Lecture 15
Wed.10.10.2018

*treb-yew-shay

GCC Lagrange and Riemann Equations for Trebuchet
(Ch. 1-5 of Unit 2 and Unit 3)

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.
- *Forces in Lagrange force equation: total, genuine, potential, and/or fictitious*

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)
- *Trebuchet Cartesian projectile coordinates are double-valued*
  - *Toroidal "rolled-up" ($q^1=\theta$, $q^2=\phi$)-manifold and "Flat" ($x=\theta$, $y=\phi$)-graph*
- *Review of covariant $\mathbf{E}_n$ and contravariant $\mathbf{E}^m$ vectors: Jacobian J vs. Kajobian K*
  - *Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)*
  - *Tangent $\{\mathbf{E}_n\}$space vs. Normal $\{\mathbf{E}^m\}$space*
  - *Covariant vs. contravariant coordinate transformations*
- *Metric g_{mn} tensor geometric relations to length, area, and volume*

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)
- *Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 77)*
- *Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 69)*
- *Riemann equation derivation for trebuchet model*
- *Riemann equation force analysis*
  - *2nd-guessing Riemann equation?*

A running collection of links to course-relevant sites and articles

Physics Web Resources

Comprehensive Harter-Soft Resource Listing

UAF Physics YouTube channel

LearnIt Physics Web Applications

"Texts"

Classical Mechanics with a Bang!

Quantum Theory for the Computer Age

Principles of Symmetry, Dynamics, and Spectroscopy

Modern Physics and its Classical Foundations

Classes

2014 AMOP

2017 Group Theory for QM

2018 AMOP

2018 Adv Mechanics

Neat external material to start the class:

- AIP publications
- AJP article on superball dynamics
- AAPT summer reading

These *are* hot off the presses:

- Sorting ultracold atoms in a three-dimensional optical lattice in a realization of Maxwell's demon - Kumar-Nature-Letters-2018
- Synthetic three-dimensional atomic structures assembled atom by atom - Berredo-Nature-Letters-2018

Slightly Older ones:

- Wave–particle duality of C60 molecules
- Optical vortex knots – One Photon at a Time

"*Relawavity*" and quantum basis of *Lagrangian* & *Hamiltonian* mechanics:

- 2-CW laser wave - BohrIt Web App
- Lagrangian vs Hamiltonian - RelaWavity Web App

- AMOP Ch 0 Space-Time Symmetry - 2019
- Seminar at Rochester Institute of Optics, Auxiliary slides, June 19, 2018

New *AnalyIt Web Application now under development* in out testing area:

https://modphys.hosted.uark.edu/testing/markup/AnalyItBJS.html

Link to default simulator of the Trebuchet for the Web application *for Lecture 15*

https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.

Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)

Trebuchet Cartesian projectile coordinates are double-valued

Toroidal "rolled-up" (q^1=θ, q^2=ϕ)-manifold and "Flat" (x=θ, y=ϕ)-graph

Review of covariant $\mathbf{E}_n$ and contravariant $\mathbf{E}^m$ vectors: Jacobian J vs. Kajobian K

Covariant metric g_{mn} vs. contravariant metric g^{mn}

Tangent $\{\mathbf{E}_n\}$space vs. Normal $\{\mathbf{E}^m\}$space

Covariant vs. contravariant coordinate transformations

Metric g_{mn} tensor geometric relations to length, area, and volume

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)

Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 15 p. 77)

Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 15 p. 69)

Riemann equation derivation for trebuchet model

Riemann equation force analysis

2nd-guessing Riemann equation?

Chapter 1. The Trebuchet: A dream problem for Galileo?

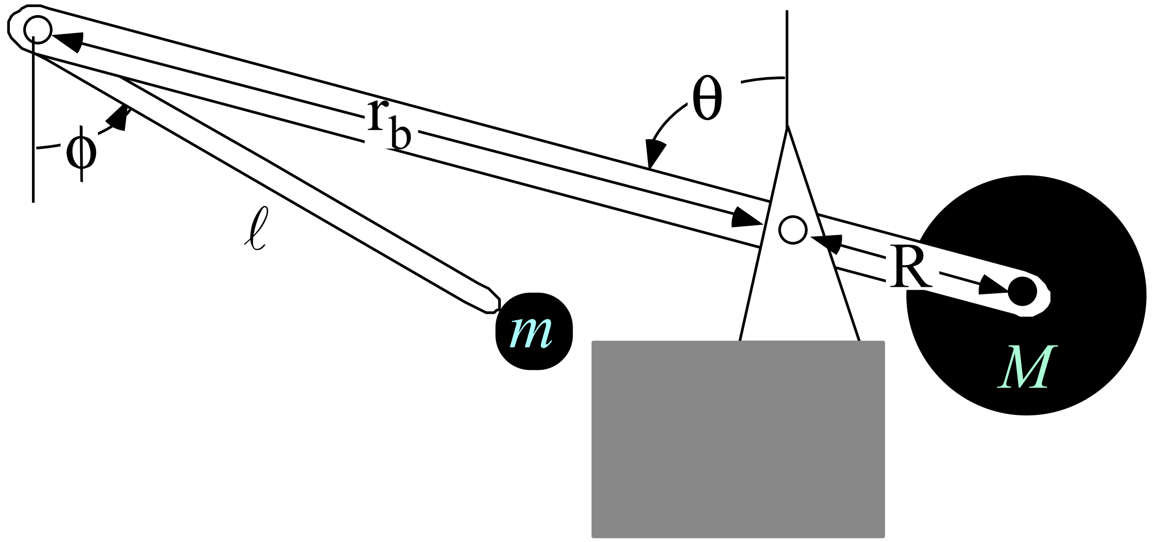
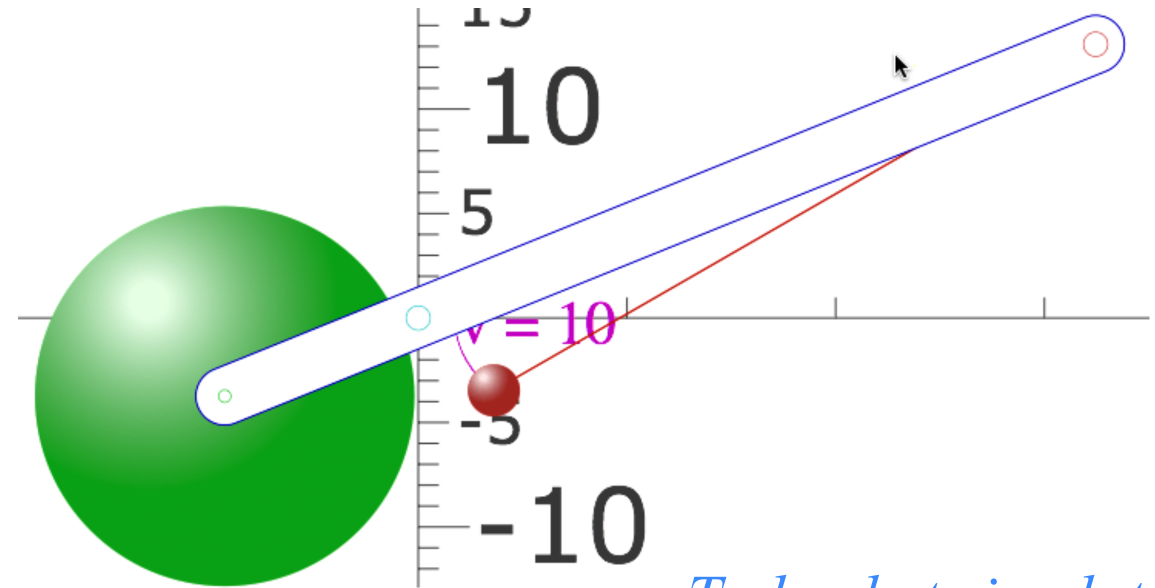
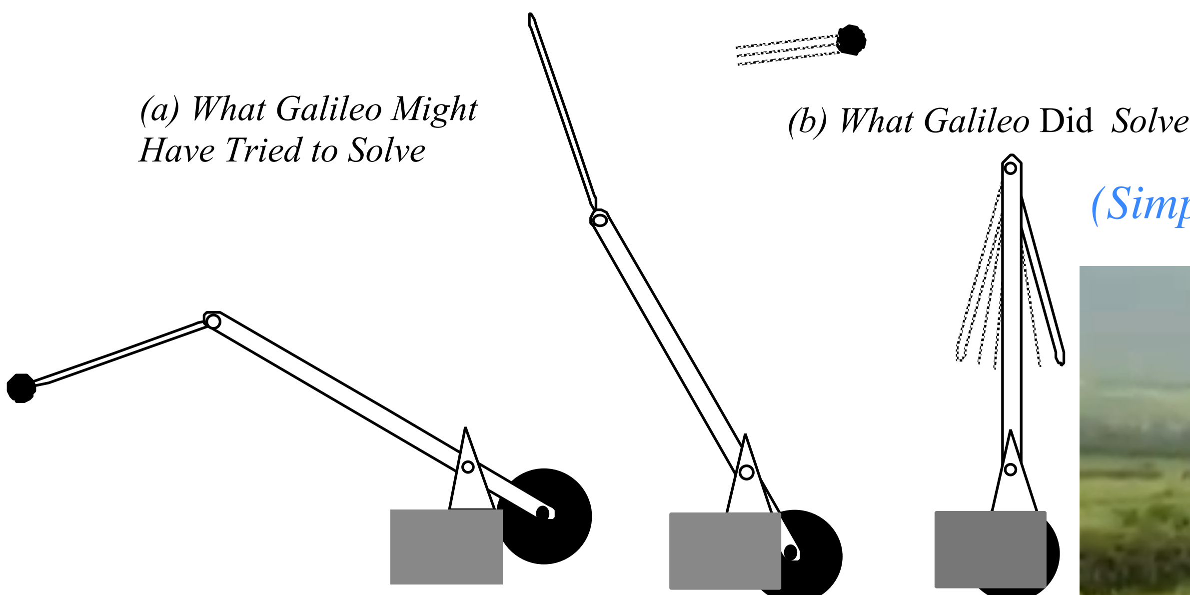


Fig. 2.1.1 An elementary ground-fixed trebuchet



Trebuchet simulator

https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html



(Simple pendulum dynamics)

Fig. 2.1.2 Galileo's (supposed) problem

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.

→ *Forces in Lagrange force equation: total, genuine, potential, and/or fictitious*

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)

Trebuchet Cartesian projectile coordinates are double-valued

Toroidal "rolled-up" ($q^1=\theta$, $q^2=\phi$)-manifold and "Flat" ($x=\theta$, $y=\phi$)-graph

Review of covariant $\mathbf{E}_n$ and contravariant $\mathbf{E}^m$ vectors: Jacobian J vs. Kajobian K

Covariant metric g_{mn} vs. contravariant metric g^{mn}

Tangent $\{\mathbf{E}_n\}$space vs. Normal $\{\mathbf{E}^m\}$space

Covariant vs. contravariant coordinate transformations

Metric g_{mn} tensor geometric relations to length, area, and volume

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)

Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 15 p. 77)

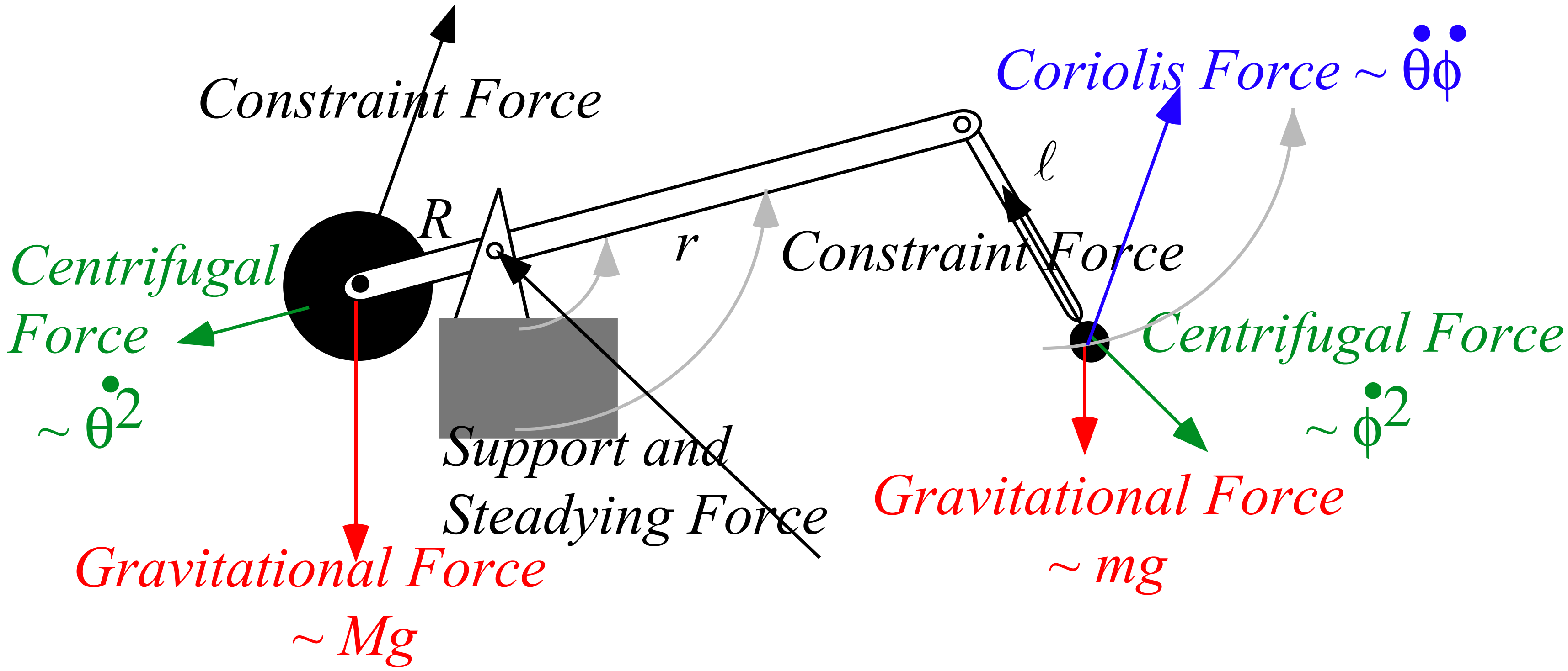
Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 15 p. 69)

Riemann equation derivation for trebuchet model

Riemann equation force analysis

2nd-guessing Riemann equation?

Forces in Lagrange force equation: total, genuine, potential, and/or fictitious



Acceleration and 'Fictitious' Forces:	Applied 'Real' Forces:	Constraint 'Internal' Forces:
Coriolis Centrifugal	Gravity Stimuli Friction...	Stresses Support... (Do not contribute. Do no work.)

$$\dot{p}_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot\theta} = \frac{\partial T}{\partial \theta} + F_\theta + 0$$

$$\dot{p}_\phi = \frac{d}{dt}\frac{\partial T}{\partial \dot\phi} = \frac{\partial T}{\partial \phi} + F_\phi + 0$$

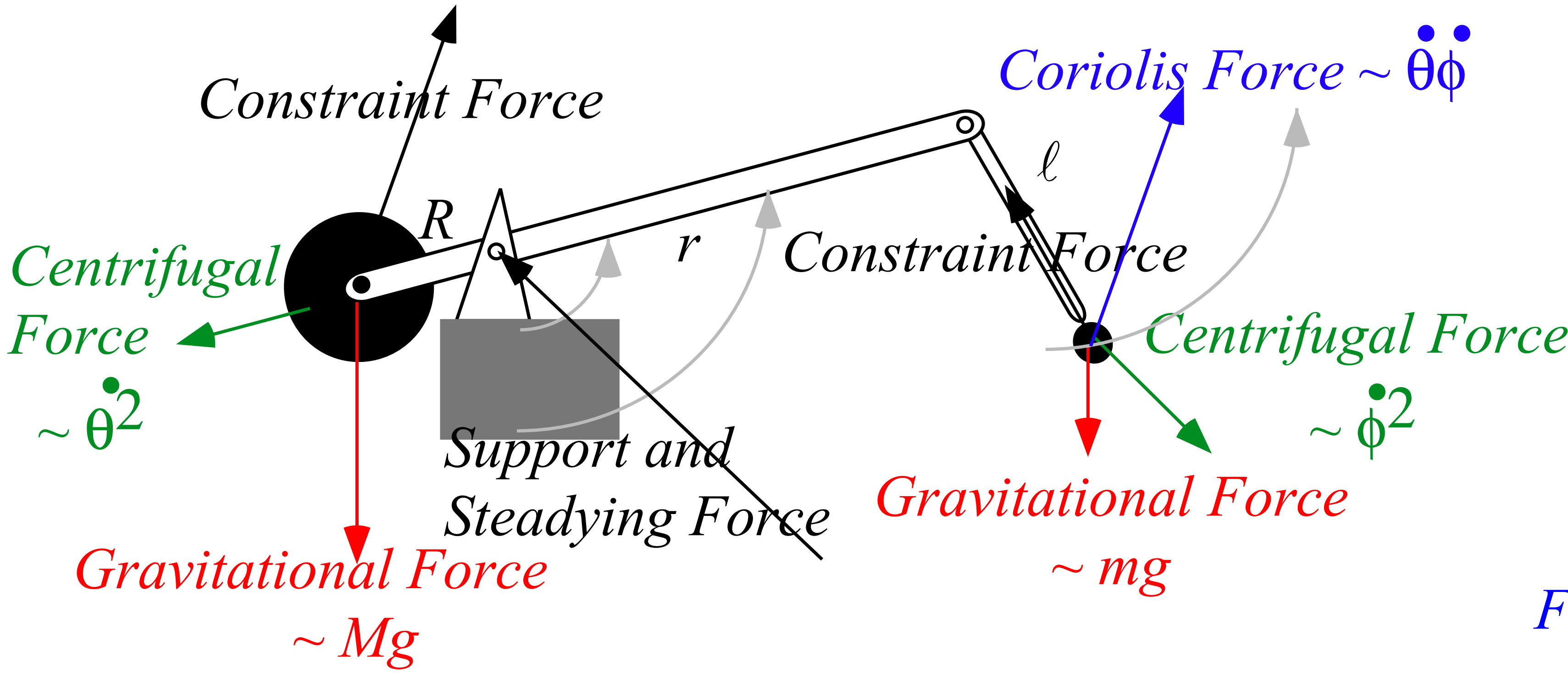
Lagrange Force equations

(See also derivation Eq. (2.4.7) on p. 23 , Unit 2)

Fig. 2.5.2 (modified)

Compare to derivation Eq (12.25a) in Ch. 12 of Unit 1 and Eq. (3.5.10) in Unit 3.

Forces in Lagrange force equation: total, genuine, potential, and/or fictitious



Acceleration and 'Fictitious' Forces:
Coriolis
Centrifugal

Applied 'Real' Forces:
Gravity
Stimuli
Friction...

Constraint 'Internal' Forces:
Stresses
Support...
(Do not contribute.
Do no work.)

For conservative forces

where: $F_\theta = -\frac{\partial V}{\partial \theta}$ and: $\frac{\partial V}{\partial \dot{\theta}} = 0$

$F_\phi = -\frac{\partial V}{\partial \phi}$ and: $\frac{\partial V}{\partial \dot{\phi}} = 0$

$$\dot{p}_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} = \frac{\partial T}{\partial \theta} + F_\theta + 0$$

$$\dot{p}_\phi = \frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} = \frac{\partial T}{\partial \phi} + F_\phi + 0$$

Lagrange Force equations
(See also derivation Eq. (2.4.7) on p. 23 , Unit 2)

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} \qquad \dot{p}_\theta = \frac{\partial L}{\partial \theta}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} \qquad \dot{p}_\phi = \frac{\partial L}{\partial \phi}$$

Lagrange Potential equations
$L = T - V$

Fig. 2.5.2
(modified)

Compare to derivation Eq (12.25a) in Ch. 12 of Unit 1 and Eq. (3.5.10) in Unit 3.

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.

Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)

Trebuchet Cartesian projectile coordinates are double-valued

Toroidal "rolled-up" ($q^1=\theta$, $q^2=\phi$)-manifold and "Flat" ($x=\theta$, $y=\phi$)-graph

Review of covariant $\mathbf{E}_n$ and contravariant $\mathbf{E}^m$ vectors: Jacobian J vs. Kajobian K

Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)

Tangent $\{\mathbf{E}_n\}$space vs. Normal $\{\mathbf{E}^m\}$space

Covariant vs. contravariant coordinate transformations

Metric g_{mn} tensor geometric relations to length, area, and volume

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)

Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 77)

Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 69)

Riemann equation derivation for trebuchet model

Riemann equation force analysis

2nd-guessing Riemann equation?

Trebuchet Cartesian projectile coordinates are double-valued

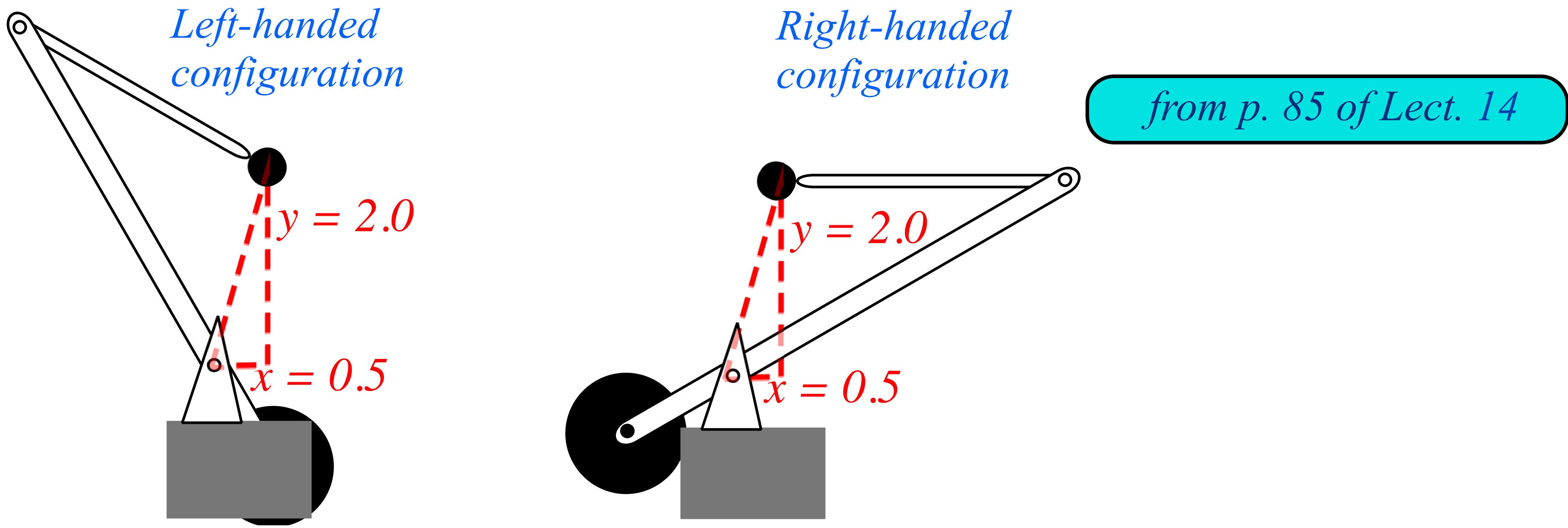


Fig. 2.2.3 Trebuchet configurations with the same coordinates x and y of projectile m.

Trebuchet Cartesian projectile coordinates are double-valued…(Belong to 2 distinct manifolds)

from p. 86 of Lect. 14

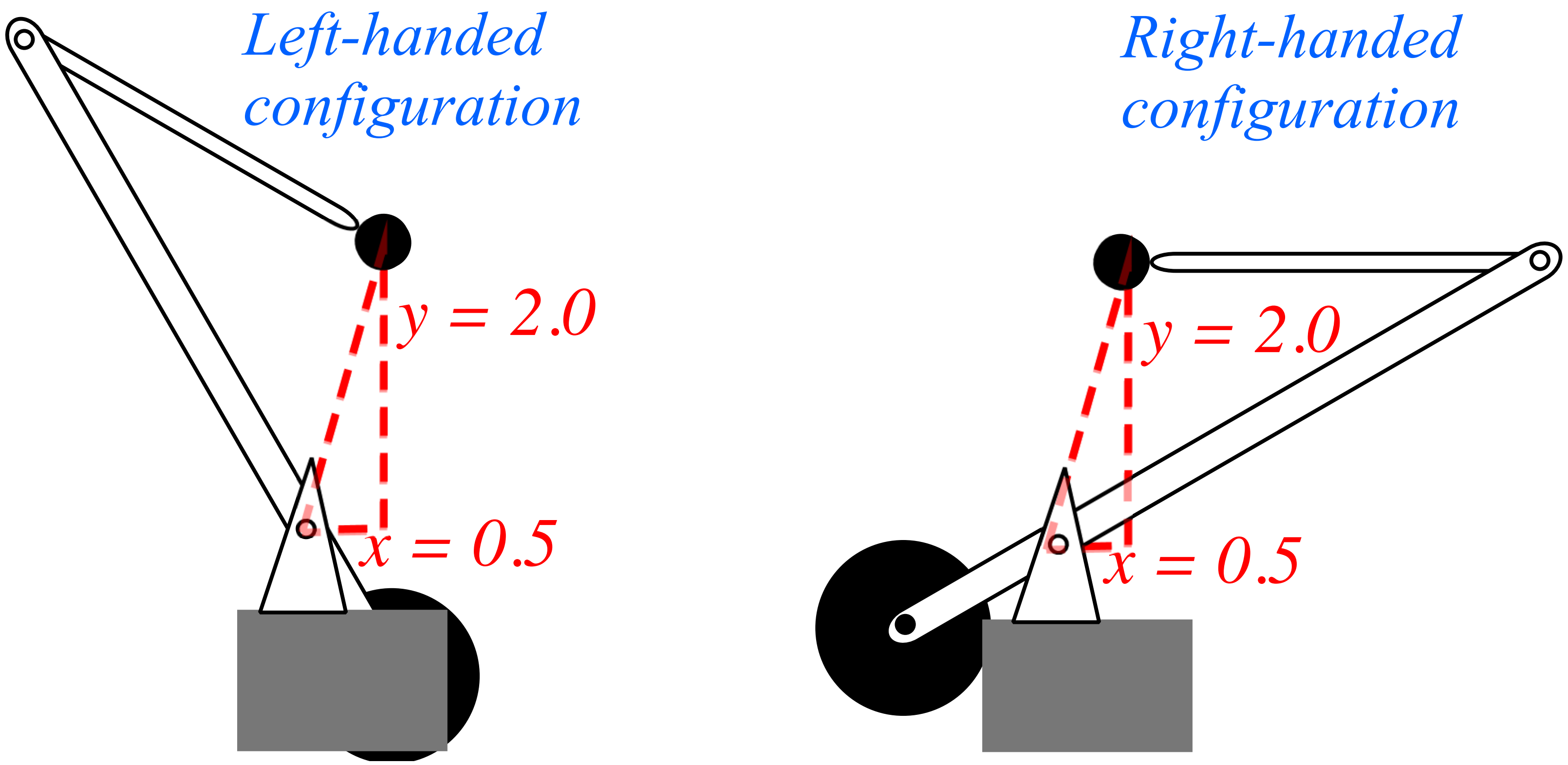


Fig. 2.2.3 Trebuchet configurations with the same coordinates x and y of projectile *m*.

So, for example, are polar coordinates … (for each angle there are two r-values)

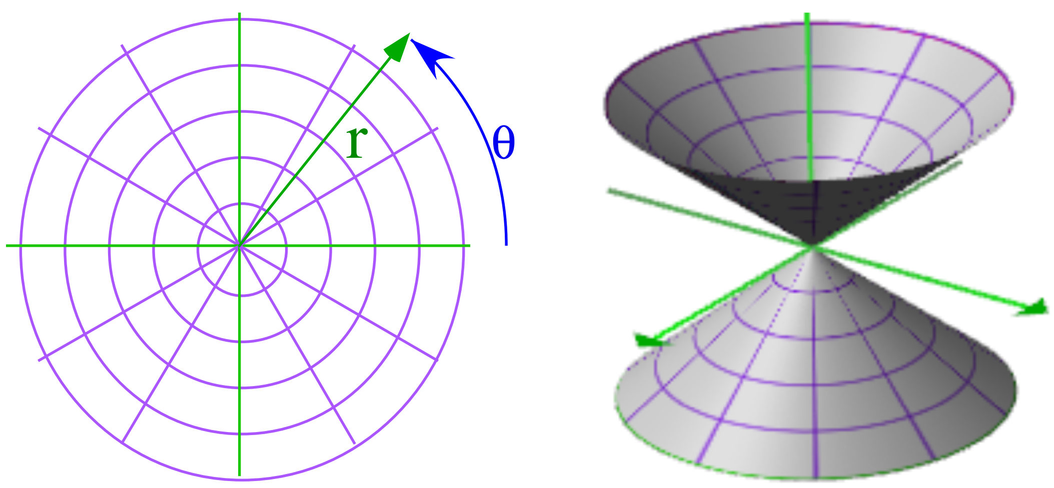


Fig. 3.1.4 Polar coordinates and possible embedding space on conical surface.

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.
Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)
Trebuchet Cartesian projectile coordinates are double-valued
Toroidal "rolled-up" ($q^1=\theta$, $q^2=\phi$)-manifold and "Flat" ($x=\theta$, $y=\phi$)-graph
Review of covariant $\mathbf{E}_n$ and contravariant $\mathbf{E}^m$ vectors: Jacobian J vs. Kajobian K
Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)
Tangent $\{\mathbf{E}_n\}$space vs. Normal $\{\mathbf{E}^m\}$space
Covariant vs. contravariant coordinate transformations
Metric g_{mn} tensor geometric relations to length, area, and volume

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)
Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 77)
Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 69)
Riemann equation derivation for trebuchet model
Riemann equation force analysis
2nd-guessing Riemann equation?

from p. 87 of Lect. 14

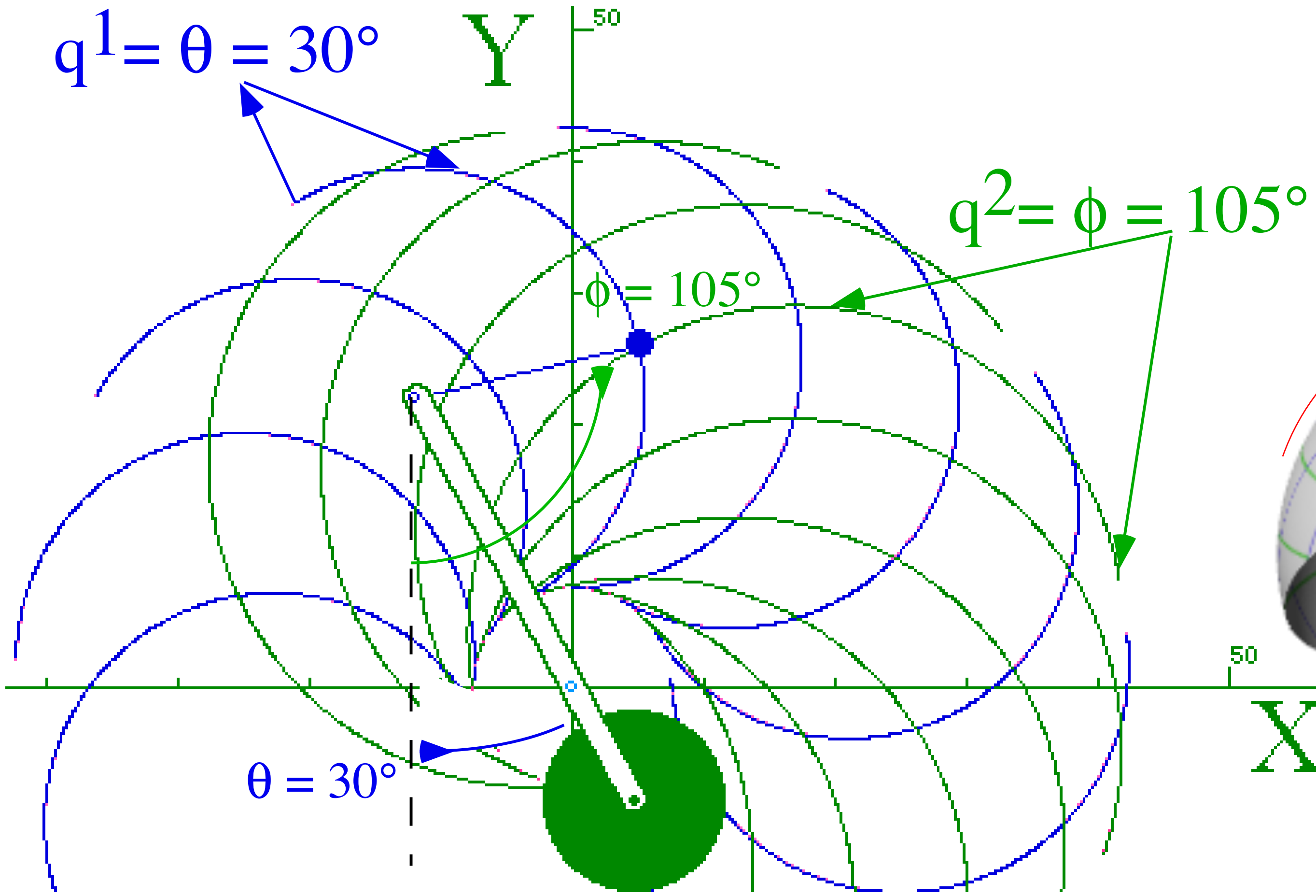


Fig. 3.1.1a (q^1=θ, q^2=φ)Coordinate manifold for trebuchet (Left handed sheet.)

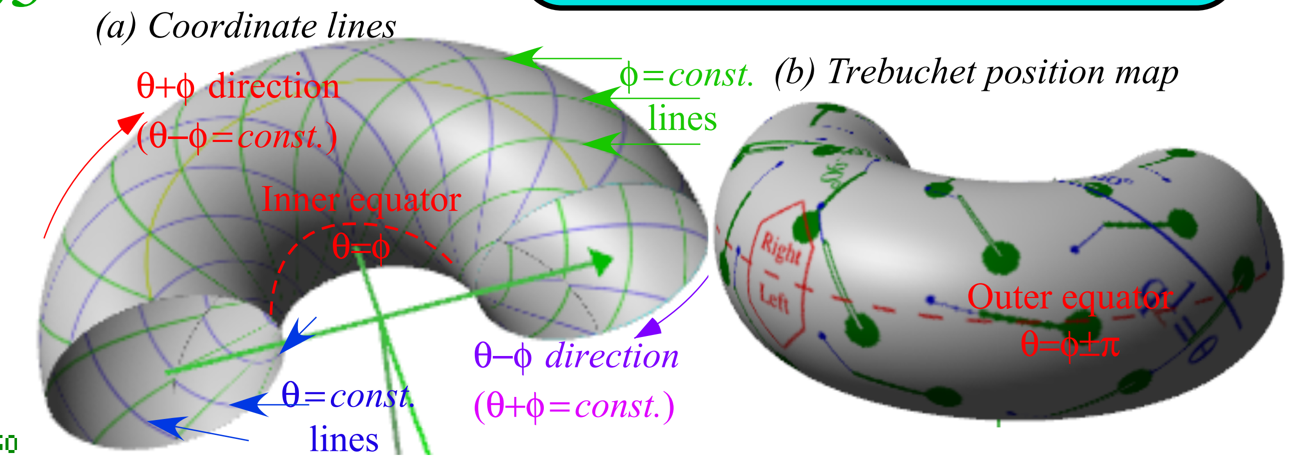


Fig. 3.1.2 Trebuchet torus.

(a) (q^1=θ, q^2=φ) coordinate lines. (b)Trebuchet position map and equators.

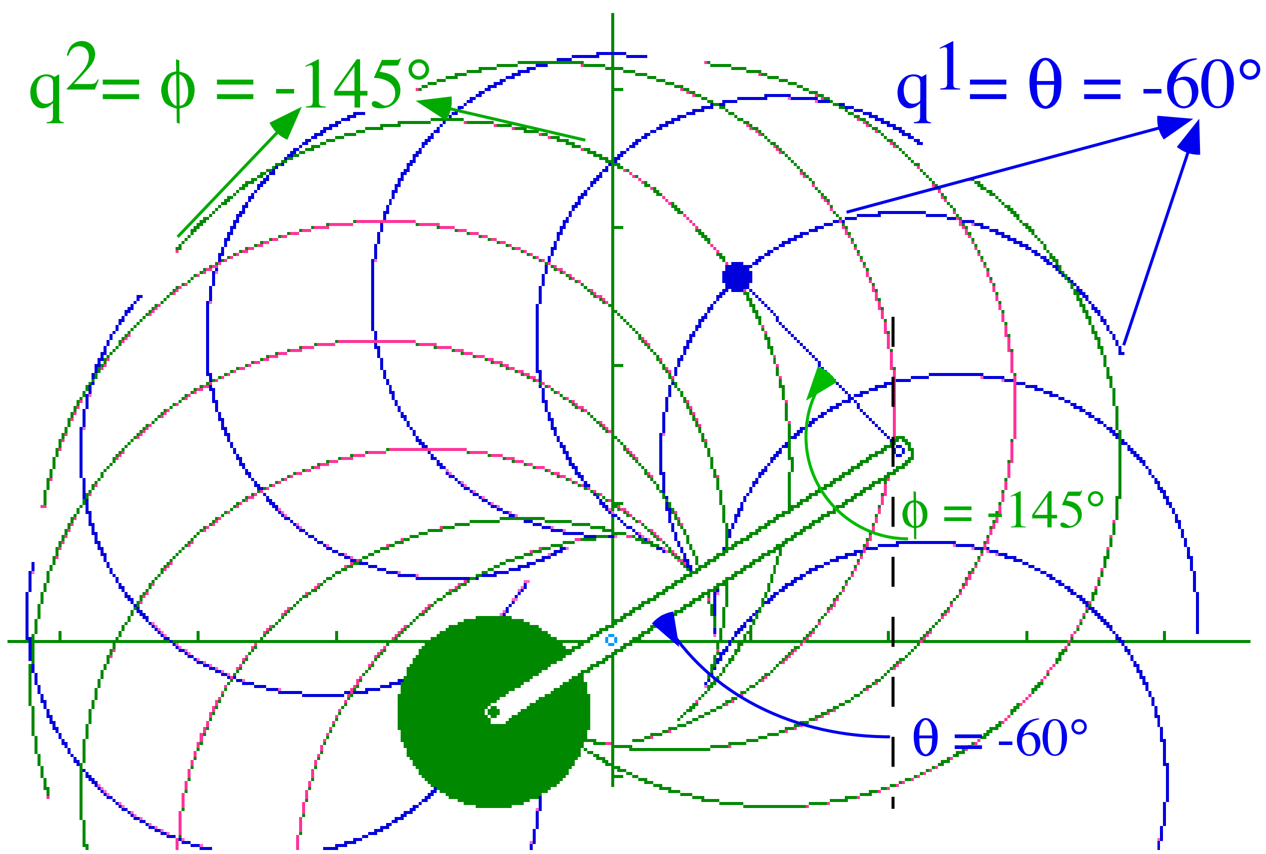


Fig. 3.1.1b (q^1=θ, q^2=φ)Coordinate manifold for trebuchet (Right handed sheet.)

“Flat” (q^1=θ, q^2=ϕ)-graph of trebuchet loci compared to “rolled-up” toroidal manifold

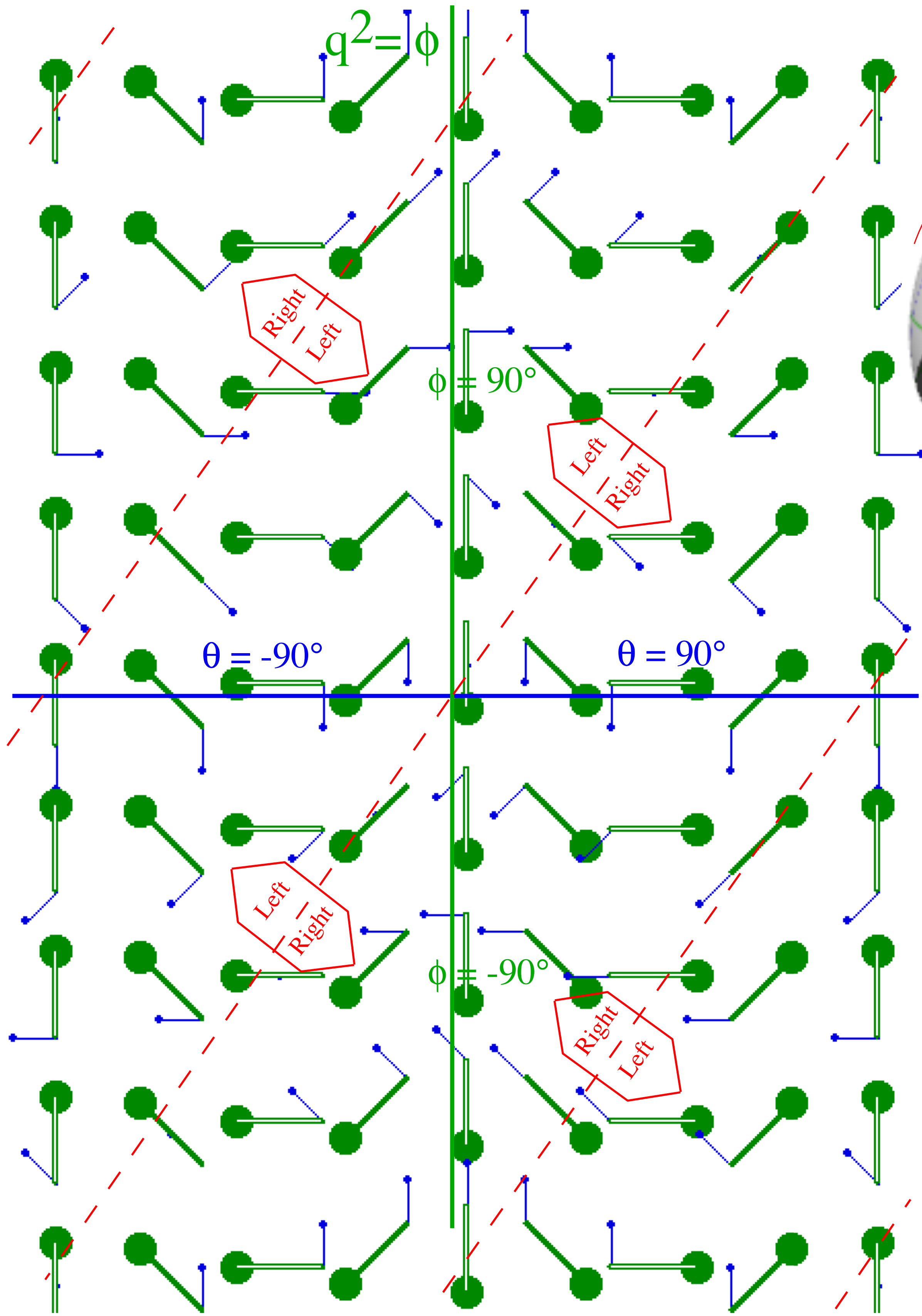


Fig. 3.1.3 "Flattened" (q^1=θ, q^2=ϕ) coordinate manifold for trebuchet

from p. 88 of Lect. 14

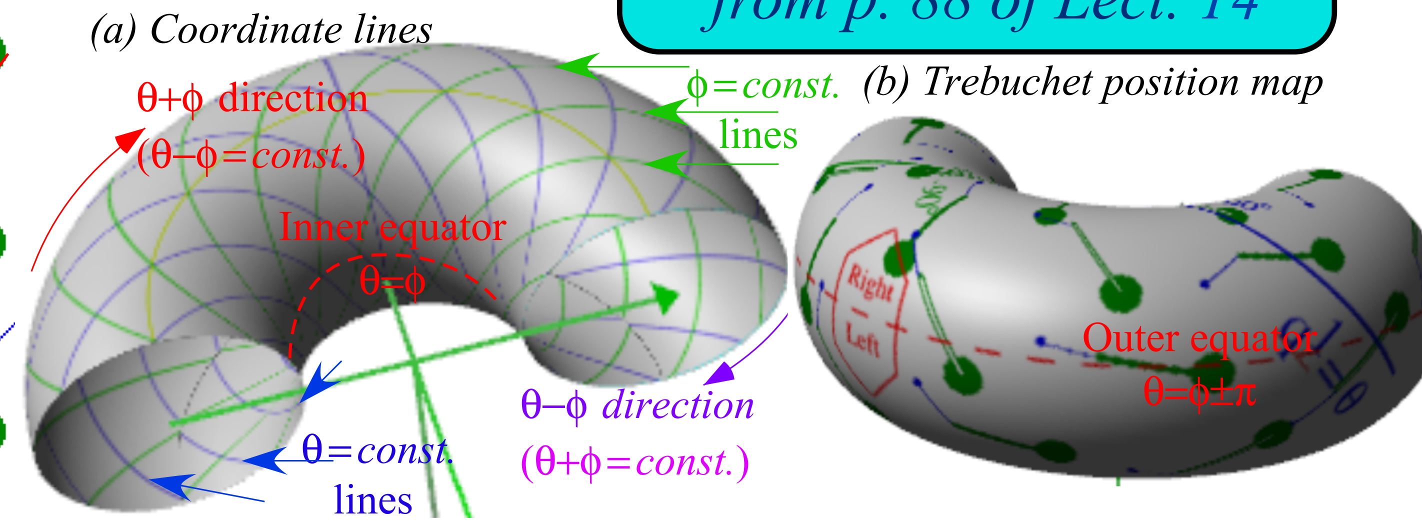


Fig. 3.1.2 Trebuchet torus.
(a) (q^1=θ, q^2=ϕ) coordinate lines.(b)Trebuchet position map and equators.

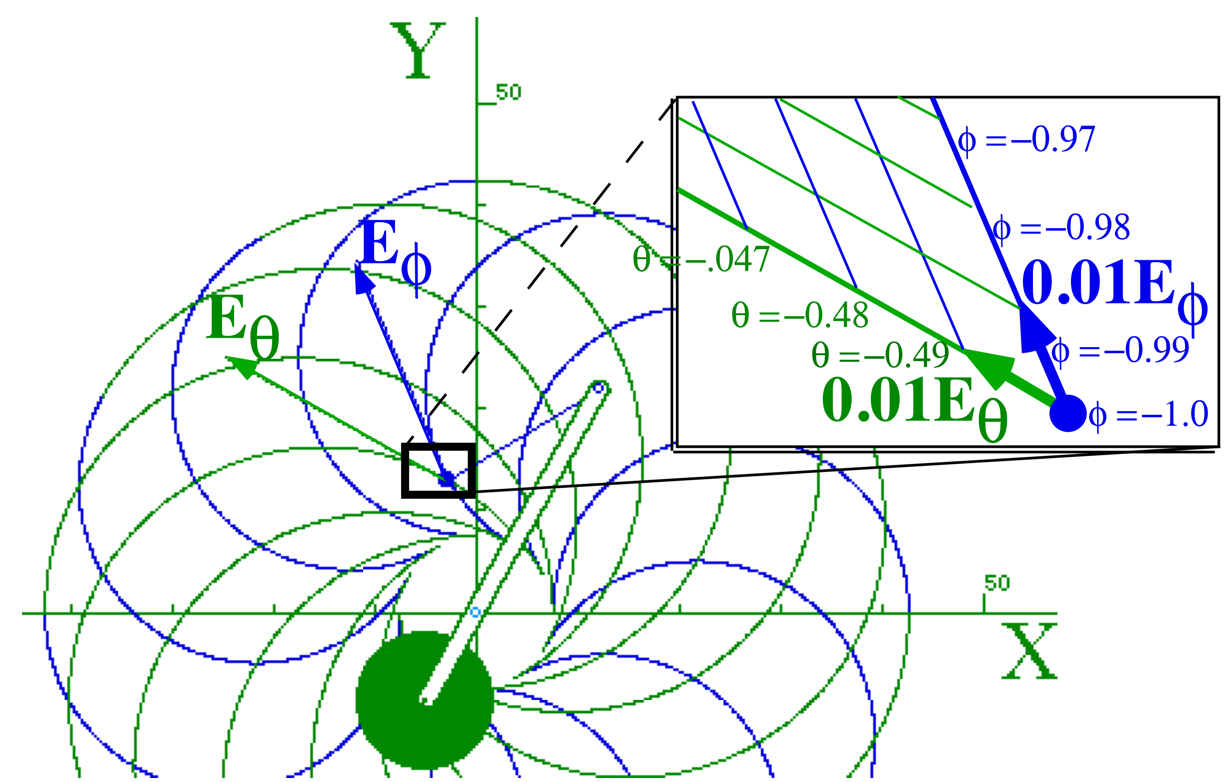


Fig. 3.2.2 Example of covariant unitary vectors and their tangent space.

Toroidal “rolled-up” ($q^1=\theta$, $q^2=\phi$)-manifold of trebuchet positions

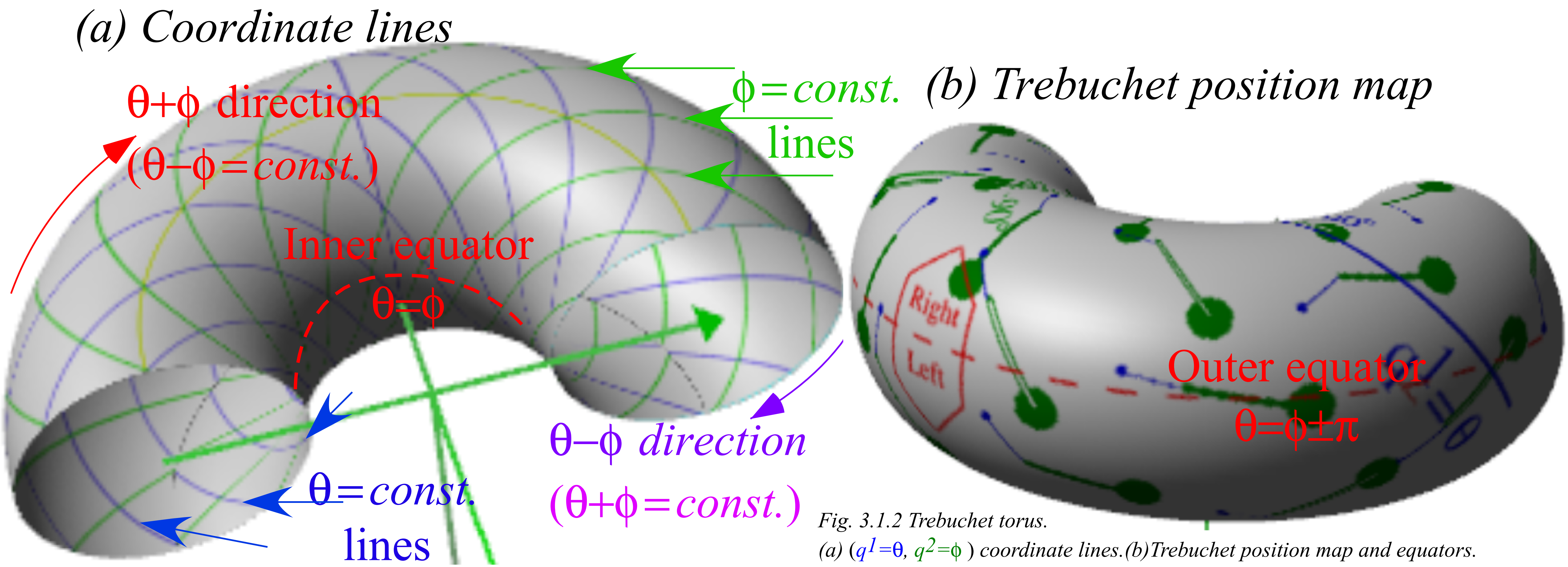


Fig. 3.1.2 Trebuchet torus.
(a) ($q^1=\theta$, $q^2=\phi$) coordinate lines.(b)Trebuchet position map and equators.

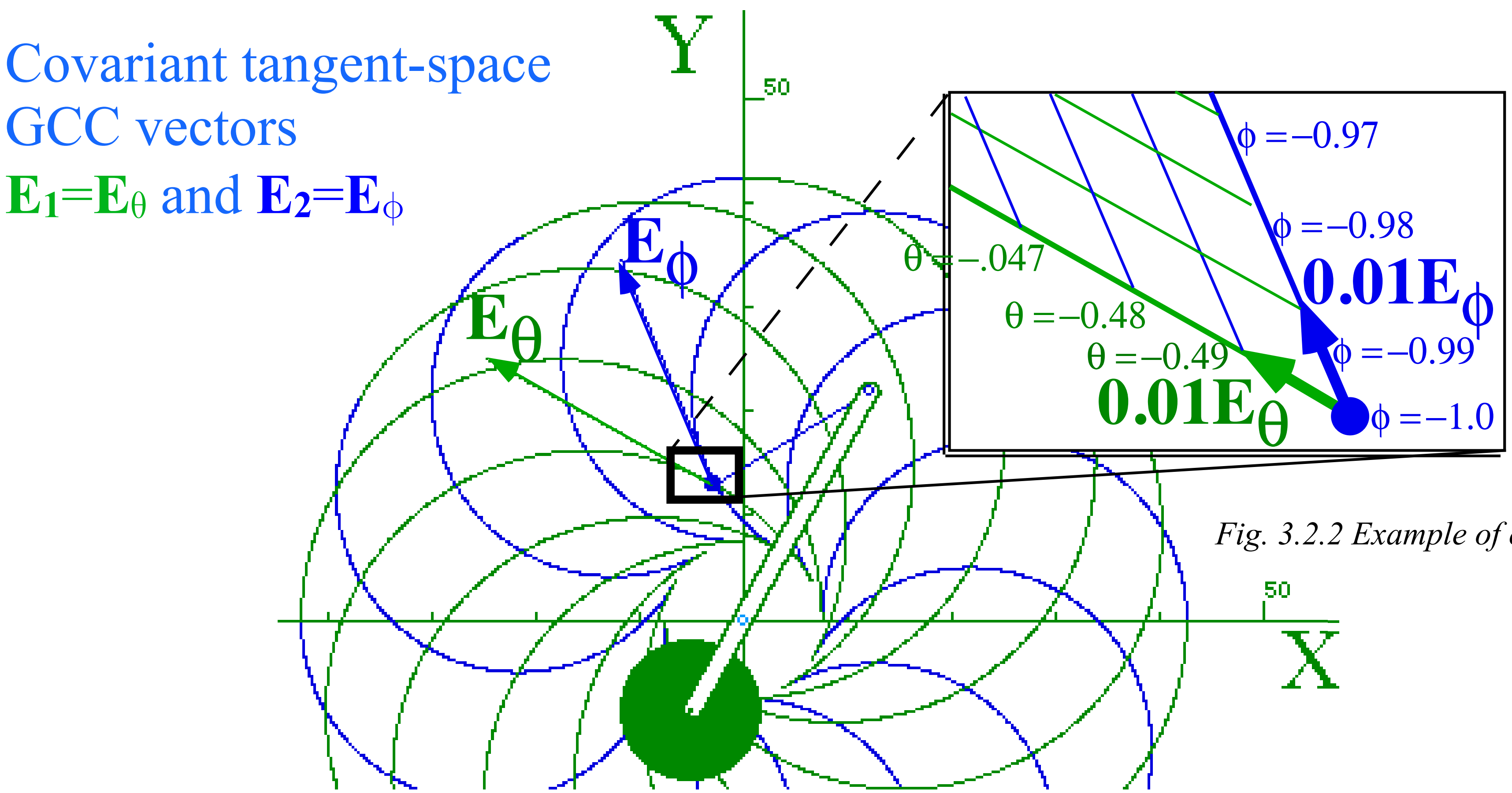


Fig. 3.2.2 Example of covariant unitary vectors and their tangent space.

Toroidal “rolled-up” (q^1=θ, q^2=ϕ)-manifold of trebuchet positions and “Flat” (q^1=θ, q^2=ϕ)-graph

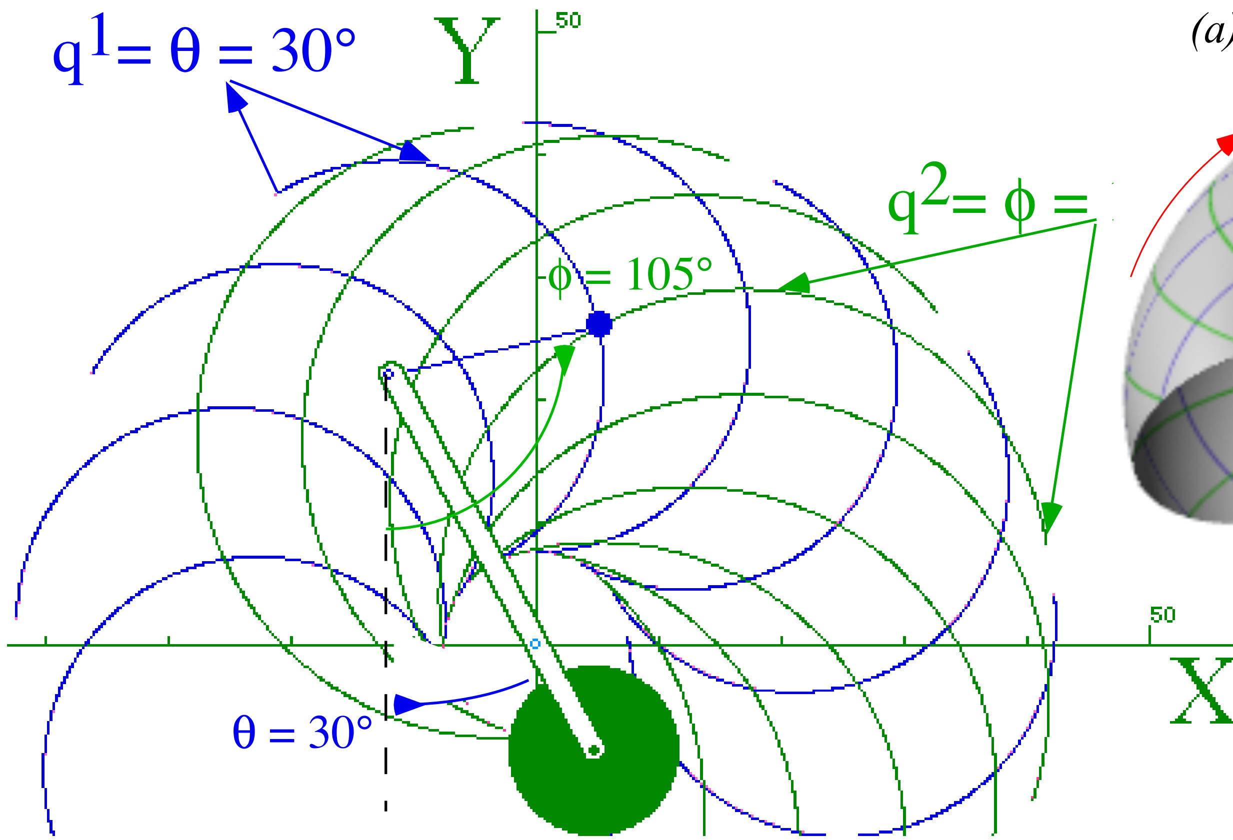


Fig. 3.1.1a (q^1=θ, q^2=ϕ)Coordinate manifold for trebuchet *(Left handed sheet.)*

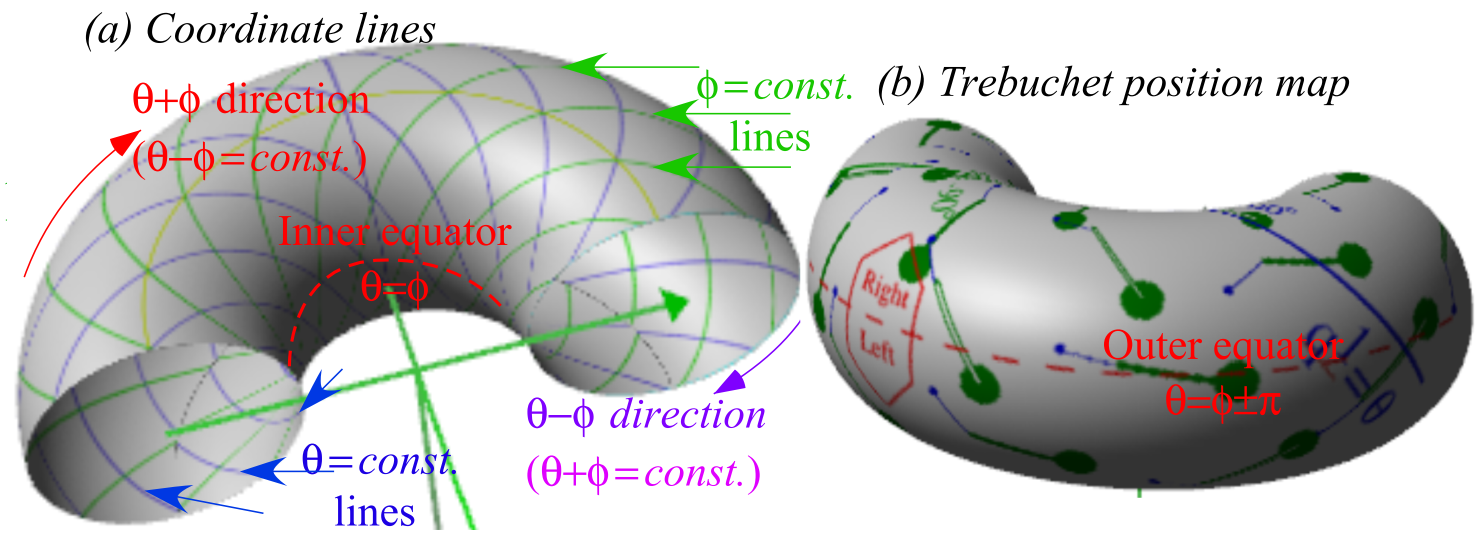
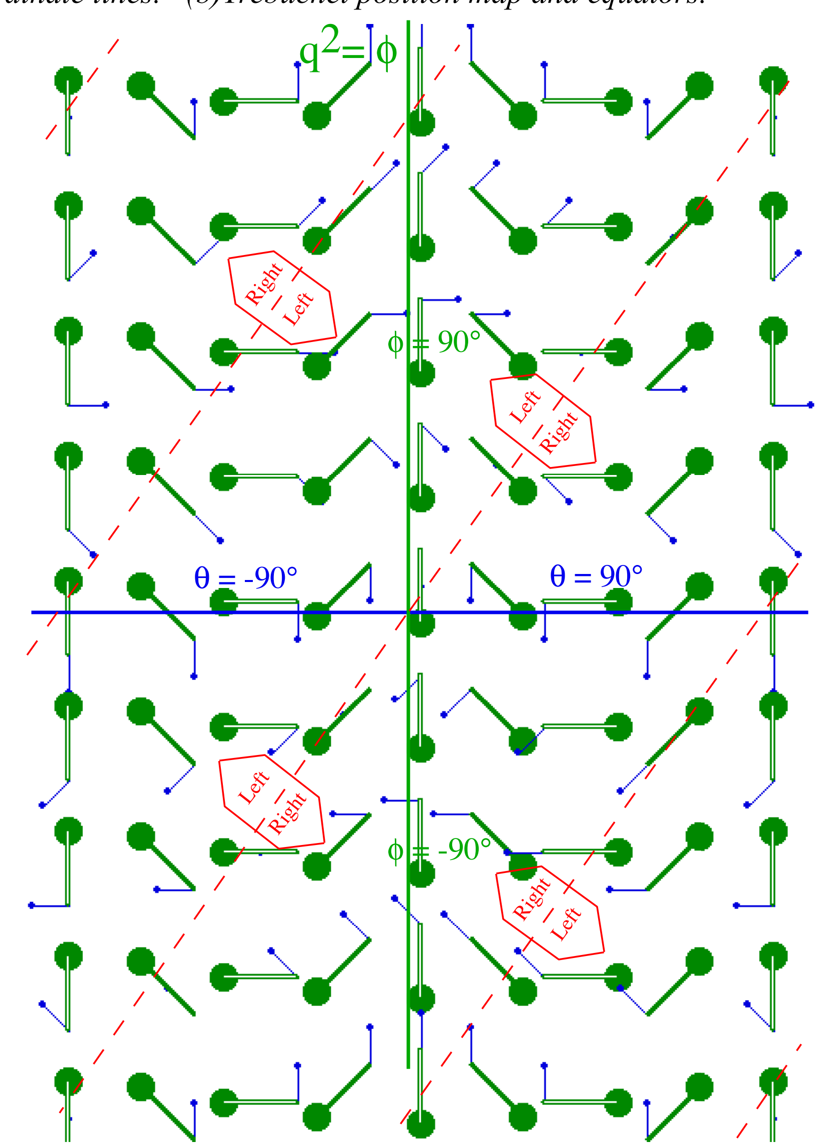


Fig. 3.1.2 Trebuchet torus.

(a) (q^1=θ, q^2=ϕ) coordinate lines. (b)Trebuchet position map and equators.



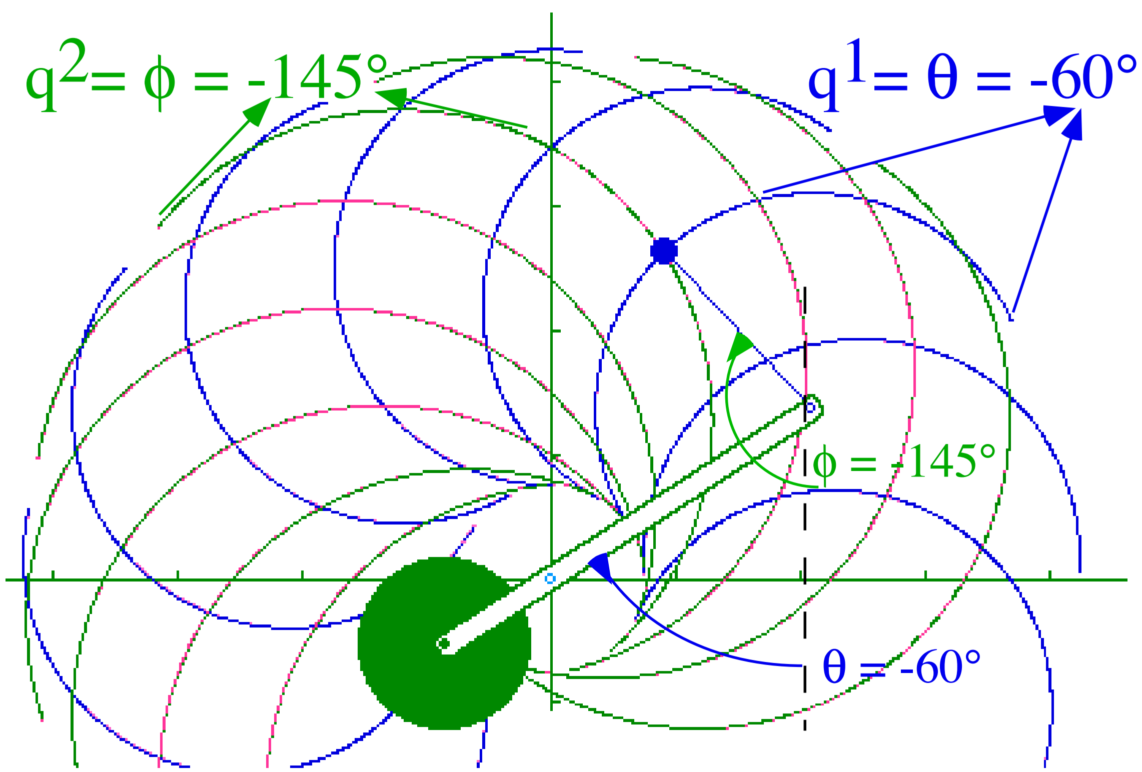


Fig. 3.1.1b (q^1=θ, q^2=ϕ)Coordinate manifold for trebuchet *(Right handed sheet.)*

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.

Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)

Trebuchet Cartesian projectile coordinates are double-valued

Toroidal "rolled-up" (q^1=θ, q^2=ϕ)-manifold and "Flat" (x=θ, y=ϕ)-graph

Review of covariant $\mathbf{E}_n$ and contravariant $\mathbf{E}^m$ vectors: Jacobian J vs. Kajobian K

Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)

Tangent $\{\mathbf{E}_n\}$space vs. Normal $\{\mathbf{E}^m\}$space

Covariant vs. contravariant coordinate transformations

Metric g_{mn} tensor geometric relations to length, area, and volume

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)

Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 77)

Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 69)

Riemann equation derivation for trebuchet model

Riemann equation force analysis

2nd-guessing Riemann equation?

A dual set of *quasi-unit vectors* show up in Jacobian J and Kajobian K. *from p. 45 of Lect. 9*

J-Columns are *covariant vectors* $\{\mathbf{E}_1=\mathbf{E}_r \;\; \mathbf{E}_2=\mathbf{E}_\phi\}$ K-Rows are *contravariant vectors* $\{\mathbf{E}^1=\mathbf{E}^r \;\; \mathbf{E}^2=\mathbf{E}^\phi\}$

$$\langle J\rangle = \begin{array}{c}\begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} \end{pmatrix} \\ \uparrow\mathbf{E}_1 \quad \uparrow\mathbf{E}_2 \end{array} = \begin{array}{c}\begin{pmatrix} \frac{\partial x}{\partial r}=\cos\phi & \frac{\partial x}{\partial \phi}=-r\sin\phi \\ \frac{\partial y}{\partial r}=\sin\phi & \frac{\partial y}{\partial \phi}=r\cos\phi \end{pmatrix} \\ \uparrow\mathbf{E}_r \qquad\qquad \uparrow\mathbf{E}_\phi \end{array}$$

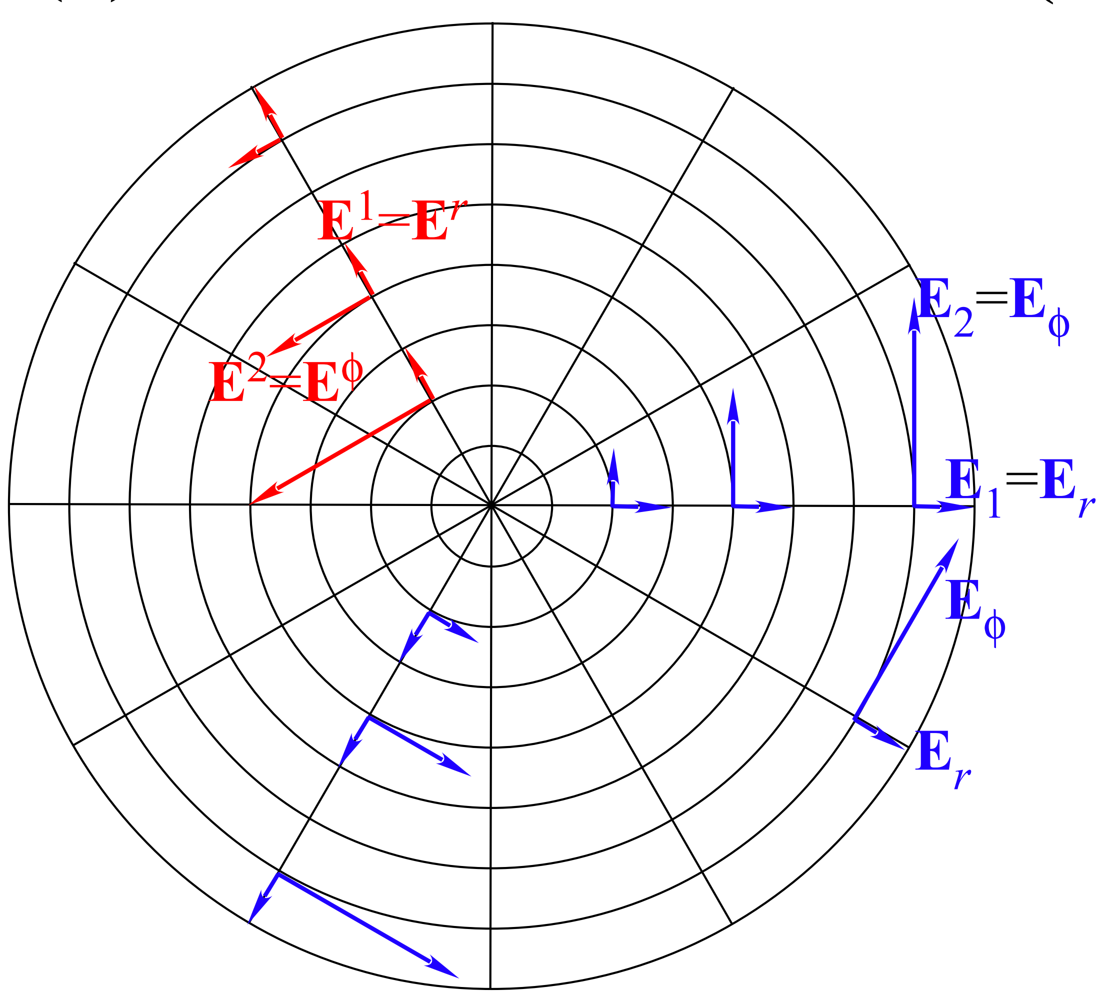
$$\langle K\rangle = \langle J^{-1}\rangle = \begin{array}{cc}\begin{pmatrix} \frac{\partial r}{\partial x}=\cos\phi & \frac{\partial r}{\partial y}=\sin\phi \\ \frac{\partial \phi}{\partial x}=\frac{-\sin\phi}{r} & \frac{\partial \phi}{\partial y}=\frac{\cos\phi}{r} \end{pmatrix} & \begin{array}{l} \leftarrow \mathbf{E}^r=\mathbf{E}^1 \\ \leftarrow \mathbf{E}^\phi=\mathbf{E}^2 \end{array}\end{array}$$

Derived from polar definition: $x=r\cos\phi$ *and* $y=r\sin\phi$

Inverse polar definition:

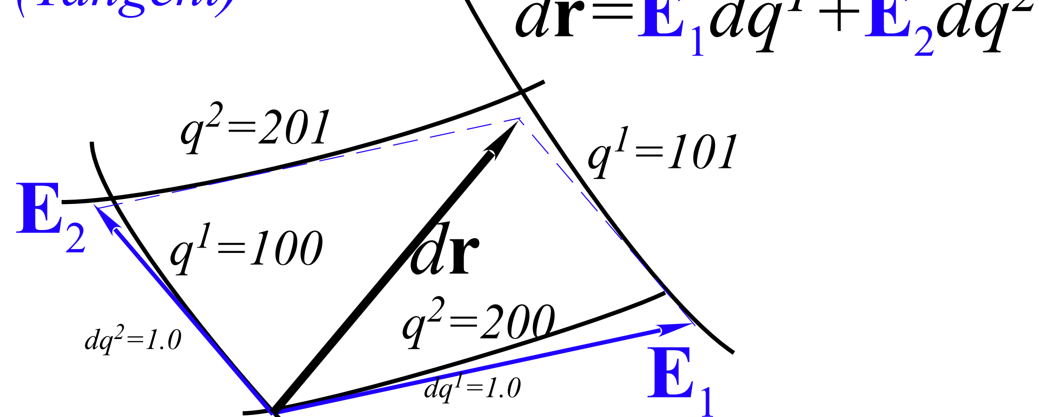
$r^2=x^2+y^2$ *and* $\phi=\mathrm{atan2}(y,x)$

(a) Polar coordinate bases



(b) *Co*variant bases $\{\mathbf{E}_1\;\mathbf{E}_2\}$

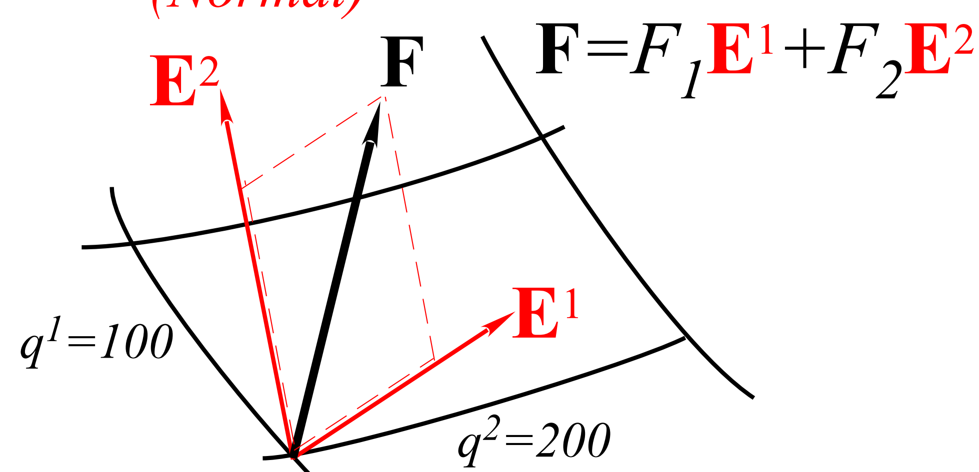
(Tangent)



NOTE:These are 2D drawings! No 3D perspective

(c) *Contra*variant bases $\{\mathbf{E}^1\;\mathbf{E}^2\}$

(Normal)



Unit 1
Fig. 12.10

Comparison: Covariant $\mathbf{E}_m = \frac{\partial \mathbf{r}}{\partial q^m}$ *vs.* *Contravariant* $\mathbf{E}^m = \frac{\partial q^m}{\partial \mathbf{r}} = \nabla q^m$

from p. 50 of Lect. 9

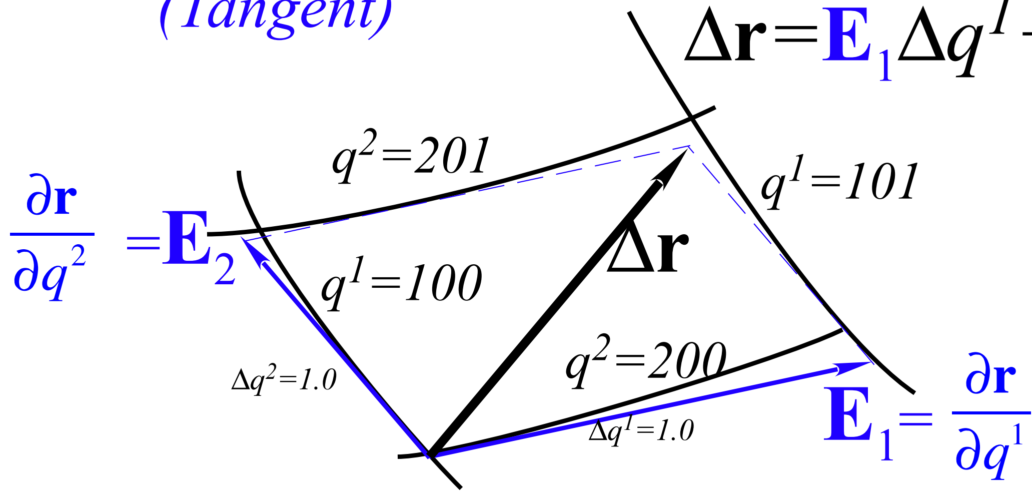
*Co*variant bases $\{\mathbf{E}_1\ \mathbf{E}_2\}$ match cell walls

geometric unit

(Tangent)

$$\Delta\mathbf{r} = \mathbf{E}_1 \Delta q^1 + \mathbf{E}_2 \Delta q^2$$

is based on chain rule: $d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial q^1} dq^1 + \frac{\partial \mathbf{r}}{\partial q^2} dq^2 = \mathbf{E}_1 dq^1 + \mathbf{E}_2 dq^2$



$\mathbf{E}_1$ *follows tangent to* q^2*=const. ...*
since only q^1 *varies in* $\frac{\partial \mathbf{r}}{\partial q^1}$
while q^2, q^3,... *remain constant*

$\mathbf{E}_m$ are convenient bases for *ex*tensive quantities like distance and velocity.

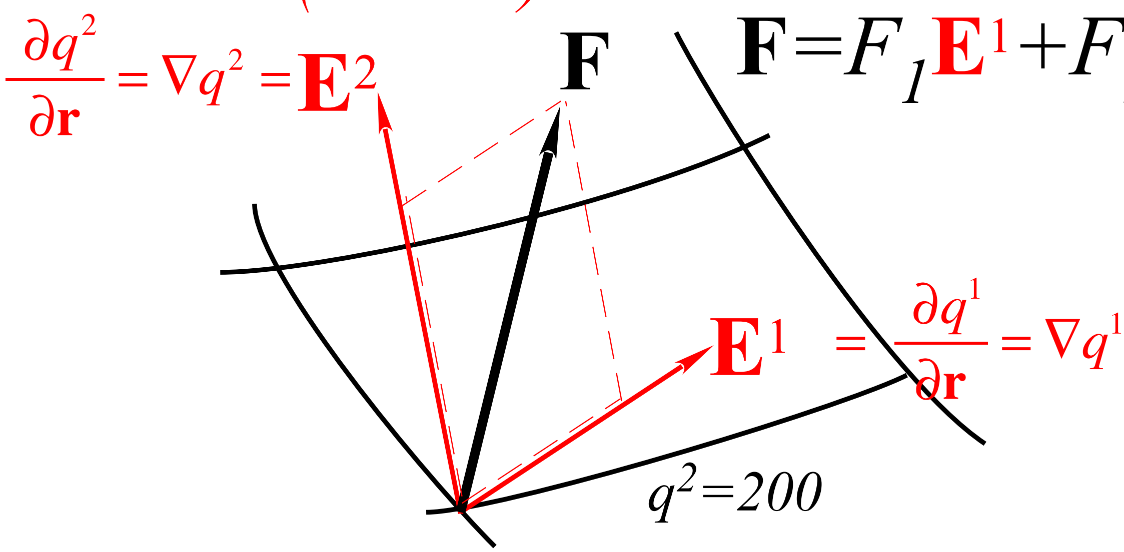
$$\mathbf{V} = V^1 \mathbf{E}_1 + V^2 \mathbf{E}_2 = V^1 \frac{\partial \mathbf{r}}{\partial q^1} + V^2 \frac{\partial \mathbf{r}}{\partial q^2}$$

*Contra*variant $\{\mathbf{E}^1\ \mathbf{E}^2\}$ match reciprocal cells

(Normal)

$$\mathbf{F} = F_1 \mathbf{E}^1 + F_2 \mathbf{E}^2$$

NOTE:These
are 2D drawings!
<u>*No*</u> 3D <u>*perspective*</u>



$\mathbf{E}^1$ *is normal to* q^1*=const. since*
gradient *of* q^1 *is vector sum* $\nabla q^1 = \begin{pmatrix} \frac{\partial q^1}{\partial x} \\ \frac{\partial q^1}{\partial y} \end{pmatrix}$
of all its partial derivatives

$\mathbf{E}^m$ are convenient bases for *in*tensive quantities like force and momentum.

$$\mathbf{F} = F_1 \mathbf{E}^1 + F_2 \mathbf{E}^2 = F_1 \frac{\partial q^1}{\partial \mathbf{r}} + F_2 \frac{\partial q^2}{\partial \mathbf{r}} = F_1 \nabla q^1 + F_2 \nabla q^2$$

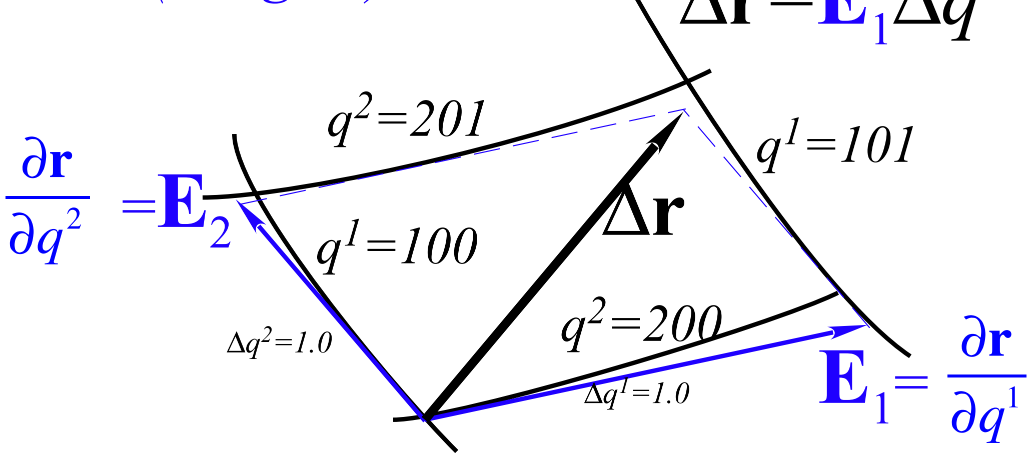
Comparison: Covariant $\mathbf{E}_m = \frac{\partial \mathbf{r}}{\partial q^m}$ vs. Contravariant $\mathbf{E}^n = \frac{\partial q^n}{\partial \mathbf{r}} = \nabla q^n$

from p. 50 of Lect. 9

Covariant bases $\{\mathbf{E}_1\ \mathbf{E}_2\}$ match geometric unit cell walls
(Tangent)

$$\Delta\mathbf{r} = \mathbf{E}_1 \Delta q^1 + \mathbf{E}_2 \Delta q^2$$

is based on chain rule: $d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial q^1} dq^1 + \frac{\partial \mathbf{r}}{\partial q^2} dq^2 = \mathbf{E}_1 dq^1 + \mathbf{E}_2 dq^2$



$\mathbf{E}_1$ follows tangent to q^2=const. ...
since only q^1 varies in $\frac{\partial \mathbf{r}}{\partial q^1}$
while q^2, q^3,... remain constant

$\mathbf{E}_m$ are convenient bases for *ex*tensive quantities like distance and velocity.

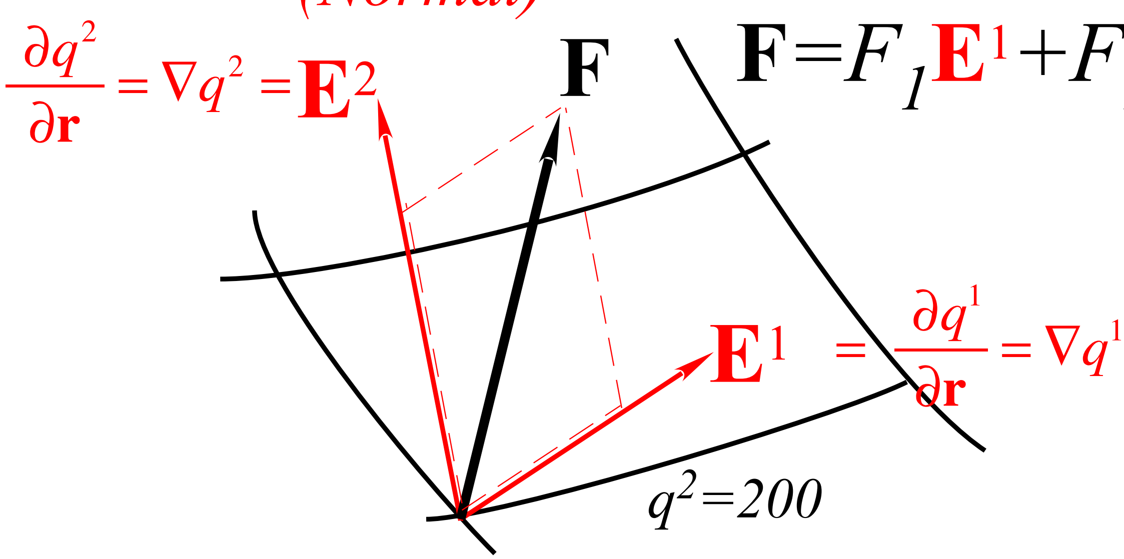
$$\mathbf{V} = V^1\mathbf{E}_1 + V^2\mathbf{E}_2 = V^1\frac{\partial \mathbf{r}}{\partial q^1} + V^2\frac{\partial \mathbf{r}}{\partial q^2}$$

Co-Contra dot products $\mathbf{E}_m \cdot \mathbf{E}^n$ are orthonormal:

$$\mathbf{E}_m \bullet \mathbf{E}^n = \frac{\partial \mathbf{r}}{\partial q^m} \bullet \frac{\partial q^n}{\partial \mathbf{r}} = \delta_m^n$$

Contravariant $\{\mathbf{E}^1\ \mathbf{E}^2\}$ match reciprocal cells
(Normal)

$$\mathbf{F} = F_1\mathbf{E}^1 + F_2\mathbf{E}^2$$



$\mathbf{E}^1$ is normal to q^1=const. since **gradient** *of q^1 is vector sum $\nabla q^1 = \begin{pmatrix} \frac{\partial q^1}{\partial x} \\ \frac{\partial q^1}{\partial y} \end{pmatrix}$ of all its partial derivatives*

$\mathbf{E}^m$ are convenient bases for *in*tensive quantities like force and momentum.

$$\mathbf{F} = F_1\mathbf{E}^1 + F_2\mathbf{E}^2 = F_1\frac{\partial q^1}{\partial \mathbf{r}} + F_2\frac{\partial q^2}{\partial \mathbf{r}} = F_1\nabla q^1 + F_2\nabla q^2$$

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.
Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)
Trebuchet Cartesian projectile coordinates are double-valued
Toroidal "rolled-up" (q^1=θ, q^2=ϕ)-manifold and "Flat" (x=θ, y=ϕ)-graph
Review of covariant $\mathbf{E}_n$ and contravariant $\mathbf{E}^m$ vectors: Jacobian J vs. Kajobian K
Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)
Tangent {$\mathbf{E}_n$}space vs. Normal {$\mathbf{E}^m$}space
Covariant vs. contravariant coordinate transformations
Metric g_{mn} tensor geometric relations to length, area, and volume

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)
Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 77)
Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 69)
Riemann equation derivation for trebuchet model
Riemann equation force analysis
2nd-guessing Riemann equation?

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

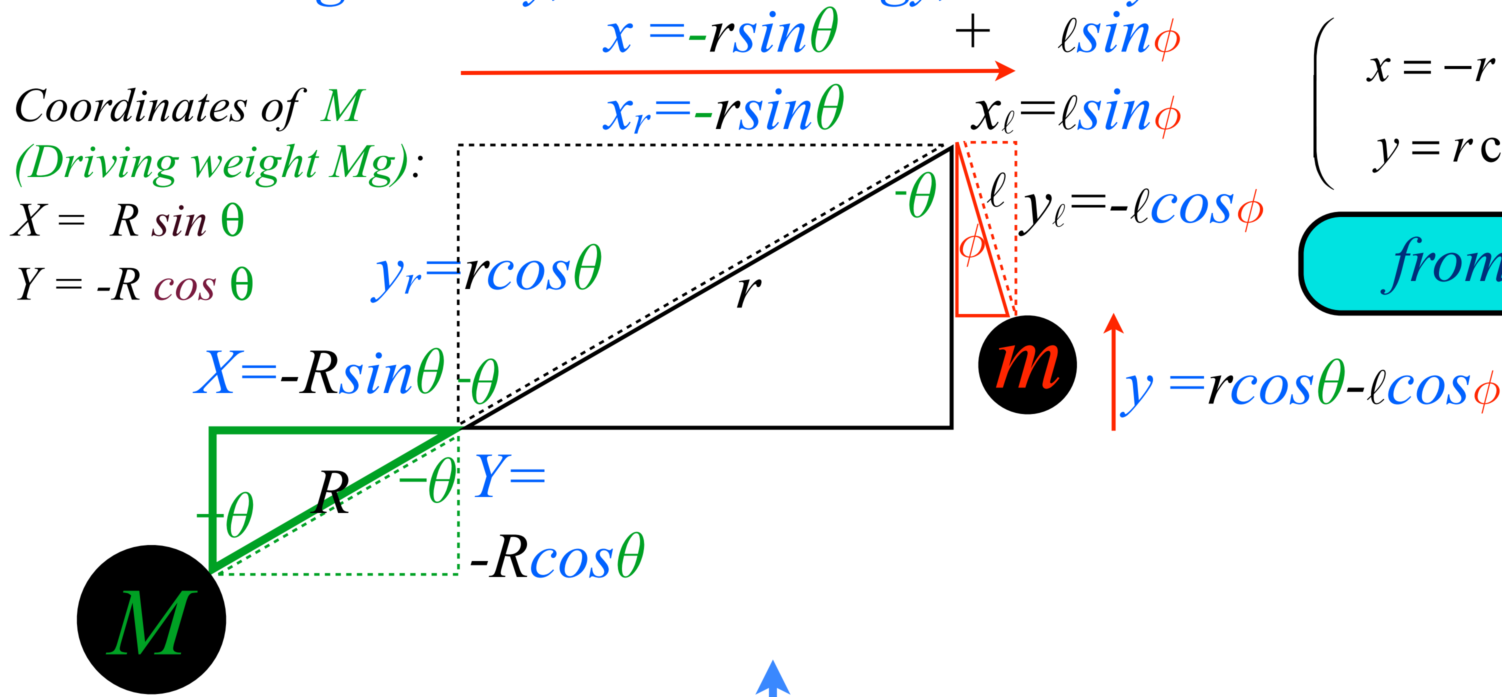
Coordinates of mass m (Payload or projectile):

$x = x_r + x_\ell = -r \sin\theta + \ell \sin\phi$

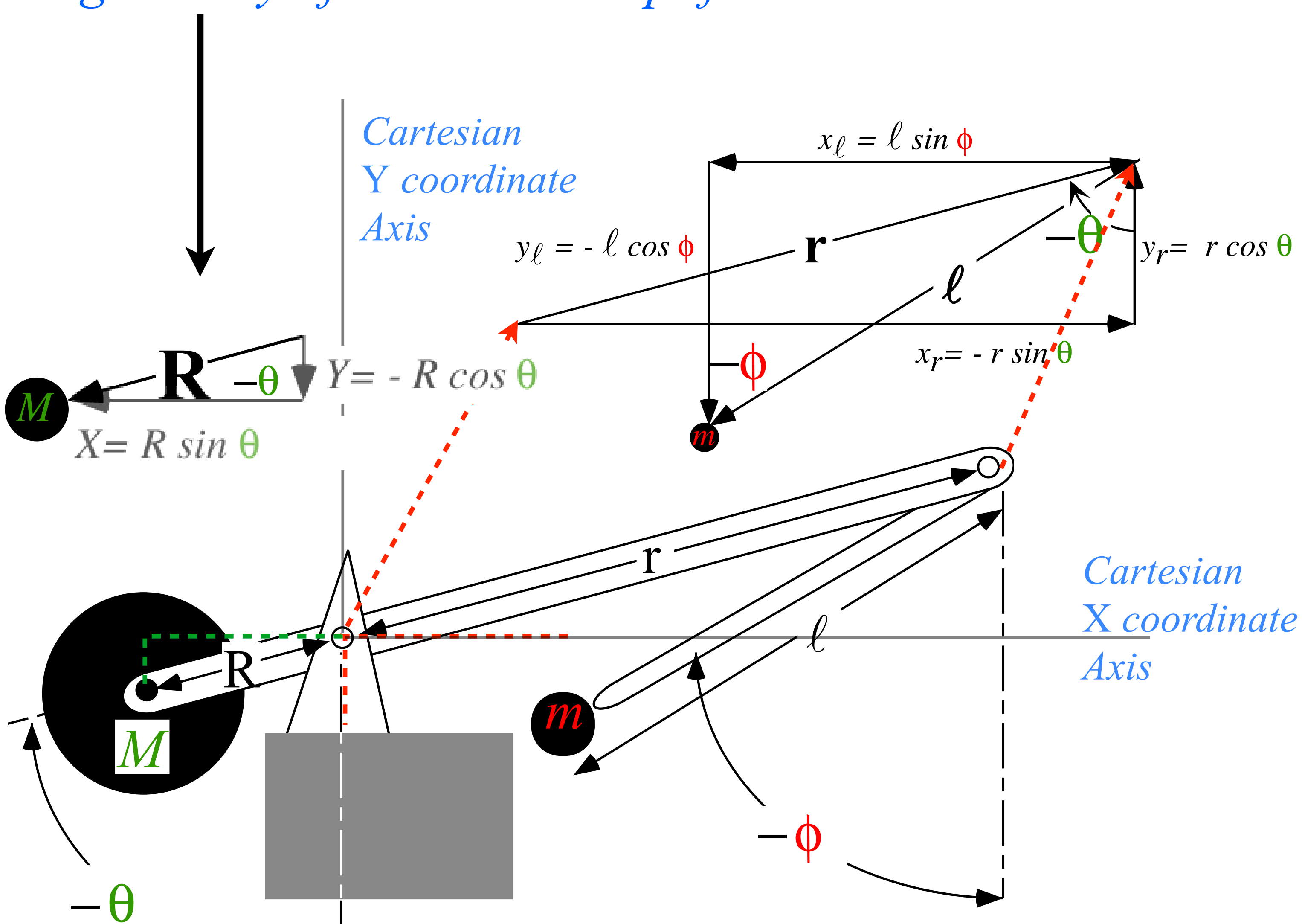
$y = y_r + y_\ell = r \cos\theta - \ell \cos\phi$

$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

from p. 20 of Lect. 14



geometry of trebuchet simplified somewhat...



Based on Fig. 2.2.1

Jacobian transformation matrix

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle =$$

$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

from p. 90 of Lect. 14

$$\begin{array}{|ccc|} \mathbf{E}_1 & \mathbf{E}_2 & \cdots \\ \hline \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \cdots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \cdots \\ \vdots & \vdots & \ddots \end{array} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix}$$

Jacobian transformation matrix

$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

from p. 90 of Lect. 14

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle = \left| \begin{array}{ccc} \mathbf{E}_1 & \mathbf{E}_2 & \cdots \\ \hline \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \cdots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \cdots \\ \vdots & \vdots & \ddots \end{array} \right| = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} = \left| \begin{array}{cc} \mathbf{E}_\theta & \mathbf{E}_\phi \\ \hline -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{array} \right|$$

Jacobian transformation matrix

$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

from p. 90 of Lect. 14

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle = \begin{array}{|ccc|} \mathbf{E}_1 & \mathbf{E}_2 & \cdots \\ \hline \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \cdots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \cdots \\ \vdots & \vdots & \ddots \end{array} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} = \begin{array}{|cc|} \mathbf{E}_\theta & \mathbf{E}_\phi \\ \hline -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{array}$$

Covariant vectors $\mathbf{E}_n$

$$\mathbf{E}_\theta = \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \end{pmatrix}, \quad \mathbf{E}_\phi = \begin{pmatrix} \ell\cos\phi \\ \ell\sin\phi \end{pmatrix}$$

Jacobian transformation matrix

$$x = -r\sin\theta + \ell\sin\phi$$
$$y = r\cos\theta - \ell\cos\phi$$

from p. 90 of Lect. 14

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle = \begin{array}{|ccc|} \mathbf{E}_1 & \mathbf{E}_2 & \cdots \\ \hline \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \cdots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \cdots \\ \vdots & \vdots & \ddots \end{array} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} = \begin{array}{|cc|} \mathbf{E}_\theta & \mathbf{E}_\phi \\ \hline -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{array}$$

Covariant vectors $\mathbf{E}_n$

$$\mathbf{E}_\theta = \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \end{pmatrix}, \quad \mathbf{E}_\phi = \begin{pmatrix} \ell\cos\phi \\ \ell\sin\phi \end{pmatrix}$$

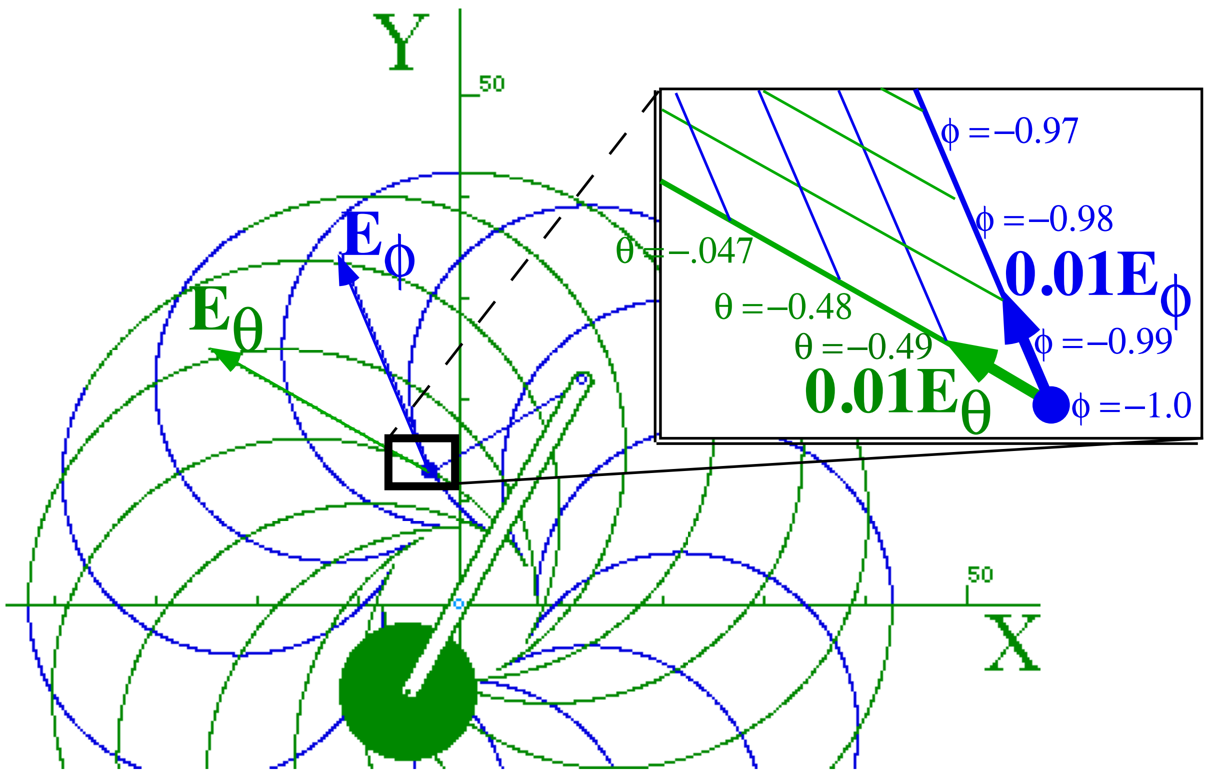


Fig. 3.2.2 Example of covariant unitary vectors and their tangent space.

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.

Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)

Trebuchet Cartesian projectile coordinates are double-valued

Toroidal "rolled-up" ($q^1=\theta$, $q^2=\phi$)-manifold and "Flat" ($x=\theta$, $y=\phi$)-graph

Review of covariant $\mathbf{E}_n$ and contravariant $\mathbf{E}^m$ vectors: Jacobian J vs. Kajobian K

Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)

Tangent $\{\mathbf{E}_n\}$space vs. Normal $\{\mathbf{E}^m\}$space

Covariant vs. contravariant coordinate transformations

Metric g_{mn} tensor geometric relations to length, area, and volume

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)

Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 77)

Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 69)

Riemann equation derivation for trebuchet model

Riemann equation force analysis

2nd-guessing Riemann equation?

Kajobian transformation matrix *versus* *Jacobian transformation matrix*

$$\left\langle \frac{\partial q^m}{\partial x^j} \right\rangle =$$

Using 2x2 inverse

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \frac{\begin{vmatrix} D & -B \\ -C & A \end{vmatrix}}{AD-BC}$$

$$\begin{array}{c|c} \begin{vmatrix} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \cdots \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \cdots \\ \vdots & \vdots & \ddots \end{vmatrix} & \begin{matrix} \mathbf{E}^1 \\ \mathbf{E}^2 \\ \vdots \end{matrix} \end{array} = \begin{pmatrix} \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{pmatrix} = \frac{\begin{array}{c|c} \begin{vmatrix} \ell \sin\phi & -\ell\cos\phi \\ r\sin\theta & -r\cos\theta \end{vmatrix} & \begin{matrix} \mathbf{E}^\theta \\ \mathbf{E}^\phi \end{matrix} \end{array}}{\ell r \sin\theta \cos\phi - \ell r \sin\phi \cos\theta}$$

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle =$$

$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

from p. 90 of Lect. 14

$$\begin{array}{c} \begin{matrix} \mathbf{E}_1 & \mathbf{E}_2 & \cdots \end{matrix} \\ \hline \begin{vmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \cdots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \cdots \\ \vdots & \vdots & \ddots \end{vmatrix} \end{array} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} = \begin{array}{c} \begin{matrix} \mathbf{E}_\theta & \mathbf{E}_\phi \end{matrix} \\ \hline \begin{vmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{vmatrix} \end{array}$$

Covariant vectors $\mathbf{E}_n$

$$\mathbf{E}_\theta = \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \end{pmatrix}, \quad \mathbf{E}_\phi = \begin{pmatrix} \ell\cos\phi \\ \ell\sin\phi \end{pmatrix}$$

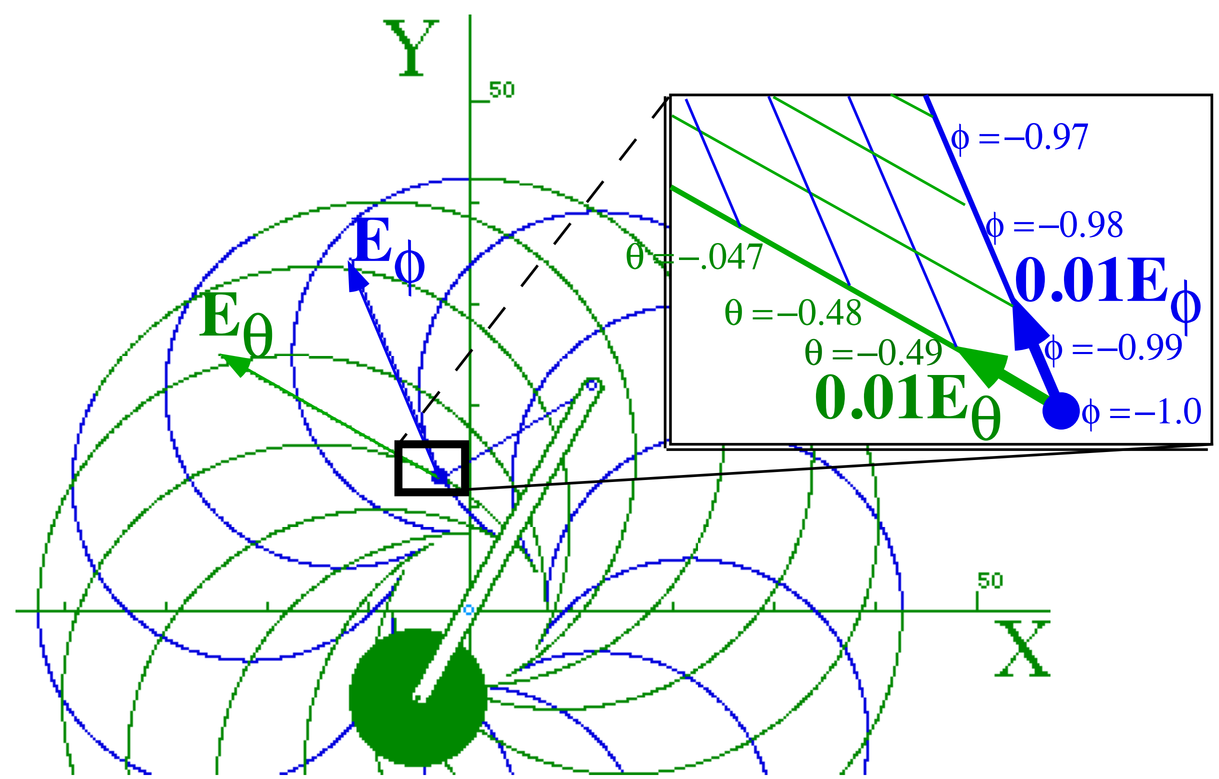


Fig. 3.2.2 Example of covariant unitary vectors and their tangent space.

Kajobian transformation matrix *versus* *Jacobian transformation matrix*

Using 2x2 inverse

$$\left(\begin{array}{cc} A & B \\ C & D \end{array}\right)^{-1} = \frac{\left|\begin{array}{cc} D & -B \\ -C & A \end{array}\right|}{AD-BC}$$

$$\left\langle \frac{\partial q^m}{\partial x^j} \right\rangle =$$

$$\left|\begin{array}{ccc|c} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \cdots & \mathbf{E}^1 \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \cdots & \mathbf{E}^2 \\ \vdots & \vdots & \ddots & \vdots \end{array}\right| = \left(\begin{array}{cc} \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{array}\right) = \frac{\left|\begin{array}{cc|c} \ell \sin\phi & -\ell\cos\phi & \mathbf{E}^\theta \\ r\sin\theta & -r\cos\theta & \mathbf{E}^\phi \end{array}\right|}{r\ell\sin(\theta-\phi)}$$

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle =$$

$$\left(\begin{array}{l} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{array}\right)$$

from p. 90 of Lect. 14

$$\left|\begin{array}{ccc} \mathbf{E}_1 & \mathbf{E}_2 & \cdots \\ \hline \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \cdots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \cdots \\ \vdots & \vdots & \ddots \end{array}\right| = \left(\begin{array}{cc} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{array}\right) = \left|\begin{array}{cc} \mathbf{E}_\theta & \mathbf{E}_\phi \\ \hline -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{array}\right|$$

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

$$\mathbf{E}^\theta = \left(\ell\sin\phi \quad -\ell\cos\phi \right) / \, r\ell\sin(\theta-\phi)$$

$$\mathbf{E}^\phi = \left(r\sin\theta \quad -r\cos\theta \right) / \, r\ell\sin(\theta-\phi)$$

$$\mathbf{E}_\theta = \left(\begin{array}{c} -r\cos\theta \\ -r\sin\theta \end{array}\right), \quad \mathbf{E}_\phi = \left(\begin{array}{c} \ell\cos\phi \\ \ell\sin\phi \end{array}\right)$$

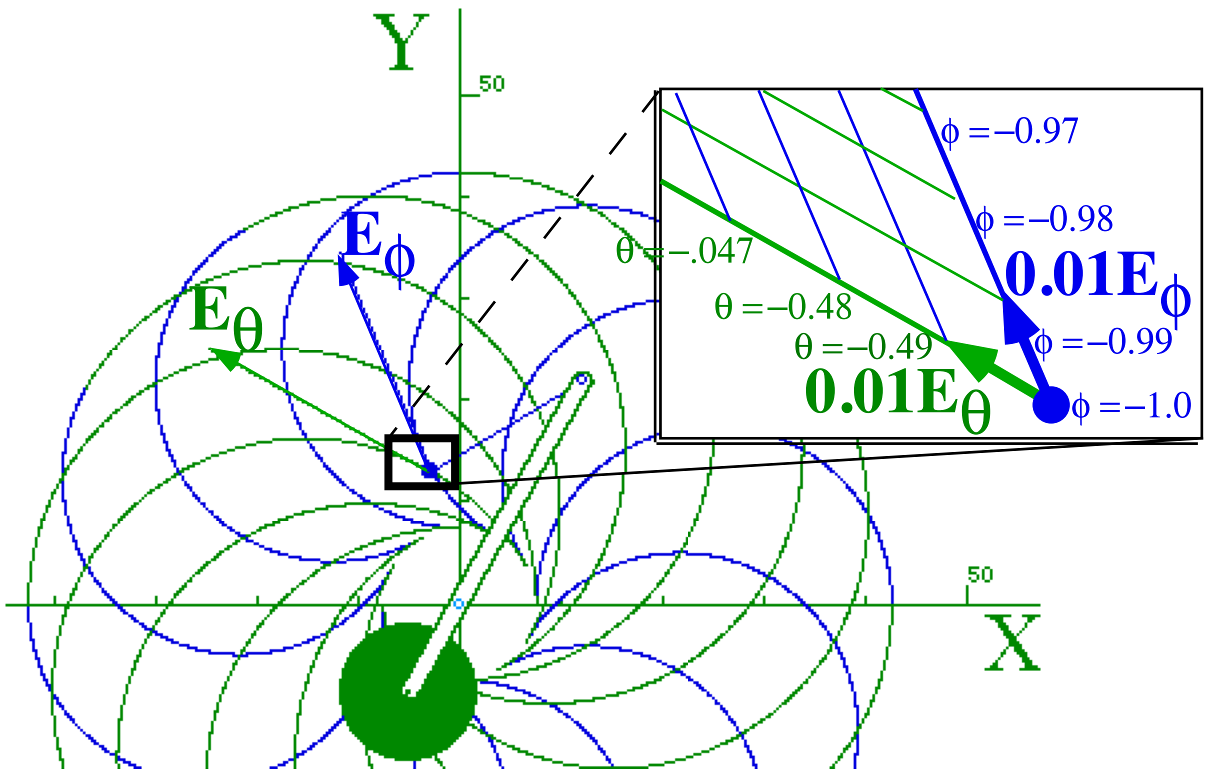


Fig. 3.2.2 Example of covariant unitary vectors and their tangent space.

Kajobian transformation matrix *versus* *Jacobian transformation matrix*

$$\left\langle \frac{\partial q^m}{\partial x^j} \right\rangle =$$

Using 2x2 inverse $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \frac{\begin{vmatrix} D & -B \\ -C & A \end{vmatrix}}{AD-BC}$

$$\begin{array}{|ccc|c} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \cdots & \mathbf{E}^1 \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \cdots & \mathbf{E}^2 \\ \vdots & \vdots & \ddots & \vdots \end{array} = \begin{pmatrix} \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} \end{pmatrix} = \frac{\begin{array}{|cc|c} \ell\sin\phi & -\ell\cos\phi & \mathbf{E}^\theta \\ r\sin\theta & -r\cos\theta & \mathbf{E}^\phi \end{array}}{r\ell\sin(\theta-\phi)}$$

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

$$\mathbf{E}^\theta = \begin{pmatrix} \ell\sin\phi & -\ell\cos\phi \end{pmatrix} / r\ell\sin(\theta-\phi)$$

$$\mathbf{E}^\phi = \begin{pmatrix} r\sin\theta & -r\cos\theta \end{pmatrix} / r\ell\sin(\theta-\phi)$$

$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

from p. 90 of Lect. 14

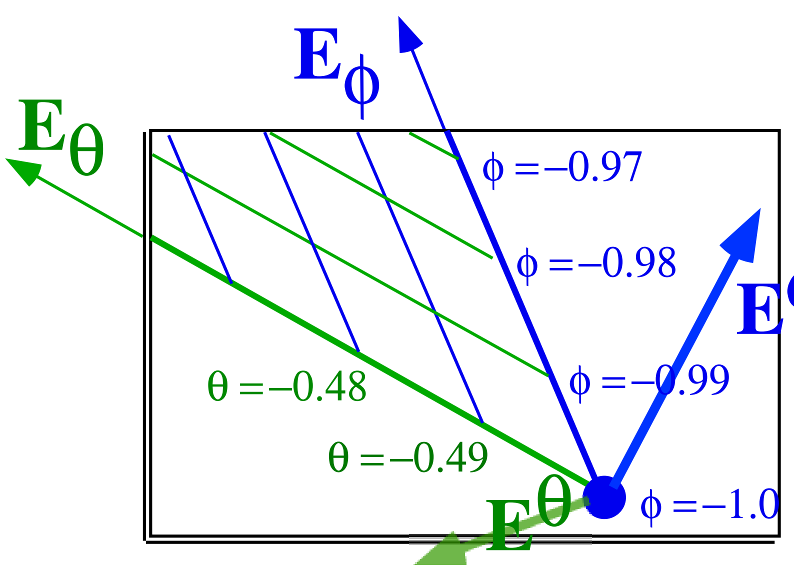
$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle =$$

$$\begin{array}{|ccc|} \mathbf{E}_1 & \mathbf{E}_2 & \cdots \\ \hline \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \cdots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \cdots \\ \vdots & \vdots & \ddots \end{array} = \begin{pmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} = \begin{array}{|cc|} \mathbf{E}_\theta & \mathbf{E}_\phi \\ \hline -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{array}$$

$$\mathbf{E}_\theta = \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \end{pmatrix}, \quad \mathbf{E}_\phi = \begin{pmatrix} \ell\cos\phi \\ \ell\sin\phi \end{pmatrix}$$

Covariant tangent-space GCC vectors

$\mathbf{E}_1 = \mathbf{E}_\theta$ and $\mathbf{E}_2 = \mathbf{E}_\phi$



Contravariant normal-space GCC vectors

$\mathbf{E}^1 = \mathbf{E}^\theta$ and $\mathbf{E}^2 = \mathbf{E}^\phi$

Fig. 3.2.3 Example of contravariant unitary vectors and their normal space.

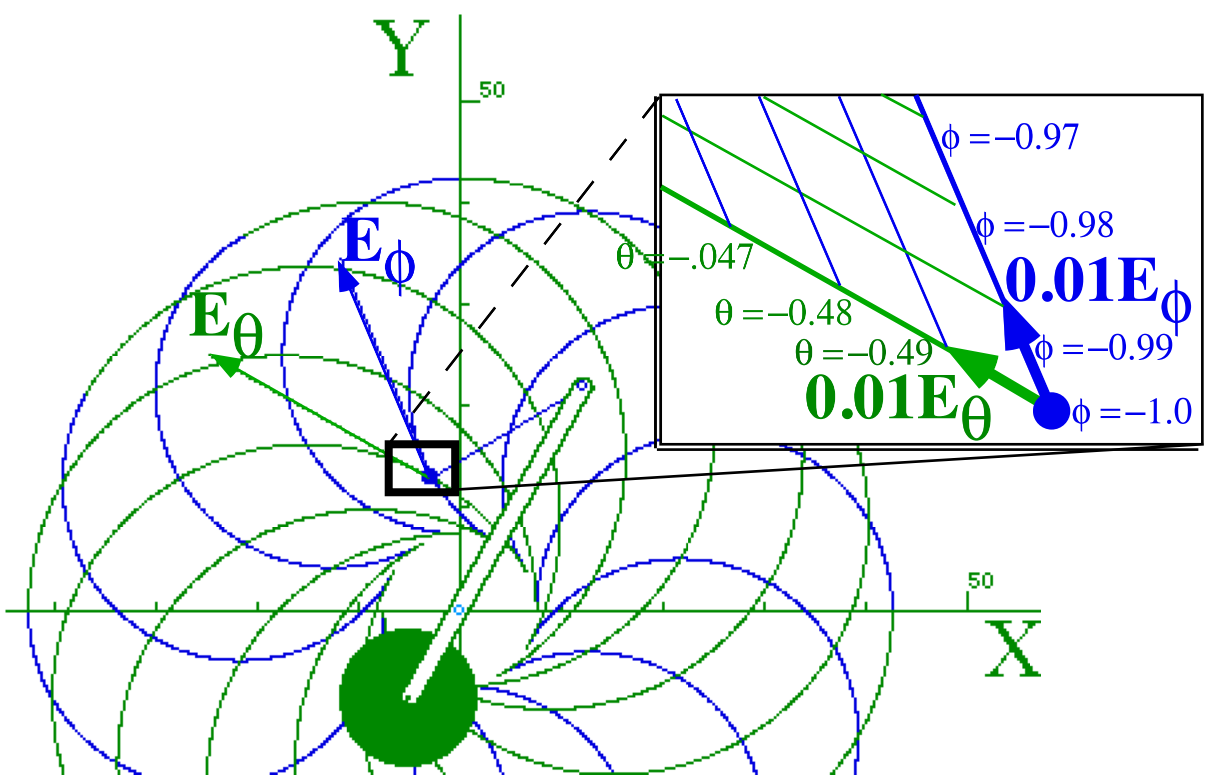


Fig. 3.2.2 Example of covariant unitary vectors and their tangent space.

Kajobian transformation matrix *versus* *Jacobian transformation matrix*

$$\left\langle \frac{\partial q^m}{\partial x^j} \right\rangle =$$

Using 2x2 inverse $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \dfrac{\begin{vmatrix} D & -B \\ -C & A \end{vmatrix}}{AD-BC}$

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle =$$

$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

from p. 90 of Lect. 14

$$\left| \begin{array}{ccc|c} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \cdots & \mathbf{E}^1 \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \cdots & \mathbf{E}^2 \\ \vdots & \vdots & \ddots & \vdots \end{array} \right| = \begin{pmatrix} \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} \end{pmatrix} = \frac{\left| \begin{array}{cc|c} \ell\sin\phi & -\ell\cos\phi & \mathbf{E}^\theta \\ r\sin\theta & -r\cos\theta & \mathbf{E}^\phi \end{array} \right|}{r\ell\sin(\theta-\phi)}$$

$$\left| \begin{array}{ccc} \mathbf{E}_1 & \mathbf{E}_2 & \cdots \\ \hline \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \cdots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \cdots \\ \vdots & \vdots & \ddots \end{array} \right| = \begin{pmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} = \left| \begin{array}{cc} \mathbf{E}_\theta & \mathbf{E}_\phi \\ \hline -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{array} \right|$$

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

$$\mathbf{E}^\theta = \begin{pmatrix} \ell\sin\phi & -\ell\cos\phi \end{pmatrix} / r\ell\sin(\theta-\phi)$$

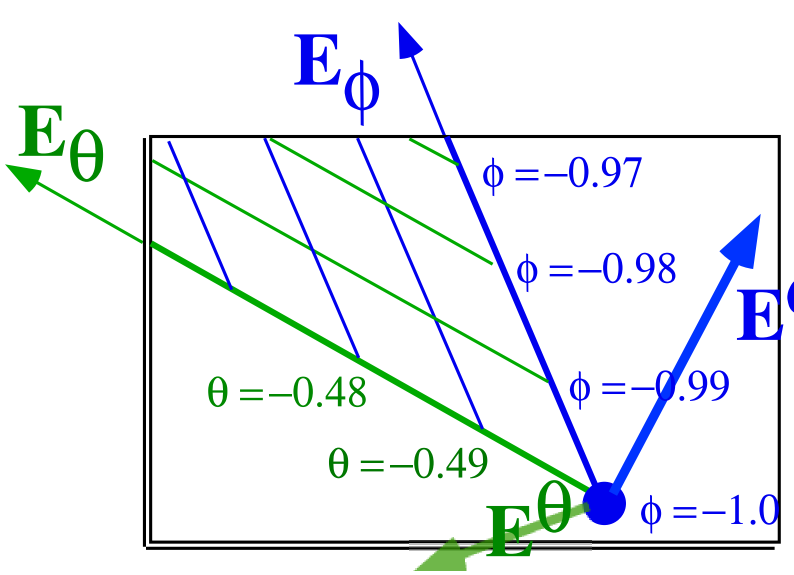
$$\mathbf{E}^\phi = \begin{pmatrix} r\sin\theta & -r\cos\theta \end{pmatrix} / r\ell\sin(\theta-\phi)$$

$$\mathbf{E}^\theta\bullet\mathbf{E}_\phi = 0 = \mathbf{E}_\theta\bullet\mathbf{E}^\phi$$

$$\mathbf{E}^\theta\bullet\mathbf{E}_\theta = 1 = \mathbf{E}_\phi\bullet\mathbf{E}^\phi$$

$$\mathbf{E}_\theta = \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \end{pmatrix}, \quad \mathbf{E}_\phi = \begin{pmatrix} \ell\cos\phi \\ \ell\sin\phi \end{pmatrix}$$

Covariant tangent-space GCC vectors
$\mathbf{E}_1 = \mathbf{E}_\theta$ and $\mathbf{E}_2 = \mathbf{E}_\phi$



Contravariant normal-space GCC vectors
$\mathbf{E}^1 = \mathbf{E}^\theta$ and $\mathbf{E}^2 = \mathbf{E}^\phi$

Fig. 3.2.3 Example of contravariant unitary vectors and their normal space.

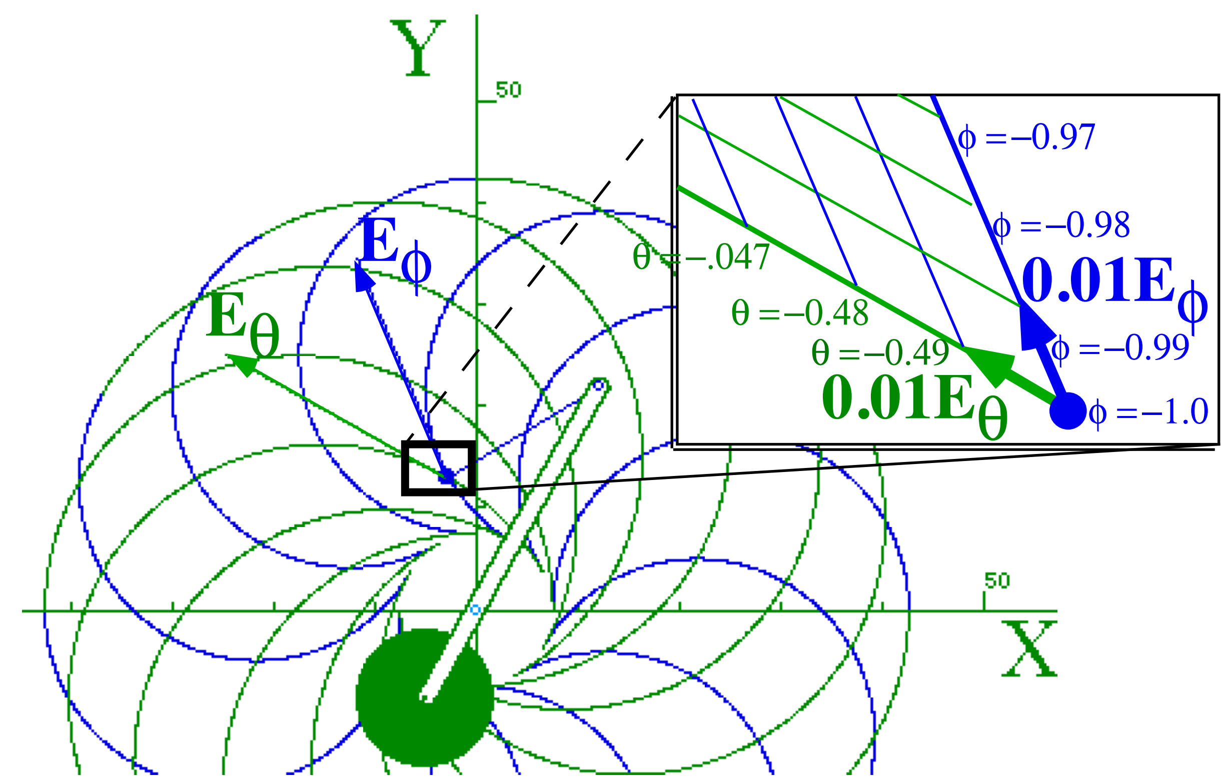


Fig. 3.2.2 Example of covariant unitary vectors and their tangent space.

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.

Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)

Trebuchet Cartesian projectile coordinates are double-valued

Toroidal "rolled-up" ($q^1=\theta$, $q^2=\phi$)-manifold and "Flat" ($x=\theta$, $y=\phi$)-graph

Review of covariant $\mathbf{E}_n$ and contravariant $\mathbf{E}^m$ vectors. Jacobian J vs. Kajobian K

Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)

Tangent $\{\mathbf{E}_n\}$space vs. Normal $\{\mathbf{E}^m\}$space

Covariant vs. contravariant coordinate transformations

Metric g_{mn} tensor geometric relations to length, area, and volume

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)

Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 77)

Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 69)

Riemann equation derivation for trebuchet model

Riemann equation force analysis

2nd-guessing Riemann equation?

Covariant g_{mn} vs. Invariant δ_m^n vs. Contravariant g^{mn}

$$\mathbf{E}_m \bullet \mathbf{E}_n = \frac{\partial \mathbf{r}}{\partial q^m} \bullet \frac{\partial \mathbf{r}}{\partial q^n} \equiv g_{mn}$$

$$\mathbf{E}_m \bullet \mathbf{E}^n = \frac{\partial \mathbf{r}}{\partial q^m} \bullet \frac{\partial q^n}{\partial \mathbf{r}} = \delta_m^n$$

$$\mathbf{E}^m \bullet \mathbf{E}^n = \frac{\partial q^m}{\partial \mathbf{r}} \bullet \frac{\partial q^n}{\partial \mathbf{r}} \equiv g^{mn}$$

Covariant metric tensor g_{mn}

Invariant Kroneker unit tensor

$$\delta_m^n \equiv \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

Contravariant metric tensor g^{mn}

from p. 55 of Lect. 9

Polar coordinate examples (again):

$$\langle J \rangle = \begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} = \cos\phi & \frac{\partial x}{\partial \phi} = -r\sin\phi \\ \frac{\partial y}{\partial r} = \sin\phi & \frac{\partial y}{\partial \phi} = r\cos\phi \end{pmatrix}$$

$$\uparrow \mathbf{E}_1 \quad \uparrow \mathbf{E}_2 \qquad \uparrow \mathbf{E}_r \qquad \uparrow \mathbf{E}_\phi$$

$$\langle K \rangle = \langle J^{-1} \rangle = \begin{pmatrix} \frac{\partial r}{\partial x} = \cos\phi & \frac{\partial r}{\partial y} = \sin\phi \\ \frac{\partial \phi}{\partial x} = \frac{-\sin\phi}{r} & \frac{\partial \phi}{\partial y} = \frac{\cos\phi}{r} \end{pmatrix} \begin{matrix} \leftarrow \mathbf{E}^r = \mathbf{E}^1 \\ \leftarrow \mathbf{E}^\phi = \mathbf{E}^2 \end{matrix}$$

Covariant g_{mn}

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r \bullet \mathbf{E}_r & \mathbf{E}_r \bullet \mathbf{E}_\phi \\ \mathbf{E}_\phi \bullet \mathbf{E}_r & \mathbf{E}_\phi \bullet \mathbf{E}_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

Invariant δ_m^n

$$\begin{pmatrix} \delta_r^r & \delta_r^\phi \\ \delta_\phi^r & \delta_\phi^\phi \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r \bullet \mathbf{E}^r & \mathbf{E}_r \bullet \mathbf{E}^\phi \\ \mathbf{E}_\phi \bullet \mathbf{E}^r & \mathbf{E}_\phi \bullet \mathbf{E}^\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Contravariant g^{mn}

$$\begin{pmatrix} g^{rr} & g^{r\phi} \\ g^{\phi r} & g^{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}^r \bullet \mathbf{E}^r & \mathbf{E}^r \bullet \mathbf{E}^\phi \\ \mathbf{E}^\phi \bullet \mathbf{E}^r & \mathbf{E}^\phi \bullet \mathbf{E}^\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}$$

Kajobian transformation matrix *versus* *Jacobian transformation matrix*

$$\left\langle \frac{\partial q^m}{\partial x^j} \right\rangle =$$

Using 2x2 inverse

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \frac{\begin{vmatrix} D & -B \\ -C & A \end{vmatrix}}{AD-BC}$$

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle =$$

$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

from p. 89 of Lect. 14

$$\left| \begin{array}{ccc|c} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \cdots & \mathbf{E}^1 \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \cdots & \mathbf{E}^2 \\ \vdots & \vdots & \ddots & \vdots \end{array} \right| = \begin{pmatrix} \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} \end{pmatrix} = \frac{\left| \begin{array}{cc|c} \ell\sin\phi & -\ell\cos\phi & \mathbf{E}^\theta \\ r\sin\theta & -r\cos\theta & \mathbf{E}^\phi \end{array} \right|}{r\ell\sin(\theta-\phi)}$$

$$\left| \begin{array}{ccc} \mathbf{E}_1 & \mathbf{E}_2 & \cdots \\ \hline \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \cdots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \cdots \\ \vdots & \vdots & \ddots \end{array} \right| = \begin{pmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} = \left| \begin{array}{cc} \mathbf{E}_\theta & \mathbf{E}_\phi \\ \hline -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{array} \right|$$

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

$$\mathbf{E}^\theta = \begin{pmatrix} \ell\sin\phi & -\ell\cos\phi \end{pmatrix} / \, r\ell\sin(\theta-\phi)$$

$$\mathbf{E}^\phi = \begin{pmatrix} r\sin\theta & -r\cos\theta \end{pmatrix} / \, r\ell\sin(\theta-\phi)$$

$$\mathbf{E}^\theta \bullet \mathbf{E}_\phi = 0 = \mathbf{E}_\theta \bullet \mathbf{E}^\phi$$

$$\mathbf{E}^\theta \bullet \mathbf{E}_\theta = 1 = \mathbf{E}_\phi \bullet \mathbf{E}^\phi$$

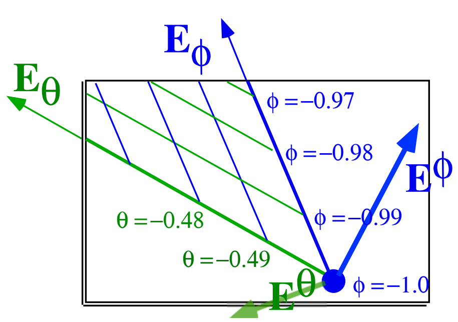
$$\mathbf{E}_\theta = \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \end{pmatrix}, \quad \mathbf{E}_\phi = \begin{pmatrix} \ell\cos\phi \\ \ell\sin\phi \end{pmatrix}$$

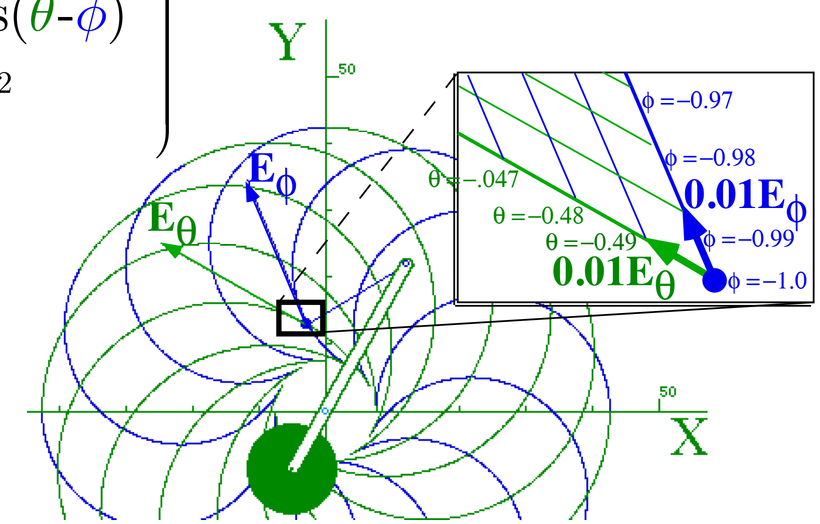
Contravariant metric $g^{mn} = \mathbf{E}^m \bullet \mathbf{E}^n = g^{nm}$ *versus* *Covariant metric* $g_{mn} = \mathbf{E}_m \bullet \mathbf{E}_n = g_{nm}$

$$\begin{pmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_\theta \bullet \mathbf{E}_\theta & \mathbf{E}_\theta \bullet \mathbf{E}_\phi \\ \mathbf{E}_\phi \bullet \mathbf{E}_\theta & \mathbf{E}_\phi \bullet \mathbf{E}_\phi \end{pmatrix}$$

$$= \begin{pmatrix} r^2 & -r\ell(\cos\theta\cos\phi + \sin\theta\sin\phi) \\ g_{\phi\theta} & \ell^2 \end{pmatrix}$$

$$= \begin{pmatrix} r^2 & -r\ell\cos(\theta-\phi) \\ g_{\phi\theta} & \ell^2 \end{pmatrix}$$





Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.

Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)

Trebuchet Cartesian projectile coordinates are double-valued

Toroidal "rolled-up" ($q^1=\theta$, $q^2=\phi$)-manifold and "Flat" ($x=\theta$, $y=\phi$)-graph

Review of covariant $\mathbf{E}_n$ and contravariant $\mathbf{E}^m$ vectors: Jacobian J vs. Kajobian K

Covariant metric g_{mn} vs contravariant metric g^{mn} (Lect. 10 p.43-49)

Tangent $\{\mathbf{E}_n\}$space vs. Normal $\{\mathbf{E}^m\}$space

Covariant vs. contravariant coordinate transformations

Metric g_{mn} tensor geometric relations to length, area, and volume

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)

Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 15 p. 77)

Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 15 p. 69)

Riemann equation derivation for trebuchet model

Riemann equation force analysis

2nd-guessing Riemann equation?

Kajobian transformation matrix *versus* *Jacobian transformation matrix*

Using 2x2 inverse

$$\left\langle \frac{\partial q^m}{\partial x^j} \right\rangle = \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \frac{\begin{vmatrix} D & -B \\ -C & A \end{vmatrix}}{AD\text{-}BC}$$

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle = \begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

$$\begin{array}{|ccc|c} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \cdots & \mathbf{E}^1 \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \cdots & \mathbf{E}^2 \\ \vdots & \vdots & \ddots & \vdots \end{array} = \begin{pmatrix} \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} \end{pmatrix} = \frac{\begin{array}{|cc|c} \ell\sin\phi & -\ell\cos\phi & \mathbf{E}^\theta \\ r\sin\theta & -r\cos\theta & \mathbf{E}^\phi \end{array}}{r\ell\sin(\theta-\phi)}$$

$$\begin{array}{|ccc|} \mathbf{E}_1 & \mathbf{E}_2 & \cdots \\ \hline \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \cdots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \cdots \\ \vdots & \vdots & \ddots \end{array} = \begin{pmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} = \begin{array}{|cc|} \mathbf{E}_\theta & \mathbf{E}_\phi \\ \hline -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{array}$$

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

$$\mathbf{E}^\theta = \begin{pmatrix} \ell\sin\phi & -\ell\cos\phi \end{pmatrix} / \, r\ell\sin(\theta-\phi)$$

$$\mathbf{E}^\phi = \begin{pmatrix} r\sin\theta & -r\cos\theta \end{pmatrix} / \, r\ell\sin(\theta-\phi)$$

$$\mathbf{E}^\theta \bullet \mathbf{E}_\phi = 0 = \mathbf{E}_\theta \bullet \mathbf{E}^\phi$$

$$\mathbf{E}^\theta \bullet \mathbf{E}_\theta = 1 = \mathbf{E}_\phi \bullet \mathbf{E}^\phi$$

$$\mathbf{E}_\theta = \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \end{pmatrix}, \quad \mathbf{E}_\phi = \begin{pmatrix} \ell\cos\phi \\ \ell\sin\phi \end{pmatrix}$$

Contravariant metric $g^{mn}=\mathbf{E}^m\bullet\mathbf{E}^n=g^{nm}$ *versus* *Covariant metric* $g_{mn}=\mathbf{E}_m\bullet\mathbf{E}_n=g_{nm}$

$$\begin{pmatrix} g^{\theta\theta} & g^{\theta\phi} \\ g^{\phi\theta} & g^{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}^\theta\bullet\mathbf{E}^\theta & \mathbf{E}^\theta\bullet\mathbf{E}^\phi \\ \mathbf{E}^\phi\bullet\mathbf{E}^\theta & \mathbf{E}^\phi\bullet\mathbf{E}^\phi \end{pmatrix}$$

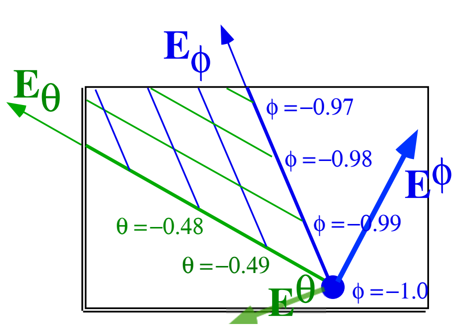
$$= \begin{pmatrix} \ell^2 & r\ell(\sin\phi\sin\theta + \cos\phi\cos\theta) \\ g^{\phi\theta} & r^2 \end{pmatrix} / r^2\ell^2\sin^2(\theta\text{-}\phi)$$

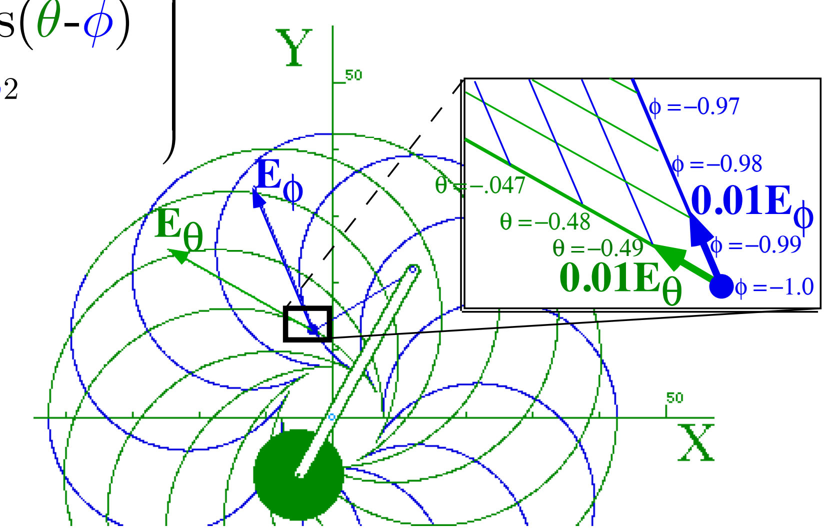
$$= \begin{pmatrix} \ell^2 & r\ell\cos(\theta\text{-}\phi) \\ g^{\phi\theta} & r^2 \end{pmatrix} / r^2\ell^2\sin^2(\theta\text{-}\phi)$$

$$\begin{pmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_\theta\bullet\mathbf{E}_\theta & \mathbf{E}_\theta\bullet\mathbf{E}_\phi \\ \mathbf{E}_\phi\bullet\mathbf{E}_\theta & \mathbf{E}_\phi\bullet\mathbf{E}_\phi \end{pmatrix}$$

$$= \begin{pmatrix} r^2 & -r\ell(\cos\theta\cos\phi + \sin\theta\sin\phi) \\ g_{\phi\theta} & \ell^2 \end{pmatrix}$$

$$= \begin{pmatrix} r^2 & -r\ell\cos(\theta\text{-}\phi) \\ g_{\phi\theta} & \ell^2 \end{pmatrix}$$





Kajobian transformation matrix versus *Jacobian transformation matrix*

$$\left\langle \frac{\partial q^m}{\partial x^j}\right\rangle = \qquad \text{Using 2x2 inverse} \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \frac{\begin{vmatrix} D & -B \\ -C & A \end{vmatrix}}{AD-BC}$$

$$\left\langle \frac{\partial x^j}{\partial q^m}\right\rangle = \qquad \begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

$$\begin{vmatrix} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \cdots \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \cdots \\ \vdots & \vdots & \ddots \end{vmatrix} \begin{matrix} \mathbf{E}^1 \\ \mathbf{E}^2 \\ \vdots \end{matrix} = \begin{pmatrix} \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} \end{pmatrix} = \frac{\begin{vmatrix} \ell\sin\phi & -\ell\cos\phi \\ r\sin\theta & -r\cos\theta \end{vmatrix} \begin{matrix} \mathbf{E}^\theta \\ \mathbf{E}^\phi \end{matrix}}{r\ell\sin(\theta-\phi)}$$

$$\begin{array}{|ccc|} \mathbf{E}_1 & \mathbf{E}_2 & \cdots \\ \hline \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \cdots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \cdots \\ \vdots & \vdots & \ddots \end{array} = \begin{pmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} = \begin{array}{|cc|} \mathbf{E}_\theta & \mathbf{E}_\phi \\ \hline -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{array}$$

Contravariant vectors $\mathbf{E}^m$ versus *Covariant vectors* $\mathbf{E}_n$

$$\mathbf{E}^\theta = \begin{pmatrix} \ell\sin\phi & -\ell\cos\phi \end{pmatrix} / \, r\ell\sin(\theta-\phi)$$

$$\mathbf{E}^\phi = \begin{pmatrix} r\sin\theta & -r\cos\theta \end{pmatrix} / \, r\ell\sin(\theta-\phi)$$

$$\mathbf{E}^\theta \bullet \mathbf{E}_\phi = 0 = \mathbf{E}_\theta \bullet \mathbf{E}^\phi$$

$$\mathbf{E}^\theta \bullet \mathbf{E}_\theta = 1 = \mathbf{E}_\phi \bullet \mathbf{E}^\phi$$

$$\mathbf{E}_\theta = \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \end{pmatrix}, \quad \mathbf{E}_\phi = \begin{pmatrix} \ell\cos\phi \\ \ell\sin\phi \end{pmatrix}$$

Contravariant metric $g^{mn} = \mathbf{E}^m \bullet \mathbf{E}^n = g^{nm}$ versus *Covariant metric* $g_{mn} = \mathbf{E}_m \bullet \mathbf{E}_n = g_{nm}$

$$\begin{pmatrix} g^{\theta\theta} & g^{\theta\phi} \\ g^{\phi\theta} & g^{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}^\theta \bullet \mathbf{E}^\theta & \mathbf{E}^\theta \bullet \mathbf{E}^\phi \\ \mathbf{E}^\phi \bullet \mathbf{E}^\theta & \mathbf{E}^\phi \bullet \mathbf{E}^\phi \end{pmatrix}$$

$$= \begin{pmatrix} \ell^2 & r\ell(\sin\phi\sin\theta + \cos\phi\cos\theta) \\ g^{\phi\theta} & r^2 \end{pmatrix} / r^2\ell^2\sin^2(\theta\text{-}\phi)$$

$$= \begin{pmatrix} \ell^2 & r\ell\cos(\theta\text{-}\phi) \\ g^{\phi\theta} & r^2 \end{pmatrix} / r^2\ell^2\sin^2(\theta\text{-}\phi)$$

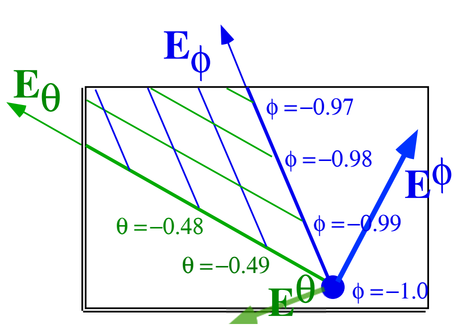
$$\begin{pmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_\theta \bullet \mathbf{E}_\theta & \mathbf{E}_\theta \bullet \mathbf{E}_\phi \\ \mathbf{E}_\phi \bullet \mathbf{E}_\theta & \mathbf{E}_\phi \bullet \mathbf{E}_\phi \end{pmatrix}$$

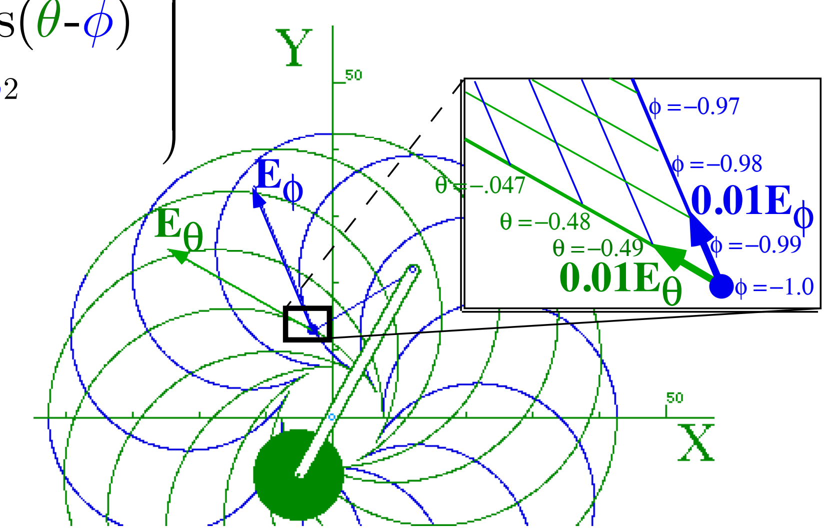
$$= \begin{pmatrix} r^2 & -r\ell(\cos\theta\cos\phi + \sin\theta\sin\phi) \\ g_{\phi\theta} & \ell^2 \end{pmatrix}$$

$$= \begin{pmatrix} r^2 & -r\ell\cos(\theta\text{-}\phi) \\ g_{\phi\theta} & \ell^2 \end{pmatrix}$$

Jacobian J^TJ*-product gives* g_{mn}

$$J^TJ = \begin{array}{c|cc} \mathbf{E}_\theta & -r\cos\theta & -r\sin\theta \\ \mathbf{E}_\phi & \ell\cos\phi & \ell\sin\phi \end{array} \; \begin{array}{|cc|} \mathbf{E}_\theta & \mathbf{E}_\phi \\ \hline -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{array} = \begin{pmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{pmatrix}$$





Kajobian transformation matrix versus Jacobian transformation matrix

$$\left\langle \frac{\partial q^m}{\partial x^j} \right\rangle =$$

Using 2x2 inverse $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \dfrac{\begin{vmatrix} D & -B \\ -C & A \end{vmatrix}}{AD-BC}$

$$\begin{array}{|ccc|c} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \cdots & \mathbf{E}^1 \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \cdots & \mathbf{E}^2 \\ \vdots & \vdots & \ddots & \vdots \end{array} = \begin{pmatrix} \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{pmatrix} = \frac{\begin{array}{|cc|c} \ell \sin\phi & -\ell\cos\phi & \mathbf{E}^\theta \\ r\sin\theta & -r\cos\theta & \mathbf{E}^\phi \end{array}}{r\ell\sin(\theta-\phi)}$$

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle = \qquad \begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

$$\begin{array}{|ccc|} \mathbf{E}_1 & \mathbf{E}_2 & \cdots \\ \hline \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \cdots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \cdots \\ \vdots & \vdots & \ddots \end{array} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} = \begin{array}{|cc|} \mathbf{E}_\theta & \mathbf{E}_\phi \\ \hline -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{array}$$

Contravariant vectors $\mathbf{E}^m$ versus Covariant vectors $\mathbf{E}_n$

$$\mathbf{E}^\theta = \begin{pmatrix} \ell\sin\phi & -\ell\cos\phi \end{pmatrix} / \, r\ell\sin(\theta-\phi)$$

$$\mathbf{E}^\phi = \begin{pmatrix} r\sin\theta & -r\cos\theta \end{pmatrix} / \, r\ell\sin(\theta-\phi)$$

$$\mathbf{E}^\theta \bullet \mathbf{E}_\phi = 0 = \mathbf{E}_\theta \bullet \mathbf{E}^\phi$$

$$\mathbf{E}^\theta \bullet \mathbf{E}_\theta = 1 = \mathbf{E}_\phi \bullet \mathbf{E}^\phi$$

$$\mathbf{E}_\theta = \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \end{pmatrix}, \quad \mathbf{E}_\phi = \begin{pmatrix} \ell\cos\phi \\ \ell\sin\phi \end{pmatrix}$$

Contravariant metric $g^{mn} = \mathbf{E}^m \bullet \mathbf{E}^n = g^{nm}$ versus Covariant metric $g_{mn} = \mathbf{E}_m \bullet \mathbf{E}_n = g_{nm}$

$$\begin{pmatrix} g^{\theta\theta} & g^{\theta\phi} \\ g^{\phi\theta} & g^{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}^\theta \bullet \mathbf{E}^\theta & \mathbf{E}^\theta \bullet \mathbf{E}^\phi \\ \mathbf{E}^\phi \bullet \mathbf{E}^\theta & \mathbf{E}^\phi \bullet \mathbf{E}^\phi \end{pmatrix}$$

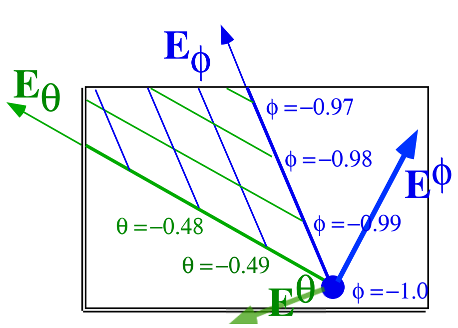
$$= \begin{pmatrix} \ell^2 & r\ell(\sin\phi\sin\theta + \cos\phi\cos\theta) \\ g^{\phi\theta} & r^2 \end{pmatrix} / r^2\ell^2\sin^2(\theta\text{-}\phi)$$

$$= \begin{pmatrix} \ell^2 & r\ell\cos(\theta\text{-}\phi) \\ g^{\phi\theta} & r^2 \end{pmatrix} / r^2\ell^2\sin^2(\theta\text{-}\phi)$$

$$\begin{pmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_\theta \bullet \mathbf{E}_\theta & \mathbf{E}_\theta \bullet \mathbf{E}_\phi \\ \mathbf{E}_\phi \bullet \mathbf{E}_\theta & \mathbf{E}_\phi \bullet \mathbf{E}_\phi \end{pmatrix}$$

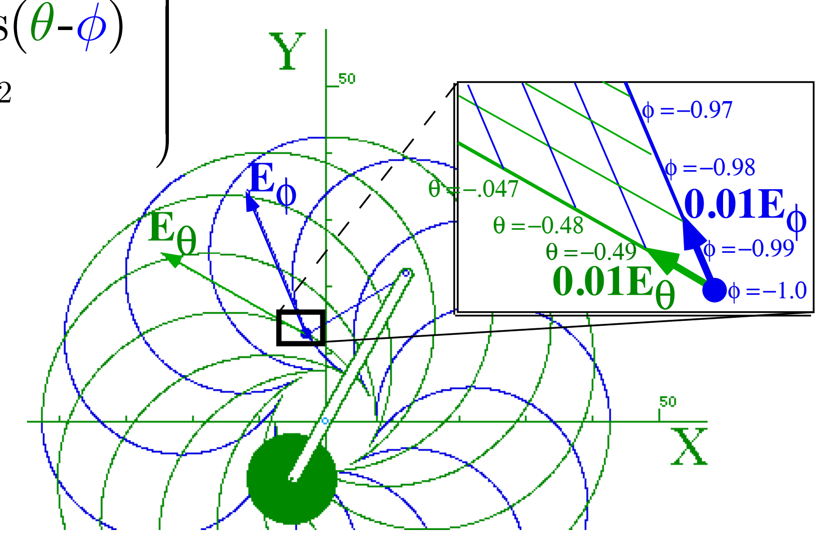
$$= \begin{pmatrix} r^2 & -r\ell(\cos\theta\cos\phi + \sin\theta\sin\phi) \\ g_{\phi\theta} & \ell^2 \end{pmatrix}$$

$$= \begin{pmatrix} r^2 & -r\ell\cos(\theta\text{-}\phi) \\ g_{\phi\theta} & \ell^2 \end{pmatrix}$$



Jacobian J^TJ-product gives g_{mn}

$$J^TJ = \begin{array}{c|cc} \mathbf{E}_\theta & -r\cos\theta & -r\sin\theta \\ \mathbf{E}_\phi & \ell\cos\phi & \ell\sin\phi \end{array} \; \begin{array}{|cc|} \mathbf{E}_\theta & \mathbf{E}_\phi \\ \hline -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{array} = \begin{pmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{pmatrix}$$



Kajobian KK^T-product would give g^{mn}

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.
Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)
Trebuchet Cartesian projectile coordinates are double-valued
Toroidal "rolled-up" ($q^1=\theta$, $q^2=\phi$)-manifold and "Flat" ($x=\theta$, $y=\phi$)-graph
Review of covariant $\mathbf{E}_n$ and contravariant $\mathbf{E}^m$ vectors: Jacobian J vs. Kajobian K
Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)
Tangent $\{\mathbf{E}_n\}$space vs. Normal $\{\mathbf{E}^m\}$space
Covariant vs. contravariant coordinate transformations
Metric g_{mn} tensor geometric relations to length, area, and volume

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)
Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 77)
Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 69)
Riemann equation derivation for trebuchet model
Riemann equation force analysis
2nd-guessing Riemann equation?

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

Any vector **U,V,...** is expressed using *either* set from any viewpoint, coordinate system, or *frame*,

$$\mathbf{U} = U^m \mathbf{E_m} = U_n \mathbf{E^n} = \bar{U}^{\bar{m}} \mathbf{\bar{E}_{\bar{m}}} = \bar{U}_{\bar{n}} \mathbf{\bar{E}^{\bar{n}}} \qquad \mathbf{V} = V^m \mathbf{E_m} = V_n \mathbf{E^n} = \bar{V}^{\bar{m}} \mathbf{\bar{E}_{\bar{m}}} = \bar{V}_{\bar{n}} \mathbf{\bar{E}^{\bar{n}}}$$

where the *Um, Vm,...are contravariant components* and the *Un , Vn ,...are covariant components*

$U^m = \mathbf{U \bullet E^m}$, $V^m = \mathbf{V \bullet E^m}$, and $\bar{U}^{\bar{m}} = \mathbf{U \bullet \bar{E}^{\bar{m}}}$, $\bar{V}^{\bar{m}} = \mathbf{V \bullet \bar{E}^{\bar{m}}}$, , $U_n = \mathbf{U \bullet E_n}$, $V_n = \mathbf{V \bullet E_n}$, and $\bar{U}_{\bar{n}} = \mathbf{U \bullet \bar{E}_{\bar{n}}}$, *etc.*

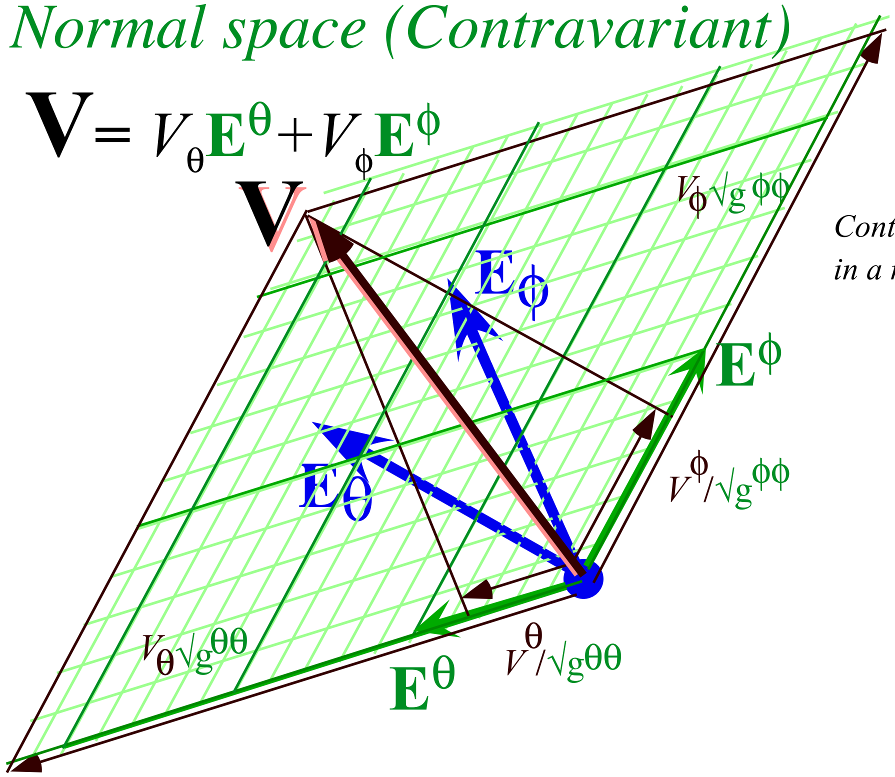


Fig. 3.3.2
Contravariant vector geometry in a normal space $(\mathbf{E}^\theta, \mathbf{E}^\phi)$.

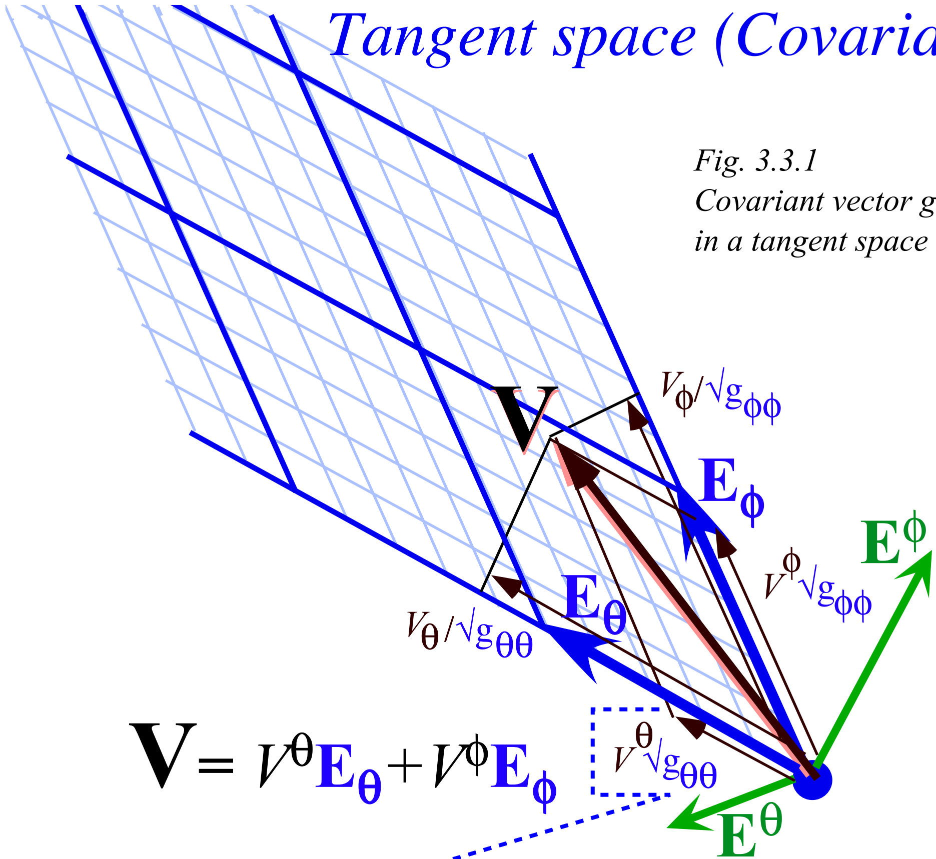


Fig. 3.3.1
Covariant vector geometry in a tangent space $(\mathbf{E}_\theta, \mathbf{E}_\phi)$.

covariant parallelogram side

$$\left| V^\theta \mathbf{E}_\theta \right| = V^\theta \sqrt{\mathbf{E}_\theta \cdot \mathbf{E}_\theta} = V^\theta \sqrt{g_{\theta\theta}}$$

Metric g_{mn} *or* g^{mn} *tensor geometric relations to length, area, and volume*

from p. 92 of Lect. 14

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

Any vector **U,V,...** is expressed using *either* set from any viewpoint, coordinate system, or *frame*,

$$\mathbf{U} = U^m\,\mathbf{E}_\mathbf{m} = U_n\,\mathbf{E}^\mathbf{n} = \bar{U}^{\bar{m}}\,\bar{\mathbf{E}}_{\bar{\mathbf{m}}} = \bar{U}_{\bar{n}}\,\bar{\mathbf{E}}^{\bar{\mathbf{n}}} \qquad \mathbf{V} = V^m\,\mathbf{E}_\mathbf{m} = V_n\,\mathbf{E}^\mathbf{n} = \bar{V}^{\bar{m}}\,\bar{\mathbf{E}}_{\bar{\mathbf{m}}} = \bar{V}_{\bar{n}}\,\bar{\mathbf{E}}^{\bar{\mathbf{n}}}$$

where the *U^m, V^m,...are contravariant components* and the *U_n , V_n ,...are covariant components*

$U^m = \mathbf{U}\bullet\mathbf{E}^\mathbf{m}$, $V^m = \mathbf{V}\bullet\mathbf{E}^\mathbf{m}$, and $\bar{U}^{\bar{m}} = \mathbf{U}\bullet\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$, $\bar{V}^{\bar{m}} = \mathbf{V}\bullet\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$, , $U_n = \mathbf{U}\bullet\mathbf{E}_\mathbf{n}$, $V_n = \mathbf{V}\bullet\mathbf{E}_\mathbf{n}$, and $\bar{U}_{\bar{n}} = \mathbf{U}\bullet\bar{\mathbf{E}}_{\bar{\mathbf{n}}}$, *etc.*

Normal space (Contravariant)

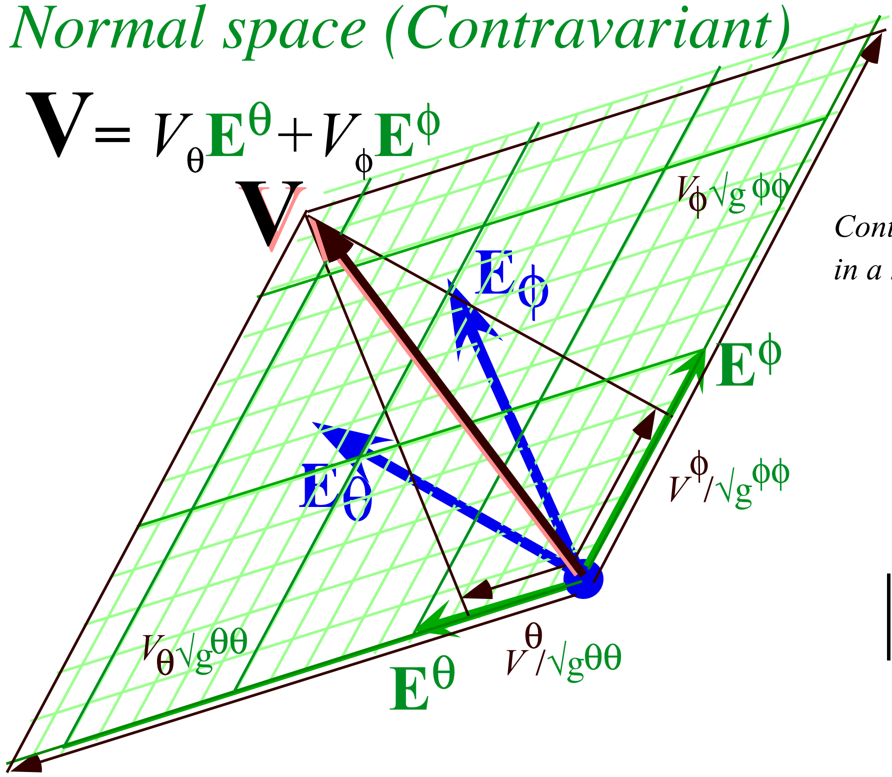


Fig. 3.3.2
Contravariant vector geometry in a normal space ($\mathbf{E}^\theta$,$\mathbf{E}^\phi$).

Tangent space (Covariant)

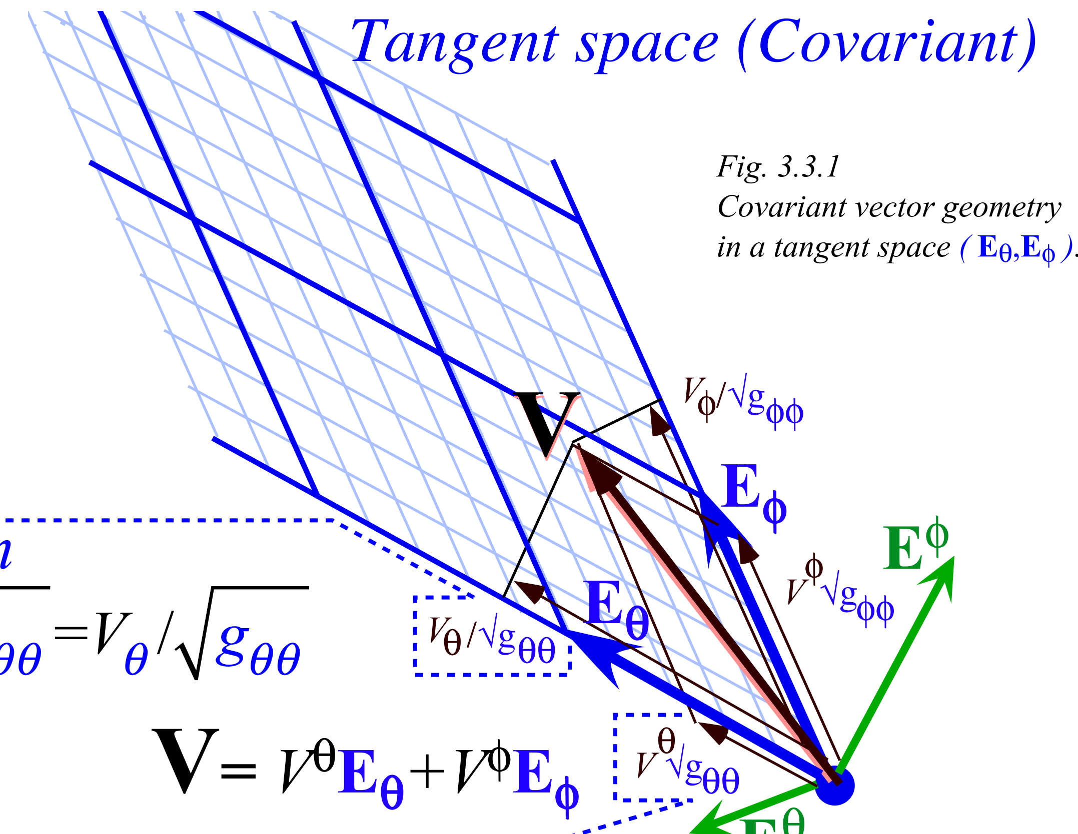


Fig. 3.3.1
Covariant vector geometry in a tangent space ($\mathbf{E}_\theta$,$\mathbf{E}_\phi$).

covariant projection

$$\left|\mathbf{V}\bullet\mathbf{E}_\theta\right| = \mathbf{V}\bullet\hat{\mathbf{E}}_\theta = \mathbf{V}\bullet\mathbf{E}_\theta/\sqrt{g_{\theta\theta}} = V_\theta/\sqrt{g_{\theta\theta}}$$

$$\mathbf{V} = V^\theta\mathbf{E}_\theta + V^\phi\mathbf{E}_\phi$$

covariant parallelogram side

$$\left|V^\theta\,\mathbf{E}_\theta\right| = V^\theta\sqrt{\mathbf{E}_\theta\cdot\mathbf{E}_\theta} = V^\theta\sqrt{g_{\theta\theta}}$$

Contravariant vector $\mathbf{E}^m$ is written in terms of covariant vectors $\mathbf{E}_n$ as would any vector $\mathbf{V} = V^n\,\mathbf{E}_n$ using dot product $V^n = \mathbf{V}\bullet\mathbf{E}^n$ and metric g_{mn} or g^{mn}...

Metric g_{mn} or g^{mn} tensor geometric relations to length, area, and volume

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

Any vector **U,V,...** is expressed using *either* set from any viewpoint, coordinate system, or *frame*,

$$\mathbf{U} = U^m\,\mathbf{E_m} = U_n\,\mathbf{E^n} = \bar{U}^{\bar{m}}\,\mathbf{\bar{E}_{\bar{m}}} = \bar{U}_{\bar{n}}\,\mathbf{\bar{E}^{\bar{n}}} \qquad \mathbf{V} = V^m\,\mathbf{E_m} = V_n\,\mathbf{E^n} = \bar{V}^{\bar{m}}\,\mathbf{\bar{E}_{\bar{m}}} = \bar{V}_{\bar{n}}\,\mathbf{\bar{E}^{\bar{n}}}$$

where the *U^m, V^m,...are contravariant components* and the *U_n , V_n ,...are covariant components*

$U^m = \mathbf{U}\bullet\mathbf{E^m}$, $V^m = \mathbf{V}\bullet\mathbf{E^m}$, and $\bar{U}^{\bar{m}} = \mathbf{U}\bullet\mathbf{\bar{E}^{\bar{m}}}$, $\bar{V}^{\bar{m}} = \mathbf{V}\bullet\mathbf{\bar{E}^{\bar{m}}}$, , $U_n = \mathbf{U}\bullet\mathbf{E_n}$, $V_n = \mathbf{V}\bullet\mathbf{E_n}$, and $\bar{U}_{\bar{n}} = \mathbf{U}\bullet\,\mathbf{\bar{E}_{\bar{n}}}$, *etc.*

Normal space (Contravariant)

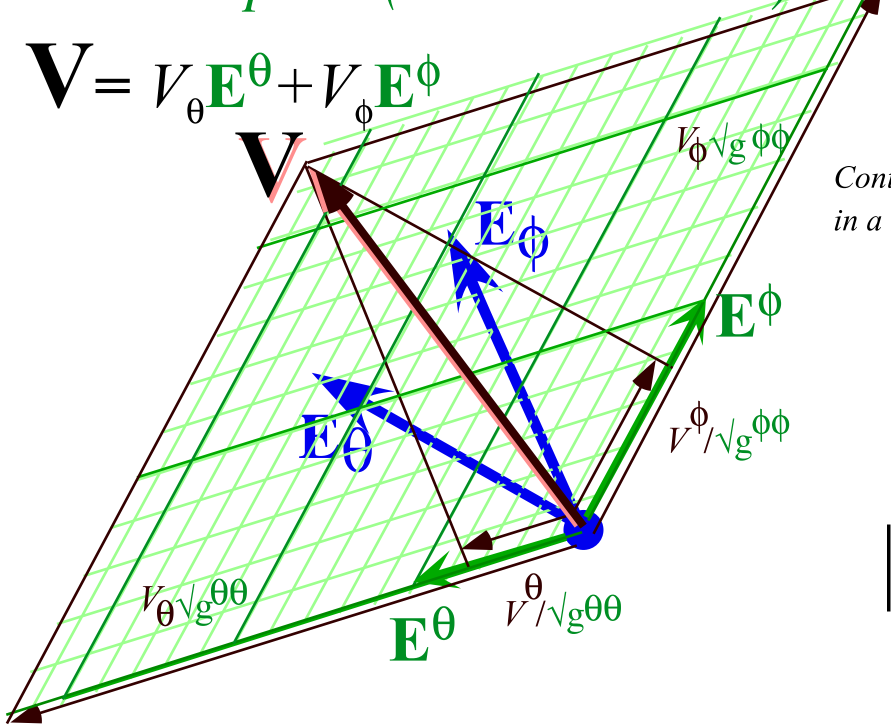


Fig. 3.3.2
Contravariant vector geometry in a normal space ($\mathbf{E}^\theta,\mathbf{E}^\phi$).

Tangent space (Covariant)

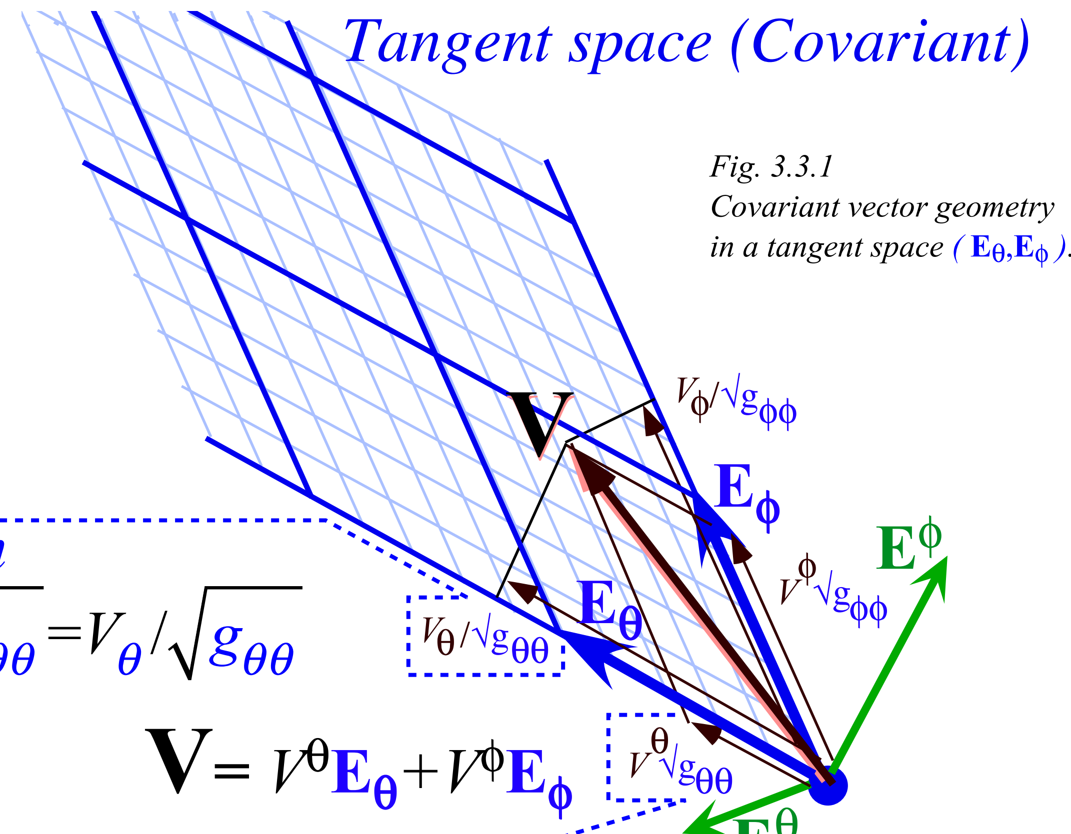


Fig. 3.3.1
Covariant vector geometry in a tangent space ($\mathbf{E}_\theta,\mathbf{E}_\phi$).

covariant projection

$$\left|\mathbf{V}\bullet\mathbf{E}_\theta\right| = \mathbf{V}\bullet\hat{\mathbf{E}}_\theta = \mathbf{V}\bullet\mathbf{E}_\theta/\sqrt{g_{\theta\theta}} = V_\theta/\sqrt{g_{\theta\theta}}$$

covariant parallelogram side

$$\left|V^\theta\,\mathbf{E}_\theta\right| = V^\theta\sqrt{\mathbf{E}_\theta\cdot\mathbf{E}_\theta} = V^\theta\sqrt{g_{\theta\theta}}$$

Contravariant vector $\mathbf{E}^m$ is written in terms of covariant vectors $\mathbf{E}_n$ as would any vector $\mathbf{V} = V^n\,\mathbf{E}_n$ using dot product $V^n = \mathbf{V}\bullet\mathbf{E}^n$ and metric g_{mn} or g^{mn}...

$$\mathbf{E}^m = (\mathbf{E}^m)^n\mathbf{E}_n \text{ implies: } (\mathbf{E}^m)^n = \mathbf{E}^m\bullet\mathbf{E}^n = g^{mn}$$

so: $\mathbf{E}^m = g^{mn}\,\mathbf{E}_n$

Contravariant vectors $\mathbf{E}^m$ *versus* Covariant vectors $\mathbf{E}_n$

Any vector **U**,**V**,... is expressed using *either* set from any viewpoint, coordinate system, or *frame*,

$$\mathbf{U} = U^m\,\mathbf{E_m} = U_n\,\mathbf{E^n} = \bar{U}^{\bar{m}}\,\bar{\mathbf{E}}_{\bar{\mathbf{m}}} = \bar{U}_{\bar{n}}\,\bar{\mathbf{E}}^{\bar{\mathbf{n}}} \qquad \mathbf{V} = V^m\,\mathbf{E_m} = V_n\,\mathbf{E^n} = \bar{V}^{\bar{m}}\,\bar{\mathbf{E}}_{\bar{\mathbf{m}}} = \bar{V}_{\bar{n}}\,\bar{\mathbf{E}}^{\bar{\mathbf{n}}}$$

where the *Um, Vm,...are contravariant components* and the *Un , Vn ,...are covariant components*

$U^m = \mathbf{U}\bullet\mathbf{E^m}$, $V^m = \mathbf{V}\bullet\mathbf{E^m}$, and $\bar{U}^{\bar{m}} = \mathbf{U}\bullet\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$, $\bar{V}^{\bar{m}} = \mathbf{V}\bullet\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$, , $\quad U_n = \mathbf{U}\bullet\mathbf{E_n}$, $V_n = \mathbf{V}\bullet\mathbf{E_n}$, and $\bar{U}_{\bar{n}} = \mathbf{U}\bullet\bar{\mathbf{E}}_{\bar{\mathbf{n}}}$, *etc.*

Normal space (Contravariant)

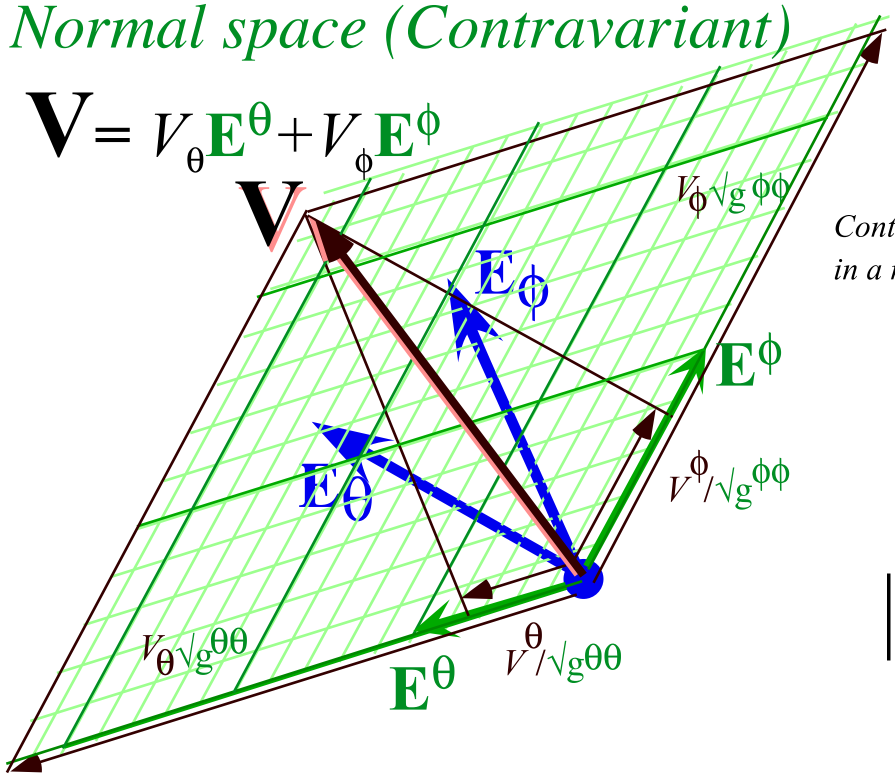


Fig. 3.3.2
Contravariant vector geometry in a normal space ($\mathbf{E}^\theta$,$\mathbf{E}^\phi$).

Tangent space (Covariant)

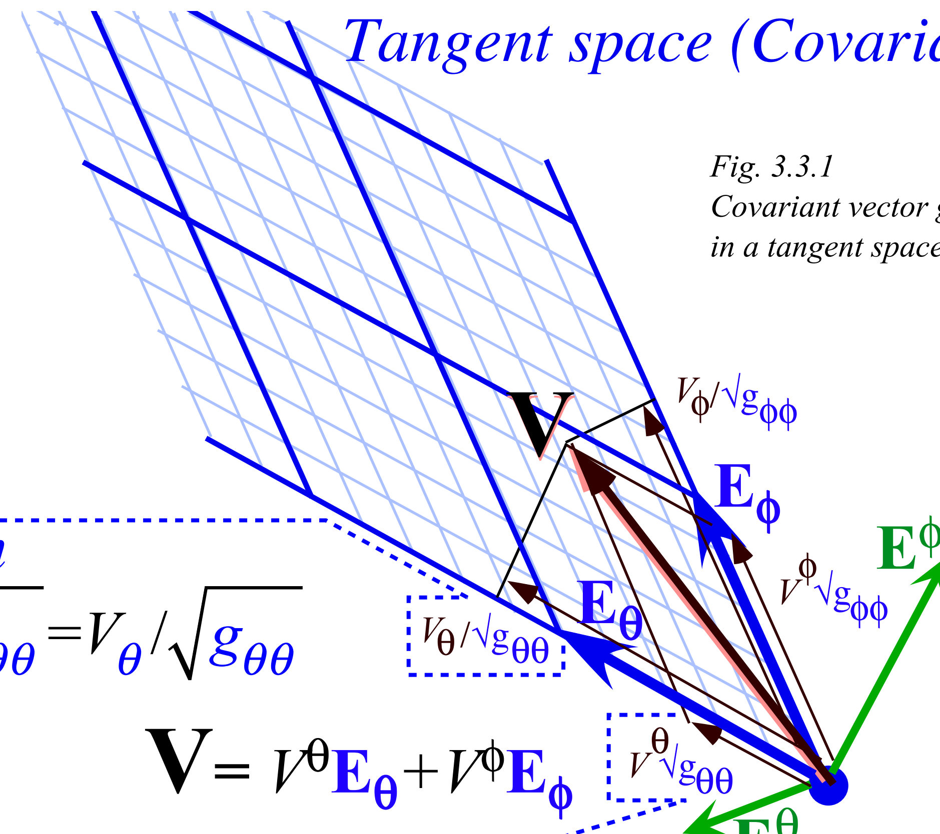


Fig. 3.3.1
Covariant vector geometry in a tangent space ($\mathbf{E}_\theta$,$\mathbf{E}_\phi$).

covariant projection

$$|\mathbf{V}\bullet\mathbf{E}_\theta| = \mathbf{V}\bullet\hat{\mathbf{E}}_\theta = \mathbf{V}\bullet\mathbf{E}_\theta/\sqrt{g_{\theta\theta}} = V_\theta/\sqrt{g_{\theta\theta}}$$

covariant parallelogram side

$$|V^\theta\,\mathbf{E}_\theta| = V^\theta\sqrt{\mathbf{E}_\theta\cdot\mathbf{E}_\theta} = V^\theta\sqrt{g_{\theta\theta}}$$

Contravariant vector $\mathbf{E}^m$ is written in terms of covariant vectors $\mathbf{E}_n$ as would any vector $\mathbf{V} = V^n\,\mathbf{E}_n$ using dot product $V^n = \mathbf{V}\bullet\mathbf{E}^n$ and metric g_{mn} or g^{mn}...

$$\mathbf{E}^m = (\mathbf{E}^m)^n\mathbf{E}_n \text{ implies: } (\mathbf{E}^m)^n = \mathbf{E}^m\bullet\mathbf{E}^n = g^{mn}$$

so: $\mathbf{E}^m = g^{mn}\,\mathbf{E}_n$

and: $\mathbf{E}_n = g_{mn}\,\mathbf{E}^m$ ...the same for covariant vectors

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.

Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)

Trebuchet Cartesian projectile coordinates are double-valued

Toroidal "rolled-up" (q^1=θ, q^2=ϕ)-manifold and "Flat" (x=θ, y=ϕ)-graph

Review of covariant $\mathbf{E}_n$ and contravariant $\mathbf{E}^m$ vectors: Jacobian J vs. Kajobian K

Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)

Tangent $\{\mathbf{E}_n\}$space vs Normal $\{\mathbf{E}^m\}$space

Covariant vs. contravariant coordinate transformations

Metric g_{mn} tensor geometric relations to length, area, and volume

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)

Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 77)

Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 69)

Riemann equation derivation for trebuchet model

Riemann equation force analysis

2nd-guessing Riemann equation?

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

Any vector **U,V,...** is expressed using *either* set from any viewpoint, coordinate system, or *frame*,

$$\mathbf{U} = U^m\,\mathbf{E_m} = U_n\,\mathbf{E^n} = \bar{U}^{\bar{m}}\,\mathbf{\bar{E}_{\bar{m}}} = \bar{U}_{\bar{n}}\,\mathbf{\bar{E}^{\bar{n}}} \qquad \mathbf{V} = V^m\,\mathbf{E_m} = V_n\,\mathbf{E^n} = \bar{V}^{\bar{m}}\,\mathbf{\bar{E}_{\bar{m}}} = \bar{V}_{\bar{n}}\,\mathbf{\bar{E}^{\bar{n}}}$$

where the *U^m, V^m,...are contravariant components* and the *U_n , V_n ...are covariant components*

$U^m = \mathbf{U}\bullet\mathbf{E^m}$, $V^m = \mathbf{V}\bullet\mathbf{E^m}$, and $\bar{U}^{\bar{m}} = \mathbf{U}\bullet\mathbf{\bar{E}^{\bar{m}}}$, $\bar{V}^{\bar{m}} = \mathbf{V}\bullet\mathbf{\bar{E}^{\bar{m}}}$, , $\quad$ $U_n = \mathbf{U}\bullet\mathbf{E_n}$, $V_n = \mathbf{V}\bullet\mathbf{E_n}$, and $\bar{U}_{\bar{n}} = \mathbf{U}\bullet\,\mathbf{\bar{E}_{\bar{n}}}$, *etc.*

Normal space (Contravariant)

$\mathbf{V} = V_\theta \mathbf{E}^\theta + V_\phi \mathbf{E}^\phi$

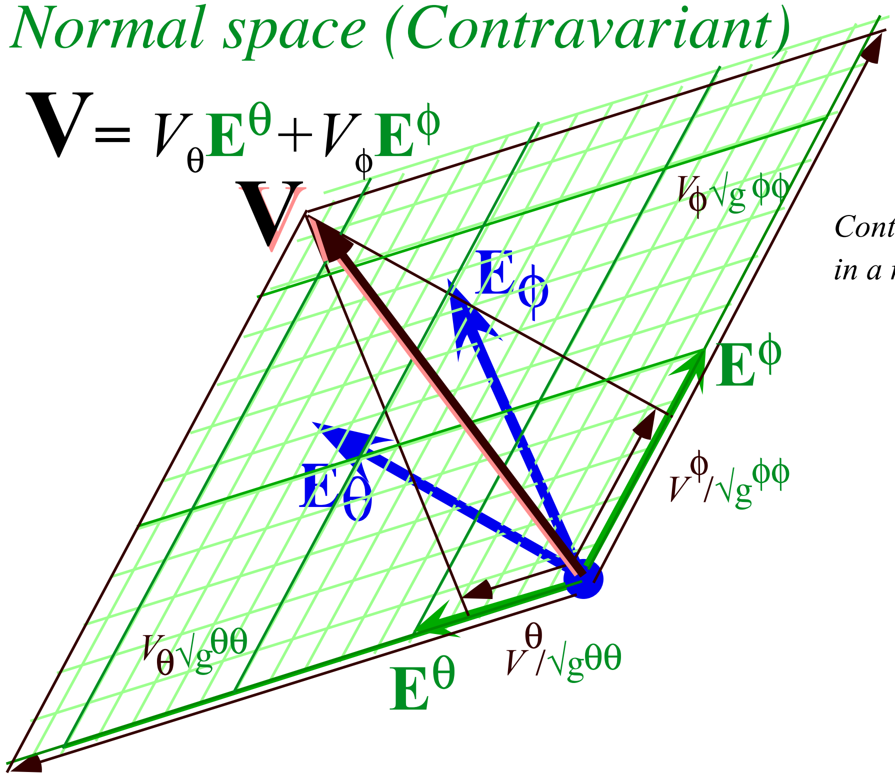


Fig. 3.3.2
Contravariant vector geometry in a normal space ($\mathbf{E}^\theta$,$\mathbf{E}^\phi$).

Tangent space (Covariant)

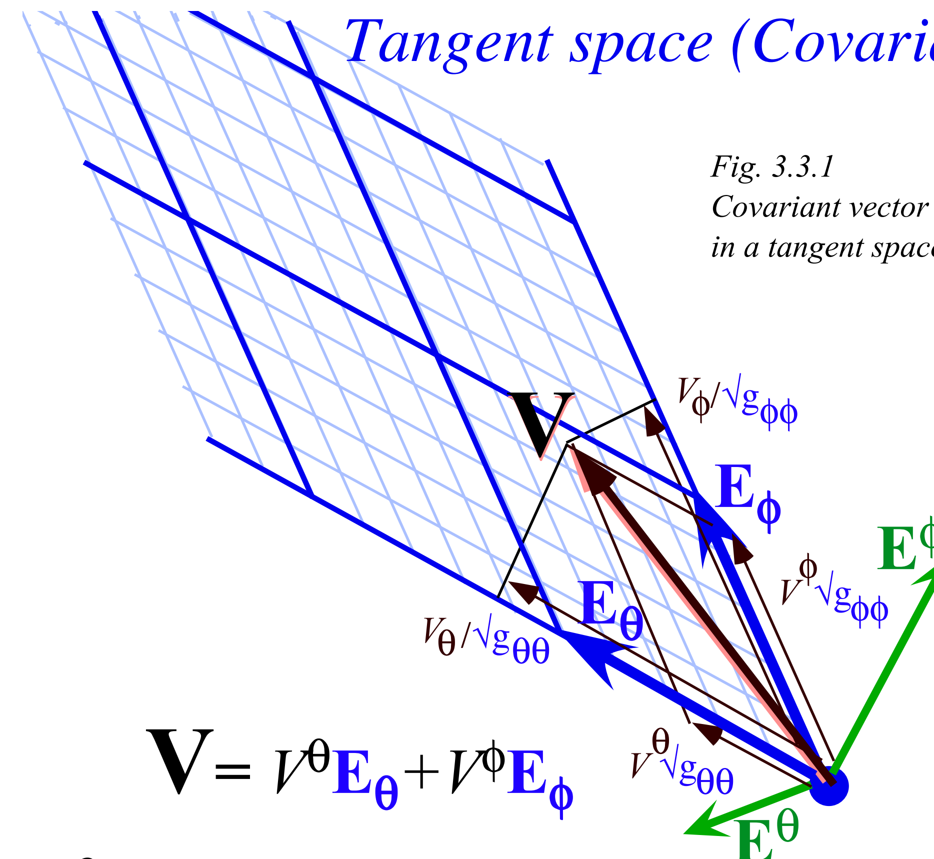


Fig. 3.3.1
Covariant vector geometry in a tangent space ($\mathbf{E}_\theta$,$\mathbf{E}_\phi$).

$\mathbf{V} = V^\theta \mathbf{E}_\theta + V^\phi \mathbf{E}_\phi$

Contravariant vector $\mathbf{E^m}$ for frame $\{q^1, q^2, \cdots\}$ is written in terms of new vectors $\mathbf{\bar{E}^{\bar{m}}}$ for a new "barred" frame $\{\bar{q}^1, \bar{q}^2, \cdots\}$ *(not drawn here)* using a *"chain-saw-sum rule"*....

$$\mathbf{E^m} = \frac{\partial q^m}{\partial \mathbf{r}} = \frac{\partial q^m}{\quad}\frac{}{\partial \mathbf{r}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\frac{\partial \bar{q}^{\bar{m}}}{\partial \mathbf{r}}\ , \quad \text{or:}\ \boxed{\mathbf{E^m} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\mathbf{\bar{E}^{\bar{m}}}}$$

from p. 93 of Lect. 14

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

Any vector **U,V,...** is expressed using *either* set from any viewpoint, coordinate system, or *frame*,

$$\mathbf{U} = U^m\,\mathbf{E_m} = U_n\,\mathbf{E^n} = \bar{U}^{\bar{m}}\,\mathbf{\bar{E}_{\bar{m}}} = \bar{U}_{\bar{n}}\,\mathbf{\bar{E}^{\bar{n}}} \qquad \mathbf{V} = V^m\,\mathbf{E_m} = V_n\,\mathbf{E^n} = \bar{V}^{\bar{m}}\,\mathbf{\bar{E}_{\bar{m}}} = \bar{V}_{\bar{n}}\,\mathbf{\bar{E}^{\bar{n}}}$$

where the *U^m, V^m,...are contravariant components* and the *U_n , V_n ,...are covariant components*

$U^m = \mathbf{U}\bullet\mathbf{E^m}$, $V^m = \mathbf{V}\bullet\mathbf{E^m}$, and $\bar{U}^{\bar{m}} = \mathbf{U}\bullet\mathbf{\bar{E}^{\bar{m}}}$, $\bar{V}^{\bar{m}} = \mathbf{V}\bullet\mathbf{\bar{E}^{\bar{m}}}$, , $\quad$ $U_n = \mathbf{U}\bullet\mathbf{E_n}$, $V_n = \mathbf{V}\bullet\mathbf{E_n}$, and $\bar{U}_{\bar{n}} = \mathbf{U}\bullet\,\mathbf{\bar{E}_{\bar{n}}}$, *etc.*

Normal space (Contravariant)

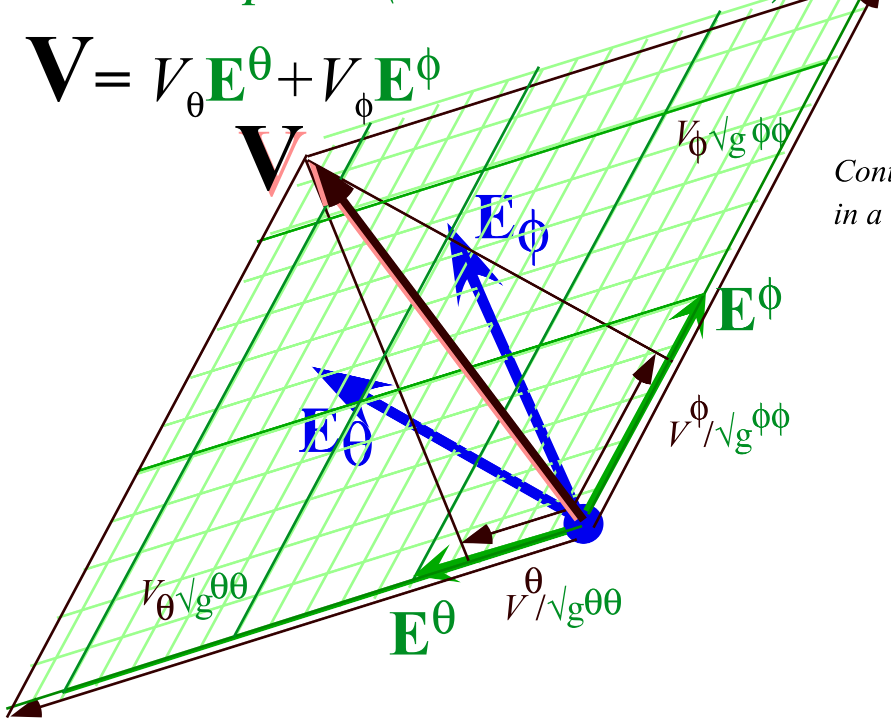


Fig. 3.3.2
Contravariant vector geometry in a normal space ($\mathbf{E}^\theta$,$\mathbf{E}^\phi$).

Tangent space (Covariant)

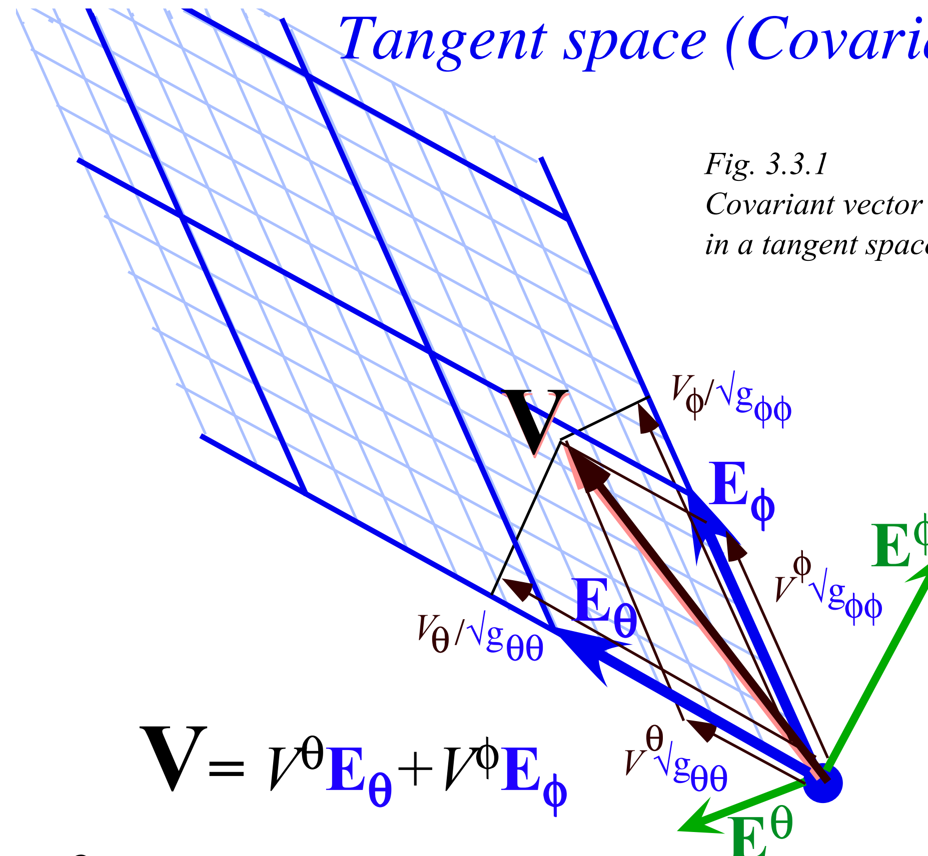


Fig. 3.3.1
Covariant vector geometry in a tangent space ($\mathbf{E}_\theta$,$\mathbf{E}_\phi$).

Contravariant vector $\mathbf{E^m}$ for frame $\{q^1, q^2, \cdots\}$ is written in terms of new vectors $\mathbf{\bar{E}^{\bar{m}}}$ for a new "barred" frame $\{\bar{q}^1, \bar{q}^2, \cdots\}$ *(not drawn here)* using a *"chain-saw-sum rule"*….

...and the same for covariant vectors $\mathbf{E_m}$ and $\mathbf{\bar{E}_{\bar{m}}}$

$$\mathbf{E^m} = \frac{\partial q^m}{\partial \mathbf{r}} = \frac{\partial q^m}{\phantom{\partial \bar{q}^{\bar{m}}}}\frac{\phantom{\partial \bar{q}^{\bar{m}}}}{\partial \mathbf{r}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\frac{\partial \bar{q}^{\bar{m}}}{\partial \mathbf{r}}\ ,\quad \text{or: } \boxed{\mathbf{E^m} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\mathbf{\bar{E}^{\bar{m}}}}$$

$$\mathbf{E_m} = \frac{\partial \mathbf{r}}{\partial q^m} = \frac{\phantom{\partial \bar{q}^{\bar{m}}}}{\partial q^m}\frac{\partial \mathbf{r}}{\phantom{\partial \bar{q}^{\bar{m}}}} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}}\ ,\quad \text{or: } \boxed{\mathbf{E_m} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\mathbf{\bar{E}_{\bar{m}}}}$$

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

Any vector **U,V,...** is expressed using *either* set from any viewpoint, coordinate system, or *frame*,

$$\mathbf{U} = U^m\,\mathbf{E_m} = U_n\,\mathbf{E^n} = \bar{U}^{\bar{m}}\,\mathbf{\bar{E}_{\bar{m}}} = \bar{U}_{\bar{n}}\,\mathbf{\bar{E}^{\bar{n}}} \qquad \mathbf{V} = V^m\,\mathbf{E_m} = V_n\,\mathbf{E^n} = \bar{V}^{\bar{m}}\,\mathbf{\bar{E}_{\bar{m}}} = \bar{V}_{\bar{n}}\,\mathbf{\bar{E}^{\bar{n}}}$$

where the *Um, Vm,...are contravariant components* and the *Un , Vn ,...are covariant components*

$U^m = \mathbf{U}\bullet\mathbf{E^m}$, $V^m = \mathbf{V}\bullet\mathbf{E^m}$, and $\bar{U}^{\bar{m}} = \mathbf{U}\bullet\mathbf{\bar{E}^{\bar{m}}}$, $\bar{V}^{\bar{m}} = \mathbf{V}\bullet\mathbf{\bar{E}^{\bar{m}}}$, , $U_n = \mathbf{U}\bullet\mathbf{E_n}$, $V_n = \mathbf{V}\bullet\mathbf{E_n}$, and $\bar{U}_{\bar{n}} = \mathbf{U}\bullet\mathbf{\bar{E}_{\bar{n}}}$, *etc.*

Normal space (Contravariant)

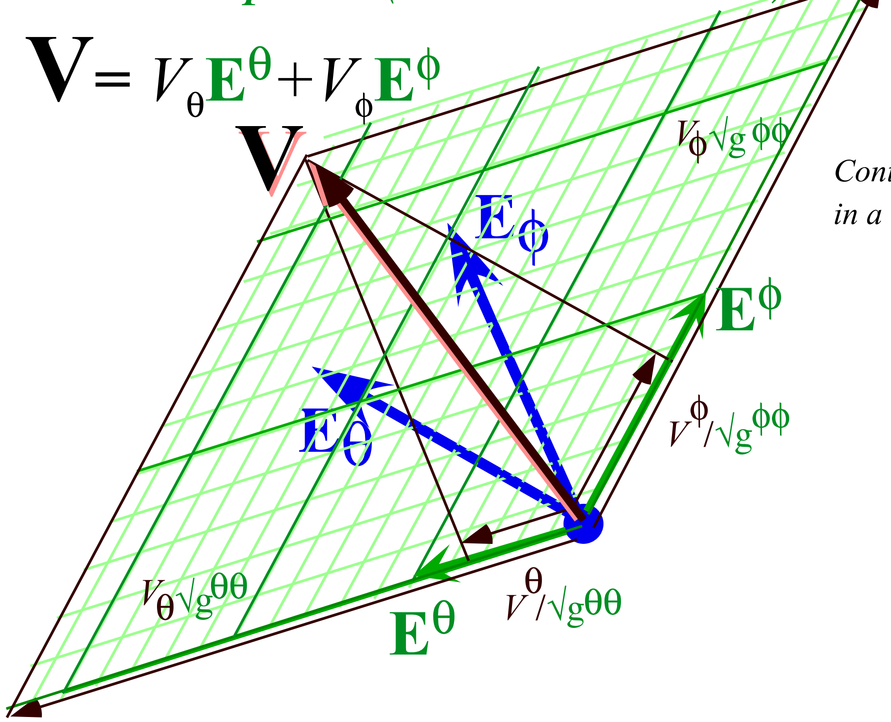


Fig. 3.3.2
Contravariant vector geometry
in a normal space ($\mathbf{E}^\theta$,$\mathbf{E}^\phi$).

Tangent space (Covariant)

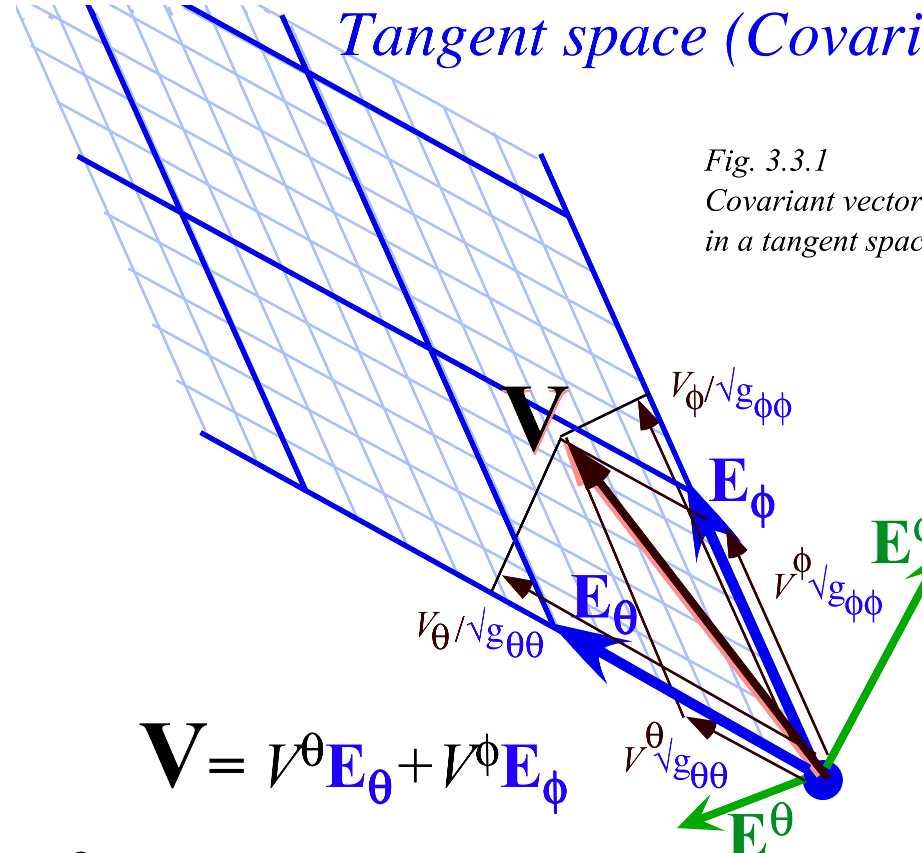


Fig. 3.3.1
Covariant vector geometry
in a tangent space ($\mathbf{E}_\theta$,$\mathbf{E}_\phi$).

Contravariant vector $\mathbf{E^m}$ for frame $\{q^1, q^2, \cdots\}$ is written in terms of new vectors $\mathbf{\bar{E}^{\bar{m}}}$ for a new "barred" frame $\{\bar{q}^1, \bar{q}^2, \cdots\}$ *(not drawn here)* using a *"chain-saw-sum rule"*….

...and the same for covariant vectors $\mathbf{E_m}$ and $\mathbf{\bar{E}_{\bar{m}}}$

$$\mathbf{E^m} = \frac{\partial q^m}{\partial \mathbf{r}} = \frac{\partial q^m}{}\frac{}{\partial \mathbf{r}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\frac{\partial \bar{q}^{\bar{m}}}{\partial \mathbf{r}}, \quad \text{or: } \boxed{\mathbf{E^m} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\mathbf{\bar{E}^{\bar{m}}}}$$

implies: $V^m = \dfrac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\bar{V}^{\bar{m}}$

$$\mathbf{E_m} = \frac{\partial \mathbf{r}}{\partial q^m} = \frac{}{\partial q^m}\frac{\partial \mathbf{r}}{} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}}, \quad \text{or: } \boxed{\mathbf{E_m} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\mathbf{\bar{E}_{\bar{m}}}}$$

implies: $V_m = \dfrac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\bar{V}_{\bar{m}}$

Contravariant vectors $\mathbf{E}^m$ *versus* Covariant vectors $\mathbf{E}_n$

Any vector **U**,**V**,... is expressed using *either* set from any viewpoint, coordinate system, or *frame*,

$$\mathbf{U} = U^m\,\mathbf{E_m} = U_n\,\mathbf{E^n} = \bar{U}^{\bar{m}}\,\mathbf{\bar{E}_{\bar{m}}} = \bar{U}_{\bar{n}}\,\mathbf{\bar{E}^{\bar{n}}} \qquad \mathbf{V} = V^m\,\mathbf{E_m} = V_n\,\mathbf{E^n} = \bar{V}^{\bar{m}}\,\mathbf{\bar{E}_{\bar{m}}} = \bar{V}_{\bar{n}}\,\mathbf{\bar{E}^{\bar{n}}}$$

where the *Um, Vm,...are contravariant components* and the *Un , Vn ,...are covariant components*

$$U^m = \mathbf{U}\bullet\mathbf{E^m},\ V^m = \mathbf{V}\bullet\mathbf{E^m},\ \text{and}\ \bar{U}^{\bar{m}} = \mathbf{U}\bullet\mathbf{\bar{E}^{\bar{m}}},\ \bar{V}^{\bar{m}} = \mathbf{V}\bullet\mathbf{\bar{E}^{\bar{m}}},\ , \qquad U_n = \mathbf{U}\bullet\mathbf{E_n},\ V_n = \mathbf{V}\bullet\mathbf{E_n},\ \text{and}\ \bar{U}_{\bar{n}} = \mathbf{U}\bullet\mathbf{\bar{E}_{\bar{n}}},\ etc.$$

Normal space (Contravariant)

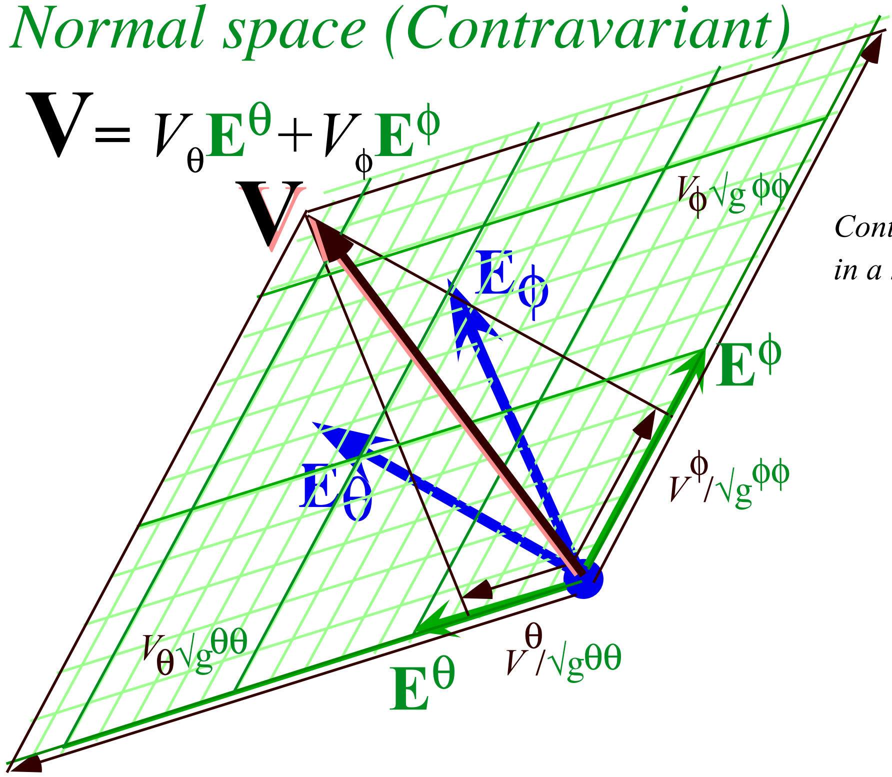


Fig. 3.3.2
Contravariant vector geometry in a normal space ($\mathbf{E}^\theta$,$\mathbf{E}^\phi$).

Tangent space (Covariant)

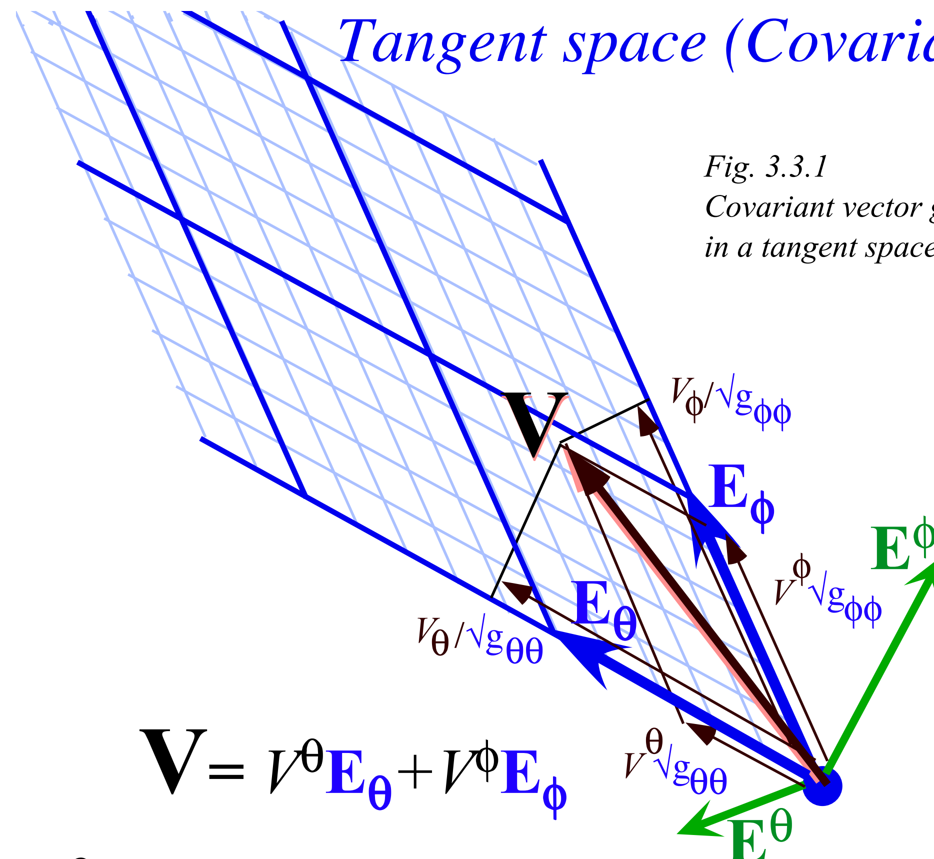


Fig. 3.3.1
Covariant vector geometry in a tangent space ($\mathbf{E}_\theta$,$\mathbf{E}_\phi$).

Contravariant vector $\mathbf{E^m}$ for frame $\{q^1, q^2, \cdots\}$ is written in terms of new vectors $\mathbf{\bar{E}^{\bar{m}}}$ for a new "barred" frame $\{\bar{q}^{\bar{1}}, \bar{q}^{\bar{2}}, \cdots\}$ *(not drawn here)* using a *"chain-saw-sum rule"*....

...and the same for covariant vectors $\mathbf{E_m}$ and $\mathbf{\bar{E}_{\bar{m}}}$

$$\mathbf{E^m} = \frac{\partial q^m}{\partial \mathbf{r}} = \frac{\partial q^m}{\quad}\frac{}{\partial \mathbf{r}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\frac{\partial \bar{q}^{\bar{m}}}{\partial \mathbf{r}}, \quad \text{or: } \boxed{\mathbf{E^m} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\mathbf{\bar{E}^{\bar{m}}}}$$

implies: $V^m = \dfrac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\bar{V}^{\bar{m}}$

$$\mathbf{E_m} = \frac{\partial \mathbf{r}}{\partial q^m} = \frac{\quad}{\partial q^m}\frac{\partial \mathbf{r}}{} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}}, \quad \text{or: } \boxed{\mathbf{E_m} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\mathbf{\bar{E}_{\bar{m}}}}$$

Dirac notation equivalents:

$$\langle m| = \langle m|\cdot\mathbf{1} = \langle m|\cdot\sum_{\bar{m}}|\bar{m}\rangle\langle\bar{m}| = \sum_{\bar{m}}\langle m|\bar{m}\rangle\langle\bar{m}|$$

implies: $\langle m|\Psi\rangle = \sum_{\bar{m}}\langle m|\bar{m}\rangle\langle\bar{m}|\Psi\rangle$

Dirac notation equivalents:

$$|m\rangle = \mathbf{1}\cdot|m\rangle = \sum_{\bar{m}}|\bar{m}\rangle\langle\bar{m}||m\rangle = \sum_{\bar{m}}\langle\bar{m}|m\rangle|\bar{m}\rangle$$

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

Any vector **U**,**V**,**...** is expressed using *either* set from any viewpoint, coordinate system, or *frame*,

$$\mathbf{U} = U^m\,\mathbf{E_m} = U_n\,\mathbf{E^n} = \bar{U}^{\bar{m}}\,\mathbf{\bar{E}_{\bar{m}}} = \bar{U}_{\bar{n}}\,\mathbf{\bar{E}^{\bar{n}}} \qquad \mathbf{V} = V^m\,\mathbf{E_m} = V_n\,\mathbf{E^n} = \bar{V}^{\bar{m}}\,\mathbf{\bar{E}_{\bar{m}}} = \bar{V}_{\bar{n}}\,\mathbf{\bar{E}^{\bar{n}}}$$

where the *Um, Vm,...are contravariant components* and the *Un , Vn ,...are covariant components*

$$U^m = \mathbf{U}\bullet\mathbf{E^m},\; V^m = \mathbf{V}\bullet\mathbf{E^m}, \text{ and } \bar{U}^{\bar{m}} = \mathbf{U}\bullet\mathbf{\bar{E}^{\bar{m}}},\; \bar{V}^{\bar{m}} = \mathbf{V}\bullet\mathbf{\bar{E}^{\bar{m}}},\,, \qquad U_n = \mathbf{U}\bullet\mathbf{E_n},\; V_n = \mathbf{V}\bullet\mathbf{E_n}, \text{ and } \bar{U}_{\bar{n}} = \mathbf{U}\bullet\mathbf{\bar{E}_{\bar{n}}}, \textit{ etc.}$$

Normal space (Contravariant)

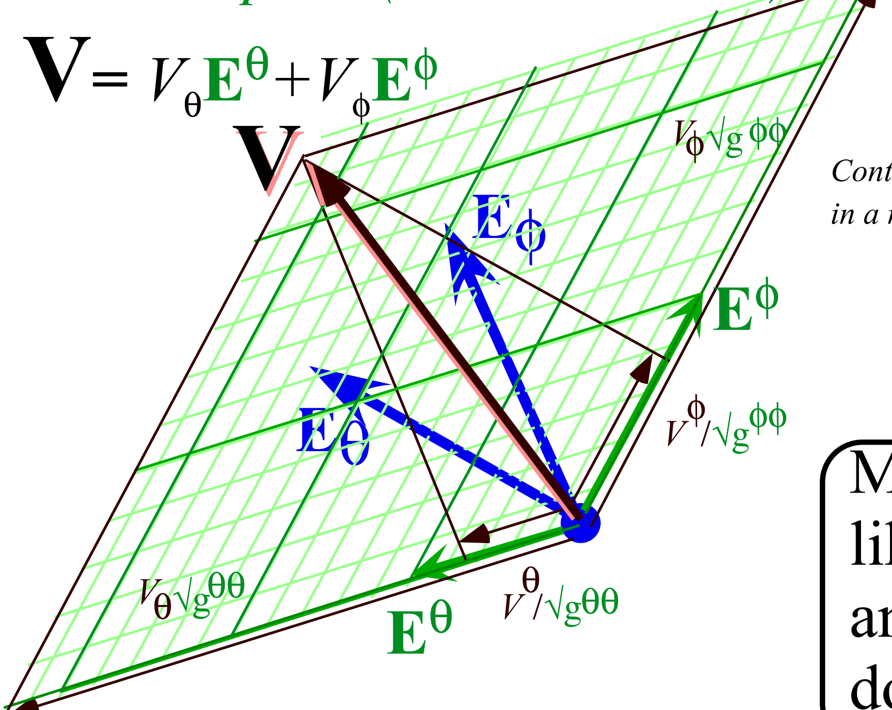


Fig. 3.3.2
Contravariant vector geometry in a normal space ($\mathbf{E}^\theta,\mathbf{E}^\phi$).

Tangent space (Covariant)

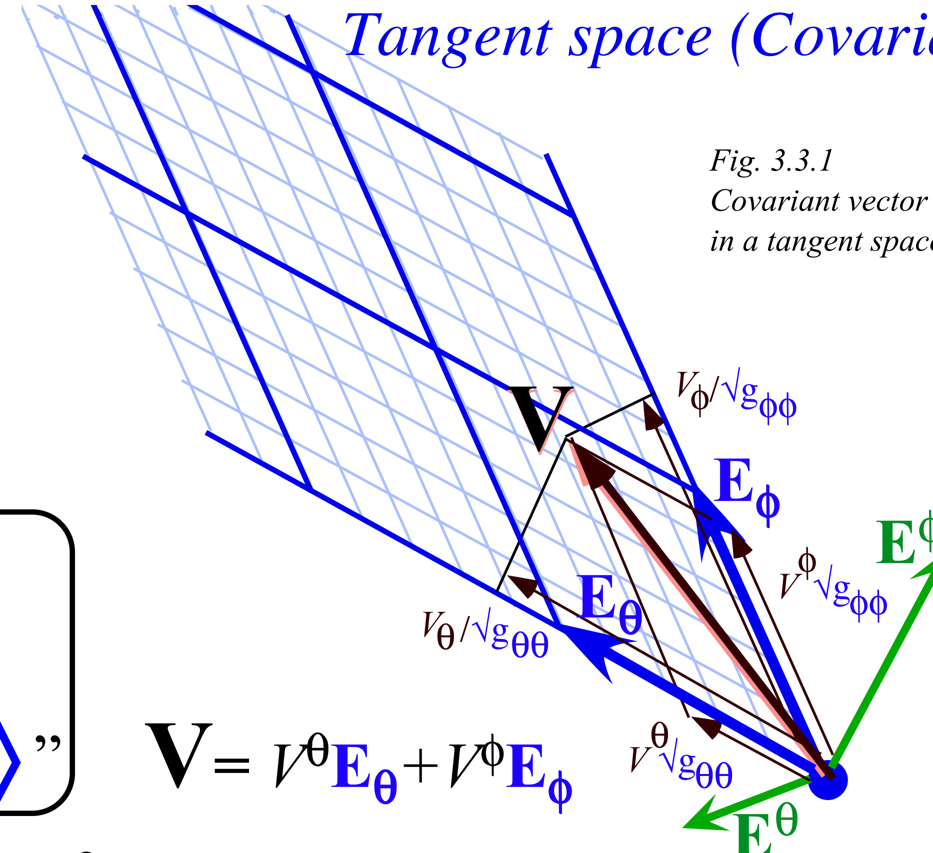


Fig. 3.3.1
Covariant vector geometry in a tangent space ($\mathbf{E}_\theta,\mathbf{E}_\phi$).

Metric relations
like: $\mathbf{E}^m = g^{mn}\,\mathbf{E}_n$
and: $\mathbf{E}_n = g_{mn}\,\mathbf{E}^m$
don't exist for "$\langle$bra-kets$\rangle$"

Contravariant vector $\mathbf{E^m}$ for frame $\{q^1, q^2, \cdots\}$ is written in terms of new vectors $\mathbf{\bar{E}^{\bar{m}}}$ for a new "barred" frame $\{\bar{q}^{\bar{1}}, \bar{q}^{\bar{2}}, \cdots\}$ *(not drawn here)* using a *"chain-saw-sum rule"*....

...and the same for covariant vectors $\mathbf{E_m}$ and $\mathbf{\bar{E}_{\bar{m}}}$

$$\mathbf{E^m} = \frac{\partial q^m}{\partial \mathbf{r}} = \frac{\partial q^m}{\quad}\frac{\quad}{\partial \mathbf{r}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\frac{\partial \bar{q}^{\bar{m}}}{\partial \mathbf{r}}, \quad \text{or: } \boxed{\mathbf{E^m} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\mathbf{\bar{E}^{\bar{m}}}} \text{ implies: } V^m = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\bar{V}^{\bar{m}}$$

$$\mathbf{E_m} = \frac{\partial \mathbf{r}}{\partial q^m} = \frac{\quad}{\partial q^m}\frac{\partial \mathbf{r}}{\quad} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}}, \quad \text{or: } \boxed{\mathbf{E_m} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\mathbf{\bar{E}_{\bar{m}}}}$$

Dirac notation equivalents:

$$\langle m| = \langle m|\cdot\mathbf{1} = \langle m|\cdot\sum_{\bar{m}}|\bar{m}\rangle\langle\bar{m}| = \sum_{\bar{m}}\langle m|\bar{m}\rangle\langle\bar{m}| \quad \text{implies: } \langle m|\Psi\rangle = \sum_{\bar{m}}\langle m|\bar{m}\rangle\langle\bar{m}|\Psi\rangle$$

Dirac notation equivalents:

$$|m\rangle = \mathbf{1}\cdot|m\rangle = \sum_{\bar{m}}|\bar{m}\rangle\langle\bar{m}||m\rangle = \sum_{\bar{m}}\langle\bar{m}|m\rangle|\bar{m}\rangle$$

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.

Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)

Trebuchet Cartesian projectile coordinates are double-valued

Toroidal "rolled-up" ($q^1=\theta$, $q^2=\phi$)-manifold and "Flat" ($x=\theta$, $y=\phi$)-graph

Review of covariant $\mathbf{E}_n$ and contravariant $\mathbf{E}^m$ vectors: Jacobian J vs. Kajobian K

Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)

Tangent $\{\mathbf{E}_n\}$space vs Normal $\{\mathbf{E}^m\}$space

Covariant vs. contravariant coordinate transformations

Metric g_{mn} tensor geometric relations to length, area, and volume

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)

Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 77)

Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 69)

Riemann equation derivation for trebuchet model

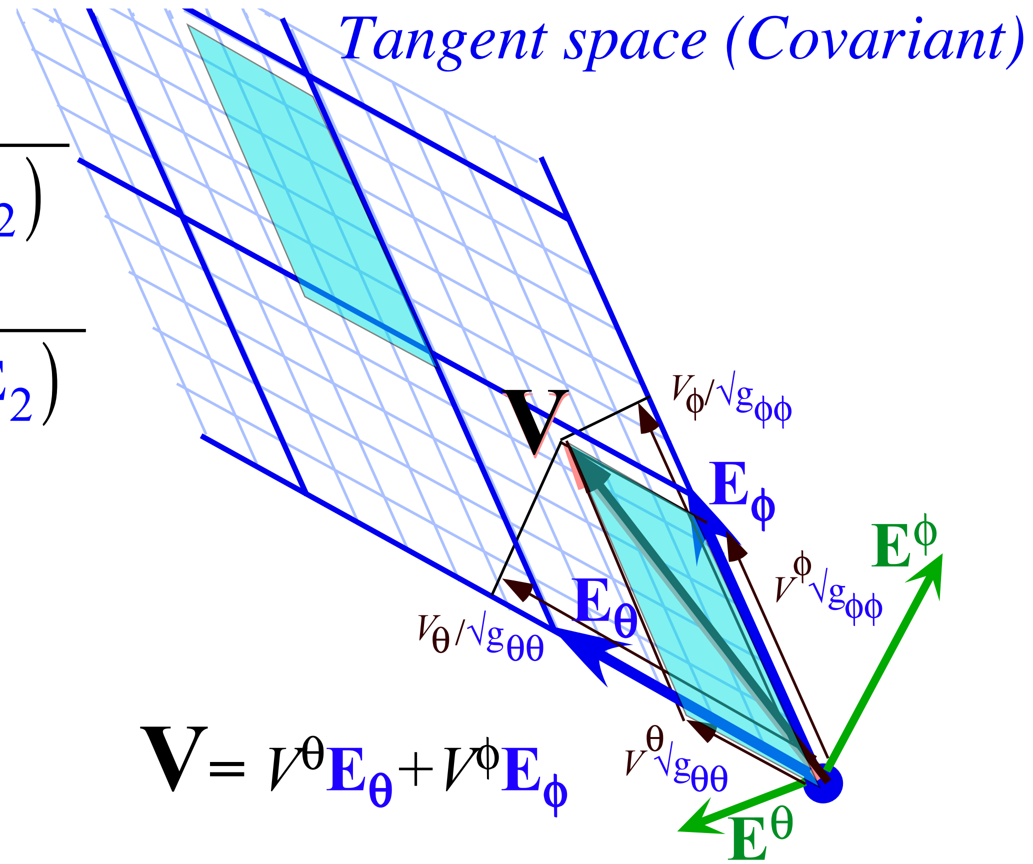
Riemann equation force analysis

2nd-guessing Riemann equation?

Tangent space (Covariant) area spanned by $V^1\mathbf{E}_1$ and $V^2\mathbf{E}_2$

$$Area\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2\right) = V^1V^2\left|\mathbf{E}_1\times\mathbf{E}_2\right| = V^1V^2\sqrt{\left(\mathbf{E}_1\times\mathbf{E}_2\right)\bullet\left(\mathbf{E}_1\times\mathbf{E}_2\right)}$$

$$Area\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2\right) = V^1V^2\sqrt{\left(\mathbf{E}_1\bullet\mathbf{E}_1\right)\left(\mathbf{E}_2\bullet\mathbf{E}_2\right)-\left(\mathbf{E}_1\bullet\mathbf{E}_2\right)\left(\mathbf{E}_1\bullet\mathbf{E}_2\right)}$$



Metric $\mathbf{g}_{mn}$ or $\mathbf{g}^{mn}$ tensor geometric relations to length, area, and volume

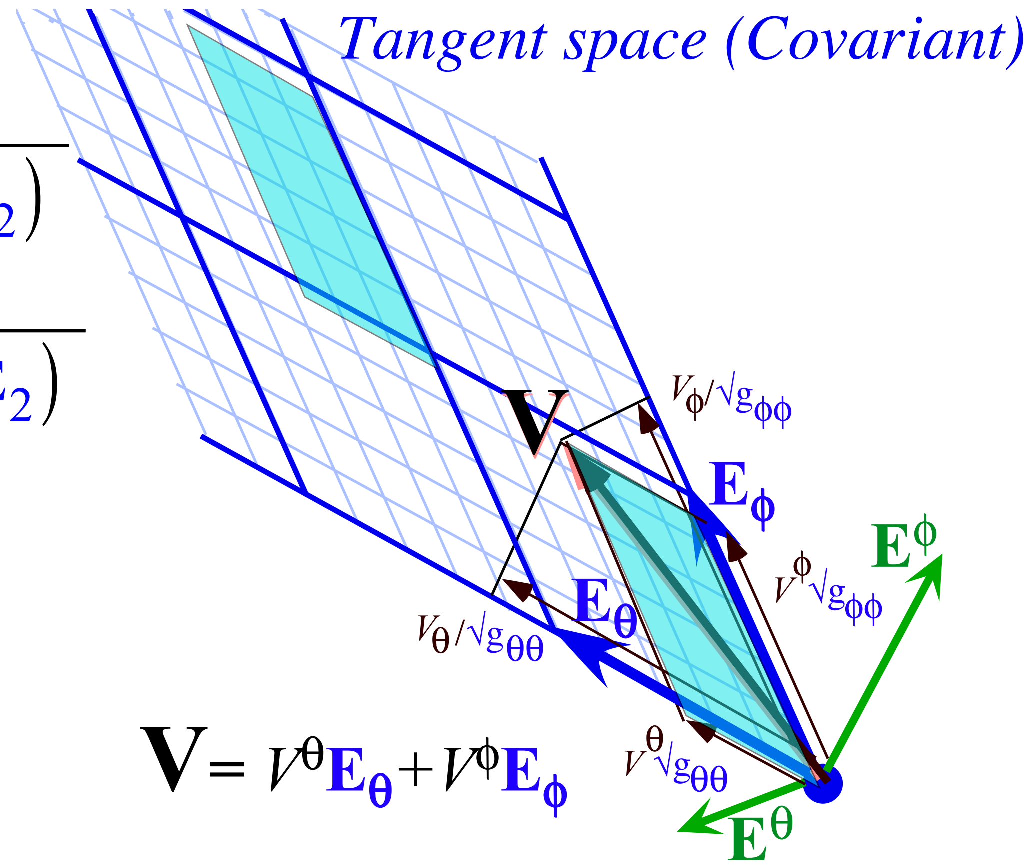
Tangent space (Covariant) area spanned by $V^1\mathbf{E}_1$ and $V^2\mathbf{E}_2$

$$Area\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2\right) = V^1V^2\left|\mathbf{E}_1\times\mathbf{E}_2\right| = V^1V^2\sqrt{\left(\mathbf{E}_1\times\mathbf{E}_2\right)\bullet\left(\mathbf{E}_1\times\mathbf{E}_2\right)}$$

$$Area\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2\right) = V^1V^2\sqrt{\left(\mathbf{E}_1\bullet\mathbf{E}_1\right)\left(\mathbf{E}_2\bullet\mathbf{E}_2\right)-\left(\mathbf{E}_1\bullet\mathbf{E}_2\right)\left(\mathbf{E}_1\bullet\mathbf{E}_2\right)}$$

$$= V^1V^2\sqrt{g_{11}g_{22}-g_{12}g_{21}}$$

where: $g_{12} = \mathbf{E}_1\bullet\mathbf{E}_2 = g_{21}$



Metric $\mathbf{g}_{mn}$ or $\mathbf{g}^{mn}$ tensor geometric relations to length, area, and volume

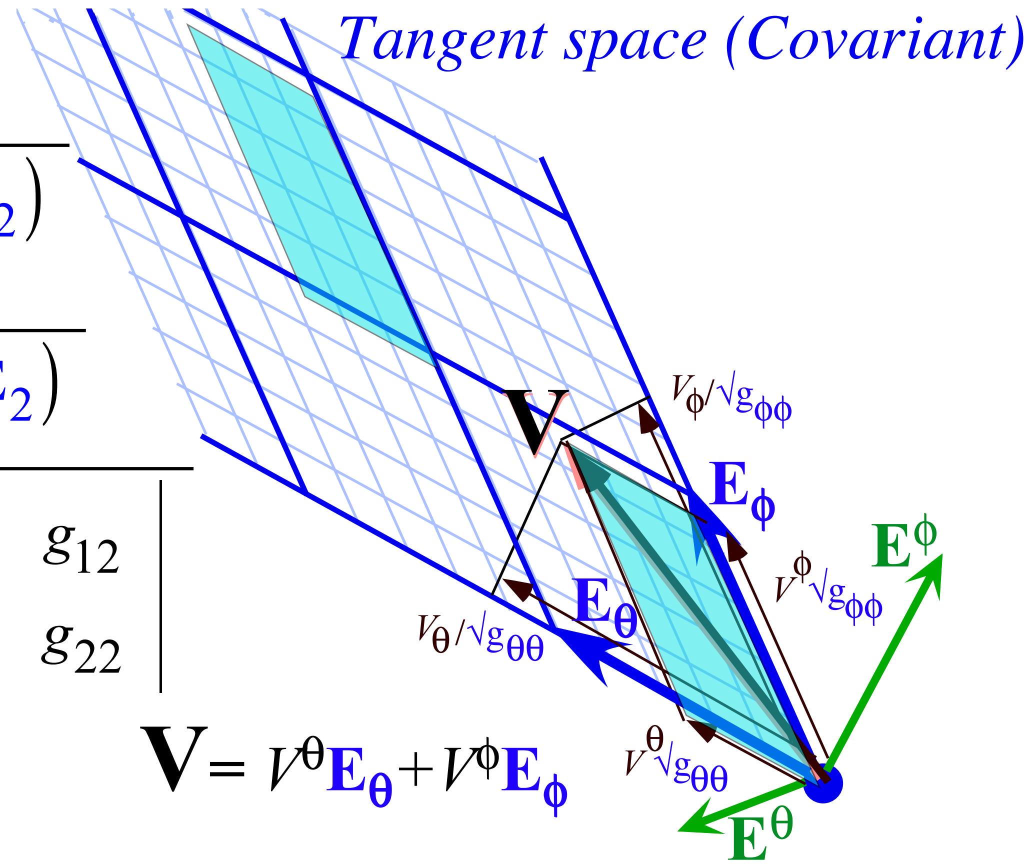
Tangent space (Covariant) area spanned by $V^1\mathbf{E}_1$ and $V^2\mathbf{E}_2$

$$Area\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2\right) = V^1V^2\left|\mathbf{E}_1\times\mathbf{E}_2\right| = V^1V^2\sqrt{\left(\mathbf{E}_1\times\mathbf{E}_2\right)\bullet\left(\mathbf{E}_1\times\mathbf{E}_2\right)}$$

$$Area\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2\right) = V^1V^2\sqrt{\left(\mathbf{E}_1\bullet\mathbf{E}_1\right)\left(\mathbf{E}_2\bullet\mathbf{E}_2\right)-\left(\mathbf{E}_1\bullet\mathbf{E}_2\right)\left(\mathbf{E}_1\bullet\mathbf{E}_2\right)}$$

$$= V^1V^2\sqrt{g_{11}g_{22}-g_{12}g_{21}} = V^1V^2\sqrt{\det\begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{vmatrix}}$$

where: $g_{12} = \mathbf{E}_1\bullet\mathbf{E}_2 = g_{21}$



Metric g_{mn} or g^{mn} tensor geometric relations to length, area, and volume

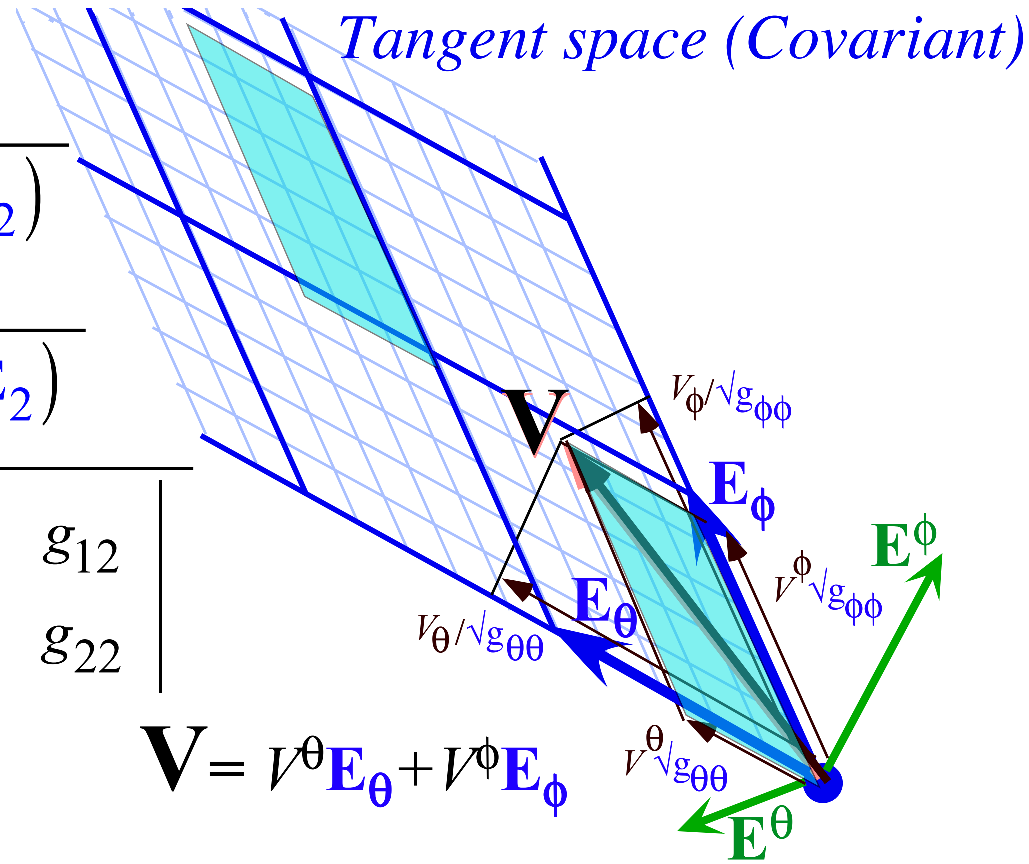
Tangent space (Covariant) area spanned by $V^1\mathbf{E}_1$ and $V^2\mathbf{E}_2$

$$Area\left(V^1\mathbf{E}_1,V^2\mathbf{E}_2\right)=V^1V^2\left|\mathbf{E}_1\times\mathbf{E}_2\right|=V^1V^2\sqrt{\left(\mathbf{E}_1\times\mathbf{E}_2\right)\bullet\left(\mathbf{E}_1\times\mathbf{E}_2\right)}$$

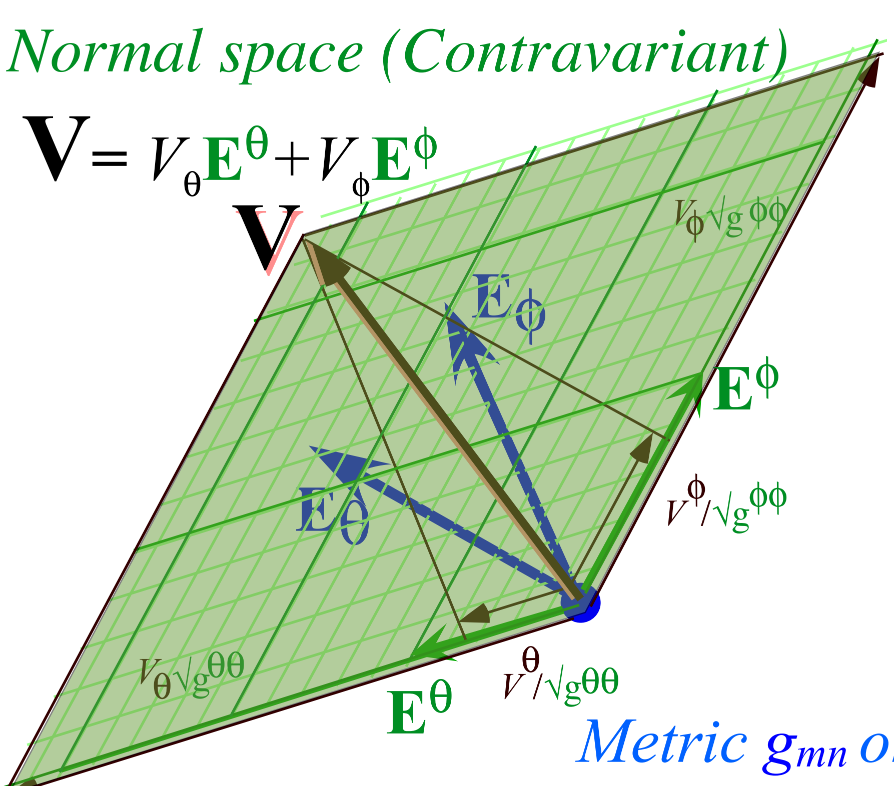
$$Area\left(V^1\mathbf{E}_1,V^2\mathbf{E}_2\right)=V^1V^2\sqrt{\left(\mathbf{E}_1\bullet\mathbf{E}_1\right)\left(\mathbf{E}_2\bullet\mathbf{E}_2\right)-\left(\mathbf{E}_1\bullet\mathbf{E}_2\right)\left(\mathbf{E}_1\bullet\mathbf{E}_2\right)}$$

$$=V^1V^2\sqrt{g_{11}g_{22}-g_{12}g_{21}}=V^1V^2\sqrt{\det\begin{vmatrix} g_{11} & g_{12}\\ g_{21} & g_{22}\end{vmatrix}}$$

where: $g_{12}=\mathbf{E}_1\bullet\mathbf{E}_2=g_{21}$



Normal space (Contravariant) area spanned by $V_1\mathbf{E}^1$ and $V_2\mathbf{E}^2$



$$Area\left(V_1\mathbf{E}^1,V_2\mathbf{E}^2\right)=V_1V_2\left|\mathbf{E}^1\times\mathbf{E}^2\right|=V_1V_2\sqrt{\left(\mathbf{E}^1\times\mathbf{E}^2\right)\bullet\left(\mathbf{E}^1\times\mathbf{E}^2\right)}$$

$$Area\left(V_1\mathbf{E}^1,V_2\mathbf{E}^2\right)=V_1V_2\sqrt{\left(\mathbf{E}^1\bullet\mathbf{E}^1\right)\left(\mathbf{E}^2\bullet\mathbf{E}^2\right)-\left(\mathbf{E}^1\bullet\mathbf{E}^2\right)\left(\mathbf{E}^1\bullet\mathbf{E}^2\right)}$$

$$=V_1V_2\sqrt{g^{11}g^{22}-g^{12}g^{21}}=V_1V_2\sqrt{\det\begin{vmatrix} g^{11} & g^{12}\\ g^{21} & g^{22}\end{vmatrix}}$$

Metric g_{mn} or g^{mn} tensor geometric relations to length, area, and volume

where: $g^{12}=\mathbf{E}^1\bullet\mathbf{E}^2=g^{21}$

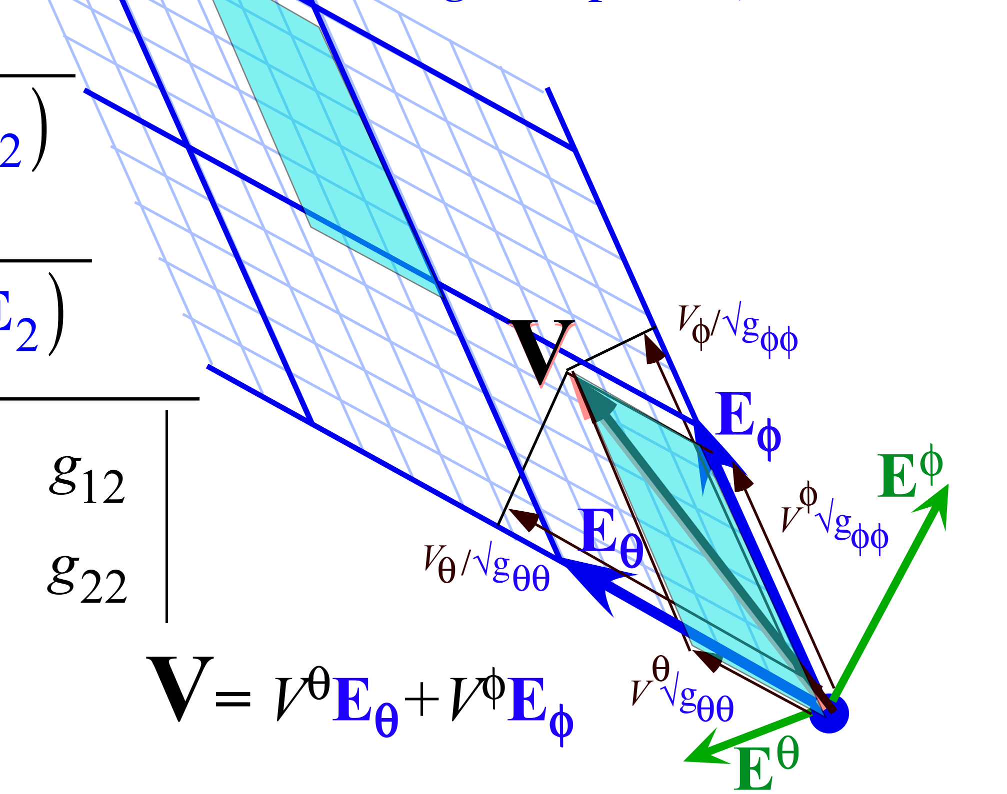
Tangent space (Covariant)

$$Area\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2\right) = V^1V^2\left|\mathbf{E}_1\times\mathbf{E}_2\right| = V^1V^2\sqrt{\left(\mathbf{E}_1\times\mathbf{E}_2\right)\bullet\left(\mathbf{E}_1\times\mathbf{E}_2\right)}$$

$$Area\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2\right) = V^1V^2\sqrt{\left(\mathbf{E}_1\bullet\mathbf{E}_1\right)\left(\mathbf{E}_2\bullet\mathbf{E}_2\right)-\left(\mathbf{E}_1\bullet\mathbf{E}_2\right)\left(\mathbf{E}_1\bullet\mathbf{E}_2\right)}$$

$$= V^1V^2\sqrt{g_{11}g_{22}-g_{12}g_{21}} = V^1V^2\sqrt{\det\begin{vmatrix} g_{11} & g_{12}\\ g_{21} & g_{22}\end{vmatrix}}$$

where: $g_{12} = \mathbf{E}_1\bullet\mathbf{E}_2 = g_{21}$



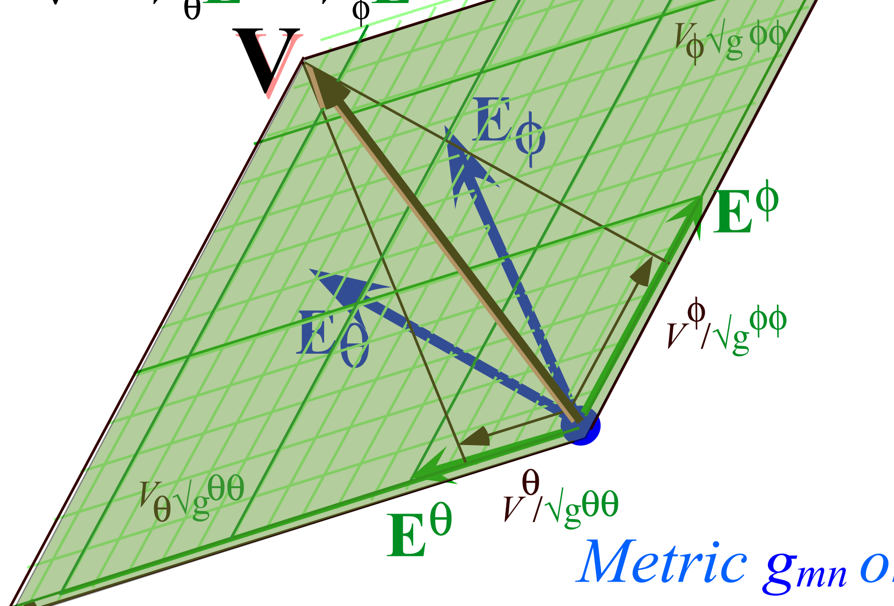
$$\mathbf{V} = V^\theta\mathbf{E}_\theta + V^\phi\mathbf{E}_\phi$$

Determinant product rule: $det|\mathbf{g}_{cov}|\cdot det|\mathbf{g}^{cont}| = 1$ since $(\mathbf{g}_{cov})^{-1} = \mathbf{g}^{cont}$ or : $\begin{pmatrix} g_{11} & g_{12}\\ g_{21} & g_{22}\end{pmatrix}\cdot\begin{pmatrix} g^{11} & g^{12}\\ g^{21} & g^{22}\end{pmatrix} = \begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}$

(Recall: $\mathbf{E}_n\bullet\mathbf{E}^n = 1$) $\quad(\mathbf{g}_{cov}) \cdot (\mathbf{g}^{cont}) = (\mathbf{1})$

Normal space (Contravariant)

$$\mathbf{V} = V_\theta\mathbf{E}^\theta + V_\phi\mathbf{E}^\phi$$



$$Area\left(V_1\mathbf{E}^1, V_2\mathbf{E}^2\right) = V_1V_2\left|\mathbf{E}^1\times\mathbf{E}^2\right| = V_1V_2\sqrt{\left(\mathbf{E}^1\times\mathbf{E}^2\right)\bullet\left(\mathbf{E}^1\times\mathbf{E}^2\right)}$$

$$Area\left(V_1\mathbf{E}^1, V_2\mathbf{E}^2\right) = V_1V_2\sqrt{\left(\mathbf{E}^1\bullet\mathbf{E}^1\right)\left(\mathbf{E}^2\bullet\mathbf{E}^2\right)-\left(\mathbf{E}^1\bullet\mathbf{E}^2\right)\left(\mathbf{E}^1\bullet\mathbf{E}^2\right)}$$

$$= V_1V_2\sqrt{g^{11}g^{22}-g^{12}g^{21}} = V_1V_2\sqrt{\det\begin{vmatrix} g^{11} & g^{12}\\ g^{21} & g^{22}\end{vmatrix}}$$

Metric g_{mn} or g^{mn} tensor geometric relations to length, area, and volume

where: $g^{12} = \mathbf{E}^1\bullet\mathbf{E}^2 = g^{21}$

3D Covariant Jacobian determinant J-columns are $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$.

$$Volume\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2, V^3\mathbf{E}_3\right) = V^1V^2V^3\left|\mathbf{E}_1 \times \mathbf{E}_2 \bullet \mathbf{E}_3\right| = V^1V^2V^3 \det\begin{vmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \frac{\partial x^1}{\partial q^3} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^2}{\partial q^3} \\ \frac{\partial x^3}{\partial q^1} & \frac{\partial x^3}{\partial q^2} & \frac{\partial x^3}{\partial q^3} \end{vmatrix}$$

Metric g_{mn} or g^{mn} tensor geometric relations to length, area, and volume

3D Covariant Jacobian determinant J-columns are $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$.

$$Volume\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2, V^3\mathbf{E}_3\right) = V^1V^2V^3\left|\mathbf{E}_1 \times \mathbf{E}_2 \bullet \mathbf{E}_3\right| = V^1V^2V^3 \det\begin{vmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \frac{\partial x^1}{\partial q^3} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^2}{\partial q^3} \\ \frac{\partial x^3}{\partial q^1} & \frac{\partial x^3}{\partial q^2} & \frac{\partial x^3}{\partial q^3} \end{vmatrix}$$

Covariant metric matrix is product of J-matrix and its transpose J^T

$$\mathbf{g}_{cov} \equiv \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} = \begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^2}{\partial q^1} & \frac{\partial x^3}{\partial q^1} \\ \frac{\partial x^1}{\partial q^2} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^3}{\partial q^2} \\ \frac{\partial x^1}{\partial q^3} & \frac{\partial x^2}{\partial q^3} & \frac{\partial x^3}{\partial q^3} \end{pmatrix} \bullet \begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \frac{\partial x^1}{\partial q^3} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^2}{\partial q^3} \\ \frac{\partial x^3}{\partial q^1} & \frac{\partial x^3}{\partial q^2} & \frac{\partial x^3}{\partial q^3} \end{pmatrix} = J^T \bullet J$$

Metric g_{mn} or g^{mn} tensor geometric relations to length, area, and volume

3D Covariant Jacobian determinant J-columns are $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$.

$$Volume\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2, V^3\mathbf{E}_3\right) = V^1V^2V^3\left|\mathbf{E}_1 \times \mathbf{E}_2 \bullet \mathbf{E}_3\right| = V^1V^2V^3 \det\begin{vmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \frac{\partial x^1}{\partial q^3} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^2}{\partial q^3} \\ \frac{\partial x^3}{\partial q^1} & \frac{\partial x^3}{\partial q^2} & \frac{\partial x^3}{\partial q^3} \end{vmatrix}$$

Covariant metric matrix is product of J-matrix and its transpose J^T

$$\mathbf{g}_{cov} \equiv \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} = \begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^2}{\partial q^1} & \frac{\partial x^3}{\partial q^1} \\ \frac{\partial x^1}{\partial q^2} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^3}{\partial q^2} \\ \frac{\partial x^1}{\partial q^3} & \frac{\partial x^2}{\partial q^3} & \frac{\partial x^3}{\partial q^3} \end{pmatrix} \bullet \begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \frac{\partial x^1}{\partial q^3} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^2}{\partial q^3} \\ \frac{\partial x^3}{\partial q^1} & \frac{\partial x^3}{\partial q^2} & \frac{\partial x^3}{\partial q^3} \end{pmatrix} = J^T \bullet J$$

Then determinant product ($det|A|\ det|B| = det|A \bullet B|$) and symmetry ($det|A^T| = det|A|$) gives:

$$Volume\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2, V^3\mathbf{E}_3\right) = V^1V^2V^3 \det\left|J\right| = V^1V^2V^3\sqrt{\det\left|\mathbf{g}_{cov}\right|}$$

Metric g_{mn} or g^{mn} tensor geometric relations to length, area, and volume

3D Contravariant Kajobian determinant K-rows are $\mathbf{E}^1$, $\mathbf{E}^2$ and $\mathbf{E}^3$.

$$Volume\left(V_1\mathbf{E}^1, V_2\mathbf{E}^2, V_3\mathbf{E}^3\right) = V_1V_2V_3\left|\mathbf{E}^1 \times \mathbf{E}^2 \bullet \mathbf{E}^3\right| = V_1V_2V_3 \det\begin{vmatrix} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \frac{\partial q^1}{\partial x^3} \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \frac{\partial q^2}{\partial x^3} \\ \frac{\partial q^3}{\partial x^1} & \frac{\partial q^3}{\partial x^2} & \frac{\partial q^3}{\partial x^3} \end{vmatrix}$$

Contravariant metric matrix is product of K-matrix and its transpose K^T

$$\mathbf{g}^{cont} \equiv \begin{pmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{pmatrix} = \begin{pmatrix} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \frac{\partial q^1}{\partial x^3} \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \frac{\partial q^2}{\partial x^3} \\ \frac{\partial q^3}{\partial x^1} & \frac{\partial q^3}{\partial x^2} & \frac{\partial q^3}{\partial x^3} \end{pmatrix} \bullet \begin{pmatrix} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^2}{\partial x^1} & \frac{\partial x^1}{\partial q^3} \\ \frac{\partial q^1}{\partial x^2} & \frac{\partial q^2}{\partial x^2} & \frac{\partial q^3}{\partial x^2} \\ \frac{\partial q^1}{\partial x^3} & \frac{\partial q^2}{\partial x^3} & \frac{\partial q^3}{\partial x^3} \end{pmatrix} = K \bullet K^T$$

Then determinant product ($det|A|\ det|B| = det|A \bullet B|$) and symmetry ($det|A^T| = det|A|$) gives:

$$Volume\left(V_1\mathbf{E}^1, V_2\mathbf{E}^2, V_3\mathbf{E}^3\right) = V_1V_2V_3 \det\left|K\right| = V_1V_2V_3\sqrt{\det\left|\mathbf{g}^{cont}\right|}$$

Metric g_{mn} or g^{mn} tensor geometric relations to length, area, and volume

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.

Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)

Trebuchet Cartesian projectile coordinates are double-valued

Toroidal "rolled-up" ($q^1=\theta$, $q^2=\phi$)-manifold and "Flat" ($x=\theta$, $y=\phi$)-graph

Review of covariant $\mathbf{E}_n$ and contravariant $\mathbf{E}^m$ vectors: Jacobian J vs. Kajobian K

Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)

Tangent $\{\mathbf{E}_n\}$space vs. Normal $\{\mathbf{E}^m\}$space

Covariant vs. contravariant coordinate transformations

Metric g_{mn} tensor geometric relations to length, area, and volume

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)

Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 77)

Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 69)

Riemann equation derivation for trebuchet model

Riemann equation force analysis

2nd-guessing Riemann equation?

Canonical momentum and γ_{mn} tensor

Review of p_θ , p_ϕ vs γ_{mn} from p. 79 of Lect. 14

Standard formulation of $p_m = \frac{\partial T}{\partial \dot{q}^m}$

$$\text{Total } KE = T = T(M) + T(m)$$

$$= \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2 \right)$$

$$= (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2 \right)$$

$$= m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

The γ_{mn} tensor/matrix formulation

$$\text{Total } KE = T = T(M) + T(m)$$

$$= \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

where: γ_{mn} tensor is

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$$

Momentum γ_{mn}-matrix theorem:

(matrix-proof on page 80 of Lect.14)

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial T}{\partial \dot{\theta}} \\ \frac{\partial T}{\partial \dot{\phi}} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \quad \text{if: } \gamma_{\phi,\theta} = \gamma_{\theta,\phi} \text{ (symmetry)}$$

$$= \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Momentum γ_{mn}-tensor theorem: (proof here)

$$p_m = \gamma_{mn}\,\dot{q}^n$$

proof:

$$\text{Given: } p_m = \frac{\partial T}{\partial \dot{q}^m} \quad \text{where: } \quad T = \frac{1}{2}\gamma_{jk}\dot{q}^j\dot{q}^k$$

$$\text{Then: } \quad p_m = \frac{\partial}{\partial \dot{q}^m}\frac{1}{2}\gamma_{jk}\dot{q}^j\dot{q}^k = \frac{1}{2}\gamma_{jk}\frac{\partial \dot{q}^j}{\partial \dot{q}^m}\dot{q}^k + \frac{1}{2}\gamma_{jk}\dot{q}^j\frac{\partial \dot{q}^k}{\partial \dot{q}^m}$$

$$= \frac{1}{2}\gamma_{jk}\delta^j_m\dot{q}^k + \frac{1}{2}\gamma_{jk}\dot{q}^j\delta^k_m = \frac{1}{2}\gamma_{mk}\dot{q}^k + \frac{1}{2}\gamma_{jm}\dot{q}^j$$

$$= \gamma_{mn}\dot{q}^n \quad \text{if: } \gamma_{mn} = \gamma_{nm}$$

QED

Lagrange equation force analysis

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$$

Dot means *total* differentiation

<u>Everything that can move</u> contributes. (<u>Very</u> easy to miss a term!)

$$\dot{p}_\theta = \frac{d}{dt} p_\theta = \frac{d}{dt}\left(\left(MR^2 + mr^2\right)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)\right)$$

[$\dot{M}, \dot{R}, \dot{m}, \dot{r}$, and $\dot{\ell}$ are (thankfully) zero]

p-dot part of Lagrange 2^{nd} equations

$$\dot{p}_\phi = \frac{d}{dt} p_\phi = \frac{d}{dt}\left(m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)\right)$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2\right)$$

$$= (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2\right)$$

$$= m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

From preceding Lagrange 1^{st} equations

Lagrange equation force analysis

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^{\mu}} - \frac{\partial T}{\partial q^{\mu}} = \dot{p}_{\mu} - \frac{\partial T}{\partial q^{\mu}} = F_{\mu}$$

Dot means *total* differentiation

Everything that can move contributes. (Very easy to miss a term!)

$$\dot{p}_{\theta} = \frac{d}{dt} p_{\theta} = \frac{d}{dt}\left(\left(MR^2 + mr^2\right)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)\right) \quad [\dot{M}, \dot{R}, \dot{m}, \dot{r}, \text{ and } \dot{\ell} \text{ are (thankfully) zero}]$$

$$= \left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) + mr\ell\dot{\phi}(\dot{\theta}-\dot{\phi})\sin(\theta-\phi)$$

p-dot part of Lagrange 2nd equations

$$\dot{p}_{\phi} = \frac{d}{dt} p_{\phi} = \frac{d}{dt}\left(m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)\right)$$

$$= m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}(\dot{\theta}-\dot{\phi})\sin(\theta-\phi)$$

$$p_{\theta} = \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2\right)$$

$$= (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

$$p_{\phi} = \frac{\partial T}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2\right)$$

$$= m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

From preceding Lagrange 1st equations

Lagrange equation force analysis

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^{\mu}} - \frac{\partial T}{\partial q^{\mu}} = \dot{p}_{\mu} - \frac{\partial T}{\partial q^{\mu}} = F_{\mu}$$

Dot means *total* differentiation

Everything that can move contributes. (Very easy to miss a term!)

$$\dot{p}_{\theta} = \frac{d}{dt} p_{\theta} = \frac{d}{dt}\left(\left(MR^2 + mr^2\right)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)\right)$$

[$\dot{M}, \dot{R}, \dot{m}, \dot{r}$, and $\dot{\ell}$ are (thankfully) zero]

$$= \left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) + mr\ell\dot{\phi}(\dot{\theta}-\dot{\phi})\sin(\theta-\phi)$$

$$= \boxed{\left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) + mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi) - mr\ell\dot{\phi}^2\sin(\theta-\phi)}$$

p-dot part of Lagrange 2nd equations

$$\dot{p}_{\phi} = \frac{d}{dt} p_{\phi} = \frac{d}{dt}\left(m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)\right)$$

$$= m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}(\dot{\theta}-\dot{\phi})\sin(\theta-\phi)$$

$$= \boxed{m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) - mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi) + mr\ell\dot{\theta}^2\sin(\theta-\phi)}$$

$$p_{\theta} = \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2\right)$$

$$= (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

$$p_{\phi} = \frac{\partial T}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2\right)$$

$$= m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

From preceding Lagrange 1st equations

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.

Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)

Trebuchet Cartesian projectile coordinates are double-valued

Toroidal "rolled-up" (q^1=θ, q^2=ϕ)-manifold and "Flat" (x=θ, y=ϕ)-graph

Review of covariant $\mathbf{E}_n$ and contravariant $\mathbf{E}^m$ vectors: Jacobian J vs. Kajobian K

Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)

Tangent {$\mathbf{E}_n$}space vs. Normal {$\mathbf{E}^m$}space

Covariant vs. contravariant coordinate transformations

Metric g_{mn} tensor geometric relations to length, area, and volume

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)

Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 77)

Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 69)

Riemann equation derivation for trebuchet model

Riemann equation force analysis

2nd-guessing Riemann equation?

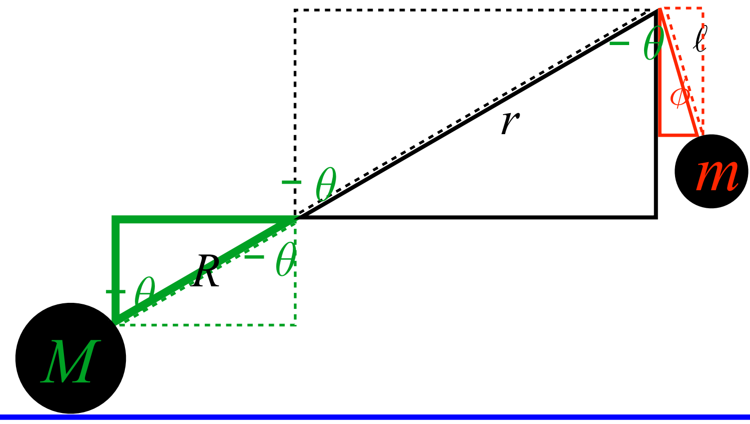
Force, Work, and Acceleration

Review of F_θ, F_ϕ vs F_x, F_y, F_X, F_Y from p. 69 of Lect. 14

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_y\, dy$$
$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

Write work-sums in columns: (Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$\begin{array}{lllll}
dW = F_X\, dX & = M\ddot{X}\,dX & = F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi & = & M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi \\
+F_Y\, dY & +M\ddot{Y}\,dY & +F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi & & +M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi \\
+F_x\, dx & +m\ddot{x}\,dx & +F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi & & +m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi \\
+F_y\, dy & +m\ddot{y}\,dy & +F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi & & +m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi
\end{array}$$



Lagrange trickery:

STEP **D** Add up first and last columns for each variable θ and ϕ for: $T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$

$$\text{Let}: F_X\frac{\partial X}{\partial\theta} + F_Y\frac{\partial Y}{\partial\theta} + F_x\frac{\partial x}{\partial\theta} + F_y\frac{\partial y}{\partial\theta} \equiv F_\theta \quad \text{Defines } F_\theta$$
$$\equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta}$$

$$\text{Let}: F_X\frac{\partial X}{\partial\phi} + F_Y\frac{\partial Y}{\partial\phi} + F_x\frac{\partial x}{\partial\phi} + F_y\frac{\partial y}{\partial\phi} \equiv F_\phi \quad \text{Defines } F_\phi$$
$$\equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi}$$

Completes derivation of Lagrange covariant-force equation for each GCC variable θ and ϕ.

$$F_X R\cos\theta + F_Y R\sin\theta - F_x r\cos\theta - F_y r\sin\theta$$
$$\equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta}$$

Add F_θ gravity given
($F_X = 0$, $F_Y = -Mg$)
($F_x = 0$, $F_y = -mg$)

$$F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta} = -MgR\sin\theta + mgr\sin\theta$$

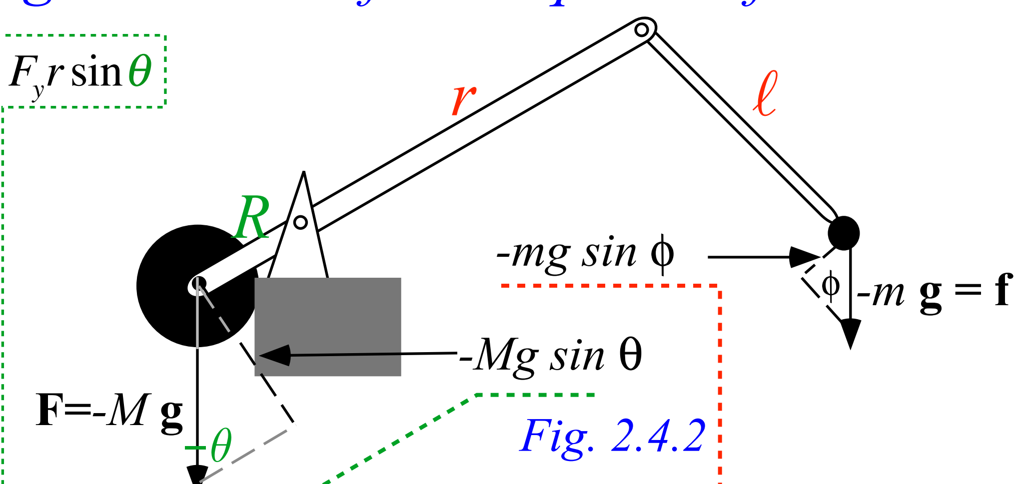


Fig. 2.4.2

Q: Are there ± sign errors here?
A: No. Beam in $-\theta$ position.

$$F_X\cdot 0 + F_Y\cdot 0 + F_x\ell\cos\phi + F_y\ell\sin\phi$$
$$\equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi}$$

Add F_ϕ gravity given
($F_X = 0$, $F_Y = -Mg$)
($F_x = 0$, $F_y = -mg$)

$$F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi} = -mg\ell\sin\phi$$

These are competing torques on main beam R...   *... and a torque on throwing lever ℓ*

Lagrange equation force analysis

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^{\mu}} - \frac{\partial T}{\partial q^{\mu}} = \dot{p}_{\mu} - \frac{\partial T}{\partial q^{\mu}} = F_{\mu}$$

Dot means *total* differentiation

Everything that can move contributes. (Very easy to miss a term!)

$$\dot{p}_{\theta} = \frac{d}{dt} p_{\theta} = \frac{d}{dt}\left(\left(MR^2 + mr^2\right)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)\right)$$

[$\dot{M}, \dot{R}, \dot{m}, \dot{r}$, and $\dot{\ell}$ are (thankfully) zero]

$$= \left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) + mr\ell\dot{\phi}(\dot{\theta}-\dot{\phi})\sin(\theta-\phi)$$

$$= \boxed{\left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) + mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi) - mr\ell\dot{\phi}^2\sin(\theta-\phi)}$$

p-dot part of Lagrange 2nd equations

$$\dot{p}_{\phi} = \frac{d}{dt} p_{\phi} = \frac{d}{dt}\left(m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)\right)$$

$$= m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}(\dot{\theta}-\dot{\phi})\sin(\theta-\phi)$$

$$= \boxed{m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) - mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi) + mr\ell\dot{\theta}^2\sin(\theta-\phi)}$$

Set equal to real (*gravity*) force F_{μ} plus *fictitious force* $\partial T/\partial q^{\mu}$ terms

$$\dot{p}_{\theta} = F_{\theta} + \frac{\partial T}{\partial \theta} = F_{\theta} + \frac{\partial}{\partial \theta}\left(\frac{1}{2}\left(MR^2 + mr^2\right)\dot{\theta}^2 + \frac{1}{2}m\ell^2\dot{\phi}^2 - mr\ell\dot{\theta}\dot{\phi}\cos(\theta-\phi)\right)$$

The rest of Lagrange 2nd equations

$$\dot{p}_{\phi} = F_{\phi} + \frac{\partial T}{\partial \phi} = F_{\phi} + \frac{\partial}{\partial \phi}\left(\frac{1}{2}\left(MR^2 + mr^2\right)\dot{\theta}^2 + \frac{1}{2}m\ell^2\dot{\phi}^2 - mr\ell\dot{\theta}\dot{\phi}\cos(\theta-\phi)\right)$$

Lagrange equation force analysis

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^{\mu}} - \frac{\partial T}{\partial q^{\mu}} = \dot{p}_{\mu} - \frac{\partial T}{\partial q^{\mu}} = F_{\mu}$$

Dot means *total* differentiation

Everything that can move contributes. (Very easy to miss a term!)

$$\dot{p}_{\theta} = \frac{d}{dt}p_{\theta} = \frac{d}{dt}\left(\left(MR^2 + mr^2\right)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)\right)$$

[$\dot{M}, \dot{R}, \dot{m}, \dot{r}$, and $\dot{\ell}$ are (thankfully) zero]

$$= \left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) + mr\ell\dot{\phi}(\dot{\theta}-\dot{\phi})\sin(\theta-\phi)$$

$$= \boxed{\left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) + mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi) - mr\ell\dot{\phi}^2\sin(\theta-\phi)}$$

p-dot part of Lagrange 2nd equations

$$\dot{p}_{\phi} = \frac{d}{dt}p_{\phi} = \frac{d}{dt}\left(m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)\right)$$

$$= m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}(\dot{\theta}-\dot{\phi})\sin(\theta-\phi)$$

$$= \boxed{m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) - mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi) + mr\ell\dot{\theta}^2\sin(\theta-\phi)}$$

Set equal to real (*gravity*) force F_{μ} plus *fictitious force* $\partial T/\partial q^{\mu}$ terms

$$\dot{p}_{\theta} = F_{\theta} + \frac{\partial T}{\partial \theta} = F_{\theta} + \frac{\partial}{\partial \theta}\left(\frac{1}{2}\left(MR^2 + mr^2\right)\dot{\theta}^2 + \frac{1}{2}m\ell^2\dot{\phi}^2 - mr\ell\dot{\theta}\dot{\phi}\cos(\theta-\phi)\right)$$

$$= \boxed{F_{\theta} + mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)}$$

The rest of Lagrange 2nd equations

$$\dot{p}_{\phi} = F_{\phi} + \frac{\partial T}{\partial \phi} = F_{\phi} + \frac{\partial}{\partial \phi}\left(\frac{1}{2}\left(MR^2 + mr^2\right)\dot{\theta}^2 + \frac{1}{2}m\ell^2\dot{\phi}^2 - mr\ell\dot{\theta}\dot{\phi}\cos(\theta-\phi)\right)$$

$$= \boxed{F_{\phi} - mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)}$$

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.

Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)

Trebuchet Cartesian projectile coordinates are double-valued

Toroidal "rolled-up" (q^1=θ, q^2=ϕ)-manifold and "Flat" (x=θ, y=ϕ)-graph

Review of covariant $\mathbf{E}_n$ and contravariant $\mathbf{E}^m$ vectors: Jacobian J vs. Kajobian K

Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)

Tangent $\{\mathbf{E}_n\}$space vs. Normal $\{\mathbf{E}^m\}$space

Covariant vs. contravariant coordinate transformations

Metric g_{mn} tensor geometric relations to length, area, and volume

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)

Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 77)

Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 69)

Riemann equation derivation for trebuchet model

Riemann equation force analysis

2nd-guessing Riemann equation?

Lagrange equation force analysis

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$$

Dot means *total* differentiation

Everything that can move contributes. (Very easy to miss a term!)

$$\dot{p}_\theta = \frac{d}{dt} p_\theta = \frac{d}{dt}\left(\left(MR^2 + mr^2\right)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)\right) \quad [\dot{M}, \dot{R}, \dot{m}, \dot{r}, \text{ and } \dot{\ell} \text{ are (thankfully) zero}]$$

$$= \left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) + mr\ell\dot{\phi}(\dot{\theta}-\dot{\phi})\sin(\theta-\phi)$$

$$= \boxed{\left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) + mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi) - mr\ell\dot{\phi}^2\sin(\theta-\phi)}$$

$$\dot{p}_\phi = \frac{d}{dt} p_\phi = \frac{d}{dt}\left(m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)\right)$$

$$= m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}(\dot{\theta}-\dot{\phi})\sin(\theta-\phi)$$

$$= \boxed{m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) - mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi) + mr\ell\dot{\theta}^2\sin(\theta-\phi)}$$

Set equal to real (*gravity*) force F_μ plus *fictitious force* $\partial T/\partial q^\mu$ terms

$$\dot{p}_\theta = F_\theta + \frac{\partial T}{\partial \theta} = F_\theta + \frac{\partial}{\partial \theta}\left(\frac{1}{2}\left(MR^2 + mr^2\right)\dot{\theta}^2 + \frac{1}{2}m\ell^2\dot{\phi}^2 - mr\ell\dot{\theta}\dot{\phi}\cos(\theta-\phi)\right)$$

$$= \boxed{F_\theta + mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)}$$

$$\dot{p}_\phi = F_\phi + \frac{\partial T}{\partial \phi} = F_\phi + \frac{\partial}{\partial \phi}\left(\frac{1}{2}\left(MR^2 + mr^2\right)\dot{\theta}^2 + \frac{1}{2}m\ell^2\dot{\phi}^2 - mr\ell\dot{\theta}\dot{\phi}\cos(\theta-\phi)\right)$$

$$= \boxed{F_\phi - mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)}$$

gravity forces F_μ from p.69 of Lect. 14 (or p.65 above)

$$F_\theta = -MgR\sin\theta + mgr\sin\theta$$

$$F_\phi = -mg\ell\sin\phi$$

Lagrange equation force analysis

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$$

Dot means *total* differentiation

Everything that can move contributes. (Very easy to miss a term!)

$$\dot{p}_\theta = \frac{d}{dt} p_\theta = \frac{d}{dt}\left(\left(MR^2 + mr^2\right)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)\right)$$

[$\dot{M}, \dot{R}, \dot{m}, \dot{r}$, and $\dot{\ell}$ are (thankfully) zero]

$$= \left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) + mr\ell\dot{\phi}(\dot{\theta}-\dot{\phi})\sin(\theta-\phi)$$

Using: $\dot{p}_\mu = F_\mu + \frac{\partial T}{\partial q^\mu}$

$$= \boxed{\left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) + mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi) - mr\ell\dot{\phi}^2\sin(\theta-\phi)} \quad \boxed{= F_\theta + mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)}$$

$$\dot{p}_\phi = \frac{d}{dt} p_\phi = \frac{d}{dt}\left(m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)\right)$$

$$= m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}(\dot{\theta}-\dot{\phi})\sin(\theta-\phi)$$

$$= \boxed{m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) - mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi) + mr\ell\dot{\theta}^2\sin(\theta-\phi)} \quad \boxed{= F_\phi - mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)}$$

Set equal to real (*gravity*) force F_μ plus *fictitious force* $\partial T/\partial q^\mu$ terms

$$\dot{p}_\theta = F_\theta + \frac{\partial T}{\partial \theta} = F_\theta + \frac{\partial}{\partial \theta}\left(\frac{1}{2}\left(MR^2 + mr^2\right)\dot{\theta}^2 + \frac{1}{2}m\ell^2\dot{\phi}^2 - mr\ell\dot{\theta}\dot{\phi}\cos(\theta-\phi)\right)$$

$$\boxed{= F_\theta + mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)}$$

$$\dot{p}_\phi = F_\phi + \frac{\partial T}{\partial \phi} = F_\phi + \frac{\partial}{\partial \phi}\left(\frac{1}{2}\left(MR^2 + mr^2\right)\dot{\theta}^2 + \frac{1}{2}m\ell^2\dot{\phi}^2 - mr\ell\dot{\theta}\dot{\phi}\cos(\theta-\phi)\right)$$

$$\boxed{= F_\phi - mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)}$$

gravity forces F_μ from p.69 of Lect. 14 (or p.65 above)

$$F_\theta = -MgR\sin\theta + mgr\sin\theta$$

$$F_\phi = -mg\ell\sin\phi$$

Lagrange equation force analysis

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$$

Dot means *total* differentiation

Everything that can move contributes. (Very easy to miss a term!)

$$\dot{p}_\theta = \frac{d}{dt} p_\theta = \frac{d}{dt}\left(\left(MR^2 + mr^2\right)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta - \phi)\right)$$

[$\dot{M}, \dot{R}, \dot{m}, \dot{r}$, and $\dot{\ell}$ are (thankfully) zero]

$$= \left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) + mr\ell\dot{\phi}(\dot{\theta}-\dot{\phi})\sin(\theta-\phi)$$

Using: $\dot{p}_\mu = F_\mu + \frac{\partial T}{\partial q^\mu}$

$$= \boxed{\left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) + \cancel{mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)} - mr\ell\dot{\phi}^2\sin(\theta-\phi)} \quad \boxed{= F_\theta + \cancel{mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)}}$$

$$\dot{p}_\phi = \frac{d}{dt} p_\phi = \frac{d}{dt}\left(m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)\right)$$

$$= m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}(\dot{\theta}-\dot{\phi})\sin(\theta-\phi)$$

$$= \boxed{m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) - \cancel{mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)} + mr\ell\dot{\theta}^2\sin(\theta-\phi)} \quad \boxed{= F_\phi - \cancel{mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)}}$$

Set equal to real (*gravity*) force F_μ plus *fictitious force* $\partial T/\partial q^\mu$ terms

$$\dot{p}_\theta = F_\theta + \frac{\partial T}{\partial \theta} = F_\theta + \frac{\partial}{\partial \theta}\left(\frac{1}{2}\left(MR^2 + mr^2\right)\dot{\theta}^2 + \frac{1}{2}m\ell^2\dot{\phi}^2 - mr\ell\dot{\theta}\dot{\phi}\cos(\theta-\phi)\right)$$

$$\boxed{= F_\theta + mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)}$$

$$\dot{p}_\phi = F_\phi + \frac{\partial T}{\partial \phi} = F_\phi + \frac{\partial}{\partial \phi}\left(\frac{1}{2}\left(MR^2 + mr^2\right)\dot{\theta}^2 + \frac{1}{2}m\ell^2\dot{\phi}^2 - mr\ell\dot{\theta}\dot{\phi}\cos(\theta-\phi)\right)$$

$$\boxed{= F_\phi - mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)}$$

gravity forces F_μ from p.69 of Lect. 14 (or p.65 above)

$$F_\theta = -MgR\sin\theta + mgr\sin\theta$$

$$F_\phi = -mg\ell\sin\phi$$

Lagrange equation force analysis $\quad \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$

Dot means *total* differentiation

Everything that can move contributes. (Very easy to miss a term!)

$$\dot{p}_\theta = \frac{d}{dt} p_\theta = \frac{d}{dt}\left(\left(MR^2 + mr^2\right)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta - \phi)\right) \quad [\dot{M}, \dot{R}, \dot{m}, \dot{r}, \text{ and } \dot{\ell} \text{ are (thankfully) zero}]$$

$$= \left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta - \phi) + mr\ell\dot{\phi}(\dot{\theta} - \dot{\phi})\sin(\theta - \phi)$$

$$= \boxed{\left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta - \phi) - mr\ell\dot{\phi}^2 \sin(\theta - \phi) = F_\theta}$$

$$\dot{p}_\phi = \frac{d}{dt} p_\phi = \frac{d}{dt}\left(m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta - \phi)\right)$$

$$= m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta - \phi) + mr\ell\dot{\theta}(\dot{\theta} - \dot{\phi})\sin(\theta - \phi)$$

$$= \boxed{m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta - \phi) + mr\ell\dot{\theta}^2 \sin(\theta - \phi) = F_\phi}$$

Set equal to real (*gravity*) force F_μ plus *fictitious force* $\partial T/\partial q^\mu$ terms

$$\dot{p}_\theta = F_\theta + \frac{\partial T}{\partial \theta} = F_\theta + \frac{\partial}{\partial \theta}\left(\frac{1}{2}\left(MR^2 + mr^2\right)\dot{\theta}^2 + \frac{1}{2}m\ell^2\dot{\phi}^2 - mr\ell\dot{\theta}\dot{\phi}\cos(\theta - \phi)\right)$$

$$= \boxed{F_\theta + mr\ell\dot{\theta}\dot{\phi}\sin(\theta - \phi)}$$

$$\dot{p}_\phi = F_\phi + \frac{\partial T}{\partial \phi} = F_\phi + \frac{\partial}{\partial \phi}\left(\frac{1}{2}\left(MR^2 + mr^2\right)\dot{\theta}^2 + \frac{1}{2}m\ell^2\dot{\phi}^2 - mr\ell\dot{\theta}\dot{\phi}\cos(\theta - \phi)\right)$$

$$= \boxed{F_\phi - mr\ell\dot{\theta}\dot{\phi}\sin(\theta - \phi)}$$

gravity forces F_μ from p.69 of Lect. 14 (or p.65 above)

$$F_\theta = -MgR\sin\theta + mgr\sin\theta$$

$$F_\phi = -mg\ell\sin\phi$$

Lagrange equation force analysis

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$$

Dot means *total* differentiation

Everything that can move contributes. (Very easy to miss a term!)

$$\dot{p}_\theta = \frac{d}{dt}p_\theta = \frac{d}{dt}\left(\left(MR^2+mr^2\right)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)\right)$$

[$\dot{M}, \dot{R}, \dot{m}, \dot{r}$, and $\dot{\ell}$ are (thankfully) zero]

$$= \left(MR^2+mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) + mr\ell\dot{\phi}(\dot{\theta}-\dot{\phi})\sin(\theta-\phi)$$

$$= \boxed{\left(MR^2+mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) - mr\ell\dot{\phi}^2\sin(\theta-\phi)} \quad \boxed{= F_\theta = -MgR\sin\theta + mgr\sin\theta}$$

$$\dot{p}_\phi = \frac{d}{dt}p_\phi = \frac{d}{dt}\left(m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)\right)$$

$$= m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}(\dot{\theta}-\dot{\phi})\sin(\theta-\phi)$$

$$= \boxed{m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}^2\sin(\theta-\phi)} \quad \boxed{= F_\phi = -mg\ell\sin\phi}$$

Set equal to real (*gravity*) force F_μ plus *fictitious force* $\partial T/\partial q^\mu$ terms

$$\dot{p}_\theta = F_\theta + \frac{\partial T}{\partial \theta} = F_\theta + \frac{\partial}{\partial \theta}\left(\frac{1}{2}\left(MR^2+mr^2\right)\dot{\theta}^2 + \frac{1}{2}m\ell^2\dot{\phi}^2 - mr\ell\dot{\theta}\dot{\phi}\cos(\theta-\phi)\right)$$

$$= \boxed{F_\theta + mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)}$$

$$\dot{p}_\phi = F_\phi + \frac{\partial T}{\partial \phi} = F_\phi + \frac{\partial}{\partial \phi}\left(\frac{1}{2}\left(MR^2+mr^2\right)\dot{\theta}^2 + \frac{1}{2}m\ell^2\dot{\phi}^2 - mr\ell\dot{\theta}\dot{\phi}\cos(\theta-\phi)\right)$$

$$= \boxed{F_\phi - mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)}$$

gravity forces F_μ from p.69 of Lect. 14 (or p.65 above)

$$F_\theta = -MgR\sin\theta + mgr\sin\theta$$

$$F_\phi = -mg\ell\sin\phi$$

Lagrange equation force analysis

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^{\mu}} - \frac{\partial T}{\partial q^{\mu}} = \dot{p}_{\mu} - \frac{\partial T}{\partial q^{\mu}} = F_{\mu}$$

$$\dot{p}_{\theta} = \boxed{\left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta - \phi) - mr\ell\dot{\phi}^2\sin(\theta - \phi)} = \boxed{F_{\theta} = -MgR\sin\theta + mgr\sin\theta}$$

$$\dot{p}_{\phi} = \boxed{m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta - \phi) + mr\ell\dot{\theta}^2\sin(\theta - \phi)} = \boxed{F_{\phi} = -mg\ell\sin\phi}$$

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.
Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)
Trebuchet Cartesian projectile coordinates are double-valued
Toroidal "rolled-up" ($q^1=\theta$, $q^2=\phi$)-manifold and "Flat" ($x=\theta$, $y=\phi$)-graph
Review of covariant $\mathbf{E}_n$ and contravariant $\mathbf{E}^m$ vectors: Jacobian J vs. Kajobian K
Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)
Tangent $\{\mathbf{E}_n\}$space vs. Normal $\{\mathbf{E}^m\}$space
Covariant vs. contravariant coordinate transformations
Metric g_{mn} tensor geometric relations to length, area, and volume

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)
Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 77)
Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 69)
→ Riemann equation derivation for trebuchet model
Riemann equation force analysis
2nd-guessing Riemann equation?

Riemann equation force analysis

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^{\mu}} - \frac{\partial T}{\partial q^{\mu}} = \dot{p}_{\mu} - \frac{\partial T}{\partial q^{\mu}} = F_{\mu}$$

Riemann equation force analysis solves for GCC accelerations $\ddot{\theta}$ *and* $\ddot{\phi}$

$$\dot{p}_{\theta} = \boxed{\left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta - \phi) - mr\ell\dot{\phi}^2\sin(\theta - \phi)} = F_{\theta} = -MgR\sin\theta + mgr\sin\theta$$

$$\dot{p}_{\phi} = \boxed{m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta - \phi) + mr\ell\dot{\theta}^2\sin(\theta - \phi)} = F_{\phi} = -mg\ell\sin\phi$$

Riemann equation force analysis

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^{\mu}}-\frac{\partial T}{\partial q^{\mu}}=\dot{p}_{\mu}-\frac{\partial T}{\partial q^{\mu}}=F_{\mu}$$

Riemann equation force analysis solves for GCC accelerations $\ddot{\theta}$ *and* $\ddot{\phi}$

$$\dot{p}_{\theta}=\boxed{\left(MR^2+mr^2\right)\ddot{\theta}-mr\ell\ddot{\phi}\cos(\theta-\phi)-mr\ell\dot{\phi}^2\sin(\theta-\phi)=F_{\theta}=-MgR\sin\theta+mgr\sin\theta}$$

$$\dot{p}_{\phi}=\boxed{m\ell^2\ddot{\phi}-mr\ell\ddot{\theta}\cos(\theta-\phi)+mr\ell\dot{\theta}^2\sin(\theta-\phi)=F_{\phi}=-mg\ell\sin\phi}$$

In matrix form:

$$\begin{pmatrix}\dot{p}_{\theta}\\ \dot{p}_{\phi}\end{pmatrix}=\begin{pmatrix}\left(MR^2+mr^2\right) & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}\begin{pmatrix}\ddot{\theta}\\ \ddot{\phi}\end{pmatrix}-\begin{pmatrix}mr\ell\dot{\phi}^2\sin(\theta-\phi)\\ -mr\ell\dot{\theta}^2\sin(\theta-\phi)\end{pmatrix}=\begin{pmatrix}F_{\theta}\\ F_{\phi}\end{pmatrix}$$

Riemann equation force analysis

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^{\mu}} - \frac{\partial T}{\partial q^{\mu}} = \dot{p}_{\mu} - \frac{\partial T}{\partial q^{\mu}} = F_{\mu}$$

Riemann equation force analysis solves for GCC accelerations $\ddot{\theta}$ *and* $\ddot{\phi}$

$$\dot{p}_{\theta} = \boxed{\left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) - mr\ell\dot{\phi}^2\sin(\theta-\phi)} = F_{\theta} = -MgR\sin\theta + mgr\sin\theta$$

$$\dot{p}_{\phi} = \boxed{m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}^2\sin(\theta-\phi)} = F_{\phi} = -mg\ell\sin\phi$$

In matrix form:

$$\begin{pmatrix} \dot{p}_{\theta} \\ \dot{p}_{\phi} \end{pmatrix} = \begin{pmatrix} \left(MR^2 + mr^2\right) & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} - \begin{pmatrix} mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ -mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix} = \begin{pmatrix} F_{\theta} \\ F_{\phi} \end{pmatrix}$$

This uses the γ_{mn} *tensor :*

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$$

$$= \begin{pmatrix} -MgR\sin\theta + mgr\sin\theta \\ -mg\ell\sin\phi \end{pmatrix}$$

Riemann equation force analysis

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$$

Riemann equation force analysis solves for GCC accelerations $\ddot{\theta}$ *and* $\ddot{\phi}$

$$\dot{p}_\theta = \boxed{\left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) - mr\ell\dot{\phi}^2\sin(\theta-\phi)} = F_\theta = -MgR\sin\theta + mgr\sin\theta$$

$$\dot{p}_\phi = \boxed{m\ell^2\ddot{\phi} \quad - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}^2\sin(\theta-\phi)} = F_\phi = -mg\ell\sin\phi$$

In matrix form:

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \left(MR^2 + mr^2\right) & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} - \begin{pmatrix} mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ -mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix} = \begin{pmatrix} F_\theta \\ F_\phi \end{pmatrix} = \begin{pmatrix} -MgR\sin\theta + mgr\sin\theta \\ -mg\ell\sin\phi \end{pmatrix}$$

This uses the γ_{mn} tensor :

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$$

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} F_\theta + mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ F_\phi - mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix}$$

Need to invert the γ_{mn}-matrix... Let's consolidate ...

Riemann equation force analysis

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^{\mu}} - \frac{\partial T}{\partial q^{\mu}} = \dot{p}_{\mu} - \frac{\partial T}{\partial q^{\mu}} = F_{\mu}$$

$$\dot{p}_{\theta} = \boxed{\left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) - mr\ell\dot{\phi}^2\sin(\theta-\phi) = F_{\theta} = -MgR\sin\theta + mgr\sin\theta}$$

$$\dot{p}_{\phi} = \boxed{m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}^2\sin(\theta-\phi) = F_{\phi} = -mg\ell\sin\phi}$$

In matrix form:

$$\begin{pmatrix} \dot{p}_{\theta} \\ \dot{p}_{\phi} \end{pmatrix} = \begin{pmatrix} \left(MR^2 + mr^2\right) & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} - \begin{pmatrix} mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ -mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix} = \begin{pmatrix} F_{\theta} \\ F_{\phi} \end{pmatrix}$$

$$= \begin{pmatrix} -MgR\sin\theta + mgr\sin\theta \\ -mg\ell\sin\phi \end{pmatrix}$$

This uses the γ_{mn} tensor :

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$$

$$\begin{pmatrix} \dot{p}_{\theta} \\ \dot{p}_{\phi} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} F_{\theta} + mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ F_{\phi} - mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix}$$

Need to invert the γ_{mn}-matrix...

Riemann equation force analysis $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$

$$\dot{p}_\theta = \boxed{\left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) - mr\ell\dot{\phi}^2\sin(\theta-\phi)} = F_\theta = -MgR\sin\theta + mgr\sin\theta$$

$$\dot{p}_\phi = \boxed{m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}^2\sin(\theta-\phi)} = F_\phi = -mg\ell\sin\phi$$

In matrix form:

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \left(MR^2 + mr^2\right) & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} - \begin{pmatrix} mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ -mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix} = \begin{pmatrix} F_\theta \\ F_\phi \end{pmatrix}$$

This uses the γ_{mn} tensor : $\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$

$$= \begin{pmatrix} -MgR\sin\theta + mgr\sin\theta \\ -mg\ell\sin\phi \end{pmatrix}$$

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} F_\theta + mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ F_\phi - mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix}$$

Need to invert the γ_{mn}-matrix...

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} = \frac{\begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2 \end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi)\right]}$$

"Super-Inertia" I_S

Riemann equation force analysis $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$ becomes $\gamma^{\mu\nu}\dot{p}_\mu = \ddot{q}^\nu$ ...

$$\dot{p}_\theta = \boxed{\left(MR^2+mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) - mr\ell\dot{\phi}^2\sin(\theta-\phi) = F_\theta = -MgR\sin\theta + mgr\sin\theta}$$

$$\dot{p}_\phi = \boxed{m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}^2\sin(\theta-\phi) = F_\phi = -mg\ell\sin\phi}$$

In matrix form:

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \left(MR^2+mr^2\right) & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} - \begin{pmatrix} mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ -mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix} = \begin{pmatrix} F_\theta \\ F_\phi \end{pmatrix} = \begin{pmatrix} -MgR\sin\theta + mgr\sin\theta \\ -mg\ell\sin\phi \end{pmatrix}$$

This uses the γ_{mn} tensor : $\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2+mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} F_\theta + mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ F_\phi - mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix}$$

Need to invert the γ_{mn}-matrix...

... and apply it...

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} = \frac{\begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2+mr^2 \end{pmatrix}}{m\ell^2\left[MR^2+mr^2\sin^2(\theta-\phi)\right]}$$

“Super-Inertia” I_S

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \boxed{\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} F_\theta + mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ F_\phi - mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix}}$$

Riemann equation form

Riemann equation force analysis $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$ becomes $\gamma^{\mu\nu}\dot{p}_\mu = \ddot{q}^\nu \ldots$

$$\dot{p}_\theta = \boxed{\left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) - mr\ell\dot{\phi}^2\sin(\theta-\phi) = F_\theta = -MgR\sin\theta + mgr\sin\theta}$$

$$\dot{p}_\phi = \boxed{m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}^2\sin(\theta-\phi) = F_\phi = -mg\ell\sin\phi}$$

In matrix form:

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \left(MR^2+mr^2\right) & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} - \begin{pmatrix} mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ -mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix} = \begin{pmatrix} F_\theta \\ F_\phi \end{pmatrix}$$

$$= \begin{pmatrix} -MgR\sin\theta + mgr\sin\theta \\ -mg\ell\sin\phi \end{pmatrix}$$

This uses the γ_{mn} tensor : $\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2+mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} F_\theta + mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ F_\phi - mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix}$$

Need to invert the γ_{mn}-matrix...

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} = \frac{\begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2+mr^2 \end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi)\right]}$$

"Super-Inertia" I_S

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1}\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \boxed{\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1}\begin{pmatrix} F_\theta + mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ F_\phi - mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix}}$$

Riemann equation form

Gravity-free case:

$F_\theta = 0 = F_\phi$

$$I_S\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = I_S\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1}\begin{pmatrix} \dot{\phi}^2 \\ -\dot{\theta}^2 \end{pmatrix} mr\ell\sin(\theta-\phi)$$

First trebuchet (~3000BCE in China) was Gravity-free...
.. powered by many Chinese warriors!

Riemann equation force analysis $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu}-\frac{\partial T}{\partial q^\mu}=\dot{p}_\mu-\frac{\partial T}{\partial q^\mu}=F_\mu$ becomes $\gamma^{\mu\nu}\dot{p}_\mu=\ddot{q}^\nu$ ...

$$\dot{p}_\theta=\boxed{\left(MR^2+mr^2\right)\ddot{\theta}-mr\ell\ddot{\phi}\cos(\theta-\phi)-mr\ell\dot{\phi}^2\sin(\theta-\phi)}=F_\theta=\boxed{-MgR\sin\theta+mgr\sin\theta}$$

$$\dot{p}_\phi=\boxed{m\ell^2\ddot{\phi}\quad-mr\ell\ddot{\theta}\cos(\theta-\phi)+mr\ell\dot{\theta}^2\sin(\theta-\phi)}=F_\phi=\boxed{-mg\ell\sin\phi}$$

In matrix form:

$$\begin{pmatrix}\dot{p}_\theta\\ \dot{p}_\phi\end{pmatrix}=\begin{pmatrix}\left(MR^2+mr^2\right) & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}\begin{pmatrix}\ddot{\theta}\\ \ddot{\phi}\end{pmatrix}-\begin{pmatrix}mr\ell\dot{\phi}^2\sin(\theta-\phi)\\ -mr\ell\dot{\theta}^2\sin(\theta-\phi)\end{pmatrix}=\begin{pmatrix}F_\theta\\ F_\phi\end{pmatrix}$$

This uses the γ_{mn} tensor : $\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}=\begin{pmatrix}MR^2+mr^2 & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}$

$$=\begin{pmatrix}-MgR\sin\theta+mgr\sin\theta\\ -mg\ell\sin\phi\end{pmatrix}$$

$$\begin{pmatrix}\dot{p}_\theta\\ \dot{p}_\phi\end{pmatrix}=\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}\begin{pmatrix}\ddot{\theta}\\ \ddot{\phi}\end{pmatrix}=\begin{pmatrix}F_\theta+mr\ell\dot{\phi}^2\sin(\theta-\phi)\\ F_\phi-mr\ell\dot{\theta}^2\sin(\theta-\phi)\end{pmatrix}$$

Need to invert the γ_{mn}-matrix...

$$I_s=m\ell^2\left[MR^2+mr^2\sin^2(\theta-\phi)\right]$$

$$\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}^{-1}=\frac{\begin{pmatrix}m\ell^2 & mr\ell\cos(\theta-\phi)\\ mr\ell\cos(\theta-\phi) & MR^2+mr^2\end{pmatrix}}{m\ell^2\left[MR^2+mr^2\sin^2(\theta-\phi)\right]}\longleftarrow \text{“Super-Inertia” } I_S$$

$$\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}^{-1}\begin{pmatrix}\dot{p}_\theta\\ \dot{p}_\phi\end{pmatrix}=\boxed{\begin{pmatrix}\ddot{\theta}\\ \ddot{\phi}\end{pmatrix}=\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}^{-1}\begin{pmatrix}F_\theta+mr\ell\dot{\phi}^2\sin(\theta-\phi)\\ F_\phi-mr\ell\dot{\theta}^2\sin(\theta-\phi)\end{pmatrix}\quad\text{Riemann equation form}}$$

Gravity-free case:

$F_\theta=0=F_\phi$

$$I_S\begin{pmatrix}\ddot{\theta}\\ \ddot{\phi}\end{pmatrix}=I_S\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}^{-1}\begin{pmatrix}\dot{\phi}^2\\ -\dot{\theta}^2\end{pmatrix}mr\ell\sin(\theta-\phi)=\begin{pmatrix}m\ell^2 & mr\ell\cos(\theta-\phi)\\ mr\ell\cos(\theta-\phi) & MR^2+mr^2\end{pmatrix}\begin{pmatrix}\dot{\phi}^2\\ -\dot{\theta}^2\end{pmatrix}mr\ell\sin(\theta-\phi)$$

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.
Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)
Trebuchet Cartesian projectile coordinates are double-valued
Toroidal "rolled-up" (q^1=θ, q^2=ϕ)-manifold and "Flat" (x=θ, y=ϕ)-graph
Review of covariant $\mathbf{E}_n$ and contravariant $\mathbf{E}^m$ vectors: Jacobian J vs. Kajobian K
Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)
Tangent {$\mathbf{E}_n$}space vs. Normal {$\mathbf{E}^m$}space
Covariant vs. contravariant coordinate transformations
Metric g_{mn} tensor geometric relations to length, area, and volume

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)
Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 77)
Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 69)
Riemann equation derivation for trebuchet model
→ *Riemann equation force analysis*
2nd-guessing Riemann equation?

Riemann equation force analysis $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu}-\frac{\partial T}{\partial q^\mu}=\dot{p}_\mu-\frac{\partial T}{\partial q^\mu}=F_\mu$ becomes $\gamma^{\mu\nu}\dot{p}_\mu=\ddot{q}^\nu \ldots$

$$\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}^{-1}\begin{pmatrix}\dot{p}_\theta\\ \dot{p}_\phi\end{pmatrix}=\boxed{\begin{pmatrix}\ddot{\theta}\\ \ddot{\phi}\end{pmatrix}=\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}^{-1}\begin{pmatrix}F_\theta+mr\ell\dot{\phi}^2\sin(\theta-\phi)\\ F_\phi-mr\ell\dot{\theta}^2\sin(\theta-\phi)\end{pmatrix}\quad \text{Riemann equation form}}$$

Gravity-free case:

$F_\theta=0=F_\phi$

$$I_s=m\ell^2\left[MR^2+mr^2\sin^2(\theta-\phi)\right]$$

$$I_S\begin{pmatrix}\ddot{\theta}\\ \ddot{\phi}\end{pmatrix}=I_S\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}^{-1}\begin{pmatrix}\dot{\phi}^2\\ -\dot{\theta}^2\end{pmatrix}mr\ell\sin(\theta-\phi)=\begin{pmatrix}m\ell^2 & mr\ell\cos(\theta-\phi)\\ mr\ell\cos(\theta-\phi) & MR^2+mr^2\end{pmatrix}\begin{pmatrix}\dot{\phi}^2\\ -\dot{\theta}^2\end{pmatrix}mr\ell\sin(\theta-\phi)$$

$Let:(\theta-\phi)=-\frac{\pi}{2}$ so: $I_S=m\ell^2\left[MR^2+mr^2\right]$ and let: $\omega\equiv\dot{\theta}=\dot{\phi}$

$$I_S\begin{pmatrix}\ddot{\theta}\\ \ddot{\phi}\end{pmatrix}=I_S\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}^{-1}\begin{pmatrix}-\dot{\phi}^2\\ \dot{\theta}^2\end{pmatrix}mr\ell=\begin{pmatrix}m\ell^2 & 0\\ 0 & MR^2+mr^2\end{pmatrix}\begin{pmatrix}-\omega^2\\ \omega^2\end{pmatrix}mr\ell$$

$$\begin{pmatrix}\ddot{\theta}\\ \ddot{\phi}\end{pmatrix}=\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}^{-1}\begin{pmatrix}-\dot{\phi}^2\\ \dot{\theta}^2\end{pmatrix}mr\ell=\frac{\begin{pmatrix}m\ell^2 & 0\\ 0 & MR^2+mr^2\end{pmatrix}}{m\ell^2\left[MR^2+mr^2\right]}\begin{pmatrix}-mr\ell\omega^2\\ mr\ell\omega^2\end{pmatrix}=\begin{pmatrix}\frac{-mr\ell\omega^2}{MR^2+mr^2}\\ \omega^2 r/\ell\end{pmatrix}$$

Trying to 2nd-guess Riemann results

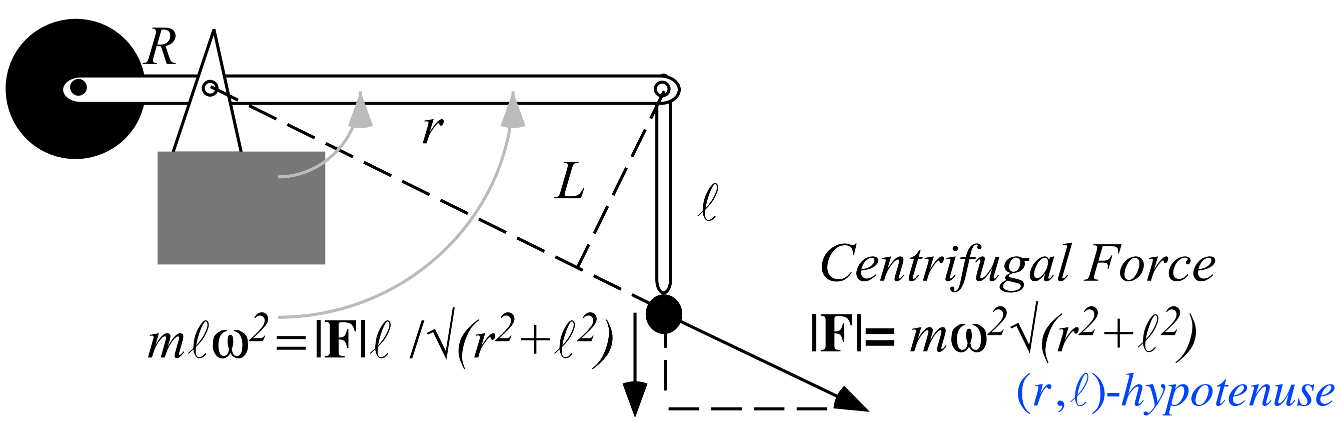


Fig. 2.5.1 Centrifugal force for a particular state of motion ($\omega\equiv\dot{\theta}=\dot{\phi},\ \theta=\frac{-\pi}{2},\ \phi=0$)

Riemann equation force analysis $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$ becomes $\gamma^{\mu\nu}\dot{p}_\mu = \ddot{q}^\nu \ldots$

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} F_\theta + mr\ell\dot{\phi}^2 \sin(\theta-\phi) \\ F_\phi - mr\ell\dot{\theta}^2 \sin(\theta-\phi) \end{pmatrix}$$

Riemann equation form

Gravity-free case:

$F_\theta = 0 = F_\phi$

$$I_s = m\ell^2\left[MR^2 + mr^2 \sin^2(\theta-\phi)\right]$$

$$I_S \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = I_S \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} \dot{\phi}^2 \\ -\dot{\theta}^2 \end{pmatrix} mr\ell\sin(\theta-\phi) = \begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2+mr^2 \end{pmatrix} \begin{pmatrix} \dot{\phi}^2 \\ -\dot{\theta}^2 \end{pmatrix} mr\ell\sin(\theta-\phi)$$

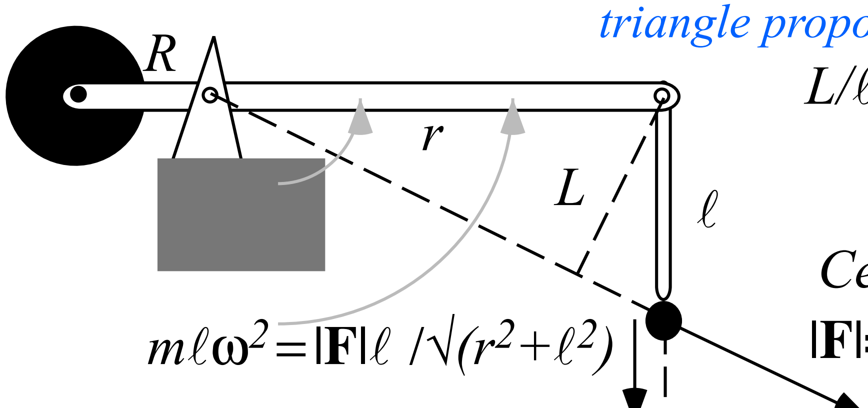
$Let: (\theta-\phi) = -\frac{\pi}{2}$ so: $I_S = m\ell^2\left[MR^2+mr^2\right]$ and let: $\omega \equiv \dot{\theta} = \dot{\phi}$

$$I_S \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = I_S \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} -\dot{\phi}^2 \\ \dot{\theta}^2 \end{pmatrix} mr\ell = \begin{pmatrix} m\ell^2 & 0 \\ 0 & MR^2+mr^2 \end{pmatrix} \begin{pmatrix} -\omega^2 \\ \omega^2 \end{pmatrix} mr\ell$$

$$\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} -\dot{\phi}^2 \\ \dot{\theta}^2 \end{pmatrix} mr\ell = \frac{\begin{pmatrix} m\ell^2 & 0 \\ 0 & MR^2+mr^2 \end{pmatrix}}{m\ell^2\left[MR^2+mr^2\right]} \begin{pmatrix} -mr\ell\omega^2 \\ mr\ell\omega^2 \end{pmatrix} = \begin{pmatrix} \frac{-mr\ell\omega^2}{MR^2+mr^2} \\ \omega^2 r/\ell \end{pmatrix}$$

Trying to 2nd-guess Riemann results

The ϕ–torque on mass m on leg ℓ due to centrifugal force is force times *moment* arm $L = r\cdot\ell/\sqrt{(r^2+\ell^2)}$.



triangle proportionality:

$L/\ell = r/\sqrt{(r^2+\ell^2)}$

Centrifugal Force

$|\mathbf{F}| = m\omega^2\sqrt{(r^2+\ell^2)}$

(r,ℓ)-hypotenuse

$m\ell\omega^2 = |\mathbf{F}|\ell/\sqrt{(r^2+\ell^2)}$

Fig. 2.5.1 Centrifugal force for a particular state of motion ($\omega \equiv \dot{\theta} = \dot{\phi},\ \theta = \frac{-\pi}{2},\ \phi = 0$)

Riemann equation force analysis $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$ *becomes* $\gamma^{\mu\nu}\dot{p}_\mu = \ddot{q}^\nu ...$

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \boxed{\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} F_\theta + mr\ell\dot{\phi}^2 \sin(\theta-\phi) \\ F_\phi - mr\ell\dot{\theta}^2 \sin(\theta-\phi) \end{pmatrix} \quad \text{Riemann equation form}}$$

Gravity-free case:

$F_\theta = 0 = F_\phi$ $\qquad I_s = m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi)\right]$

$$I_S\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = I_S\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} \dot{\phi}^2 \\ -\dot{\theta}^2 \end{pmatrix} mr\ell\sin(\theta-\phi) = \begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2 \end{pmatrix}\begin{pmatrix} \dot{\phi}^2 \\ -\dot{\theta}^2 \end{pmatrix} mr\ell\sin(\theta-\phi)$$

$Let: (\theta-\phi) = -\frac{\pi}{2}$ so: $I_S = m\ell^2\left[MR^2 + mr^2\right]$ and let: $\omega \equiv \dot{\theta} = \dot{\phi}$

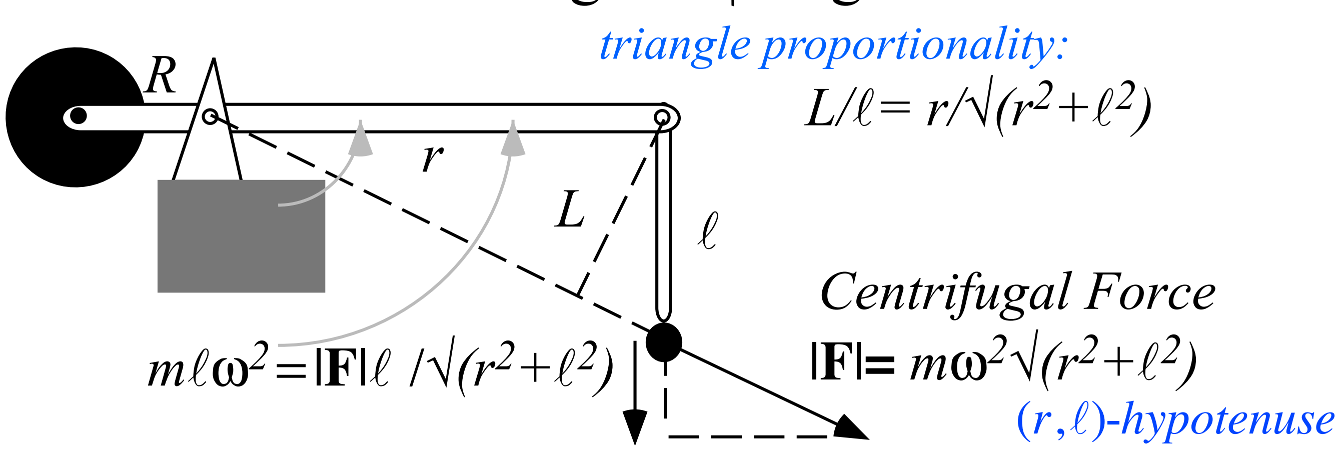
$$I_S\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = I_S\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} -\dot{\phi}^2 \\ \dot{\theta}^2 \end{pmatrix} mr\ell = \begin{pmatrix} m\ell^2 & 0 \\ 0 & MR^2 + mr^2 \end{pmatrix}\begin{pmatrix} -\omega^2 \\ \omega^2 \end{pmatrix} mr\ell$$

$$\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} -\dot{\phi}^2 \\ \dot{\theta}^2 \end{pmatrix} mr\ell = \frac{\begin{pmatrix} m\ell^2 & 0 \\ 0 & MR^2 + mr^2 \end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\right]}\begin{pmatrix} -mr\ell\omega^2 \\ mr\ell\omega^2 \end{pmatrix} = \begin{pmatrix} \frac{-mr\ell\omega^2}{MR^2 + mr^2} \\ \omega^2 r/\ell \end{pmatrix}$$

Trying to 2nd-guess Riemann results (Gravity-free case)

The ϕ–torque on mass m on leg ℓ due to centrifugal force is force times *moment* arm $L = r\cdot\ell/\sqrt{(r^2+\ell^2)}$.

This is the rate of change of ϕ-angular momentum around the pivot at the top of ℓ.



$$m\ell^2\ddot{\phi} = FL = m\omega^2\sqrt{r^2+\ell^2}\,\frac{r\ell}{\sqrt{r^2+\ell^2}} = m\omega^2 r\ell$$

or: $\ddot{\phi} = FL/m\ell^2 = \omega^2 r/\ell$

Fig. 2.5.1 Centrifugal force for a particular state of motion ($\omega \equiv \dot{\theta} = \dot{\phi},\ \theta = \frac{-\pi}{2},\ \phi = 0$)

Move to top of page...

Riemann equation force analysis $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^{\mu}} - \frac{\partial T}{\partial q^{\mu}} = \dot{p}_{\mu} - \frac{\partial T}{\partial q^{\mu}} = F_{\mu}$ *becomes* $\gamma^{\mu\nu}\dot{p}_{\mu} = \ddot{q}^{\nu}$ ...

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} \dot{p}_{\theta} \\ \dot{p}_{\phi} \end{pmatrix} = \boxed{\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} F_{\theta} + mr\ell\dot{\phi}^{2}\sin(\theta-\phi) \\ F_{\phi} - mr\ell\dot{\theta}^{2}\sin(\theta-\phi) \end{pmatrix} \quad \textit{Riemann equation form}}$$

Gravity-free case:

$$I_{s} = m\ell^{2}\left[MR^{2} + mr^{2}\sin^{2}(\theta-\phi)\right]$$

$F_{\theta}=0=F_{\phi}$

$$I_{S}\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = I_{S}\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1}\begin{pmatrix} \dot{\phi}^{2} \\ -\dot{\theta}^{2} \end{pmatrix} mr\ell\sin(\theta-\phi) = \begin{pmatrix} m\ell^{2} & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^{2}+mr^{2} \end{pmatrix}\begin{pmatrix} \dot{\phi}^{2} \\ -\dot{\theta}^{2} \end{pmatrix} mr\ell\sin(\theta-\phi)$$

Let $:(\theta-\phi) = -\frac{\pi}{2}$ so: $I_{S} = m\ell^{2}\left[MR^{2}+mr^{2}\right]$ and let: $\omega \equiv \dot{\theta} = \dot{\phi}$

$$I_{S}\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = I_{S}\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1}\begin{pmatrix} -\dot{\phi}^{2} \\ \dot{\theta}^{2} \end{pmatrix} mr\ell = \begin{pmatrix} m\ell^{2} & 0 \\ 0 & MR^{2}+mr^{2} \end{pmatrix}\begin{pmatrix} -\omega^{2} \\ \omega^{2} \end{pmatrix} mr\ell$$

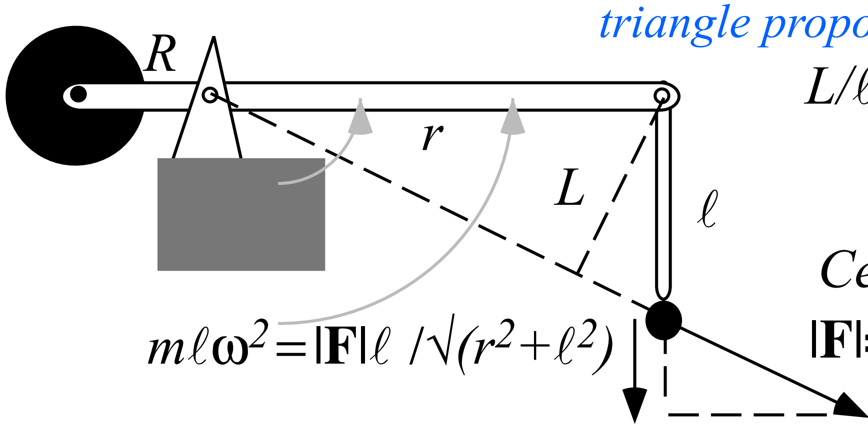
$$\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1}\begin{pmatrix} -\dot{\phi}^{2} \\ \dot{\theta}^{2} \end{pmatrix} mr\ell = \frac{\begin{pmatrix} m\ell^{2} & 0 \\ 0 & MR^{2}+mr^{2} \end{pmatrix}}{m\ell^{2}\left[MR^{2}+mr^{2}\right]}\begin{pmatrix} -mr\ell\omega^{2} \\ mr\ell\omega^{2} \end{pmatrix} = \begin{pmatrix} \frac{-mr\ell\omega^{2}}{MR^{2}+mr^{2}} \\ \omega^{2}r/\ell \end{pmatrix}$$

Trying to 2nd-guess Riemann results (Gravity-free case)

Move to top of page...

The ϕ–torque on mass m on leg ℓ due to centrifugal force is force times *moment* arm $L=r\cdot\ell/\sqrt{(r^2+\ell^2)}$.

This is the rate of change of ϕ-angular momentum around the pivot at the top of ℓ.



triangle proportionality:

$L/\ell = r/\sqrt{(r^2+\ell^2)}$

Centrifugal Force

$|\mathbf{F}| = m\omega^2\sqrt{(r^2+\ell^2)}$

(r,ℓ)-hypotenuse

$$m\ell^{2}\ddot{\phi} = FL = m\omega^{2}\sqrt{r^{2}+\ell^{2}}\,\frac{r\ell}{\sqrt{r^{2}+\ell^{2}}} = m\omega^{2}r\ell$$

or: $\ddot{\phi} = FL/m\ell^{2} = \omega^{2}r/\ell$

Fig. 2.5.1 Centrifugal force for a particular state of motion ($\omega \equiv \dot{\theta} = \dot{\phi},\ \theta = \frac{-\pi}{2},\ \phi = 0$)

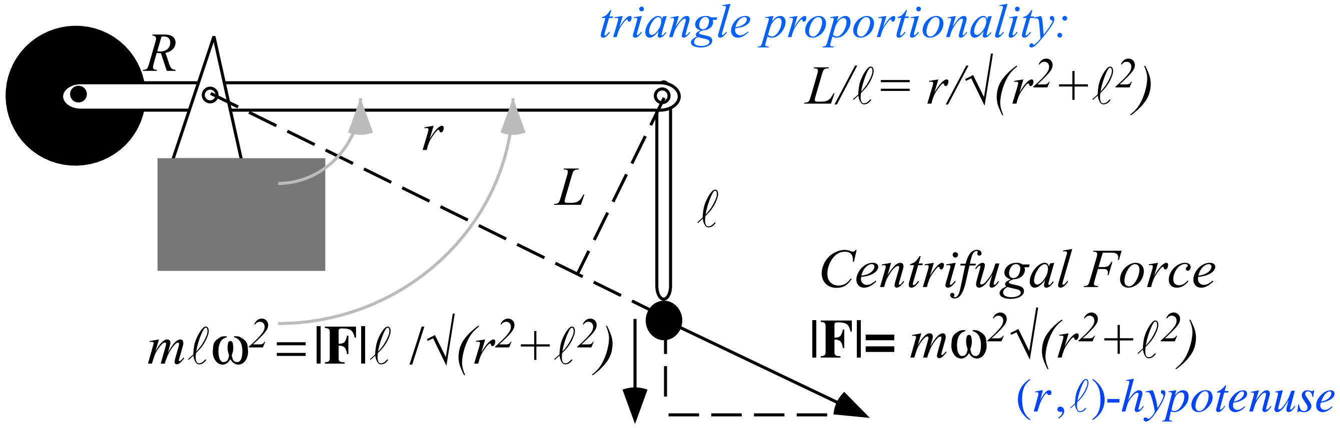
Move to top of page...

Trying to 2nd-guess Riemann results (Gravity-free case)

Move to top of page...

The ϕ–torque on mass m on leg ℓ due to centrifugal force is force times *moment* arm $L=r\cdot\ell/\sqrt{(r^2+\ell^2)}$.

This is the rate of change of ϕ-angular momentum around the pivot at the top of ℓ.



$$m\ell^2\ddot{\phi} = FL = m\omega^2\sqrt{r^2+\ell^2}\,\frac{r\ell}{\sqrt{r^2+\ell^2}} = m\omega^2 r\ell$$

or: $\ddot{\phi} = FL/m\ell^2 = \omega^2 r/\ell$

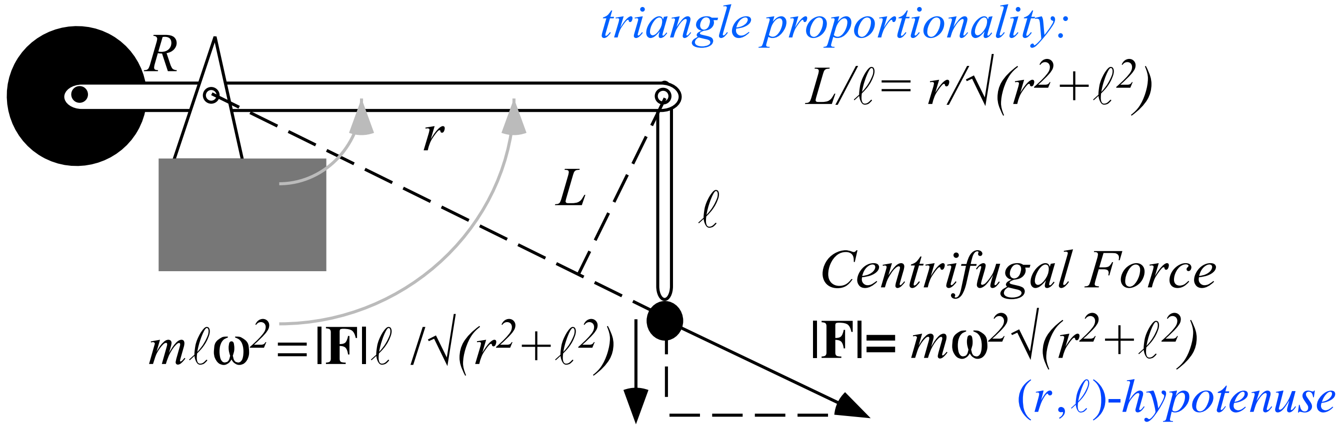
Fig. 2.5.1 Centrifugal force for a particular state of motion ($\omega \equiv \dot{\theta} = \dot{\phi},\ \theta = \frac{-\pi}{2},\ \phi = 0$)

Move to top of page...

Trying to 2nd-guess Riemann results (Gravity-free case)

The ϕ–torque on mass m on leg ℓ due to centrifugal force is force times *moment* arm $L=r\cdot\ell/\sqrt{(r^2+\ell^2)}$.

This is the rate of change of ϕ-angular momentum around the pivot at the top of ℓ.



$$m\ell^2\ddot{\phi} = FL = m\omega^2\sqrt{r^2+\ell^2}\,\frac{r\ell}{\sqrt{r^2+\ell^2}} = m\omega^2 r\ell$$

or: $\ddot{\phi} = FL\,/\,m\ell^2 = \omega^2 r\,/\,\ell$

Fig. 2.5.1 Centrifugal force for state of motion ($\omega \equiv \dot{\theta} = \dot{\phi}$, $\theta = \frac{-\pi}{2}$, $\phi = 0$)

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.

Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)

Trebuchet Cartesian projectile coordinates are double-valued

Toroidal "rolled-up" ($q^1=\theta$, $q^2=\phi$)-manifold and "Flat" ($x=\theta$, $y=\phi$)-graph

Review of covariant $\mathbf{E}_n$ and contravariant $\mathbf{E}^m$ vectors: Jacobian J vs. Kajobian K

Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)

Tangent $\{\mathbf{E}_n\}$space vs. Normal $\{\mathbf{E}^m\}$space

Covariant vs. contravariant coordinate transformations

Metric g_{mn} tensor geometric relations to length, area, and volume

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)

Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 77)

Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 69)

Riemann equation derivation for trebuchet model

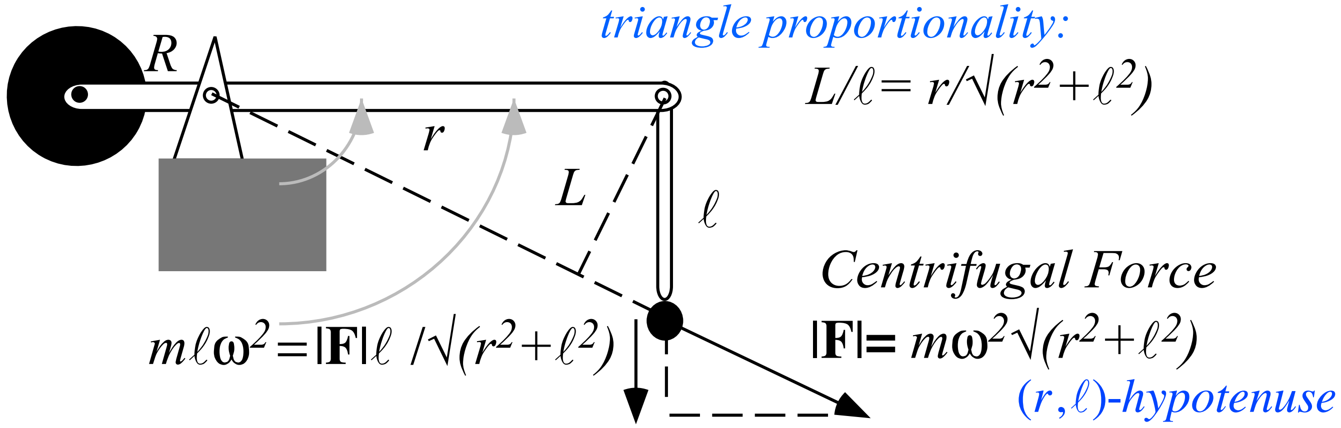
Riemann equation force analysis

→ *2nd-guessing Riemann equation?*

Trying to 2nd-guess Riemann results (Gravity-free case)

The ϕ–torque on mass m on leg ℓ due to centrifugal force is force times *moment* arm $L=r\cdot\ell/\sqrt{(r^2+\ell^2)}$.

This is the rate of change of ϕ-angular momentum around the pivot at the top of ℓ.



$$m\ell^2\ddot{\phi} = FL = m\omega^2\sqrt{r^2+\ell^2}\,\frac{r\ell}{\sqrt{r^2+\ell^2}} = m\omega^2 r\ell$$

or: $\ddot{\phi} = FL/m\ell^2 = \omega^2 r/\ell$

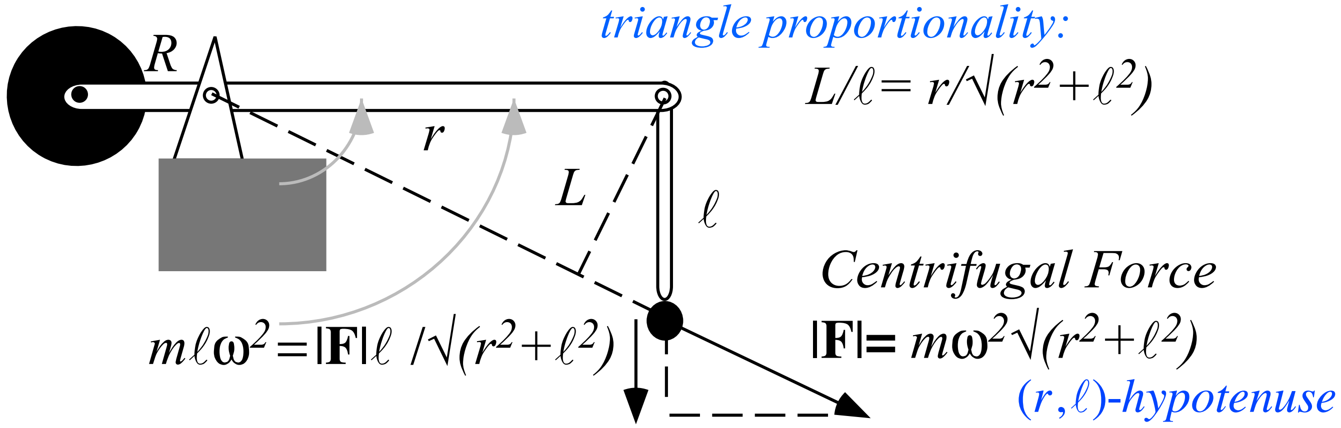
Now... 2nd-guess Riemann results:
$$\begin{pmatrix}\ddot{\theta}\\ \ddot{\phi}\end{pmatrix} = \begin{pmatrix}\dfrac{-mr\ell\omega^2}{MR^2+mr^2}\\ \omega^2 r/\ell\end{pmatrix}$$

Fig. 2.5.1 Centrifugal force for state of motion ($\omega \equiv \dot{\theta} = \dot{\phi}$, $\theta = \frac{-\pi}{2}$, $\phi = 0$)

Trying to 2nd-guess Riemann results (Gravity-free case)

The ϕ–torque on mass m on leg ℓ due to centrifugal force is force times *moment* arm $L=r\cdot\ell/\sqrt{(r^2+\ell^2)}$.

This is the rate of change of ϕ-angular momentum around the pivot at the top of ℓ.



$$m\ell^2\ddot{\phi} = FL = m\omega^2\sqrt{r^2+\ell^2}\,\frac{r\ell}{\sqrt{r^2+\ell^2}} = m\omega^2 r\ell$$

or: $\ddot{\phi} = FL/m\ell^2 = \omega^2 r/\ell$

Now... 2nd-guess Riemann results:

$$\begin{pmatrix}\ddot{\theta}\\ \ddot{\phi}\end{pmatrix} = \begin{pmatrix}\dfrac{-mr\ell\omega^2}{MR^2+mr^2}\\ \omega^2 r/\ell\end{pmatrix}$$

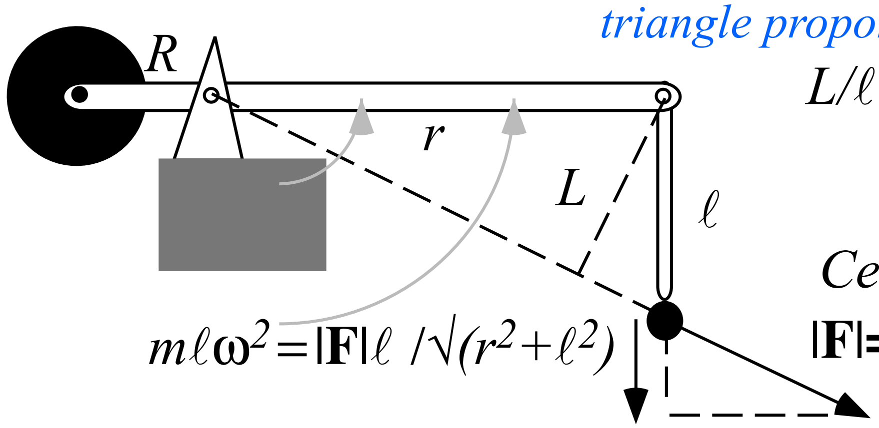
Fig. 2.5.1 Centrifugal force for state of motion ($\omega \equiv \dot{\theta} = \dot{\phi},\ \theta = \frac{-\pi}{2},\ \phi = 0$)

It may seem paradoxical that the θ-coordinate for main r-arm feels any torque or acceleration at all.

Trying to 2nd-guess Riemann results (Gravity-free case)

The ϕ–torque on mass m on leg ℓ due to centrifugal force is force times *moment* arm $L=r\cdot\ell/\sqrt{(r^2+\ell^2)}$.

This is the rate of change of ϕ-angular momentum around the pivot at the top of ℓ.



triangle proportionality:

$L/\ell= r/\sqrt{(r^2+\ell^2)}$

$$m\ell^2\ddot{\phi} = FL = m\omega^2\sqrt{r^2+\ell^2}\,\frac{r\ell}{\sqrt{r^2+\ell^2}} = m\omega^2 r\ell$$

or: $\ddot{\phi} = FL/m\ell^2 = \omega^2 r/\ell$

Fig. 2.5.1 Centrifugal force for state of motion $\left(\omega\equiv\dot{\theta}=\dot{\phi},\ \theta=\frac{-\pi}{2},\ \phi=0\right)$

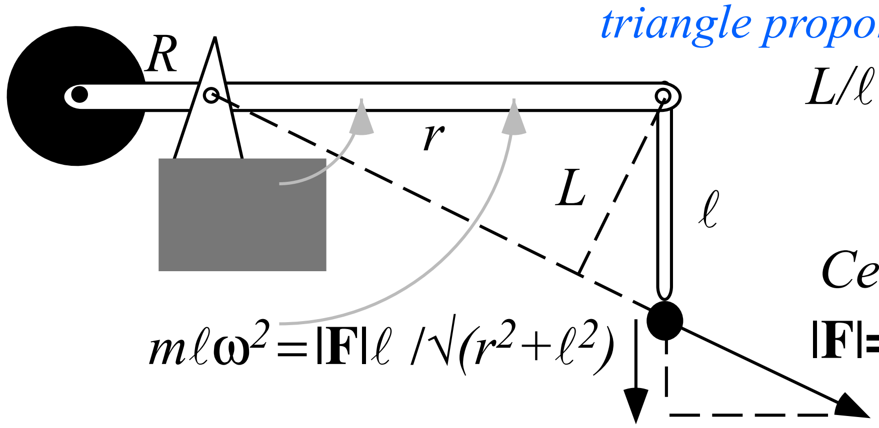
Now... 2nd-guess Riemann results: $\begin{pmatrix}\ddot{\theta}\\ \ddot{\phi}\end{pmatrix} = \begin{pmatrix}\dfrac{-mr\ell\omega^2}{MR^2+mr^2}\\ \omega^2 r/\ell\end{pmatrix}$

It may seem paradoxical that the θ-coordinate for main r-arm feels any torque or acceleration at all.
Indeed, if the device is rigid there can be none since the centrifugal force has no moment;
(Its line of action hits the θ-axis of the R-arm.)

Trying to 2nd-guess Riemann results (Gravity-free case)

The ϕ–torque on mass m on leg ℓ due to centrifugal force is force times *moment* arm $L=r\cdot\ell/\sqrt{(r^2+\ell^2)}$.

This is the rate of change of ϕ-angular momentum around the pivot at the top of ℓ.



triangle proportionality:

$L/\ell= r/\sqrt{(r^2+\ell^2)}$

Centrifugal Force

$|\mathbf{F}|= m\omega^2\sqrt{(r^2+\ell^2)}$

(r,ℓ)-hypotenuse

$$m\ell^2\ddot{\phi} = FL = m\omega^2\sqrt{r^2+\ell^2}\,\frac{r\ell}{\sqrt{r^2+\ell^2}} = m\omega^2 r\ell$$

or: $\ddot{\phi} = FL/m\ell^2 = \omega^2 r/\ell$

Fig. 2.5.1 Centrifugal force for state of motion $(\omega\equiv\dot{\theta}=\dot{\phi},\ \theta=\frac{-\pi}{2},\ \phi=0)$

Now... 2nd-guess Riemann results:
$$\begin{pmatrix}\ddot{\theta}\\ \ddot{\phi}\end{pmatrix} = \begin{pmatrix}\dfrac{-mr\ell\omega^2}{MR^2+mr^2}\\ \omega^2 r/\ell\end{pmatrix}$$

It may seem paradoxical that the θ-coordinate for main r-arm feels any torque or acceleration at all. Indeed, if the device is rigid there can be none since the centrifugal force has no moment;

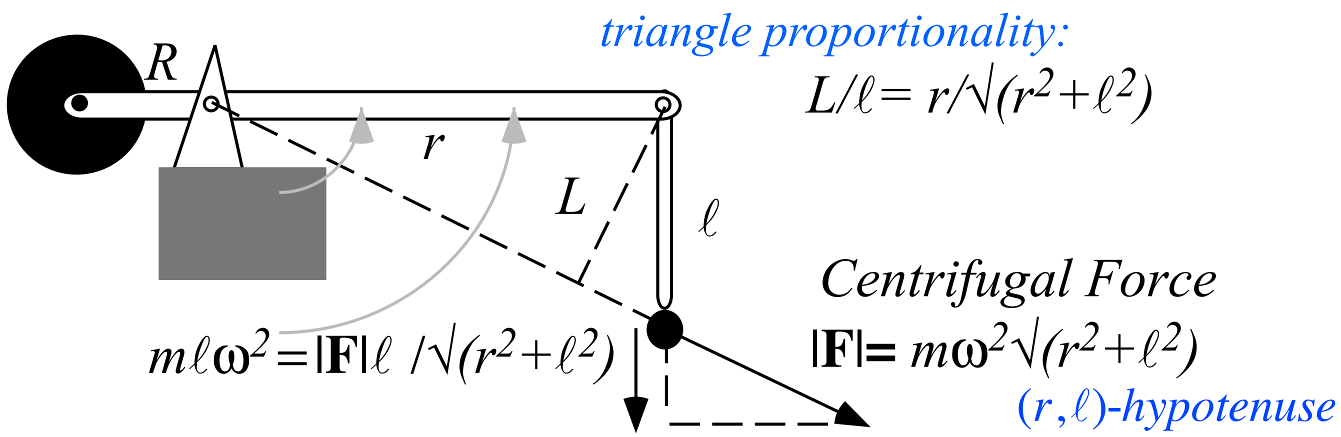
(Its line of action hits the θ-axis of the R-arm.)

However, this device isn't rigid. The ℓ-leg pivot is frictionless and can only transmit a component $m\cdot\ell\omega^2$ of force along ℓ.

Trying to 2nd-guess Riemann results (Gravity-free case)

The ϕ–torque on mass m on leg ℓ due to centrifugal force is force times *moment* arm $L=r\cdot\ell/\sqrt{(r^2+\ell^2)}$.

This is the rate of change of ϕ-angular momentum around the pivot at the top of ℓ.



$$m\ell^2\ddot{\phi} = FL = m\omega^2\sqrt{r^2+\ell^2}\,\frac{r\ell}{\sqrt{r^2+\ell^2}} = m\omega^2 r\ell$$

or: $\ddot{\phi} = FL\,/\,m\ell^2 = \omega^2 r\,/\,\ell$

Fig. 2.5.1 Centrifugal force for state of motion $\left(\omega \equiv \dot{\theta} = \dot{\phi},\ \theta = \frac{-\pi}{2},\ \phi = 0\right)$

Now... 2nd-guess Riemann results: $\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} \dfrac{-mr\ell\omega^2}{MR^2+mr^2} \\ \omega^2 r\,/\,\ell \end{pmatrix}$

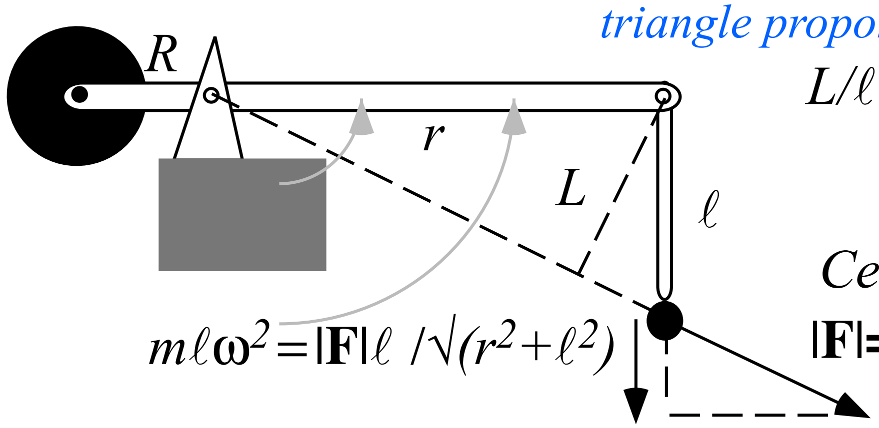
It may seem paradoxical that the θ-coordinate for main r-arm feels any torque or acceleration at all.
Indeed, if the device is rigid there can be none since the centrifugal force has no moment;
(Its line of action hits the θ-axis of the R-arm.)

However, this device isn't rigid. The ℓ-leg pivot is frictionless and can only transmit a component $m\cdot\ell\omega^2$ of force along ℓ.
This causes a negative torque $-mr\ell\,\omega^2$ on the big r-arm.

Trying to 2nd-guess Riemann results (Gravity-free case)

The ϕ–torque on mass m on leg ℓ due to centrifugal force is force times *moment* arm $L=r\cdot\ell/\sqrt{(r^2+\ell^2)}$.

This is the rate of change of ϕ-angular momentum around the pivot at the top of ℓ.



triangle proportionality:

$L/\ell= r/\sqrt{(r^2+\ell^2)}$

Centrifugal Force

$|\mathbf{F}|= m\omega^2\sqrt{(r^2+\ell^2)}$

(r,ℓ)-hypotenuse

$$m\ell^2\ddot{\phi} = FL = m\omega^2\sqrt{r^2+\ell^2}\,\frac{r\ell}{\sqrt{r^2+\ell^2}} = m\omega^2 r\ell$$

or: $\ddot{\phi} = FL\,/\,m\ell^2 = \omega^2 r\,/\,\ell$

Fig. 2.5.1 Centrifugal force for state of motion ($\omega\equiv\dot{\theta}=\dot{\phi}$, $\theta=\frac{-\pi}{2}$, $\phi=0$)

Now... 2nd-guess Riemann results:
$$\begin{pmatrix}\ddot{\theta}\\ \ddot{\phi}\end{pmatrix} = \begin{pmatrix}\dfrac{-mr\ell\omega^2}{MR^2+mr^2}\\ \omega^2 r\,/\,\ell\end{pmatrix}$$

It may seem paradoxical that the θ-coordinate for main r-arm feels any torque or acceleration at all.
Indeed, if the device is rigid there can be none since the centrifugal force has no moment;
(Its line of action hits the θ-axis of the R-arm.)

However, this device isn't rigid. The ℓ-leg pivot is frictionless and can only transmit a component $m\cdot\ell\omega^2$ of force along ℓ.
This causes a negative torque $-mr\ell\,\omega^2$ on the big r-arm.
It reduces θ-angular momentum to exactly cancel the rate of increase in ϕ-momentum.

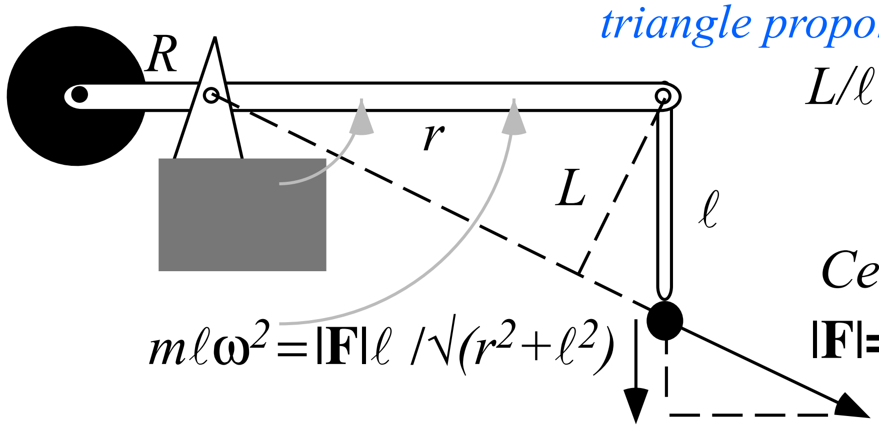
$$\left(MR^2+mr^2\right)\ddot{\theta} = -mr\ell\omega^2$$

Checks with $\ddot{\theta}$ Riemann equation

Trying to 2nd-guess Riemann results (Gravity-free case)

The ϕ–torque on mass m on leg ℓ due to centrifugal force is force times *moment* arm $L=r\cdot\ell/\sqrt{(r^2+\ell^2)}$.

This is the rate of change of ϕ-angular momentum around the pivot at the top of ℓ.



triangle proportionality:

$L/\ell = r/\sqrt{(r^2+\ell^2)}$

Centrifugal Force

$|\mathbf{F}| = m\omega^2\sqrt{(r^2+\ell^2)}$

(r,ℓ)-hypotenuse

$$m\ell^2\ddot{\phi} = FL = m\omega^2\sqrt{r^2+\ell^2}\,\frac{r\ell}{\sqrt{r^2+\ell^2}} = m\omega^2 r\ell$$

or: $\ddot{\phi} = FL/m\ell^2 = \omega^2 r/\ell$

Fig. 2.5.1 Centrifugal force for state of motion ($\omega \equiv \dot{\theta} = \dot{\phi}$, $\theta = \frac{-\pi}{2}$, $\phi = 0$)

Now... 2nd-guess Riemann results:

$$\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} \dfrac{-mr\ell\omega^2}{MR^2+mr^2} \\ \omega^2 r/\ell \end{pmatrix}$$

It may seem paradoxical that the θ-coordinate for main r-arm feels any torque or acceleration at all.
Indeed, if the device is rigid there can be none since the centrifugal force has no moment;
(Its line of action hits the θ-axis of the R-arm.)

However, this device isn't rigid. The ℓ-leg pivot is frictionless and can only transmit a component $m\cdot\ell\omega^2$ of force along ℓ.

This causes a negative torque $-mr\ell\,\omega^2$ on the big r-arm.

It reduces θ-angular momentum to exactly cancel the rate of increase in ϕ-momentum.

$$\left(MR^2 + mr^2\right)\ddot{\theta} = -mr\ell\omega^2$$

Checks with $\ddot{\theta}$ Riemann equation

Note the time derivative of total momentum is zero if outside torques are zero.(twirling skater analogy)

$$\dot{p}_\theta + \dot{p}_\phi = 0, \text{ if } F_\theta = 0 = F_\phi$$