Lecture 17
Mon 10.22.2018

Reimann-Christoffel equations and covariant derivative
(Ch. 4-7 of Unit 3)

Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;}^{k}$

Christoffel g-derivative formula
What's a tensor? What's not?

General Riemann equations of motion (No explicit t-dependence and fixed GCC)

Riemann-forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)
Christoffel relation to Coriolis coefficients
Mechanics of ideal fluid vortex

Separation of GCC Equations: Effective Potentials

Small (n_ρ:m_ϕ)-periodic and quasi-periodic oscillations
2D Spherical pendulum "Bowl-Bowling" and the "I-Ball"
(n_ρ:m_ϕ)=(2:1) vs (1:1) periodic and quasi-periodic orbits

Cycloidal ruler&compass geometry

(To be applied later to mechanics in electromagnetic fields.)

A running collection of links to course-relevant sites and articles

Physics Web Resources

Comprehensive Harter-Soft Resource Listing

UAF Physics YouTube channel

LearnIt Physics Web Applications

"Texts"

Classical Mechanics with a Bang!

Quantum Theory for the Computer Age

Principles of Symmetry, Dynamics, and Spectroscopy

Modern Physics and its Classical Foundations

Classes

2014 AMOP

2017 Group Theory for QM

2018 AMOP

2018 Adv Mechanics

Neat external material to start the class:

- AIP publications
- AJP article on superball dynamics
- AAPT summer reading

These *are* hot off the presses:

- Sorting ultracold atoms in a three-dimensional optical lattice in a realization of Maxwell's demon - Kumar-Nature-Letters-2018
- Synthetic three-dimensional atomic structures assembled atom by atom - Berredo-Nature-Letters-2018

Slightly Older ones:

- Wave–particle duality of C60 molecules
- Optical vortex knots – One Photon at a Time

"*Relawavity*" and quantum basis of *Lagrangian* & *Hamiltonian* mechanics:

- 2-CW laser wave - BohrIt Web App
- Lagrangian vs Hamiltonian - RelaWavity Web App

- AMOP Ch 0 Space-Time Symmetry - 2019
- Seminar at Rochester Institute of Optics, Auxiliary slides, June 19, 2018

Previous Links from Lectures 14-16

- http://thearmchaircritic.blogspot.com/2011/11/punkin-chunkin.html
- http://www.sussexcountyonline.com/news/photos/punkinchunkin.html
- http://www.twcenter.net/forums/showthread.php?358315-Shooting-range-for-medieval-siege-weapons-Anybody-knows
- https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html
- https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html?scenario=MontezumasRevenge
- https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html?scenario=SeigeOfKenilworth
- https://modphys.hosted.uark.edu/pdfs/Journal_Pdfs/Trebuchet-SciAm_273_66_July_1995_chevedden1.pdf

Links to supplement Lecture 17

- '*Simple*' *Pendulum* Sim: https://modphys.hosted.uark.edu/markup/PendulumWeb.html
- '*Cycloid*' *Pendulum*: https://modphys.hosted.uark.edu/markup/CycloidulumWeb.html
- *Google* search on: "Satelite view of Patricia" (Images)
- *Physics Girl* Channel - Fun with Vortex Rings in the Pool: https://www.youtube.com/watch?v=72LWr7BU8Ao
- *iBall* demo - Quasi-periodicity: https://youtu.be/_jntDtULxDc

AnalyIt Web Application, posted 10/22/2018 in our *testing* area:
https://modphys.hosted.uark.edu/testing/markup/AnalyItBJS.html

→ *Covariant derivative and Christoffel Coefficients* $\Gamma_{ij;k}$ *and* $\Gamma_{ij;}{}^{k}$

Christoffel g-derivative formula
What's a tensor? What's not?

Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ *and* $\Gamma_{ij;}{}^{k}$

GCC q^m derivatives of vectors $\mathbf{U}$ are due to:

(1) changing U^m components *(2)* curving GCC vectors $\mathbf{E_n}$.

$$\frac{\partial \mathbf{U}}{\partial q^i} = \frac{\partial}{\partial q^i}\left(U^j \mathbf{E}_j\right) = \boxed{\frac{\partial U^m}{\partial q^i}}\left(\mathbf{E}_m\right) + U^n \boxed{\frac{\partial \mathbf{E}_n}{\partial q^i}}$$

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(Note funny semi-colon ; notation)

Derivative of $\mathbf{E}_n$ is expressed using $\mathbf{E}^\ell$ or else $\mathbf{E}_m$

$$\frac{\partial \mathbf{E_n}}{\partial q^i} = \Gamma_{in;\ell}\,\mathbf{E}^\ell$$

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Christoffel coefficients $\Gamma_{ij;k}$ *of the first kind*

defined by:

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$\Gamma_{in;\ell} = g_{m\ell}\Gamma_{in;}^{\ m}$ since: $\mathbf{E}_\ell = g_{m\ell}\mathbf{E}^m$

$\Gamma_{in;}^{\ m} = g^{m\ell}\Gamma_{in;\ell}$ since: $\mathbf{E}^m = g^{m\ell}\mathbf{E}_\ell$

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i,n to n,i symmetry guaranteed here

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Q: Do we need a third kind of Γ-coefficient or a Λ-coefficient?

$$\frac{\partial \mathbf{E}^n}{\partial q^i} = \Lambda^n_{im}\mathbf{E}^m \text{ , where: } \Lambda^n_{im} = \frac{\partial \mathbf{E}^n}{\partial q^i} \bullet \mathbf{E}_m$$

(to differentiate *contra*variant-$\mathbf{E}^n$ or *co*variant U_n)

Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;}{}^{k}$

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Christoffel coefficients $\Gamma_{ij;}{}^{k}$ the second kind defined by:

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A: NO! That Λ-coefficient is just a Γ-coefficient with a (-).

$$0 = \frac{\partial\left(\delta^n_m\right)}{\partial q^i} = \frac{\partial\left(\mathbf{E}^n \bullet \mathbf{E}_m\right)}{\partial q^i} = \frac{\partial \mathbf{E}^n}{\partial q^i} \bullet \mathbf{E}_m + \mathbf{E}^n \bullet \frac{\partial \mathbf{E}_m}{\partial q^i}$$

So: $\Lambda^n_{im} = -\Gamma^n_{im}$

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Any vector derivative can be expressed using $\Gamma_{ij;}{}^{k}$ in terms of $\mathbf{E}_m$

$$\frac{\partial \mathbf{U}}{\partial q^i} = \left(\frac{\partial U^m}{\partial q^i} + U^n \Gamma_{in;}{}^{m}\right)\mathbf{E}_m$$

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Any vector derivative can be expressed using $\Gamma_{ij;}^{k}$ in terms of $\mathbf{E}_m$ or $\mathbf{E}^m$

$$\frac{\partial \mathbf{U}}{\partial q^i} = \left(\frac{\partial U^m}{\partial q^i} + U^n\Gamma_{in;}^{\ \ \ m}\right)\mathbf{E}_m = \left(\frac{\partial U_m}{\partial q^i} - U_n\Gamma_{im;}^{\ \ \ n}\right)\mathbf{E}^m$$

$$\frac{\partial \mathbf{E}^n}{\partial q^i}\bullet\mathbf{E}_m = -\mathbf{E}^n\bullet\frac{\partial \mathbf{E}_m}{\partial q^i}$$

So: $\Lambda_{im}^{n} = -\Gamma_{im}^{n}$

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$$= U^m_{\;;i}\,\mathbf{E}_m = U_{m;i}\,\mathbf{E}^m$$

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So: $\Lambda^n_{im} = -\Gamma^n_{im}$

(Note more funny semi-colon ; notation)

Defining *covariant derivative* $U^m_{\;;i}$ *of a contravariant component* U^m

$$U^m_{\;\;;i} = \frac{\partial U^m}{\partial q^i} + U^n \Gamma_{in;}^{\;\;\;m} \equiv \frac{\partial \mathbf{U}}{\partial q^i}\bullet \mathbf{E}^m$$

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$$= U^m{}_{;i}\,\mathbf{E}_m = U_{m;i}\,\mathbf{E}^m$$

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Defining *covariant derivative* $U^m{}_{;i}$ *of a contravariant component* U^m

$$U^m{}_{;i} = \frac{\partial U^m}{\partial q^i} + U^n \Gamma_{in;}{}^{m} \equiv \frac{\partial \mathbf{U}}{\partial q^i} \bullet \mathbf{E}^m$$

...and *covariant derivative* $U_{m;i}$ *of a covariant component* U_m

$$U_{m;i} = \frac{\partial U_m}{\partial q^i} - U_n \Gamma_{im;}{}^{n} \equiv \frac{\partial \mathbf{U}}{\partial q^i} \bullet \mathbf{E}_m$$

Intrinsic derivatives:

(Mathematicians being cute)

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Defining *intrinsic derivative of contravariant vector components*.

$$\frac{\delta V^k}{\delta t} = \frac{dV^k}{dt} + \Gamma_{mn}{}^{k} V^m \dot{q}^n = \frac{\partial V^k}{\partial q^n}\dot{q}^n + \Gamma_{mn}{}^{k} V^m \dot{q}^n$$

$$= V^k{}_{;n}\dot{q}^n$$

$$F^k = \frac{\delta p^k}{\delta t}$$ *"Cute" contravariant Newton II*

Defining *intrinsic derivative of covariant vector components*.

$$\frac{\delta V_k}{\delta t} = \frac{dV_k}{dt} - \Gamma_{kn}{}^{m} V_m \dot{q}^n = \frac{\partial V_k}{\partial q^n}\dot{q}^n - \Gamma_{kn}{}^{m} V_m \dot{q}^n$$

$$= V_{k\,;n}\dot{q}^n$$

$$F_k = \frac{\delta p_k}{\delta t}$$ *"Cute" covariant Newton II*

Tensor chain rules replace Cartesian chain rules.

$$\frac{\delta V^k}{\delta t} = V^k{}_{;n}\dot{q}^n$$ replaces: $$\frac{dV^k}{dt} = \frac{\partial V^k}{\partial q^n}\dot{q}^n$$

where: $$V^k{}_{;n} = \frac{\partial V^k}{\partial q^n} + \Gamma_{mn}{}^{k} V^m$$

$$\frac{\delta V_k}{\delta t} = V_{k\,;n}\dot{q}^n$$ replaces: $$\frac{dV_k}{dt} = \frac{\partial V_k}{\partial q^n}\dot{q}^n$$

where: $$V_{k\,;n} = \frac{\partial V_k}{\partial q^n} - \Gamma_{kn}{}^{m} V_m$$

contravariant *versus* *covariant*

Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ *and* $\Gamma_{ij;}{}^{k}$

→ *Christoffel g-derivative formula*
What's a tensor? What's not?

Christoffel ($g_{mn}=\mathbf{E_m}\bullet\mathbf{E_n}$)*-derivative formula*

$$\frac{\partial\left(\mathbf{E_m}\bullet\mathbf{E_n}\right)}{\partial q^i}=\frac{\partial\mathbf{E_m}}{\partial q^i}\bullet\mathbf{E_n}+\mathbf{E_m}\bullet\frac{\partial\mathbf{E_n}}{\partial q^i}$$

$$\frac{\partial g_{mn}}{\partial q^i}=\quad\Gamma_{i\,m;\,n}\quad+\quad\Gamma_{i\,n;\,m}$$

Christoffel ($g_{mn}=\mathbf{E_m}\cdot\mathbf{E_n}$)*-derivative formula*

$$\frac{\partial\left(\mathbf{E_m}\bullet\mathbf{E_n}\right)}{\partial q^i}=\frac{\partial\mathbf{E_m}}{\partial q^i}\bullet\mathbf{E_n}+\mathbf{E_m}\bullet\frac{\partial\mathbf{E_n}}{\partial q^i}$$

$$\frac{\partial g_{mn}}{\partial q^i}=\Gamma_{im;n}+\Gamma_{in;m}$$

$$\frac{\partial g_{mi}}{\partial q^n}=\Gamma_{nm;i}+\Gamma_{in;m}\qquad\text{(switched i}\leftrightarrow\text{n)}$$

$$\frac{\partial g_{in}}{\partial q^m}=\Gamma_{im;n}+\Gamma_{mn;i}\qquad\text{(switched i}\leftrightarrow\text{m)}$$

$$\textit{Christoffel symmetry }\Gamma_{in;\alpha}=\Gamma_{ni;\alpha}$$

due to:

$$\frac{\partial\mathbf{E_n}}{\partial q^i}=\frac{\partial^2\mathbf{r}}{\partial q^i\,\partial q^n}=\frac{\partial^2\mathbf{r}}{\partial q^n\,\partial q^i}=\frac{\partial\mathbf{E_i}}{\partial q^n}$$

Christoffel ($g_{mn}=\mathbf{E_m}\cdot\mathbf{E_n}$)*-derivative formula*

$$\frac{\partial\left(\mathbf{E_m}\bullet\mathbf{E_n}\right)}{\partial q^i}=\frac{\partial\mathbf{E_m}}{\partial q^i}\bullet\mathbf{E_n}+\mathbf{E_m}\bullet\frac{\partial\mathbf{E_n}}{\partial q^i}$$

$$\frac{\partial g_{mn}}{\partial q^i}=\Gamma_{im;n}+\Gamma_{in;m}$$

$$-\frac{\partial g_{mi}}{\partial q^n}=-\Gamma_{nm;i}-\Gamma_{in;m}\quad\text{(switched i}\leftrightarrow\text{n)}$$

$$\frac{\partial g_{in}}{\partial q^m}=\Gamma_{im;n}+\Gamma_{mn;i}\quad\text{(switched i}\leftrightarrow\text{m)}$$

Christoffel symmetry $\Gamma_{in;\alpha}=\Gamma_{ni;\alpha}$ *due to:*

$$\frac{\partial\mathbf{E_n}}{\partial q^i}=\frac{\partial^2\mathbf{r}}{\partial q^i\,\partial q^n}=\frac{\partial^2\mathbf{r}}{\partial q^n\,\partial q^i}=\frac{\partial\mathbf{E_i}}{\partial q^n}$$

Christoffel ($g_{mn}=\mathbf{E_m}\cdot\mathbf{E_n}$)*-derivative formula*

$$\frac{\partial\left(\mathbf{E_m}\bullet\mathbf{E_n}\right)}{\partial q^i}=\frac{\partial\mathbf{E_m}}{\partial q^i}\bullet\mathbf{E_n}+\mathbf{E_m}\bullet\frac{\partial\mathbf{E_n}}{\partial q^i}$$

$$\frac{\partial g_{mn}}{\partial q^i}=\Gamma_{im;n}+\cancel{\Gamma_{in;m}}$$

$$-\frac{\partial g_{mi}}{\partial q^n}=-\cancel{\Gamma_{nm;i}}-\cancel{\Gamma_{in;m}}\quad\text{(switched i}\leftrightarrow\text{n)}$$

$$\frac{\partial g_{in}}{\partial q^m}=\Gamma_{im;n}+\cancel{\Gamma_{mn;i}}\quad\text{(switched i}\leftrightarrow\text{m)}$$

Christoffel symmetry $\Gamma_{in;\alpha}=\Gamma_{ni;\alpha}$

due to:

$$\frac{\partial\mathbf{E_n}}{\partial q^i}=\frac{\partial^2\mathbf{r}}{\partial q^i\,\partial q^n}=\frac{\partial^2\mathbf{r}}{\partial q^n\,\partial q^i}=\frac{\partial\mathbf{E_i}}{\partial q^n}$$

Christoffel ($g_{mn}=\mathbf{E_m}\cdot\mathbf{E_n}$)*-derivative formula*

$$\frac{\partial\left(\mathbf{E_m}\bullet\mathbf{E_n}\right)}{\partial q^i}=\frac{\partial\mathbf{E_m}}{\partial q^i}\bullet\mathbf{E_n}+\mathbf{E_m}\bullet\frac{\partial\mathbf{E_n}}{\partial q^i}$$

$$\frac{\partial g_{mn}}{\partial q^i}=\Gamma_{im;n}+\cancel{\Gamma_{in;m}}$$

$$-\frac{\partial g_{mi}}{\partial q^n}=\cancel{-\Gamma_{nm;i}}\cancel{-\Gamma_{in;m}}$$ (switched i ↔ n)

$$\frac{\partial g_{in}}{\partial q^m}=\Gamma_{im;n}+\cancel{\Gamma_{mn;i}}$$ (switched i ↔ m)

Christoffel symmetry $\Gamma_{in;\alpha}=\Gamma_{ni;\alpha}$
due to:

$$\frac{\partial\mathbf{E_n}}{\partial q^i}=\frac{\partial^2\mathbf{r}}{\partial q^i\,\partial q^n}=\frac{\partial^2\mathbf{r}}{\partial q^n\,\partial q^i}=\frac{\partial\mathbf{E_i}}{\partial q^n}$$

Gives the Christoffel formula

$$\boxed{\Gamma_{im;n}=\frac{1}{2}\left(\frac{\partial g_{mn}}{\partial q^i}+\frac{\partial g_{in}}{\partial q^m}-\frac{\partial g_{im}}{\partial q^n}\right)}$$

Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;}{}^{k}$

Christoffel g-derivative formula

→ *What's a tensor? What's not?*

What's a tensor? What's not? $g_{mn} = \mathbf{E_m} \bullet \mathbf{E_n}$ is example of 2^{nd} rank covariant tensor

$$\frac{\partial(\mathbf{E_m} \bullet \mathbf{E_n})}{\partial q^i} = \frac{\partial \mathbf{E_m}}{\partial q^i} \bullet \mathbf{E_n} + \mathbf{E_m} \bullet \frac{\partial \mathbf{E_n}}{\partial q^i}$$

$$\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \cancel{\Gamma_{in;m}}$$

$$-\frac{\partial g_{mi}}{\partial q^n} = -\cancel{\Gamma_{nm;i}} - \cancel{\Gamma_{in;m}}$$ (switched i ↔ n)

$$\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \cancel{\Gamma_{mn;i}}$$ (switched i ↔ m)

… Christoffel coefficients…

$$\boxed{\Gamma_{im;n} = \frac{1}{2}\left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n}\right)}$$

What's a tensor? What's not? $g_{mn} = \mathbf{E_m} \bullet \mathbf{E_n}$ is example of 2nd rank covariant tensor whose transformation to "bar-frame" is *cov-cov* product found by chain-saw-sum

$$\frac{\partial(\mathbf{E_m} \bullet \mathbf{E_n})}{\partial q^i} = \frac{\partial \mathbf{E_m}}{\partial q^i} \bullet \mathbf{E_n} + \mathbf{E_m} \bullet \frac{\partial \mathbf{E_n}}{\partial q^i}$$

$$\bar{g}_{\bar{m}\bar{n}} = \bar{\mathbf{E}}_{\bar{\mathbf{m}}} \bullet \bar{\mathbf{E}}_{\bar{\mathbf{n}}} = \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}} \bullet \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{n}}} = \frac{\quad}{\partial \bar{q}^{\bar{m}}} \frac{\partial \mathbf{r}}{\quad} \bullet \frac{\partial \mathbf{r}}{\quad} \frac{\quad}{\partial \bar{q}^{\bar{n}}}$$

$$\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} \cancel{+ \Gamma_{in;m}}$$

$$-\frac{\partial g_{mi}}{\partial q^n} = \cancel{-\Gamma_{nm;i}} \cancel{-\Gamma_{in;m}}$$ (switched i ↔ n)

$$\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} \cancel{+ \Gamma_{mn;i}}$$ (switched i ↔ m)

… Christoffel coefficients…

$$\boxed{\Gamma_{im;n} = \frac{1}{2}\left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n}\right)}$$

What's a tensor? What's not? $g_{mn} = \mathbf{E_m} \bullet \mathbf{E_n}$ is example of 2^{nd} rank covariant tensor whose transformation to "bar-frame" is *cov-cov* product found by chain-saw-sum

$$\bar{g}_{\bar{m}\bar{n}} = \mathbf{\bar{E}_{\bar{m}}} \bullet \mathbf{\bar{E}_{\bar{n}}} = \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}} \bullet \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{n}}} = \frac{}{\partial \bar{q}^{\bar{m}}} \frac{\partial \mathbf{r}}{} \bullet \frac{\partial \mathbf{r}}{} \frac{}{\partial \bar{q}^{\bar{n}}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} \frac{\partial \mathbf{r}}{\partial q^m} \bullet \frac{\partial \mathbf{r}}{\partial q^n} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}}$$

$$\frac{\partial(\mathbf{E_m} \bullet \mathbf{E_n})}{\partial q^i} = \frac{\partial \mathbf{E_m}}{\partial q^i} \bullet \mathbf{E_n} + \mathbf{E_m} \bullet \frac{\partial \mathbf{E_n}}{\partial q^i}$$

$$\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \cancel{\Gamma_{in;m}}$$

$$-\frac{\partial g_{mi}}{\partial q^n} = -\cancel{\Gamma_{nm;i}} - \cancel{\Gamma_{in;m}} \quad \text{(switched i} \leftrightarrow \text{n)}$$

$$\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \cancel{\Gamma_{mn;i}} \quad \text{(switched i} \leftrightarrow \text{m)}$$

... Christoffel coefficients...

$$\boxed{\Gamma_{im;n} = \frac{1}{2}\left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n} \right)}$$

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$$\frac{\partial\left(\mathbf{E_m}\bullet\mathbf{E_n}\right)}{\partial q^i}=\frac{\partial\mathbf{E_m}}{\partial q^i}\bullet\mathbf{E_n}+\mathbf{E_m}\bullet\frac{\partial\mathbf{E_n}}{\partial q^i}$$

$$\frac{\partial g_{mn}}{\partial q^i}=\Gamma_{im;n}+\cancel{\Gamma_{in;m}}$$

$$-\frac{\partial g_{mi}}{\partial q^n}=-\cancel{\Gamma_{nm;i}}-\cancel{\Gamma_{in;m}}\qquad\text{(switched i}\leftrightarrow\text{n)}$$

$$\frac{\partial g_{in}}{\partial q^m}=\Gamma_{im;n}+\cancel{\Gamma_{mn;i}}\qquad\text{(switched i}\leftrightarrow\text{m)}$$

$$g_{\bar m\bar n}=\mathbf{\bar E_{\bar m}}\bullet\mathbf{\bar E_{\bar n}}=\frac{\partial\mathbf{r}}{\partial\bar q^{\bar m}}\bullet\frac{\partial\mathbf{r}}{\partial\bar q^{\bar n}}=\frac{\quad}{\partial\bar q^{\bar m}}\frac{\partial\mathbf{r}}{\quad}\bullet\frac{\partial\mathbf{r}}{\quad}\frac{\quad}{\partial\bar q^{\bar n}}=\frac{\partial q^m}{\partial\bar q^{\bar m}}\frac{\partial\mathbf{r}}{\partial q^m}\bullet\frac{\partial\mathbf{r}}{\partial q^n}\frac{\partial q^n}{\partial\bar q^{\bar n}}$$

$$=\frac{\partial q^m}{\partial\bar q^{\bar m}}\;g_{mn}\;\frac{\partial q^n}{\partial\bar q^{\bar n}}$$

$$\boxed{\bar g_{\bar m\bar n}=g_{mn}\frac{\partial q^m}{\partial\bar q^{\bar m}}\frac{\partial q^n}{\partial\bar q^{\bar n}}}$$

…but Christoffel coefficients…

$$\boxed{\Gamma_{im;n}=\frac{1}{2}\left(\frac{\partial g_{mn}}{\partial q^i}+\frac{\partial g_{in}}{\partial q^m}-\frac{\partial g_{im}}{\partial q^n}\right)}$$

…are NOT tensorial

What's a tensor? What's not? $g_{mn} = \mathbf{E_m} \bullet \mathbf{E_n}$ is example of 2nd rank covariant tensor whose transformation to "bar-frame" is *cov-cov* product found by chain-saw-sum

$$\frac{\partial(\mathbf{E_m} \bullet \mathbf{E_n})}{\partial q^i} = \frac{\partial \mathbf{E_m}}{\partial q^i} \bullet \mathbf{E_n} + \mathbf{E_m} \bullet \frac{\partial \mathbf{E_n}}{\partial q^i}$$

$$\bar{g}_{\bar{m}\bar{n}} = \bar{\mathbf{E}}_{\bar{\mathbf{m}}} \bullet \bar{\mathbf{E}}_{\bar{\mathbf{n}}} = \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}} \bullet \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{n}}} = \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}} \bullet \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{n}}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} \frac{\partial \mathbf{r}}{\partial q^m} \bullet \frac{\partial \mathbf{r}}{\partial q^n} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}}$$

$$= \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} \; g_{mn} \; \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}}$$

$$\boxed{\bar{g}_{\bar{m}\bar{n}} = g_{mn} \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}}}$$

$$\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \cancel{\Gamma_{in;m}}$$

$$-\frac{\partial g_{mi}}{\partial q^n} = -\cancel{\Gamma_{nm;i}} - \cancel{\Gamma_{in;m}} \quad \text{(switched i} \leftrightarrow \text{n)}$$

$$\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \cancel{\Gamma_{mn;i}} \quad \text{(switched i} \leftrightarrow \text{m)}$$

…but Christoffel coefficients…

$$\boxed{\Gamma_{im;n} = \frac{1}{2}\left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n}\right)}$$

…are NOT tensorial

C-saw-sum gets "bar-frame" view $\bar{U}^{\bar{m}}{}_{;\bar{n}} = \frac{\partial \mathbf{U}}{\partial \bar{q}^{\bar{n}}} \bullet \bar{\mathbf{E}}^{\bar{\mathbf{m}}}$ from covar-derivative $U^m{}_{;n} = \frac{\partial \mathbf{U}}{\partial q^n} \bullet \mathbf{E^m}$ of contravar-U^m:

$$\boxed{\bar{U}^{\bar{m}}{}_{;\bar{n}} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} U^m{}_{;n}}$$

How to derive this?

From p.13-14:

$$\frac{\partial \mathbf{U}}{\partial q^n} = \left(\frac{\partial U^m}{\partial q^n} + U^k \Gamma_{kn;}{}^m\right)\mathbf{E}_m$$

$$= U^m{}_{;n} \mathbf{E}_m$$

What's a tensor? What's not? $g_{mn}=\mathbf{E_m}\bullet\mathbf{E_n}$ is example of 2nd rank covariant tensor whose transformation to "bar-frame" is *cov-cov* product found by chain-saw-sum

$$\frac{\partial(\mathbf{E_m}\bullet\mathbf{E_n})}{\partial q^i}=\frac{\partial\mathbf{E_m}}{\partial q^i}\bullet\mathbf{E_n}+\mathbf{E_m}\bullet\frac{\partial\mathbf{E_n}}{\partial q^i}$$

$$\bar{g}_{\bar{m}\bar{n}}=\mathbf{\bar{E}_{\bar{m}}}\bullet\mathbf{\bar{E}_{\bar{n}}}=\frac{\partial\mathbf{r}}{\partial\bar{q}^{\bar{m}}}\bullet\frac{\partial\mathbf{r}}{\partial\bar{q}^{\bar{n}}}=\frac{\partial\mathbf{r}}{\partial\bar{q}^{\bar{m}}}\bullet\frac{\partial\mathbf{r}}{\partial\bar{q}^{\bar{n}}}=\frac{\partial q^m}{\partial\bar{q}^{\bar{m}}}\frac{\partial\mathbf{r}}{\partial q^m}\bullet\frac{\partial\mathbf{r}}{\partial q^n}\frac{\partial q^n}{\partial\bar{q}^{\bar{n}}}$$

$$=\frac{\partial q^m}{\partial\bar{q}^{\bar{m}}}\;g_{mn}\;\frac{\partial q^n}{\partial\bar{q}^{\bar{n}}}$$

$$\frac{\partial g_{mn}}{\partial q^i}=\Gamma_{im;n}\;\cancel{+\;\Gamma_{in;m}}$$

$$-\;\frac{\partial g_{mi}}{\partial q^n}=\cancel{-\Gamma_{nm;i}}\;\cancel{-\;\Gamma_{in;m}}\qquad\text{(switched i}\leftrightarrow\text{n)}$$

$$\frac{\partial g_{in}}{\partial q^m}=\Gamma_{im;n}\;\cancel{+\;\Gamma_{mn;i}}\qquad\text{(switched i}\leftrightarrow\text{m)}$$

$$\boxed{\bar{g}_{\bar{m}\bar{n}}=g_{mn}\frac{\partial q^m}{\partial\bar{q}^{\bar{m}}}\frac{\partial q^n}{\partial\bar{q}^{\bar{n}}}}$$

…but Christoffel coefficients…

$$\boxed{\Gamma_{im;n}=\frac{1}{2}\left(\frac{\partial g_{mn}}{\partial q^i}+\frac{\partial g_{in}}{\partial q^m}-\frac{\partial g_{im}}{\partial q^n}\right)}$$

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C-saw-sum gets "bar-frame" view $\bar{U}^{\bar{m}}{}_{;\bar{n}}=\frac{\partial\mathbf{U}}{\partial\bar{q}^{\bar{n}}}\bullet\mathbf{\bar{E}^{\bar{m}}}$ from covar-derivative $U^m{}_{;n}=\frac{\partial\mathbf{U}}{\partial q^n}\bullet\mathbf{E^m}$ of contravar-U^m:

$$\bar{U}^{\bar{m}}{}_{;\bar{n}}=\frac{\partial\mathbf{U}}{\partial\bar{q}^{\bar{n}}}\bullet\mathbf{\bar{E}}^{\bar{m}}=\frac{\partial\mathbf{U}}{\partial\bar{q}^{\bar{n}}}\bullet\mathbf{\bar{E}}^{\bar{m}}=\frac{\partial q^n}{\partial\bar{q}^{\bar{n}}}\frac{\partial\mathbf{U}}{\partial q^n}\bullet\mathbf{\bar{E}}^{\bar{m}}$$

$$\boxed{\bar{U}^{\bar{m}}{}_{;\bar{n}}=\frac{\partial\bar{q}^{\bar{m}}}{\partial q^m}\frac{\partial q^n}{\partial\bar{q}^{\bar{n}}}U^m{}_{;n}}$$

How to derive this?

From p.13-16:

$$\frac{\partial\mathbf{U}}{\partial q^n}=\underbrace{\left(\frac{\partial U^m}{\partial q^n}+U^k\Gamma_{kn;}{}^m\right)}_{U^m{}_{;n}}\mathbf{E}_m$$

$$=U^m{}_{;n}\mathbf{E}_m$$

What's a tensor? What's not? $g_{mn}=\mathbf{E_m}\bullet\mathbf{E_n}$ is example of 2nd rank covariant tensor whose transformation to "bar-frame" is *cov-cov* product found by chain-saw-sum

$$\frac{\partial(\mathbf{E_m}\bullet\mathbf{E_n})}{\partial q^i}=\frac{\partial\mathbf{E_m}}{\partial q^i}\bullet\mathbf{E_n}+\mathbf{E_m}\bullet\frac{\partial\mathbf{E_n}}{\partial q^i}$$

$$\bar g_{\bar m\bar n}=\bar{\mathbf{E}}_{\bar{\mathbf m}}\bullet\bar{\mathbf{E}}_{\bar{\mathbf n}}=\frac{\partial\mathbf r}{\partial\bar q^{\bar m}}\bullet\frac{\partial\mathbf r}{\partial\bar q^{\bar n}}=\frac{\qquad}{\partial\bar q^{\bar m}}\frac{\partial\mathbf r}{\qquad}\bullet\frac{\partial\mathbf r}{\qquad}\frac{\qquad}{\partial\bar q^{\bar n}}=\frac{\partial q^m}{\partial\bar q^{\bar m}}\frac{\partial\mathbf r}{\partial q^m}\bullet\frac{\partial\mathbf r}{\partial q^n}\frac{\partial q^n}{\partial\bar q^{\bar n}}$$

$$=\frac{\partial q^m}{\partial\bar q^{\bar m}}\;g_{mn}\;\frac{\partial q^n}{\partial\bar q^{\bar n}}$$

$$\frac{\partial g_{mn}}{\partial q^i}=\Gamma_{im;n}+\cancel{\Gamma_{in;m}}$$

$$-\frac{\partial g_{mi}}{\partial q^n}=-\cancel{\Gamma_{nm;i}}-\cancel{\Gamma_{in;m}}\quad\text{(switched i}\leftrightarrow\text{n)}$$

$$\frac{\partial g_{in}}{\partial q^m}=\Gamma_{im;n}+\cancel{\Gamma_{mn;i}}\quad\text{(switched i}\leftrightarrow\text{m)}$$

$$\boxed{\bar g_{\bar m\bar n}=g_{mn}\frac{\partial q^m}{\partial\bar q^{\bar m}}\frac{\partial q^n}{\partial\bar q^{\bar n}}}$$

…but Christoffel coefficients…

$$\boxed{\Gamma_{im;n}=\frac{1}{2}\left(\frac{\partial g_{mn}}{\partial q^i}+\frac{\partial g_{in}}{\partial q^m}-\frac{\partial g_{im}}{\partial q^n}\right)}$$

…are NOT tensorial

C-saw-sum gets "bar-frame" view $\bar U^{\bar m}{}_{;\bar n}=\frac{\partial\mathbf U}{\partial\bar q^{\bar n}}\bullet\bar{\mathbf E}^{\bar{\mathbf m}}$ from covar-derivative $U^m{}_{;n}=\frac{\partial\mathbf U}{\partial q^n}\bullet\mathbf E^{\mathbf m}$ of contravar-U^m:

$$\bar U^{\bar m}{}_{;\bar n}=\frac{\partial\mathbf U}{\partial\bar q^{\bar n}}\bullet\bar{\mathbf E}^{\bar m}=\frac{\qquad}{\partial\bar q^{\bar n}}\frac{\partial\mathbf U}{\qquad}\bullet\bar{\mathbf E}^{\bar m}=\frac{\partial q^n}{\partial\bar q^{\bar n}}\frac{\partial\mathbf U}{\partial q^n}\bullet\bar{\mathbf E}^{\bar m}=\frac{\partial q^n}{\partial\bar q^{\bar n}}\frac{\partial\mathbf U}{\partial q^n}\bullet\frac{\partial\bar q^{\bar m}}{\partial q^m}\mathbf E^m=\frac{\partial q^n}{\partial\bar q^{\bar n}}\frac{\partial\bar q^{\bar m}}{\partial q^m}\frac{\partial\mathbf U}{\partial q^n}\bullet\mathbf E^m$$

$$\boxed{\bar U^{\bar m}{}_{;\bar n}=\frac{\partial\bar q^{\bar m}}{\partial q^m}\frac{\partial q^n}{\partial\bar q^{\bar n}}U^m{}_{;n}}$$

contravariant $\mathbf E$ transform

$$\bar{\mathbf E}^{\bar m}=\frac{\partial\bar q^{\bar m}}{\partial\mathbf r}=\frac{\partial\bar q^{\bar m}}{\qquad}\frac{\qquad}{\partial\mathbf r}=\frac{\partial\bar q^{\bar m}}{\partial q^m}\frac{\partial q^m}{\partial\mathbf r}=\frac{\partial\bar q^{\bar m}}{\partial q^m}\mathbf E^m$$

How to derive this?
…by C-saw sums…

From p.13-16:

$$\frac{\partial\mathbf U}{\partial q^n}=\Bigg(\underbrace{\frac{\partial U^m}{\partial q^n}+U^k\Gamma_{kn;}{}^m}\Bigg)\mathbf E_m$$

$$=\qquad U^m{}_{;n}\mathbf E_m$$

What's a tensor? What's not? $g_{mn} = \mathbf{E_m} \bullet \mathbf{E_n}$ is example of 2nd rank covariant tensor whose transformation to "bar-frame" is *cov-cov* product found by chain-saw-sum

$$\bar{g}_{\bar{m}\bar{n}} = \mathbf{E}_{\bar{\mathbf{m}}} \bullet \mathbf{E}_{\bar{\mathbf{n}}} = \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}} \bullet \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{n}}} = \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}} \bullet \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{n}}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} \frac{\partial \mathbf{r}}{\partial q^m} \bullet \frac{\partial \mathbf{r}}{\partial q^n} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} g_{mn} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}}$$

$$\boxed{\bar{g}_{\bar{m}\bar{n}} = g_{mn} \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}}}$$

$$\frac{\partial(\mathbf{E_m} \bullet \mathbf{E_n})}{\partial q^i} = \frac{\partial \mathbf{E_m}}{\partial q^i} \bullet \mathbf{E_n} + \mathbf{E_m} \bullet \frac{\partial \mathbf{E_n}}{\partial q^i}$$

$$\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m}$$

$$-\frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{nm;i} - \Gamma_{in;m} \quad \text{(switched i} \leftrightarrow \text{n)}$$

$$\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i} \quad \text{(switched i} \leftrightarrow \text{m)}$$

...but Christoffel coefficients...

$$\boxed{\Gamma_{im;n} = \frac{1}{2}\left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n} \right)}$$

...are NOT tensorial

C-saw-sum gets "bar-frame" view $\bar{U}_{\bar{m};\bar{n}} = \frac{\partial \mathbf{U}}{\partial \bar{q}^{\bar{n}}} \bullet \mathbf{E}_{\bar{\mathbf{m}}}$ from covar-derivative $U_{m;n} = \frac{\partial \mathbf{U}}{\partial q^n} \bullet \mathbf{E_m}$ of covar-U_m:

$$\bar{U}_{\bar{m};\bar{n}} = \frac{\partial \mathbf{U}}{\partial \bar{q}^{\bar{n}}} \bullet \mathbf{E}_{\bar{m}} = \frac{\partial \mathbf{U}}{\partial \bar{q}^{\bar{n}}} \bullet \mathbf{E}_{\bar{m}} = \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} \frac{\partial \mathbf{U}}{\partial q^n} \bullet \mathbf{E}_{\bar{m}} = \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} \frac{\partial \mathbf{U}}{\partial q^n} \bullet \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} \mathbf{E}_m = \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} \frac{\partial \mathbf{U}}{\partial q^n} \bullet \mathbf{E}_m$$

$$\boxed{\bar{U}_{\bar{m};\bar{n}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} U_{m;n}}$$

How to derive this?
...by C-saw sums...

covariant **E** transform

$$\mathbf{E}_{\bar{m}} = \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}} = \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} \frac{\partial \mathbf{r}}{\partial q^m} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} \mathbf{E}_m$$

From p.13-16:

$$\frac{\partial \mathbf{U}}{\partial q^n} = \underbrace{\left(\frac{\partial U^m}{\partial q^n} + U^k \Gamma_{kn;}^{\ \ \ m} \right)}_{U^m_{;n}} \mathbf{E}_m = \underbrace{\left(\frac{\partial U_m}{\partial q^n} - U_k \Gamma_{mn;}^{\ \ \ k} \right)}_{U_{m;n}} \mathbf{E}^m$$

$$= U^m_{;n} \mathbf{E}_m = U_{m;n} \mathbf{E}^m$$

What's a tensor? What's not?

$$\frac{\partial(\mathbf{E_m}\bullet\mathbf{E_n})}{\partial q^i}=\frac{\partial\mathbf{E_m}}{\partial q^i}\bullet\mathbf{E_n}+\mathbf{E_m}\bullet\frac{\partial\mathbf{E_n}}{\partial q^i}$$

$$\frac{\partial g_{mn}}{\partial q^i}=\Gamma_{im;n}+\Gamma_{in;m}$$

$$-\frac{\partial g_{mi}}{\partial q^n}=-\Gamma_{nm;i}-\Gamma_{in;m}$$

$$\frac{\partial g_{in}}{\partial q^m}=\Gamma_{im;n}+\Gamma_{mn;i}$$

…but Christoffel coefficients…

$$\boxed{\Gamma_{im;n}=\frac{1}{2}\left(\frac{\partial g_{mn}}{\partial q^i}+\frac{\partial g_{in}}{\partial q^m}-\frac{\partial g_{im}}{\partial q^n}\right)}$$

…are NOT tensorial

Chain-saw-sums transform a "bar-frame" view $\bar{U}^{\bar{m}}{}_{;\bar{n}}=\frac{\partial\bar{\mathbf{U}}}{\partial\bar{q}^{\bar{n}}}\bullet\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$ of covariant derivative $U^m{}_{;n}=\frac{\partial\mathbf{U}}{\partial q^n}\bullet\mathbf{E}^{\mathbf{m}}$

$$\bar{U}^{\bar{m}}{}_{;\bar{n}}=\frac{\partial\mathbf{U}}{\partial\bar{q}^{\bar{n}}}\bullet\bar{\mathbf{E}}^{\bar{m}}=\frac{\partial\mathbf{U}}{\partial\bar{q}^{\bar{n}}}\bullet\bar{\mathbf{E}}^{\bar{m}}=\frac{\partial q^n}{\partial\bar{q}^{\bar{n}}}\frac{\partial\mathbf{U}}{\partial q^n}\bullet\bar{\mathbf{E}}^{\bar{m}}=\frac{\partial q^n}{\partial\bar{q}^{\bar{n}}}\frac{\partial\mathbf{U}}{\partial q^n}\bullet\frac{\partial\bar{q}^{\bar{m}}}{\partial q^m}\mathbf{E}^m=\frac{\partial q^n}{\partial\bar{q}^{\bar{n}}}\frac{\partial\bar{q}^{\bar{m}}}{\partial q^m}\frac{\partial\mathbf{U}}{\partial q^n}\bullet\mathbf{E}^m$$

$$\boxed{\bar{U}^{\bar{m}}{}_{;\bar{n}}=\frac{\partial\bar{q}^{\bar{m}}}{\partial q^m}\frac{\partial q^n}{\partial\bar{q}^{\bar{n}}}U^m{}_{;n}}$$

The transformation of $U^m{}_{;n}=\frac{\partial U^m}{\partial q^n}+U^\ell\Gamma_{n\ell;}{}^m$ is that of mixed 2nd-rank tensor $T^m{}_n$

$$\boxed{T^{\bar{m}}{}_{\bar{n}}=\frac{\partial\bar{q}^{\bar{m}}}{\partial q^m}\frac{\partial q^n}{\partial\bar{q}^{\bar{n}}}T^m{}_n}$$

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The transformation of $U^m{}_{,n}=\frac{\partial U^m}{\partial q^n}$ is NOT so simple. At first it looks possible. $\frac{\partial\bar{U}^{\bar{m}}}{\partial\bar{q}^{\bar{n}}}=\frac{\quad}{\partial\bar{q}^{\bar{n}}}\frac{\partial\bar{U}^{\bar{m}}}{\quad}=\frac{\partial q^n}{\partial\bar{q}^{\bar{n}}}\frac{\partial\bar{U}^{\bar{m}}}{\partial q^n}$

But, still need to write $\frac{\partial\bar{U}^{\bar{m}}}{\partial q^n}$ in terms of $\frac{\partial U^m}{\partial q^n}$.

$$\frac{\partial\bar{U}^{\bar{m}}}{\partial\bar{q}^{\bar{n}}}=\frac{\partial q^n}{\partial\bar{q}^{\bar{n}}}\frac{\partial}{\partial q^n}\Bigg(\overbrace{\frac{\partial\bar{q}^{\bar{m}}}{\partial q^m}U^m}^{\text{standard contra-tran: }\bar{U}^{\bar{m}}}\Bigg)=\frac{\partial\bar{q}^{\bar{m}}}{\partial q^m}\frac{\partial q^n}{\partial\bar{q}^{\bar{n}}}\frac{\partial U^m}{\partial q^n}+U^m\frac{\partial}{\partial q^n}\left(\frac{\partial\bar{q}^{\bar{m}}}{\partial q^m}\right)$$

What's a tensor? What's not?

$$\frac{\partial(\mathbf{E_m}\bullet\mathbf{E_n})}{\partial q^i}=\frac{\partial\mathbf{E_m}}{\partial q^i}\bullet\mathbf{E_n}+\mathbf{E_m}\bullet\frac{\partial\mathbf{E_n}}{\partial q^i}$$

$$\frac{\partial g_{mn}}{\partial q^i}=\Gamma_{im;n}+\cancel{\Gamma_{in;m}}$$

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$$\boxed{\Gamma_{im;n}=\frac{1}{2}\left(\frac{\partial g_{mn}}{\partial q^i}+\frac{\partial g_{in}}{\partial q^m}-\frac{\partial g_{im}}{\partial q^n}\right)}$$

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$$\bar{U}^{\bar{m}}{}_{;\bar{n}}=\frac{\partial\bar{\mathbf{U}}}{\partial\bar{q}^{\bar{n}}}\bullet\bar{\mathbf{E}}^{\bar{m}}=\frac{\partial\bar{\mathbf{U}}}{\partial\bar{q}^{\bar{n}}}\bullet\bar{\mathbf{E}}^{\bar{m}}=\frac{\partial q^n}{\partial\bar{q}^{\bar{n}}}\frac{\partial\bar{\mathbf{U}}}{\partial q^n}\bullet\bar{\mathbf{E}}_{\bar{m}}=\frac{\partial q^n}{\partial\bar{q}^{\bar{n}}}\frac{\partial\bar{\mathbf{U}}}{\partial q^n}\bullet\frac{\partial q^m}{\partial\bar{q}^{\bar{m}}}\mathbf{E}_m$$

$$\boxed{\bar{U}^{\bar{m}}{}_{;\bar{n}}=\frac{\partial\bar{q}^{\bar{m}}}{\partial q^m}\frac{\partial q^n}{\partial\bar{q}^{\bar{n}}}U^m{}_{;n}}$$

The transformation of $U^m{}_{;n}=\frac{\partial U^m}{\partial q^n}+U^\ell\Gamma_{n\ell;}{}^m$ is that of mixed 2nd-rank tensor $T^m{}_n$

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The transformation of $U^m{}_{,n}=\frac{\partial U^m}{\partial q^n}$ is NOT so simple. At first it looks possible. $\frac{\partial\bar{U}^{\bar{m}}}{\partial\bar{q}^{\bar{n}}}=\frac{\partial\bar{U}^{\bar{m}}}{\partial\bar{q}^{\bar{n}}}=\frac{\partial q^n}{\partial\bar{q}^{\bar{n}}}\frac{\partial\bar{U}^{\bar{m}}}{\partial q^n}$

But, still need to write $\frac{\partial\bar{U}^{\bar{m}}}{\partial q^n}$ in terms of $\frac{\partial U^m}{\partial q^n}$.

$$\frac{\partial\bar{U}^{\bar{m}}}{\partial\bar{q}^{\bar{n}}}=\frac{\partial q^n}{\partial\bar{q}^{\bar{n}}}\frac{\partial}{\partial q^n}\Bigg(\overbrace{\frac{\partial\bar{q}^{\bar{m}}}{\partial q^m}U^m}^{\text{standard contra-tran: }\bar{U}^{\bar{m}}}\Bigg)=\underbrace{\frac{\partial\bar{q}^{\bar{m}}}{\partial q^m}\frac{\partial q^n}{\partial\bar{q}^{\bar{n}}}\frac{\partial U^m}{\partial q^n}}+\underbrace{U^m\frac{\partial}{\partial q^n}\left(\frac{\partial\bar{q}^{\bar{m}}}{\partial q^m}\right)}$$

$\frac{\partial\bar{U}^{\bar{m}}}{\partial\bar{q}^{\bar{n}}}=\frac{\partial\bar{q}^{\bar{m}}}{\partial q^m}\frac{\partial q^n}{\partial\bar{q}^{\bar{n}}}\frac{\partial U^m}{\partial q^n}$ holds if and only if $\frac{\partial}{\partial q^n}\left(\frac{\partial\bar{q}^{\bar{m}}}{\partial q^m}\right)\equiv 0$

1st term is OK, but 2nd term is zero only if Jacobian is constant matrix!

What's a tensor? What's not?

From p.13-14:

$$\frac{\partial \mathbf{U}}{\partial q^n} = \underbrace{\left(\frac{\partial U^m}{\partial q^n} + U^k \Gamma_{kn;}{}^m\right)}_{U^m{}_{;n}} \mathbf{E}_m = U^m{}_{;n}\mathbf{E}_m$$

$$\frac{\partial(\mathbf{E_m} \bullet \mathbf{E_n})}{\partial q^i} = \frac{\partial \mathbf{E_m}}{\partial q^i} \bullet \mathbf{E_n} + \mathbf{E_m} \bullet \frac{\partial \mathbf{E_n}}{\partial q^i}$$

$$\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \cancel{\Gamma_{in;m}}$$

$$-\frac{\partial g_{mi}}{\partial q^n} = -\cancel{\Gamma_{nm;i}} - \cancel{\Gamma_{in;m}}$$

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$$\boxed{\Gamma_{im;n} = \frac{1}{2}\left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n}\right)}$$

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$$\boxed{\bar{U}^{\bar{m}}{}_{;\bar{n}} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} U^m{}_{;n}}$$

The transformation of $U^m{}_{;n} = \frac{\partial U^m}{\partial q^n} + U^\ell \Gamma_{n\ell;}{}^m$ is that of mixed 2nd-rank tensor $T^m{}_n$

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But, still need to write $\frac{\partial \bar{U}^{\bar{m}}}{\partial q^n}$ in terms of $\frac{\partial U^m}{\partial q^n}$.

$$\frac{\partial \bar{U}^{\bar{m}}}{\partial \bar{q}^{\bar{n}}} = \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} \frac{\partial}{\partial q^n} \overbrace{\left(\frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} U^m\right)}^{\text{standard contra-tran: } \bar{U}^{\bar{m}}} = \underbrace{\frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} \frac{\partial U^m}{\partial q^n}} + U^m \frac{\partial}{\partial q^n} \underbrace{\left(\frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\right)}$$

$\frac{\partial \bar{U}^{\bar{m}}}{\partial \bar{q}^{\bar{n}}} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} \frac{\partial U^m}{\partial q^n}$ holds if and only if $\frac{\partial}{\partial q^n}\left(\frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\right) \equiv 0$

1st term is OK, but 2nd term is zero only if Jacobian is constant matrix!

Otherwise, $U^m{}_{,n}$ needs "Christoffel correction" $U^\ell \Gamma_{n\ell;}{}^m$. That $U^\ell \Gamma_{n\ell;}{}^m$ <u>*cannot*</u> be a $T_n{}^m$-tensor either!

General Riemann equations of motion *(No explicit t-dependence and fixed GCC)*

Riemann-forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

Christoffel relation to Coriolis coefficients

Mechanics of ideal fluid vortex

Riemann equations of motion *(No explicit t-dependence and fixed GCC)*

Kinetic metric γ_{mn} is a covariant tensor transform of an original Cartesian inertia tensor M_{ij}

$$\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n}$$ Converts Cartesian kinetic energy $T = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k$ to GCC $T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$

γ_{mn} replaces g_{mn}
when mass matrix
M_{ij} is involved

$$T = \frac{1}{2} M_{jk} \left(\frac{\partial x^j}{\partial q^m} \dot{q}^m + \cancel{\left\{ \frac{\partial x^j}{\partial t} \right\}} \right) \left(\frac{\partial x^k}{\partial q^n} \dot{q}^n + \cancel{\left\{ \frac{\partial x^k}{\partial t} \right\}} \right)$$

All explicit-t-dependent terms are zero

Riemann equations of motion *(No explicit t-dependence and fixed GCC)*

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Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell} = \frac{1}{2} \frac{d}{dt} \frac{\partial \left(\gamma_{mn} \dot{q}^m \dot{q}^n \right)}{\partial \dot{q}^\ell} - \frac{1}{2} \frac{\partial \left(\gamma_{mn} \dot{q}^m \dot{q}^n \right)}{\partial q^\ell}$$

$$T = \frac{1}{2} M_{jk} \left(\frac{\partial x^j}{\partial q^m} \dot{q}^m + \cancel{\left\{ \frac{\partial x^j}{\partial t} \right\}} \right) \left(\frac{\partial x^k}{\partial q^n} \dot{q}^n + \cancel{\left\{ \frac{\partial x^k}{\partial t} \right\}} \right)$$

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All explicit-t-dependent terms are zero
(Time must be included as a dimension)

1st term involves ***covariant momentum*** p_ℓ.

$$p_\ell \equiv \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2} \frac{\partial \left(\gamma_{mn} \dot{q}^m \dot{q}^n \right)}{\partial \dot{q}^\ell} = \gamma_{\ell n} \dot{q}^n$$

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$$F_\ell = \frac{d}{dt}\boxed{\frac{\partial T}{\partial \dot{q}^\ell}} - \frac{\partial T}{\partial q^\ell} = \frac{1}{2}\frac{d}{dt}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} - \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial q^\ell}$$

$$T = \frac{1}{2}M_{jk}\left(\frac{\partial x^j}{\partial q^m}\dot{q}^m + \cancel{\left\{\frac{\partial x^j}{\partial t}\right\}}\right)\left(\frac{\partial x^k}{\partial q^n}\dot{q}^n + \cancel{\left\{\frac{\partial x^k}{\partial t}\right\}}\right)$$

All explicit-t-dependent terms are zero
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1st term involves *covariant momentum* $\boxed{p_\ell}$.

$$p_\ell \equiv \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} = \gamma_{\ell n}\dot{q}^n$$

Inverse *contravariant kinetic metric* γ^{mn} gives velocity $\dot{q}^n$

$$\dot{q}^n = p_\ell \gamma^{\ell n} \equiv p^n$$

Riemann equations of motion *(No explicit t-dependence and fixed GCC)*

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Lagrange equations for fixed GCC convert to tensor form

$$T = \frac{1}{2}M_{jk}\left(\frac{\partial x^j}{\partial q^m}\dot{q}^m + \cancel{\left\{\frac{\partial x^j}{\partial t}\right\}}\right)\left(\frac{\partial x^k}{\partial q^n}\dot{q}^n + \cancel{\left\{\frac{\partial x^k}{\partial t}\right\}}\right)$$

All explicit-t-dependent terms are *<u>zero</u>*
(Time must be included as a *<u>dimension</u>*)

$$F_\ell = \frac{d}{dt}\boxed{\frac{\partial T}{\partial \dot{q}^\ell}} - \frac{\partial T}{\partial q^\ell} = \frac{1}{2}\frac{d}{dt}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} - \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial q^\ell}$$

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$$\dot{q}^n = p_\ell\gamma^{\ell n} \equiv p^n$$

Canonical Lagrange equations valid for *<u>all</u>* GCC, fixed or explicit in time *t:*

$$\boxed{F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell}}$$

Lagrange Force Equation
The "4-wheel-drive garbage truck"

Riemann equations of motion *(No explicit t-dependence and fixed GCC)*

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Converts Cartesian kinetic energy $T = \frac{1}{2} M_{jk} \dot{x}^j \dot{q}^k$ to GCC $T = \frac{1}{2}\gamma_{mn}\dot{q}^m \dot{q}^n$

Lagrange equations for fixed GCC convert to tensor form

$$T = \frac{1}{2} M_{jk}\left(\frac{\partial x^j}{\partial q^m}\dot{q}^m + \cancel{\left\{\frac{\partial x^j}{\partial t}\right\}}\right)\left(\frac{\partial x^k}{\partial q^n}\dot{q}^n + \cancel{\left\{\frac{\partial x^k}{\partial t}\right\}}\right)$$

All explicit-t-dependent terms are zero
(Time must be included as a dimension)

$$F_\ell = \frac{d}{dt}\boxed{\frac{\partial T}{\partial \dot{q}^\ell}} - \frac{\partial T}{\partial q^\ell} = \frac{1}{2}\frac{d}{dt}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} - \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial q^\ell}$$

1st term involves ***covariant momentum*** $\boxed{p_\ell}$.

$$p_\ell \equiv \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} = \gamma_{\ell n}\dot{q}^n$$

Inverse ***contravariant kinetic metric*** γ^{mn} gives velocity $\dot{q}^n$

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Canonical Lagrange equations valid for *all* GCC, fixed or explicit in time *t:*

$$\boxed{F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell}}$$

Lagrange Force Equation
The "4-wheel-drive garbage truck"

Following is for fixed GCC only:

$$F_\ell = \boxed{\frac{d}{dt}\left(\gamma_{\ell n}\dot{q}^n\right)} - \frac{1}{2}\frac{\partial \gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n = \boxed{\gamma_{\ell n}\ddot{q}^n + \dot{q}^n\frac{d\gamma_{\ell n}}{dt}} - \frac{1}{2}\frac{\partial \gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n$$

Riemann equations of motion *(No explicit t-dependence and fixed GCC)*

Kinetic metric γ_{mn} is a covariant tensor transform of an original Cartesian inertia tensor M_{ij}

$$\gamma_{mn} = M_{jk}\frac{\partial x^j}{\partial q^m}\frac{\partial x^k}{\partial q^n}$$ Converts Cartesian kinetic energy $T=\frac{1}{2}M_{jk}\dot{x}^j\dot{q}^k$ to GCC $T=\frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$

Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt}\boxed{\frac{\partial T}{\partial \dot{q}^\ell}} - \frac{\partial T}{\partial q^\ell} = \frac{1}{2}\frac{d}{dt}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} - \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial q^\ell}$$

$$T=\frac{1}{2}M_{jk}\left(\frac{\partial x^j}{\partial q^m}\dot{q}^m + \left\{\cancel{\frac{\partial x^j}{\partial t}}\right\}\right)\left(\frac{\partial x^k}{\partial q^n}\dot{q}^n + \left\{\cancel{\frac{\partial x^k}{\partial t}}\right\}\right)$$

All explicit-t-dependent terms are zero
(Time must be included as a dimension)

1[st] term involves *covariant momentum* p_ℓ.

$$p_\ell \equiv \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} = \gamma_{\ell n}\dot{q}^n$$

Inverse *contravariant kinetic metric* γ^{mn} gives velocity $\dot{q}^n$

$$\dot{q}^n = p_\ell \gamma^{\ell n} \equiv p^n$$

Canonical Lagrange equations valid for *all* GCC, fixed or explicit in time *t*:

$$\boxed{F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell}}$$

Lagrange Force Equation
The "4-wheel-drive garbage truck"

Following is for fixed GCC only:

$$F_\ell = \frac{d}{dt}\left(\gamma_{\ell n}\dot{q}^n\right) - \frac{1}{2}\frac{\partial \gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n = \gamma_{\ell n}\ddot{q}^n + \dot{q}^n\frac{d\gamma_{\ell n}}{dt} - \frac{1}{2}\frac{\partial \gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n$$

Time derivative of kinetic metric is expanded by chain rule.

$$\frac{d\gamma_{\ell n}}{dt} = \frac{\partial \gamma_{\ell n}}{\partial q^m}\dot{q}^m$$

Riemann equations of motion *(No explicit t-dependence and fixed GCC)*

Kinetic metric γ_{mn} is a covariant tensor transform of an original Cartesian inertia tensor M_{ij}

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Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt}\boxed{\frac{\partial T}{\partial \dot{q}^\ell}} - \frac{\partial T}{\partial q^\ell} = \frac{1}{2}\frac{d}{dt}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} - \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial q^\ell}$$

$$T=\frac{1}{2}M_{jk}\left(\frac{\partial x^j}{\partial q^m}\dot{q}^m + \cancel{\left\{\frac{\partial x^j}{\partial t}\right\}}\right)\left(\frac{\partial x^k}{\partial q^n}\dot{q}^n + \cancel{\left\{\frac{\partial x^k}{\partial t}\right\}}\right)$$

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Lagrange Force Equation
The "4-wheel-drive garbage truck"

Following is for fixed GCC only:

$$F_\ell = \frac{d}{dt}\left(\gamma_{\ell n}\dot{q}^n\right) - \frac{1}{2}\frac{\partial \gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n = \gamma_{\ell n}\ddot{q}^n + \dot{q}^n\boxed{\frac{d\gamma_{\ell n}}{dt}} - \frac{1}{2}\frac{\partial \gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n$$

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Riemann equations of motion *(No explicit t-dependence and fixed GCC)*

Kinetic metric γ_{mn} is a covariant tensor transform of an original Cartesian inertia tensor M_{ij}

$$\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n}$$

Converts Cartesian kinetic energy $T = \frac{1}{2} M_{jk} \dot{x}^j \dot{q}^k$ to GCC $T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$

Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt} \boxed{\frac{\partial T}{\partial \dot{q}^\ell}} - \frac{\partial T}{\partial q^\ell} = \frac{1}{2} \frac{d}{dt} \frac{\partial \left(\gamma_{mn} \dot{q}^m \dot{q}^n \right)}{\partial \dot{q}^\ell} - \frac{1}{2} \frac{\partial \left(\gamma_{mn} \dot{q}^m \dot{q}^n \right)}{\partial q^\ell}$$

$$T = \frac{1}{2} M_{jk} \left(\frac{\partial x^j}{\partial q^m} \dot{q}^m + \left\{ \cancel{\frac{\partial x^j}{\partial t}} \right\} \right) \left(\frac{\partial x^k}{\partial q^n} \dot{q}^n + \left\{ \cancel{\frac{\partial x^k}{\partial t}} \right\} \right)$$

All explicit-t-dependent terms are *<u>zero</u>*
(Time must be included as a *<u>dimension</u>)*

1st term involves *covariant momentum* $\boxed{p_\ell}$.

$$p_\ell \equiv \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2} \frac{\partial \left(\gamma_{mn} \dot{q}^m \dot{q}^n \right)}{\partial \dot{q}^\ell} = \gamma_{\ell n} \dot{q}^n$$

Inverse *contravariant kinetic metric* γ^{mn} gives velocity $\dot{q}^n$

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Canonical Lagrange equations valid for *<u>all</u>* GCC, fixed or explicit in time *t:*

$$\boxed{F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell} = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell}}$$

Lagrange Force Equation
The "4-wheel-drive garbage truck"

Following is for <u>fixed</u> GCC only:

$$F_\ell = \frac{d}{dt} \left(\gamma_{\ell n} \dot{q}^n \right) - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n = \gamma_{\ell n} \ddot{q}^n + \dot{q}^n \boxed{\frac{d\gamma_{\ell n}}{dt}} - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n$$

Time derivative of kinetic metric is expanded by chain rule.

$$\boxed{\frac{d\gamma_{\ell n}}{dt}} = \frac{\partial \gamma_{\ell n}}{\partial q^m} \dot{q}^m$$

$$F_\ell = \gamma_{\ell n} \ddot{q}^n + \dot{q}^n \boxed{\frac{\partial \gamma_{\ell n}}{\partial q^m}} \dot{q}^m - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n$$

$$F_\ell = \gamma_{\ell n} \ddot{q}^n + \frac{1}{2} \left[\boxed{\frac{\partial \gamma_{n\ell}}{\partial q^m} + \frac{\partial \gamma_{\ell n}}{\partial q^m}} - \frac{\partial \gamma_{mn}}{\partial q^\ell} \right] \dot{q}^m \dot{q}^n$$

Rearrange to expose
Christoffel coefficients (from p 24):

$$\boxed{\Gamma_{im;n} = \frac{1}{2} \left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{mi}}{\partial q^n} \right)}$$

Riemann equations of motion *(No explicit t-dependence and fixed GCC)*

Kinetic metric γ_{mn} is a covariant tensor transform of an original Cartesian inertia tensor M_{ij}

$$\gamma_{mn} = M_{jk}\frac{\partial x^j}{\partial q^m}\frac{\partial x^k}{\partial q^n}$$ Converts Cartesian kinetic energy $T = \frac{1}{2}M_{jk}\dot{x}^j\dot{q}^k$ to GCC $T = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$

Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt}\boxed{\frac{\partial T}{\partial \dot{q}^\ell}} - \frac{\partial T}{\partial q^\ell} = \frac{1}{2}\frac{d}{dt}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} - \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial q^\ell}$$

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Canonical Lagrange equations valid for *all* GCC, fixed or explicit in time *t:*

$$\boxed{F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell}}$$

Lagrange Force Equation
The "4-wheel-drive garbage truck"

Following is for fixed GCC only:

$$F_\ell = \frac{d}{dt}\left(\gamma_{\ell n}\dot{q}^n\right) - \frac{1}{2}\frac{\partial \gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n = \gamma_{\ell n}\ddot{q}^n + \dot{q}^n\boxed{\frac{d\gamma_{\ell n}}{dt}} - \frac{1}{2}\frac{\partial \gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n$$

Time derivative of kinetic metric is expanded by chain rule.

$$\boxed{\frac{d\gamma_{\ell n}}{dt}} = \frac{\partial \gamma_{\ell n}}{\partial q^m}\dot{q}^m$$

$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \dot{q}^n\boxed{\frac{\partial \gamma_{\ell n}}{\partial q^m}}\dot{q}^m - \frac{1}{2}\frac{\partial \gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n$$

$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \frac{1}{2}\left[\boxed{\frac{\partial \gamma_{n\ell}}{\partial q^m} + \frac{\partial \gamma_{\ell n}}{\partial q^m}} - \frac{\partial \gamma_{mn}}{\partial q^\ell}\right]\dot{q}^m\dot{q}^n$$

$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \frac{1}{2}\left[\frac{\partial \gamma_{n\ell}}{\partial q^m} + \frac{\partial \gamma_{\ell m}}{\partial q^n} - \frac{\partial \gamma_{mn}}{\partial q^\ell}\right]\dot{q}^m\dot{q}^n$$

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Riemann equations of motion *(No explicit t-dependence and fixed GCC)*

Kinetic metric γ_{mn} is a covariant tensor transform of an original Cartesian inertia tensor M_{ij}

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Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt}\boxed{\frac{\partial T}{\partial \dot{q}^\ell}} - \frac{\partial T}{\partial q^\ell} = \frac{1}{2}\frac{d}{dt}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} - \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial q^\ell}$$

$$T = \frac{1}{2}M_{jk}\left(\frac{\partial x^j}{\partial q^m}\dot{q}^m + \cancel{\left\{\frac{\partial x^j}{\partial t}\right\}}\right)\left(\frac{\partial x^k}{\partial q^n}\dot{q}^n + \cancel{\left\{\frac{\partial x^k}{\partial t}\right\}}\right)$$

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Canonical Lagrange equations valid for *all* GCC, fixed or explicit in time *t:*

$$\boxed{F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell}}$$

Lagrange Force Equation
The "4-wheel-drive garbage truck"

Following is for fixed GCC only:

$$F_\ell = \frac{d}{dt}\left(\gamma_{\ell n}\dot{q}^n\right) - \frac{1}{2}\frac{\partial\gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n = \gamma_{\ell n}\ddot{q}^n + \dot{q}^n\boxed{\frac{d\gamma_{\ell n}}{dt}} - \frac{1}{2}\frac{\partial\gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n$$

Time derivative of kinetic metric is expanded by chain rule.

$$\boxed{\frac{d\gamma_{\ell n}}{dt}} = \frac{\partial\gamma_{\ell n}}{\partial q^m}\dot{q}^m$$

$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \dot{q}^n\boxed{\frac{\partial\gamma_{\ell n}}{\partial q^m}}\dot{q}^m - \frac{1}{2}\frac{\partial\gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n$$

$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \frac{1}{2}\left[\frac{\partial\gamma_{n\ell}}{\partial q^m} + \frac{\partial\gamma_{\ell n}}{\partial q^m} - \frac{\partial\gamma_{mn}}{\partial q^\ell}\right]\dot{q}^m\dot{q}^n$$

$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \frac{1}{2}\left[\frac{\partial\gamma_{n\ell}}{\partial q^m} + \frac{\partial\gamma_{\ell m}}{\partial q^n} - \frac{\partial\gamma_{mn}}{\partial q^\ell}\right]\dot{q}^m\dot{q}^n$$

Rearrange to expose
Christoffel coefficients (from p 24):

$$\boxed{\Gamma_{im;n} = \frac{1}{2}\left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{mi}}{\partial q^n}\right)}$$

This gives *covariant Riemann equations*

$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \Gamma_{mn;\ell}\dot{q}^m\dot{q}^n$$

Riemann equations of motion *(No explicit t-dependence and fixed GCC)*

Kinetic metric γ_{mn} is a covariant tensor transform of an original Cartesian inertia tensor M_{ij}

$$\gamma_{mn} = M_{jk}\frac{\partial x^j}{\partial q^m}\frac{\partial x^k}{\partial q^n}$$ Converts Cartesian kinetic energy $T = \frac{1}{2}M_{jk}\dot{x}^j\dot{q}^k$ to GCC $T = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$

Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell} = \frac{1}{2}\frac{d}{dt}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} - \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial q^\ell}$$

$$T = \frac{1}{2}M_{jk}\left(\frac{\partial x^j}{\partial q^m}\dot{q}^m + \cancel{\left\{\frac{\partial x^j}{\partial t}\right\}}\right)\left(\frac{\partial x^k}{\partial q^n}\dot{q}^n + \cancel{\left\{\frac{\partial x^k}{\partial t}\right\}}\right)$$

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1st term involves *covariant momentum* p_ℓ.

$$p_\ell \equiv \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} = \gamma_{\ell n}\dot{q}^n$$

Inverse *contravariant kinetic metric* γ^{mn} gives velocity $\dot{q}^n$

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Canonical Lagrange equations valid for *all* GCC, fixed or explicit in time *t:*

$$\boxed{F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell}}$$

Lagrange Force Equation
The "4-wheel-drive garbage truck"

Following is for fixed GCC only:

$$F_\ell = \frac{d}{dt}\left(\gamma_{\ell n}\dot{q}^n\right) - \frac{1}{2}\frac{\partial \gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n = \gamma_{\ell n}\ddot{q}^n + \dot{q}^n\frac{d\gamma_{\ell n}}{dt} - \frac{1}{2}\frac{\partial \gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n$$

Time derivative of kinetic metric is expanded by chain rule.

$$\frac{d\gamma_{\ell n}}{dt} = \frac{\partial \gamma_{\ell n}}{\partial q^m}\dot{q}^m$$

$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \dot{q}^n\frac{\partial \gamma_{\ell n}}{\partial q^m}\dot{q}^m - \frac{1}{2}\frac{\partial \gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n$$

Rearrange to expose Christoffel coefficients:

$$\boxed{\Gamma_{im;n} = \frac{1}{2}\left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{mi}}{\partial q^n}\right)}$$

$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \frac{1}{2}\left[\frac{\partial \gamma_{n\ell}}{\partial q^m} + \frac{\partial \gamma_{\ell n}}{\partial q^m} - \frac{\partial \gamma_{mn}}{\partial q^\ell}\right]\dot{q}^m\dot{q}^n$$

$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \frac{1}{2}\left[\frac{\partial \gamma_{n\ell}}{\partial q^m} + \frac{\partial \gamma_{\ell m}}{\partial q^n} - \frac{\partial \gamma_{mn}}{\partial q^\ell}\right]\dot{q}^m\dot{q}^n$$

This gives *covariant Riemann equations*

$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \Gamma_{mn;\ell}\dot{q}^m\dot{q}^n$$

and *contravariant Riemann equations*.

$$F^k = \ddot{q}^k + \Gamma^k_{mn}\dot{q}^m\dot{q}^n$$

General Riemann equations of motion *(No explicit t-dependence and fixed GCC)*

→ *Riemann-forms in cylindrical polar OCC* ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

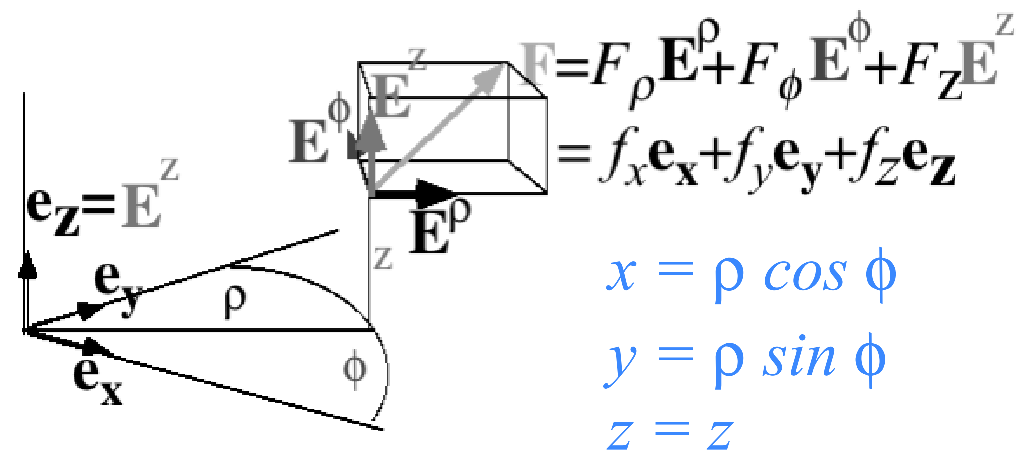
Christoffel relation to Coriolis coefficients

Mechanics of ideal fluid vortex

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

$$\langle J\rangle = \begin{pmatrix} \frac{\partial x}{\partial \rho}=\cos\phi & \frac{\partial x}{\partial \phi}=-\rho\sin\phi & 0 \\ \frac{\partial y}{\partial \rho}=\sin\phi & \frac{\partial y}{\partial \phi}=\rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix}, \quad \langle K\rangle = \begin{array}{cl} \begin{pmatrix} \frac{\partial \rho}{\partial x}=\cos\phi & \frac{\partial \rho}{\partial y}=\sin\phi & 0 \\ \frac{\partial \phi}{\partial x}=\frac{-\sin\phi}{\rho} & \frac{\partial \phi}{\partial y}=\frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix} & \begin{array}{l} \leftarrow \mathbf{E}^{\rho} \\ \leftarrow \mathbf{E}^{\phi} \\ \leftarrow \mathbf{E}^{z} \end{array} \end{array}$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_{\rho} & \mathbf{E}_{\phi} & \mathbf{E}_{z} \end{array} \qquad = \langle J^{-1}\rangle$$



$x = \rho \cos \phi$

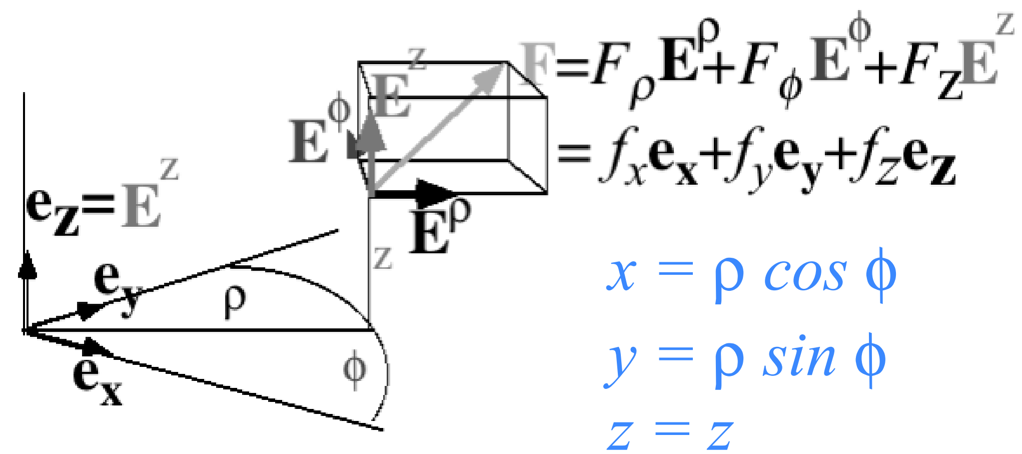
$y = \rho \sin \phi$

$z = z$

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

$$\langle J\rangle = \begin{array}{c} \begin{pmatrix} \frac{\partial x}{\partial \rho}=\cos\phi & \frac{\partial x}{\partial \phi}=-\rho\sin\phi & 0 \\ \frac{\partial y}{\partial \rho}=\sin\phi & \frac{\partial y}{\partial \phi}=\rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{array} \end{array}, \quad \langle K\rangle = \begin{array}{cc} \begin{pmatrix} \frac{\partial \rho}{\partial x}=\cos\phi & \frac{\partial \rho}{\partial y}=\sin\phi & 0 \\ \frac{\partial \phi}{\partial x}=\frac{-\sin\phi}{\rho} & \frac{\partial \phi}{\partial y}=\frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix} & \begin{array}{c} \leftarrow \mathbf{E}^\rho \\ \leftarrow \mathbf{E}^\phi \\ \leftarrow \mathbf{E}^z \end{array} \end{array}$$

$$= \langle J^{-1}\rangle$$



$x = \rho \cos \phi$

$y = \rho \sin \phi$

$z = z$

Covariant forces

$$F_\rho = f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = \quad f_x \cos\phi + f_y \sin\phi \quad + 0$$

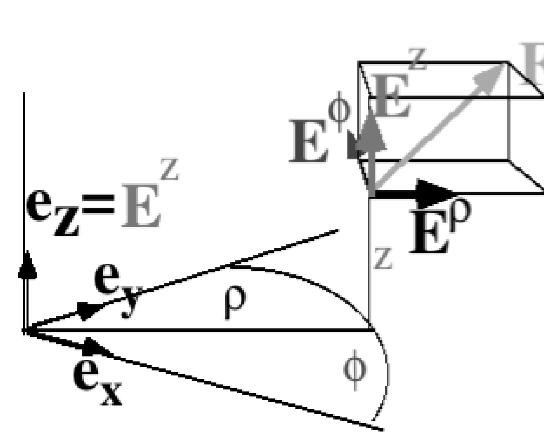
$$F_\phi = f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin\phi + f_y \rho \cos\phi + 0$$

$$F_z = f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = \qquad 0 \qquad + \qquad 0 \quad + f_z$$

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

$$\langle J\rangle = \begin{array}{c} \begin{pmatrix} \frac{\partial x}{\partial \rho}=\cos\phi & \frac{\partial x}{\partial \phi}=-\rho\sin\phi & 0 \\ \frac{\partial y}{\partial \rho}=\sin\phi & \frac{\partial y}{\partial \phi}=\rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{array} \end{array}, \quad \langle K\rangle = \begin{array}{cc} \begin{pmatrix} \frac{\partial \rho}{\partial x}=\cos\phi & \frac{\partial \rho}{\partial y}=\sin\phi & 0 \\ \frac{\partial \phi}{\partial x}=\frac{-\sin\phi}{\rho} & \frac{\partial \phi}{\partial y}=\frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix} & \begin{array}{c} \leftarrow \mathbf{E}^\rho \\ \leftarrow \mathbf{E}^\phi \\ \leftarrow \mathbf{E}^z \end{array} \end{array}$$

$$= \langle J^{-1}\rangle$$



$$\mathbf{F} = F_\rho \mathbf{E}^\rho + F_\phi \mathbf{E}^\phi + F_z \mathbf{E}^z = f_x \mathbf{e}_x + f_y \mathbf{e}_y + f_z \mathbf{e}_z$$

$x = \rho \cos \phi$

$y = \rho \sin \phi$

$z = z$

Covariant forces

$$F_\rho = f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = \quad f_x \cos\phi + f_y \sin\phi \quad + 0$$

$$F_\phi = f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin\phi + f_y \rho \cos\phi + 0$$

$$F_z = f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = \quad 0 \quad + \quad 0 \quad + f_z$$

Covariant kinetic metric

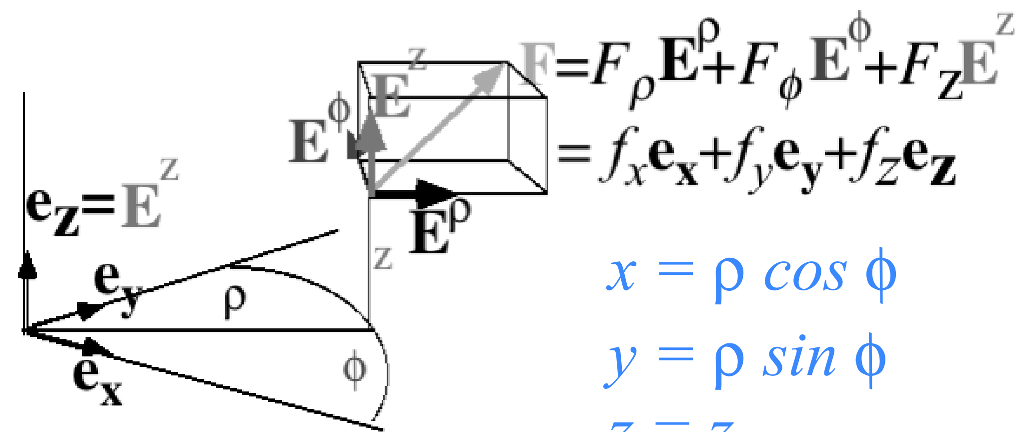
$$\gamma_{\rho\rho} = m \frac{\partial x_j}{\partial \rho} \frac{\partial x^j}{\partial \rho} = m \mathbf{E}_\rho \bullet \mathbf{E}_\rho = m\left(\cos^2\phi + \sin^2\phi\right) = m$$

$$\gamma_{\phi\phi} = m \frac{\partial x_j}{\partial \phi} \frac{\partial x^j}{\partial \phi} = m \mathbf{E}_\phi \bullet \mathbf{E}_\phi = m\left(\rho^2\cos^2\phi + \rho^2\sin^2\phi\right) = m\rho^2$$

$$\gamma_{zz} = m \frac{\partial x_j}{\partial z} \frac{\partial x^j}{\partial z} = m \mathbf{E}_z \bullet \mathbf{E}_z = m$$

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

$$\langle J\rangle = \begin{array}{c} \begin{pmatrix} \frac{\partial x}{\partial \rho}=\cos\phi & \frac{\partial x}{\partial \phi}=-\rho\sin\phi & 0 \\ \frac{\partial y}{\partial \rho}=\sin\phi & \frac{\partial y}{\partial \phi}=\rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{array} \end{array}, \quad \langle K\rangle = \begin{array}{c} \begin{array}{cc} \begin{pmatrix} \frac{\partial \rho}{\partial x}=\cos\phi & \frac{\partial \rho}{\partial y}=\sin\phi & 0 \\ \frac{\partial \phi}{\partial x}=\frac{-\sin\phi}{\rho} & \frac{\partial \phi}{\partial y}=\frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix} & \begin{array}{c} \leftarrow \mathbf{E}^\rho \\ \leftarrow \mathbf{E}^\phi \\ \leftarrow \mathbf{E}^z \end{array} \end{array} \\ =\langle J^{-1}\rangle \end{array}$$



$x = \rho \cos \phi$

$y = \rho \sin \phi$

$z = z$

Covariant forces

$$F_\rho = f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = \quad f_x \cos\phi + f_y \sin\phi \quad + 0$$

$$F_\phi = f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin\phi + f_y \rho\cos\phi + 0$$

$$F_z = f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = \quad 0 \quad + \quad 0 \quad + f_z$$

Covariant kinetic metric

$$\gamma_{\rho\rho} = m \frac{\partial x_j}{\partial \rho}\frac{\partial x^j}{\partial \rho} = m\mathbf{E}_\rho \bullet \mathbf{E}_\rho = m\left(\cos^2\phi + \sin^2\phi\right) = m$$

$$\gamma_{\phi\phi} = m \frac{\partial x_j}{\partial \phi}\frac{\partial x^j}{\partial \phi} = m\mathbf{E}_\phi \bullet \mathbf{E}_\phi = m\left(\rho^2\cos^2\phi + \rho^2\sin^2\phi\right) = m\rho^2$$

$$\gamma_{zz} = m \frac{\partial x_j}{\partial z}\frac{\partial x^j}{\partial z} = m\mathbf{E}_z \bullet \mathbf{E}_z = m$$

Contravariant kinetic metric

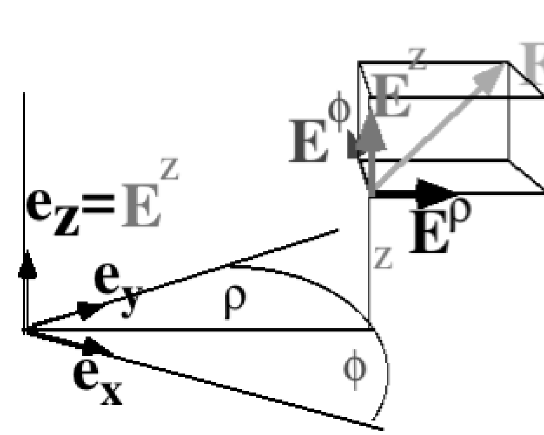
$$\gamma^{\rho\rho} = 1/m$$

$$\gamma^{\phi\phi} = 1/\left(m\rho^2\right)$$

$$\gamma^{zz} = 1/m$$

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

$$
\begin{array}{c}
\langle J \rangle = \begin{pmatrix} \frac{\partial x}{\partial \rho} = \cos\phi & \frac{\partial x}{\partial \phi} = -\rho\sin\phi & 0 \\ \frac{\partial y}{\partial \rho} = \sin\phi & \frac{\partial y}{\partial \phi} = \rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix} \\
\qquad\quad \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{array}
\end{array}
,\quad
\langle K \rangle = \begin{array}{c} \begin{pmatrix} \frac{\partial \rho}{\partial x} = \cos\phi & \frac{\partial \rho}{\partial y} = \sin\phi & 0 \\ \frac{\partial \phi}{\partial x} = \frac{-\sin\phi}{\rho} & \frac{\partial \phi}{\partial y} = \frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix} \begin{array}{l} \leftarrow \mathbf{E}^\rho \\ \leftarrow \mathbf{E}^\phi \\ \leftarrow \mathbf{E}^z \end{array} \\ = \langle J^{-1} \rangle \end{array}
$$



$$\mathbf{F} = F_\rho \mathbf{E}^\rho + F_\phi \mathbf{E}^\phi + F_z \mathbf{E}^z$$
$$= f_x \mathbf{e_x} + f_y \mathbf{e_y} + f_z \mathbf{e_z}$$

$x = \rho \cos \phi$

$y = \rho \sin \phi$

$z = z$

Covariant forces

$$F_\rho = f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = \quad f_x \cos\phi + f_y \sin\phi \quad + 0$$

$$F_\phi = f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin\phi + f_y \rho \cos\phi + 0$$

$$F_z = f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = \quad 0 \quad + \quad 0 \quad + f_z$$

Covariant kinetic metric

$$\gamma_{\rho\rho} = m \frac{\partial x_j}{\partial \rho} \frac{\partial x^j}{\partial \rho} = m \mathbf{E}_\rho \bullet \mathbf{E}_\rho = m\left(\cos^2\phi + \sin^2\phi\right) = m$$

$$\gamma_{\phi\phi} = m \frac{\partial x_j}{\partial \phi} \frac{\partial x^j}{\partial \phi} = m \mathbf{E}_\phi \bullet \mathbf{E}_\phi = m\left(\rho^2 \cos^2\phi + \rho^2 \sin^2\phi\right) = m\rho^2$$

$$\gamma_{zz} = m \frac{\partial x_j}{\partial z} \frac{\partial x^j}{\partial z} = m \mathbf{E}_z \bullet \mathbf{E}_z = m$$

Contravariant kinetic metric

$$\gamma^{\rho\rho} = 1/m$$

$$\gamma^{\phi\phi} = 1/\left(m\rho^2\right)$$

$$\gamma^{zz} = 1/m$$

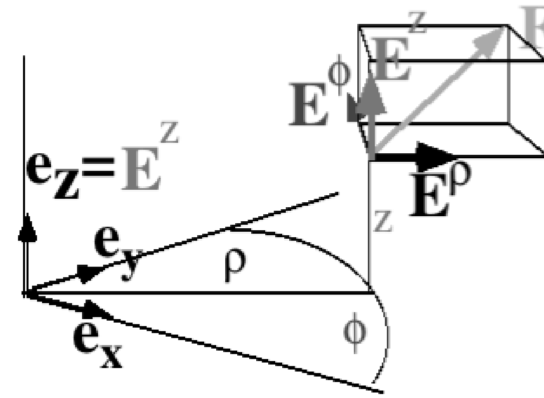
Lagrangian

$$T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2$$

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

$$\langle J\rangle = \begin{array}{c} \begin{pmatrix} \frac{\partial x}{\partial\rho}=\cos\phi & \frac{\partial x}{\partial\phi}=-\rho\sin\phi & 0 \\ \frac{\partial y}{\partial\rho}=\sin\phi & \frac{\partial y}{\partial\phi}=\rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{array} \end{array}, \quad \langle K\rangle = \begin{pmatrix} \frac{\partial\rho}{\partial x}=\cos\phi & \frac{\partial\rho}{\partial y}=\sin\phi & 0 \\ \frac{\partial\phi}{\partial x}=\frac{-\sin\phi}{\rho} & \frac{\partial\phi}{\partial y}=\frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix} \begin{array}{l} \leftarrow \mathbf{E}^\rho \\ \leftarrow \mathbf{E}^\phi \\ \leftarrow \mathbf{E}^z \end{array}$$

$$= \langle J^{-1}\rangle$$



$$\mathbf{F}=F_\rho\mathbf{E}^\rho+F_\phi\mathbf{E}^\phi+F_z\mathbf{E}^z$$
$$=f_x\mathbf{e_x}+f_y\mathbf{e_y}+f_z\mathbf{e_z}$$

$x = \rho\cos\phi$

$y = \rho\sin\phi$

$z = z$

Covariant forces

$$F_\rho = f_x\frac{\partial x}{\partial\rho}+f_y\frac{\partial y}{\partial\rho}+f_z\frac{\partial z}{\partial\rho} = \quad f_x\cos\phi+f_y\sin\phi \quad +0$$

$$F_\phi = f_x\frac{\partial x}{\partial\phi}+f_y\frac{\partial y}{\partial\phi}+f_z\frac{\partial z}{\partial\phi} = -f_x\rho\sin\phi+f_y\rho\cos\phi+0$$

$$F_z = f_x\frac{\partial x}{\partial z}+f_y\frac{\partial y}{\partial z}+f_z\frac{\partial z}{\partial z} = \quad 0 \quad + \quad 0 \quad +f_z$$

Covariant kinetic metric

$$\gamma_{\rho\rho} = m\frac{\partial x_j}{\partial\rho}\frac{\partial x^j}{\partial\rho} = m\mathbf{E}_\rho\bullet\mathbf{E}_\rho = m\left(\cos^2\phi+\sin^2\phi\right) = m$$

$$\gamma_{\phi\phi} = m\frac{\partial x_j}{\partial\phi}\frac{\partial x^j}{\partial\phi} = m\mathbf{E}_\phi\bullet\mathbf{E}_\phi = m\left(\rho^2\cos^2\phi+\rho^2\sin^2\phi\right) = m\rho^2$$

$$\gamma_{zz} = m\frac{\partial x_j}{\partial z}\frac{\partial x^j}{\partial z} = m\mathbf{E}_z\bullet\mathbf{E}_z = m$$

Contravariant kinetic metric

$$\gamma^{\rho\rho} = 1/m$$
$$\gamma^{\phi\phi} = 1/\left(m\rho^2\right)$$
$$\gamma^{zz} = 1/m$$

Lagrangian

$$T = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n = \frac{1}{2}m\dot{\rho}^2+\frac{1}{2}m\rho^2\dot{\phi}^2+\frac{1}{2}m\dot{z}^2$$

Covariant momenta

$$p_\rho = \frac{\partial T}{\partial\dot{\rho}} = \gamma_{\rho\rho}\dot{\rho} = m\dot{\rho} \qquad p_\phi = \frac{\partial T}{\partial\dot{\phi}} = \gamma_{\phi\phi}\dot{\phi} = m\rho^2\dot{\phi} \qquad p_z = \frac{\partial T}{\partial\dot{z}} = \gamma_{zz}\dot{z} = m\dot{z}$$

Contravariant momenta

$$p^\rho = \dot{\rho}$$
$$p^\phi = \dot{\phi}$$
$$p^z = \dot{z}$$

General Riemann equations of motion *(No explicit t-dependence and fixed GCC)*

Riemann-forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

→ *Christoffel relation to Coriolis coefficients*

Mechanics of ideal fluid vortex

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

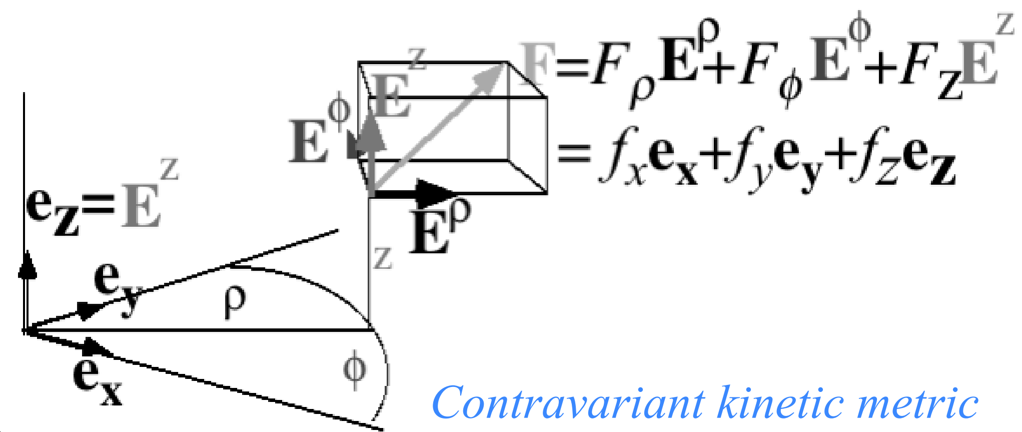
$$\langle J\rangle = \begin{pmatrix} \frac{\partial x}{\partial\rho}=\cos\phi & \frac{\partial x}{\partial\phi}=-\rho\sin\phi & 0 \\ \frac{\partial y}{\partial\rho}=\sin\phi & \frac{\partial y}{\partial\phi}=\rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix}, \quad \langle K\rangle = \begin{pmatrix} \frac{\partial\rho}{\partial x}=\cos\phi & \frac{\partial\rho}{\partial y}=\sin\phi & 0 \\ \frac{\partial\phi}{\partial x}=\frac{-\sin\phi}{\rho} & \frac{\partial\phi}{\partial y}=\frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix} \begin{array}{l} \leftarrow \mathbf{E}^\rho \\ \leftarrow \mathbf{E}^\phi \\ \leftarrow \mathbf{E}^z \end{array}$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{array} \qquad = \langle J^{-1}\rangle$$

$x = \rho \cos \phi$

$y = \rho \sin \phi$

$z = z$



Covariant forces

$$F_\rho = f_x\frac{\partial x}{\partial\rho} + f_y\frac{\partial y}{\partial\rho} + f_z\frac{\partial z}{\partial\rho} = \quad f_x\cos\phi + f_y\sin\phi \quad +0$$

$$F_\phi = f_x\frac{\partial x}{\partial\phi} + f_y\frac{\partial y}{\partial\phi} + f_z\frac{\partial z}{\partial\phi} = -f_x\rho\sin\phi + f_y\rho\cos\phi + 0$$

$$F_z = f_x\frac{\partial x}{\partial z} + f_y\frac{\partial y}{\partial z} + f_z\frac{\partial z}{\partial z} = \quad 0 \quad + \quad 0 \quad + f_z$$

Covariant kinetic metric

$$\gamma_{\rho\rho} = m\frac{\partial x_j}{\partial\rho}\frac{\partial x^j}{\partial\rho} = m\mathbf{E}_\rho \bullet \mathbf{E}_\rho = m\left(\cos^2\phi + \sin^2\phi\right) = m$$

$$\gamma_{\phi\phi} = m\frac{\partial x_j}{\partial\phi}\frac{\partial x^j}{\partial\phi} = m\mathbf{E}_\phi \bullet \mathbf{E}_\phi = m\left(\rho^2\cos^2\phi + \rho^2\sin^2\phi\right) = m\rho^2$$

$$\gamma_{zz} = m\frac{\partial x_j}{\partial z}\frac{\partial x^j}{\partial z} = m\mathbf{E}_z \bullet \mathbf{E}_z = m$$

Contravariant kinetic metric

$$\gamma^{\rho\rho} = 1/m$$

$$\gamma^{\phi\phi} = 1/\left(m\rho^2\right)$$

$$\gamma^{zz} = 1/m$$

Lagrangian

$$T = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\phi}^2 + \frac{1}{2}m\dot{z}^2$$

Covariant momenta

$$p_\rho = \frac{\partial T}{\partial\dot{\rho}} = \gamma_{\rho\rho}\dot{\rho} = m\dot{\rho} \qquad p_\phi = \frac{\partial T}{\partial\dot{\phi}} = \gamma_{\phi\phi}\dot{\phi} = m\rho^2\dot{\phi} \qquad p_z = \frac{\partial T}{\partial\dot{z}} = \gamma_{zz}\dot{z} = m\dot{z}$$

Contravariant momenta

$$p^\rho = \dot{\rho}$$

$$p^\phi = \dot{\phi}$$

$$p^z = \dot{z}$$

Comparing Lagrange and the Riemann *covariant force* equations

$$\boxed{F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell}} = \boxed{\gamma_{\ell n}\ddot{q}^n + \Gamma_{mn;\ell}\dot{q}^m\dot{q}^n}$$

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

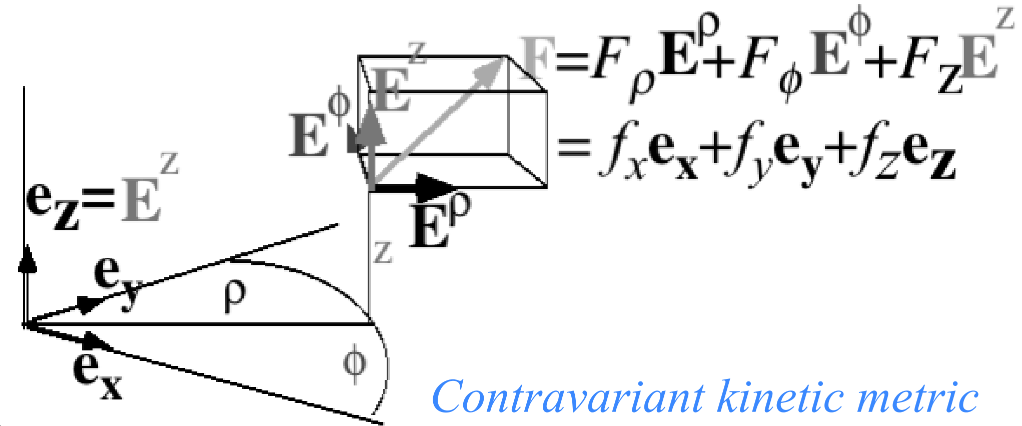
$$\langle J\rangle = \begin{pmatrix} \frac{\partial x}{\partial \rho}=\cos\phi & \frac{\partial x}{\partial \phi}=-\rho\sin\phi & 0 \\ \frac{\partial y}{\partial \rho}=\sin\phi & \frac{\partial y}{\partial \phi}=\rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix}, \quad \langle K\rangle = \begin{pmatrix} \frac{\partial \rho}{\partial x}=\cos\phi & \frac{\partial \rho}{\partial y}=\sin\phi & 0 \\ \frac{\partial \phi}{\partial x}=\frac{-\sin\phi}{\rho} & \frac{\partial \phi}{\partial y}=\frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix} \begin{matrix} \leftarrow \mathbf{E}^\rho \\ \leftarrow \mathbf{E}^\phi \\ \leftarrow \mathbf{E}^z \end{matrix}$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{matrix} \qquad\qquad = \langle J^{-1}\rangle$$

$x = \rho \cos \phi$

$y = \rho \sin \phi$

$z = z$



Covariant forces

$$F_\rho = f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = \quad f_x \cos\phi + f_y \sin\phi \quad +0$$

$$F_\phi = f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin\phi + f_y \rho\cos\phi + 0$$

$$F_z = f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = \quad 0 \quad + \quad 0 \quad + f_z$$

Covariant kinetic metric

$$\gamma_{\rho\rho} = m \frac{\partial x_j}{\partial \rho}\frac{\partial x^j}{\partial \rho} = m\mathbf{E}_\rho \bullet \mathbf{E}_\rho = m\left(\cos^2\phi + \sin^2\phi\right) = m$$

$$\gamma_{\phi\phi} = m \frac{\partial x_j}{\partial \phi}\frac{\partial x^j}{\partial \phi} = m\mathbf{E}_\phi \bullet \mathbf{E}_\phi = m\left(\rho^2\cos^2\phi + \rho^2\sin^2\phi\right) = m\rho^2$$

$$\gamma_{zz} = m \frac{\partial x_j}{\partial z}\frac{\partial x^j}{\partial z} = m\mathbf{E}_z \bullet \mathbf{E}_z = m$$

Contravariant kinetic metric

$$\gamma^{\rho\rho} = 1/m$$

$$\gamma^{\phi\phi} = 1/\left(m\rho^2\right)$$

$$\gamma^{zz} = 1/m$$

Lagrangian

$$T = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\phi}^2 + \frac{1}{2}m\dot{z}^2$$

Covariant momenta

$$p_\rho = \frac{\partial T}{\partial \dot{\rho}} = \gamma_{\rho\rho}\dot{\rho} = m\dot{\rho} \qquad p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \gamma_{\phi\phi}\dot{\phi} = m\rho^2\dot{\phi} \qquad p_z = \frac{\partial T}{\partial \dot{z}} = \gamma_{zz}\dot{z} = m\dot{z}$$

Contravariant momenta

$$p^\rho = \dot{\rho}$$

$$p^\phi = \dot{\phi}$$

$$p^z = \dot{z}$$

Comparing Lagrange and the Riemann *covariant force* equations

$$\boxed{F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell}} = \boxed{\gamma_{\ell n}\ddot{q}^n + \Gamma_{mn;\ell}\dot{q}^m\dot{q}^n}$$

Note: This a much more efficient way to derive Γ-coefficients than the g-formula.

Only three non-zero Christoffel coefficients appear, and only two are independent.

$$F_\rho = \frac{dp_\rho}{dt} - \frac{\partial T}{\partial \rho} = \gamma_{\rho\rho}\ddot{\rho} + \Gamma_{mn;\rho}\dot{q}^m\dot{q}^n \qquad F_\phi = \frac{dp_\phi}{dt} - \frac{\partial T}{\partial \phi} = \gamma_{\phi\phi}\ddot{\phi} + \Gamma_{mn;\phi}\dot{q}^m\dot{q}^n$$

Christoffel g-formula

(from p. 24 to p. 37):

$$\boxed{\Gamma_{im;n} = \frac{1}{2}\left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{mi}}{\partial q^n}\right)}$$

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

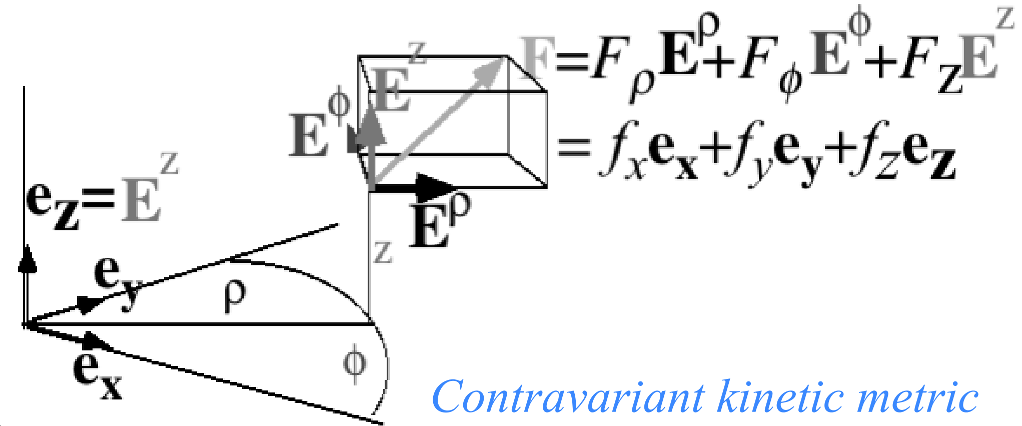
$$\langle J\rangle = \begin{pmatrix} \frac{\partial x}{\partial\rho}=\cos\phi & \frac{\partial x}{\partial\phi}=-\rho\sin\phi & 0 \\ \frac{\partial y}{\partial\rho}=\sin\phi & \frac{\partial y}{\partial\phi}=\rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix}, \quad \langle K\rangle = \begin{pmatrix} \frac{\partial\rho}{\partial x}=\cos\phi & \frac{\partial\rho}{\partial y}=\sin\phi & 0 \\ \frac{\partial\phi}{\partial x}=\frac{-\sin\phi}{\rho} & \frac{\partial\phi}{\partial y}=\frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix} \begin{array}{l} \leftarrow \mathbf{E}^\rho \\ \leftarrow \mathbf{E}^\phi \\ \leftarrow \mathbf{E}^z \end{array}$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{array} \qquad = \langle J^{-1}\rangle$$

$x = \rho\cos\phi$

$y = \rho\sin\phi$

$z = z$



Covariant forces

$$F_\rho = f_x\frac{\partial x}{\partial\rho} + f_y\frac{\partial y}{\partial\rho} + f_z\frac{\partial z}{\partial\rho} = \quad f_x\cos\phi + f_y\sin\phi \quad +0$$

$$F_\phi = f_x\frac{\partial x}{\partial\phi} + f_y\frac{\partial y}{\partial\phi} + f_z\frac{\partial z}{\partial\phi} = -f_x\rho\sin\phi + f_y\rho\cos\phi + 0$$

$$F_z = f_x\frac{\partial x}{\partial z} + f_y\frac{\partial y}{\partial z} + f_z\frac{\partial z}{\partial z} = \quad 0 \quad + \quad 0 \quad +f_z$$

Covariant kinetic metric

$$\gamma_{\rho\rho} = m\frac{\partial x_j}{\partial\rho}\frac{\partial x^j}{\partial\rho} = m\mathbf{E}_\rho\bullet\mathbf{E}_\rho = m\left(\cos^2\phi + \sin^2\phi\right) = m$$

$$\gamma_{\phi\phi} = m\frac{\partial x_j}{\partial\phi}\frac{\partial x^j}{\partial\phi} = m\mathbf{E}_\phi\bullet\mathbf{E}_\phi = m\left(\rho^2\cos^2\phi + \rho^2\sin^2\phi\right) = m\rho^2$$

$$\gamma_{zz} = m\frac{\partial x_j}{\partial z}\frac{\partial x^j}{\partial z} = m\mathbf{E}_z\bullet\mathbf{E}_z = m$$

Contravariant kinetic metric

$$\gamma^{\rho\rho} = 1/m$$

$$\gamma^{\phi\phi} = 1/\left(m\rho^2\right)$$

$$\gamma^{zz} = 1/m$$

Lagrangian

$$T = \frac{1}{2}\gamma_{mn}\dot q^m\dot q^n = \frac{1}{2}m\dot\rho^2 + \frac{1}{2}m\rho^2\dot\phi^2 + \frac{1}{2}m\dot z^2$$

Covariant momenta

$$p_\rho = \frac{\partial T}{\partial\dot\rho} = \gamma_{\rho\rho}\dot\rho = m\dot\rho \qquad p_\phi = \frac{\partial T}{\partial\dot\phi} = \gamma_{\phi\phi}\dot\phi = m\rho^2\dot\phi \qquad p_z = \frac{\partial T}{\partial\dot z} = \gamma_{zz}\dot z = m\dot z$$

Contravariant momenta

$$p^\rho = \dot\rho$$

$$p^\phi = \dot\phi$$

$$p^z = \dot z$$

Comparing Lagrange and the Riemann *covariant force* equations

$$\boxed{F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell}} = \boxed{\gamma_{\ell n}\ddot q^n + \Gamma_{mn;\ell}\,\dot q^m\dot q^n}$$

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Only three non-zero Christoffel coefficients appear, and only two are independent.

$$F_\rho = \frac{dp_\rho}{dt} - \frac{\partial T}{\partial\rho} = \gamma_{\rho\rho}\ddot\rho + \Gamma_{mn;\rho}\,\dot q^m\dot q^n$$

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Christoffel g-formula (from p. 24 to p. 37):

$$\boxed{\Gamma_{im;n} = \frac{1}{2}\left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{mi}}{\partial q^n}\right)}$$

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

$$\langle J\rangle = \begin{pmatrix} \frac{\partial x}{\partial \rho}=\cos\phi & \frac{\partial x}{\partial \phi}=-\rho\sin\phi & 0 \\ \frac{\partial y}{\partial \rho}=\sin\phi & \frac{\partial y}{\partial \phi}=\rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix}, \quad \langle K\rangle = \begin{pmatrix} \frac{\partial \rho}{\partial x}=\cos\phi & \frac{\partial \rho}{\partial y}=\sin\phi & 0 \\ \frac{\partial \phi}{\partial x}=\frac{-\sin\phi}{\rho} & \frac{\partial \phi}{\partial y}=\frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix} \begin{matrix} \leftarrow \mathbf{E}^\rho \\ \leftarrow \mathbf{E}^\phi \\ \leftarrow \mathbf{E}^z \end{matrix}$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{matrix} \qquad = \langle J^{-1}\rangle$$

$x = \rho \cos \phi$
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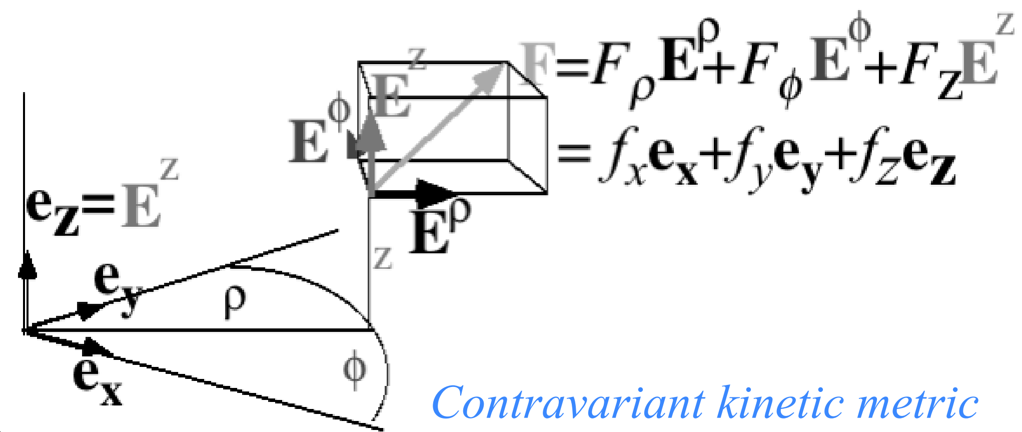

Covariant forces

$$F_\rho = f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = \quad f_x \cos\phi + f_y \sin\phi \quad +0$$

$$F_\phi = f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho\sin\phi + f_y \rho\cos\phi + 0$$

$$F_z = f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = \quad 0 \quad + \quad 0 \quad + f_z$$

Covariant kinetic metric

$$\gamma_{\rho\rho} = m \frac{\partial x_j}{\partial \rho}\frac{\partial x^j}{\partial \rho} = m\mathbf{E}_\rho \bullet \mathbf{E}_\rho = m\left(\cos^2\phi + \sin^2\phi\right) = m$$

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Contravariant kinetic metric

$$\gamma^{\rho\rho} = 1/m$$

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Lagrangian

$$T = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\phi}^2 + \frac{1}{2}m\dot{z}^2$$

Covariant momenta

$$p_\rho = \frac{\partial T}{\partial \dot{\rho}} = \gamma_{\rho\rho}\dot{\rho} \qquad p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \gamma_{\phi\phi}\dot{\phi} \qquad p_z = \frac{\partial T}{\partial \dot{z}} = \gamma_{zz}\dot{z}$$

$$= m\dot{\rho} \qquad\qquad = m\rho^2\dot{\phi} \qquad\qquad = m\dot{z}$$

Contravariant momenta

$$p^\rho = \dot{\rho}$$

$$p^\phi = \dot{\phi}$$

$$p^z = \dot{z}$$

Comparing Lagrange and the Riemann *covariant force* equations

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$$= \frac{d\left(m\rho^2\dot{\phi}\right)}{dt} - 0 = m\rho^2\ddot{\phi} + 2m\rho\dot{\rho}\dot{\phi} \quad \text{so: } \Gamma_{\rho\phi;\phi} = m\rho = \Gamma_{\phi\rho;\phi}$$

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

$$\langle J\rangle = \begin{array}{c} \begin{pmatrix} \frac{\partial x}{\partial \rho}=\cos\phi & \frac{\partial x}{\partial \phi}=-\rho\sin\phi & 0 \\ \frac{\partial y}{\partial \rho}=\sin\phi & \frac{\partial y}{\partial \phi}=\rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{array} \end{array}, \quad \langle K\rangle = \begin{array}{cc} \begin{pmatrix} \frac{\partial \rho}{\partial x}=\cos\phi & \frac{\partial \rho}{\partial y}=\sin\phi & 0 \\ \frac{\partial \phi}{\partial x}=\frac{-\sin\phi}{\rho} & \frac{\partial \phi}{\partial y}=\frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix} & \begin{array}{c} \leftarrow \mathbf{E}^\rho \\ \leftarrow \mathbf{E}^\phi \\ \leftarrow \mathbf{E}^z \end{array} \end{array} = \langle J^{-1}\rangle$$

$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$
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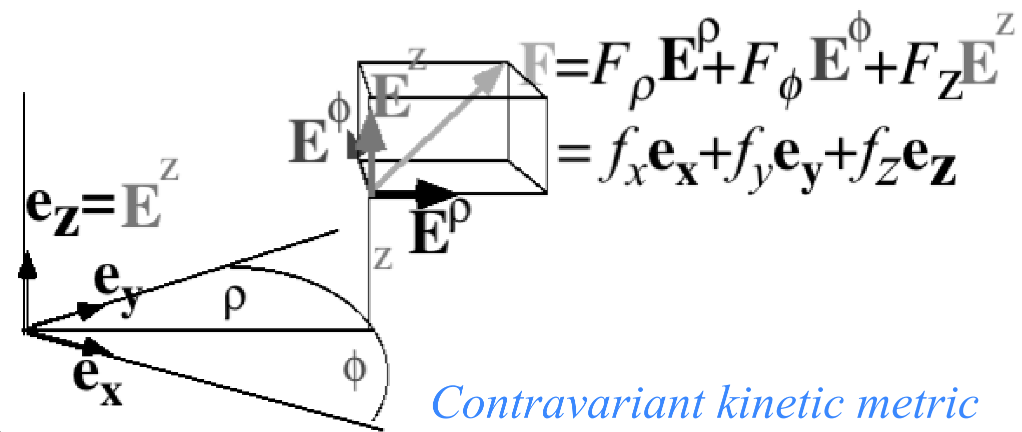

Covariant forces

$$F_\rho = f_x\frac{\partial x}{\partial \rho}+f_y\frac{\partial y}{\partial \rho}+f_z\frac{\partial z}{\partial \rho} = \quad f_x\cos\phi + f_y\sin\phi \ +0$$

$$F_\phi = f_x\frac{\partial x}{\partial \phi}+f_y\frac{\partial y}{\partial \phi}+f_z\frac{\partial z}{\partial \phi} = -f_x\rho\sin\phi + f_y\rho\cos\phi + 0$$

$$F_z = f_x\frac{\partial x}{\partial z}+f_y\frac{\partial y}{\partial z}+f_z\frac{\partial z}{\partial z} = \quad 0 \quad + \quad 0 \quad + f_z$$

Covariant kinetic metric

$$\gamma_{\rho\rho} = m\frac{\partial x_j}{\partial \rho}\frac{\partial x^j}{\partial \rho} = m\mathbf{E}_\rho\bullet\mathbf{E}_\rho = m\left(\cos^2\phi+\sin^2\phi\right) = m$$

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Contravariant kinetic metric

$$\gamma^{\rho\rho} = 1/m$$
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Lagrangian

$$T = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n = \frac{1}{2}m\dot{\rho}^2+\frac{1}{2}m\rho^2\dot{\phi}^2+\frac{1}{2}m\dot{z}^2$$

Covariant momenta

$$p_\rho = \frac{\partial T}{\partial \dot{\rho}} = \gamma_{\rho\rho}\dot{\rho} = m\dot{\rho} \qquad p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \gamma_{\phi\phi}\dot{\phi} = m\rho^2\dot{\phi} \qquad p_z = \frac{\partial T}{\partial \dot{z}} = \gamma_{zz}\dot{z} = m\dot{z}$$

Contravariant momenta

$$p^\rho = \dot{\rho}$$
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Comparing Lagrange and the Riemann *covariant force* equations

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$$= \frac{d\left(m\rho^2\dot{\phi}\right)}{dt} - 0 = m\rho^2\ddot{\phi}+2m\rho\dot{\rho}\dot{\phi} \quad \text{so: } \Gamma_{\rho\phi;\phi} = m\rho = \Gamma_{\phi\rho;\phi}$$

*Contra*variant equations are *acceleration* equations. $F^k = \gamma^{jk}F_j = \ddot{q}^k + \Gamma^k_{mn}\dot{q}^m\dot{q}^n$

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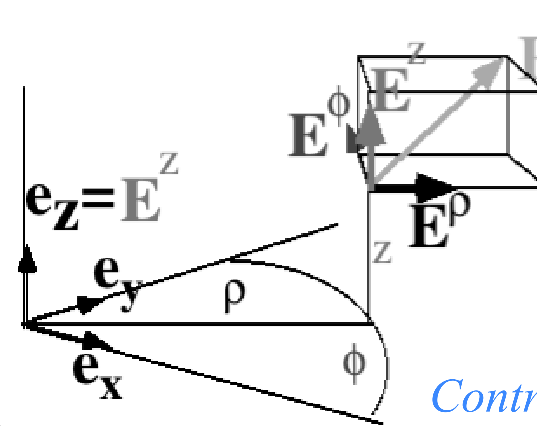

$$\mathbf{F} = F_\rho \mathbf{E}^\rho + F_\phi \mathbf{E}^\phi + F_z \mathbf{E}^z$$
$$= f_x \mathbf{e}_x + f_y \mathbf{e}_y + f_z \mathbf{e}_z$$

Covariant forces

$$F_\rho = f_x\frac{\partial x}{\partial\rho} + f_y\frac{\partial y}{\partial\rho} + f_z\frac{\partial z}{\partial\rho} = \quad f_x\cos\phi + f_y\sin\phi \quad +0$$

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Covariant kinetic metric

$$\gamma_{\rho\rho} = m\frac{\partial x_j}{\partial\rho}\frac{\partial x^j}{\partial\rho} = m\mathbf{E}_\rho \bullet \mathbf{E}_\rho = m\left(\cos^2\phi + \sin^2\phi\right) = m$$

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Contravariant kinetic metric

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Lagrangian

$$T = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\phi}^2 + \frac{1}{2}m\dot{z}^2$$

Covariant momenta

$$p_\rho = \frac{\partial T}{\partial\dot{\rho}} = \gamma_{\rho\rho}\dot{\rho} = m\dot{\rho} \qquad p_\phi = \frac{\partial T}{\partial\dot{\phi}} = \gamma_{\phi\phi}\dot{\phi} = m\rho^2\dot{\phi} \qquad p_z = \frac{\partial T}{\partial\dot{z}} = \gamma_{zz}\dot{z} = m\dot{z}$$

Contravariant momenta

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Comparing Lagrange and the Riemann *covariant force* equations

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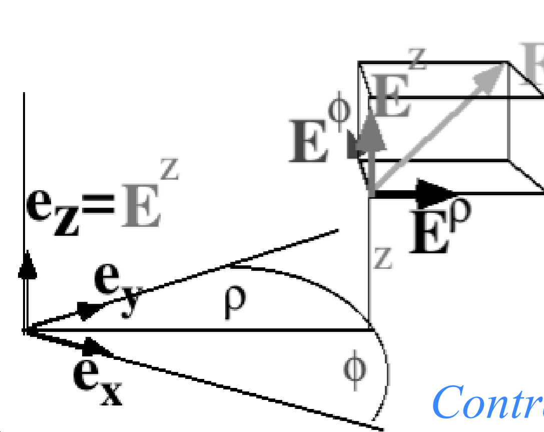

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Covariant forces

$$F_\rho = f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = \quad f_x \cos\phi + f_y \sin\phi \quad + 0$$

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Covariant momenta

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Contravariant momenta

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Comparing Lagrange and the Riemann *covariant force* equations

$$\boxed{F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell}} = \boxed{\gamma_{\ell n}\ddot{q}^n + \Gamma_{mn;\ell}\dot{q}^m\dot{q}^n}$$

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$$= \frac{d\left(m\rho^2\dot{\phi}\right)}{dt} - 0 = m\rho^2\ddot{\phi} + 2m\rho\dot{\rho}\dot{\phi} \quad \text{so: } \Gamma_{\rho\phi;\phi} = m\rho = \Gamma_{\phi\rho;\phi}$$

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$$F^\phi = \gamma^{\phi\phi}F_\phi = \ddot{q}^\rho + \Gamma^\rho_{mn}\dot{q}^m\dot{q}^n$$

$$= \ddot{\phi} + 2\dot{\rho}\dot{\phi}/\rho \quad \text{so: } \Gamma^\phi_{\rho\phi} = 1/\rho = \Gamma^\phi_{\phi\rho} \qquad \gamma^{\phi\phi} = 1/\left(m\rho^2\right)$$

$$\ddot{\rho} = F^\rho + \rho\dot{\phi}^2 \qquad \textit{(Centrifugal acceleration)}$$

$$\ddot{\phi} = F^\phi - 2\dot{\rho}\dot{\phi}/\rho \qquad \textit{(Coriolis acceleration)}$$

Rewriting GCC Lagrange equations :

(Review of Lecture 11)

$$\dot{p}_r \equiv \frac{dp_r}{dt} = M\,\ddot{r} = M\,r\,\dot{\phi}^2 - \frac{\partial U}{\partial r}$$

Centrifugal (center-fleeing) force equals total Centripetal (center-pulling) force

$$\dot{p}_\phi \equiv \frac{dp_\phi}{dt} = 2Mr\dot{r}\dot{\phi} + Mr^2\ddot{\phi} = 0 - \frac{\partial U}{\partial \phi}$$

Torque relates to two distinct parts: Coriolis and angular acceleration

Angular momentum p_ϕ is conserved if potential U has no explicit ϕ-dependence

Conventional forms

radial force: $M\,\ddot{r} = M\,r\,\dot{\phi}^2 - \frac{\partial U}{\partial r}$ *angular force or torque:* $Mr^2\ddot{\phi} = -2Mr\dot{r}\dot{\phi} - \frac{\partial U}{\partial \phi}$

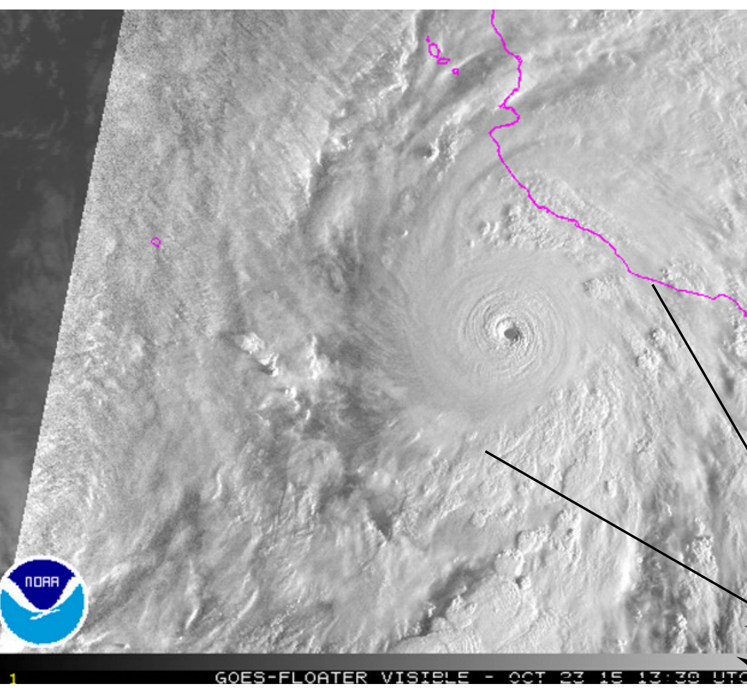
Field-free (U=0)

radial acceleration: $\ddot{r} = r\,\dot{\phi}^2$ *angular acceleration:* $\ddot{\phi} = -2\frac{\dot{r}\dot{\phi}}{r}$

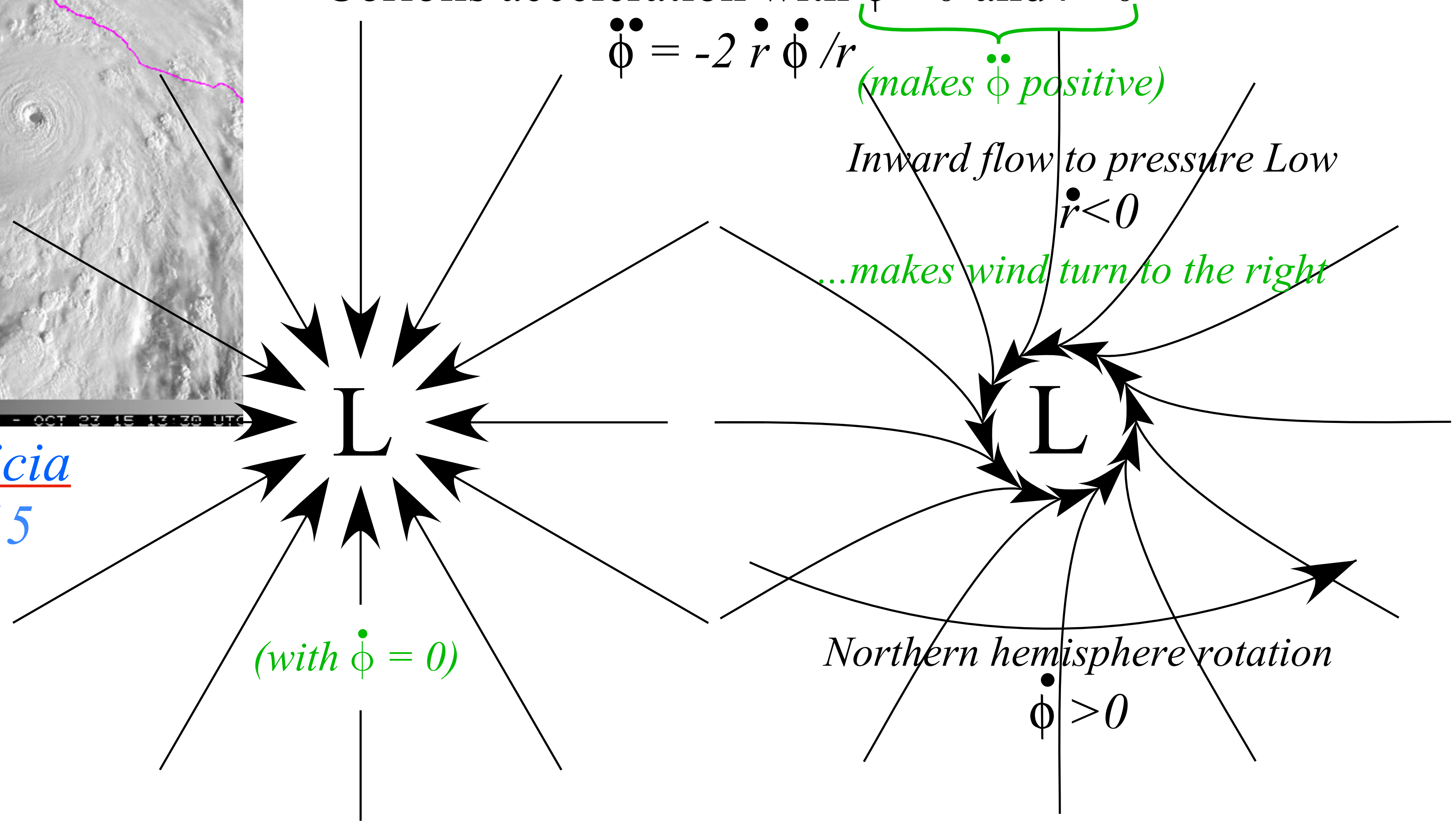
Because Earth rotation is <u>counter</u>-clockwise (positive) in North

Coriolis acceleration with $\dot{\phi} > 0$ and $\dot{r} < 0$

$\ddot{\phi} = -2\,\dot{r}\,\dot{\phi}\,/r$ *(makes $\ddot{\phi}$ positive)*

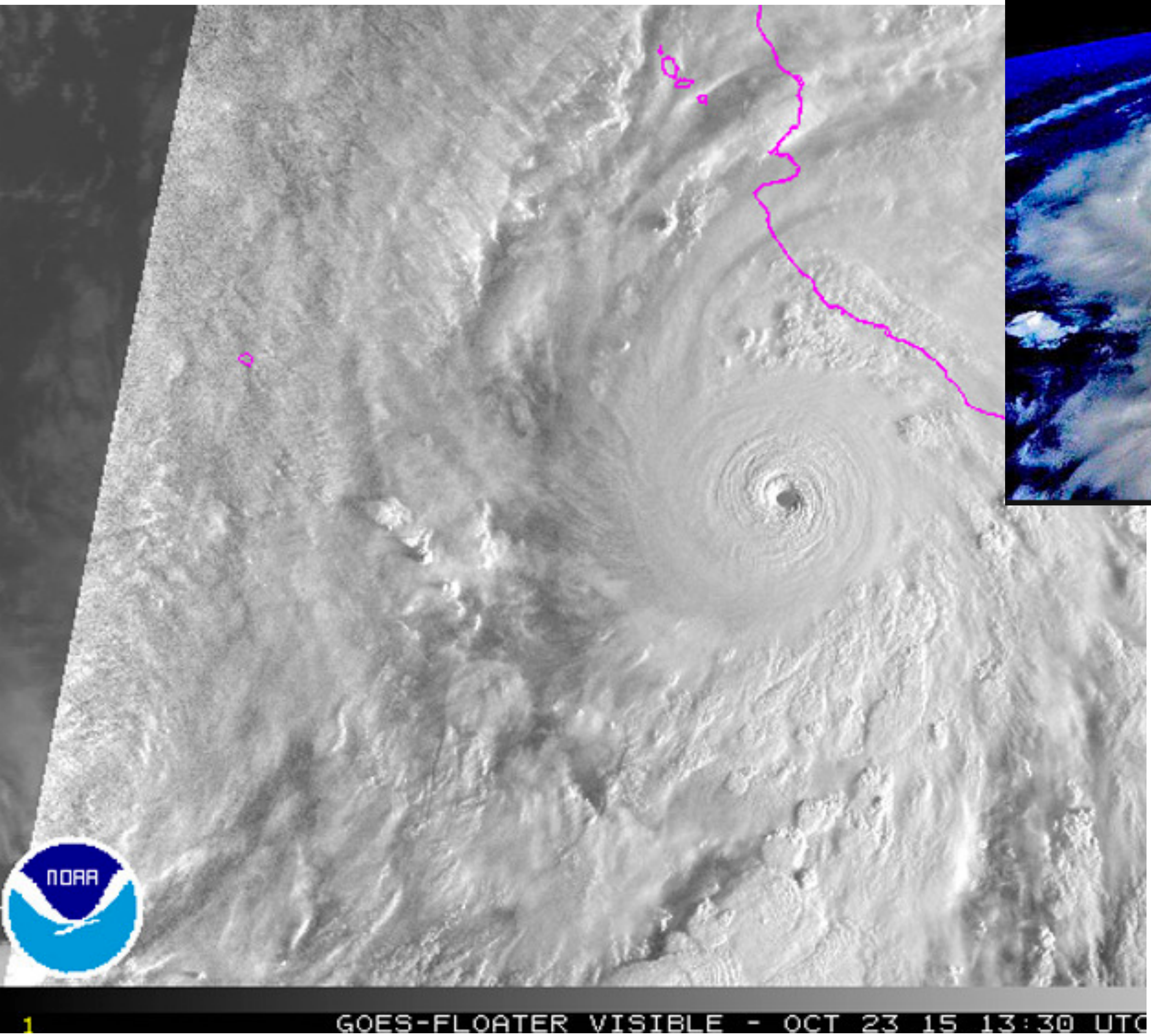


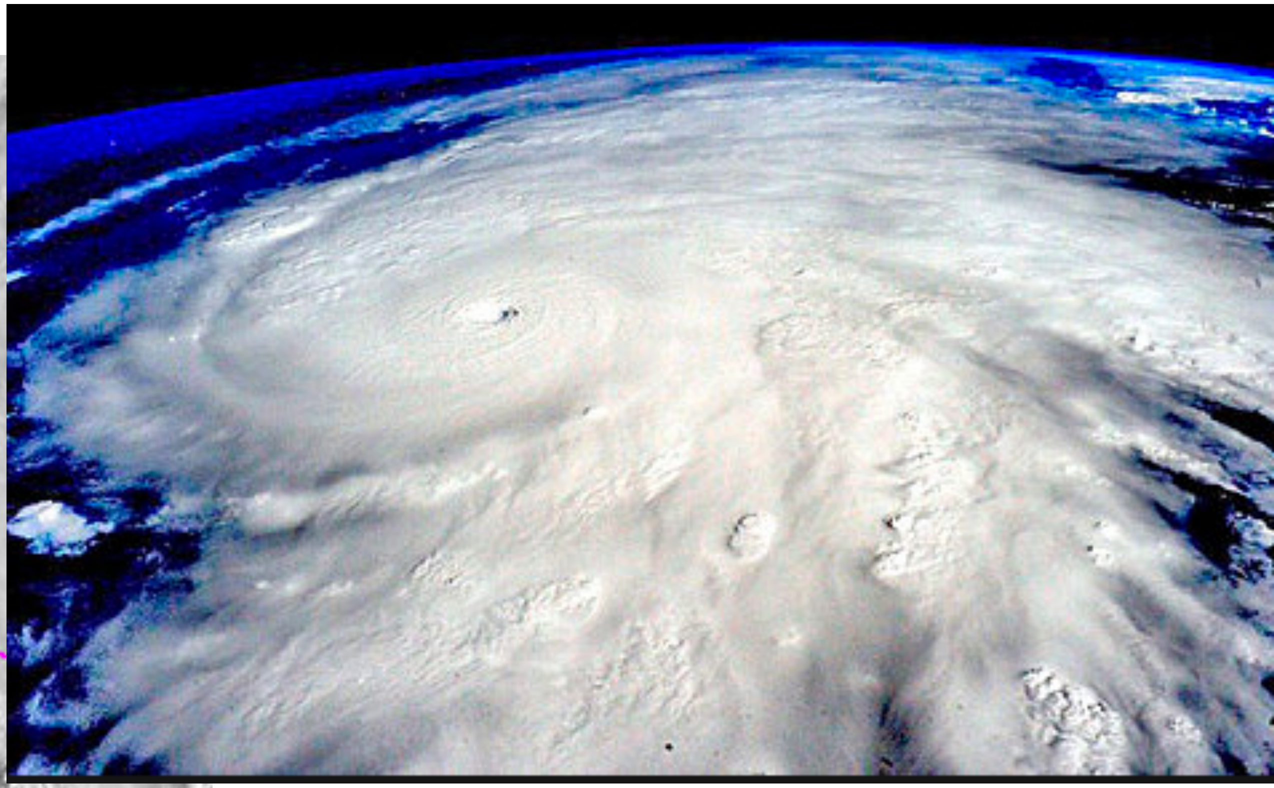
Hurricane Patricia
October 23, 2015



Effect on <u>Northern</u> Hemisphere local weather

Cyclonic flow around lows





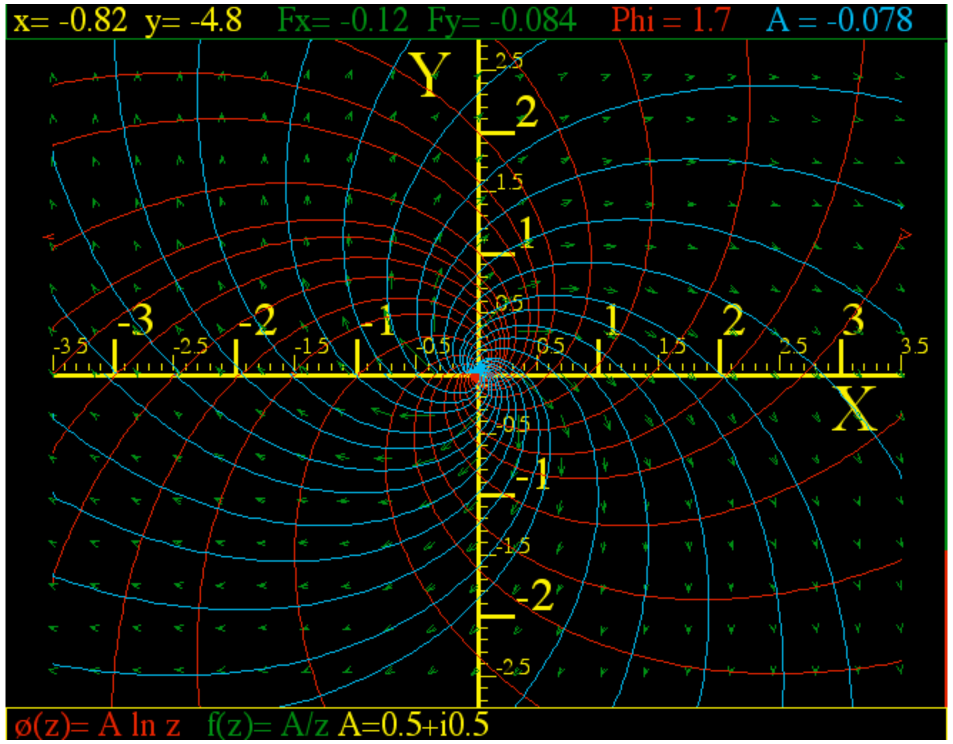
Hurricane Patricia
October 23, 2015

https://www.google.com/search?q=Satellite+view+of+Patricia&biw=1811&bih=1247&tbm=isch&tbo=u&source=univ&sa=X&ved=0CD0QsARqFQoTCLb17N728sgCFdA0iAodl4kMsg

Riemann-forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

Christoffel relation to Coriolis coefficients

→ *Mechanics of ideal fluid vortex*

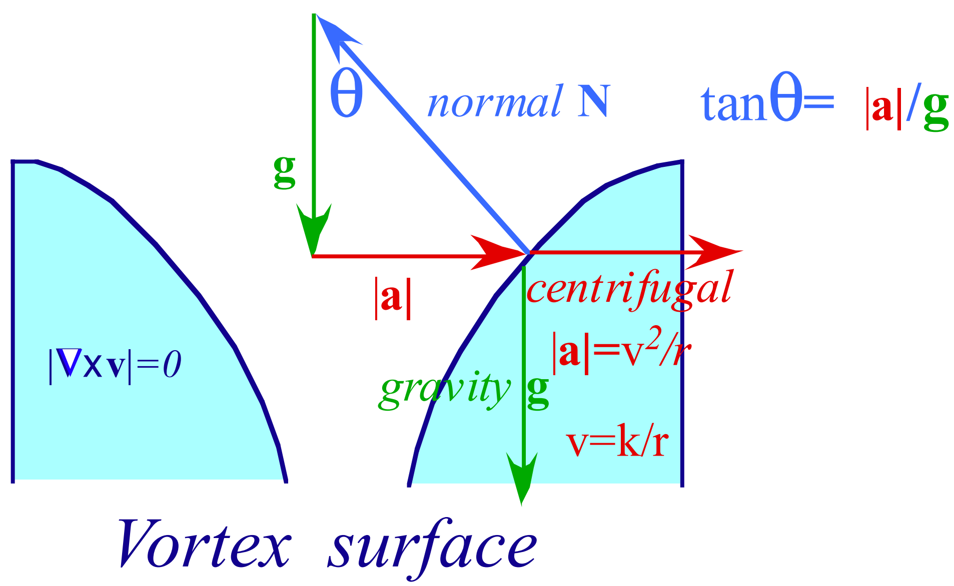


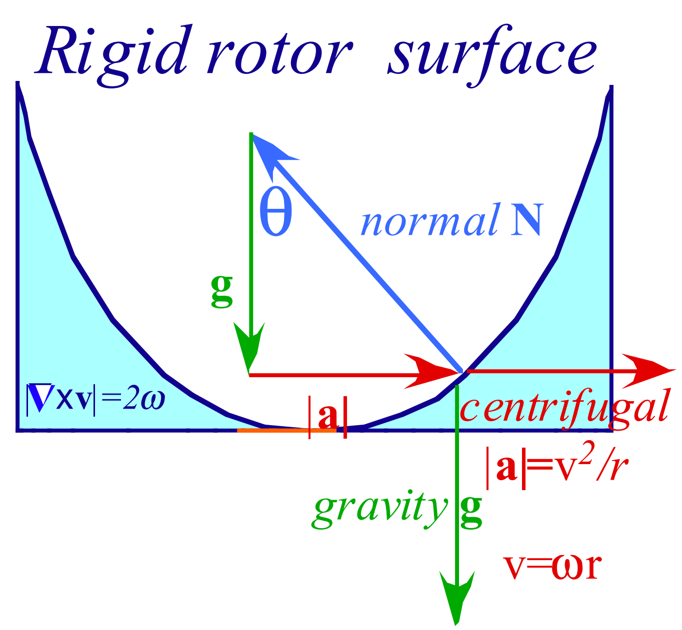
Mechanics of ideal fluid vortex

Rotating fluid surface has normal acceleration **N** *be sum of gravity* **g** *and centrifugal* $\mathbf{a}=\mathbf{e}_a\, v^2/r$

Case 1: Vortex with velocity field $v=k/r$

Case 2: Rigidly rotating fluid with velocity field $v=\omega\, r$





Mechanics of ideal fluid vortex

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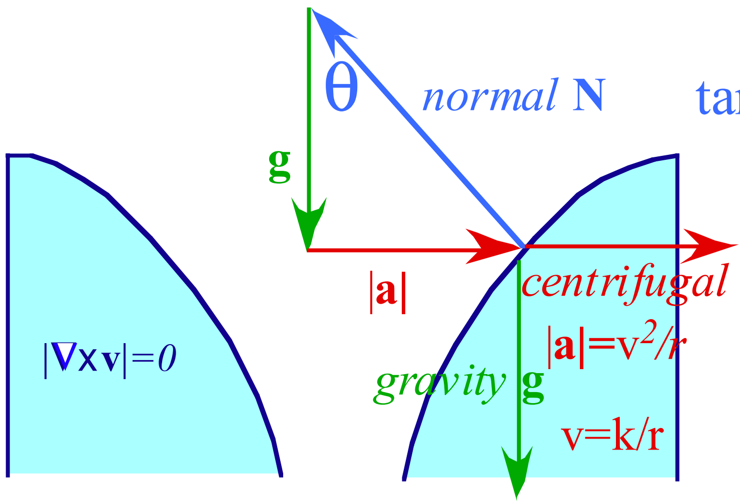
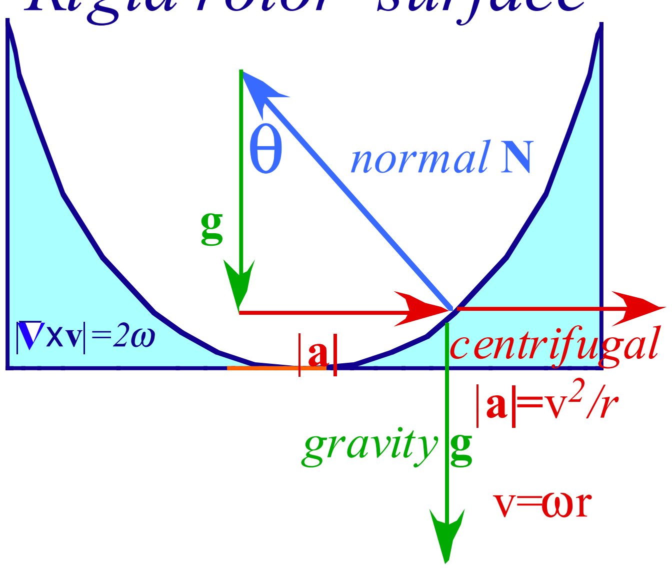
In either case slope θ of normal **N** *is:*

$$\tan\theta = \frac{dz}{dr} = \frac{|a|}{g} = \frac{v^2/r}{g}$$



Vortex surface

Rigid rotor surface



Mechanics of ideal fluid vortex

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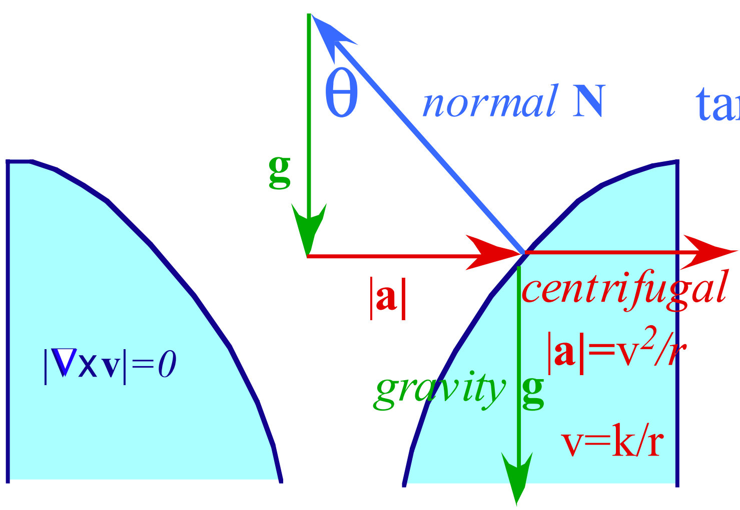
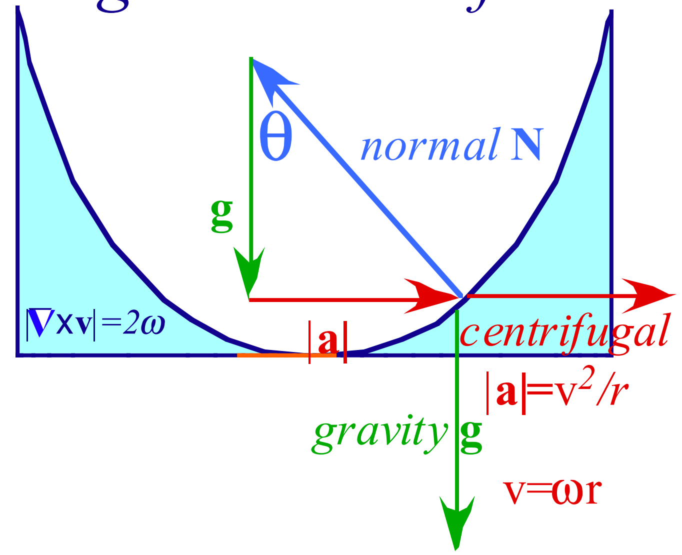


$\tan\theta = |\mathbf{a}|/\mathbf{g}$

Rigid rotor surface



Vortex surface

$$\tan\theta = \frac{dz}{dr} = \frac{|a|}{g} = \frac{\mathrm{v}^2/r}{g} = \frac{k^2}{gr^3}$$

$$\tan\theta = \frac{dz}{dr} = \frac{|a|}{g} = \frac{\mathrm{v}^2/r}{g} = \frac{\omega^2}{g} r^1$$

Mechanics of ideal fluid vortex

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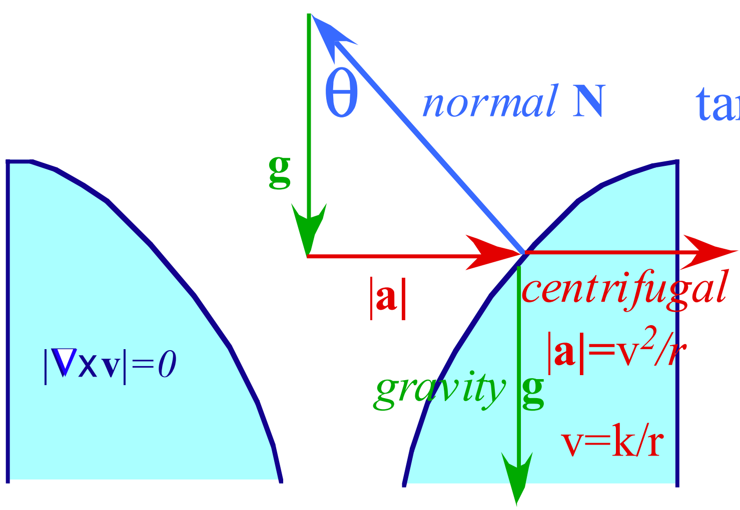
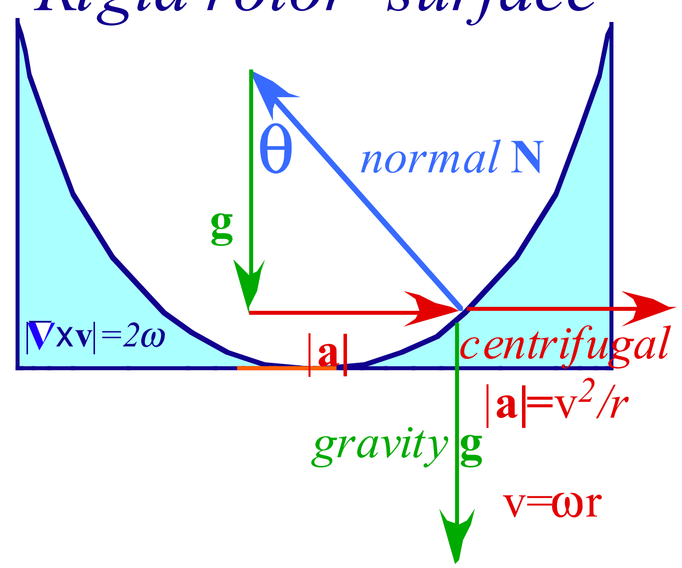


tanθ= |**a**|/**g**

Vortex surface

Rigid rotor surface



$$\tan\theta = \frac{dz}{dr} = \frac{|a|}{g} = \frac{\mathrm{v}^2/r}{g} = \frac{k^2}{gr^3}$$

Integrating:

$$z(r) = \int \frac{dz}{dr}\,dr = \int \frac{k^2}{gr^3}\,dr = -\frac{k^2}{2gr^2}$$

$$\tan\theta = \frac{dz}{dr} = \frac{|a|}{g} = \frac{\mathrm{v}^2/r}{g} = \frac{\omega^2}{g} r^1$$

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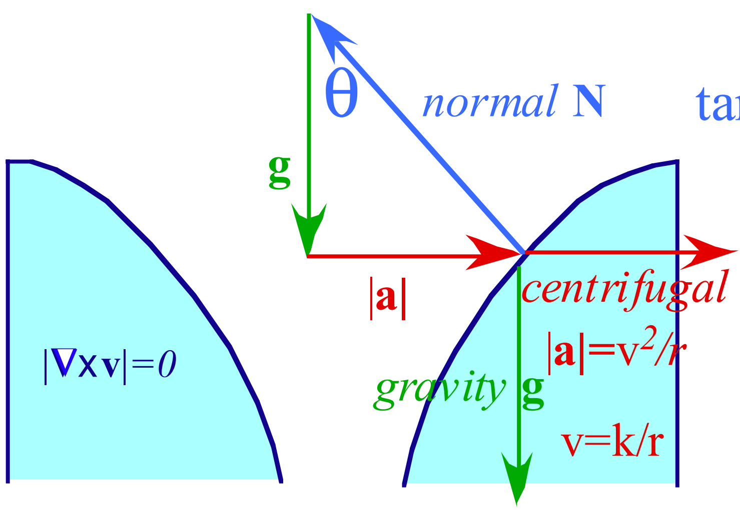
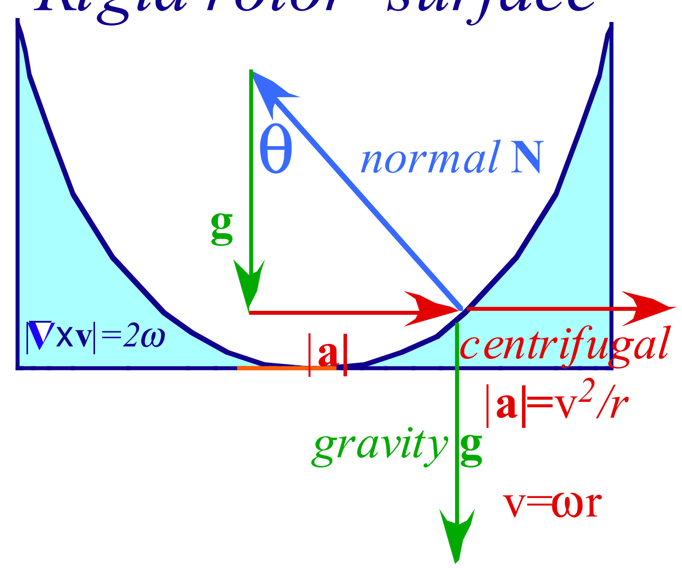


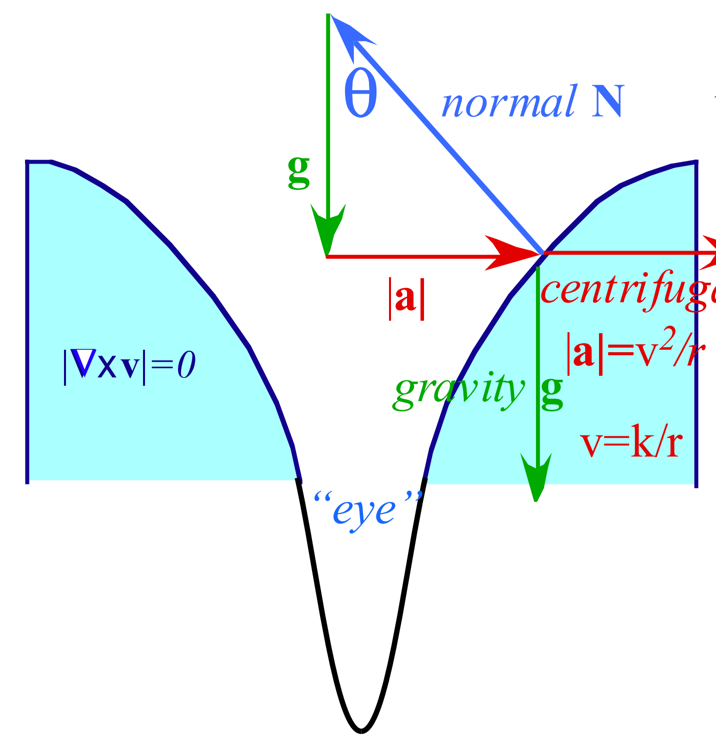
$\tan\theta = |\mathbf{a}|/\mathbf{g}$

Vortex surface

Rigid rotor surface



Ideal vortex without drain has a parabolic "eye"



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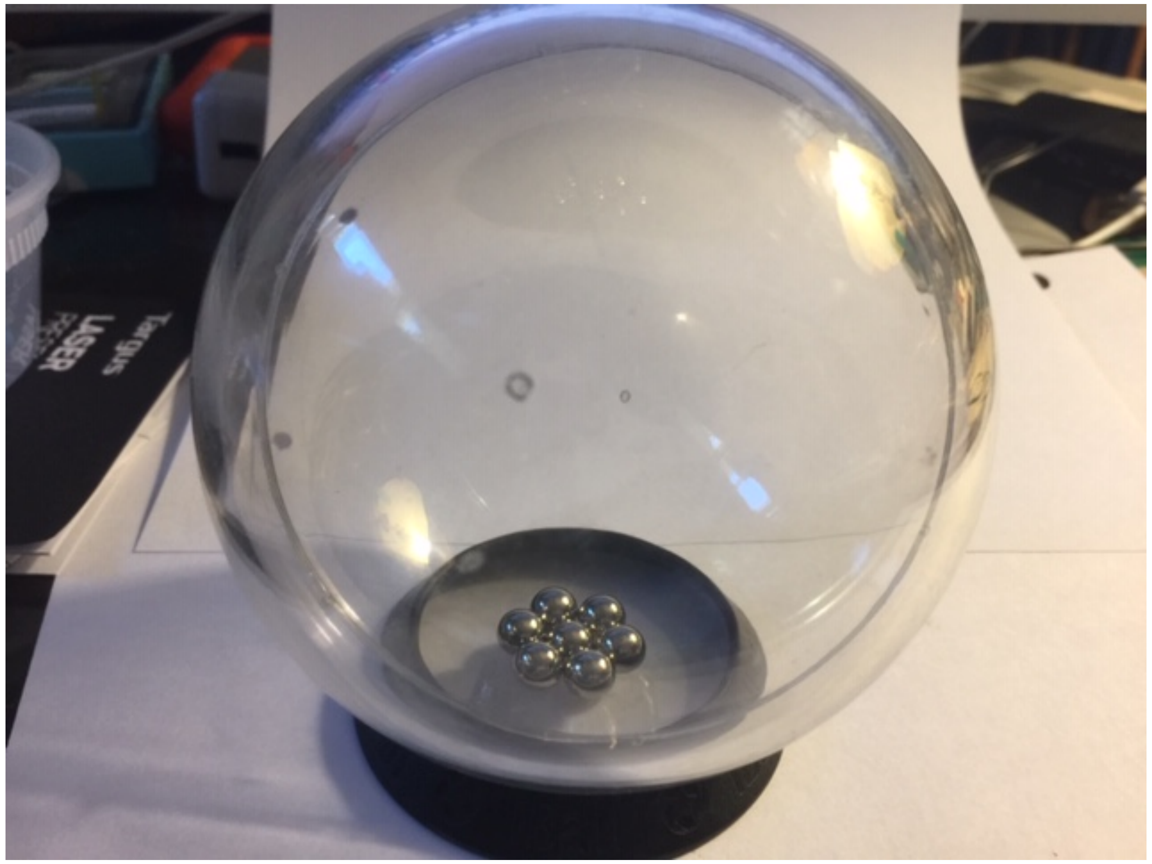
Integrating:

$$z(r) = \int \frac{dz}{dr}dr = \int \frac{\omega^2}{g}r^1\,dr = \frac{\omega^2}{2g}r^2$$

Somewhat analogous to the "Sophomore-Physics Earth"

Pool vortex movie



→ *Separation of GCC Equations: Effective Potentials*

Small (n_ρ:m_ϕ)-periodic and quasi-periodic oscillations

2D Spherical pendulum or "Bowl-Bowling" and the "I-Ball"

(n_ρ:m_ϕ)=(2:1) vs (1:1) periodic and quasi-periodic orbits

Demo at YouTube

Separation of GCC Equations: Effective Potentials

$$H = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n + V = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\phi}^2 + \frac{1}{2}m\dot{z}^2 + V$$

(Numerically correct ONLY!)

$$= \frac{1}{2}\gamma^{mn}p_m p_n + V = \frac{1}{2m}p_\rho^2 + \frac{1}{2m\rho^2}p_\phi^2 + \frac{1}{2m}p_z^2 + V$$

(Formally **and** Numerically correct)

Separation of GCC Equations: Effective Potentials (For isotropic $H(r,p_r,\phi,p_\phi)$)

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If potential V is *isotropic* (cylindrical) function of radius ρ. ($V = V(\rho)$)

H has no explicit ϕ–dependence and the ϕ–momenta is constant.

$$m\rho^2\dot{\phi} = p_\phi = const. = \mu$$

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If *H* has no explicit *z*–dependence then the *z*–momenta is constant, too.

$$m\dot{z} = p_z = const. = k$$

Separation of GCC Equations: Effective Potentials

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(Let k =0)

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(Let k =0)

Symmetry reduces problem to a one-dimensional form.

$$H = \frac{1}{2m}p_\rho^2 + V^{eff}(\rho) = E = const.$$

Separation of GCC Equations: Effective Potentials

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An *effective potential* $V^{eff}(\rho)$ has a *centrifugal barrier*.

$$V^{eff}(\rho) = \frac{\mu^2}{2m\rho^2} + V(\rho)$$ (*centrifugal barrier* prevents ρ from going to zero.)

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Velocity relations:

$$\dot{\phi} = \mu / \left(m\rho^2\right)$$

$$\dot{\rho} = \frac{d\rho}{dt} = \frac{\partial H}{\partial p_\rho} = \frac{p_\rho}{m} = \pm\sqrt{\frac{2}{m}\left(E - V^{eff}(\rho)\right)}$$

Separation of GCC Equations: Effective Potentials

$$H = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n + V = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\phi}^2 + \frac{1}{2}m\dot{z}^2 + V$$
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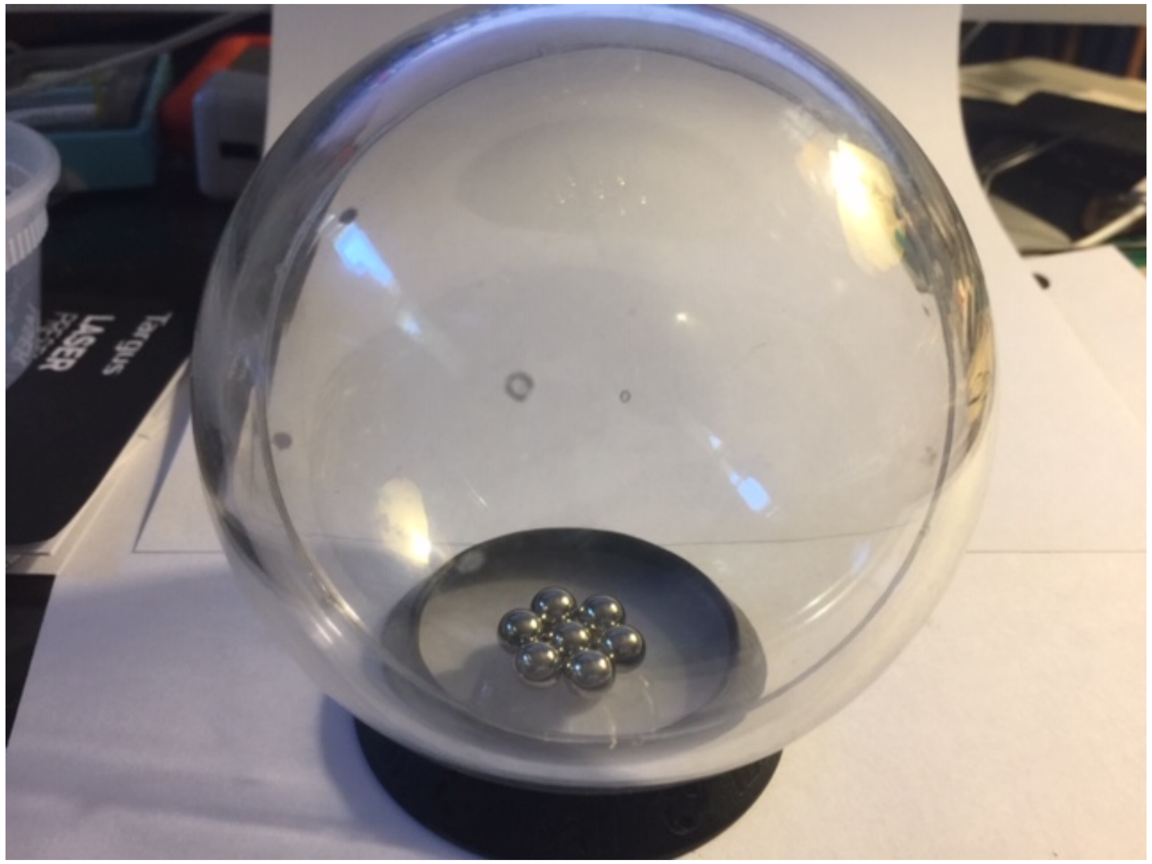
Equations solved by a *quadrature integral* for time versus radius.

$$\int_{t_0}^{t_1} dt = \int_{\rho_0}^{\rho_1} \frac{d\rho}{\sqrt{\frac{2}{m}\left(E - V^{eff}(\rho)\right)}} = \left(\text{Travel time } \rho_0 \text{ to } \rho_1\right) = t_1 - t_0$$



Separation of GCC Equations: Effective Potentials

→ *Small (n_ρ:m_ϕ)-periodic and quasi-periodic oscillations*

2D Spherical pendulum or "Bowl-Bowling" and the "I-Ball"

(n_ρ:m_ϕ)=(2:1) vs (1:1) periodic and quasi-periodic orbits

Demo at YouTube

Small radial oscillations

Stable minimal-energy radius will satisfy a zero-slope equation.

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_0} = 0 \,, \quad \text{with: } \left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_0} > 0 \,.$$

A Taylor series around this minimum can be used to estimate orbit properties for small oscillations.

$$V^{eff}(\rho) = V^{eff}(\rho_0) + 0 + \frac{1}{2}(\rho - \rho_0)^2 \left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_0}$$

Stable flat $\left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_0} > 0$ Unstable flat $\left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_0} < 0$

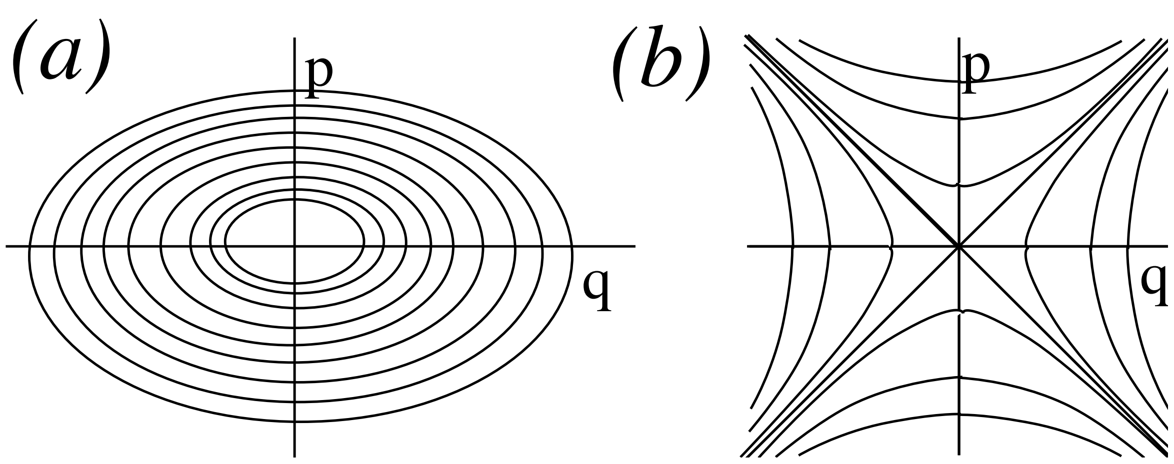


Fig. 2.7.4 Phase paths around fixed points (a) Stable point (b) Unstable saddle point

Small radial oscillations

Stable minimal-energy radius will satisfy a zero-slope equation.

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}} = 0 \ , \quad \text{with:} \ \left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}} > 0 \ .$$

A Taylor series around this minimum can be used to estimate orbit properties for small oscillations.

$$V^{eff}(\rho) = V^{eff}(\rho_{stable}) + 0 + \frac{1}{2}(\rho - \rho_{stable})^2 \left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}}$$

An effective "spring constant" at the stable point giving approximate frequency of oscillation.

$$k^{eff} = \left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}} \qquad \omega_{\rho_{stable}} = \sqrt{\frac{k^{eff}}{m}} = \sqrt{\frac{1}{m}\left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}}}$$

Small radial oscillations

Stable minimal-energy radius will satisfy a zero-slope equation.

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}} = 0\ , \quad \text{with:} \quad \left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}} > 0\ .$$

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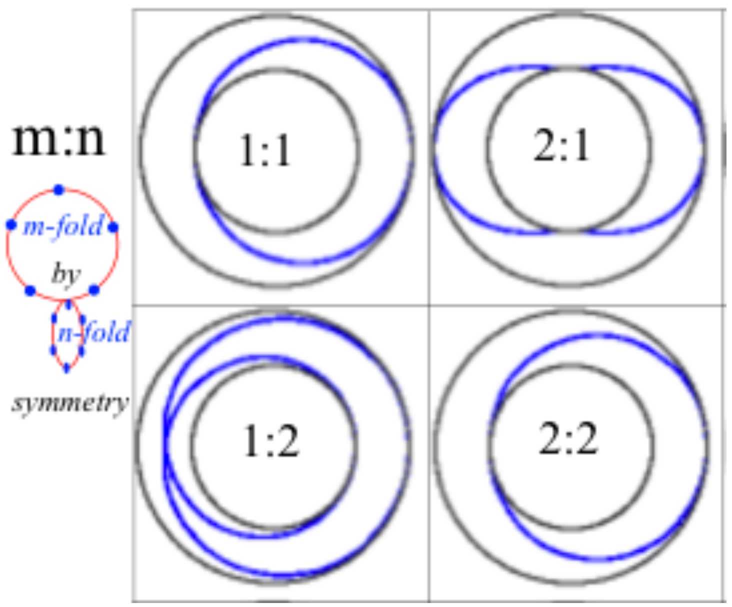
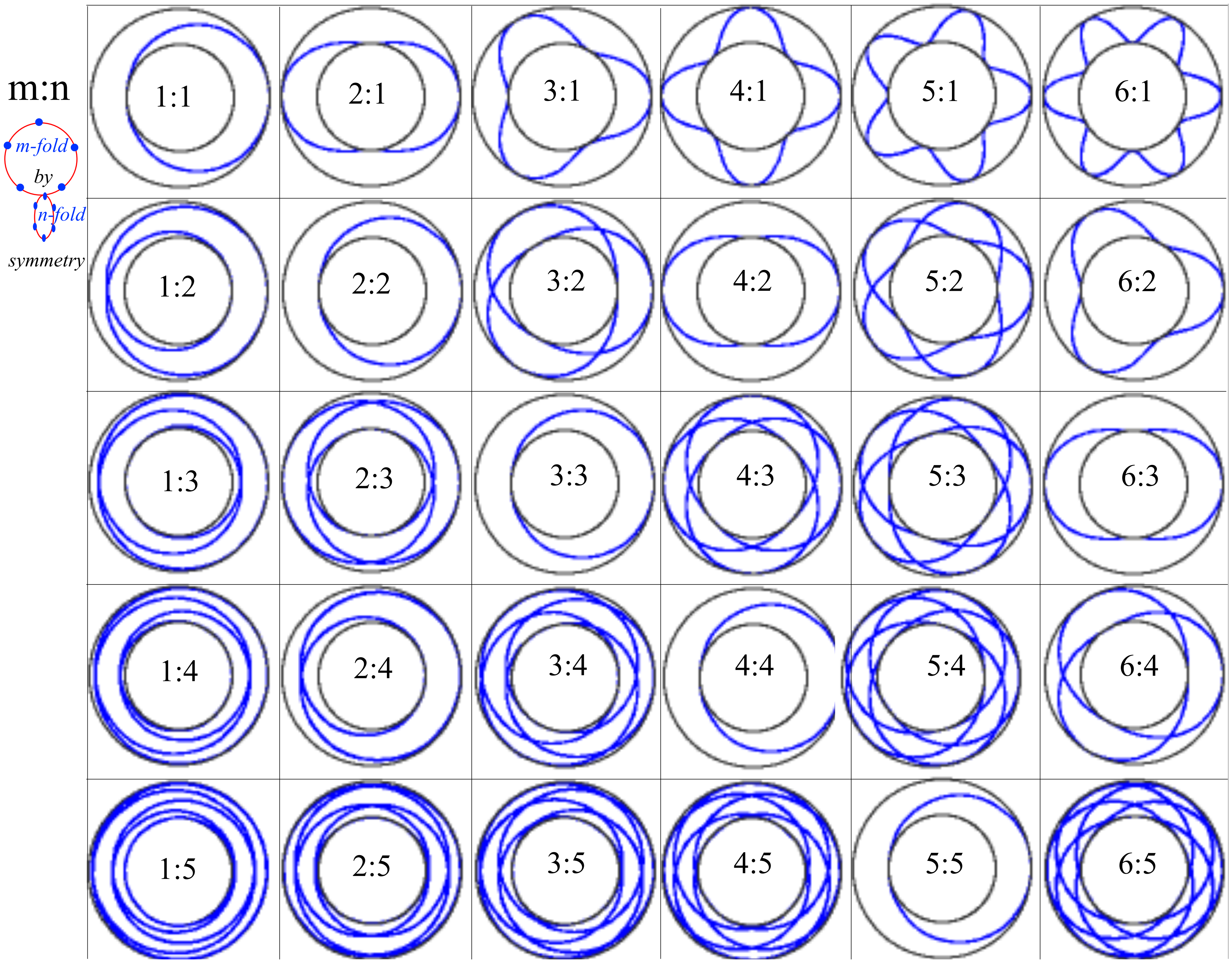
Some generic shapes resulting from various ratios $n_\rho : n_\phi$





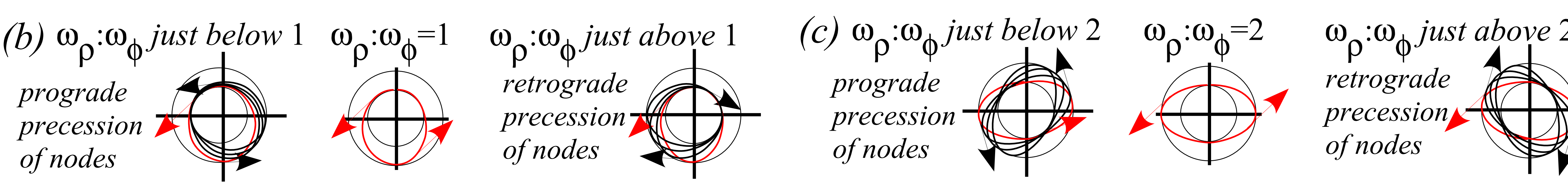
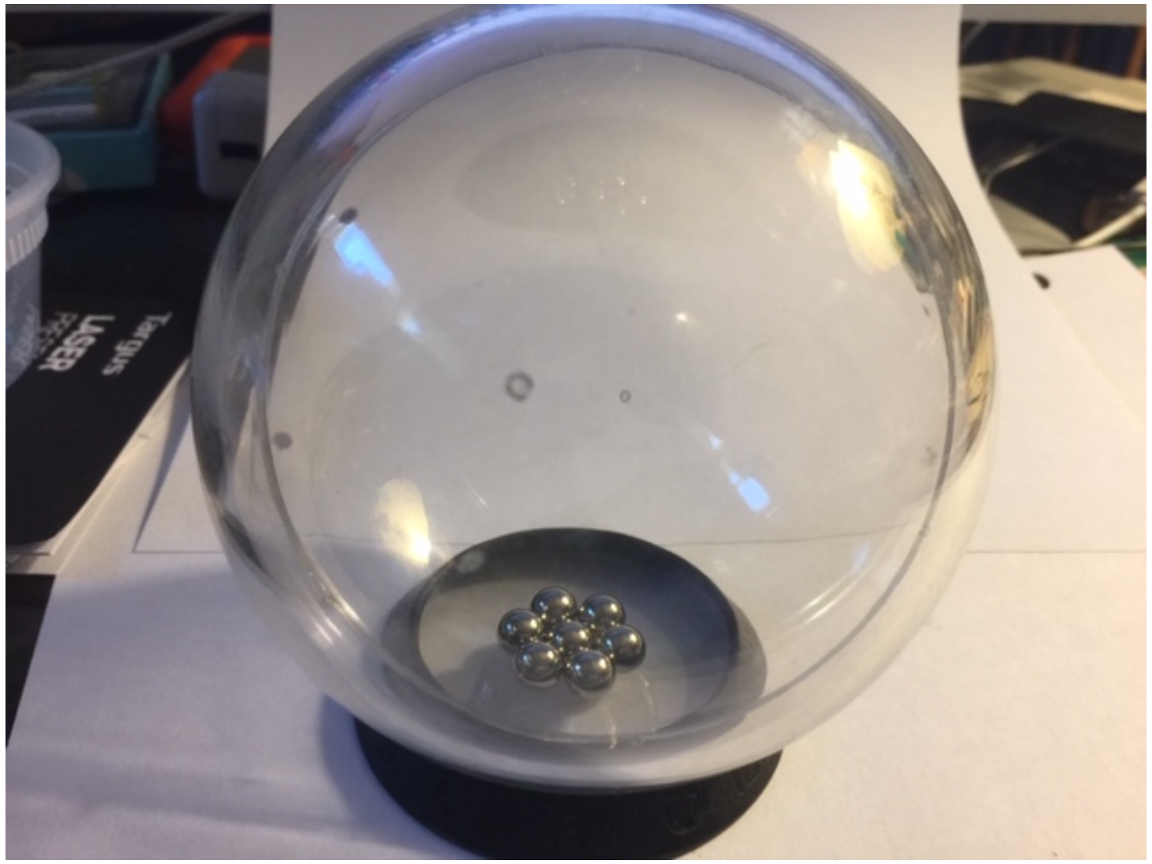


Fig. 3.8.1 from *CMwBang!* text



Separation of GCC Equations: Effective Potentials

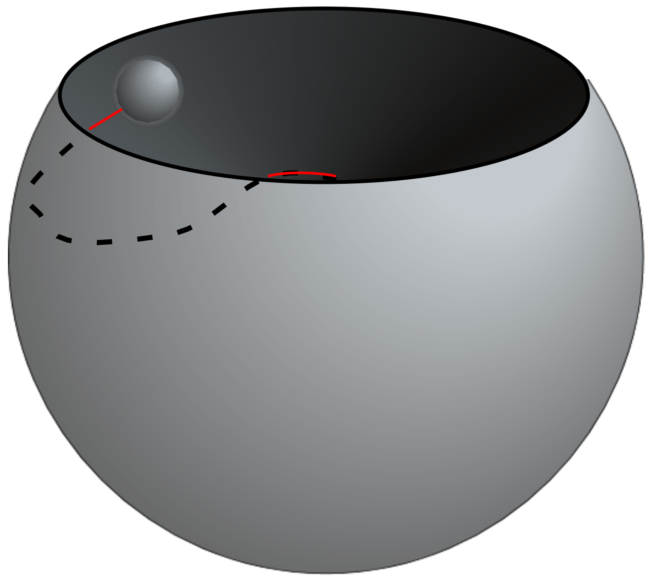
Small (n_ρ:m_ϕ)-periodic and quasi-periodic oscillations

2D Spherical pendulum or "Bowl-Bowling" and the "I-Ball"

(n_ρ:m_ϕ)=(2:1) vs (1:1) periodic and quasi-periodic orbits

Demo at YouTube

2D Spherical pendulum or "Bowl-Bowling" and the "I-Ball"



Spherical coordinates: $\{q^1=r, q^2=\theta, q^3=\phi\}$ obvious choice:

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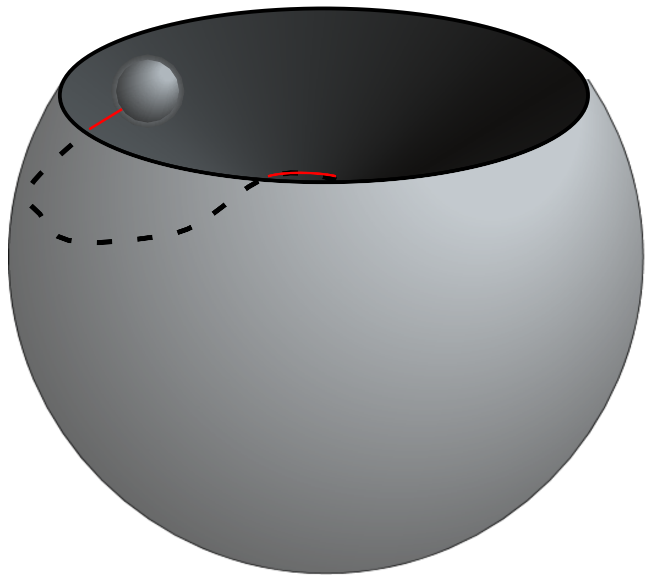

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Jacobian matrices and determinants:

Reduced to cylindrical coordinates:

$$J=\begin{array}{c} \begin{array}{ccc} \mathbf{E}_r & \mathbf{E}_\theta & \mathbf{E}_\phi \end{array} \\ \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{pmatrix} \end{array} = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} \xrightarrow[\substack{\theta=\pi/2 \\ r=\rho}]{} \begin{pmatrix} \cos\phi & 0 & -\rho\sin\phi \\ \sin\phi & 0 & \rho\cos\phi \\ 0 & -\rho & 0 \end{pmatrix}$$

$$\det J=\det J^{\mathrm{T}}=\frac{\partial\{xyz\}}{\partial\{r\theta\phi\}}=r^2\sin\theta \xrightarrow[\substack{\theta=\pi/2 \\ r=\rho}]{} \rho^2$$

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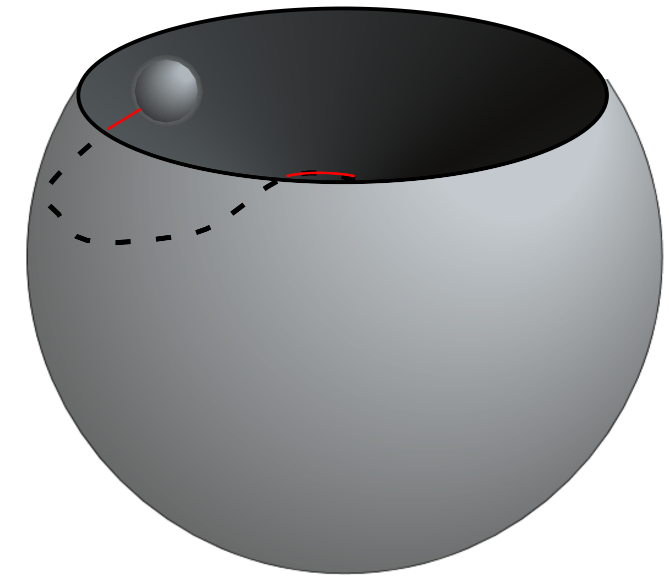

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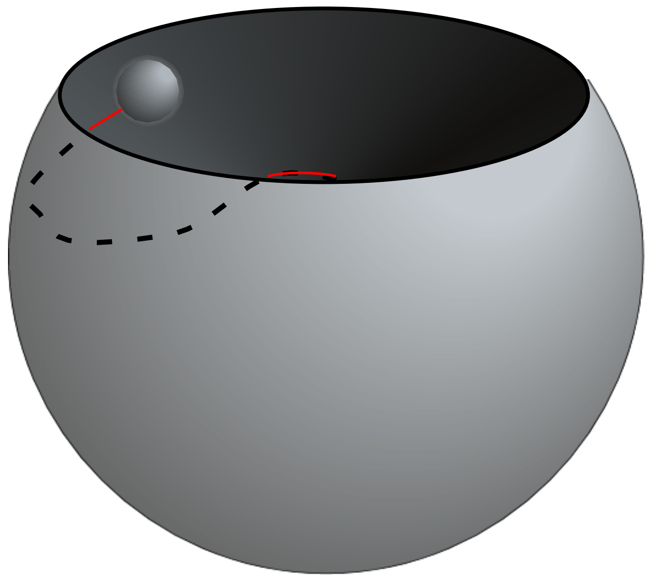

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(Lagrangian form) *(Hamiltonian form)*

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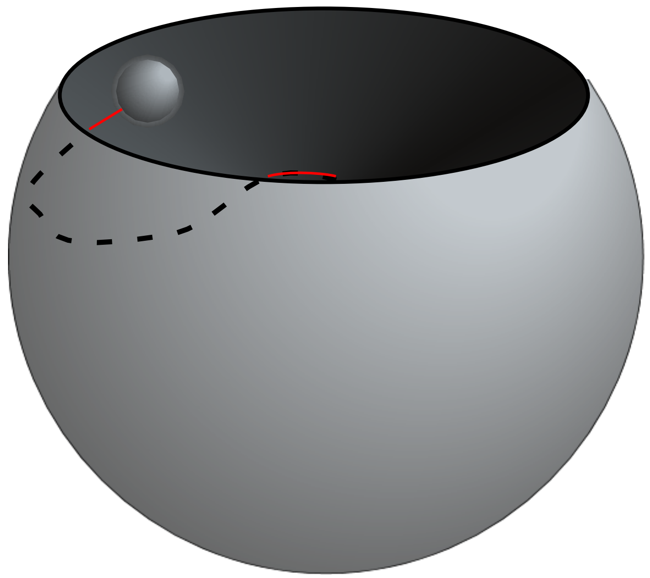
2D Spherical pendulum or "Bowl-Bowling"



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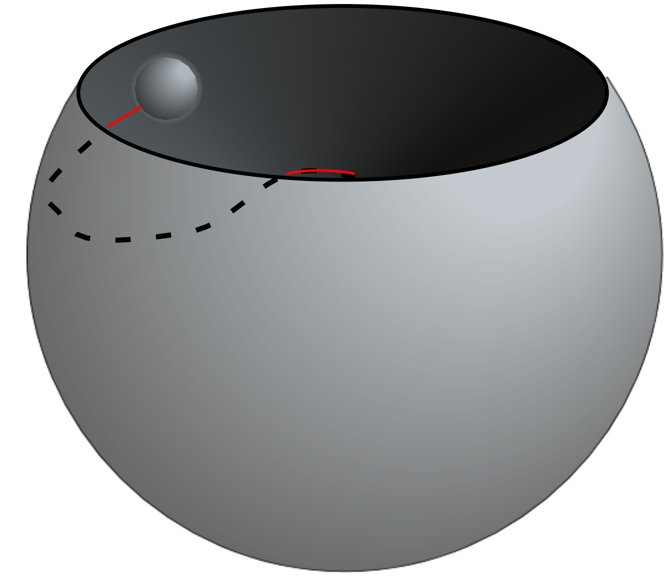
Spherical coordinates with constant radius r implies conserved azimuthal momentum:

$$p_\phi\equiv\frac{\partial T}{\partial\dot\phi}=m(R^2\sin^2\theta)\dot\phi\ =const.$$

2D Spherical pendulum or "Bowl-Bowling"



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Total Energy from Hamiltonian $E=T+V(gravity)=const.$:

$$E=\frac{mR^2}{2}\dot\theta^2+V^{effective}(\theta)=\frac{mR^2}{2}\dot\theta^2+\frac{p_\phi^2}{2mR^2\sin^2\theta}+mgR\cos\theta\quad=\quad\alpha\dot\theta^2+\frac{\delta}{\sin^2\theta}+\gamma\cos\theta$$

Let : $\quad\alpha=\frac{mR^2}{2},\quad\delta=\frac{p_\phi^2}{2mR^2},\quad\gamma=mgR\quad$ where: $\quad p_\phi=mR^2\sin^2\theta(\dot\phi\;)$

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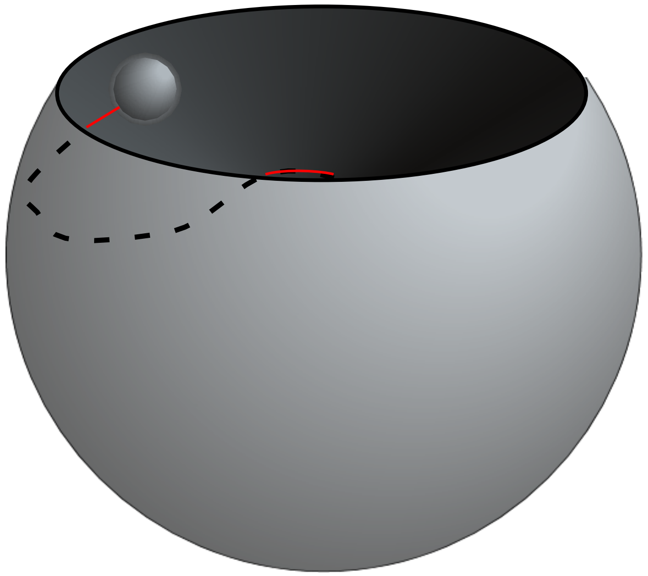

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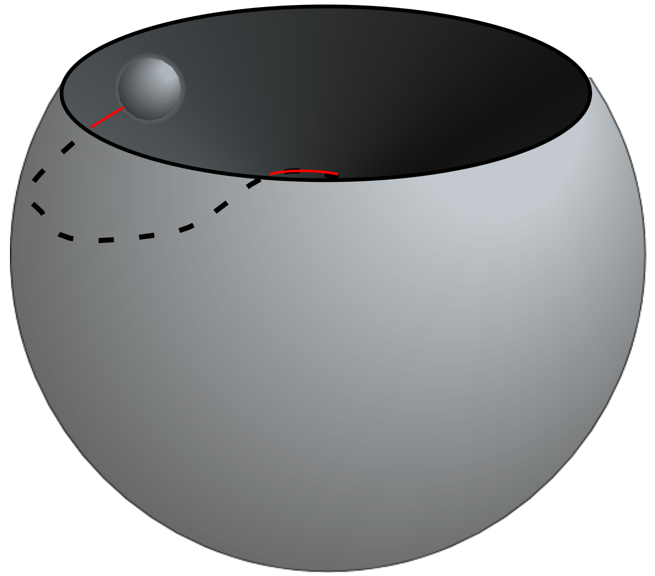

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Equilibrium point of stable orbit

$$\frac{dV^{effective}(\theta)}{d\theta}=\frac{-2\delta\cos\theta}{\sin^3\theta}-\gamma\sin\theta=0=\frac{-2p_\phi^2\cos\theta}{2mR^2\sin^3\theta}-mgR\sin\theta$$

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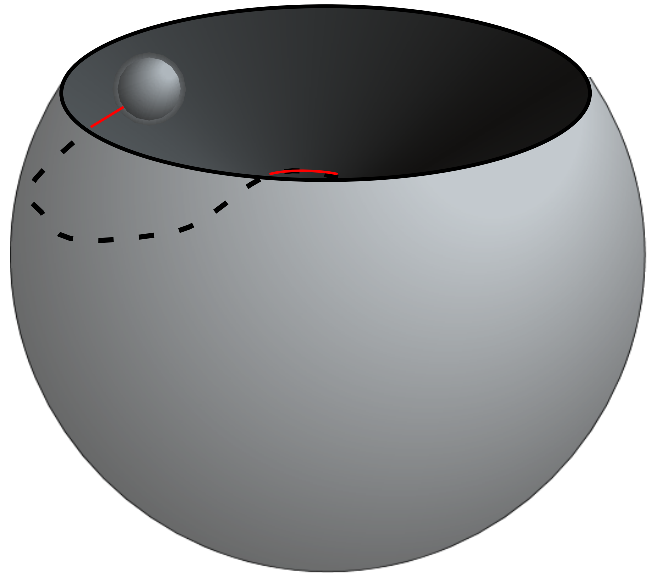

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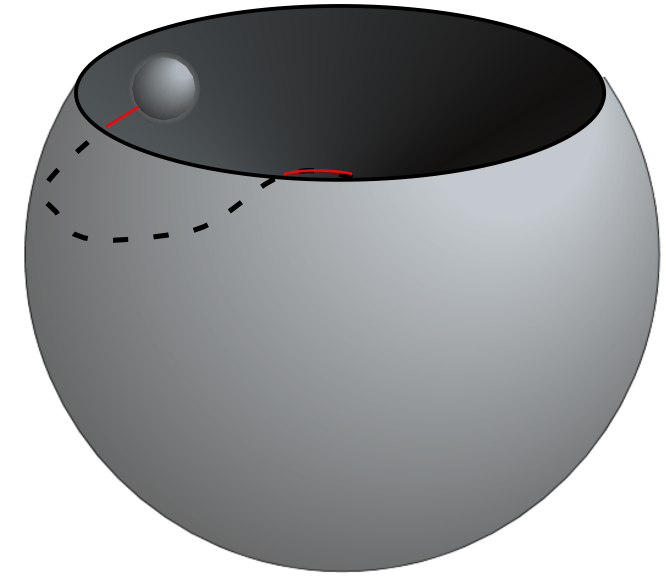
Equilibrium point of stable orbit and small oscillation frequency near equilibrium:

$$\frac{dV^{effective}(\theta)}{d\theta} = \frac{-2\delta\cos\theta}{\sin^3\theta} - \gamma\sin\theta = 0 = \frac{-2p_\phi^2\cos\theta}{2mR^2\sin^3\theta} - mgR\sin\theta$$

$$\left(\omega_\theta^{equil}\right)^2 = \frac{1}{mR^2}\left.\frac{d^2V^{effective}(\theta)}{d\theta^2}\right|_{equil}$$

2D Spherical pendulum or "Bowl-Bowling"



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$$0=(mR^2\sin\theta)\dot{\phi}^2\cos\theta-mgR\sin\theta$$ or: $$\dot{\phi}_{equil}^2=-\frac{g}{R\cos\theta_{equil}}$$

(Polar angle librational frequency ω_θ^{equil} is related to azimuthal frequency $\dot{\phi}_{equil}^2$.)

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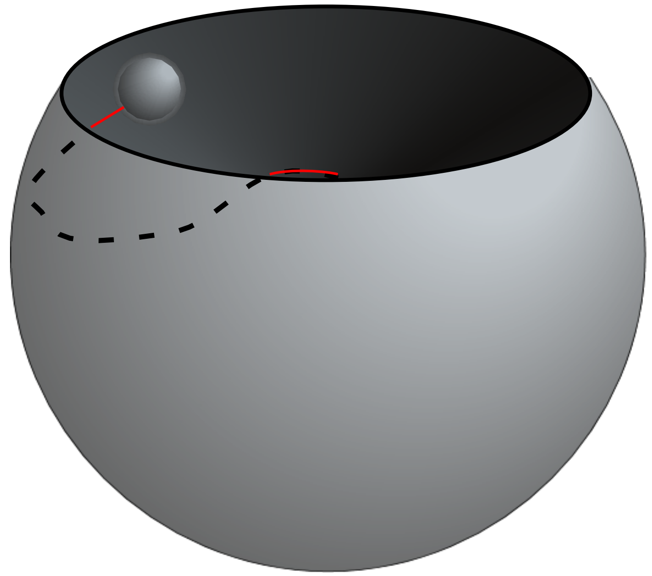

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V-Derivative for small oscillation frequency:

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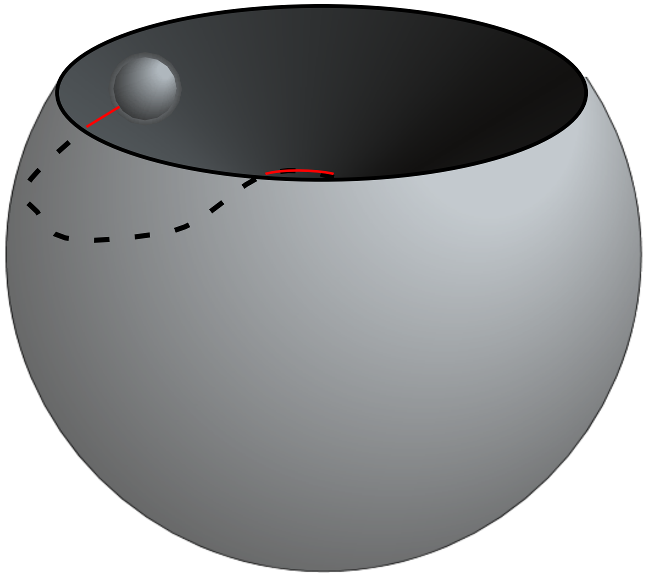
$$=-mgR\cos\theta+mR^2\dot{\phi}^2\left(1+2\cos^2\theta\right)$$

2D Spherical pendulum or "Bowl-Bowling"



Total Energy from Hamiltonian *E=T+V(gravity)=const. :*

$$E=\frac{mR^2}{2}\dot{\theta}^2+V^{effective}(\theta)=\alpha\dot{\theta}^2+\frac{\delta}{\sin^2\theta}+\gamma\cos\theta$$

Let : $\alpha=\frac{mR^2}{2}$, $\delta=\frac{p_\phi^2}{2mR^2}$, $\gamma=mgR$ where: $p_\phi=mR^2\sin^2\theta(\dot{\phi}\ )$

Equilibrium point of stable orbit and small oscillation frequency near equilibrium:

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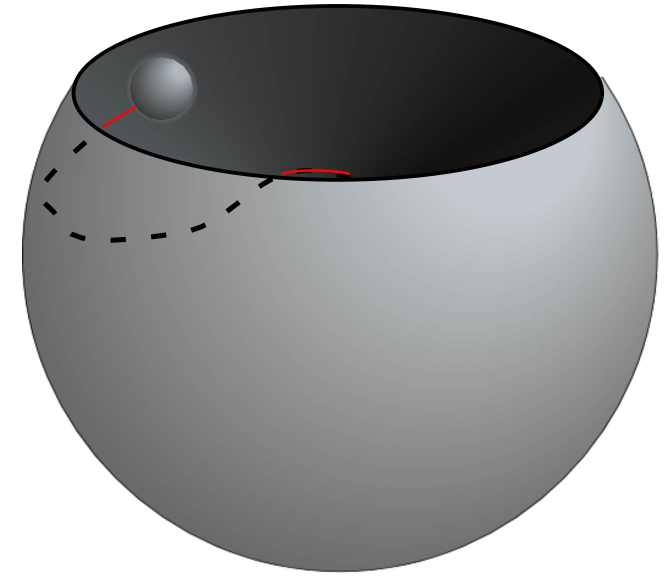

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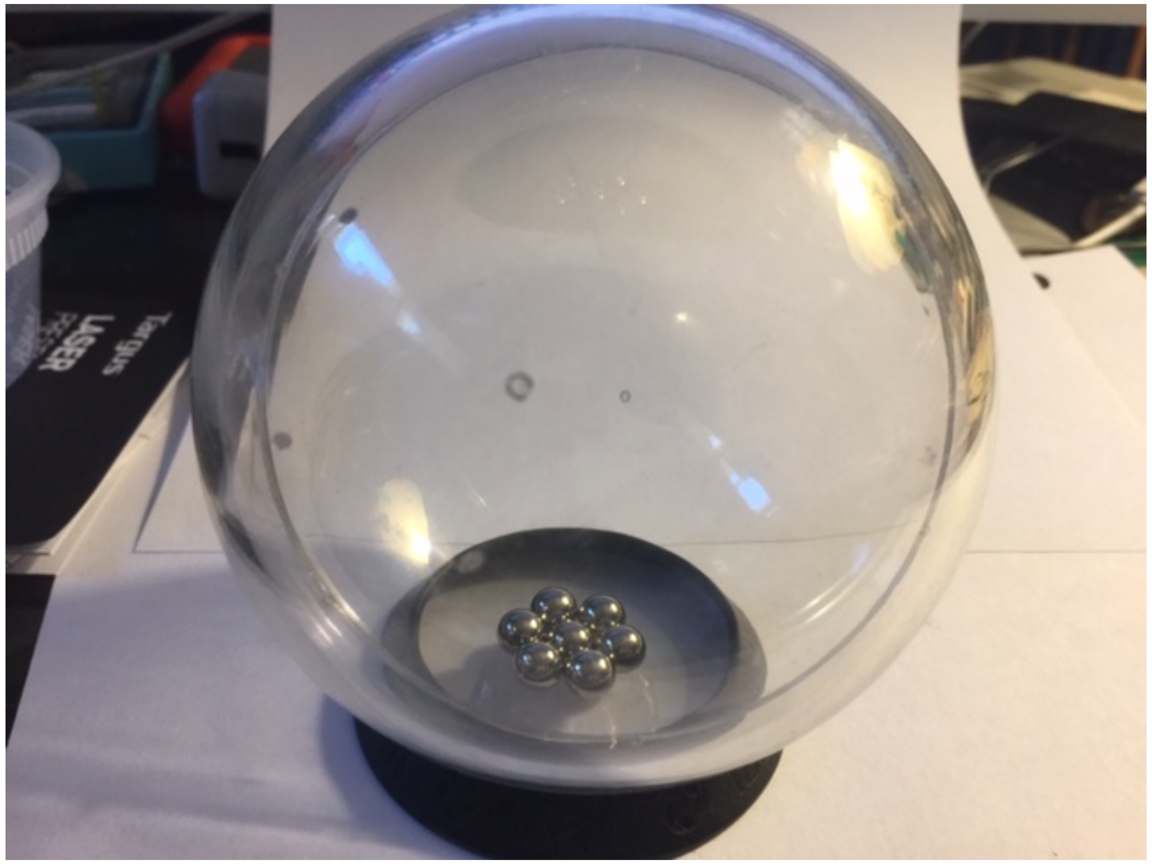


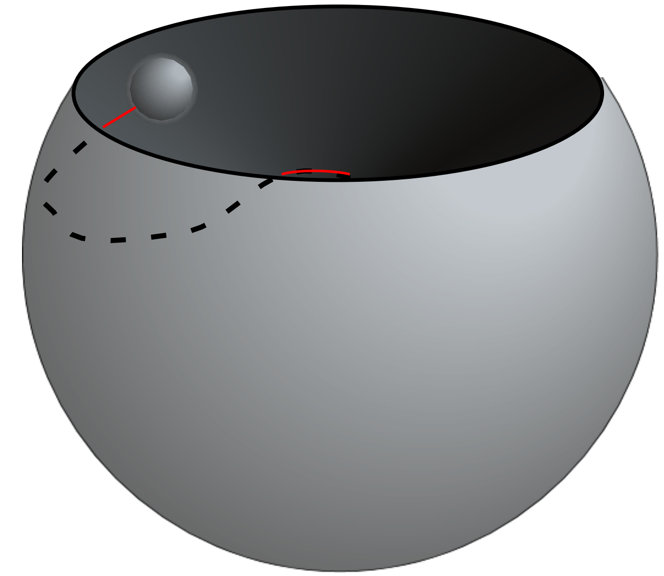
Separation of GCC Equations: Effective Potentials

Small (n_ρ:m_ϕ)-periodic and quasi-periodic oscillations

→ *2D Spherical pendulum or "Bowl-Bowling" and the "I-Ball"*

(n_ρ:m_ϕ)=(2:1) vs (1:1) periodic and quasi-periodic orbits

2D Spherical pendulum or "Bowl-Bowling"



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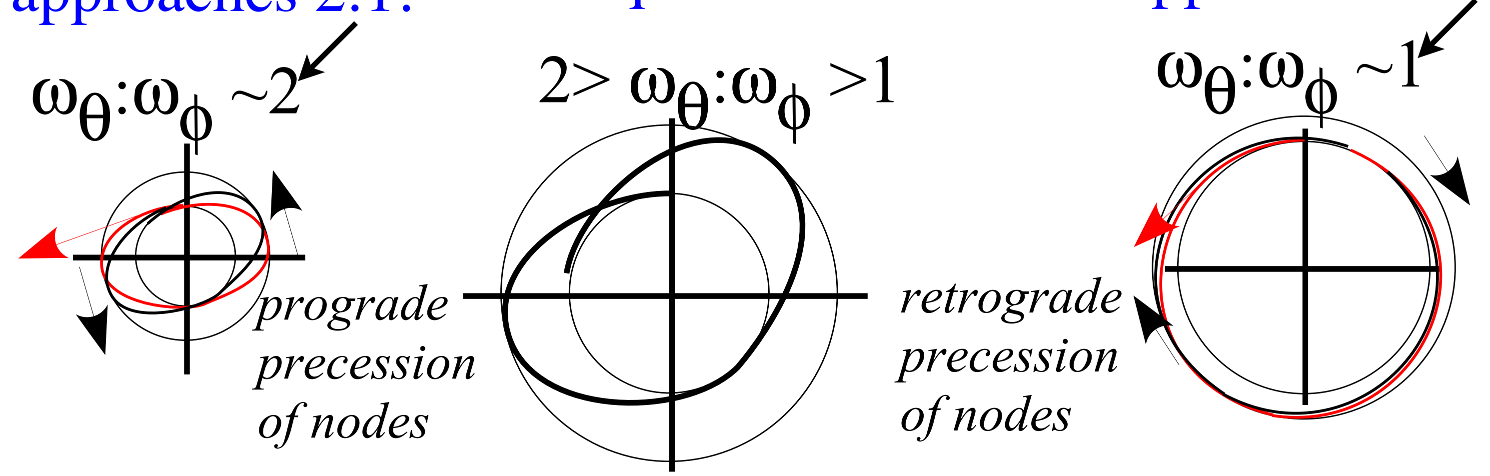
At bottom $\theta\to\pi$ ratio of in-out ω_θ to circle ω_ϕ approaches 2:1.

At equator $\theta\to\pi/2$ ratio approaches 1:1



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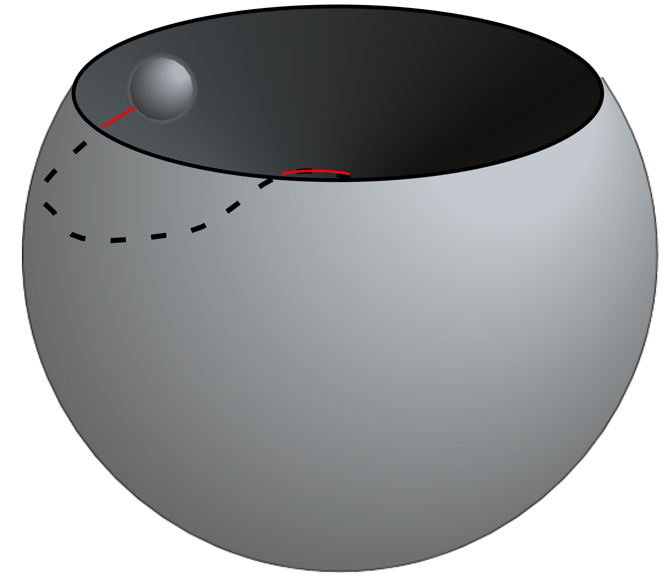

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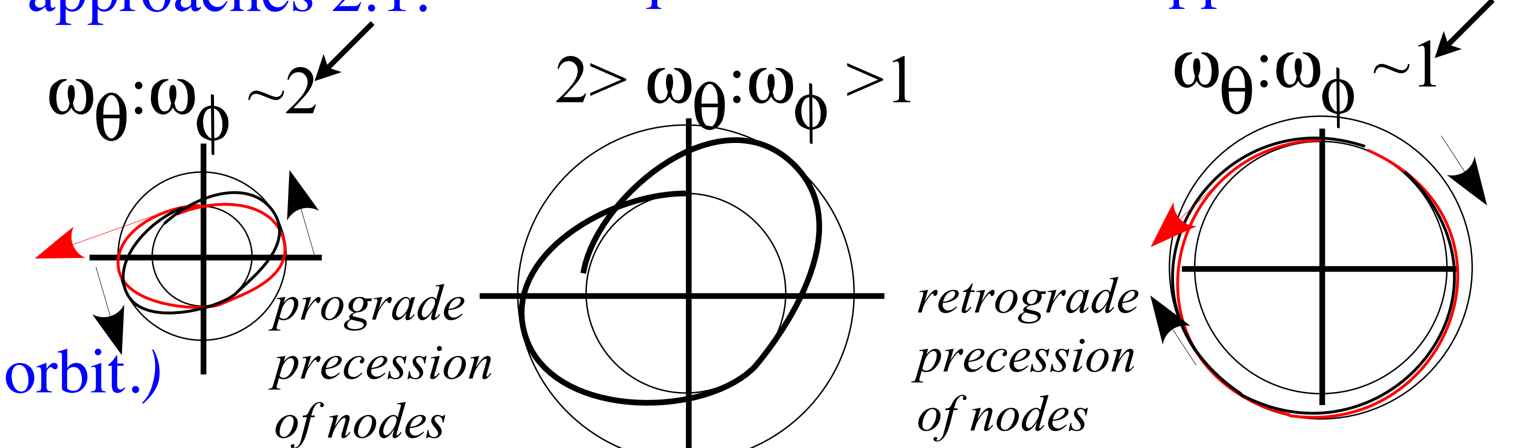
At bottom $\theta\to\pi$ ratio of in-out ω_θ to circle ω_ϕ approaches 2:1.

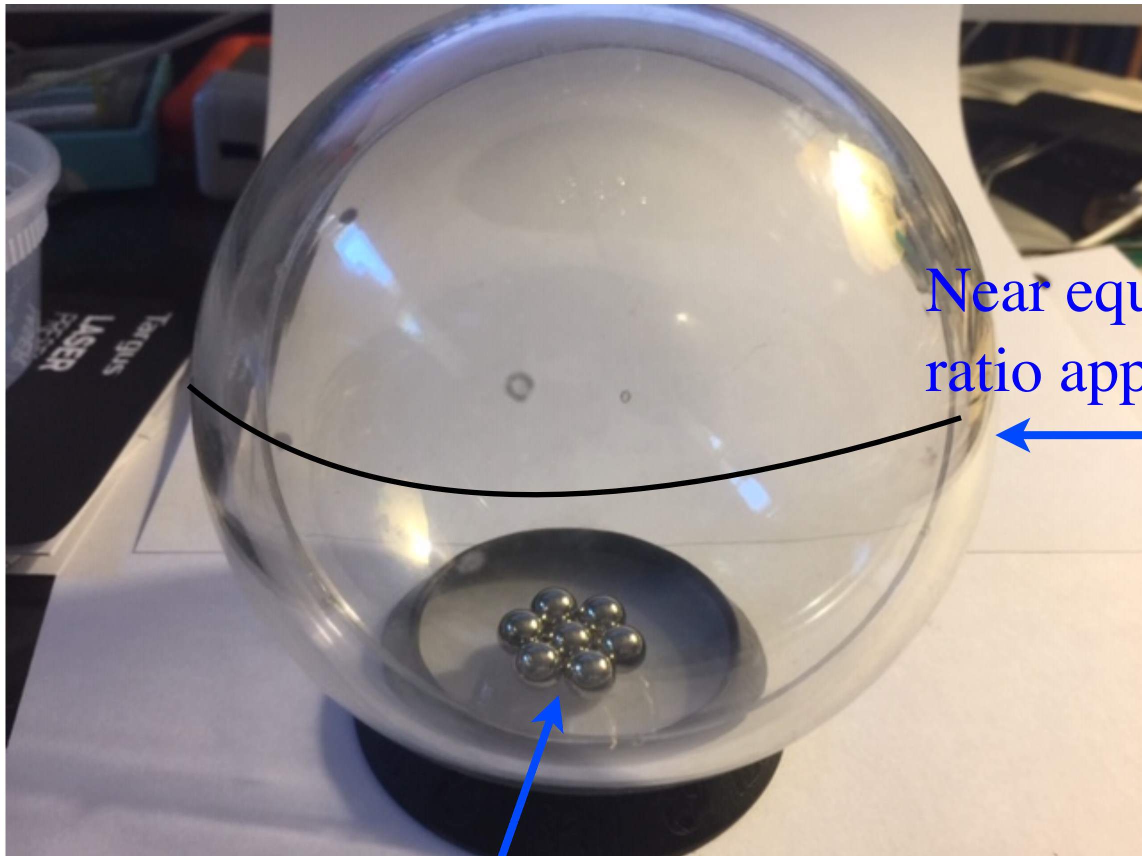
At equator $\theta\to\pi/2$ ratio approaches 1:1

Ratio is between 2 and 1

(Usually irrational non-closed orbit).

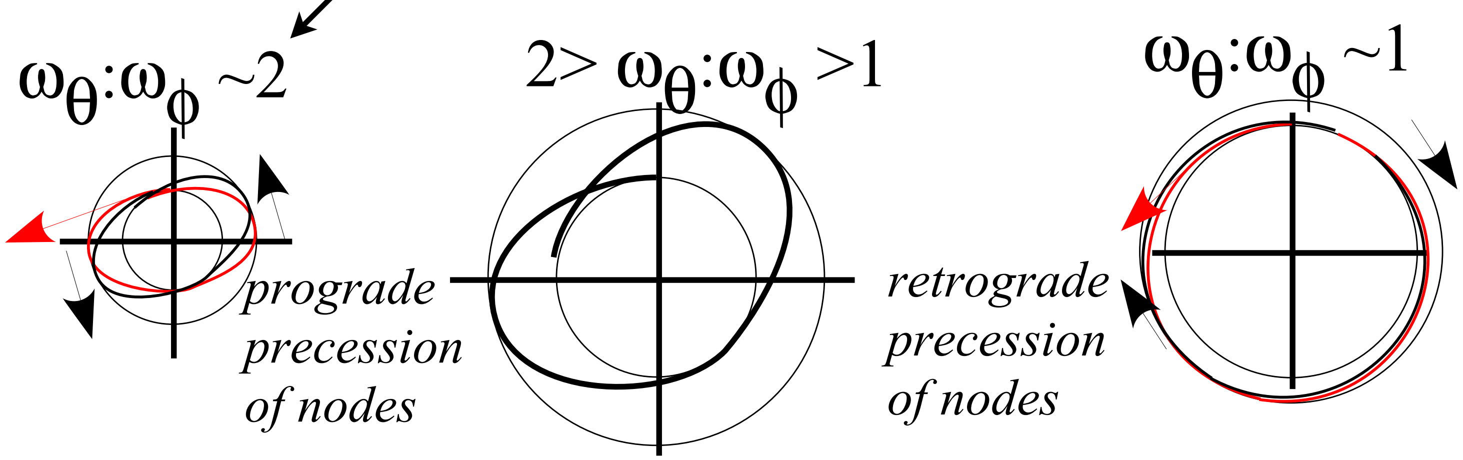
(2:1 is like 2D IHO, but 1:1 is like coulomb orbit.)





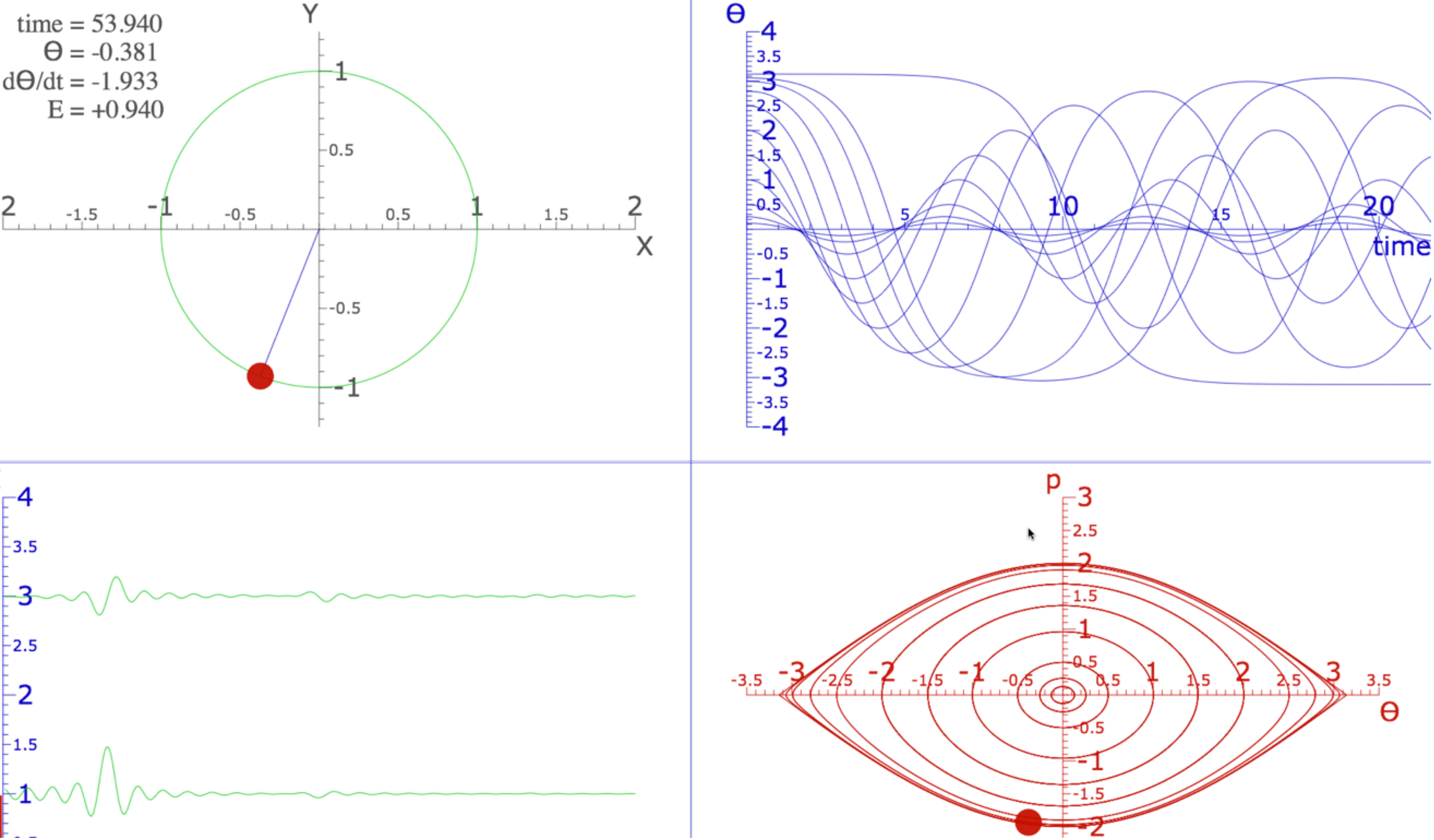
Near equator $\theta \to \pi/2$
ratio approaches 1:1.

Near bottom $\theta \to \pi$
ratio of in-out ω_θ to circle ω_ϕ
approaches 2:1.

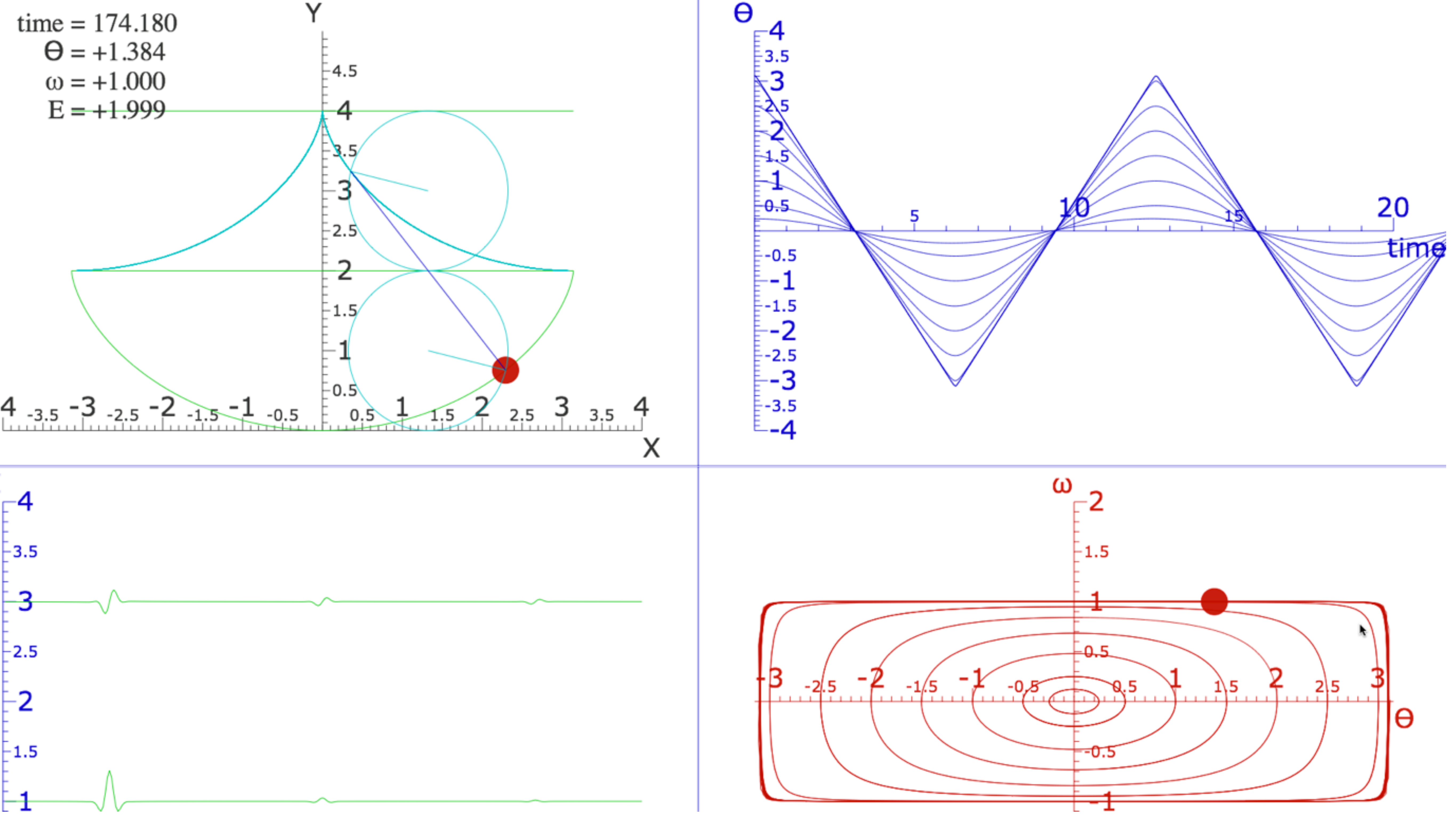


>> Linked also at YouTube

→ *Cycloidal ruler&compass geometry*

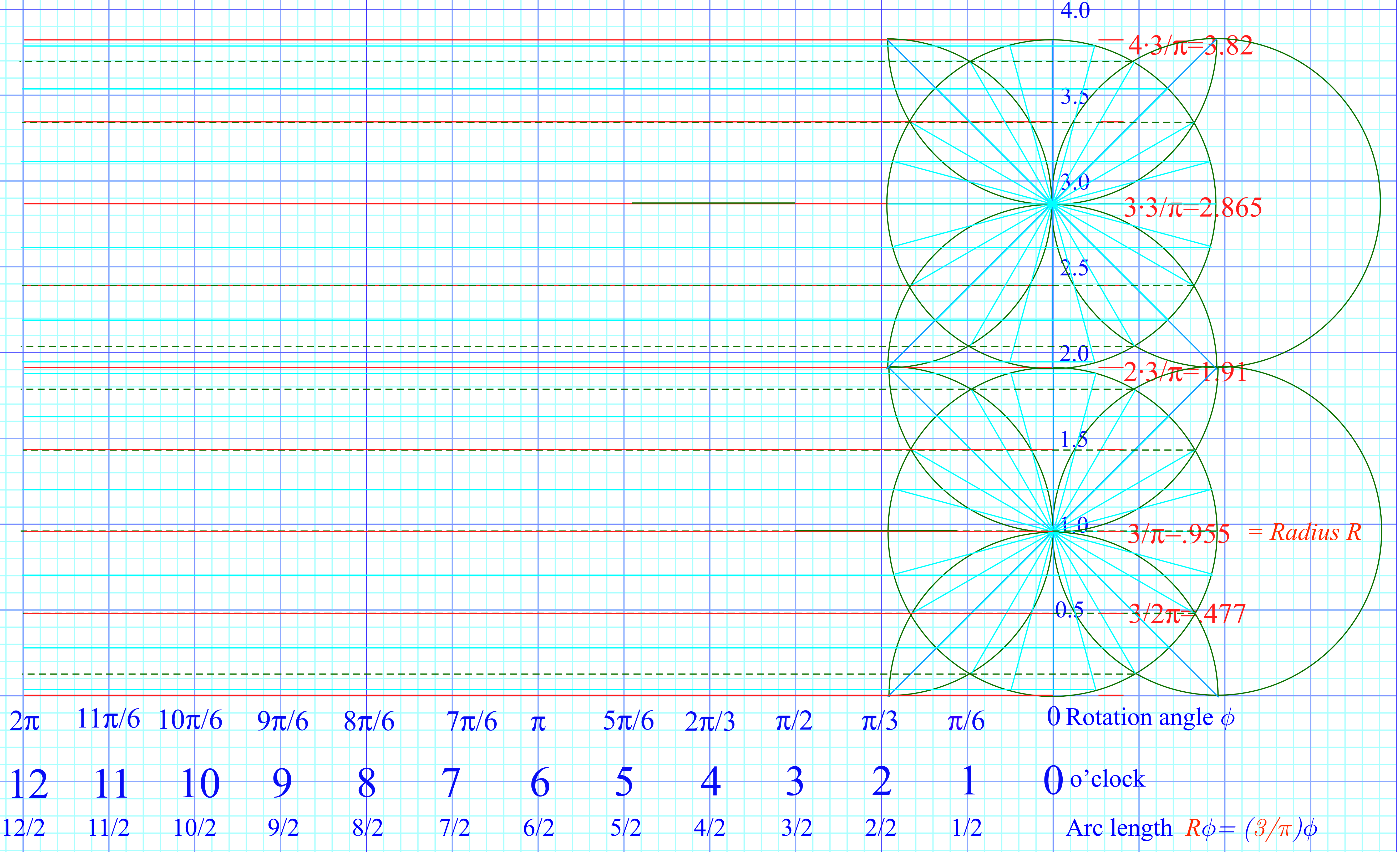


https://modphys.hosted.uark.edu/markup/PendulumWeb.html



https://modphys.hosted.uark.edu/markup/CycloidulumWeb.html

Here the radius is plotted as an irrational $R=3/\pi=0.955$ length so rolling by rational angle $\phi = m\pi/n$ is a rational length of rolled -out circumference $R\phi = (3/\pi)m\pi/n = 3m/n$. Diameter is $2R=6/\pi=1.91$



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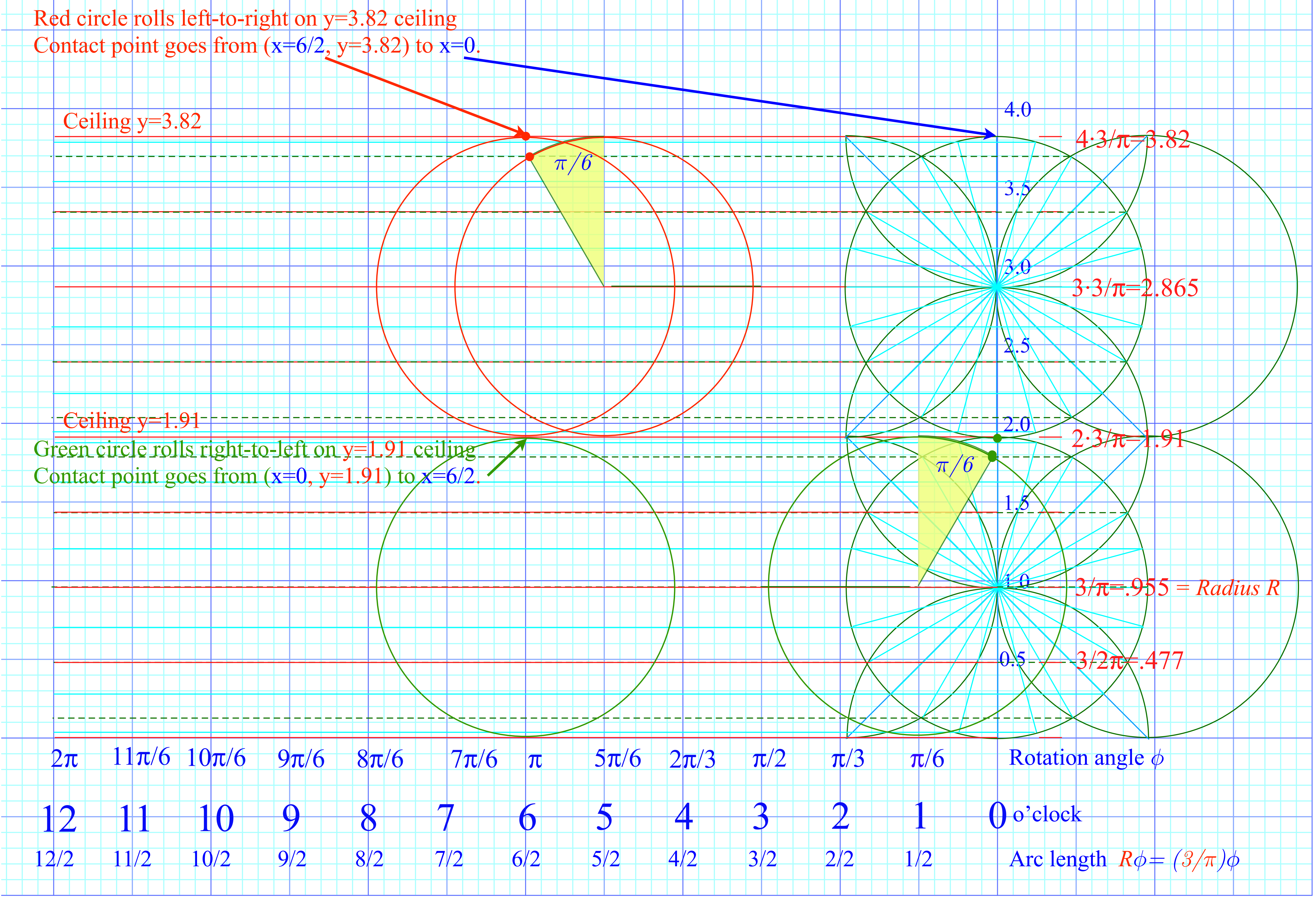

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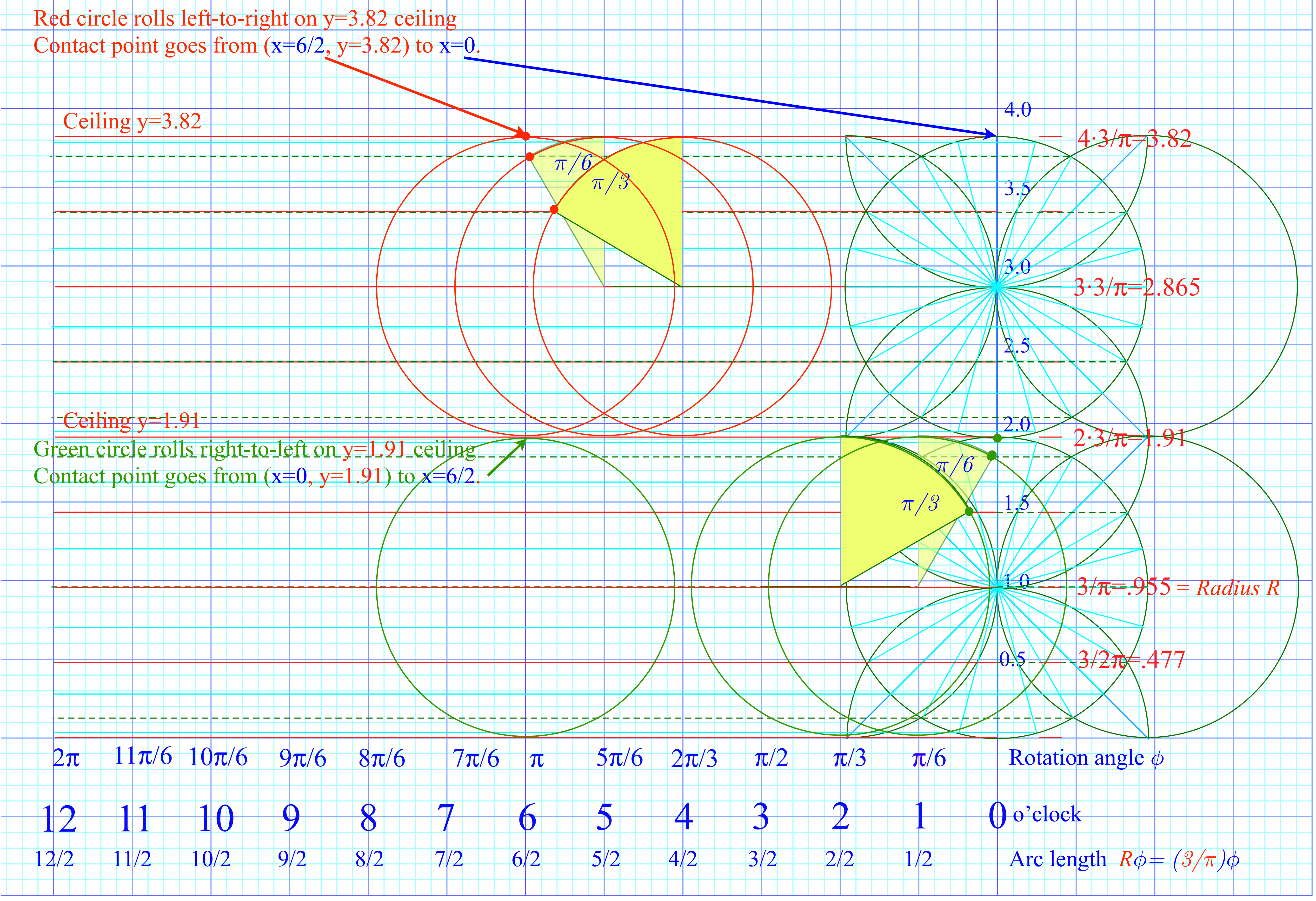

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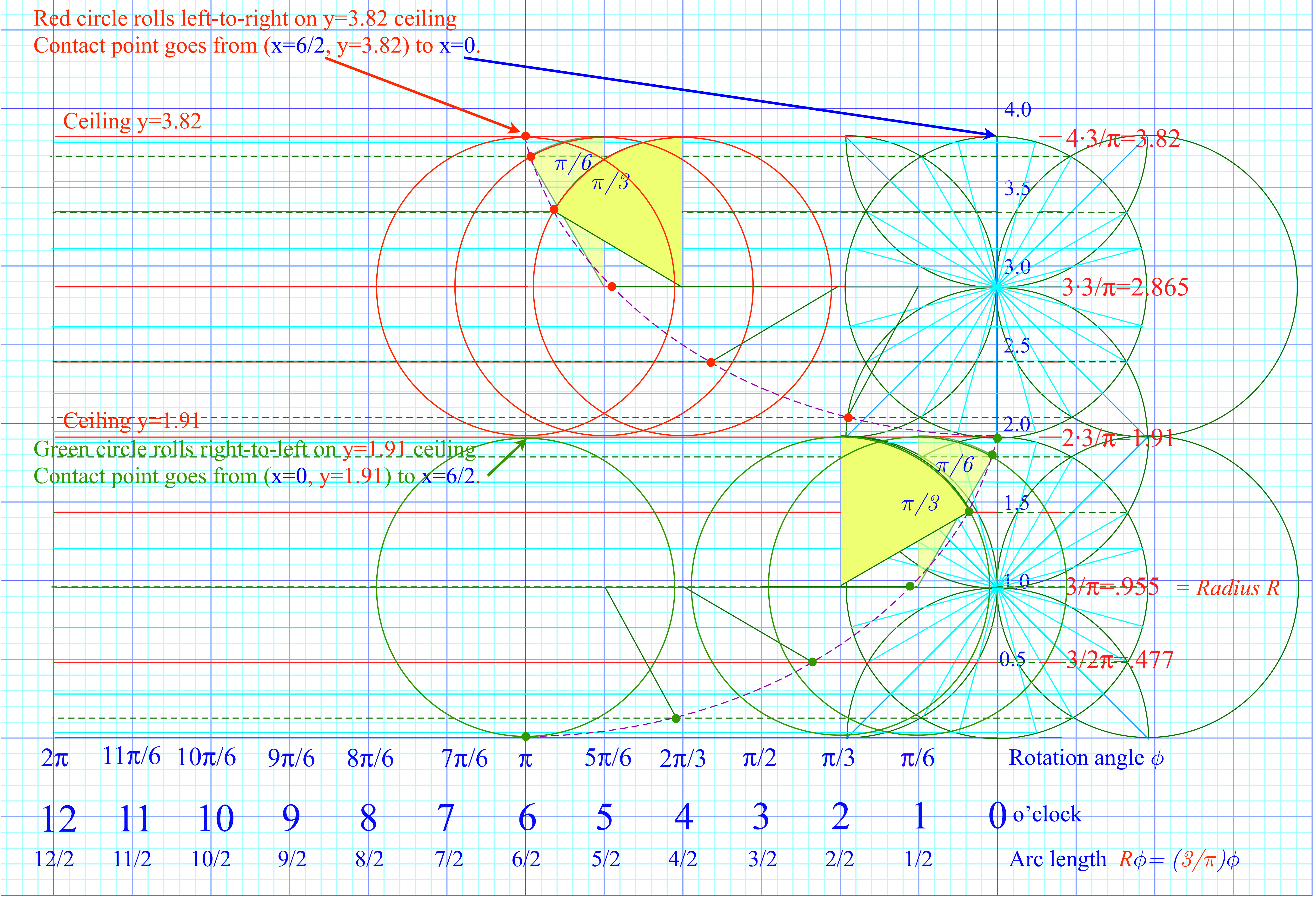

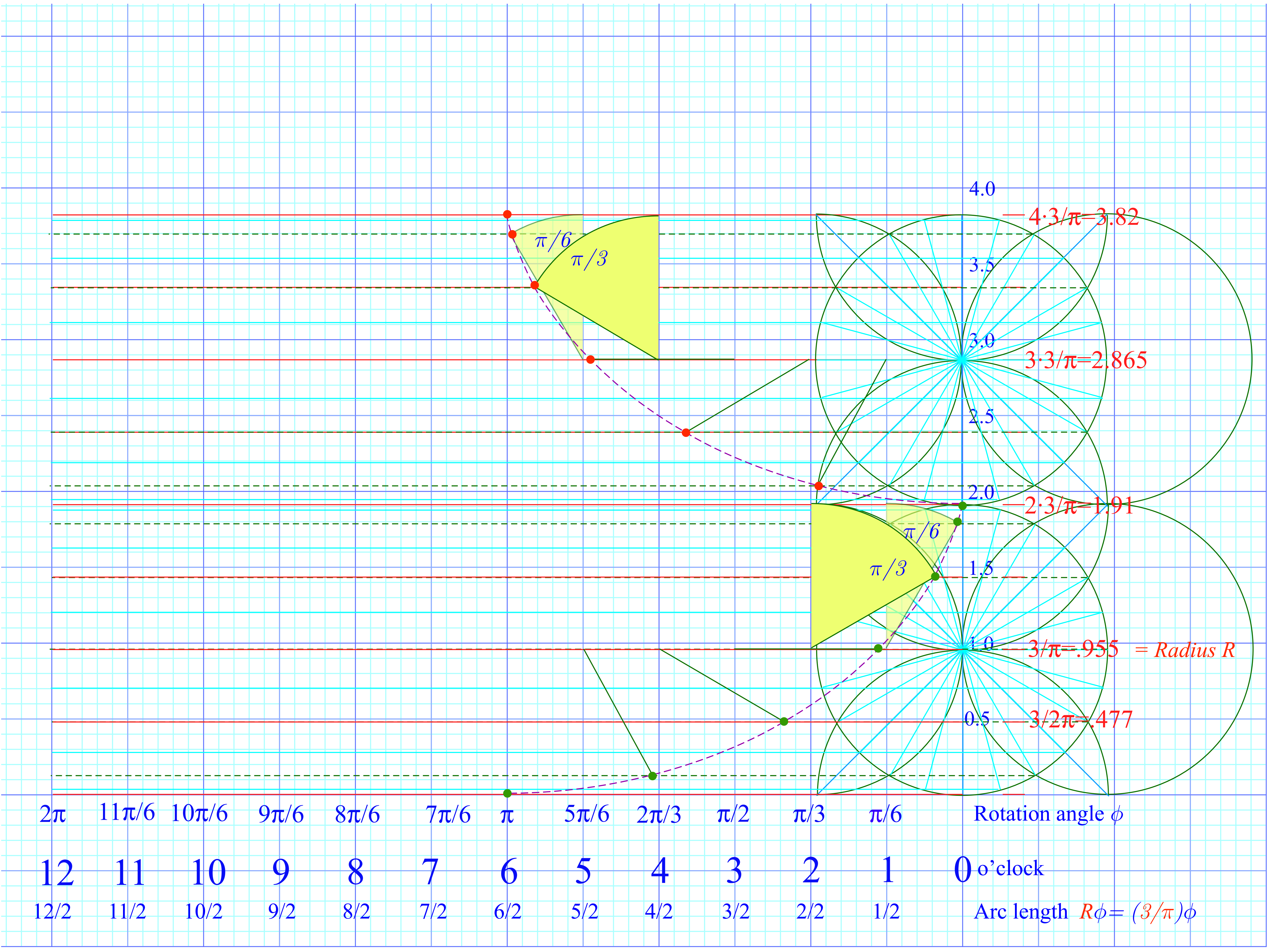
4.0
3.5
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0.5
4·3/π=3.82
3·3/π=2.865
2·3/π=1.91
3/π=.955 = Radius R
3/2π=.477
π/6
π/3
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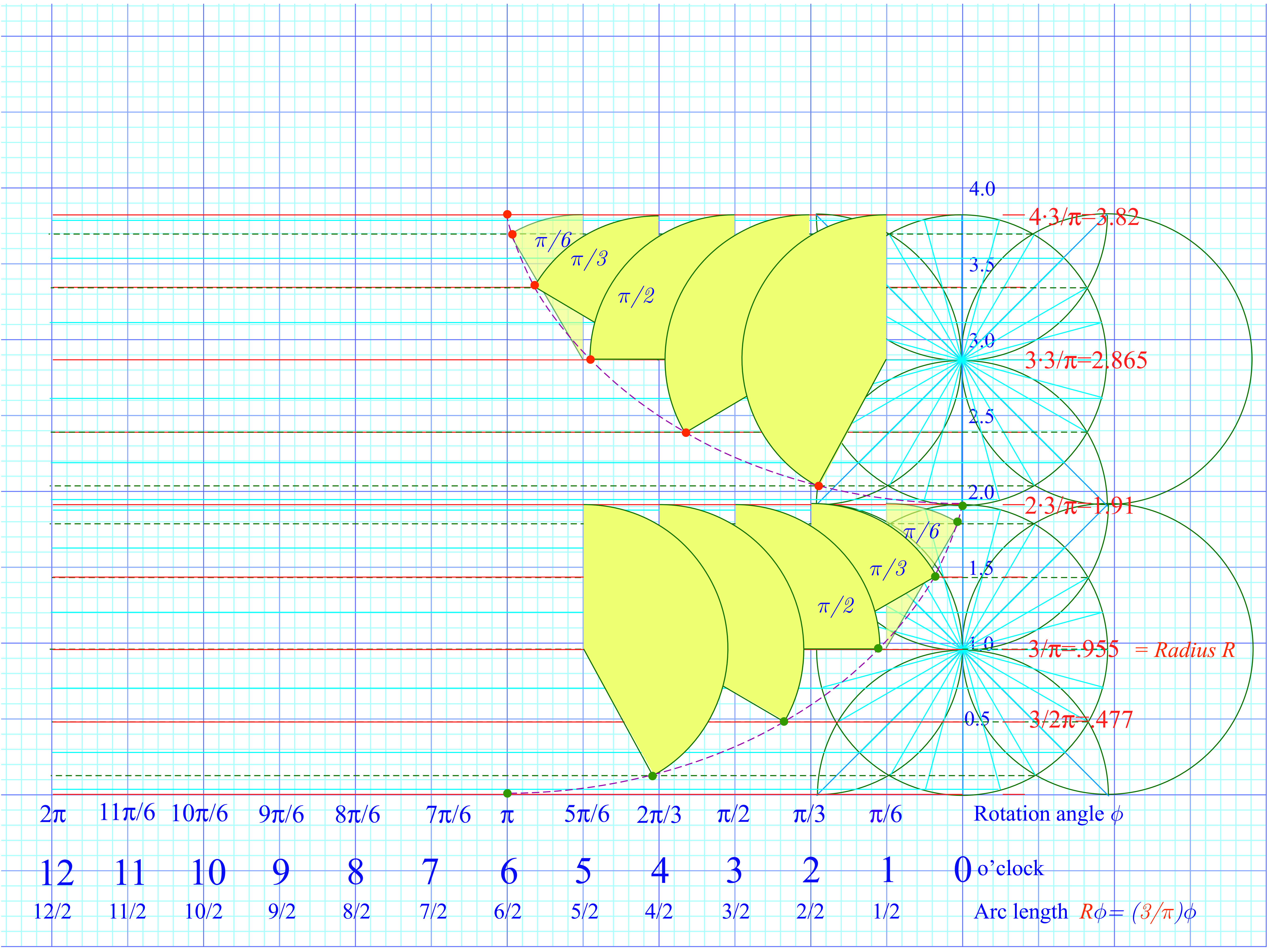
π/3
2π 11π/6 10π/6 9π/6 8π/6 7π/6 π 5π/6 2π/3 π/2 π/3 π/6 Rotation angle ϕ
12 11 10 9 8 7 6 5 4 3 2 1 0 o'clock
12/2 11/2 10/2 9/2 8/2 7/2 6/2 5/2 4/2 3/2 2/2 1/2 Arc length Rϕ= (3/π)ϕ



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5π/6
2π/3
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π/3
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Rotation angle ϕ
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4/2
3/2
2/2
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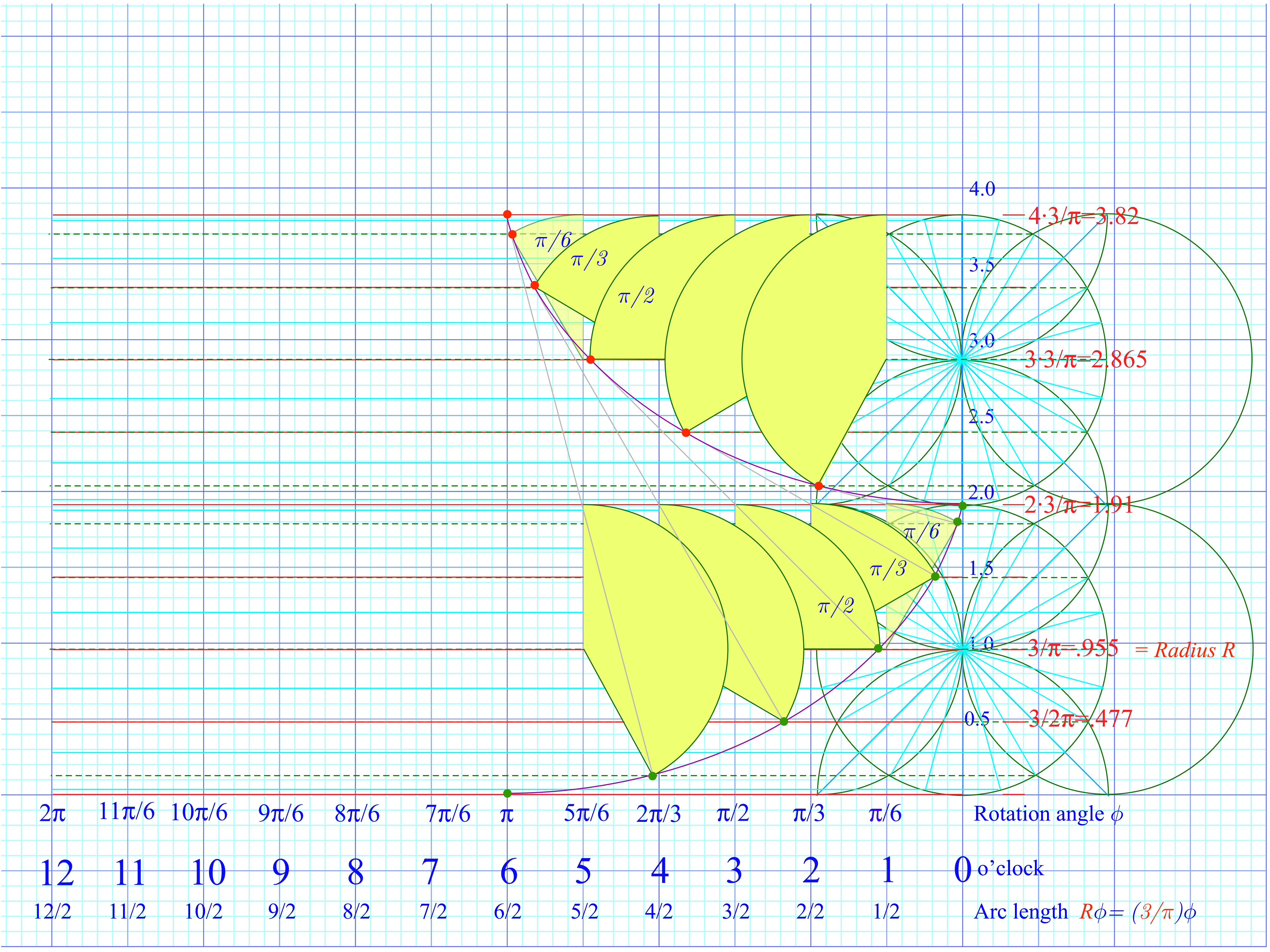

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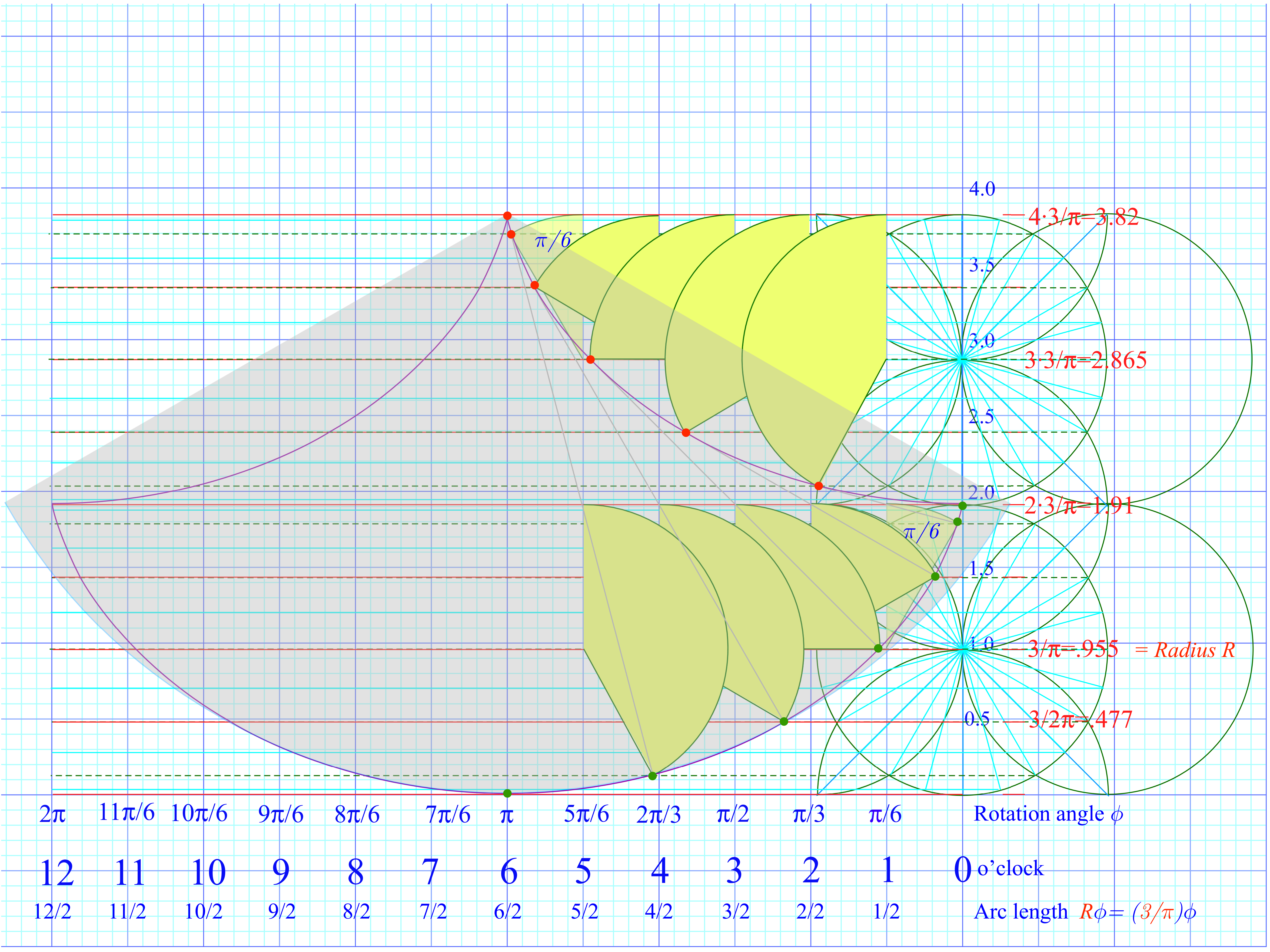

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