partial \dot{q}^k}{\partial \dot{q}^m}$$

$$= \frac{1}{2} \gamma_{jk} \delta_m^j \dot{q}^k + \frac{1}{2} \gamma_{jk} \dot{q}^j \delta_m^k$$

Canonical momentum and γ_{mn} tensor

Standard formulation of $p_m = \frac{\partial T}{\partial \dot{q}^m}$

$$\text{Total KE} = T = T(\mathbf{M}) + T(\mathbf{m})$$

$$= \frac{1}{2} \left[(MR^2 + mr^2) \dot{\theta}^2 - 2mrl \cos(\theta - \phi) \dot{\theta} \dot{\phi} + ml^2 \dot{\phi}^2 \right]$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}} \left(\frac{1}{2} (MR^2 + mr^2) \dot{\theta}^2 - mrl \cos(\theta - \phi) \dot{\theta} \dot{\phi} + \frac{1}{2} ml^2 \dot{\phi}^2 \right)$$

$$= (MR^2 + mr^2) \dot{\theta} - mrl \dot{\phi} \cos(\theta - \phi)$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}} \left(\frac{1}{2} (MR^2 + mr^2) \dot{\theta}^2 - mrl \cos(\theta - \phi) \dot{\theta} \dot{\phi} + \frac{1}{2} ml^2 \dot{\phi}^2 \right)$$

$$= ml^2 \dot{\phi} - mrl \dot{\theta} \cos(\theta - \phi)$$

The γ_{mn} tensor/matrix formulation

$$\text{Total KE} = T = T(\mathbf{M}) + T(\mathbf{m})$$

$$= \frac{1}{2} \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

where: γ_{mn} tensor is
$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mrl \cos(\theta - \phi) \\ -mrl \cos(\theta - \phi) & ml^2 \end{pmatrix}$$

Momentum γ_{mn} -matrix theorem: (matrix-proof on page 43)

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial T}{\partial \dot{\theta}} \\ \frac{\partial T}{\partial \dot{\phi}} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \quad \text{if: } \gamma_{\phi,\theta} = \gamma_{\theta,\phi} \text{ (symmetry)}$$

$$= \begin{pmatrix} MR^2 + mr^2 & -mrl \cos(\theta - \phi) \\ -mrl \cos(\theta - \phi) & ml^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Momentum γ_{mn} -tensor theorem: (proof here)

$$p_m = \gamma_{mn} \dot{q}^n$$

proof: Given: $p_m = \frac{\partial T}{\partial \dot{q}^m}$ where: $T = \frac{1}{2} \gamma_{jk} \dot{q}^j \dot{q}^k$

$$\text{Then: } p_m = \frac{\partial}{\partial \dot{q}^m} \frac{1}{2} \gamma_{jk} \dot{q}^j \dot{q}^k = \frac{1}{2} \gamma_{jk} \frac{\partial \dot{q}^j}{\partial \dot{q}^m} \dot{q}^k + \frac{1}{2} \gamma_{jk} \dot{q}^j \frac{\partial \dot{q}^k}{\partial \dot{q}^m}$$

$$= \frac{1}{2} \gamma_{jk} \delta_m^j \dot{q}^k + \frac{1}{2} \gamma_{jk} \dot{q}^j \delta_m^k = \frac{1}{2} \gamma_{mk} \dot{q}^k + \frac{1}{2} \gamma_{jm} \dot{q}^j$$

Canonical momentum and γ_{mn} tensor

Standard formulation of $p_m = \frac{\partial T}{\partial \dot{q}^m}$

$$\text{Total KE} = T = T(M) + T(m)$$

$$= \frac{1}{2} \left[(MR^2 + mr^2) \dot{\theta}^2 - 2mrl \cos(\theta - \phi) \dot{\theta} \dot{\phi} + ml^2 \dot{\phi}^2 \right]$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}} \left(\frac{1}{2} (MR^2 + mr^2) \dot{\theta}^2 - mrl \cos(\theta - \phi) \dot{\theta} \dot{\phi} + \frac{1}{2} ml^2 \dot{\phi}^2 \right)$$

$$= (MR^2 + mr^2) \dot{\theta} - mrl \dot{\phi} \cos(\theta - \phi)$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}} \left(\frac{1}{2} (MR^2 + mr^2) \dot{\theta}^2 - mrl \cos(\theta - \phi) \dot{\theta} \dot{\phi} + \frac{1}{2} ml^2 \dot{\phi}^2 \right)$$

$$= ml^2 \dot{\phi} - mrl \dot{\theta} \cos(\theta - \phi)$$

The γ_{mn} tensor/matrix formulation

$$\text{Total KE} = T = T(M) + T(m)$$

$$= \frac{1}{2} \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

where: γ_{mn} tensor is
$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mrl \cos(\theta - \phi) \\ -mrl \cos(\theta - \phi) & ml^2 \end{pmatrix}$$

Momentum γ_{mn} -matrix theorem: (matrix-proof on page 43)

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial T}{\partial \dot{\theta}} \\ \frac{\partial T}{\partial \dot{\phi}} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \quad \text{if: } \gamma_{\phi,\theta} = \gamma_{\theta,\phi} \text{ (symmetry)}$$

$$= \begin{pmatrix} MR^2 + mr^2 & -mrl \cos(\theta - \phi) \\ -mrl \cos(\theta - \phi) & ml^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Momentum γ_{mn} -tensor theorem: (proof here)

$$p_m = \gamma_{mn} \dot{q}^n$$

proof: Given: $p_m = \frac{\partial T}{\partial \dot{q}^m}$ where: $T = \frac{1}{2} \gamma_{jk} \dot{q}^j \dot{q}^k$

$$\text{Then: } p_m = \frac{\partial}{\partial \dot{q}^m} \frac{1}{2} \gamma_{jk} \dot{q}^j \dot{q}^k = \frac{1}{2} \gamma_{jk} \frac{\partial \dot{q}^j}{\partial \dot{q}^m} \dot{q}^k + \frac{1}{2} \gamma_{jk} \dot{q}^j \frac{\partial \dot{q}^k}{\partial \dot{q}^m}$$

$$= \frac{1}{2} \gamma_{jk} \delta_m^j \dot{q}^k + \frac{1}{2} \gamma_{jk} \dot{q}^j \delta_m^k = \frac{1}{2} \gamma_{mk} \dot{q}^k + \frac{1}{2} \gamma_{jm} \dot{q}^j$$

$$= \gamma_{mn} \dot{q}^n \quad \text{if: } \gamma_{mn} = \gamma_{nm} \quad \text{QED}$$

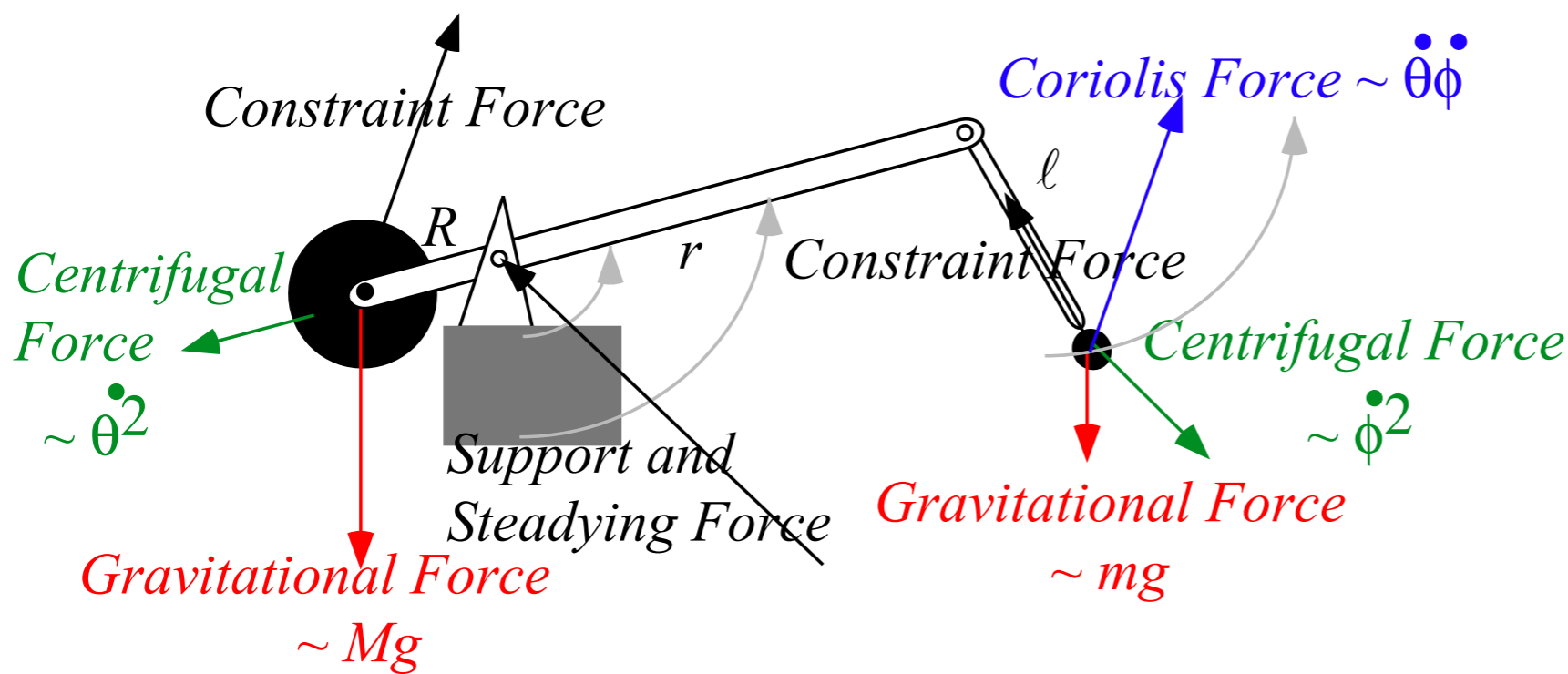
Momentum γ_{mn} -matrix theorem: (matrix-proof here on page 43)

$$\begin{aligned}
 \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} &= \begin{pmatrix} \frac{\partial T}{\partial \dot{\theta}} \\ \frac{\partial T}{\partial \dot{\phi}} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \frac{\partial}{\partial \dot{\theta}} \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \cdot \gamma \cdot \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} + \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \cdot \gamma \cdot \frac{\partial}{\partial \dot{\theta}} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \\ \frac{\partial}{\partial \dot{\phi}} \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \cdot \gamma \cdot \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} + \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \cdot \gamma \cdot \frac{\partial}{\partial \dot{\phi}} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \end{pmatrix} \\
 &= \frac{1}{2} \begin{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \gamma \cdot \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} + \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \cdot \gamma \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \end{pmatrix} \cdot \gamma \cdot \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} + \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \cdot \gamma \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} \\
 &= \frac{1}{2} \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\phi,\theta} \\ \gamma_{\theta,\phi} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \\
 &= \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \quad \text{if: } \gamma_{\phi,\theta} = \gamma_{\theta,\phi} \text{ (symmetry)} \\
 &= \begin{pmatrix} MR^2 + mr^2 & -mr\ell \cos(\theta - \phi) \\ -mr\ell \cos(\theta - \phi) & m\ell^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \quad \text{QED}
 \end{aligned}$$

Summary of Lagrange equations and force analysis (Mostly Unit 2.)

 *Forces: total, genuine, potential, and/or fictitious*

Forces: total, genuine, potential, and/or fictitious



Acceleration and 'Fictitious' Forces:

Forces:

*Coriolis
Centrifugal*

Applied 'Real' Forces:

*Gravity
Stimuli
Friction...*

Constraint 'Internal' Forces:

*Stresses
Support...*

*(Do not contribute.
Do no work.)*

$$\dot{p}_\theta = \frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial T}{\partial \theta} + F_\theta + 0$$

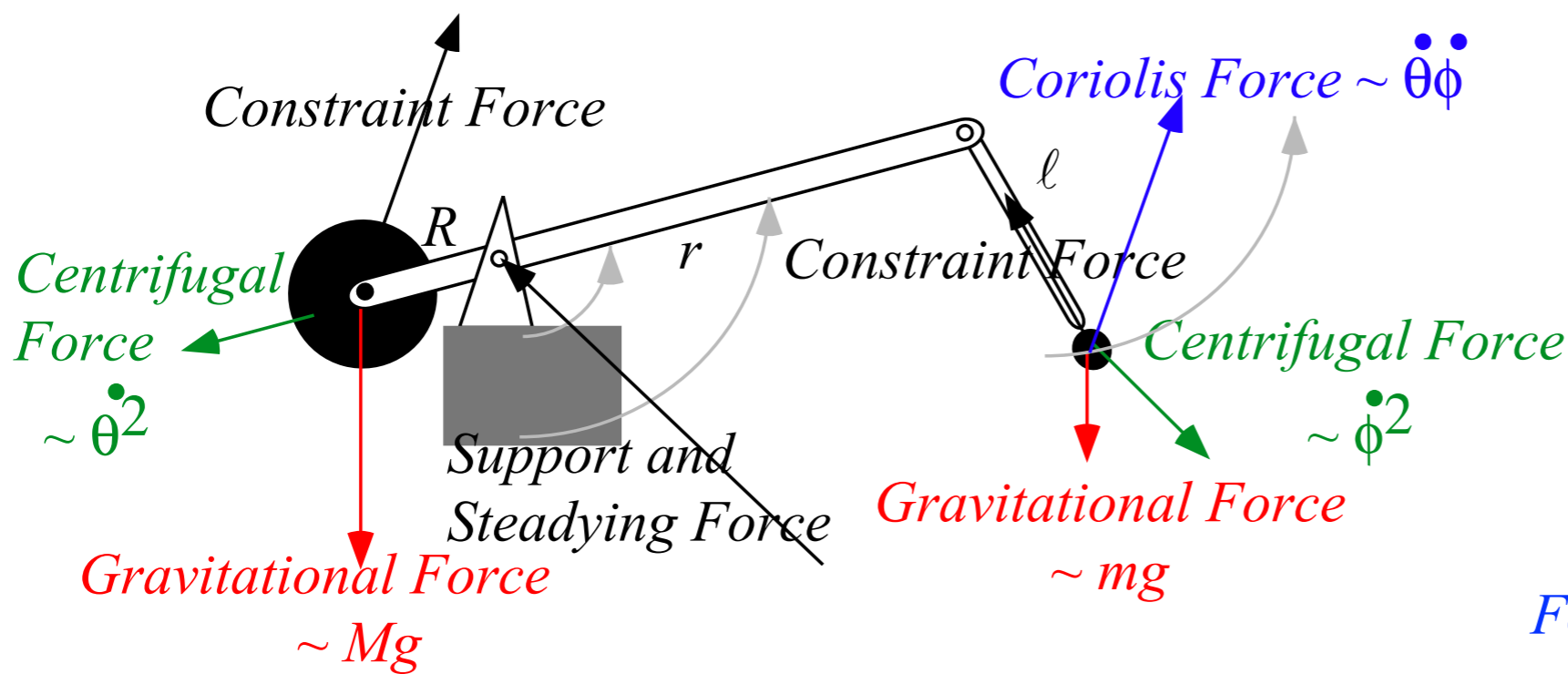
$$\dot{p}_\phi = \frac{d}{dt} \frac{\partial T}{\partial \dot{\phi}} = \frac{\partial T}{\partial \phi} + F_\phi + 0$$

Lagrange Force equations
(See also derivation Eq. (2.4.7) on p. 23, Unit 2)

Fig. 2.5.2 (modified)

Compare to derivation Eq (12.25a) in Ch. 12 of Unit 1 and Eq. (3.5.10) in Unit 3.

Forces: total, genuine, potential, and/or fictitious



Acceleration and 'Fictitious' Forces:

*Coriolis
Centrifugal*

*Applied 'Real' Forces:
Gravity
Stimuli
Friction...*

*Constraint 'Internal' Forces:
Stresses
Support...
(Do not contribute.
Do no work.)*

For conservative forces

where: $F_{\theta} = -\frac{\partial V}{\partial \theta}$ and: $\frac{\partial V}{\partial \dot{\theta}} = 0$
 $F_{\phi} = -\frac{\partial V}{\partial \phi}$ and: $\frac{\partial V}{\partial \dot{\phi}} = 0$

$$\dot{p}_{\theta} = \frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial T}{\partial \theta} + F_{\theta} + 0$$

$$\dot{p}_{\phi} = \frac{d}{dt} \frac{\partial T}{\partial \dot{\phi}} = \frac{\partial T}{\partial \phi} + F_{\phi} + 0$$

Lagrange Force equations
 (See also derivation Eq. (2.4.7) on p. 23, Unit 2)

$$p_{\theta} = \frac{\partial L}{\partial \dot{\theta}} \quad \dot{p}_{\theta} = \frac{\partial L}{\partial \theta}$$

$$p_{\phi} = \frac{\partial L}{\partial \dot{\phi}} \quad \dot{p}_{\phi} = \frac{\partial L}{\partial \phi}$$

Lagrange Potential equations
 $L = T - V$

Fig. 2.5.2 (modified)

Compare to derivation Eq (12.25a) in Ch. 12 of Unit 1 and Eq. (3.5.10) in Unit 3.

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)



Multivalued functionality and connections

Covariant and contravariant relations

Tangent space vs. Normal space

Metric g_{mn} tensor geometric relations to length, area, and volume

Trebuchet Cartesian projectile coordinates are double-valued

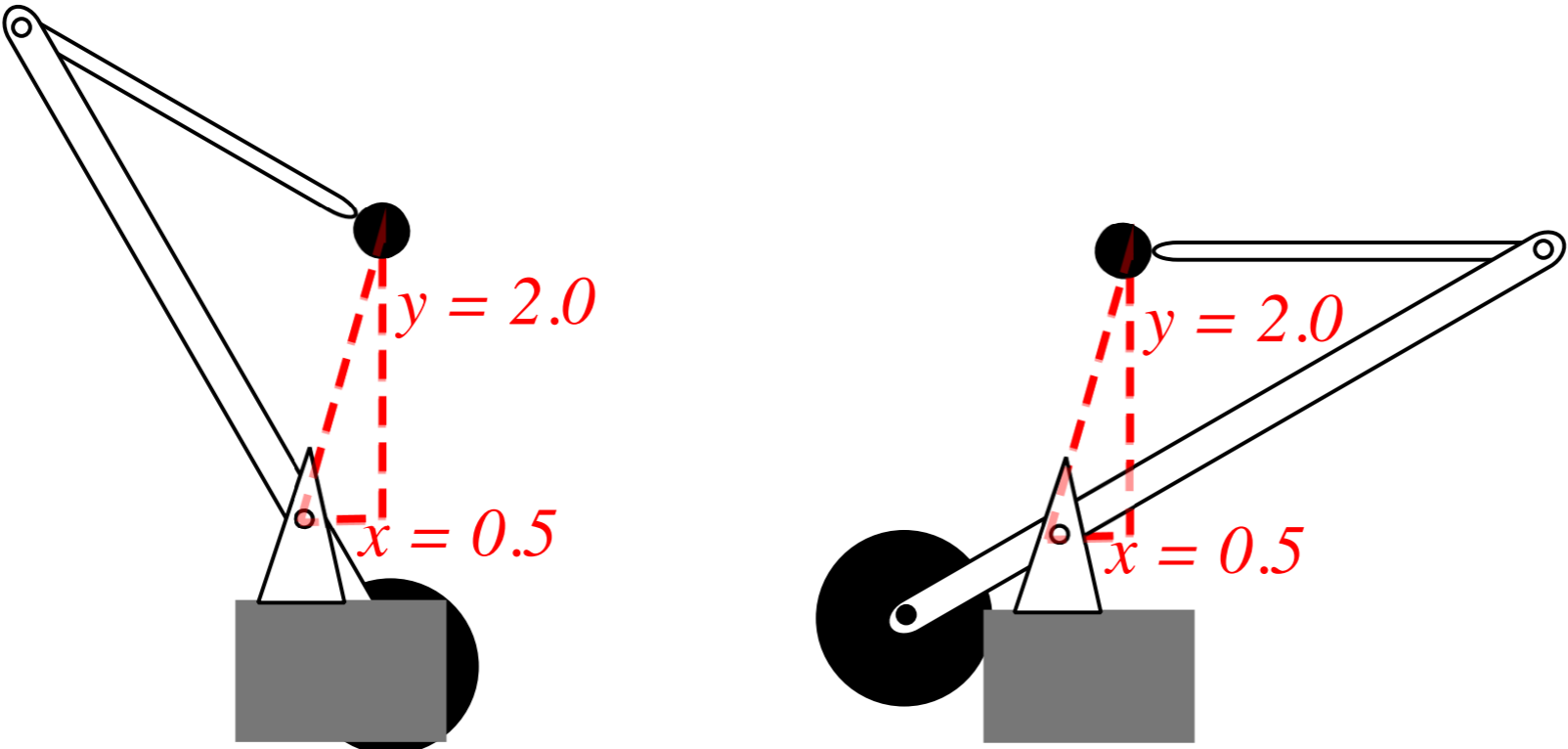


Fig. 2.2.3 Trebuchet configurations with the same coordinates x and y of projectile m .

Trebuchet Cartesian projectile coordinates are double-valued... (Belong to 2 distinct manifolds)

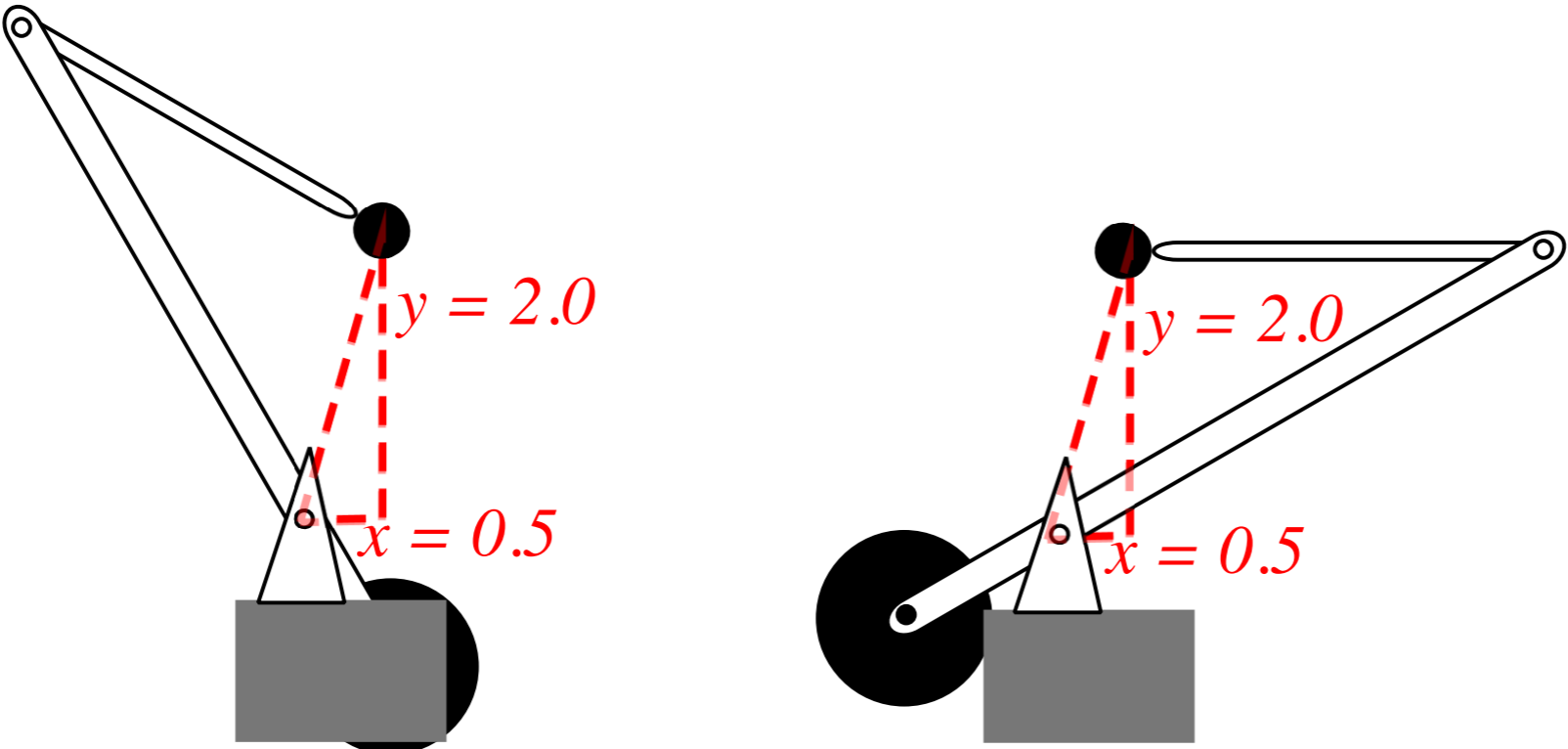


Fig. 2.2.3 Trebuchet configurations with the same coordinates x and y of projectile m .

So, for example, are polar coordinates ... (for each angle there are two r -values)

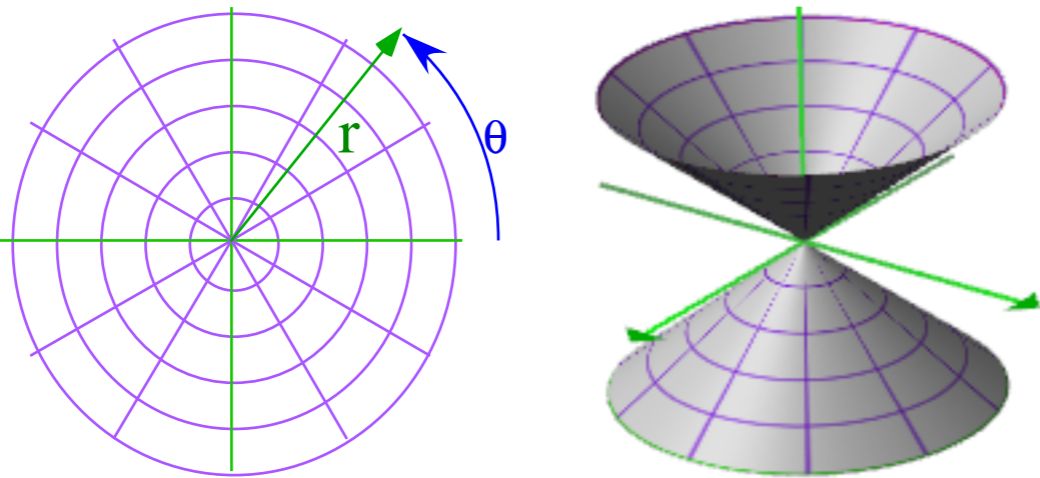


Fig. 3.1.4 Polar coordinates and possible embedding space on conical surface.

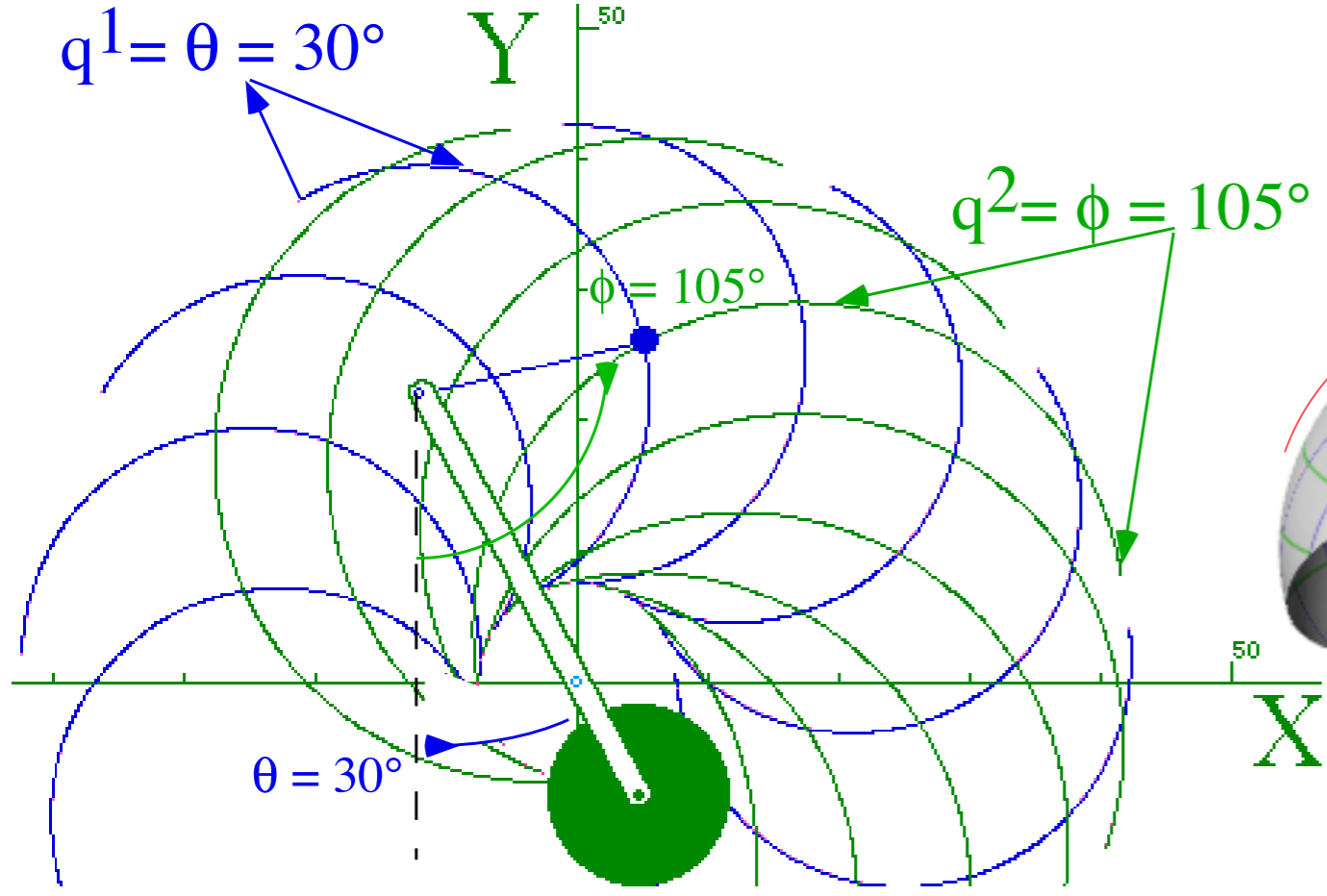


Fig. 3.1.1a ($q^1 = \theta, q^2 = \phi$) Coordinate manifold for trebuchet (Left handed sheet.)

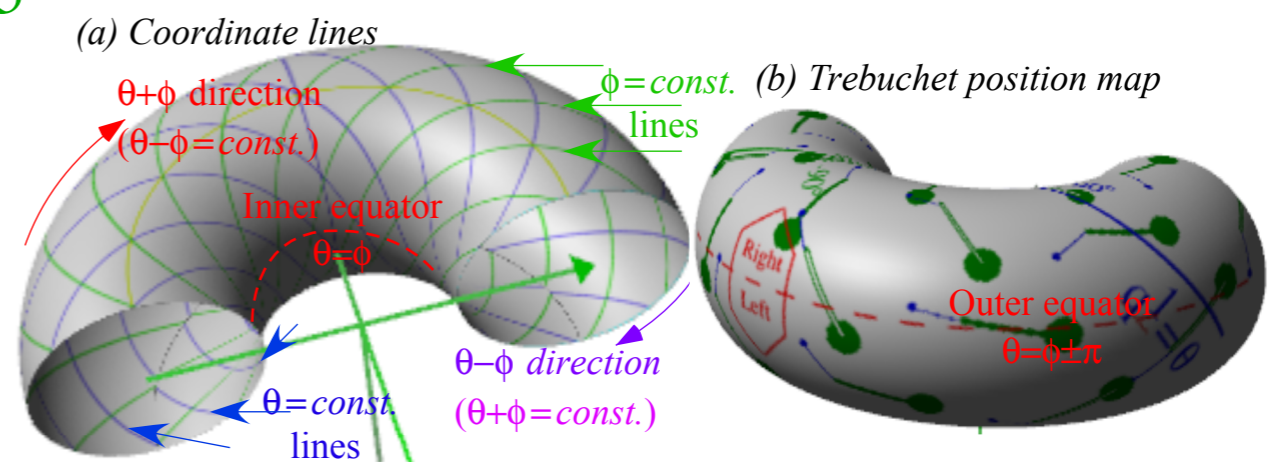


Fig. 3.1.2 Trebuchet torus.
 (a) ($q^1 = \theta, q^2 = \phi$) coordinate lines. (b) Trebuchet position map and equators.

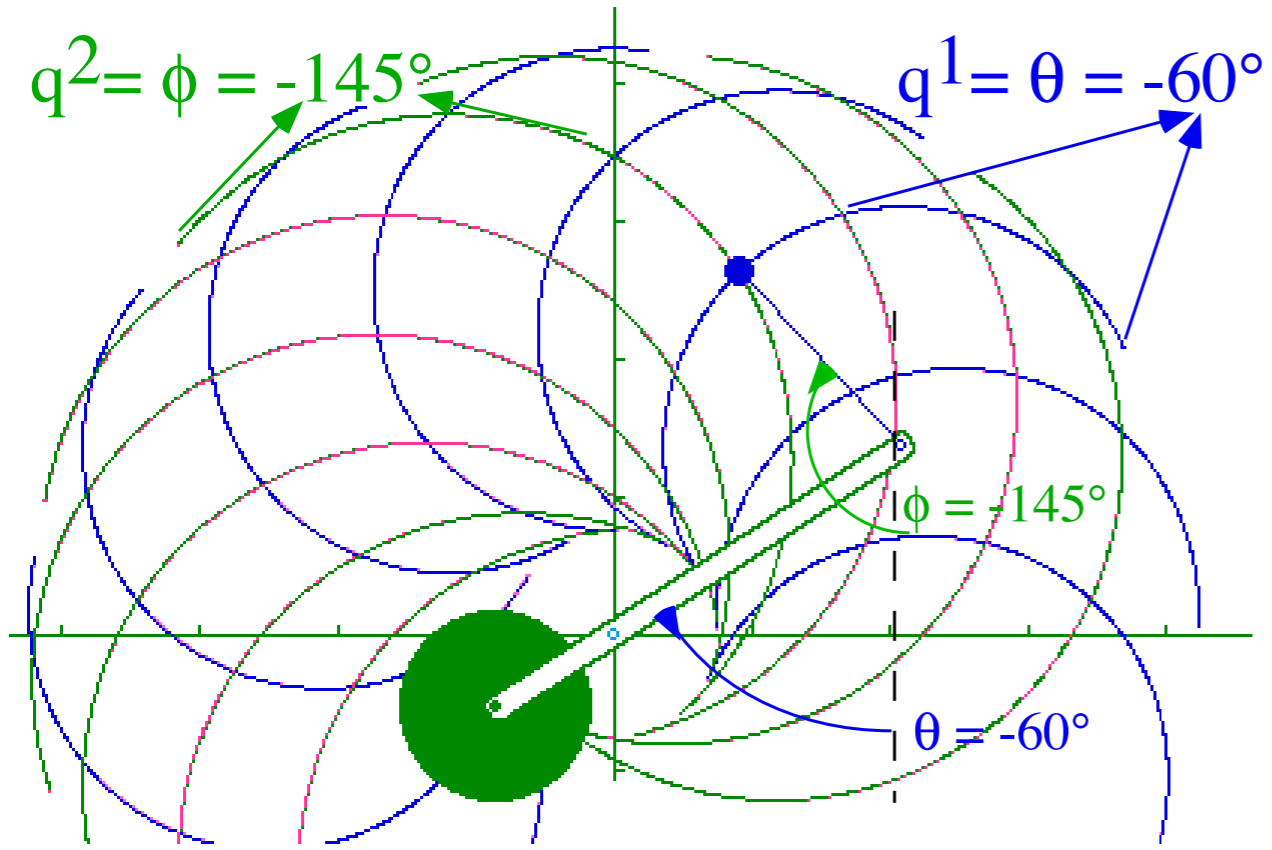


Fig. 3.1.1b ($q^1 = \theta, q^2 = \phi$) Coordinate manifold for trebuchet (Right handed sheet.)

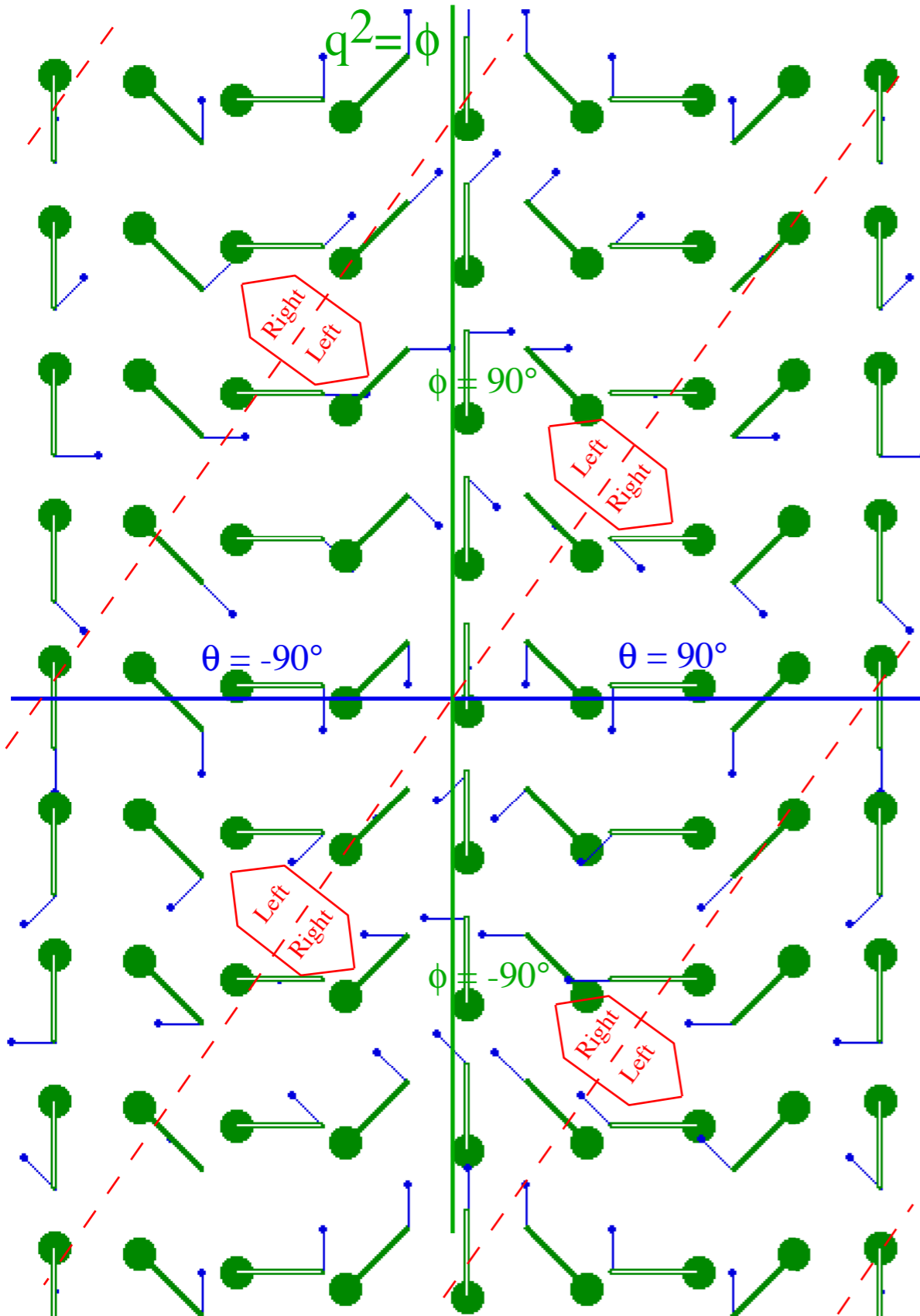


Fig. 3.1.3 "Flattened" ($q^1=\theta, q^2=\phi$) coordinate manifold for trebuchet

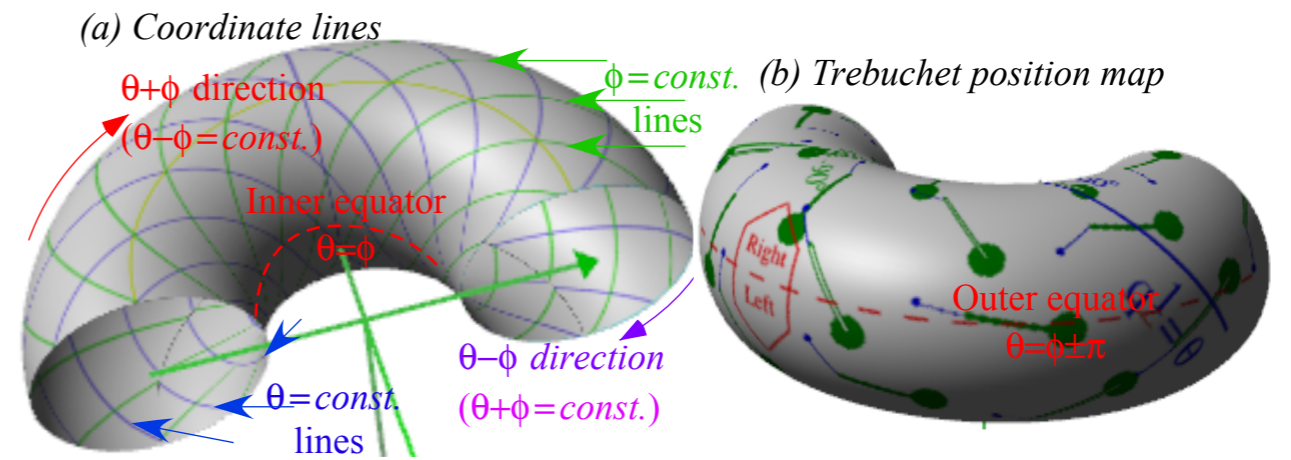


Fig. 3.1.2 Trebuchet torus.

(a) ($q^1=\theta, q^2=\phi$) coordinate lines. (b) Trebuchet position map and equators.

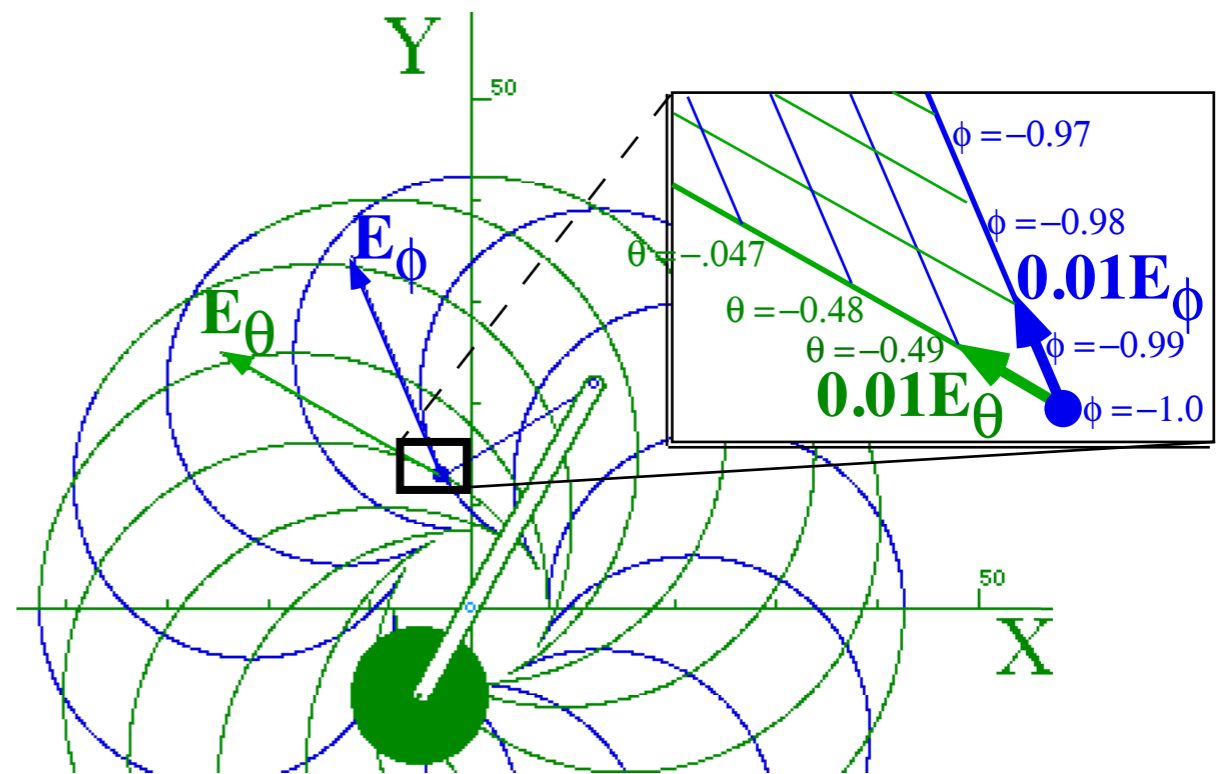


Fig. 3.2.2 Example of covariant unitary vectors and their tangent space.

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)

Multivalued functionality and connections

 *Covariant and contravariant relations*

Tangent space vs. Normal space

Metric g_{mn} tensor geometric relations to length, area, and volume

Kajobian transformation matrix

$$\left\langle \frac{\partial q^m}{\partial x^j} \right\rangle =$$

$$\begin{vmatrix} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \dots \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \dots \\ \vdots & \vdots & \ddots \end{vmatrix} \begin{matrix} \mathbf{E}^1 \\ \mathbf{E}^2 \\ \vdots \end{matrix} = \begin{pmatrix} \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{pmatrix} = \frac{\begin{vmatrix} l \sin \phi & -l \cos \phi \\ r \sin \theta & -r \cos \theta \end{vmatrix}}{rl \sin(\theta - \phi)} \begin{matrix} \mathbf{E}^\theta \\ \mathbf{E}^\phi \end{matrix}$$

Contravariant vectors \mathbf{E}^m

$$\mathbf{E}^\theta = \begin{pmatrix} l \sin \phi & -l \cos \phi \end{pmatrix} / rl \sin(\theta - \phi)$$

$$\mathbf{E}^\phi = \begin{pmatrix} r \sin \theta & -r \cos \theta \end{pmatrix} / rl \sin(\theta - \phi)$$

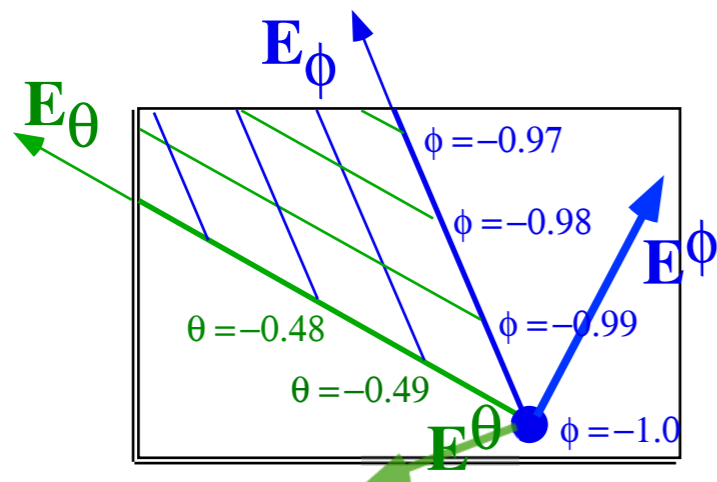


Fig. 3.2.3 Example of contravariant unitary vectors and their normal space.

versus

Jacobian transformation matrix

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle =$$

$$\begin{vmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \dots \\ \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \dots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \dots \\ \vdots & \vdots & \ddots \end{vmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} = \begin{vmatrix} \mathbf{E}_\theta & \mathbf{E}_\phi \\ -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{vmatrix}$$

Covariant vectors \mathbf{E}_n

$$\mathbf{E}_\theta = \begin{pmatrix} -r \cos \theta \\ -r \sin \theta \end{pmatrix}, \quad \mathbf{E}_\phi = \begin{pmatrix} l \cos \phi \\ l \sin \phi \end{pmatrix}$$

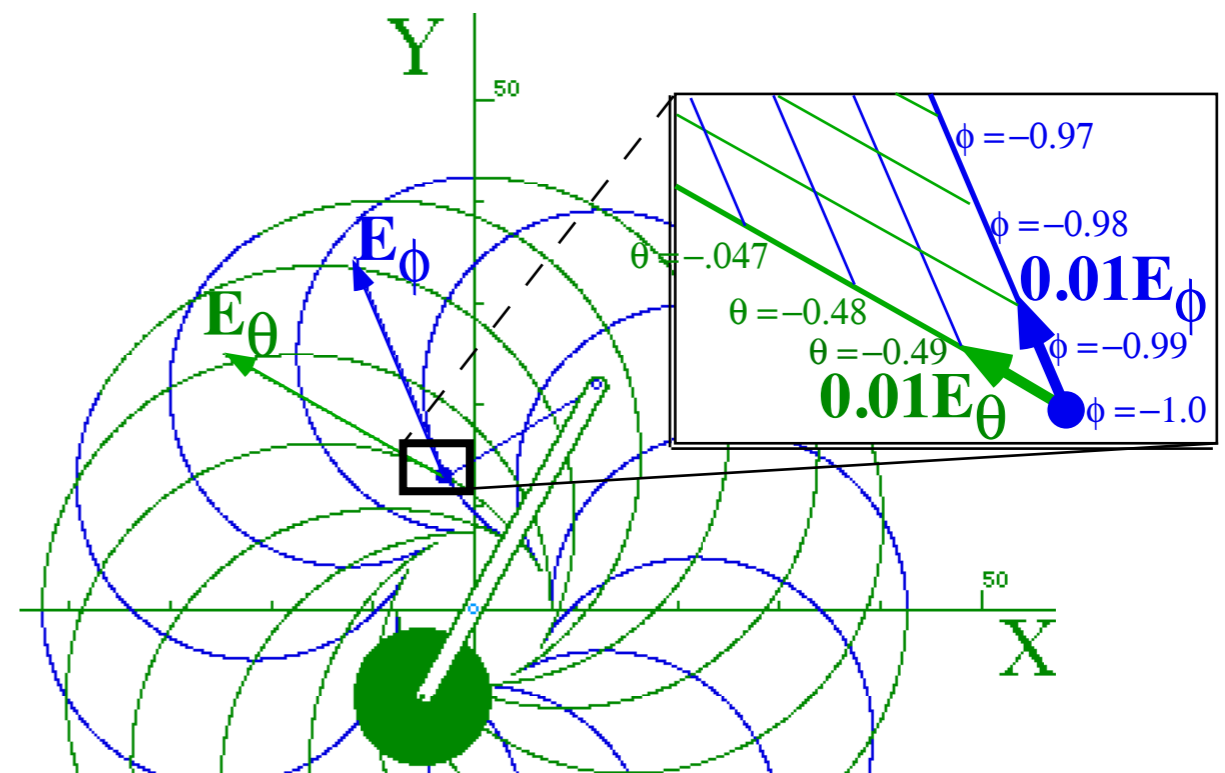


Fig. 3.2.2 Example of covariant unitary vectors and their tangent space.

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)

Multivalued functionality and connections

Covariant and contravariant relations

 *Tangent space vs. Normal space*

Metric g_{mn} tensor geometric relations to length, area, and volume

Contravariant vectors \mathbf{E}^m

versus

Covariant vectors \mathbf{E}_n

Any vector $\mathbf{U}, \mathbf{V}, \dots$ is expressed using *either* set from any viewpoint, coordinate system, or *frame*,

$$\mathbf{U} = U^m \mathbf{E}_m = U_n \mathbf{E}^n = \bar{U}^{\bar{m}} \bar{\mathbf{E}}_{\bar{m}} = \bar{U}_{\bar{n}} \bar{\mathbf{E}}^{\bar{n}}$$

$$\mathbf{V} = V^m \mathbf{E}_m = V_n \mathbf{E}^n = \bar{V}^{\bar{m}} \bar{\mathbf{E}}_{\bar{m}} = \bar{V}_{\bar{n}} \bar{\mathbf{E}}^{\bar{n}}$$

where the U^m, V^m, \dots are *contravariant components*

and the U_n, V_n, \dots are *covariant components*

$$U^m = \mathbf{U} \cdot \mathbf{E}^m, \quad V^m = \mathbf{V} \cdot \mathbf{E}^m, \quad \text{and} \quad \bar{U}^{\bar{m}} = \mathbf{U} \cdot \bar{\mathbf{E}}^{\bar{m}}, \quad \bar{V}^{\bar{m}} = \mathbf{V} \cdot \bar{\mathbf{E}}^{\bar{m}},$$

$$U_n = \mathbf{U} \cdot \mathbf{E}_n, \quad V_n = \mathbf{V} \cdot \mathbf{E}_n, \quad \text{and} \quad \bar{U}_{\bar{n}} = \mathbf{U} \cdot \bar{\mathbf{E}}_{\bar{n}}, \quad \text{etc.}$$

Normal space (Contravariant)

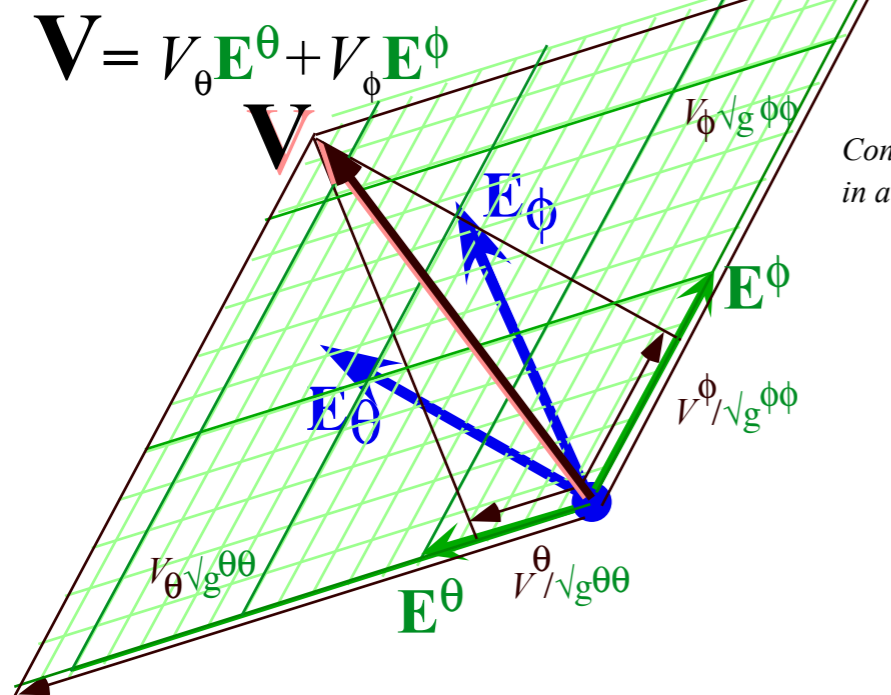


Fig. 3.3.2
Contravariant vector geometry
in a normal space ($\mathbf{E}^\theta, \mathbf{E}^\phi$).

Tangent space (Covariant)

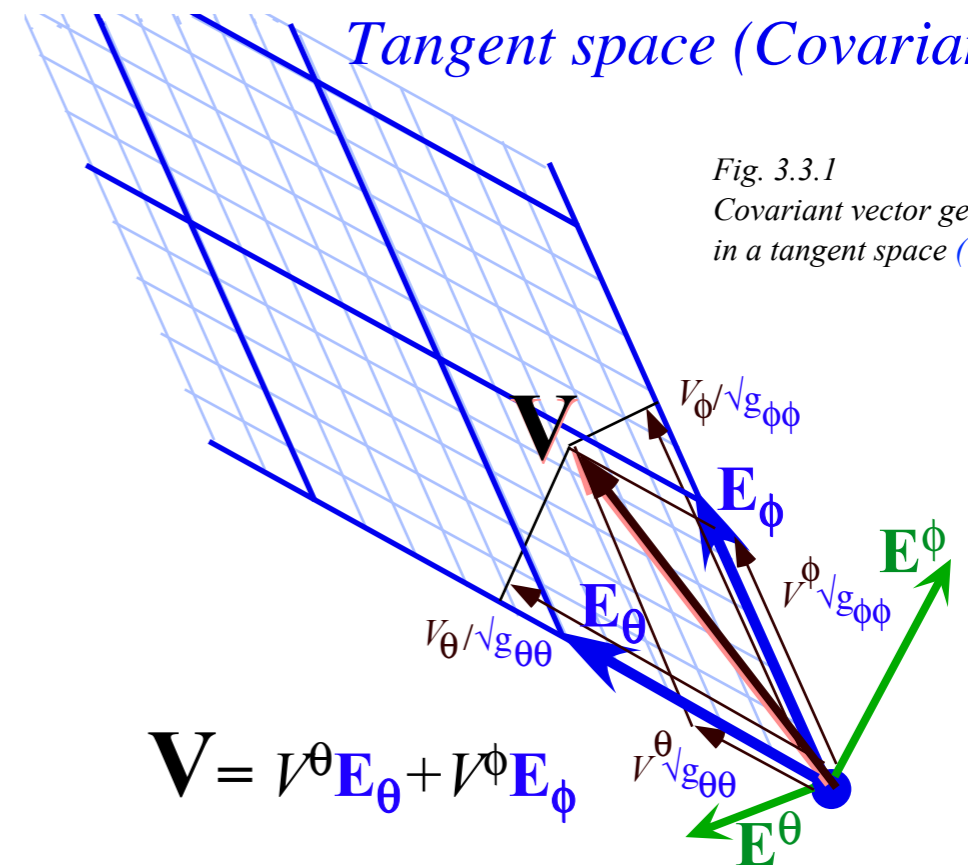


Fig. 3.3.1
Covariant vector geometry
in a tangent space ($\mathbf{E}_\theta, \mathbf{E}_\phi$).

Contravariant vectors \mathbf{E}^m

versus

Covariant vectors \mathbf{E}_n

Any vector $\mathbf{U}, \mathbf{V}, \dots$ is expressed using *either* set from any viewpoint, coordinate system, or *frame*,

$$\mathbf{U} = U^m \mathbf{E}_m = U_n \mathbf{E}^n = \bar{U}^{\bar{m}} \bar{\mathbf{E}}_{\bar{m}} = \bar{U}_{\bar{n}} \bar{\mathbf{E}}^{\bar{n}}$$

$$\mathbf{V} = V^m \mathbf{E}_m = V_n \mathbf{E}^n = \bar{V}^{\bar{m}} \bar{\mathbf{E}}_{\bar{m}} = \bar{V}_{\bar{n}} \bar{\mathbf{E}}^{\bar{n}}$$

where the U^m, V^m, \dots are *contravariant components*

and the U_n, V_n, \dots are *covariant components*

$$U^m = \mathbf{U} \cdot \mathbf{E}^m, \quad V^m = \mathbf{V} \cdot \mathbf{E}^m, \quad \text{and} \quad \bar{U}^{\bar{m}} = \mathbf{U} \cdot \bar{\mathbf{E}}^{\bar{m}}, \quad \bar{V}^{\bar{m}} = \mathbf{V} \cdot \bar{\mathbf{E}}^{\bar{m}},$$

$$U_n = \mathbf{U} \cdot \mathbf{E}_n, \quad V_n = \mathbf{V} \cdot \mathbf{E}_n, \quad \text{and} \quad \bar{U}_{\bar{n}} = \mathbf{U} \cdot \bar{\mathbf{E}}_{\bar{n}}, \quad \text{etc.}$$

Normal space (Contravariant)

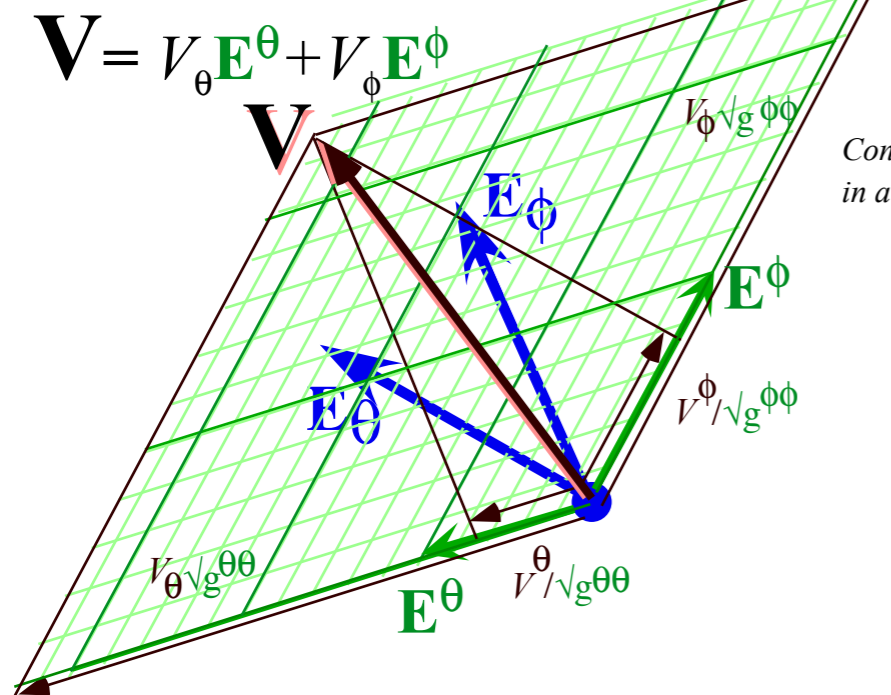


Fig. 3.3.2
Contravariant vector geometry
in a normal space ($\mathbf{E}^\theta, \mathbf{E}^\phi$).

Tangent space (Covariant)

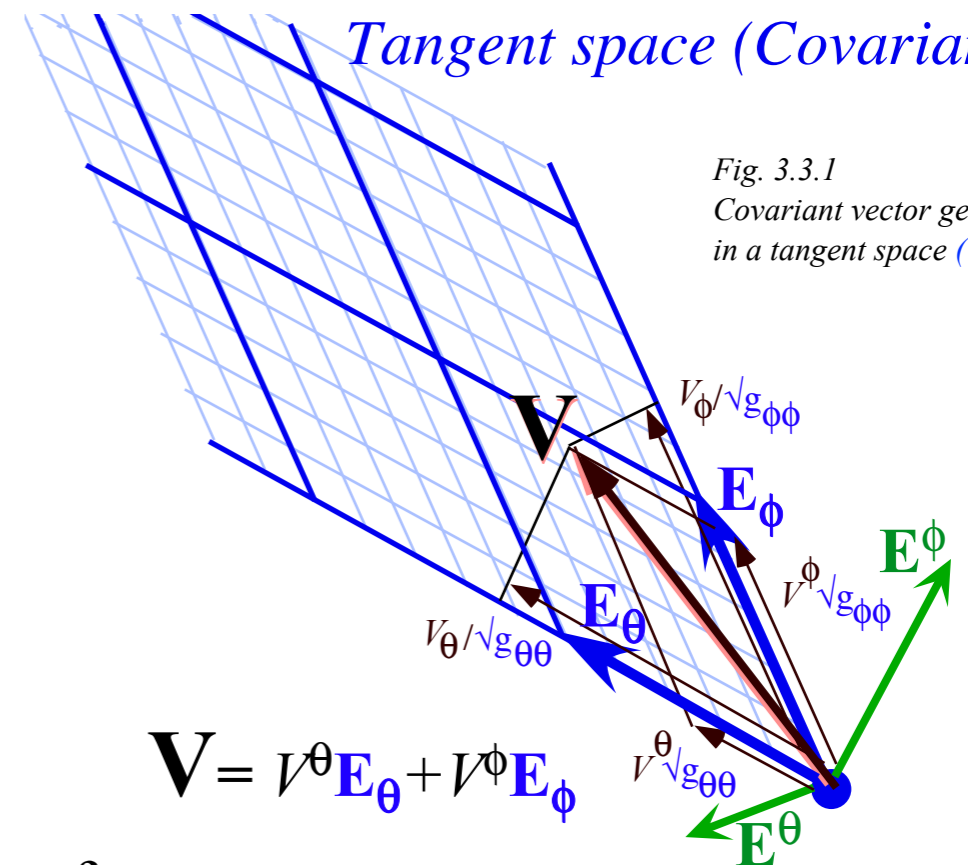


Fig. 3.3.1
Covariant vector geometry
in a tangent space ($\mathbf{E}_\theta, \mathbf{E}_\phi$).

Contravariant vector \mathbf{E}^m for frame $\{q^1, q^2, \dots\}$ is written in terms of new vectors $\bar{\mathbf{E}}^{\bar{m}}$ for a new "barred" frame $\{\bar{q}^{\bar{1}}, \bar{q}^{\bar{2}}, \dots\}$ using a "chain-saw-sum rule"....

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Contravariant vectors \mathbf{E}^m

versus

Covariant vectors \mathbf{E}_n

Any vector $\mathbf{U}, \mathbf{V}, \dots$ is expressed using *either* set from any viewpoint, coordinate system, or *frame*,

$$\mathbf{U} = U^m \mathbf{E}_m = U_n \mathbf{E}^n = \bar{U}^{\bar{m}} \bar{\mathbf{E}}_{\bar{m}} = \bar{U}_{\bar{n}} \bar{\mathbf{E}}^{\bar{n}}$$

$$\mathbf{V} = V^m \mathbf{E}_m = V_n \mathbf{E}^n = \bar{V}^{\bar{m}} \bar{\mathbf{E}}_{\bar{m}} = \bar{V}_{\bar{n}} \bar{\mathbf{E}}^{\bar{n}}$$

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$$U^m = \mathbf{U} \cdot \mathbf{E}^m, \quad V^m = \mathbf{V} \cdot \mathbf{E}^m, \quad \text{and} \quad \bar{U}^{\bar{m}} = \mathbf{U} \cdot \bar{\mathbf{E}}^{\bar{m}}, \quad \bar{V}^{\bar{m}} = \mathbf{V} \cdot \bar{\mathbf{E}}^{\bar{m}},$$

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Normal space (Contravariant)

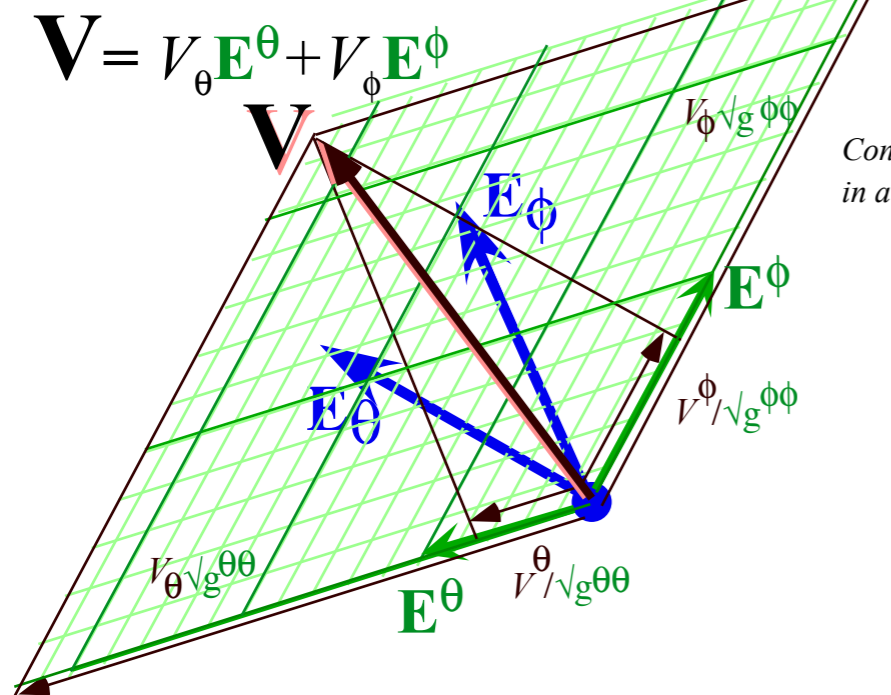


Fig. 3.3.2
Contravariant vector geometry
in a normal space ($\mathbf{E}^\theta, \mathbf{E}^\phi$).

Tangent space (Covariant)

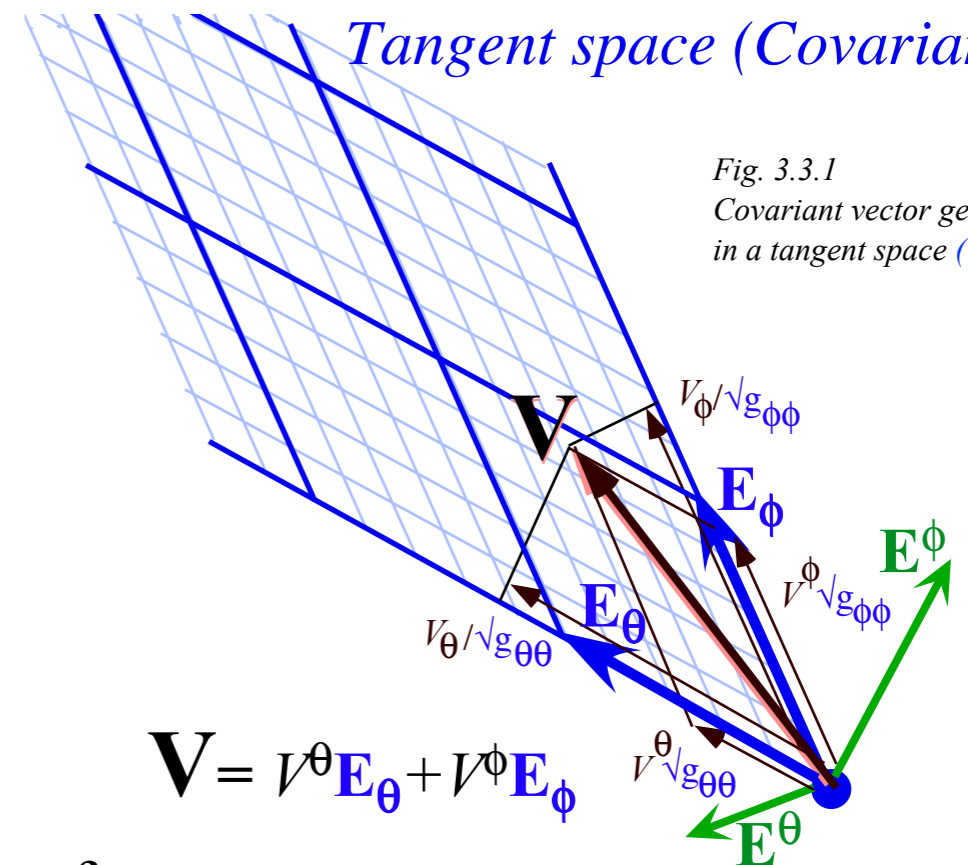


Fig. 3.3.1
Covariant vector geometry
in a tangent space ($\mathbf{E}_\theta, \mathbf{E}_\phi$).

Contravariant vector \mathbf{E}^m for frame $\{q^1, q^2, \dots\}$ is written in terms of new vectors $\bar{\mathbf{E}}^{\bar{m}}$ for a new "barred" frame $\{\bar{q}^{\bar{1}}, \bar{q}^{\bar{2}}, \dots\}$ using a "chain-saw-sum rule"....

...and the same for covariant vectors \mathbf{E}_m and $\bar{\mathbf{E}}_{\bar{m}}$

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$$\mathbf{E}_m = \frac{\partial \mathbf{r}}{\partial q^m} = \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}} \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}, \quad \text{or:} \quad \boxed{\mathbf{E}_m = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} \bar{\mathbf{E}}_{\bar{m}}}$$

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Normal space (Contravariant)

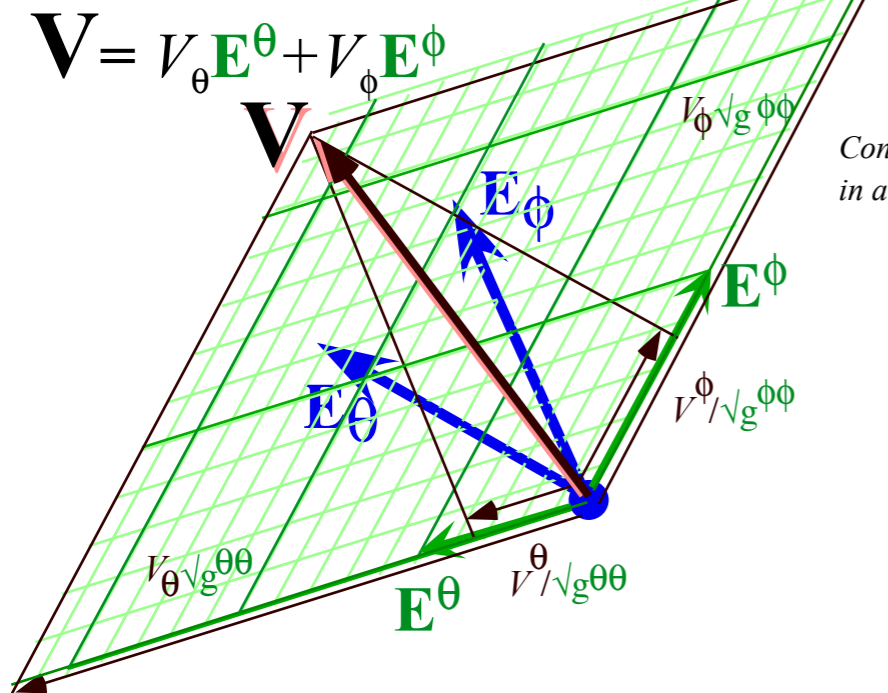


Fig. 3.3.2
Contravariant vector geometry
in a normal space ($\mathbf{E}^\theta, \mathbf{E}^\phi$).

Tangent space (Covariant)

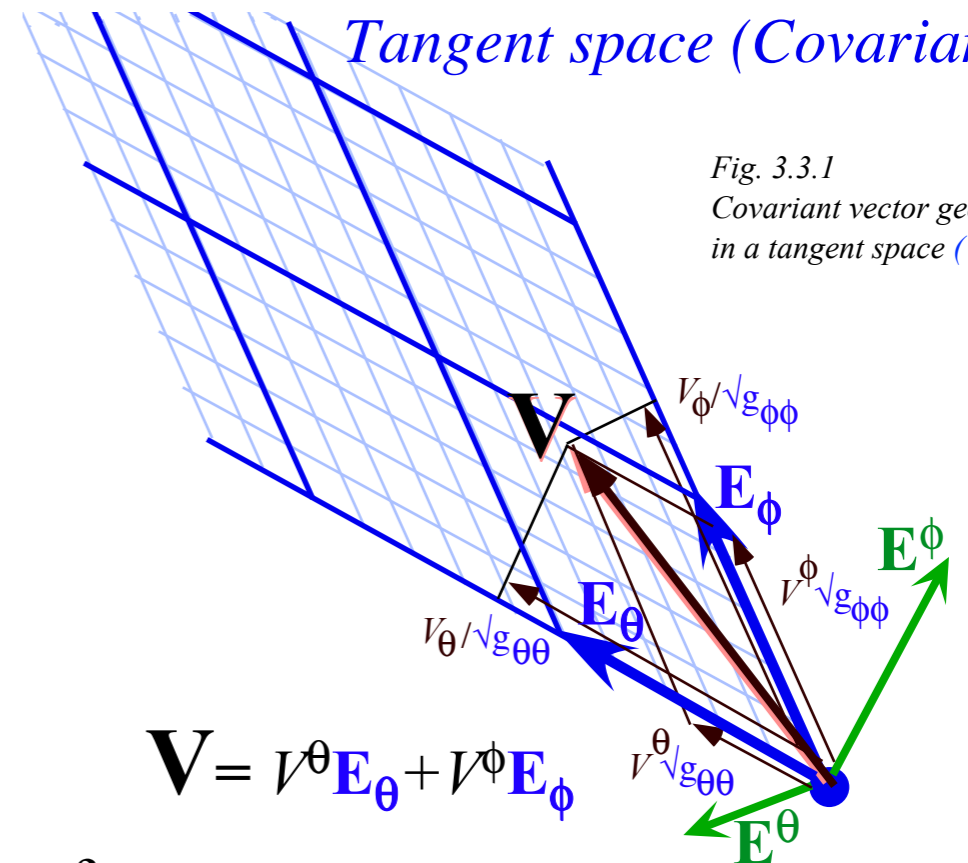


Fig. 3.3.1
Covariant vector geometry
in a tangent space ($\mathbf{E}_\theta, \mathbf{E}_\phi$).

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implies: $V^m = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} \bar{V}^{\bar{m}}$

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versus

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Normal space (Contravariant)

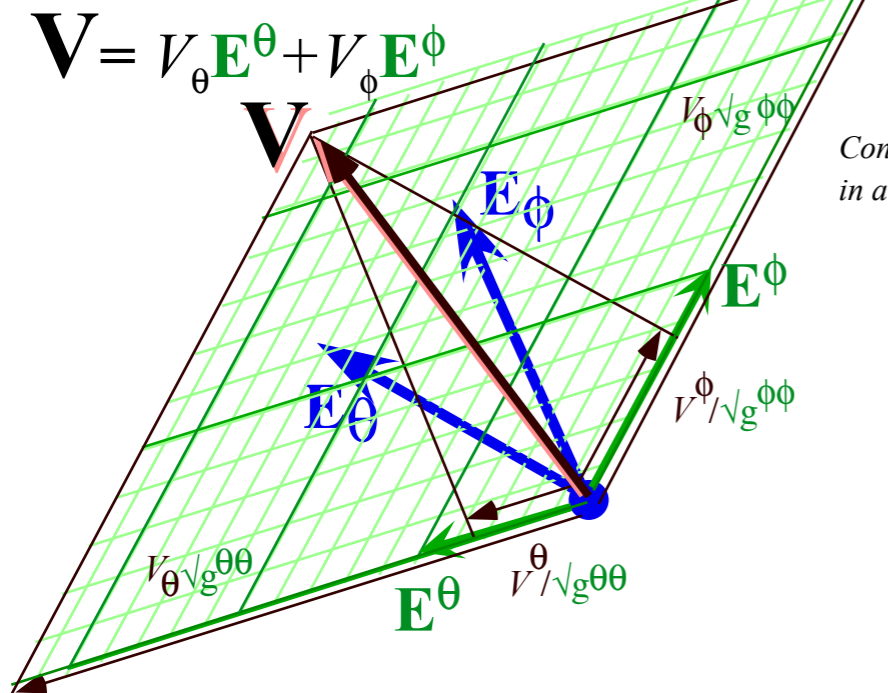


Fig. 3.3.2
Contravariant vector geometry
in a normal space ($\mathbf{E}^\theta, \mathbf{E}^\phi$).

covariant projection

$$|\mathbf{V} \cdot \mathbf{E}_\theta| = \mathbf{V} \cdot \hat{\mathbf{E}}_\theta = \mathbf{V} \cdot \mathbf{E}_\theta / \sqrt{g_{\theta\theta}} = V_\theta / \sqrt{g_{\theta\theta}}$$

Tangent space (Covariant)

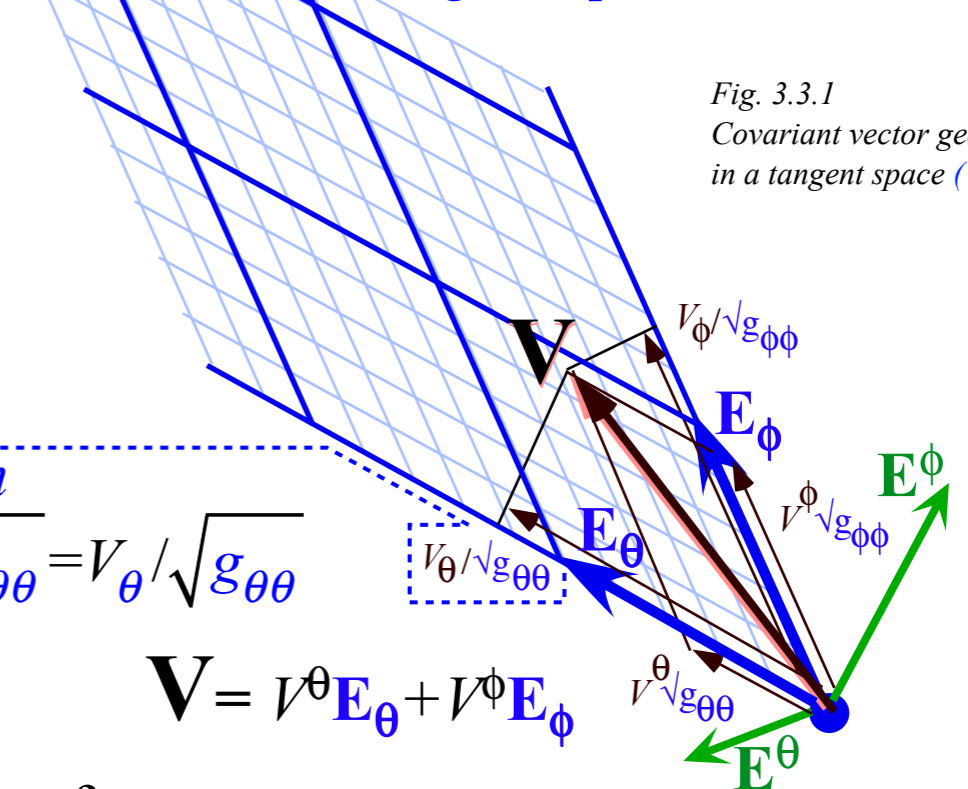


Fig. 3.3.1
Covariant vector geometry
in a tangent space ($\mathbf{E}_\theta, \mathbf{E}_\phi$).

$$\mathbf{V} = V^\theta \mathbf{E}_\theta + V^\phi \mathbf{E}_\phi$$

Contravariant vector \mathbf{E}^m for frame $\{q^1, q^2, \dots\}$ is written in terms of new vectors $\bar{\mathbf{E}}^{\bar{m}}$ for a new "barred" frame $\{\bar{q}^{\bar{1}}, \bar{q}^{\bar{2}}, \dots\}$ using a "chain-saw-sum rule"....

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implies: $V^m = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} \bar{V}^{\bar{m}}$

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Dirac notation equivalents:

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$$\langle m | = \langle m | \cdot \mathbf{1} = \langle m | \cdot \sum_{\bar{m}} |\bar{m}\rangle \langle \bar{m}| = \sum_{\bar{m}} \langle m | \bar{m}\rangle \langle \bar{m}| \quad \text{implies:} \quad \langle m | \Psi\rangle = \sum_{\bar{m}} \langle m | \bar{m}\rangle \langle \bar{m} | \Psi\rangle$$

$$|m\rangle = \mathbf{1} \cdot |m\rangle = \sum_{\bar{m}} |\bar{m}\rangle \langle \bar{m} | m\rangle = \sum_{\bar{m}} \langle \bar{m} | m\rangle |\bar{m}\rangle$$

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)

Multivalued functionality and connections

Covariant and contravariant relations

Tangent space vs. Normal space

 *Metric g_{mn} tensor geometric relations to length, area, and volume*

Metric tensor \mathbf{g} covariant (and contravariant) metric components g_{mn} (and g^{mn})

$$g_{mn} = \mathbf{E}_m \bullet \mathbf{E}_n = g_{nm} , \quad g^{mn} = \mathbf{E}^m \bullet \mathbf{E}^n = g^{nm} .$$

Metric tensor \mathbf{g} covariant (and contravariant) metric components g_{mn} (and g^{mn})

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"Mixed" covariant-contravariant metric components

$$g_m^n = \mathbf{E}_m \bullet \mathbf{E}^n = g_m^n = \mathbf{E}_m \bullet \mathbf{E}^n = \delta_m^n = \begin{cases} 0 & \text{if: } m \neq n \\ 1 & \text{if: } m = n \end{cases} \quad \text{Caution: } \delta_{mn} \text{ is } g_{mn} \text{ and not } \delta_n^m \text{ in GCC.}$$

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Metric coefficients express covariant unitary vectors in terms of contras and vice-versa

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$$\mathbf{E}_m = g_{mn} \mathbf{E}^n, \quad \mathbf{E}^m = g^{mn} \mathbf{E}_n.$$

Co-and-Contra vector and tensor components are related by g -transformation. (So are g 's themselves.)

$$V_m = g_{mn} V^n, \quad V^m = g^{mn} V_n, \quad T^{mm'} = g^{mn} g^{m'n'} T_{nn'}, \text{ etc.}$$

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Co-and-Contra vector and tensor components are related by g -transformation. (So are g 's themselves.)

$$V_m = g_{mn} V^n, \quad V^m = g^{mn} V_n, \quad T^{mm'} = g^{mn} g^{m'n'} V_{nn'}, \text{ etc.}$$

Diagonal square roots $\sqrt{g_{mm}}$ are the lengths of the covariant unitary vectors. $|\mathbf{E}_m| = \sqrt{\mathbf{E}_m \bullet \mathbf{E}_m} = \sqrt{g_{mm}}$
 $|\mathbf{E}^m| = \sqrt{\mathbf{E}^m \bullet \mathbf{E}^m} = \sqrt{g^{mm}}$

tangent space area spanned by $V^1\mathbf{E}_1$ and $V^2\mathbf{E}_2$

$$Area(V^1\mathbf{E}_1, V^2\mathbf{E}_2) = V^1V^2 |\mathbf{E}_1 \times \mathbf{E}_2| = V^1V^2 \sqrt{(\mathbf{E}_1 \times \mathbf{E}_2) \cdot (\mathbf{E}_1 \times \mathbf{E}_2)}$$

$$\begin{aligned} Area(V^1\mathbf{E}_1, V^2\mathbf{E}_2) &= V^1V^2 \sqrt{(\mathbf{E}_1 \cdot \mathbf{E}_1)(\mathbf{E}_2 \cdot \mathbf{E}_2) - (\mathbf{E}_1 \cdot \mathbf{E}_2)(\mathbf{E}_1 \cdot \mathbf{E}_2)} \\ &= V^1V^2 \sqrt{g_{11}g_{22} - g_{12}g_{21}} = V^1V^2 \sqrt{\det \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix}} \end{aligned}$$

3D Jacobian determinant J -columns are \mathbf{E}_1 , \mathbf{E}_2 and \mathbf{E}_3 .

$$Volume(V^1\mathbf{E}_1, V^2\mathbf{E}_2, V^3\mathbf{E}_3) = V^1V^2V^3 |\mathbf{E}_1 \times \mathbf{E}_2 \cdot \mathbf{E}_3| = V^1V^2V^3 \det \begin{vmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \frac{\partial x^1}{\partial q^3} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^2}{\partial q^3} \\ \frac{\partial x^3}{\partial q^1} & \frac{\partial x^3}{\partial q^2} & \frac{\partial x^3}{\partial q^3} \end{vmatrix}$$

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} = \begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^2}{\partial q^1} & \frac{\partial x^3}{\partial q^1} \\ \frac{\partial x^1}{\partial q^2} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^3}{\partial q^2} \\ \frac{\partial x^1}{\partial q^3} & \frac{\partial x^2}{\partial q^3} & \frac{\partial x^3}{\partial q^3} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \frac{\partial x^1}{\partial q^3} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^2}{\partial q^3} \\ \frac{\partial x^3}{\partial q^1} & \frac{\partial x^3}{\partial q^2} & \frac{\partial x^3}{\partial q^3} \end{pmatrix} = J^T \cdot J$$

Determinant product ($\det|A| \det|B| = \det|A \cdot B|$) and symmetry ($\det|A^T| = \det|A|$) gives

$$Volume(V^1\mathbf{E}_1, V^2\mathbf{E}_2, V^3\mathbf{E}_3) = V^1V^2V^3 \det|J| = V^1V^2V^3 \sqrt{\det|g|}$$

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