Lecture 18
Wed. 10.24.2018

Electromagnetic Lagrangian and charge-field mechanics (Ch. 2.8 of Unit 2)

Charge mechanics in electromagnetic fields

- *Vector analysis for particle-in-(**A**,Φ)-potential*
- *Lagrangian for particle-in-(**A**,Φ)-potential*
- *Hamiltonian for particle-in-(**A**,Φ)-potential*
  - *Canonical momentum in (**A**,Φ) potential*
  - *Hamiltonian formulation*
  - *Hamilton's equations*

Crossed E and B field mechanics

- *Classical Hall-effect and cyclotron orbit orbit equations*
  - *Vector theory vs. complex variable theory*
- *Mechanical analog of cyclotron and FBI rule*
  - *Cycloidal ruler&compass geometry*
  - *Cycloidal geometry of flying levers*
  - *Practical poolhall application*

This mechanical analog of (E_x,B_z) field mimics ***A****-field with tabletop* ***v****-field*

A running collection of links to course-relevant sites and articles

Physics Web Resources

Comprehensive Harter-Soft Resource Listing

UAF Physics YouTube channel

LearnIt Physics Web Applications

Neat external material to start the class:
- AIP publications
- AJP article on superball dynamics
- AAPT summer reading

“Texts”

Classical Mechanics with a Bang!

Quantum Theory for the Computer Age

Principles of Symmetry, Dynamics, and Spectroscopy

Modern Physics and its Classical Foundations

Classes

2014 AMOP

2017 Group Theory for QM

2018 AMOP

2018 Adv Mechanics

These *are* hot off the presses:
- Sorting ultracold atoms in a three-dimensional optical lattice in a realization of Maxwell’s demon - Kumar-Nature-Letters-2018
- Synthetic three-dimensional atomic structures assembled atom by atom - Berredo-Nature-Letters-2018

Slightly Older ones:
- Wave–particle duality of C60 molecules
- Optical vortex knots – One Photon at a Time

“*Relawavity*” and quantum basis of *Lagrangian* & *Hamiltonian* mechanics:
- 2-CW laser wave - BohrIt Web App
- Lagrangian vs Hamiltonian - RelaWavity Web App

AMOP Ch 0 Space-Time Symmetry - 2019
Seminar at Rochester Institute of Optics, Auxiliary slides, June 19, 2018

Previous Links from Lectures 14-17

http://thearmchaircritic.blogspot.com/2011/11/punkin-chunkin.html
http://www.sussexcountyonline.com/news/photos/punkinchunkin.html
http://www.twcenter.net/forums/showthread.php?358315-Shooting-range-for-medieval-siege-weapons-Anybody-knows
https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html
https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html?scenario=MontezumasRevenge
https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html?scenario=SeigeOfKenilworth
https://modphys.hosted.uark.edu/pdfs/Journal_Pdfs/Trebuchet-SciAm_273_66_July_1995_chevedden1.pdf
‘*Simple*’ *Pendulum* Sim: https://modphys.hosted.uark.edu/markup/PendulumWeb.html
‘*Cycloid*’ *Pendulum*: https://modphys.hosted.uark.edu/markup/CycloidulumWeb.html
Google search on: "Satelite view of Patricia" (Images)
Physics Girl Channel - Fun with Vortex Rings in the Pool: https://www.youtube.com/watch?v=72LWr7BU8Ao
iBall demo - Quasi-periodicity: https://youtu.be/_jntDtULxDc

Links to supplement Lecture 18

https://modphys.hosted.uark.edu/markup/CoulItWeb.html?scenario=SynchrotronMotion
https://modphys.hosted.uark.edu/markup/CoulItWeb.html?scenario=SynchrotronMotion2
Mechanical Analog to EM Motion (YouTube video) - https://youtu.be/hTd5FTJ-vRk

AnalyIt Web Application, posted 10/22/2018 in our *testing* area:
https://modphys.hosted.uark.edu/testing/markup/AnalyItBJS.html

Charge mechanics in electromagnetic fields

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Vector analysis for particle-in-(A,Φ)-potential

So-called *pondermotive* form for Newton's *F=ma* equation for a mass *m* of charge *e*.

electronic charge:
e =-1.602176·10^{-19}Coulombs

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e(\mathbf{E}+\mathbf{v}\times\mathbf{B})$$

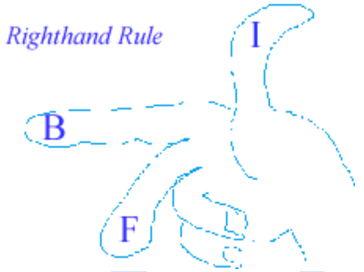
Electric field **E** and magnetic field **B**

scalar potential field $\Phi=\Phi(\mathbf{r},t)$

vector potential field $\mathbf{A}=\mathbf{A}(\mathbf{r},t)$

$$\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B}=\nabla\times\mathbf{A}$$



$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

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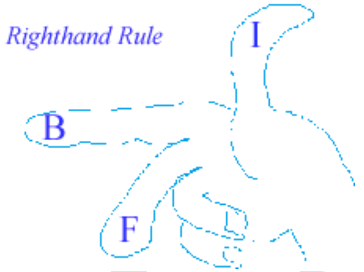

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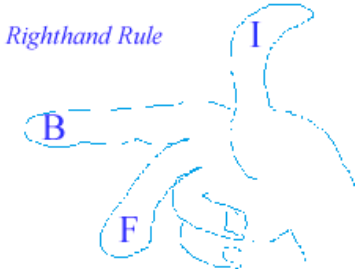

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Doing a double-cross

ε_{ijk}-Tensor analysis of $\mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{A})$

$$\left[\mathbf{v}\times(\nabla\times\mathbf{A})\right]_k=\varepsilon_{kij}v_i\left(\nabla\times\mathbf{A}\right)_j$$

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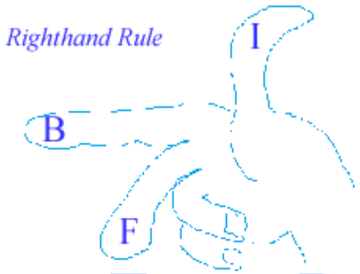

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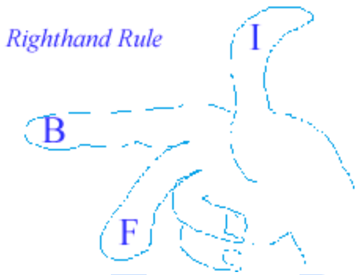

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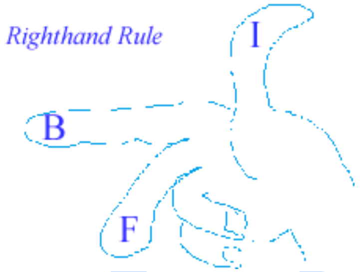

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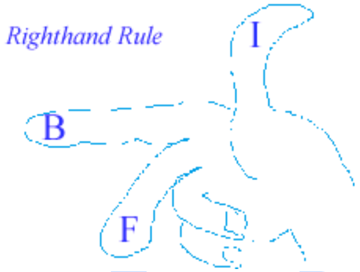

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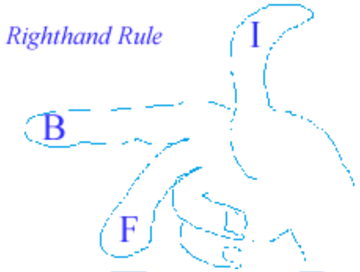

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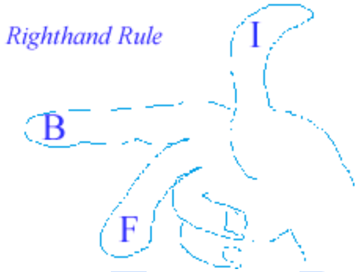

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&=\quad\left(\partial_k A_b\right)v_b\quad-v_a\left(\partial_a A_k\right)\\
&=\partial_k\left(A_b v_b\right)-\left(\partial_k v_b\right)A_b-v_a\left(\partial_a A_k\right)
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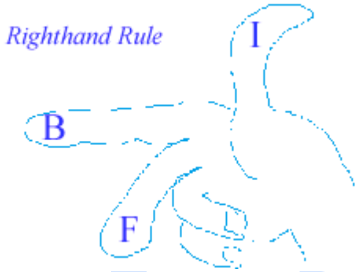

F=*q***v**×**B**=**I**×**B**

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$$=\quad\left(\partial_k A_b\right)v_b\quad -v_a\left(\partial_a A_k\right)=(\nabla\mathbf{A})\bullet\mathbf{v}-\mathbf{v}\bullet\nabla\mathbf{A}$$

$$=\partial_k\left(A_b v_b\right)-\left(\partial_k v_b\right)A_b-v_a\left(\partial_a A_k\right)=\nabla(\mathbf{A}\bullet\mathbf{v})-(\nabla\mathbf{v})\bullet\mathbf{A}-\mathbf{v}\bullet\nabla\mathbf{A}$$

Applying Levi-Civita ε-identity:

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Converting back to Gibbs's **bold** notation involves *tensors* like ∇**A** and ∇**v**.

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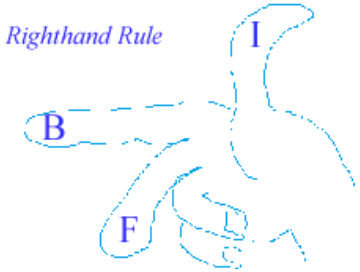

$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

Doing a double-cross

ε_{ijk}-Tensor analysis of $\mathbf{v}\times(\nabla\times\mathbf{A})$

$$\varepsilon_{ijk} = \begin{cases} +1 \text{ for } \textit{even} \text{ permutaion of } i<j<k \\ 0 \quad \text{if any of } i,j,k \text{ are equal} \\ -1 \text{ for } \textit{odd} \text{ permutaion of } i<j<k \end{cases}$$

$$\varepsilon_{ijk} = \varepsilon_{ikj} = \varepsilon_{kij} = 1$$

$$= -\varepsilon_{jik} = -\varepsilon_{jki} = -\varepsilon_{kji}$$

$$\left[\mathbf{v}\times(\nabla\times\mathbf{A})\right]_k = \varepsilon_{kij}v_i\left(\nabla\times\mathbf{A}\right)_j$$

$$= \varepsilon_{kij}v_i\left(\varepsilon_{abj}\left(\partial_a A_b\right)\right)$$

$$= \varepsilon_{kij}\varepsilon_{abj} \quad v_i\left(\partial_a A_b\right)$$

$$= (\delta_{ka}\delta_{ib} - \delta_{kb}\delta_{ia})v_i\left(\partial_a A_b\right)$$

$$= \delta_{ka}\delta_{ib}v_i\left(\partial_a A_b\right) - \delta_{kb}\delta_{ia}v_i\left(\partial_a A_b\right)$$

$$= \quad v_b\left(\partial_k A_b\right) \quad - v_a\left(\partial_a A_k\right)$$

$$= \quad \left(\partial_k A_b\right)v_b \quad - v_a\left(\partial_a A_k\right) = (\nabla\mathbf{A})\bullet\mathbf{v} - \mathbf{v}\bullet\nabla\mathbf{A}$$

$$= \partial_k\left(A_b v_b\right) - \left(\partial_k v_b\right)A_b - v_a\left(\partial_a A_k\right) = \nabla(\mathbf{A}\bullet\mathbf{v}) - (\nabla\mathbf{v})\bullet\mathbf{A} - \mathbf{v}\bullet\nabla\mathbf{A}$$

Applying Levi-Civita ε-identity:

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Converting back to Gibbs’s **bold** notation involves *tensors* like $\nabla\mathbf{A}$ and $\nabla\mathbf{v}$.

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Vector analysis for particle-in-($\mathbf{A}$,Φ)-potential

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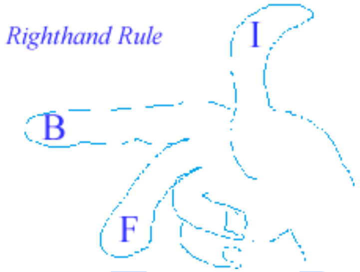

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Summary of Vector analysis for particle-in-($\mathbf{A}$,Φ)-potential

Tensor index notation helps to distinguish $\left(\nabla\mathbf{A}\right)\bullet\mathbf{v}$, $\mathbf{v}\bullet\left(\nabla\mathbf{A}\right)$, and $\nabla(\mathbf{A}\bullet\mathbf{v}) = (\nabla\mathbf{A})\bullet\mathbf{v} + (\nabla\mathbf{v})\bullet\mathbf{A}$.

$$\left[(\nabla\mathbf{A})\bullet\mathbf{v}\right]_k = \frac{\partial A_j}{\partial x_k} v_j$$

$$=(\partial_k A_j) v_j$$

$$\left[\mathbf{v}\bullet(\nabla\mathbf{A})\right]_k = v_j \frac{\partial A_k}{\partial x_j}$$

$$=(v_j \partial_j A_k)$$

$$[\nabla(\mathbf{A}\bullet\mathbf{v})]_k = [(\nabla\mathbf{A})\bullet\mathbf{v} + (\nabla\mathbf{v})\bullet\mathbf{A}]_k$$

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Charge mechanics in electromagnetic fields

- *Vector analysis for particle-in-(**A**,Φ)-potential*
- → *Lagrangian for particle-in-(**A**,Φ)-potential*
- *Hamiltonian for particle-in-(**A**,Φ)-potential*
  - *Canonical momentum in (**A**,Φ) potential*
  - *Hamiltonian formulation*
  - *Hamilton's equations*

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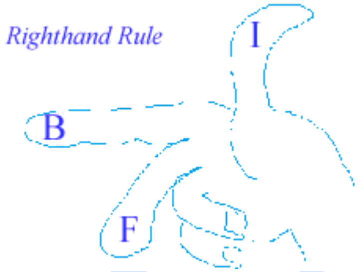

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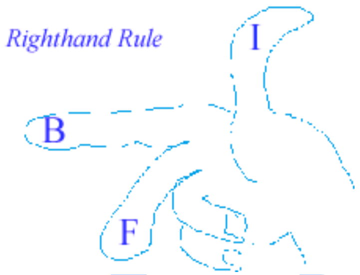

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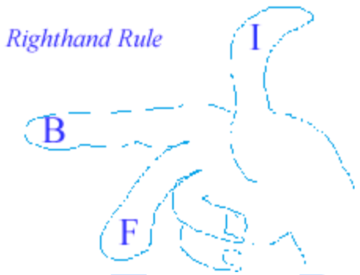

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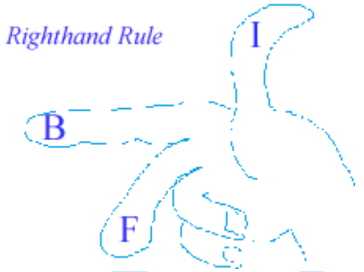

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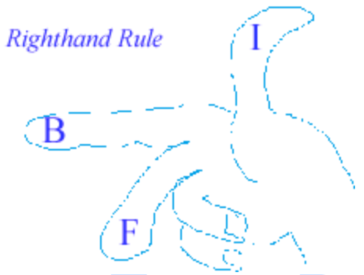

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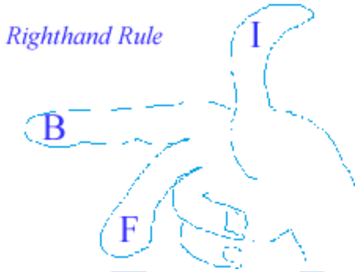

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Inserting Φ-term that $\partial_{\mathbf{v}}$ zeros :

This step requires that : $\left(\frac{\partial}{\partial\mathbf{v}}\left(e\Phi\right)=0\right)$ $\left(\text{and}:\frac{\partial}{\partial\mathbf{v}}\left(\mathbf{v}\bullet e\mathbf{A}\right)=e\mathbf{A}\right)$

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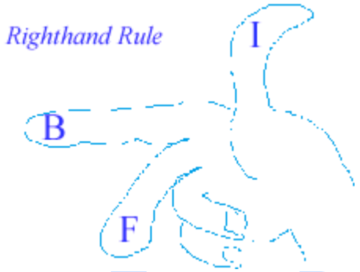

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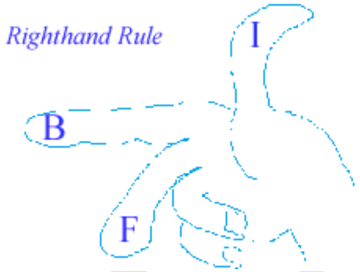

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Chain rule expansion of vector potential total *t*-derivative:

$$\frac{d\mathbf{A}}{dt}=\frac{\partial\mathbf{A}}{\partial x}\dot{x}+\frac{\partial\mathbf{A}}{\partial y}\dot{y}+\frac{\partial\mathbf{A}}{\partial z}\dot{z}+\frac{\partial\mathbf{A}}{\partial t}=\boxed{\frac{\partial\mathbf{A}}{\partial t}}+(\mathbf{v}\bullet\nabla)\mathbf{A}$$

$$m\frac{d\mathbf{v}}{dt}=e\left[-\nabla\Phi+\nabla(\mathbf{v}\bullet\mathbf{A})-\frac{\partial\mathbf{A}}{\partial t}-(\mathbf{v}\bullet\nabla)\mathbf{A}\right]=e\left[-\nabla(\Phi-\mathbf{v}\bullet\mathbf{A})-\frac{d\mathbf{A}}{dt}\right]$$

$$m\frac{d\mathbf{v}}{dt}=\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}$$

$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}=\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\left(e\Phi-\mathbf{v}\bullet e\mathbf{A}\right)-\nabla\left(e\Phi-\mathbf{v}\bullet e\mathbf{A}\right)$$

$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\left(e\Phi-\mathbf{v}\bullet e\mathbf{A}\right)=-e\frac{d\mathbf{A}}{dt}$$

$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\underbrace{\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}-(e\Phi-\mathbf{v}\bullet e\mathbf{A})\right)}=\frac{\partial}{\partial\mathbf{r}}\underbrace{\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}-(e\Phi-\mathbf{v}\bullet e\mathbf{A})\right)}$$

$$\frac{\partial}{\partial\mathbf{r}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}\right)=0$$

$$\frac{d}{dt}\frac{\partial L}{\partial\mathbf{v}}=\frac{\partial L}{\partial\mathbf{r}}$$

Lagrangian for particle-in-($\mathbf{A}$,Φ)-potential

So-called *pondermotive* form for Newton's *F=ma* equation for a mass *m* of charge *e*.

electronic charge:
e =-1.602176·10^{-19}Coulombs

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e(\mathbf{E}+\mathbf{v}\times\mathbf{B})$$

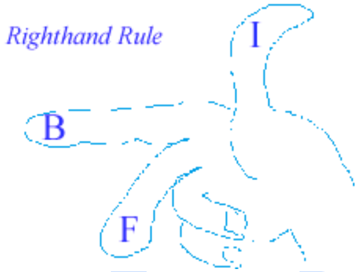
Electric field **E** and magnetic field **B**

scalar potential field $\mathbf{\Phi}=\mathbf{\Phi}(\mathbf{r},t)$

vector potential field $\mathbf{A}=\mathbf{A}(\mathbf{r},t)$

$$\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B}=\nabla\times\mathbf{A}$$



$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e\left[-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}+\mathbf{v}\times\left(\nabla\times\mathbf{A}\right)\right]$$

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e\left[-\nabla\Phi-\boxed{\frac{\partial\mathbf{A}}{\partial t}}+\nabla(\mathbf{v}\bullet\mathbf{A})-\boxed{(\mathbf{v}\bullet\nabla)\mathbf{A}}\right]$$

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Lagrangian has a *linear* velocity term *e***v•A** in addition to the usual quadratic *KE=mv^2/2* and *PE=e*Φ.

$$L=L(\mathbf{r},\mathbf{v},t)=\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}-\left(e\Phi(\mathbf{r},t)-\mathbf{v}\bullet e\mathbf{A}(\mathbf{r},t)\right)$$

Charge mechanics in electromagnetic fields

*Vector analysis for particle-in-(**A**,Φ)-potential*

*Lagrangian for particle-in-(**A**,Φ)-potential*

*Hamiltonian for particle-in-(**A**,Φ)-potential*

→ *Canonical momentum in (**A**,Φ) potential*

Hamiltonian formulation

Hamilton's equations

*Hamiltonian for particle-in-(**A**,Φ)-potential*

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Canonical momentum is defined by L's **v**-derivative

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v} - \left(e\Phi(\mathbf{r},t) - \mathbf{v}\bullet e\mathbf{A}(\mathbf{r},t)\right)\right)$$

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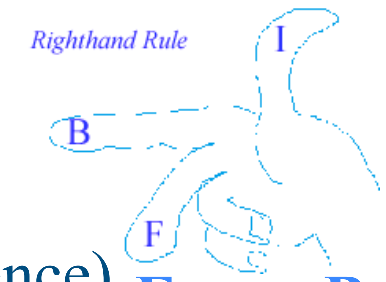
*Canonical momentum in (**A**,Φ) potential*

Canonical momentum is defined by *L*'s **v**-derivative (…scalar Φ has no **v**-dependence)



$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v} - \big(e\Phi(\mathbf{r},t) - \mathbf{v}\bullet e\mathbf{A}(\mathbf{r},t)\big)\right) = \frac{\partial}{\partial \mathbf{v}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v} - e\Phi(\mathbf{r},t)\right)_{For\ \mathbf{A}(\mathbf{r},t)=0}$$

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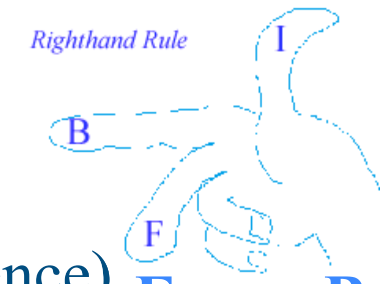

F=*q***v**×**B**=**I**×**B**

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if magnetic (vector) potential $\mathbf{A}=\mathbf{A}(\mathbf{r},t)$ is zero everywhere.

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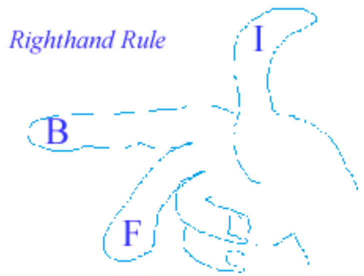

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Hamiltonian for particle-in-(A,Φ)-potential

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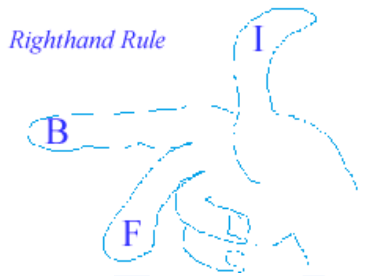

F=qv×B=I×B

Canonical momentum in (A,Φ) potential

Canonical momentum is defined by L's **v**-derivative

$$\mathbf{p}=\frac{\partial L}{\partial\mathbf{v}}=\frac{\partial}{\partial\mathbf{v}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}-\left(e\Phi(\mathbf{r},t)-\mathbf{v}\bullet e\mathbf{A}(\mathbf{r},t)\right)\right)=\frac{\partial}{\partial\mathbf{v}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}-e\Phi(\mathbf{r},t)\right)_{For\ \mathbf{A}(\mathbf{r},t)=0}$$

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Then canonical momentum is usual form: $\mathbf{p}=m\mathbf{v}$ $\quad(For\ \mathbf{A}(\mathbf{r},t)=0)$

Otherwise vector potential term $-\mathbf{v}\cdot e\mathbf{A}$ leads to an extraordinary *canonical momentum:* $\mathbf{p}=m\mathbf{v}+e\mathbf{A}(\mathbf{r},t)$.
Particle momentum $m\mathbf{v}$ is not canonical, but related to *canonical* $\mathbf{p}$ as follows: $m\mathbf{v}=\mathbf{p}-e\mathbf{A}(\mathbf{r},t)$

Charge mechanics in electromagnetic fields

*Vector analysis for particle-in-(**A**,Φ)-potential*

*Lagrangian for particle-in-(**A**,Φ)-potential*

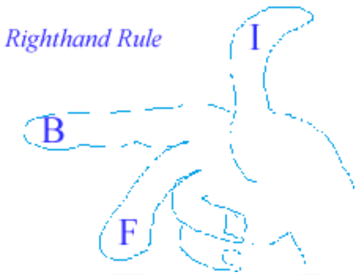
*Hamiltonian for particle-in-(**A**,Φ)-potential*

- *Canonical momentum in (**A**,Φ) potential*
- → *Hamiltonian formulation*
- *Hamilton's equations*

Hamiltonian for charged particle in fields

The Hamiltonian function of the Legendre-Poincare form is the following.

$$H=\sum_{\mu}\dot{q}^{\mu}p_{\mu}-L=\mathbf{v}\bullet\mathbf{p}-L=\mathbf{v}\bullet\left(m\mathbf{v}+e\mathbf{A}(\mathbf{r},t)\right)-\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}-\left(e\Phi(\mathbf{r},t)-\mathbf{v}\bullet e\mathbf{A}(\mathbf{r},t)\right)\right)$$



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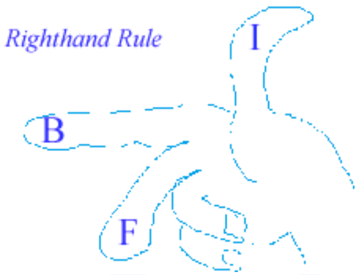
(Only correct numerically!)



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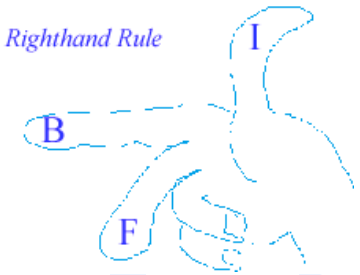

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Hamiltonian for charged particle in fields

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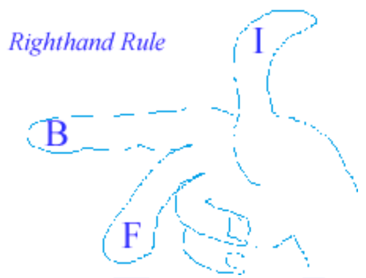

F=q**v**×**B**=**I**×**B**

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Hamiltonian for charged particle in fields

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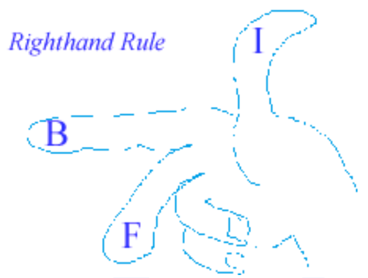

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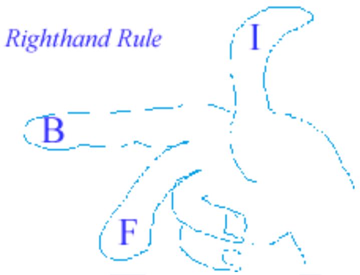
(Correct formally and numerically)

Hamiltonian for charged particle in fields



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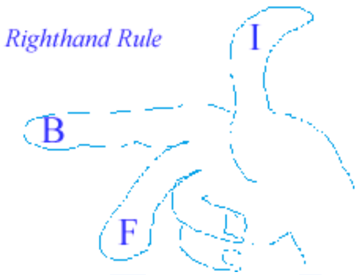

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Hamilton's equations for charged particle in fields

Hamiltonian is explicit function of *momentum* **p**. Must replace velocity **v** using $m\mathbf{v}=\mathbf{p} - e\mathbf{A}(\mathbf{r},t)$.

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(Correct formally and numerically)

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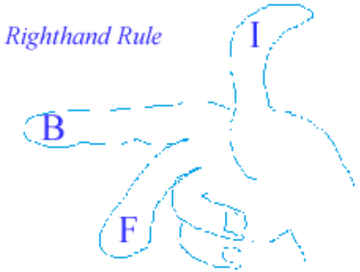
(Expanded)



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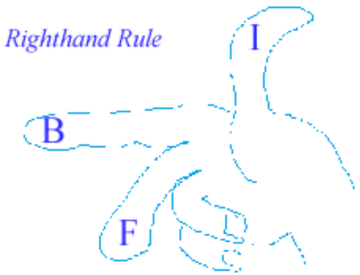

$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

Hamilton's **v** equation: $$\mathbf{v} = \dot{\mathbf{r}} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\mathbf{p} - e\mathbf{A}(\mathbf{r},t)}{m}$$

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(Correct formally and numerically)

$$H = \frac{\mathbf{p}\bullet\mathbf{p}}{2m} - \frac{e}{2m}\left(\mathbf{p}\bullet\mathbf{A} + \mathbf{A}\bullet\mathbf{p}\right) + \frac{e^2}{2m}\mathbf{A}\bullet\mathbf{A} + e\Phi(\mathbf{r},t)$$

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$m\mathbf{v}+e\mathbf{A}(\mathbf{r},t)=\mathbf{p}$ ··················· $\dot{p}_a=m\dot{v}_a+e\dot{A}_a=+\frac{\left(p_\mu-eA_\mu\right)}{m}e\frac{\partial A_\mu}{\partial x_a}-e\frac{\partial\Phi}{\partial x_a}$

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$$m\dot{\mathbf{v}}=e\left(\mathbf{v}\times\left(\nabla\times\mathbf{A}\right)+\mathbf{E}\right)=e(\mathbf{v}\times\mathbf{B}+\mathbf{E})$$

$$\mathbf{B}=\nabla\times\mathbf{A}$$

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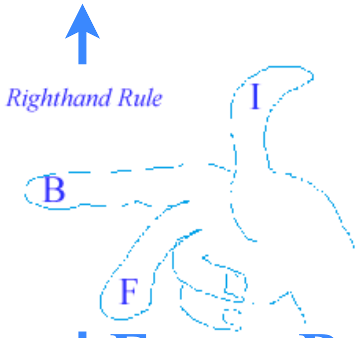

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...and now

we come back

full circle...

$$m\dot{v}_a = e\left(\underbrace{v_\mu \frac{\partial A_\mu}{\partial x_a} - v_\mu \frac{\partial A_a}{\partial x_\mu}} + E_a \right)$$

$$m\dot{\mathbf{v}} = e\left(\mathbf{v}\times\left(\nabla\times\mathbf{A}\right) + \mathbf{E} \right) = e(\mathbf{v}\times\mathbf{B} + \mathbf{E})$$

$$\mathbf{B} = \nabla\times\mathbf{A}$$

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for particle mechanics

Crossed E and B field mechanics

→ *Classical Hall-effect and cyclotron orbit orbit equations*

Vector theory vs. complex variable theory

Mechanical analog of cyclotron and FBI rule

Cycloid geometry and flying sticks

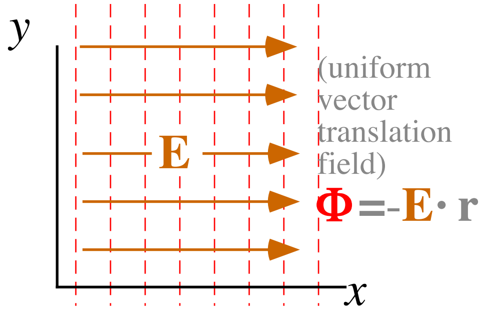
Practical poolhall application

Crossed E and B field mechanics

A constant **E** field has a scalar potential field **Φ** with constant gradient.

$$\Phi(\mathbf{r}) = -\mathbf{E}\bullet\mathbf{r}, \qquad -\nabla\Phi(\mathbf{r}) = -\nabla\left(-\mathbf{E}\bullet\mathbf{r}\right) = \mathbf{E} = const.$$

Fig. 2.4.1.



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A constant **B** field has a vector potential field **A** that resembles a disc spinning counter-clockwise around the **B** axis.

$$\mathbf{A}(\mathbf{r}) = \frac{1}{2}\mathbf{B}\times\mathbf{r}, \qquad \nabla\times\mathbf{A}(\mathbf{r}) = \nabla\times\left(\frac{1}{2}\mathbf{B}\times\mathbf{r}\right) = \mathbf{B} = const.$$

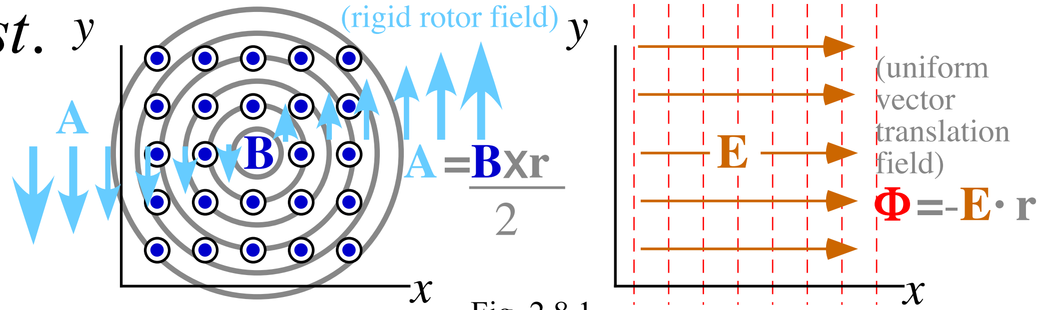

Fig. 2.8.1.

This mechanical analog of (E_x,B_z) field mimics ***A****-field with tabletop* ***v****-field*

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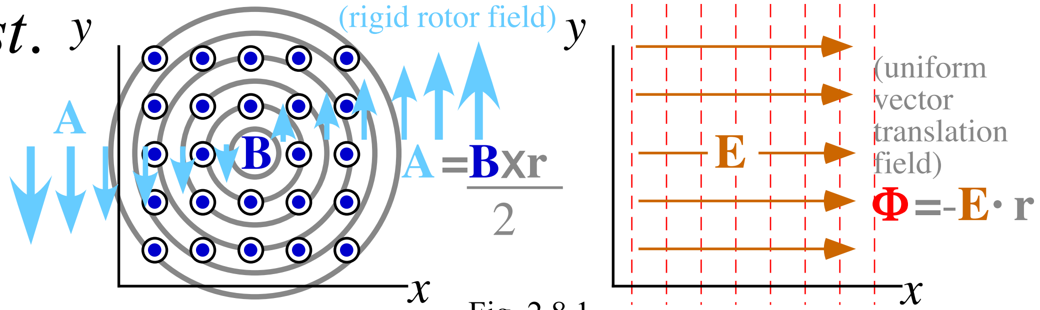
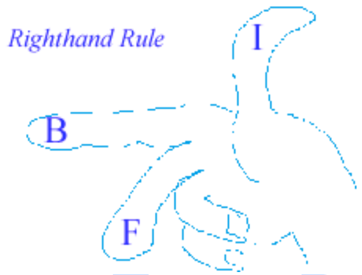


Fig. 2.8.1.

Newtonian electromagnetic equations of motion:

$$m\dot{\mathbf{v}} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v} \times \frac{e}{m}\mathbf{B}.$$



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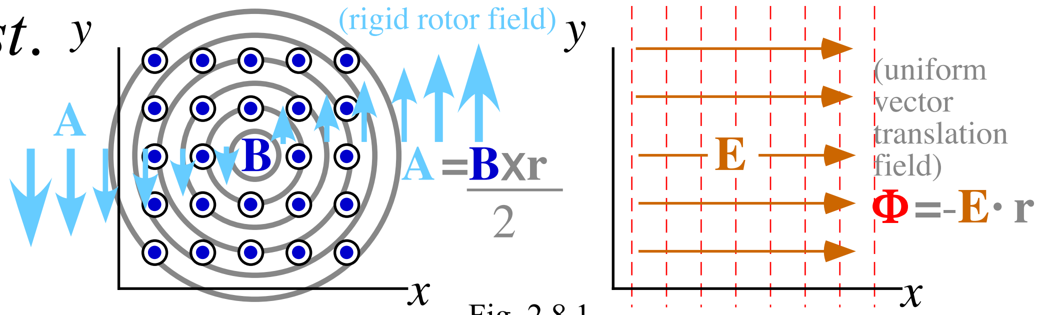

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$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \varepsilon + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z$$

$$\varepsilon_x = \frac{e}{m}E_x \qquad \varepsilon_y = \frac{e}{m}E_y \qquad B = \frac{e}{m}B_z$$

Shorthand Labeling

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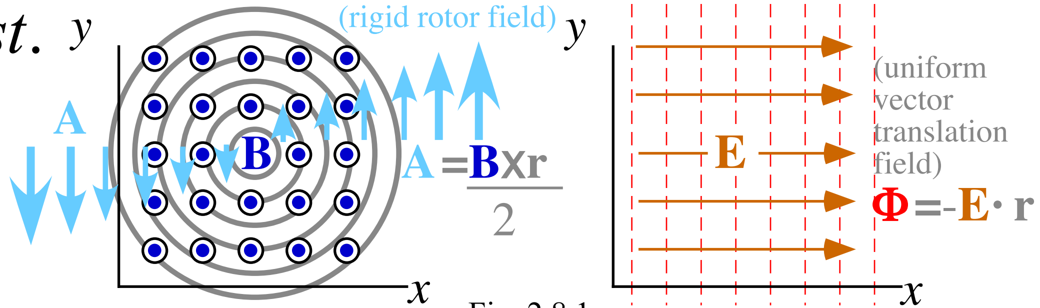

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Shorthand Labeling

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Crossed E and B field mechanics

Classical Hall-effect and cyclotron orbit equations

→ *Vector theory vs. complex variable theory*

Mechanical analog of cyclotron and FBI rule

Cycloid geometry and flying sticks

Practical poolhall application

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A constant **E** field has a scalar potential field **Φ** with constant gradient.

$$\Phi(\mathbf{r}) = -\mathbf{E}\bullet\mathbf{r}, \qquad -\nabla\Phi(\mathbf{r}) = -\nabla\left(-\mathbf{E}\bullet\mathbf{r}\right) = \mathbf{E} = const.$$

A constant **B** field has a vector potential field **A** that resembles a disc spinning counter-clockwise around the **B** axis.

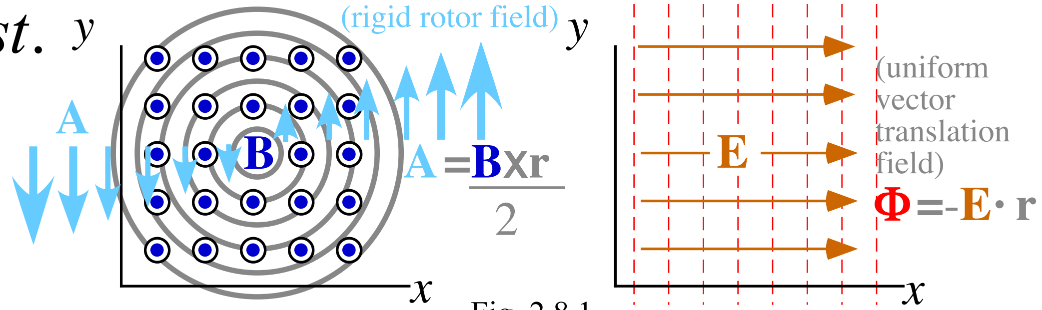

Fig. 2.8.1.

$$\mathbf{A}(\mathbf{r}) = \frac{1}{2}\mathbf{B}\times\mathbf{r}, \qquad \nabla\times\mathbf{A}(\mathbf{r}) = \nabla\times\left(\frac{1}{2}\mathbf{B}\times\mathbf{r}\right) = \mathbf{B} = const.$$

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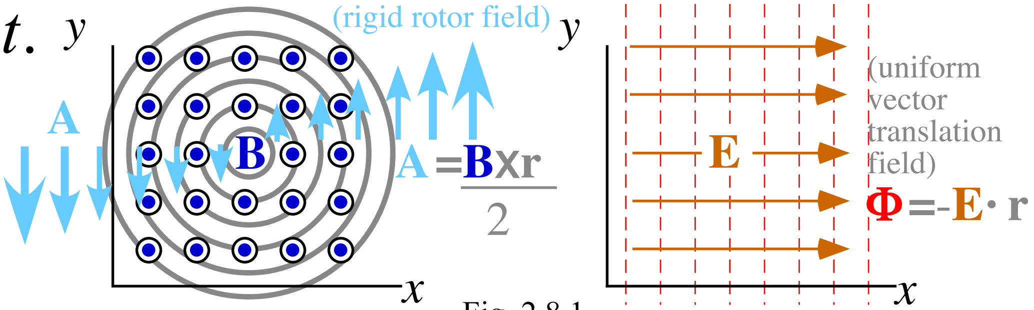

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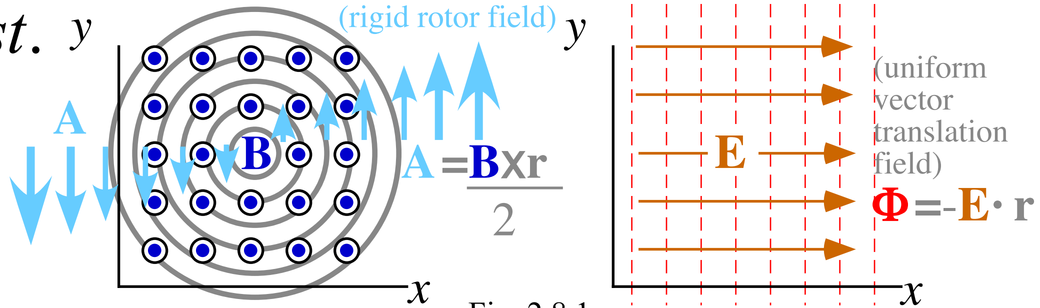

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where : $\boxed{\beta = -\frac{\varepsilon}{iB} = i\frac{\varepsilon}{B}}$

Move last part of this calculation UP↑

Crossed E and B field mechanics (Solution by complex variables)

$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \varepsilon + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z$$

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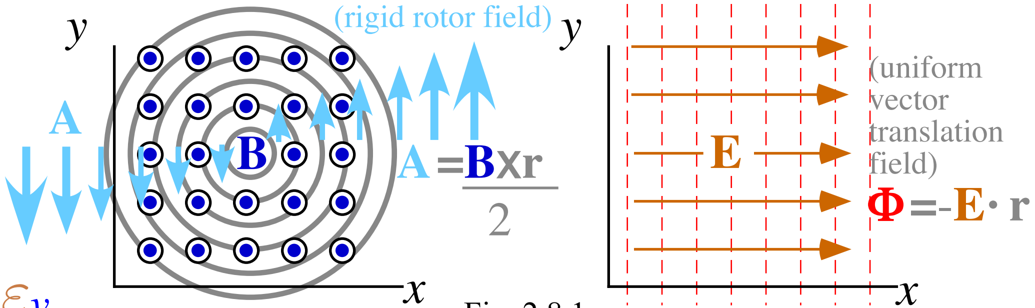
Shorthand Labeling



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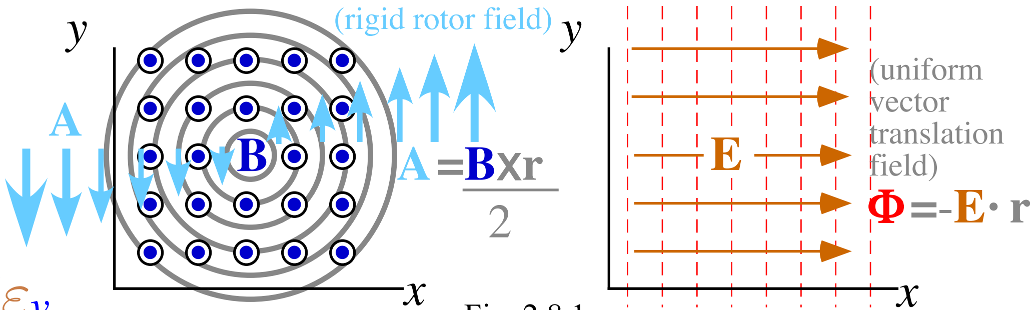

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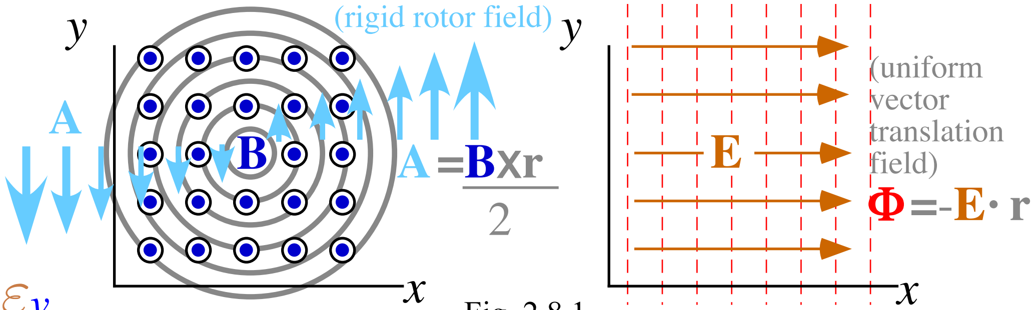

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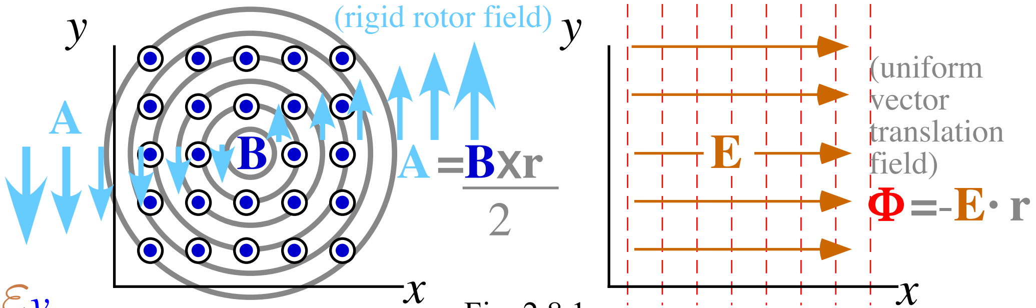
$$\boxed{\begin{array}{l}\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \varepsilon + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z \\ \varepsilon_x = \frac{e}{m}E_x \qquad \varepsilon_y = \frac{e}{m}E_y \qquad B = \frac{e}{m}B_z \\ \textit{Shorthand Labeling}\end{array}}$$



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$$v(t)+\beta = V(t) = e^{-iB\cdot t}V(0) = e^{-iB\cdot t}(v(0)+\beta) \quad \text{or:} \quad v(t)= e^{-iB\cdot t}(v(0)+\beta)-\beta = e^{-iB\cdot t}(v(0)+i\frac{\varepsilon}{B})-i\frac{\varepsilon}{B}$$

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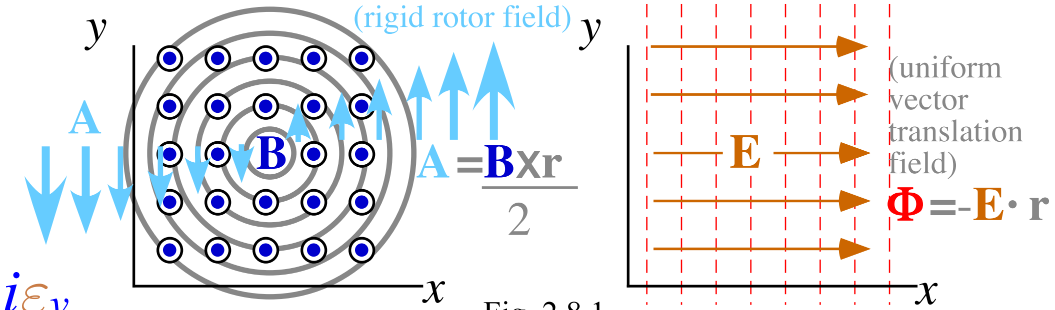
Shorthand Labeling



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Expanding e^{-iBt}, $v=v_x+iv_y$, and $\varepsilon=\varepsilon_x+i\varepsilon_y$ reveals x (Real) and y (Imaginary) components

$$\begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix} = \begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix} \begin{pmatrix} v_x(0) - \frac{\varepsilon_y}{B} \\ v_y(0) + \frac{\varepsilon_x}{B} \end{pmatrix} + \begin{pmatrix} \frac{\varepsilon_y}{B} \\ -\frac{\varepsilon_x}{B} \end{pmatrix}$$

vector form

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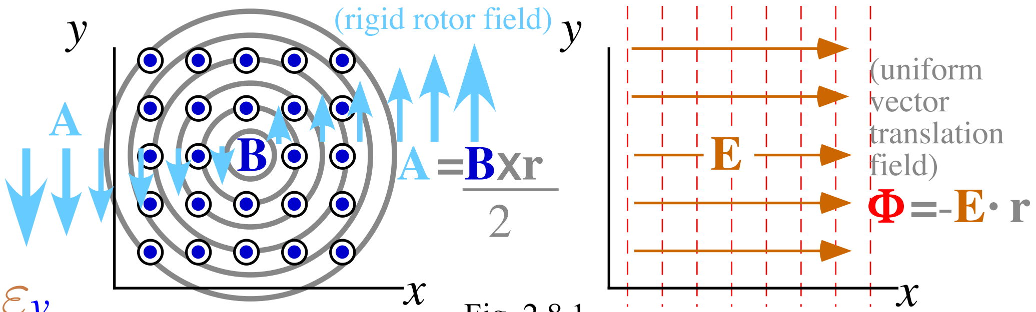

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Integrating $v(t)$ yields complex coordinate $q=x+iy$ affected by both ε_x and ε_y.

vector form

Crossed E and B field mechanics (Solution by complex variables)

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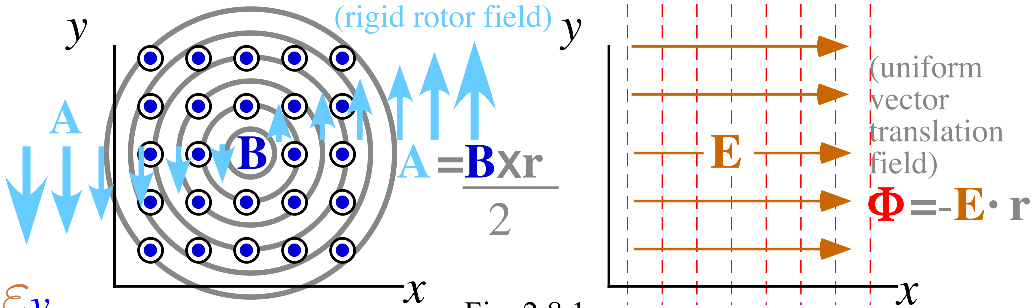

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Complex variable velocity: $v=v_x+iv_y$ and electric field: $\varepsilon=\varepsilon_x+i\varepsilon_y$

$$\dot{v}_x + i\dot{v}_y = \varepsilon_x + i\varepsilon_y - iBv_x + Bv_y = \varepsilon_x + i\varepsilon_y - iB(v_x + iv_y)$$

$$\dot{v} = \varepsilon - iBv \qquad \text{with replacements : } \hat{\mathbf{e}}_{\mathbf{x}} \rightarrow 1 \quad \text{and : } \hat{\mathbf{e}}_{\mathbf{y}} \rightarrow i = \sqrt{-1}$$

A velocity transformation $V(t)=v(t)+\beta$ cancels constant ε-field to give an equation: $\dot{V}=(const.)V$

$$\dot{V}(t) = \dot{v}(t) + \dot{\beta} = \varepsilon - iBv = \varepsilon - iB(V(t) - \beta) = -iBV(t) \qquad \text{where : } \beta = -\frac{\varepsilon}{iB} = i\frac{\varepsilon}{B}$$

An exponential $V(t)=e^{-iBt}V(0)$ solution results: e^{-iBt} is a clockwise 2D rotation.

complex form

$$v(t) + \beta = V(t) = e^{-iB\cdot t}V(0) = e^{-iB\cdot t}(v(0)+\beta) \quad \text{or:} \quad v(t) = e^{-iB\cdot t}(v(0)+\beta) - \beta = e^{-iB\cdot t}(v(0) + i\frac{\varepsilon}{B}) - i\frac{\varepsilon}{B}$$

Expanding e^{-iBt}, $v=v_x+iv_y$, and $\varepsilon=\varepsilon_x+i\varepsilon_y$ reveals x (Real) and y (Imaginary) components

$$\begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix} = \begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix} \begin{pmatrix} v_x(0) - \frac{\varepsilon_y}{B} \\ v_y(0) + \frac{\varepsilon_x}{B} \end{pmatrix} + \begin{pmatrix} \frac{\varepsilon_y}{B} \\ -\frac{\varepsilon_x}{B} \end{pmatrix}$$

vector form

Integrating $v(t)$ yields complex coordinate $q=x+iy$ affected by both ε_x and ε_y.

$$q(t) = \int v(t)\,dt = \frac{e^{-iB\cdot t}}{-iB}(v(0) + i\frac{\varepsilon}{B}) - i\frac{\varepsilon}{B}\cdot t + Const. \quad \text{where: } Const. = q(0) - (\frac{v(0)}{-iB} - \frac{\varepsilon}{B^2})$$

complex form

Crossed E and B field mechanics (Solution by complex variables)

$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \varepsilon + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z$$

$$\varepsilon_x = \tfrac{e}{m}E_x \qquad \varepsilon_y = \tfrac{e}{m}E_y \qquad B = \tfrac{e}{m}B_z$$

Shorthand Labeling

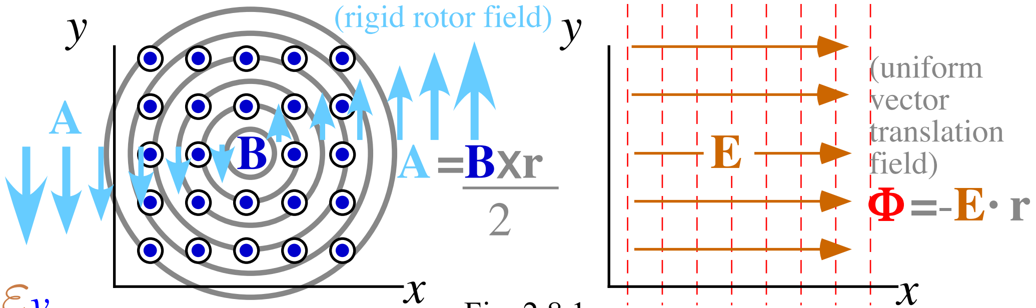

Fig. 2.8.1.

Complex variable velocity: $v=v_x+iv_y$ and electric field: $\varepsilon=\varepsilon_x+i\varepsilon_y$

$$\dot{v}_x + i\dot{v}_y = \varepsilon_x + i\varepsilon_y - iBv_x + Bv_y = \varepsilon_x + i\varepsilon_y - iB(v_x + iv_y)$$

$$\dot{v} \quad = \quad \varepsilon \quad - \quad iBv \qquad \text{with replacements}: \hat{\mathbf{e}}_\mathbf{x} \to 1 \qquad \text{and}: \hat{\mathbf{e}}_\mathbf{y} \to i = \sqrt{-1}$$

A velocity transformation $V(t)=v(t)+\beta$ cancels constant ε-field to give an equation: $\boxed{\dot{V}=(const.)V}$

$$\dot{V}(t)=\dot{v}(t)+\dot{\beta} = \varepsilon - iBv = \varepsilon - iB(V(t)-\beta) = -iBV(t) \qquad \text{where}: \boxed{\beta = -\frac{\varepsilon}{iB} = i\frac{\varepsilon}{B}}$$

An exponential $V(t)=e^{-iBt}V(0)$ solution results: e^{-iBt} is a clockwise 2D rotation.

complex form

$$v(t)+\beta = V(t) = e^{-iB\cdot t}V(0) = e^{-iB\cdot t}(v(0)+\beta) \quad \text{or:} \quad v(t) = e^{-iB\cdot t}(v(0)+\beta)-\beta = e^{-iB\cdot t}(v(0)+i\frac{\varepsilon}{B}) - i\frac{\varepsilon}{B}$$

Expanding e^{-iBt}, $v=v_x+iv_y$, and $\varepsilon=\varepsilon_x+i\varepsilon_y$ reveals x (Real) and y (Imaginary) components

$$\begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix} = \begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix} \begin{pmatrix} v_x(0) - \frac{\varepsilon_y}{B} \\ v_y(0) + \frac{\varepsilon_x}{B} \end{pmatrix} + \begin{pmatrix} \frac{\varepsilon_y}{B} \\ -\frac{\varepsilon_x}{B} \end{pmatrix}$$

vector form

Integrating $v(t)$ yields complex coordinate $q=x+iy$ affected by both ε_x and ε_y.

$$q(t) = \int v(t)\,dt = \frac{e^{-iB\cdot t}}{-iB}(v(0)+i\frac{\varepsilon}{B}) - i\frac{\varepsilon}{B}\cdot t + Const. \qquad \text{where: } Const. = q(0) - (\frac{v(0)}{-iB} - \frac{\varepsilon}{B^2})$$

complex form

$$x(t)+iy(t) = e^{-iB\cdot t}(i\frac{v(0)}{B} - \frac{\varepsilon}{B^2}) - i\frac{\varepsilon}{B}\cdot t + x(0) + iy(0) - i\frac{v(0)}{B} + \frac{\varepsilon}{B^2}$$

Crossed E and B field mechanics (Solution by complex variables)

$$\boxed{\begin{array}{l}\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \varepsilon + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z \\ \varepsilon_x = \frac{e}{m}E_x \qquad \varepsilon_y = \frac{e}{m}E_y \qquad B = \frac{e}{m}B_z \\ \textit{Shorthand Labeling}\end{array}}$$

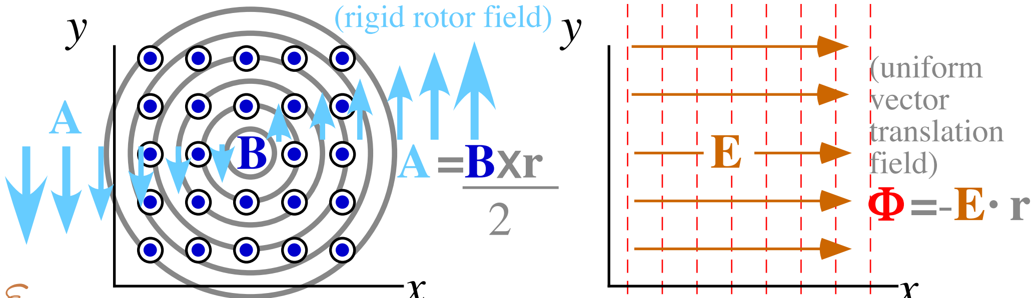

Fig. 2.8.1.

Complex variable velocity: $v=v_x+iv_y$ *and electric field:* $\varepsilon=\varepsilon_x+i\varepsilon_y$

$$\dot{v}_x + i\dot{v}_y = \varepsilon_x + i\varepsilon_y - iBv_x + Bv_y = \varepsilon_x + i\varepsilon_y - iB(v_x + iv_y)$$

$$\dot{v} \quad = \quad \varepsilon \quad - \quad iBv \qquad \text{with replacements : } \hat{\mathbf{e}}_\mathbf{x} \rightarrow 1 \qquad \text{and : } \hat{\mathbf{e}}_\mathbf{y} \rightarrow i = \sqrt{-1}$$

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$$\dot{V}(t) = \dot{v}(t) + \dot{\beta} = \varepsilon - iBv = \varepsilon - iB(V(t) - \beta) = -iBV(t) \qquad \text{where : } \boxed{\beta = -\frac{\varepsilon}{iB} = i\frac{\varepsilon}{B}}$$

An exponential $V(t)=e^{-iBt}V(0)$ solution results: e^{-iBt} is a clockwise 2D rotation.

complex form

$$v(t) + \beta = V(t) = e^{-iB\cdot t}V(0) = e^{-iB\cdot t}(v(0) + \beta) \quad \text{or:} \quad v(t) = e^{-iB\cdot t}(v(0) + \beta) - \beta = e^{-iB\cdot t}(v(0) + i\frac{\varepsilon}{B}) - i\frac{\varepsilon}{B}$$

Expanding e^{-iBt}, $v=v_x+iv_y$, and $\varepsilon=\varepsilon_x+i\varepsilon_y$ reveals x (Real) and y (Imaginary) components

$$\begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix} = \begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix} \begin{pmatrix} v_x(0) - \frac{\varepsilon_y}{B} \\ v_y(0) + \frac{\varepsilon_x}{B} \end{pmatrix} + \begin{pmatrix} \frac{\varepsilon_y}{B} \\ -\frac{\varepsilon_x}{B} \end{pmatrix}$$

vector form

Integrating $v(t)$ yields complex coordinate $q=x+iy$ affected by both ε_x and ε_y.

$$q(t) = \int v(t)\,dt = \frac{e^{-iB\cdot t}}{-iB}(v(0) + i\frac{\varepsilon}{B}) - i\frac{\varepsilon}{B}\cdot t + Const. \quad \text{where: } Const. = q(0) - (\frac{v(0)}{-iB} - \frac{\varepsilon}{B^2}) \quad \textit{complex form}$$

$$x(t) + iy(t) = e^{-iB\cdot t}(i\frac{v(0)}{B} - \frac{\varepsilon}{B^2}) - i\frac{\varepsilon}{B}\cdot t + x(0) + iy(0) - i\frac{v(0)}{B} + \frac{\varepsilon}{B^2}$$

Move last part of this calculation UP↑

Crossed E and B field mechanics (Solution by complex variables)

$$\dot{V}(t)=\dot{v}(t)+\dot{\beta}=\varepsilon-iBv=\varepsilon-iB(V(t)-\beta)=-iBV(t) \qquad \text{where : } \boxed{\beta=-\frac{\varepsilon}{iB}=i\frac{\varepsilon}{B}}$$

An exponential $V(t)=e^{-iBt}V(0)$ solution results: e^{-iBt} is a clockwise 2D rotation.

complex form

$$v(t)+\beta=V(t)=e^{-iB\cdot t}V(0)=e^{-iB\cdot t}(v(0)+\beta) \quad \text{or:} \quad v(t)=e^{-iB\cdot t}(v(0)+\beta)-\beta=e^{-iB\cdot t}(v(0)+i\frac{\varepsilon}{B})-i\frac{\varepsilon}{B}$$

Expanding e^{-iBt}, $v=v_x+iv_y$, and $\varepsilon=\varepsilon_x+i\varepsilon_y$ reveals x (Real) and y (Imaginary) components

$$\begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix}=\begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix}\begin{pmatrix} v_x(0)-\frac{\varepsilon_y}{B} \\ v_y(0)+\frac{\varepsilon_x}{B} \end{pmatrix}+\begin{pmatrix} \frac{\varepsilon_y}{B} \\ -\frac{\varepsilon_x}{B} \end{pmatrix}$$

Integrating $v(t)$ yields complex coordinate $q=x+iy$ affected by both ε_x and ε_y.

vector form

$$q(t)=\int v(t)\,dt=\frac{e^{-iB\cdot t}}{-iB}(v(0)+i\frac{\varepsilon}{B})-i\frac{\varepsilon}{B}\cdot t+Const. \quad \text{where: } Const.=q(0)-(\frac{v(0)}{-iB}-\frac{\varepsilon}{B^2})$$

complex form

$$x(t)+iy(t)=e^{-iB\cdot t}(i\frac{v(0)}{B}-\frac{\varepsilon}{B^2})-i\frac{\varepsilon}{B}\cdot t+x(0)+iy(0)-i\frac{v(0)}{B}+\frac{\varepsilon}{B^2}$$

complex form

Crossed E and B field mechanics (Solution by complex variables)

$$\dot{V}(t)=\dot{v}(t)+\dot{\beta}=\varepsilon-iBv=\varepsilon-iB(V(t)-\beta)=-iBV(t) \qquad \text{where :} \quad \beta=-\frac{\varepsilon}{iB}=i\frac{\varepsilon}{B}$$

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$$v(t)+\beta=V(t)=e^{-iB\cdot t}V(0)=e^{-iB\cdot t}(v(0)+\beta) \quad \text{or:} \quad v(t)=e^{-iB\cdot t}(v(0)+\beta)-\beta=e^{-iB\cdot t}(v(0)+i\frac{\varepsilon}{B})-i\frac{\varepsilon}{B}$$

Expanding e^{-iBt}, $v=v_x+iv_y$, and $\varepsilon=\varepsilon_x+i\varepsilon_y$ reveals x (Real) and y (Imaginary) components

$$\begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix}=\begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix}\begin{pmatrix} v_x(0)-\frac{\varepsilon_y}{B} \\ v_y(0)+\frac{\varepsilon_x}{B} \end{pmatrix}+\begin{pmatrix} \frac{\varepsilon_y}{B} \\ -\frac{\varepsilon_x}{B} \end{pmatrix}$$

vector form

Integrating $v(t)$ yields complex coordinate $q=x+iy$ affected by both ε_x and ε_y.

$$q(t)=\int v(t)\,dt=\frac{e^{-iB\cdot t}}{-iB}(v(0)+i\frac{\varepsilon}{B})-i\frac{\varepsilon}{B}\cdot t+Const. \quad \text{where: } Const.=q(0)-(\frac{v(0)}{-iB}-\frac{\varepsilon}{B^2})$$ *complex form*

$$x(t)+iy(t)=e^{-iB\cdot t}(i\frac{v(0)}{B}-\frac{\varepsilon}{B^2})-i\frac{\varepsilon}{B}\cdot t+x(0)+iy(0)-i\frac{v(0)}{B}+\frac{\varepsilon}{B^2}$$ *complex form*

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}=\begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix}\begin{pmatrix} -\frac{v_y(0)}{B}-\frac{\varepsilon_x}{B^2} \\ \frac{v_x(0)}{B}-\frac{\varepsilon_y}{B^2} \end{pmatrix}+\begin{pmatrix} \frac{\varepsilon_y}{B}t \\ -\frac{\varepsilon_x}{B}t \end{pmatrix}+\begin{pmatrix} x(0)+\frac{v_y(0)}{B}+\frac{\varepsilon_x}{B^2} \\ y(0)-\frac{v_x(0)}{B}+\frac{\varepsilon_y}{B^2} \end{pmatrix}$$ *vector form*

Crossed E and B field mechanics (Solution by complex variables)

$$\dot{V}(t)=\dot{v}(t)+\dot{\beta}=\varepsilon-iBv=\varepsilon-iB(V(t)-\beta)=-iBV(t)$$

where : $\boxed{\beta=-\frac{\varepsilon}{iB}=i\frac{\varepsilon}{B}}$

An exponential $V(t)=e^{-iBt}V(0)$ solution results: e^{-iBt} is a clockwise 2D rotation.

$$v(t)+\beta=V(t)=e^{-iB\cdot t}V(0)=e^{-iB\cdot t}(v(0)+\beta)$$

or:

$$v(t)=e^{-iB\cdot t}(v(0)+\beta)-\beta=e^{-iB\cdot t}\left(v(0)+i\frac{\varepsilon}{B}\right)-i\frac{\varepsilon}{B}$$

complex form

Expanding e^{-iBt}, $v=v_x+iv_y$, and $\varepsilon=\varepsilon_x+i\varepsilon_y$ reveals x (Real) and y (Imaginary) components

$$\begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix}=\begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix}\begin{pmatrix} v_x(0)-\frac{\varepsilon_y}{B} \\ v_y(0)+\frac{\varepsilon_x}{B} \end{pmatrix}+\begin{pmatrix} \frac{\varepsilon_y}{B} \\ -\frac{\varepsilon_x}{B} \end{pmatrix}$$

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Integrating $v(t)$ yields complex coordinate $q=x+iy$ affected by both ε_x and ε_y.

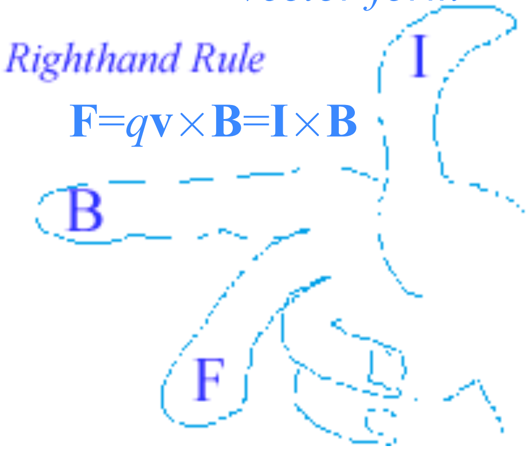
$$q(t)=\int v(t)\,dt=\frac{e^{-iB\cdot t}}{-iB}\left(v(0)+i\frac{\varepsilon}{B}\right)-i\frac{\varepsilon}{B}\cdot t+Const. \quad \text{where: } Const.=q(0)-\left(\frac{v(0)}{-iB}-\frac{\varepsilon}{B^2}\right)$$

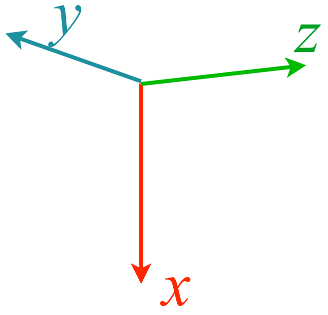
complex form

$$x(t)+iy(t)=e^{-iB\cdot t}\left(i\frac{v(0)}{B}-\frac{\varepsilon}{B^2}\right)-i\frac{\varepsilon}{B}\cdot t+x(0)+iy(0)-i\frac{v(0)}{B}+\frac{\varepsilon}{B^2}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}=\begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix}\begin{pmatrix} -\frac{v_y(0)}{B}-\frac{\varepsilon_x}{B^2} \\ \frac{v_x(0)}{B}-\frac{\varepsilon_y}{B^2} \end{pmatrix}+\begin{pmatrix} \frac{\varepsilon_y}{B}t \\ -\frac{\varepsilon_x}{B}t \end{pmatrix}+\begin{pmatrix} x(0)+\frac{v_y(0)}{B}+\frac{\varepsilon_x}{B^2} \\ y(0)-\frac{v_x(0)}{B}+\frac{\varepsilon_y}{B^2} \end{pmatrix}$$

vector form





Crossed E and B field mechanics (Solution by complex variables)

$$\boxed{\dot{V}(t)=\dot{v}(t)+\dot{\beta}=\varepsilon-iBv=\varepsilon-iB(V(t)-\beta)=-iBV(t)}$$

where : $\boxed{\beta=-\frac{\varepsilon}{iB}=i\frac{\varepsilon}{B}}$

An exponential $V(t)=e^{-iBt}V(0)$ solution results: e^{-iBt} is a clockwise 2D rotation.

$$v(t)+\beta=V(t)=e^{-iB\cdot t}V(0)=e^{-iB\cdot t}(v(0)+\beta)$$

or:

$$v(t)=e^{-iB\cdot t}(v(0)+\beta)-\beta=e^{-iB\cdot t}(v(0)+i\frac{\varepsilon}{B})-i\frac{\varepsilon}{B}$$

complex form

Expanding e^{-iBt}, $v=v_x+iv_y$, and $\varepsilon=\varepsilon_x+i\varepsilon_y$ reveals x (Real) and y (Imaginary) components

$$\begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix}=\begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix}\begin{pmatrix} v_x(0)-\frac{\varepsilon_y}{B} \\ v_y(0)+\frac{\varepsilon_x}{B} \end{pmatrix}+\begin{pmatrix} \frac{\varepsilon_y}{B} \\ -\frac{\varepsilon_x}{B} \end{pmatrix}$$

vector form

Integrating $v(t)$ yields complex coordinate $q=x+iy$ affected by both ε_x and ε_y.

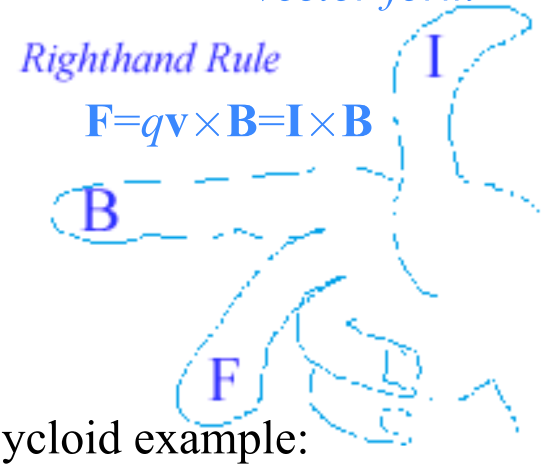
$$q(t)=\int v(t)\,dt=\frac{e^{-iB\cdot t}}{-iB}(v(0)+i\frac{\varepsilon}{B})-i\frac{\varepsilon}{B}\cdot t+Const. \quad \text{where: } Const.=q(0)-(\frac{v(0)}{-iB}-\frac{\varepsilon}{B^2})$$

complex form

$$x(t)+iy(t)=\quad e^{-iB\cdot t}\quad (i\frac{v(0)}{B}-\frac{\varepsilon}{B^2})\quad -i\frac{\varepsilon}{B}\cdot t\quad +x(0)+iy(0)-i\frac{v(0)}{B}+\frac{\varepsilon}{B^2}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}=\begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix}\begin{pmatrix} -\frac{v_y(0)}{B}-\frac{\varepsilon_x}{B^2} \\ \frac{v_x(0)}{B}-\frac{\varepsilon_y}{B^2} \end{pmatrix}+\begin{pmatrix} \frac{\varepsilon_y}{B}t \\ -\frac{\varepsilon_x}{B}t \end{pmatrix}+\begin{pmatrix} x(0)+\frac{v_y(0)}{B}+\frac{\varepsilon_x}{B^2} \\ y(0)-\frac{v_x(0)}{B}+\frac{\varepsilon_y}{B^2} \end{pmatrix}$$

vector form



Righthand Rule

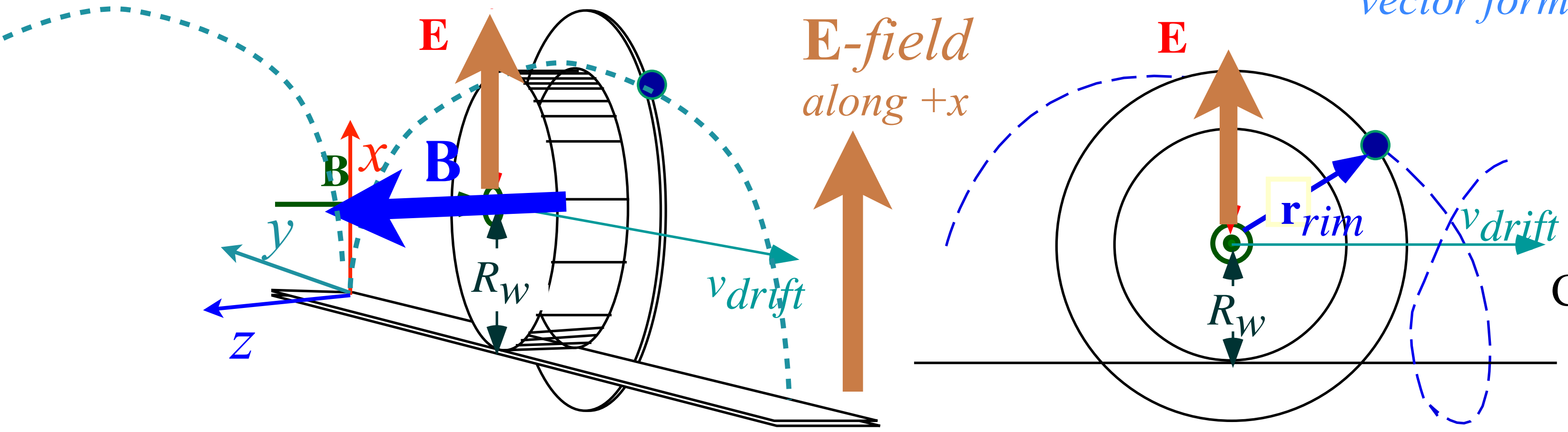
F=q**v**×**B**=**I**×**B**

Cycloid example:
initial $(x(0)\,,\,y(0))=(0,0)$
and $(v_x(0),v_y(0))=(0,0)$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}=$$

is on rim of a wheel of radius $R_W=E/B^2$

$$\begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix}\begin{pmatrix} -\frac{E}{B^2} \\ 0 \end{pmatrix}+\begin{pmatrix} 0 \\ -\frac{E}{B}t \end{pmatrix}+\begin{pmatrix} \frac{E}{B^2} \\ 0 \end{pmatrix}$$



Constant v_{drift} in -y-direction

$v_{drift}=-E/B$

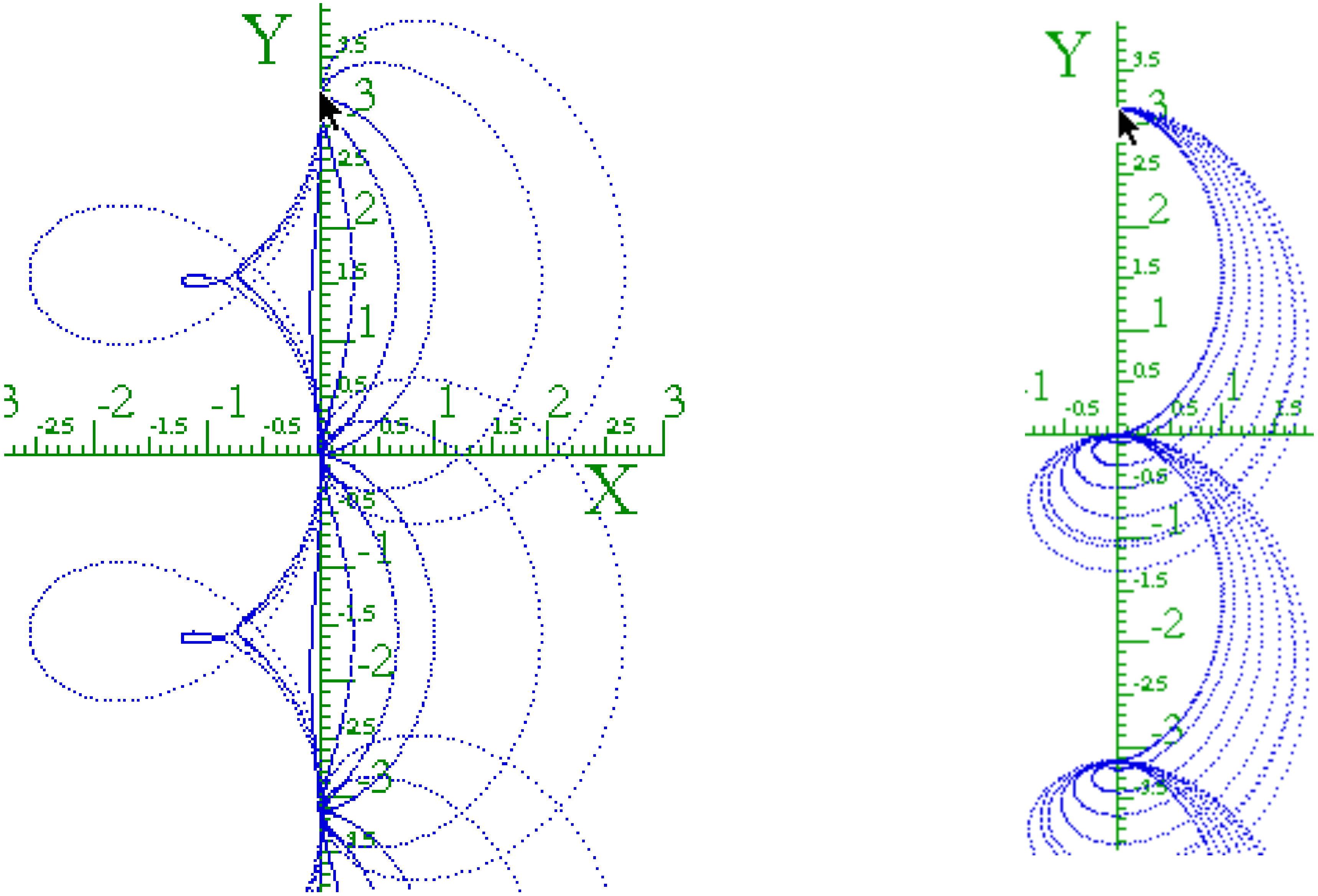


Fig. 2.8.2 Trajectories of unit charge and mass in magnetic and electric fields (E=1/2, B=1)

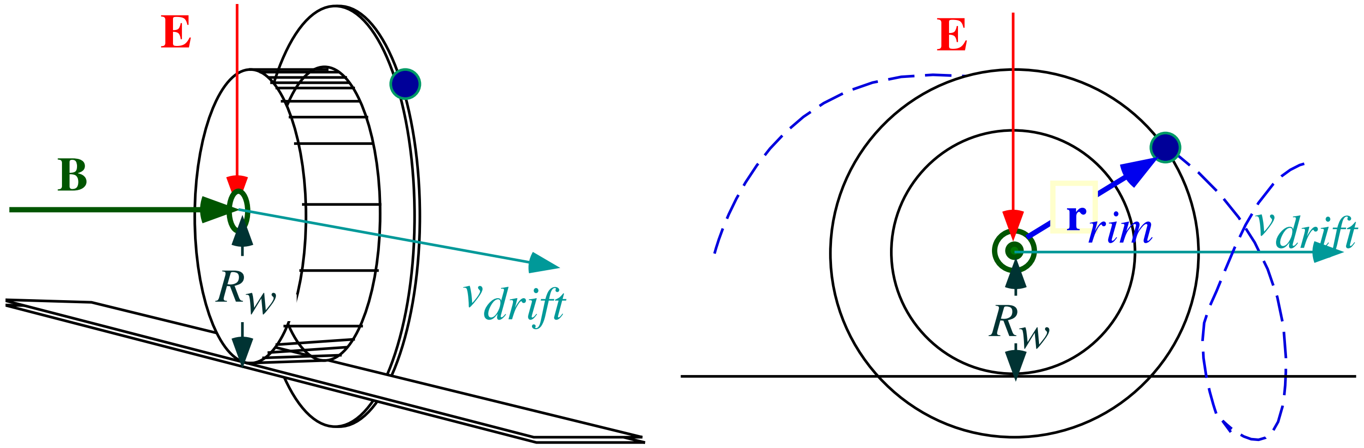
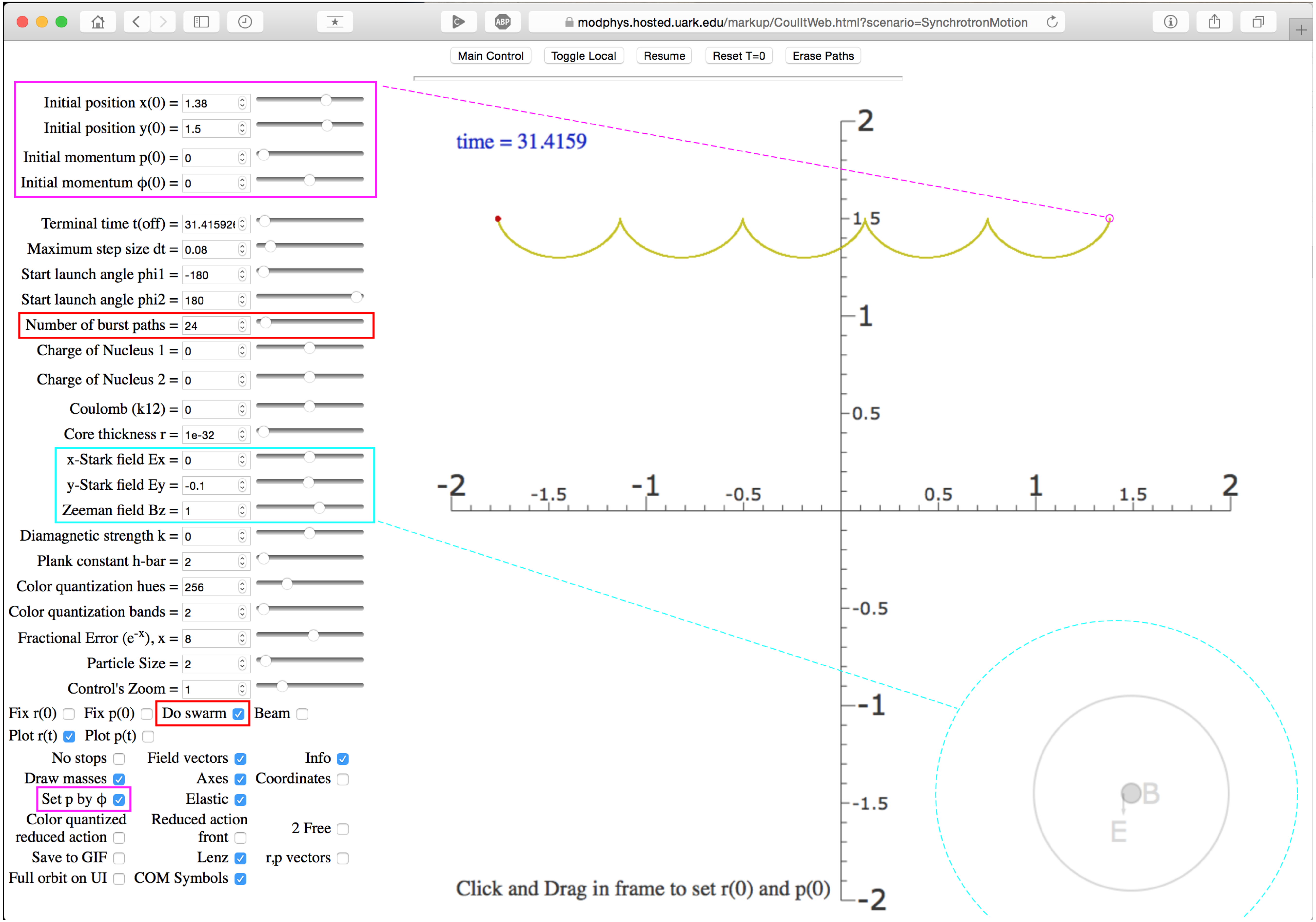


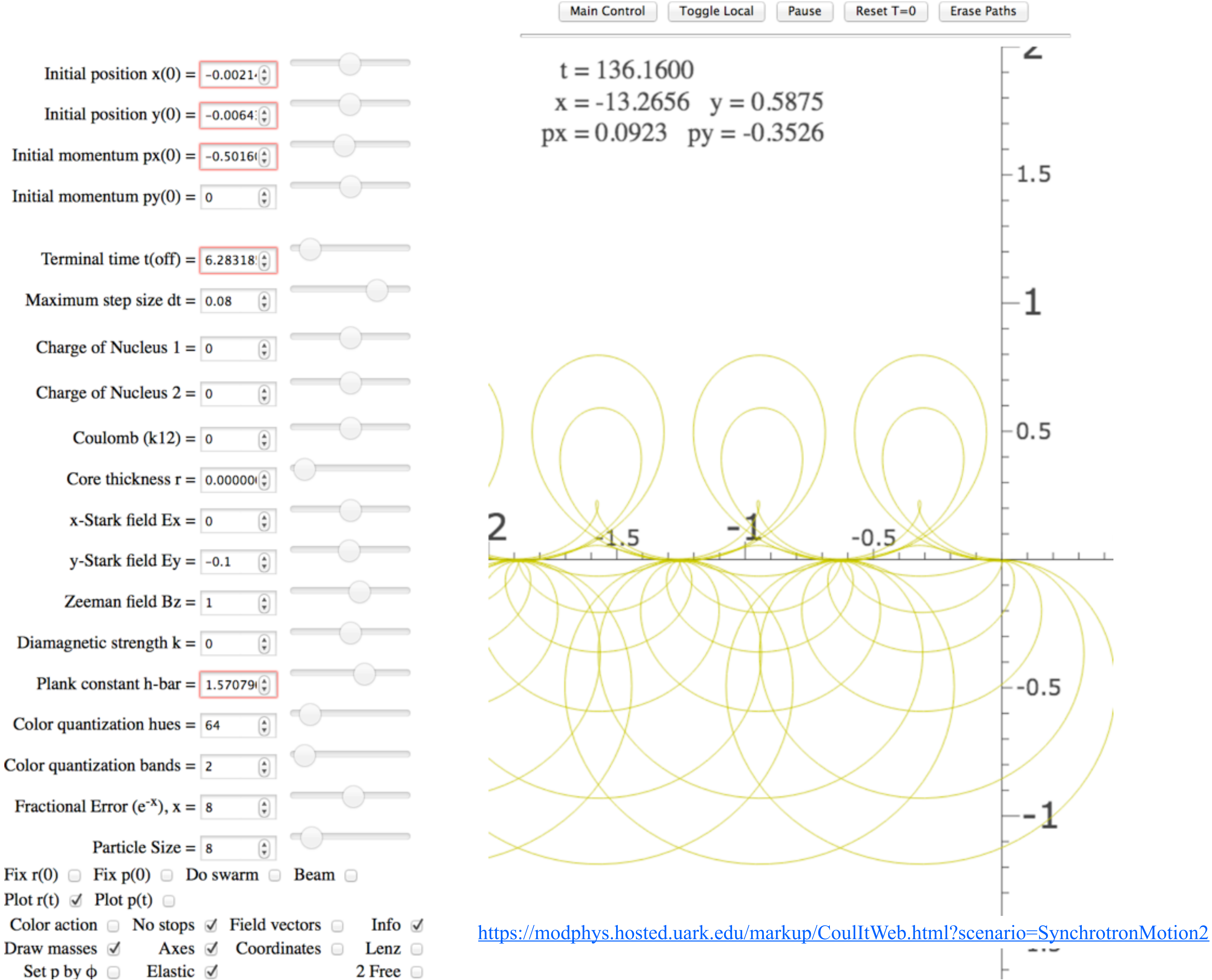
Fig. 2.8.3 Rolling railroad wheel and rim analogy for cyclotron orbits



Initial position x(0) = 1.38
Initial position y(0) = 1.5
Initial momentum p(0) = 0
Initial momentum ϕ(0) = 0
Terminal time t(off) = 31.415926
Maximum step size dt = 0.08
Start launch angle phi1 = -180
Start launch angle phi2 = 180
Number of burst paths = 24
Charge of Nucleus 1 = 0
Charge of Nucleus 2 = 0
Coulomb (k12) = 0
Core thickness r = 1e-32
x-Stark field Ex = 0
y-Stark field Ey = -0.1
Zeeman field Bz = 1
Diamagnetic strength k = 0
Plank constant h-bar = 2
Color quantization hues = 256
Color quantization bands = 2
Fractional Error (e^{-x}), x = 8
Particle Size = 2
Control's Zoom = 1

Fix r(0) ☐ Fix p(0) ☐ Do swarm ☑ Beam ☐
Plot r(t) ☑ Plot p(t) ☐
No stops ☐ Field vectors ☑ Info ☑
Draw masses ☑ Axes ☑ Coordinates ☐
Set p by ϕ ☑ Elastic ☑
Color quantized reduced action ☐ Reduced action front ☐ 2 Free ☐
Save to GIF ☐ Lenz ☑ r,p vectors ☐
Full orbit on UI ☐ COM Symbols ☑

https://modphys.hosted.uark.edu/markup/CoulItWeb.html?scenario=SynchrotronMotion



t = 136.1600
x = -13.2656 y = 0.5875
px = 0.0923 py = -0.3526

Initial position x(0) = -0.0021

Initial position y(0) = -0.0064

Initial momentum px(0) = -0.5016

Initial momentum py(0) = 0

Terminal time t(off) = 6.28318

Maximum step size dt = 0.08

Charge of Nucleus 1 = 0

Charge of Nucleus 2 = 0

Coulomb (k12) = 0

Core thickness r = 0.00000

x-Stark field Ex = 0

y-Stark field Ey = -0.1

Zeeman field Bz = 1

Diamagnetic strength k = 0

Plank constant h-bar = 1.57079

Color quantization hues = 64

Color quantization bands = 2

Fractional Error (e^{-x}), x = 8

Particle Size = 8

Fix r(0) ☐ Fix p(0) ☐ Do swarm ☐ Beam ☐
Plot r(t) ☑ Plot p(t) ☐
Color action ☐ No stops ☑ Field vectors ☐ Info ☑
Draw masses ☑ Axes ☑ Coordinates ☐ Lenz ☐
Set p by ϕ ☐ Elastic ☑ 2 Free ☐

https://modphys.hosted.uark.edu/markup/CoulItWeb.html?scenario=SynchrotronMotion2

Crossed E and B field mechanics

Classical Hall-effect and cyclotron orbit equations

Vector theory vs. complex variable theory

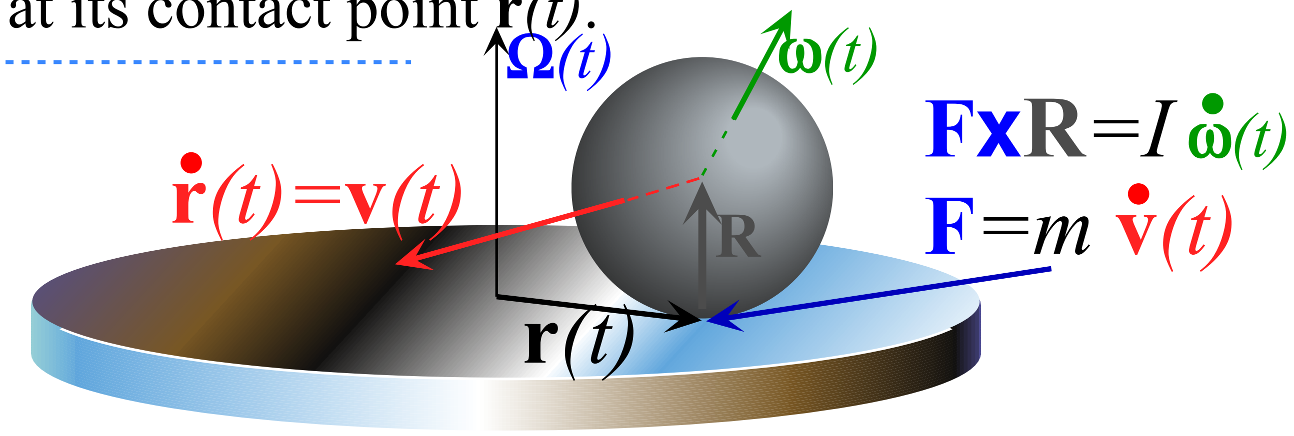
→ *Mechanical analog of cyclotron and FBI rule*

Cycloid geometry and flying sticks

Practical poolhall application

Mechanical analog of cyclotron and FBI rule

Velocity vector of the ball contact point ($\mathbf{v}(t)$-$\boldsymbol{\omega}(t)\times\mathrm{R}$) equals table surface velocity $\boldsymbol{\Omega}\times\mathbf{r}(t)$ at its contact point $\mathbf{r}(t)$.



$$\mathbf{F}\mathbf{x}\mathrm{R}=I\,\dot{\boldsymbol{\omega}}(t)$$

$$\mathbf{F}=m\,\dot{\mathbf{v}}(t)$$

Turntable turning at constant angular velocity $\boldsymbol{\Omega}=\Omega\hat{\mathbf{z}}$.

YouTube Video of Analog to Syncrotron Motion

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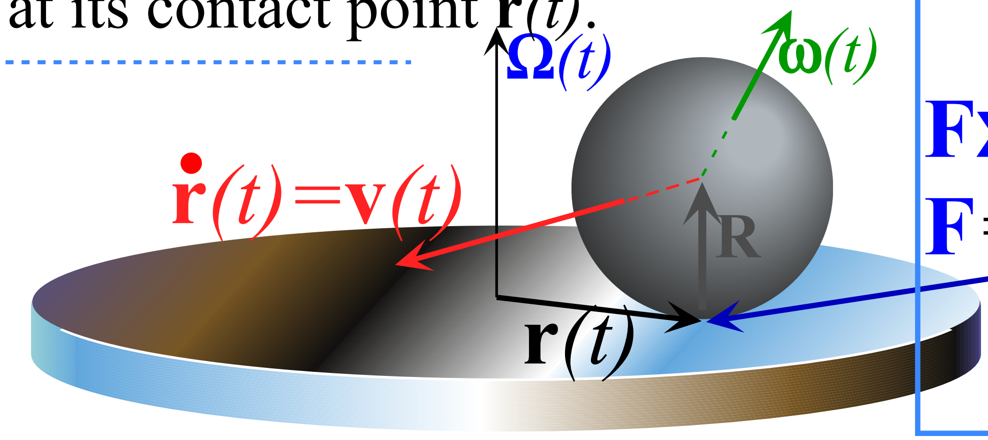

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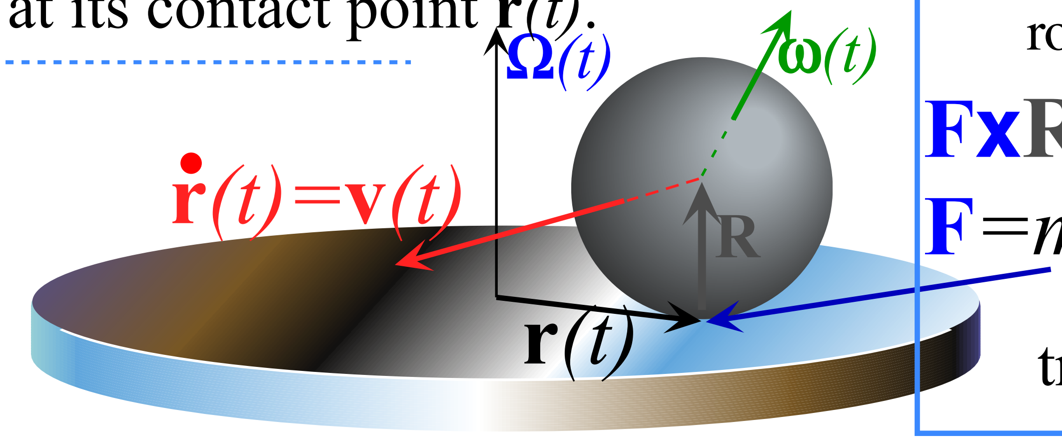

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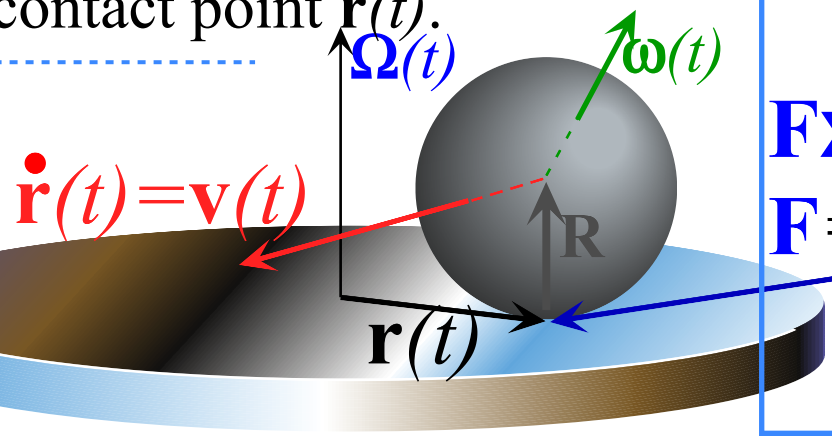

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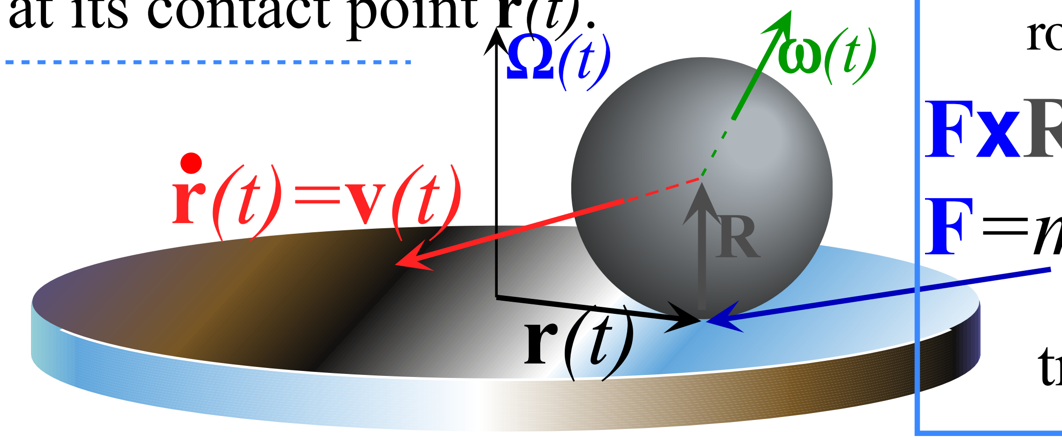

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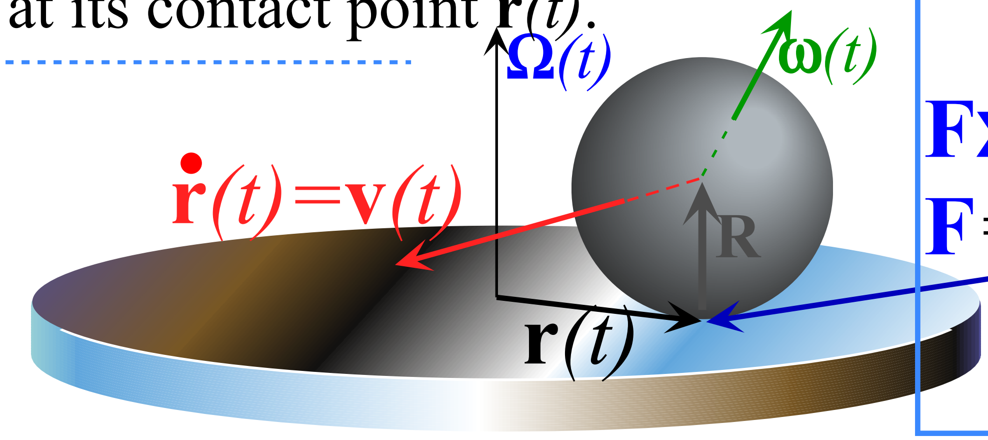

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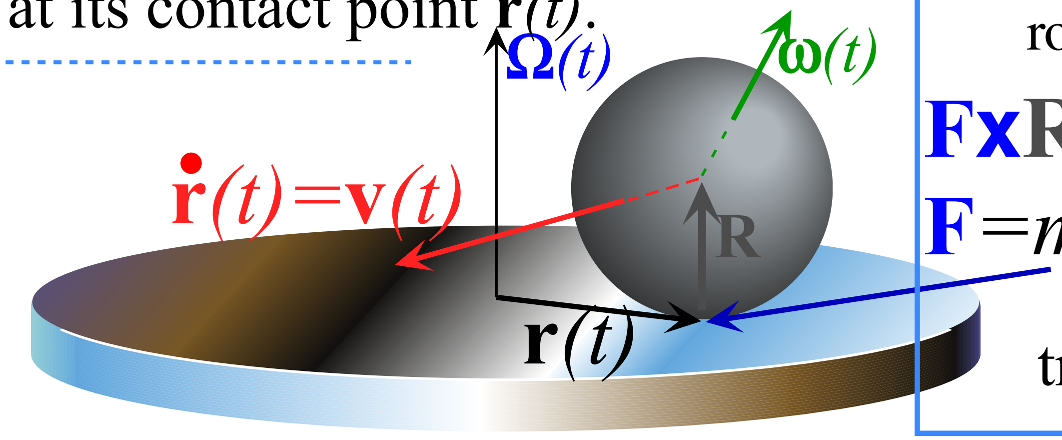

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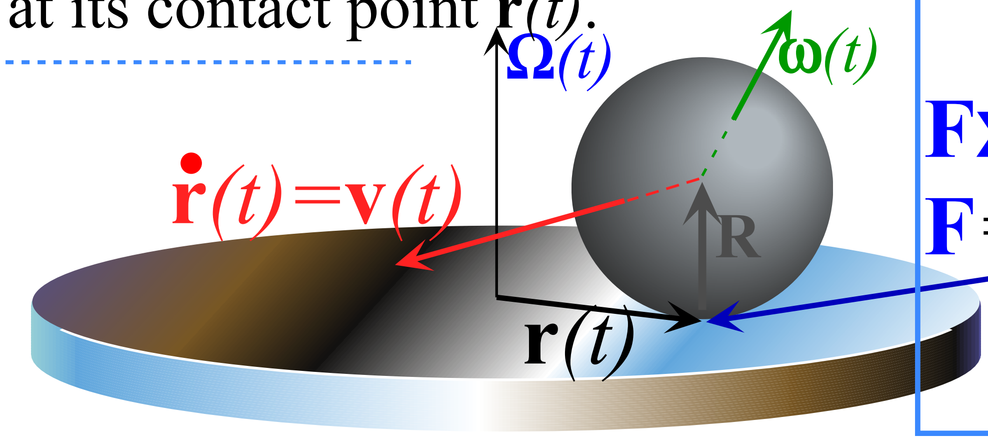

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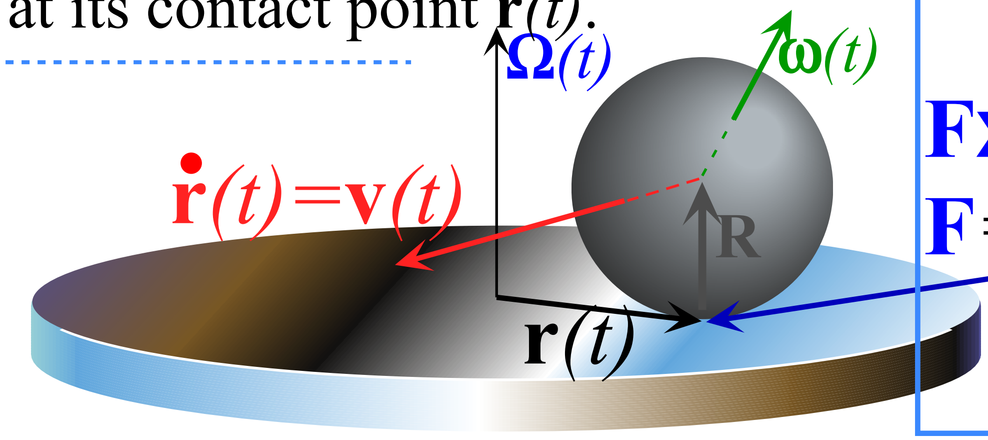

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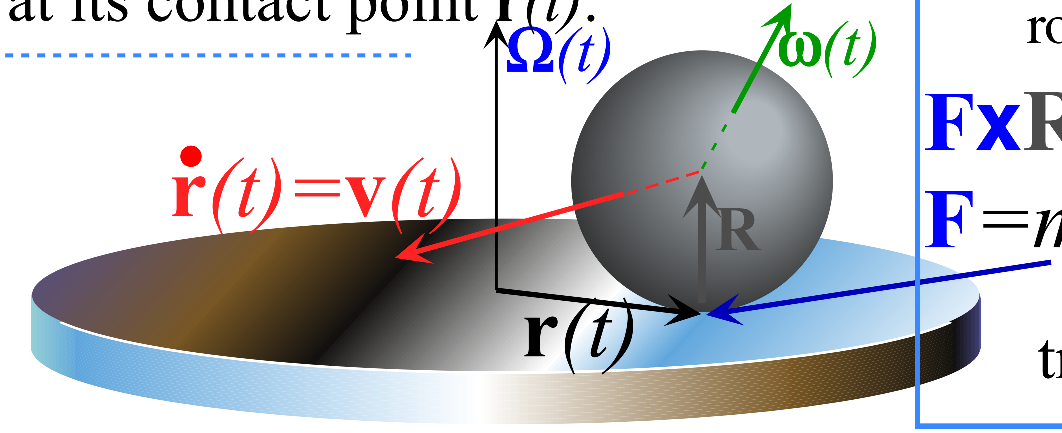

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$\mathbf{F}=\mathbf{B}\times\mathbf{v}$ mechanical analog:

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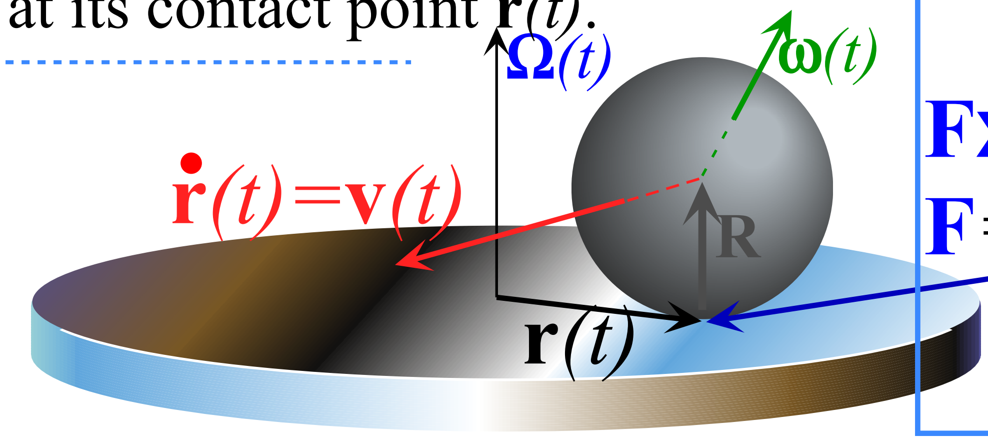

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ma=eB×v mechanical analog:

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Mechanical analog cyclotron frequency

$$\omega=\frac{e}{m}B=\frac{\boldsymbol{\Omega}}{1+\dfrac{mR^2}{I}}$$

$\omega=\frac{2}{7}\boldsymbol{\Omega}$ for: $\frac{I}{mR^2}=\frac{2}{5}$ ●

$=\frac{2}{5}\boldsymbol{\Omega}$ for: $\frac{I}{mR^2}=\frac{2}{3}$ ○

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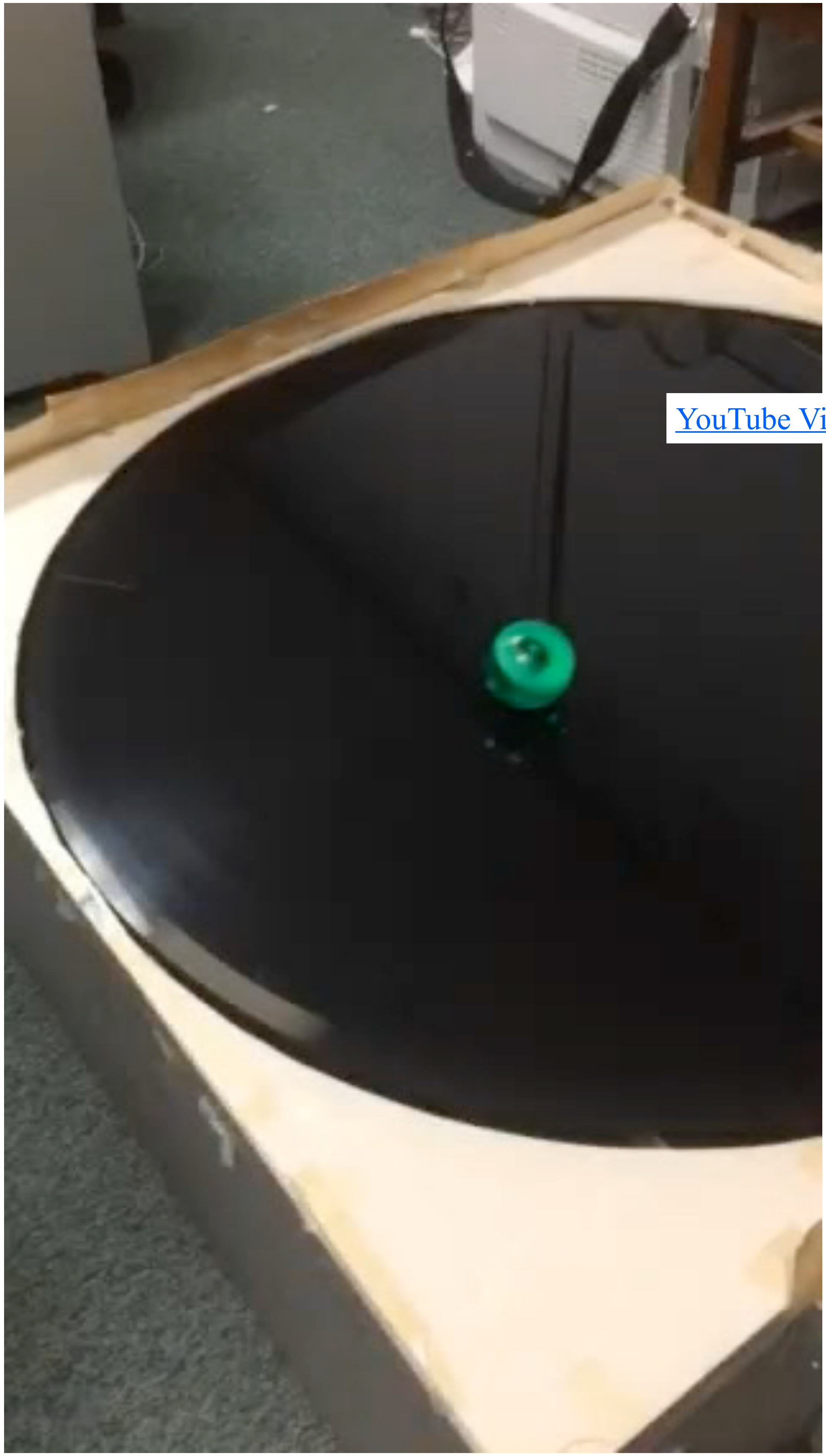

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Solid ball has 2 orbits
as table turns 7 rotations

Mechanical analog
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Crossed E and B field mechanics

Classical Hall-effect and cyclotron orbits

Vector theory vs. complex variable theory

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Cycloid geometry and flying sticks

Practical poolhall application

If you hammer a stick at a point h meters from its center
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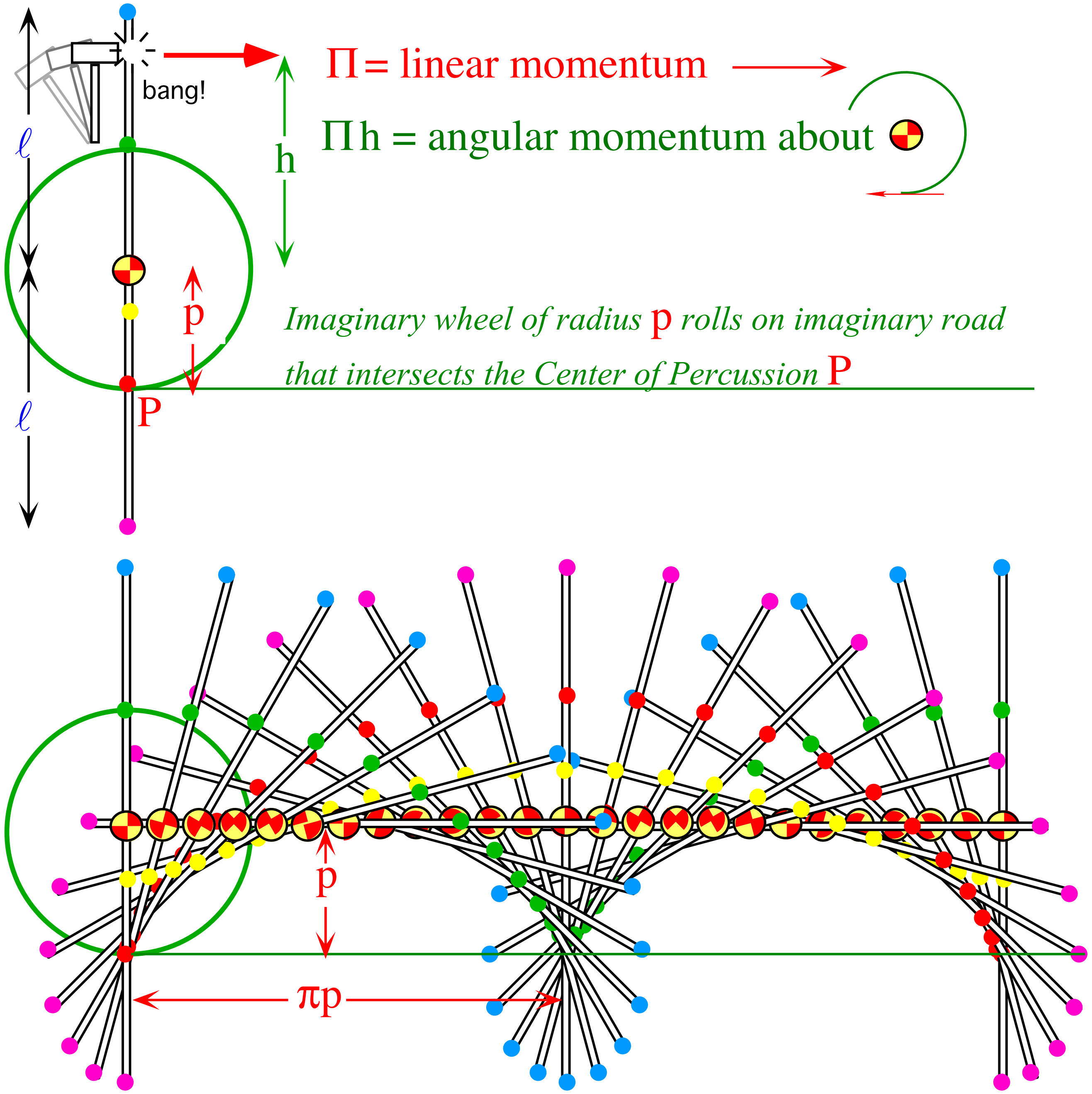

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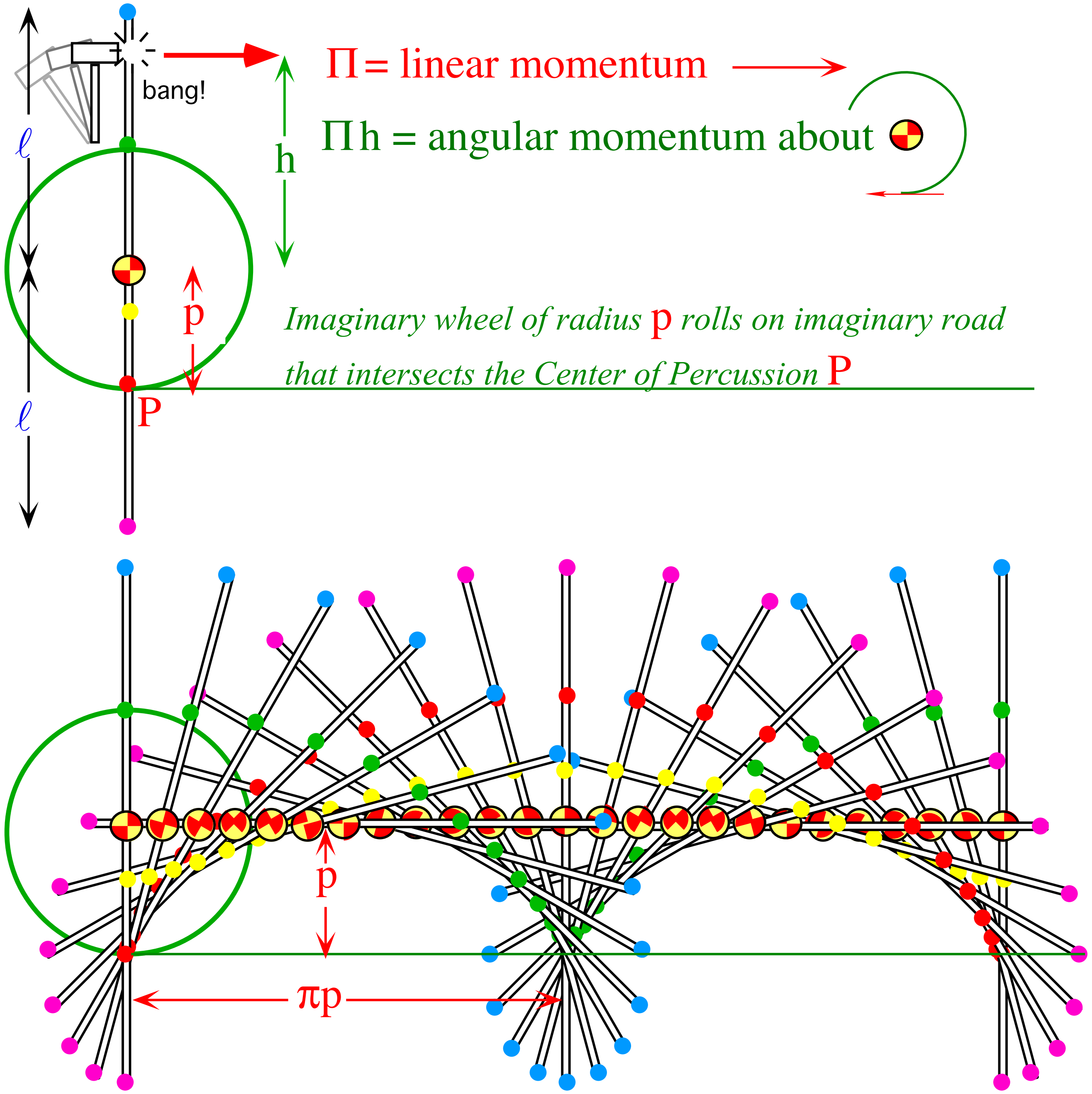
$$I = \int_0^{\ell} \rho r^2\, dr = \frac{\rho r^3}{3}\Big|_0^{\ell} = \frac{\rho \ell^3}{3} = M\,\frac{\ell^2}{3}$$

$$M = \rho \ell$$



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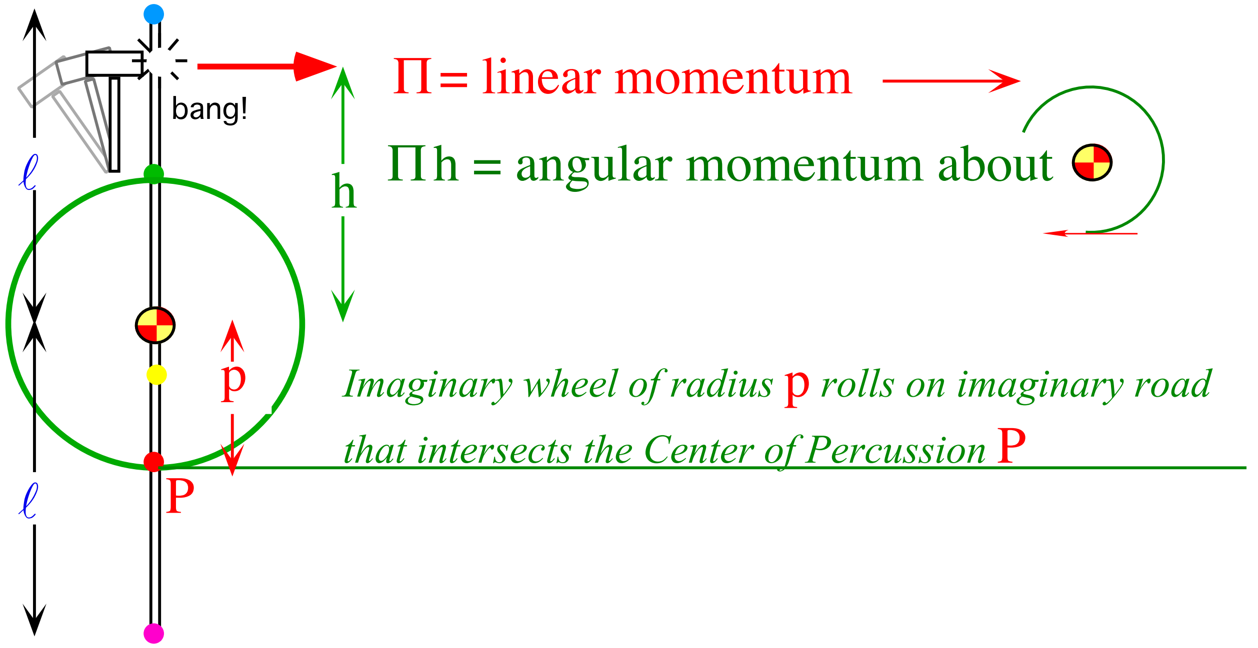

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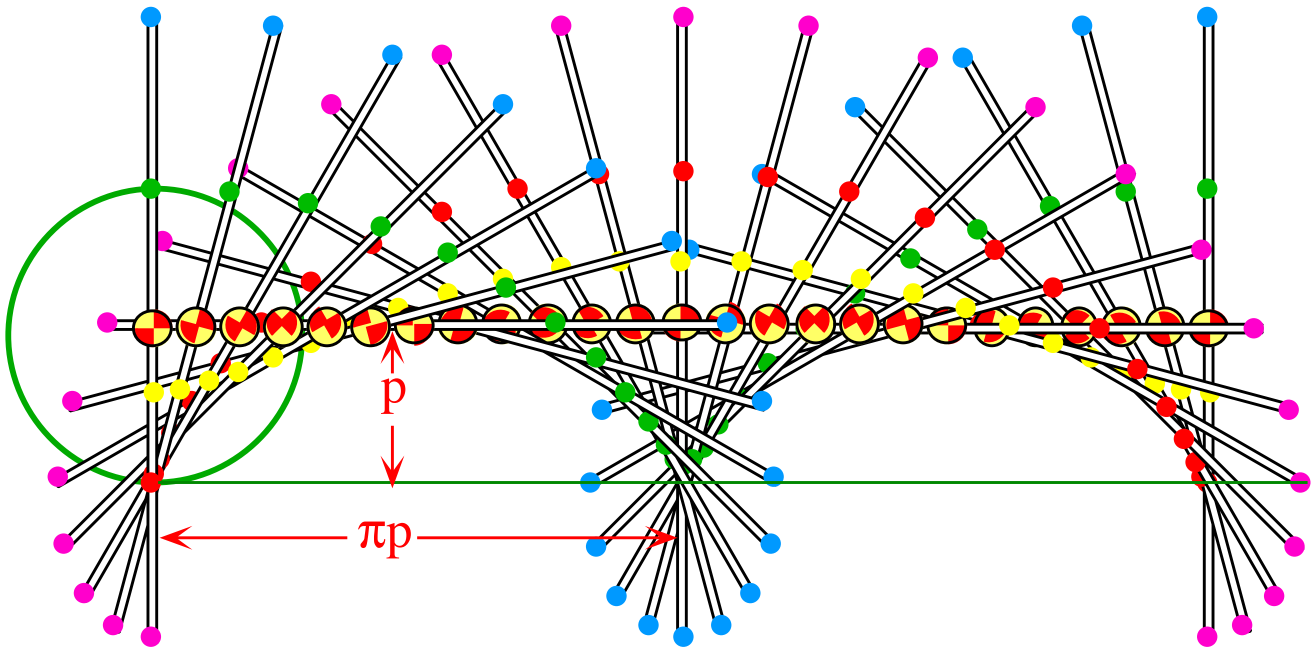

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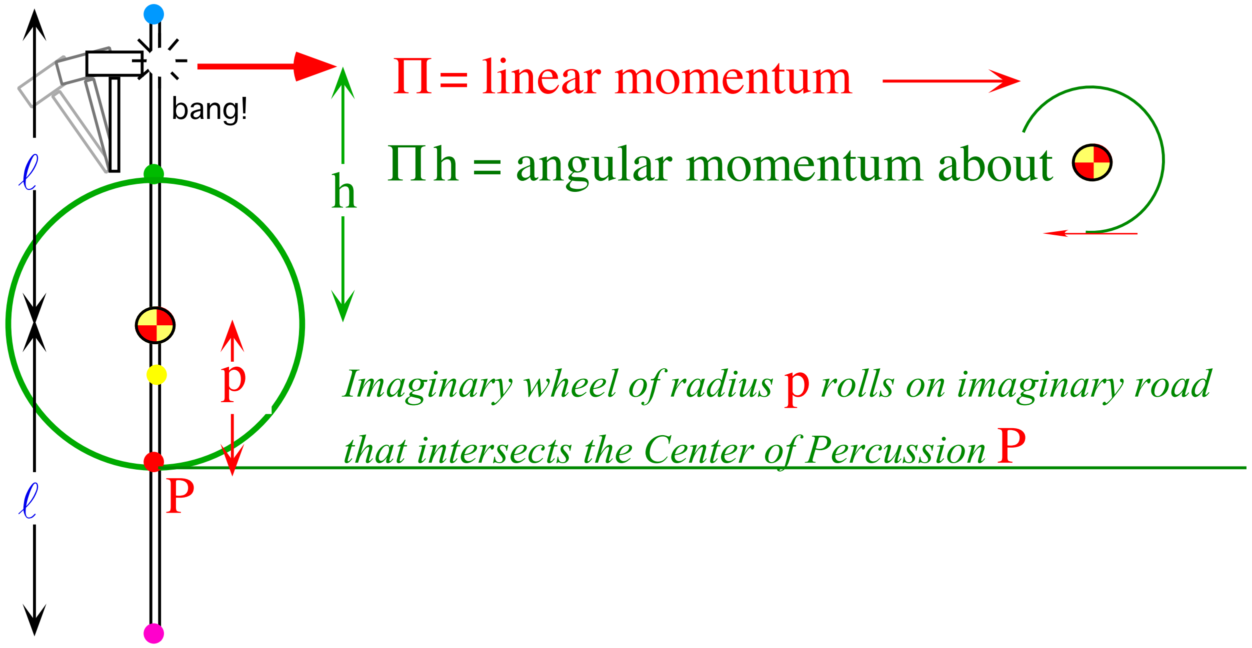

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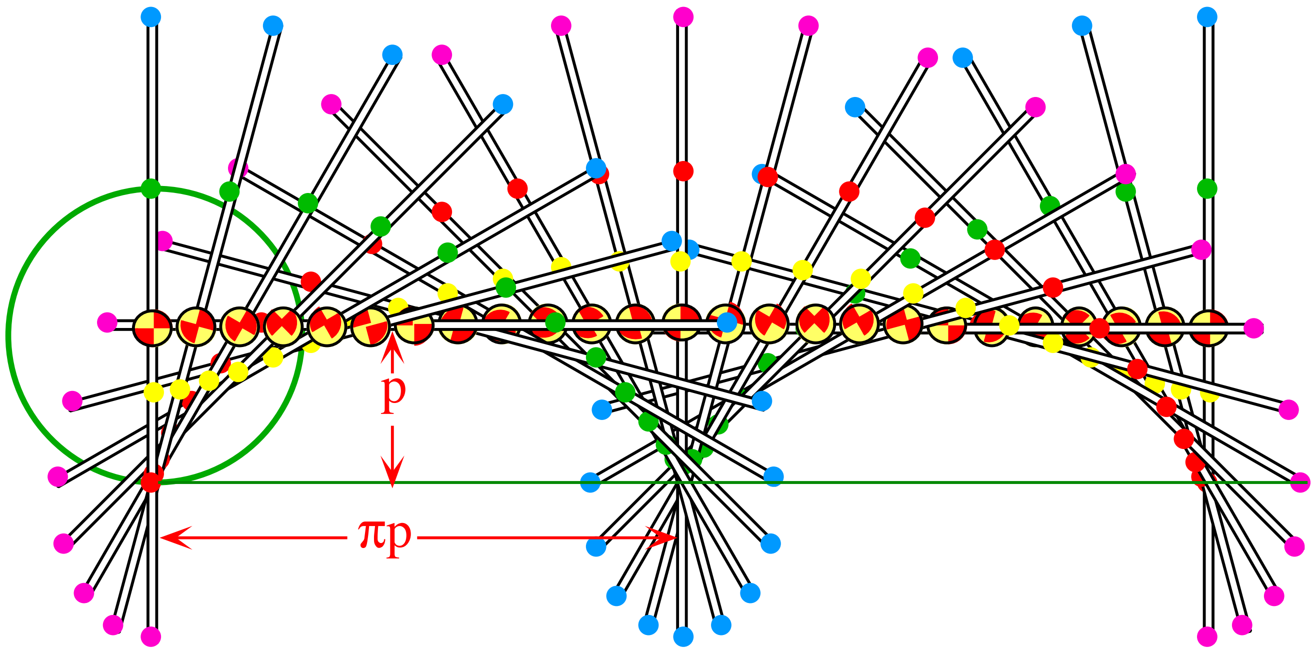

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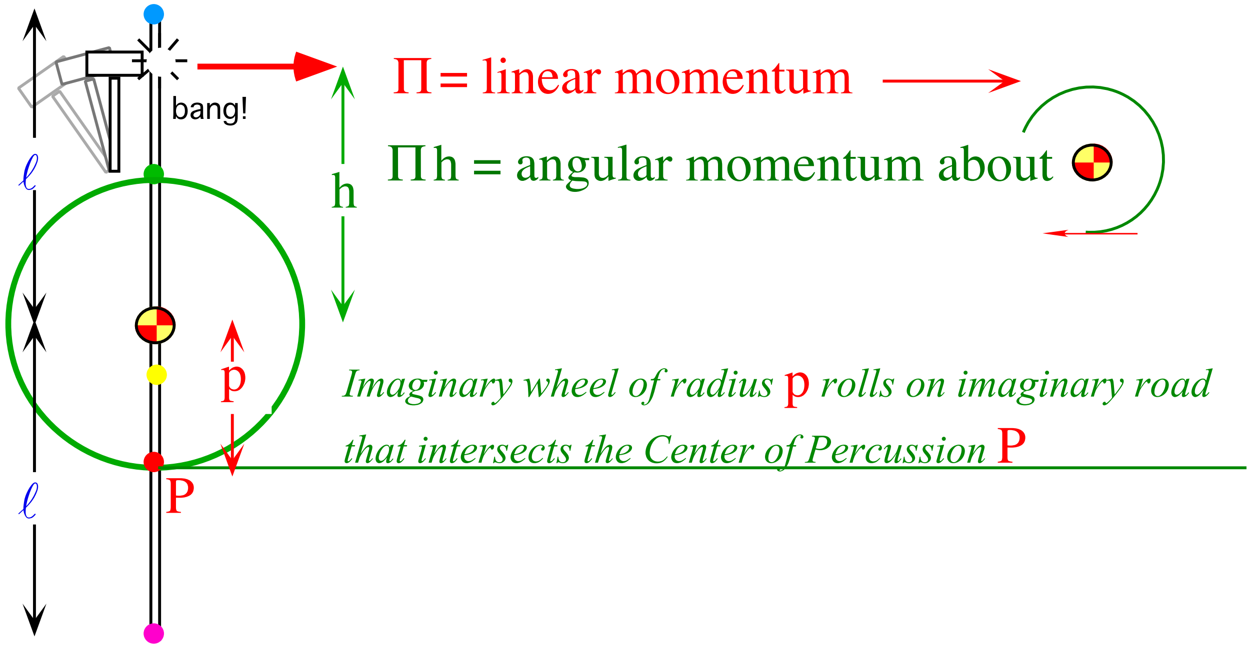

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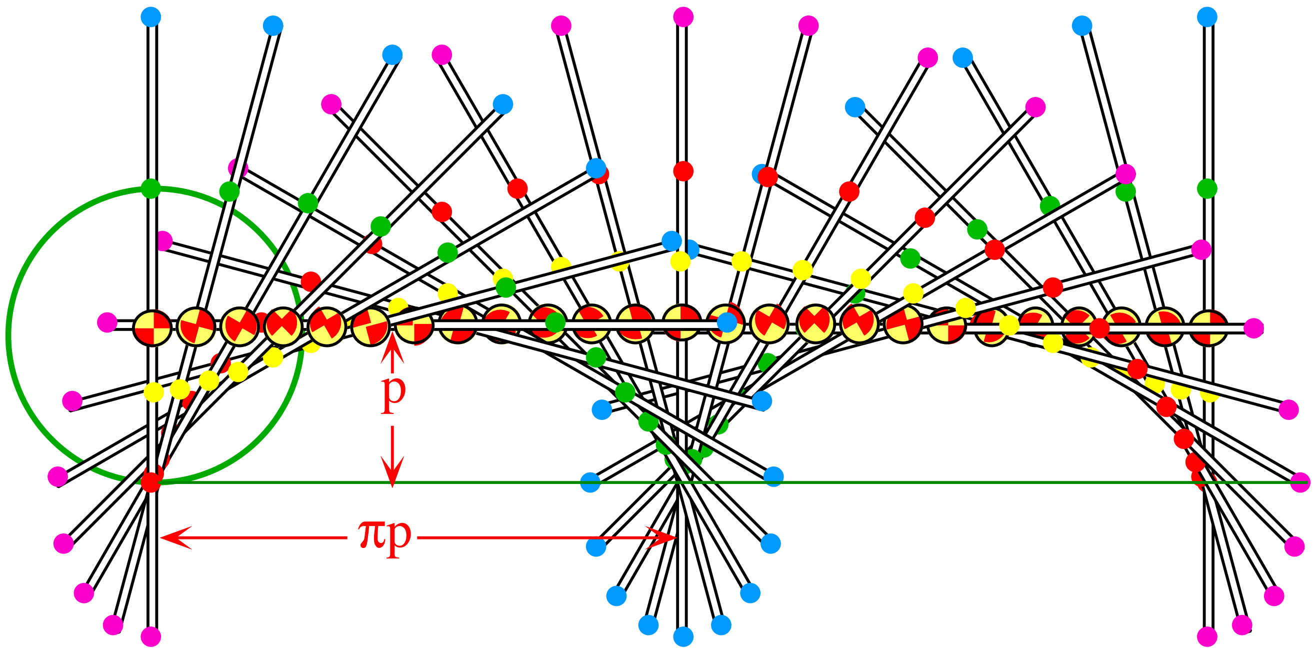

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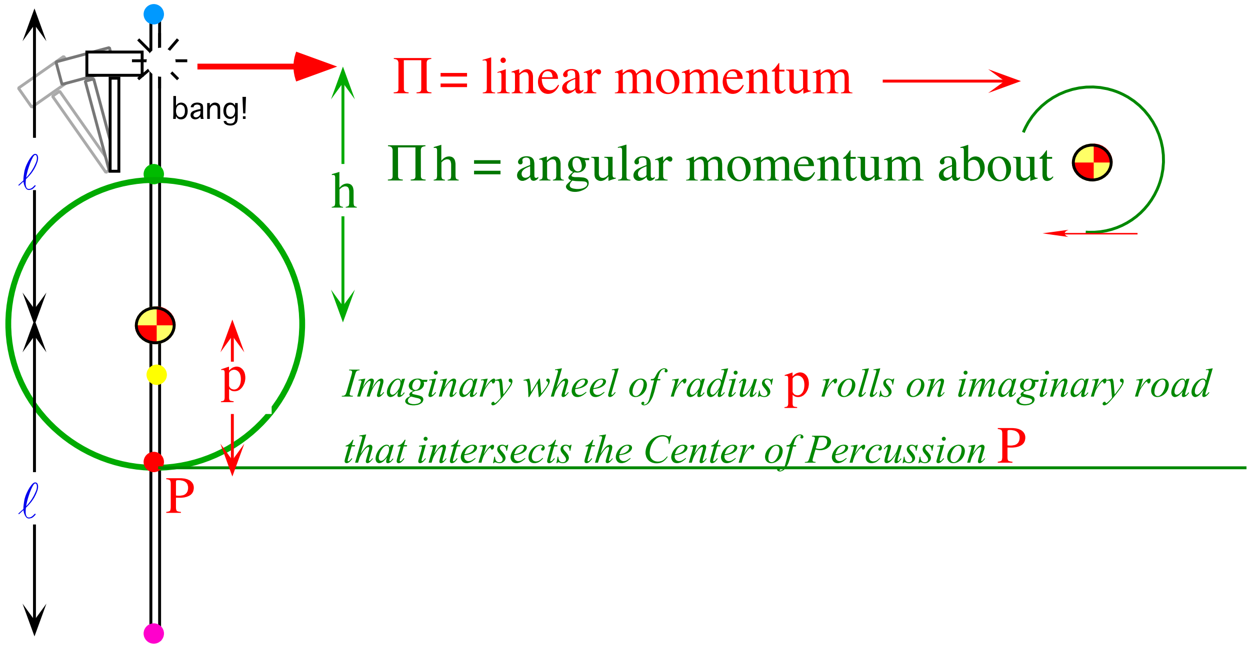

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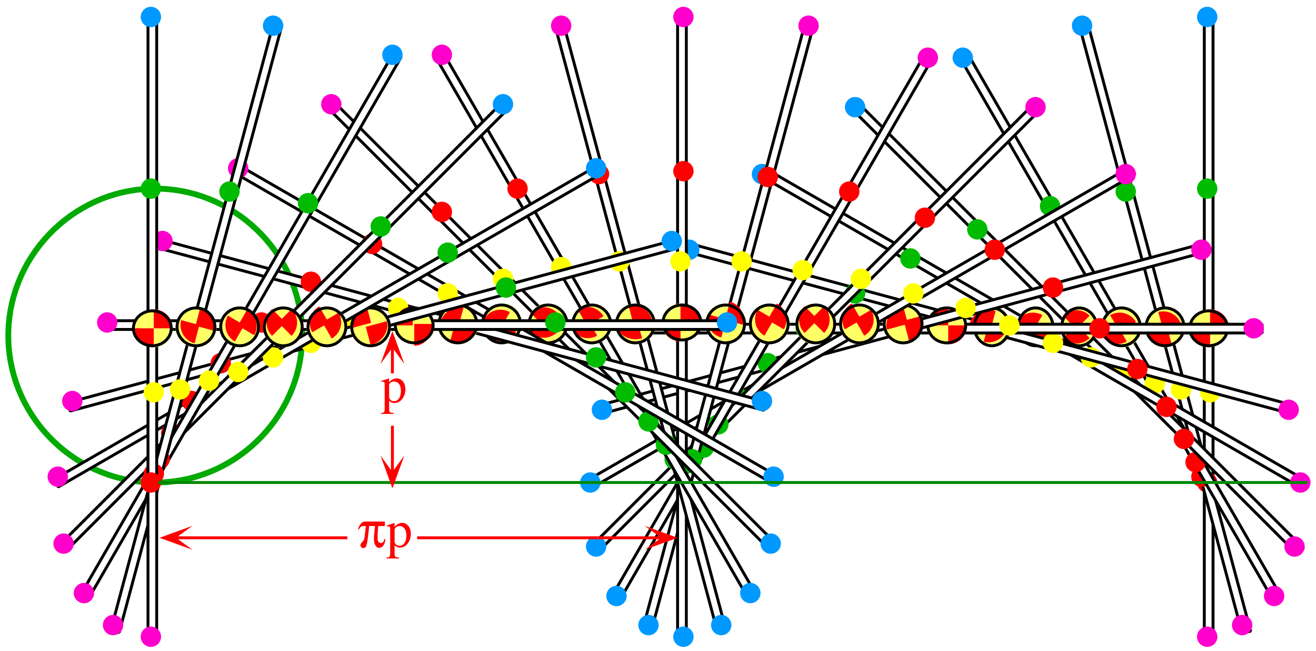

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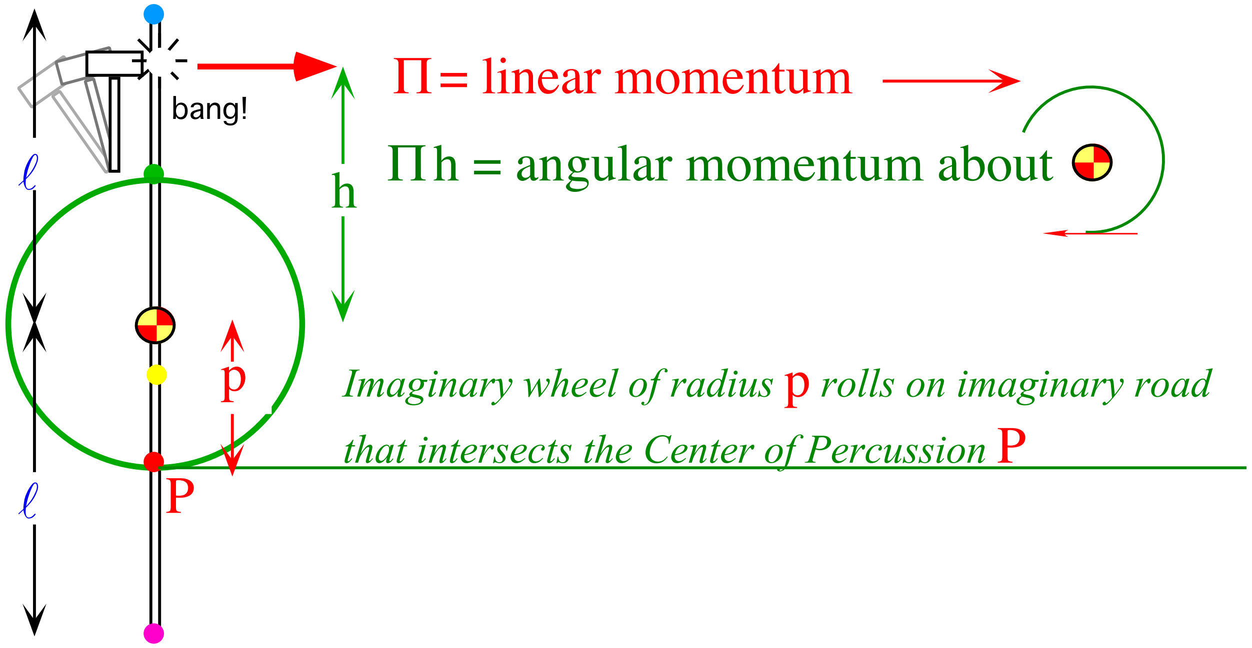

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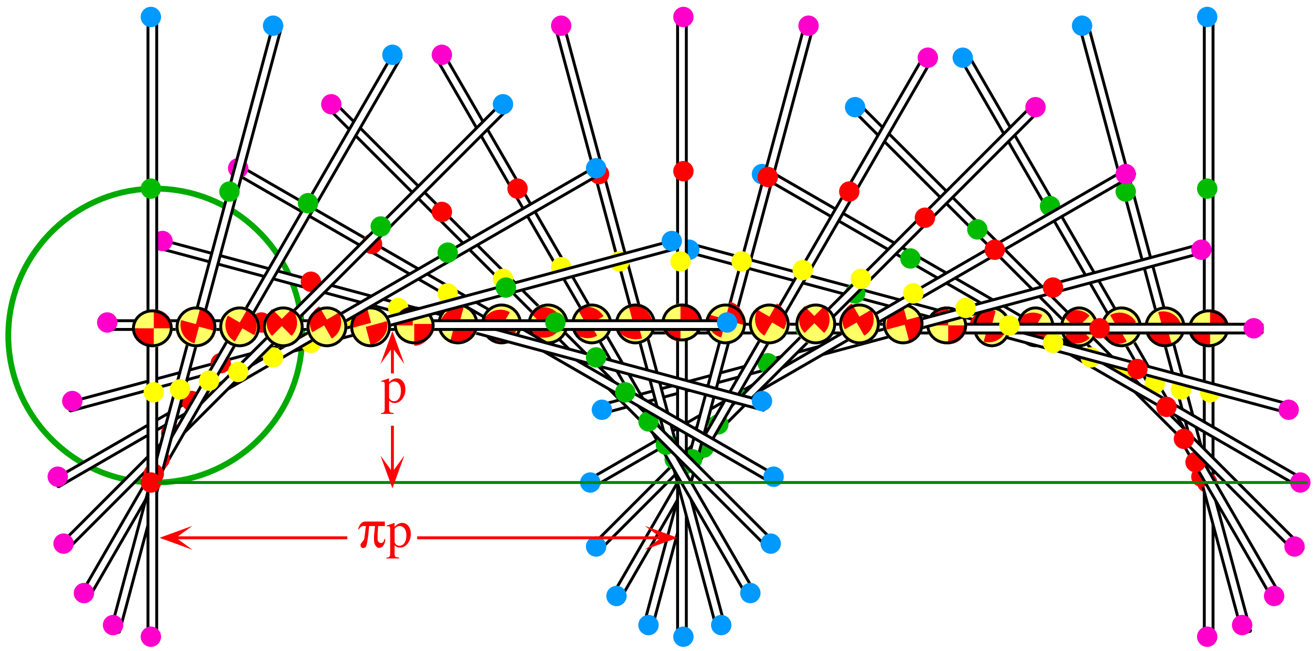

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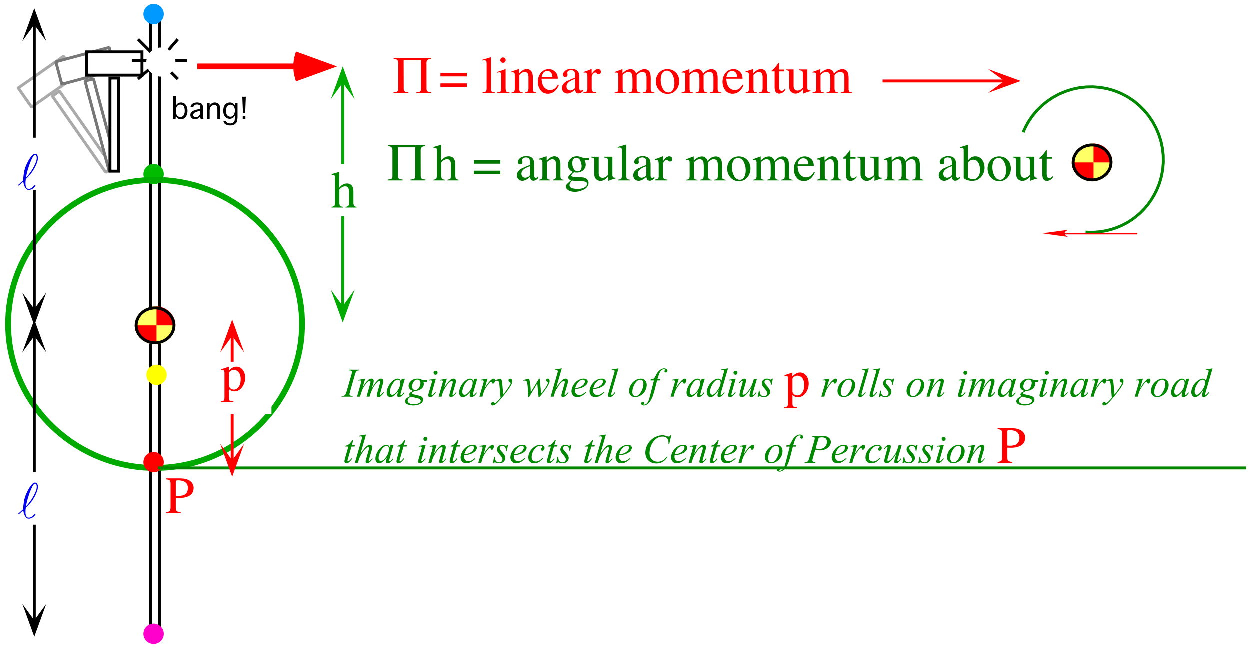

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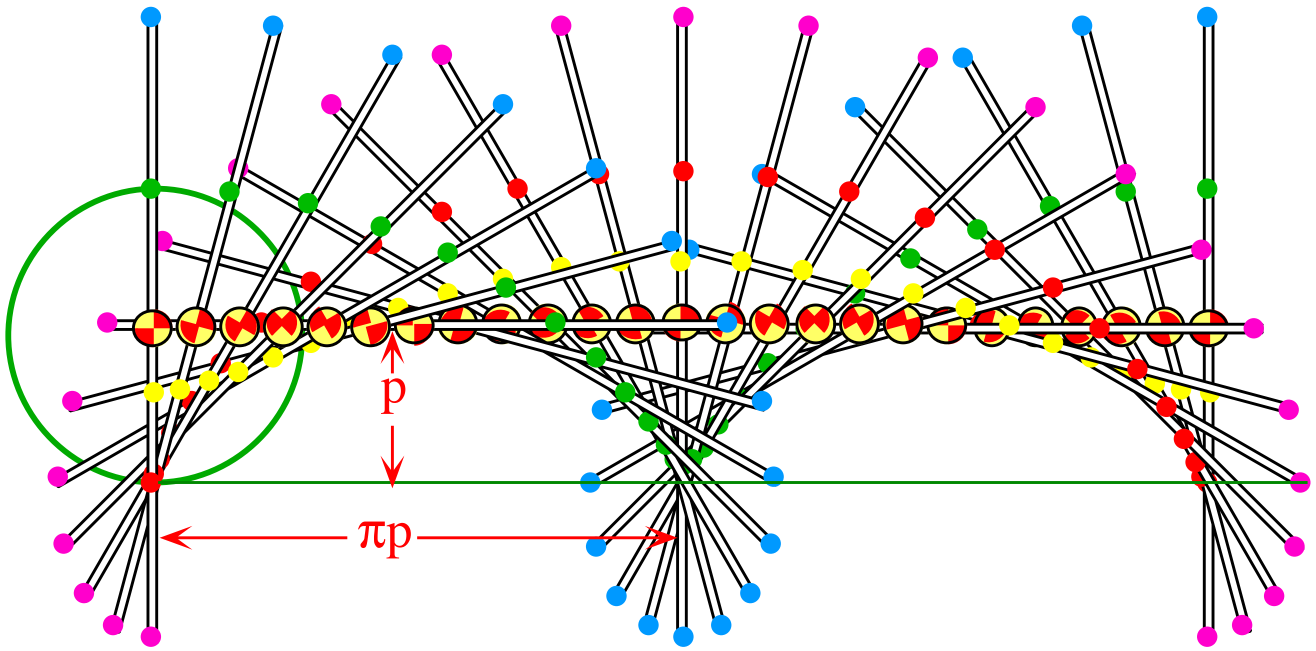

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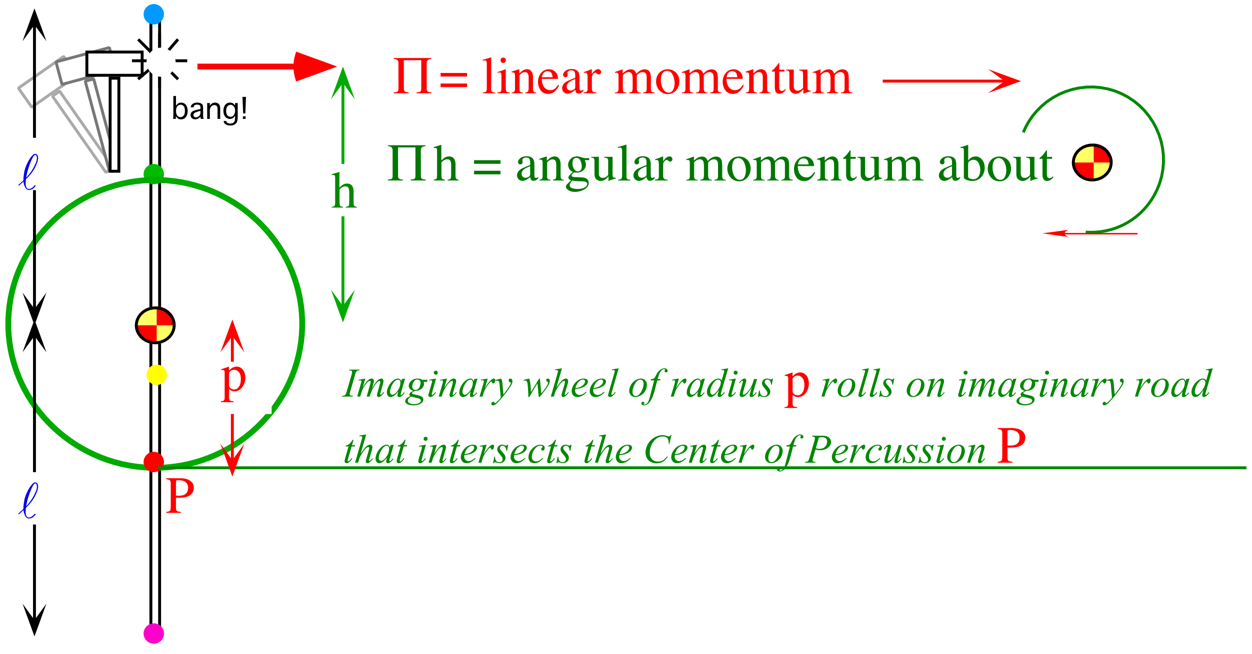

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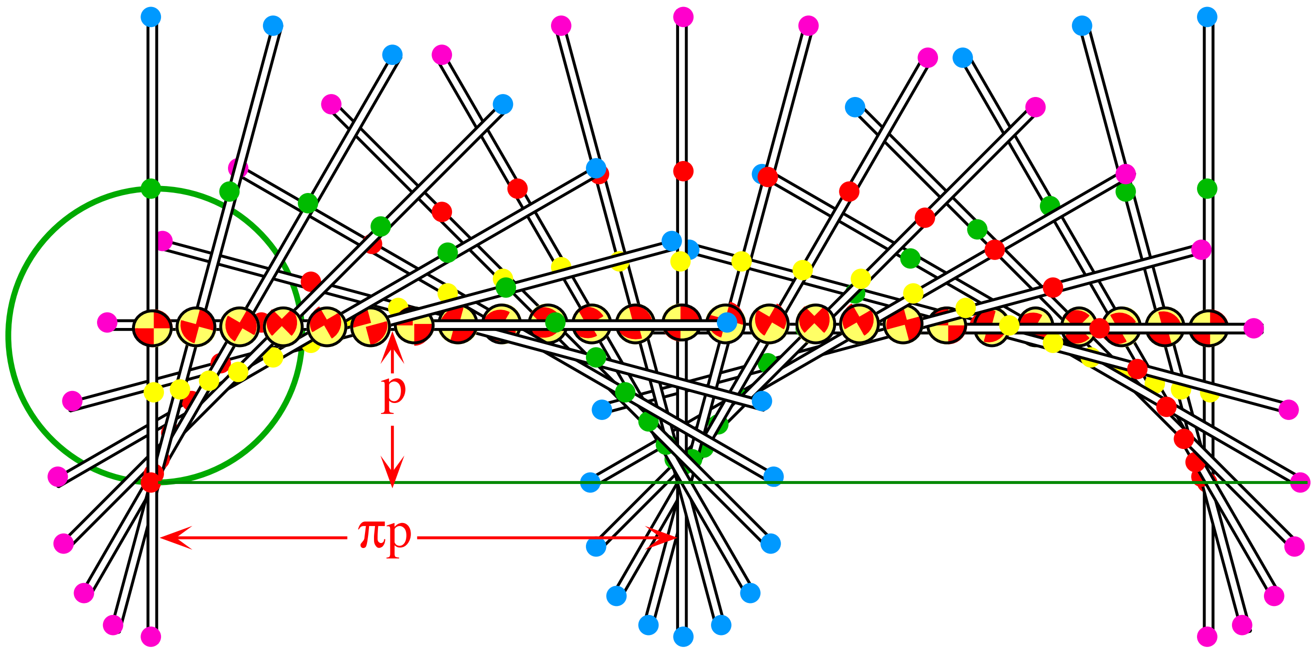

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The *percussion radius* $p = \ell^2/3h$ is of the CoP point
that has no velocity just after hammer hits at h.

Crossed E and B field mechanics

Classical Hall-effect and cyclotron orbits

Vector theory vs. complex variable theory

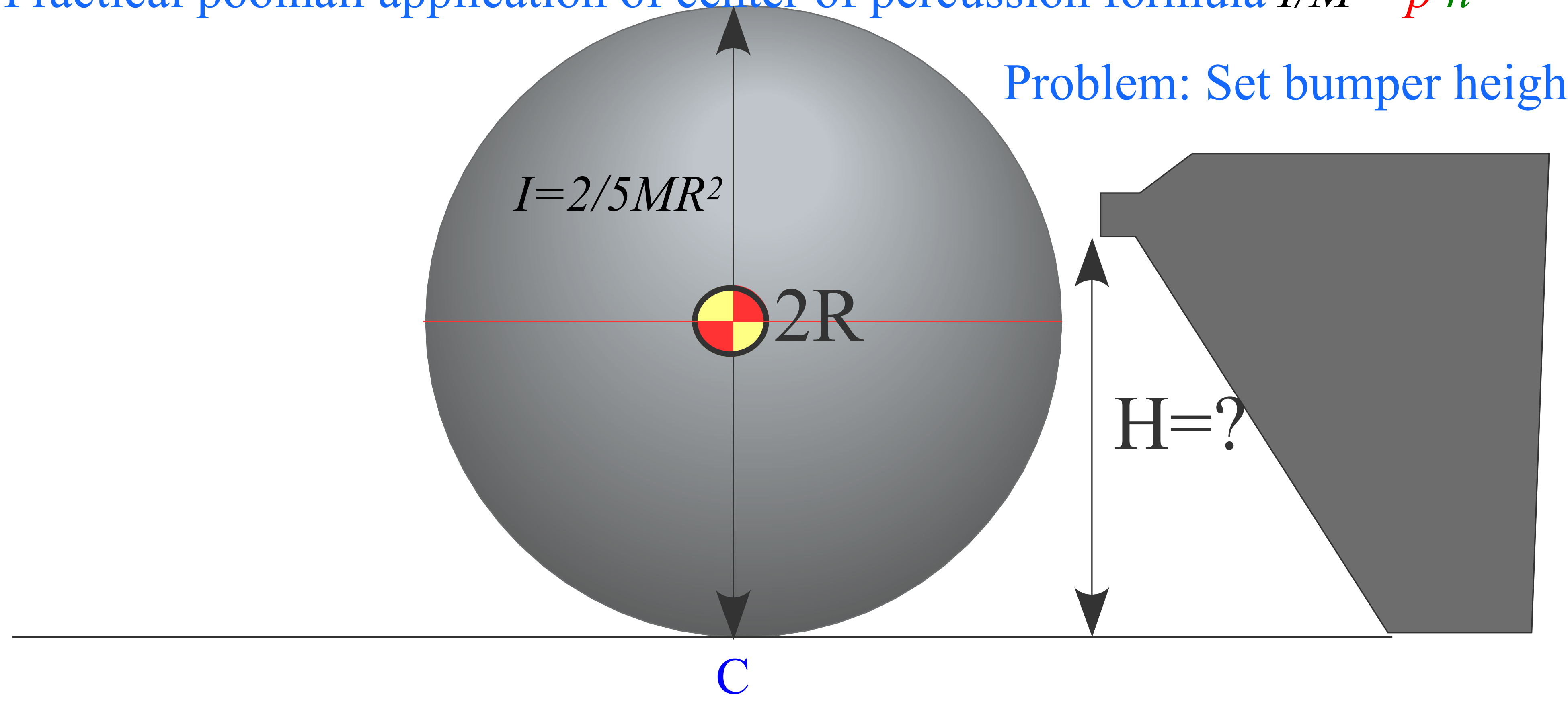
Mechanical analog of cyclotron and FBI rule

Cycloid geometry and flying sticks

→ Practical poolhall application

Practical poolhall application of center of percussion formula $I/M = p \cdot h$

Problem: Set bumper height H so ball does not skid.



Practical poolhall application of center of percussion formula $I/M = p\cdot h$

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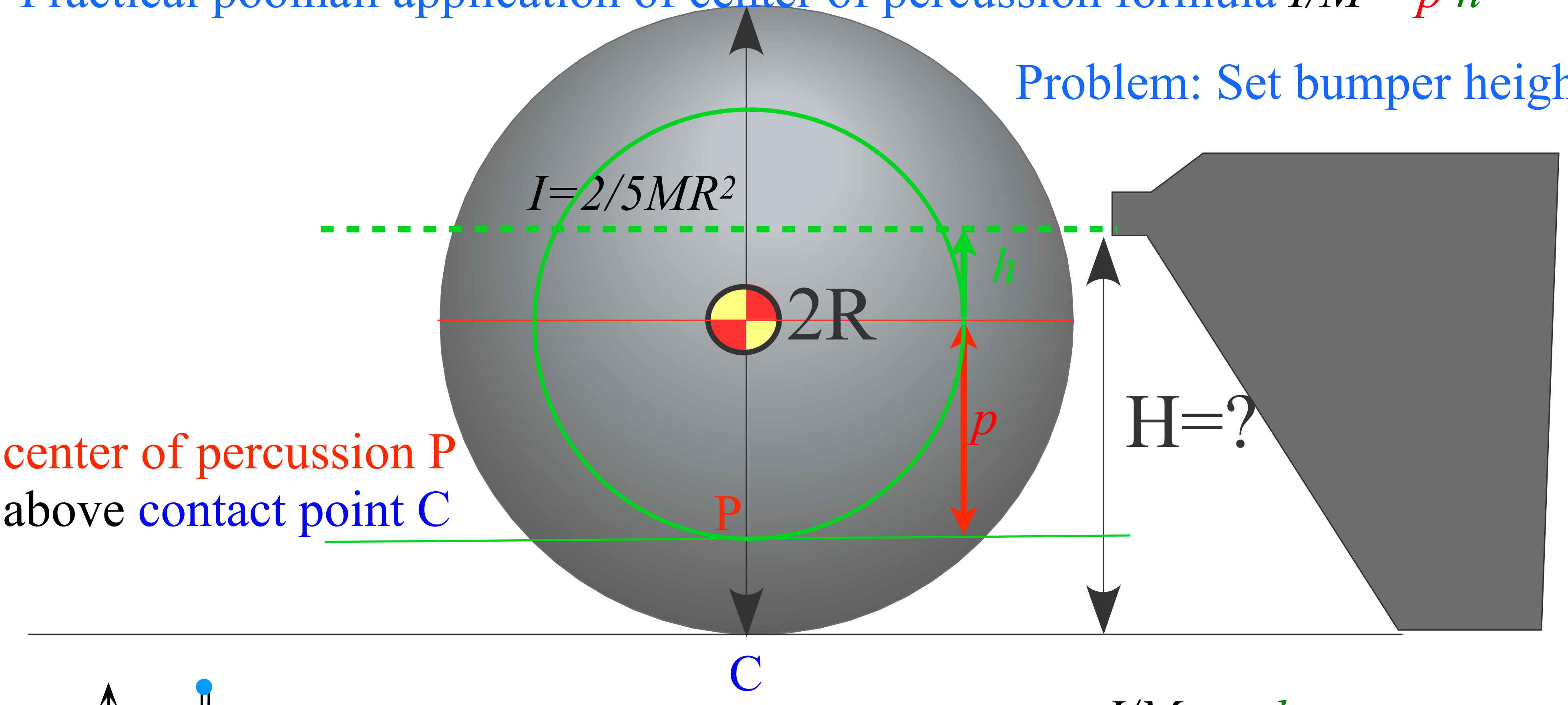
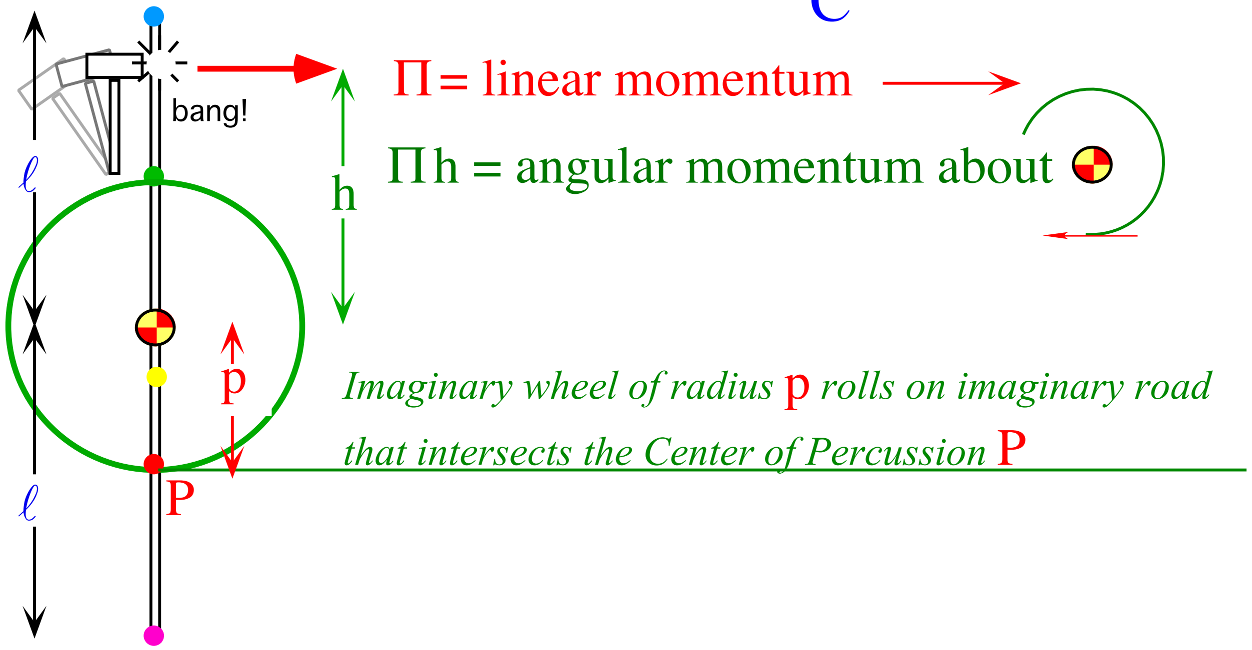
Where should bumper height H be set to make ball contact point C at the center of percussion P?



center of percussion P
above contact point C

$I/M = p\cdot h$



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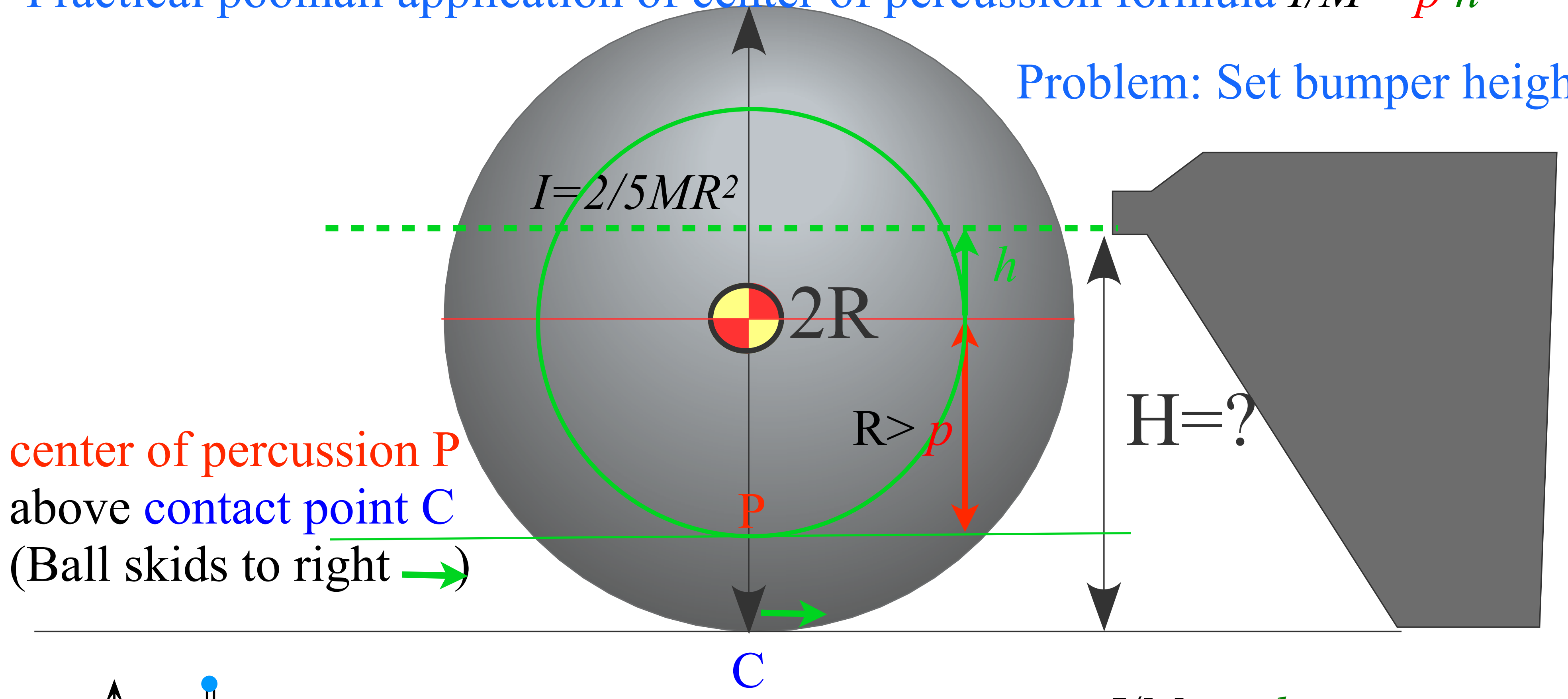
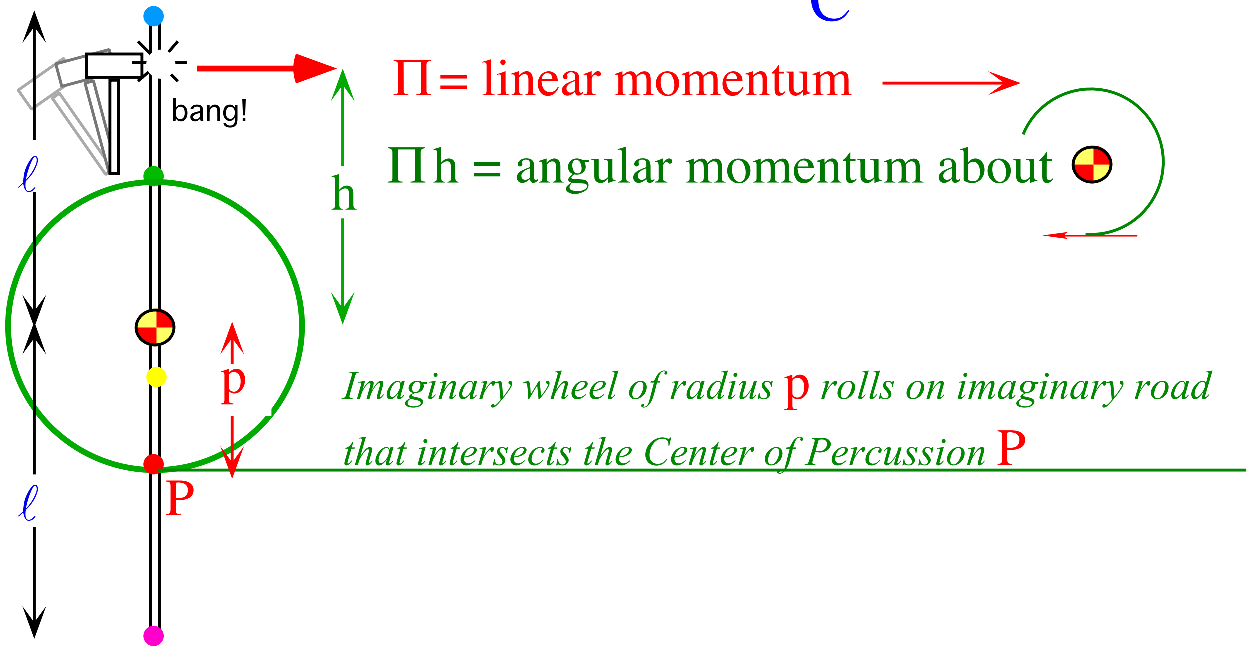


Where should bumper height H be set to make ball contact point C at the center of percussion P?

center of percussion P
above contact point C
(Ball skids to right →)



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Practical poolhall application of center of percussion formula $I/M = p\cdot h$

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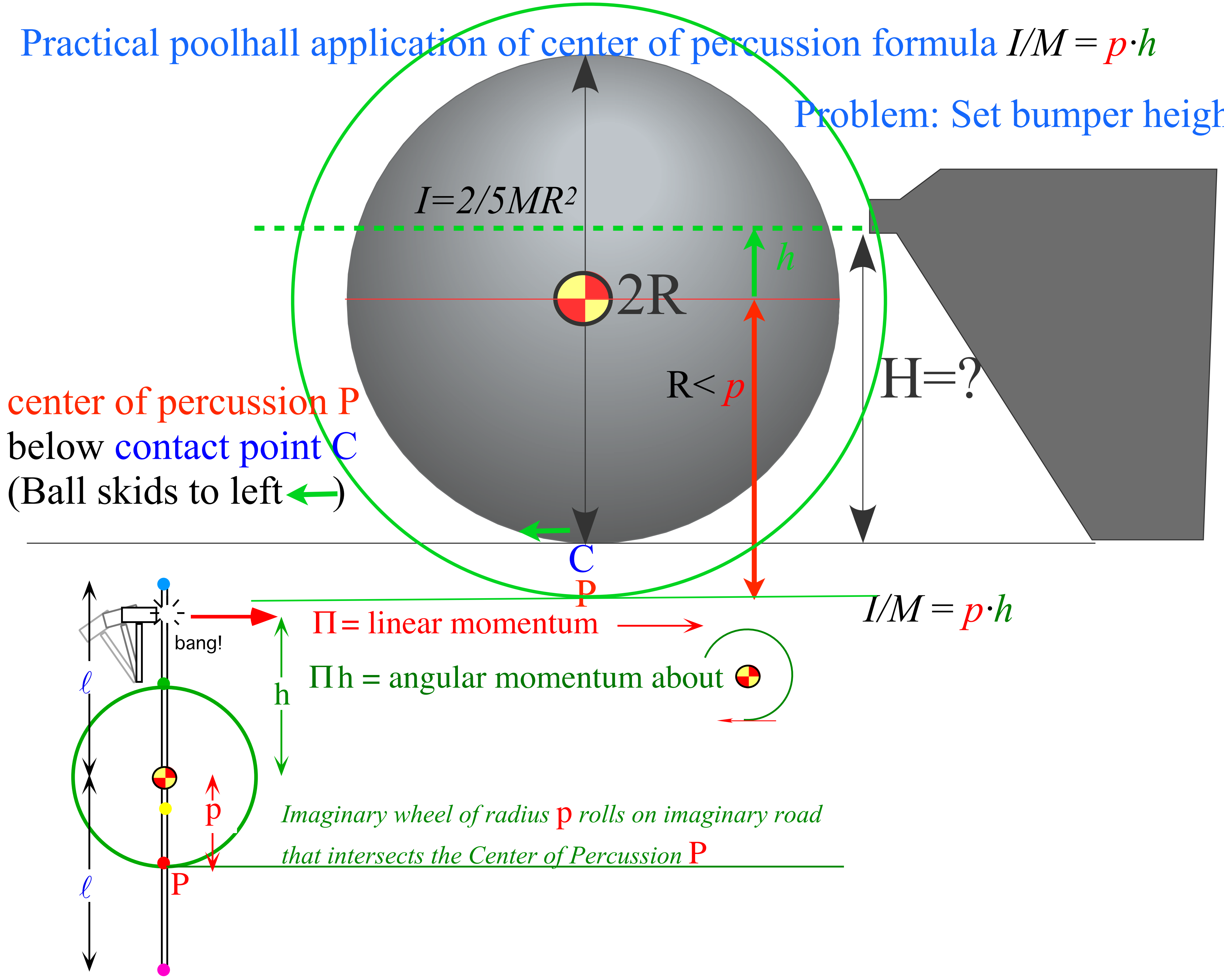

center of percussion P
below contact point C
(Ball skids to left ←)

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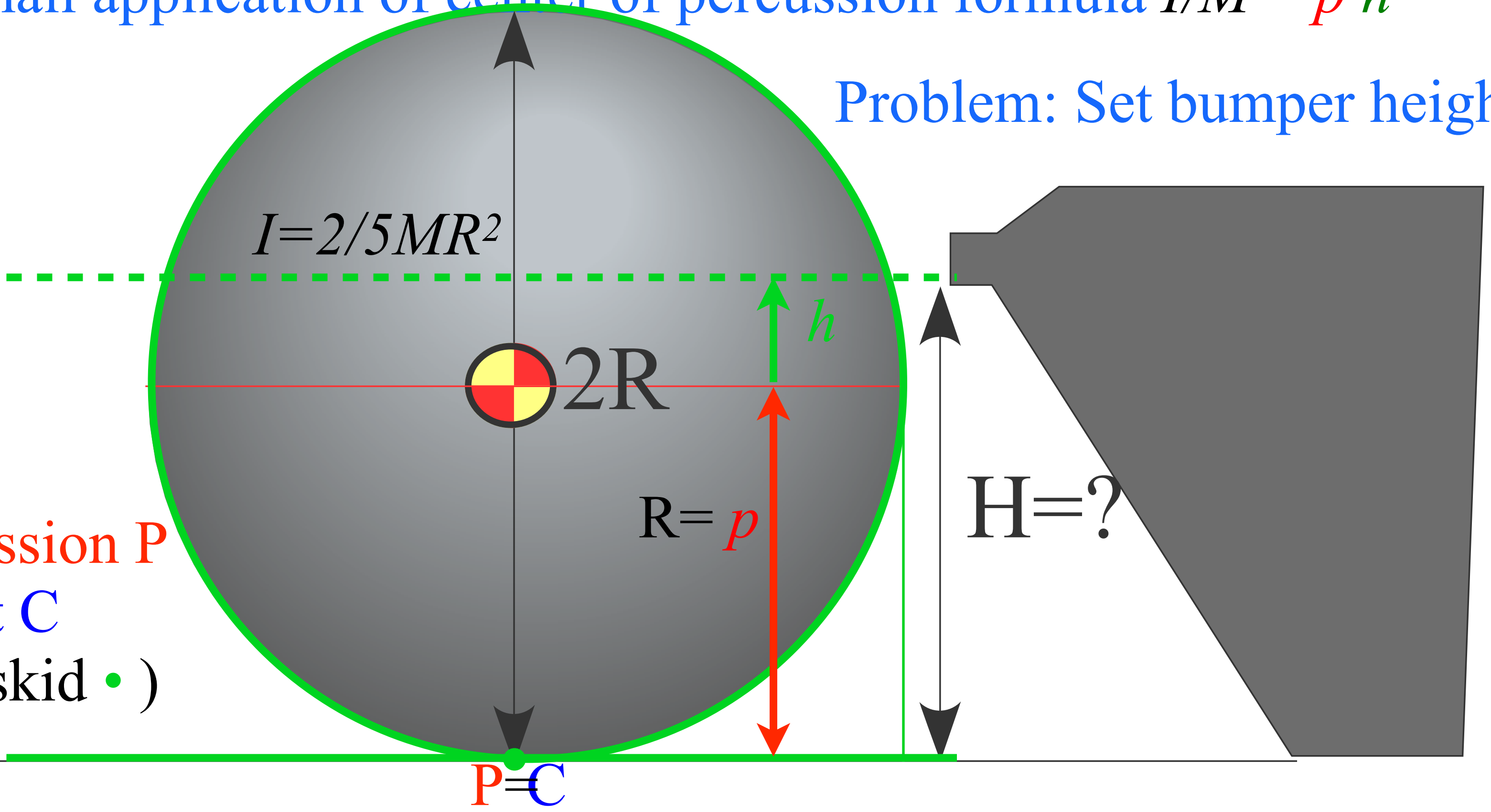
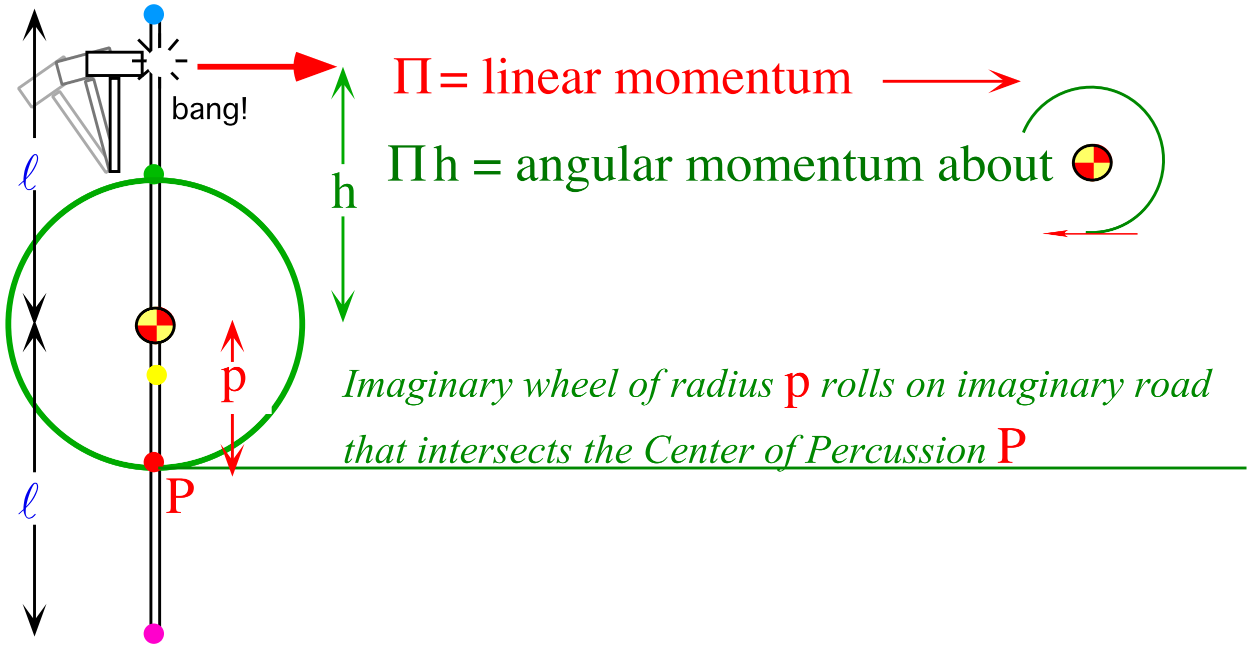


Where should bumper height H be set to make ball contact point C at the center of percussion P?

center of percussion P
at contact point C
(Ball does not skid •)



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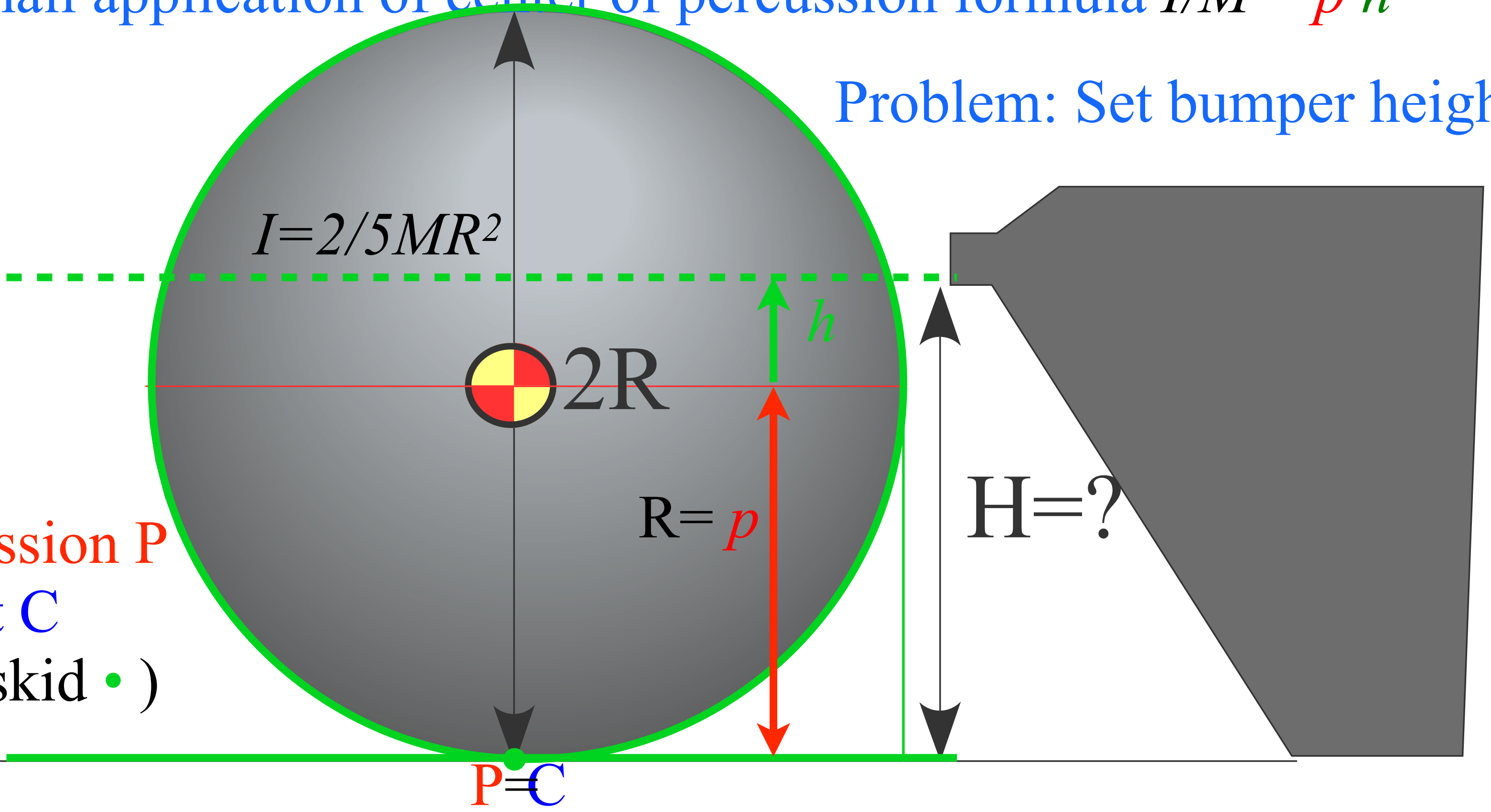

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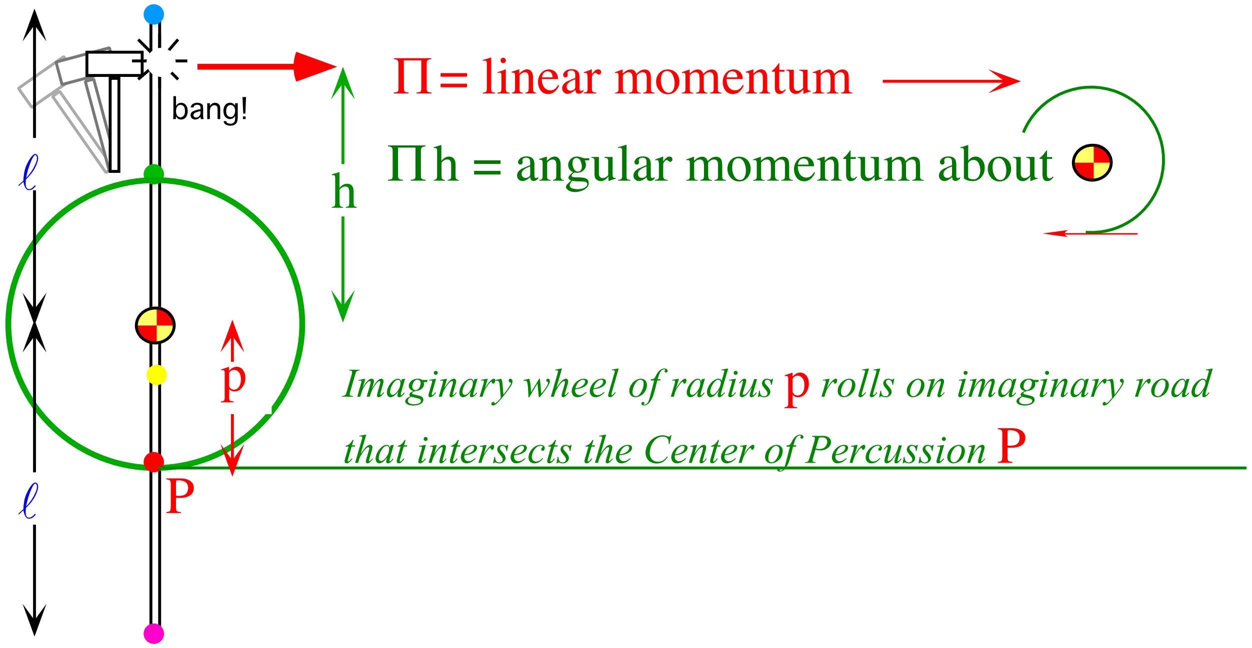

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$h = I/Mp = I/MR$ (For R= p)

$= 2/5MR^2/MR$

$= 2/5R$

For: H= $R+h = 7/10(2R)$ ball does not skid.

Thats all folks!