

Lecture 21

Wed. 10.31.2018

Introduction to coupled oscillation and eigenmodes

(Ch. 2-4 of Unit 4)

2D harmonic oscillator equations

Lagrangian and matrix forms and Reciprocity symmetry

2D harmonic oscillator equation eigensolutions

Geometric method

Matrix-algebraic eigensolutions with example $M = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

Secular equation

Hamilton-Cayley equation and projectors

Idempotent projectors (how eigenvalues \Rightarrow eigenvectors)

Operator orthonormality and Completeness (Idempotent means: $\mathbf{P} \cdot \mathbf{P} = \mathbf{P}$)

Spectral Decompositions

Functional spectral decomposition

Orthonormality vs. Completeness vis-a`-vis Operator vs. State

Lagrange functional interpolation formula

Diagonalizing Transformations (D-Ttran) from projectors

2D-HO eigensolution example with bilateral (B-Type) symmetry

Mixed mode beat dynamics and fixed $\pi/2$ phase

2D-HO eigensolution example with asymmetric (A-Type) symmetry

Initial state projection, mixed mode beat dynamics with variable phase



A running collection of links to course-relevant sites and articles

Physics Web Resources

[Comprehensive Harter-Soft Resource Listing](#)

[UAF Physics YouTube channel](#)

[LearnIt Physics Web Applications](#)

Neat external material to start the class:

[AIP publications](#)

[AJP article on superball dynamics](#)

[AAPT summer reading](#)

These are hot off the presses:

[Sorting ultracold atoms in a three-dimensional optical lattice in a realization of Maxwell's demon - Kumar-Nature-Letters-2018](#)

[Synthetic three-dimensional atomic structures assembled atom by atom - Berredo-Nature-Letters-2018](#)

Slightly Older ones:

[Wave-particle duality of C60 molecules](#)

[Optical vortex knots – One Photon at a Time](#)

“Relawavity” and quantum basis of *Lagrangian & Hamiltonian* mechanics:

[2-CW laser wave - BohrIt Web App](#)

[Lagrangian vs Hamiltonian - RelaWavity Web App](#)

[AMOP Ch 0 Space-Time Symmetry - 2019](#)

[Seminar at Rochester Institute of Optics, Auxiliary slides, June 19, 2018](#)

Older Links from Lectures 14-20

<http://thearmchaircritic.blogspot.com/2011/11/punkin-chunkin.html>

<http://www.sussexcountyonline.com/news/photos/punkinchunkin.html>

<http://www.twcenter.net/forums/showthread.php?358315-Shooting-range-for-medieval-siege-weapons-Anybody-knows>

<https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html>

<https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html?scenario=MontezumasRevenge>

<https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html?scenario=SeigeOfKenilworth>

https://modphys.hosted.uark.edu/pdfs/Journal_Pdfs/Trebuchet-SciAm_273_66_July_1995_chevedden1.pdf

‘Simple’ Pendulum Sim: <https://modphys.hosted.uark.edu/markup/PendulumWeb.html>

‘Cycloid’ Pendulum: <https://modphys.hosted.uark.edu/markup/CycloidulumWeb.html>

Google search on: “Satelite view of Patricia” (Images)

Physics Girl Channel - Fun with Vortex Rings in the Pool: <https://www.youtube.com/watch?v=72lWr7BU8Ao>

iBall demo - Quasi-periodicity: https://youtu.be/_jntDtULxDc

<https://modphys.hosted.uark.edu/markup/CoulltWeb.html?scenario=SynchrotronMotion>

<https://modphys.hosted.uark.edu/markup/CoulltWeb.html?scenario=SynchrotronMotion2>

Mechanical Analog to EM Motion (YouTube video) - <https://youtu.be/hTd5FTJ-vRk>

Coullt Web Simulation: [Bound-state motion in parabolic coordinates](#)

Coullt Web Simulation: [Bound-state motion in hyperbolic coordinates](#)

OscillIt Web App: Simulations of various types of resonance: [18](#), [27](#), [31](#), [35](#), [38](#), [39](#)

[Smith Chart](#)

<http://nobelprize.org/>

“Texts”

[Classical Mechanics with a Bang!](#)

[Quantum Theory for the Computer Age](#)

[Principles of Symmetry, Dynamics, and Spectroscopy](#)

[Modern Physics and its Classical Foundations](#)

Classes

[2014 AMOP](#)

[2017 Group Theory for QM](#)

[2018 AMOP](#)

[2018 Adv Mechanics](#)

AnalyIt Web Application, posted 10/22/2018 in our *testing area*:

<https://modphys.hosted.uark.edu/testing/markup/AnalyItBJS.html>

Links to supplement Lecture 21

BoxIt Web App:

[A-Type w/Cosine](#)

[AB-Type w/Cosine](#)

[A-Type w/Freq ratios](#)

[AB-Type 2:1 Freq ratio](#)

[Wiki on Pafnuty Chebyshev](#)

➔ *2D harmonic oscillator equations*
Lagrangian and matrix forms and Reciprocity symmetry



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2D harmonic oscillators

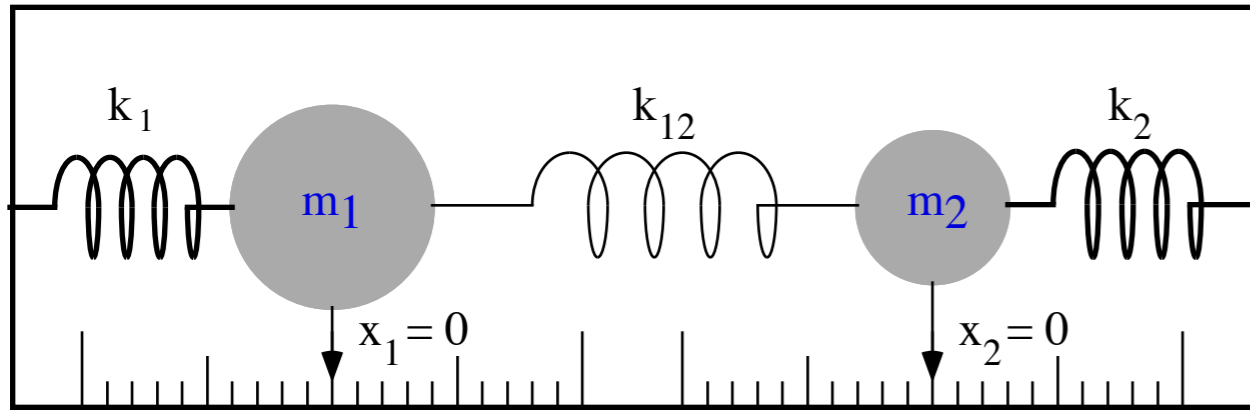


Fig. 3.3.1 Two 1-dimensional coupled oscillators

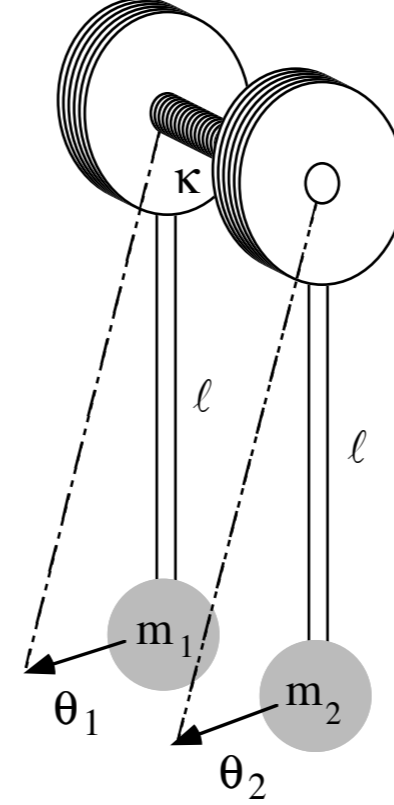
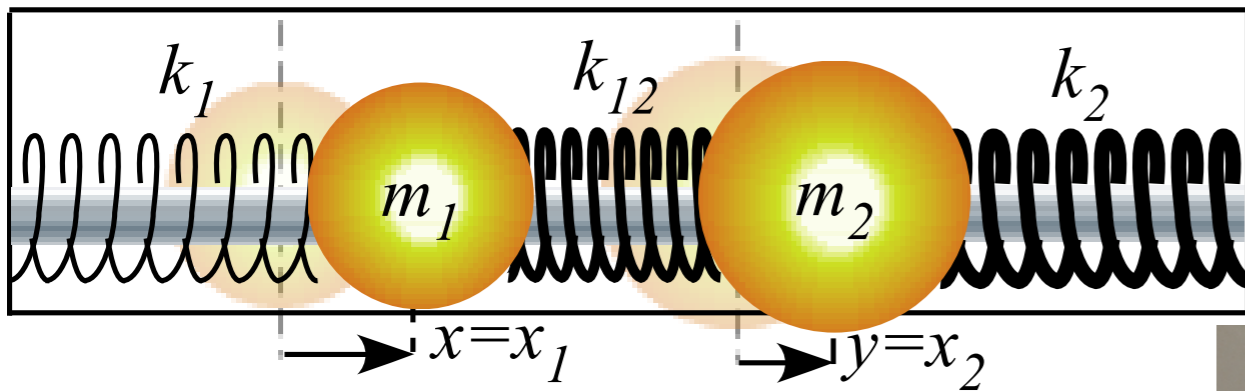


Fig. 3.3.2 Coupled pendulums

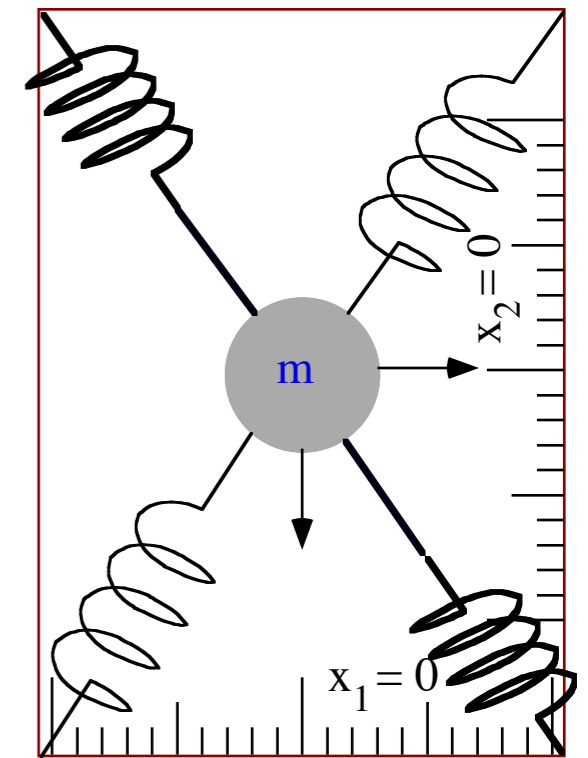


Fig. 3.3.3 One 2-dimensional coupled oscillator



2D harmonic oscillator energy

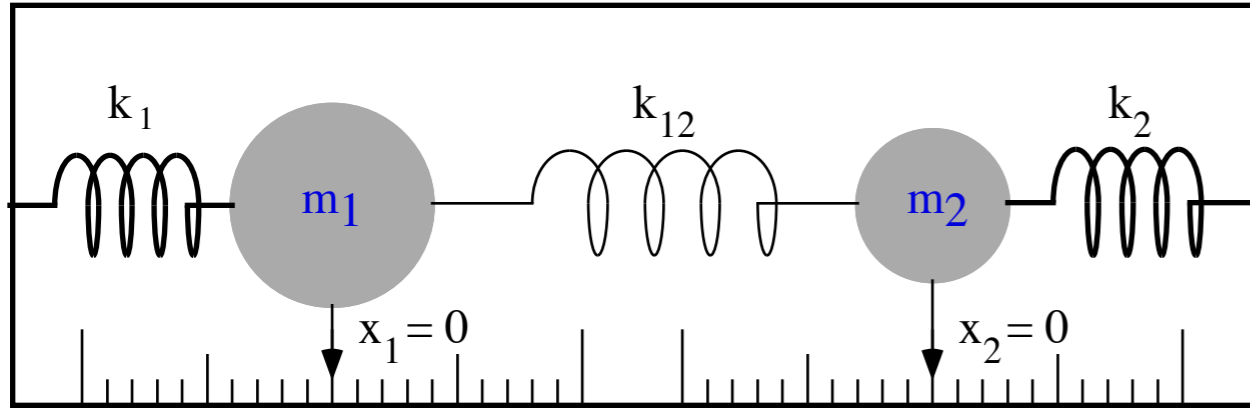
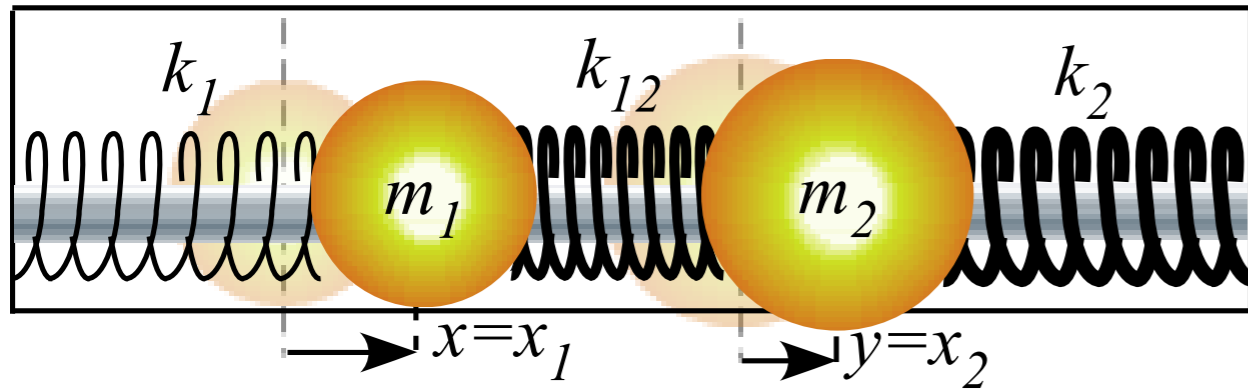


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2D HO kinetic energy $T(v_1, v_2)$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

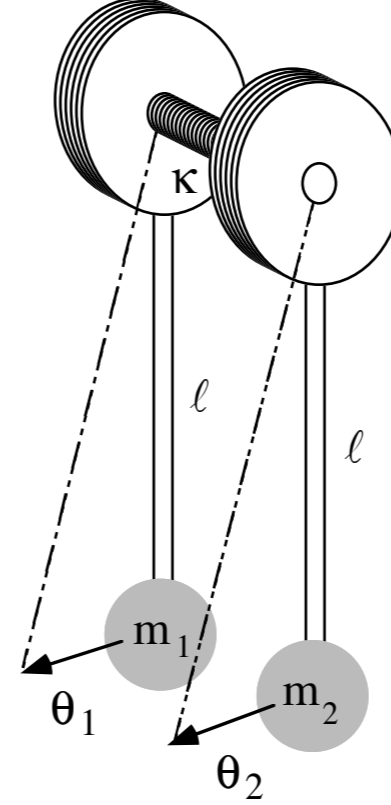


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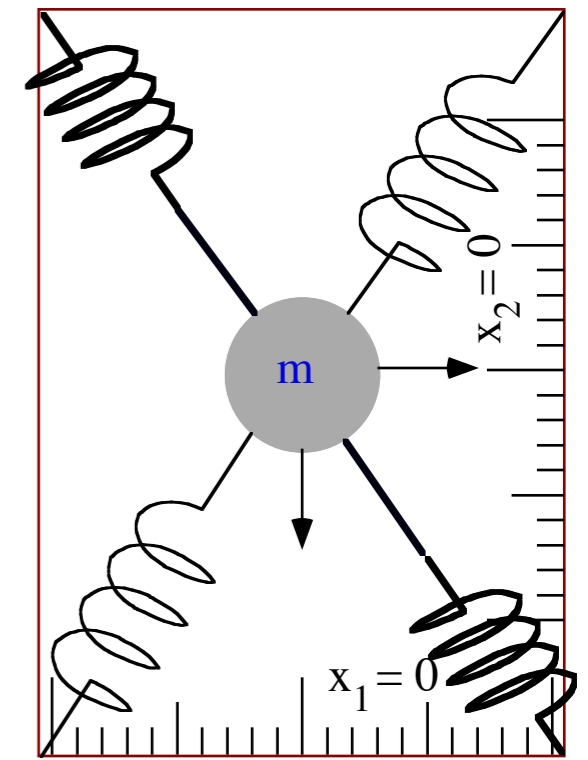


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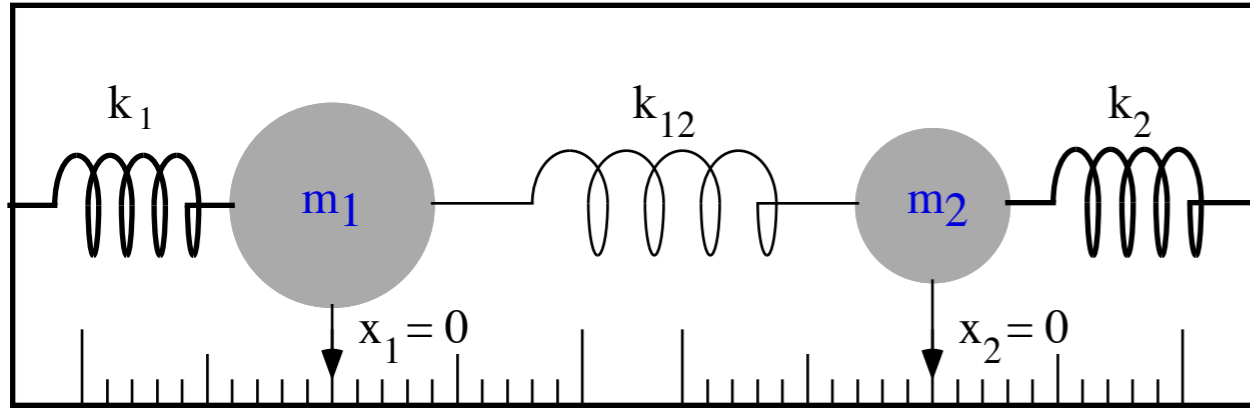


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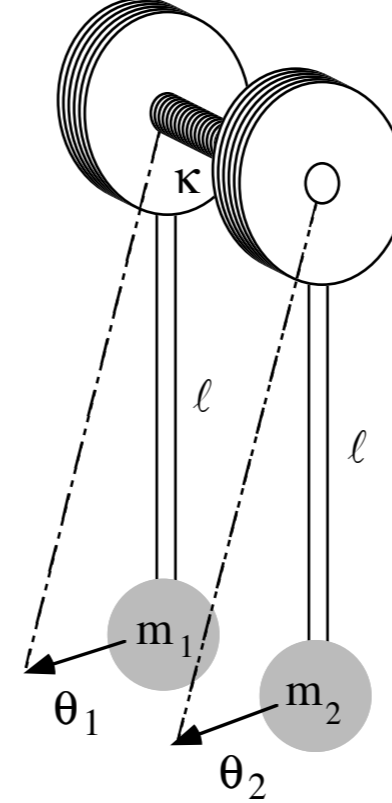
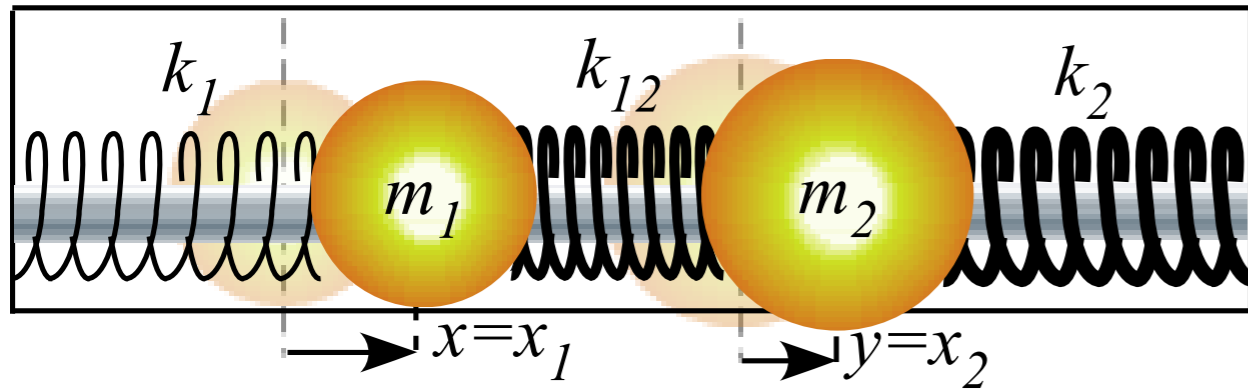


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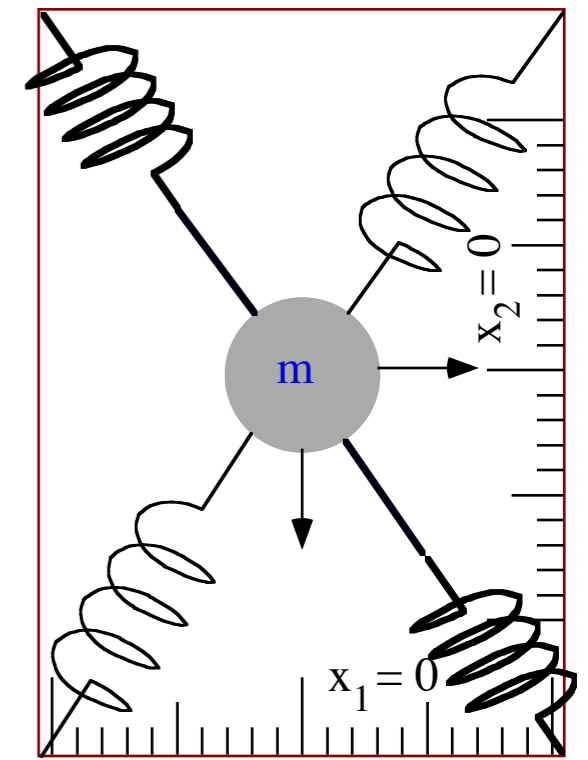


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2D HO potential energy $V(x_1, x_2)$

$$\begin{aligned} V &= \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2 + \frac{1}{2} k_{12} (x_1 - x_2)^2 \\ &= \frac{1}{2} (k_1 + k_{12}) x_1^2 - k_{12} x_1 x_2 + \frac{1}{2} (k_2 + k_{12}) x_2^2 \end{aligned}$$

Lagrangian $L=T-V$

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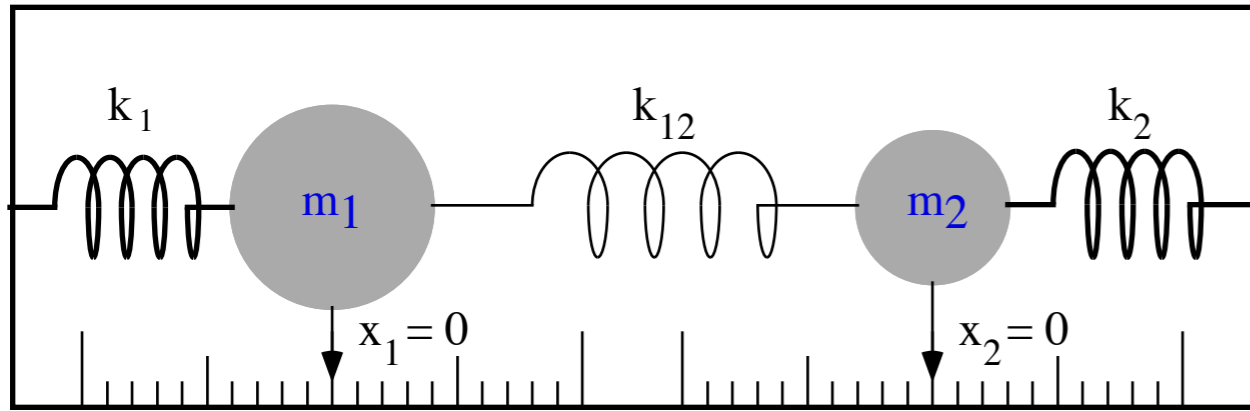


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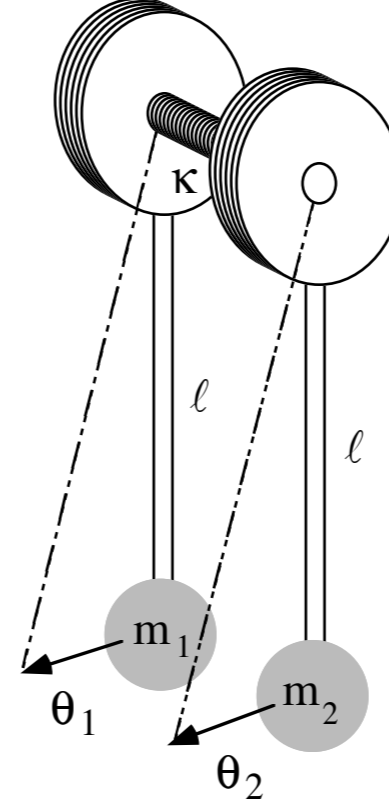
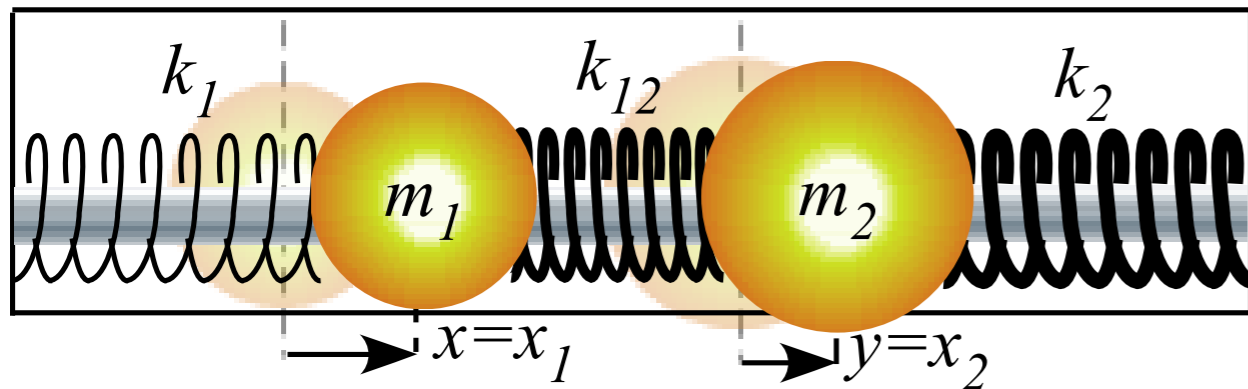


Fig. 3.3.2 Coupled pendulums

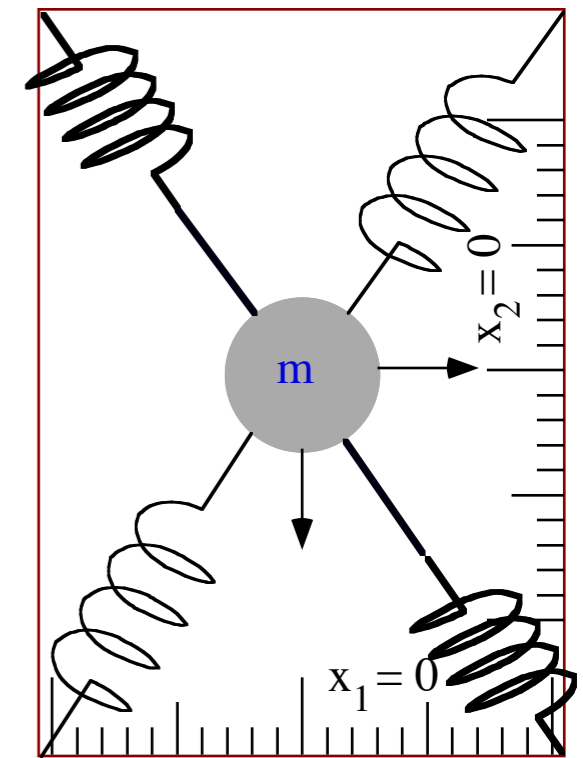


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$$\begin{aligned} V &= \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2 + \frac{1}{2} k_{12} (x_1 - x_2)^2 \\ &= \frac{1}{2} (k_1 + k_{12}) x_1^2 - k_{12} x_1 x_2 + \frac{1}{2} (k_2 + k_{12}) x_2^2 \end{aligned}$$

Lagrangian $L=T-V$

2D HO Lagrange equations

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_1} \right) = m_1 \ddot{x}_1 = F_1 = - \frac{\partial V}{\partial x_1} = - (k_1 + k_{12}) x_1 + k_{12} x_2$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_2} \right) = m_2 \ddot{x}_2 = F_2 = - \frac{\partial V}{\partial x_2} = k_{12} x_1 - (k_2 + k_{12}) x_2$$

2D harmonic oscillator equations

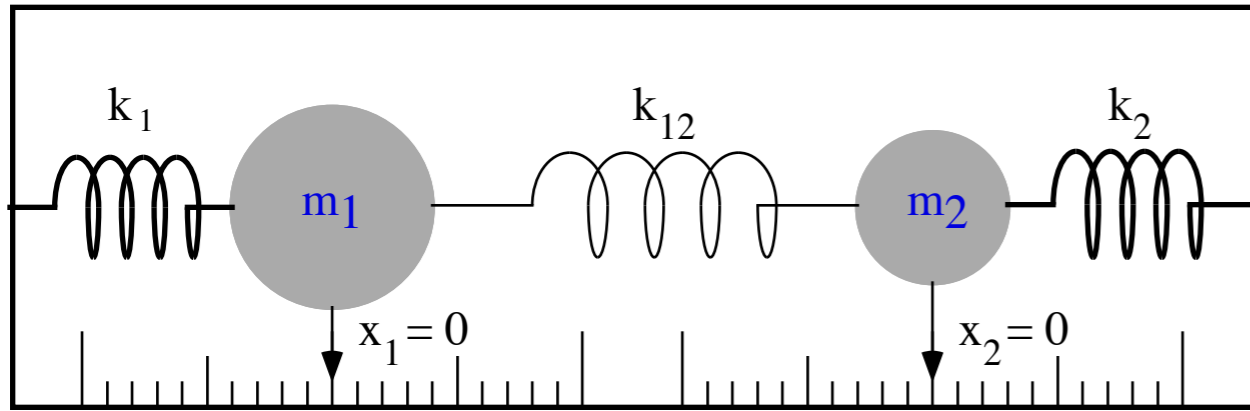


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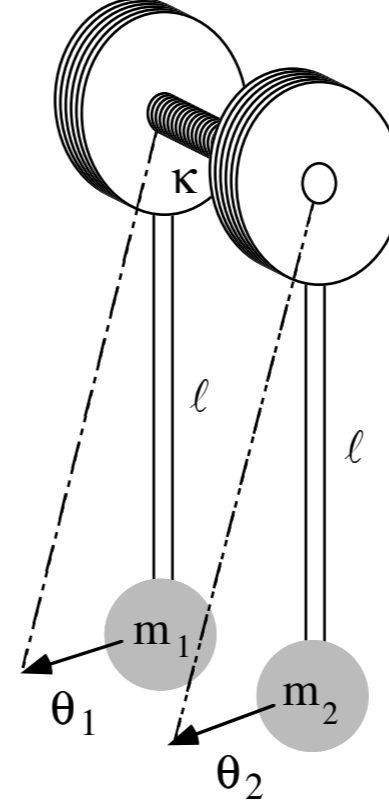
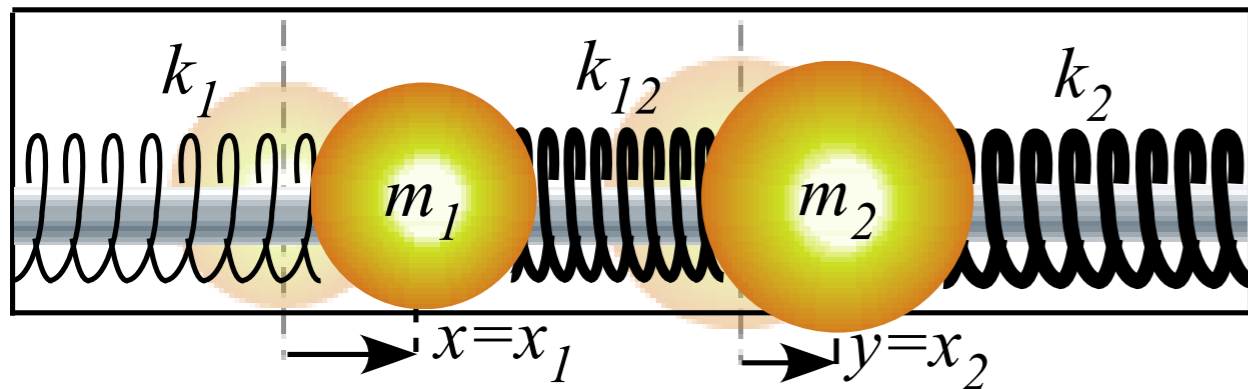


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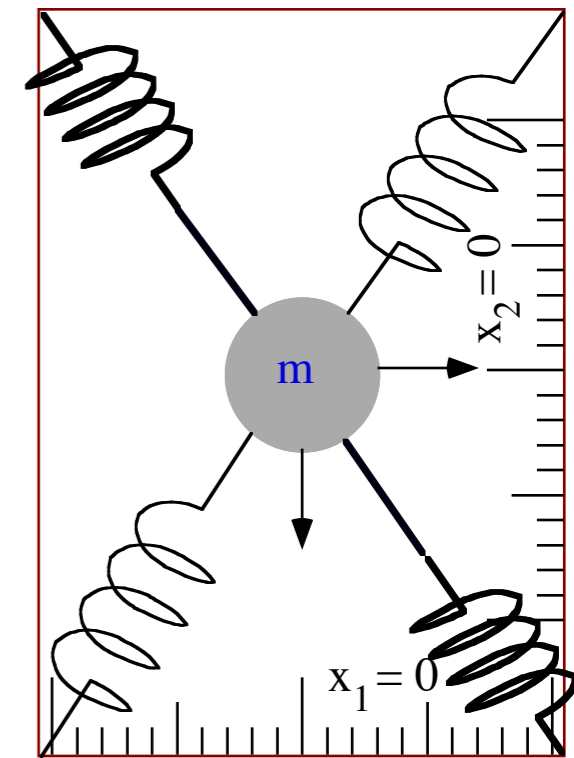


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$$V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2 + \frac{1}{2} k_{12} (x_1 - x_2)^2$$

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2D HO Lagrange equations

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_1} \right) = m_1 \ddot{x}_1 = F_1 = - \frac{\partial V}{\partial x_1} = - (k_1 + k_{12}) x_1 + k_{12} x_2$$

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2D HO Matrix operator equations

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = - \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

2D harmonic oscillator equations

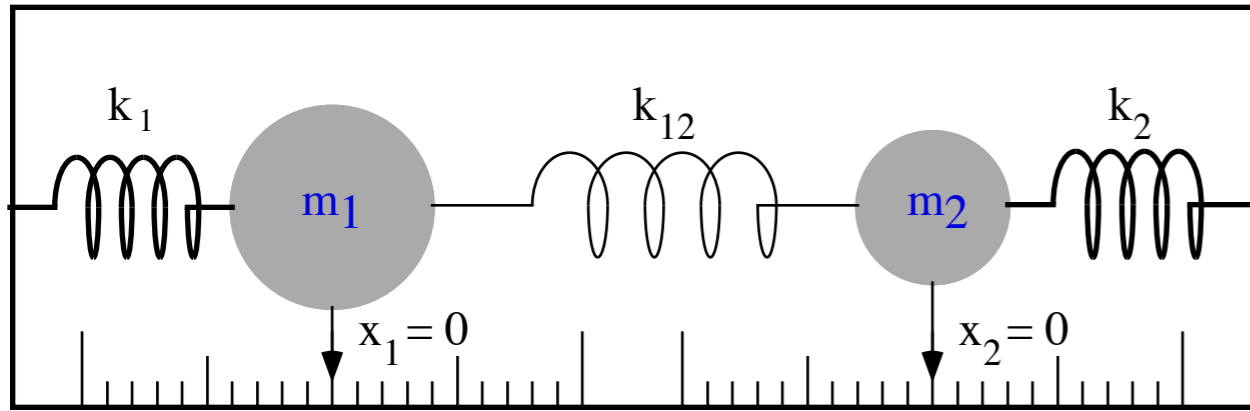


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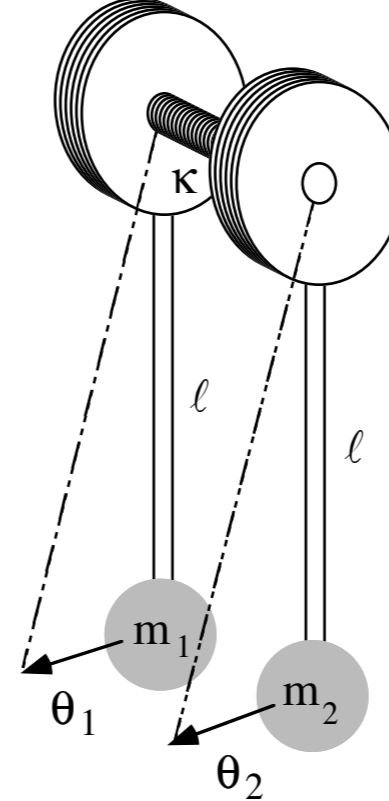
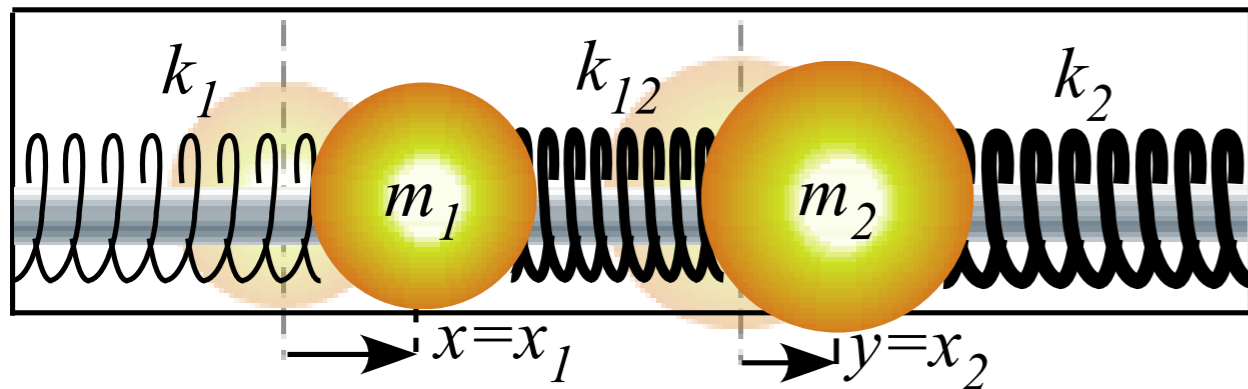


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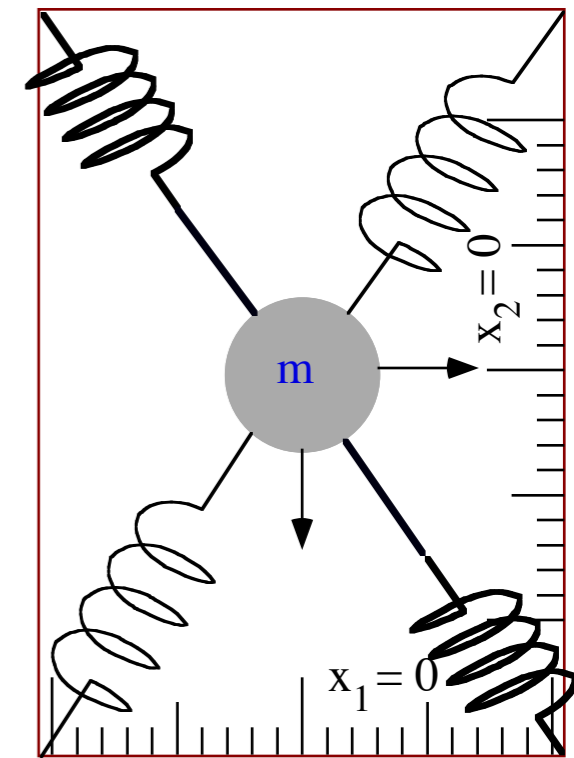


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$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_1} \right) = m_1 \ddot{x}_1 = F_1 = - \frac{\partial V}{\partial x_1} = - (k_1 + k_{12}) x_1 + k_{12} x_2$$

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Matrix operator notation:

$$\mathbf{M} \cdot |\ddot{\mathbf{x}}\rangle = - \mathbf{K} \cdot |\mathbf{x}\rangle$$

2D harmonic oscillator equations

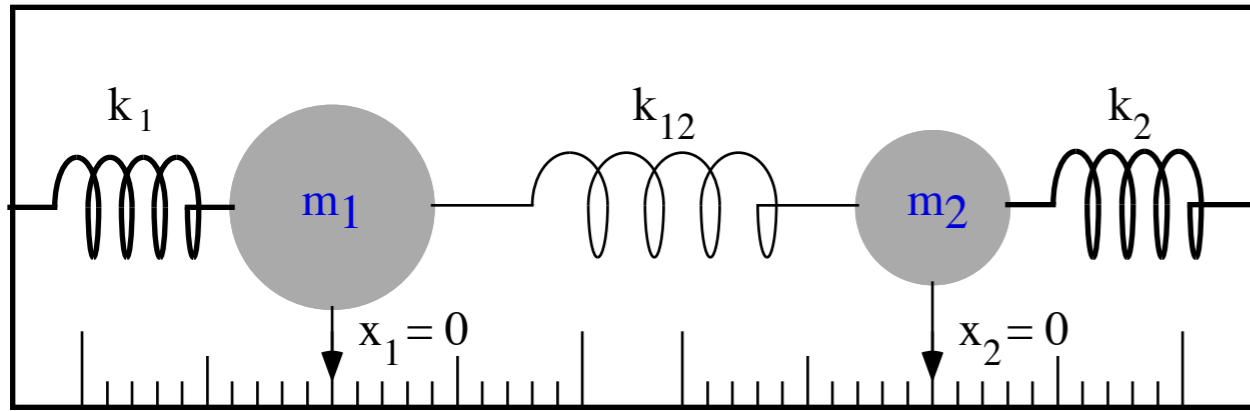


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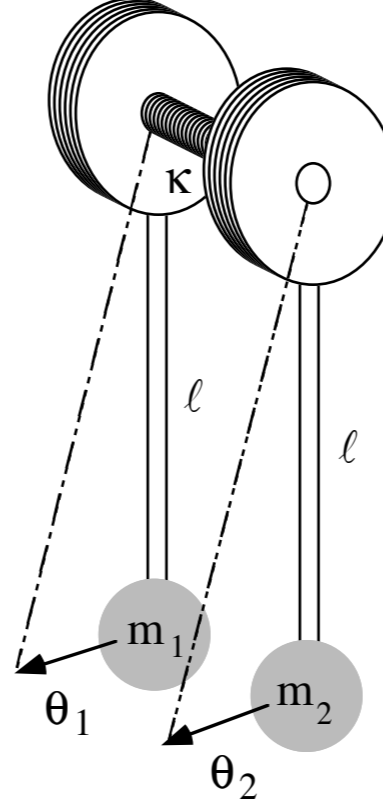
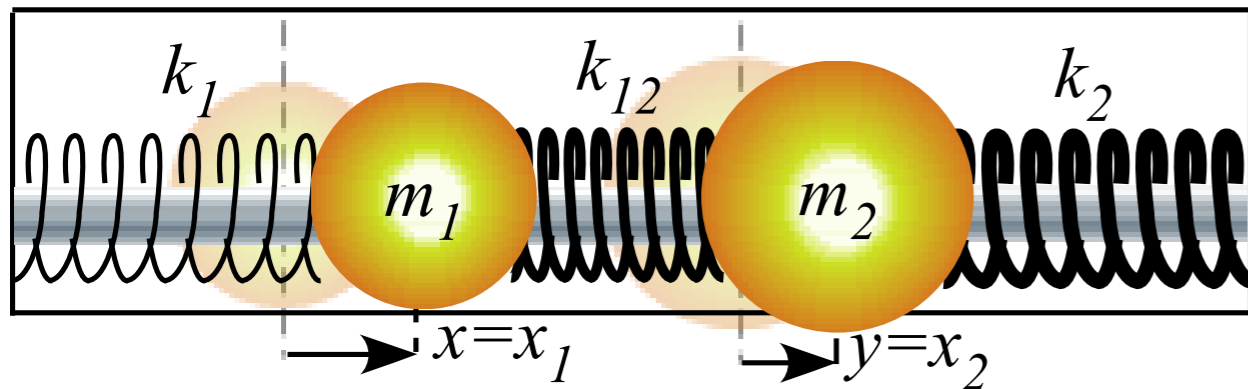


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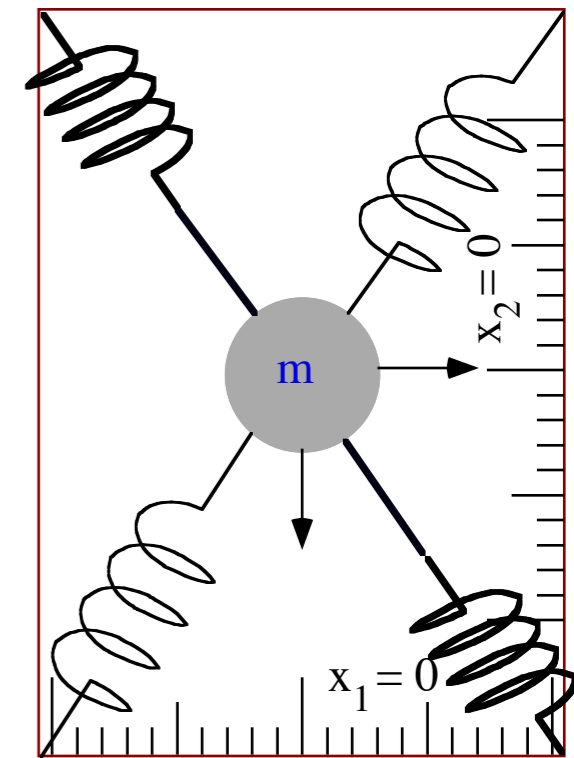


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2D HO kinetic energy $T(v_1, v_2)$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$= \frac{1}{2} \langle \dot{\mathbf{x}} | \mathbf{M} | \dot{\mathbf{x}} \rangle$$

2D HO potential energy $V(x_1, x_2)$

$$V = \frac{1}{2} (k_1 + k_{12}) x_1^2 - k_{12} x_1 x_2 + \frac{1}{2} (k_2 + k_{12}) x_2^2$$

$$= \frac{1}{2} \langle \mathbf{x} | \mathbf{K} | \mathbf{x} \rangle$$

Lagrangian $L=T-V$

where: $\mathbf{K} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix}$

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Matrix equations and reciprocity symmetry

General form of 2D-HO equation of motion has force matrix components: $\kappa_{11} = k_1 + k_{12}$, $\kappa_{22} = k_2 + k_{12}$

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} m_1 \ddot{x}_1 \\ m_2 \ddot{x}_2 \end{pmatrix} = - \begin{pmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Off-diagonal force constants satisfy *Reciprocity Relations*: $-\kappa_{12} = k_{12} = \frac{\partial F_1}{\partial x_2} = -\frac{\partial^2 V}{\partial x_2 \partial x_1} = -\frac{\partial^2 V}{\partial x_1 \partial x_2} = \frac{\partial F_2}{\partial x_1} = k_{21} = -\kappa_{21}$

Rescaling and symmetrization

Each coordinate (x_1, x_2) is rescaled $(q_1 = s_1 x_1, q_2 = s_2 x_2)$ to symmetrize mass factors on \ddot{q}_j -terms.

$$\begin{aligned} -\frac{m_1}{s_1} \ddot{q}_1 &= \kappa_{11} \frac{q_1}{s_1} + \kappa_{12} \frac{q_2}{s_2} & -\ddot{q}_1 &= \frac{\kappa_{11}}{m_1} q_1 + \frac{\kappa_{12} s_1}{m_1 s_2} q_2 \equiv \mathbf{K}_{11} q_1 + \mathbf{K}_{12} q_2 \\ -\frac{m_2}{s_2} \ddot{q}_2 &= \kappa_{12} \frac{q_1}{s_1} + \kappa_{22} \frac{q_2}{s_2} & -\ddot{q}_2 &= \frac{\kappa_{12} s_2}{m_2 s_1} q_1 + \frac{\kappa_{22}}{m_2} q_2 \equiv \mathbf{K}_{21} q_1 + \mathbf{K}_{22} q_2 \end{aligned}$$

New constants K_{ij} have pseudo-reciprocity symmetry for a special scale factor ratio: $\frac{s_2}{s_1} = \sqrt{\frac{m_2}{m_1}}$

$$\mathbf{K}_{21} = \frac{\kappa_{12} s_2}{m_2 s_1} = \mathbf{K}_{12} = \frac{\kappa_{12} s_1}{m_1 s_2} = \frac{-k_{12}}{\sqrt{m_1 m_2}}$$

Diagonal constants K_{jj} are not affected by scaling. To be equal requires: $\frac{\kappa_{11}}{m_1} = \frac{\kappa_{22}}{m_2}$ or: $\frac{\kappa_{11}}{\kappa_{22}} = \frac{m_1}{m_2}$

$$\mathbf{K}_{11} = \frac{\kappa_{11}}{m_1} = \frac{k_1 + k_{12}}{m_1} \quad \mathbf{K}_{22} = \frac{\kappa_{22}}{m_2} = \frac{k_2 + k_{12}}{m_2}$$

Caution is advised since such forced symmetry may give modes with imaginary frequency.

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1. May rewrite equation $\mathbf{M} \cdot |\ddot{\mathbf{x}}\rangle = -\mathbf{K} \cdot |\mathbf{x}\rangle$ in acceleration matrix form: $|\ddot{\mathbf{x}}\rangle = -\mathbf{A}|\mathbf{x}\rangle$ where: $\mathbf{A} = \mathbf{M}^{-1} \cdot \mathbf{K}$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = - \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}^{-1} \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = - \begin{pmatrix} \frac{k_1 + k_{12}}{m_1} & \frac{-k_{12}}{m_1} \\ \frac{-k_{12}}{m_2} & \frac{k_2 + k_{12}}{m_2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

2. Need to find *eigenvectors* $|\mathbf{e}_1\rangle, |\mathbf{e}_2\rangle, \dots$ of acceleration matrix such that: $\mathbf{A}|\mathbf{e}_n\rangle = \varepsilon_n|\mathbf{e}_n\rangle = \omega_n^2|\mathbf{e}_n\rangle$

Then equations decouple to: $|\ddot{\mathbf{e}}_n\rangle = -\mathbf{A}|\mathbf{e}_n\rangle = -\varepsilon_n|\mathbf{e}_n\rangle = -\omega_n^2|\mathbf{e}_n\rangle$ where ε_n is an *eigenvalue*

and ω_n is an *eigenfrequency*

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To introduce eigensolutions we take a simple case of unit masses ($m_1=1=m_2$)

So equation of motion is simply: $|\ddot{\mathbf{x}}\rangle = -\mathbf{K}|\mathbf{x}\rangle$

Eigenvectors $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$ are in special directions where $|\ddot{\mathbf{x}}\rangle = -\mathbf{K}|\mathbf{x}\rangle$ is in same direction as $|\mathbf{x}\rangle$

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➔ *Geometric method*



Matrix-algebraic eigensolutions with example $M = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

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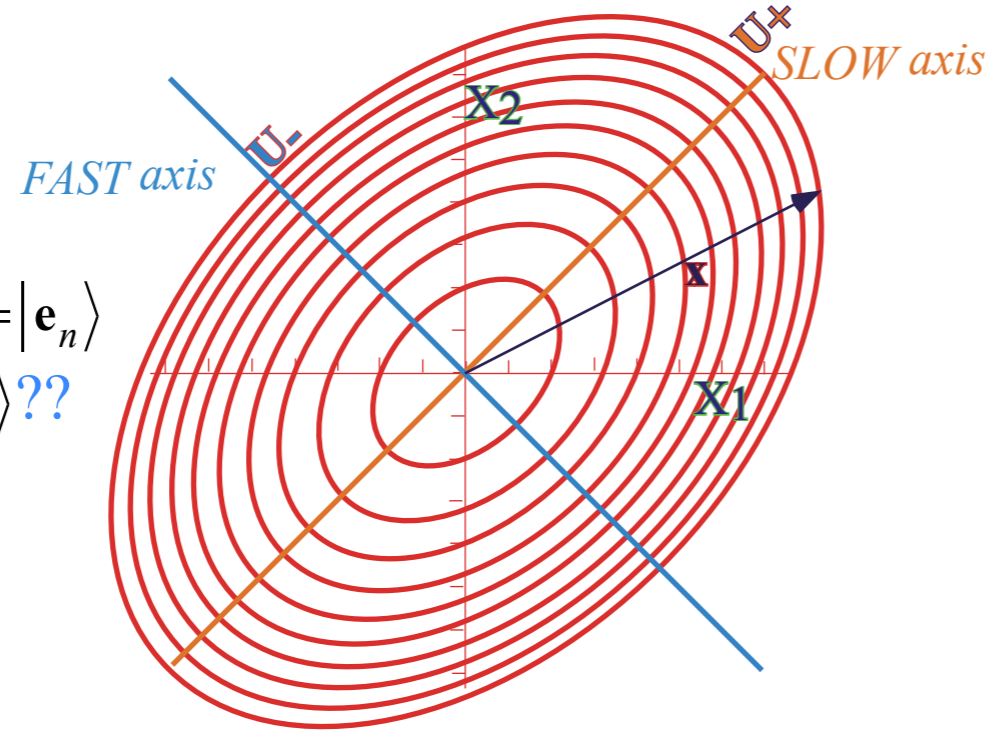
2D-HO eigensolution example with asymmetric (A-Type) symmetry

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2D HO potential energy $V(x_1, x_2)$ quadratic form defines layers of elliptical V -contours

$$V = \frac{1}{2}(k_1 + k_{12})x_1^2 - k_{12}x_1x_2 + \frac{1}{2}(k_2 + k_{12})x_2^2 = \frac{1}{2}\langle \mathbf{x} | \mathbf{K} | \mathbf{x} \rangle = \frac{1}{2}\mathbf{x} \cdot \mathbf{K} \cdot \mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(a) PE Contours



What direction $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$
is the same as $\mathbf{K}|\mathbf{x}\rangle$??

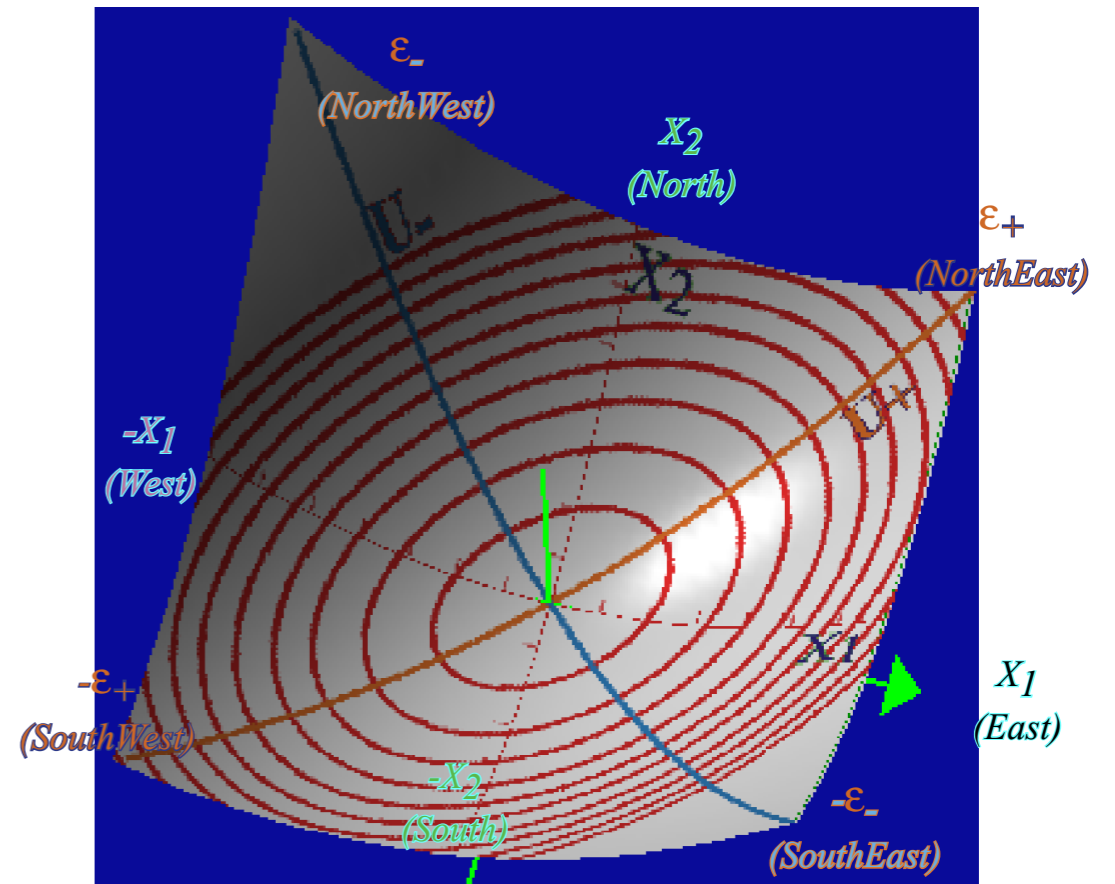
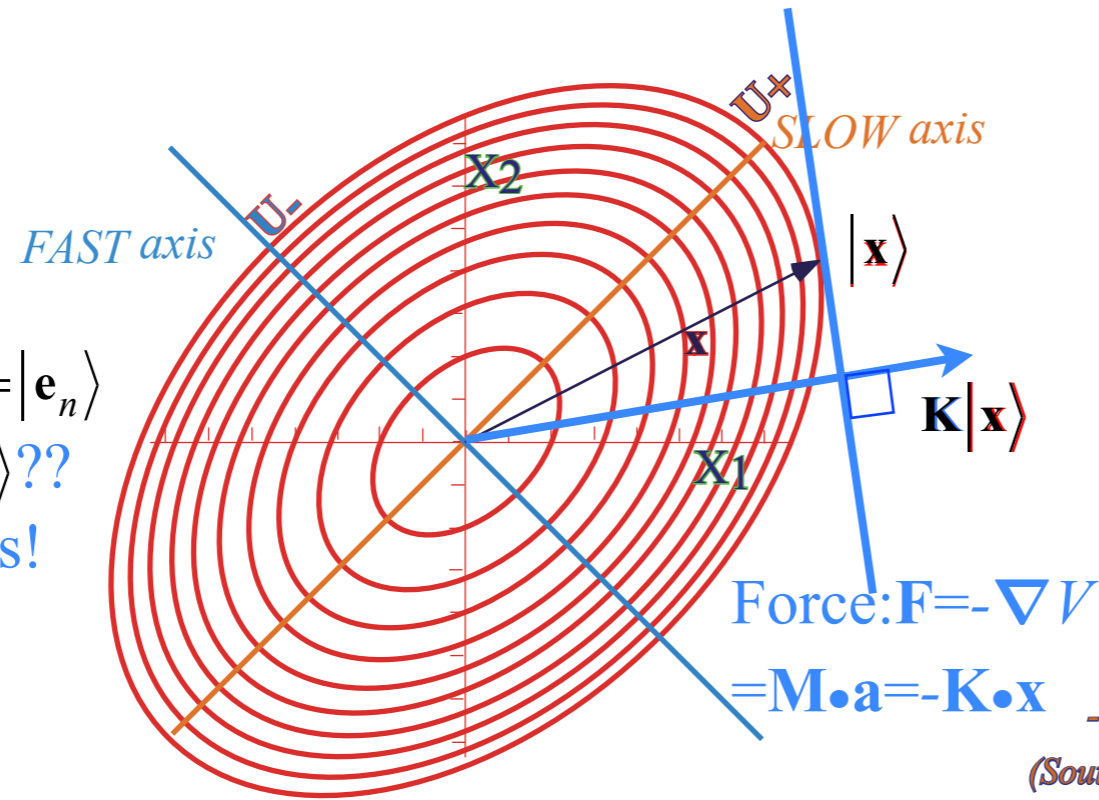


Fig. 3.3.4 Plot of potential function $V(x_1, x_2)$ showing elliptical $V(x_1, x_2) = \text{const.}$ level curves.

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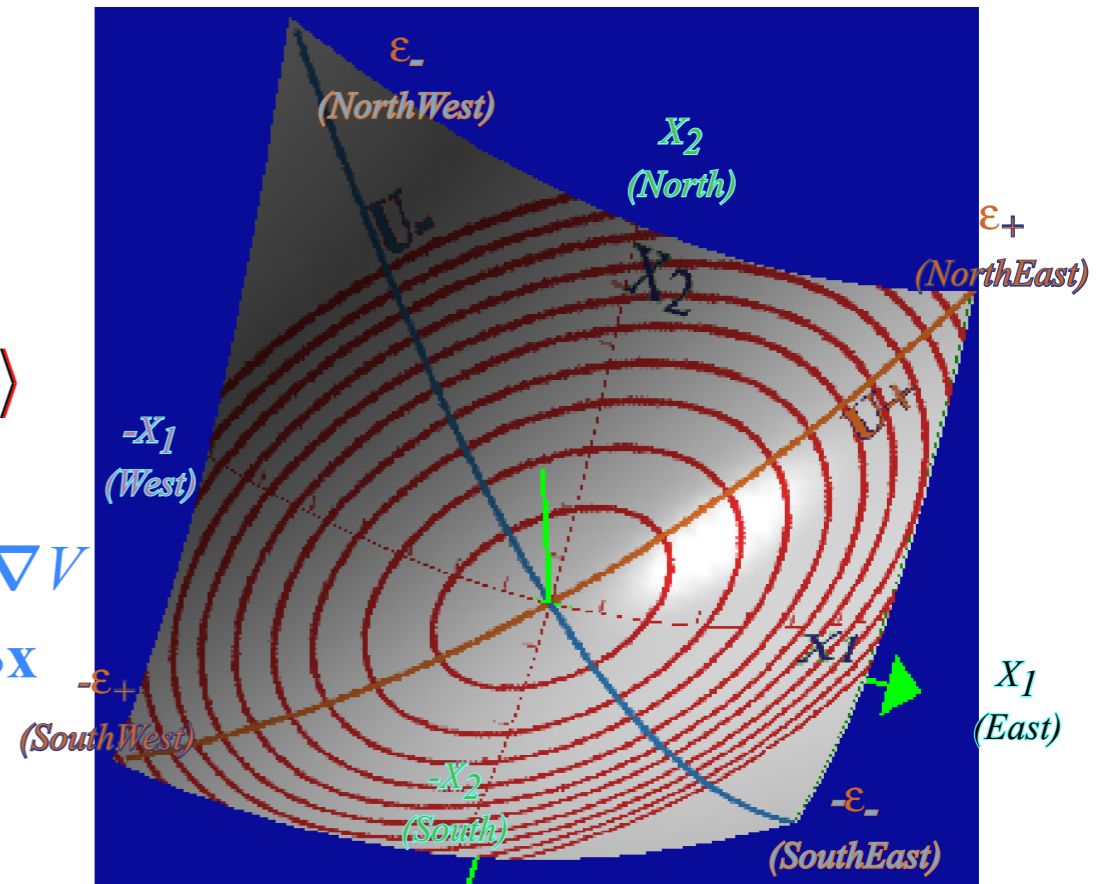
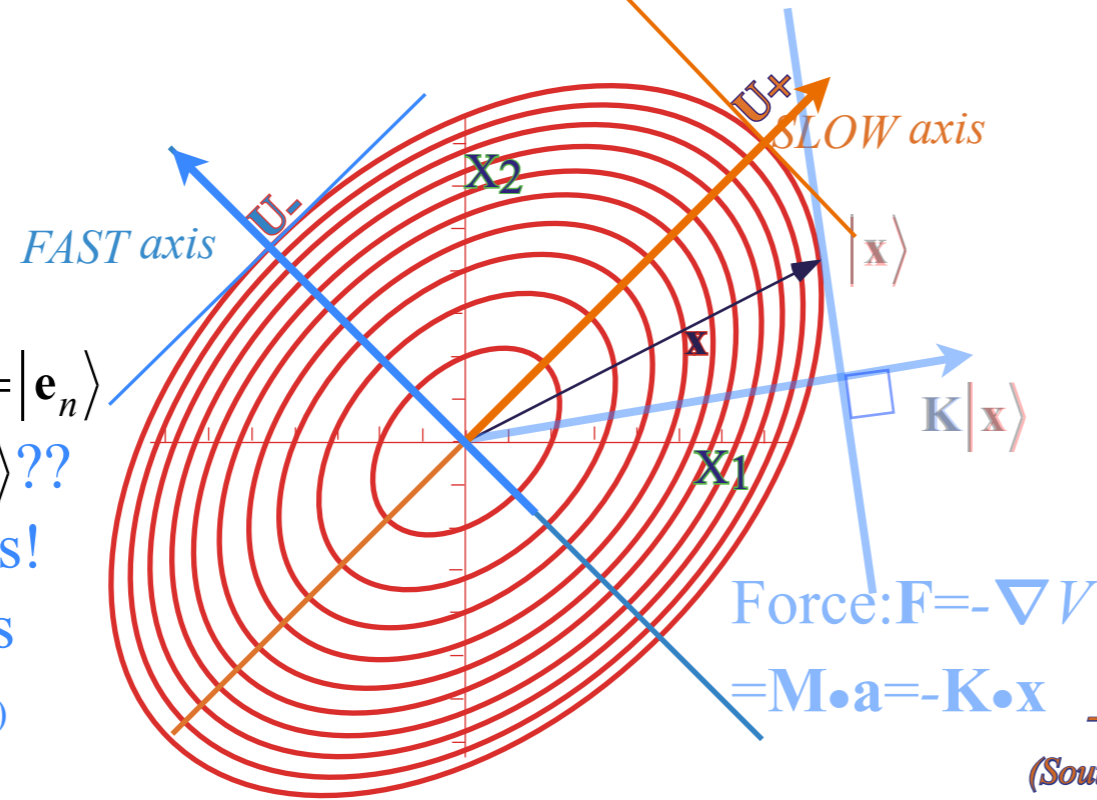


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What direction $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$ is the same as $\mathbf{K}|\mathbf{x}\rangle$??
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 Only extremal axes work. (major or minor axes)

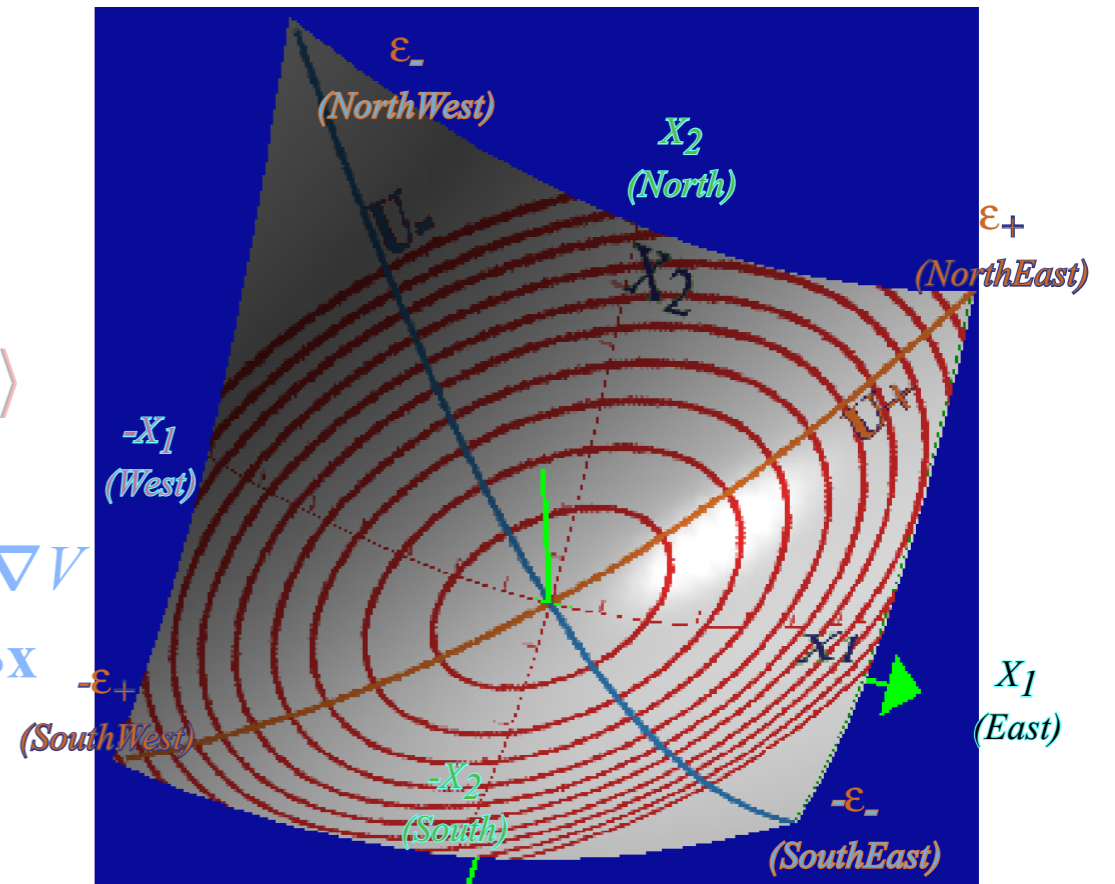
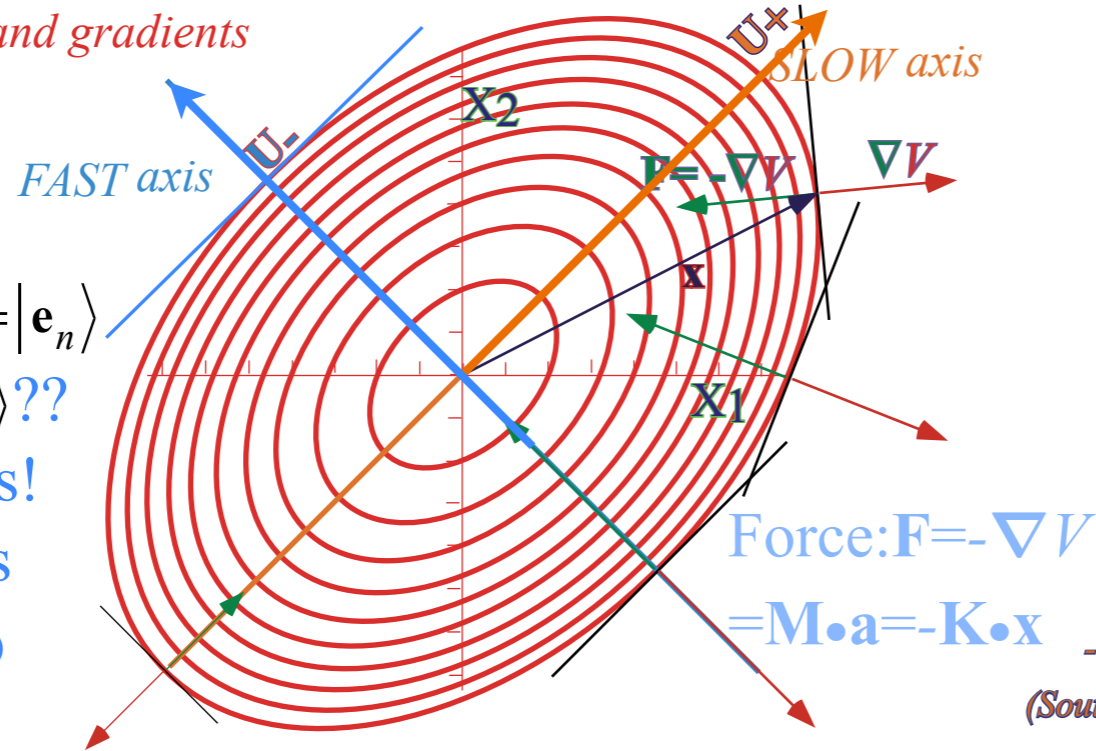


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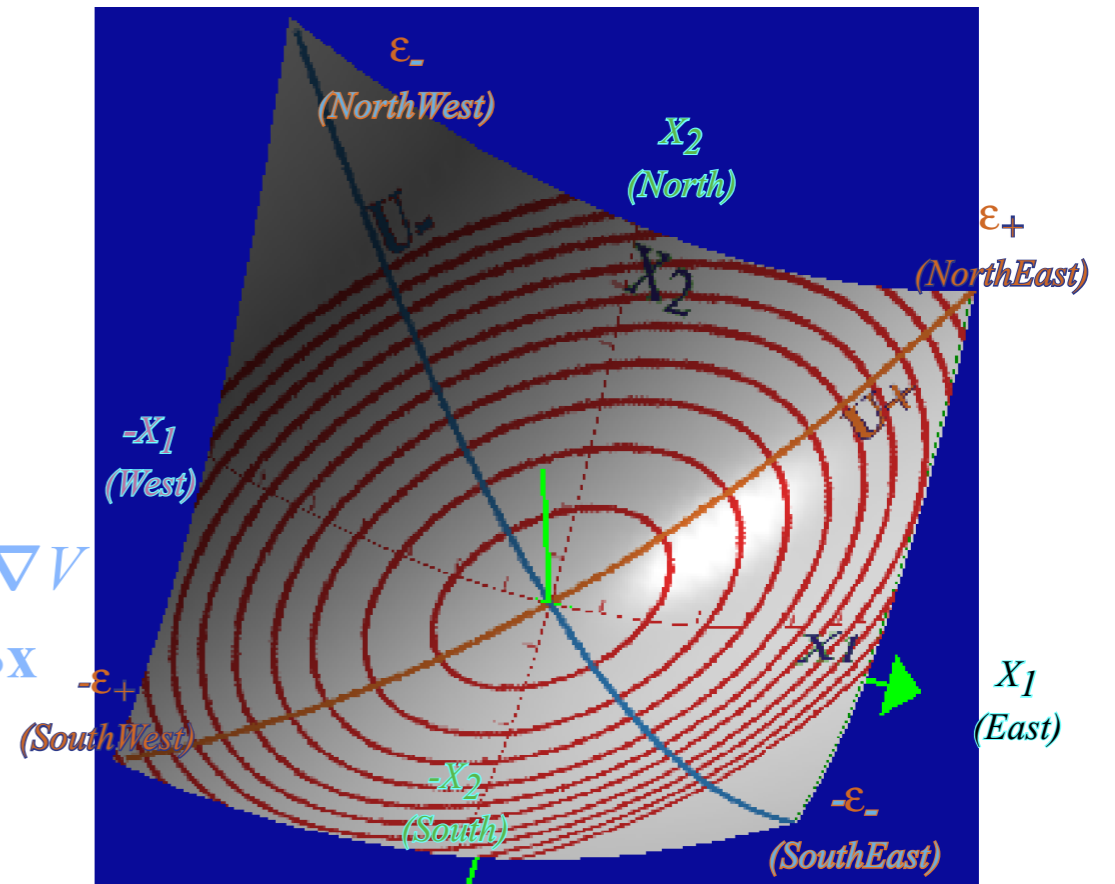
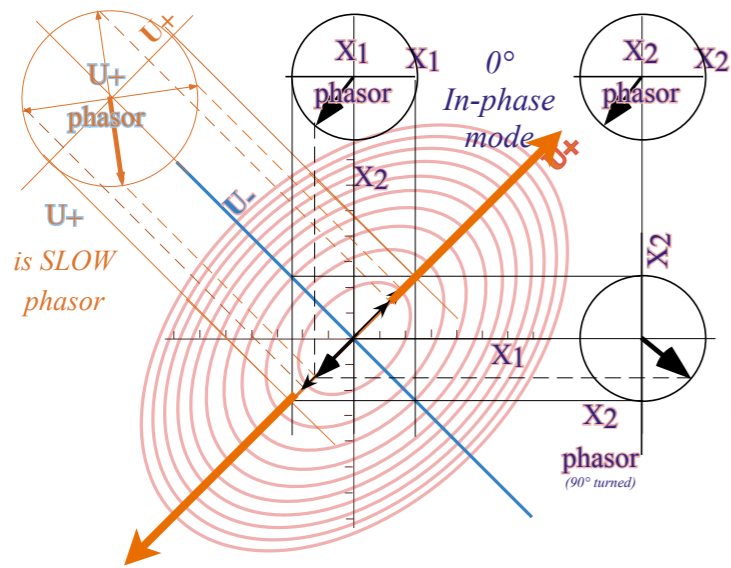
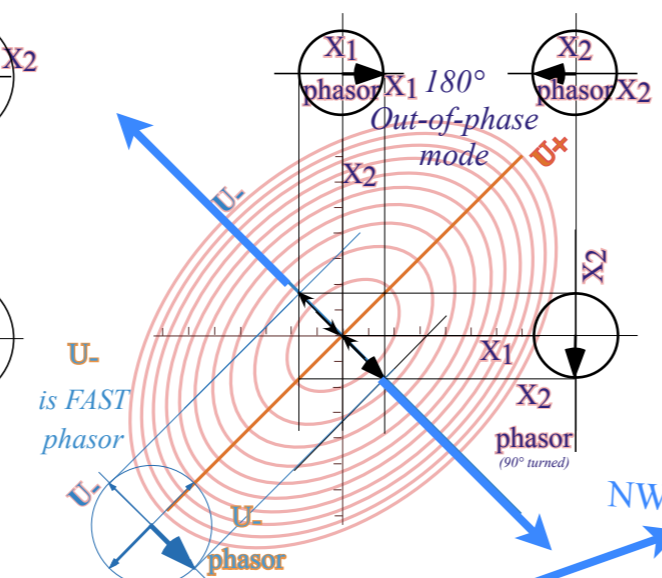


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(b) Symmetric $U+$ Coordinate SLOW Mode



(c) Anti-symmetric $U-$ Coordinate FAST Mode



With Bilateral symmetry ($k_1 = k = k_2$) the extremal axes lie at $\pm 45^\circ$

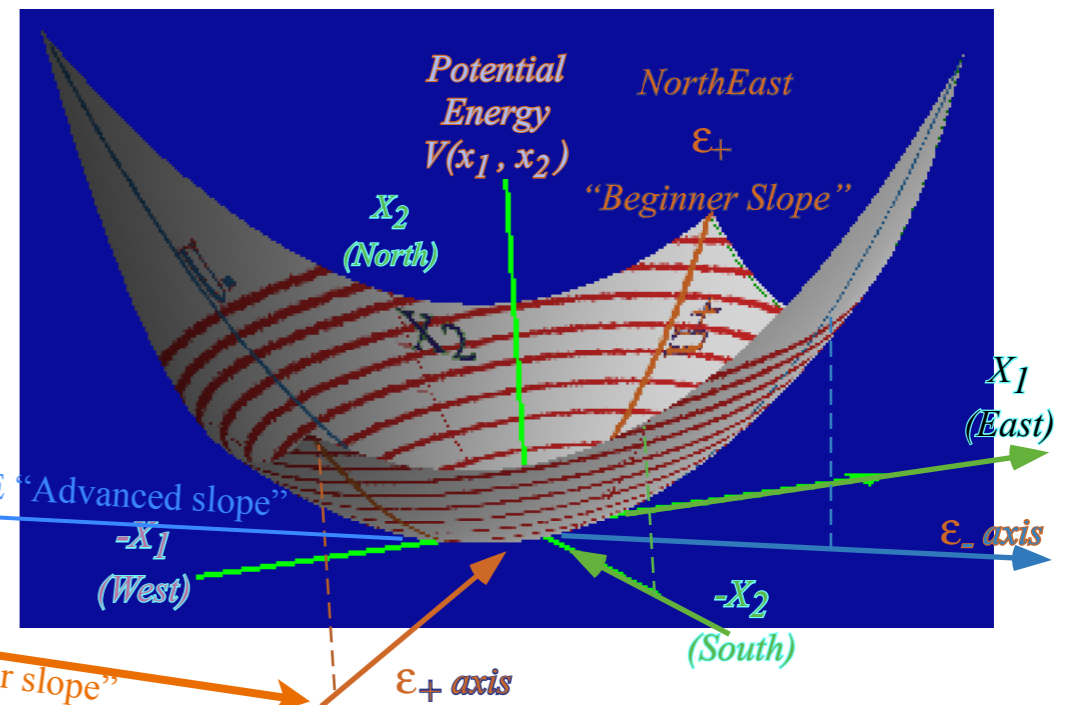
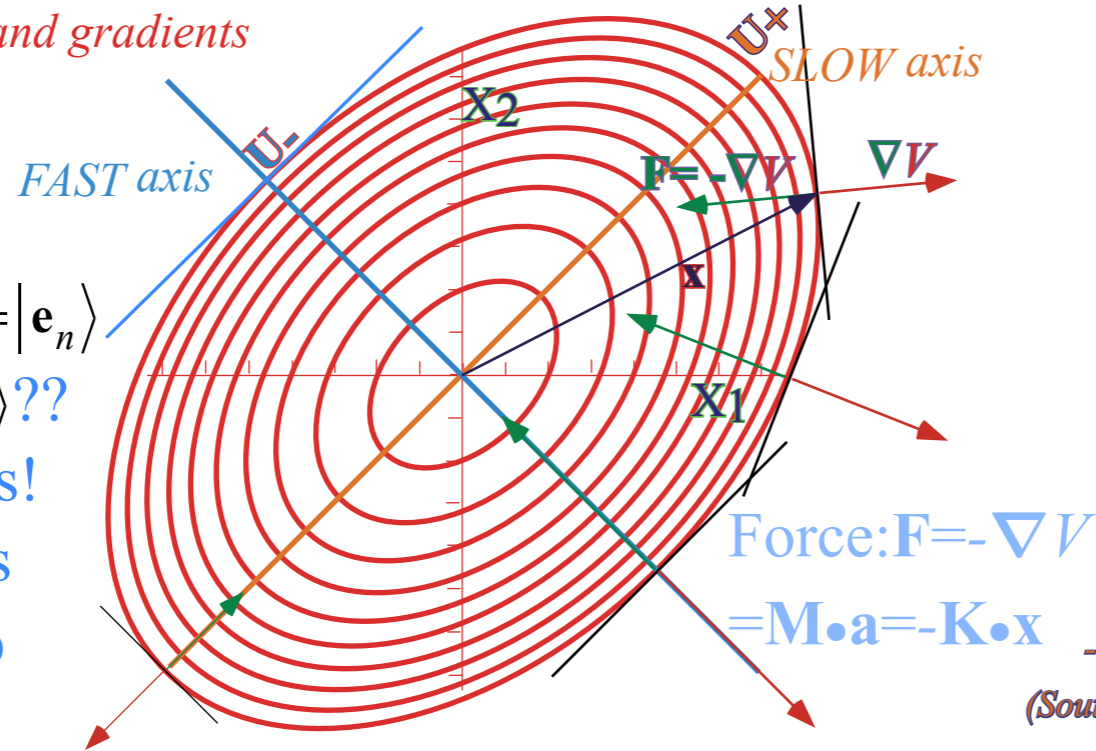


Fig. 3.3.5 Topography lines of potential function $V(x_1, x_2)$ and orthogonal ϵ_+ and ϵ_- normal mode slopes

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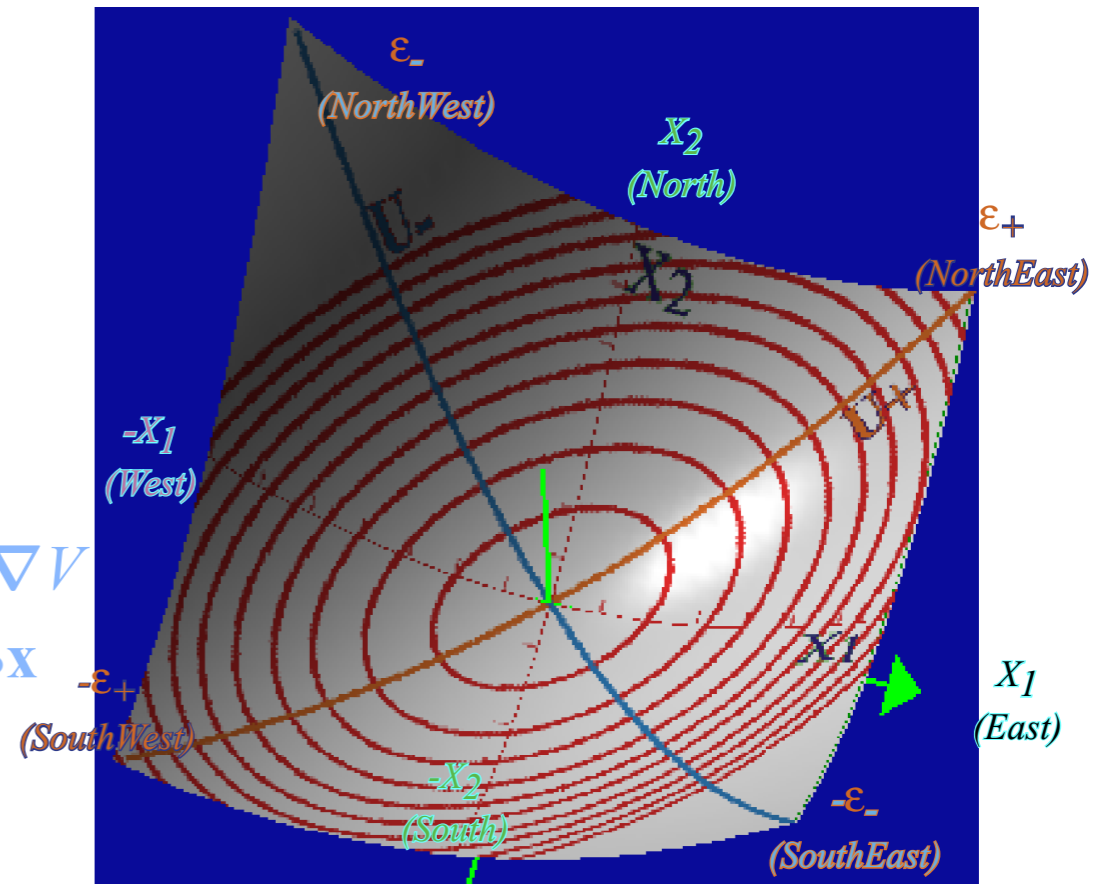
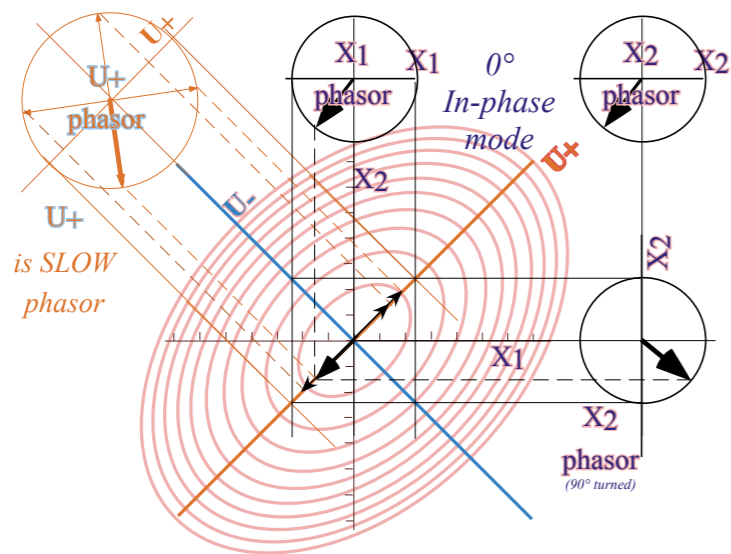
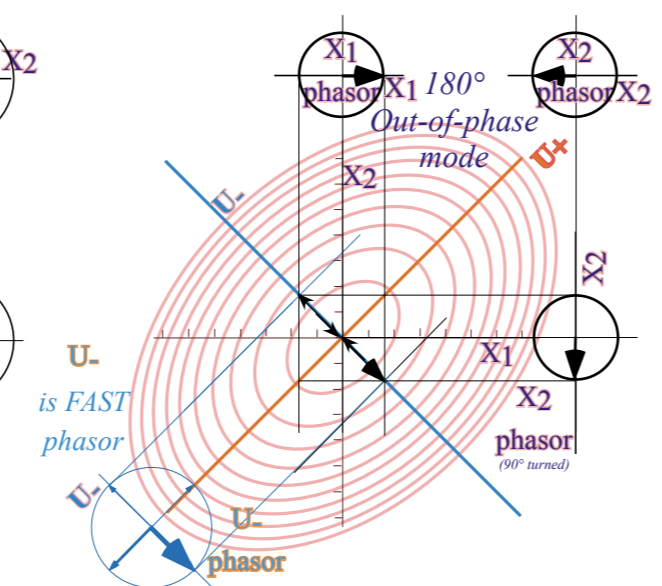


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ε_k is *eigenvalue* associated with eigenvector $|\varepsilon_k\rangle$ direction.

A change of basis to $\{|\varepsilon_1\rangle, |\varepsilon_2\rangle, \dots, |\varepsilon_n\rangle\}$ called *diagonalization* gives

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First step in finding eigenvalues: Solve *secular equation*

$$\det|\mathbf{M} - \epsilon \mathbf{1}| = 0 = (-1)^n (\epsilon^n + a_1 \epsilon^{n-1} + a_2 \epsilon^{n-2} + \dots + a_{n-1} \epsilon + a_n)$$

where:

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Only possible non-zero $\{x, y\}$ if denominator is zero, too!

$$0 = \det|\mathbf{M} - \epsilon \cdot \mathbf{1}| = \det \left[\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} - \epsilon \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] = \det \begin{pmatrix} 4-\epsilon & 1 \\ 3 & 2-\epsilon \end{pmatrix}$$

$$0 = (4-\epsilon)(2-\epsilon) - 1 \cdot 3 = 8 - 6\epsilon + \epsilon^2 - 1 \cdot 3 = \epsilon^2 - 6\epsilon + 5$$

$$0 = \epsilon^2 - \text{Trace}(\mathbf{M})\epsilon + \det(\mathbf{M})$$

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Secular equation has n -factors, one for each eigenvalue.

$$\det|\mathbf{M} - \epsilon \mathbf{1}| = 0 = (-1)^n (\epsilon - \epsilon_1)(\epsilon - \epsilon_2) \dots (\epsilon - \epsilon_n)$$

$$\mathbf{M}|\epsilon\rangle = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \epsilon \begin{pmatrix} x \\ y \end{pmatrix} \text{ or: } \begin{pmatrix} 4 - \epsilon & 1 \\ 3 & 2 - \epsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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$$x = \frac{\det \begin{pmatrix} 0 & 1 \\ 0 & 2 - \epsilon \end{pmatrix}}{\det \begin{pmatrix} 4 - \epsilon & 1 \\ 3 & 2 - \epsilon \end{pmatrix}} \quad \text{and} \quad y = \frac{\det \begin{pmatrix} 4 - \epsilon & 0 \\ 3 & 0 \end{pmatrix}}{\det \begin{pmatrix} 4 - \epsilon & 1 \\ 3 & 2 - \epsilon \end{pmatrix}}$$

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Secular equation

→ *Hamilton-Cayley equation* **←** *projectors*

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Matrix-algebraic method for finding eigenvector and eigenvalues With example matrix $\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

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ϵ_k is *eigenvalue* associated with eigenvector $|\epsilon_k\rangle$ direction.

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$$\begin{pmatrix} \langle \epsilon_1 | \mathbf{M} | \epsilon_1 \rangle & \langle \epsilon_1 | \mathbf{M} | \epsilon_2 \rangle & \dots & \langle \epsilon_1 | \mathbf{M} | \epsilon_n \rangle \\ \langle \epsilon_2 | \mathbf{M} | \epsilon_1 \rangle & \langle \epsilon_2 | \mathbf{M} | \epsilon_2 \rangle & \dots & \langle \epsilon_2 | \mathbf{M} | \epsilon_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \epsilon_n | \mathbf{M} | \epsilon_1 \rangle & \langle \epsilon_n | \mathbf{M} | \epsilon_2 \rangle & \dots & \langle \epsilon_n | \mathbf{M} | \epsilon_n \rangle \end{pmatrix} = \begin{pmatrix} \epsilon_1 & 0 & \dots & 0 \\ 0 & \epsilon_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \epsilon_n \end{pmatrix}$$

First step in finding eigenvalues: Solve *secular equation*

$$\det|\mathbf{M} - \epsilon \mathbf{1}| = 0 = (-1)^n (\epsilon^n + a_1 \epsilon^{n-1} + a_2 \epsilon^{n-2} + \dots + a_{n-1} \epsilon + a_n)$$

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Secular equation has n -factors, one for each eigenvalue.

$$\det|\mathbf{M} - \epsilon \mathbf{1}| = 0 = (-1)^n (\epsilon - \epsilon_1)(\epsilon - \epsilon_2) \dots (\epsilon - \epsilon_n)$$

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Only possible non-zero $\{x, y\}$ if denominator is zero, too!

$$0 = \det|\mathbf{M} - \epsilon \cdot \mathbf{1}| = \det \left[\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} - \epsilon \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] = \det \begin{pmatrix} 4 - \epsilon & 1 \\ 3 & 2 - \epsilon \end{pmatrix}$$

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Replace j^{th} HC-factor by $(\mathbf{1})$ to make *projection operators*

$$\begin{aligned} \mathbf{p}_1 &= (\mathbf{1})(\mathbf{M} - \epsilon_2\mathbf{1}) \dots (\mathbf{M} - \epsilon_n\mathbf{1}) \\ \mathbf{p}_2 &= (\mathbf{M} - \epsilon_1\mathbf{1})(\mathbf{1}) \dots (\mathbf{M} - \epsilon_n\mathbf{1}) \\ &\vdots \\ \mathbf{p}_n &= (\mathbf{M} - \epsilon_1\mathbf{1})(\mathbf{M} - \epsilon_2\mathbf{1}) \dots (\mathbf{1}) \end{aligned}$$

$$\mathbf{M}|\epsilon\rangle = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \epsilon \begin{pmatrix} x \\ y \end{pmatrix} \text{ or: } \begin{pmatrix} 4-\epsilon & 1 \\ 3 & 2-\epsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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$$\vdots \quad \epsilon_j \neq \epsilon_k \neq \dots$$

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Notice \mathbf{p}_k commutes with \mathbf{M} ,..

since $\mathbf{M}^1, \mathbf{M}^2, \dots$ commute with \mathbf{M} .

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$$\mathbf{p}_j \mathbf{p}_k = \mathbf{p}_j \prod_{m \neq k} (\mathbf{M} - \varepsilon_m \mathbf{1}) = \prod_{m \neq k} (\mathbf{p}_j \mathbf{M} - \varepsilon_m \mathbf{p}_j \mathbf{1}) \quad \mathbf{M} \mathbf{p}_k = \varepsilon_k \mathbf{p}_k = \mathbf{p}_k \mathbf{M}$$

Multiplication properties of \mathbf{p}_j :

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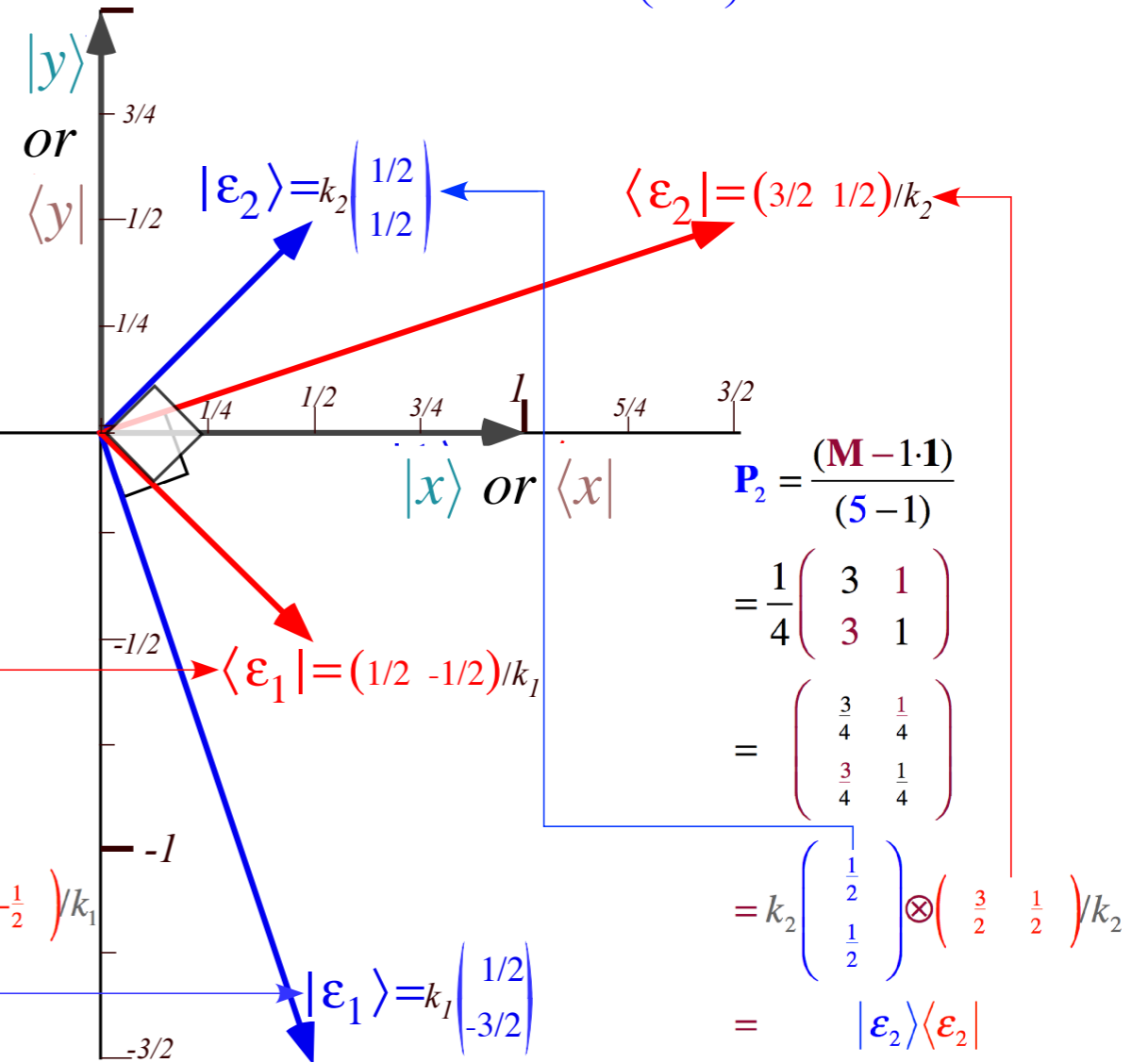
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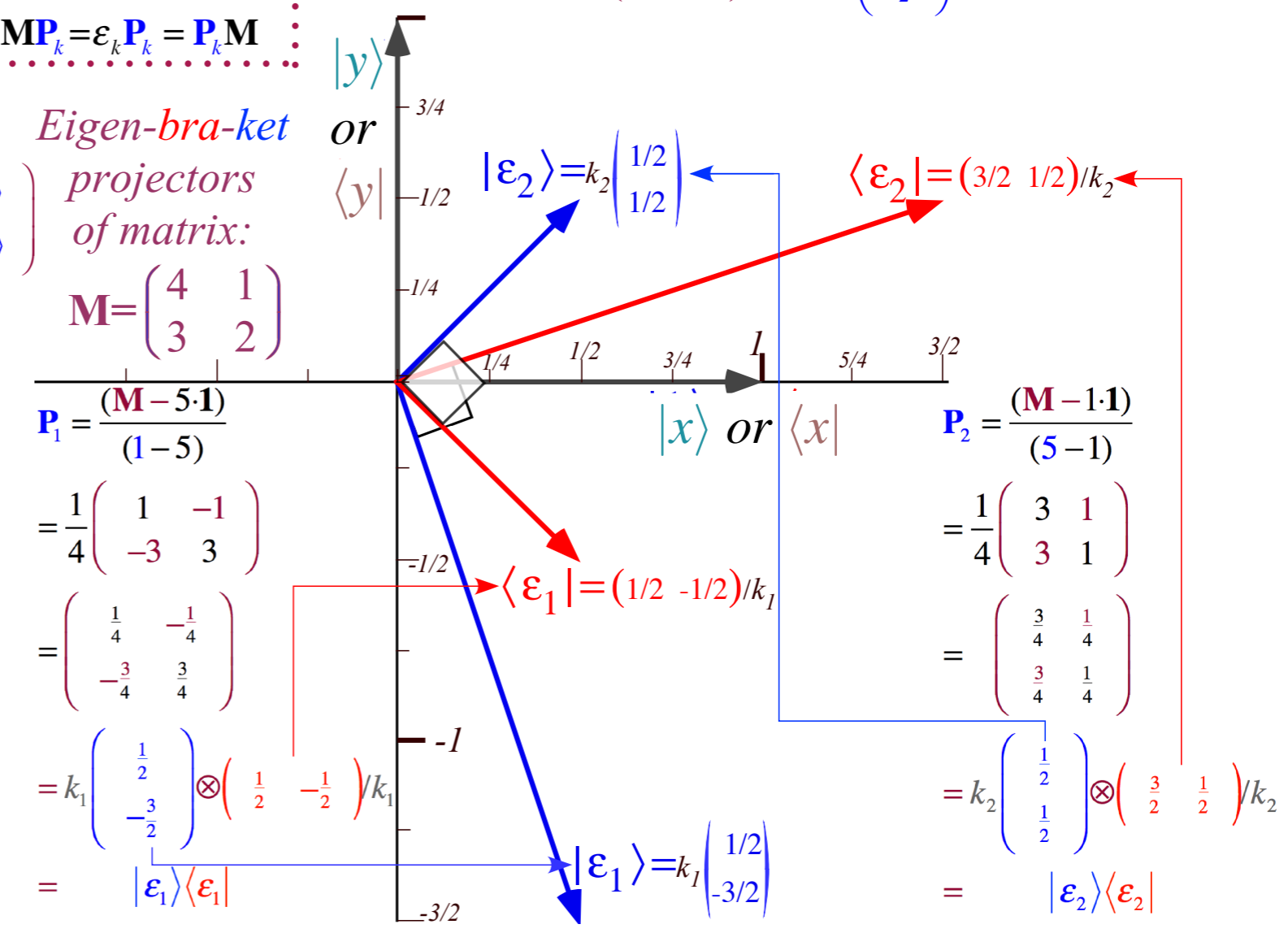
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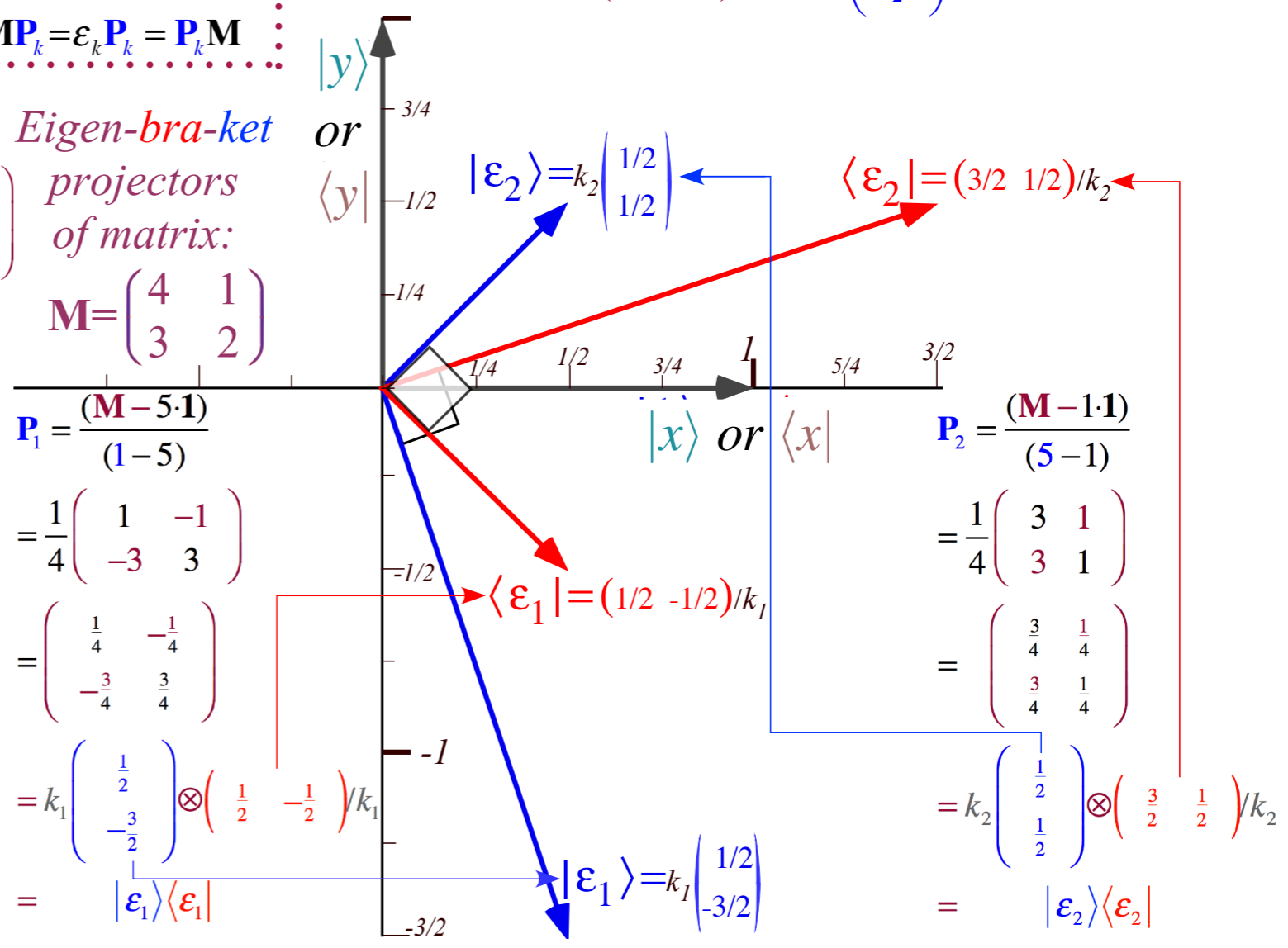
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...and the P_j satisfy a *Completeness Relation*:
 $1 = P_1 + P_2 + \dots + P_n$
 $= |\epsilon_1\rangle \langle \epsilon_1| + |\epsilon_2\rangle \langle \epsilon_2| + \dots + |\epsilon_n\rangle \langle \epsilon_n|$

$$P_1 + P_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |\epsilon_1\rangle \langle \epsilon_1| + |\epsilon_2\rangle \langle \epsilon_2|$$



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$$P_j P_k = P_j \prod_{m \neq k} (M - \epsilon_m \mathbf{1}) = \prod_{m \neq k} (P_j M - \epsilon_m P_j \mathbf{1}) \quad M P_k = \epsilon_k P_k = P_k M$$

Multiplication properties of P_j :

$$P_j P_k = \prod_{m \neq k} (\epsilon_j P_j - \epsilon_m P_j) = P_j \prod_{m \neq k} (\epsilon_j - \epsilon_m) = \begin{cases} \mathbf{0} & \text{if } j \neq k \\ P_k \prod_{m \neq k} (\epsilon_k - \epsilon_m) & \text{if } j = k \end{cases}$$

Last step:

make *Idempotent Projectors*: $P_k = \frac{P_k}{\prod_{m \neq k} (\epsilon_k - \epsilon_m)} = \frac{\prod_{m \neq k} (M - \epsilon_m \mathbf{1})}{\prod_{m \neq k} (\epsilon_k - \epsilon_m)}$
(Idempotent means: P · P = P)

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The P_j are *Mutually Ortho-Normal* as are bra-ket $\langle \epsilon_j |$ and $|\epsilon_j \rangle$ inside P_j 's

$$\begin{pmatrix} \langle \epsilon_1 | \epsilon_1 \rangle & \langle \epsilon_1 | \epsilon_2 \rangle \\ \langle \epsilon_2 | \epsilon_1 \rangle & \langle \epsilon_2 | \epsilon_2 \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

...and the P_j satisfy a *Completeness Relation*:

$$\mathbf{1} = P_1 + P_2 + \dots + P_n = |\epsilon_1 \rangle \langle \epsilon_1| + |\epsilon_2 \rangle \langle \epsilon_2| + \dots + |\epsilon_n \rangle \langle \epsilon_n|$$

$$P_1 + P_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |\epsilon_1 \rangle \langle \epsilon_1| + |\epsilon_2 \rangle \langle \epsilon_2|$$

Eigen-operators $M P_k = \epsilon_k P_k$ then give *Spectral Decomposition* of operator M

$$M = M P_1 + M P_2 + \dots + M P_n = \epsilon_1 P_1 + \epsilon_2 P_2 + \dots + \epsilon_n P_n$$

$$P_1 = (M - 5 \cdot \mathbf{1}) = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} \quad P_1 P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

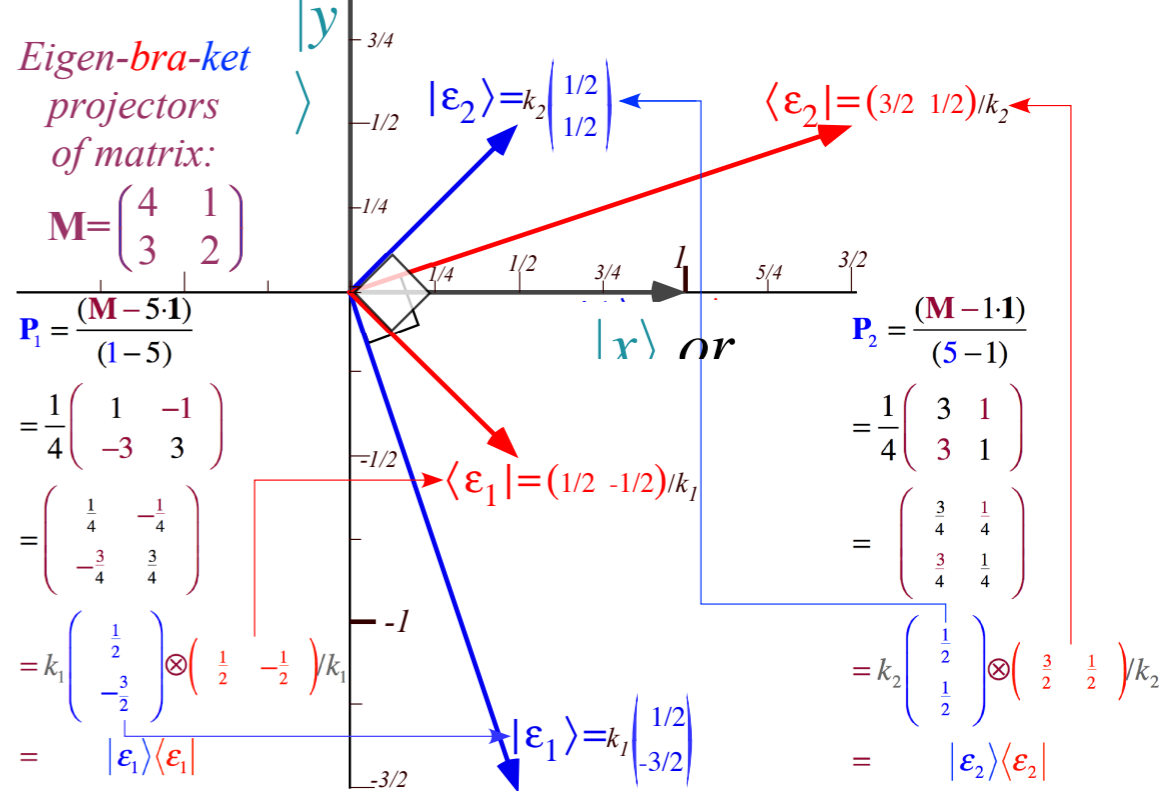
$$P_2 = (M - 1 \cdot \mathbf{1}) = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$

Factoring bra-kets into "Ket-Bras:

$$P_1 = \frac{(M - 5 \cdot \mathbf{1})}{(1 - 5)} = \frac{1}{4} \begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1 \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}{k_1} = |\epsilon_1 \rangle \langle \epsilon_1|$$

"Gauge" scale factors that only affect plots

$$P_2 = \frac{(M - 1 \cdot \mathbf{1})}{(5 - 1)} = \frac{1}{4} \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = k_2 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}}{k_2} = |\epsilon_2 \rangle \langle \epsilon_2|$$



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$$\mathbf{p}_j \mathbf{p}_k = \mathbf{p}_j \prod_{m \neq k} (\mathbf{M} - \varepsilon_m \mathbf{1}) = \prod_{m \neq k} (\mathbf{p}_j \mathbf{M} - \varepsilon_m \mathbf{p}_j \mathbf{1}) \quad \mathbf{M} \mathbf{p}_k = \varepsilon_k \mathbf{p}_k = \mathbf{p}_k \mathbf{M}$$

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$$\begin{pmatrix} \langle \varepsilon_1 | \varepsilon_1 \rangle & \langle \varepsilon_1 | \varepsilon_2 \rangle \\ \langle \varepsilon_2 | \varepsilon_1 \rangle & \langle \varepsilon_2 | \varepsilon_2 \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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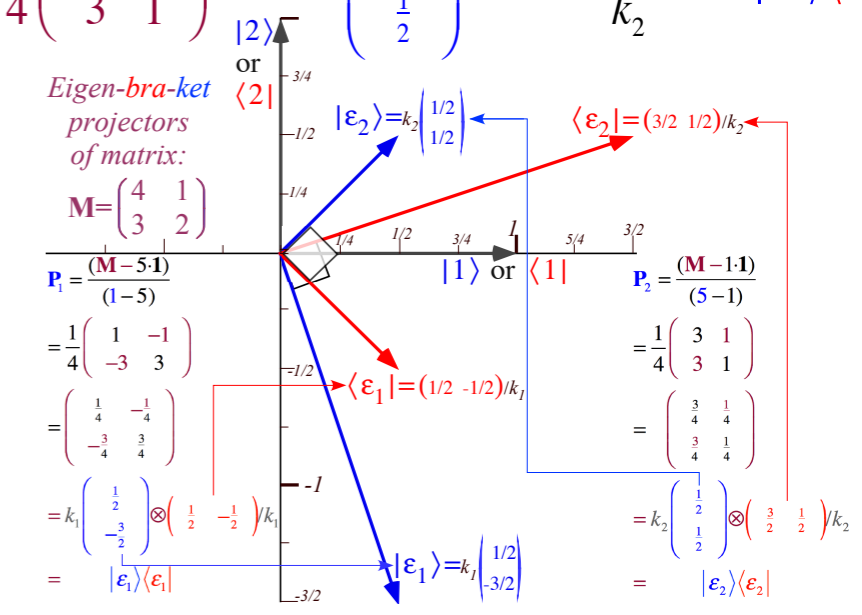
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2D harmonic oscillator equations

Lagrangian and matrix forms and Reciprocity symmetry

2D harmonic oscillator equation eigensolutions

Geometric method

Matrix-algebraic eigensolutions with example $M = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

Secular equation

Hamilton-Cayley equation and projectors

Idempotent projectors (how eigenvalues \Rightarrow eigenvectors)

Operator orthonormality and Completeness (Idempotent means: $\mathbf{P} \cdot \mathbf{P} = \mathbf{P}$)

Spectral Decompositions

➔ *Functional spectral decomposition* **←**

Orthonormality vs. Completeness vis-a`-vis Operator vs. State

Lagrange functional interpolation formula

Diagonalizing Transformations (D-Ttran) from projectors

2D-HO eigensolution example with bilateral (B-Type) symmetry

Mixed mode beat dynamics and fixed $\pi/2$ phase

2D-HO eigensolution example with asymmetric (A-Type) symmetry

Initial state projection, mixed mode beat dynamics with variable phase

Matrix and operator Spectral Decompositions

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Example:

$$\mathbf{M}^{50} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = 1^{50} \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{3}{4} \end{pmatrix} + 5^{50} \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1+3 \cdot 5^{50} & 5^{50}-1 \\ 3 \cdot 5^{50}-3 & 5^{50}+3 \end{pmatrix}$$

Matrix and operator Spectral Decompositions

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$$\mathbf{P}_1 = \frac{(\mathbf{M} - 5 \cdot \mathbf{1})}{(1 - 5)} = \frac{1}{4} \begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1 \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}{k_1} = |\varepsilon_1 \rangle \langle \varepsilon_1 |$$

$$\mathbf{P}_2 = \frac{(\mathbf{M} - 1 \cdot \mathbf{1})}{(5 - 1)} = \frac{1}{4} \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = k_2 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}}{k_2} = |\varepsilon_2 \rangle \langle \varepsilon_2 |$$

Examples:

$$\mathbf{M}^{50} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = 1^{50} \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{3}{4} \end{pmatrix} + 5^{50} \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 + 3 \cdot 5^{50} & 5^{50} - 1 \\ 3 \cdot 5^{50} - 3 & 5^{50} + 3 \end{pmatrix}$$

$$\sqrt{\mathbf{M}} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = \pm \sqrt{1} \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{3}{4} \end{pmatrix} \pm \sqrt{5} \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

2D harmonic oscillator equations

Lagrangian and matrix forms and Reciprocity symmetry

2D harmonic oscillator equation eigensolutions

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Matrix-algebraic eigensolutions with example $M = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

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Hamilton-Cayley equation and projectors

Idempotent projectors (how eigenvalues \Rightarrow eigenvectors)

Operator orthonormality and Completeness (Idempotent means: $\mathbf{P} \cdot \mathbf{P} = \mathbf{P}$)

Spectral Decompositions

Functional spectral decomposition

➔ *Orthonormality vs. Completeness vis-a`-vis Operator vs. State* **←**

Lagrange functional interpolation formula

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2D-HO eigensolution example with bilateral (B-Type) symmetry

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Initial state projection, mixed mode beat dynamics with variable phase

Orthonormality vs. Completeness

$$\mathbf{P}_j \mathbf{P}_k = \mathbf{P}_j \prod_{m \neq k} (\mathbf{M} - \varepsilon_m \mathbf{1}) = \prod_{m \neq k} (\mathbf{P}_j \mathbf{M} - \varepsilon_m \mathbf{P}_j \mathbf{1})$$

$$\mathbf{M} \mathbf{P}_k = \varepsilon_k \mathbf{P}_k = \mathbf{P}_k \mathbf{M}$$

$$\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$$

Multiplication properties of \mathbf{P}_j :

$$\mathbf{P}_j \mathbf{P}_k = \prod_{m \neq k} (\varepsilon_j \mathbf{P}_j - \varepsilon_m \mathbf{P}_j) = \mathbf{P}_j \prod_{m \neq k} (\varepsilon_j - \varepsilon_m) = \begin{cases} \mathbf{0} & \text{if } j \neq k \\ \mathbf{P}_k \prod_{m \neq k} (\varepsilon_k - \varepsilon_m) & \text{if } j = k \end{cases}$$

$$\mathbf{P}_1 = (\mathbf{M} - 5 \cdot \mathbf{1}) = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}$$

$$\mathbf{P}_1 \mathbf{P}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathbf{P}_2 = (\mathbf{M} - 1 \cdot \mathbf{1}) = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$

Last step:

make *Idempotent Projectors*: $\mathbf{P}_k = \frac{\mathbf{P}_k}{\prod_{m \neq k} (\varepsilon_k - \varepsilon_m)} = \frac{\prod_{m \neq k} (\mathbf{M} - \varepsilon_m \mathbf{1})}{\prod_{m \neq k} (\varepsilon_k - \varepsilon_m)}$
(Idempotent means: $\mathbf{P} \cdot \mathbf{P} = \mathbf{P}$)

Factoring bra-kets into "Ket-Bras:

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"Gauge" scale factors that only affect plots

$$\mathbf{P}_2 = \frac{(\mathbf{M} - 1 \cdot \mathbf{1})}{(5 - 1)} = \frac{1}{4} \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = k_2 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}}{k_2} = |\varepsilon_2\rangle \langle \varepsilon_2|$$

$$\mathbf{P}_j \mathbf{P}_k = \begin{cases} \mathbf{0} & \text{if } j \neq k \\ \mathbf{P}_k & \text{if } j = k \end{cases}$$

implies: $\mathbf{M} \mathbf{P}_k = \varepsilon_k \mathbf{P}_k = \mathbf{P}_k \mathbf{M}$
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The \mathbf{P}_j are *Mutually Ortho-Normal* as are bra-ket $\langle \varepsilon_j |$ and $|\varepsilon_j \rangle$ inside \mathbf{P}_j 's

Eigen-bra-ket projectors of matrix:

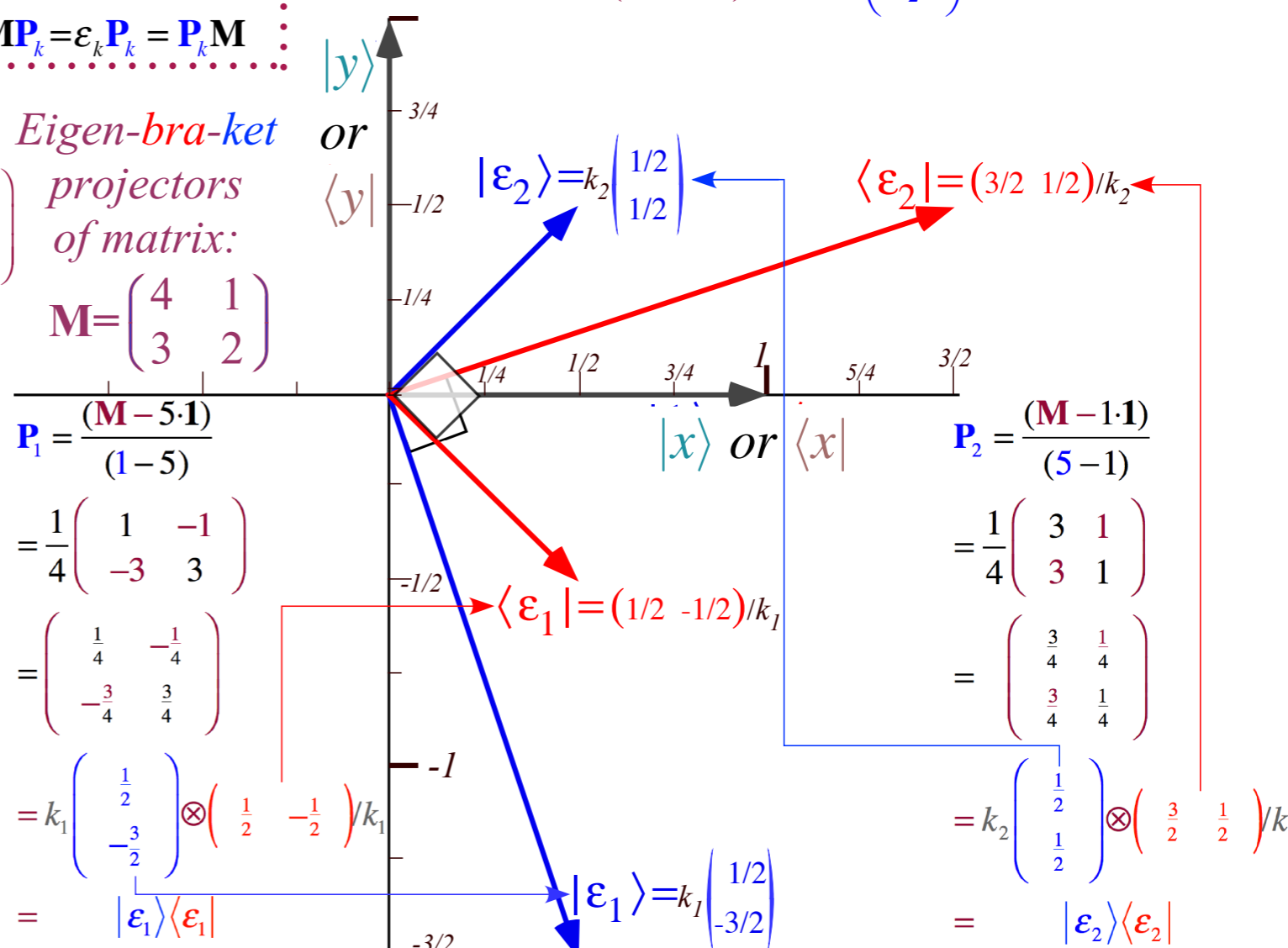
$$\begin{pmatrix} \langle \varepsilon_1 | \varepsilon_1 \rangle & \langle \varepsilon_1 | \varepsilon_2 \rangle \\ \langle \varepsilon_2 | \varepsilon_1 \rangle & \langle \varepsilon_2 | \varepsilon_2 \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$$

...and the \mathbf{P}_j satisfy a *Completeness Relation*:

$$\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + \dots + \mathbf{P}_n = |\varepsilon_1\rangle \langle \varepsilon_1| + |\varepsilon_2\rangle \langle \varepsilon_2| + \dots + |\varepsilon_n\rangle \langle \varepsilon_n|$$

$$\mathbf{P}_1 + \mathbf{P}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |\varepsilon_1\rangle \langle \varepsilon_1| + |\varepsilon_2\rangle \langle \varepsilon_2|$$



$\{|x\rangle, |y\rangle\}$ -orthonormality with $\{|\varepsilon_1\rangle, |\varepsilon_2\rangle\}$ -completeness

$$\langle x | y \rangle = \delta_{x,y} = \langle x | \mathbf{1} | y \rangle = \langle x | \varepsilon_1 \rangle \langle \varepsilon_1 | y \rangle + \langle x | \varepsilon_2 \rangle \langle \varepsilon_2 | y \rangle.$$

$\{|\varepsilon_1\rangle, |\varepsilon_2\rangle\}$ -orthonormality with $\{|x\rangle, |y\rangle\}$ -completeness

$$\langle \varepsilon_i | \varepsilon_j \rangle = \delta_{i,j} = \langle \varepsilon_i | \mathbf{1} | \varepsilon_j \rangle = \langle \varepsilon_i | x \rangle \langle x | \varepsilon_j \rangle + \langle \varepsilon_i | y \rangle \langle y | \varepsilon_j \rangle$$

Orthonormality vs. Completeness vis-a-vis Operator vs. State

Operator expressions for orthonormality appear quite different from expressions for completeness.

$$\mathbf{P}_j \mathbf{P}_k = \delta_{jk} \mathbf{P}_k = \begin{cases} \mathbf{0} & \text{if } j \neq k \\ \mathbf{P}_k & \text{if } j = k \end{cases}$$

$$\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + \dots + \mathbf{P}_n$$

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$$|\varepsilon_j\rangle \langle \varepsilon_j | \varepsilon_k\rangle \langle \varepsilon_k| = \delta_{jk} |\varepsilon_k\rangle \langle \varepsilon_k| \quad \text{or:} \quad \langle \varepsilon_j | \varepsilon_k\rangle = \delta_{jk}$$

$$\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + \dots + \mathbf{P}_n$$

$$\mathbf{1} = |\varepsilon_1\rangle \langle \varepsilon_1| + |\varepsilon_2\rangle \langle \varepsilon_2| + \dots + |\varepsilon_n\rangle \langle \varepsilon_n|$$

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$$\mathbf{1} = |\epsilon_1\rangle\langle\epsilon_1| + |\epsilon_2\rangle\langle\epsilon_2| + \dots + |\epsilon_n\rangle\langle\epsilon_n|$$

State vector representations of orthonormality are quite **similar** to representations of completeness.

Like 2-sides of the same coin.

$\{|x\rangle, |y\rangle\}$ -orthonormality with $\{|\epsilon_1\rangle, |\epsilon_2\rangle\}$ -completeness

$$\langle x|y\rangle = \delta_{x,y} = \langle x|\mathbf{1}|y\rangle = \langle x|\epsilon_1\rangle\langle\epsilon_1|y\rangle + \langle x|\epsilon_2\rangle\langle\epsilon_2|y\rangle.$$

$\{|\epsilon_1\rangle, |\epsilon_2\rangle\}$ -orthonormality with $\{|x\rangle, |y\rangle\}$ -completeness

$$\langle\epsilon_i|\epsilon_j\rangle = \delta_{i,j} = \langle\epsilon_i|\mathbf{1}|\epsilon_j\rangle = \langle\epsilon_i|x\rangle\langle x|\epsilon_j\rangle + \langle\epsilon_i|y\rangle\langle y|\epsilon_j\rangle$$

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$$\mathbf{1} = |\epsilon_1\rangle \langle \epsilon_1| + |\epsilon_2\rangle \langle \epsilon_2| + \dots + |\epsilon_n\rangle \langle \epsilon_n|$$

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$$\langle x | y \rangle = \delta(x, y) = \psi_1(x) \psi_1^*(y) + \psi_2(x) \psi_2^*(y) + \dots$$

Dirac δ -function

$\{|\epsilon_1\rangle, |\epsilon_2\rangle\}$ -orthonormality with $\{|x\rangle, |y\rangle\}$ -completeness

$$\langle \epsilon_i | \epsilon_j \rangle = \delta_{i,j} = \langle \epsilon_i | \mathbf{1} | \epsilon_j \rangle = \langle \epsilon_i | x \rangle \langle x | \epsilon_j \rangle + \langle \epsilon_i | y \rangle \langle y | \epsilon_j \rangle$$

However Schrodinger wavefunction notation $\psi(x) = \langle x | \psi \rangle$ shows quite a difference...

Orthonormality vs. Completeness vis-a-vis Operator vs. State

Operator expressions for orthonormality appear quite **different** from expressions for completeness.

$$\mathbf{P}_j \mathbf{P}_k = \delta_{jk} \mathbf{P}_k = \begin{cases} \mathbf{0} & \text{if } j \neq k \\ \mathbf{P}_k & \text{if } j = k \end{cases} \quad \mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + \dots + \mathbf{P}_n$$

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$$\langle \epsilon_i | \epsilon_j \rangle = \delta_{i,j} = \dots + \psi_i^*(x) \psi_j(x) + \psi_2(y) \psi_2^*(y) + \dots \rightarrow \int dx \psi_i^*(x) \psi_j(x)$$

However Schrodinger wavefunction notation $\psi(x) = \langle x | \psi \rangle$ shows quite a difference...

...particularly in the orthonormality integral.

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A Proof of Projector Completeness (Truer-than-true by Lagrange interpolation)

Compare matrix *completeness relation* and *functional spectral decompositions*

$$\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + \dots + \mathbf{P}_n = \sum_{\varepsilon_k} \mathbf{P}_k = \sum_{\varepsilon_k} \frac{\prod_{m \neq k} (\mathbf{M} - \varepsilon_m \mathbf{1})}{\prod_{m \neq k} (\varepsilon_k - \varepsilon_m)} \quad f(\mathbf{M}) = f(\varepsilon_1)\mathbf{P}_1 + f(\varepsilon_2)\mathbf{P}_2 + \dots + f(\varepsilon_n)\mathbf{P}_n = \sum_{\varepsilon_k} f(\varepsilon_k)\mathbf{P}_k = \sum_{\varepsilon_k} f(\varepsilon_k) \frac{\prod_{m \neq k} (\mathbf{M} - \varepsilon_m \mathbf{1})}{\prod_{m \neq k} (\varepsilon_k - \varepsilon_m)}$$

with *Lagrange interpolation formula* of function $f(x)$ approximated by its value at N points x_1, x_2, \dots, x_N .

$$L(f(x)) = \sum_{k=1}^N f(x_k) \cdot P_k(x) \quad \text{where: } P_k(x) = \frac{\prod_{j \neq k}^N (x - x_j)}{\prod_{j \neq k}^N (x_k - x_j)}$$

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Each polynomial term $P_m(x)$ has zeros at each point $x=x_j$ *except* where $x=x_m$. Then $P_m(x_m)=1$.

So at each of these points this L-approximation becomes exact: $L(f(x_j))=f(x_j)$.

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If $f(x)$ happens to be a polynomial of degree $N-1$ or less, then $L(f(x))=f(x)$ may be exact everywhere.

$$1 = \sum_{m=1}^N P_m(x)$$

$$x = \sum_{m=1}^N x_m P_m(x)$$

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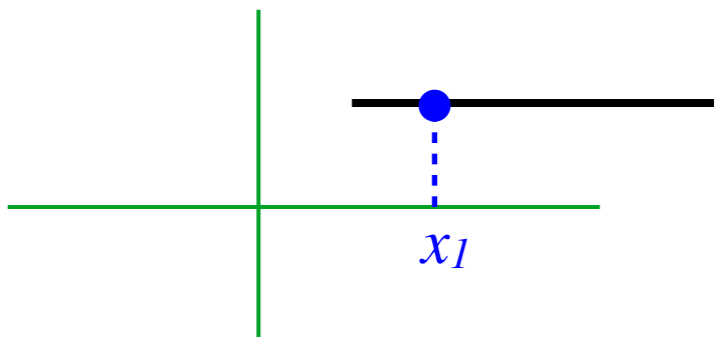
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One point determines a constant level line,



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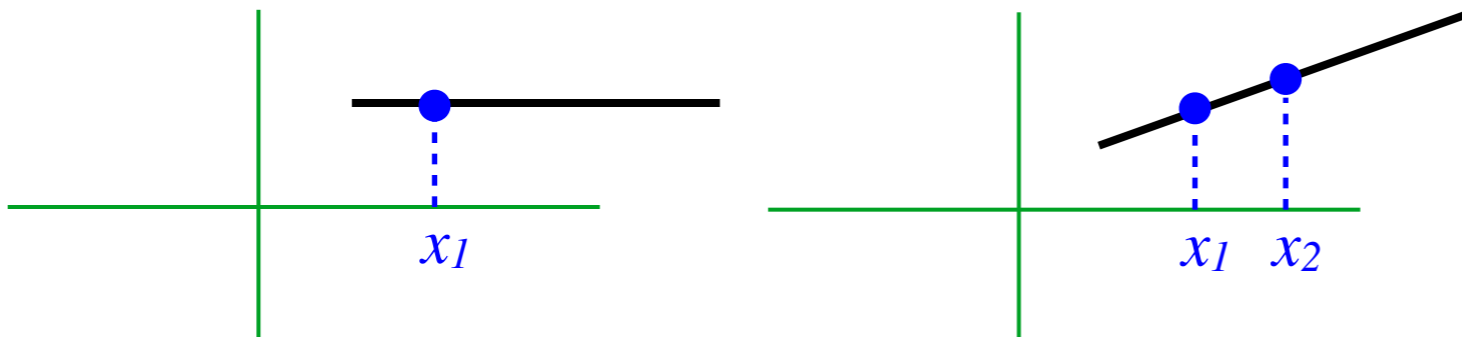
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One point determines a constant level line, two separate points uniquely determine a sloping line,



A Proof of Projector Completeness (Truer-than-true)

Compare matrix *completeness relation* and *functional spectral decompositions*

$$\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + \dots + \mathbf{P}_n = \sum_{\epsilon_k} \mathbf{P}_k = \sum_{\epsilon_k} \frac{\prod_{m \neq k} (\mathbf{M} - \epsilon_m \mathbf{1})}{\prod_{m \neq k} (\epsilon_k - \epsilon_m)} \quad f(\mathbf{M}) = f(\epsilon_1)\mathbf{P}_1 + f(\epsilon_2)\mathbf{P}_2 + \dots + f(\epsilon_n)\mathbf{P}_n = \sum_{\epsilon_k} f(\epsilon_k)\mathbf{P}_k = \sum_{\epsilon_k} f(\epsilon_k) \frac{\prod_{m \neq k} (\mathbf{M} - \epsilon_m \mathbf{1})}{\prod_{m \neq k} (\epsilon_k - \epsilon_m)}$$

with *Lagrange interpolation formula* of function $f(x)$ approximated by its value at N points x_1, x_2, \dots, x_N .

$$L(f(x)) = \sum_{k=1}^N f(x_k) \cdot P_k(x) \quad \text{where: } P_k(x) = \frac{\prod_{j \neq k} (x - x_j)}{\prod_{j \neq k} (x_k - x_j)}$$

Each polynomial term $P_m(x)$ has zeros at each point $x=x_j$ *except* where $x=x_m$. Then $P_m(x_m)=1$.

So at each of these points this L-approximation becomes exact: $L(f(x_j))=f(x_j)$.

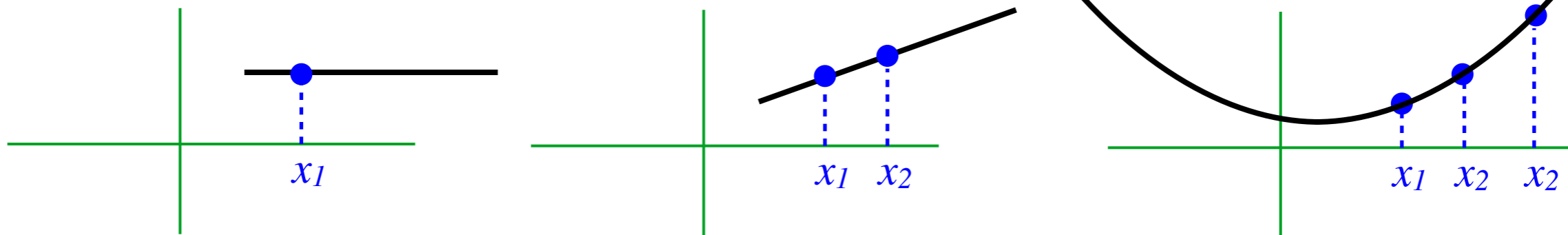
If $f(x)$ happens to be a polynomial of degree $N-1$ or less, then $L(f(x))=f(x)$ may be exact everywhere.

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However, only *select* values ε_k work for eigen-forms $\mathbf{M}\mathbf{P}_k = \varepsilon_k \mathbf{P}_k$ or orthonormality $\mathbf{P}_j \mathbf{P}_k = \delta_{jk} \mathbf{P}_k$.

2D harmonic oscillator equations

Lagrangian and matrix forms and Reciprocity symmetry

2D harmonic oscillator equation eigensolutions

Geometric method

Matrix-algebraic eigensolutions with example $M = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

Secular equation

Hamilton-Cayley equation and projectors

Idempotent projectors (how eigenvalues \Rightarrow eigenvectors)

Operator orthonormality and Completeness (Idempotent means: $\mathbf{P} \cdot \mathbf{P} = \mathbf{P}$)

Spectral Decompositions

Functional spectral decomposition

Orthonormality vs. Completeness vis-a`-vis Operator vs. State

Lagrange functional interpolation formula

➔ *Diagonalizing Transformations (D-Ttran) from projectors* **←**

2D-HO eigensolution example with bilateral (B-Type) symmetry

Mixed mode beat dynamics and fixed $\pi/2$ phase

2D-HO eigensolution example with asymmetric (A-Type) symmetry

Initial state projection, mixed mode beat dynamics with variable phase

Diagonalizing Transformations (D-Ttran) from projectors

Given our eigenvectors and their Projectors.

$$\mathbf{P}_1 = \frac{(\mathbf{M} - 5 \cdot \mathbf{1})}{(1 - 5)} = \frac{1}{4} \begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1 \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}{k_1} = |\epsilon_1\rangle\langle\epsilon_1|$$
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Load distinct bras $\langle\epsilon_1|$ and $\langle\epsilon_2|$ into d-tran **rows**, kets $|\epsilon_1\rangle$ and $|\epsilon_2\rangle$ into inverse d-tran **columns**.

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$$\begin{pmatrix} \langle\epsilon_1|x\rangle & \langle\epsilon_1|y\rangle \\ \langle\epsilon_2|x\rangle & \langle\epsilon_2|y\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}, \quad \begin{pmatrix} \langle x|\epsilon_1\rangle & \langle x|\epsilon_2\rangle \\ \langle y|\epsilon_1\rangle & \langle y|\epsilon_2\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

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Use *Dirac labeling for all components* so transformation is OK

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Check inverse-d-tran is really inverse of *your* d-tran.

$$\begin{pmatrix} \langle\varepsilon_1|1\rangle & \langle\varepsilon_1|2\rangle \\ \langle\varepsilon_2|1\rangle & \langle\varepsilon_2|2\rangle \end{pmatrix} \cdot \begin{pmatrix} \langle 1|\varepsilon_1\rangle & \langle 1|\varepsilon_2\rangle \\ \langle 2|\varepsilon_1\rangle & \langle 2|\varepsilon_2\rangle \end{pmatrix} = \begin{pmatrix} \langle\varepsilon_1|1|\varepsilon_1\rangle & \langle\varepsilon_1|1|\varepsilon_2\rangle \\ \langle\varepsilon_2|1|\varepsilon_1\rangle & \langle\varepsilon_2|1|\varepsilon_2\rangle \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Diagonalizing Transformations (D-Tran) from projectors

Given our eigenvectors and their Projectors.

$$\mathbf{P}_1 = \frac{(\mathbf{M} - 5 \cdot \mathbf{1})}{(1-5)} = \frac{1}{4} \begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1 \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}{k_1} = |\epsilon_1\rangle\langle\epsilon_1|$$

$$\mathbf{P}_2 = \frac{(\mathbf{M} - 1 \cdot \mathbf{1})}{(5-1)} = \frac{1}{4} \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = k_2 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}}{k_2} = |\epsilon_2\rangle\langle\epsilon_2|$$

Load distinct bras $\langle\epsilon_1|$ and $\langle\epsilon_2|$ into d-tran **rows**, kets $|\epsilon_1\rangle$ and $|\epsilon_2\rangle$ into inverse d-tran **columns**.

$$\left\{ \langle\epsilon_1| = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \langle\epsilon_2| = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix} \right\}, \quad \left\{ |\epsilon_1\rangle = \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix}, |\epsilon_2\rangle = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right\}$$

$(\epsilon_1, \epsilon_2) \leftarrow (1, 2)$ *d-Tran matrix*

$(1, 2) \leftarrow (\epsilon_1, \epsilon_2)$ **INVERSE** *d-Tran matrix*

$$\begin{pmatrix} \langle\epsilon_1|x\rangle & \langle\epsilon_1|y\rangle \\ \langle\epsilon_2|x\rangle & \langle\epsilon_2|y\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}, \quad \begin{pmatrix} \langle x|\epsilon_1\rangle & \langle x|\epsilon_2\rangle \\ \langle y|\epsilon_1\rangle & \langle y|\epsilon_2\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

Use *Dirac labeling for all components* so transformation is OK

$$\begin{pmatrix} \langle\epsilon_1|x\rangle & \langle\epsilon_1|y\rangle \\ \langle\epsilon_2|x\rangle & \langle\epsilon_2|y\rangle \end{pmatrix} \cdot \begin{pmatrix} \langle x|\mathbf{K}|x\rangle & \langle x|\mathbf{K}|y\rangle \\ \langle y|\mathbf{K}|x\rangle & \langle y|\mathbf{K}|y\rangle \end{pmatrix} \cdot \begin{pmatrix} \langle x|\epsilon_1\rangle & \langle x|\epsilon_2\rangle \\ \langle y|\epsilon_1\rangle & \langle y|\epsilon_2\rangle \end{pmatrix} = \begin{pmatrix} \langle\epsilon_1|\mathbf{K}|\epsilon_1\rangle & \langle\epsilon_1|\mathbf{K}|\epsilon_2\rangle \\ \langle\epsilon_2|\mathbf{K}|\epsilon_1\rangle & \langle\epsilon_2|\mathbf{K}|\epsilon_2\rangle \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$$

Check inverse-d-tran is really inverse of *your* d-tran. In standard quantum matrices inverses are “easy”

$$\begin{pmatrix} \langle\epsilon_1|x\rangle & \langle\epsilon_1|y\rangle \\ \langle\epsilon_2|x\rangle & \langle\epsilon_2|y\rangle \end{pmatrix} \cdot \begin{pmatrix} \langle x|\epsilon_1\rangle & \langle x|\epsilon_2\rangle \\ \langle y|\epsilon_1\rangle & \langle y|\epsilon_2\rangle \end{pmatrix} = \begin{pmatrix} \langle\epsilon_1|\mathbf{1}|\epsilon_1\rangle & \langle\epsilon_1|\mathbf{1}|\epsilon_2\rangle \\ \langle\epsilon_2|\mathbf{1}|\epsilon_1\rangle & \langle\epsilon_2|\mathbf{1}|\epsilon_2\rangle \end{pmatrix} \begin{pmatrix} \langle\epsilon_1|x\rangle & \langle\epsilon_1|y\rangle \\ \langle\epsilon_2|x\rangle & \langle\epsilon_2|y\rangle \end{pmatrix} = \begin{pmatrix} \langle x|\epsilon_1\rangle & \langle x|\epsilon_2\rangle \\ \langle y|\epsilon_1\rangle & \langle y|\epsilon_2\rangle \end{pmatrix}^\dagger = \begin{pmatrix} \langle x|\epsilon_1\rangle^* & \langle y|\epsilon_1\rangle^* \\ \langle x|\epsilon_2\rangle^* & \langle y|\epsilon_2\rangle^* \end{pmatrix} = \begin{pmatrix} \langle x|\epsilon_1\rangle & \langle x|\epsilon_2\rangle \\ \langle y|\epsilon_1\rangle & \langle y|\epsilon_2\rangle \end{pmatrix}^{-1}$$

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2D harmonic oscillator equations

Lagrangian and matrix forms and Reciprocity symmetry

2D harmonic oscillator equation eigensolutions

Geometric method

Matrix-algebraic eigensolutions with example $M = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

Secular equation

Hamilton-Cayley equation and projectors

Idempotent projectors (how eigenvalues \Rightarrow eigenvectors)

Operator orthonormality and Completeness (Idempotent means: $\mathbf{P} \cdot \mathbf{P} = \mathbf{P}$)

Spectral Decompositions

Functional spectral decomposition

Orthonormality vs. Completeness vis-a`-vis Operator vs. State

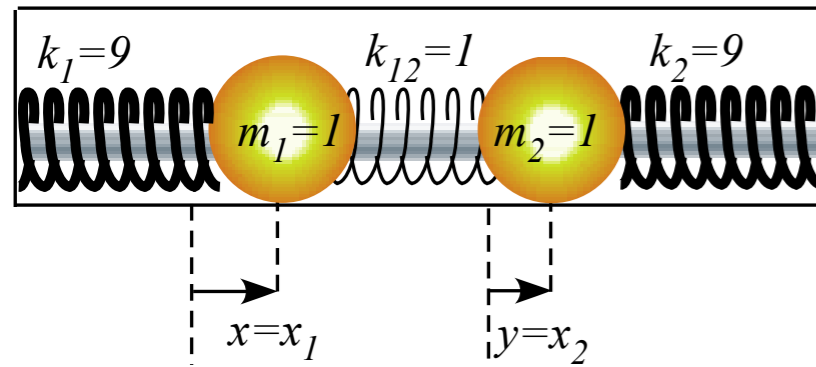
Lagrange functional interpolation formula

Diagonalizing Transformations (D-Ttran) from projectors

→ *2D-HO eigensolution example with bilateral (B-Type) symmetry* **←**
Mixed mode beat dynamics and fixed $\pi/2$ phase

2D-HO eigensolution example with asymmetric (A-Type) symmetry
Initial state projection, mixed mode beat dynamics with variable phase

Analyzing 2D-HO beats and mixed mode eigen-solutions



$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} = \begin{pmatrix} 10 & -1 \\ -1 & 10 \end{pmatrix}$$

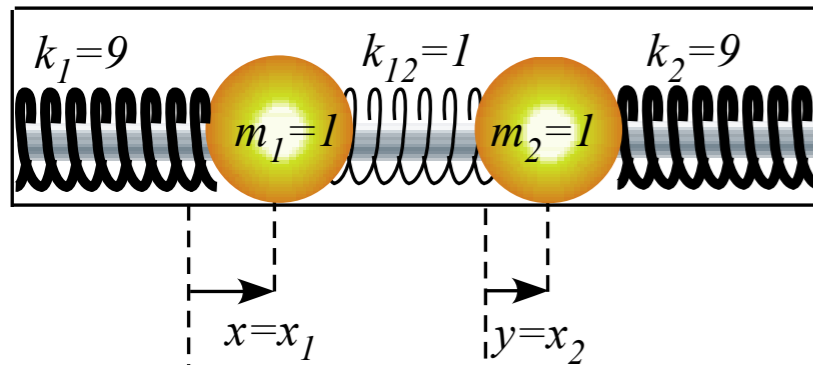
$\text{Det}(\mathbf{K}) = 10 \cdot 10 - 1 = 99$
 $\text{Trace}(\mathbf{K}) = 10 + 10 = 20$

The \mathbf{K} secular equation $K^2 - \text{Trace}(\mathbf{K})K + \text{Det}(\mathbf{K}) = K^2 - 20K + 99 = 0 = (K - 9)(K - 11) = (K - K_1)(K - K_2)$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\epsilon_k)^2$

$$K_1 = \omega_0^2(\epsilon_1) = 9, \quad K_2 = \omega_0^2(\epsilon_2) = 11,$$

Analyzing 2D-HO beats and mixed mode eigen-solutions



$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} = \begin{pmatrix} 10 & -1 \\ -1 & 10 \end{pmatrix}$$

$\text{Det}(\mathbf{K}) = 10 \cdot 10 - 1 = 99$
 $\text{Trace}(\mathbf{K}) = 10 + 10 = 20$

The \mathbf{K} secular equation $K^2 - \text{Trace}(\mathbf{K})K + \text{Det}(\mathbf{K}) = K^2 - 20K + 99 = 0 = (K - 9)(K - 11) = (K - K_1)(K - K_2)$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\epsilon_k)^2$

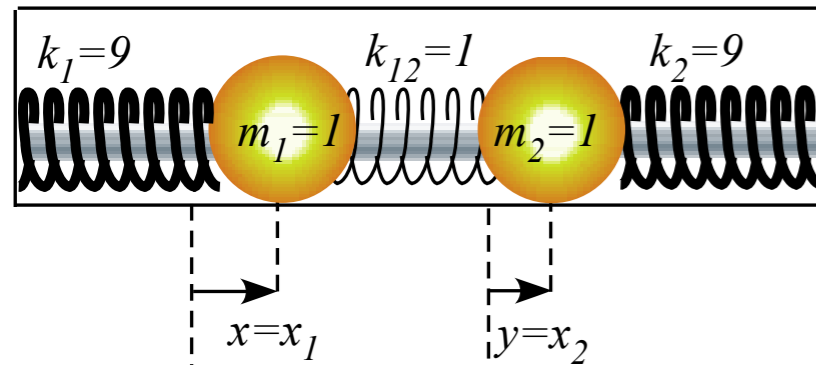
$$K_1 = \omega_0^2(\epsilon_1) = 9, \quad K_2 = \omega_0^2(\epsilon_2) = 11,$$

Eigen-projectors \mathbf{P}_k

$$\mathbf{P}_1 = \frac{\begin{pmatrix} K_{11} - K_2 & K_{12} \\ K_{12} & K_{22} - K_2 \end{pmatrix}}{K_1 - K_2} = \frac{\begin{pmatrix} 10 - 11 & -1 \\ -1 & 10 - 11 \end{pmatrix}}{9 - 11} = \frac{\begin{pmatrix} 1 & +1 \\ +1 & 1 \end{pmatrix}}{2}$$

$$\mathbf{P}_2 = \frac{\begin{pmatrix} K_{11} - K_1 & K_{12} \\ K_{12} & K_{22} - K_1 \end{pmatrix}}{K_2 - K_1} = \frac{\begin{pmatrix} 10 - 9 & -1 \\ -1 & 10 - 9 \end{pmatrix}}{11 - 9} = \frac{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}{2}$$

Analyzing 2D-HO beats and mixed mode eigen-solutions



$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} = \begin{pmatrix} 10 & -1 \\ -1 & 10 \end{pmatrix}$$

$$\text{Det}(\mathbf{K}) = 10 \cdot 10 - 1 = 99$$

$$\text{Trace}(\mathbf{K}) = 10 + 10 = 20$$

The \mathbf{K} secular equation $K^2 - \text{Trace}(\mathbf{K})K + \text{Det}(\mathbf{K}) = K^2 - 20K + 99 = 0 = (K - 9)(K - 11) = (K - K_1)(K - K_2)$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\epsilon_k)^2$

$$K_1 = \omega_0^2(\epsilon_1) = 9, \quad K_2 = \omega_0^2(\epsilon_2) = 11,$$

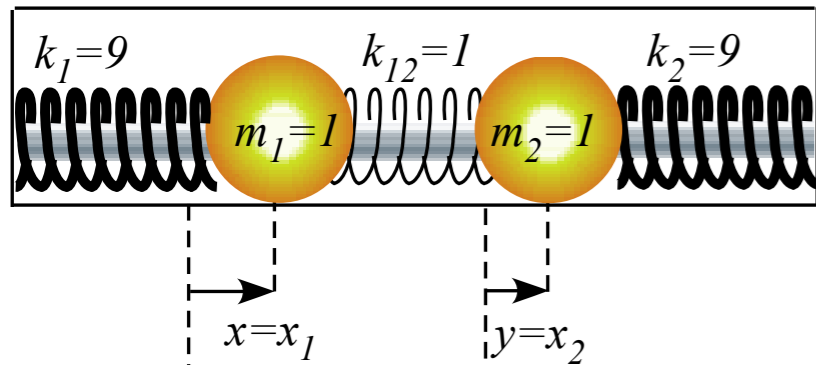
Eigen-projectors \mathbf{P}_k

$$\begin{aligned} \mathbf{P}_1 &= \frac{\begin{pmatrix} K_{11} - K_2 & K_{12} \\ K_{12} & K_{22} - K_2 \end{pmatrix}}{K_1 - K_2} = \frac{\begin{pmatrix} 10 - 11 & -1 \\ -1 & 10 - 11 \end{pmatrix}}{9 - 11} = \frac{\begin{pmatrix} 1 & +1 \\ +1 & 1 \end{pmatrix}}{2} \\ &= \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = |\epsilon_1\rangle\langle\epsilon_1| \end{aligned}$$

$$\begin{aligned} \mathbf{P}_2 &= \frac{\begin{pmatrix} K_{11} - K_1 & K_{12} \\ K_{12} & K_{22} - K_1 \end{pmatrix}}{K_2 - K_1} = \frac{\begin{pmatrix} 10 - 9 & -1 \\ -1 & 10 - 9 \end{pmatrix}}{11 - 9} = \frac{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}{2} \\ &= \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} = |\epsilon_2\rangle\langle\epsilon_2| \end{aligned}$$

Eigenbra vectors: $\langle\epsilon_1| = \left(1/\sqrt{2} \quad +1/\sqrt{2} \right), \quad \langle\epsilon_2| = \left(1/\sqrt{2} \quad -1/\sqrt{2} \right)$

Analyzing 2D-HO beats and mixed mode eigen-solutions



$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} = \begin{pmatrix} 10 & -1 \\ -1 & 10 \end{pmatrix}$$

$$\text{Det}(\mathbf{K}) = 10 \cdot 10 - 1 = 99$$

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The \mathbf{K} secular equation $K^2 - \text{Trace}(\mathbf{K})K + \text{Det}(\mathbf{K}) = K^2 - 20K + 99 = 0 = (K - 9)(K - 11) = (K - K_1)(K - K_2)$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\epsilon_k)^2$

$$K_1 = \omega_0^2(\epsilon_1) = 9, \quad K_2 = \omega_0^2(\epsilon_2) = 11,$$

Eigen-projectors \mathbf{P}_k

$$\mathbf{P}_1 = \frac{\begin{pmatrix} K_{11} - K_2 & K_{12} \\ K_{12} & K_{22} - K_2 \end{pmatrix}}{K_1 - K_2} = \frac{\begin{pmatrix} 10 - 11 & -1 \\ -1 & 10 - 11 \end{pmatrix}}{9 - 11} = \frac{\begin{pmatrix} 1 & +1 \\ +1 & 1 \end{pmatrix}}{2}$$

$$= \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = |\epsilon_1\rangle\langle\epsilon_1|$$

$$\mathbf{P}_2 = \frac{\begin{pmatrix} K_{11} - K_1 & K_{12} \\ K_{12} & K_{22} - K_1 \end{pmatrix}}{K_2 - K_1} = \frac{\begin{pmatrix} 10 - 9 & -1 \\ -1 & 10 - 9 \end{pmatrix}}{11 - 9} = \frac{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}{2}$$

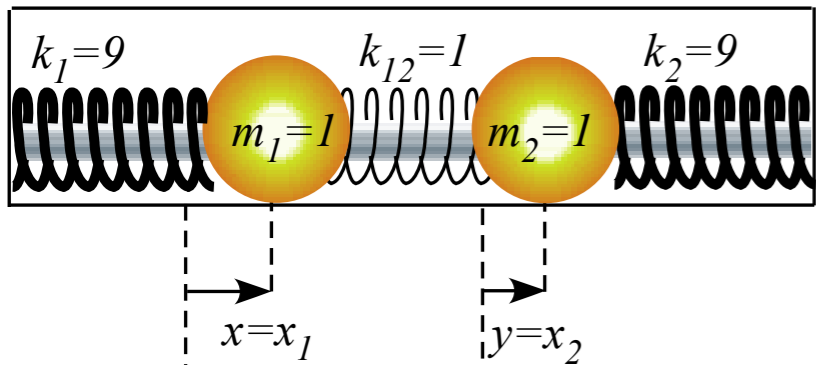
$$= \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} = |\epsilon_2\rangle\langle\epsilon_2|$$

Eigenbra vectors: $\langle\epsilon_1| = \left(1/\sqrt{2} \quad +1/\sqrt{2} \right)$, $\langle\epsilon_2| = \left(1/\sqrt{2} \quad -1/\sqrt{2} \right)$

Mixed mode dynamics

$$|x(t)\rangle = |\epsilon_1\rangle \langle\epsilon_1|x(0)\rangle e^{-i\omega_1 t} + |\epsilon_2\rangle \langle\epsilon_2|x(0)\rangle e^{-i\omega_2 t}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \langle\epsilon_1|x(0)\rangle e^{-i\omega_1 t} + \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \langle\epsilon_2|x(0)\rangle e^{-i\omega_2 t}$$



Analyzing 2D-HO beats and mixed mode eigen-solutions

$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} = \begin{pmatrix} 10 & -1 \\ -1 & 10 \end{pmatrix}$$

$$\text{Det}(\mathbf{K}) = 10 \cdot 10 - 1 = 99$$

$$\text{Trace}(\mathbf{K}) = 10 + 10 = 20$$

The \mathbf{K} secular equation $K^2 - \text{Trace}(\mathbf{K})K + \text{Det}(\mathbf{K}) = K^2 - 20K + 99 = 0 = (K - 9)(K - 11) = (K - K_1)(K - K_2)$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\epsilon_k)^2$

$$K_1 = \omega_0^2(\epsilon_1) = 9, \quad K_2 = \omega_0^2(\epsilon_2) = 11,$$

Eigen-projectors \mathbf{P}_k

$$\mathbf{P}_1 = \frac{\begin{pmatrix} K_{11} - K_2 & K_{12} \\ K_{12} & K_{22} - K_2 \end{pmatrix}}{K_1 - K_2} = \frac{\begin{pmatrix} 10 - 11 & -1 \\ -1 & 10 - 11 \end{pmatrix}}{9 - 11} = \frac{\begin{pmatrix} 1 & +1 \\ +1 & 1 \end{pmatrix}}{2}$$

$$= \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = |\epsilon_1\rangle\langle\epsilon_1|$$

$$\mathbf{P}_2 = \frac{\begin{pmatrix} K_{11} - K_1 & K_{12} \\ K_{12} & K_{22} - K_1 \end{pmatrix}}{K_2 - K_1} = \frac{\begin{pmatrix} 10 - 9 & -1 \\ -1 & 10 - 9 \end{pmatrix}}{11 - 9} = \frac{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}{2}$$

$$= \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} = |\epsilon_2\rangle\langle\epsilon_2|$$

Eigenbra vectors: $\langle\epsilon_1| = \left(1/\sqrt{2} \quad +1/\sqrt{2} \right)$, $\langle\epsilon_2| = \left(1/\sqrt{2} \quad -1/\sqrt{2} \right)$

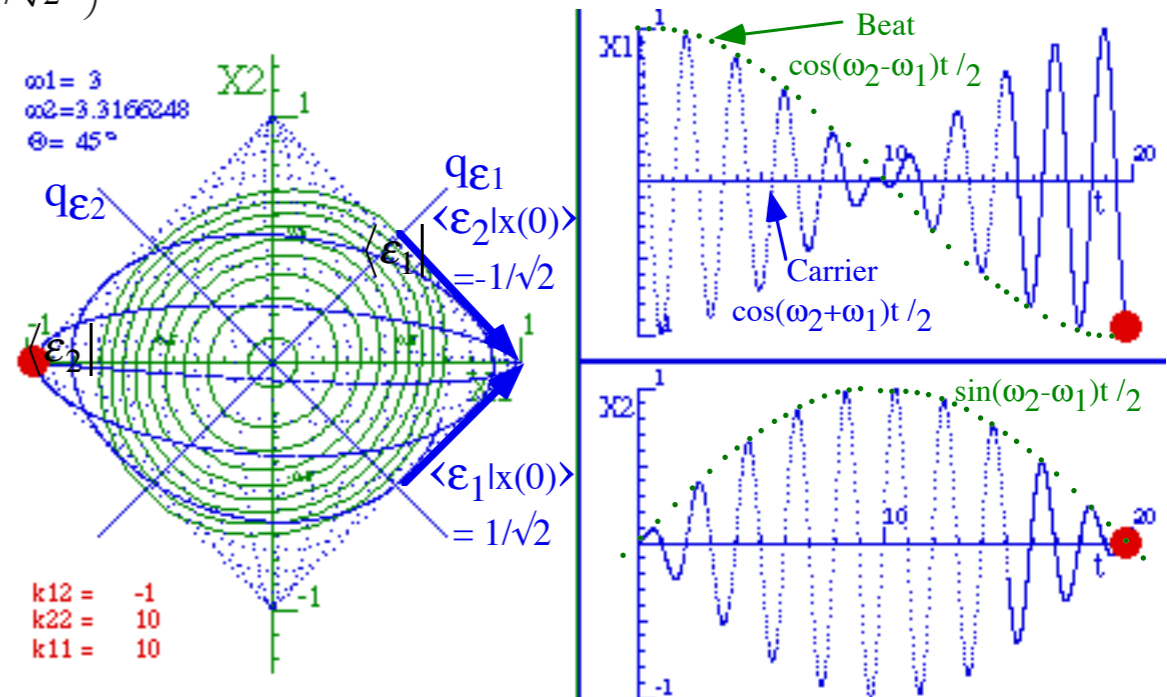
Mixed mode dynamics

$$|x(t)\rangle = |\epsilon_1\rangle \langle\epsilon_1|x(0)\rangle e^{-i\omega_1 t} + |\epsilon_2\rangle \langle\epsilon_2|x(0)\rangle e^{-i\omega_2 t}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \langle\epsilon_1|x(0)\rangle e^{-i\omega_1 t} + \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \langle\epsilon_2|x(0)\rangle e^{-i\omega_2 t}$$

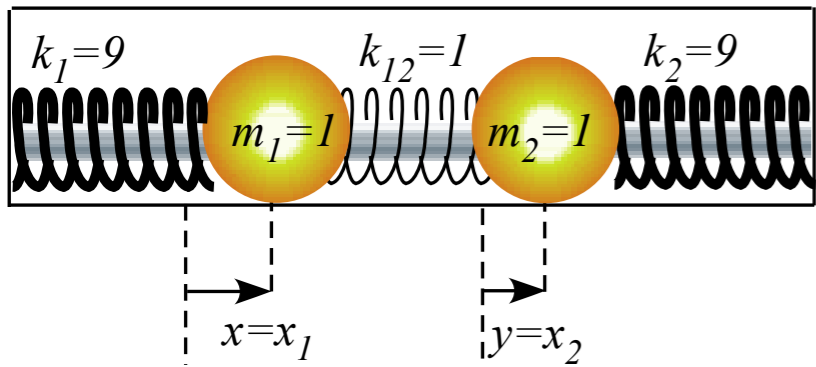
100% modulation (SWR=0) $\frac{e^{ia} + e^{ib}}{2} = e^{\frac{i(a+b)}{2}} \frac{e^{\frac{i(a-b)}{2}} + e^{-\frac{i(a-b)}{2}}}{2}$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} \frac{e^{-i\omega_1 t} + e^{-i\omega_2 t}}{2} \\ \frac{e^{-i\omega_1 t} - e^{-i\omega_2 t}}{2} \end{pmatrix} = \frac{e^{-\frac{i(\omega_1 + \omega_2)t}}{2}} \begin{pmatrix} e^{-\frac{i(\omega_1 - \omega_2)t}{2}} + e^{\frac{i(\omega_1 - \omega_2)t}{2}} \\ e^{-\frac{i(\omega_1 - \omega_2)t}{2}} - e^{\frac{i(\omega_1 - \omega_2)t}{2}} \end{pmatrix}$$



BoxIt (Beating) Web Simulation

Fig. 3.3.9 Beats in weakly coupled symmetric oscillators with equal mode magnitudes.



Analyzing 2D-HO beats and mixed mode eigen-solutions

$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} = \begin{pmatrix} 10 & -1 \\ -1 & 10 \end{pmatrix}$$

$$\text{Det}(\mathbf{K}) = 10 \cdot 10 - 1 = 99$$

$$\text{Trace}(\mathbf{K}) = 10 + 10 = 20$$

The \mathbf{K} secular equation $K^2 - \text{Trace}(\mathbf{K})K + \text{Det}(\mathbf{K}) = K^2 - 20K + 99 = 0 = (K - 9)(K - 11) = (K - K_1)(K - K_2)$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\epsilon_k)^2$

$$K_1 = \omega_0^2(\epsilon_1) = 9, \quad K_2 = \omega_0^2(\epsilon_2) = 11,$$

Eigen-projectors \mathbf{P}_k

$$\mathbf{P}_1 = \frac{\begin{pmatrix} K_{11} - K_2 & K_{12} \\ K_{12} & K_{22} - K_2 \end{pmatrix}}{K_1 - K_2} = \frac{\begin{pmatrix} 10 - 11 & -1 \\ -1 & 10 - 11 \end{pmatrix}}{9 - 11} = \frac{\begin{pmatrix} 1 & +1 \\ +1 & 1 \end{pmatrix}}{2}$$

$$= \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = |\epsilon_1\rangle\langle\epsilon_1|$$

$$\mathbf{P}_2 = \frac{\begin{pmatrix} K_{11} - K_1 & K_{12} \\ K_{12} & K_{22} - K_1 \end{pmatrix}}{K_2 - K_1} = \frac{\begin{pmatrix} 10 - 9 & -1 \\ -1 & 10 - 9 \end{pmatrix}}{11 - 9} = \frac{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}{2}$$

$$= \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} = |\epsilon_2\rangle\langle\epsilon_2|$$

Eigenbra vectors: $\langle\epsilon_1| = \left(1/\sqrt{2} \quad +1/\sqrt{2} \right)$, $\langle\epsilon_2| = \left(1/\sqrt{2} \quad -1/\sqrt{2} \right)$

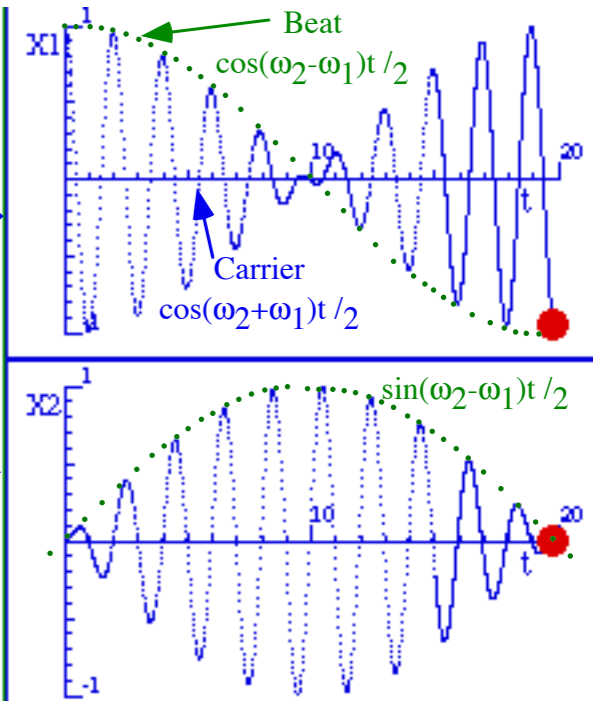
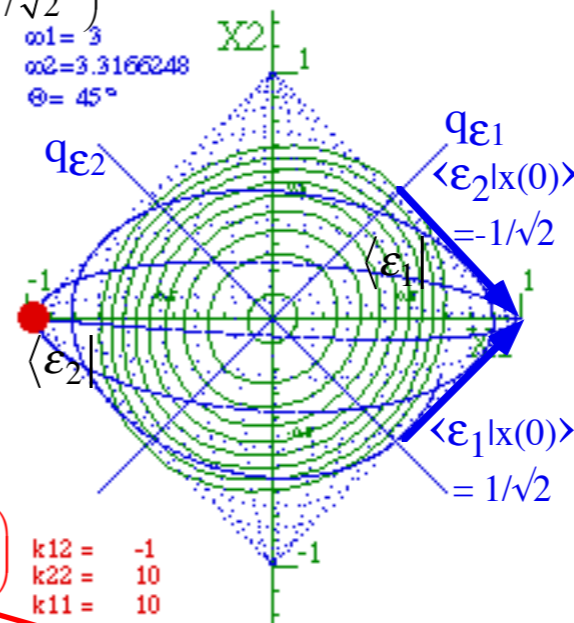
Mixed mode dynamics

$$|x(t)\rangle = |\epsilon_1\rangle \langle\epsilon_1|x(0)\rangle e^{-i\omega_1 t} + |\epsilon_2\rangle \langle\epsilon_2|x(0)\rangle e^{-i\omega_2 t}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \langle\epsilon_1|x(0)\rangle e^{-i\omega_1 t} + \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \langle\epsilon_2|x(0)\rangle e^{-i\omega_2 t}$$

100% modulation (SWR=0) $\frac{e^{ia} + e^{ib}}{2} = e^{i\frac{a+b}{2}} \frac{e^{i\frac{a-b}{2}} + e^{-i\frac{a-b}{2}}}{2} = e^{i\frac{a+b}{2}} \cos\left(\frac{a-b}{2}\right)$

$$\begin{matrix} k_{12} = & -1 \\ k_{22} = & 10 \\ k_{11} = & 10 \end{matrix}$$



BoxIt (Beating) Web Simulation

Note: Beat frequency is half-difference of eigenvalues

Note the i phase

Fig. 3.3.9 Beats in weakly coupled symmetric oscillators with equal mode magnitudes.

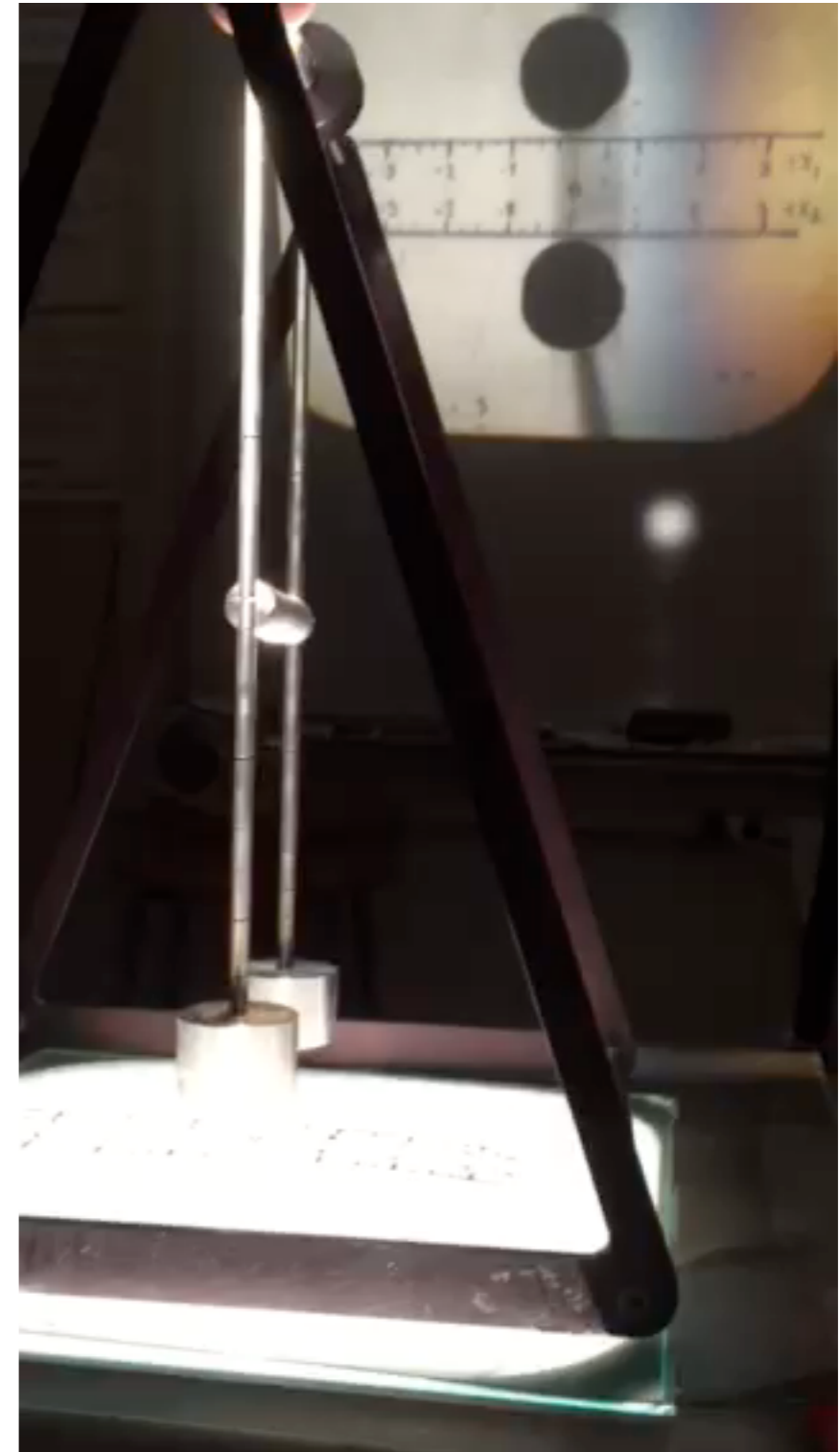
Videos of Coupled Pendula aided by Overhead Projector



[View on YouTube](#) 

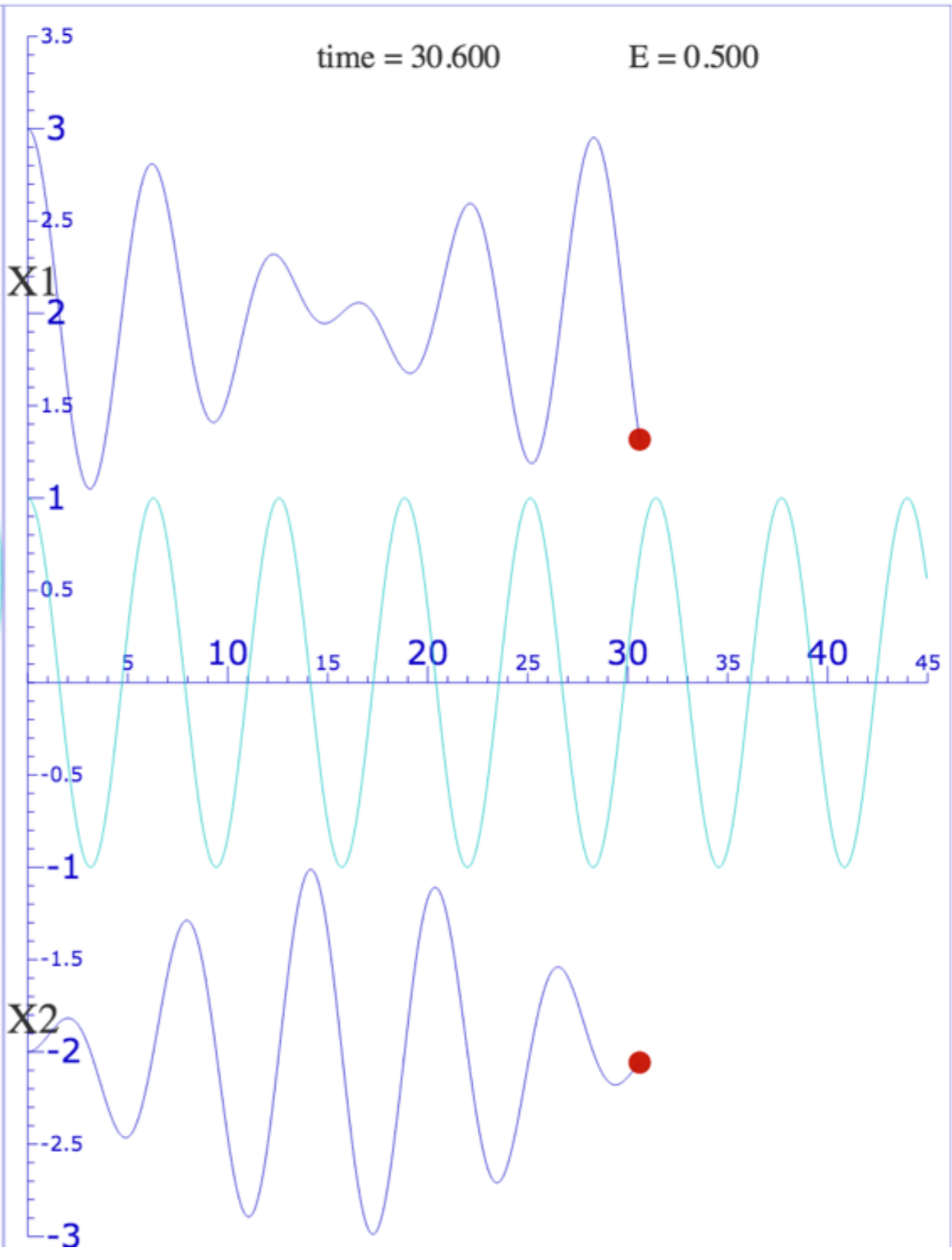
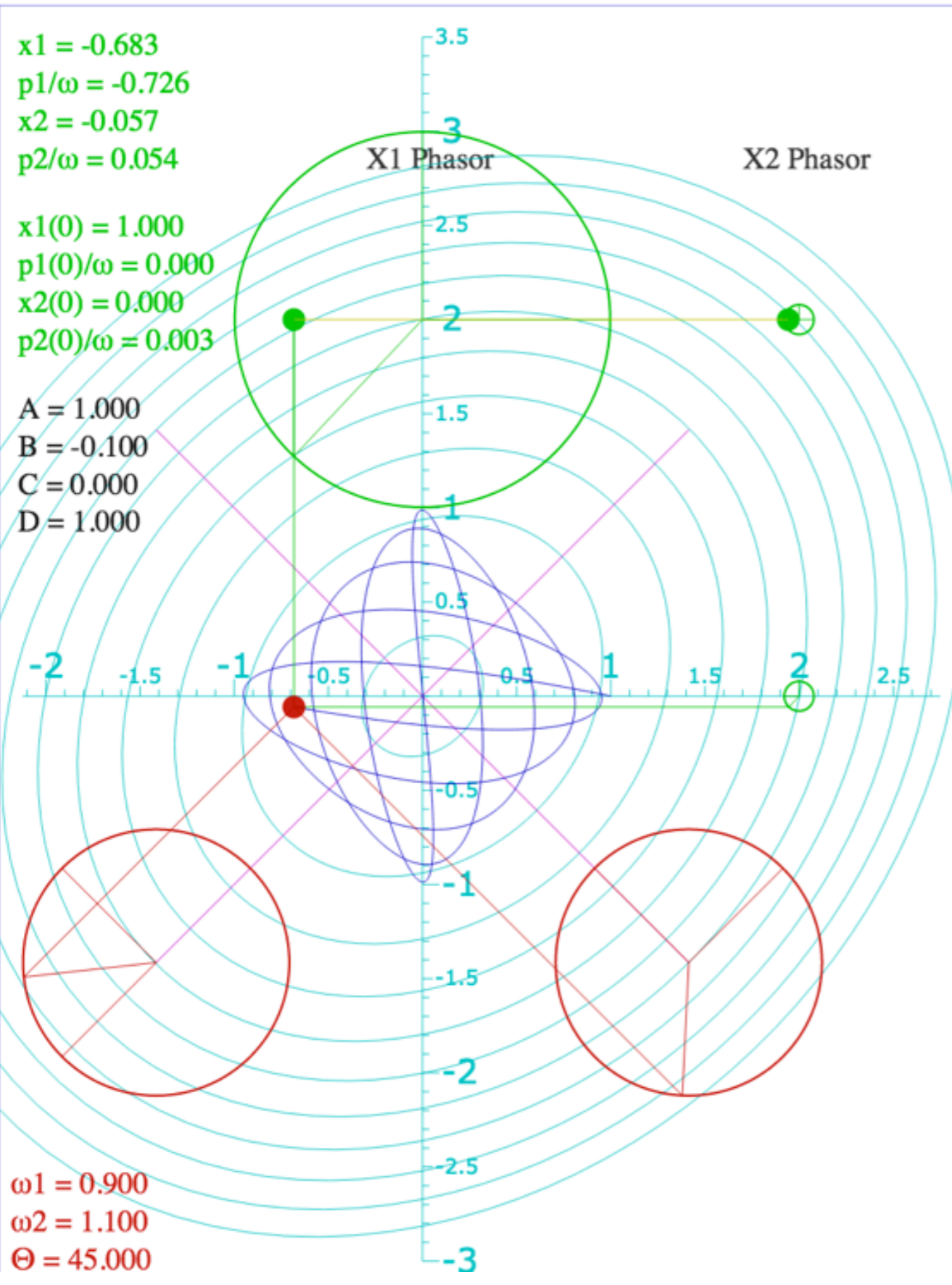
*Launch embedded videos
using your browser/App
or*

⇐ view on YouTube ⇒



[View on YouTube](#) 

Stronger coupling on the left, illustrated indirectly by a darker looking spring on screen



BoxIt (Beating) Web Simulation ($A=1, B=-0.1, C=0, D=1$)

Controls Resume Reset T=0 Erase Paths

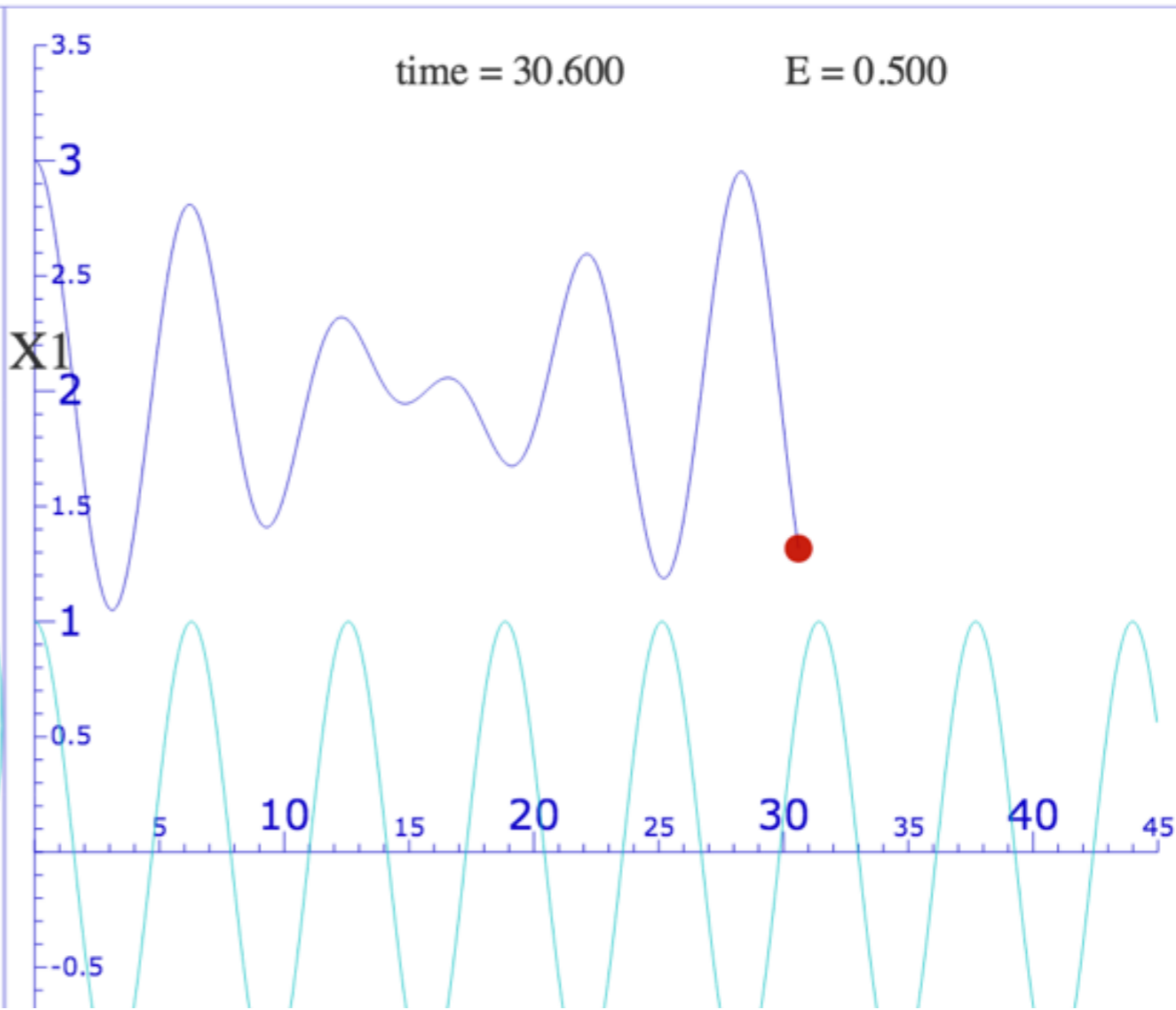
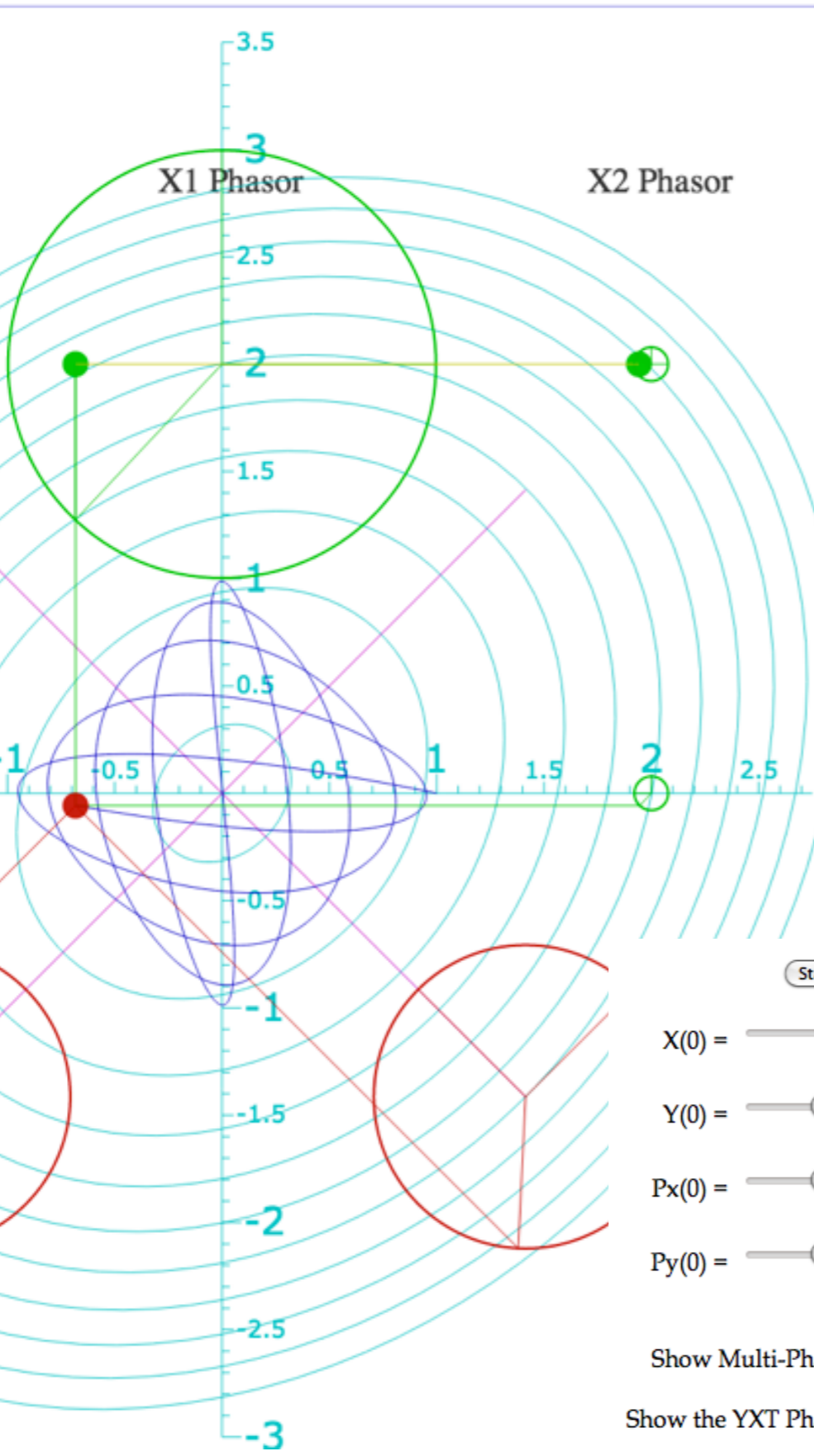
Speed x10^

x1 = -0.683
p1/ω = -0.726
x2 = -0.057
p2/ω = 0.054

x1(0) = 1.000
p1(0)/ω = 0.000
x2(0) = 0.000
p2(0)/ω = 0.003

A = 1.000
B = -0.100
C = 0.000
D = 1.000

ω1 = 0.900
ω2 = 1.100
Θ = 45.000



time = 30.600 E = 0.500

Start Resume Reset T=0 Erase Paths

Speed x10^

X(0) = A = Number of Derivatives =

Y(0) = B =

Px(0) = C =

Py(0) = D =

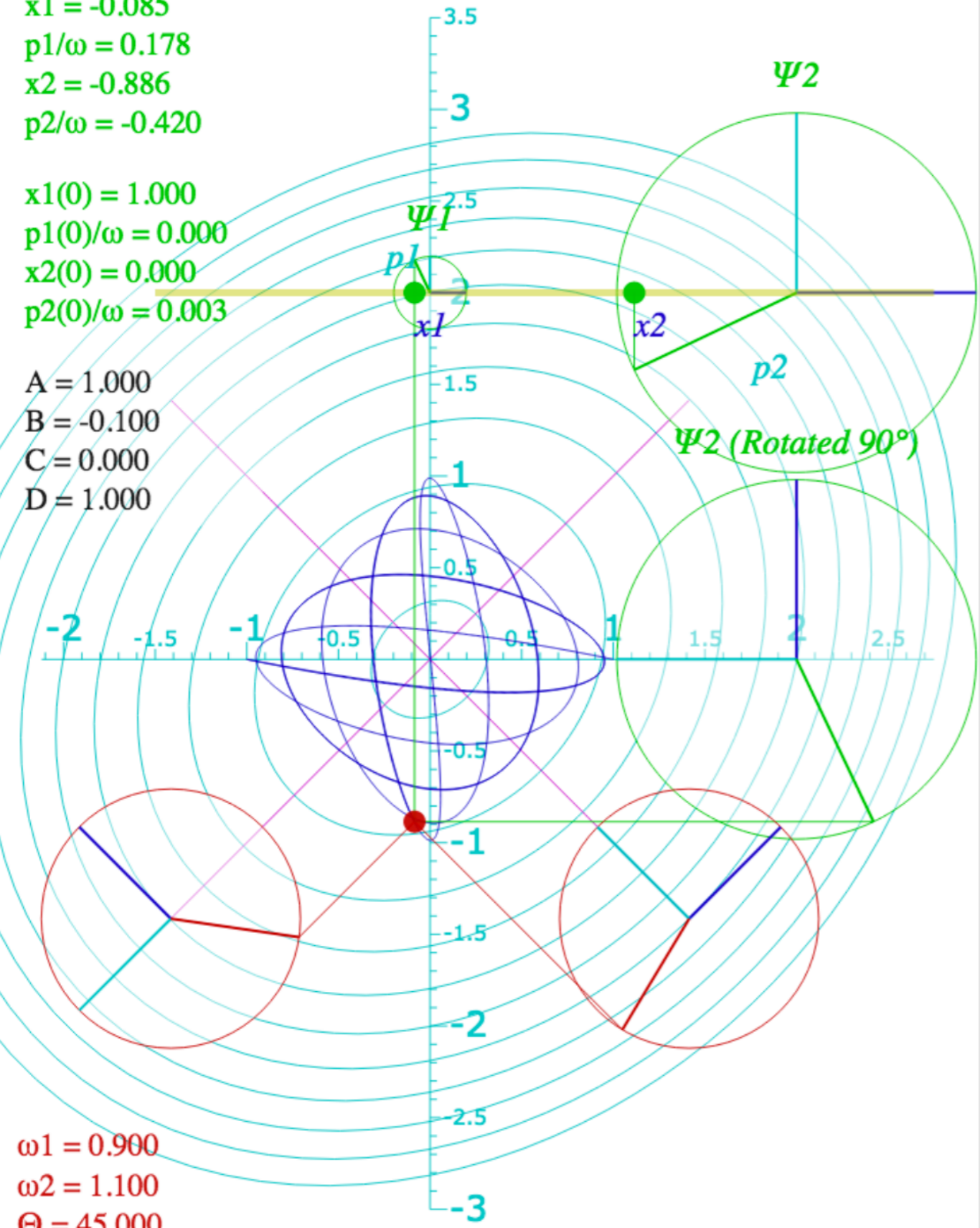
- Show Multi-Phasor View
- Show the YXT Phasor View
- Draw Main Phasors
- Draw Vector Heads
- wantVectorHeads, wantTimeRateTangents
- Draw PE Levels
- Draw Box Lines
- Draw Modal Phasors
- Draw Time Rate Tangents
- Left Phasor Rides on Right Phasor
- Left Phasor Rides on Right Phasor
- Normalize Phasors
- Print ω1:ω2 fractions

$x1 = -0.085$
 $p1/\omega = 0.178$
 $x2 = -0.886$
 $p2/\omega = -0.420$

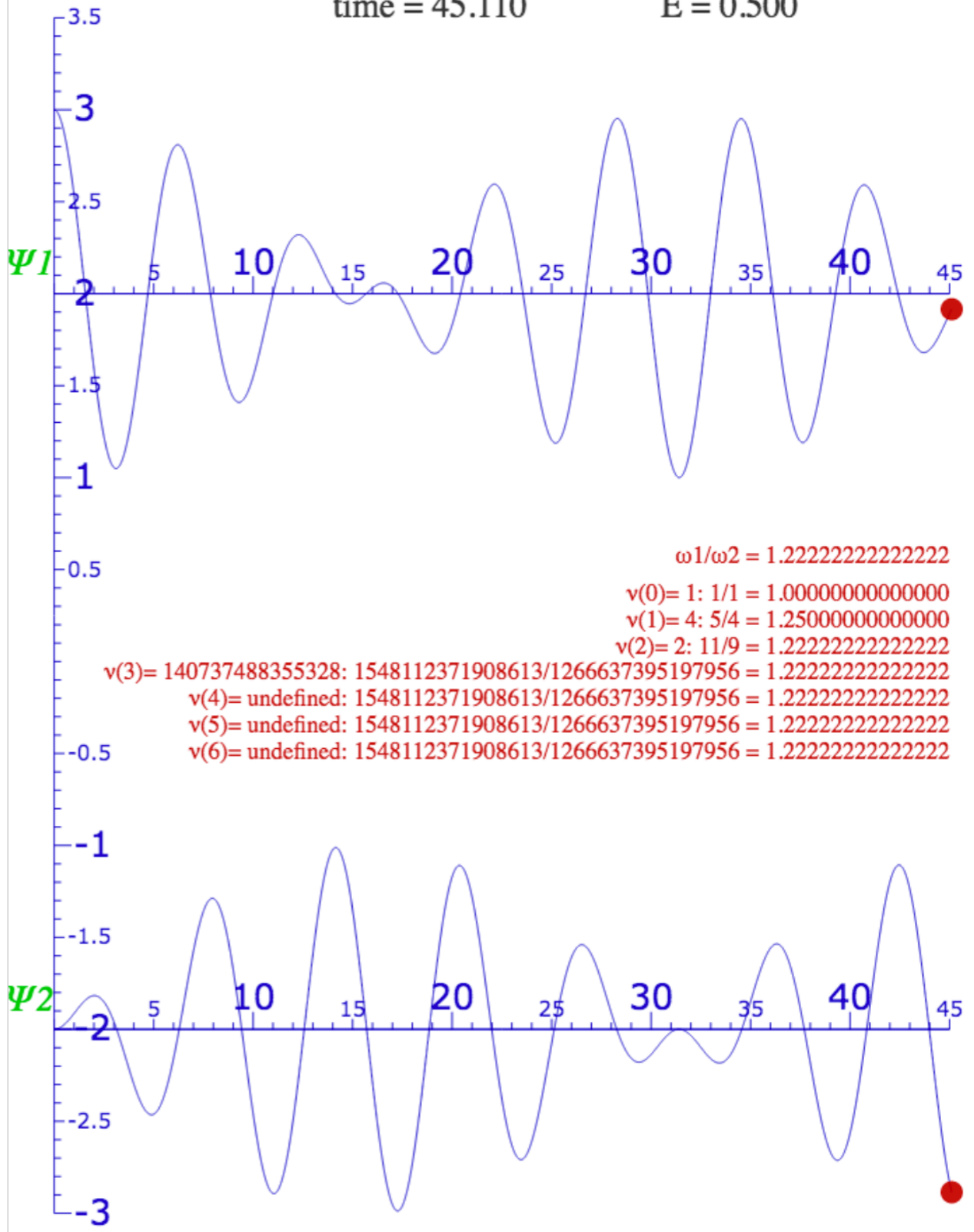
$x1(0) = 1.000$
 $p1(0)/\omega = 0.000$
 $x2(0) = 0.000$
 $p2(0)/\omega = 0.003$

$A = 1.000$
 $B = -0.100$
 $C = 0.000$
 $D = 1.000$

$\omega1 = 0.900$
 $\omega2 = 1.100$
 $\Theta = 45.000$



time = 45.110 E = 0.500



$\omega1/\omega2 = 1.22222222222222$
 $v(0) = 1: 1/1 = 1.00000000000000$
 $v(1) = 4: 5/4 = 1.25000000000000$
 $v(2) = 2: 11/9 = 1.22222222222222$
 $v(3) = 140737488355328: 1548112371908613/1266637395197956 = 1.22222222222222$
 $v(4) = \text{undefined}: 1548112371908613/1266637395197956 = 1.22222222222222$
 $v(5) = \text{undefined}: 1548112371908613/1266637395197956 = 1.22222222222222$
 $v(6) = \text{undefined}: 1548112371908613/1266637395197956 = 1.22222222222222$

BoxIt (Beating) Web Simulation ($A=1, B=-0.1, C=0, D=1$) with frequency ratios

Approximating decimal frequencies $\omega=\alpha$ using successive levels of *continued fractions*.

$$\alpha = n_0 + \frac{1}{n_1 + \frac{1}{n_2 + \frac{1}{n_3 + \frac{1}{n_4 + \ddots}}}}$$

Pi ($\pi=3.14159265\dots$) converges rather quickly by cf.

$$A_0 = \alpha = 3.14159265\dots$$

$$n_0 = INT(A_0) = 3$$

$$\pi \cong = 3.000\dots$$

$$A_1 = \frac{1}{A_0 - n_0} = 7.06\dots$$

$$n_1 = INT(A_1) = 7$$

$$\pi \cong 3 + \frac{1}{7} = \frac{22}{7} = 3.1428$$

$$A_2 = \frac{1}{A_1 - n_1} = 15.99\dots$$

$$n_2 = INT(A_2) = 15$$

$$\pi \cong 3 + \frac{1}{7 + \frac{1}{15}} = \frac{333}{106} = 3.141509$$

$$A_3 = \frac{1}{A_2 - n_2} = 1.003\dots$$

$$n_3 = INT(A_3) = 1$$

$$\pi \cong 3 + \frac{1}{7 + \frac{1}{15 + 1}} = \frac{355}{113} = 3.14159292$$

Not so much for the *Golden Mean* $G=(1+\sqrt{5})/2=1.618\dots$

$$A_0 = G = 1.618033989\dots$$

$$n_0 = INT(A_0) = 1$$

$$G \cong = 1.000\dots$$

$$A_1 = \frac{1}{A_0 - n_0} = 1.6180\dots$$

$$n_1 = INT(A_1) = 1$$

$$G \cong 1 + \frac{1}{1} = \frac{2}{1} = 2.000$$

$$A_2 = \frac{1}{A_1 - n_1} = 1.6180\dots$$

$$n_2 = INT(A_2) = 1$$

$$G \cong 1 + \frac{1}{1 + \frac{1}{1}} = \frac{3}{2} = 1.500$$

$$A_3 = \frac{1}{A_2 - n_2} = 1.6180\dots$$

$$n_3 = INT(A_3) = 1$$

$$G \cong 1 + \frac{1}{1 + \frac{1}{1 + 1}} = \frac{5}{3} = 1.666\dots$$

2D harmonic oscillator equations

Lagrangian and matrix forms and Reciprocity symmetry

2D harmonic oscillator equation eigensolutions

Geometric method

Matrix-algebraic eigensolutions with example $M = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

Secular equation

Hamilton-Cayley equation and projectors

Idempotent projectors (how eigenvalues \Rightarrow eigenvectors)

Operator orthonormality and Completeness (Idempotent means: $\mathbf{P} \cdot \mathbf{P} = \mathbf{P}$)

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Functional spectral decomposition

Orthonormality vs. Completeness vis-a`-vis Operator vs. State

Lagrange functional interpolation formula

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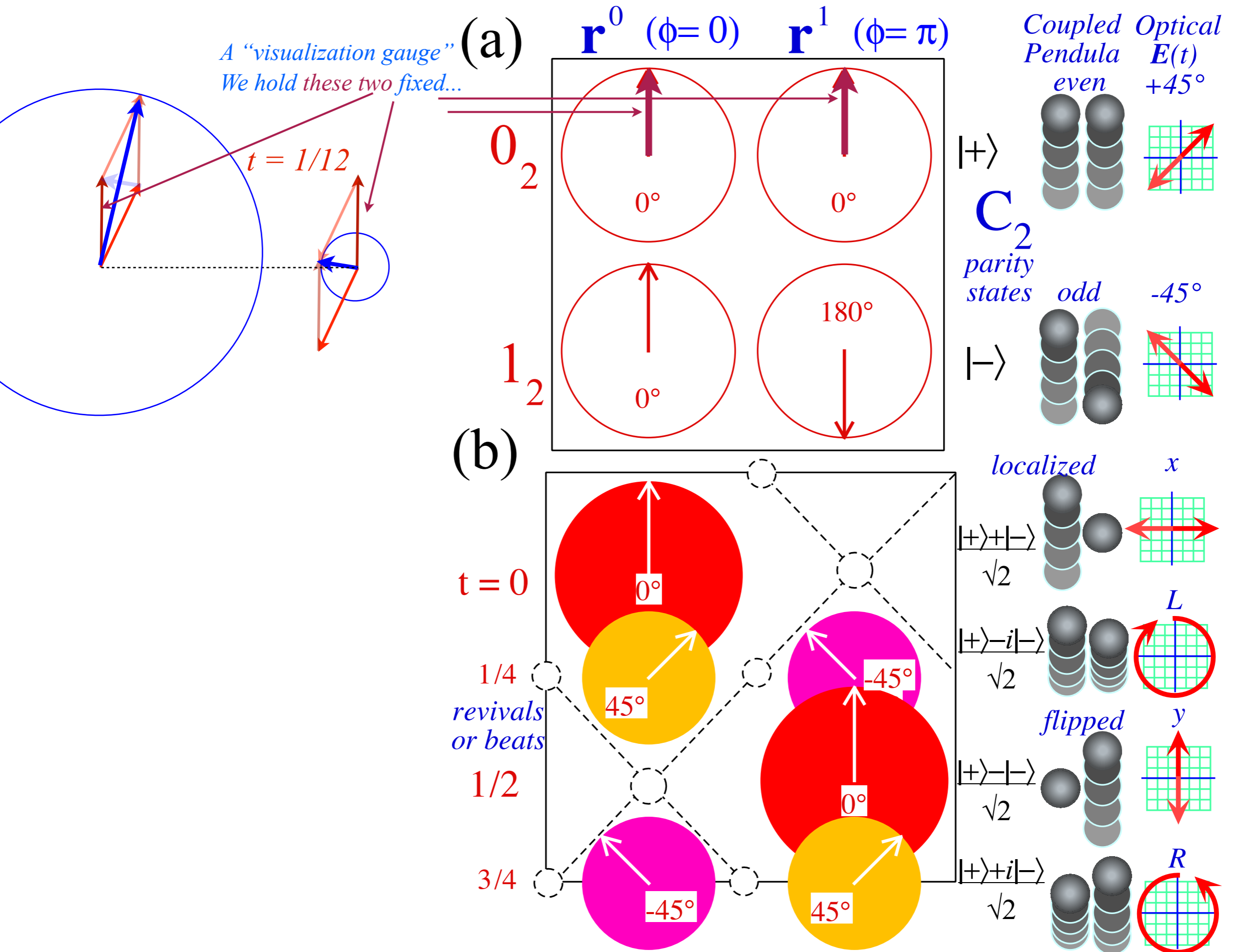
2D-HO eigensolution example with bilateral (B-Type) symmetry

➔ Mixed mode beat dynamics and fixed $\pi/2$ phase ←

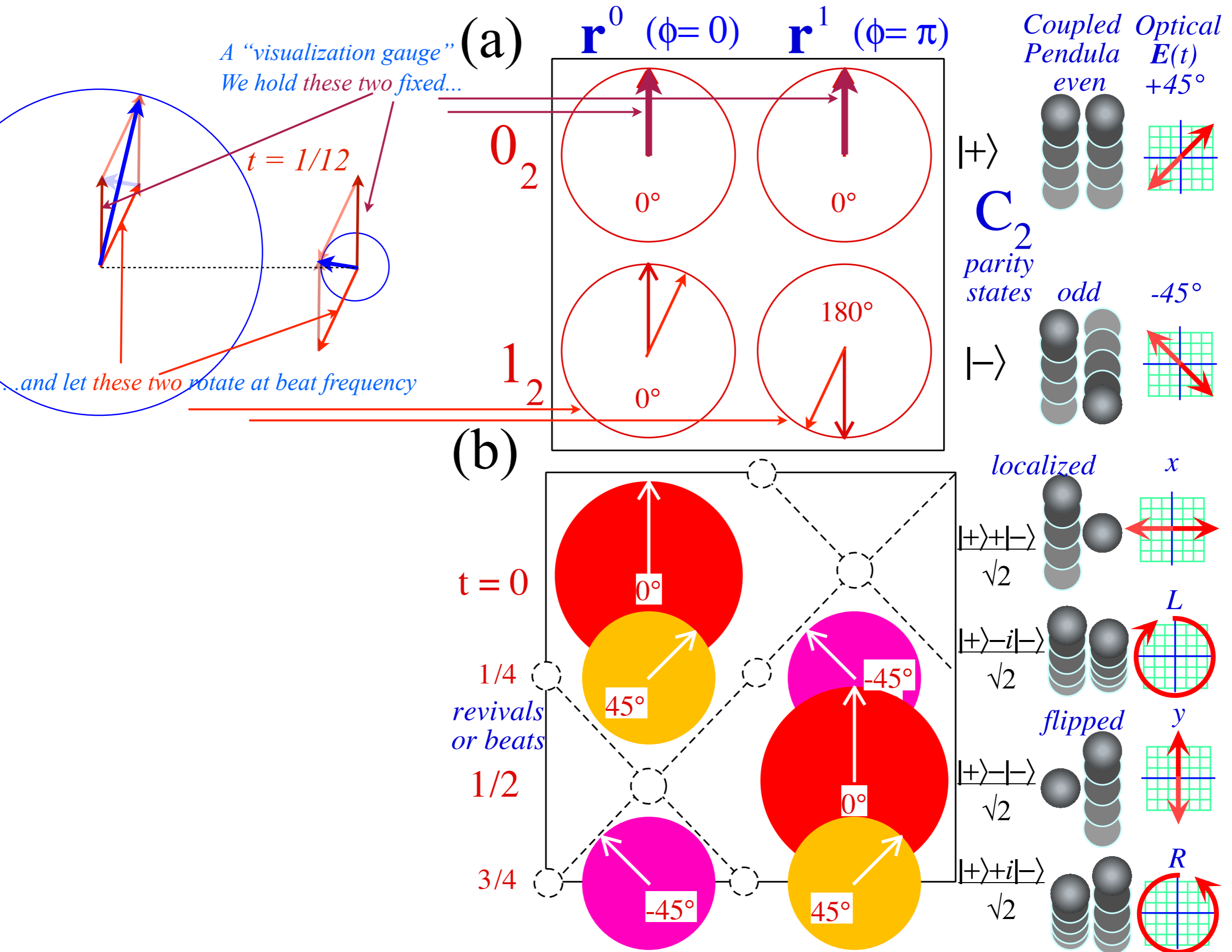
2D-HO eigensolution example with asymmetric (A-Type) symmetry

Initial state projection, mixed mode beat dynamics with variable phase

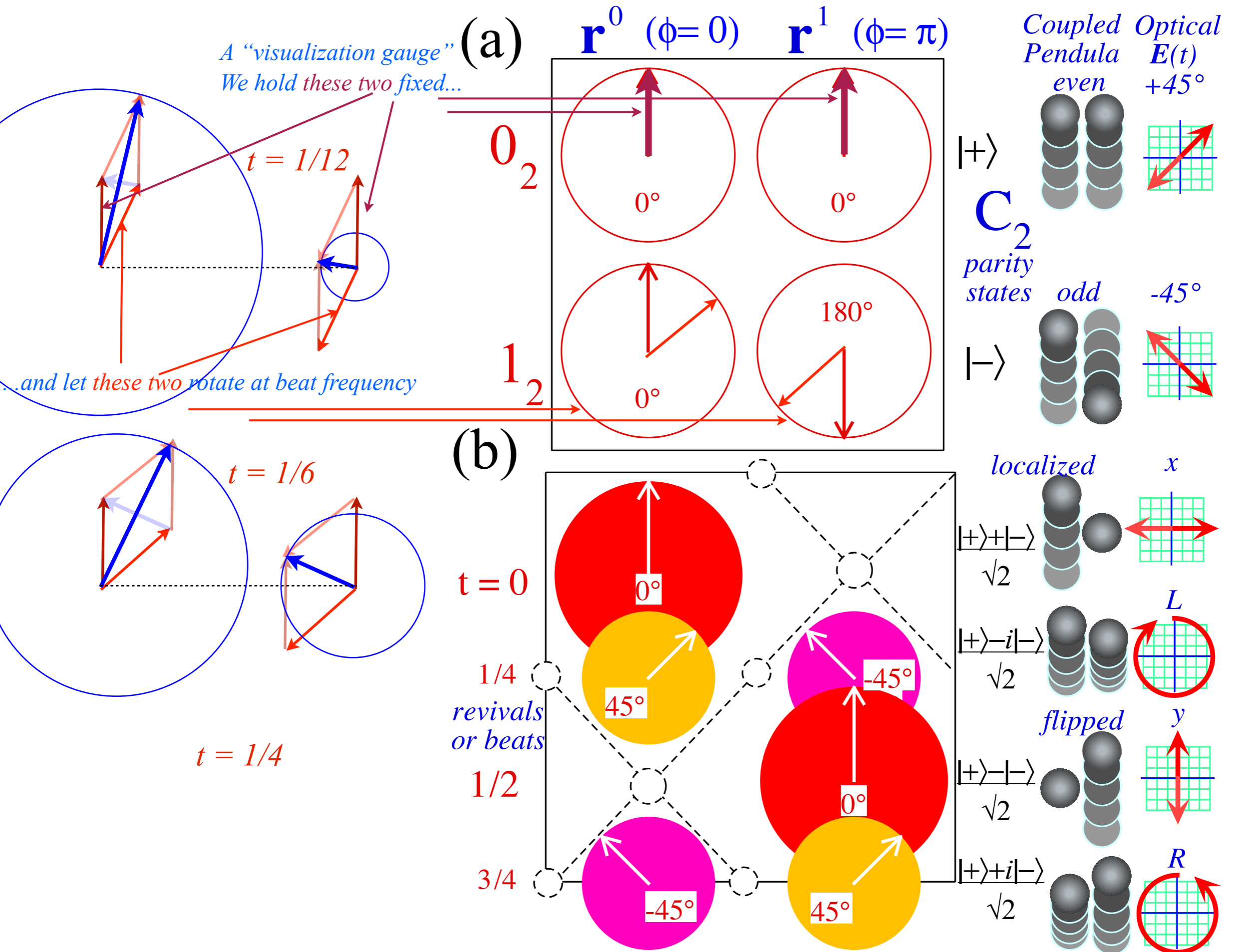
2D-HO beats and mixed mode geometry



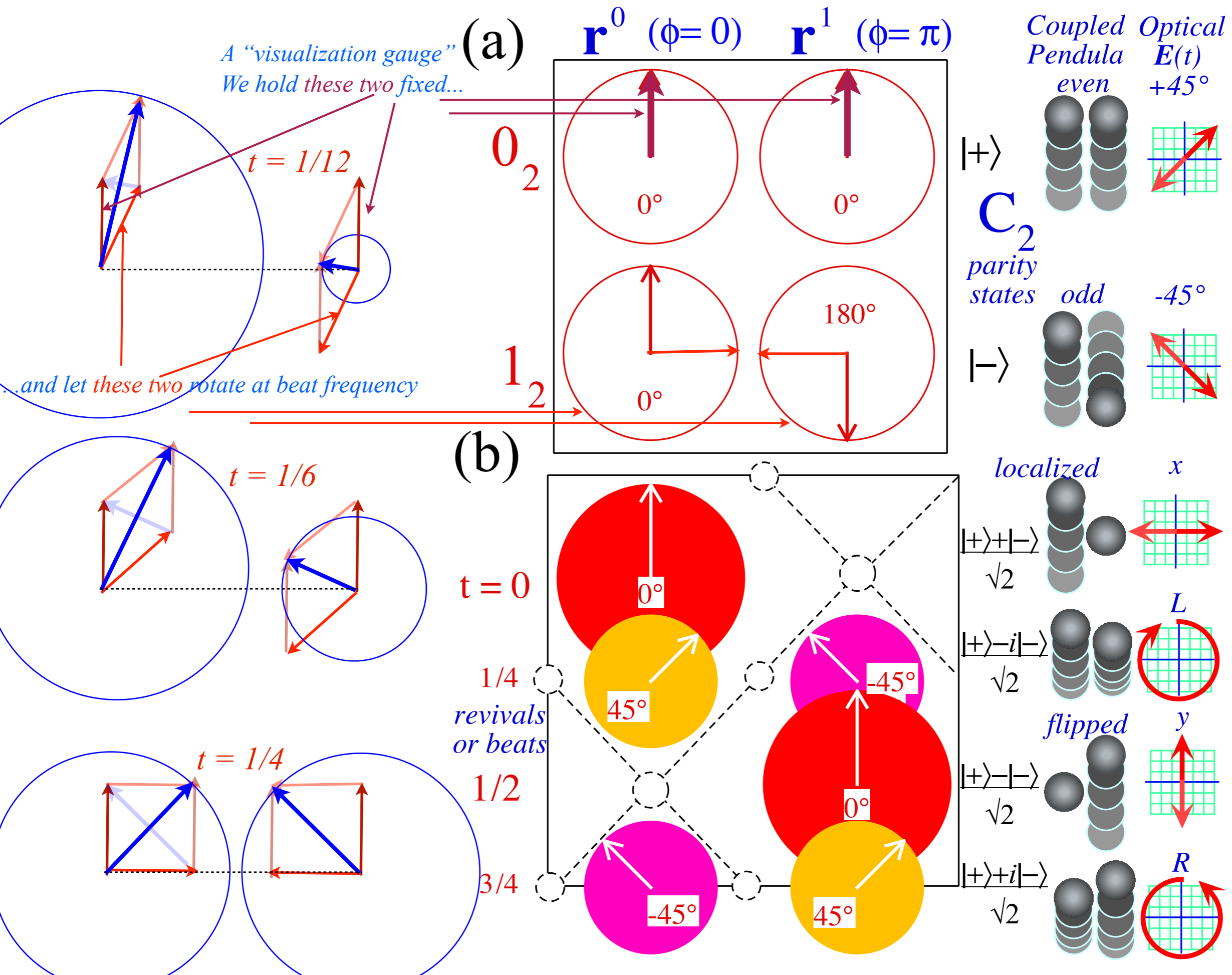
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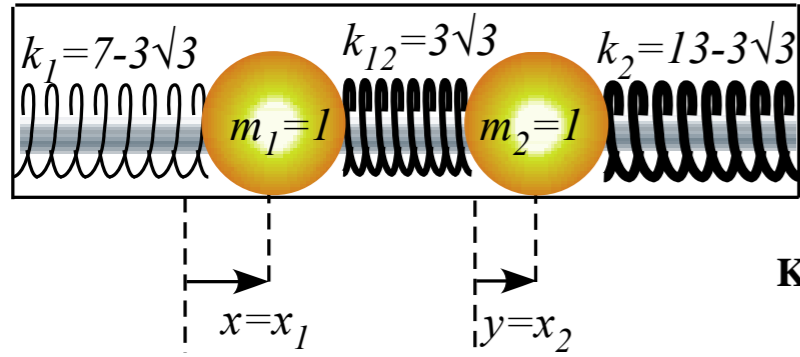
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➔ *2D-HO eigensolution example with asymmetric (A-Type) symmetry* **←**
Initial state projection, mixed mode beat dynamics with variable phase

Spectral decomposition of 2D-HO mode dynamics for lower symmetry

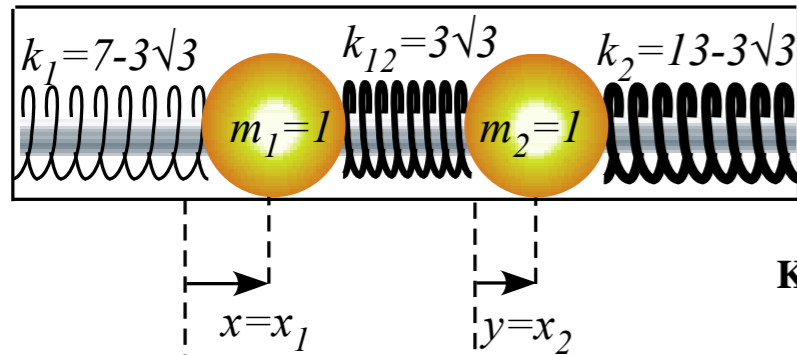


$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} = \begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13 \end{pmatrix}$$

$$\text{Det}(\mathbf{K}) = 7 \cdot 13 - 27 = 91 - 27 = 64$$

$$\text{Trace}(\mathbf{K}) = 7 + 13 = 20$$

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



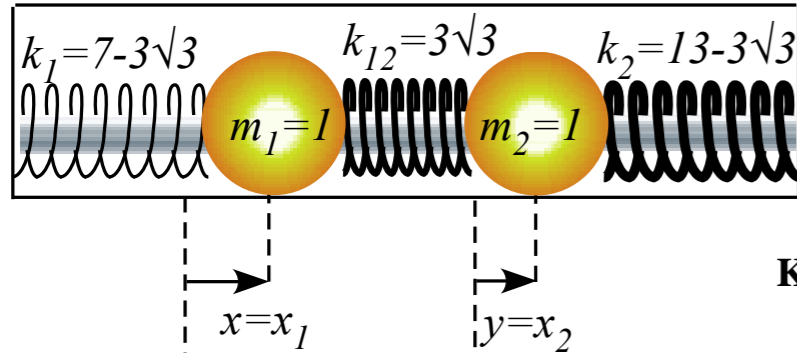
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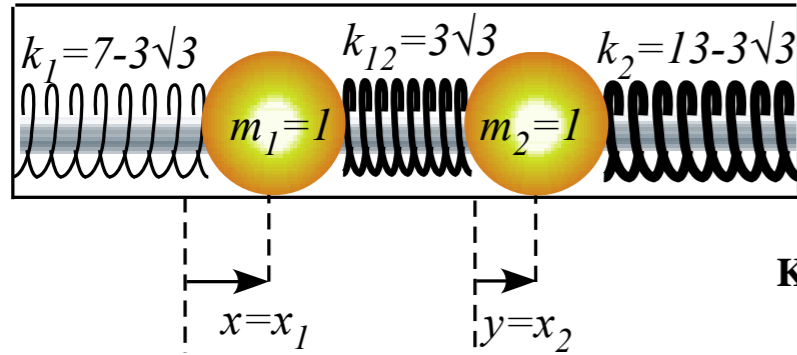


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The \mathbf{K} secular equation $K^2 - \text{Trace}(\mathbf{K})K + \text{Det}(\mathbf{K}) = K^2 - 20K + 64 = 0 = (K - 4)(K - 16)$

$\text{Det}(\mathbf{K}) = 7 \cdot 13 - 27 = 91 - 27 = 64$
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Spectral decomposition of 2D-HO mode dynamics for lower symmetry



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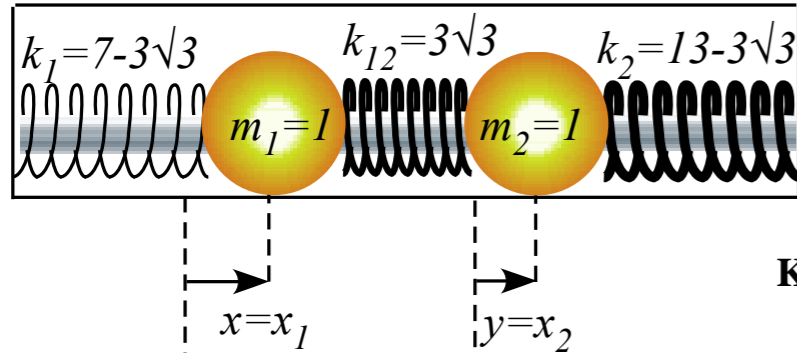
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Eigenvalues K_k and squared eigenfrequencies $\omega_0(\epsilon_k)^2$

$$K_1 = \omega_0^2(\epsilon_1) = 4, \quad K_2 = \omega_0^2(\epsilon_2) = 16,$$

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



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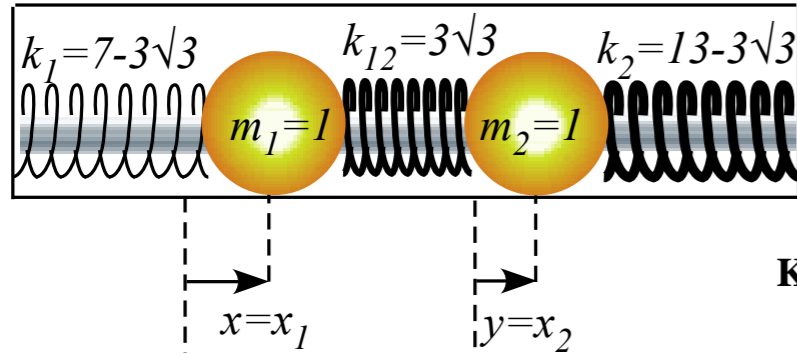
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Eigen-projectors \mathbf{P}_k

$$\begin{aligned} \mathbf{P}_1 &= \frac{\begin{pmatrix} K_{11} - K_2 & K_{12} \\ K_{12} & K_{22} - K_2 \end{pmatrix}}{K_1 - K_2} = \frac{\begin{pmatrix} 7 - 16 & -3\sqrt{3} \\ -3\sqrt{3} & 13 - 16 \end{pmatrix}}{4 - 16} = \frac{\begin{pmatrix} 9 & +3\sqrt{3} \\ +3\sqrt{3} & 3 \end{pmatrix}}{12} \\ &= \frac{\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}}{4} = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 & 1/2 \end{pmatrix} = |\epsilon_1\rangle\langle\epsilon_1| \end{aligned}$$

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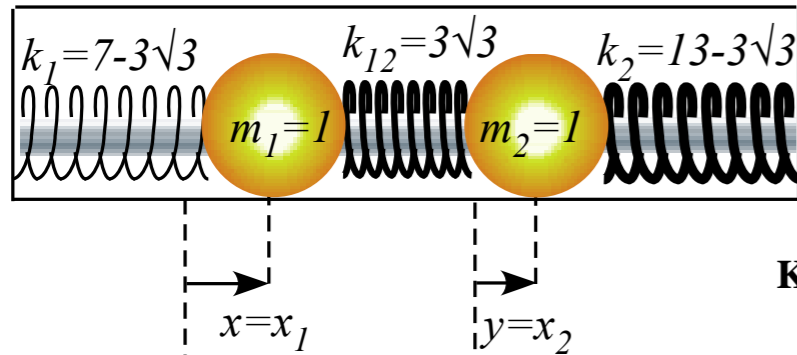
$$\mathbf{P}_1 = \frac{\begin{pmatrix} K_{11} - K_2 & K_{12} \\ K_{12} & K_{22} - K_2 \end{pmatrix}}{K_1 - K_2} = \frac{\begin{pmatrix} 7 - 16 & -3\sqrt{3} \\ -3\sqrt{3} & 13 - 16 \end{pmatrix}}{4 - 16} = \frac{\begin{pmatrix} 9 & +3\sqrt{3} \\ +3\sqrt{3} & 3 \end{pmatrix}}{12}$$

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Spectral decomposition of 2D-HO mode dynamics for lower symmetry



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Eigenbra vectors: $\langle\epsilon_1| = \left(\sqrt{3}/2 \quad 1/2 \right), \quad \langle\epsilon_2| = \left(-1/2 \quad \sqrt{3}/2 \right)$

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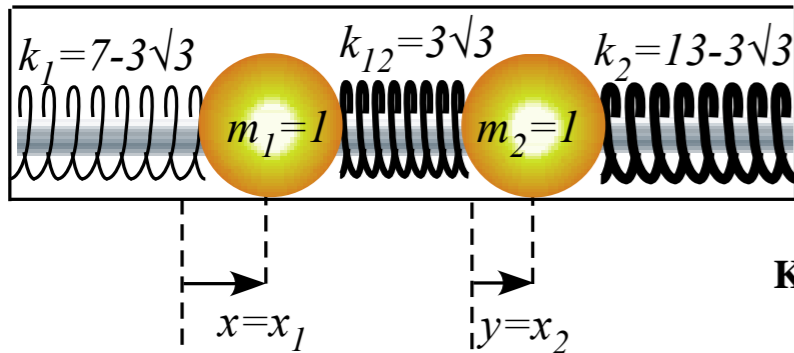
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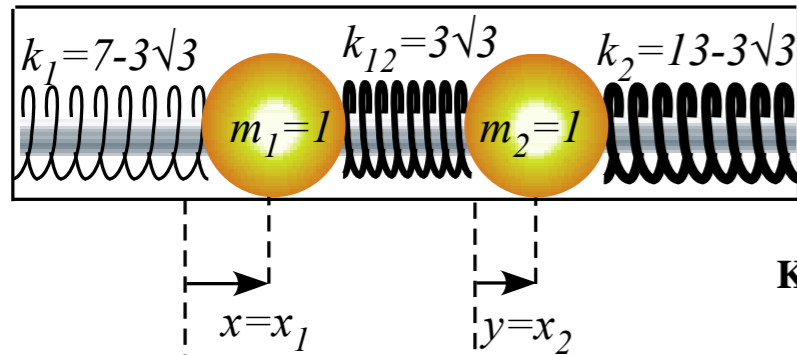
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Spectral decomposition of initial state $\mathbf{x}(0) = (1, 0)$:

$$\mathbf{1} \cdot \mathbf{x}(0) = (\mathbf{P}_1 + \mathbf{P}_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \otimes \begin{pmatrix} \sqrt{3} & 1/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix} \otimes \begin{pmatrix} -1/2 & \sqrt{3} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



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$$= \frac{\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}}{4} = \begin{pmatrix} \sqrt{3}/2 & \\ & 1/2 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 & 1/2 \end{pmatrix} = |\epsilon_1\rangle\langle\epsilon_1|$$

$$\mathbf{P}_2 = \frac{\begin{pmatrix} K_{11} - K_1 & K_{12} \\ K_{12} & K_{22} - K_1 \end{pmatrix}}{K_2 - K_1} = \frac{\begin{pmatrix} 7 - 4 & -3\sqrt{3} \\ -3\sqrt{3} & 13 - 4 \end{pmatrix}}{16 - 4} = \frac{\begin{pmatrix} 3 & -3\sqrt{3} \\ -3\sqrt{3} & 9 \end{pmatrix}}{12}$$

$$= \frac{\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix}}{4} = \begin{pmatrix} -1/2 & \\ & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} -1/2 & \sqrt{3}/2 \end{pmatrix} = |\epsilon_2\rangle\langle\epsilon_2|$$

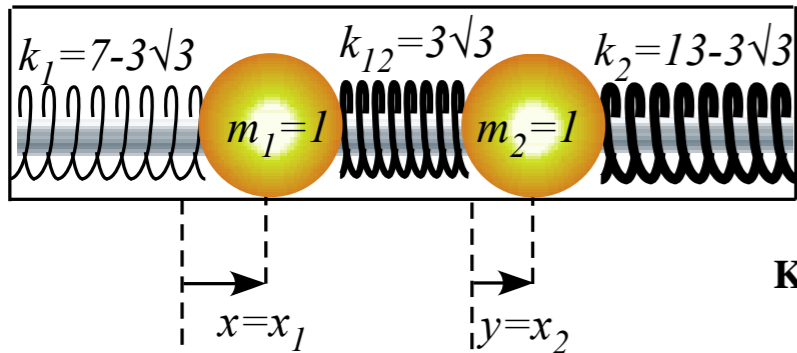
Eigenbra vectors: $\langle\epsilon_1| = \begin{pmatrix} \sqrt{3}/2 & 1/2 \end{pmatrix}$, $\langle\epsilon_2| = \begin{pmatrix} -1/2 & \sqrt{3}/2 \end{pmatrix}$

Spectral decomposition of initial state $\mathbf{x}(0) = (1, 0)$:

$$\mathbf{1} \cdot \mathbf{x}(0) = (\mathbf{P}_1 + \mathbf{P}_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \otimes \begin{pmatrix} \sqrt{3}/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix} \otimes \begin{pmatrix} -1/2 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} + \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix} \quad (\text{Note projection onto eigen-axes})$$

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} = \begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13 \end{pmatrix}$$

The \mathbf{K} secular equation $K^2 - \text{Trace}(\mathbf{K})K + \text{Det}(\mathbf{K}) = K^2 - 20K + 64 = 0 = (K - 4)(K - 16)$

$\text{Det}(\mathbf{K}) = 7 \cdot 13 - 27 = 91 - 27 = 64$
 $\text{Trace}(\mathbf{K}) = 7 + 13 = 20$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\epsilon_k)^2$ $K_1 = \omega_0^2(\epsilon_1) = 4$, $K_2 = \omega_0^2(\epsilon_2) = 16$,

Eigen-projectors \mathbf{P}_k

$$\mathbf{P}_1 = \frac{\begin{pmatrix} K_{11} - K_2 & K_{12} \\ K_{12} & K_{22} - K_2 \end{pmatrix}}{K_1 - K_2} = \frac{\begin{pmatrix} 7 - 16 & -3\sqrt{3} \\ -3\sqrt{3} & 13 - 16 \end{pmatrix}}{4 - 16} = \frac{\begin{pmatrix} 9 & +3\sqrt{3} \\ +3\sqrt{3} & 3 \end{pmatrix}}{12}$$

$$= \frac{\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}}{4} = \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix} = |\epsilon_1\rangle\langle\epsilon_1|$$

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$$= \frac{\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix}}{4} = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix} = |\epsilon_2\rangle\langle\epsilon_2|$$

Eigenbra vectors: $\langle\epsilon_1| = \begin{pmatrix} \sqrt{3}/2 & 1/2 \end{pmatrix}$, $\langle\epsilon_2| = \begin{pmatrix} -1/2 & \sqrt{3}/2 \end{pmatrix}$

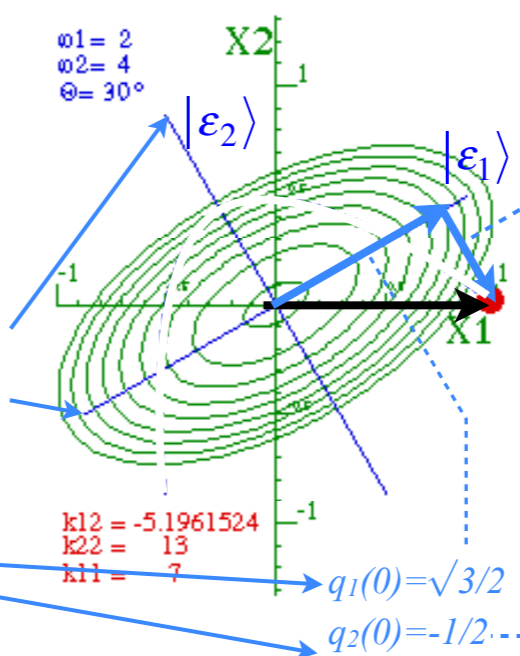
Spectral decomposition of initial state $\mathbf{x}(0) = (1, 0)$:

$$\mathbf{1} \cdot \mathbf{x}(0) = (\mathbf{P}_1 + \mathbf{P}_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \otimes \begin{pmatrix} \sqrt{3}/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix} \otimes \begin{pmatrix} -1/2 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} + \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}$$

$$\begin{pmatrix} q_1(t) = \frac{\sqrt{3}}{2} \cos 2t, & q_2(t) = -\frac{1}{2} \cos 4t \end{pmatrix}$$

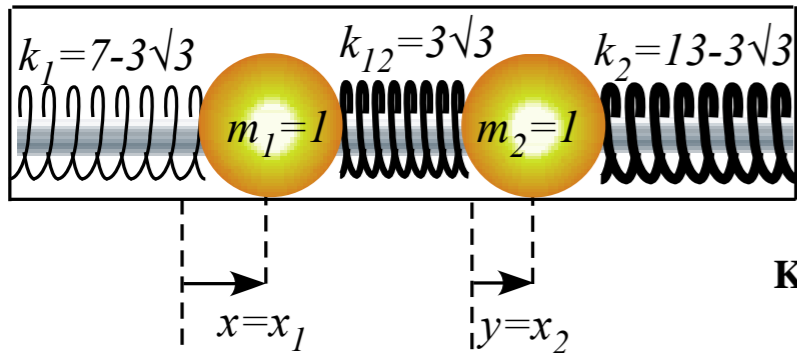
$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



(Note projection of $\mathbf{x}(0)$ onto eigen-axes)

$q_1(0) = \sqrt{3}/2$
 $q_2(0) = -1/2$

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} = \begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13 \end{pmatrix}$$

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$$= \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \left(\frac{\sqrt{3}}{2}\right) + \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix} \left(-\frac{1}{2}\right)$$

$$\begin{pmatrix} q_1(t) = \frac{\sqrt{3}}{2} \cos 2t, & q_2(t) = -\frac{1}{2} \cos 4t \end{pmatrix}$$

Using $\cos 4t = 2 \cos^2 2t - 1$ derives a parabolic trajectory!

$$q_2(t) = -\frac{1}{2} (2 \cos^2 2t - 1) = -\frac{4}{3} [q_1(t)]^2 + \frac{1}{2}$$

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

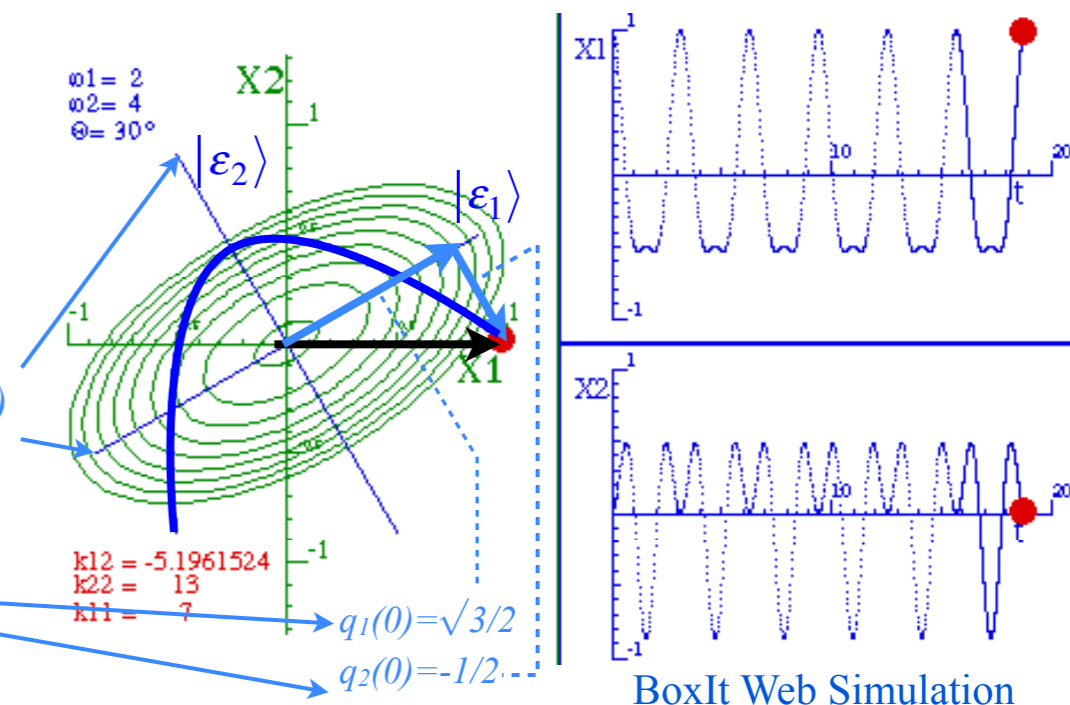
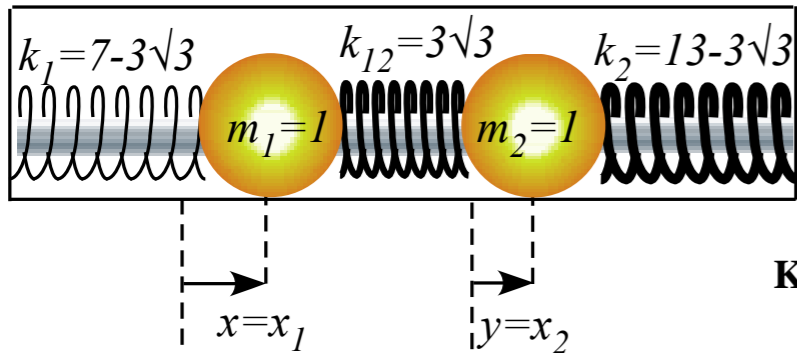


Fig. 3.3.6 Normal coordinate axes, coupled oscillator trajectories and equipotential ($V=\text{const.}$) ovals for an integral 1:2 eigenfrequency ratio ($\omega_0(\epsilon_1)=2.0$, $\omega_0(\epsilon_2)=4.0$) and zero initial velocity.

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Spectral decomposition of 2D-HO mode dynamics for lower symmetry



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$$\mathbf{P}_2 = \frac{\begin{pmatrix} K_{11} - K_1 & K_{12} \\ K_{12} & K_{22} - K_1 \end{pmatrix}}{K_2 - K_1} = \frac{\begin{pmatrix} 7 - 4 & -3\sqrt{3} \\ -3\sqrt{3} & 13 - 4 \end{pmatrix}}{16 - 4} = \frac{\begin{pmatrix} 3 & -3\sqrt{3} \\ -3\sqrt{3} & 9 \end{pmatrix}}{12}$$

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$$= \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \begin{pmatrix} \sqrt{3} \\ 1/2 \end{pmatrix} + \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} -1/2 \\ \sqrt{3} \end{pmatrix}$$

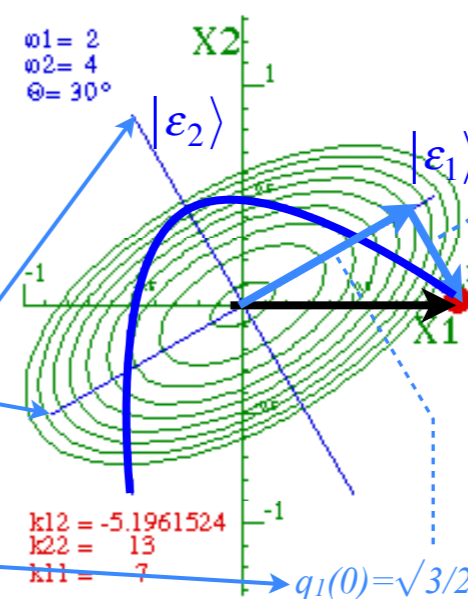
(Note projection of $\mathbf{x}(0)$ onto eigen-axes)

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$$q_2(t) = -\frac{1}{2} (2 \cos^2 2t - 1) = -\frac{4}{3} [q_1(t)]^2 + \frac{1}{2}$$

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[Pafnuty Chebyshev](#)



Pafnuty Lvovich Chebyshev was a Russian mathematician. His name can be alternatively transliterated as Chebychev, Chebysheff, Chebyshev, Tchebychev or Tchebycheff, or Tschebyschew or Tschebyschew. Wikipedia

Born: May 16, 1821, Borovsk
 Died: December 8, 1894, Saint Petersburg

Example of a Tschebycheff Polynomial order 2