

Lecture 16
Wed. 10.17.2018

Hamilton Equations for Trebuchet and Related Things

(Ch. 5-9 of Unit 2)

Review of Hamiltonian equation derivation (Elementary trebuchet)

Hamiltonian definition from Lagrangian and γ_{mn} tensor

Hamilton's equations and Poincare invariant relations

Hamiltonian expression and contravariant γ^{mn} tensor

Hamiltonian energy and momentum conservation and symmetry coordinates

Coordinate transformation helps reduce symmetric Hamiltonian

Free-space trebuchet kinematics by symmetry

Algebraic approximate approach

Direct approximate approach and Superball analogy

Trebuchet vs Flinger and sports kinematics

The multiple approaches to Mechanics (and physics in general)

A simplified trebuchet approximation (but unfinished)

A running collection of links to course-relevant sites and articles

Physics Web Resources

[Comprehensive Harter-Soft Resource Listing](#)

[UAF Physics YouTube channel](#)

[LearnIt Physics Web Applications](#)

Neat external material to start the class:

[AIP publications](#)

[AJP article on superball dynamics](#)

[AAPT summer reading](#)

These *are* hot off the presses:

[Sorting ultracold atoms in a three-dimensional optical lattice in a realization of Maxwell's demon - Kumar-Nature-Letters-2018](#)

[Synthetic three-dimensional atomic structures assembled atom by atom - Berredo-Nature-Letters-2018](#)

Slightly Older ones:

[Wave-particle duality of C60 molecules](#)

[Optical vortex knots – One Photon at a Time](#)

“*Relawavity*” and quantum basis of *Lagrangian & Hamiltonian* mechanics:

[2-CW laser wave - BohrIt Web App](#)

[Lagrangian vs Hamiltonian - RelaWavity Web App](#)

[AMOP Ch 0 Space-Time Symmetry - 2019](#)

[Seminar at Rochester Institute of Optics, Auxiliary slides, June 19, 2018](#)

New *Analyt Web Application now under development* in out testing area:

<https://modphys.hosted.uark.edu/testing/markup/AnalyItBJS.html>

“Texts”

[Classical Mechanics with a Bang!](#)

[Quantum Theory for the Computer Age](#)

[Principles of Symmetry, Dynamics, and Spectroscopy](#)

[Modern Physics and its Classical Foundations](#)

Classes

[2014 AMOP](#)

[2017 Group Theory for QM](#)

[2018 AMOP](#)

[2018 Adv Mechanics](#)

Unique sorted Links for Lectures 14-16

<http://thearmchaircritic.blogspot.com/2011/11/punkin-chunkin.html>

<http://www.sussexcountyonline.com/news/photos/punkinchunkin.html>

<http://www.twcenter.net/forums/showthread.php?358315-Shooting-range-for-medieval-siege-weapons-Anybody-knows>

<https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html>

<https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html?scenario=MontezumasRevenge>

<https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html?scenario=SeigeOfKenilworth>

https://modphys.hosted.uark.edu/pdfs/Journal_Pdfs/Trebuchet-SciAm_273_66_July_1995_chevedden1.pdf

Chapter 1. The Trebuchet: A dream problem for Galileo?

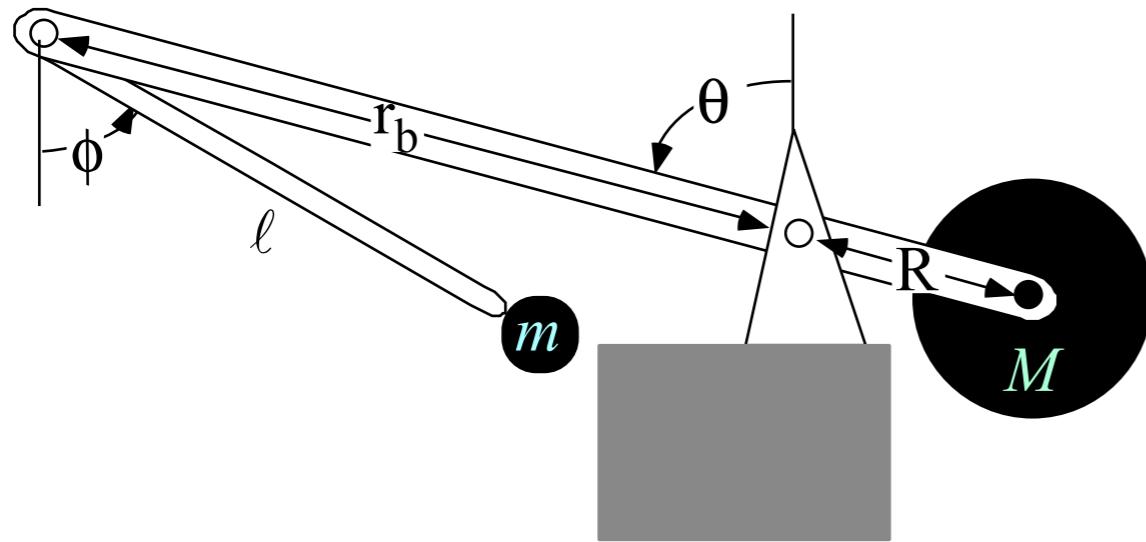
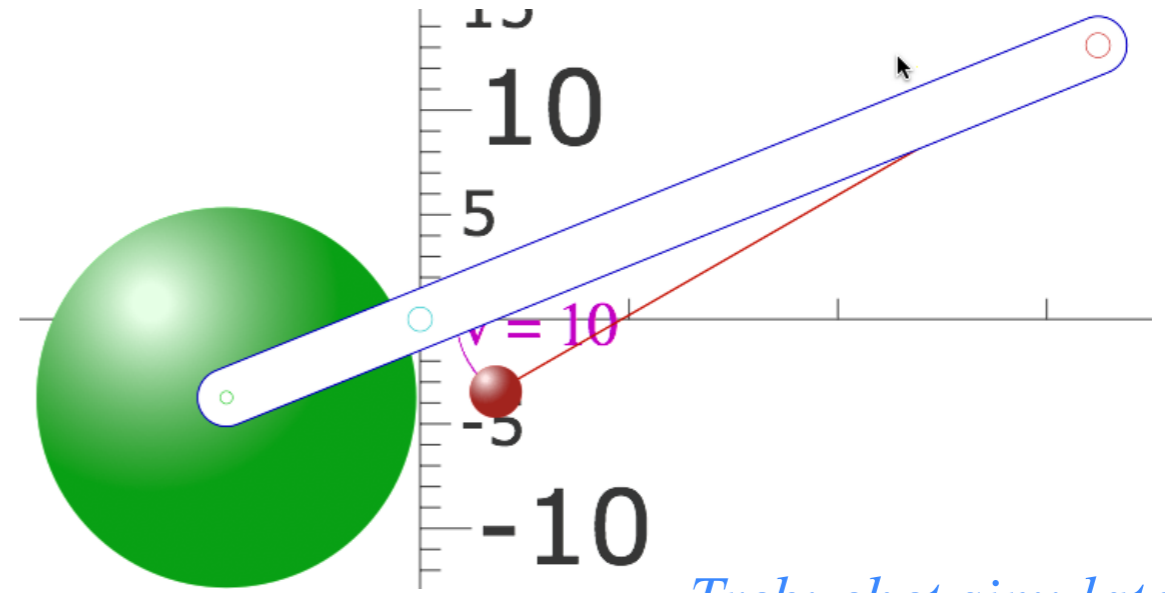
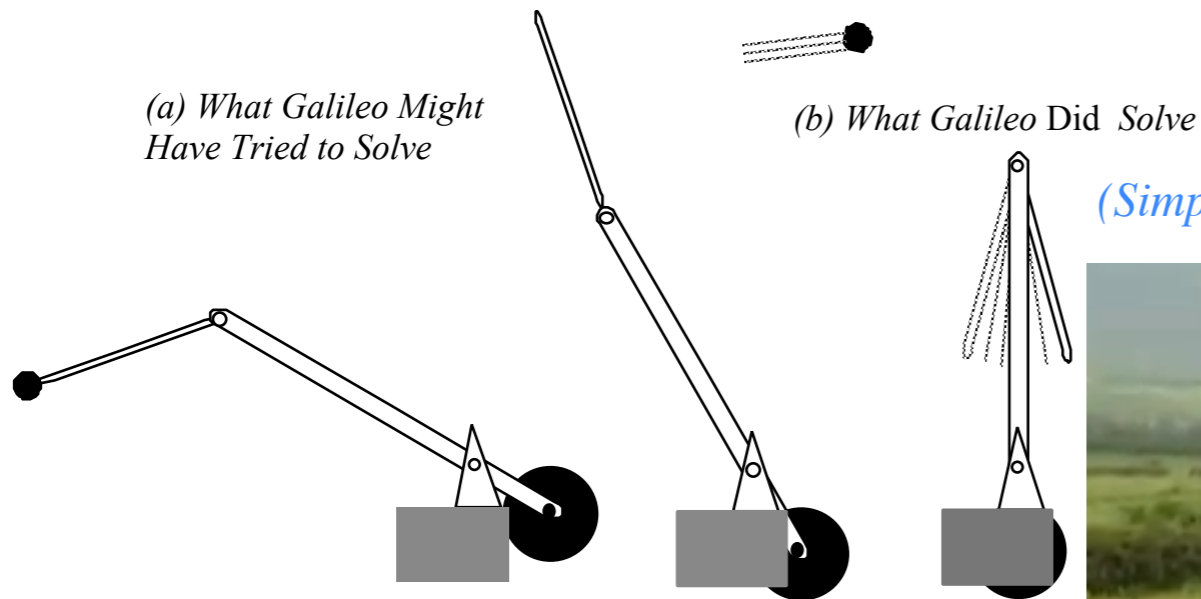


Fig. 2.1.1 An elementary ground-fixed trebuchet



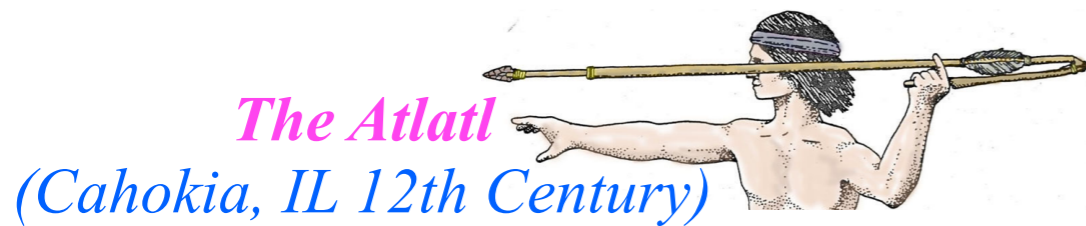
Trebuchet simulator

<https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html>



(Simple pendulum dynamics)


Fig. 2.1.2 Galileo's (supposed) problem



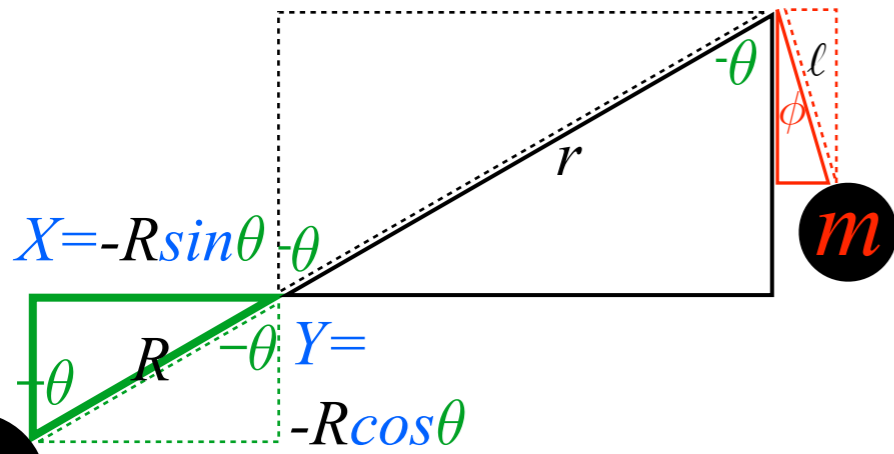
The Atlatl
(Cahokia, IL 12th Century)



Review of Hamiltonian equation derivation (Elementary trebuchet)

 *Hamiltonian definition from Lagrangian and γ_{mn} tensor*
Hamilton's equations and Poincare invariant relations
Hamiltonian expression and contravariant γ^{mn} tensor

$$\text{Total KE} = T = \frac{1}{2} [M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2] = \frac{1}{2} [(MR^2 + mr^2)\dot{\theta}^2 - 2mrl \cos(\theta - \phi)\dot{\theta}\dot{\phi} + ml^2\dot{\phi}^2]$$



$$T = \frac{1}{2} \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} MR^2 + mr^2 & -mrl \cos(\theta - \phi) \\ -mrl \cos(\theta - \phi) & ml^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

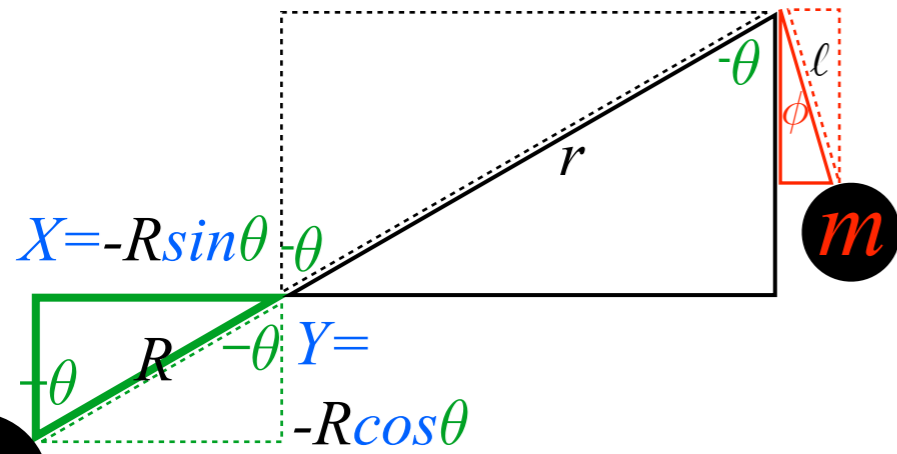
$$\begin{pmatrix} p_{\theta} \\ p_{\phi} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \quad \text{Dynamic metric tensor } \gamma_{mn} \text{ in GCC } \theta \text{ and } \phi$$

M

Lagrangian function of GCC and velocities: $L(\theta, \phi, \dot{\theta}, \dot{\phi}, t) = T(\theta, \phi, \dot{\theta}, \dot{\phi}, t) - V(\theta, \phi, t)$

$$dL(\theta, \phi, \dot{\theta}, \dot{\phi}, t) = \frac{\partial L}{\partial \theta} d\theta + \frac{\partial L}{\partial \phi} d\phi + \frac{\partial L}{\partial \dot{\theta}} d\dot{\theta} + \frac{\partial L}{\partial \dot{\phi}} d\dot{\phi} + \frac{\partial L}{\partial t} dt \quad \text{1st differential chain}$$

$$\text{Total KE} = T = \frac{1}{2} [M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2] = \frac{1}{2} [(MR^2 + mr^2)\dot{\theta}^2 - 2mrl \cos(\theta - \phi)\dot{\theta}\dot{\phi} + ml^2\dot{\phi}^2]$$



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$$\begin{pmatrix} p_{\theta} \\ p_{\phi} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \quad \text{Dynamic metric tensor}$$

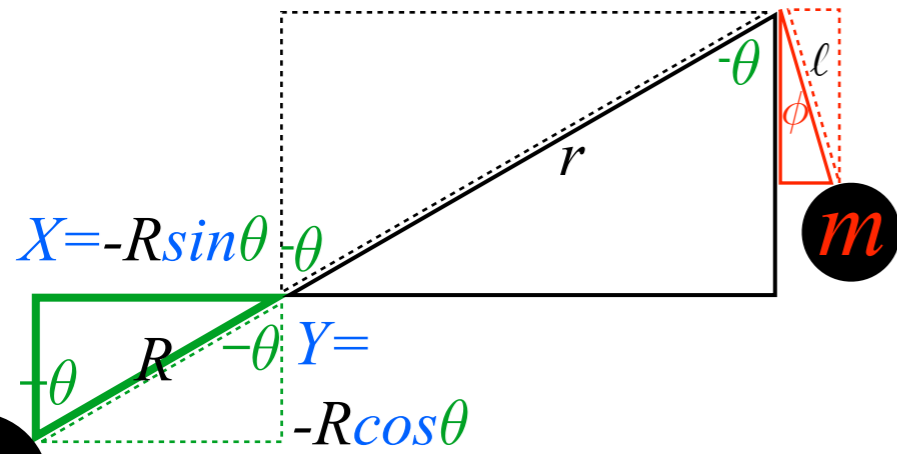
γ_{mn}
in GCC θ and ϕ

Lagrangian function of GCC and velocities: $L(\theta, \phi, \dot{\theta}, \dot{\phi}, t) = T(\theta, \phi, \dot{\theta}, \dot{\phi}, t) - V(\theta, \phi, t)$

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$$\frac{dL}{dt} \equiv \dot{L}(\theta, \phi, \dot{\theta}, \dot{\phi}, t) = \frac{dL}{dt} = \frac{\partial L}{\partial \theta} \frac{d\theta}{dt} + \frac{\partial L}{\partial \phi} \frac{d\phi}{dt} + \frac{\partial L}{\partial \dot{\theta}} \frac{d\dot{\theta}}{dt} + \frac{\partial L}{\partial \dot{\phi}} \frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t} \quad \text{velocity chain}$$

$$\text{Total KE} = T = \frac{1}{2} [M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2] = \frac{1}{2} [(MR^2 + mr^2)\dot{\theta}^2 - 2mrl \cos(\theta - \phi)\dot{\theta}\dot{\phi} + ml^2\dot{\phi}^2]$$



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Dynamic metric tensor

γ_{mn}
in GCC θ and ϕ

Lagrangian function of GCC and velocities: $L(\theta, \phi, \dot{\theta}, \dot{\phi}, t) = T(\theta, \phi, \dot{\theta}, \dot{\phi}, t) - V(\theta, \phi, t)$

$$dL(\theta, \phi, \dot{\theta}, \dot{\phi}, t) = \frac{\partial L}{\partial \theta} d\theta + \frac{\partial L}{\partial \phi} d\phi + \frac{\partial L}{\partial \dot{\theta}} d\dot{\theta} + \frac{\partial L}{\partial \dot{\phi}} d\dot{\phi} + \frac{\partial L}{\partial t} dt$$

1st differential chain

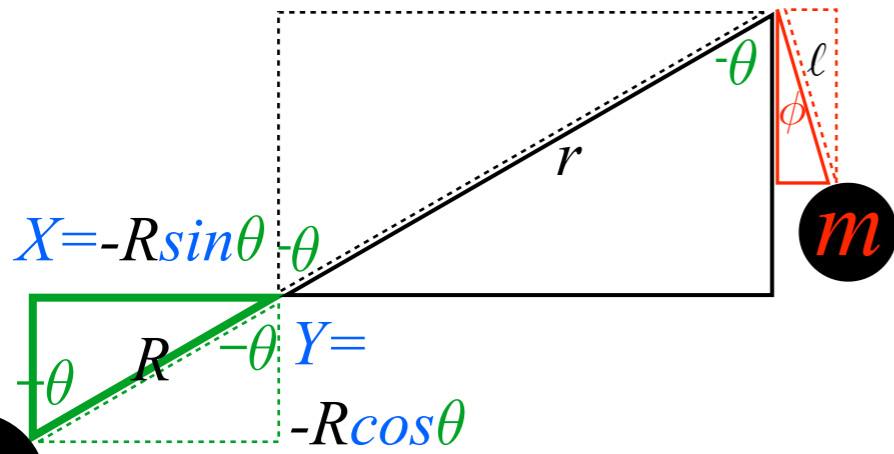
$$\dot{L}(\theta, \phi, \dot{\theta}, \dot{\phi}, t) = \frac{dL}{dt} = \frac{\partial L}{\partial \theta} \frac{d\theta}{dt} + \frac{\partial L}{\partial \phi} \frac{d\phi}{dt} + \frac{\partial L}{\partial \dot{\theta}} \frac{d\dot{\theta}}{dt} + \frac{\partial L}{\partial \dot{\phi}} \frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

velocity chain

$$\dot{L}(\theta, \phi, \dot{\theta}, \dot{\phi}, t) = \frac{dL}{dt} = \dot{p}_{\theta} \frac{d\theta}{dt} + \dot{p}_{\phi} \frac{d\phi}{dt} + p_{\theta} \frac{d\dot{\theta}}{dt} + p_{\phi} \frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

Lagrange equations

$$\text{Total KE} = T = \frac{1}{2} [M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2] = \frac{1}{2} [(MR^2 + mr^2)\dot{\theta}^2 - 2mrl \cos(\theta - \phi)\dot{\theta}\dot{\phi} + ml^2\dot{\phi}^2]$$



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velocity chain

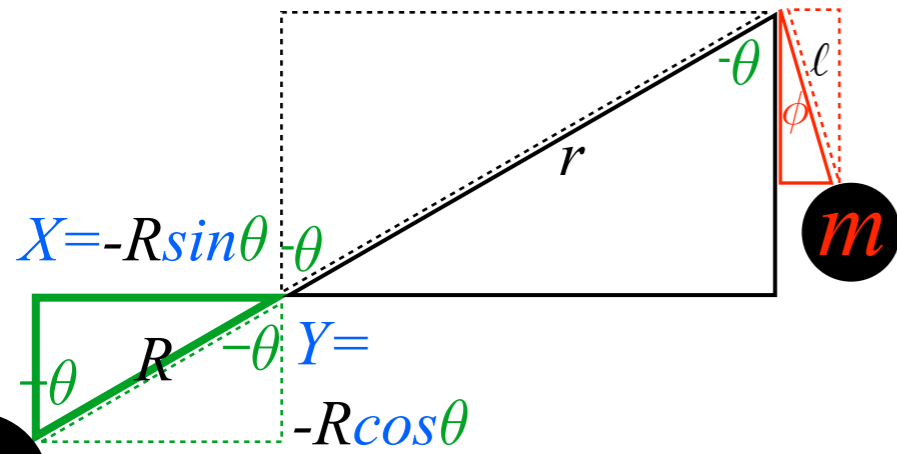
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(Consolidating)

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Lagrange equations

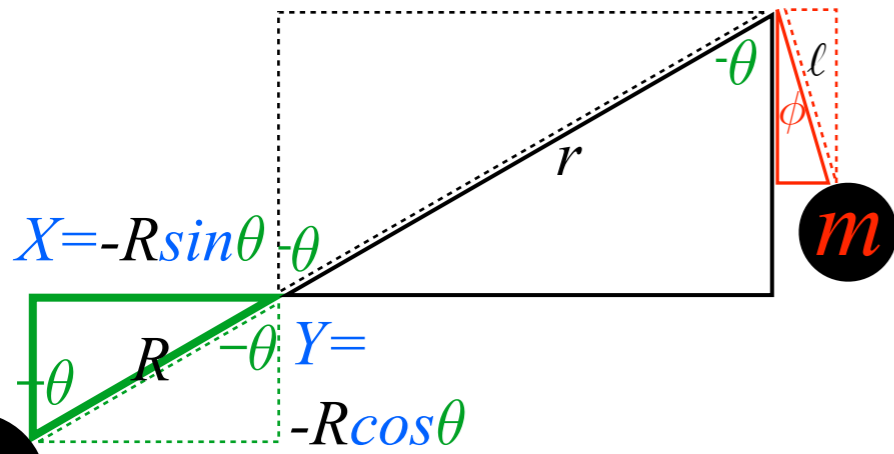
$$= \frac{dL}{dt} = \frac{d}{dt} (p_{\theta} \dot{\theta} + p_{\phi} \dot{\phi}) + \frac{\partial L}{\partial t}$$

(Consolidating)

$$\frac{d}{dt} (p_{\theta} \dot{\theta} + p_{\phi} \dot{\phi} - L) = -\frac{\partial L}{\partial t}$$

(Rearranging)

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Dynamic metric tensor

γ_{mn}
in GCC θ and ϕ

Lagrangian function of GCC and velocities: $L(\theta, \phi, \dot{\theta}, \dot{\phi}, t) = T(\theta, \phi, \dot{\theta}, \dot{\phi}, t) - V(\theta, \phi, t)$

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Lagrange equations

$$= \frac{dL}{dt} = \frac{d}{dt} (p_{\theta} \dot{\theta} + p_{\phi} \dot{\phi}) + \frac{\partial L}{\partial t}$$

(Consolidating)

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(Rearranging)

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

Defining the
Hamiltonian function

Hamiltonian function of GCC and momenta: $H(\theta, \phi, p_{\theta}, p_{\phi}, t) = p_{\theta} \dot{\theta} + p_{\phi} \dot{\phi} - L$

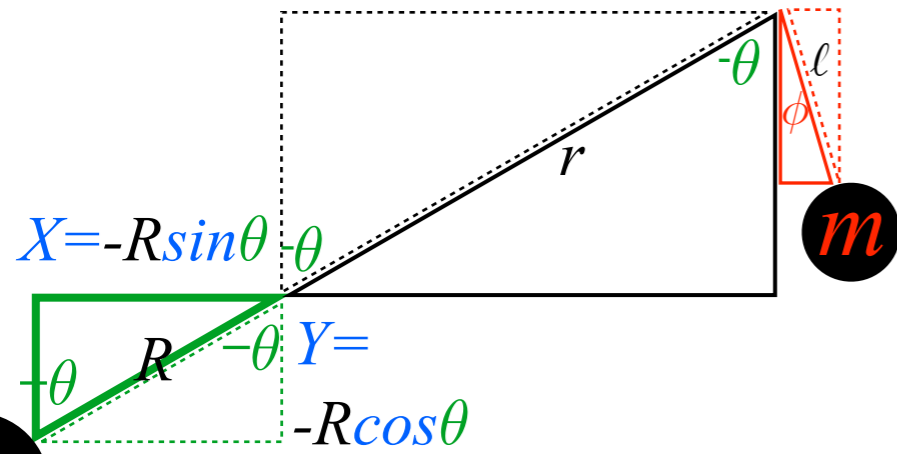
Review of Hamiltonian equation derivation (Elementary trebuchet)

Hamiltonian definition from Lagrangian and γ_{mn} tensor

 *Hamilton's equations and Poincare invariant relations*

Hamiltonian expression and contravariant γ^{mn} tensor

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Dynamic metric tensor
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velocity chain

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Lagrange equations

$$= \frac{dL}{dt} = \frac{d}{dt} (p_{\theta} \dot{\theta} + p_{\phi} \dot{\phi}) + \frac{\partial L}{\partial t}$$

(Consolidating)

$$\frac{d}{dt} (p_{\theta} \dot{\theta} + p_{\phi} \dot{\phi} - L) = -\frac{\partial L}{\partial t}$$

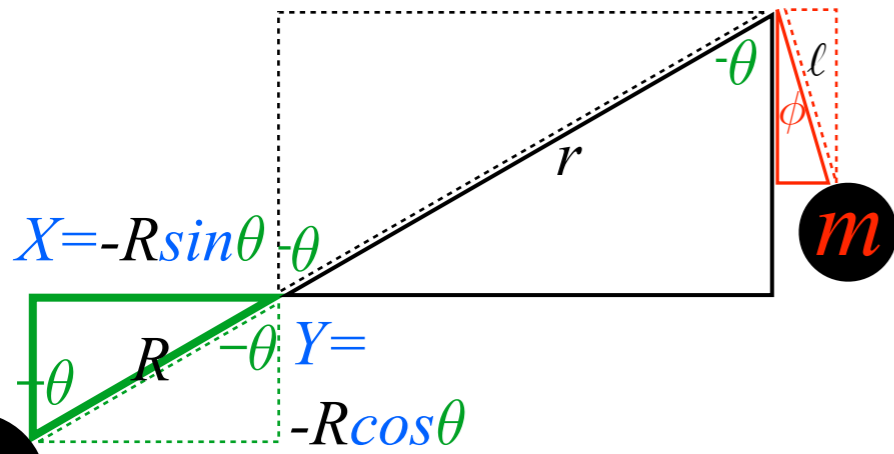
(Rearranging)

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

Defining the
Hamiltonian function

Hamiltonian function of GCC and momenta: $H(\theta, \phi, p_{\theta}, p_{\phi}, t) = p_{\theta} \dot{\theta} + p_{\phi} \dot{\phi} - L$

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Dynamic metric tensor
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in GCC θ and ϕ

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$$\dot{L}(\theta, \phi, \dot{\theta}, \dot{\phi}, t) = \frac{dL}{dt} = \frac{\partial L}{\partial \theta} \frac{d\theta}{dt} + \frac{\partial L}{\partial \phi} \frac{d\phi}{dt} + \frac{\partial L}{\partial \dot{\theta}} \frac{d\dot{\theta}}{dt} + \frac{\partial L}{\partial \dot{\phi}} \frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

velocity chain

$$\dot{L}(\theta, \phi, \dot{\theta}, \dot{\phi}, t) = \frac{dL}{dt} = \dot{p}_{\theta} \frac{d\theta}{dt} + \dot{p}_{\phi} \frac{d\phi}{dt} + p_{\theta} \frac{d\dot{\theta}}{dt} + p_{\phi} \frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

Lagrange equations

$$= \frac{dL}{dt} = \frac{d}{dt} (p_{\theta} \dot{\theta} + p_{\phi} \dot{\phi}) + \frac{\partial L}{\partial t}$$

(Consolidating)

$$\frac{d}{dt} (p_{\theta} \dot{\theta} + p_{\phi} \dot{\phi} - L) = -\frac{\partial L}{\partial t}$$

(Rearranging)

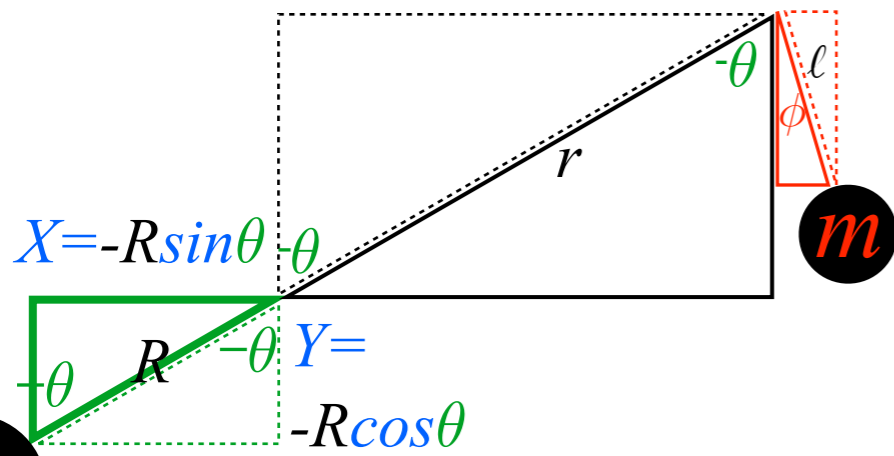
Defining the
Hamiltonian function

Hamiltonian function of GCC and momenta: $H(\theta, \phi, p_{\theta}, p_{\phi}, t) = p_{\theta} \dot{\theta} + p_{\phi} \dot{\phi} - L$ by assumed Lagrange functionality

$$\frac{\partial H}{\partial \theta} = -\frac{\partial L}{\partial \theta} = -\dot{p}_{\theta}$$

by Lagrange equations

$$\text{Total KE} = T = \frac{1}{2} [M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2] = \frac{1}{2} [(MR^2 + mr^2)\dot{\theta}^2 - 2mrl \cos(\theta - \phi)\dot{\theta}\dot{\phi} + ml^2\dot{\phi}^2]$$



$$T = \frac{1}{2} \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} MR^2 + mr^2 & -mrl \cos(\theta - \phi) \\ -mrl \cos(\theta - \phi) & ml^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Dynamic metric tensor
 γ_{mn}
in GCC θ and ϕ

Lagrangian function of GCC and velocities: $L(\theta, \phi, \dot{\theta}, \dot{\phi}, t) = T(\theta, \phi, \dot{\theta}, \dot{\phi}, t) - V(\theta, \phi, t)$

$$dL(\theta, \phi, \dot{\theta}, \dot{\phi}, t) = \frac{\partial L}{\partial \theta} d\theta + \frac{\partial L}{\partial \phi} d\phi + \frac{\partial L}{\partial \dot{\theta}} d\dot{\theta} + \frac{\partial L}{\partial \dot{\phi}} d\dot{\phi} + \frac{\partial L}{\partial t} dt$$

$$\dot{L}(\theta, \phi, \dot{\theta}, \dot{\phi}, t) = \frac{dL}{dt} = \frac{\partial L}{\partial \theta} \frac{d\theta}{dt} + \frac{\partial L}{\partial \phi} \frac{d\phi}{dt} + \frac{\partial L}{\partial \dot{\theta}} \frac{d\dot{\theta}}{dt} + \frac{\partial L}{\partial \dot{\phi}} \frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

velocity chain

$$\dot{L}(\theta, \phi, \dot{\theta}, \dot{\phi}, t) = \frac{dL}{dt} = \dot{p}_\theta \frac{d\theta}{dt} + \dot{p}_\phi \frac{d\phi}{dt} + p_\theta \frac{d\dot{\theta}}{dt} + p_\phi \frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

Lagrange equations

$$= \frac{dL}{dt} = \frac{d}{dt} (p_\theta \dot{\theta} + p_\phi \dot{\phi}) + \frac{\partial L}{\partial t}$$

(Consolidating)

$$\frac{d}{dt} (p_\theta \dot{\theta} + p_\phi \dot{\phi} - L) = -\frac{\partial L}{\partial t}$$

(Rearranging)

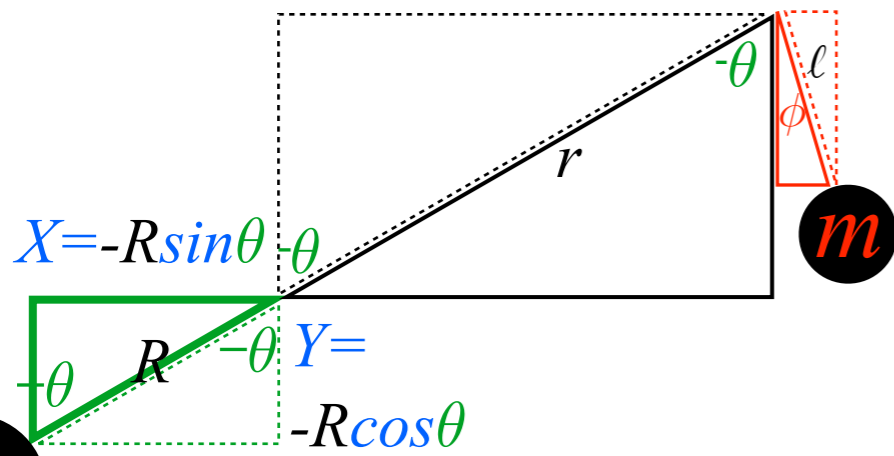
Defining the Hamiltonian function

Hamiltonian function of GCC and momenta: $H(\theta, \phi, p_\theta, p_\phi, t) = \boxed{p_\theta \dot{\theta} + p_\phi \dot{\phi} - L}$ by assumed Lagrange functionality

$$\frac{\partial H}{\partial \theta} = -\frac{\partial L}{\partial \theta} = -\dot{p}_\theta \quad \frac{\partial H}{\partial p_\theta} = \dot{\theta} - \frac{\partial L}{\partial p_\theta} = \dot{\theta}$$

by Lagrange equations

$$\text{Total KE} = T = \frac{1}{2} [M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2] = \frac{1}{2} [(MR^2 + mr^2)\dot{\theta}^2 - 2mrl \cos(\theta - \phi)\dot{\theta}\dot{\phi} + ml^2\dot{\phi}^2]$$



$$T = \frac{1}{2} \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} MR^2 + mr^2 & -mrl \cos(\theta - \phi) \\ -mrl \cos(\theta - \phi) & ml^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Dynamic metric tensor
 γ_{mn}
in GCC θ and ϕ

Lagrangian function of GCC and velocities: $L(\theta, \phi, \dot{\theta}, \dot{\phi}, t) = T(\theta, \phi, \dot{\theta}, \dot{\phi}, t) - V(\theta, \phi, t)$

$$dL(\theta, \phi, \dot{\theta}, \dot{\phi}, t) = \frac{\partial L}{\partial \theta} d\theta + \frac{\partial L}{\partial \phi} d\phi + \frac{\partial L}{\partial \dot{\theta}} d\dot{\theta} + \frac{\partial L}{\partial \dot{\phi}} d\dot{\phi} + \frac{\partial L}{\partial t} dt$$

$$\dot{L}(\theta, \phi, \dot{\theta}, \dot{\phi}, t) = \frac{dL}{dt} = \frac{\partial L}{\partial \theta} \frac{d\theta}{dt} + \frac{\partial L}{\partial \phi} \frac{d\phi}{dt} + \frac{\partial L}{\partial \dot{\theta}} \frac{d\dot{\theta}}{dt} + \frac{\partial L}{\partial \dot{\phi}} \frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

velocity chain

$$\dot{L}(\theta, \phi, \dot{\theta}, \dot{\phi}, t) = \frac{dL}{dt} = \dot{p}_\theta \frac{d\theta}{dt} + \dot{p}_\phi \frac{d\phi}{dt} + p_\theta \frac{d\dot{\theta}}{dt} + p_\phi \frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

Lagrange equations

$$= \frac{dL}{dt} = \frac{d}{dt} (p_\theta \dot{\theta} + p_\phi \dot{\phi}) + \frac{\partial L}{\partial t}$$

(Consolidating)

$$\frac{d}{dt} (p_\theta \dot{\theta} + p_\phi \dot{\phi} - L) = -\frac{\partial L}{\partial t}$$

(Rearranging)

Defining the Hamiltonian function

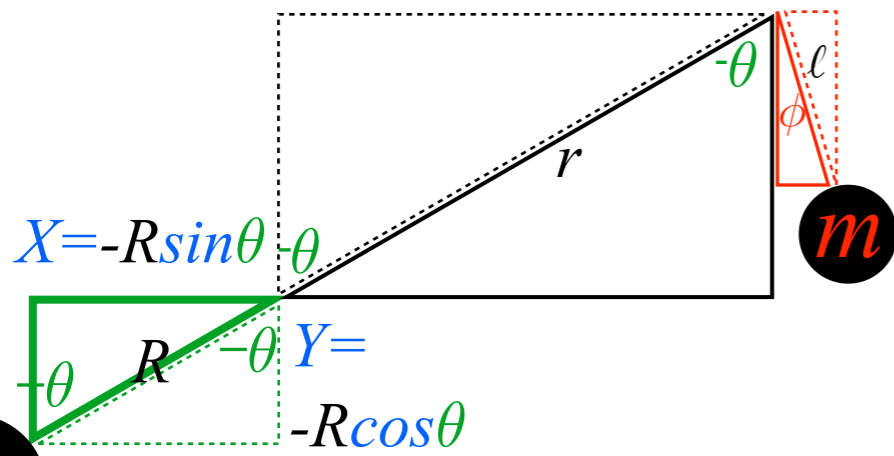
Hamiltonian function of GCC and momenta: $H(\theta, \phi, p_\theta, p_\phi, t) = [p_\theta \dot{\theta} + p_\phi \dot{\phi} - L]$ by assumed Lagrange functionality

$$\frac{\partial H}{\partial \theta} = -\frac{\partial L}{\partial \theta} = -\dot{p}_\theta \quad \frac{\partial H}{\partial p_\theta} = \dot{\theta} - \frac{\partial L}{\partial p_\theta} = \dot{\theta}$$

$$\frac{\partial H}{\partial \dot{\theta}} = p_\theta - \frac{\partial L}{\partial \dot{\theta}} = 0$$

by Lagrange equations

$$\text{Total KE} = T = \frac{1}{2} [M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2] = \frac{1}{2} [(MR^2 + mr^2)\dot{\theta}^2 - 2mrl \cos(\theta - \phi)\dot{\theta}\dot{\phi} + ml^2\dot{\phi}^2]$$



$$T = \frac{1}{2} \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} MR^2 + mr^2 & -mrl \cos(\theta - \phi) \\ -mrl \cos(\theta - \phi) & ml^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Dynamic metric tensor
 γ_{mn}
in GCC θ and ϕ

Lagrangian function of GCC and velocities: $L(\theta, \phi, \dot{\theta}, \dot{\phi}, t) = T(\theta, \phi, \dot{\theta}, \dot{\phi}, t) - V(\theta, \phi, t)$

$$dL(\theta, \phi, \dot{\theta}, \dot{\phi}, t) = \frac{\partial L}{\partial \theta} d\theta + \frac{\partial L}{\partial \phi} d\phi + \frac{\partial L}{\partial \dot{\theta}} d\dot{\theta} + \frac{\partial L}{\partial \dot{\phi}} d\dot{\phi} + \frac{\partial L}{\partial t} dt$$

$$\dot{L}(\theta, \phi, \dot{\theta}, \dot{\phi}, t) = \frac{dL}{dt} = \frac{\partial L}{\partial \theta} \frac{d\theta}{dt} + \frac{\partial L}{\partial \phi} \frac{d\phi}{dt} + \frac{\partial L}{\partial \dot{\theta}} \frac{d\dot{\theta}}{dt} + \frac{\partial L}{\partial \dot{\phi}} \frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

velocity chain

$$\dot{L}(\theta, \phi, \dot{\theta}, \dot{\phi}, t) = \frac{dL}{dt} = \dot{p}_\theta \frac{d\theta}{dt} + \dot{p}_\phi \frac{d\phi}{dt} + p_\theta \frac{d\dot{\theta}}{dt} + p_\phi \frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

Lagrange equations

$$= \frac{dL}{dt} = \frac{d}{dt} (p_\theta \dot{\theta} + p_\phi \dot{\phi}) + \frac{\partial L}{\partial t}$$

(Consolidating)

$$\frac{d}{dt} (p_\theta \dot{\theta} + p_\phi \dot{\phi} - L) = -\frac{\partial L}{\partial t}$$

(Rearranging)

Defining the
Hamiltonian function

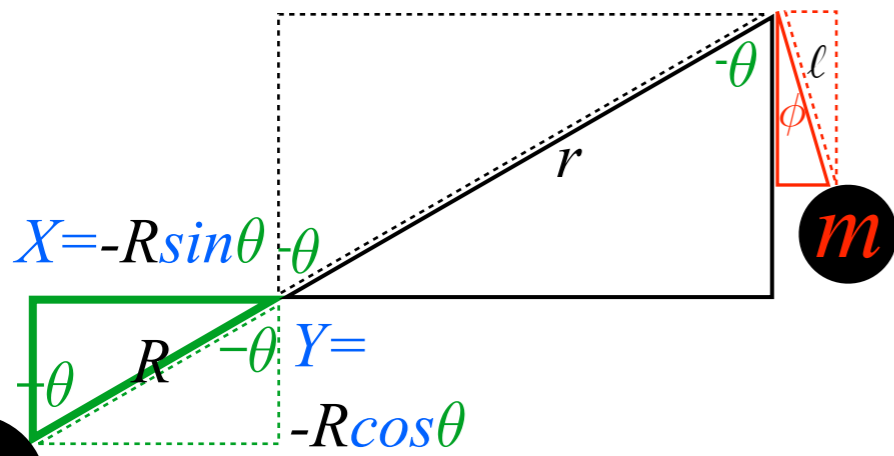
Hamiltonian function of GCC and momenta: $H(\theta, \phi, p_\theta, p_\phi, t) = [p_\theta \dot{\theta} + p_\phi \dot{\phi} - L]$ by assumed Lagrange functionality

$$\frac{\partial H}{\partial \theta} = -\dot{p}_\theta \quad \frac{\partial H}{\partial p_\theta} = \dot{\theta} \quad \frac{\partial H}{\partial \dot{\theta}} = 0 \quad \frac{\partial H}{\partial \phi} = -\dot{p}_\phi \quad \frac{\partial H}{\partial p_\phi} = \dot{\phi} \quad \frac{\partial H}{\partial \dot{\phi}} = 0$$

Hamilton's equations

by Lagrange equations

$$\text{Total KE} = T = \frac{1}{2} [M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2] = \frac{1}{2} [(MR^2 + mr^2)\dot{\theta}^2 - 2mrl \cos(\theta - \phi)\dot{\theta}\dot{\phi} + ml^2\dot{\phi}^2]$$



$$T = \frac{1}{2} \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} MR^2 + mr^2 & -mrl \cos(\theta - \phi) \\ -mrl \cos(\theta - \phi) & ml^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Dynamic metric tensor
 γ_{mn}
in GCC θ and ϕ

Lagrangian function of GCC and velocities: $L(\theta, \phi, \dot{\theta}, \dot{\phi}, t) = T(\theta, \phi, \dot{\theta}, \dot{\phi}, t) - V(\theta, \phi, t)$

$$dL(\theta, \phi, \dot{\theta}, \dot{\phi}, t) = \frac{\partial L}{\partial \theta} d\theta + \frac{\partial L}{\partial \phi} d\phi + \frac{\partial L}{\partial \dot{\theta}} d\dot{\theta} + \frac{\partial L}{\partial \dot{\phi}} d\dot{\phi} + \frac{\partial L}{\partial t} dt$$

$$\dot{L}(\theta, \phi, \dot{\theta}, \dot{\phi}, t) = \frac{dL}{dt} = \frac{\partial L}{\partial \theta} \frac{d\theta}{dt} + \frac{\partial L}{\partial \phi} \frac{d\phi}{dt} + \frac{\partial L}{\partial \dot{\theta}} \frac{d\dot{\theta}}{dt} + \frac{\partial L}{\partial \dot{\phi}} \frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

velocity chain

$$\dot{L}(\theta, \phi, \dot{\theta}, \dot{\phi}, t) = \frac{dL}{dt} = \dot{p}_\theta \frac{d\theta}{dt} + \dot{p}_\phi \frac{d\phi}{dt} + p_\theta \frac{d\dot{\theta}}{dt} + p_\phi \frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

Lagrange equations

$$= \frac{dL}{dt} = \frac{d}{dt} (p_\theta \dot{\theta} + p_\phi \dot{\phi}) + \frac{\partial L}{\partial t}$$

(Consolidating)

$$\frac{d}{dt} (p_\theta \dot{\theta} + p_\phi \dot{\phi} - L) = -\frac{\partial L}{\partial t}$$

(Rearranging)

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

Defining the Hamiltonian function

Hamiltonian function of GCC and momenta: $H(\theta, \phi, p_\theta, p_\phi, t) = p_\theta \dot{\theta} + p_\phi \dot{\phi} - L$ *Poincare-Legendre relation*

$$\frac{\partial H}{\partial \theta} = -\dot{p}_\theta \quad \frac{\partial H}{\partial p_\theta} = \dot{\theta} \quad \frac{\partial H}{\partial \dot{\theta}} \equiv 0$$

$$\frac{\partial H}{\partial \phi} = -\dot{p}_\phi \quad \frac{\partial H}{\partial p_\phi} = \dot{\phi} \quad \frac{\partial H}{\partial \dot{\phi}} \equiv 0$$

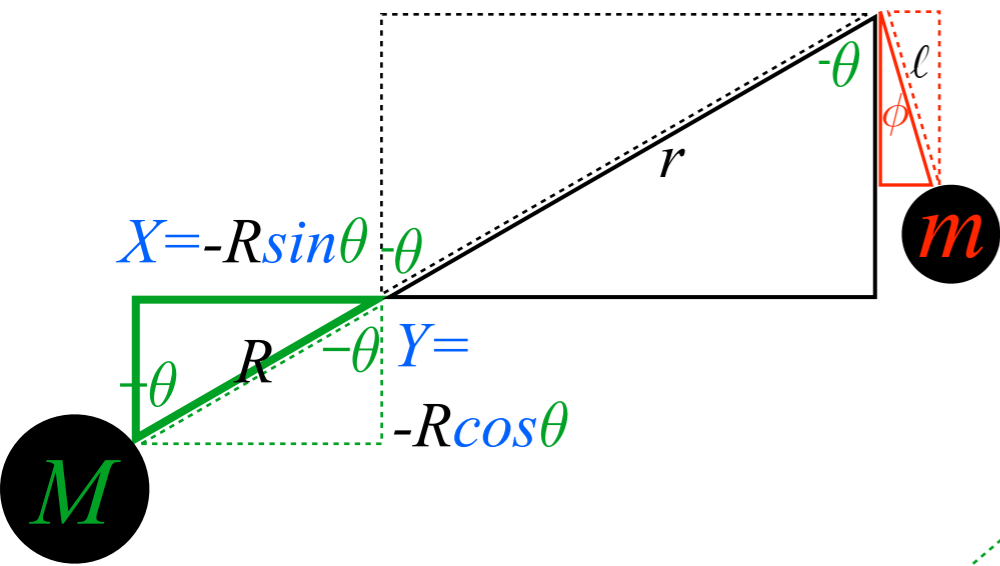
Hamilton's equations

Review of Hamiltonian equation derivation (Elementary trebuchet)

Hamiltonian definition from Lagrangian and γ_{mn} tensor

Hamilton's equations and Poincare invariant relations

 *Hamiltonian expression and contravariant γ^{mn} tensor*



$$\text{Total KE} = T = \frac{1}{2} [M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2] = \frac{1}{2} [(MR^2 + mr^2)\dot{\theta}^2 - 2mrl \cos(\theta - \phi)\dot{\theta}\dot{\phi} + ml^2\dot{\phi}^2]$$

$$T = \frac{1}{2} \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} MR^2 + mr^2 & -mrl \cos(\theta - \phi) \\ -mrl \cos(\theta - \phi) & ml^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta\theta} & \gamma_{\theta\phi} \\ \gamma_{\phi\theta} & \gamma_{\phi\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \quad \text{Covariant metric tensor} \quad \gamma_{mn}$$

$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} \quad \text{Contravariant metric tensor} \quad \gamma^{mn}$$

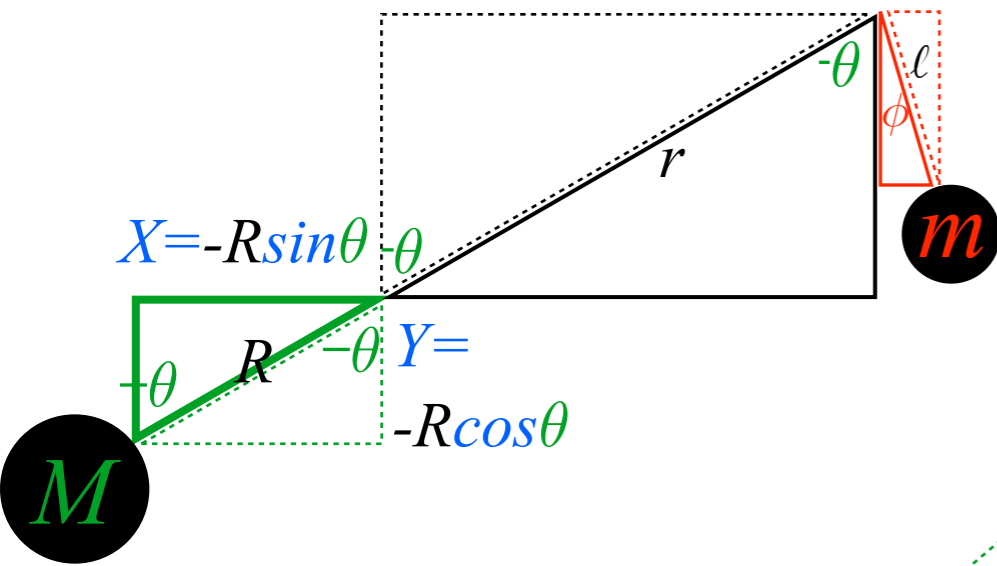
$$T = \frac{1}{2} \begin{pmatrix} p_\theta & p_\phi \end{pmatrix} \begin{pmatrix} m\ell^2 & mrl \cos(\theta - \phi) \\ mrl \cos(\theta - \phi) & MR^2 + mr^2 \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \frac{1}{2} \gamma^{mn} p_m p_n$$

Lagrangian function of GCC and velocities: $L(\theta, \phi, \dot{\theta}, \dot{\phi}, t) = T(\theta, \phi, \dot{\theta}, \dot{\phi}, t) - V(\theta, \phi, t)$

Hamiltonian function of GCC and momenta: $H(\theta, \phi, p_\theta, p_\phi, t) = p_\theta \dot{\theta} + p_\phi \dot{\phi} - L$ *Poincare-Legendre relation*

$$H = p_\theta \dot{\theta} + p_\phi \dot{\phi} - T + V$$

$$H = (\gamma_{\theta\theta} \dot{\theta} + \gamma_{\theta\phi} \dot{\phi}) \dot{\theta} + (\gamma_{\phi\theta} \dot{\theta} + \gamma_{\phi\phi} \dot{\phi}) \dot{\phi} - \frac{1}{2} (\gamma_{\theta\theta} \dot{\theta} \dot{\theta} + \gamma_{\theta\phi} \dot{\theta} \dot{\phi} + \gamma_{\phi\theta} \dot{\theta} \dot{\phi} + \gamma_{\phi\phi} \dot{\phi} \dot{\phi}) + V$$



$$\text{Total KE} = T = \frac{1}{2} [M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2] = \frac{1}{2} [(MR^2 + mr^2)\dot{\theta}^2 - 2mrl \cos(\theta - \phi)\dot{\theta}\dot{\phi} + ml^2\dot{\phi}^2]$$

$$T = \frac{1}{2} \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} MR^2 + mr^2 & -mrl \cos(\theta - \phi) \\ -mrl \cos(\theta - \phi) & ml^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

$$\begin{pmatrix} p_{\theta} \\ p_{\phi} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta\theta} & \gamma_{\theta\phi} \\ \gamma_{\phi\theta} & \gamma_{\phi\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \quad \text{Covariant metric tensor} \quad \gamma_{mn}$$

$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix} \begin{pmatrix} p_{\theta} \\ p_{\phi} \end{pmatrix} \quad \text{Contravariant metric tensor} \quad \gamma^{mn}$$

$$T = \frac{1}{2} \begin{pmatrix} p_{\theta} & p_{\phi} \end{pmatrix} \begin{pmatrix} ml^2 & mrl \cos(\theta - \phi) \\ mrl \cos(\theta - \phi) & MR^2 + mr^2 \end{pmatrix} \begin{pmatrix} p_{\theta} \\ p_{\phi} \end{pmatrix} = \frac{1}{2} \gamma^{mn} p_m p_n$$

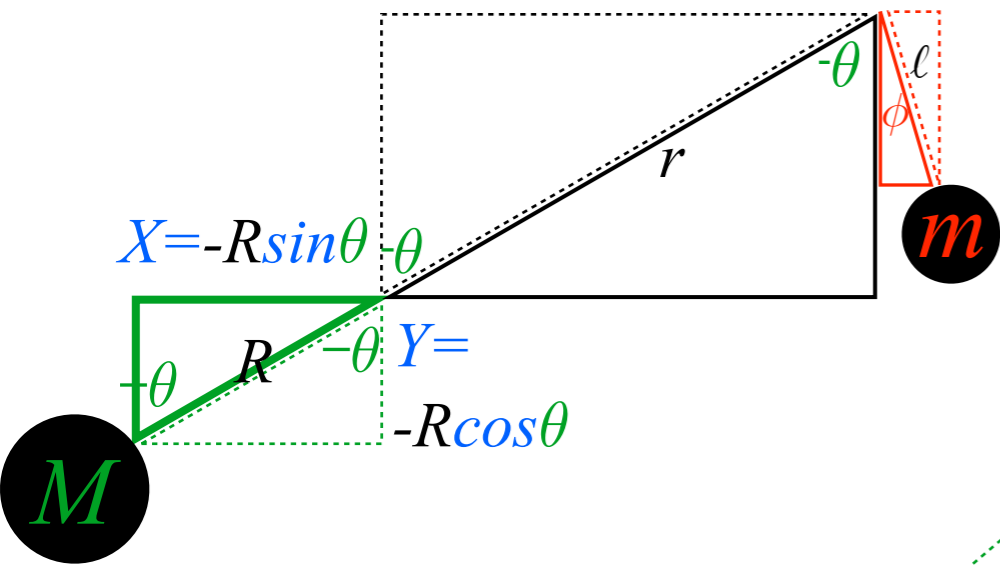
Lagrangian function of GCC and velocities: $L(\theta, \phi, \dot{\theta}, \dot{\phi}, t) = T(\theta, \phi, \dot{\theta}, \dot{\phi}, t) - V(\theta, \phi, t)$

Hamiltonian function of GCC and momenta: $H(\theta, \phi, p_{\theta}, p_{\phi}, t) = p_{\theta}\dot{\theta} + p_{\phi}\dot{\phi} - L$ *Poincare-Legendre relation*

$$H = p_{\theta}\dot{\theta} + p_{\phi}\dot{\phi} - T + V$$

$$H = (\gamma_{\theta\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi})\dot{\theta} + (\gamma_{\phi\theta}\dot{\theta} + \gamma_{\phi\phi}\dot{\phi})\dot{\phi} - \frac{1}{2}(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}) + V$$

$$H = \frac{1}{2}(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}) + V = T + V \quad (\text{Only correct numerically!})$$



$$\text{Total KE} = T = \frac{1}{2} [M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2] = \frac{1}{2} [(MR^2 + mr^2)\dot{\theta}^2 - 2mrl \cos(\theta - \phi)\dot{\theta}\dot{\phi} + ml^2\dot{\phi}^2]$$

$$T = \frac{1}{2} \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} MR^2 + mr^2 & -mrl \cos(\theta - \phi) \\ -mrl \cos(\theta - \phi) & ml^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

$$\begin{pmatrix} p_{\theta} \\ p_{\phi} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta\theta} & \gamma_{\theta\phi} \\ \gamma_{\phi\theta} & \gamma_{\phi\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \quad \text{Covariant metric tensor} \quad \gamma_{mn}$$

$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix} \begin{pmatrix} p_{\theta} \\ p_{\phi} \end{pmatrix} \quad \text{Contravariant metric tensor} \quad \gamma^{mn}$$

$$T = \frac{1}{2} \begin{pmatrix} p_{\theta} & p_{\phi} \end{pmatrix} \begin{pmatrix} m\ell^2 & mrl \cos(\theta - \phi) \\ mrl \cos(\theta - \phi) & MR^2 + mr^2 \end{pmatrix} \begin{pmatrix} p_{\theta} \\ p_{\phi} \end{pmatrix} = \frac{1}{2} \gamma^{mn} p_m p_n$$

Lagrangian function of GCC and velocities: $L(\theta, \phi, \dot{\theta}, \dot{\phi}, t) = T(\theta, \phi, \dot{\theta}, \dot{\phi}, t) - V(\theta, \phi, t)$

Hamiltonian function of GCC and momenta: $H(\theta, \phi, p_{\theta}, p_{\phi}, t) = p_{\theta}\dot{\theta} + p_{\phi}\dot{\phi} - L$ *Poincare-Legendre relation*

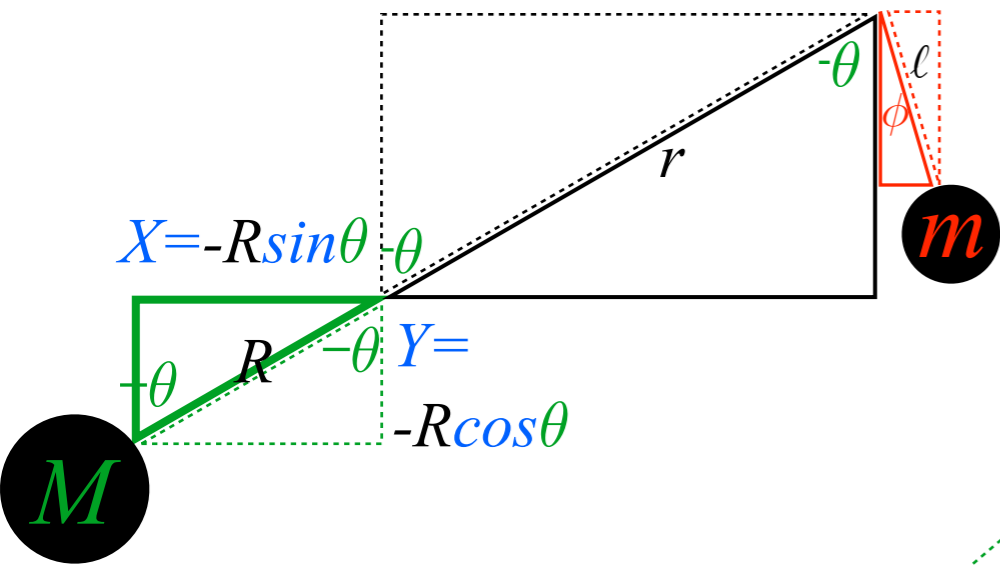
$$H = p_{\theta}\dot{\theta} + p_{\phi}\dot{\phi} - T + V$$

$$H = (\gamma_{\theta\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi})\dot{\theta} + (\gamma_{\phi\theta}\dot{\theta} + \gamma_{\phi\phi}\dot{\phi})\dot{\phi} - \frac{1}{2}(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}) + V$$

$$H = \frac{1}{2}(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}) + V = T + V \equiv E \quad \text{(Only correct numerically!)}$$

Constant Energy if no explicit t-dependence

Hamiltonian must be explicit in momenta p_m



$$\text{Total KE} = T = \frac{1}{2} [M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2] = \frac{1}{2} [(MR^2 + mr^2)\dot{\theta}^2 - 2mrl \cos(\theta - \phi)\dot{\theta}\dot{\phi} + ml^2\dot{\phi}^2]$$

$$T = \frac{1}{2} \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} MR^2 + mr^2 & -mrl \cos(\theta - \phi) \\ -mrl \cos(\theta - \phi) & ml^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta\theta} & \gamma_{\theta\phi} \\ \gamma_{\phi\theta} & \gamma_{\phi\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \quad \text{Covariant metric tensor} \quad \gamma_{mn}$$

$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} \quad \text{Contravariant metric tensor} \quad \gamma^{mn}$$

$$T = \frac{1}{2} \begin{pmatrix} p_\theta & p_\phi \end{pmatrix} \begin{pmatrix} ml^2 & mrl \cos(\theta - \phi) \\ mrl \cos(\theta - \phi) & MR^2 + mr^2 \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \frac{1}{2} \gamma^{mn} p_m p_n$$

Lagrangian function of GCC and velocities: $L(\theta, \phi, \dot{\theta}, \dot{\phi}, t) = T(\theta, \phi, \dot{\theta}, \dot{\phi}, t) - V(\theta, \phi, t)$

Hamiltonian function of GCC and momenta: $H(\theta, \phi, p_\theta, p_\phi, t) = p_\theta \dot{\theta} + p_\phi \dot{\phi} - L$ *Poincare-Legendre relation*

$$H = p_\theta \dot{\theta} + p_\phi \dot{\phi} - T + V$$

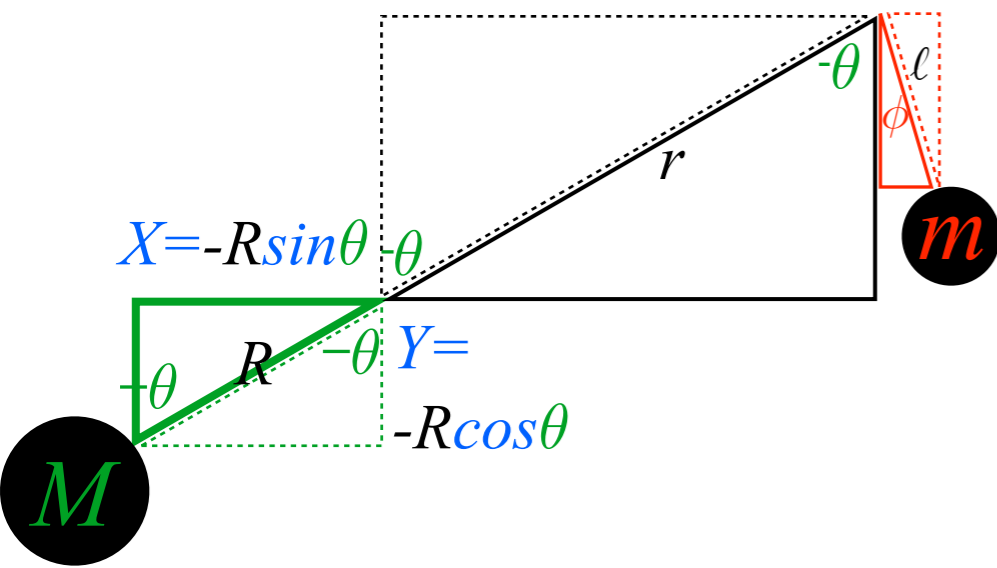
$$H = (\gamma_{\theta\theta} \dot{\theta} + \gamma_{\theta\phi} \dot{\phi}) \dot{\theta} + (\gamma_{\phi\theta} \dot{\theta} + \gamma_{\phi\phi} \dot{\phi}) \dot{\phi} - \frac{1}{2} (\gamma_{\theta\theta} \dot{\theta} \dot{\theta} + \gamma_{\theta\phi} \dot{\phi} \dot{\theta} + \gamma_{\phi\theta} \dot{\theta} \dot{\phi} + \gamma_{\phi\phi} \dot{\phi} \dot{\phi}) + V$$

$$H = \frac{1}{2} (\gamma_{\theta\theta} \dot{\theta} \dot{\theta} + \gamma_{\theta\phi} \dot{\phi} \dot{\theta} + \gamma_{\phi\theta} \dot{\theta} \dot{\phi} + \gamma_{\phi\phi} \dot{\phi} \dot{\phi}) + V = T + V \equiv E \quad \text{(Only correct numerically!)}$$

$$H = \frac{1}{2} (\gamma^{\theta\theta} p_\theta p_\theta + \gamma^{\theta\phi} p_\theta p_\phi + \gamma^{\phi\theta} p_\phi p_\theta + \gamma^{\phi\phi} p_\phi p_\phi) + V = T + V \quad \text{(Correct formally and numerically)}$$

Constant Energy if no explicit t-dependence

Hamiltonian must be explicit in momenta p_m



$$\text{Total KE} = T = \frac{1}{2} [M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2] = \frac{1}{2} [(MR^2 + mr^2)\dot{\theta}^2 - 2mrl \cos(\theta - \phi)\dot{\theta}\dot{\phi} + ml^2\dot{\phi}^2]$$

$$T = \frac{1}{2} \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} MR^2 + mr^2 & -mrl \cos(\theta - \phi) \\ -mrl \cos(\theta - \phi) & ml^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta\theta} & \gamma_{\theta\phi} \\ \gamma_{\phi\theta} & \gamma_{\phi\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \quad \text{Covariant metric tensor} \quad \gamma_{mn}$$

$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} \quad \text{Contravariant metric tensor} \quad \gamma^{mn}$$

$$T = \frac{1}{2} \begin{pmatrix} p_\theta & p_\phi \end{pmatrix} \begin{pmatrix} ml^2 & mrl \cos(\theta - \phi) \\ mrl \cos(\theta - \phi) & MR^2 + mr^2 \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \frac{1}{2} \gamma^{mn} p_m p_n$$

Lagrangian function of GCC and velocities: $L(\theta, \phi, \dot{\theta}, \dot{\phi}, t) = T(\theta, \phi, \dot{\theta}, \dot{\phi}, t) - V(\theta, \phi, t)$

Hamiltonian function of GCC and momenta: $H(\theta, \phi, p_\theta, p_\phi, t) = p_\theta \dot{\theta} + p_\phi \dot{\phi} - L$ *Poincare-Legendre relation*

$$H = p_\theta \dot{\theta} + p_\phi \dot{\phi} - T + V$$

$$H = (\gamma_{\theta\theta} \dot{\theta} + \gamma_{\theta\phi} \dot{\phi}) \dot{\theta} + (\gamma_{\phi\theta} \dot{\theta} + \gamma_{\phi\phi} \dot{\phi}) \dot{\phi} - \frac{1}{2} (\gamma_{\theta\theta} \dot{\theta}^2 + \gamma_{\theta\phi} \dot{\theta} \dot{\phi} + \gamma_{\phi\theta} \dot{\theta} \dot{\phi} + \gamma_{\phi\phi} \dot{\phi}^2) + V$$

$$H = \frac{1}{2} (\gamma_{\theta\theta} \dot{\theta}^2 + \gamma_{\theta\phi} \dot{\theta} \dot{\phi} + \gamma_{\phi\theta} \dot{\theta} \dot{\phi} + \gamma_{\phi\phi} \dot{\phi}^2) + V = T + V \equiv E \quad \text{(Only correct numerically!)}$$

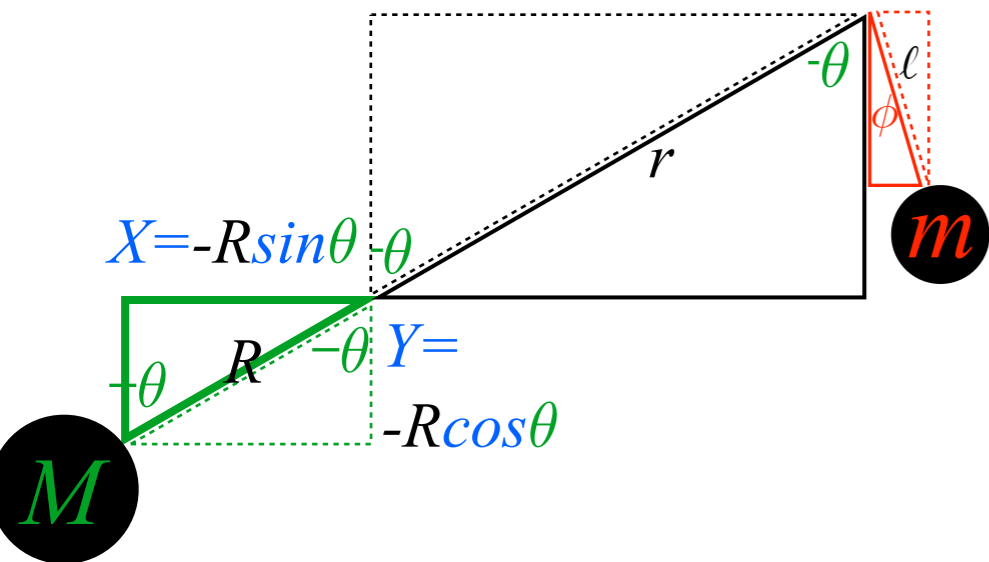
Constant Energy if no explicit t-dependence

Hamiltonian must be explicit in momenta p_m

$$H = \frac{1}{2} (\gamma^{\theta\theta} p_\theta p_\theta + \gamma^{\theta\phi} p_\theta p_\phi + \gamma^{\phi\theta} p_\phi p_\theta + \gamma^{\phi\phi} p_\phi p_\phi) + V = T + V \quad \text{(Correct formally and numerically)}$$

$$H = \frac{ml^2 p_\theta p_\theta + 2mrl \cos(\theta - \phi) p_\theta p_\phi + (MR^2 + mr^2) p_\phi p_\phi}{2ml^2 [MR^2 + mr^2 \sin^2(\theta - \phi)]} + V$$

Hamilton equations for elementary trebuchet



$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix} \begin{pmatrix} p_{\theta} \\ p_{\phi} \end{pmatrix} \quad \text{Contravariant metric tensor}$$

$$T = \frac{1}{2} \begin{pmatrix} p_{\theta} & p_{\phi} \end{pmatrix} \begin{pmatrix} ml^2 & mrl \cos(\theta - \phi) \\ mrl \cos(\theta - \phi) & MR^2 + mr^2 \end{pmatrix} \begin{pmatrix} p_{\theta} \\ p_{\phi} \end{pmatrix} = \frac{1}{2} \gamma^{mn} p_m p_n$$

$$ml^2 [MR^2 + mr^2 \sin^2(\theta - \phi)]$$

$$\begin{aligned} \dot{\theta} &= \gamma^{\theta\theta} p_{\theta} + \gamma^{\theta\phi} p_{\phi} \\ \dot{\phi} &= \gamma^{\phi\theta} p_{\theta} + \gamma^{\phi\phi} p_{\phi} \end{aligned}$$

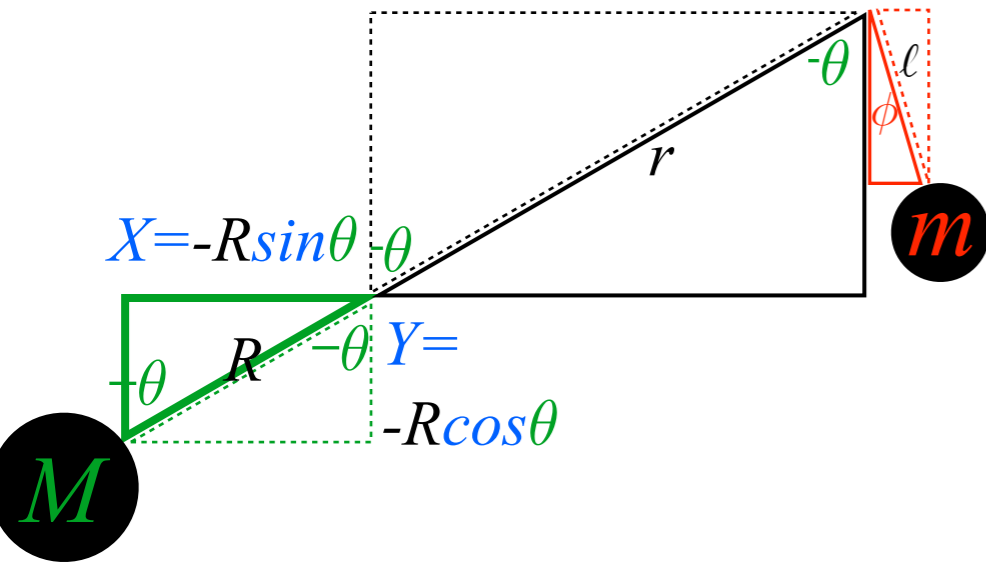
$$\begin{aligned} p_{\theta} &= \gamma_{\theta\theta} \dot{\theta} + \gamma_{\theta\phi} \dot{\phi} \\ p_{\phi} &= \gamma_{\phi\theta} \dot{\theta} + \gamma_{\phi\phi} \dot{\phi} \end{aligned}$$

Coordinate equations

$$\frac{\partial H}{\partial p_{\theta}} = \dot{\theta} = \gamma^{\phi\theta} p_{\theta} + \gamma^{\phi\phi} p_{\phi}$$

$$\frac{\partial H}{\partial p_{\phi}} = \dot{\phi} = \gamma^{\theta\theta} p_{\theta} + \gamma^{\theta\phi} p_{\phi}$$

Hamilton equations for elementary trebuchet



$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix} \begin{pmatrix} p_{\theta} \\ p_{\phi} \end{pmatrix} \quad \text{Contravariant metric tensor}$$

$$T = \frac{1}{2} \begin{pmatrix} p_{\theta} & p_{\phi} \end{pmatrix} \begin{pmatrix} ml^2 & mrl \cos(\theta - \phi) \\ mrl \cos(\theta - \phi) & MR^2 + mr^2 \end{pmatrix} \begin{pmatrix} p_{\theta} \\ p_{\phi} \end{pmatrix} = \frac{1}{2} \gamma^{mn} p_m p_n$$

$$\begin{aligned} \dot{\theta} &= \gamma^{\theta\theta} p_{\theta} + \gamma^{\theta\phi} p_{\phi} \\ \dot{\phi} &= \gamma^{\phi\theta} p_{\theta} + \gamma^{\phi\phi} p_{\phi} \end{aligned}$$

$$\begin{aligned} p_{\theta} &= \gamma_{\theta\theta} \dot{\theta} + \gamma_{\theta\phi} \dot{\phi} \\ p_{\phi} &= \gamma_{\phi\theta} \dot{\theta} + \gamma_{\phi\phi} \dot{\phi} \end{aligned}$$

Coordinate equations

$$\frac{\partial H}{\partial p_{\theta}} = \dot{\theta} = \gamma^{\phi\theta} p_{\theta} + \gamma^{\phi\phi} p_{\phi}$$

$$\frac{\partial H}{\partial p_{\phi}} = \dot{\phi} = \gamma^{\theta\theta} p_{\theta} + \gamma^{\theta\phi} p_{\phi}$$

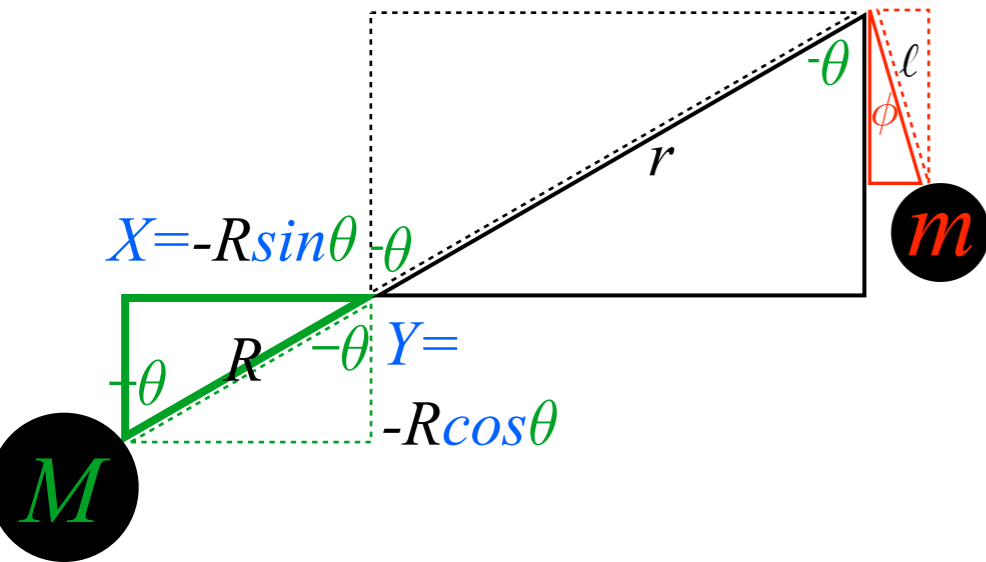
Momentum/force equations

$$\begin{aligned} \dot{p}_{\theta} &= -\frac{\partial H}{\partial \theta} = \frac{\partial L}{\partial \theta} = \frac{\partial T}{\partial \theta} - \frac{\partial V}{\partial \theta} \\ &= mrl \dot{\theta} \dot{\phi} \sin(\theta - \phi) + F_{\theta} \end{aligned}$$

(May just use Lagrange results...
...but to be formally correct...
...must convert contra-velocities
to covariant momenta!)

$$\begin{aligned} \dot{p}_{\phi} &= -\frac{\partial H}{\partial \phi} = \frac{\partial L}{\partial \phi} = \frac{\partial T}{\partial \phi} - \frac{\partial V}{\partial \phi} \\ &= -mrl \dot{\theta} \dot{\phi} \sin(\theta - \phi) + F_{\phi} \end{aligned}$$

Hamilton equations for elementary trebuchet



$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix} \begin{pmatrix} p_{\theta} \\ p_{\phi} \end{pmatrix} \quad \text{Contravariant metric tensor}$$

$$T = \frac{1}{2} \begin{pmatrix} p_{\theta} & p_{\phi} \end{pmatrix} \begin{pmatrix} ml^2 & mrl \cos(\theta - \phi) \\ mrl \cos(\theta - \phi) & MR^2 + mr^2 \end{pmatrix} \begin{pmatrix} p_{\theta} \\ p_{\phi} \end{pmatrix} = \frac{1}{2} \gamma^{mn} p_m p_n$$

$$\begin{aligned} \dot{\theta} &= \gamma^{\theta\theta} p_{\theta} + \gamma^{\theta\phi} p_{\phi} \\ \dot{\phi} &= \gamma^{\phi\theta} p_{\theta} + \gamma^{\phi\phi} p_{\phi} \end{aligned}$$

$$\begin{aligned} p_{\theta} &= \gamma_{\theta\theta} \dot{\theta} + \gamma_{\theta\phi} \dot{\phi} \\ p_{\phi} &= \gamma_{\phi\theta} \dot{\theta} + \gamma_{\phi\phi} \dot{\phi} \end{aligned}$$

Coordinate equations

$$\frac{\partial H}{\partial p_{\theta}} = \dot{\theta} = \gamma^{\phi\theta} p_{\theta} + \gamma^{\phi\phi} p_{\phi}$$

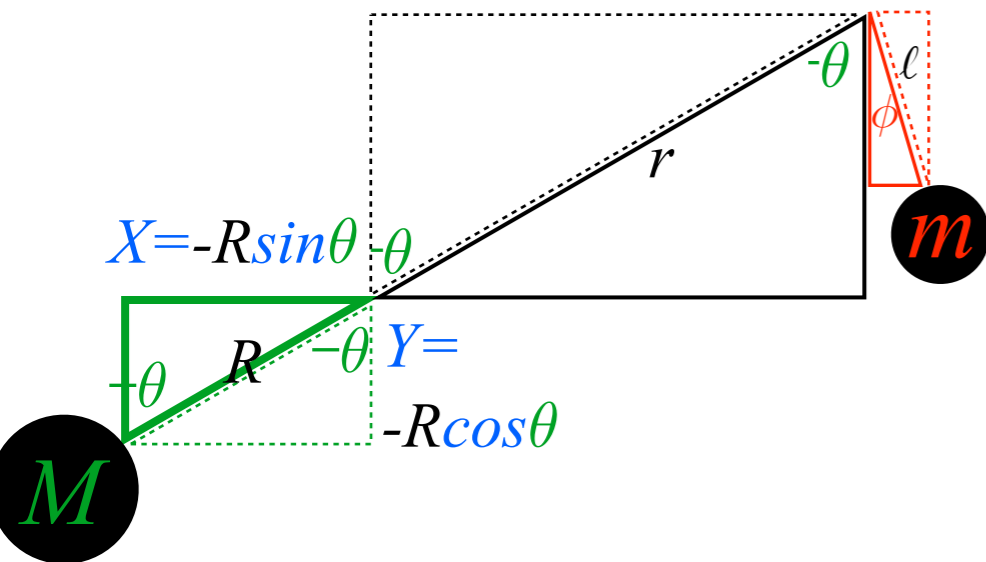
$$\frac{\partial H}{\partial p_{\phi}} = \dot{\phi} = \gamma^{\theta\theta} p_{\theta} + \gamma^{\theta\phi} p_{\phi}$$

Momentum/force equations

$$\begin{aligned} \dot{p}_{\theta} &= -\frac{\partial H}{\partial \theta} = \frac{\partial L}{\partial \theta} = \frac{\partial T}{\partial \theta} - \frac{\partial V}{\partial \theta} \quad (\text{May just use Lagrange results...}) \\ &= mrl \dot{\theta} \dot{\phi} \sin(\theta - \phi) + F_{\theta} \quad (\text{...but to be formally correct...}) \\ &= mrl (\gamma^{\theta\theta} p_{\theta} + \gamma^{\theta\phi} p_{\phi}) (\gamma^{\phi\theta} p_{\theta} + \gamma^{\phi\phi} p_{\phi}) \sin(\theta - \phi) + F_{\theta} \quad (\text{...must convert contra-velocities to covariant momenta!}) \end{aligned}$$

$$\begin{aligned} \dot{p}_{\phi} &= -\frac{\partial H}{\partial \phi} = \frac{\partial L}{\partial \phi} = \frac{\partial T}{\partial \phi} - \frac{\partial V}{\partial \phi} \\ &= -mrl \dot{\theta} \dot{\phi} \sin(\theta - \phi) + F_{\phi} \\ &= -mrl (\gamma^{\theta\theta} p_{\theta} + \gamma^{\theta\phi} p_{\phi}) (\gamma^{\phi\theta} p_{\theta} + \gamma^{\phi\phi} p_{\phi}) \sin(\theta - \phi) + F_{\phi} \end{aligned}$$

Hamilton equations for elementary trebuchet



$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} \quad \text{Contravariant metric tensor}$$

$$T = \frac{1}{2} \begin{pmatrix} p_\theta & p_\phi \end{pmatrix} \begin{pmatrix} ml^2 & mrl \cos(\theta - \phi) \\ mrl \cos(\theta - \phi) & MR^2 + mr^2 \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \frac{1}{2} \gamma^{mn} p_m p_n$$

$$\begin{aligned} \dot{\theta} &= \gamma^{\theta\theta} p_\theta + \gamma^{\theta\phi} p_\phi \\ \dot{\phi} &= \gamma^{\phi\theta} p_\theta + \gamma^{\phi\phi} p_\phi \end{aligned}$$

$$\begin{aligned} p_\theta &= \gamma_{\theta\theta} \dot{\theta} + \gamma_{\theta\phi} \dot{\phi} \\ p_\phi &= \gamma_{\phi\theta} \dot{\theta} + \gamma_{\phi\phi} \dot{\phi} \end{aligned}$$

Coordinate equations

$$\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \gamma^{\phi\theta} p_\theta + \gamma^{\phi\phi} p_\phi$$

$$\frac{\partial H}{\partial p_\phi} = \dot{\phi} = \gamma^{\theta\theta} p_\theta + \gamma^{\theta\phi} p_\phi$$

Momentum/force equations

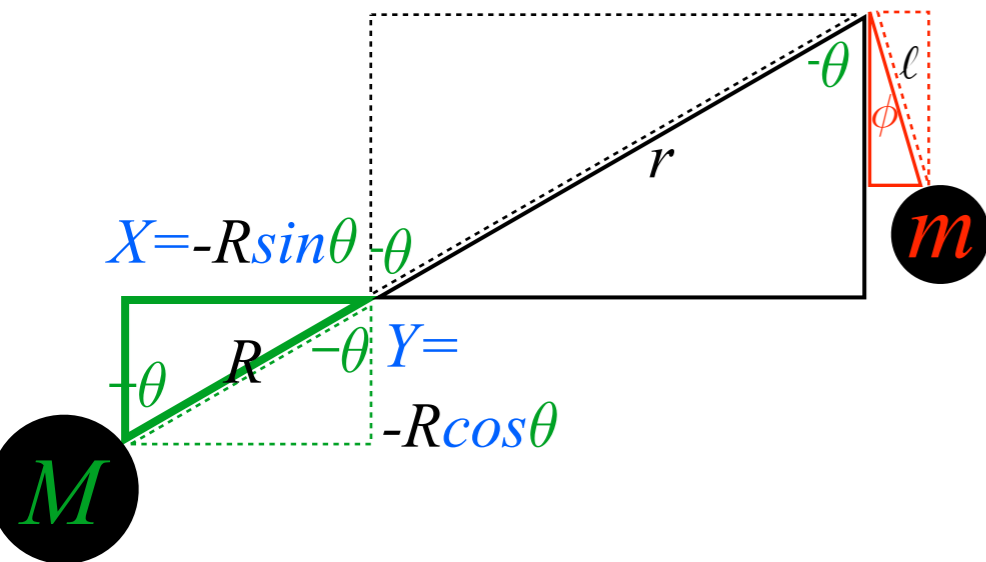
$$\begin{aligned} \dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = \frac{\partial L}{\partial \theta} = \frac{\partial T}{\partial \theta} - \frac{\partial V}{\partial \theta} \quad (\text{May just use Lagrange results...}) \\ &= mr\ell \dot{\theta} \dot{\phi} \sin(\theta - \phi) + F_\theta \quad (\text{...but to be formally correct...}) \\ &\quad (\text{...must convert contra-velocities to covariant momenta!}) \end{aligned}$$

$$\begin{aligned} \dot{p}_\phi &= -\frac{\partial H}{\partial \phi} = \frac{\partial L}{\partial \phi} = \frac{\partial T}{\partial \phi} - \frac{\partial V}{\partial \phi} \\ &= -mr\ell \dot{\theta} \dot{\phi} \sin(\theta - \phi) + F_\phi \end{aligned}$$

$$\begin{aligned} &= mr\ell (\gamma^{\theta\theta} p_\theta + \gamma^{\theta\phi} p_\phi) (\gamma^{\phi\theta} p_\theta + \gamma^{\phi\phi} p_\phi) \sin(\theta - \phi) + F_\theta \\ &= mr\ell (\gamma^{\theta\theta} \gamma^{\phi\theta} p_\theta^2 + [\gamma^{\theta\theta} \gamma^{\phi\phi} + (\gamma^{\theta\phi})^2] p_\phi p_\theta + \gamma^{\theta\phi} \gamma^{\phi\phi} p_\phi^2) \sin(\theta - \phi) + F_\theta \end{aligned}$$

$$\begin{aligned} &= -mr\ell (\gamma^{\theta\theta} p_\theta + \gamma^{\theta\phi} p_\phi) (\gamma^{\phi\theta} p_\theta + \gamma^{\phi\phi} p_\phi) \sin(\theta - \phi) + F_\phi \\ &= -[\text{messy factor}] \sin(\theta - \phi) + F_\phi \end{aligned}$$

Hamilton equations for elementary trebuchet



$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix} \begin{pmatrix} p_{\theta} \\ p_{\phi} \end{pmatrix} \quad \text{Contravariant metric tensor}$$

$$T = \frac{1}{2} \begin{pmatrix} p_{\theta} & p_{\phi} \end{pmatrix} \begin{pmatrix} ml^2 & mrl \cos(\theta - \phi) \\ mrl \cos(\theta - \phi) & MR^2 + mr^2 \end{pmatrix} \begin{pmatrix} p_{\theta} \\ p_{\phi} \end{pmatrix} = \frac{1}{2} \gamma^{mn} p_m p_n$$

$$\begin{aligned} \dot{\theta} &= \gamma^{\theta\theta} p_{\theta} + \gamma^{\theta\phi} p_{\phi} \\ \dot{\phi} &= \gamma^{\phi\theta} p_{\theta} + \gamma^{\phi\phi} p_{\phi} \end{aligned}$$

$$\begin{aligned} p_{\theta} &= \gamma_{\theta\theta} \dot{\theta} + \gamma_{\theta\phi} \dot{\phi} \\ p_{\phi} &= \gamma_{\phi\theta} \dot{\theta} + \gamma_{\phi\phi} \dot{\phi} \end{aligned}$$

Coordinate equations

$$\frac{\partial H}{\partial p_{\theta}} = \dot{\theta} = \gamma^{\phi\theta} p_{\theta} + \gamma^{\phi\phi} p_{\phi}$$

$$\frac{\partial H}{\partial p_{\phi}} = \dot{\phi} = \gamma^{\theta\theta} p_{\theta} + \gamma^{\theta\phi} p_{\phi}$$

Momentum/force equations

$$\begin{aligned} \dot{p}_{\theta} &= -\frac{\partial H}{\partial \theta} = \frac{\partial L}{\partial \theta} = \frac{\partial T}{\partial \theta} - \frac{\partial V}{\partial \theta} \\ &= mrl \dot{\theta} \dot{\phi} \sin(\theta - \phi) + F_{\theta} \end{aligned}$$

$$\begin{aligned} \dot{p}_{\phi} &= -\frac{\partial H}{\partial \phi} = \frac{\partial L}{\partial \phi} = \frac{\partial T}{\partial \phi} - \frac{\partial V}{\partial \phi} \\ &= -mrl \dot{\theta} \dot{\phi} \sin(\theta - \phi) + F_{\phi} \end{aligned}$$

$$= mrl (\gamma^{\theta\theta} p_{\theta} + \gamma^{\theta\phi} p_{\phi}) (\gamma^{\phi\theta} p_{\theta} + \gamma^{\phi\phi} p_{\phi}) \sin(\theta - \phi) + F_{\theta}$$

$$= -mrl (\gamma^{\theta\theta} p_{\theta} + \gamma^{\theta\phi} p_{\phi}) (\gamma^{\phi\theta} p_{\theta} + \gamma^{\phi\phi} p_{\phi}) \sin(\theta - \phi) + F_{\phi}$$

$$= mrl (\gamma^{\theta\theta} \gamma^{\phi\theta} p_{\theta}^2 + [\gamma^{\theta\theta} \gamma^{\phi\phi} + (\gamma^{\theta\phi})^2] p_{\phi} p_{\theta} + \gamma^{\theta\phi} \gamma^{\phi\phi} p_{\phi}^2) \sin(\theta - \phi) + F_{\theta}$$

$$= -[\text{messy factor}] \sin(\theta - \phi) + F_{\phi}$$

A lesson on Hamiltonian “elegance”...

...may be very elegant formally...but may not be so elegant algebraically!

Hamiltonian energy and momentum conservation and symmetry coordinates

→ *Coordinate transformation helps reduce symmetric Hamiltonian*

Free-space trebuchet kinematics by symmetry

Algebraic approach

Direct approach and Superball analogy

Trebuchet vs Flinger and sports kinematics

Many approaches to Mechanics

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta + \pi/2$

Jacobian Lemma-1 definition:

$$\phi_B = -\theta + \phi - \pi/2$$

$$\begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta_B}{\partial \theta} & \frac{\partial \theta_B}{\partial \phi} \\ \frac{\partial \phi_B}{\partial \theta} & \frac{\partial \phi_B}{\partial \phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Lemma-1 from Lect.9 p.13

$$\begin{pmatrix} \frac{\partial \dot{\theta}_B}{\partial \dot{\theta}} & \frac{\partial \dot{\theta}_B}{\partial \dot{\phi}} \\ \frac{\partial \dot{\phi}_B}{\partial \dot{\theta}} & \frac{\partial \dot{\phi}_B}{\partial \dot{\phi}} \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta_B}{\partial \theta} & \frac{\partial \theta_B}{\partial \phi} \\ \frac{\partial \phi_B}{\partial \theta} & \frac{\partial \phi_B}{\partial \phi} \end{pmatrix}$$

(Used in Lect.14 p.60)

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B .

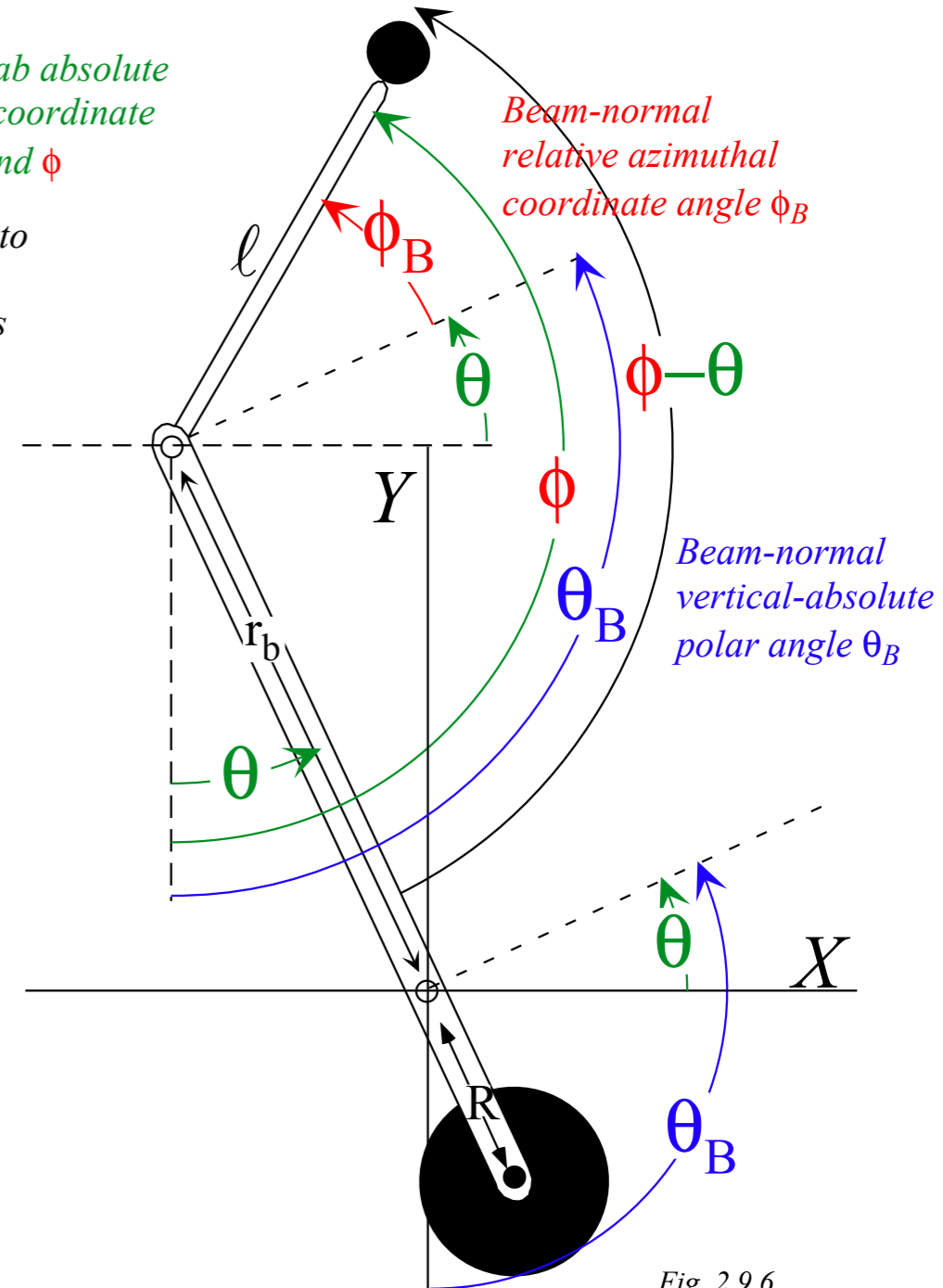


Fig. 2.9.6

Lab (θ, ϕ) and beam-normal (θ_B, ϕ_B) relative coordinates for trebuchet. (Each value is positive.)

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta + \pi/2$

Jacobian Lemma-1 definition:

$$\phi_B = -\theta + \phi - \pi/2$$

$$\begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta_B}{\partial \theta} & \frac{\partial \theta_B}{\partial \phi} \\ \frac{\partial \phi_B}{\partial \theta} & \frac{\partial \phi_B}{\partial \phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Kajobian of inverse transform $\phi_B = \phi - \theta - \pi/2$ and $\theta = \theta_B - \pi/2$

$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta}{\partial \theta_B} & \frac{\partial \theta}{\partial \phi_B} \\ \frac{\partial \phi}{\partial \theta_B} & \frac{\partial \phi}{\partial \phi_B} \end{pmatrix} \begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix} \quad \phi = \theta_B + \phi_B$$

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B .

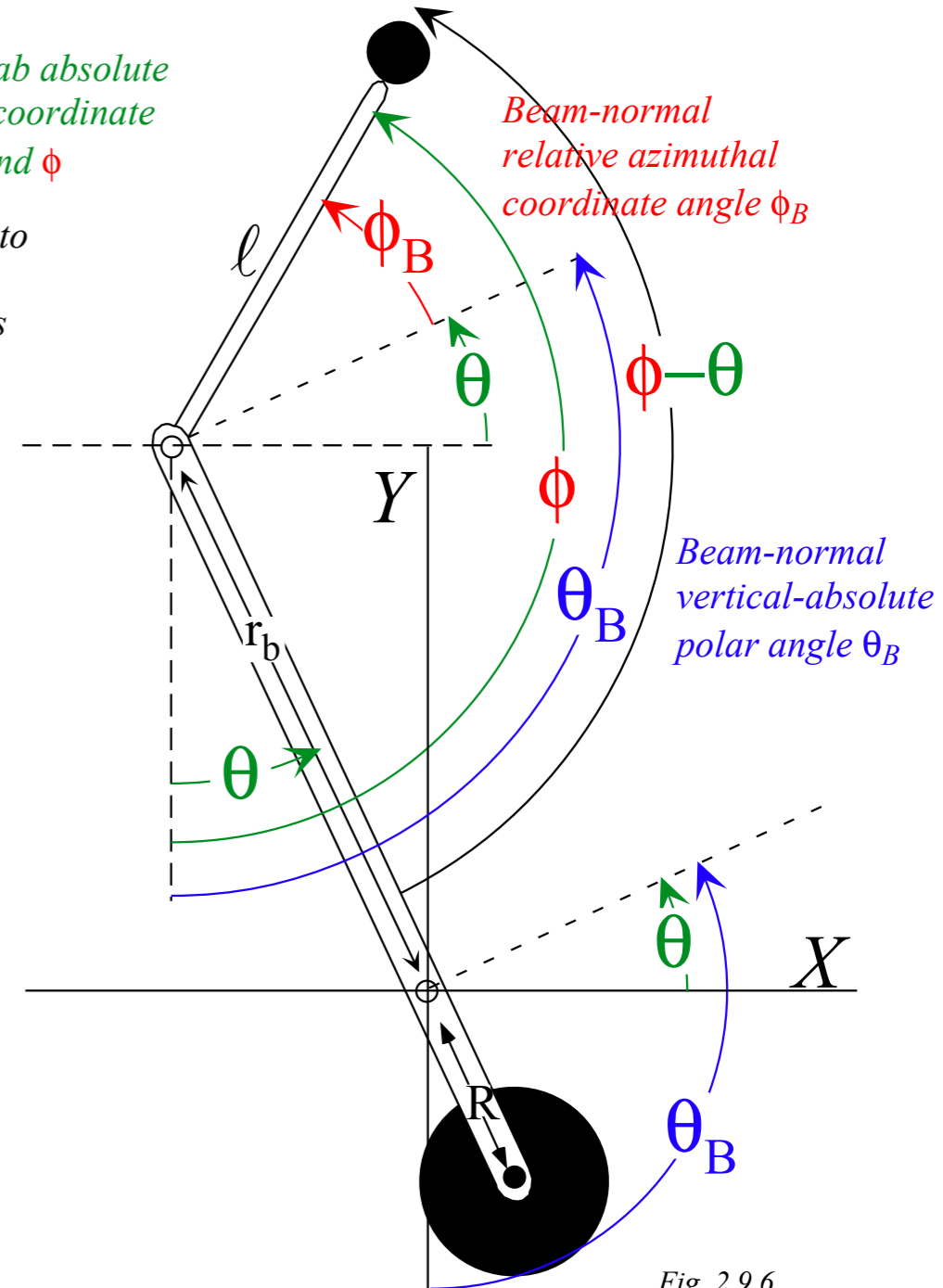


Fig. 2.9.6

Lab (θ, ϕ) and beam-normal (θ_B, ϕ_B) relative coordinates for trebuchet. (Each value is positive.)

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta + \pi/2$

Jacobian Lemma-1 definition:

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$$\begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta_B}{\partial \theta} & \frac{\partial \theta_B}{\partial \phi} \\ \frac{\partial \phi_B}{\partial \theta} & \frac{\partial \phi_B}{\partial \phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Kajobian of inverse transform $\phi_B = \phi - \theta - \pi/2$ and $\theta = \theta_B - \pi/2$

$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta}{\partial \theta_B} & \frac{\partial \theta}{\partial \phi_B} \\ \frac{\partial \phi}{\partial \theta_B} & \frac{\partial \phi}{\partial \phi_B} \end{pmatrix} \begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix}$$

$$\phi = \theta_B + \phi_B$$

Be careful with momentum!
Poincare invariance is crucial!

Poincare invariant must remain invariant

$$p_\theta \dot{\theta} + p_\phi \dot{\phi} = \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = p_\theta^B \dot{\theta}_B + p_\phi^B \dot{\phi}_B$$

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B .

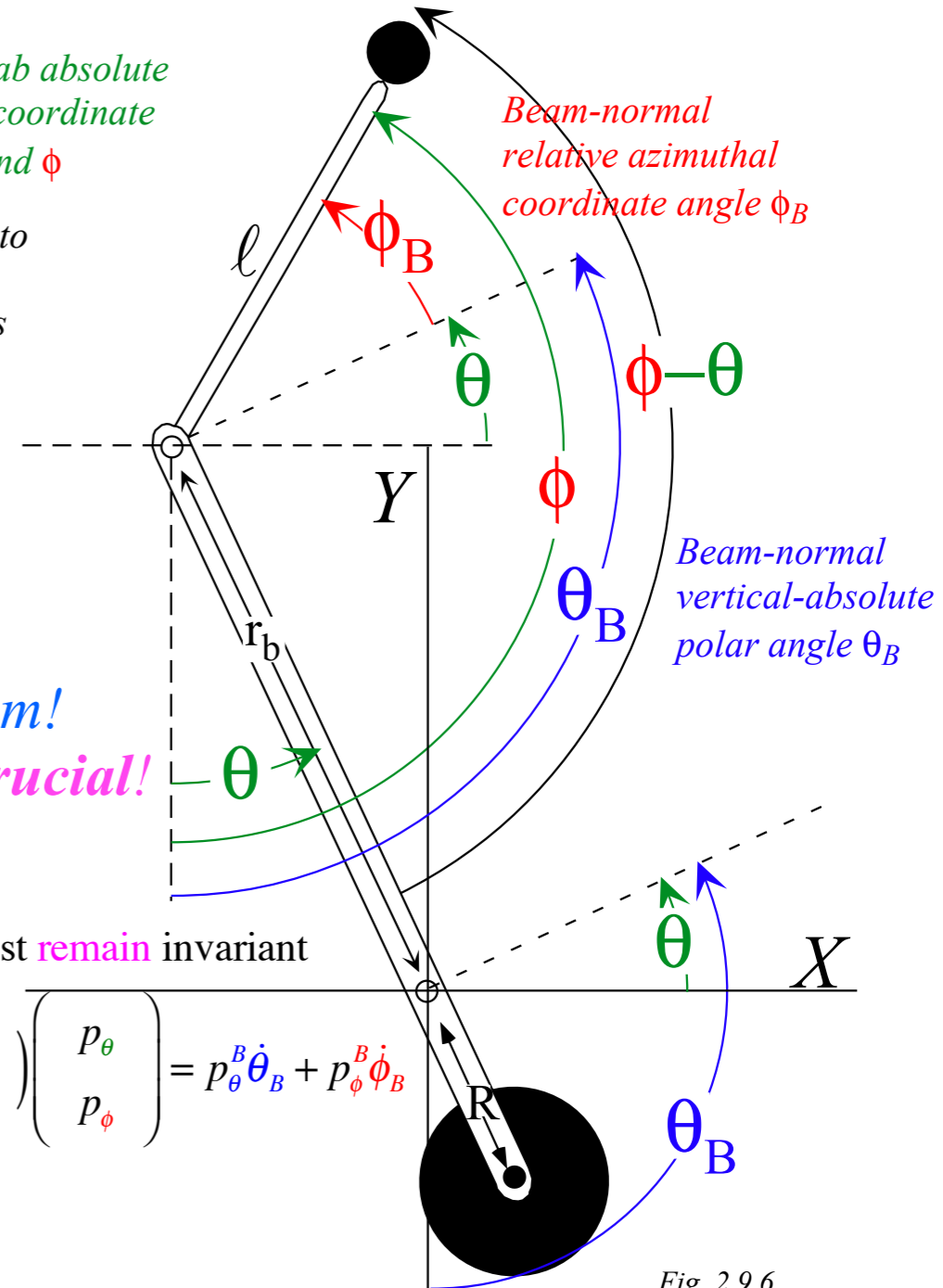


Fig. 2.9.6

Lab (θ, ϕ) and beam-normal (θ_B, ϕ_B) relative coordinates for trebuchet. (Each value is positive.)

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta + \pi/2$

Jacobian Lemma-1 definition:

$$\begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta_B}{\partial \theta} & \frac{\partial \theta_B}{\partial \phi} \\ \frac{\partial \phi_B}{\partial \theta} & \frac{\partial \phi_B}{\partial \phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Kajobian of inverse transform $\phi_B = \phi - \theta - \pi/2$ and $\theta = \theta_B - \pi/2$

$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta}{\partial \theta_B} & \frac{\partial \theta}{\partial \phi_B} \\ \frac{\partial \phi}{\partial \theta_B} & \frac{\partial \phi}{\partial \phi_B} \end{pmatrix} \begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix}$$

$$\phi_B = -\theta + \phi - \pi/2$$

$$\phi = \theta_B + \phi_B$$

p_m transform is **TRANSPOSE INVERSE** to q^m

$$\begin{pmatrix} p_{\theta}^B \\ p_{\phi}^B \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta}{\partial \theta_B} & \frac{\partial \theta}{\partial \phi_B} \\ \frac{\partial \phi}{\partial \theta_B} & \frac{\partial \phi}{\partial \phi_B} \end{pmatrix} \begin{pmatrix} p_{\theta} \\ p_{\phi} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{\theta} \\ p_{\phi} \end{pmatrix}$$

$$\begin{pmatrix} p_{\theta} \\ p_{\phi} \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta_B}{\partial \theta} & \frac{\partial \theta_B}{\partial \phi} \\ \frac{\partial \phi_B}{\partial \theta} & \frac{\partial \phi_B}{\partial \phi} \end{pmatrix} \begin{pmatrix} p_{\theta}^B \\ p_{\phi}^B \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{\theta}^B \\ p_{\phi}^B \end{pmatrix}$$

Be careful with momentum!
Poincare invariance is **crucial!**

Poincare invariant must **remain** invariant

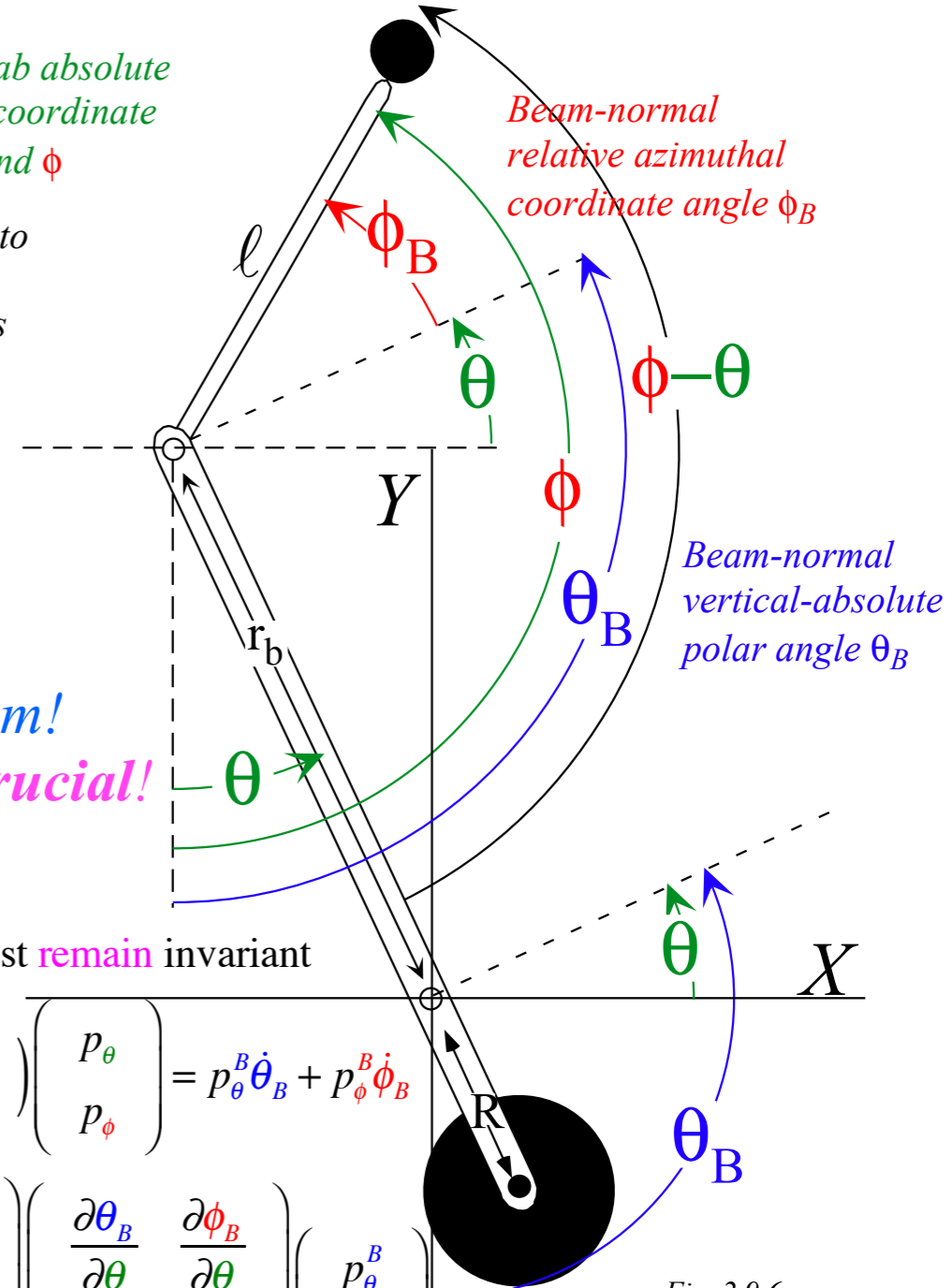
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Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B .



Beam-normal relative azimuthal coordinate angle ϕ_B

Beam-normal vertical-absolute polar angle θ_B

Fig. 2.9.6

Lab (θ, ϕ) and beam-normal (θ_B, ϕ_B) relative coordinates for trebuchet. (Each value is positive.)

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p_m transform is **TRANSPOSE INVERSE** to q^m

$$\begin{pmatrix} p_\theta^B \\ p_\phi^B \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta}{\partial \theta_B} & \frac{\partial \theta}{\partial \phi_B} \\ \frac{\partial \phi}{\partial \theta_B} & \frac{\partial \phi}{\partial \phi_B} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

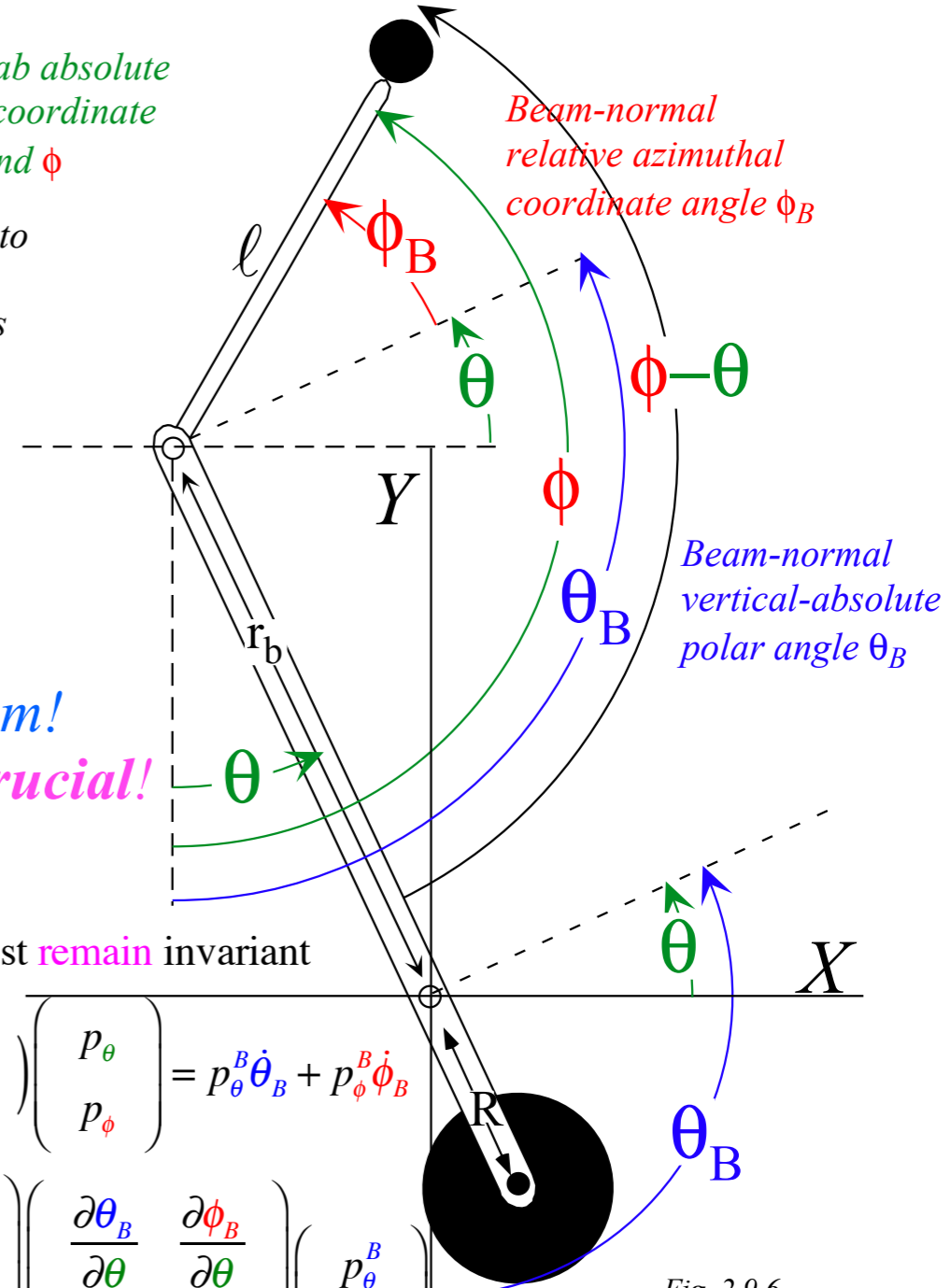
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Resulting momentum transform: $p_\theta = p_\theta^B - p_\phi^B$
 $p_\phi = p_\phi^B$

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

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+V

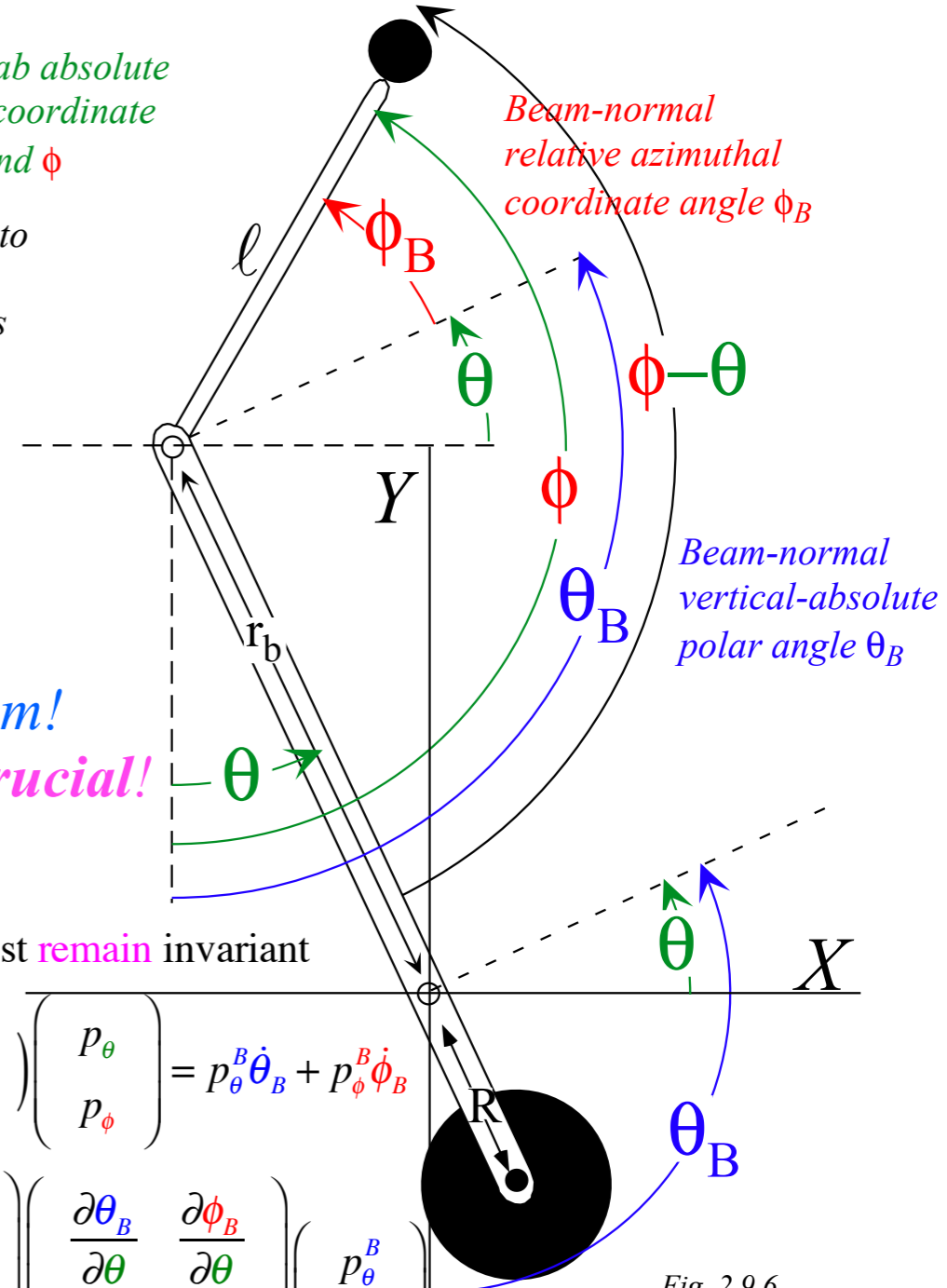
$$H = \frac{m\ell^2 p_{\theta} p_{\theta} + (MR^2 + mr^2) p_{\phi} p_{\phi} + 2mr\ell p_{\theta} p_{\phi} \cos(\theta - \phi)}{2m\ell^2 [MR^2 + mr^2 \sin^2(\theta - \phi)]} + V$$

Original (ϕ, θ) Hamiltonian

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

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Poincare invariant must **remain** invariant

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Lab (θ, ϕ) and beam-normal (θ_B, ϕ_B) relative coordinates for trebuchet. (Each value is positive.)

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Poincare invariant must **remain** invariant

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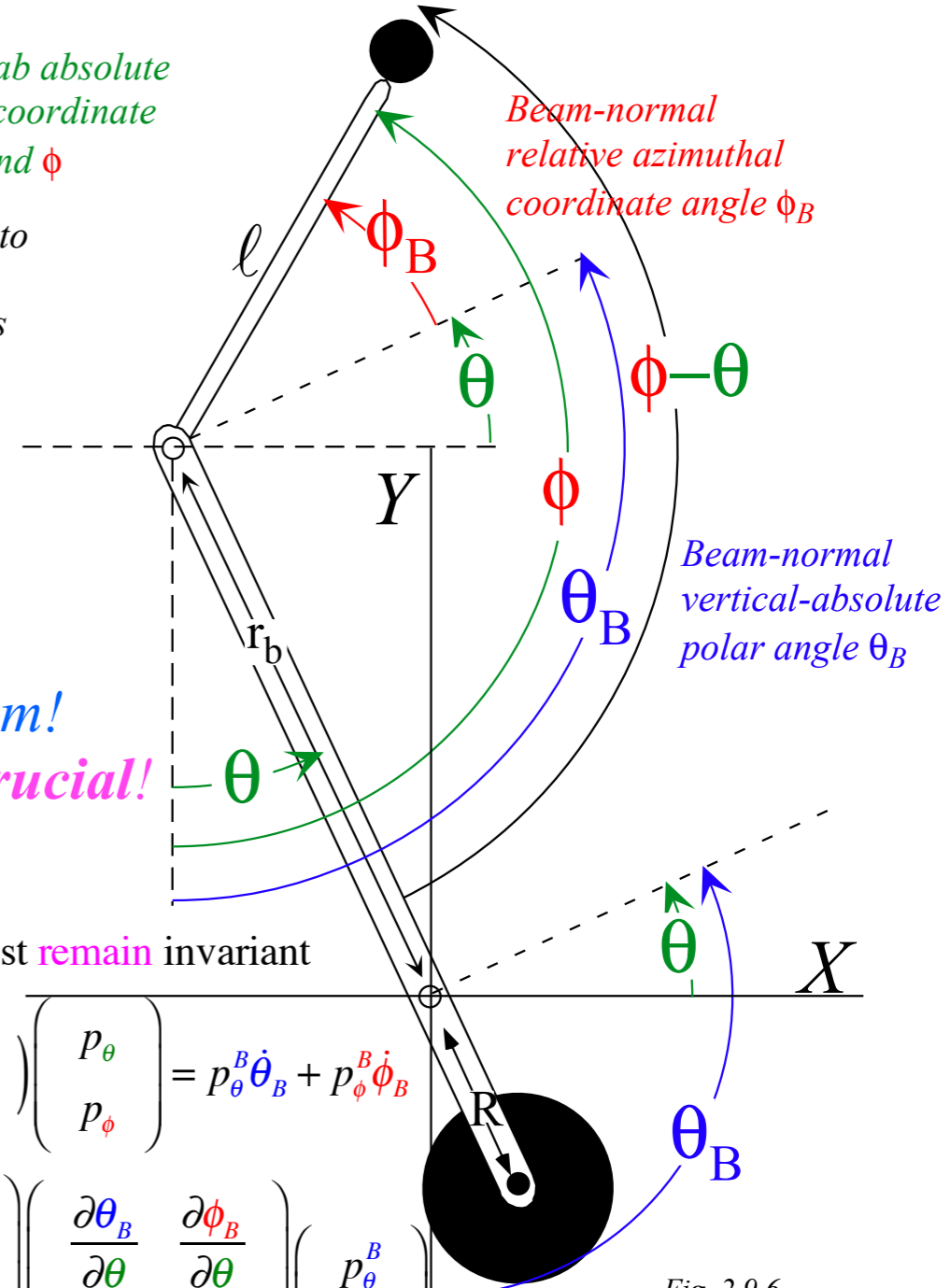


Fig. 2.9.6

Lab (θ, ϕ) and beam-normal (θ_B, ϕ_B) relative coordinates for trebuchet. (Each value is positive.)

Original (ϕ, θ) Hamiltonian

(Use $\phi_B = \pi/2 - (\theta - \phi)$)

Transformed (ϕ_B, θ_B) Hamiltonian

$$H = \frac{m\ell^2 p_{\theta} p_{\theta} + (MR^2 + mr^2) p_{\phi} p_{\phi} + 2mr\ell p_{\theta} p_{\phi} \cos(\theta - \phi)}{2m\ell^2 [MR^2 + mr^2 \sin^2(\theta - \phi)]} + V$$

$$H = \frac{m\ell^2 (p_{\theta}^B - p_{\phi}^B)^2 + (MR^2 + mr^2) (p_{\phi}^B)^2 - 2mr\ell p_{\phi}^B (p_{\theta}^B - p_{\phi}^B) \sin \phi_B}{m\ell^2 [MR^2 + mr^2 \cos^2 \phi_B]} + V$$

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$$\phi = \theta_B + \phi_B$$

p_m transform is **TRANSPOSE INVERSE** to q^m

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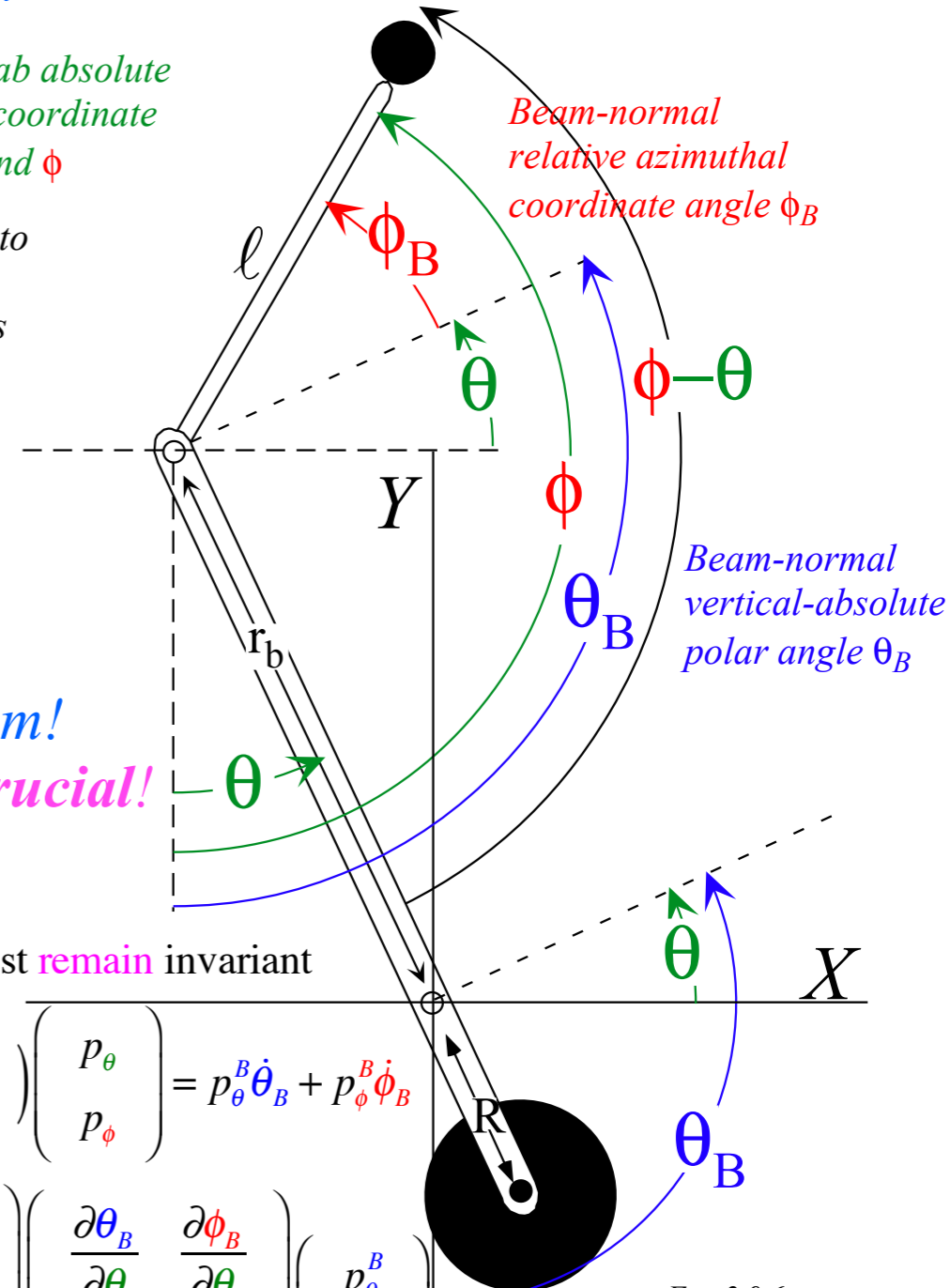


Fig. 2.9.6

Lab (θ, ϕ) and beam-normal (θ_B, ϕ_B) relative coordinates for trebuchet. (Each value is positive.)

$$F_\theta = -MgR \sin \theta + mgr \sin \theta$$

$$F_\phi = -mgl \sin \phi$$

$$H = \frac{m\ell^2 p_\theta p_\theta + (MR^2 + mr^2) p_\phi p_\phi + 2mr\ell p_\theta p_\phi \cos(\theta - \phi)}{2m\ell^2 [MR^2 + mr^2 \sin^2(\theta - \phi)]} + (MR - mr)g \cos \theta + mgl \cos \phi$$

$$\theta - \phi = -\pi/2 - \phi_B$$

$$H = \frac{m\ell^2 (p_\theta^B - p_\phi^B)^2 + (MR^2 + mr^2) (p_\phi^B)^2 - 2mr\ell p_\phi^B (p_\theta^B - p_\phi^B) \sin \phi_B}{m\ell^2 [MR^2 + mr^2 \cos^2 \phi_B]} - (MR - mr)g \sin \theta_B - mgl \cos(\phi_B + \theta_B)$$

Be careful with momentum!
Poincare invariance is **crucial!**

Hamiltonian energy and momentum conservation and symmetry coordinates

Coordinate transformation helps reduce symmetric Hamiltonian

Free-space trebuchet kinematics by symmetry

 *Algebraic approach*

Direct approach and Superball analogy

Trebuchet vs Flinger and sports kinematics

Many approaches to Mechanics

$$H = \frac{m\ell^2 (p_\theta^B - p_\phi^B)^2 + (MR^2 + mr^2)(p_\phi^B)^2 - 2mr\ell p_\phi^B (p_\theta^B - p_\phi^B) \sin \phi_B}{m\ell^2 [MR^2 + mr^2 \cos^2 \phi_B]} - (MR - mr)g \sin \theta_B - mgl \cos(\phi_B + \theta_B)$$

(Assume zero-gravity)

For zero-gravity H is not a function of θ_B

so : $\dot{p}_\theta^B = \frac{\partial H}{\partial \theta_B} = 0$ and : $p_\theta^B = \Lambda = \text{const.}$

H is not an explicit function of t so : $H = \text{const.} = E$

Previous lab absolute trebuchet coordinate angles θ and ϕ

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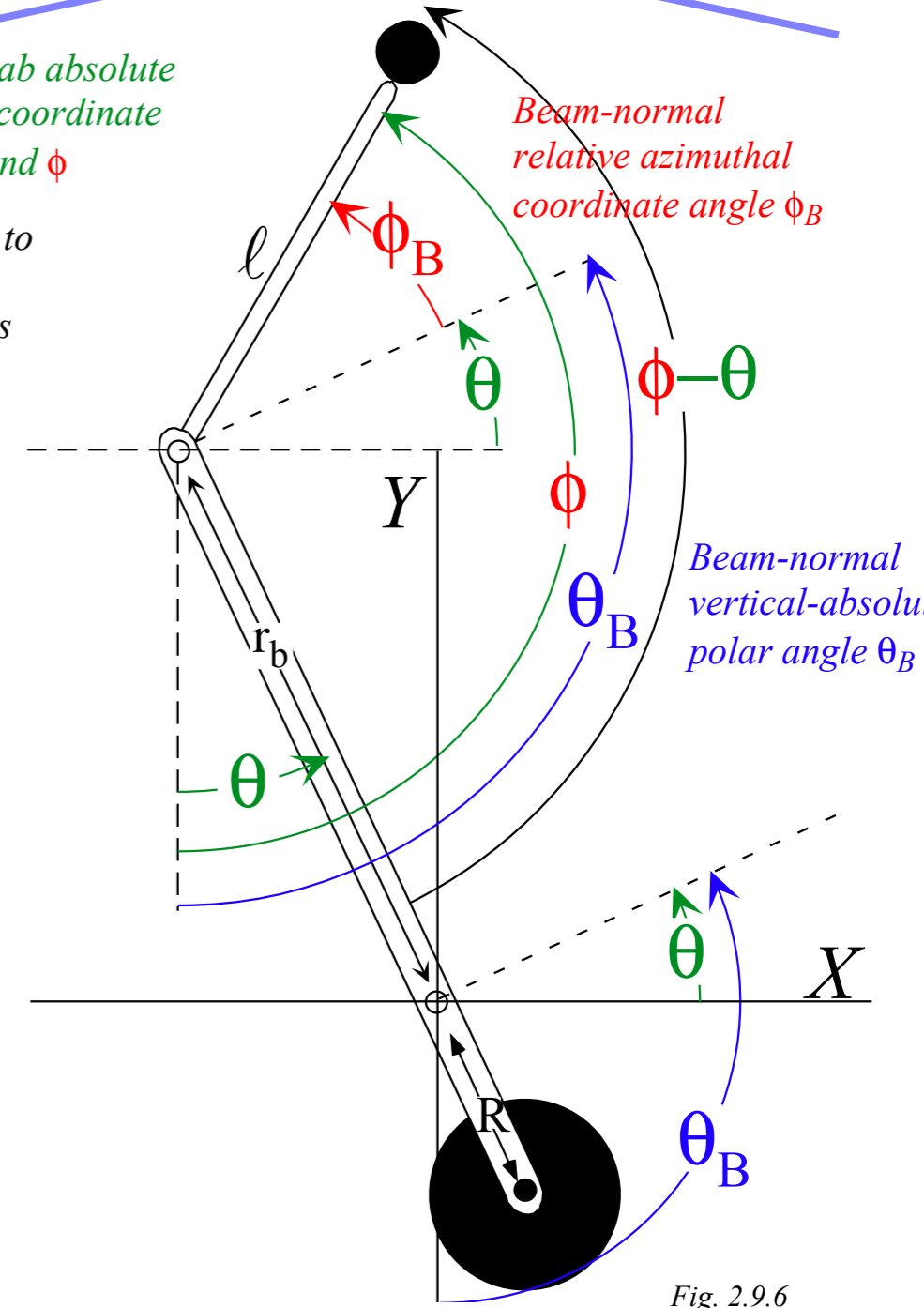


Fig. 2.9.6
Lab (θ, ϕ) and beam-normal (θ_B, ϕ_B) relative coordinates for trebuchet.
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H is not an explicit function of t so : $H = \text{const.} = E$

$$H = \frac{m\ell^2 \left(\Lambda - p_\phi^B \right)^2 + \left(MR^2 + mr^2 \right) \left(p_\phi^B \right)^2 - 2mr\ell p_\phi^B \left(\Lambda - p_\phi^B \right) \sin \phi_B}{m\ell^2 \left[MR^2 + mr^2 \cos^2 \phi_B \right]} = \text{const.} = E$$

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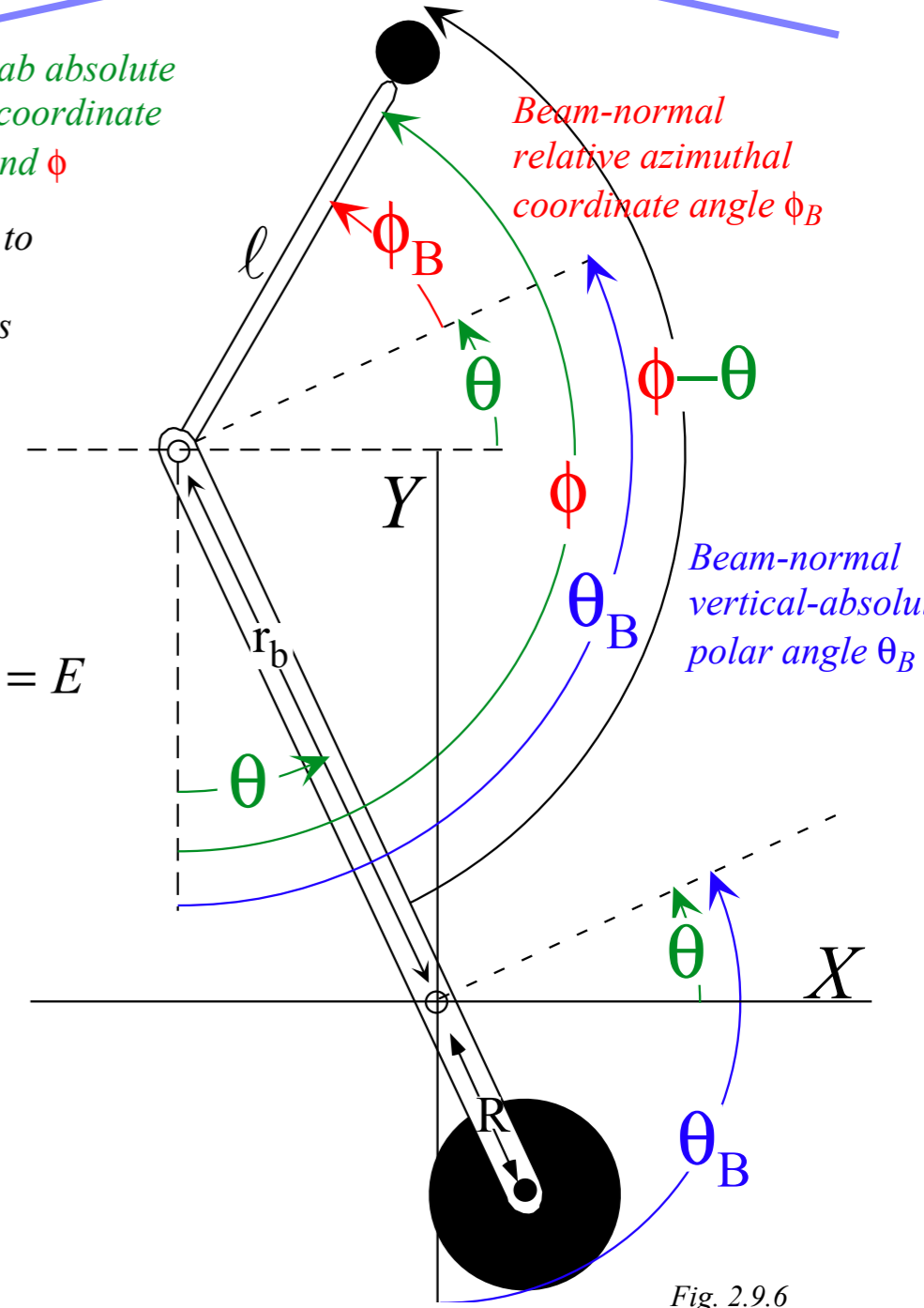


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(Assume zero-gravity)

For zero-gravity H is not a function of θ_B

so : $\dot{p}_\theta^B = \frac{\partial H}{\partial \theta_B} = 0$ and : $p_\theta^B = \Lambda = \text{const.}$

H is not an explicit function of t so : $H = \text{const.} = E$

$$H = \frac{m\ell^2 \left(\Lambda - p_\phi^B \right)^2 + \left(MR^2 + mr^2 \right) \left(p_\phi^B \right)^2 - 2mr\ell p_\phi^B \left(\Lambda - p_\phi^B \right) \sin \phi_B}{m\ell^2 \left[MR^2 + mr^2 \cos^2 \phi_B \right]} = \text{const.} = E$$

Rewrite $H=E$ as a quadratic equation in p_ϕ :

$$m\ell^2 \left(\Lambda^2 - 2\Lambda(p_\phi^B) + (p_\phi^B)^2 \right) + \left(MR^2 + mr^2 \right) \left(p_\phi^B \right)^2 - 2mr\ell(p_\phi^B) \left(\Lambda - p_\phi^B \right) \sin \phi_B = Em\ell^2 \left[MR^2 + mr^2 \cos^2 \phi_B \right]$$

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B .

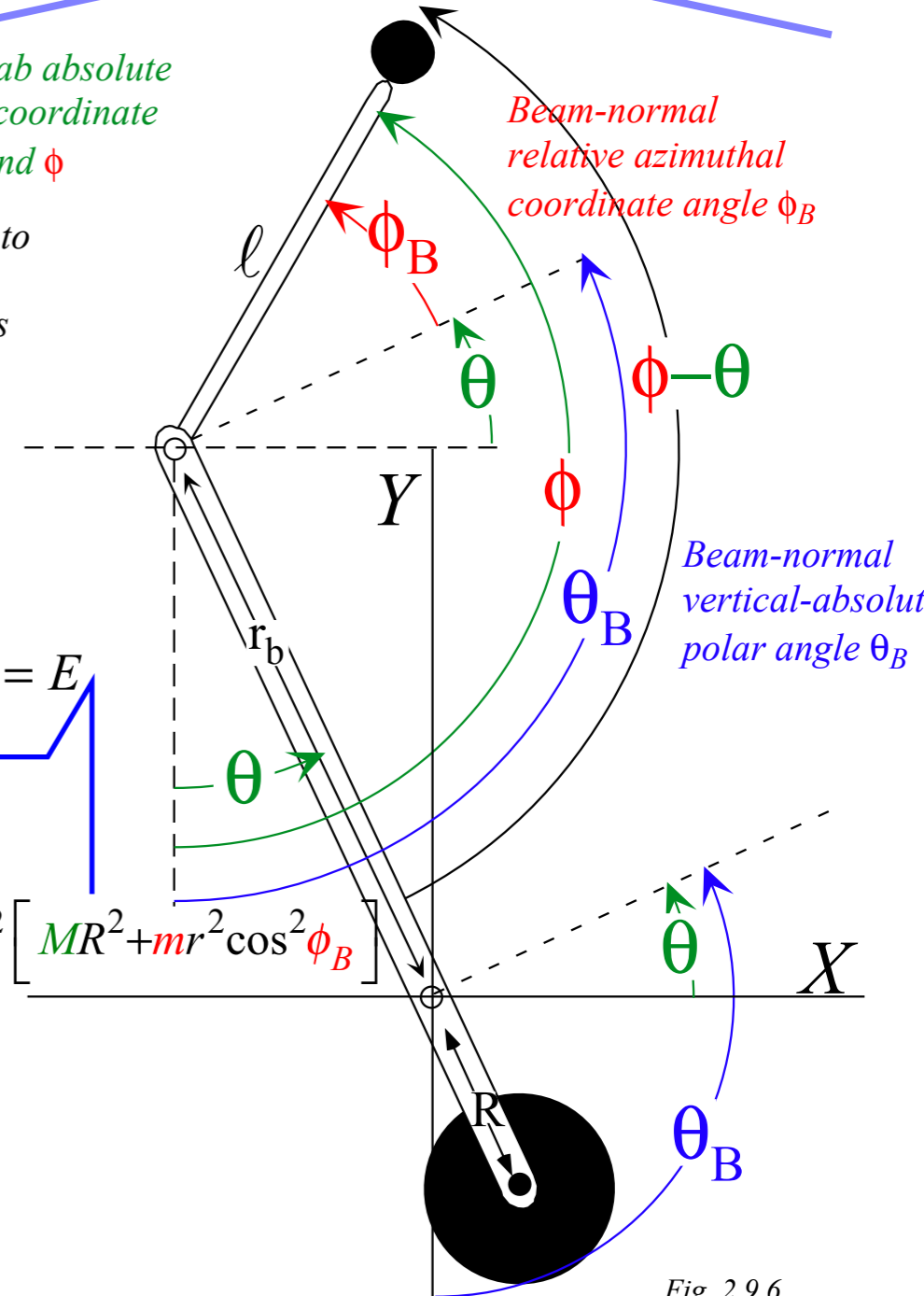


Fig. 2.9.6

Lab (θ, ϕ) and beam-normal (θ_B, ϕ_B) relative coordinates for trebuchet. (Each value is positive.)

Throwing-momentum p_ϕ^B is a function of beam-relative angle ϕ_B , total E , and $\Lambda = p_\theta^B$.

$$H = \frac{m\ell^2 (p_\theta^B - p_\phi^B)^2 + (MR^2 + mr^2)(p_\phi^B)^2 - 2mr\ell p_\phi^B (p_\theta^B - p_\phi^B) \sin \phi_B}{m\ell^2 [MR^2 + mr^2 \cos^2 \phi_B]} - (MR - mr)g \sin \theta_B - mgl \cos(\phi_B + \theta_B)$$

(Assume zero-gravity)

For zero-gravity H is not a function of θ_B

so : $\dot{p}_\theta^B = \frac{\partial H}{\partial \theta_B} = 0$ and : $p_\theta^B = \Lambda = \text{const.}$

H is not an explicit function of t so : $H = \text{const.} = E$

$$H = \frac{m\ell^2 (\Lambda - p_\phi^B)^2 + (MR^2 + mr^2)(p_\phi^B)^2 - 2mr\ell p_\phi^B (\Lambda - p_\phi^B) \sin \phi_B}{m\ell^2 [MR^2 + mr^2 \cos^2 \phi_B]} = \text{const.} = E$$

Rewrite $H=E$ as a quadratic equation in p_ϕ :

$$m\ell^2 (\Lambda^2 - 2\Lambda(p_\phi^B) + (p_\phi^B)^2) + (MR^2 + mr^2)(p_\phi^B)^2 - 2mr\ell(p_\phi^B)(\Lambda - p_\phi^B) \sin \phi_B = Em\ell^2 [MR^2 + mr^2 \cos^2 \phi_B]$$

$$m\ell^2 \Lambda^2 - 2m\ell^2 \Lambda(p_\phi^B) + (m\ell^2 + 2mr\ell \sin \phi_B + MR^2 + mr^2)(p_\phi^B)^2 - 2mr\ell \Lambda \sin \phi_B (p_\phi^B) = Em\ell^2 [MR^2 + mr^2 \cos^2 \phi_B]$$

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B .

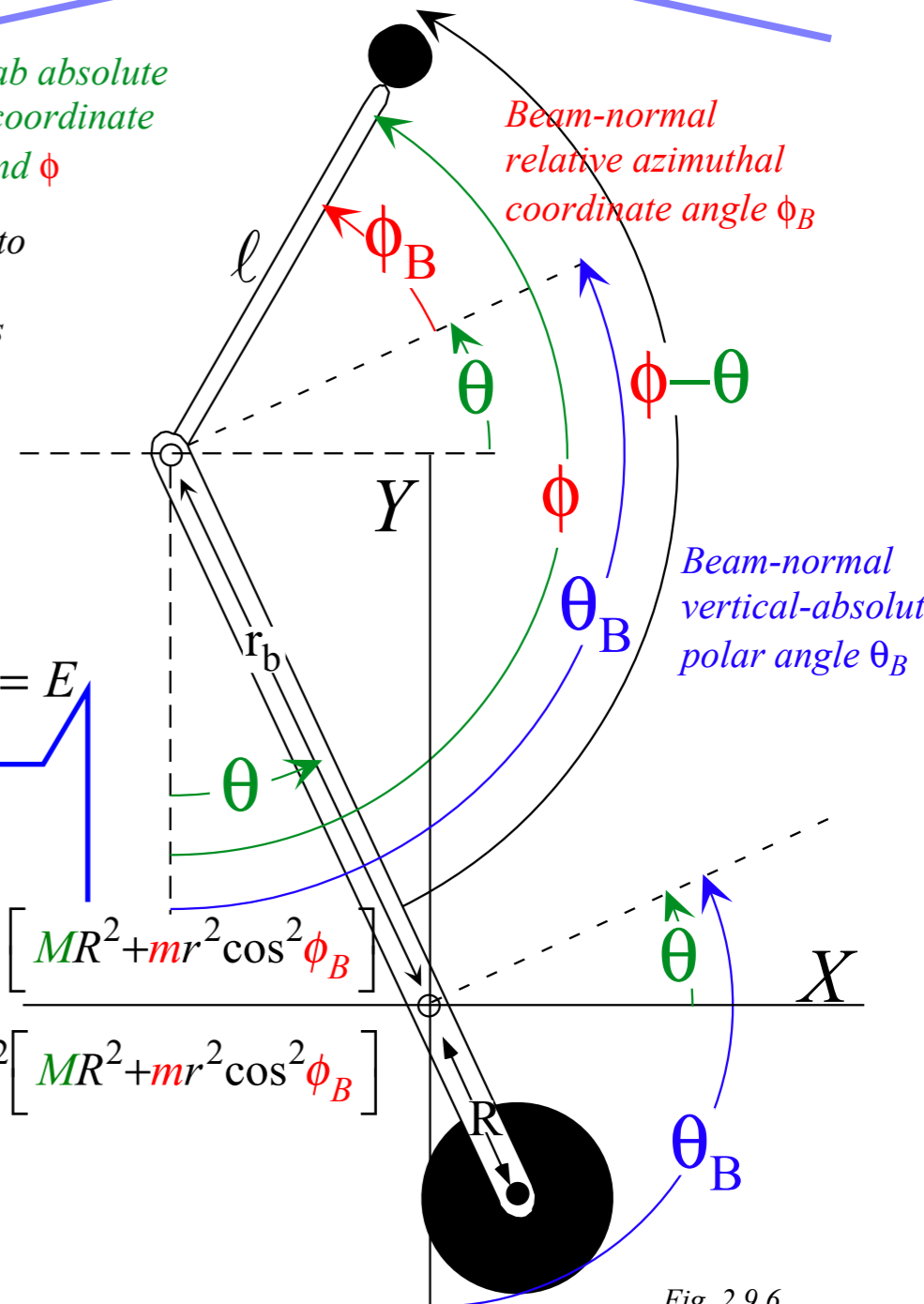


Fig. 2.9.6

Lab (θ, ϕ) and beam-normal (θ_B, ϕ_B) relative coordinates for trebuchet. (Each value is positive.)

Throwing-momentum p_ϕ^B is a function of beam-relative angle ϕ_B , total E , and $\Lambda = p_\theta^B$.

$$H = \frac{m\ell^2 (p_\theta^B - p_\phi^B)^2 + (MR^2 + mr^2)(p_\phi^B)^2 - 2mr\ell p_\phi^B (p_\theta^B - p_\phi^B) \sin \phi_B}{m\ell^2 [MR^2 + mr^2 \cos^2 \phi_B]} - (MR - mr)g \sin \theta_B - mgl \cos(\phi_B + \theta_B)$$

(Assume zero-gravity)

For zero-gravity H is not a function of θ_B

so : $\dot{p}_\theta^B = \frac{\partial H}{\partial \theta_B} = 0$ and : $p_\theta^B = \Lambda = \text{const.}$

H is not an explicit function of t so : $H = \text{const.} = E$

$$H = \frac{m\ell^2 (\Lambda - p_\phi^B)^2 + (MR^2 + mr^2)(p_\phi^B)^2 - 2mr\ell p_\phi^B (\Lambda - p_\phi^B) \sin \phi_B}{m\ell^2 [MR^2 + mr^2 \cos^2 \phi_B]} = \text{const.} = E$$

Rewrite $H=E$ as a quadratic equation in p_ϕ :

$$m\ell^2 (\Lambda^2 - 2\Lambda(p_\phi^B) + (p_\phi^B)^2) + (MR^2 + mr^2)(p_\phi^B)^2 - 2mr\ell(p_\phi^B)(\Lambda - p_\phi^B) \sin \phi_B = Em\ell^2 [MR^2 + mr^2 \cos^2 \phi_B]$$

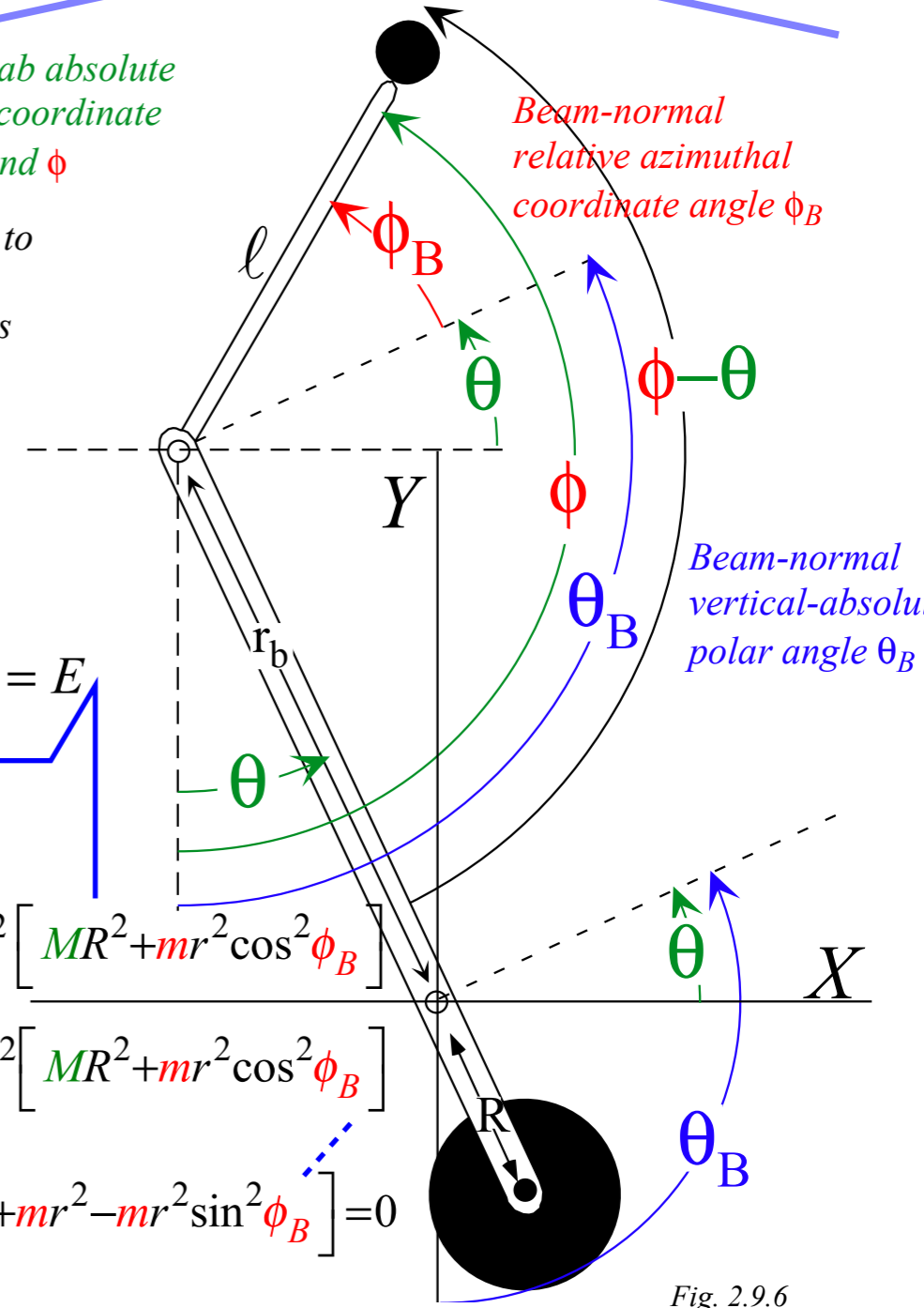
$$m\ell^2 \Lambda^2 - 2m\ell^2 \Lambda(p_\phi^B) + (m\ell^2 + 2mr\ell \sin \phi_B + MR^2 + mr^2)(p_\phi^B)^2 - 2mr\ell \Lambda \sin \phi_B (p_\phi^B) = Em\ell^2 [MR^2 + mr^2 \cos^2 \phi_B]$$

$$(m\ell^2 + 2mr\ell \sin \phi_B + MR^2 + mr^2)(p_\phi^B)^2 - 2\Lambda(mr\ell \sin \phi_B + m\ell^2)(p_\phi^B) + m\ell^2 \Lambda^2 - Em\ell^2 [MR^2 + mr^2 - mr^2 \sin^2 \phi_B] = 0$$

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B .



Beam-normal relative azimuthal coordinate angle ϕ_B

Beam-normal vertical-absolute polar angle θ_B

Fig. 2.9.6

Lab (θ, ϕ) and beam-normal (θ_B, ϕ_B) relative coordinates for trebuchet. (Each value is positive.)

Throwing-momentum p_ϕ^B is a function of beam-relative angle ϕ_B , total E , and $\Lambda = p_\theta^B$.

$$H = \frac{m\ell^2 (p_\theta^B - p_\phi^B)^2 + (MR^2 + mr^2)(p_\phi^B)^2 - 2mr\ell p_\phi^B (p_\theta^B - p_\phi^B) \sin \phi_B}{m\ell^2 [MR^2 + mr^2 \cos^2 \phi_B]} - (MR - mr)g \sin \theta_B - mgl \cos(\phi_B + \theta_B)$$

(Assume zero-gravity)

For zero-gravity H is not a function of θ_B

so : $\dot{p}_\theta^B = \frac{\partial H}{\partial \theta_B} = 0$ and : $p_\theta^B = \Lambda = \text{const.}$

H is not an explicit function of t so : $H = \text{const.} = E$

$$H = \frac{m\ell^2 (\Lambda - p_\phi^B)^2 + (MR^2 + mr^2)(p_\phi^B)^2 - 2mr\ell p_\phi^B (\Lambda - p_\phi^B) \sin \phi_B}{m\ell^2 [MR^2 + mr^2 \cos^2 \phi_B]} = \text{const.} = E$$

Rewrite $H=E$ as a quadratic equation in p_ϕ^B :

$$m\ell^2 (\Lambda^2 - 2\Lambda(p_\phi^B) + (p_\phi^B)^2) + (MR^2 + mr^2)(p_\phi^B)^2 - 2mr\ell(p_\phi^B)(\Lambda - p_\phi^B) \sin \phi_B = Em\ell^2 [MR^2 + mr^2 \cos^2 \phi_B]$$

$$m\ell^2 \Lambda^2 - 2m\ell^2 \Lambda(p_\phi^B) + (m\ell^2 + 2mr\ell \sin \phi_B + MR^2 + mr^2)(p_\phi^B)^2 - 2mr\ell \Lambda \sin \phi_B (p_\phi^B) = Em\ell^2 [MR^2 + mr^2 \cos^2 \phi_B]$$

$$(m\ell^2 + 2mr\ell \sin \phi_B + MR^2 + mr^2)(p_\phi^B)^2 - 2\Lambda(mr\ell \sin \phi_B + m\ell^2)(p_\phi^B) + m\ell^2 \Lambda^2 - Em\ell^2 [MR^2 + mr^2 - mr^2 \sin^2 \phi_B] = 0$$

$$(1 + 2(r/\ell) \sin \phi_B + J)(p_\phi^B)^2 - 2\Lambda((r/\ell) \sin \phi_B + 1)(p_\phi^B) + \Lambda^2 - E[I - mr^2 \sin^2 \phi_B] = 0$$

Throwing-momentum p_ϕ^B is a function of beam-relative angle ϕ_B , total E , and $\Lambda = p_\theta^B$.

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B .

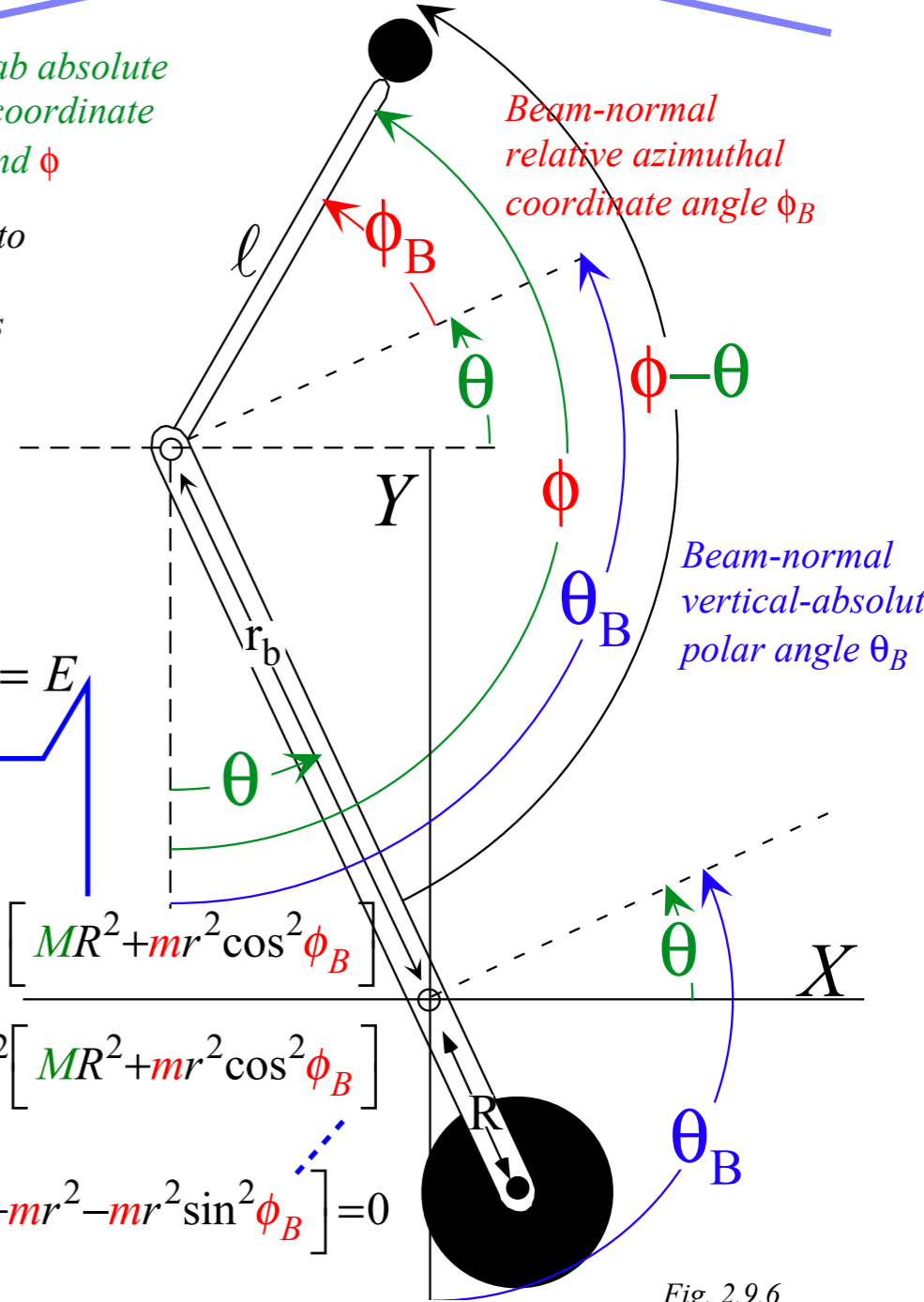


Fig. 2.9.6

Lab (θ, ϕ) and beam-normal (θ_B, ϕ_B) relative coordinates for trebuchet.

(Each value is positive.)

with: $J = \frac{MR^2 + mr^2}{m\ell^2}$, $I = MR^2 + mr^2$

$$H = \frac{m\ell^2 \left(p_\theta^B - p_\phi^B \right)^2 + \left(MR^2 + mr^2 \right) \left(p_\phi^B \right)^2 - 2mr\ell p_\phi^B \left(p_\theta^B - p_\phi^B \right) \sin \phi_B}{m\ell^2 \left[MR^2 + mr^2 \cos^2 \phi_B \right]} - \left(MR - mr \right) g \sin \theta_B - mgl \cos \left(\phi_B + \theta_B \right)$$

(Assume zero-gravity)

For zero-gravity H is not a function of θ_B

so : $\dot{p}_\theta^B = \frac{\partial H}{\partial \theta_B} = 0$ and : $p_\theta^B = \Lambda = \text{const.}$

H is not an explicit function of t so : $H = \text{const.} = E$

$$H = \frac{m\ell^2 \left(\Lambda - p_\phi^B \right)^2 + \left(MR^2 + mr^2 \right) \left(p_\phi^B \right)^2 - 2mr\ell p_\phi^B \left(\Lambda - p_\phi^B \right) \sin \phi_B}{m\ell^2 \left[MR^2 + mr^2 \cos^2 \phi_B \right]} = \text{const.} = E$$

Rewrite $H=E$ as a quadratic equation in p_ϕ :

$$m\ell^2 \left(\Lambda^2 - 2\Lambda(p_\phi^B) + (p_\phi^B)^2 \right) + \left(MR^2 + mr^2 \right) (p_\phi^B)^2 - 2mr\ell(p_\phi^B) \left(\Lambda - p_\phi^B \right) \sin \phi_B = Em\ell^2 \left[MR^2 + mr^2 \cos^2 \phi_B \right]$$

$$m\ell^2 \Lambda^2 - 2m\ell^2 \Lambda(p_\phi^B) + \left(m\ell^2 + 2mr\ell \sin \phi_B + MR^2 + mr^2 \right) (p_\phi^B)^2 - 2mr\ell \Lambda \sin \phi_B (p_\phi^B) = Em\ell^2 \left[MR^2 + mr^2 \cos^2 \phi_B \right]$$

$$\left(m\ell^2 + 2mr\ell \sin \phi_B + MR^2 + mr^2 \right) (p_\phi^B)^2 - 2\Lambda \left(mr\ell \sin \phi_B + m\ell^2 \right) (p_\phi^B) + m\ell^2 \Lambda^2 - Em\ell^2 \left[MR^2 + mr^2 - mr^2 \sin^2 \phi_B \right] = 0$$

$$\left(1 + 2(r/\ell) \sin \phi_B + J \right) (p_\phi^B)^2 - 2\Lambda \left((r/\ell) \sin \phi_B + 1 \right) (p_\phi^B) + \Lambda^2 - E \left[I - mr^2 \sin^2 \phi_B \right] = 0 \quad \left(\text{using quadratic solution: } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \right)$$

Throwing-momentum p_ϕ^B is a function of beam-relative angle ϕ_B , total E , and $\Lambda = p_\theta^B$.

$$p_\phi^B = \frac{2\Lambda \left((r/\ell) \sin \phi_B + 1 \right) \pm \sqrt{4\Lambda^2 \left((r/\ell) \sin \phi_B + 1 \right)^2 - 4 \left(1 + 2(r/\ell) \sin \phi_B + J \right) \left(\Lambda^2 - E \left[I - mr^2 \sin^2 \phi_B \right] \right)}}{2 \left(1 + 2(r/\ell) \sin \phi_B + J \right)} \quad \text{with: } J = \frac{MR^2 + mr^2}{m\ell^2}, \quad I = MR^2 + mr^2$$

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B .

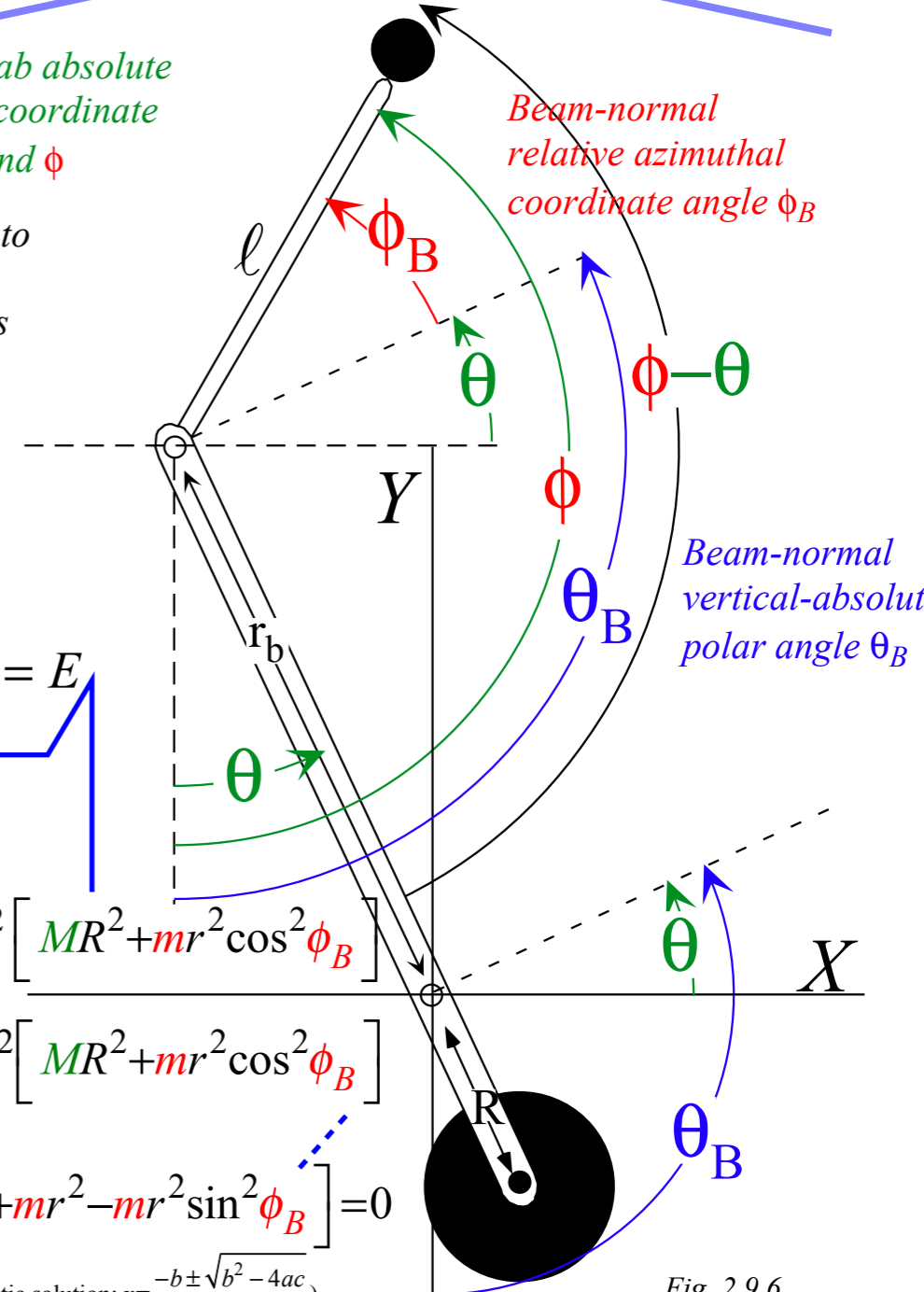


Fig. 2.9.6

Lab (θ, ϕ) and beam-normal (θ_B, ϕ_B) relative coordinates for trebuchet.


(Each value is positive.)

Hamiltonian energy and momentum conservation and symmetry coordinates

Coordinate transformation helps reduce symmetric Hamiltonian

Free-space trebuchet kinematics by symmetry

Algebraic approach

 *Direct approach and Superball analogy*

Trebuchet vs Flinger and sports kinematics

Many approaches to Mechanics

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Energy for zero-gravity

$$\text{Total KE} = T = \frac{1}{2} \left[(MR^2 + mr^2) \dot{\theta}^2 - 2mrl \cos(\theta - \phi) \dot{\theta} \dot{\phi} + m\ell^2 \dot{\phi}^2 \right]$$

$$p_{\theta} = \frac{\partial T}{\partial \dot{\theta}} = (MR^2 + mr^2) \dot{\theta} - mrl \dot{\phi} \cos(\theta - \phi)$$

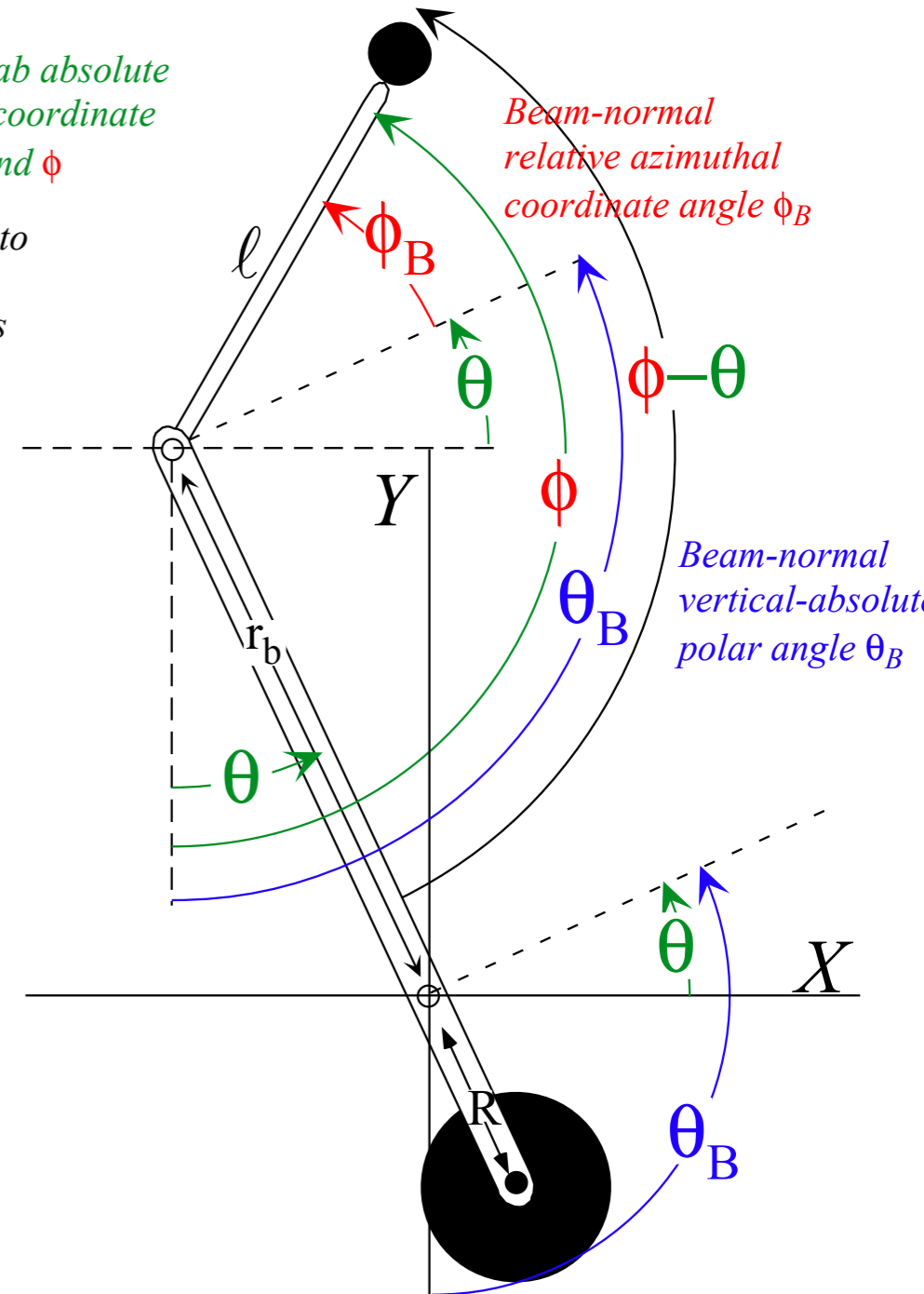
$$p_{\phi} = \frac{\partial T}{\partial \dot{\phi}} = m\ell^2 \dot{\phi} - mrl \dot{\theta} \cos(\theta - \phi)$$

(Assume zero-gravity)

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B .



Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Energy for zero-gravity

$$\text{Total KE} = T = \frac{1}{2} \left[(MR^2 + mr^2) \dot{\theta}^2 - 2mrl \cos(\theta - \phi) \dot{\theta} \dot{\phi} + ml^2 \dot{\phi}^2 \right]$$

$$p_{\theta} = \frac{\partial T}{\partial \dot{\theta}} = (MR^2 + mr^2) \dot{\theta} - mrl \dot{\phi} \cos(\theta - \phi)$$

$$p_{\phi} = \frac{\partial T}{\partial \dot{\phi}} = ml^2 \dot{\phi} - mrl \dot{\theta} \cos(\theta - \phi)$$

Transform to beam-relative coordinates and momenta

$$\theta = \theta_B - \pi/2$$

$$\phi = \theta_B + \phi_B$$

$$\theta - \phi = -\phi_B - \pi/2$$

$$\theta_B = \theta + \pi/2$$

$$\phi_B = -\theta + \phi - \pi/2$$

$$p_{\theta} = p_{\theta}^B - p_{\phi}^B$$

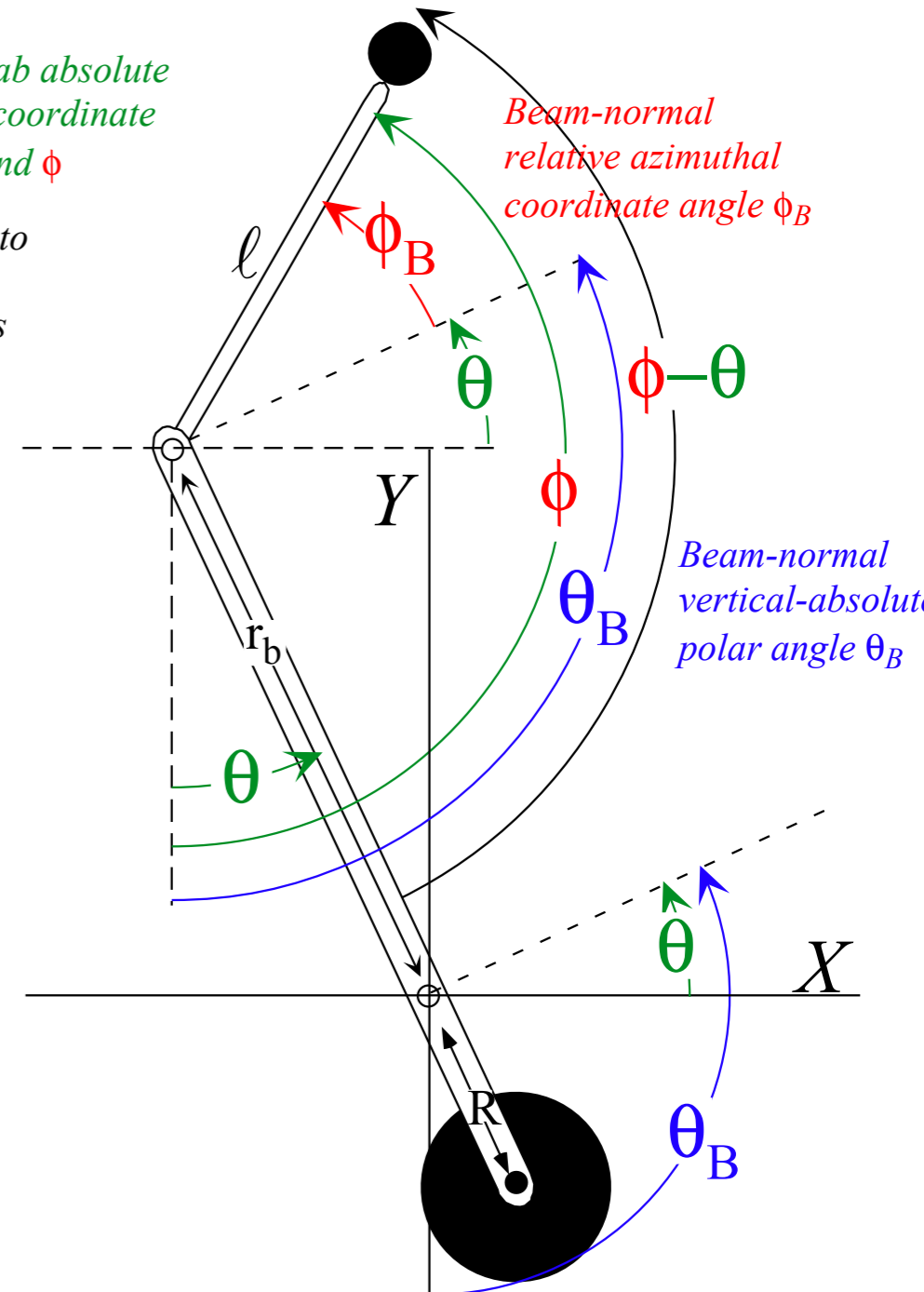
$$p_{\phi} = p_{\phi}^B$$

(Assume zero-gravity)

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B .



Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Energy for zero-gravity

$$\text{Total KE} = T = \frac{1}{2} \left[(MR^2 + mr^2) \dot{\theta}^2 - 2mrl \cos(\theta - \phi) \dot{\theta} \dot{\phi} + ml^2 \dot{\phi}^2 \right]$$

$$p_{\theta} = \frac{\partial T}{\partial \dot{\theta}} = (MR^2 + mr^2) \dot{\theta} - mrl \dot{\phi} \cos(\theta - \phi)$$

$$p_{\phi} = \frac{\partial T}{\partial \dot{\phi}} = ml^2 \dot{\phi} - mrl \dot{\theta} \cos(\theta - \phi)$$

Transform to beam-relative coordinates and momenta

$$\theta = \theta_B - \pi/2$$

$$\phi = \theta_B + \phi_B$$

$$\theta - \phi = -\phi_B - \pi/2$$

$$\theta_B = \theta + \pi/2$$

$$\phi_B = -\theta + \phi - \pi/2$$

$$p_{\theta} = p_{\theta}^B - p_{\phi}^B$$

$$p_{\phi} = p_{\phi}^B$$

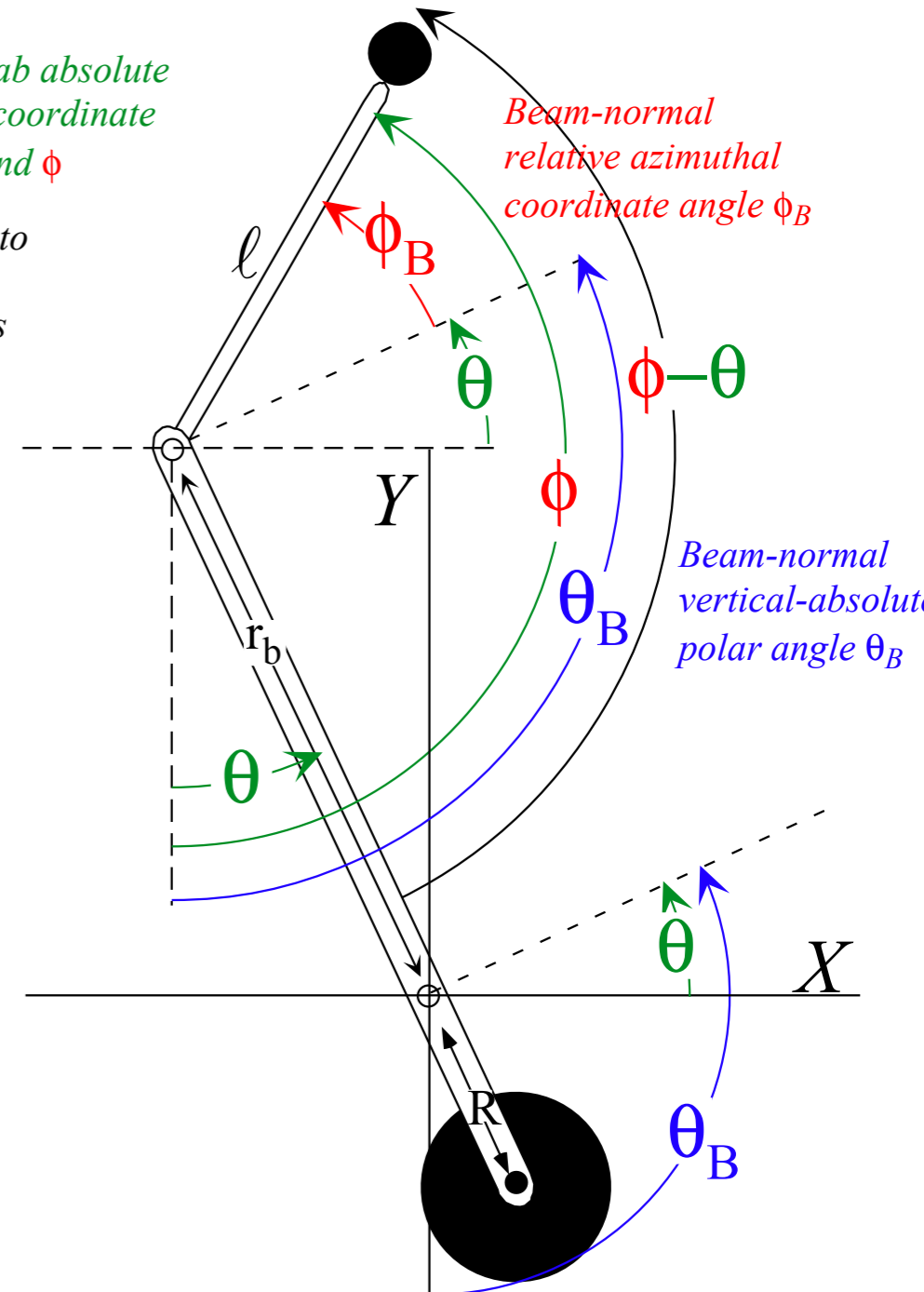
$$2E = (MR^2 + mr^2) \dot{\theta}^2 + 2mrl \dot{\phi} \dot{\theta} \sin \phi_B + ml^2 \dot{\phi}^2 = \text{const.}$$

(Assume zero-gravity)

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B .



Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Energy for zero-gravity

$$\text{Total KE} = T = \frac{1}{2} \left[(MR^2 + mr^2) \dot{\theta}^2 - 2mr\ell \cos(\theta - \phi) \dot{\theta} \dot{\phi} + m\ell^2 \dot{\phi}^2 \right]$$

$$p_{\theta} = \frac{\partial T}{\partial \dot{\theta}} = (MR^2 + mr^2) \dot{\theta} - mr\ell \dot{\phi} \cos(\theta - \phi)$$

$$p_{\phi} = \frac{\partial T}{\partial \dot{\phi}} = m\ell^2 \dot{\phi} - mr\ell \dot{\theta} \cos(\theta - \phi)$$

Transform to beam-relative coordinates and momenta

$$\theta = \theta_B - \pi/2$$

$$\phi = \theta_B + \phi_B$$

$$\theta - \phi = -\phi_B - \pi/2$$

$$\theta_B = \theta + \pi/2$$

$$\phi_B = -\theta + \phi - \pi/2$$

$$p_{\theta} = p_{\theta}^B - p_{\phi}^B$$

$$p_{\phi} = p_{\phi}^B$$

$$2E = (MR^2 + mr^2) \dot{\theta}^2 + 2mr\ell \dot{\phi} \dot{\theta} \sin \phi_B + m\ell^2 \dot{\phi}^2 = \text{const.}$$

$$p_{\theta}^B = \Lambda = \text{const.} = p_{\theta} + p_{\phi}$$

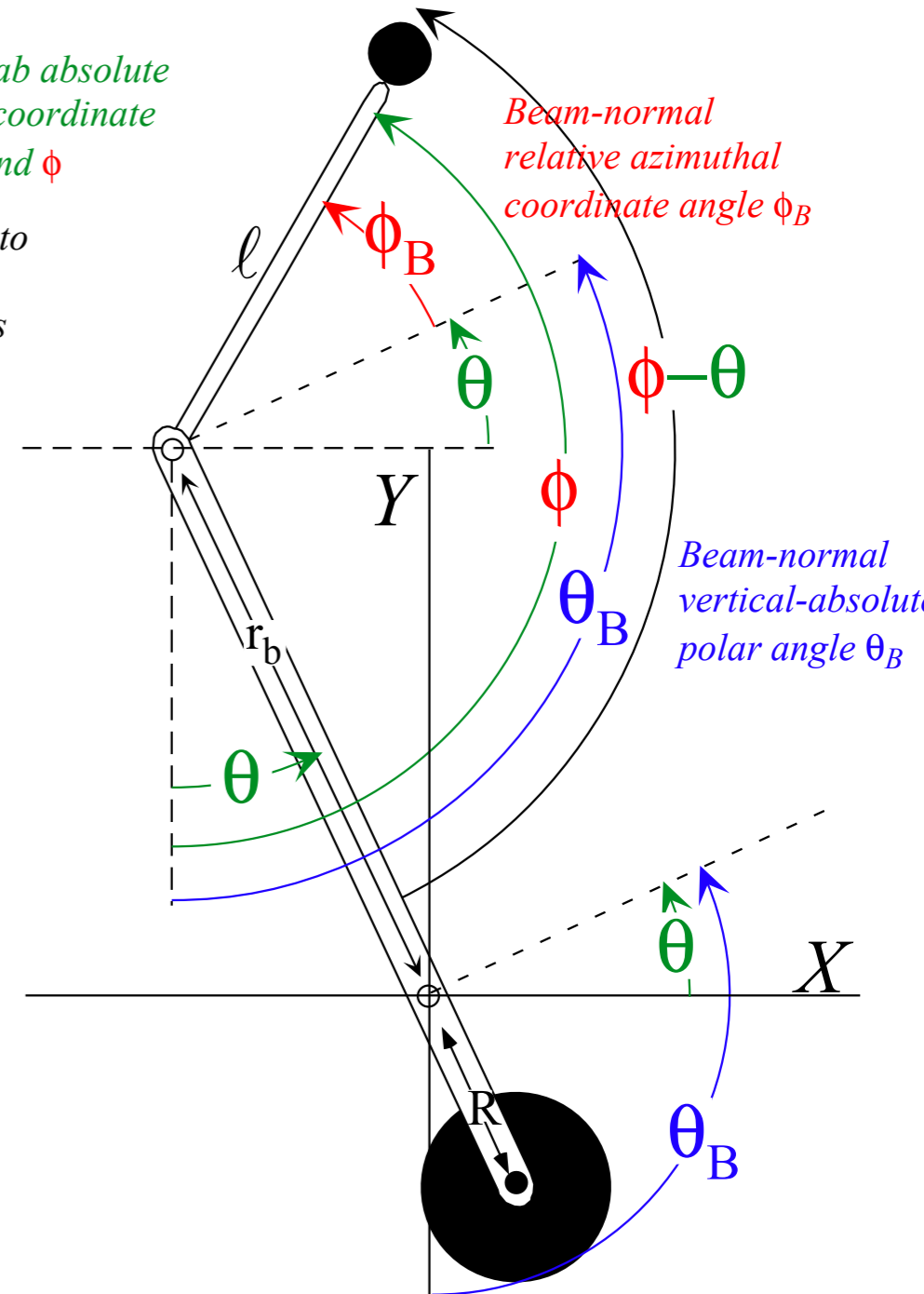
$$= \left((MR^2 + mr^2) \dot{\theta} + mr\ell \dot{\phi} \sin \phi_B \right) + \left(m\ell^2 \dot{\phi} + mr\ell \dot{\theta} \sin \phi_B \right)$$

(Assume zero-gravity)

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B .



Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

(Assume zero-gravity)

Energy for zero-gravity

$$\text{Total KE} = T = \frac{1}{2} \left[(MR^2 + mr^2) \dot{\theta}^2 - 2mr\ell \cos(\theta - \phi) \dot{\theta} \dot{\phi} + m\ell^2 \dot{\phi}^2 \right]$$

$$p_{\theta} = \frac{\partial T}{\partial \dot{\theta}} = (MR^2 + mr^2) \dot{\theta} - mr\ell \dot{\phi} \cos(\theta - \phi)$$

$$p_{\phi} = \frac{\partial T}{\partial \dot{\phi}} = m\ell^2 \dot{\phi} - mr\ell \dot{\theta} \cos(\theta - \phi)$$

Transform to beam-relative coordinates and momenta

$$\begin{aligned} \theta &= \theta_B - \pi/2 & \theta_B &= \theta + \pi/2 \\ \phi &= \theta_B + \phi_B & \phi_B &= -\theta + \phi - \pi/2 \\ \theta - \phi &= -\phi_B - \pi/2 \end{aligned}$$

$$\begin{aligned} p_{\theta} &= p_{\theta}^B - p_{\phi}^B \\ p_{\phi} &= p_{\phi}^B \end{aligned}$$

$$2E = (MR^2 + mr^2) \dot{\theta}^2 + 2mr\ell \dot{\phi} \dot{\theta} \sin \phi_B + m\ell^2 \dot{\phi}^2 = \text{const.}$$

$$\begin{aligned} p_{\theta}^B &= \Lambda = \text{const.} = p_{\theta} + p_{\phi} \\ &= \left((MR^2 + mr^2) \dot{\theta} + mr\ell \dot{\phi} \sin \phi_B \right) + \left(m\ell^2 \dot{\phi} + mr\ell \dot{\theta} \sin \phi_B \right) \end{aligned}$$

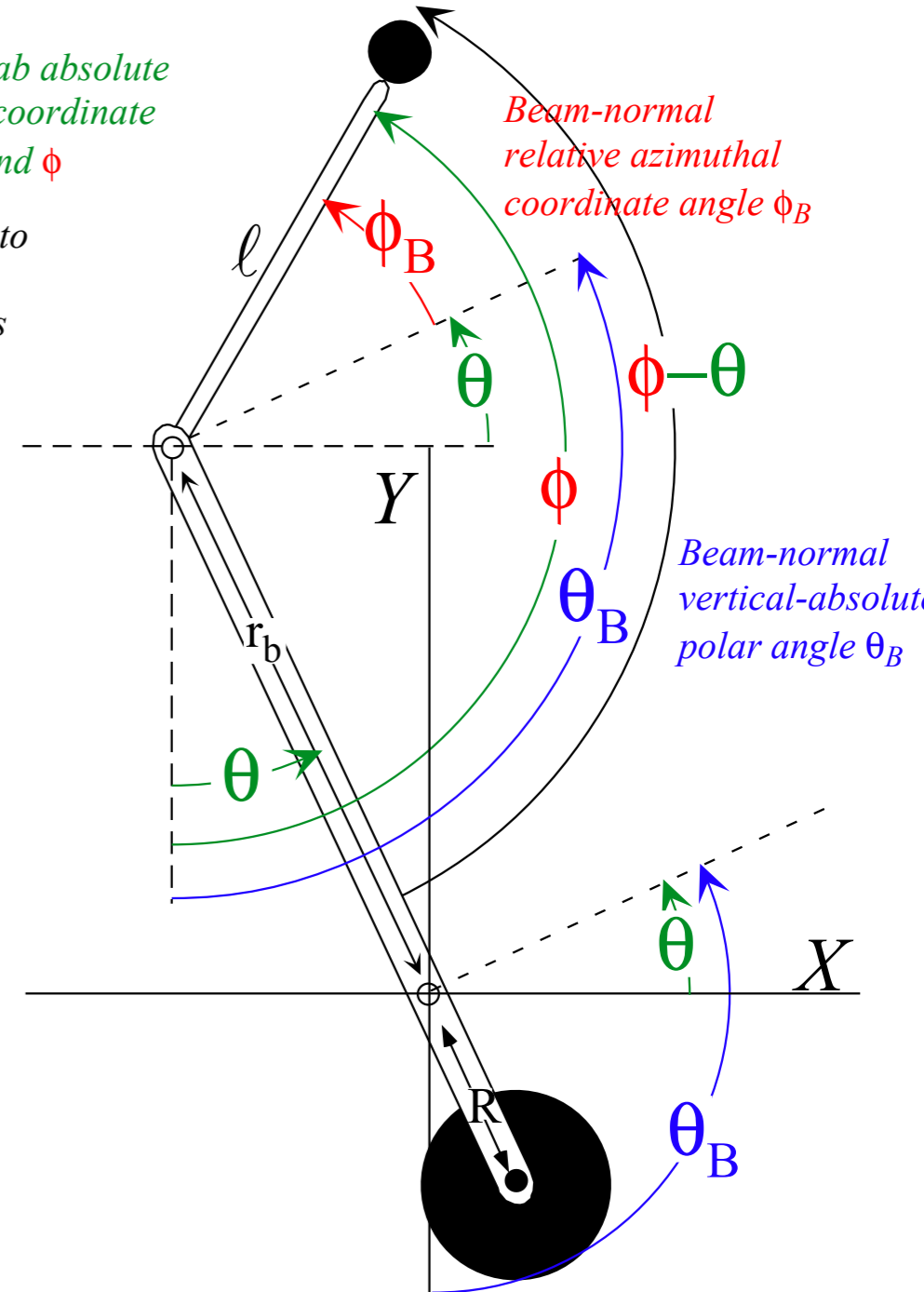
Case of equal arms $r = \ell$ (easier algebra)

$$\left. \begin{aligned} 2E &= MR^2 \dot{\theta}^2 + mr^2 \left(\dot{\theta}^2 + 2\dot{\phi} \dot{\theta} \sin \phi_B + \dot{\phi}^2 \right) \\ \Lambda &= MR^2 \dot{\theta} + mr^2 (1 + \sin \phi_B) (\dot{\theta} + \dot{\phi}) \end{aligned} \right\} \text{(For: } r = \ell)$$

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

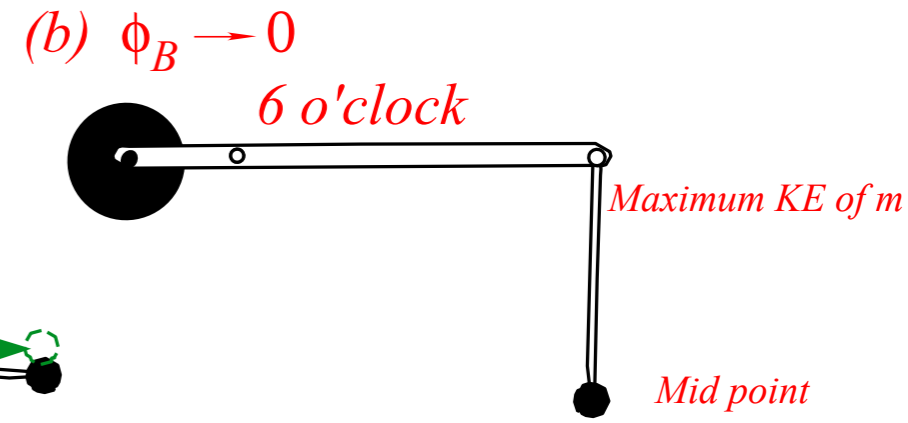
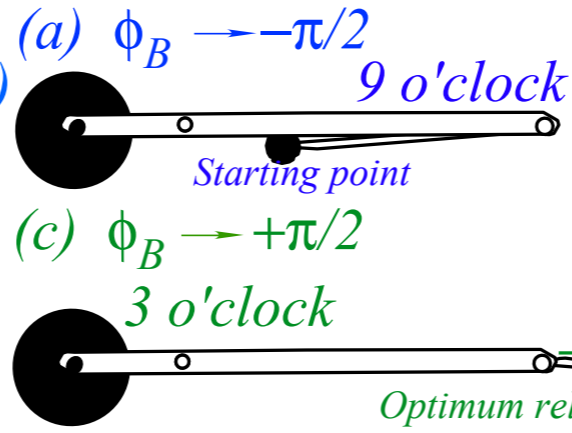
new angles θ_B and ϕ_B .



Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r = \ell$ (easier algebra)

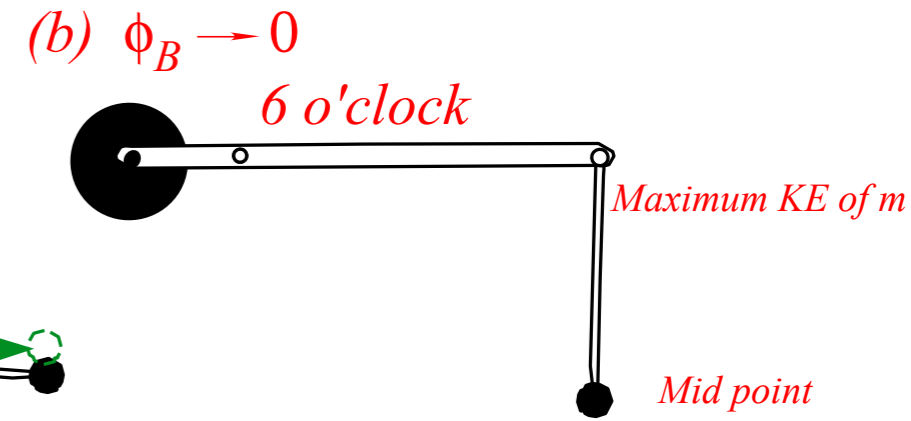
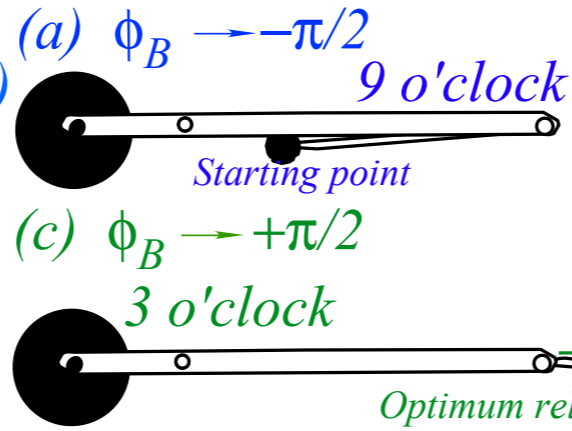
$$\left. \begin{aligned} 2E &= MR^2\dot{\theta}^2 + mr^2(\dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B + \dot{\phi}^2) \\ \Lambda &= MR^2\dot{\theta} + mr^2(1 + \sin\phi_B)(\dot{\theta} + \dot{\phi}) \end{aligned} \right\} \begin{array}{l} \text{For:} \\ r = \ell \end{array}$$



Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r = \ell$ (easier algebra)

$$\left. \begin{aligned} 2E &= MR^2 \dot{\theta}^2 + mr^2 (\dot{\theta}^2 + 2\dot{\phi}\dot{\theta} \sin \phi_B + \dot{\phi}^2) \\ \Lambda &= MR^2 \dot{\theta} + mr^2 (1 + \sin \phi_B) (\dot{\theta} + \dot{\phi}) \end{aligned} \right\} \text{For: } r = \ell$$



Start at 9 o'clock with $\phi_B \sim -90^\circ$ (beam r and throwing arm ℓ rotating together)

$$\left. \begin{aligned} \phi_B &= \frac{-\pi}{2} \\ \sin \phi_B &= -1 \end{aligned} \right\} \begin{cases} 2E = MR^2 \dot{\theta}_{-\pi/2}^2 + mr^2 (\dot{\phi}_{-\pi/2} - \dot{\theta}_{-\pi/2})^2 \\ \Lambda = MR^2 \dot{\theta}_{-\pi/2} \end{cases}$$

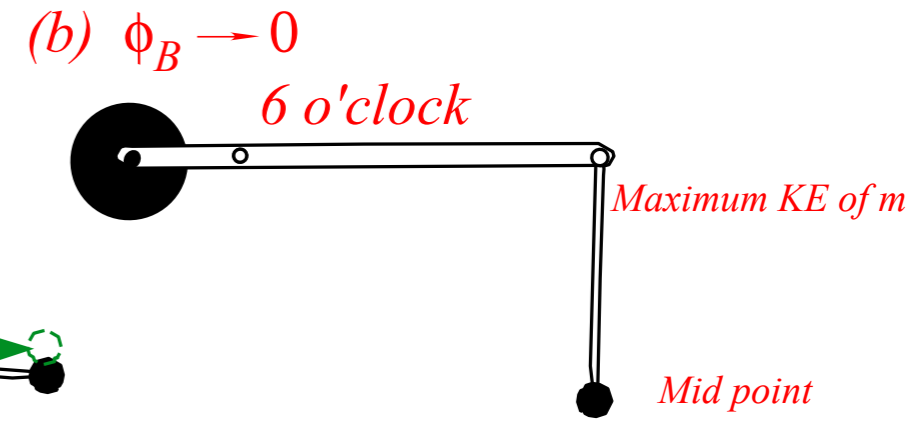
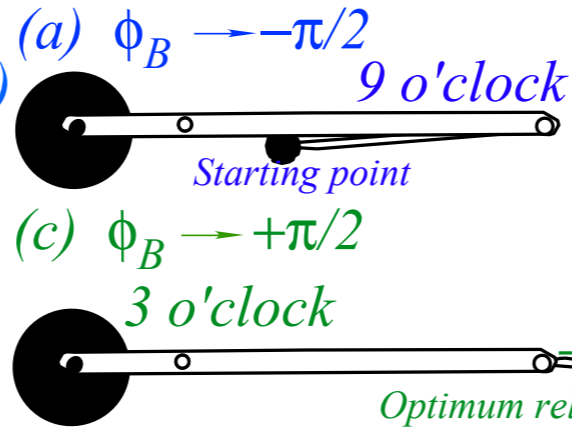
Conserved

$$\text{or: } \begin{cases} \text{initial } 2E \\ 2E = MR^2 \omega^2 \\ \Lambda = MR^2 \omega \\ \text{initial } \Lambda \end{cases} \quad \text{For: } \dot{\theta}_{-\pi/2} = \omega = \dot{\phi}_{-\pi/2}$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r = \ell$ (easier algebra)

$$\left. \begin{aligned} 2E &= MR^2 \dot{\theta}^2 + mr^2 (\dot{\theta}^2 + 2\dot{\phi}\dot{\theta} \sin \phi_B + \dot{\phi}^2) \\ \Lambda &= MR^2 \dot{\theta} + mr^2 (1 + \sin \phi_B) (\dot{\theta} + \dot{\phi}) \end{aligned} \right\} \text{For: } r = \ell$$



Start at 9 o'clock with $\phi_B \sim -90^\circ$ (beam r and throwing arm ℓ rotating together)

$$\left. \begin{aligned} \phi_B &= \frac{-\pi}{2} \\ \sin \phi_B &= -1 \end{aligned} \right\} \begin{cases} 2E = MR^2 \dot{\theta}_{-\pi/2}^2 + mr^2 (\dot{\phi}_{-\pi/2} - \dot{\theta}_{-\pi/2})^2 \\ \Lambda = MR^2 \dot{\theta}_{-\pi/2} \end{cases} \text{Conserved}$$

or: $\begin{cases} \text{initial } 2E \\ 2E = MR^2 \omega^2 \\ \Lambda = MR^2 \omega \\ \text{initial } \Lambda \end{cases} \text{ For: } \dot{\theta}_{-\pi/2} = \omega = \dot{\phi}_{-\pi/2}$

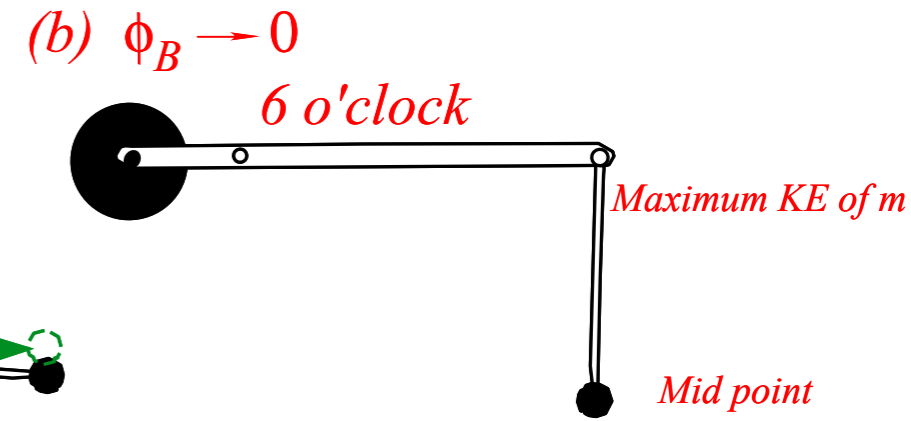
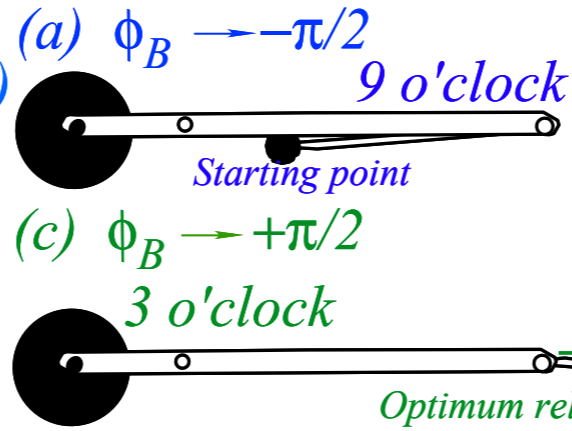
Move to 6 o'clock with $\phi_B \sim 0^\circ$ (beam r slowing, throwing arm ℓ accelerating)

$$\left. \begin{aligned} \phi_B &= 0 \\ \sin \phi_B &= 0 \end{aligned} \right\} \begin{cases} 2E = MR^2 \dot{\theta}_0^2 + mr^2 (\dot{\phi}_0^2 + \dot{\theta}_0^2) \\ \Lambda = MR^2 \dot{\theta}_0 + mr^2 (\dot{\phi}_0 + \dot{\theta}_0) \end{cases}$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r = \ell$ (easier algebra)

$$\left. \begin{aligned} 2E &= MR^2 \dot{\theta}^2 + mr^2 (\dot{\theta}^2 + 2\dot{\phi}\dot{\theta} \sin \phi_B + \dot{\phi}^2) \\ \Lambda &= MR^2 \dot{\theta} + mr^2 (1 + \sin \phi_B)(\dot{\theta} + \dot{\phi}) \end{aligned} \right\} \text{For: } r = \ell$$



Start at 9 o'clock with $\phi_B \sim -90^\circ$ (beam r and throwing arm ℓ rotating together)

$$\left. \begin{aligned} \phi_B &= \frac{-\pi}{2} \\ \sin \phi_B &= -1 \end{aligned} \right\} \begin{cases} 2E = MR^2 \dot{\theta}_{-\pi/2}^2 + mr^2 (\dot{\phi}_{-\pi/2} - \dot{\theta}_{-\pi/2})^2 \\ \Lambda = MR^2 \dot{\theta}_{-\pi/2} \end{cases} \text{Conserved} \quad \text{or: } \begin{cases} \text{initial } 2E \\ 2E = MR^2 \omega^2 \\ \Lambda = MR^2 \omega \\ \text{initial } \Lambda \end{cases} \quad \text{For: } \dot{\theta}_{-\pi/2} = \omega = \dot{\phi}_{-\pi/2}$$

Move to 6 o'clock with $\phi_B \sim 0^\circ$ (beam r slowing, throwing arm ℓ accelerating)

$$\left. \begin{aligned} \phi_B &= 0 \\ \sin \phi_B &= 0 \end{aligned} \right\} \begin{cases} 2E = MR^2 \dot{\theta}_0^2 + mr^2 (\dot{\phi}_0^2 + \dot{\theta}_0^2) \\ \Lambda = MR^2 \dot{\theta}_0 + mr^2 (\dot{\phi}_0 + \dot{\theta}_0) \end{cases}$$

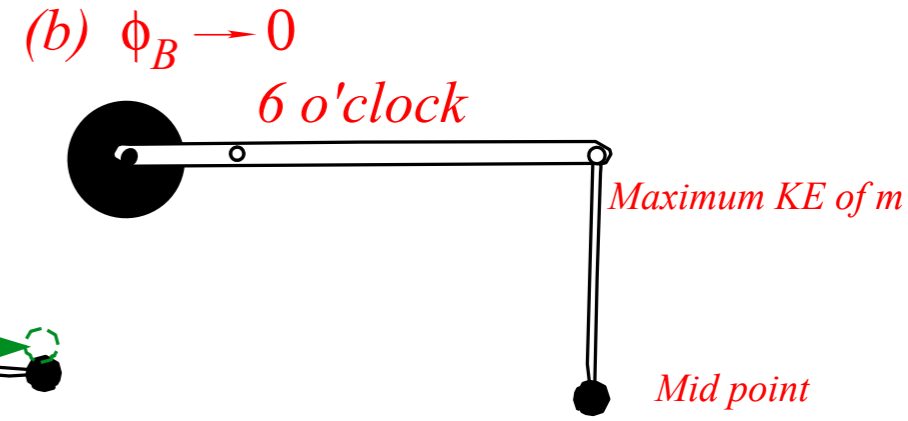
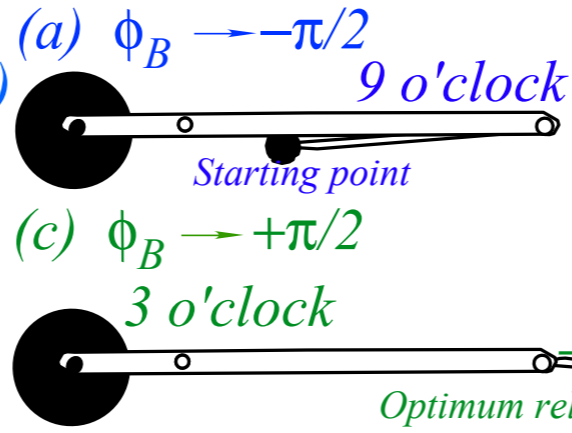
Move to 3 o'clock with $\phi_B \sim +90^\circ$ (beam r slowed, throwing arm ℓ releasing)

$$\left. \begin{aligned} \phi_B &= \pi/2 \\ \sin \phi_B &= +1 \end{aligned} \right\} \begin{cases} 2E = MR^2 \dot{\theta}_{\pi/2}^2 + mr^2 (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2})^2 = \text{initial } 2E \\ \Lambda = MR^2 \dot{\theta}_{\pi/2} + 2mr^2 (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}) = \text{initial } \Lambda \end{cases} \text{Conserved}$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r = \ell$ (easier algebra)

$$\left. \begin{aligned} 2E &= MR^2 \dot{\theta}^2 + mr^2 (\dot{\theta}^2 + 2\dot{\phi}\dot{\theta} \sin \phi_B + \dot{\phi}^2) \\ \Lambda &= MR^2 \dot{\theta} + mr^2 (1 + \sin \phi_B) (\dot{\theta} + \dot{\phi}) \end{aligned} \right\} \text{For: } r = \ell$$



Start at 9 o'clock with $\phi_B \sim -90^\circ$ (beam r and throwing arm ℓ rotating together)

$$\left. \begin{aligned} \phi_B &= \frac{-\pi}{2} \\ \sin \phi_B &= -1 \end{aligned} \right\} \begin{cases} 2E = MR^2 \dot{\theta}_{-\pi/2}^2 + mr^2 (\dot{\phi}_{-\pi/2} - \dot{\theta}_{-\pi/2})^2 \\ \Lambda = MR^2 \dot{\theta}_{-\pi/2} \end{cases} \text{Conserved initial } 2E \text{ and } \Lambda$$

or: $\begin{cases} 2E = MR^2 \omega^2 \\ \Lambda = MR^2 \omega \end{cases}$ For: $\dot{\theta}_{-\pi/2} = \omega = \dot{\phi}_{-\pi/2}$

Move to 6 o'clock with $\phi_B \sim 0^\circ$ (beam r slowing, throwing arm ℓ accelerating)

$$\left. \begin{aligned} \phi_B &= 0 \\ \sin \phi_B &= 0 \end{aligned} \right\} \begin{cases} 2E = MR^2 \dot{\theta}_0^2 + mr^2 (\dot{\phi}_0^2 + \dot{\theta}_0^2) \\ \Lambda = MR^2 \dot{\theta}_0 + mr^2 (\dot{\phi}_0 + \dot{\theta}_0) \end{cases}$$

Move to 3 o'clock with $\phi_B \sim +90^\circ$ (beam r slowed, throwing arm ℓ releasing)

$$\left. \begin{aligned} \phi_B &= \pi/2 \\ \sin \phi_B &= +1 \end{aligned} \right\} \begin{cases} 2E = MR^2 \dot{\theta}_{\pi/2}^2 + mr^2 (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2})^2 = MR^2 \omega^2 \\ \Lambda = MR^2 \dot{\theta}_{\pi/2} + 2mr^2 (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}) = MR^2 \omega \end{cases} \text{Conserved initial } 2E \text{ and } \Lambda$$

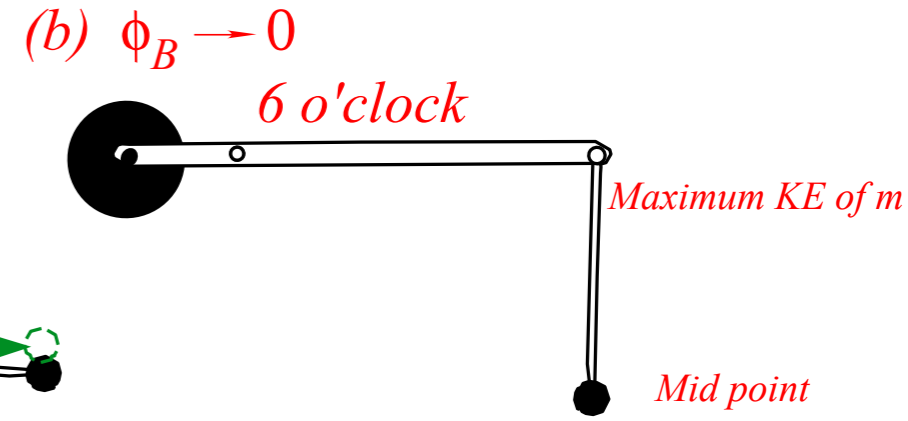
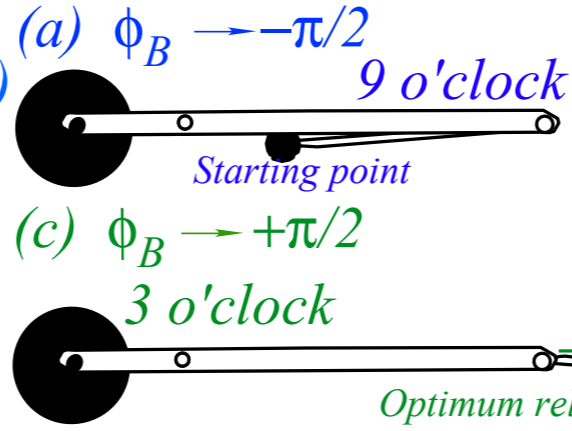
$$KE(m) = \frac{mr^2}{2} (\dot{\phi}^2 + \dot{\theta}^2 + 2\dot{\phi}\dot{\theta} \sin \phi_B)$$

$$= \begin{cases} \frac{mr^2}{2} (\dot{\phi} - \dot{\theta})^2 & \left(\text{For: } \phi_B = -\frac{\pi}{2} \right) \\ \frac{mr^2}{2} (\dot{\phi}^2 + \dot{\theta}^2) & \left(\text{For: } \phi_B = 0 \right) \\ \frac{mr^2}{2} (\dot{\phi} + \dot{\theta})^2 & \left(\text{For: } \phi_B = \frac{\pi}{2} \right) \end{cases}$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r = \ell$ (easier algebra)

$$\left. \begin{aligned} 2E &= MR^2 \dot{\theta}^2 + mr^2 (\dot{\theta}^2 + 2\dot{\phi}\dot{\theta} \sin \phi_B + \dot{\phi}^2) \\ \Lambda &= MR^2 \dot{\theta} + mr^2 (1 + \sin \phi_B) (\dot{\theta} + \dot{\phi}) \end{aligned} \right\} \text{For: } r = \ell$$



Start at 9 o'clock with $\phi_B \sim -90^\circ$ (beam r and throwing arm ℓ rotating together)

$$\left. \begin{aligned} \phi_B &= \frac{-\pi}{2} \\ \sin \phi_B &= -1 \end{aligned} \right\} \begin{cases} 2E = MR^2 \dot{\theta}_{-\pi/2}^2 + mr^2 (\dot{\phi}_{-\pi/2} - \dot{\theta}_{-\pi/2})^2 \\ \Lambda = MR^2 \dot{\theta}_{-\pi/2} \end{cases} \text{Conserved} \\ \text{or: } \begin{cases} \text{initial } 2E \\ 2E = MR^2 \omega^2 \\ \Lambda = MR^2 \omega \\ \text{initial } \Lambda \end{cases} \text{For: } \dot{\theta}_{-\pi/2} = \omega = \dot{\phi}_{-\pi/2}$$

Move to 6 o'clock with $\phi_B \sim 0^\circ$ (beam r slowing, throwing arm ℓ accelerating)

$$\left. \begin{aligned} \phi_B &= 0 \\ \sin \phi_B &= 0 \end{aligned} \right\} \begin{cases} 2E = MR^2 \dot{\theta}_0^2 + mr^2 (\dot{\phi}_0^2 + \dot{\theta}_0^2) \\ \Lambda = MR^2 \dot{\theta}_0 + mr^2 (\dot{\phi}_0 + \dot{\theta}_0) \end{cases}$$

Move to 3 o'clock with $\phi_B \sim +90^\circ$ (beam r slowed, throwing arm ℓ releasing)

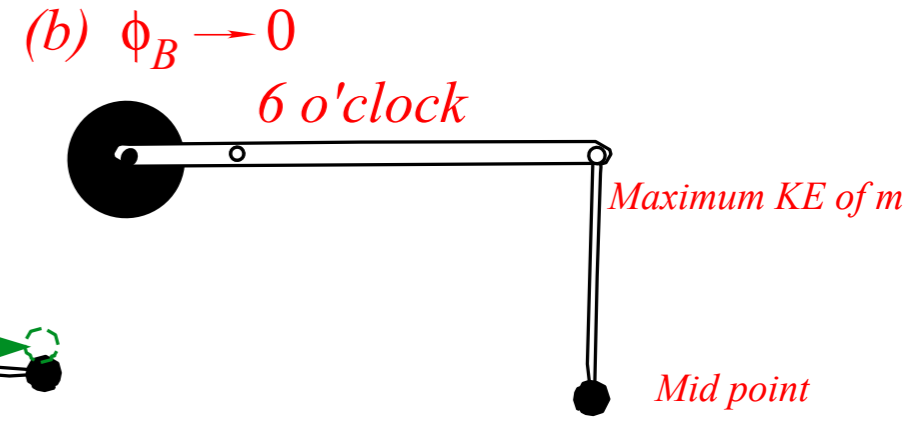
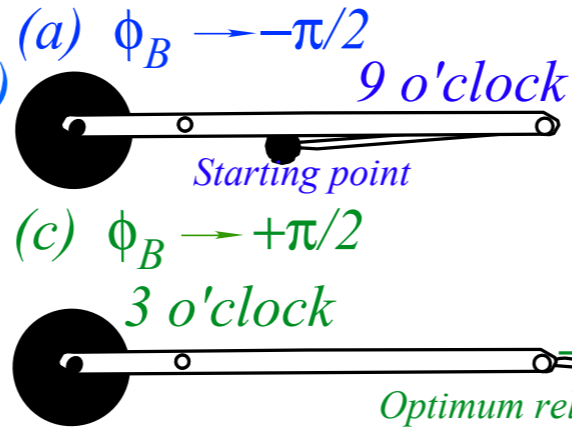
$$\left. \begin{aligned} \phi_B &= \pi/2 \\ \sin \phi_B &= +1 \end{aligned} \right\} \begin{cases} 2E = MR^2 \dot{\theta}_{\pi/2}^2 + mr^2 (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2})^2 \\ \Lambda = MR^2 \dot{\theta}_{\pi/2} + 2mr^2 (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}) \end{cases} \text{Conserved} \\ \begin{aligned} &= MR^2 \omega^2 \longrightarrow (\omega^2 - \dot{\theta}_{\pi/2}^2) = \frac{mr^2}{MR^2} (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2})^2 \\ &= MR^2 \omega \longrightarrow (\omega - \dot{\theta}_{\pi/2}) = \frac{2mr^2}{MR^2} (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}) \end{aligned}$$

$$KE(m) = \frac{mr^2}{2} (\dot{\phi}^2 + \dot{\theta}^2 + 2\dot{\phi}\dot{\theta} \sin \phi_B) = \begin{cases} \frac{mr^2}{2} (\dot{\phi} - \dot{\theta})^2 & \left(\text{For: } \phi_B = -\frac{\pi}{2} \right) \\ \frac{mr^2}{2} (\dot{\phi}^2 + \dot{\theta}^2) & \left(\text{For: } \phi_B = 0 \right) \\ \frac{mr^2}{2} (\dot{\phi} + \dot{\theta})^2 & \left(\text{For: } \phi_B = \frac{\pi}{2} \right) \end{cases}$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r = \ell$ (easier algebra)

$$\left. \begin{aligned} 2E &= MR^2 \dot{\theta}^2 + mr^2 (\dot{\theta}^2 + 2\dot{\phi}\dot{\theta} \sin \phi_B + \dot{\phi}^2) \\ \Lambda &= MR^2 \dot{\theta} + mr^2 (1 + \sin \phi_B) (\dot{\theta} + \dot{\phi}) \end{aligned} \right\} \text{For: } r = \ell$$



Start at 9 o'clock with $\phi_B \sim -90^\circ$ (beam r and throwing arm ℓ rotating together)

$$\left. \begin{aligned} \phi_B &= \frac{-\pi}{2} \\ \sin \phi_B &= -1 \end{aligned} \right\} \begin{cases} 2E = MR^2 \dot{\theta}_{-\pi/2}^2 + mr^2 (\dot{\phi}_{-\pi/2} - \dot{\theta}_{-\pi/2})^2 \\ \Lambda = MR^2 \dot{\theta}_{-\pi/2} \end{cases} \text{Conserved} \quad \text{or: } \begin{cases} \text{initial } 2E \\ 2E = MR^2 \omega^2 \\ \Lambda = MR^2 \omega \\ \text{initial } \Lambda \end{cases} \quad \text{For: } \dot{\theta}_{-\pi/2} = \omega = \dot{\phi}_{-\pi/2}$$

Move to 6 o'clock with $\phi_B \sim 0^\circ$ (beam r slowing, throwing arm ℓ accelerating)

$$\left. \begin{aligned} \phi_B &= 0 \\ \sin \phi_B &= 0 \end{aligned} \right\} \begin{cases} 2E = MR^2 \dot{\theta}_0^2 + mr^2 (\dot{\phi}_0^2 + \dot{\theta}_0^2) \\ \Lambda = MR^2 \dot{\theta}_0 + mr^2 (\dot{\phi}_0 + \dot{\theta}_0) \end{cases}$$

Move to 3 o'clock with $\phi_B \sim +90^\circ$ (beam r slowed, throwing arm ℓ releasing)

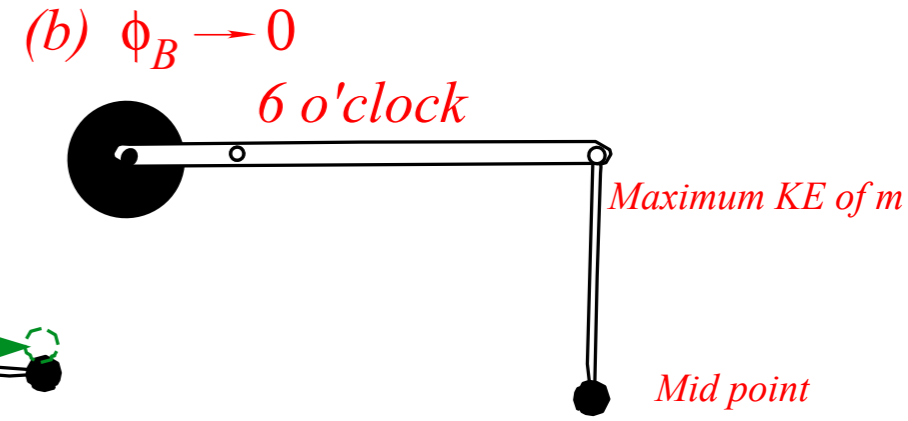
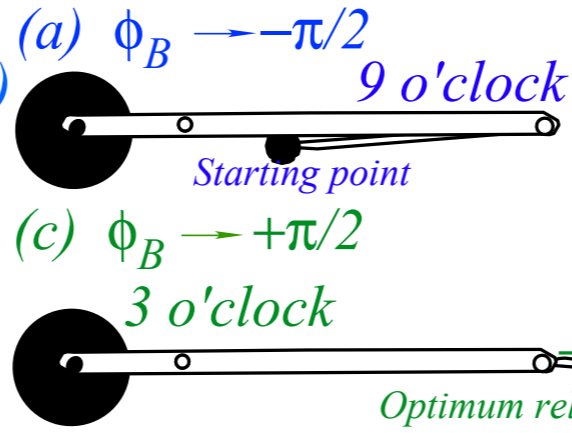
$$\left. \begin{aligned} \phi_B &= \pi/2 \\ \sin \phi_B &= +1 \end{aligned} \right\} \begin{cases} 2E = MR^2 \dot{\theta}_{\pi/2}^2 + mr^2 (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2})^2 \\ \Lambda = MR^2 \dot{\theta}_{\pi/2} + 2mr^2 (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}) \end{cases} \begin{array}{l} \text{Conserved} \\ \text{initial } 2E \\ = MR^2 \omega^2 \end{array} \begin{array}{l} \xrightarrow{\text{divide } 2E} \\ (\omega^2 - \dot{\theta}_{\pi/2}^2) = \frac{mr^2}{MR^2} (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2})^2 \\ \xrightarrow{\text{by } \Lambda} \\ (\omega - \dot{\theta}_{\pi/2}) = \frac{2mr^2}{MR^2} (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}) \end{array} \rightarrow (\omega + \dot{\theta}_{\pi/2}) = \frac{1}{2} (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2})$$

$$KE(m) = \frac{mr^2}{2} (\dot{\phi}^2 + \dot{\theta}^2 + 2\dot{\phi}\dot{\theta} \sin \phi_B) = \begin{cases} \frac{mr^2}{2} (\dot{\phi} - \dot{\theta})^2 & \left(\text{For: } \phi_B = -\frac{\pi}{2} \right) \\ \frac{mr^2}{2} (\dot{\phi}^2 + \dot{\theta}^2) & \left(\text{For: } \phi_B = 0 \right) \\ \frac{mr^2}{2} (\dot{\phi} + \dot{\theta})^2 & \left(\text{For: } \phi_B = \frac{\pi}{2} \right) \end{cases}$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r = \ell$ (easier algebra)

$$\left. \begin{aligned} 2E &= MR^2 \dot{\theta}^2 + mr^2 (\dot{\theta}^2 + 2\dot{\phi}\dot{\theta} \sin \phi_B + \dot{\phi}^2) \\ \Lambda &= MR^2 \dot{\theta} + mr^2 (1 + \sin \phi_B) (\dot{\theta} + \dot{\phi}) \end{aligned} \right\} \text{For: } r = \ell$$



Start at 9 o'clock with $\phi_B \sim -90^\circ$ (beam r and throwing arm ℓ rotating together)

$$\left. \begin{aligned} \phi_B &= \frac{-\pi}{2} \\ \sin \phi_B &= -1 \end{aligned} \right\} \begin{cases} 2E = MR^2 \dot{\theta}_{-\pi/2}^2 + mr^2 (\dot{\phi}_{-\pi/2} - \dot{\theta}_{-\pi/2})^2 \\ \Lambda = MR^2 \dot{\theta}_{-\pi/2} \end{cases} \text{Conserved}$$

initial $2E$

$$\text{or: } \begin{cases} 2E = MR^2 \omega^2 \\ \Lambda = MR^2 \omega \end{cases} \text{For: } \dot{\theta}_{-\pi/2} = \omega = \dot{\phi}_{-\pi/2}$$

initial Λ

Move to 6 o'clock with $\phi_B \sim 0^\circ$ (beam r slowing, throwing arm ℓ accelerating)

$$\left. \begin{aligned} \phi_B &= 0 \\ \sin \phi_B &= 0 \end{aligned} \right\} \begin{cases} 2E = MR^2 \dot{\theta}_0^2 + mr^2 (\dot{\phi}_0^2 + \dot{\theta}_0^2) \\ \Lambda = MR^2 \dot{\theta}_0 + mr^2 (\dot{\phi}_0 + \dot{\theta}_0) \end{cases}$$

Move to 3 o'clock with $\phi_B \sim +90^\circ$ (beam r slowed, throwing arm ℓ releasing)

$$\left. \begin{aligned} \phi_B &= \pi/2 \\ \sin \phi_B &= +1 \end{aligned} \right\} \begin{cases} 2E = MR^2 \dot{\theta}_{\pi/2}^2 + mr^2 (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2})^2 \\ \Lambda = MR^2 \dot{\theta}_{\pi/2} + 2mr^2 (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}) \end{cases} \text{Conserved}$$

initial $2E$

$$\begin{aligned} (\omega^2 - \dot{\theta}_{\pi/2}^2) &= \frac{mr^2}{MR^2} (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2})^2 && \text{divide } 2E \\ (\omega - \dot{\theta}_{\pi/2}) &= \frac{2mr^2}{MR^2} (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}) && \text{by } \Lambda \end{aligned}$$

$$(\omega + \dot{\theta}_{\pi/2}) = \frac{1}{2} (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2})$$

$$\dot{\phi}_{\pi/2} = \dot{\theta}_{\pi/2} + 2\omega$$

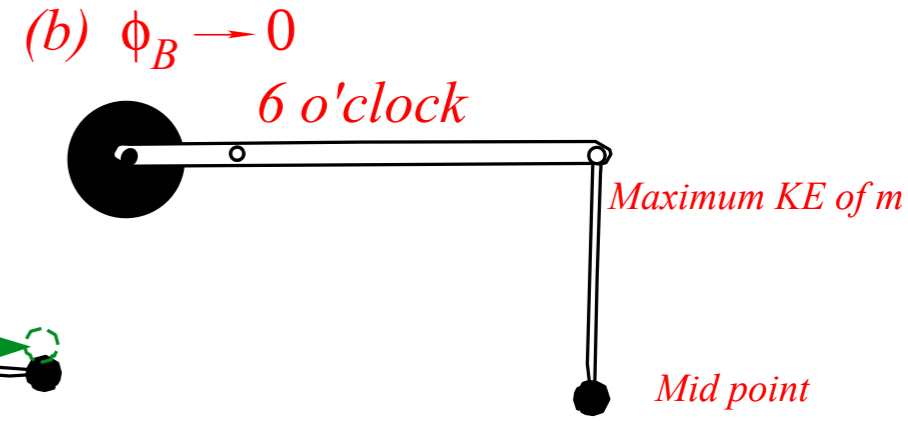
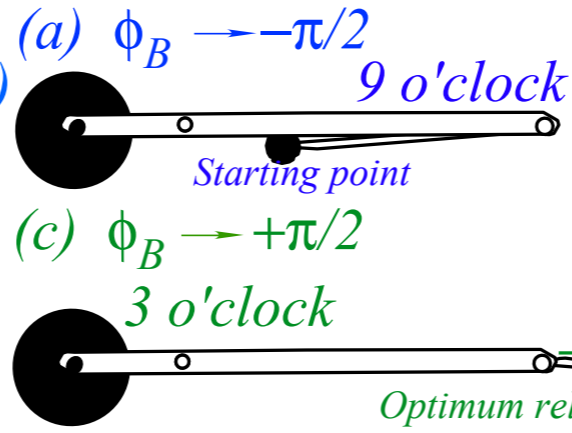
$$KE(m) = \frac{mr^2}{2} (\dot{\phi}^2 + \dot{\theta}^2 + 2\dot{\phi}\dot{\theta} \sin \phi_B)$$

$$= \begin{cases} \frac{mr^2}{2} (\dot{\phi} - \dot{\theta})^2 & \left(\text{For: } \phi_B = -\frac{\pi}{2} \right) \\ \frac{mr^2}{2} (\dot{\phi}^2 + \dot{\theta}^2) & \left(\text{For: } \phi_B = 0 \right) \\ \frac{mr^2}{2} (\dot{\phi} + \dot{\theta})^2 & \left(\text{For: } \phi_B = \frac{\pi}{2} \right) \end{cases}$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r = \ell$ (easier algebra)

$$\left. \begin{aligned} 2E &= MR^2 \dot{\theta}^2 + mr^2 (\dot{\theta}^2 + 2\dot{\phi}\dot{\theta} \sin \phi_B + \dot{\phi}^2) \\ \Lambda &= MR^2 \dot{\theta} + mr^2 (1 + \sin \phi_B) (\dot{\theta} + \dot{\phi}) \end{aligned} \right\} \text{For: } r = \ell$$



Start at 9 o'clock with $\phi_B \sim -90^\circ$ (beam r and throwing arm ℓ rotating together)

$$\left. \begin{aligned} \phi_B &= \frac{-\pi}{2} \\ \sin \phi_B &= -1 \end{aligned} \right\} \begin{cases} 2E = MR^2 \dot{\theta}_{-\pi/2}^2 + mr^2 (\dot{\phi}_{-\pi/2} - \dot{\theta}_{-\pi/2})^2 \\ \Lambda = MR^2 \dot{\theta}_{-\pi/2} \end{cases} \text{Conserved}$$

initial $2E$

$$\text{or: } \begin{cases} 2E = MR^2 \omega^2 \\ \Lambda = MR^2 \omega \end{cases} \text{For: } \dot{\theta}_{-\pi/2} = \omega = \dot{\phi}_{-\pi/2}$$

initial Λ

Move to 6 o'clock with $\phi_B \sim 0^\circ$ (beam r slowing, throwing arm ℓ accelerating)

$$\left. \begin{aligned} \phi_B &= 0 \\ \sin \phi_B &= 0 \end{aligned} \right\} \begin{cases} 2E = MR^2 \dot{\theta}_0^2 + mr^2 (\dot{\phi}_0^2 + \dot{\theta}_0^2) \\ \Lambda = MR^2 \dot{\theta}_0 + mr^2 (\dot{\phi}_0 + \dot{\theta}_0) \end{cases}$$

Move to 3 o'clock with $\phi_B \sim +90^\circ$ (beam r slowed, throwing arm ℓ releasing)

$$\left. \begin{aligned} \phi_B &= \pi/2 \\ \sin \phi_B &= +1 \end{aligned} \right\} \begin{cases} 2E = MR^2 \dot{\theta}_{\pi/2}^2 + mr^2 (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2})^2 \\ \Lambda = MR^2 \dot{\theta}_{\pi/2} + 2mr^2 (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}) \end{cases} \text{Conserved}$$

initial $2E$

$$\begin{aligned} (\omega^2 - \dot{\theta}_{\pi/2}^2) &= \frac{mr^2}{MR^2} (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2})^2 && \text{divide } 2E \\ (\omega - \dot{\theta}_{\pi/2}) &= \frac{2mr^2}{MR^2} (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}) && \text{by } \Lambda \end{aligned}$$

$$(\omega + \dot{\theta}_{\pi/2}) = \frac{1}{2} (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2})$$

$$\dot{\phi}_{\pi/2} = \dot{\theta}_{\pi/2} + 2\omega$$

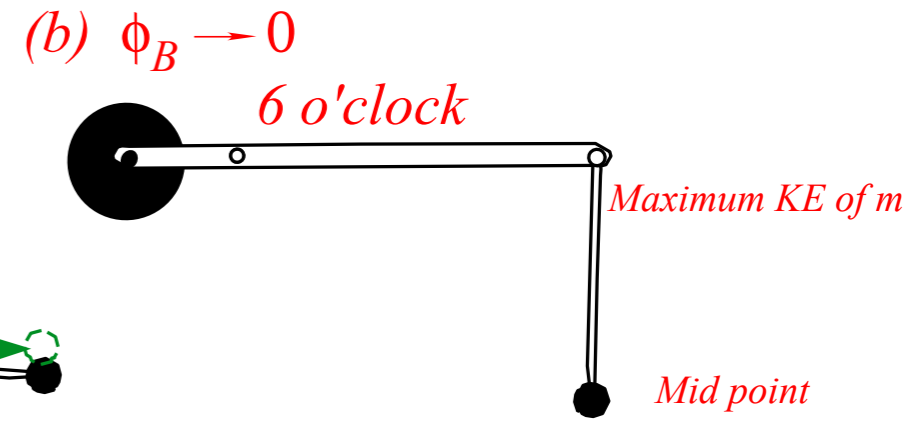
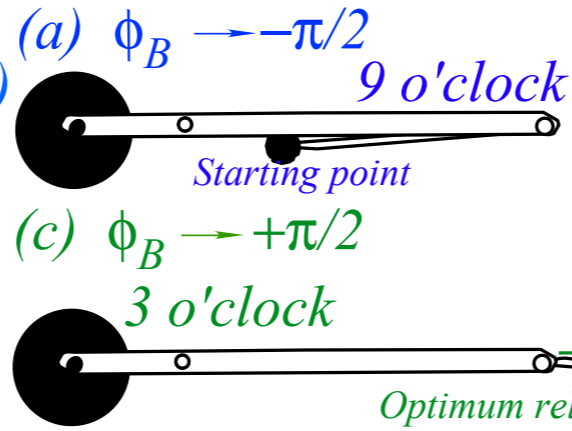
$$KE(m) = \frac{mr^2}{2} (\dot{\phi}^2 + \dot{\theta}^2 + 2\dot{\phi}\dot{\theta} \sin \phi_B)$$

$$= \begin{cases} \frac{mr^2}{2} (\dot{\phi} - \dot{\theta})^2 & \left(\text{For: } \phi_B = -\frac{\pi}{2} \right) \\ \frac{mr^2}{2} (\dot{\phi}^2 + \dot{\theta}^2) & \left(\text{For: } \phi_B = 0 \right) \\ \frac{mr^2}{2} (\dot{\phi} + \dot{\theta})^2 & \left(\text{For: } \phi_B = \frac{\pi}{2} \right) \end{cases}$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r = \ell$ (easier algebra)

$$\left. \begin{aligned} 2E &= MR^2 \dot{\theta}^2 + mr^2 (\dot{\theta}^2 + 2\dot{\phi}\dot{\theta} \sin \phi_B + \dot{\phi}^2) \\ \Lambda &= MR^2 \dot{\theta} + mr^2 (1 + \sin \phi_B) (\dot{\theta} + \dot{\phi}) \end{aligned} \right\} \text{For: } r = \ell$$



Start at 9 o'clock with $\phi_B \sim -90^\circ$ (beam r and throwing arm ℓ rotating together)

$$\left. \begin{aligned} \phi_B &= \frac{-\pi}{2} \\ \sin \phi_B &= -1 \end{aligned} \right\} \begin{cases} 2E = MR^2 \dot{\theta}_{-\pi/2}^2 + mr^2 (\dot{\phi}_{-\pi/2} - \dot{\theta}_{-\pi/2})^2 \\ \Lambda = MR^2 \dot{\theta}_{-\pi/2} \end{cases} \text{Conserved} \\ \text{or: } \begin{cases} \text{initial } 2E \\ 2E = MR^2 \omega^2 \\ \Lambda = MR^2 \omega \\ \text{initial } \Lambda \end{cases} \text{For: } \dot{\theta}_{-\pi/2} = \omega = \dot{\phi}_{-\pi/2}$$

Move to 6 o'clock with $\phi_B \sim 0^\circ$ (beam r slowing, throwing arm ℓ accelerating)

$$\left. \begin{aligned} \phi_B &= 0 \\ \sin \phi_B &= 0 \end{aligned} \right\} \begin{cases} 2E = MR^2 \dot{\theta}_0^2 + mr^2 (\dot{\phi}_0^2 + \dot{\theta}_0^2) \\ \Lambda = MR^2 \dot{\theta}_0 + mr^2 (\dot{\phi}_0 + \dot{\theta}_0) \end{cases}$$

$$KE(m) = \frac{mr^2}{2} (\dot{\phi}^2 + \dot{\theta}^2 + 2\dot{\phi}\dot{\theta} \sin \phi_B) = \begin{cases} \frac{mr^2}{2} (\dot{\phi} - \dot{\theta})^2 & \text{(For: } \phi_B = -\frac{\pi}{2}) \\ \frac{mr^2}{2} (\dot{\phi}^2 + \dot{\theta}^2) & \text{(For: } \phi_B = 0) \\ \frac{mr^2}{2} (\dot{\phi} + \dot{\theta})^2 & \text{(For: } \phi_B = \frac{\pi}{2}) \end{cases}$$

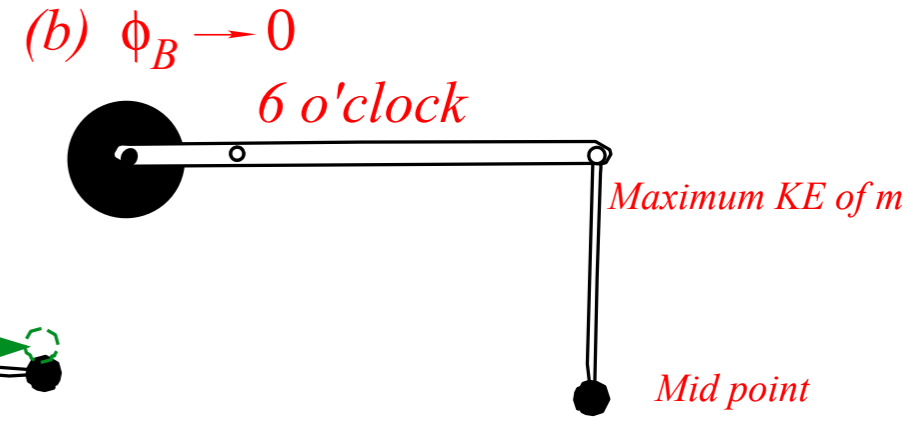
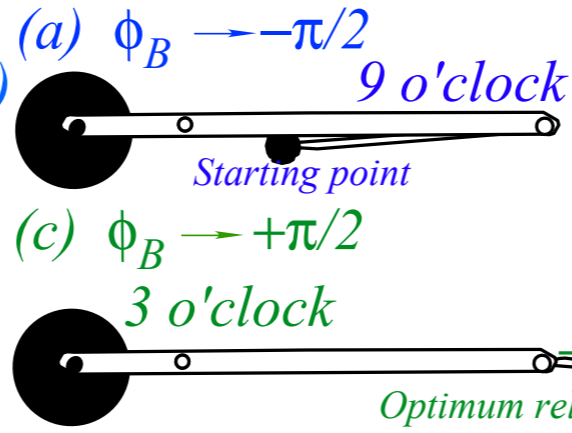
Move to 3 o'clock with $\phi_B \sim +90^\circ$ (beam r slowed, throwing arm ℓ releasing)

$$\left. \begin{aligned} \phi_B &= \pi/2 \\ \sin \phi_B &= +1 \end{aligned} \right\} \begin{cases} 2E = MR^2 \dot{\theta}_{\pi/2}^2 + mr^2 (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2})^2 \\ \Lambda = MR^2 \dot{\theta}_{\pi/2} + 2mr^2 (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}) \end{cases} \text{Conserved} \\ \begin{aligned} &= MR^2 \omega^2 \xrightarrow{\text{initial } 2E} (\omega^2 - \dot{\theta}_{\pi/2}^2) = \frac{mr^2}{MR^2} (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2})^2 \\ & \xrightarrow{\text{divide } 2E} (\omega - \dot{\theta}_{\pi/2}) = \frac{2mr^2}{MR^2} (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}) \\ & \xrightarrow{\text{by } \Lambda} \omega - \dot{\theta}_{\pi/2} = \frac{2mr^2}{MR^2} (2\omega + 2\dot{\theta}_{\pi/2}) \\ & \xrightarrow{\text{substitute}} \omega - \dot{\theta}_{\pi/2} = \frac{2mr^2}{MR^2} (2\omega + 2\dot{\theta}_{\pi/2}) \end{aligned} \quad \begin{aligned} &(\omega + \dot{\theta}_{\pi/2}) = \frac{1}{2} (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}) \\ & \downarrow \\ & \dot{\phi}_{\pi/2} = \dot{\theta}_{\pi/2} + 2\omega \end{aligned}$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r = \ell$ (easier algebra)

$$\left. \begin{aligned} 2E &= MR^2 \dot{\theta}^2 + mr^2 (\dot{\theta}^2 + 2\dot{\phi}\dot{\theta} \sin \phi_B + \dot{\phi}^2) \\ \Lambda &= MR^2 \dot{\theta} + mr^2 (1 + \sin \phi_B) (\dot{\theta} + \dot{\phi}) \end{aligned} \right\} \text{For: } r = \ell$$



Start at 9 o'clock with $\phi_B \sim -90^\circ$ (beam r and throwing arm ℓ rotating together)

$$\left. \begin{aligned} \phi_B &= \frac{-\pi}{2} \\ \sin \phi_B &= -1 \end{aligned} \right\} \begin{cases} 2E = MR^2 \dot{\theta}_{-\pi/2}^2 + mr^2 (\dot{\phi}_{-\pi/2} - \dot{\theta}_{-\pi/2})^2 \\ \Lambda = MR^2 \dot{\theta}_{-\pi/2} \end{cases} \text{Conserved}$$

or: $\begin{cases} \text{initial } 2E \\ 2E = MR^2 \omega^2 \\ \Lambda = MR^2 \omega \\ \text{initial } \Lambda \end{cases} \text{ For: } \dot{\theta}_{-\pi/2} = \omega = \dot{\phi}_{-\pi/2}$

Move to 6 o'clock with $\phi_B \sim 0^\circ$ (beam r slowing, throwing arm ℓ accelerating)

$$\left. \begin{aligned} \phi_B &= 0 \\ \sin \phi_B &= 0 \end{aligned} \right\} \begin{cases} 2E = MR^2 \dot{\theta}_0^2 + mr^2 (\dot{\phi}_0^2 + \dot{\theta}_0^2) \\ \Lambda = MR^2 \dot{\theta}_0 + mr^2 (\dot{\phi}_0 + \dot{\theta}_0) \end{cases}$$

$$KE(m) = \frac{mr^2}{2} (\dot{\phi}^2 + \dot{\theta}^2 + 2\dot{\phi}\dot{\theta} \sin \phi_B)$$

$$= \begin{cases} \frac{mr^2}{2} (\dot{\phi} - \dot{\theta})^2 & \left(\text{For: } \phi_B = -\frac{\pi}{2} \right) \\ \frac{mr^2}{2} (\dot{\phi}^2 + \dot{\theta}^2) & \left(\text{For: } \phi_B = 0 \right) \\ \frac{mr^2}{2} (\dot{\phi} + \dot{\theta})^2 & \left(\text{For: } \phi_B = \frac{\pi}{2} \right) \end{cases}$$

Move to 3 o'clock with $\phi_B \sim +90^\circ$ (beam r slowed, throwing arm ℓ releasing)

$$\left. \begin{aligned} \phi_B &= \pi/2 \\ \sin \phi_B &= +1 \end{aligned} \right\} \begin{cases} 2E = MR^2 \dot{\theta}_{\pi/2}^2 + mr^2 (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2})^2 \\ \Lambda = MR^2 \dot{\theta}_{\pi/2} + 2mr^2 (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}) \end{cases} \text{Conserved}$$

initial $2E = MR^2 \omega^2$ → $(\omega^2 - \dot{\theta}_{\pi/2}^2) = \frac{mr^2}{MR^2} (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2})^2$ (divide 2E)

initial $\Lambda = MR^2 \omega$ → $(\omega - \dot{\theta}_{\pi/2}) = \frac{2mr^2}{MR^2} (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2})$ (by Λ)

substitute → $(\omega + \dot{\theta}_{\pi/2}) = \frac{1}{2} (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2})$

→ $\dot{\phi}_{\pi/2} = \dot{\theta}_{\pi/2} + 2\omega$ (circled in red)

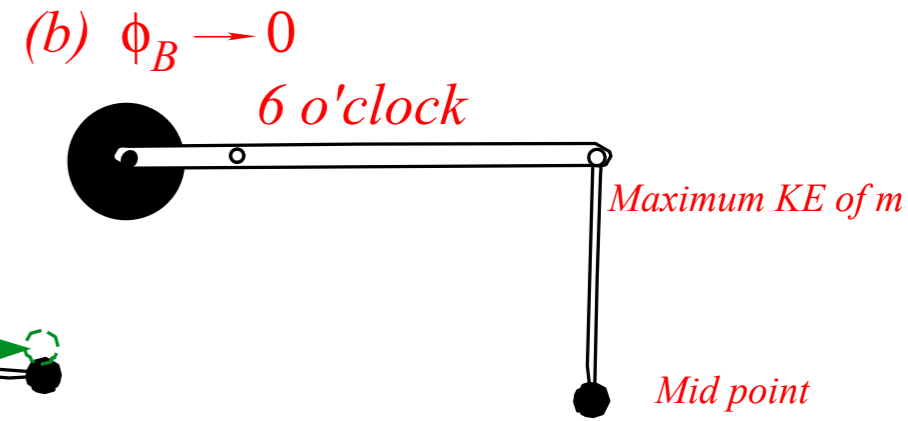
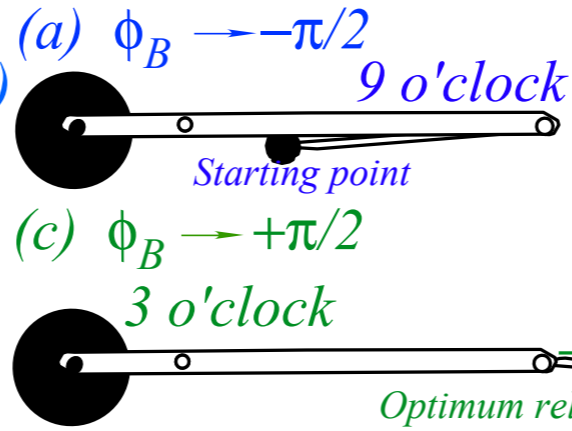
→ $\omega - \dot{\theta}_{\pi/2} = \frac{2mr^2}{MR^2} (2\omega + 2\dot{\theta}_{\pi/2})$

→ $\omega - \frac{4mr^2}{MR^2} \omega = \dot{\theta}_{\pi/2} + \frac{4mr^2}{MR^2} \dot{\theta}_{\pi/2}$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r = \ell$ (easier algebra)

$$\left. \begin{aligned} 2E &= MR^2 \dot{\theta}^2 + mr^2 (\dot{\theta}^2 + 2\dot{\phi}\dot{\theta} \sin \phi_B + \dot{\phi}^2) \\ \Lambda &= MR^2 \dot{\theta} + mr^2 (1 + \sin \phi_B) (\dot{\theta} + \dot{\phi}) \end{aligned} \right\} \text{For: } r = \ell$$



Start at 9 o'clock with $\phi_B \sim -90^\circ$ (beam r and throwing arm ℓ rotating together)

$$\left. \begin{aligned} \phi_B &= \frac{-\pi}{2} \\ \sin \phi_B &= -1 \end{aligned} \right\} \begin{cases} 2E = MR^2 \dot{\theta}_{-\pi/2}^2 + mr^2 (\dot{\phi}_{-\pi/2} - \dot{\theta}_{-\pi/2})^2 \\ \Lambda = MR^2 \dot{\theta}_{-\pi/2} \end{cases} \text{Conserved initial } 2E \text{ or } \begin{cases} 2E = MR^2 \omega^2 \\ \Lambda = MR^2 \omega \end{cases} \text{For: } \dot{\theta}_{-\pi/2} = \omega = \dot{\phi}_{-\pi/2}$$

Move to 6 o'clock with $\phi_B \sim 0^\circ$ (beam r slowing, throwing arm ℓ accelerating)

$$\left. \begin{aligned} \phi_B &= 0 \\ \sin \phi_B &= 0 \end{aligned} \right\} \begin{cases} 2E = MR^2 \dot{\theta}_0^2 + mr^2 (\dot{\phi}_0^2 + \dot{\theta}_0^2) \\ \Lambda = MR^2 \dot{\theta}_0 + mr^2 (\dot{\phi}_0 + \dot{\theta}_0) \end{cases}$$

$$KE(m) = \frac{mr^2}{2} (\dot{\phi}^2 + \dot{\theta}^2 + 2\dot{\phi}\dot{\theta} \sin \phi_B) = \begin{cases} \frac{mr^2}{2} (\dot{\phi} - \dot{\theta})^2 & \text{(For: } \phi_B = -\frac{\pi}{2}) \\ \frac{mr^2}{2} (\dot{\phi}^2 + \dot{\theta}^2) & \text{(For: } \phi_B = 0) \\ \frac{mr^2}{2} (\dot{\phi} + \dot{\theta})^2 & \text{(For: } \phi_B = \frac{\pi}{2}) \end{cases}$$

Move to 3 o'clock with $\phi_B \sim +90^\circ$ (beam r slowed, throwing arm ℓ releasing)

$$\left. \begin{aligned} \phi_B &= \pi/2 \\ \sin \phi_B &= +1 \end{aligned} \right\} \begin{cases} 2E = MR^2 \dot{\theta}_{\pi/2}^2 + mr^2 (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2})^2 \\ \Lambda = MR^2 \dot{\theta}_{\pi/2} + 2mr^2 (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}) \end{cases} \text{Conserved initial } 2E \text{ or } \begin{cases} 2E = MR^2 \omega^2 \\ \Lambda = MR^2 \omega \end{cases}$$

$$\begin{aligned} &\xrightarrow{\text{divide } 2E} (\omega^2 - \dot{\theta}_{\pi/2}^2) = \frac{mr^2}{MR^2} (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2})^2 \\ &\xrightarrow{\text{by } \Lambda} (\omega - \dot{\theta}_{\pi/2}) = \frac{2mr^2}{MR^2} (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}) \\ &\xrightarrow{\text{substitute}} \omega - \dot{\theta}_{\pi/2} = \frac{2mr^2}{MR^2} (2\omega + 2\dot{\theta}_{\pi/2}) \\ &\xrightarrow{\text{solve}} \omega - \frac{4mr^2}{MR^2} \omega = \dot{\theta}_{\pi/2} + \frac{4mr^2}{MR^2} \dot{\theta}_{\pi/2} \end{aligned}$$

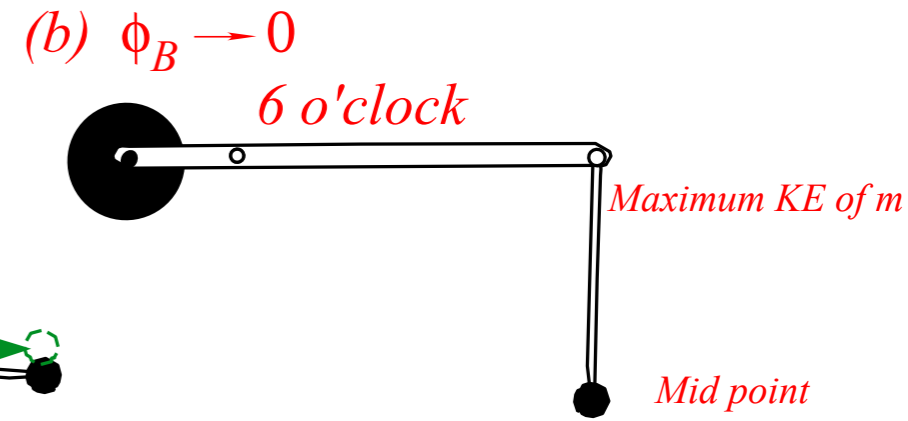
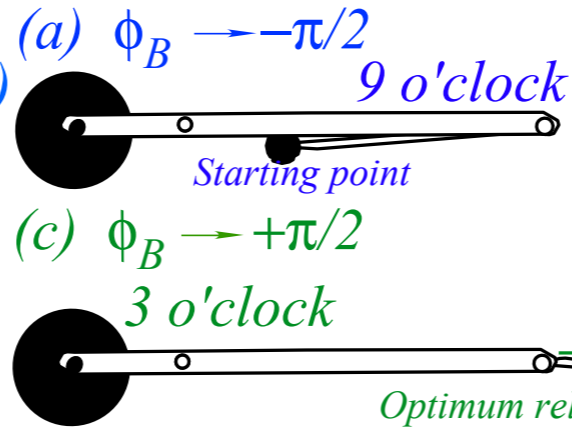
$$\dot{\phi}_{\pi/2} = \dot{\theta}_{\pi/2} + 2\omega$$

$$\dot{\theta}_{\pi/2} = \frac{1 - \frac{4mr^2}{MR^2}}{1 + \frac{4mr^2}{MR^2}} \omega$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r = \ell$ (easier algebra)

$$\left. \begin{aligned} 2E &= MR^2 \dot{\theta}^2 + mr^2 (\dot{\theta}^2 + 2\dot{\phi}\dot{\theta} \sin \phi_B + \dot{\phi}^2) \\ \Lambda &= MR^2 \dot{\theta} + mr^2 (1 + \sin \phi_B) (\dot{\theta} + \dot{\phi}) \end{aligned} \right\} \text{For: } r = \ell$$



Start at 9 o'clock with $\phi_B \sim -90^\circ$ (beam r and throwing arm ℓ rotating together)

$$\left. \begin{aligned} \phi_B = \frac{-\pi}{2} \\ \sin \phi_B = -1 \end{aligned} \right\} \begin{cases} 2E = MR^2 \dot{\theta}_{-\pi/2}^2 + mr^2 (\dot{\phi}_{-\pi/2} - \dot{\theta}_{-\pi/2})^2 \\ \Lambda = MR^2 \dot{\theta}_{-\pi/2} \end{cases} \text{Conserved}$$

or: $\begin{cases} \text{initial } 2E \\ 2E = MR^2 \omega^2 \\ \Lambda = MR^2 \omega \\ \text{initial } \Lambda \end{cases} \text{For: } \dot{\theta}_{-\pi/2} = \omega = \dot{\phi}_{-\pi/2}$

Move to 6 o'clock with $\phi_B \sim 0^\circ$ (beam r slowing, throwing arm ℓ accelerating)

$$\left. \begin{aligned} \phi_B = 0 \\ \sin \phi_B = 0 \end{aligned} \right\} \begin{cases} 2E = MR^2 \dot{\theta}_0^2 + mr^2 (\dot{\phi}_0^2 + \dot{\theta}_0^2) \\ \Lambda = MR^2 \dot{\theta}_0 + mr^2 (\dot{\phi}_0 + \dot{\theta}_0) \end{cases}$$

$$KE(m) = \frac{mr^2}{2} (\dot{\phi}^2 + \dot{\theta}^2 + 2\dot{\phi}\dot{\theta} \sin \phi_B)$$

$$= \begin{cases} \frac{mr^2}{2} (\dot{\phi} - \dot{\theta})^2 & \left(\text{For: } \phi_B = -\frac{\pi}{2} \right) \\ \frac{mr^2}{2} (\dot{\phi}^2 + \dot{\theta}^2) & \left(\text{For: } \phi_B = 0 \right) \\ \frac{mr^2}{2} (\dot{\phi} + \dot{\theta})^2 & \left(\text{For: } \phi_B = \frac{\pi}{2} \right) \end{cases}$$

Move to 3 o'clock with $\phi_B \sim +90^\circ$ (beam r slowed, throwing arm ℓ releasing)

$$\left. \begin{aligned} \phi_B = \pi/2 \\ \sin \phi_B = +1 \end{aligned} \right\} \begin{cases} 2E = MR^2 \dot{\theta}_{\pi/2}^2 + mr^2 (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2})^2 = MR^2 \omega^2 \\ \Lambda = MR^2 \dot{\theta}_{\pi/2} + 2mr^2 (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}) = MR^2 \omega \end{cases} \text{Conserved}$$

initial $2E$ \rightarrow $(\omega^2 - \dot{\theta}_{\pi/2}^2) = \frac{mr^2}{MR^2} (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2})^2$ (divide $2E$)

initial Λ \rightarrow $(\omega - \dot{\theta}_{\pi/2}) = \frac{2mr^2}{MR^2} (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2})$ (by Λ)

substitute \rightarrow $(\omega + \dot{\theta}_{\pi/2}) = \frac{1}{2} (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2})$

$\dot{\phi}_{\pi/2} = \dot{\theta}_{\pi/2} + 2\omega$

Large $M \gg m$ case (Beam nearly constant ω)

$$\dot{\phi}_{\pi/2} = \omega + 2\omega = 3\omega$$

$$\dot{\theta}_{\pi/2} = \frac{1-0}{1+0} \omega = \omega$$

$$\omega - \dot{\theta}_{\pi/2} = \frac{2mr^2}{MR^2} (2\omega + 2\dot{\theta}_{\pi/2})$$

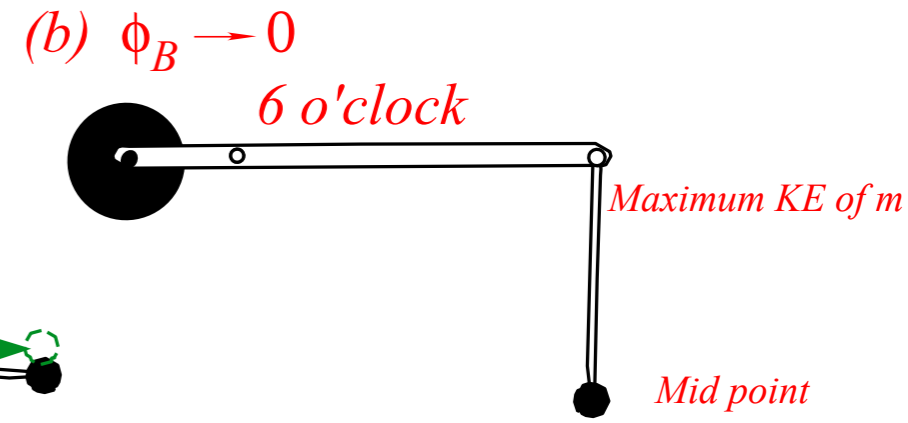
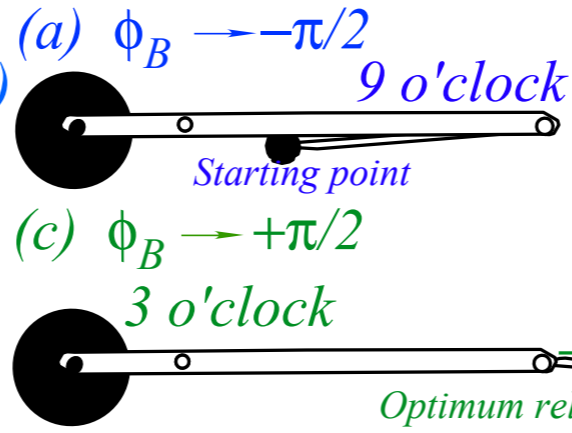
$$\omega - \frac{4mr^2}{MR^2} \omega = \dot{\theta}_{\pi/2} + \frac{4mr^2}{MR^2} \dot{\theta}_{\pi/2}$$

solve \rightarrow $\dot{\theta}_{\pi/2} = \frac{1 - \frac{4mr^2}{MR^2}}{1 + \frac{4mr^2}{MR^2}} \omega$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r = \ell$ (easier algebra)

$$\left. \begin{aligned} 2E &= MR^2 \dot{\theta}^2 + mr^2 (\dot{\theta}^2 + 2\dot{\phi}\dot{\theta} \sin \phi_B + \dot{\phi}^2) \\ \Lambda &= MR^2 \dot{\theta} + mr^2 (1 + \sin \phi_B) (\dot{\theta} + \dot{\phi}) \end{aligned} \right\} \text{For: } r = \ell$$



Start at 9 o'clock with $\phi_B \sim -90^\circ$ (beam r and throwing arm ℓ rotating together)

$$\left. \begin{aligned} \phi_B = \frac{-\pi}{2} \\ \sin \phi_B = -1 \end{aligned} \right\} \begin{cases} 2E = MR^2 \dot{\theta}_{-\pi/2}^2 + mr^2 (\dot{\phi}_{-\pi/2} - \dot{\theta}_{-\pi/2})^2 \\ \Lambda = MR^2 \dot{\theta}_{-\pi/2} \end{cases} \text{Conserved}$$

initial $2E$
initial Λ

or: $\begin{cases} 2E = MR^2 \omega^2 \\ \Lambda = MR^2 \omega \end{cases}$ For: $\dot{\theta}_{-\pi/2} = \omega = \dot{\phi}_{-\pi/2}$

Move to 6 o'clock with $\phi_B \sim 0^\circ$ (beam r slowing, throwing arm ℓ accelerating)

$$\left. \begin{aligned} \phi_B = 0 \\ \sin \phi_B = 0 \end{aligned} \right\} \begin{cases} 2E = MR^2 \dot{\theta}_0^2 + mr^2 (\dot{\phi}_0^2 + \dot{\theta}_0^2) \\ \Lambda = MR^2 \dot{\theta}_0 + mr^2 (\dot{\phi}_0 + \dot{\theta}_0) \end{cases}$$

$$KE(m) = \frac{mr^2}{2} (\dot{\phi}^2 + \dot{\theta}^2 + 2\dot{\phi}\dot{\theta} \sin \phi_B)$$

$$= \begin{cases} \frac{mr^2}{2} (\dot{\phi} - \dot{\theta})^2 & \left(\text{For: } \phi_B = -\frac{\pi}{2} \right) \\ \frac{mr^2}{2} (\dot{\phi}^2 + \dot{\theta}^2) & \left(\text{For: } \phi_B = 0 \right) \\ \frac{mr^2}{2} (\dot{\phi} + \dot{\theta})^2 & \left(\text{For: } \phi_B = \frac{\pi}{2} \right) \end{cases}$$

Move to 3 o'clock with $\phi_B \sim +90^\circ$ (beam r slowed, throwing arm ℓ releasing)

$$\left. \begin{aligned} \phi_B = \pi/2 \\ \sin \phi_B = +1 \end{aligned} \right\} \begin{cases} 2E = MR^2 \dot{\theta}_{\pi/2}^2 + mr^2 (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2})^2 = MR^2 \omega^2 \\ \Lambda = MR^2 \dot{\theta}_{\pi/2} + 2mr^2 (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}) = MR^2 \omega \end{cases} \text{Conserved}$$

initial $2E$
initial Λ

$$\begin{aligned} (\omega^2 - \dot{\theta}_{\pi/2}^2) &= \frac{mr^2}{MR^2} (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2})^2 && \text{divide } 2E \\ (\omega - \dot{\theta}_{\pi/2}) &= \frac{2mr^2}{MR^2} (\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}) && \text{by } \Lambda \\ \omega - \dot{\theta}_{\pi/2} &= \frac{2mr^2}{MR^2} (2\omega + 2\dot{\theta}_{\pi/2}) && \text{substitute } \dot{\phi}_{\pi/2} = \dot{\theta}_{\pi/2} + 2\omega \\ \omega - \frac{4mr^2}{MR^2} \omega &= \dot{\theta}_{\pi/2} + \frac{4mr^2}{MR^2} \dot{\theta}_{\pi/2} && \text{solve} \\ \dot{\theta}_{\pi/2} &= \frac{1 - \frac{4mr^2}{MR^2}}{1 + \frac{4mr^2}{MR^2}} \omega \end{aligned}$$

Large $M \gg m$ case

$$\dot{\phi}_{\pi/2} = \omega + 2\omega = 3\omega$$

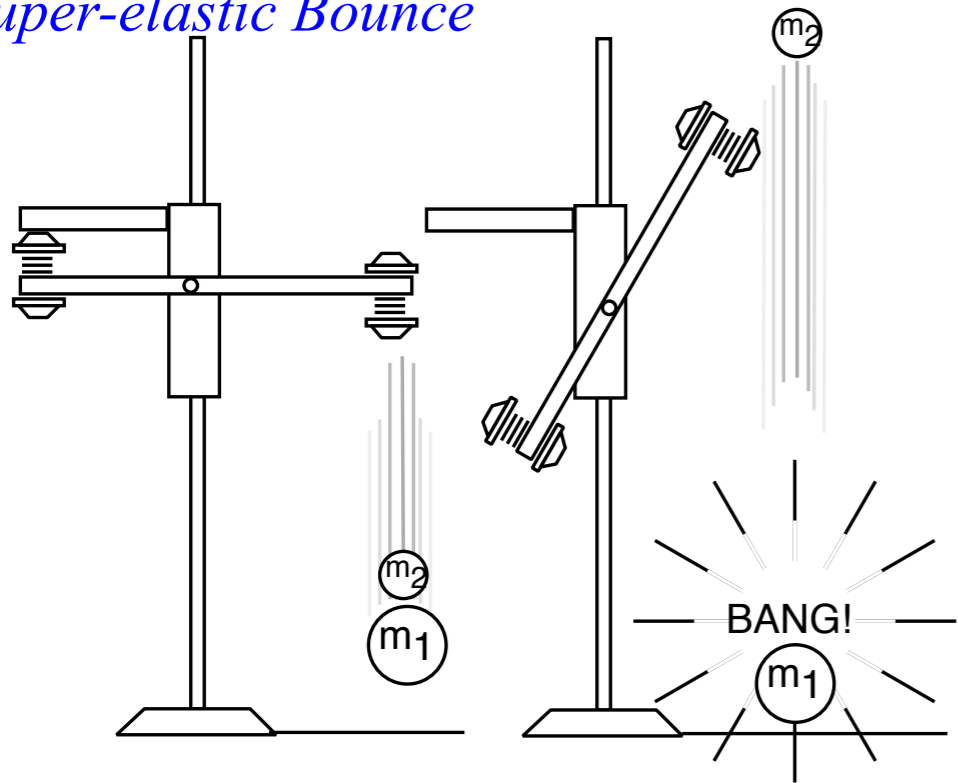
$$\dot{\theta}_{\pi/2} = \frac{1-0}{1+0} \omega = \omega$$

Optimum $MR^2 = 4mr^2$ case

$$\dot{\phi}_{\pi/2} = 0 + 2\omega = 2\omega$$

$$\dot{\theta}_{\pi/2} = \frac{1-1}{1+1} \omega = 0$$

Super-elastic Bounce

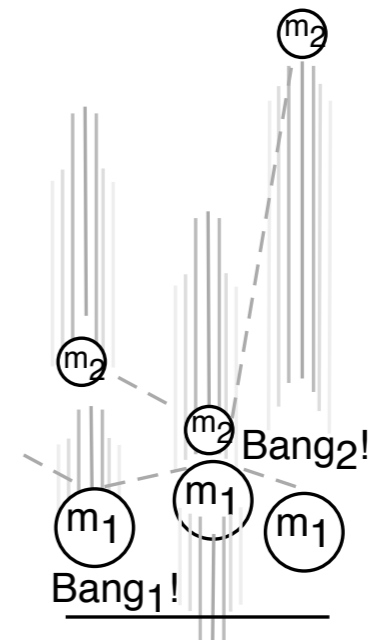


Analogous Superball Models

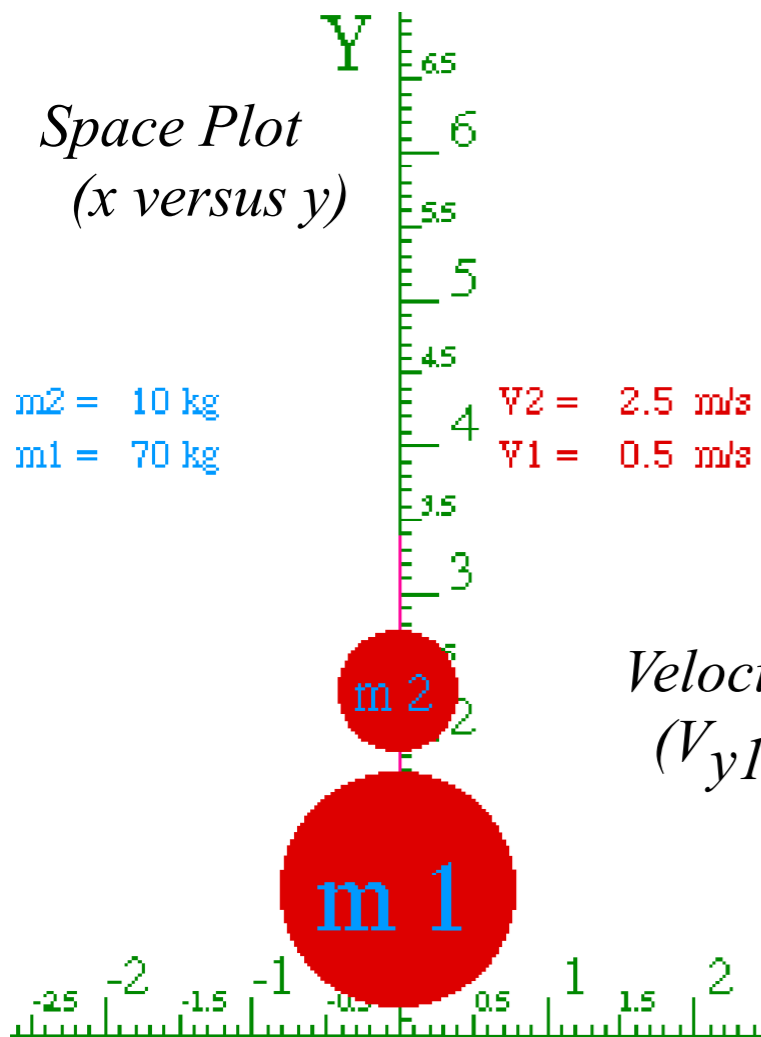
Similar in some ways to trebuchet models

Class of W. G. Harter, "Velocity Amplification in Collision Experiments Involving Superballs," *Am. J. Phys.* 39, 656 (1971) (A class project)

2-Bang Model



Space Plot (x versus y)



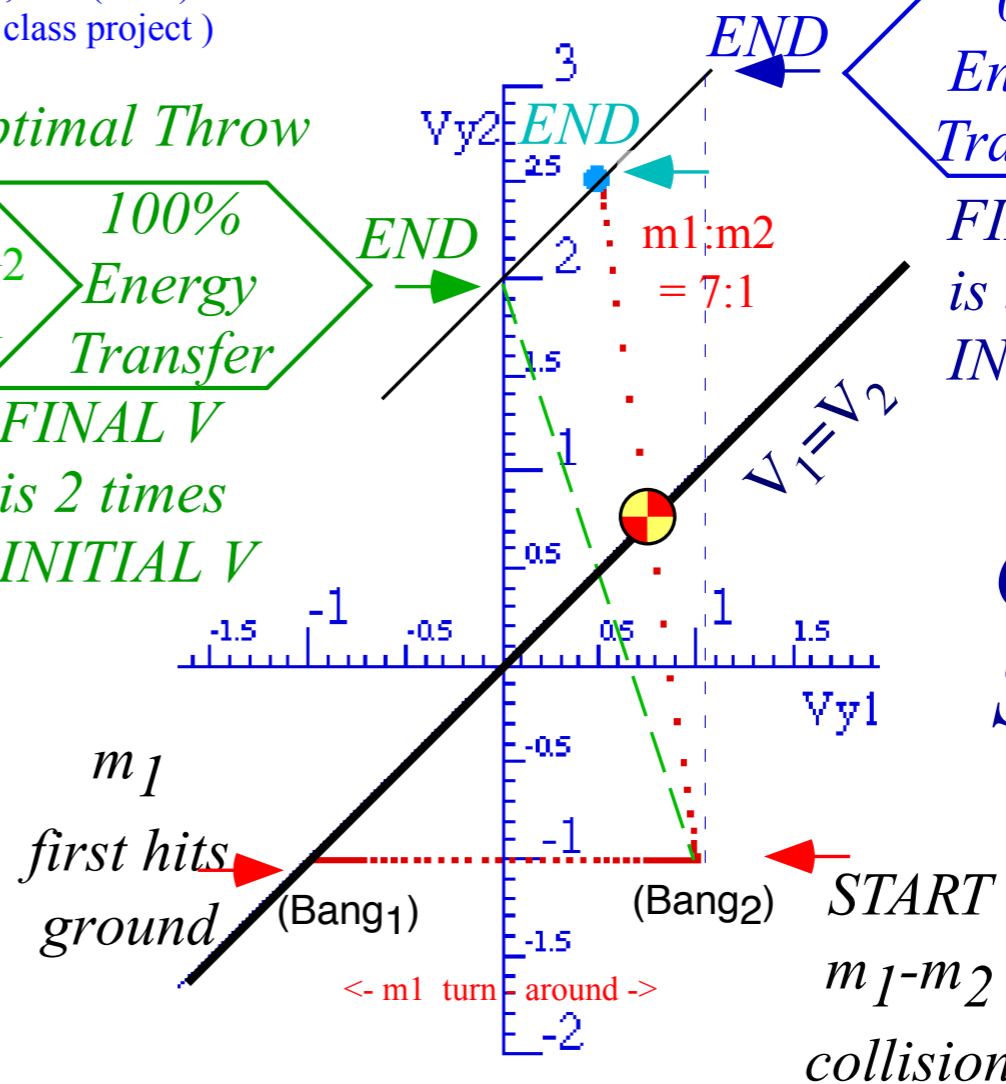
Velocity Plot (V_{y1} versus V_{y2})

Optimal Throw

100% Energy Transfer

$m_1:m_2 = 3:1$

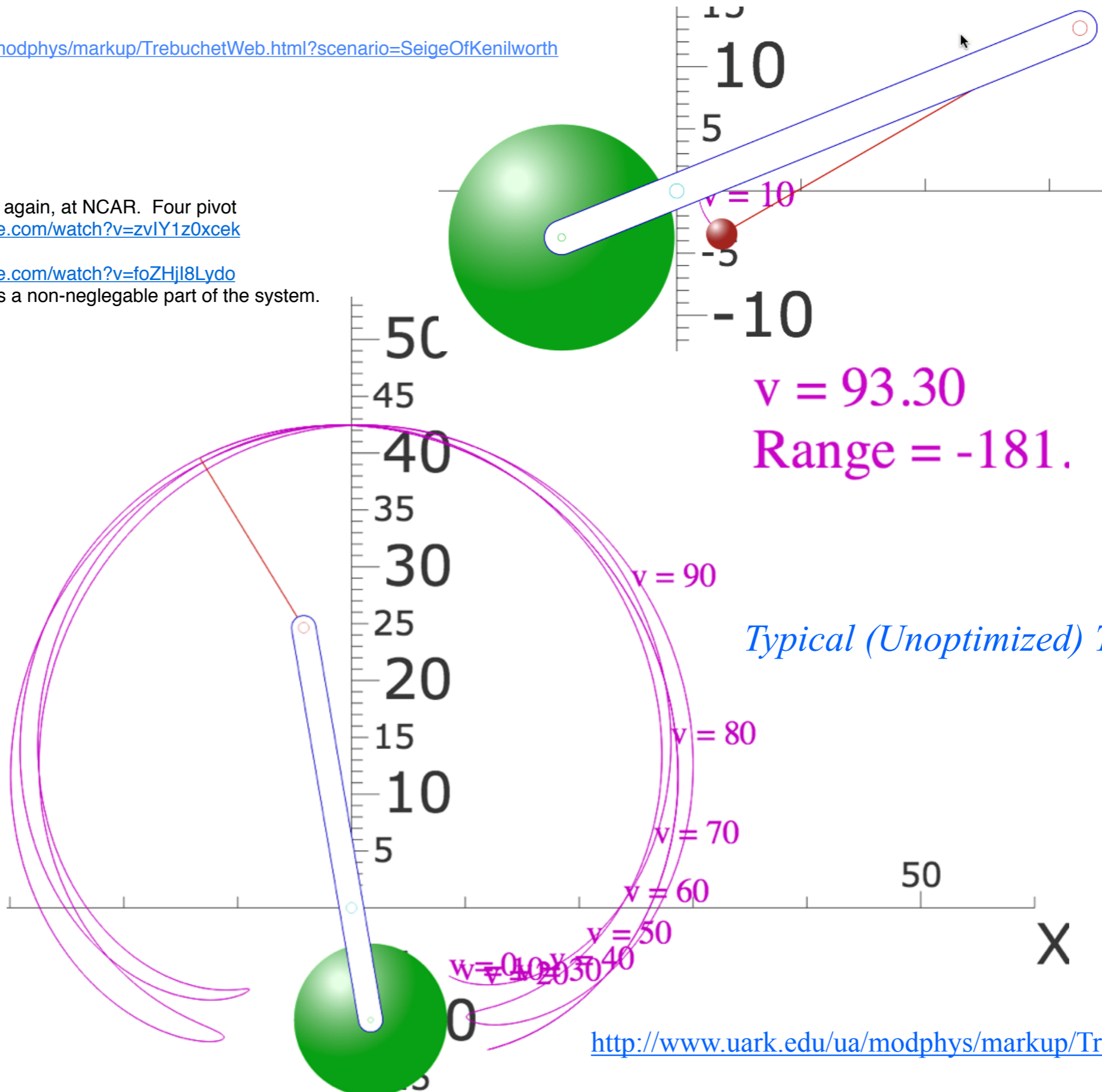
FINAL V is 2 times INITIAL V



Graphic Solution

<http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html?scenario=SeigeOfKenilworth>

Here is the system, again, at NCAR. Four pivot
<https://www.youtube.com/watch?v=zvIY1z0xcek>
Another rig:
<https://www.youtube.com/watch?v=foZHjI8Lydo>
Ha! the clock face is a non-neglegable part of the system.

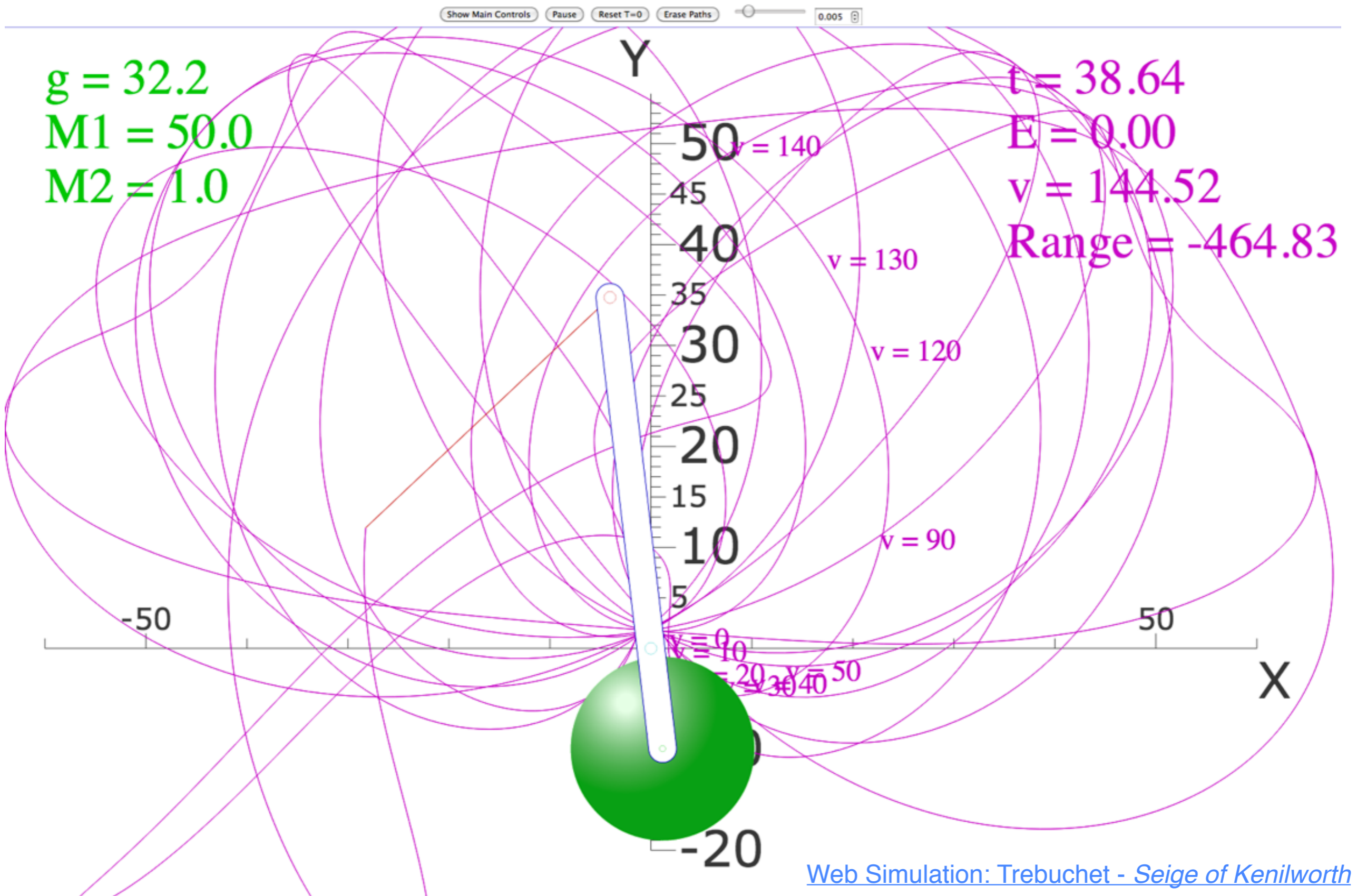


$v = 93.30$
Range = -181.

Typical (Unoptimized) Trebuchet

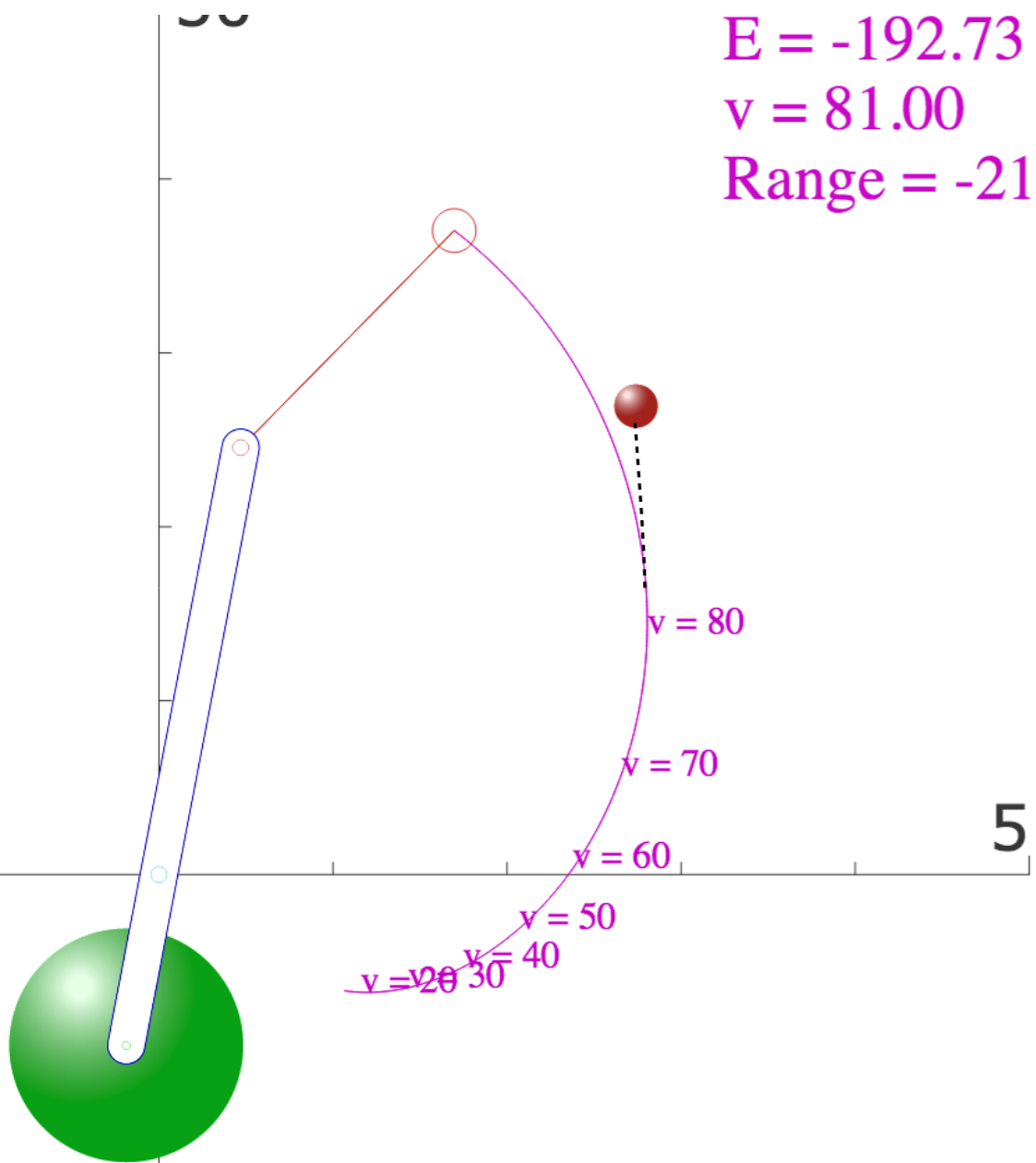
<http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html>

*Trebuchet in Siege of Kenilworth 1215 ACE
(Re-enactment shown on NOVA-TV 2005)*



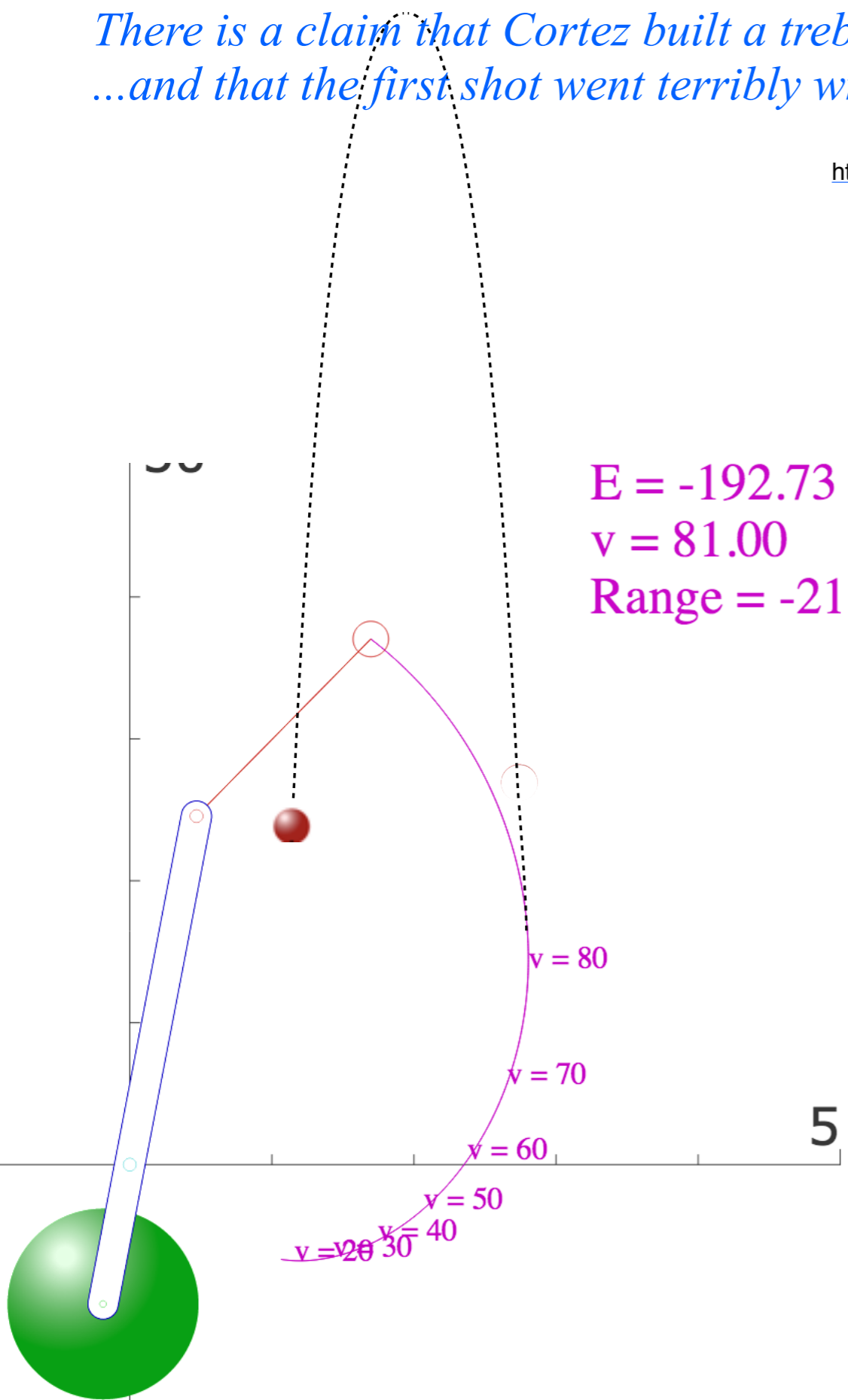
There is a claim that Cortez built a trebuchet to lay siege on Montezuma around 1500...

<http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html?scenario=MontezumasRevenge>



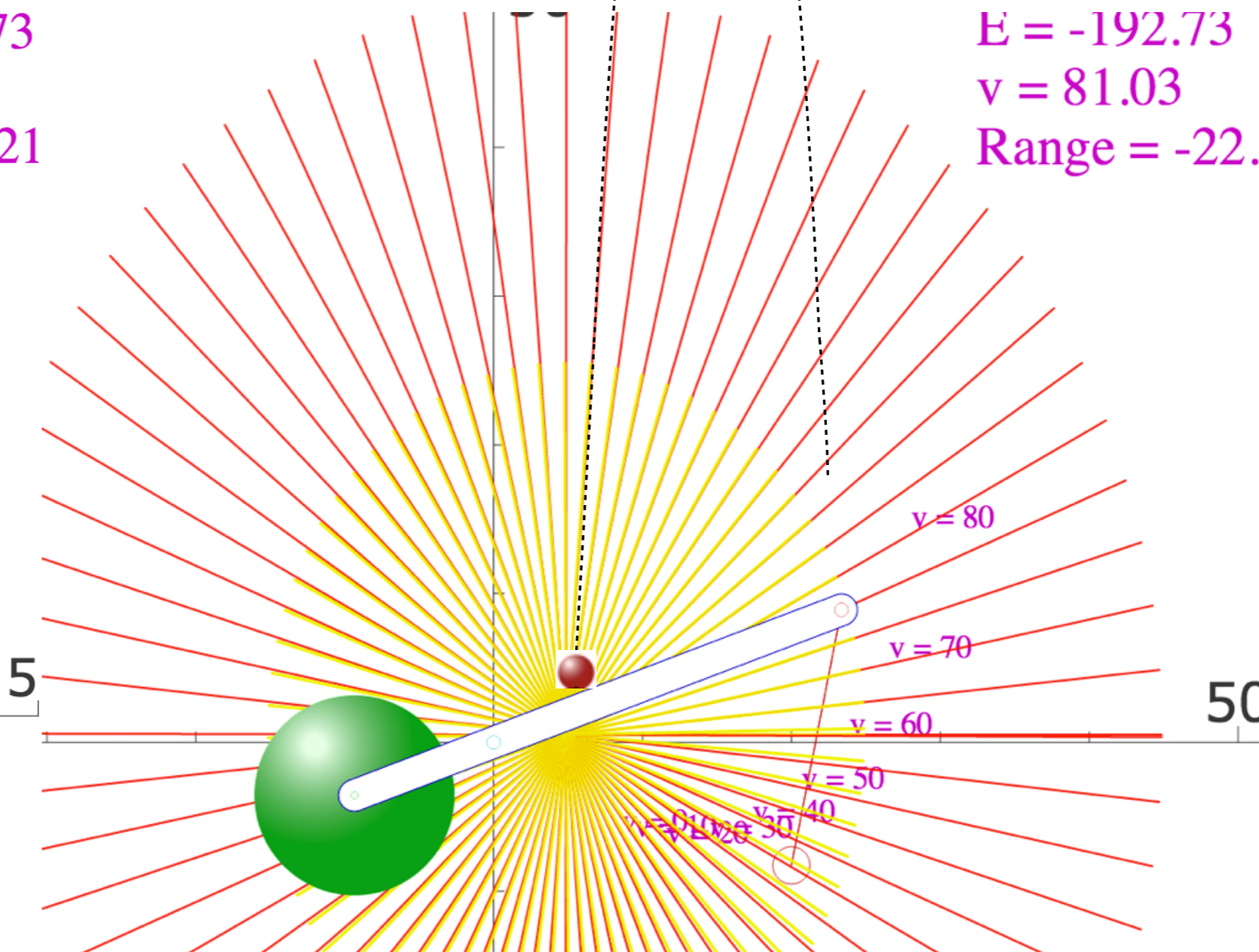
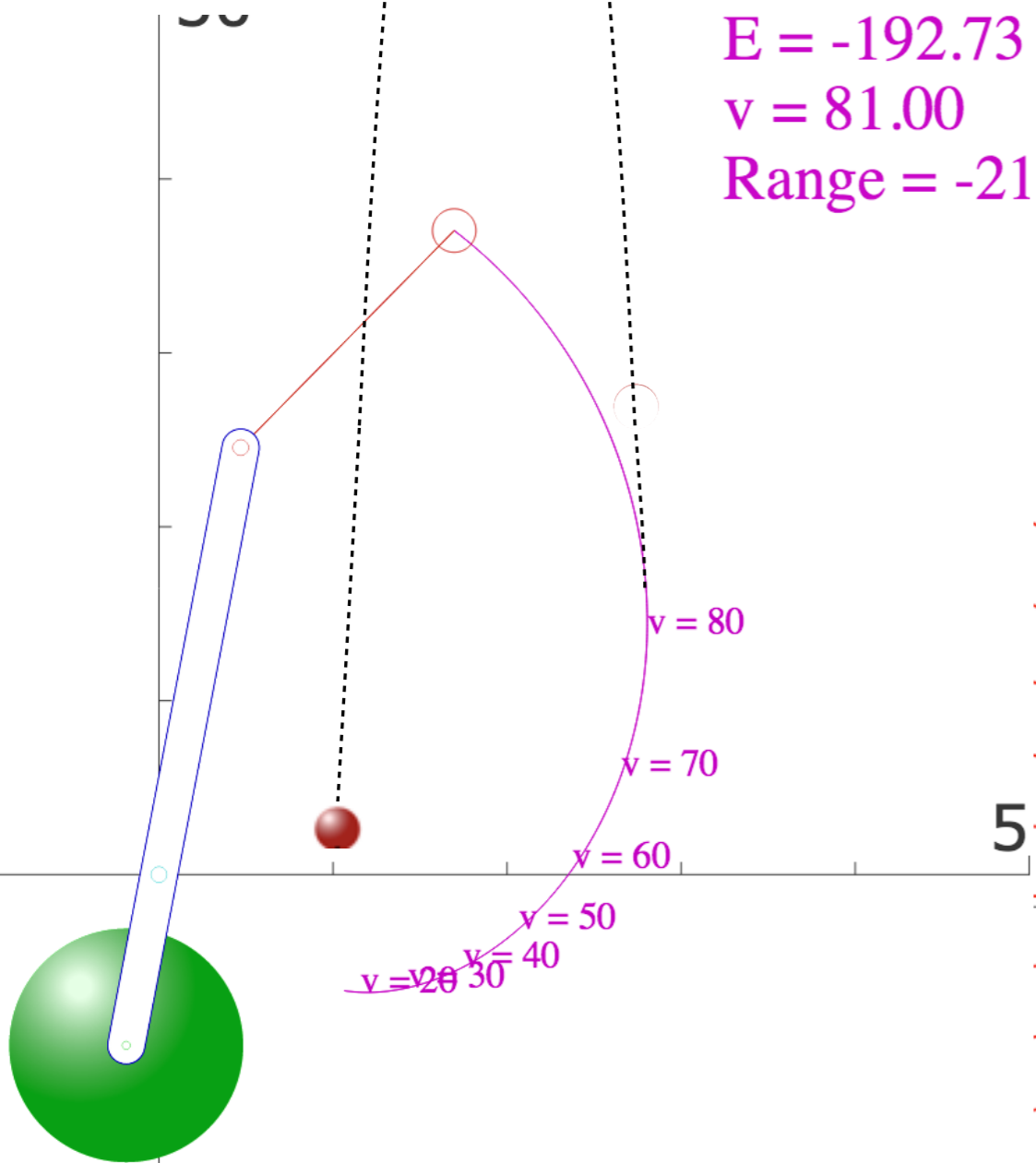
*There is a claim that Cortez built a trebuchet to lay siege on Montezuma around 1500...
...and that the first shot went terribly wrong...*


<http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html?scenario=MontezumasRevenge>



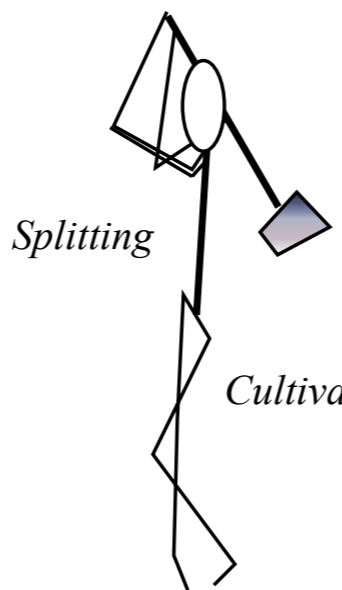
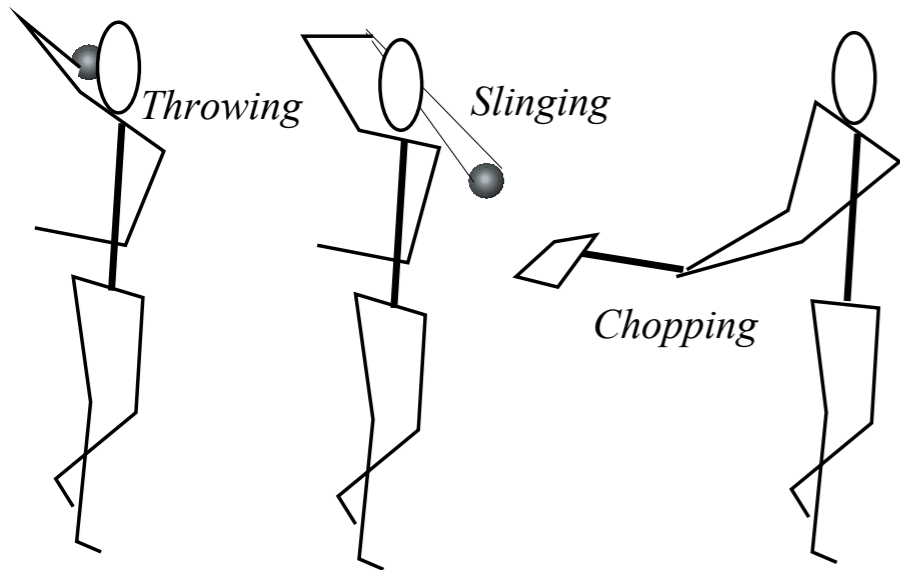
*There is a claim that Cortez built a trebuchet to lay siege on Montezuma around 1500...
...and that the first shot went terribly wrong...*

*...if this story is true, then it gives new meaning
to the expression "Montezuma's Revenge"...*

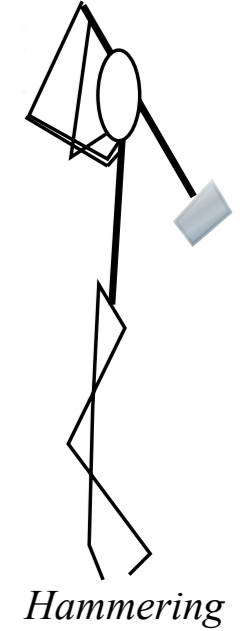
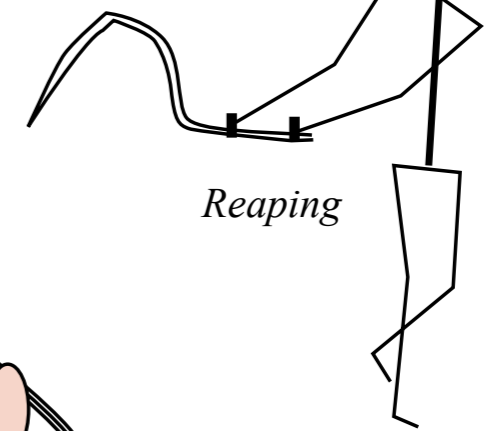
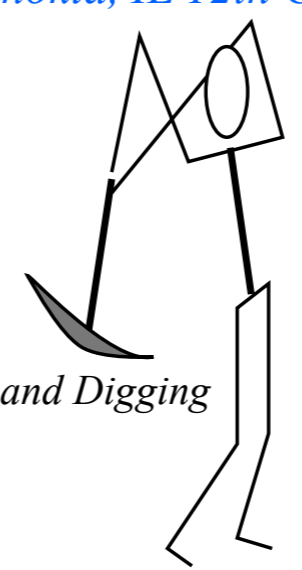
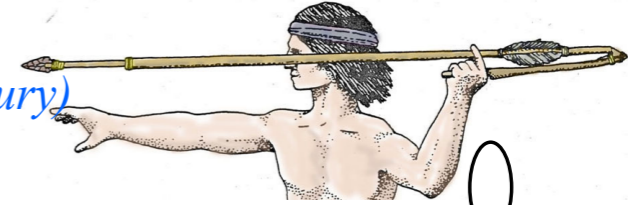


Hamiltonian energy and momentum conservation and symmetry coordinates
Coordinate transformation helps reduce symmetric Hamiltonian
Free-space trebuchet kinematics by symmetry
Algebraic approach
Direct approach and Superball analogy
 *Trebuchet vs Flinger and sports kinematics*
Many approaches to Mechanics

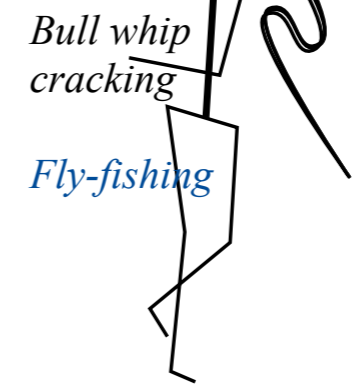
Early Human Agriculture and Infrastructure Building Activity



The Atlatl (Cahokia, IL 12th Century)

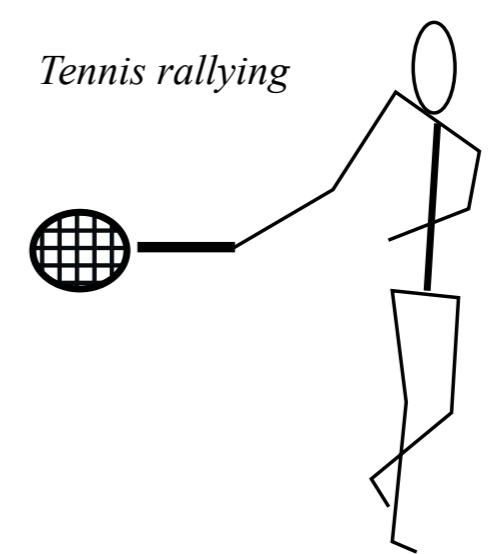
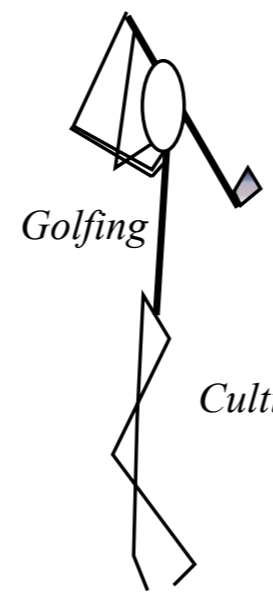
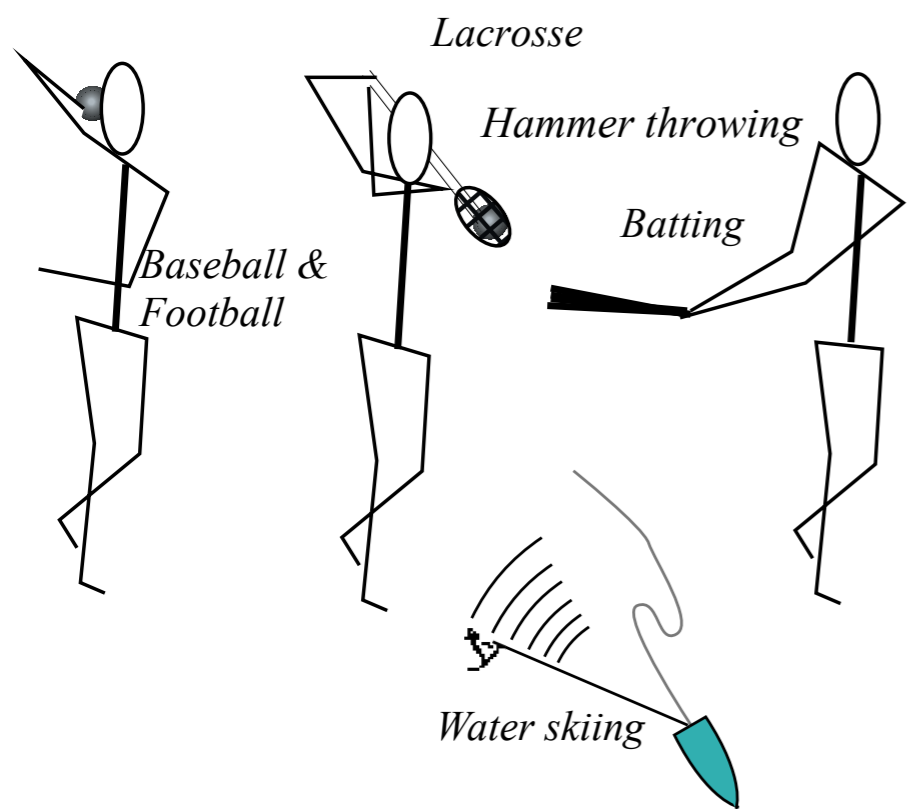


What Trebuchet mechanics is really good for...

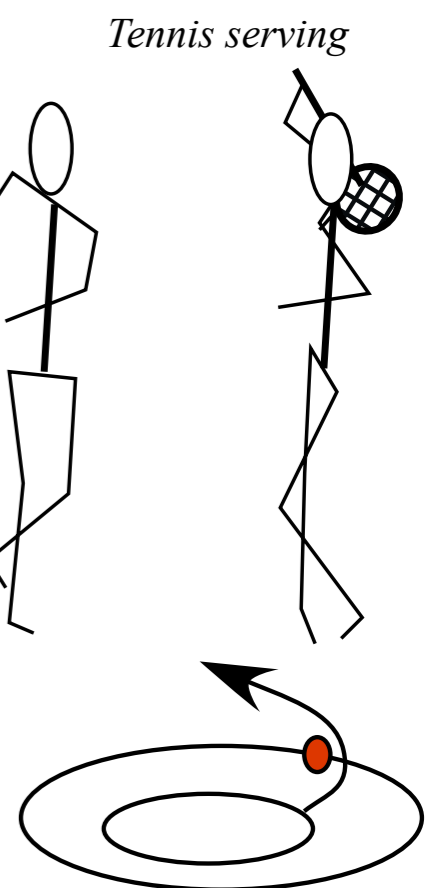


“Ring-The-Bell” (at the Fair)

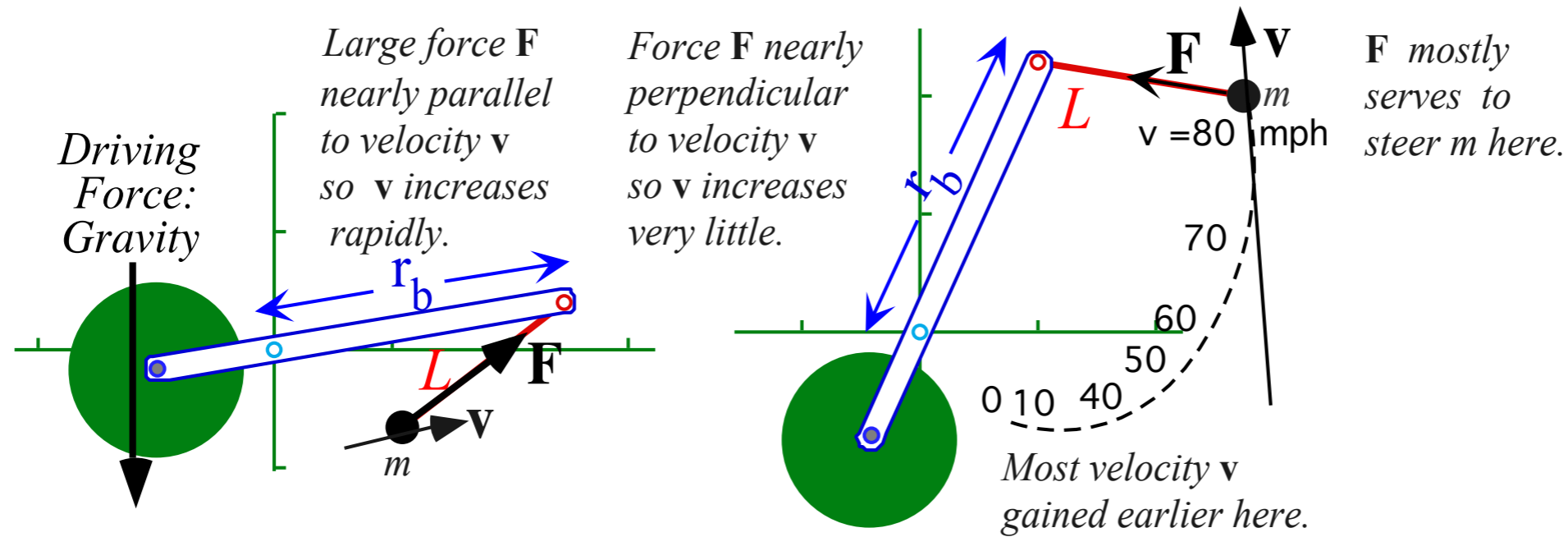
Later Human Recreational Activity



Space Probe “Planetary Slingshot”



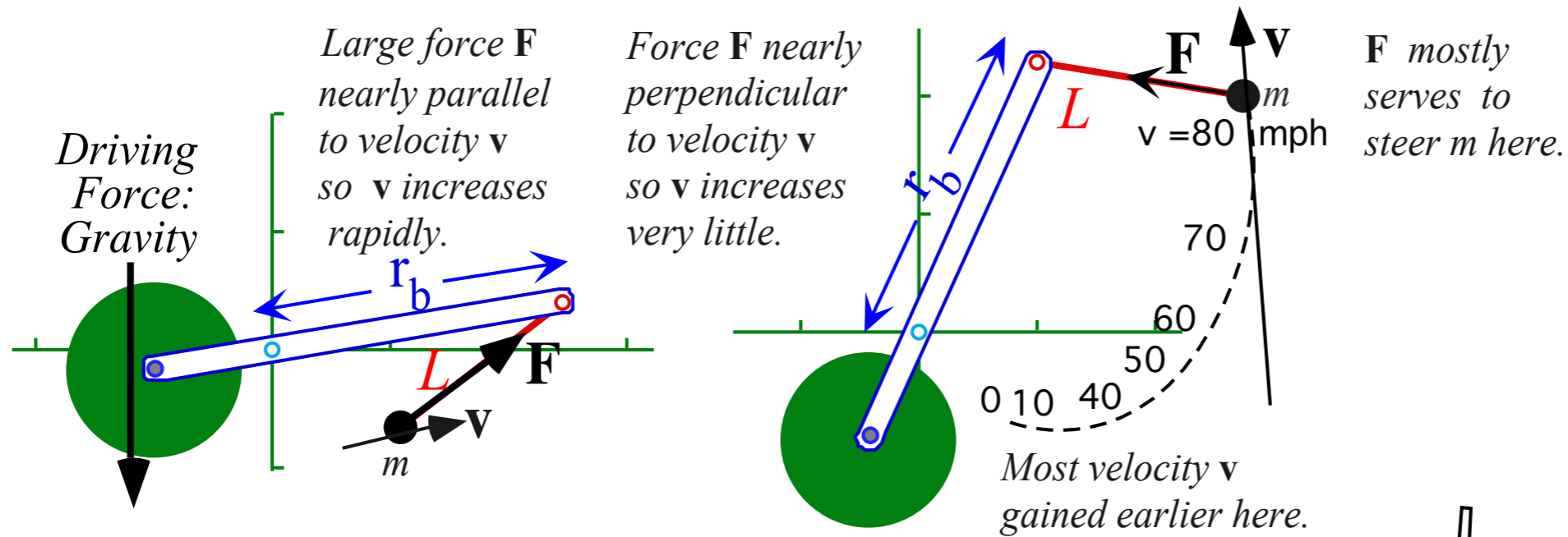
Trebuchet analogy with racquet swing - What we learn



Early on
(Gain the energy/momentum)

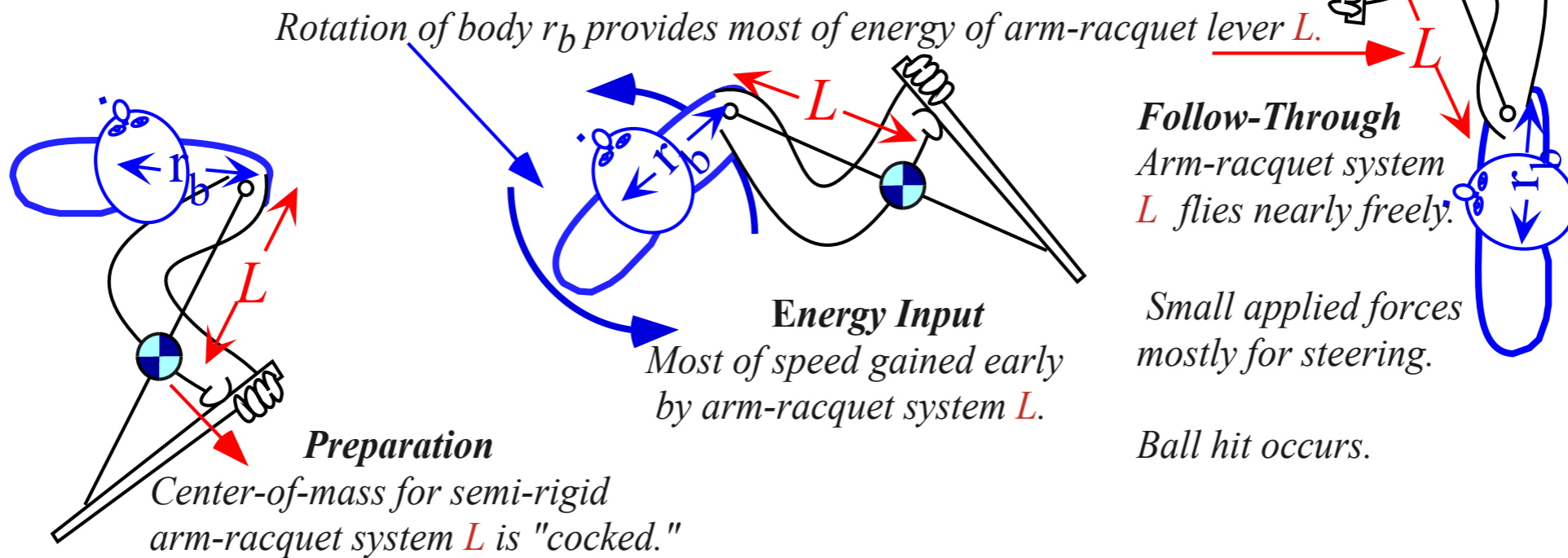
Later on
(Steer or guide)

Trebuchet analogy with racquet swing - What we learn



Early on
(Gain the energy/momentum)

Later on
(Steer or guide)

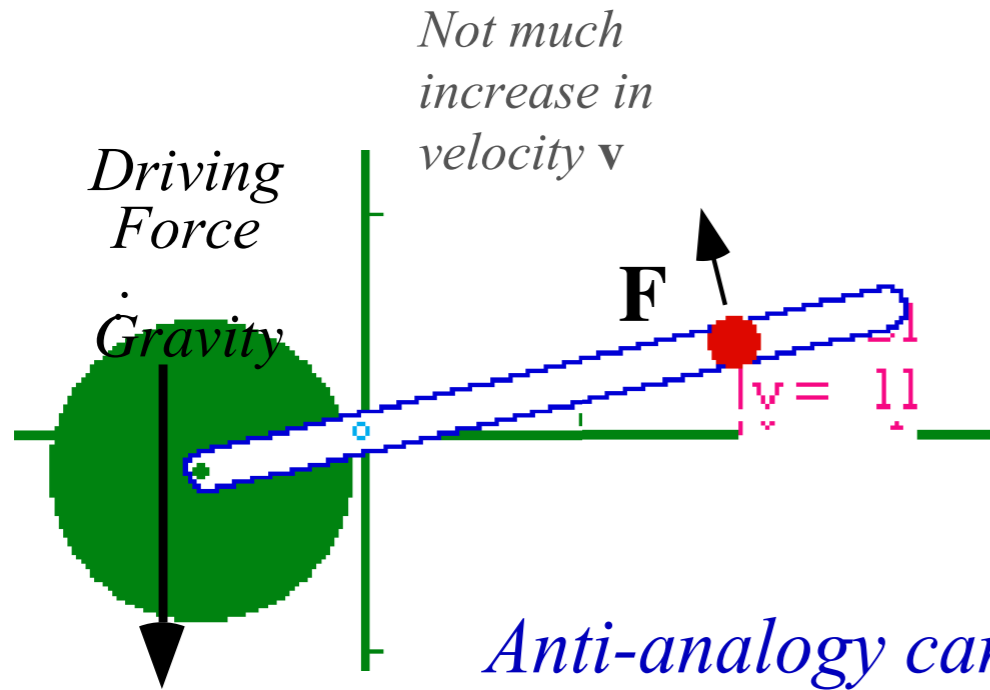


An Opposite to Trebuchet Mechanics- The "Flinger"

<http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html?scenario=AnimateFlinger>

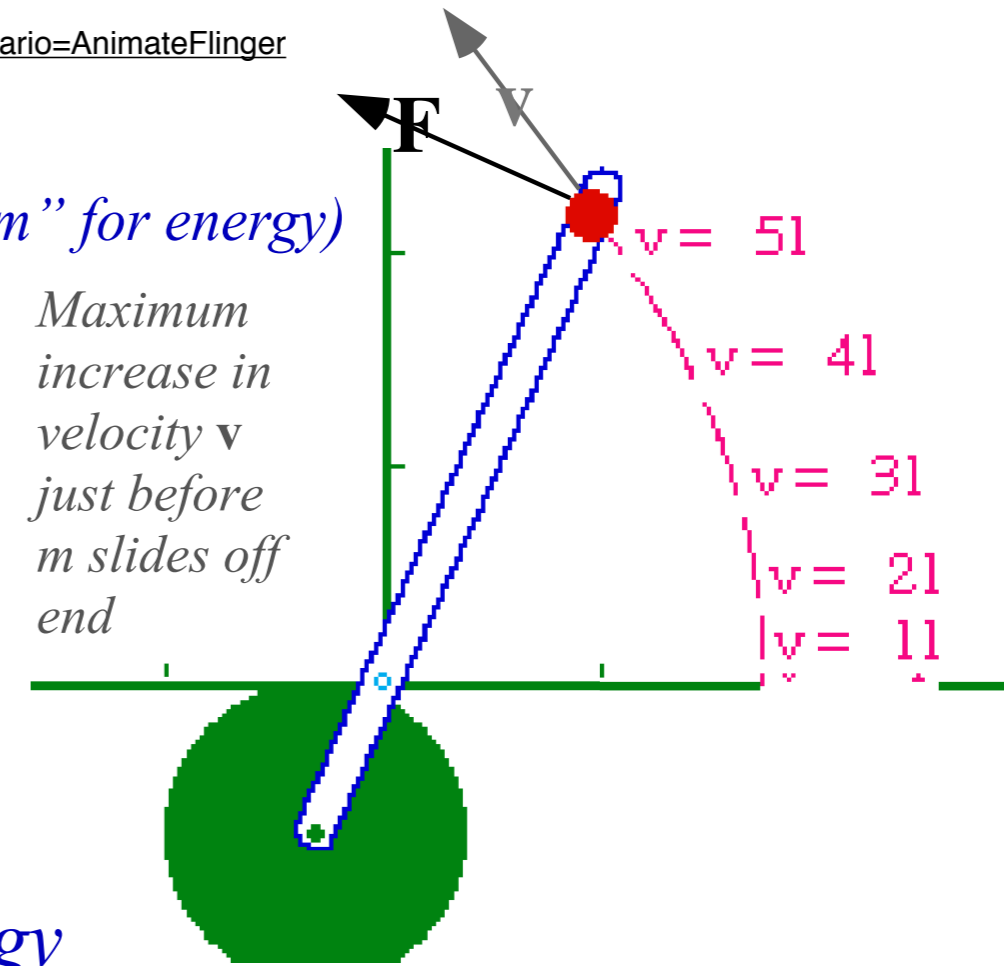
Early on

(Not much happening)



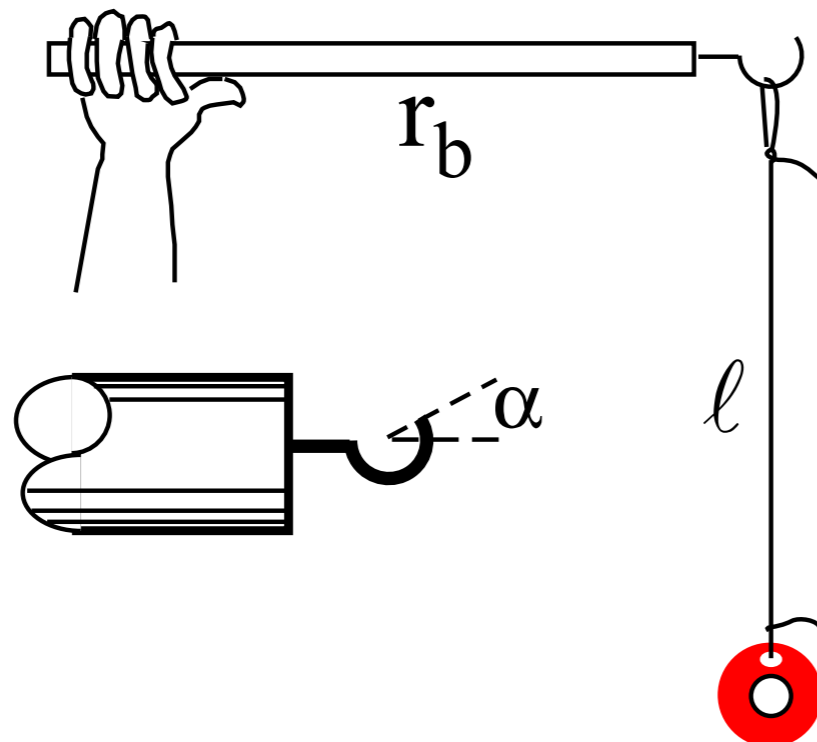
Later on

(Last-minute "cram" for energy)

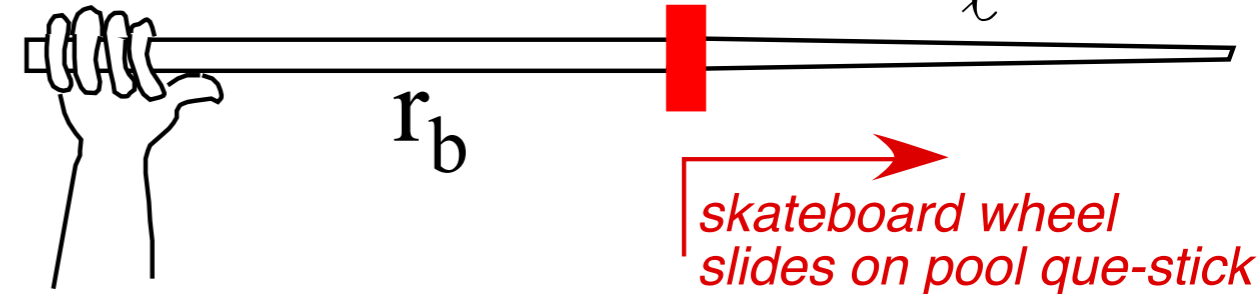


Anti-analogy can be useful pedagogy

Trebuchet-like experiment



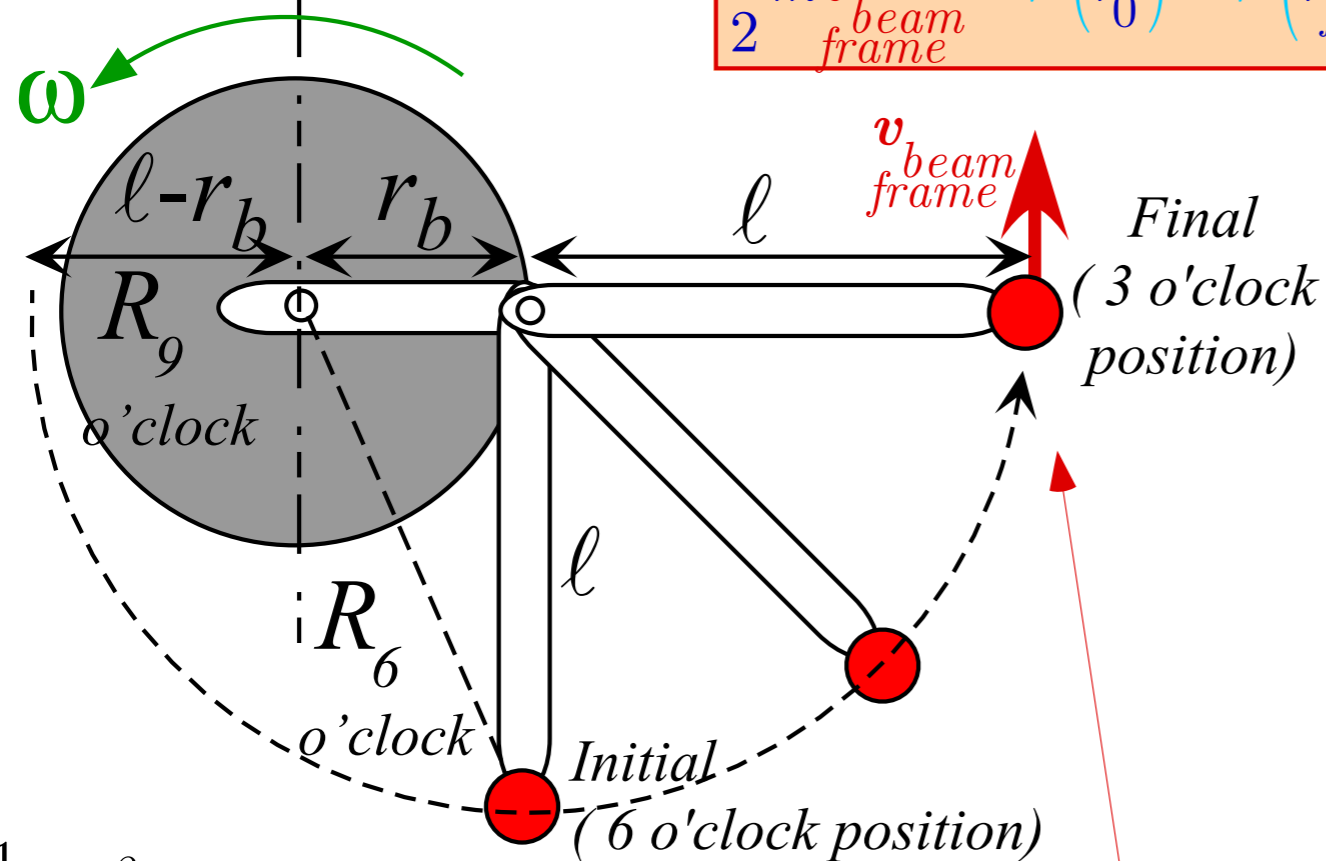
Flinger experiment



skateboard wheel swings

Trebuchet model in rotating beam frame

Assume: Constant beam ω

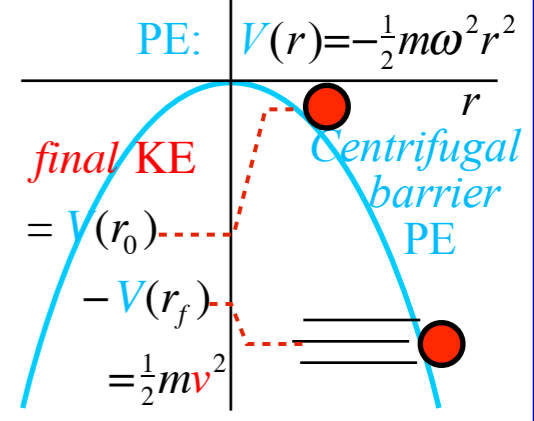
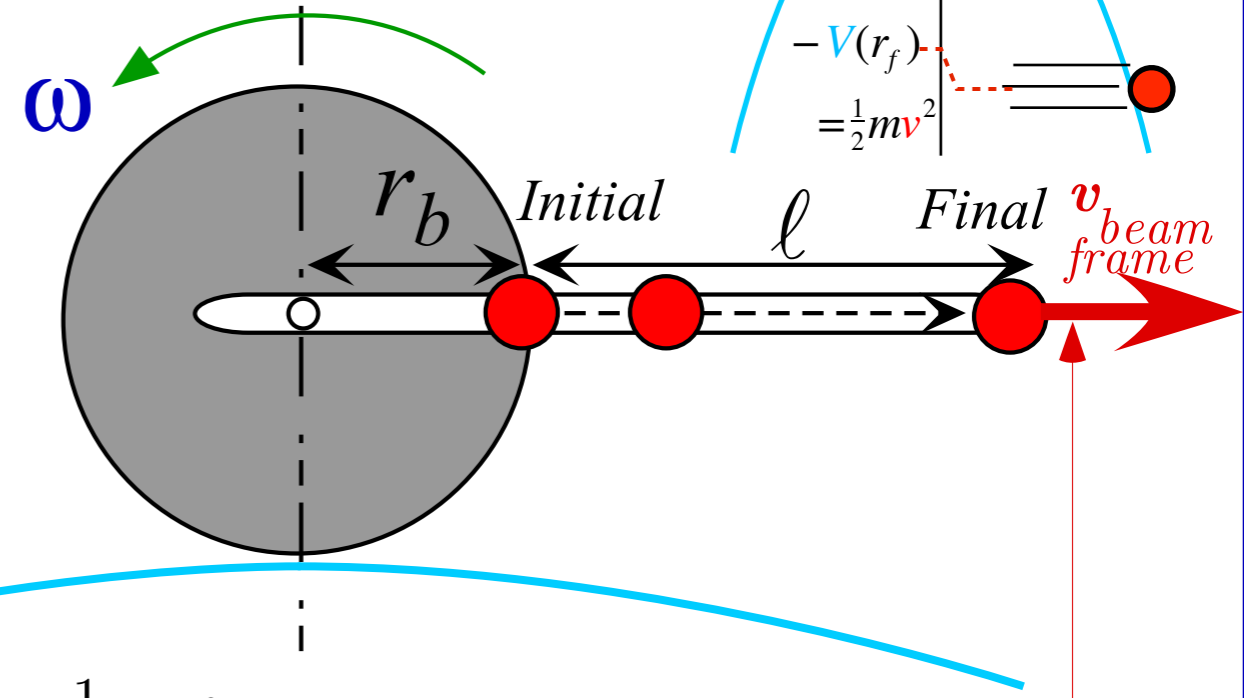


$$\frac{1}{2} m v_{\text{beam frame}}^2 = V(r_0) - V(r_f) = \frac{1}{2} m \omega^2 r_f^2 - \frac{1}{2} m \omega^2 r_0^2$$

$$\frac{1}{2} m v_{\text{beam frame}}^2 (\text{trebuchet}) = \text{Final Trebuchet } KE_{\text{beam frame}}$$

Flinger model in rotating beam frame

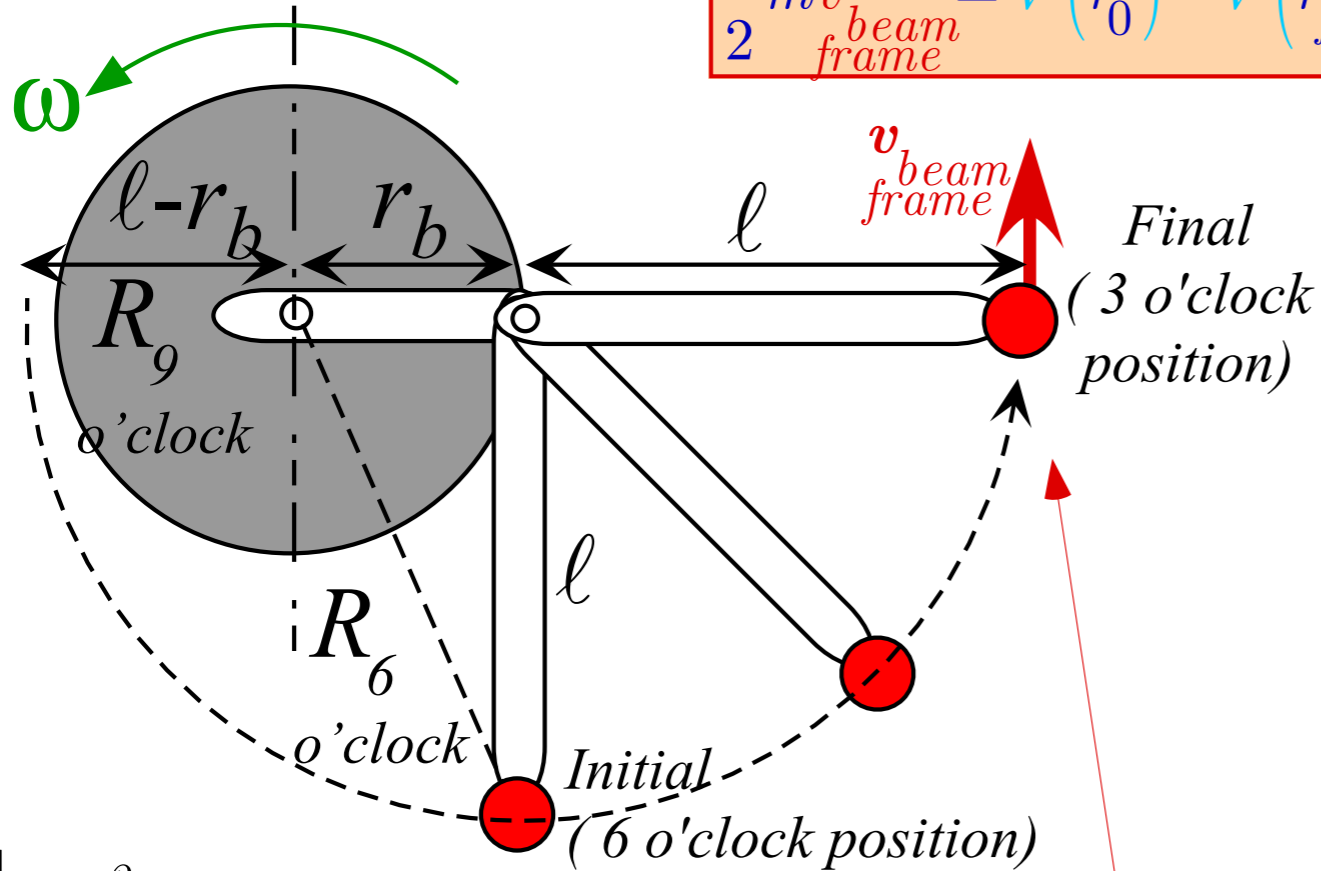
Assume: Constant beam ω



$$\frac{1}{2} m v_{\text{beam frame}}^2 (\text{flinger}) = \text{Final Flinger } KE_{\text{beam frame}}$$

Trebuchet model in rotating beam frame

Assume: Constant beam ω



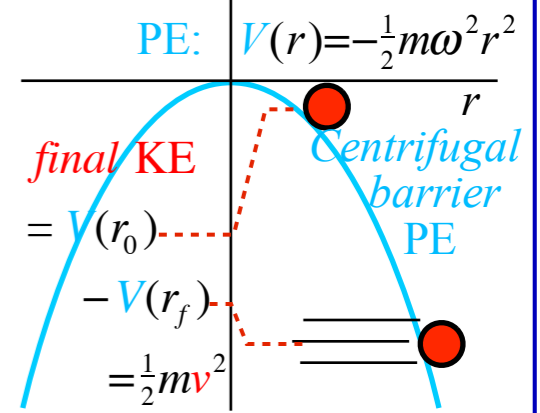
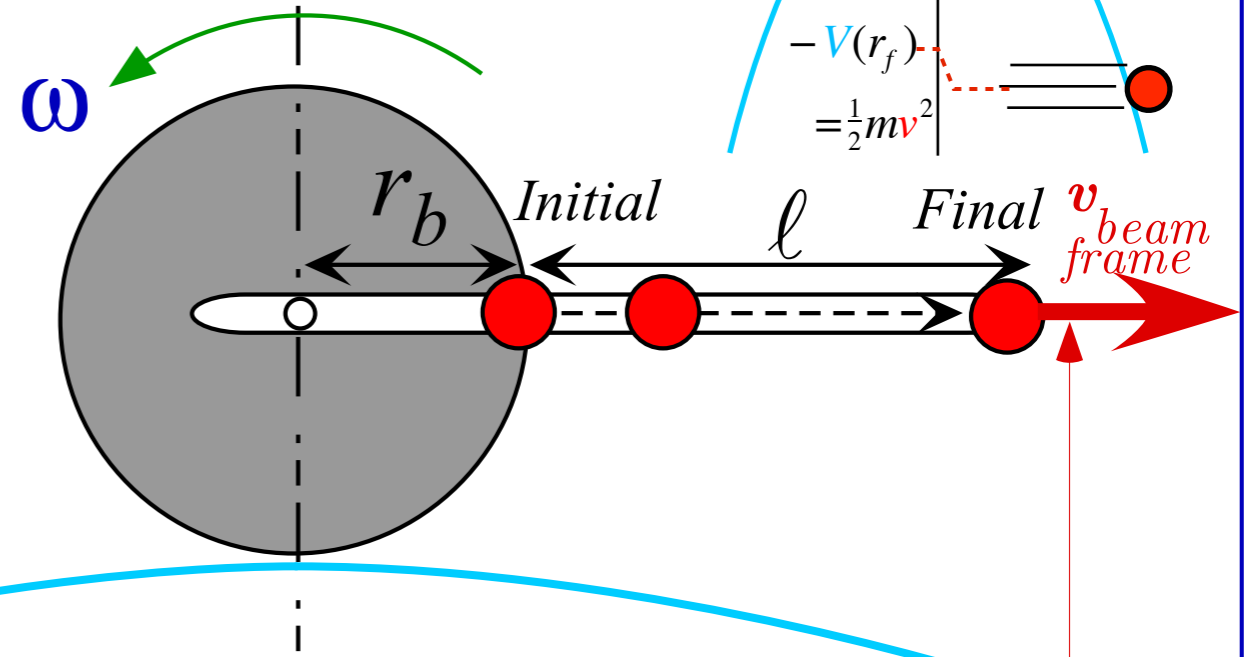
$$\frac{1}{2} m v_{\text{beam frame}}^2 = V(r_0) - V(r_f) = \frac{1}{2} m \omega^2 r_f^2 - \frac{1}{2} m \omega^2 r_0^2$$

$$\frac{1}{2} m v_{\text{beam frame}}^2 (\text{trebuchet}) = \text{Final Trebuchet } KE_{\text{beam frame}}$$

$$\frac{1}{2} m \omega^2 (r_b + l)^2 - \frac{1}{2} m \omega^2 (r_b^2 + l^2) = \frac{1}{2} m \omega^2 (2r_b l)$$

Flinger model in rotating beam frame

Assume: Constant beam ω

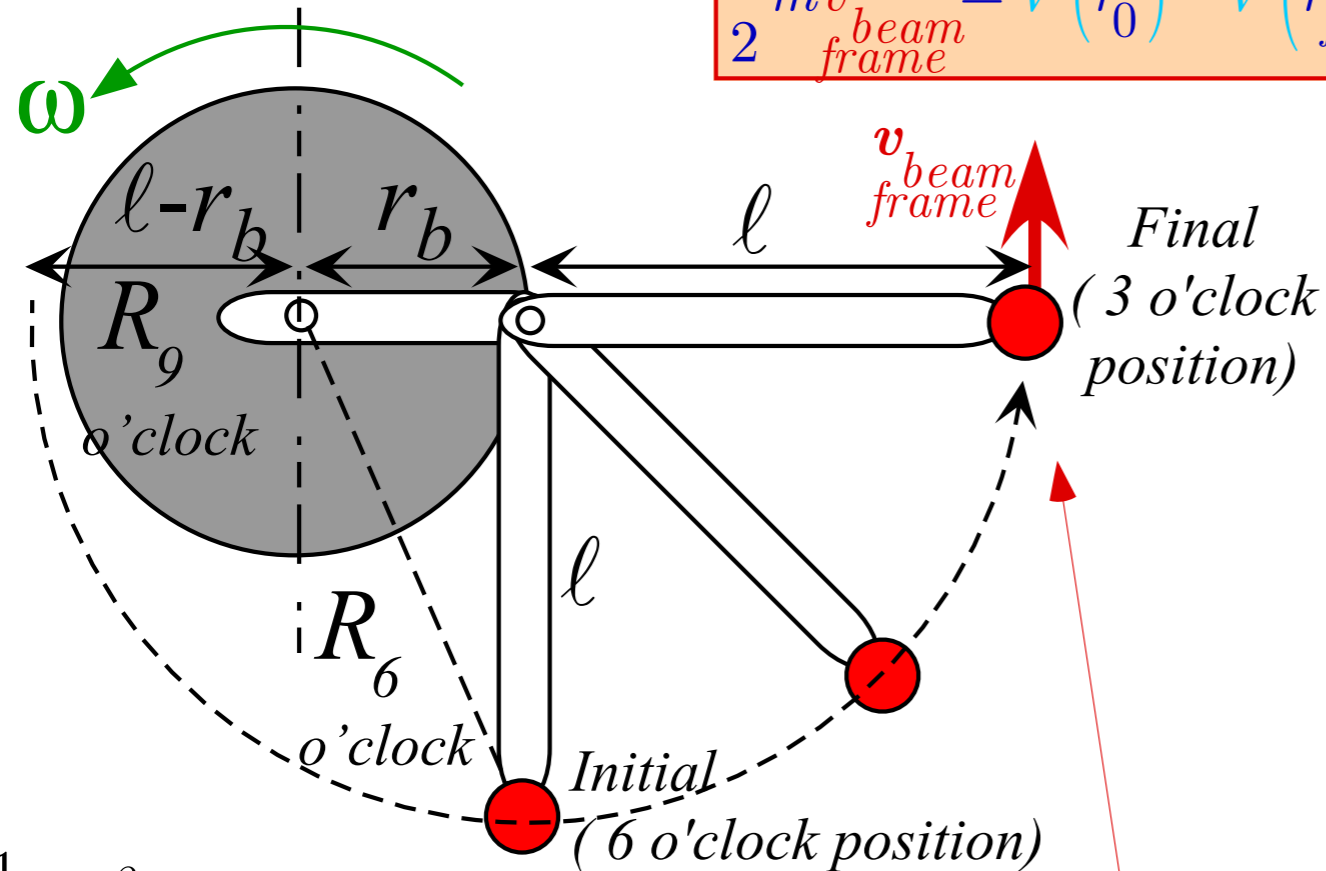


$$\frac{1}{2} m v_{\text{beam frame}}^2 (\text{flinger}) = \text{Final Flinger } KE_{\text{beam frame}}$$

$$\frac{1}{2} m \omega^2 (r_b + l)^2 - \frac{1}{2} m \omega^2 r_b^2 = \frac{1}{2} m \omega^2 l (2r_b + l)$$

Trebuchet model in rotating beam frame

Assume: Constant beam ω



$$\frac{1}{2} m v_{\text{beam frame}}^2 = V(r_0) - V(r_f) = \frac{1}{2} m \omega^2 r_f^2 - \frac{1}{2} m \omega^2 r_0^2$$

$$\frac{1}{2} m v_{\text{beam frame}}^2 (\text{trebuchet}) = \text{Final Trebuchet } KE_{\text{beam frame}}$$

$$\frac{1}{2} m \omega^2 (r_b + l)^2 - \frac{1}{2} m \omega^2 (r_b^2 + l^2) = \frac{1}{2} m \omega^2 (2r_b l)$$

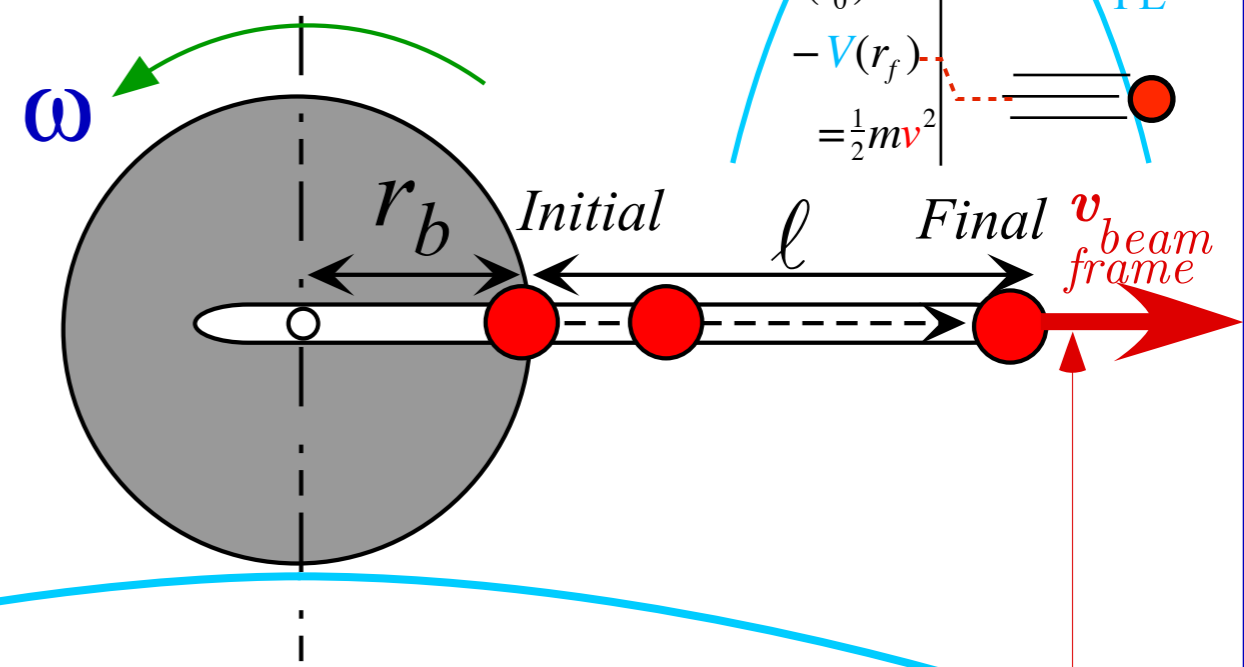
Final
3 o'clock
Initial
6 o'clock

$$R_6^2 = r_b^2 + l^2$$

o'clock

Flinger model in rotating beam frame

Assume: Constant beam ω



$$\frac{1}{2} m v_{\text{beam frame}}^2 (\text{flinger}) = \text{Final Flinger } KE_{\text{beam frame}}$$

$$\frac{1}{2} m \omega^2 (r_b + l)^2 - \frac{1}{2} m \omega^2 r_b^2 = \frac{1}{2} m \omega^2 l (2r_b + l)$$

Final
3 o'clock
Initial
3 o'clock

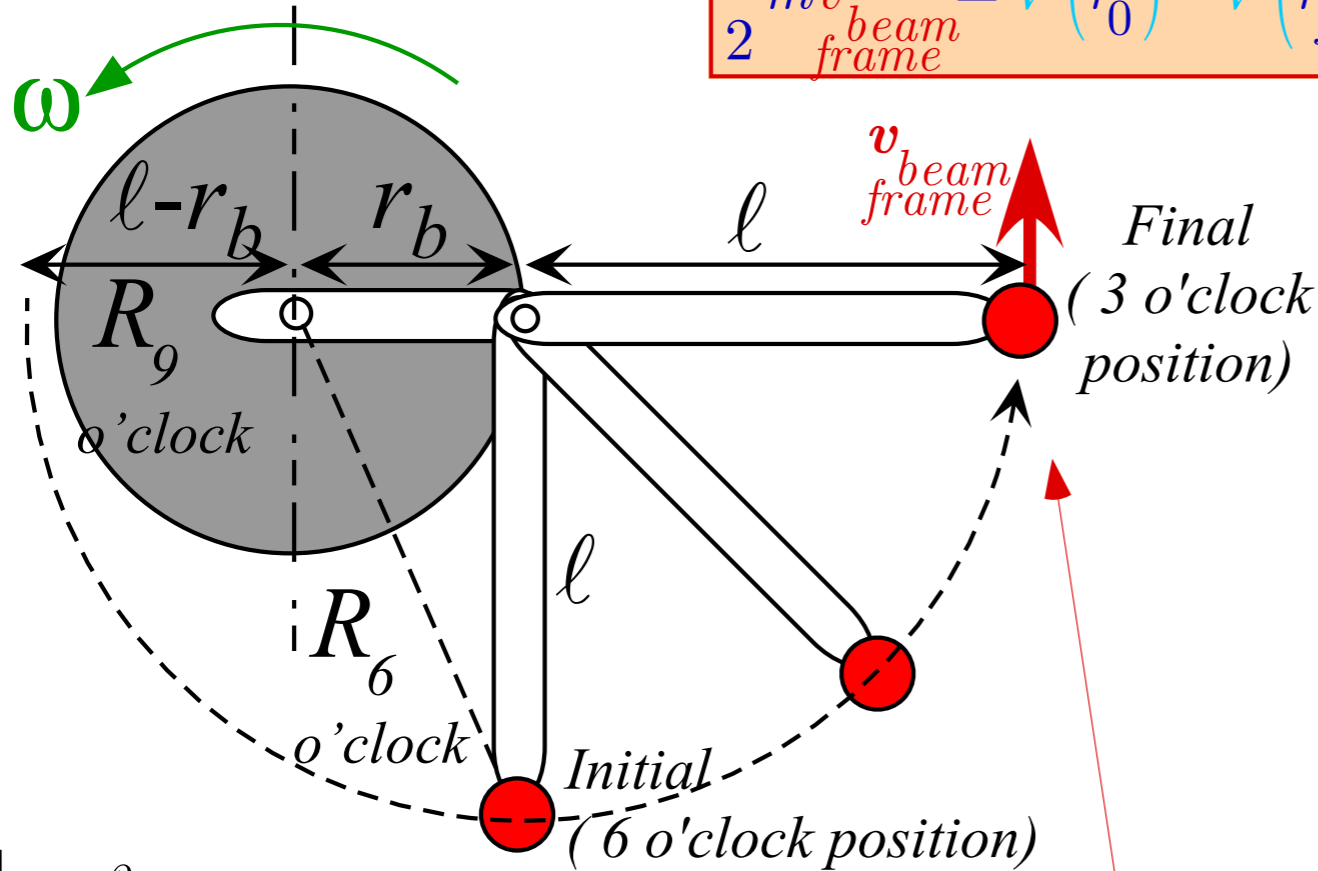
Flinger KE is $\frac{m \omega^2}{2} l^2$ more than 6 o'clock trebuchet but misdirected

Trebuchet model in rotating beam frame

Flinger model in rotating beam frame

$$\frac{1}{2} m v_{\text{beam frame}}^2 = V(r_0) \quad V(r_f) = \frac{1}{2} m \omega^2 r_f^2 \quad \frac{1}{2} m \omega^2 r_0^2$$

Assume: Constant beam ω



$$\frac{1}{2} m v_{\text{beam frame}}^2 (\text{trebuchet}) = \text{Final Trebuchet } KE_{\text{beam frame}}$$

$$\frac{1}{2} m \omega^2 (r_b + \ell)^2 \text{ (Final 3 o'clock)} - \frac{1}{2} m \omega^2 (r_b^2 + \ell^2) \text{ (Initial 6 o'clock)} = \frac{1}{2} m \omega^2 (2r_b \ell)$$

$$R_6^2 = r_b^2 + \ell^2$$

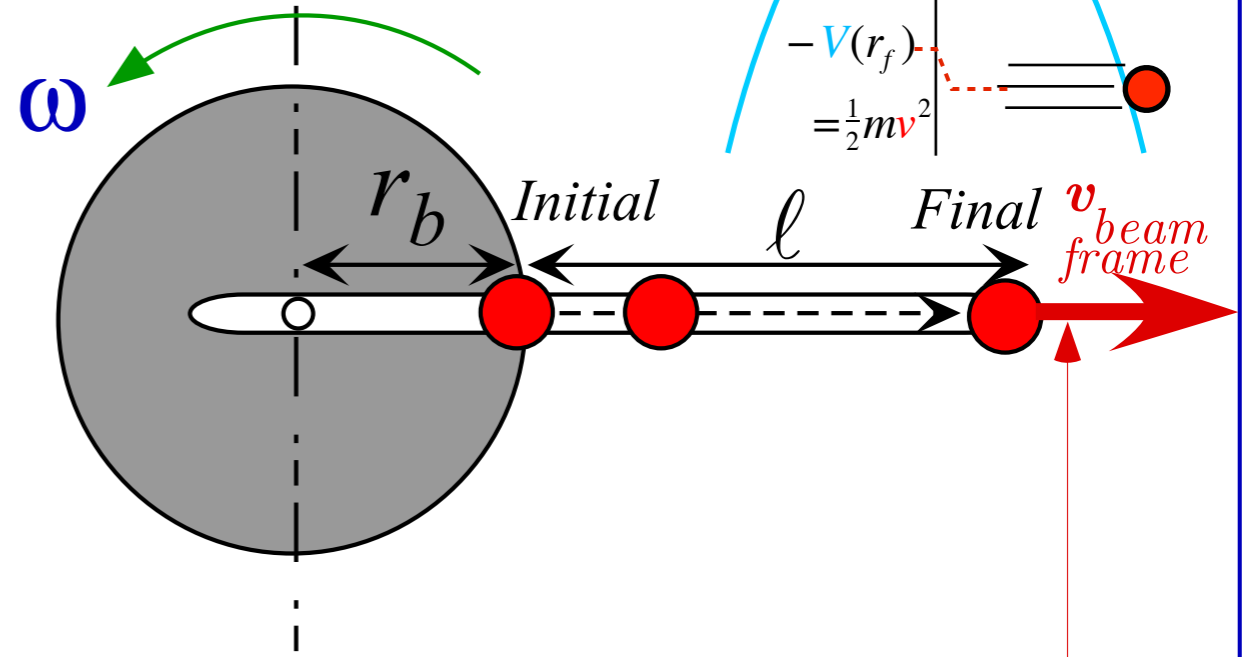
o'clock

$$\text{Initial 9 o'clock} = \frac{1}{2} m \omega^2 (4r_b \ell)$$

$$R_9^2 = r_b^2 + \ell^2 - 2r_b \ell$$

o'clock

Assume: Constant beam ω

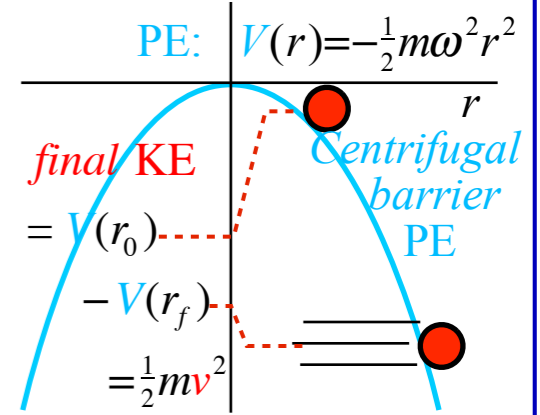


$$\frac{1}{2} m v_{\text{beam frame}}^2 (\text{flinger}) = \text{Final Flinger } KE_{\text{beam frame}}$$

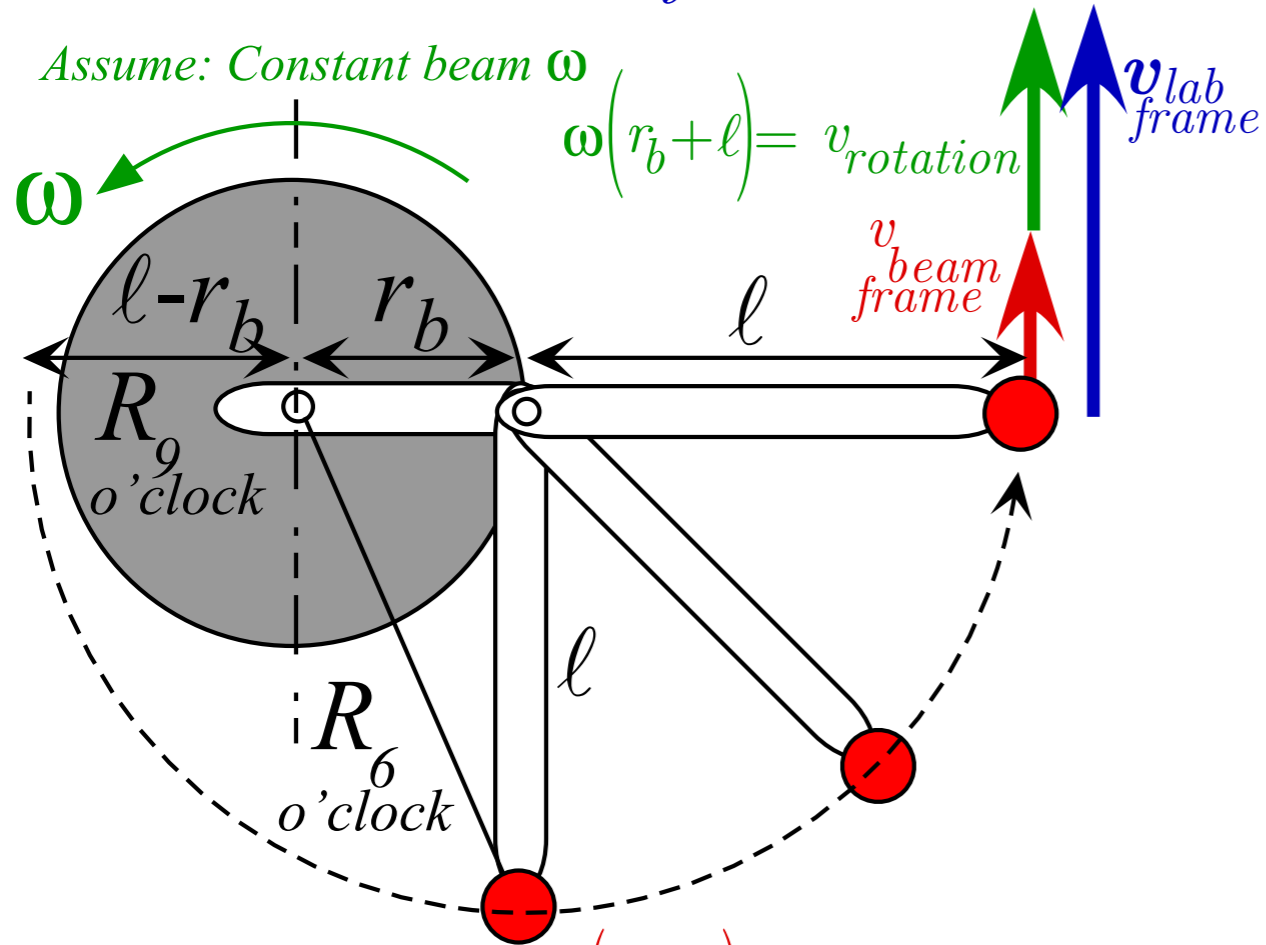
$$\frac{1}{2} m \omega^2 (r_b + \ell)^2 \text{ (Final 3 o'clock)} - \frac{1}{2} m \omega^2 r_b^2 \text{ (Initial 3 o'clock)} = \frac{1}{2} m \omega^2 \ell (2r_b + \ell)$$

Flinger KE is $\frac{m\omega^2}{2} \ell^2$ more than 6 o'clock trebuchet but misdirected

Flinger KE is $\frac{m\omega^2}{2} (2r_b \ell - \ell^2)$ less than 9 o'clock trebuchet and misdirected



Trebuchet model in lab frame



$$v_{beam\ frame}^2(\text{trebuchet}) = \begin{cases} \omega^2 (2r_b l) & \text{half-cocked 6 o'clock} \\ \omega^2 (4r_b l) & \text{fully-cocked 9 o'clock} \end{cases}$$

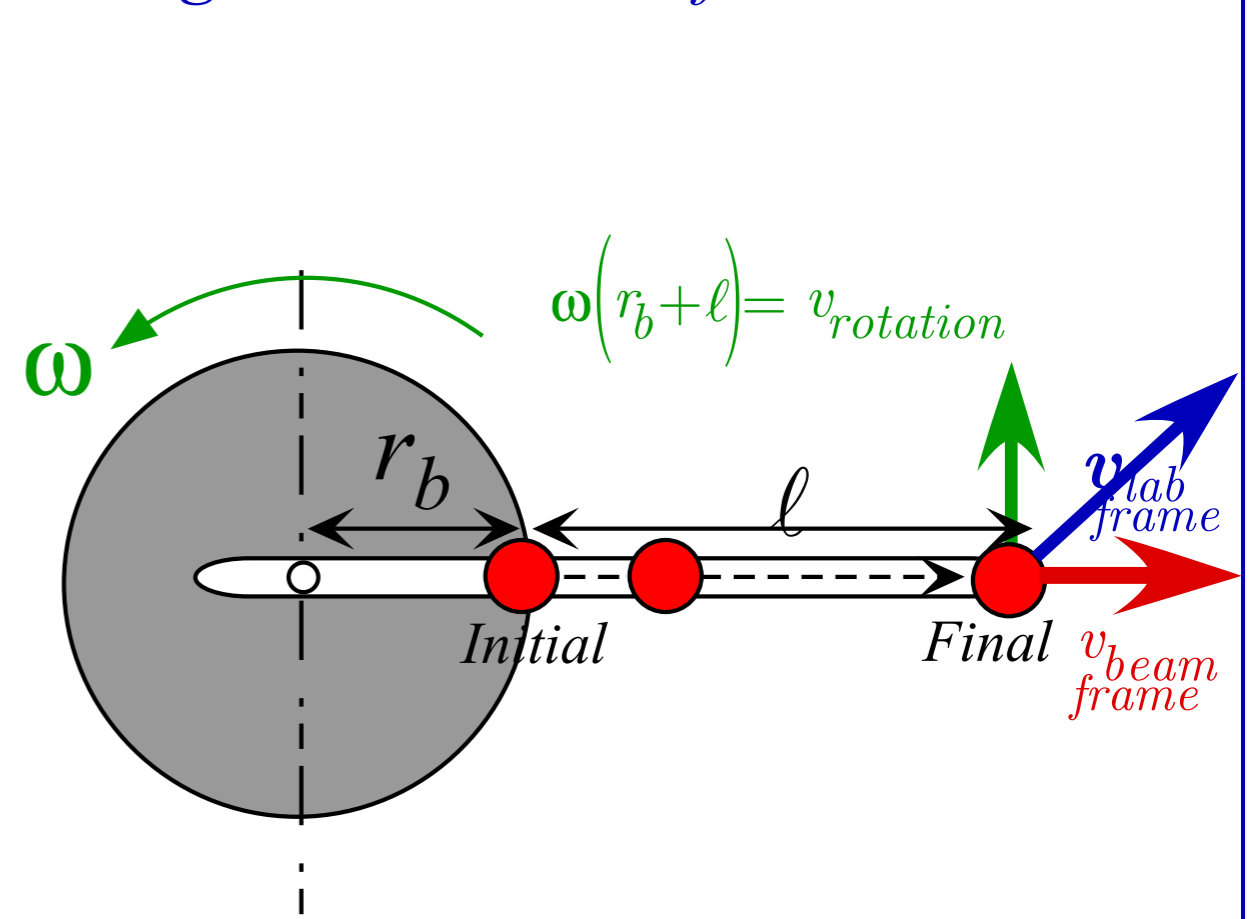
$$v_{lab\ frame}(\text{trebuchet}) =$$

$$\begin{cases} \omega(r_b + l + \sqrt{2lr_b}) & \text{half-cocked 6 o'clock} \\ \omega(r_b + l + 2\sqrt{lr_b}) & \text{fully-cocked 9 o'clock} \end{cases}$$

$$= \begin{cases} 5.00\omega & = 5.16\omega & = 5.00\omega \\ 5.82\omega & = 6.00\omega & = 5.82\omega \end{cases}$$

$$(r_b = 2, l = 1), (r_b = 1.5, l = 1.5), (r_b = 1, l = 2)$$

Flinger model in lab frame



$$v_{beam\ frame}^2(\text{flinger}) = \omega^2 l (2r_b + l)$$

$$v_{lab\ frame}(\text{flinger}) =$$

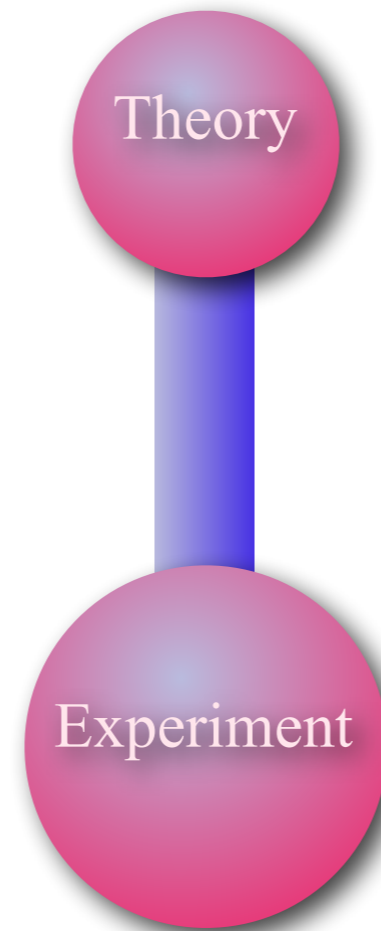
$$= \omega \sqrt{(r_b + l)^2 + l(2r_b + l)} = \omega \sqrt{2(r_b + l)^2 - r_b^2}$$

(compare)

$$= 3.74\omega \quad = 3.96\omega \quad = 4.12\omega$$

$$(r_b = 2, l = 1), (r_b = 1.5, l = 1.5), (r_b = 1, l = 2)$$

Physics used to be pretty much bi-polar...



Now that situation is changing...

Many Approaches to Mechanics (Trebuchet Equations)

Each has advantages and disadvantages

- U.S. Approach

Quick'n dirty

Newton F=Ma Equations

Cartesian coordinates

- French Approach

Tres elegant

Lagrange Equations

in Generalized Coordinates

$$F_\ell = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell}$$

- German Approach

Pride and Precision

Riemann Christoffel Equations

in Differential Manifolds

$$F^k = \ddot{q}^k + \Gamma_{mn}^k \dot{q}^m \dot{q}^n$$

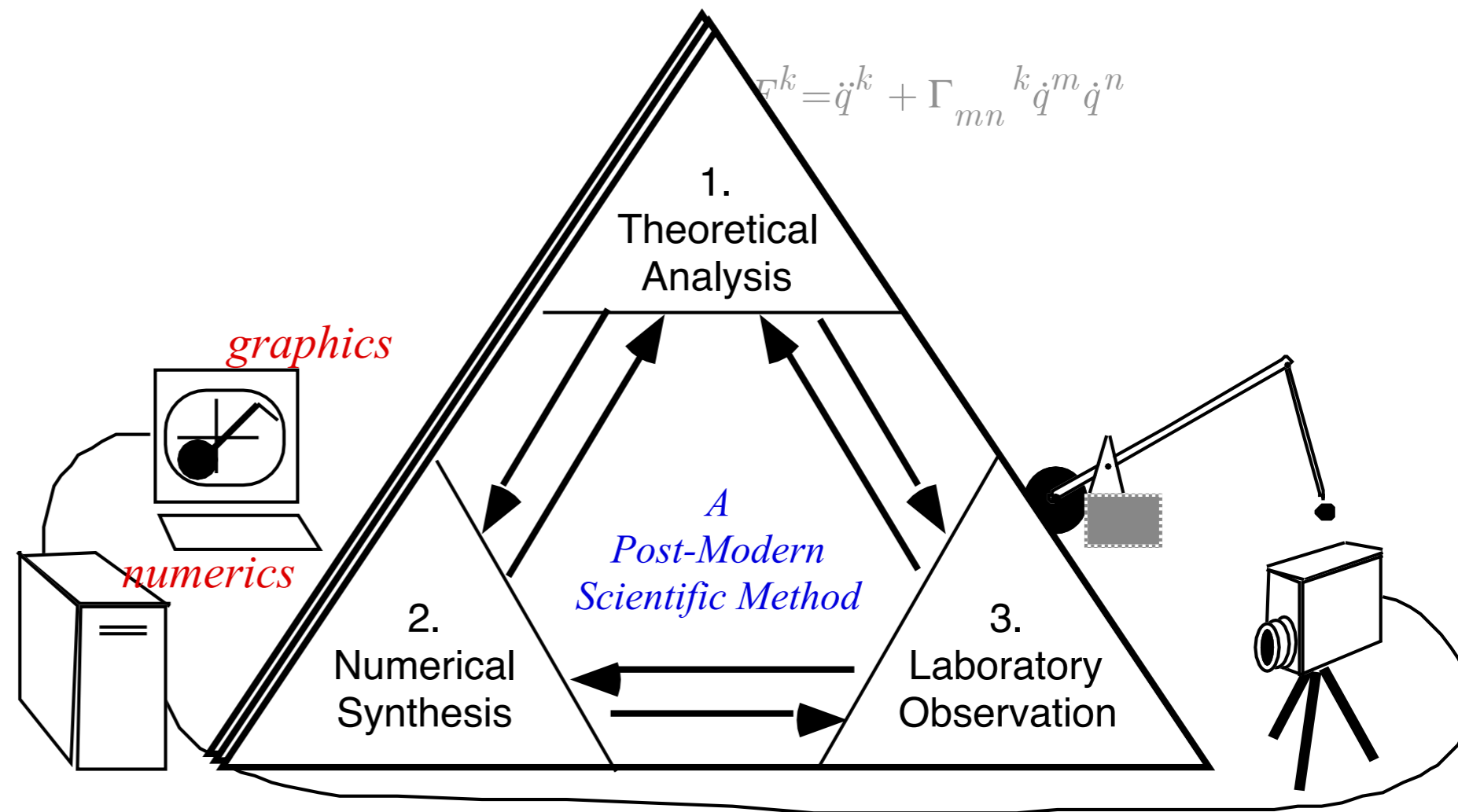
- Anglo-Irish Approach

Powerfully Creative

Hamilton's Equations

Phase Space $\dot{p}_j = -\frac{\partial H}{\partial q^j}, \quad \dot{q}^k = \frac{\partial H}{\partial p^k}.$

- Unified Approach

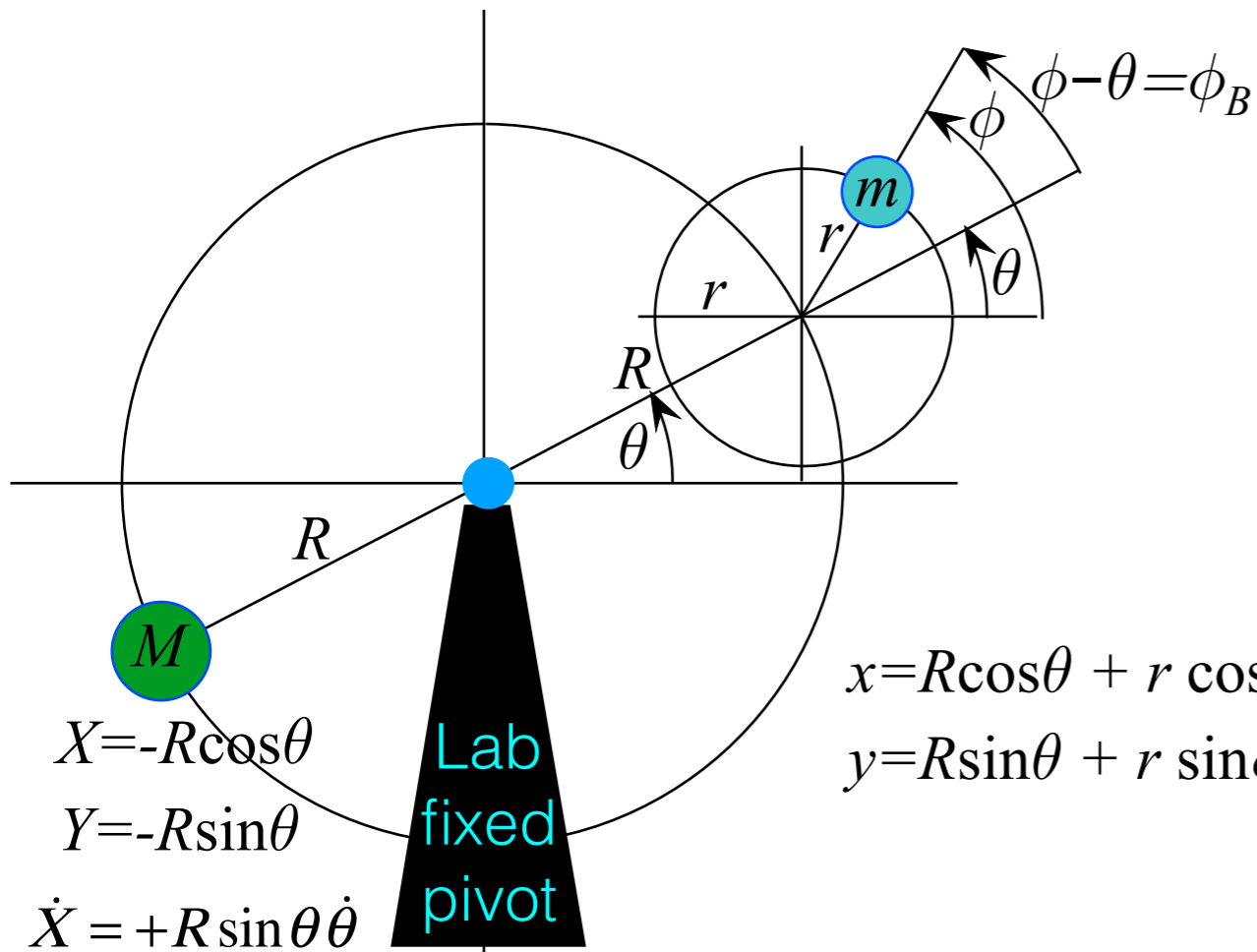


All approaches have one thing in common:

The Art of Approximation

Physics lives and dies by the art of approximate models and analogs.

Hamilton-Jacobi-Poincare: $dS = Ldt = p_\mu dq^\mu - Hdt$
 $p_\mu = \frac{\partial S}{\partial q^\mu}, -H = \frac{\partial S}{\partial t}$



$$X = -R \cos \theta$$

$$Y = -R \sin \theta$$

$$\dot{X} = +R \sin \theta \dot{\theta}$$

$$\dot{Y} = +R \cos \theta \dot{\theta}$$

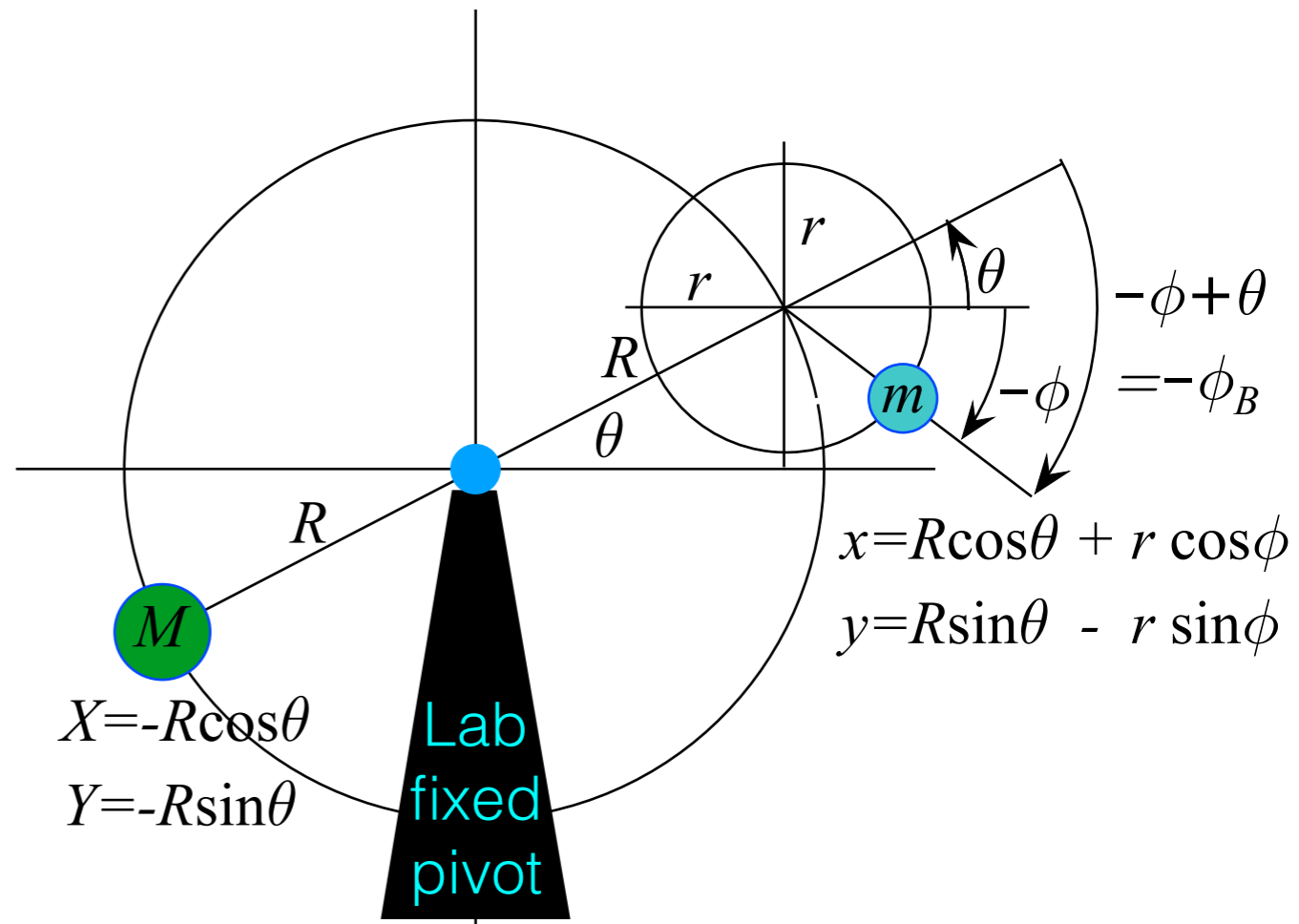
Lab
fixed
pivot

$$x = R \cos \theta + r \cos \phi$$

$$y = R \sin \theta + r \sin \phi$$

$$\dot{x} = -R \sin \theta \dot{\theta} - r \sin \phi \dot{\phi}$$

$$\dot{y} = +R \cos \theta \dot{\theta} + r \cos \phi \dot{\phi}$$



$$X = -R \cos \theta$$

$$Y = -R \sin \theta$$

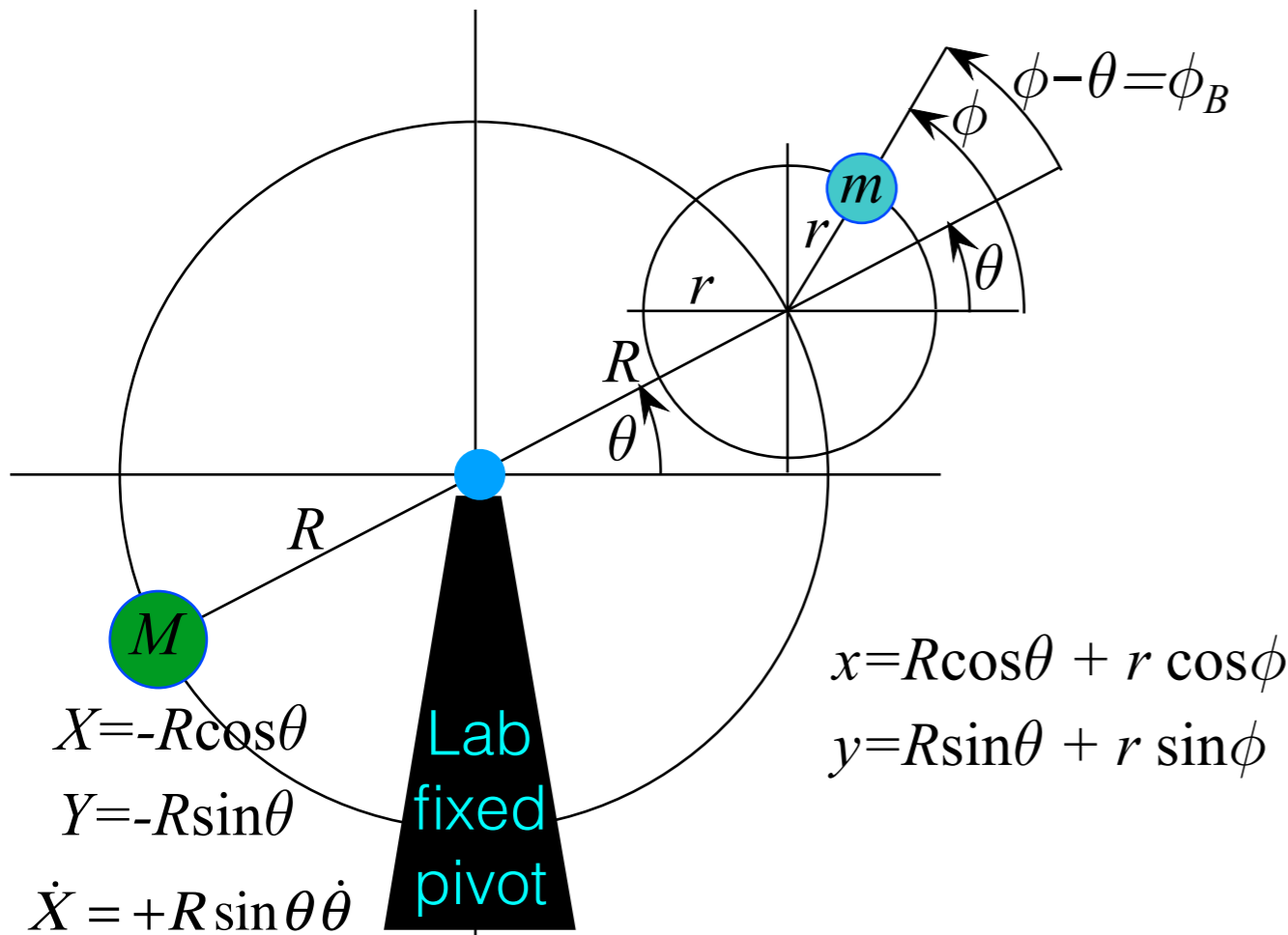
Lab
fixed
pivot

$$x = R \cos \theta + r \cos \phi$$

$$y = R \sin \theta - r \sin \phi$$

$$\dot{X}^2 = +R^2 \sin^2 \theta \dot{\theta}^2$$

$$\dot{Y}^2 = +R^2 \cos^2 \theta \dot{\theta}^2$$



$$X = -R \cos \theta$$

$$Y = -R \sin \theta$$

$$\dot{X} = +R \sin \theta \dot{\theta}$$

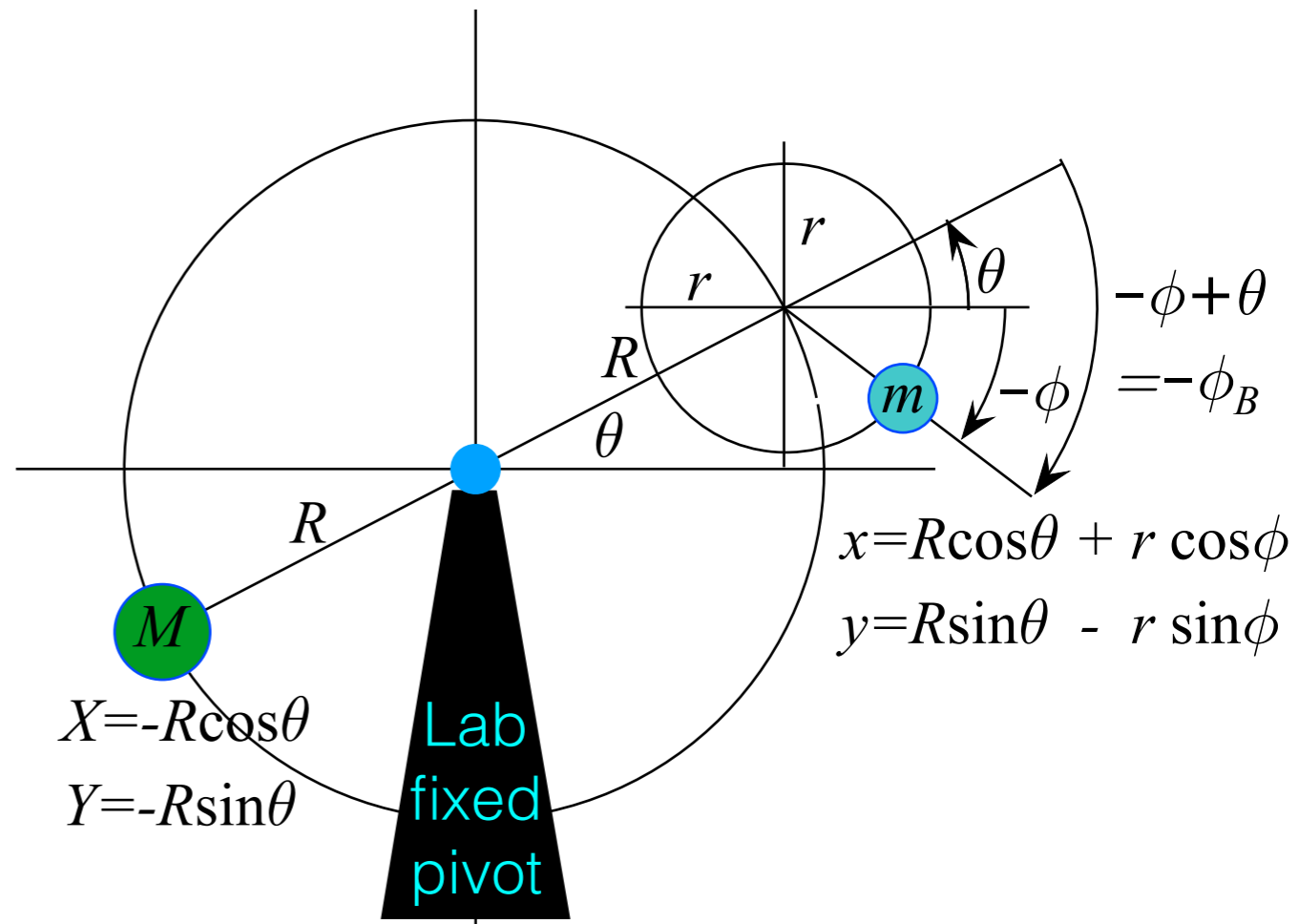
$$\dot{Y} = +R \cos \theta \dot{\theta}$$

$$x = R \cos \theta + r \cos \phi$$

$$y = R \sin \theta + r \sin \phi$$

$$\dot{x} = -R \sin \theta \dot{\theta} - r \sin \phi \dot{\phi}$$

$$\dot{y} = +R \cos \theta \dot{\theta} + r \cos \phi \dot{\phi}$$



$$X = -R \cos \theta$$

$$Y = -R \sin \theta$$

$$\dot{X} = +R \sin \theta \dot{\theta}$$

$$\dot{Y} = +R \cos \theta \dot{\theta}$$

$$x = R \cos \theta + r \cos \phi$$

$$y = R \sin \theta - r \sin \phi$$

$$\dot{x} = -R \sin \theta \dot{\theta} - r \sin \phi \dot{\phi}$$

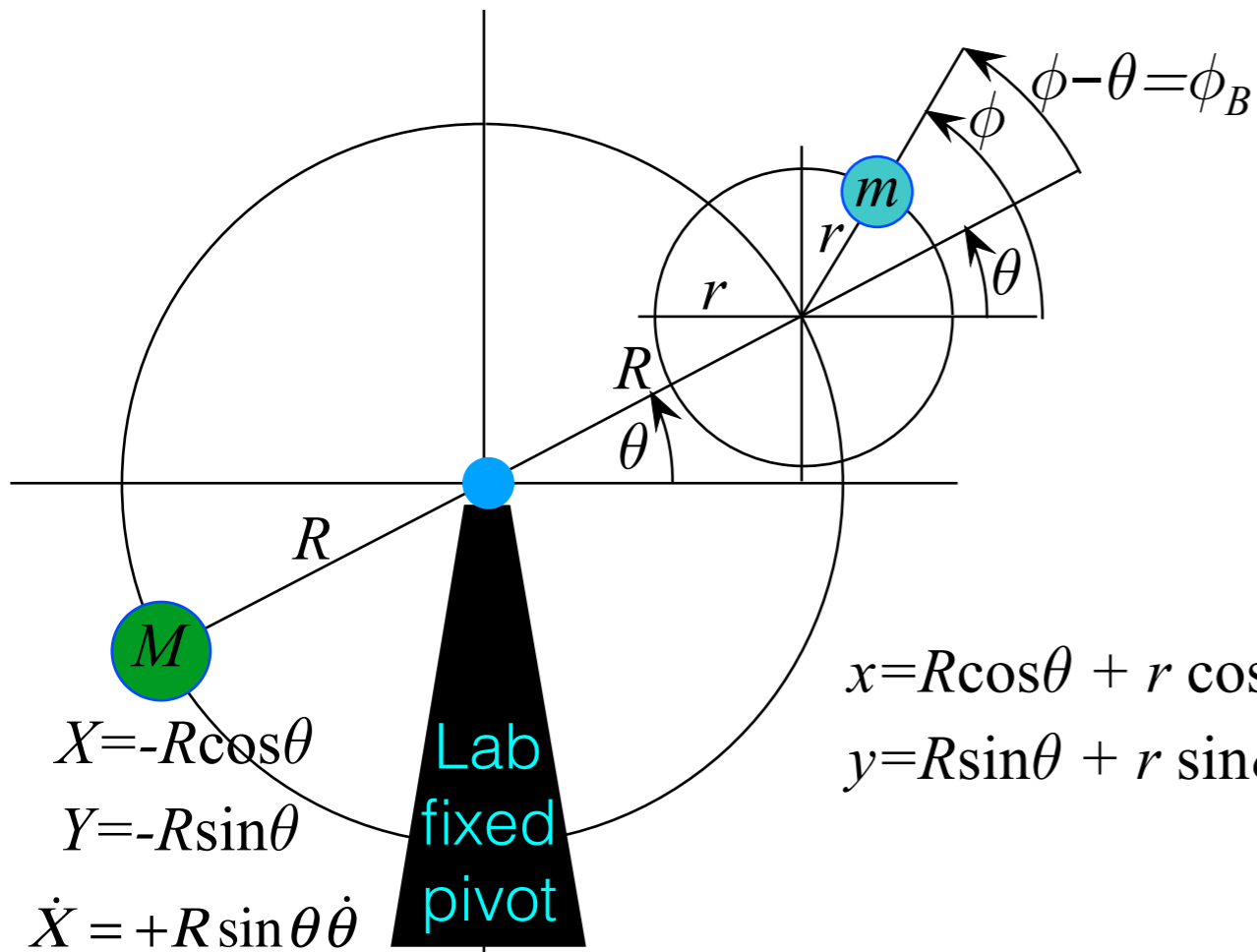
$$\dot{y} = +R \cos \theta \dot{\theta} + r \cos \phi \dot{\phi}$$

$$\dot{X}^2 = +R^2 \sin^2 \theta \dot{\theta}^2$$

$$\dot{Y}^2 = +R^2 \cos^2 \theta \dot{\theta}^2$$

$$\dot{x}^2 = (-R \sin \theta \dot{\theta} - r \sin \phi \dot{\phi})(-R \sin \theta \dot{\theta} - r \sin \phi \dot{\phi}) = R^2 \sin^2 \theta \dot{\theta}^2 + 2Rr \sin \theta \sin \phi \dot{\theta} \dot{\phi} + r^2 \sin^2 \phi \dot{\phi}^2$$

$$\dot{y}^2 = (R \cos \theta \dot{\theta} + r \cos \phi \dot{\phi})(+R \cos \theta \dot{\theta} + r \cos \phi \dot{\phi}) = R^2 \cos^2 \theta \dot{\theta}^2 + 2Rr \cos \theta \cos \phi \dot{\theta} \dot{\phi} + r^2 \cos^2 \phi \dot{\phi}^2$$



$$X = -R \cos \theta$$

$$Y = -R \sin \theta$$

$$\dot{X} = +R \sin \theta \dot{\theta}$$

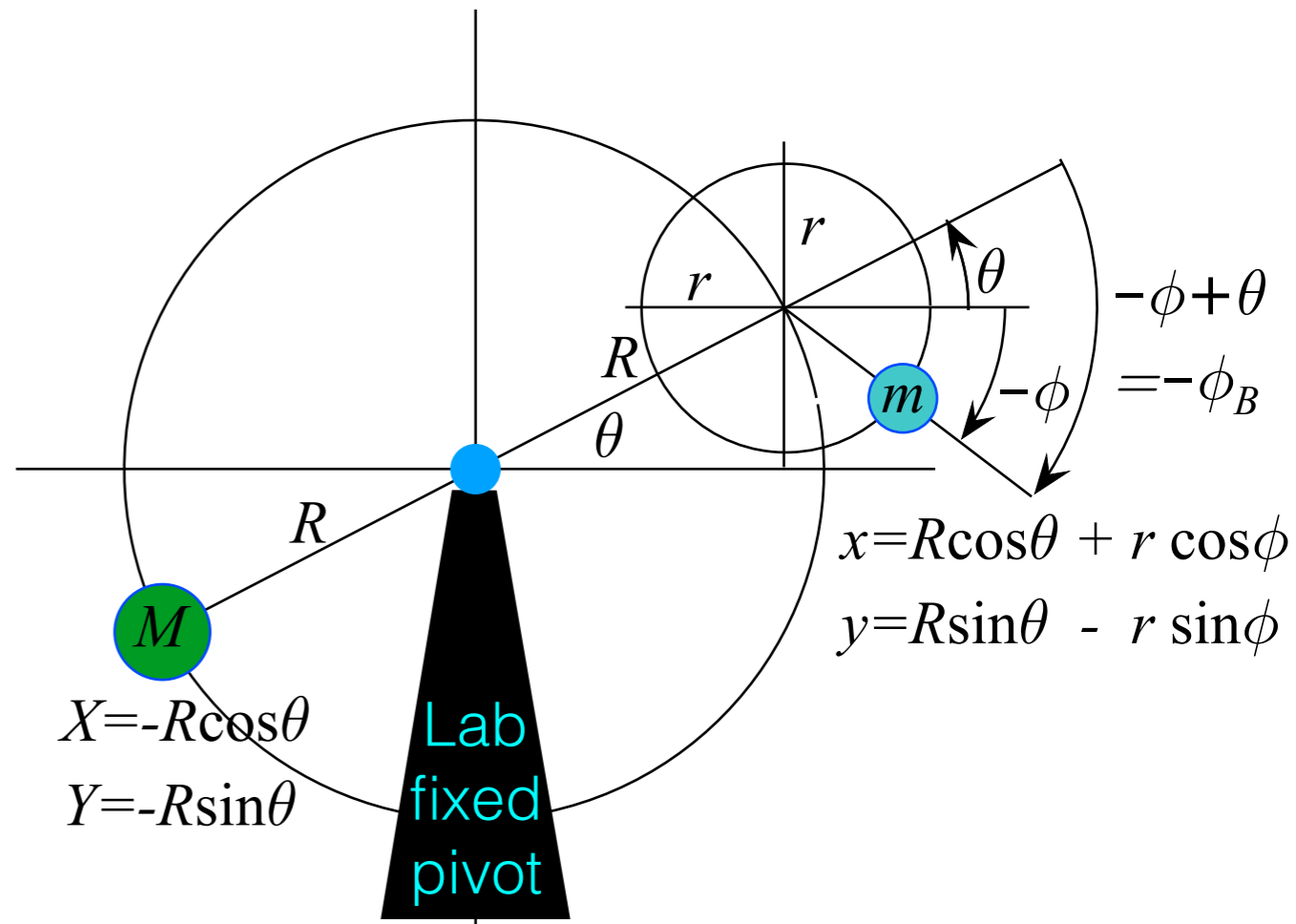
$$\dot{Y} = +R \cos \theta \dot{\theta}$$

$$x = R \cos \theta + r \cos \phi$$

$$y = R \sin \theta + r \sin \phi$$

$$\dot{x} = -R \sin \theta \dot{\theta} - r \sin \phi \dot{\phi}$$

$$\dot{y} = +R \cos \theta \dot{\theta} + r \cos \phi \dot{\phi}$$



$$X = -R \cos \theta$$

$$Y = -R \sin \theta$$

$$x = R \cos \theta + r \cos \phi$$

$$y = R \sin \theta - r \sin \phi$$

$$\dot{x}^2 + \dot{y}^2 = R^2 \dot{\theta}^2 + 2Rr \cos(\theta - \phi) \dot{\theta} \dot{\phi} + r^2 \dot{\phi}^2$$

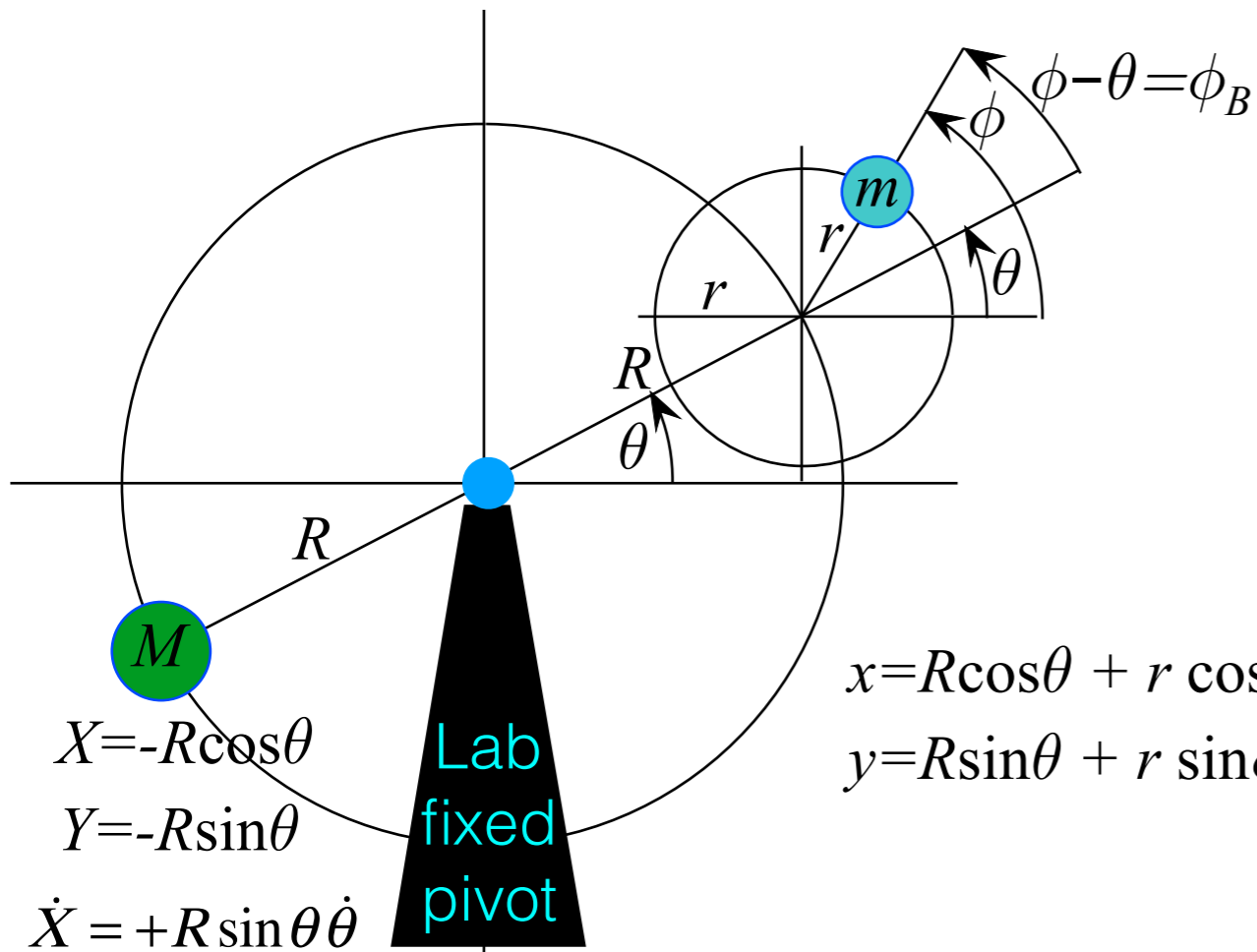
$$\dot{X}^2 = +R^2 \sin^2 \theta \dot{\theta}^2$$

$$\dot{Y}^2 = +R^2 \cos^2 \theta \dot{\theta}^2$$

$$\dot{X}^2 + \dot{Y}^2 = R^2 \dot{\theta}^2$$

$$\dot{x}^2 = (-R \sin \theta \dot{\theta} - r \sin \phi \dot{\phi})(-R \sin \theta \dot{\theta} - r \sin \phi \dot{\phi}) = R^2 \sin^2 \theta \dot{\theta}^2 + 2Rr \sin \theta \sin \phi \dot{\theta} \dot{\phi} + r^2 \sin^2 \phi \dot{\phi}^2$$

$$\dot{y}^2 = (R \cos \theta \dot{\theta} + r \cos \phi \dot{\phi})(+R \cos \theta \dot{\theta} + r \cos \phi \dot{\phi}) = R^2 \cos^2 \theta \dot{\theta}^2 + 2Rr \cos \theta \cos \phi \dot{\theta} \dot{\phi} + r^2 \cos^2 \phi \dot{\phi}^2$$



$$X = -R \cos \theta$$

$$Y = -R \sin \theta$$

$$\dot{X} = +R \sin \theta \dot{\theta}$$

$$\dot{Y} = +R \cos \theta \dot{\theta}$$

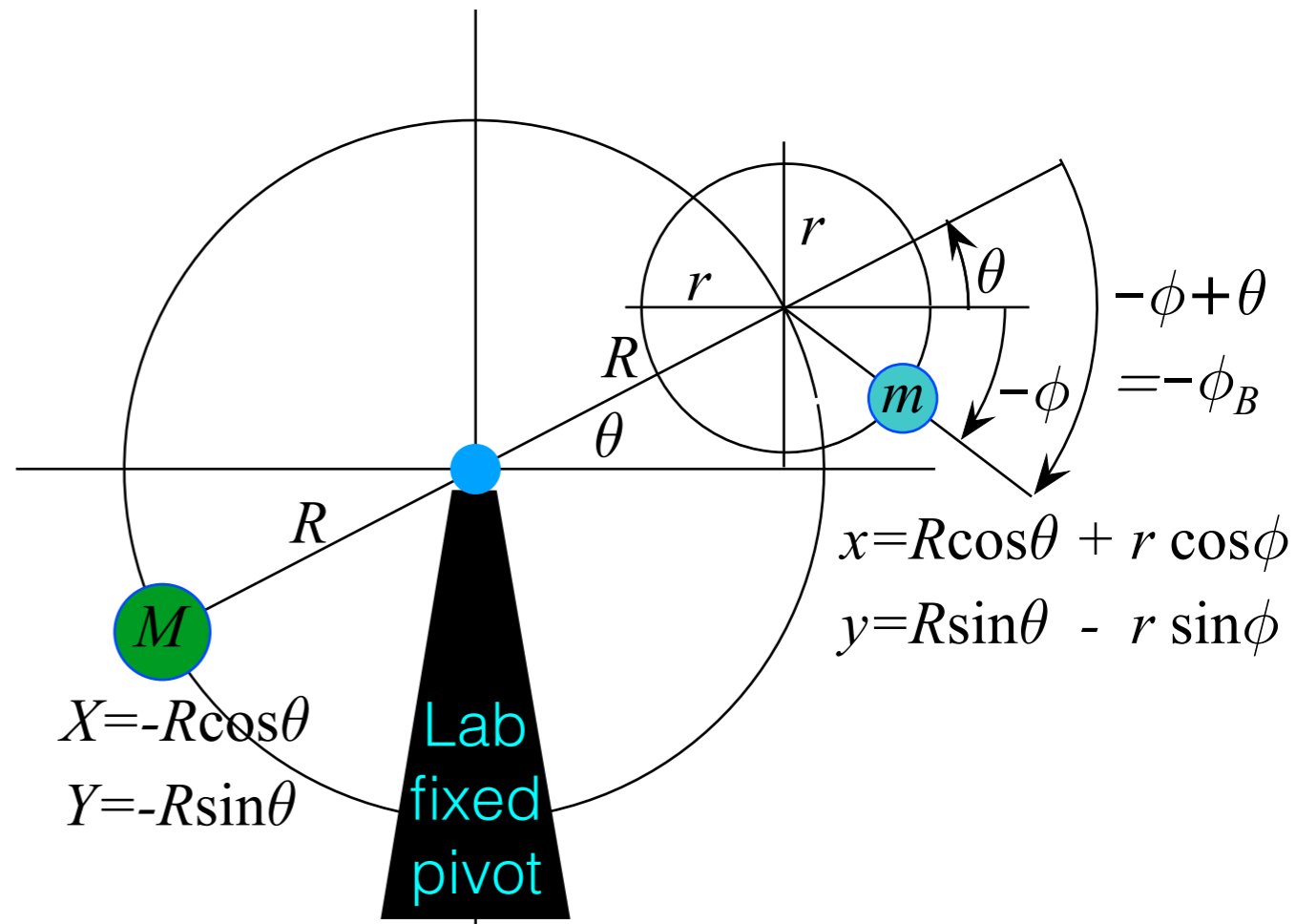
Lab
fixed
pivot

$$x = R \cos \theta + r \cos \phi$$

$$y = R \sin \theta + r \sin \phi$$

$$\dot{x} = -R \sin \theta \dot{\theta} - r \sin \phi \dot{\phi}$$

$$\dot{y} = +R \cos \theta \dot{\theta} + r \cos \phi \dot{\phi}$$



$$X = -R \cos \theta$$

$$Y = -R \sin \theta$$

$$\dot{X} = +R \sin \theta \dot{\theta}$$

$$\dot{Y} = +R \cos \theta \dot{\theta}$$

Lab
fixed
pivot

$$x = R \cos \theta + r \cos \phi$$

$$y = R \sin \theta - r \sin \phi$$

$$\dot{x}^2 + \dot{y}^2 = R^2 \dot{\theta}^2 + 2Rr \cos(\theta - \phi) \dot{\theta} \dot{\phi} + r^2 \dot{\phi}^2$$

$$\dot{X}^2 = +R^2 \sin^2 \theta \dot{\theta}^2$$

$$\dot{Y}^2 = +R^2 \cos^2 \theta \dot{\theta}^2$$

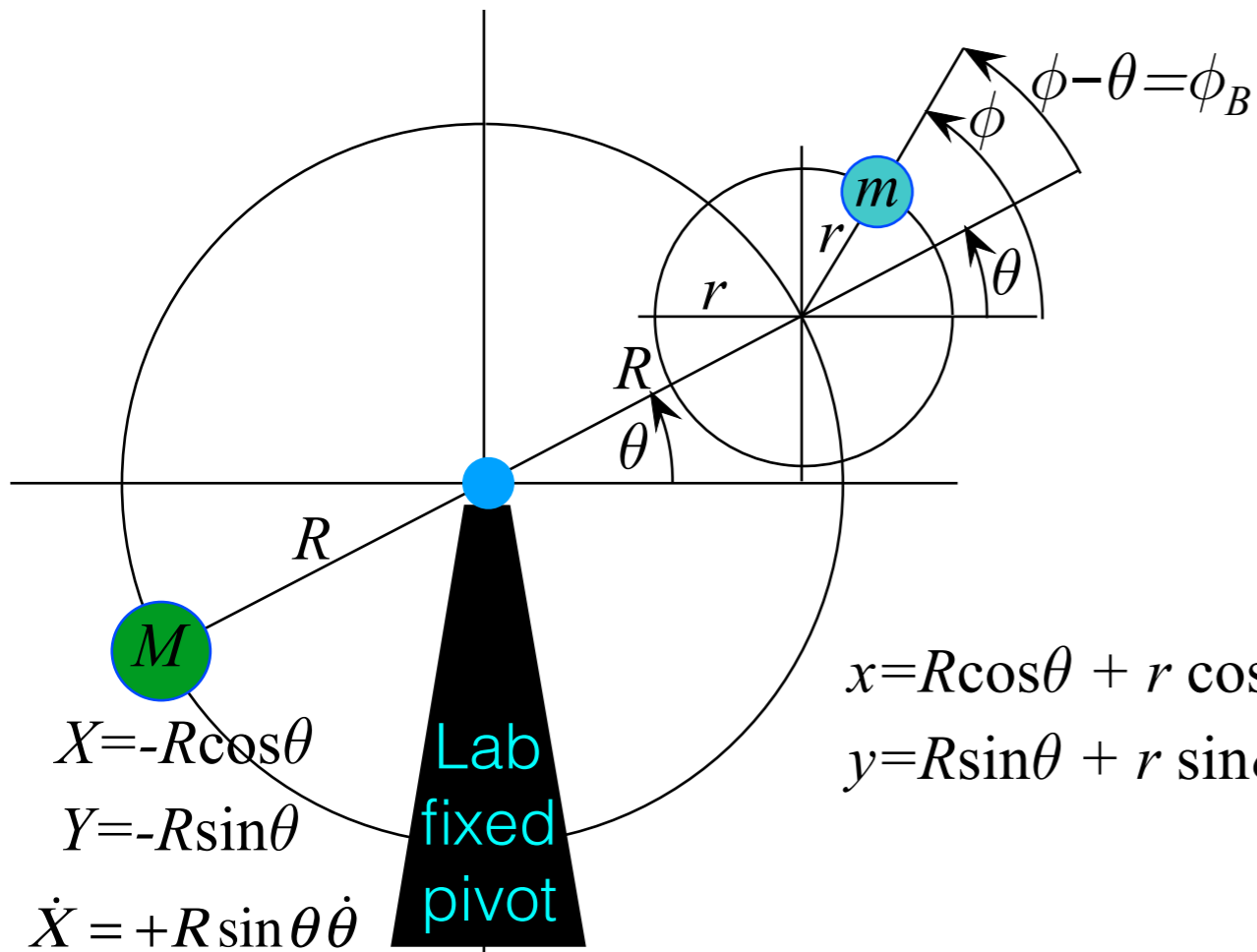
$$\dot{X}^2 + \dot{Y}^2 = R^2 \dot{\theta}^2$$

$$\dot{x}^2 = (-R \sin \theta \dot{\theta} - r \sin \phi \dot{\phi})(-R \sin \theta \dot{\theta} - r \sin \phi \dot{\phi}) = R^2 \sin^2 \theta \dot{\theta}^2 + 2Rr \sin \theta \sin \phi \dot{\theta} \dot{\phi} + r^2 \sin^2 \phi \dot{\phi}^2$$

$$\dot{y}^2 = (R \cos \theta \dot{\theta} + r \cos \phi \dot{\phi})(+R \cos \theta \dot{\theta} + r \cos \phi \dot{\phi}) = R^2 \cos^2 \theta \dot{\theta}^2 + 2Rr \cos \theta \cos \phi \dot{\theta} \dot{\phi} + r^2 \cos^2 \phi \dot{\phi}^2$$

KE of driver M plus projectile m

$$T = \frac{1}{2} M (\dot{X}^2 + \dot{Y}^2) + \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) = \frac{1}{2} M (R^2 \dot{\theta}^2) + \frac{1}{2} m (R^2 \dot{\theta}^2 + 2Rr \cos(\theta - \phi) \dot{\theta} \dot{\phi} + r^2 \dot{\phi}^2)$$



$$X = -R \cos \theta$$

$$Y = -R \sin \theta$$

$$\dot{X} = +R \sin \theta \dot{\theta}$$

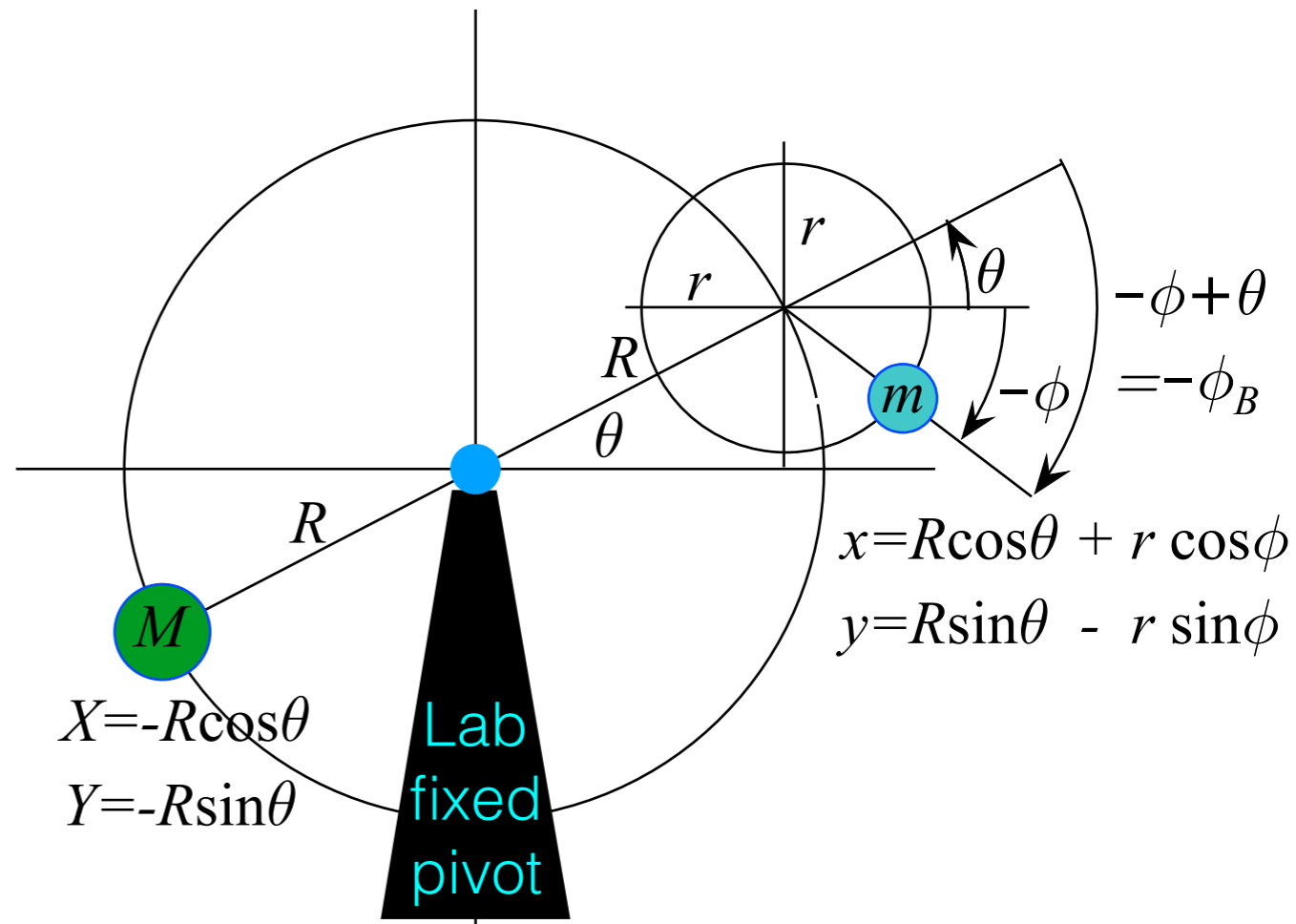
$$\dot{Y} = +R \cos \theta \dot{\theta}$$

$$x = R \cos \theta + r \cos \phi$$

$$y = R \sin \theta + r \sin \phi$$

$$\dot{x} = -R \sin \theta \dot{\theta} - r \sin \phi \dot{\phi}$$

$$\dot{y} = +R \cos \theta \dot{\theta} + r \cos \phi \dot{\phi}$$



$$X = -R \cos \theta$$

$$Y = -R \sin \theta$$

$$x = R \cos \theta + r \cos \phi$$

$$y = R \sin \theta - r \sin \phi$$

$$\dot{x}^2 + \dot{y}^2 = R^2 \dot{\theta}^2 + 2Rr \cos(\theta - \phi) \dot{\theta} \dot{\phi} + r^2 \dot{\phi}^2$$

$$\dot{X}^2 = +R^2 \sin^2 \theta \dot{\theta}^2$$

$$\dot{Y}^2 = +R^2 \cos^2 \theta \dot{\theta}^2$$

$$\dot{X}^2 + \dot{Y}^2 = R^2 \dot{\theta}^2$$

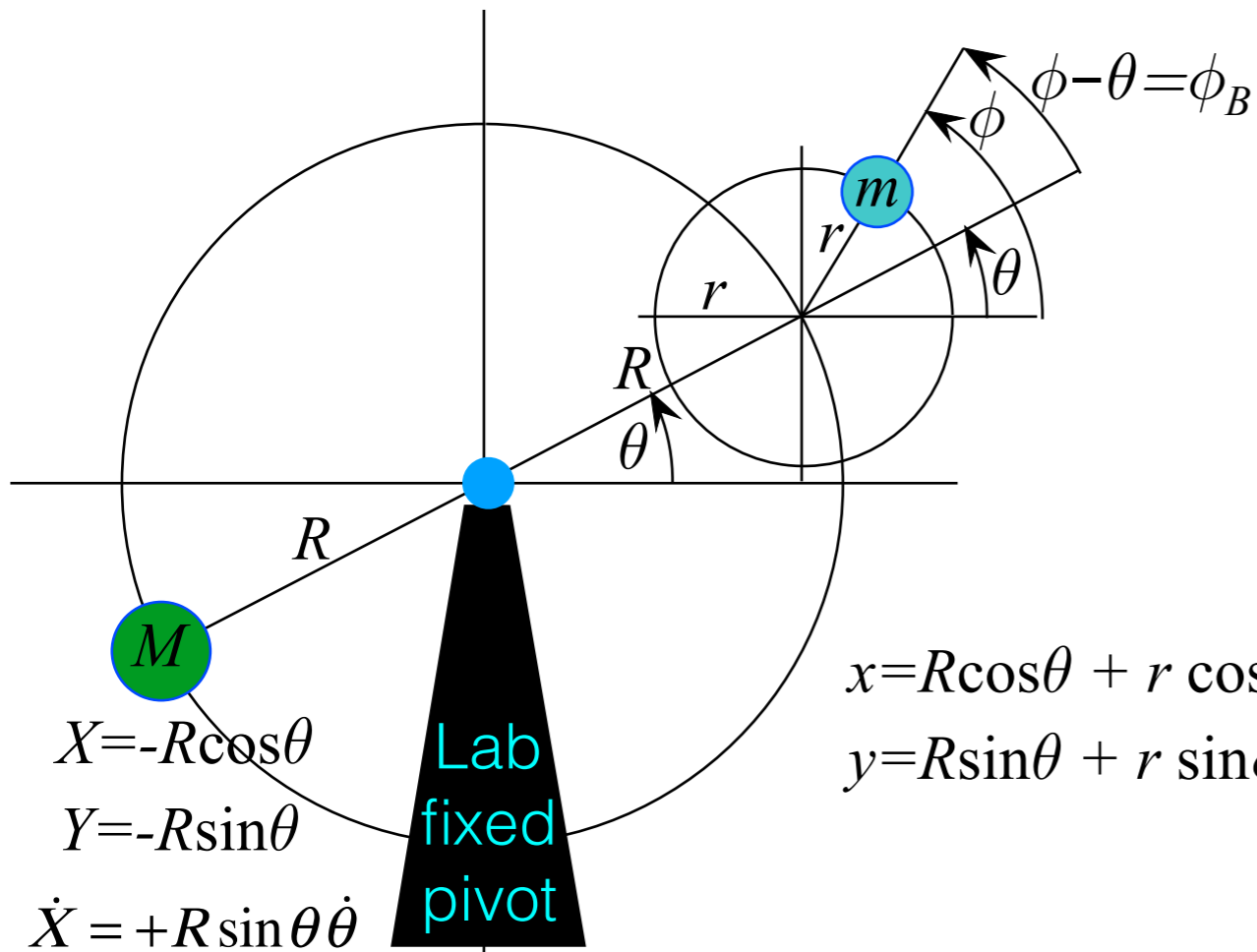
$$\dot{x}^2 = (-R \sin \theta \dot{\theta} - r \sin \phi \dot{\phi})(-R \sin \theta \dot{\theta} - r \sin \phi \dot{\phi}) = R^2 \sin^2 \theta \dot{\theta}^2 + 2Rr \sin \theta \sin \phi \dot{\theta} \dot{\phi} + r^2 \sin^2 \phi \dot{\phi}^2$$

$$\dot{y}^2 = (R \cos \theta \dot{\theta} + r \cos \phi \dot{\phi})(+R \cos \theta \dot{\theta} + r \cos \phi \dot{\phi}) = R^2 \cos^2 \theta \dot{\theta}^2 + 2Rr \cos \theta \cos \phi \dot{\theta} \dot{\phi} + r^2 \cos^2 \phi \dot{\phi}^2$$

KE of driver M plus projectile m is explicit function of beam-relative coordinate $-\phi_B = (\theta - \phi)$ only

$$T = \frac{1}{2} M (\dot{X}^2 + \dot{Y}^2) + \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) = \frac{1}{2} M (R^2 \dot{\theta}^2) + \frac{1}{2} m (R^2 \dot{\theta}^2 + 2Rr \cos(\theta - \phi) \dot{\theta} \dot{\phi} + r^2 \dot{\phi}^2)$$

$$T = \frac{1}{2} M (\dot{X}^2 + \dot{Y}^2) + \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) = \frac{1}{2} (M + m) (R^2 \dot{\theta}^2) + mRr \cos(\theta - \phi) \dot{\theta} \dot{\phi} + \frac{1}{2} m r^2 \dot{\phi}^2$$



$$X = -R \cos \theta$$

$$Y = -R \sin \theta$$

$$\dot{X} = +R \sin \theta \dot{\theta}$$

$$\dot{Y} = +R \cos \theta \dot{\theta}$$

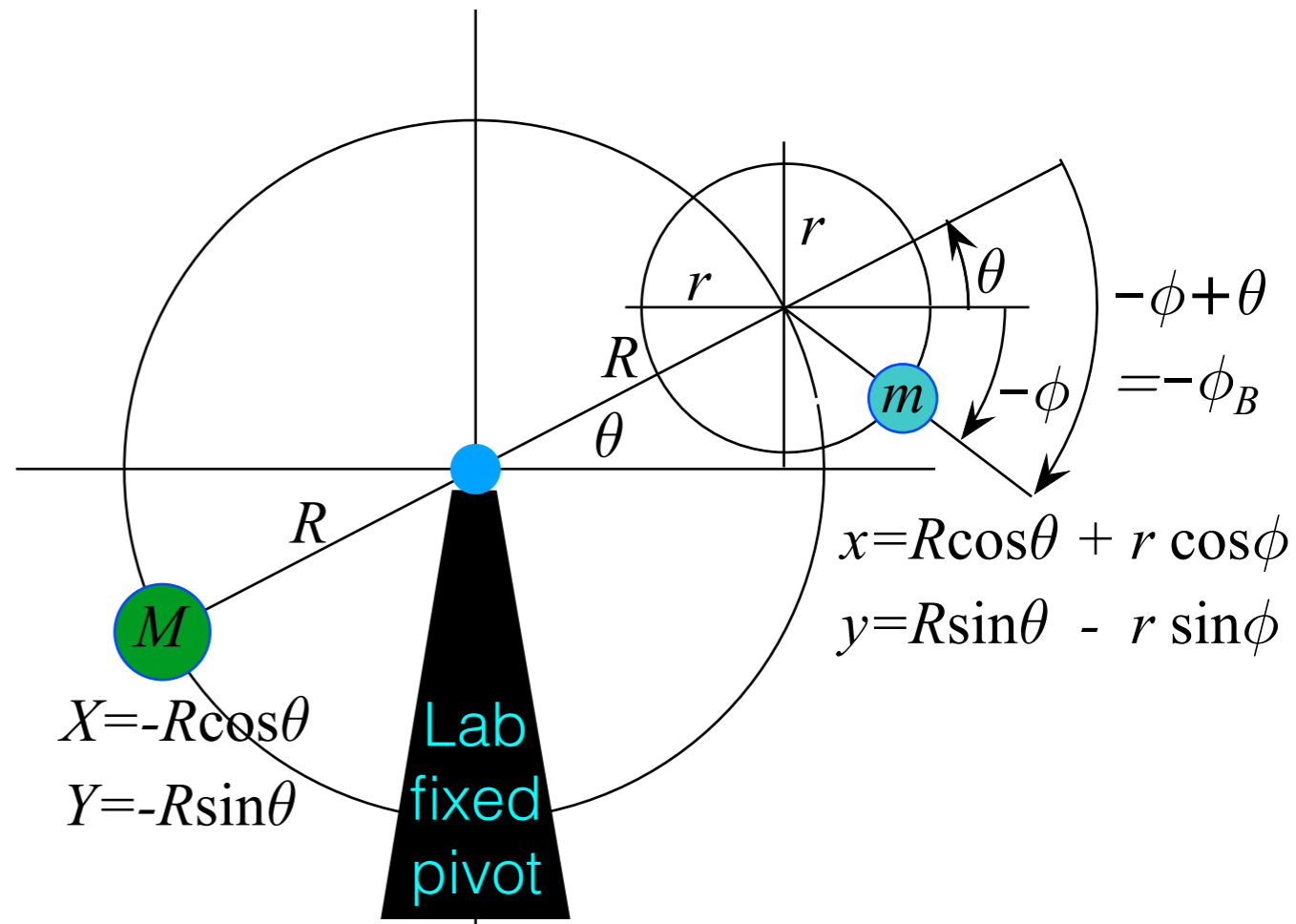
Lab
fixed
pivot

$$x = R \cos \theta + r \cos \phi$$

$$y = R \sin \theta + r \sin \phi$$

$$\dot{x} = -R \sin \theta \dot{\theta} - r \sin \phi \dot{\phi}$$

$$\dot{y} = +R \cos \theta \dot{\theta} + r \cos \phi \dot{\phi}$$



$$X = -R \cos \theta$$

$$Y = -R \sin \theta$$

Lab
fixed
pivot

$$x = R \cos \theta + r \cos \phi$$

$$y = R \sin \theta - r \sin \phi$$

$$\dot{X} = +R \sin \theta \dot{\theta}$$

$$\dot{Y} = +R \cos \theta \dot{\theta}$$

$$-\phi + \theta = -\phi_B$$

$$\dot{X}^2 = +R^2 \sin^2 \theta \dot{\theta}^2$$

$$\dot{Y}^2 = +R^2 \cos^2 \theta \dot{\theta}^2$$

$$\dot{X}^2 + \dot{Y}^2 = R^2 \dot{\theta}^2$$

$$\dot{x}^2 = (-R \sin \theta \dot{\theta} - r \sin \phi \dot{\phi})(-R \sin \theta \dot{\theta} - r \sin \phi \dot{\phi}) = R^2 \sin^2 \theta \dot{\theta}^2 + 2Rr \sin \theta \sin \phi \dot{\theta} \dot{\phi} + r^2 \sin^2 \phi \dot{\phi}^2$$

$$\dot{y}^2 = (R \cos \theta \dot{\theta} + r \cos \phi \dot{\phi})(+R \cos \theta \dot{\theta} + r \cos \phi \dot{\phi}) = R^2 \cos^2 \theta \dot{\theta}^2 + 2Rr \cos \theta \cos \phi \dot{\theta} \dot{\phi} + r^2 \cos^2 \phi \dot{\phi}^2$$

$$\dot{x}^2 + \dot{y}^2 = R^2 \dot{\theta}^2 + 2Rr \cos(\theta - \phi) \dot{\theta} \dot{\phi} + r^2 \dot{\phi}^2$$

KE of driver M plus projectile m is explicit function of beam-relative coordinate $-\phi_B = (\theta - \phi)$ only

$$T = \frac{1}{2} M (\dot{X}^2 + \dot{Y}^2) + \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) = \frac{1}{2} M (R^2 \dot{\theta}^2) + \frac{1}{2} m (R^2 \dot{\theta}^2 + 2Rr \cos(\theta - \phi) \dot{\theta} \dot{\phi} + r^2 \dot{\phi}^2)$$

$$T = \frac{1}{2} M (\dot{X}^2 + \dot{Y}^2) + \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) = \frac{1}{2} (M + m) (R^2 \dot{\theta}^2) + mRr \cos(\theta - \phi) \dot{\theta} \dot{\phi} + \frac{1}{2} m r^2 \dot{\phi}^2$$

For symmetric ($R=r$) trebuchet: $T = R^2 \left[\frac{M+m}{2} \dot{\theta}^2 + \frac{m}{2} \dot{\phi}^2 + m \dot{\theta} \dot{\phi} \cos \phi_B \right]$

Force, Work, and Acceleration

$$dW = F_X dX + F_Y dY + F_x dx + F_y dy$$

$$= M\ddot{X} dX + M\ddot{Y} dY + m\ddot{x} dx + m\ddot{y} dy$$

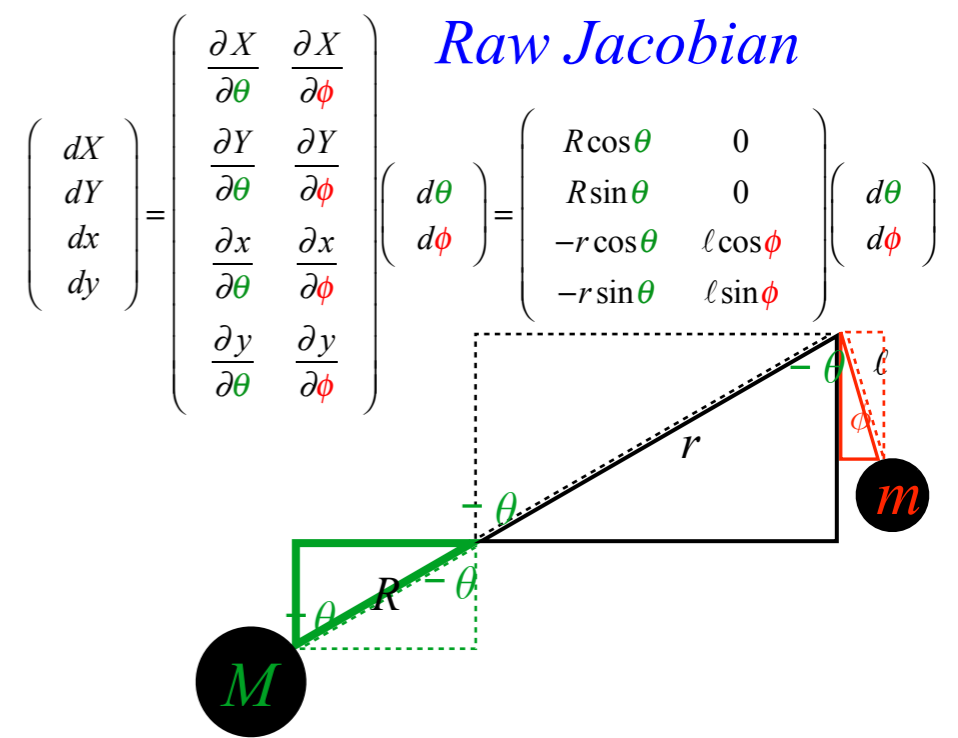
Write work-sums in columns: (Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$dW = F_X dX = M\ddot{X} dX = F_X \frac{\partial X}{\partial \theta} d\theta + F_X \frac{\partial X}{\partial \phi} d\phi = M\ddot{X} \frac{\partial X}{\partial \theta} d\theta + M\ddot{X} \frac{\partial X}{\partial \phi} d\phi$$

$$+ F_Y dY + M\ddot{Y} dY + F_Y \frac{\partial Y}{\partial \theta} d\theta + F_Y \frac{\partial Y}{\partial \phi} d\phi + M\ddot{Y} \frac{\partial Y}{\partial \theta} d\theta + M\ddot{Y} \frac{\partial Y}{\partial \phi} d\phi$$

$$+ F_x dx + m\ddot{x} dx + F_x \frac{\partial x}{\partial \theta} d\theta + F_x \frac{\partial x}{\partial \phi} d\phi + m\ddot{x} \frac{\partial x}{\partial \theta} d\theta + m\ddot{x} \frac{\partial x}{\partial \phi} d\phi$$

$$+ F_y dy + m\ddot{y} dy + F_y \frac{\partial y}{\partial \theta} d\theta + F_y \frac{\partial y}{\partial \phi} d\phi + m\ddot{y} \frac{\partial y}{\partial \theta} d\theta + m\ddot{y} \frac{\partial y}{\partial \phi} d\phi$$



Lagrange trickery:

STEP D Add up first and last columns for each variable θ and ϕ for: $T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$

Let: $F_X \frac{\partial X}{\partial \theta} + F_Y \frac{\partial Y}{\partial \theta} + F_x \frac{\partial x}{\partial \theta} + F_y \frac{\partial y}{\partial \theta} \equiv F_\theta = \frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta}$

Let: $F_X \frac{\partial X}{\partial \phi} + F_Y \frac{\partial Y}{\partial \phi} + F_x \frac{\partial x}{\partial \phi} + F_y \frac{\partial y}{\partial \phi} \equiv F_\phi = \frac{d}{dt} \frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi}$

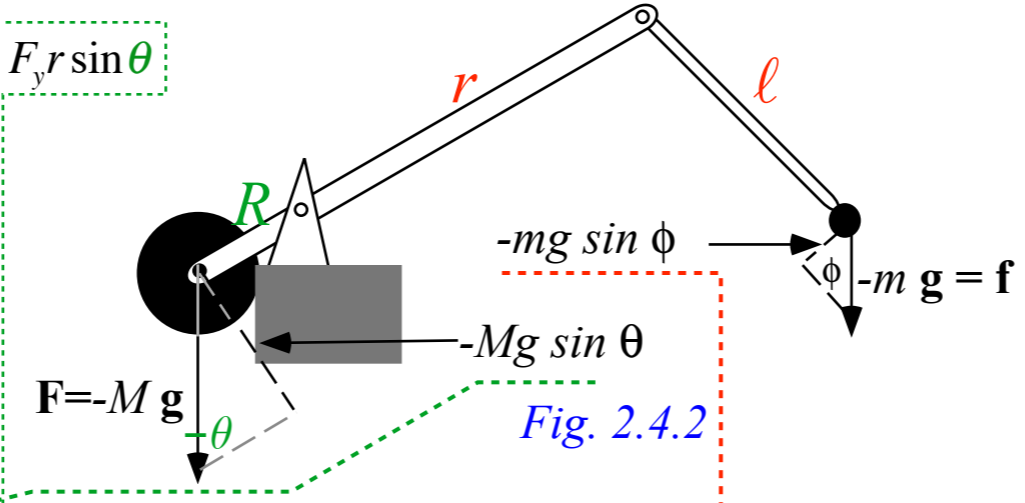
Completes derivation of Lagrange covariant-force equation for each GCC variable θ and ϕ .

$$F_X R \cos \theta + F_Y R \sin \theta - F_x r \cos \theta - F_y r \sin \theta$$

$$\equiv F_\theta = \frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta}$$

Add F_θ gravity given
 $(F_X = 0, F_Y = -Mg)$
 $(F_x = 0, F_y = -mg)$

$$F_\theta = \frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta} = -MgR \sin \theta + mgr \sin \theta$$



$$F_X \cdot 0 + F_Y \cdot 0 + F_x \ell \cos \phi + F_y \ell \sin \phi$$

$$\equiv F_\phi = \frac{d}{dt} \frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi}$$

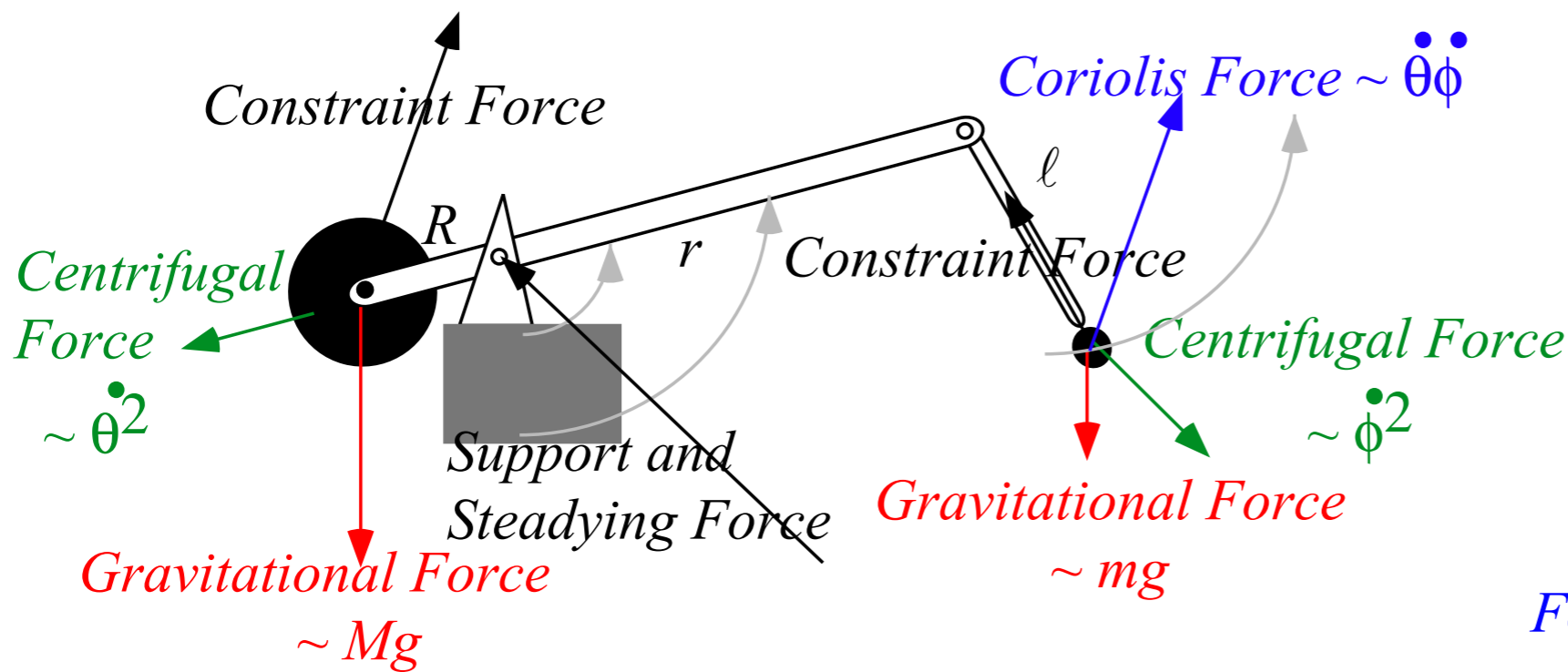
Add F_ϕ gravity given
 $(F_X = 0, F_Y = -Mg)$
 $(F_x = 0, F_y = -mg)$

$$F_\phi = \frac{d}{dt} \frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi} = -mgl \sin \phi$$

These are competing torques on main beam R...

... and a torque on throwing lever ℓ

Forces: total, genuine, potential, and/or fictitious



Acceleration and 'Fictitious' Forces:

*Coriolis
Centrifugal*

*Applied 'Real' Forces:
Gravity
Stimuli
Friction...*

*Constraint 'Internal' Forces:
Stresses
Support...
(Do not contribute.
Do no work.)*

For conservative forces

where: $F_{\theta} = -\frac{\partial V}{\partial \theta}$ and: $\frac{\partial V}{\partial \dot{\theta}} = 0$
 $F_{\phi} = -\frac{\partial V}{\partial \phi}$ and: $\frac{\partial V}{\partial \dot{\phi}} = 0$

$$\dot{p}_{\theta} = \frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial T}{\partial \theta} + F_{\theta} + 0$$

$$\dot{p}_{\phi} = \frac{d}{dt} \frac{\partial T}{\partial \dot{\phi}} = \frac{\partial T}{\partial \phi} + F_{\phi} + 0$$

Lagrange Force equations
 (See also derivation Eq. (2.4.7) on p. 23, Unit 2)

$$p_{\theta} = \frac{\partial L}{\partial \dot{\theta}} \quad \dot{p}_{\theta} = \frac{\partial L}{\partial \theta}$$

$$p_{\phi} = \frac{\partial L}{\partial \dot{\phi}} \quad \dot{p}_{\phi} = \frac{\partial L}{\partial \phi}$$

Lagrange Potential equations
 $L = T - V$

Fig. 2.5.2 (modified)

Compare to derivation Eq (12.25a) in Ch. 12 of Unit 1 and Eq. (3.5.10) in Unit 3.