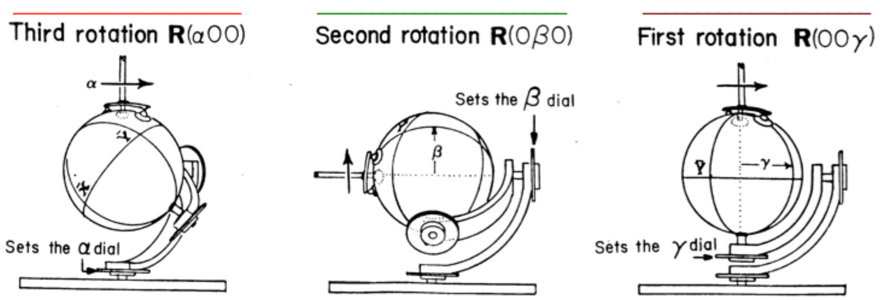


Lecture 22
Mon. 11.05.2018

Introduction to Spinor-Vector resonance dynamics

(Ch. 2-4 of Unit 4 Ch. 6-7 of Unit 6)

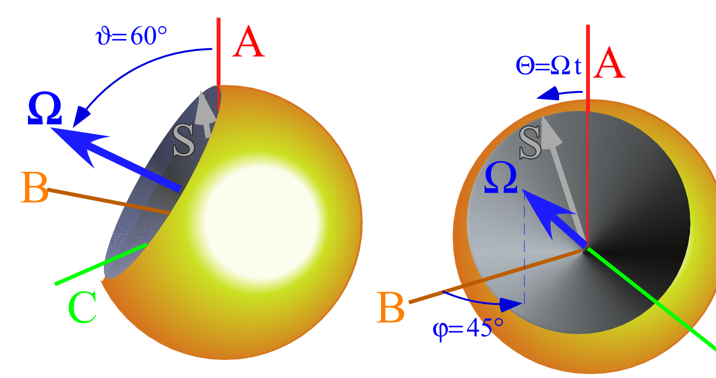
Review: 2D harmonic oscillator equations with Lagrangian and matrix forms

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

- *Hamilton-Pauli spinor symmetry (* $\boldsymbol{\sigma}$*-expansion in ABCD-Types)* $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$
- *Derive* $\boldsymbol{\sigma}$*-exponential time evolution (or revolution) operator* $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$
  - *Spinor arithmetic like complex arithmetic*
  - *Spinor vector algebra like complex vector algebra*
  - *Spinor exponentials like complex exponentials (“Crazy-Thing”-Theorem)*

- *Geometry of evolution (or revolution) operator* $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$
  - *The “mysterious” factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space*
    - *2D Spinor vs 3D vector rotation*
    - *NMR Hamiltonian: 3D Spin Moment* **m** *in* **B** *field*

Euler’s state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$
- *Spin-1 (3D-real vector) case*
- *Spin-1/2 (2D-complex spinor) case*



3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states
- *Asymmetry* $S_A=S_Z$, *Balance* $S_B=S_X$, *and Chirality* $S_C=S_Y$
- *Polarization ellipse and spinor state dynamics*

The “Great Spectral Avoided-Crossing” and A-to-B-to-A symmetry breaking

A running collection of links to course-relevant sites and articles

Physics Web Resources

Comprehensive Harter-Soft Resource Listing

UAF Physics YouTube channel

LearnIt Physics Web Applications

"Texts"

Classical Mechanics with a Bang!

Quantum Theory for the Computer Age

Principles of Symmetry, Dynamics, and Spectroscopy

Modern Physics and its Classical Foundations

Classes

2014 AMOP

2017 Group Theory for QM

2018 AMOP

2018 Adv Mechanics

Neat external material to start the class:
- AIP publications
- AJP article on superball dynamics
- AAPT summer reading

These *are* hot off the presses:
- Sorting ultracold atoms in a 3D optical lattice in a realization of Maxwell's demon - Kumar-Nature-Letters-2018
- Synthetic three-dimensional atomic structures assembled atom by atom - Berredo-Nature-Letters-2018

Slightly Older ones:
- Wave–particle duality of C60 molecules
- Optical vortex knots – One Photon at a Time

AMOP Ch 0 Space-Time Symmetry - 2019
- Seminar at Rochester Institute of Optics, Auxiliary slides, June 19, 2018

Springer AMO Handbook - Ch 32 - Harter-Reimer-2019

"*Relawavity*" and quantum basis of *Lagrangian* & *Hamiltonian* mechanics:
- 2-CW laser wave - BohrIt Web App
- Lagrangian vs Hamiltonian - RelaWavity Web App

Older Links from Lectures 14-20

http://thearmchaircritic.blogspot.com/2011/11/punkin-chunkin.html
http://www.sussexcountyonline.com/news/photos/punkinchunkin.html
Shooting-range-for-medieval-siege-weapons-Anybody-knows
https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html
https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html?scenario=MontezumasRevenge
https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html?scenario=SeigeOfKenilworth
The trebuchet, Chevedden, Sci Am 1995
'*Simple*' *Pendulum* Sim: https://modphys.hosted.uark.edu/markup/PendulumWeb.html
'*Cycloid*' *Pendulum*: https://modphys.hosted.uark.edu/markup/CycloidulumWeb.html
Google search on: "Satelite view of Patricia" (Images)
Physics Girl Channel - Fun with Vortex Rings in the Pool
iBall demo - Quasi-periodicity: https://youtu.be/_jntDtULxDc
https://modphys.hosted.uark.edu/markup/CoulItWeb.html?scenario=SynchrotronMotion
https://modphys.hosted.uark.edu/markup/CoulItWeb.html?scenario=SynchrotronMotion2
Mechanical Analog to EM Motion (YouTube video) - https://youtu.be/hTd5FTJ-vRk
CoulIt Web Simulation: Bound-state motion in parabolic coordinates
CoulIt Web Simulation: Bound-state motion in hyperbolic coordinates
OscillIt Web App: Simulations of various types of resonance: 18, 27, 31, 35, 38, 39
Smith Chart
http://nobelprize.org/

AnalyIt Web Application, posted 10/22/2018 in our *testing* area:
https://modphys.hosted.uark.edu/testing/markup/AnalyItBJS.html

Links to supplement Lecture 21

BoxIt Web App:
- Pure A-Type w/Cosine
- Pure B-Type w/Cosine
- Pure B-Type w/Freq ratios
- Mixed AB-Type 2:1 Freq ratio

Wiki on Pafnuty Chebyshev

Links to supplement Lecture 22

Advanced Atomic and Molecular Optical Physics 2018 Class #9, pages: 5, 61
BoxIt Web Simulations
- Pure A-Type A=4.9, B=0 ,C=0, & D=4.0
- Pure B-Type: A=4.0, B=-0.2, C=0, & D=4.0
- Pure C-Type A,D=4.055, B=0, C=0.1
- Mixed AB-Type w/Cosine
- Mixed AB Type A=4.0, BU2=0.866…, CU2=0, & D=1.0 w/Stokes & Freq rats

Classical Mechanics with a Bang! 2018
- Lectures 8, 9, 23 page 93
- Text Unit 6, page=27

ColorU2 for the Web - in development
Group Theory for Quantum Mechanics - 2017 Lectures: 6, 7, 8,
and the combined 9-10
Quantum Theory for the Computer Age Unit 3 Ch.7-10, page=90
Web based 3D & XR (x∈{A,M,V}, R=Reality) https://www.babylonjs.com/
Web based 3D graphics WebGL API (Graphics Layer modeled after OpenGL)

2D harmonic oscillator equations

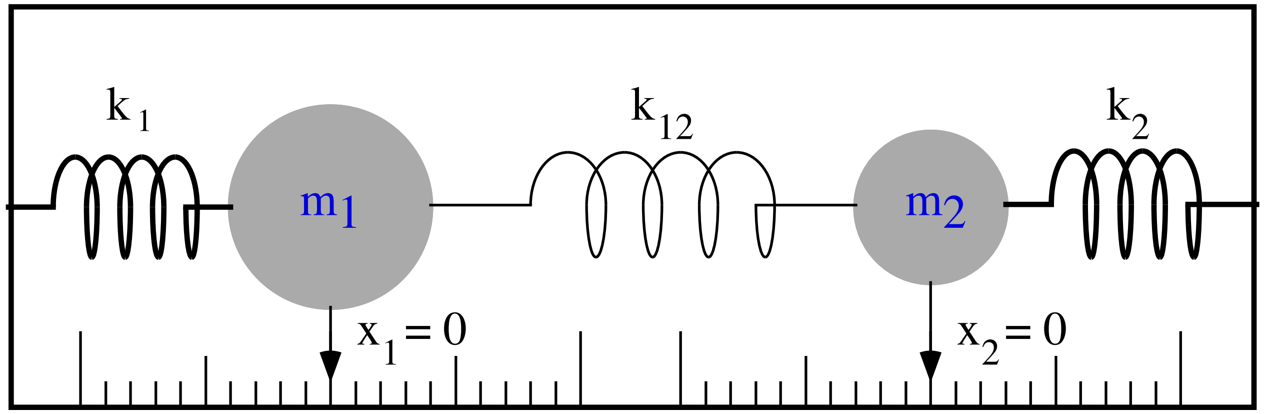
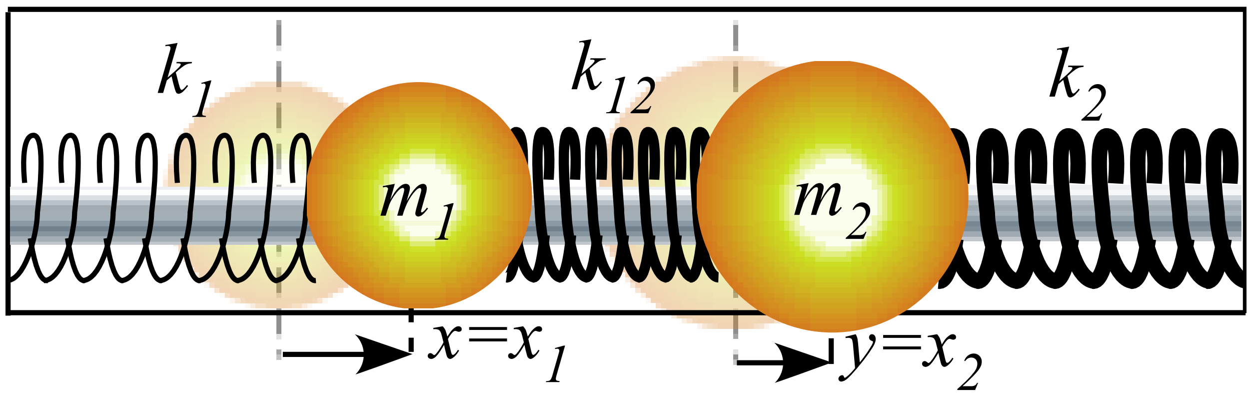


Fig. 3.3.1 Two 1-dimensional coupled oscillators



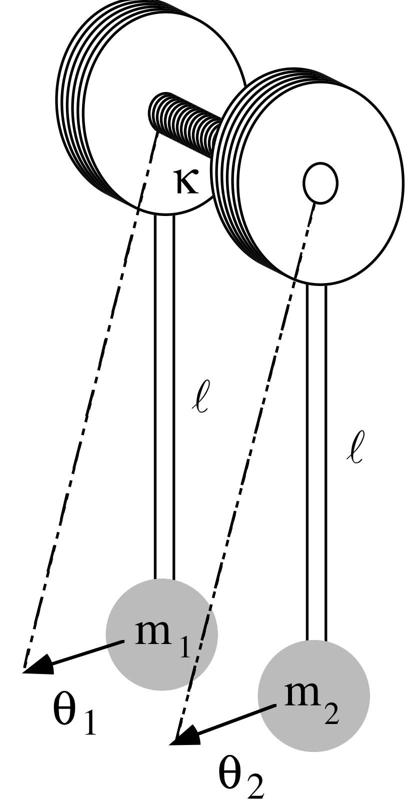


Fig. 3.3.2 Coupled pendulums

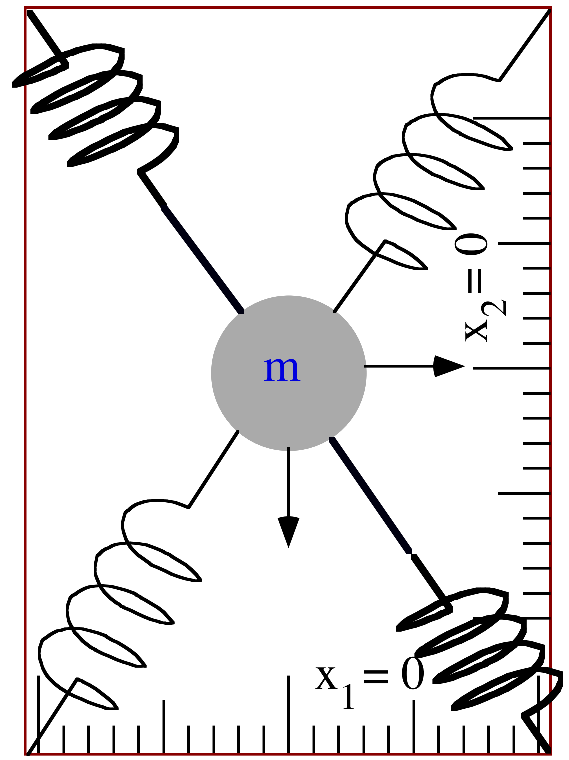


Fig. 3.3.3 One 2-dimensional coupled oscillator

(Review of Lect. 21)

2D HO kinetic energy $T(v_1, v_2)$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$= \frac{1}{2} \langle \dot{\mathbf{x}} | \mathbf{M} | \dot{\mathbf{x}} \rangle$$

2D HO potential energy $V(x_1, x_2)$

$$V = \frac{1}{2}(k_1 + k_{12}) x_1^2 - k_{12} x_1 x_2 + \frac{1}{2}(k_2 + k_{12}) x_2^2$$

$$= \frac{1}{2} \langle \mathbf{x} | \mathbf{K} | \mathbf{x} \rangle \quad \text{where: } \mathbf{K} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix}$$

Lagrangian $L = T - V$

2D HO Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = m_1 \ddot{x}_1 = F_1 = -\frac{\partial V}{\partial x_1} = -(k_1 + k_{12}) x_1 + k_{12} x_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = m_2 \ddot{x}_2 = F_2 = -\frac{\partial V}{\partial x_2} = k_{12} x_1 - (k_2 + k_{12}) x_2$$

2D HO Matrix operator equations

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Matrix operator notation:

$$\mathbf{M} \bullet | \ddot{\mathbf{x}} \rangle = - \mathbf{K} \bullet | \mathbf{x} \rangle$$

2D harmonic oscillator equation solutions *(Review of Lect. 21)*

1. May rewrite equation $\mathbf{M}\bullet|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$ in *acceleration* matrix form: $|\ddot{\mathbf{x}}\rangle = -\mathbf{A}|\mathbf{x}\rangle$ where: $\mathbf{A} = \mathbf{M}^{-1}\bullet\mathbf{K}$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}^{-1} \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\begin{pmatrix} \frac{k_1 + k_{12}}{m_1} & \frac{-k_{12}}{m_1} \\ \frac{-k_{12}}{m_2} & \frac{k_2 + k_{12}}{m_2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

2. Need to find *eigenvectors* $|\mathbf{e}_1\rangle, |\mathbf{e}_2\rangle, \ldots$ of acceleration matrix such that: $\mathbf{A}|\mathbf{e}_n\rangle = \varepsilon_n|\mathbf{e}_n\rangle = \omega_n^2|\mathbf{e}_n\rangle$

Then equations decouple to: $|\ddot{\mathbf{e}}_n\rangle = -\mathbf{A}|\mathbf{e}_n\rangle = -\varepsilon_n|\mathbf{e}_n\rangle = -\omega_n^2|\mathbf{e}_n\rangle$ where ε_n is an *eigenvalue*

and ω_n is an *eigenfrequency*

Note eigenvalue is <u>square</u> of eigenfrequency

To introduce eigensolutions we take a simple case of unit masses *(m_1=1=m_2)*

So equation of motion is simply: $|\ddot{\mathbf{x}}\rangle = -\mathbf{K}|\mathbf{x}\rangle$

Eigenvectors $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$ are in special directions where $|\ddot{\mathbf{x}}\rangle = -\mathbf{K}|\mathbf{x}\rangle$ is in same direction as $|\mathbf{x}\rangle$

➤ *ANALOGY: 2-State Schrodinger:* $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

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The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

2D Spinor vs 3D vector rotation

NMR Hamiltonian: 3D Spin Moment **m** *in* **B** *field*

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2_t \mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

$$i\hbar|\dot{\Psi}(t)\rangle = \mathbf{H}|\Psi(t)\rangle \qquad |\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$$

First start with 2-by-2 Hermitian (self-conjugate) matrix

$$\mathbf{H} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H}^\dagger$$

H_{jk} matrix must obey: $(H_{jk})^* = H_{kj}$

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that operates on 2-D complex Dirac ket vector $|\Psi\rangle$.

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Both have 4 parameters ($2^2 = 2+2$)

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$$i\hbar|\dot{\Psi}(t)\rangle=\mathbf{H}|\Psi(t)\rangle \qquad |\ddot{\mathbf{x}}\rangle=-\mathbf{K}\bullet|\mathbf{x}\rangle$$

First start with 2-by-2 Hermitian (self-conjugate) matrix

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$$\dot{x}_1 = Ap_1 + Bp_2 - Cx_2 \qquad \dot{p}_1 = -Ax_1 - Bx_2 - Cp_2$$
$$\dot{x}_2 = Bp_1 + Dp_2 + Cx_1 \qquad \dot{p}_2 = -Bx_1 - Dx_2 + Cp_1$$

$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$$

$$i\frac{\partial}{\partial t}\begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}\begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

$$\begin{pmatrix} i\dot{x}_1 - \dot{p}_1 \\ i\dot{x}_2 - \dot{p}_2 \end{pmatrix} = \begin{pmatrix} Ax_1 + Bx_2 + Cp_2 + iAp_1 + iBp_2 - iCx_2 \\ Bx_1 + Dx_2 - Cp_1 + iBp_1 + iDp_2 + iCx_1 \end{pmatrix}$$

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$$i\hbar|\dot{\Psi}(t)\rangle=\mathbf{H}|\Psi(t)\rangle \qquad |\ddot{\mathbf{x}}\rangle=-\mathbf{K}\bullet|\mathbf{x}\rangle$$

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Then start with classical Hamiltonian. (Designed to give same result.)

$$H_c=\frac{A}{2}\left(p_1^2+x_1^2\right)+B\left(x_1x_2+p_1p_2\right)+C\left(x_1p_2-x_2p_1\right)+\frac{D}{2}\left(p_2^2+x_2^2\right)$$

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Then Hamilton's equations of motion are the following.

For constant $A, B, C,$ *and* D

$$\dot{x}_1 = \frac{\partial H_c}{\partial p_1} = Ap_1 + Bp_2 - Cx_2 \qquad \dot{p}_1 = -\frac{\partial H_c}{\partial x_1} = -\left(Ax_1 + Bx_2 + Cp_2\right)$$

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QM vs. Classical Equations are identical

Then start with classical Hamiltonian. (Designed to give same result.)

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$$H_c = \frac{A}{2}\left(p_1^2 + x_1^2\right) + B\left(x_1x_2 + p_1p_2\right) + C\left(x_1p_2 - x_2p_1\right) + \frac{D}{2}\left(p_2^2 + x_2^2\right)$$

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For constant $A, B, C,$ *and* D

$$\dot{x}_1 = \frac{\partial H_c}{\partial p_1} = Ap_1 + Bp_2 - Cx_2 \qquad \dot{p}_1 = -\frac{\partial H_c}{\partial x_1} = -\left(Ax_1 + Bx_2 + Cp_2\right)$$
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➔ *Hamilton-Pauli spinor symmetry (* $\boldsymbol{\sigma}$*-expansion in ABCD-Types)* $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$

Derive $\boldsymbol{\sigma}$*-exponential time evolution (or revolution) operator* $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

Spinor arithmetic like complex arithmetic

Spinor vector algebra like complex vector algebra

Spinor exponentials like complex exponentials (“Crazy-Thing”-Theorem)

Geometry of evolution (or revolution) operator $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

The “mysterious” factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

2D Spinor vs 3D vector rotation

NMR Hamiltonian: 3D Spin Moment **m** *in* **B** *field*

ABCD Symmetry operator analysis and U(2) spinors

Decompose the Hamiltonian operator **H** into four *ABCD symmetry operators*
(Labeled to provide dynamic mnemonics as well as colorful analogies)

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = A\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + D\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = A\mathbf{e}_{11} + B\boldsymbol{\sigma}_B + C\boldsymbol{\sigma}_C + D\mathbf{e}_{22}$$

$$= \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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...current-carrier...

Symmetry archetypes: *A (Asymmetric-diagonal)*| *B (Bilateral-balanced)*| *C (Chiral-circular-complex-Coriolis-cyclotron-curly...)*

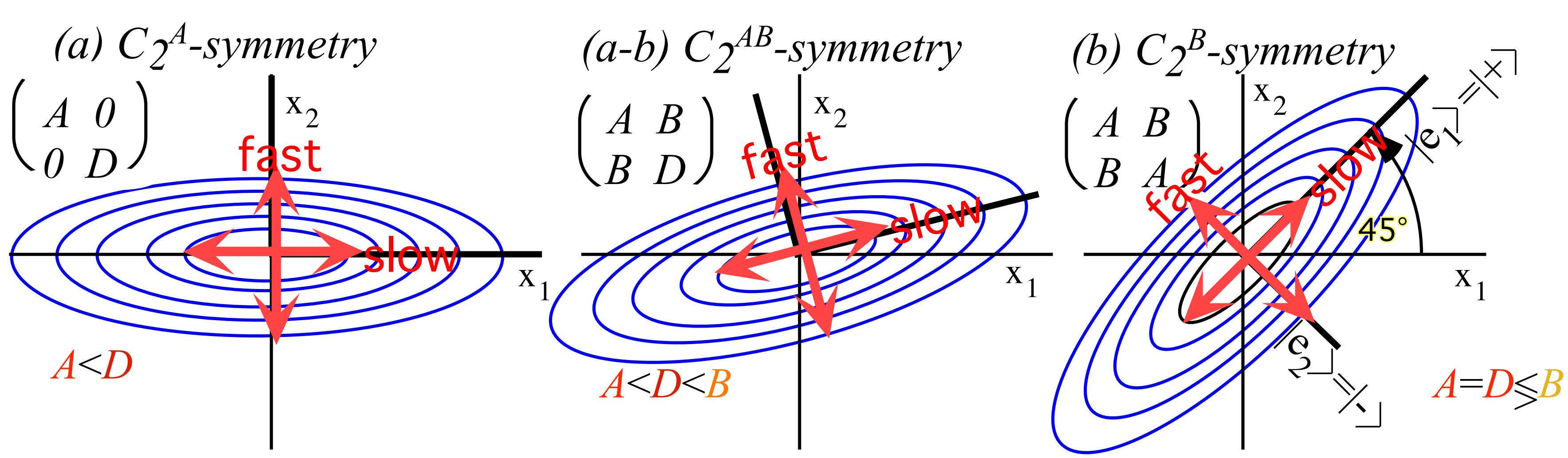

Fig. 3.4.1 Potentials for (a) $C_2{}^A$-asymmetric-diagonal, (ab) $C_2{}^{AB}$-mixed , (b) $C_2{}^B$-bilateral U(2)system.

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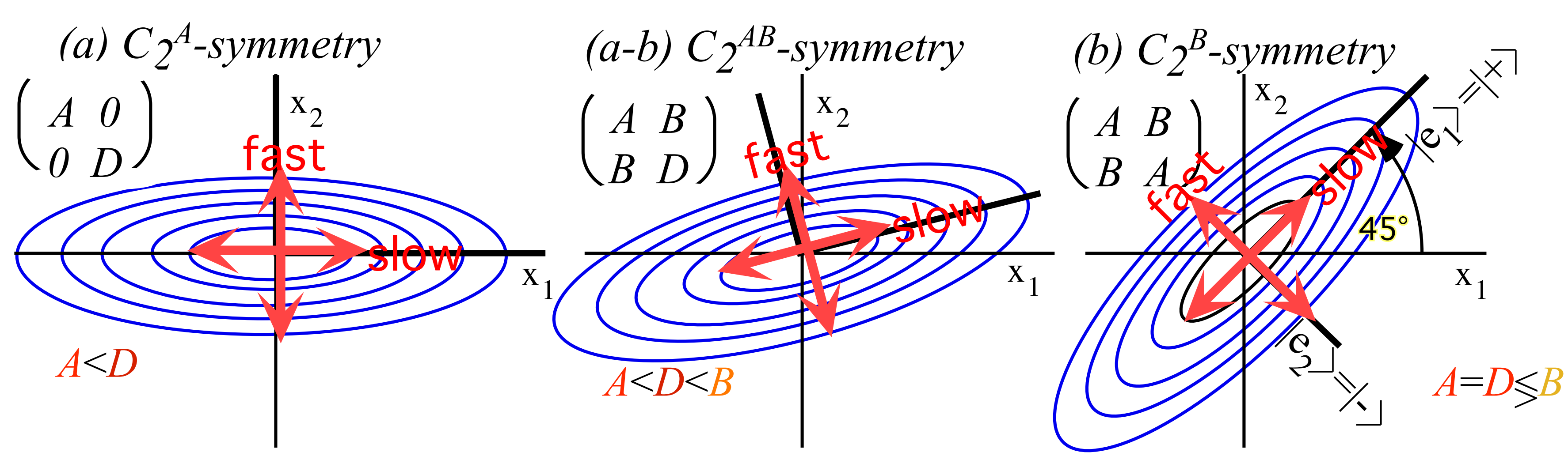
The $\{\sigma_1, \sigma_A, \sigma_B, \sigma_C\}$ are best known as *Pauli-spin operators* $\{\sigma_1=\sigma_0, \sigma_B=\sigma_X, \sigma_C=\sigma_Y, \sigma_A=\sigma_Z\}$ developed in 1927.

Wolfgang Pauli (1927) Zur Quantenmechanik des magnetischen Elektrons *Zeitschrift für Physik* (43) 601-623



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$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = A\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + D\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = A\mathbf{e}_{11} + B\boldsymbol{\sigma}_B + C\boldsymbol{\sigma}_C + D\mathbf{e}_{22}$$

$$= \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{H} = \frac{A-D}{2}\,\boldsymbol{\sigma}_A + B\,\boldsymbol{\sigma}_B + C\,\boldsymbol{\sigma}_C + \frac{A+D}{2}\,\boldsymbol{\sigma}_0$$

...current-carrier...

Symmetry archetypes: *A (Asymmetric-diagonal)*| *B (Bilateral-balanced)*| *C (Chiral-circular-complex-Coriolis-cyclotron-curly...)*

The $\{\sigma_1, \sigma_A, \sigma_B, \sigma_C\}$ are best known as *Pauli-spin operators* $\{\sigma_1=\sigma_0, \sigma_B=\sigma_X, \sigma_C=\sigma_Y, \sigma_A=\sigma_Z\}$ developed in 1927.

Wolfgang Pauli (1927) Zur Quantenmechanik des magnetischen Elektrons *Zeitschrift für Physik* (43) 601-623

In 1843 Hamilton invents *quaternions* $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. σ_μ related by i-factor: $\{\sigma_1=\mathbf{1}=\sigma_0, i\sigma_B=\mathbf{i}=i\sigma_X, i\sigma_C=\mathbf{j}=i\sigma_Y, i\sigma_A=\mathbf{k}=i\sigma_Z\}$.

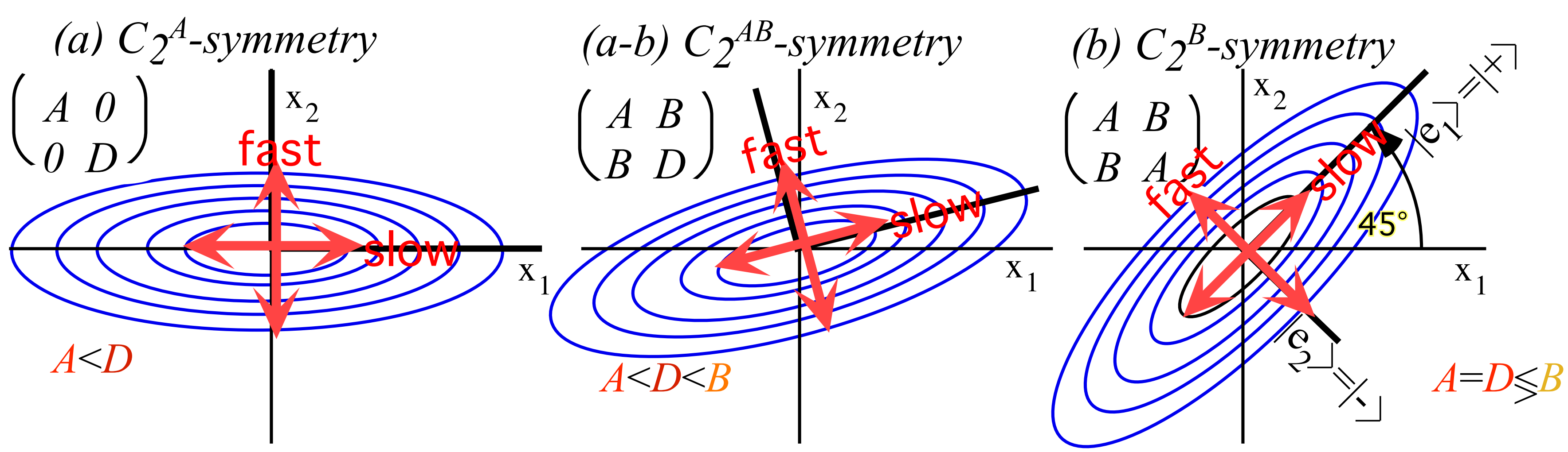

Fig. 3.4.1 Potentials for (a) C_2^A-asymmetric-diagonal, (ab) C_2^{AB}-mixed , (b) C_2^B-bilateral U(2)system.

ABCD Symmetry operator analysis and U(2) spinors

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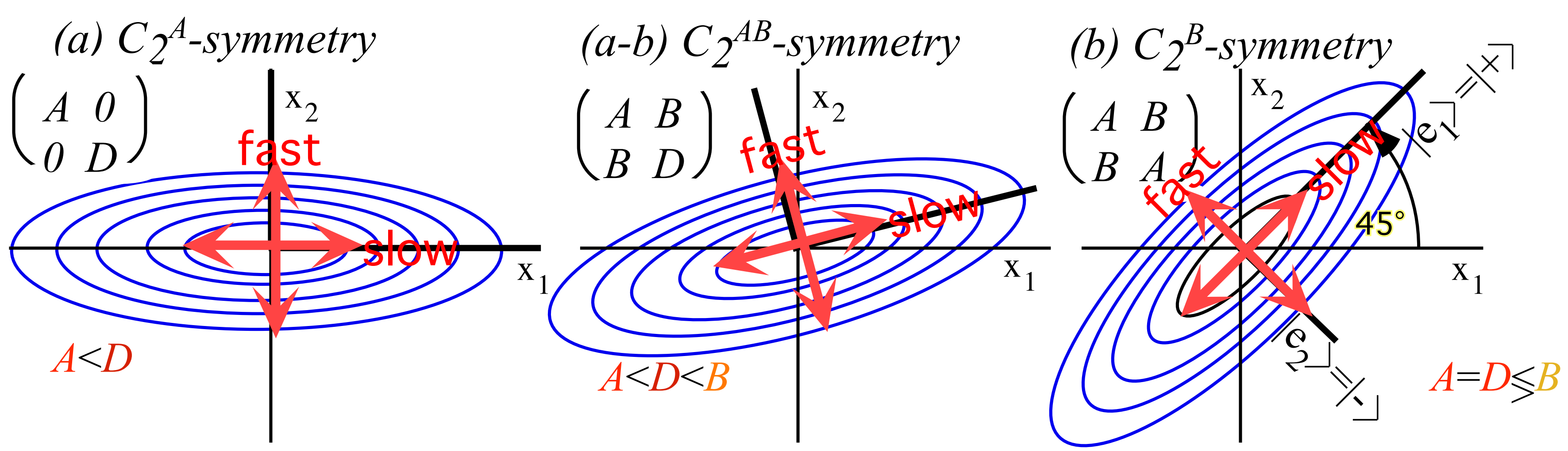

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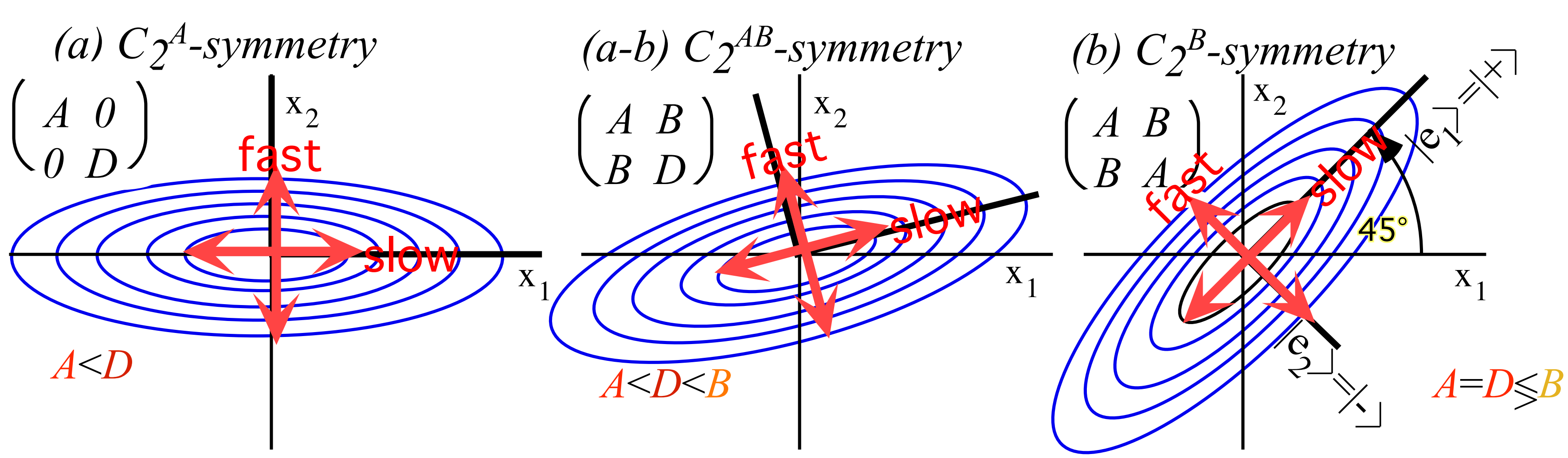
Each Pauli σ_μ squares to *positive*-**1** ($\sigma_X^2 = \sigma_Y^2 = \sigma_Z^2 = +\mathbf{1}$) (Each makes a cyclic C_2 group $C_2^A = \{\mathbf{1}, \sigma_A\}$, $C_2^B = \{\mathbf{1}, \sigma_B\}$, or $C_2^C = \{\mathbf{1}, \sigma_C\}$.)



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ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2{}_t\mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry (**σ***-expansion in ABCD-Types)* $\mathbf{H} = \omega_\mu \boldsymbol{\sigma}_\mu$

➡ *Derive* **σ***-exponential time evolution (or revolution) operator* $\mathbf{U} = e^{-i\mathbf{H}t} = e^{-i\boldsymbol{\sigma}_\mu \omega_\mu t}$

➡ *Spinor arithmetic* *like* *complex arithmetic*

Spinor vector algebra *like* *complex vector algebra*

Spinor exponentials *like* *complex exponentials* *("Crazy-Thing"-Theorem)*

Geometry of evolution (or revolution) operator $\mathbf{U} = e^{-i\mathbf{H}t} = e^{-i\boldsymbol{\sigma}_\mu \omega_\mu t}$

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

2D Spinor vs 3D vector rotation

NMR Hamiltonian: 3D Spin Moment **m** *in* **B** *field*

OBJECTIVE: Evaluate and (*most* important!) *visualize* matrix-exponent solutions.

Need to convert this to a 2x2 matrix

$$|\Psi(t)\rangle = e^{-i\mathbf{H}\cdot t}|\Psi(0)\rangle$$

Hamilton generalized Euler's expansion $e^{-i\omega t} = \cos\omega t - i\sin\omega t$ so matrix exponential becomes powerful.

$$e^{-i\mathbf{H}\cdot t} = e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}\cdot t} = e^{-i\frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot t - iB\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cdot t - iC\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cdot t - i\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cdot t}$$

$\sigma_A = \sigma_Z$ $\quad \sigma_B = \sigma_X$ $\quad \sigma_C = \sigma_Y$

$$= e^{-i\sigma_\varphi \varphi} e^{-i\omega_0 \cdot t} = e^{-i\vec{\sigma}\bullet\vec{\omega}\cdot t} e^{-i\omega_0 \cdot t} \text{ where: } \vec{\varphi} = \begin{pmatrix} \varphi_A \\ \varphi_B \\ \varphi_C \end{pmatrix} = \vec{\omega}\cdot t = \begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\cdot t = \begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix}\cdot t \text{ and: } \omega_0 = \frac{A+D}{2}$$

ABCD Time evolution operator

For constant A,B,C, and D

Symmetry relations make spinors $\sigma_X = \sigma_B$, $\sigma_Y = \sigma_C$, and $\sigma_Z = \sigma_A$ or quaternions $\mathbf{i} = -i\sigma_X$, $\mathbf{j} = -i\sigma_Y$, and $\mathbf{k} = -i\sigma_Z$ powerful.

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$$= \begin{matrix} a_X^2\mathbf{1} & \cancel{+a_X a_Y\sigma_X\sigma_Y} & \cancel{+a_X a_Z\sigma_X\sigma_Z} \\ \cancel{-a_X a_Y\sigma_X\sigma_Y} & +a_Y^2\mathbf{1} & \cancel{+a_Y a_Z\sigma_Y\sigma_Z} \\ \cancel{-a_X a_Z\sigma_X\sigma_Z} & \cancel{-a_Y a_Z\sigma_Y\sigma_Z} & +a_Z^2\mathbf{1} \end{matrix} = (a_X^2 + a_Y^2 + a_Z^2)\mathbf{1} = \mathbf{1}$$

So: $\sigma_a^2 = \mathbf{1}$

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➤ *Spinor vector algebra like complex vector algebra*

Spinor exponentials like complex exponentials ("Crazy-Thing"-Theorem)

Geometry of evolution (or revolution) operator $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

2D Spinor vs 3D vector rotation

NMR Hamiltonian: 3D Spin Moment **m** *in* **B** *field*

OBJECTIVE: Evaluate and (*most* important!) *visualize* matrix-exponent solutions.

Need to convert this to a 2x2 matrix

$$|\Psi(t)\rangle = e^{-i\mathbf{H}\cdot t}|\Psi(0)\rangle$$

Hamilton generalized Euler's expansion $e^{-i\omega t} = \cos\omega t - i\sin\omega t$ so matrix exponential becomes powerful.

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$$e^{-i\mathbf{H}\cdot t} = e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}\cdot t} = e^{-i\frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot t - iB\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cdot t - iC\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cdot t - i\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cdot t}$$

$\sigma_A = \sigma_Z$ $\sigma_B = \sigma_X$ $\sigma_C = \sigma_Y$

$$= e^{-i\sigma_\varphi \varphi} e^{-i\omega_0 \cdot t} = e^{-i\vec{\sigma}\bullet\vec{\omega}\cdot t} e^{-i\omega_0 \cdot t} \text{ where: } \vec{\varphi} = \begin{pmatrix} \varphi_A \\ \varphi_B \\ \varphi_C \end{pmatrix} = \vec{\omega}\cdot t = \begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\cdot t = \begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix}\cdot t \text{ and: } \omega_0 = \frac{A+D}{2}$$

For constant A,B,C, and D

Symmetry relations make spinors $\sigma_X = \sigma_B$, $\sigma_Y = \sigma_C$, and $\sigma_Z = \sigma_A$ or quaternions $\mathbf{i} = -i\sigma_X$, $\mathbf{j} = -i\sigma_Y$, and $\mathbf{k} = -i\sigma_Z$ powerful.

σ-products do dot • *and* cross × products by symmetries: $\sigma_X\sigma_Y = i\sigma_Z = -\sigma_Y\sigma_X$, $\sigma_Z\sigma_X = i\sigma_Y = -\sigma_X\sigma_Z$, $\sigma_Y\sigma_Z = i\sigma_X = -\sigma_Z\sigma_Y$

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$$\sigma_a\sigma_b = (\sigma\bullet\mathbf{a})(\sigma\bullet\mathbf{b}) = (a_X\sigma_X + a_Y\sigma_Y + a_Z\sigma_Z)(b_X\sigma_X + b_Y\sigma_Y + b_Z\sigma_Z)$$

$$\begin{aligned} = \; & a_Xb_X\mathbf{1} && +a_Xb_Y\sigma_X\sigma_Y && -a_Xb_Z\sigma_Z\sigma_X && && +i(a_Yb_Z - a_Zb_Y)\sigma_X \\ & -a_Yb_X\sigma_X\sigma_Y && +a_Yb_Y\mathbf{1} && +a_Yb_Z\sigma_Y\sigma_Z && = (a_Xb_X + a_Yb_Y + a_Zb_Z)\mathbf{1} && +i(a_Zb_X - a_Xb_Z)\sigma_Y \\ & +a_Zb_X\sigma_Z\sigma_X && -a_Zb_X\sigma_Y\sigma_Z && +a_Zb_Z\mathbf{1} && && +i(a_Xb_Y - a_Yb_X)\sigma_Z \end{aligned}$$

$$\sigma_Z \cdot \sigma_X: \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\sigma_Y$$

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$$= \begin{matrix} a_X b_X \mathbf{1} & +a_X b_Y \sigma_X\sigma_Y & -a_X b_Z \sigma_Z\sigma_X \\ -a_Y b_X \sigma_X\sigma_Y & +a_Y b_Y \mathbf{1} & +a_Y b_Z \sigma_Y\sigma_Z \\ +a_Z b_X \sigma_Z\sigma_X & -a_Z b_X \sigma_Y\sigma_Z & +a_Z b_Z \mathbf{1} \end{matrix} = (a_X b_X + a_Y b_Y + a_Z b_Z)\mathbf{1} \begin{matrix} +i(a_Y b_Z - a_Z b_Y)\sigma_X \\ +i(a_Z b_X - a_X b_Z)\sigma_Y \\ +i(a_X b_Y - a_Y b_X)\sigma_Z \end{matrix}$$

Write the product in Gibbs notation. (This is where Gibbs *got* his {**i,j,k**} notation!)

$$\sigma_a\sigma_b = (\sigma\bullet\mathbf{a})(\sigma\bullet\mathbf{b}) = \quad (\mathbf{a}\bullet\mathbf{b})\mathbf{1} \quad + \quad i(\mathbf{a}\times\mathbf{b})\bullet\sigma$$

$$\underset{\sigma_Z}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \cdot \underset{\sigma_X}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\sigma_Y$$

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(Recall (1.10.29). in complex variable Chapter 10 in Unit 1.)

$$A^*B = (A_X + iA_Y)^*(B_X + iB_Y) = (A_X - iA_Y)(B_X + iB_Y)$$
$$= (A_XB_X + A_YB_Y) + i(A_XB_Y - A_YB_X) = (\mathbf{A}\bullet\mathbf{B}) + i(\mathbf{A}\times\mathbf{B})_Z$$

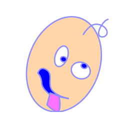
ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$*-expansion in ABCD-Types)* $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$

Derive $\boldsymbol{\sigma}$*-exponential time evolution (or revolution) operator* $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

Spinor arithmetic like complex arithmetic

Spinor vector algebra like complex vector algebra



Spinor exponentials like complex exponentials (“Crazy-Thing”-Theorem)

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Hamilton replaces *(-i)* with $-i\sigma_\varphi$ in the $e^{-i\varphi}$ power series above to get a sequence of terms just like it.

$$(-i\sigma_\varphi)^0 = +\mathbf{1},\ (-i\sigma_\varphi)^1 = -i\sigma_\varphi,\ (-i\sigma_\varphi)^2 = -\mathbf{1},\ (-i\sigma_\varphi)^3 = +i\sigma_\varphi,\ (-i\sigma_\varphi)^4 = +\mathbf{1},\ (-i\sigma_\varphi)^5 = -i\sigma_\varphi,\ etc.$$

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Note even powers of *(-i)* are ±*1* and odd powers of *(-i)* are ±*i*.: $(-i)^0 = +1,\ (-i)^1 = -i,\ (-i)^2 = -1,\ (-i)^3 = +i,\ (-i)^4 = +1,\ (-i)^5 = -i,\ etc.$

Hamilton replaces *(-i)* with $-i\sigma_\varphi$ in the $e^{-i\varphi}$ power series above to get a sequence of terms just like it.

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This allows Hamilton to generalize Euler's rotation $e^{-i\varphi}$ to $e^{-i\sigma_\varphi\varphi}$ for any $\sigma_\varphi\varphi = (\boldsymbol{\sigma}\bullet\vec{\varphi}) = \varphi_A\boldsymbol{\sigma}_A + \varphi_B\boldsymbol{\sigma}_B + \varphi_Z\boldsymbol{\sigma}_Z \doteq (\boldsymbol{\sigma}\bullet\hat{\varphi})\varphi$

$$e^{-i\varphi} = 1\ \cos\varphi - i\ \sin\varphi$$

generalizes to:

$$e^{-i\sigma_\varphi\varphi} = \mathbf{1}\cos\varphi - i\ \sigma_\varphi \sin\varphi$$

OBJECTIVE: Evaluate and (*most* important!) *visualize* matrix-exponent solutions.

Need to convert this to a 2x2 matrix → $|\Psi(t)\rangle = e^{-i\mathbf{H}\cdot t}|\Psi(0)\rangle$ **DONE!**

Hamilton generalized Euler's expansion $e^{-i\omega t} = \cos\omega t - i\sin\omega t$ so matrix exponential becomes powerful.

$$e^{-i\mathbf{H}\cdot t} = e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D\end{pmatrix}\cdot t} = e^{-i\frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\cdot t - iB\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\cdot t - iC\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\cdot t - i\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\cdot t}$$

$\sigma_A = \sigma_Z$, $\sigma_B = \sigma_X$, $\sigma_C = \sigma_Y$

$$= e^{-i\sigma_\varphi \varphi} e^{-i\omega_0\cdot t} = e^{-i\vec{\sigma}\bullet\vec{\omega}\cdot t} e^{-i\omega_0 \cdot t} \quad \text{where: } \vec{\varphi} = \begin{pmatrix}\varphi_A \\ \varphi_B \\ \varphi_C\end{pmatrix} = \vec{\omega}\cdot t = \begin{pmatrix}\omega_A \\ \omega_B \\ \omega_C\end{pmatrix}\cdot t = \begin{pmatrix}\frac{A-D}{2} \\ B \\ C\end{pmatrix}\cdot t \quad \text{and: } \omega_0 = \frac{A+D}{2}$$

$$= (\mathbf{1}\cos\varphi - i\sigma_\varphi \sin\varphi)e^{-i\omega_0 \cdot t}$$

ABCD Time evolution operator

For constant A,B,C, and D

Symmetry relations make spinors $\sigma_X = \sigma_B$, $\sigma_Y = \sigma_C$, and $\sigma_Z = \sigma_A$ or quaternions **i**$=-i\sigma_X$, **j**$=-i\sigma_Y$, and **k**$=-i\sigma_Z$ powerful.

Hamilton is able to generalize Euler's complex rotation operators $e^{+i\varphi}$ and $e^{-i\varphi}$. (Recall (1.10.17).)

$$e^{-i\varphi} = 1 + (-i\varphi) + \frac{1}{2!}(-i\varphi)^2 + \frac{1}{3!}(-i\varphi)^3 + \frac{1}{4!}(-i\varphi)^4 \cdots = [1 - \frac{1}{2!}\varphi^2 + \frac{1}{4!}\varphi^4 \cdots] = [\cos\varphi]$$
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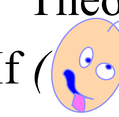
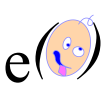
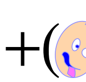
The Crazy Thing Theorem:

If ()^2= -**1**

Then:

e()$^\varphi$ =**1**cosφ+()sinφ

This allows Hamilton to generalize Euler's rotation $e^{-i\varphi}$ to $e^{-i\sigma_\varphi\varphi}$ for any $\sigma_\varphi\varphi = (\boldsymbol{\sigma}\bullet\vec{\varphi}) = \varphi_A\boldsymbol{\sigma}_A + \varphi_B\boldsymbol{\sigma}_B + \varphi_Z\boldsymbol{\sigma}_Z \doteq (\boldsymbol{\sigma}\bullet\hat{\varphi})\varphi$

$e^{-i\varphi} = 1\ \cos\varphi\ -\ i\ \sin\varphi$ *generalizes to:* $e^{-i\sigma_\varphi\varphi} = \mathbf{1}\cos\varphi\ -\ i\ \sigma_\varphi \sin\varphi$

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$\sigma_A = \sigma_Z$ $\quad$ $\sigma_B = \sigma_X$ $\quad$ $\sigma_C = \sigma_Y$

$$= e^{-i\sigma_\varphi \varphi} e^{-i\omega_0 \cdot t} = e^{-i\vec{\sigma}\bullet\vec{\omega}\cdot t} e^{-i\omega_0 \cdot t} \quad \text{where: } \vec{\varphi} = \begin{pmatrix} \varphi_A \\ \varphi_B \\ \varphi_C \end{pmatrix} = \vec{\omega}\cdot t = \begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\cdot t = \begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix}\cdot t \quad \text{and: } \omega_0 = \frac{A+D}{2}$$

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Hamilton is able to generalize Euler's complex rotation operators $e^{+i\varphi}$ and $e^{-i\varphi}$. (Recall (1.10.17).)

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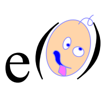
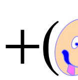
The Crazy Thing Theorem:

If ()^2= -**1**

Then:

e()φ =**1**cosφ+()sinφ

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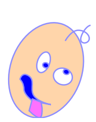
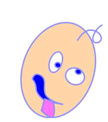
Here:  = -*i*

Crazy thing is just -√-1

Here:  $= -i\sigma_\varphi = -i(\boldsymbol{\sigma}\bullet\hat{\varphi}) = -i\frac{(\boldsymbol{\sigma}\bullet\vec{\varphi})}{\varphi}$

This -iσ$_\varphi$ is a REALLY Crazy thing!

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2{}_t\mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$*-expansion in ABCD-Types)* $\mathbf{H} = \omega_\mu \boldsymbol{\sigma}_\mu$

Derive $\boldsymbol{\sigma}$*-exponential time evolution (or revolution) operator* $\mathbf{U} = e^{-i\mathbf{H}t} = e^{-i\boldsymbol{\sigma}_\mu \omega_\mu t}$

Spinor arithmetic like complex arithmetic
Spinor vector algebra like complex vector algebra
Spinor exponentials like complex exponentials (“Crazy-Thing”-Theorem)

➤ *Geometry of evolution (or revolution) operator* $\mathbf{U} = e^{-i\mathbf{H}t} = e^{-i\boldsymbol{\sigma}_\mu \omega_\mu t}$

The “mysterious” factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space
2D Spinor vs 3D vector rotation
NMR Hamiltonian: 3D Spin Moment **m** *in* **B** *field*

OBJECTIVE: Evaluate and (*most* important!) *visualize* matrix-exponent solutions.

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For constant A,B,C, and D

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$\sigma_A = \sigma_Z$ $\quad\sigma_B = \sigma_X$ $\quad\sigma_C = \sigma_Y$

where: $\vec{\varphi} = \begin{pmatrix} \varphi_A \\ \varphi_B \\ \varphi_C \end{pmatrix} = \vec{\omega}\cdot t = \begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\cdot t = \begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix}\cdot t$ and: $\omega_0 = \frac{A+D}{2}$

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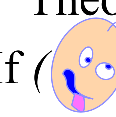
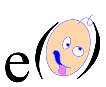
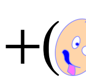
$$= \begin{pmatrix} \cos\varphi_A - i\sin\varphi_A & 0 \\ 0 & \cos\varphi_A - i\sin\varphi_A\end{pmatrix} = \begin{pmatrix} e^{-i\varphi_A} & 0 \\ 0 & e^{i\varphi_A}\end{pmatrix}$$

Example 1: A or Z rotation

The Crazy Thing Theorem:

If ()^2 = -**1**

Then:

e$^{(\text{ }  \text{ })\varphi}$ = **1**$\cos\varphi$ + ()$\sin\varphi$

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$\sigma_A = \sigma_Z$ $\sigma_B = \sigma_X$ $\sigma_C = \sigma_Y$

$$e^{-i\varphi} = 1\ \cos\varphi \ -\ i\ \sin\varphi$$

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$$= \begin{pmatrix} \cos\varphi_A - i\sin\varphi_A & 0 \\ 0 & \cos\varphi_A - i\sin\varphi_A \end{pmatrix} = \begin{pmatrix} e^{-i\varphi_A} & 0 \\ 0 & e^{i\varphi_A}\end{pmatrix}$$

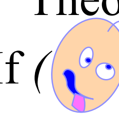
Example 1: A or Z rotation

$$e^{-i\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\varphi_C} = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\cos\varphi_C - i\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\sin\varphi_C$$

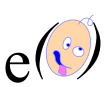
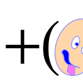
$$= \begin{pmatrix} \cos\varphi_C & -\sin\varphi_C \\ \sin\varphi_C & \cos\varphi_C \end{pmatrix}$$

Example 2: C or Y rotation

The Crazy Thing Theorem:

If ()^2 = -**1**

Then:

e$^{(\text{face})\varphi}$  = **1** cos φ + () sin φ

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$$= \begin{pmatrix} \cos\varphi_C & -\sin\varphi_C \\ \sin\varphi_C & \cos\varphi_C \end{pmatrix}$$

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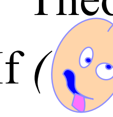
If ()^2= -**1**

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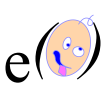
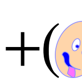
e()$^{\varphi}$ =**1**cosφ+()sinφ

Let: $\vec{\varphi} = \vec{\omega}\cdot t$ be general crank vector or $\vec{\omega}$-axis of rotation

Example 3: Any $\varphi = \omega t$-axial rotation

$$e^{-i(\sigma\bullet\vec{\varphi})t} = \mathbf{1}\cos\varphi - i\sigma_\varphi\sin\varphi = \mathbf{1}\cos\varphi - i\,(\sigma\bullet\hat{\varphi})\sin\varphi$$

$$= \mathbf{1}\cos\varphi \quad - i\,(\sigma_A\hat{\varphi}_A)\sin\varphi \quad - i\,(\sigma_B\hat{\varphi}_B)\sin\varphi \quad - i\,(\sigma_C\hat{\varphi}_C)\sin\varphi$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\varphi - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\hat{\varphi}_A\sin\varphi - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\hat{\varphi}_B\sin\varphi - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\hat{\varphi}_C\sin\varphi$$

$$= \begin{pmatrix} \cos\varphi - i\hat{\varphi}_A\sin\varphi & (-i\hat{\varphi}_B - \hat{\varphi}_C)\sin\varphi \\ (-i\hat{\varphi}_B + \hat{\varphi}_C)\sin\varphi & \cos\varphi + i\hat{\varphi}_A\sin\varphi \end{pmatrix}$$

Polar-to-Cartesian unit $\varphi = \omega\cdot t$-crank axis

$$\begin{pmatrix} \hat{\varphi}_A \\ \hat{\varphi}_B \\ \hat{\varphi}_C \end{pmatrix} = \begin{pmatrix} \cos\vartheta \\ \cos\varphi\sin\vartheta \\ \sin\varphi\sin\vartheta \end{pmatrix} = \frac{\begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix}}{\sqrt{\varphi_A^2 + \varphi_B^2 + \varphi_C^2}}$$

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$$e^{-i\mathbf{H}\cdot t} = e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}\cdot t} = e^{-i\frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot t - iB\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cdot t - iC\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cdot t - i\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cdot t} = e^{-i(\omega_0\sigma_0 + \vec{\omega}\bullet\vec{\sigma})\cdot t} = e^{-i\omega_0\cdot t}\left(\mathbf{1}\cos\omega t - i\sigma_{\varphi}\sin\omega\cdot t\right)$$

$\sigma_A = \sigma_Z$ $\quad\sigma_B = \sigma_X$ $\quad\sigma_C = \sigma_Y$

$$e^{-i\varphi} = 1\cos\varphi - i\sin\varphi$$

generalizes to:

$$e^{-i\sigma_{\varphi}\varphi} = \mathbf{1}\cos\varphi - i\,\sigma_{\varphi}\sin\varphi$$

where: $\vec{\varphi} = \begin{pmatrix} \varphi_A \\ \varphi_B \\ \varphi_C \end{pmatrix} = \vec{\omega}\cdot t = \begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\cdot t = \begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix}\cdot t$ and: $\omega_0 = \frac{A+D}{2}$

Example 1: A or Z rotation

$$e^{-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\varphi_A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\varphi_A - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sin\varphi_A$$

$$= \begin{pmatrix} \cos\varphi_A - i\sin\varphi_A & 0 \\ 0 & \cos\varphi_A - i\sin\varphi_A \end{pmatrix} = \begin{pmatrix} e^{-i\varphi_A} & 0 \\ 0 & e^{i\varphi_A} \end{pmatrix}$$

Example 2: C or Y rotation

$$e^{-i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\varphi_C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\varphi_C - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\sin\varphi_C$$

$$= \begin{pmatrix} \cos\varphi_C & -\sin\varphi_C \\ \sin\varphi_C & \cos\varphi_C \end{pmatrix}$$

We test these operators by making them rotate each other....

OBJECTIVE: Evaluate and (*most* important!) *visualize* matrix-exponent solutions.

ABCD Time evolution operator

For constant A,B,C, and D

Need to convert this to a 2x2 matrix

$$|\Psi(t)\rangle = e^{-i\mathbf{H}\cdot t}|\Psi(0)\rangle$$ *Really* DONE!

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$$e^{-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\varphi_A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\varphi_A - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sin\varphi_A$$

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Example 1: A or Z rotation

$$e^{-i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\varphi_C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\varphi_C - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\sin\varphi_C$$

$$= \begin{pmatrix} \cos\varphi_C & -\sin\varphi_C \\ \sin\varphi_C & \cos\varphi_C \end{pmatrix}$$

Example 2: C or Y rotation

3D axis vector $\vec{\varphi} = \vec{\omega}\cdot t$ corresponds to generator $\sigma_\varphi = \sigma_A\hat{\varphi}_A + \sigma_B\hat{\varphi}_B + \sigma_C\hat{\varphi}_C$ of *rotation* $e^{-i\sigma_\varphi\varphi} = R(\vec{\varphi}) = \mathbf{1}\cos\varphi - i\sigma_\varphi\sin\varphi$ about axis $\vec{\varphi}$.

Any 2-by-2 σ_μ-matrix may be rotated by any $R(\vec{\varphi})$ matrix acting *twice* (fore-and-aft^{-1}) to give: $\sigma_\mu^{(\vec{\varphi}\text{-rotated})} = R(\vec{\varphi})\sigma_\mu R^{-1}(\vec{\varphi}) = R(\vec{\varphi})\sigma_\mu R^\dagger(\vec{\varphi})$

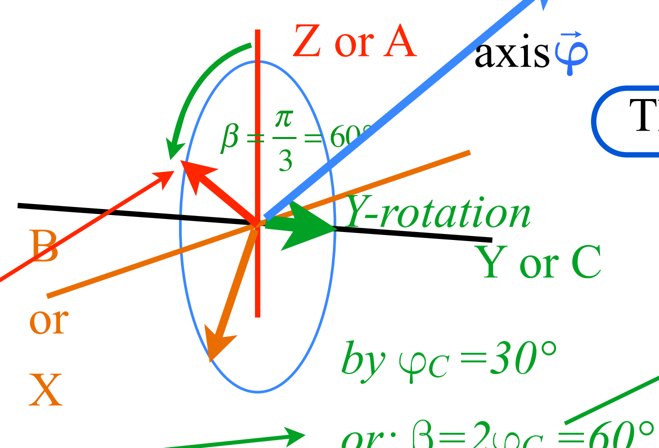
$$\mathbf{R}(\varphi_C)\ \cdot\ \boldsymbol{\sigma}_A\ \cdot\ \mathbf{R}^{-1}(\varphi_C)$$

$$= \begin{pmatrix} \cos\varphi_C & -\sin\varphi_C \\ \sin\varphi_C & \cos\varphi_C \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \cos\varphi_C & \sin\varphi_C \\ -\sin\varphi_C & \cos\varphi_C \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\varphi_C - \sin^2\varphi_C & 2\sin\varphi_C\cos\varphi_C \\ 2\sin\varphi_C\cos\varphi_C & \sin^2\varphi_C - \cos^2\varphi_C \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cos 2\varphi_C + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin 2\varphi_C$$

$$= \boldsymbol{\sigma}_A\ \cos 2\varphi_C + \boldsymbol{\sigma}_B\ \sin 2\varphi_C$$



The *3D*-rotation is by 2φ, *twice* the *2D* angle φ.

$$\hat{\varphi} = \begin{pmatrix} \hat{\varphi}_A \\ \hat{\varphi}_B \\ \hat{\varphi}_C \end{pmatrix} = \begin{pmatrix} \varphi_A \\ \varphi_B \\ \varphi_C \end{pmatrix}\frac{1}{\sqrt{\varphi_A^2 + \varphi_B^2 + \varphi_C^2}} = \begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\frac{1}{\sqrt{\omega_A^2 + \omega_B^2 + \omega_C^2}}$$

OBJECTIVE: Evaluate and (*most* important!) *visualize* matrix-exponent solutions.

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generalizes to:

$$e^{-i\sigma_\varphi \varphi} = \mathbf{1}\cos\varphi \ - \ i\ \sigma_\varphi \sin\varphi$$

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$$e^{-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\varphi_A} = \begin{pmatrix}1&0\\0&1\end{pmatrix}\cos\varphi_A - i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\sin\varphi_A$$

$$= \begin{pmatrix}\cos\varphi_A - i\sin\varphi_A & 0\\ 0 & \cos\varphi_A - i\sin\varphi_A\end{pmatrix} = \begin{pmatrix}e^{-i\varphi_A} & 0\\ 0 & e^{i\varphi_A}\end{pmatrix}$$

Example 1: A or Z rotation

$$e^{-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\varphi_C} = \begin{pmatrix}1&0\\0&1\end{pmatrix}\cos\varphi_C - i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\sin\varphi_C$$

$$= \begin{pmatrix}\cos\varphi_C & -\sin\varphi_C\\ \sin\varphi_C & \cos\varphi_C\end{pmatrix}$$

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3D axis vector $\vec{\varphi} = \vec{\omega}\cdot t$ corresponds to generator $\sigma_\varphi = \boldsymbol{\sigma}_A\hat{\varphi}_A + \boldsymbol{\sigma}_B\hat{\varphi}_B + \boldsymbol{\sigma}_C\hat{\varphi}_C$ of *rotation* $e^{-i\sigma_\varphi\varphi} = R(\vec{\varphi}) = \mathbf{1}\cos\varphi - i\sigma_\varphi\sin\varphi$ about axis $\vec{\varphi}$.

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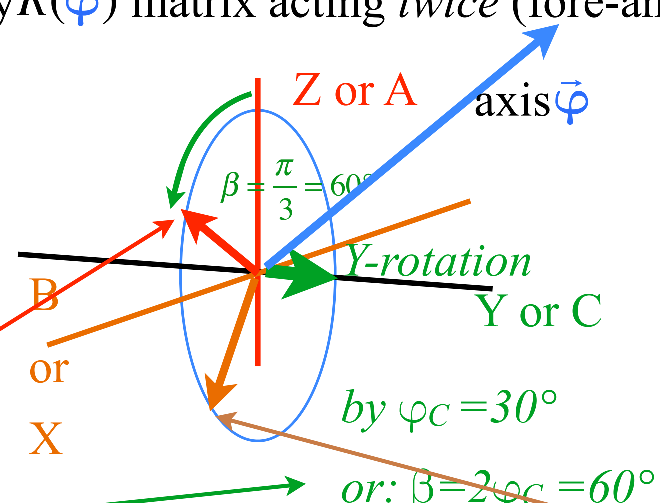
$$\mathbf{R}(\varphi_C)\cdot\boldsymbol{\sigma}_A\cdot\mathbf{R}^{-1}(\varphi_C)$$

$$= \begin{pmatrix}\cos\varphi_C & -\sin\varphi_C\\ \sin\varphi_C & \cos\varphi_C\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\cos\varphi_C & \sin\varphi_C\\ -\sin\varphi_C & \cos\varphi_C\end{pmatrix}$$

$$= \begin{pmatrix}\cos^2\varphi_C - \sin^2\varphi_C & 2\sin\varphi_C\cos\varphi_C\\ 2\sin\varphi_C\cos\varphi_C & \sin^2\varphi_C - \cos^2\varphi_C\end{pmatrix}$$

$$= \begin{pmatrix}1&0\\0&-1\end{pmatrix}\cos 2\varphi_C + \begin{pmatrix}0&1\\1&0\end{pmatrix}\sin 2\varphi_C$$

$$= \boldsymbol{\sigma}_A\ \cos 2\varphi_C + \boldsymbol{\sigma}_B\ \sin 2\varphi_C$$



$$\mathbf{R}(\varphi_C)\cdot\boldsymbol{\sigma}_B\cdot\mathbf{R}^{-1}(\varphi_C)$$

$$= \begin{pmatrix}\cos\varphi_C & -\sin\varphi_C\\ \sin\varphi_C & \cos\varphi_C\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\cos\varphi_C & \sin\varphi_C\\ -\sin\varphi_C & \cos\varphi_C\end{pmatrix}$$

$$= \begin{pmatrix}-2\sin\varphi_C\cos\varphi_C & \cos^2\varphi_C - \sin^2\varphi_C\\ \cos^2\varphi_C - \sin^2\varphi_C & 2\sin\varphi_C\cos\varphi_C\end{pmatrix}$$

$$= \begin{pmatrix}-1&0\\0&1\end{pmatrix}\sin 2\varphi_C + \begin{pmatrix}0&1\\1&0\end{pmatrix}\cos 2\varphi_C$$

$$= -\boldsymbol{\sigma}_A\ \sin 2\varphi_C + \boldsymbol{\sigma}_B\ \cos 2\varphi_C$$

The *3D*-rotation is by 2φ, *twice* the *2D* angle φ.

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$*-expansion in ABCD-Types)* $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$

Derive $\boldsymbol{\sigma}$*-exponential time evolution (or revolution) operator* $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

Spinor arithmetic	*like*	*complex arithmetic*
Spinor vector algebra	*like*	*complex vector algebra*
Spinor exponentials	*like*	*complex exponentials (“Crazy-Thing”-Theorem)*

Geometry of evolution (or revolution) operator $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

➔ *The “mysterious” factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space*

2D Spinor vs 3D vector rotation

NMR Hamiltonian: 3D Spin Moment **m** *in* **B** *field*

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

Notation for 2D Spinor space

$$\mathbf{H} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \omega_0 \sigma_0 + \omega_A \sigma_A + \omega_B \sigma_B + \omega_C \sigma_C = \omega_0\sigma_0 + \vec{\boldsymbol{\omega}} \bullet \vec{\boldsymbol{\sigma}} = \omega_0 \mathbf{1} + \omega \boldsymbol{\sigma}_\omega$$

Symmetry archetypes: *A (Asymmetric-diagonal)*| *B (Bilateral-balanced)*| *C (Chiral-circular-complex...)*

The $\{\sigma_1, \sigma_A, \sigma_B, \sigma_C\}$ are the well known *Pauli-spin operators* $\{\sigma_1 = \boldsymbol{\sigma}_0, \sigma_B = \sigma_X, \sigma_C = \sigma_Y, \sigma_A = \sigma_Z\}$

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$$\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}=\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+\frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}+B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}+C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Notation for 2D Spinor space

$$=\omega_0\sigma_0+\omega_A\sigma_A+\omega_B\sigma_B+\omega_C\sigma_C=\omega_0\sigma_0+\vec{\omega}\bullet\vec{\sigma}=\omega_0\mathbf{1}+\omega\boldsymbol{\sigma}_\omega$$

$$=\Omega_0\mathbf{1}+\Omega_A\mathbf{S}_A+\Omega_B\mathbf{S}_B+\Omega_C\mathbf{S}_C=\Omega_0\mathbf{1}+\vec{\Omega}\bullet\vec{\mathbf{S}}$$

$$=\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+(A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}+2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}+2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}$$

Notation for 3D Vector space

0th component unchanged

components A, B, C switch 1/2-factor from ω-velocity to S-momentum

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$$=\omega_0\sigma_0+\omega_A\sigma_A+\omega_B\sigma_B+\omega_C\sigma_C=\omega_0\sigma_0+\vec{\omega}\bullet\vec{\sigma}=\omega_0\mathbf{1}+\omega\boldsymbol{\sigma}_\omega$$

Notation for 2D Spinor space

$$=\Omega_0\mathbf{1}+\Omega_A\mathbf{S}_A+\Omega_B\mathbf{S}_B+\Omega_C\mathbf{S}_C=\Omega_0\mathbf{1}+\vec{\Omega}\bullet\vec{\mathbf{S}}$$

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(Often labeled $\{\mathsf{J}_X, \mathsf{J}_Y, \mathsf{J}_Z\}$)

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$$\mathbf{H} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Notation for 2D Spinor space

$$= \omega_0 \sigma_0 + \omega_A \sigma_A + \omega_B \sigma_B + \omega_C \sigma_C = \omega_0\sigma_0 + \vec{\omega}\bullet\vec{\sigma} = \omega_0 \mathbf{1} + \omega\boldsymbol{\sigma}_\omega$$

$$= \Omega_0 \mathbf{1} + \Omega_A \mathbf{S}_A + \Omega_B \mathbf{S}_B + \Omega_C \mathbf{S}_C = \Omega_0 \mathbf{1} + \vec{\Omega}\bullet\vec{\mathbf{S}}$$

$$= \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} + 2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + 2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}$$

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0th component unchanged

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Notation for 2D Spinor space

$$=\omega_0\ \sigma_0\ +\ \omega_A\ \sigma_A\ +\omega_B\ \sigma_B\ +\omega_C\ \sigma_C\ =\omega_0\sigma_0+\vec{\omega}\bullet\vec{\sigma}=\omega_0\mathbf{1}+\omega\boldsymbol{\sigma}_\omega$$

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$$=e^{-i(\Omega_0\mathbf{1}+\vec{\Omega}\bullet\vec{\mathbf{S}})\cdot t}\ =e^{-i\Omega_0\cdot t}e^{-i\,\vec{\Omega}\cdot t\bullet\mathbf{S}}\qquad =e^{-i\Omega_0\cdot t}\left(\mathbf{1}\cos\frac{\Omega\cdot t}{2}-i\boldsymbol{\sigma}_\omega\sin\frac{\Omega\cdot t}{2}\right)$$

Notation for 3D Vector space

where: $\vec{\boldsymbol{\Theta}}=\vec{\boldsymbol{\Omega}}\cdot t=\begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix}\cdot t=\begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}\cdot t$ and: $\Omega_0=\frac{A+D}{2}$

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

$$\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}=\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+\frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}+B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}+C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$=\omega_0\sigma_0+\omega_A\sigma_A+\omega_B\sigma_B+\omega_C\sigma_C=\omega_0\sigma_0+\vec{\omega}\bullet\vec{\sigma}=\omega_0\mathbf{1}+\omega\boldsymbol{\sigma}_\omega$$

Notation for 2D Spinor space

$$=\Omega_0\mathbf{1}+\Omega_A\mathbf{S}_A+\Omega_B\mathbf{S}_B+\Omega_C\mathbf{S}_C=\Omega_0\mathbf{1}+\vec{\Omega}\bullet\vec{\mathbf{S}}$$

$$=\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+(A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}+2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}+2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}$$

Notation for 3D Vector space

0^{th} component unchanged

components A, B, C switch 1/2-factor from ω-velocity to S-momentum

Symmetry archetypes: A (Asymmetric-diagonal)| B (Bilateral-balanced)| C (Chiral-circular-complex...)

"Crank" vector

The $\{\sigma_1, \sigma_A, \sigma_B, \sigma_C\}$ are the well known *Pauli-spin operators* $\{\sigma_1=\sigma_0, \sigma_B=\sigma_X, \sigma_C=\sigma_Y, \sigma_A=\sigma_Z\}$

The $\{\mathbf{1}, \mathbf{S}_A, \mathbf{S}_B, \mathbf{S}_C\}$ are the *Jordan-Angular-Momentum operators* $\{\mathbf{1}=\sigma_0, \mathbf{S}_B=\mathbf{S}_X, \mathbf{S}_C=\mathbf{S}_Y, \mathbf{S}_A=\mathbf{S}_Z\}$ (Often labeled $\{\mathbf{J}_X, \mathbf{J}_Y, \mathbf{J}_Z\}$)

$$\vec{\varphi}=\begin{pmatrix} \varphi_A \\ \varphi_B \\ \varphi_C \end{pmatrix}=\vec{\omega}\cdot t=\begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\cdot t=\begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix}\cdot t$$

Notation for 2D Spinor space

where: $\vec{\varphi}=\vec{\omega}\cdot t=\begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\cdot t=\begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix}\cdot t$ and: $\omega_0=\frac{A+D}{2}$

$$e^{-i\mathbf{H}\cdot t}=e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}t}=e^{-i(\omega_0\sigma_0+\vec{\omega}\bullet\vec{\sigma})\cdot t}=e^{-i\omega_0\cdot t}e^{-i\,\vec{\omega}\bullet\vec{\sigma}\cdot t}=e^{-i\omega_0\cdot t}e^{-i\boldsymbol{\sigma}_\omega\omega\cdot t}=e^{-i\omega_0\cdot t}\left(\mathbf{1}\cos\omega\cdot t-i\boldsymbol{\sigma}_\omega\sin\omega\cdot t\right)$$

$$=e^{-i(\Omega_0\mathbf{1}+\vec{\Omega}\bullet\vec{\mathbf{S}})\cdot t}=e^{-i\Omega_0\cdot t}e^{-i\,\vec{\Omega}\cdot t\bullet\mathbf{S}}=e^{-i\Omega_0\cdot t}\left(\mathbf{1}\cos\frac{\Omega\cdot t}{2}-i\boldsymbol{\sigma}_\omega\sin\frac{\Omega\cdot t}{2}\right)$$

"Crank" vector

$$\vec{\Theta}=\begin{pmatrix} \Theta_A \\ \Theta_B \\ \Theta_C \end{pmatrix}=\vec{\Omega}\cdot t=\begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix}\cdot t=\begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}\cdot t$$

Notation for 3D Vector space

where: $\vec{\Theta}=\vec{\Omega}\cdot t=\begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix}\cdot t=\begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}\cdot t$ and: $\Omega_0=\frac{A+D}{2}$

BoxIt (B-Type) Simulation
A=4.0, B=-0.2, C=0, D=4.0

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2{}_t\mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$*-expansion in ABCD-Types)* $\mathbf{H} = \omega_\mu \boldsymbol{\sigma}_\mu$

Derive $\boldsymbol{\sigma}$*-exponential time evolution (or revolution) operator* $\mathbf{U} = e^{-i\mathbf{H}t} = e^{-i\boldsymbol{\sigma}_\mu \omega_\mu t}$

Spinor arithmetic like complex arithmetic

Spinor vector algebra like complex vector algebra

Spinor exponentials like complex exponentials (“Crazy-Thing”-Theorem)

Geometry of evolution (or revolution) operator $\mathbf{U} = e^{-i\mathbf{H}t} = e^{-i\boldsymbol{\sigma}_\mu \omega_\mu t}$

The “mysterious” factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

➔ *2D Spinor vs 3D vector rotation*

NMR Hamiltonian: 3D Spin Moment **m** *in* **B** *field*

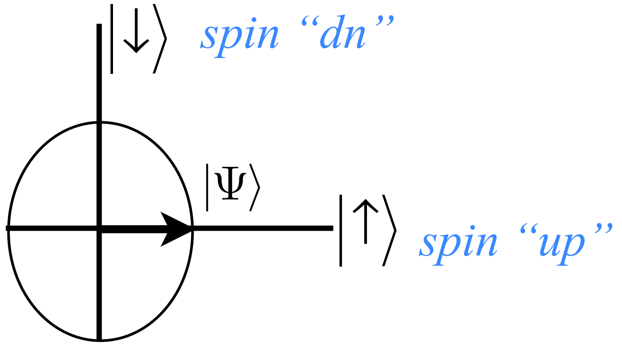
The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

U(2):2D Spinor {|↑⟩,|↓⟩}-space (complex)

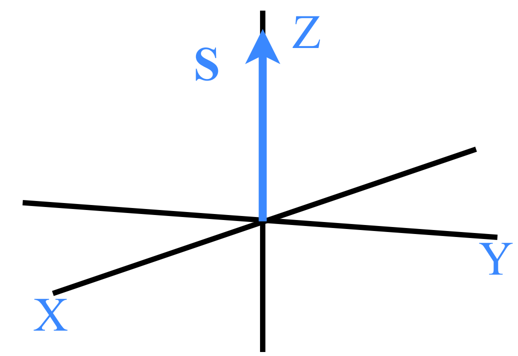
R(3):3D Spin Vector {S_X, S_Y, S_Z}-space (real)

State vector $|\Psi\rangle = |\uparrow\rangle\langle\uparrow|\Psi\rangle + |\downarrow\rangle\langle\downarrow|\Psi\rangle$

Spin vector $\mathbf{S} = |X\rangle\langle X|\mathbf{S}\rangle + |Y\rangle\langle Y|\mathbf{S}\rangle + |Z\rangle\langle Z|\mathbf{S}\rangle$



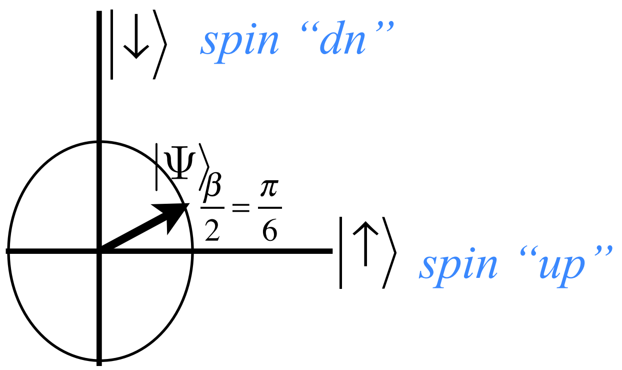
$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}$$



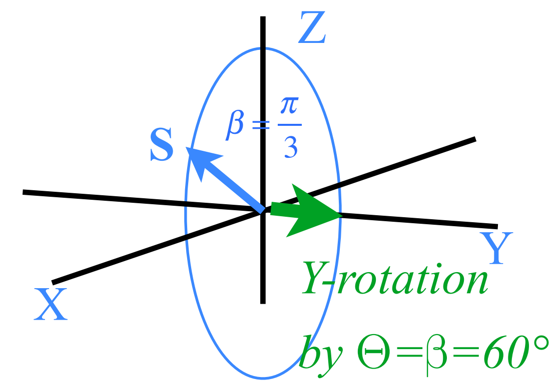
$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$$

$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix} = \begin{pmatrix}\cos\frac{\beta}{2} & -\sin\frac{\beta}{2}\\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}$$

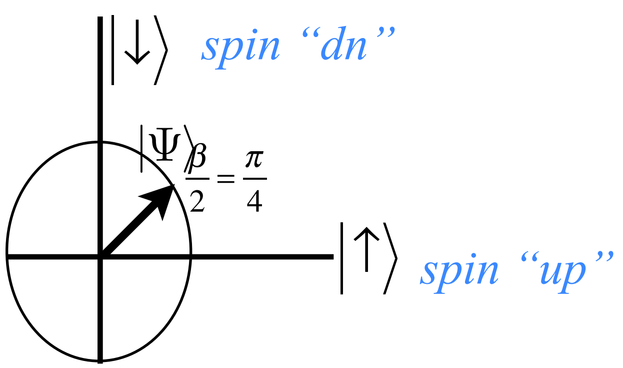
$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix} = \begin{pmatrix}\cos\beta & 0 & \sin\beta\\ 0 & 1 & 0\\ -\sin\beta & 0 & \cos\beta\end{pmatrix}\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$$



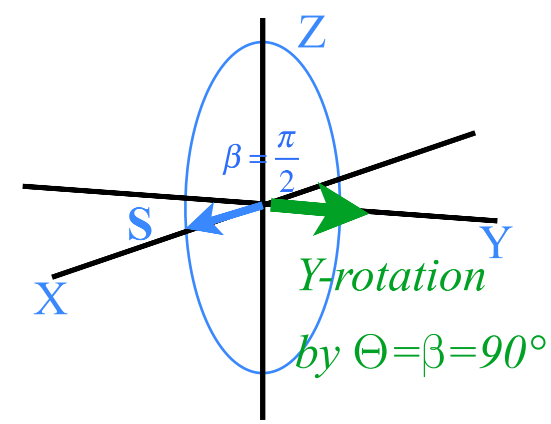
$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix} = \begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}$$



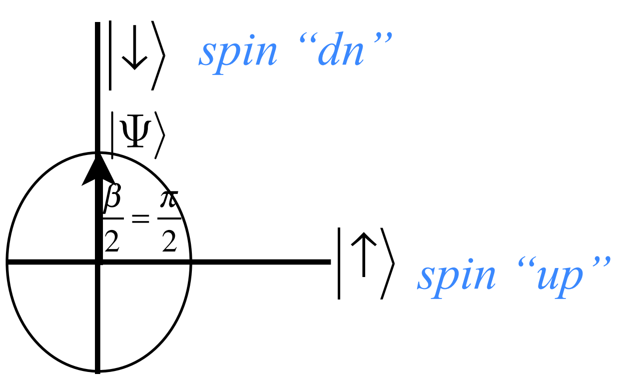
$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix} = \begin{pmatrix}\sin\beta\\ 0\\ \cos\beta\end{pmatrix} = \begin{pmatrix}\sqrt{3}/2\\ 0\\ 1/2\end{pmatrix}$$



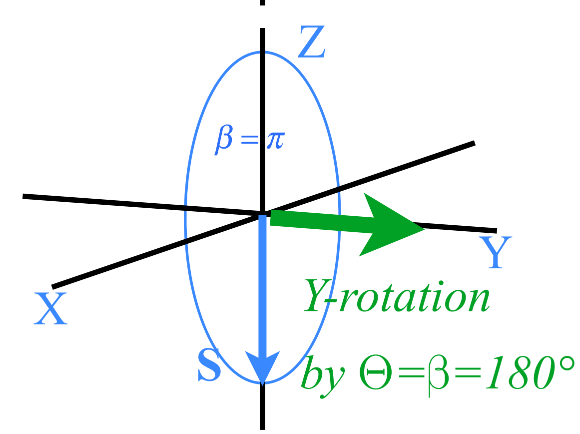
$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix} = \begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$$



$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix} = \begin{pmatrix}\sin\beta\\ 0\\ \cos\beta\end{pmatrix} = \begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$$



$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix} = \begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix}$$



$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix} = \begin{pmatrix}\sin\beta\\ 0\\ \cos\beta\end{pmatrix} = \begin{pmatrix}0\\ 0\\ -1\end{pmatrix}$$

Life in 2D Spinor space is "Half-Fast"

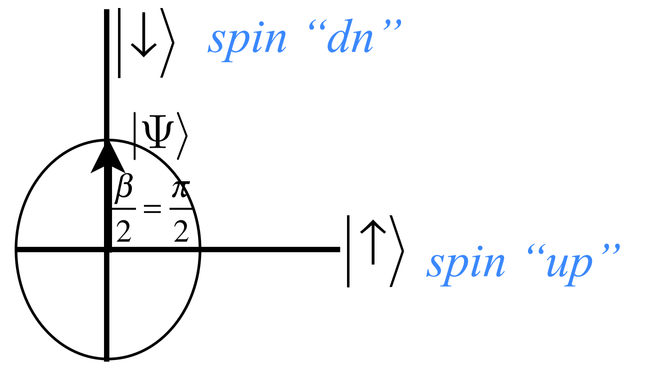
The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

U(2):2D Spinor {|↑⟩,|↓⟩}-space (complex)

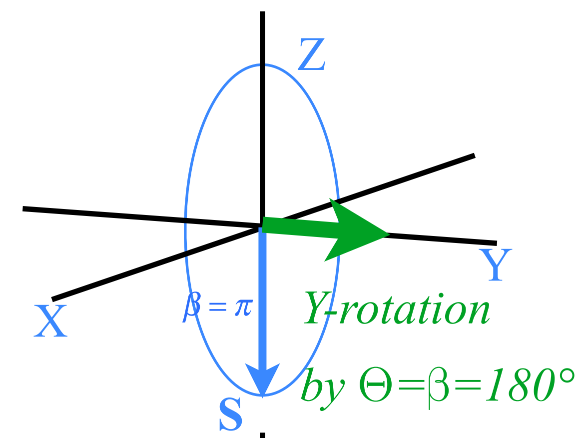
R(3):3D Spin Vector {Sx, Sy, Sz}-space (real)

State vector $|\Psi\rangle = |\uparrow\rangle\langle\uparrow|\Psi\rangle + |\downarrow\rangle\langle\downarrow|\Psi\rangle$

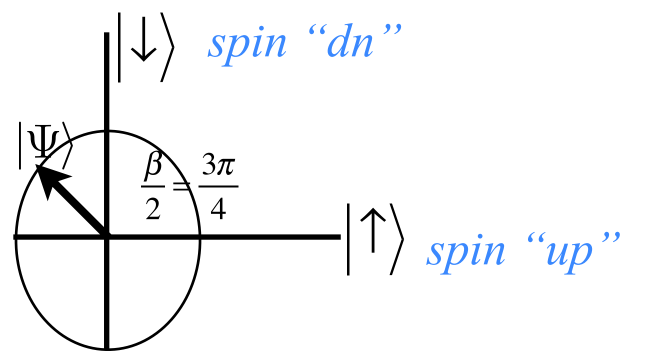
Spin vector $\mathbf{S} = |X\rangle\langle X|\mathbf{S}\rangle + |Y\rangle\langle Y|\mathbf{S}\rangle + |Z\rangle\langle Z|\mathbf{S}\rangle$



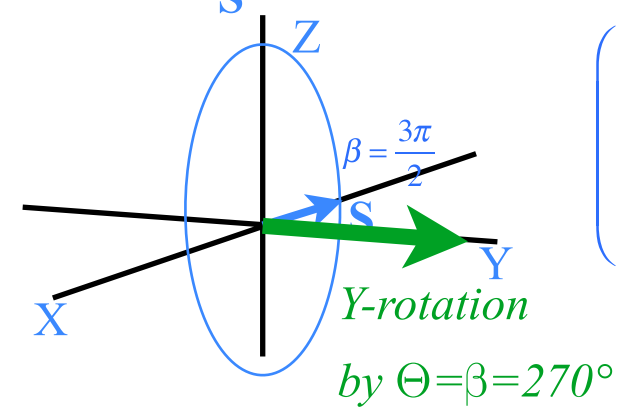
$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix} = \begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix}$$



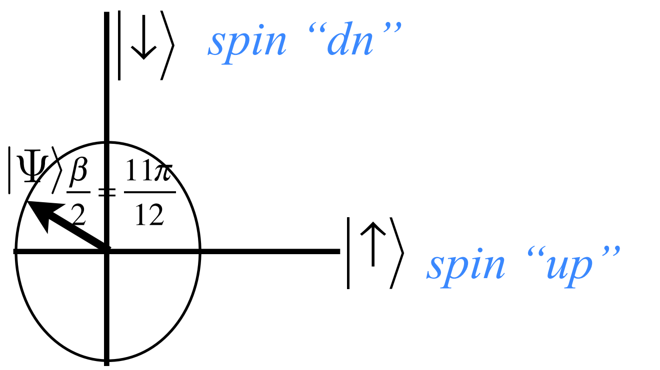
$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix} = \begin{pmatrix}\sin\beta\\ 0\\ \cos\beta\end{pmatrix} = \begin{pmatrix}0\\ 0\\ -1\end{pmatrix}$$



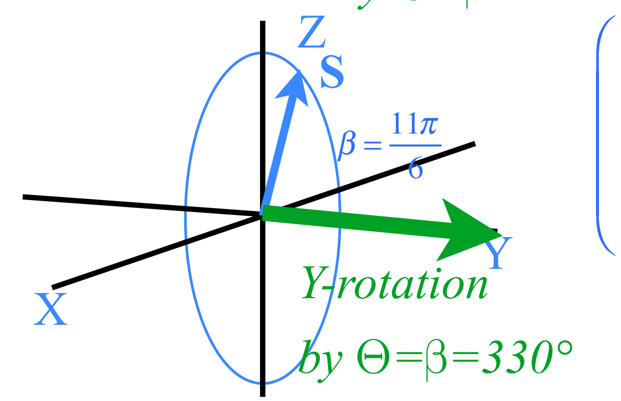
$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix} = \begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix}\frac{-1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$$



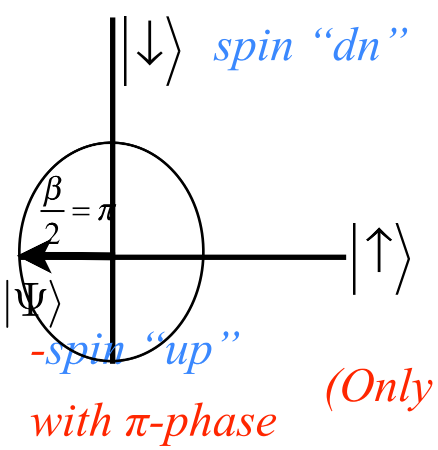
$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix} = \begin{pmatrix}\sin\beta\\ 0\\ \cos\beta\end{pmatrix} = \begin{pmatrix}-1\\ 0\\ 0\end{pmatrix}$$



$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix} = \begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix}\frac{-\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}$$

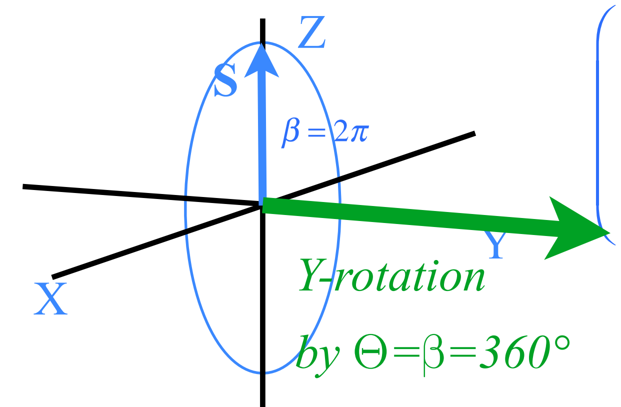


$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix} = \begin{pmatrix}\sin\beta\\ 0\\ \cos\beta\end{pmatrix} =$$



$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix} = \begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix}-1\\ 0\end{pmatrix}$$

(Only "half-way" home after 2π =360° rotation)



$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix} = \begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$$

Life in 2D Spinor space is "Half-Fast" and needs Θ=4π=720° to return to original state

ANALOGY: *2-State Schrodinger:* $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$*-expansion in ABCD-Types)* $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$

Derive $\boldsymbol{\sigma}$*-exponential time evolution (or revolution) operator* $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

Spinor arithmetic like complex arithmetic

Spinor vector algebra like complex vector algebra

Spinor exponentials like complex exponentials

Geometry of evolution (or revolution) operator $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

2D Spinor vs 3D vector rotation

➔ *NMR Hamiltonian: 3D Spin Moment* **m** *in* **B** *field*

Hamiltonian for NMR: 3D Spin Moment Vector $\mathbf{m}=(m_x, m_y, m_z,)$ *in field* $\mathbf{B}=(B_x, B_y, B_z)$

$$\mathbf{H}=\mathbf{m}\bullet\mathbf{B}=g\,\sigma\bullet\mathbf{B}=\begin{pmatrix} gB_Z & gB_X-igB_Y \\ gB_X+igB_Y & -gB_Z \end{pmatrix}=gB_Z\begin{pmatrix}1&0\\0&-1\end{pmatrix}+gB_X\begin{pmatrix}0&1\\1&0\end{pmatrix}+gB_Y\begin{pmatrix}0&-i\\i&0\end{pmatrix}$$

$$= gB_Z\ \sigma_A + gB_X\ \sigma_X + gB_Y\ \sigma_Y = \vec{\omega}\bullet\vec{\sigma}=\omega\sigma_\omega$$

Notation for 2D Spinor space

Symmetry archetypes: *A (Asymmetric-diagonal)*| *B (Bilateral-balanced)*| *C (Chiral-circular-complex...)*

The $\{\sigma_I, \sigma_A, \sigma_B, \sigma_C\}$ are the well known *Pauli-spin operators* $\{\sigma_I=\sigma_0, \sigma_B=\sigma_X, \sigma_C=\sigma_Y, \sigma_A=\sigma_Z\}$

Hamiltonian for NMR: 3D Spin Moment Vector $\mathbf{m}=(m_x, m_y, m_z)$ *in field* $\mathbf{B}=(B_x, B_y, B_z)$

$$\mathbf{H}=\mathbf{m}\bullet\mathbf{B}=g\,\sigma\bullet\mathbf{B}=\begin{pmatrix} gB_Z & gB_X-igB_Y \\ gB_X+igB_Y & -gB_Z \end{pmatrix}=gB_Z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}+gB_X\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}+gB_Y\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

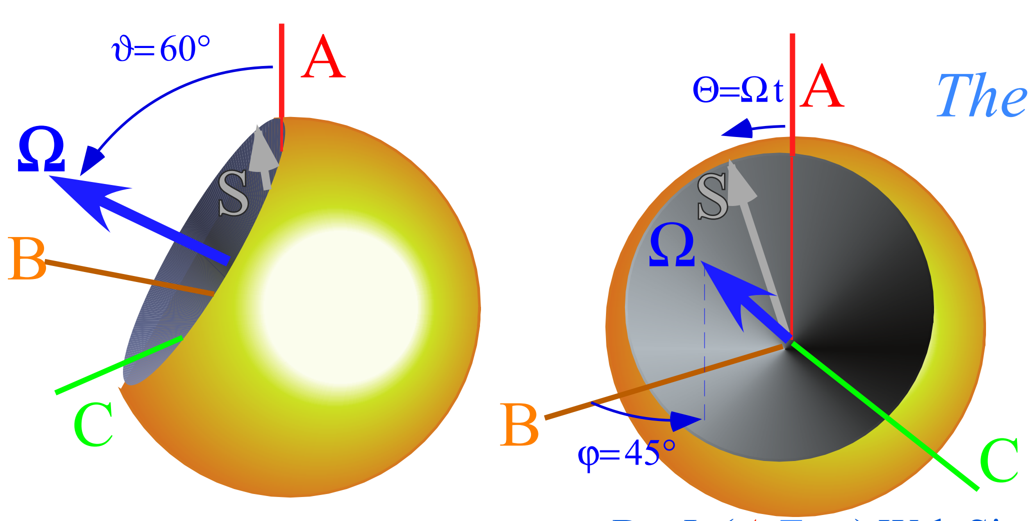
$$= gB_Z\ \sigma_A + gB_X\ \sigma_X + gB_Y\ \sigma_Y = \vec{\omega}\bullet\vec{\sigma}=\omega\sigma_\omega$$

Notation for 2D Spinor space

Symmetry archetypes: *A (Asymmetric-diagonal)*| *B (Bilateral-balanced)*| *C (Chiral-circular-complex...)*

The $\{\sigma_1, \sigma_A, \sigma_B, \sigma_C\}$ are the well known *Pauli-spin operators* $\{\sigma_1=\sigma_0, \sigma_B=\sigma_X, \sigma_C=\sigma_Y, \sigma_A=\sigma_Z\}$

Notation for 3D Vector space



The driving $\mathbf{\Theta}=\mathbf{\Omega}t$ *vector is defined by the ABCD of Hamiltonian* **H**.

The driven spin vector **S** *defines the state. But, how?*

Fig. 3.4.2 Two views of Hamilton crank vector $\Omega(\varphi,\vartheta)$ *whirling Stokes state vector* ***S*** *in ABC-space.*

BoxIt (*A-Type*) Web Simulation: A=4.9, B=C=0, D=4.0

BoxIt (*B-Type*) *Simulation A=4.0, B=-0.2, C=0, D=4.0*

BoxIt (*C-Type*) *Simulation A=4.055, B=0, C=0.1, D=4.055*

Euler's state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

➤ *Spin-1 (3D-real vector) case*

Spin-1/2 (2D-complex spinor) case

Euler's state definition using rotations **R**$(\alpha,0,0)$*,* **R**$(0,\beta,0)$*,and* **R**$(0,0,\gamma)$

Spin-1 (3D-real vector) case

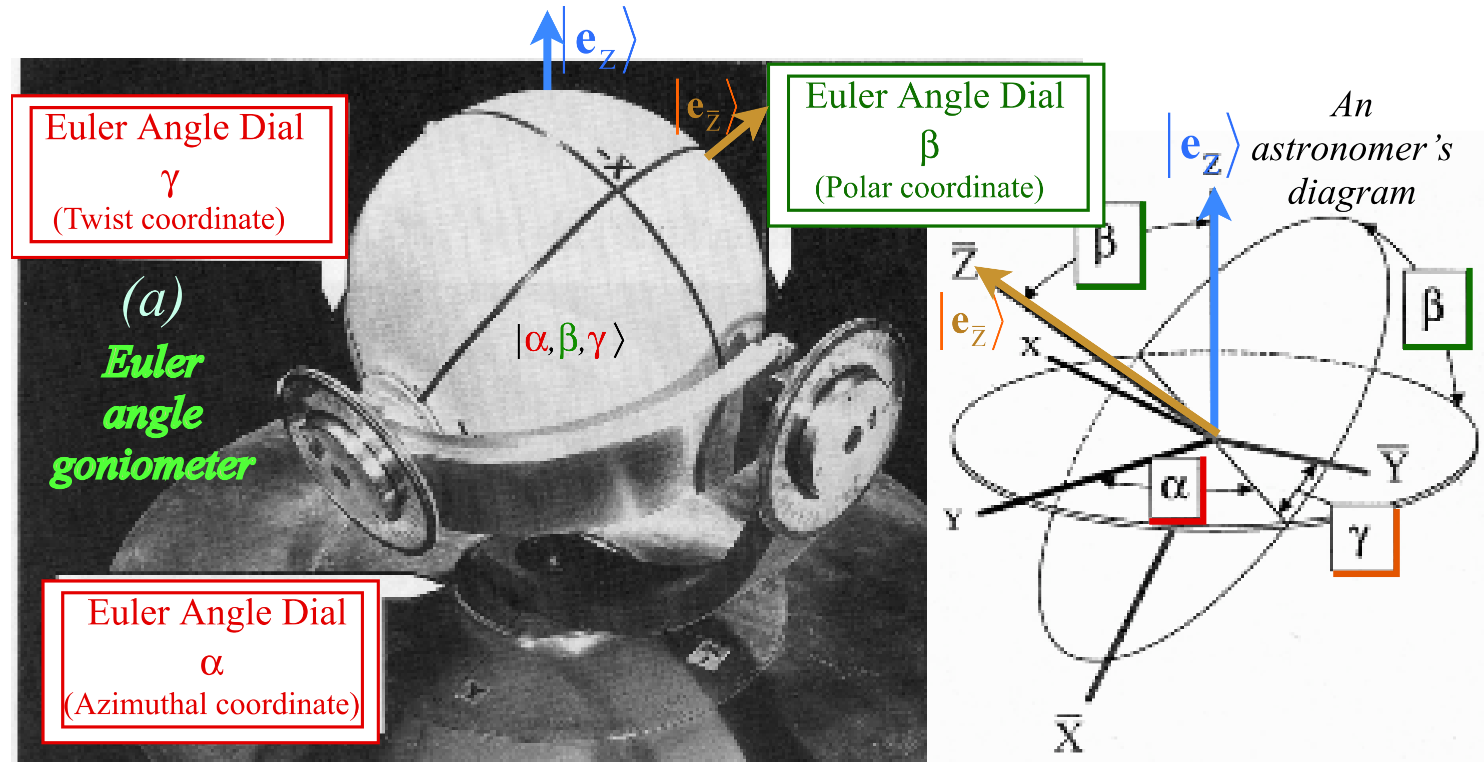
See also Alternate treatments & Supplemental references

Euler Angle machine: *CMwB Unit 6, pg 23*

Previously in our own Lectures. *8 & 9*

QTofCA Unit 3 Ch. 10A-B

Group Theory in QM 5093 *Lectures 6, 7, 8, and 9-10*



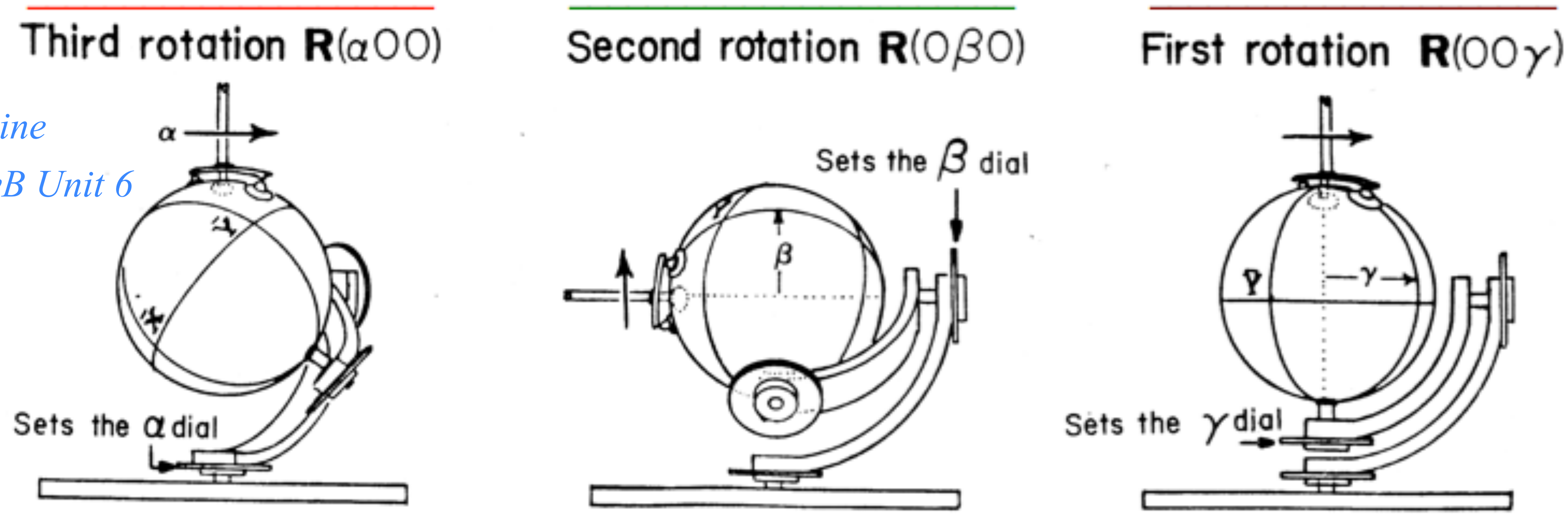
Development has begun on a web based version of this tool, but much of the App is at present (10/2/2018), in an indeterminate state.
We plan to use *Babylon.JS*, as a shim to buttress the WebGL (web graphics layer)!

Web based U(2) Calculator - Euler State

Euler's rotation state definition using rotations **R**$(\alpha,0,0)$, **R**$(0,\beta,0)$, *and* **R**$(0,0,\gamma)$

Spin-1 (3D-real vector) case

Euler Angle machine discussed in CMwB Unit 6



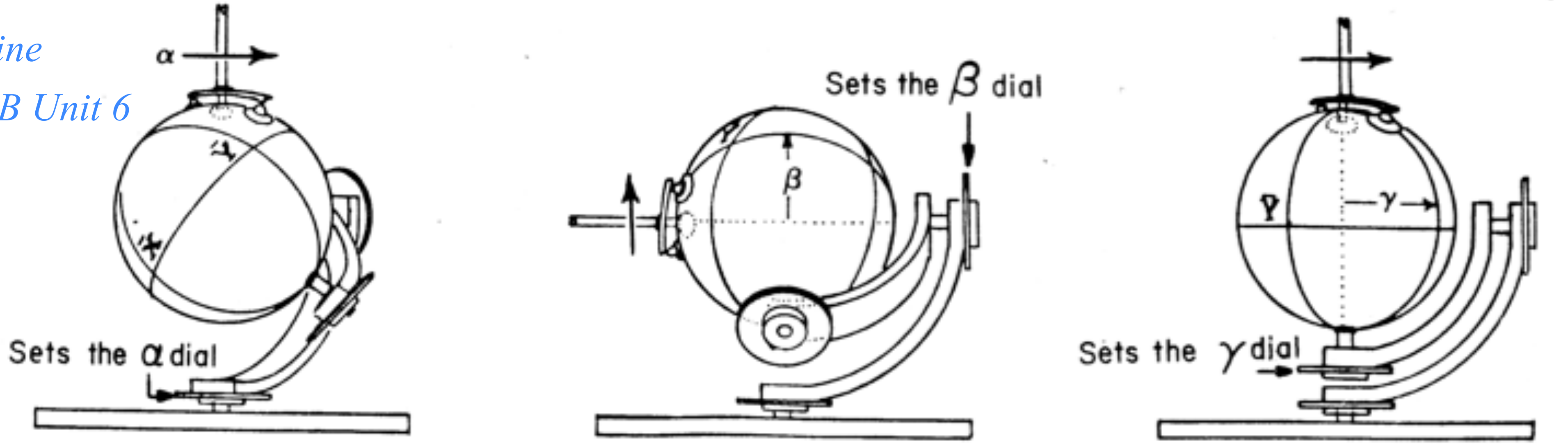
$$\langle R(\alpha\beta\gamma)\rangle = \langle R(\alpha 00)\rangle \quad \langle R(0\beta 0)\rangle \quad \langle R(00\gamma)\rangle$$

Euler's rotation state definition using rotations $\mathbf{R}(\alpha,0,0)$*,* $\mathbf{R}(0,\beta,0)$*, and* $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case

Euler Angle machine discussed in CMwB Unit 6

Third rotation $\mathbf{R}(\alpha 00)$ | Second rotation $\mathbf{R}(0\beta 0)$ | First rotation $\mathbf{R}(00\gamma)$



$$\langle R(\alpha\beta\gamma)\rangle = \langle R(\alpha 00)\rangle \quad \langle R(0\beta 0)\rangle \quad \langle R(00\gamma)\rangle$$

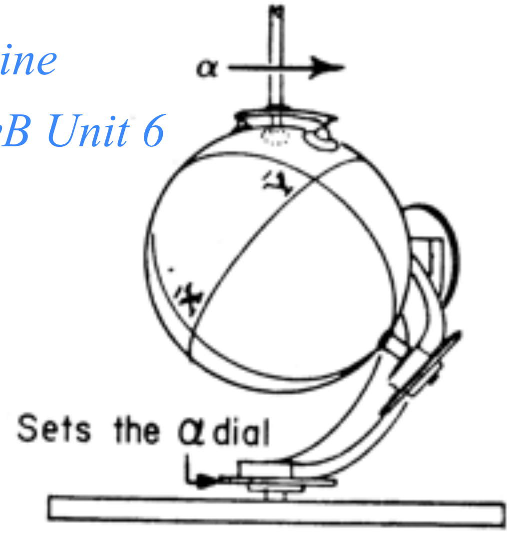
$$= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Euler's rotation state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

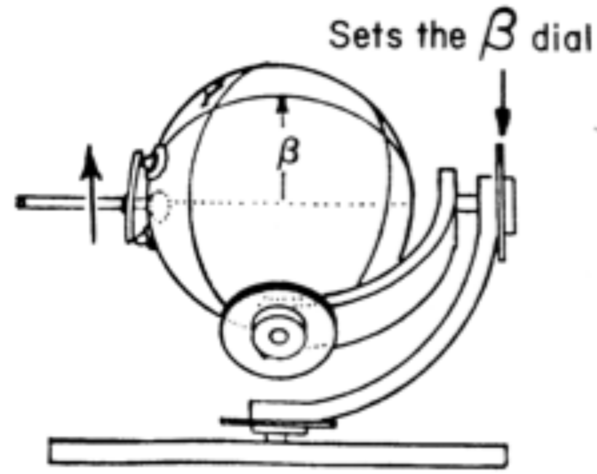
Spin-1 (3D-real vector) case

Euler Angle machine discussed in CMwB Unit 6

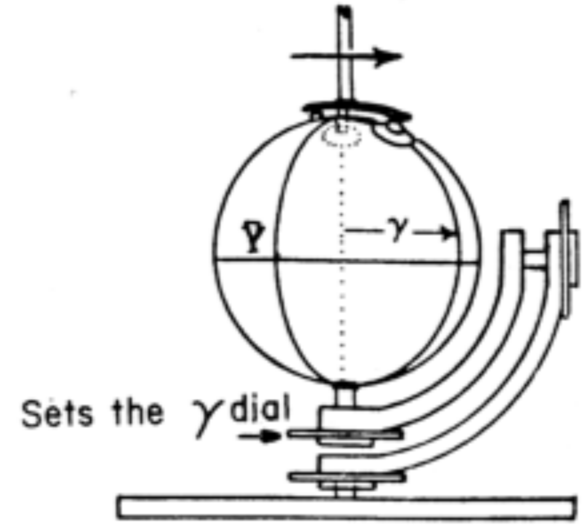
Third rotation $\mathbf{R}(\alpha 0 0)$



Second rotation $\mathbf{R}(0 \beta 0)$



First rotation $\mathbf{R}(0 0 \gamma)$



$$\langle R(\alpha\beta\gamma)\rangle = \langle R(\alpha 00)\rangle \quad \langle R(0\beta 0)\rangle \quad \langle R(00\gamma)\rangle$$

$$= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$|\mathbf{e}_{\bar{X}}\rangle = R(\alpha\beta\gamma)|\mathbf{e}_X\rangle \qquad |\mathbf{e}_{\bar{Y}}\rangle = R(\alpha\beta\gamma)|\mathbf{e}_Y\rangle \qquad |\mathbf{e}_{\bar{Z}}\rangle = R(\alpha\beta\gamma)|\mathbf{e}_Z\rangle$$

$$\left(\langle \mathbf{e}_A | R(\alpha\beta\gamma) | \mathbf{e}_B\rangle\right) = \begin{matrix} \langle \mathbf{e}_X| \\ \langle \mathbf{e}_Y| \\ \langle \mathbf{e}_Z| \end{matrix} \begin{pmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma & \boxed{\cos\alpha\sin\beta} \\ \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \boxed{\sin\alpha\sin\beta} \\ -\cos\gamma\sin\beta & \sin\gamma\sin\beta & \boxed{\cos\beta} \end{pmatrix}$$

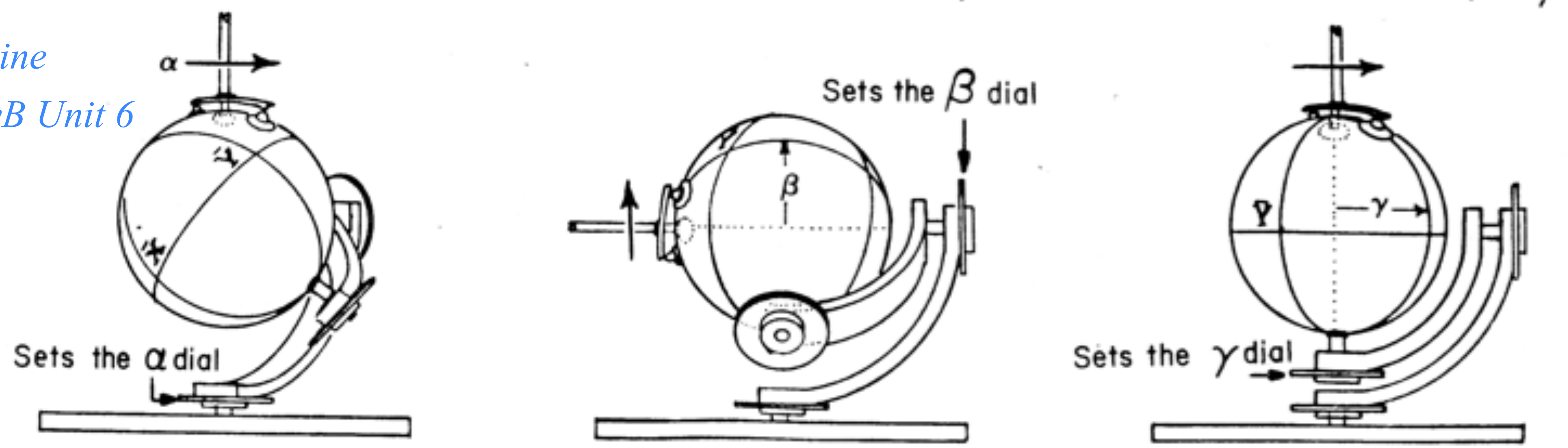
Note lab-frame polar coordinates of Z-body vector $|\mathbf{e}_{\bar{Z}}\rangle$

Euler's rotation state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case

Third rotation $\mathbf{R}(\alpha 00)$ | Second rotation $\mathbf{R}(0\beta 0)$ | First rotation $\mathbf{R}(00\gamma)$

Euler Angle machine discussed in CMwB Unit 6



$$\langle R(\alpha\beta\gamma)\rangle = \langle R(\alpha 00)\rangle \quad \langle R(0\beta 0)\rangle \quad \langle R(00\gamma)\rangle$$

$$= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$|\mathbf{e}_{\bar{X}}\rangle = R(\alpha\beta\gamma)|\mathbf{e}_X\rangle \qquad |\mathbf{e}_{\bar{Y}}\rangle = R(\alpha\beta\gamma)|\mathbf{e}_Y\rangle \qquad |\mathbf{e}_{\bar{Z}}\rangle = R(\alpha\beta\gamma)|\mathbf{e}_Z\rangle$$

$$\left(\langle \mathbf{e}_A | R(\alpha\beta\gamma) | \mathbf{e}_B\rangle\right) = \begin{matrix} \langle \mathbf{e}_X| \\ \langle \mathbf{e}_Y| \\ \langle \mathbf{e}_Z| \end{matrix} \begin{pmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta \\ \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta \\ -\cos\gamma\sin\beta & \sin\gamma\sin\beta & \cos\beta \end{pmatrix}$$

Note lab-frame polar coordinates of Z-body vector $|\mathbf{e}_{\bar{Z}}\rangle$

...and body-frame polar coordinates of Z-lab $|\mathbf{e}_Z\rangle$

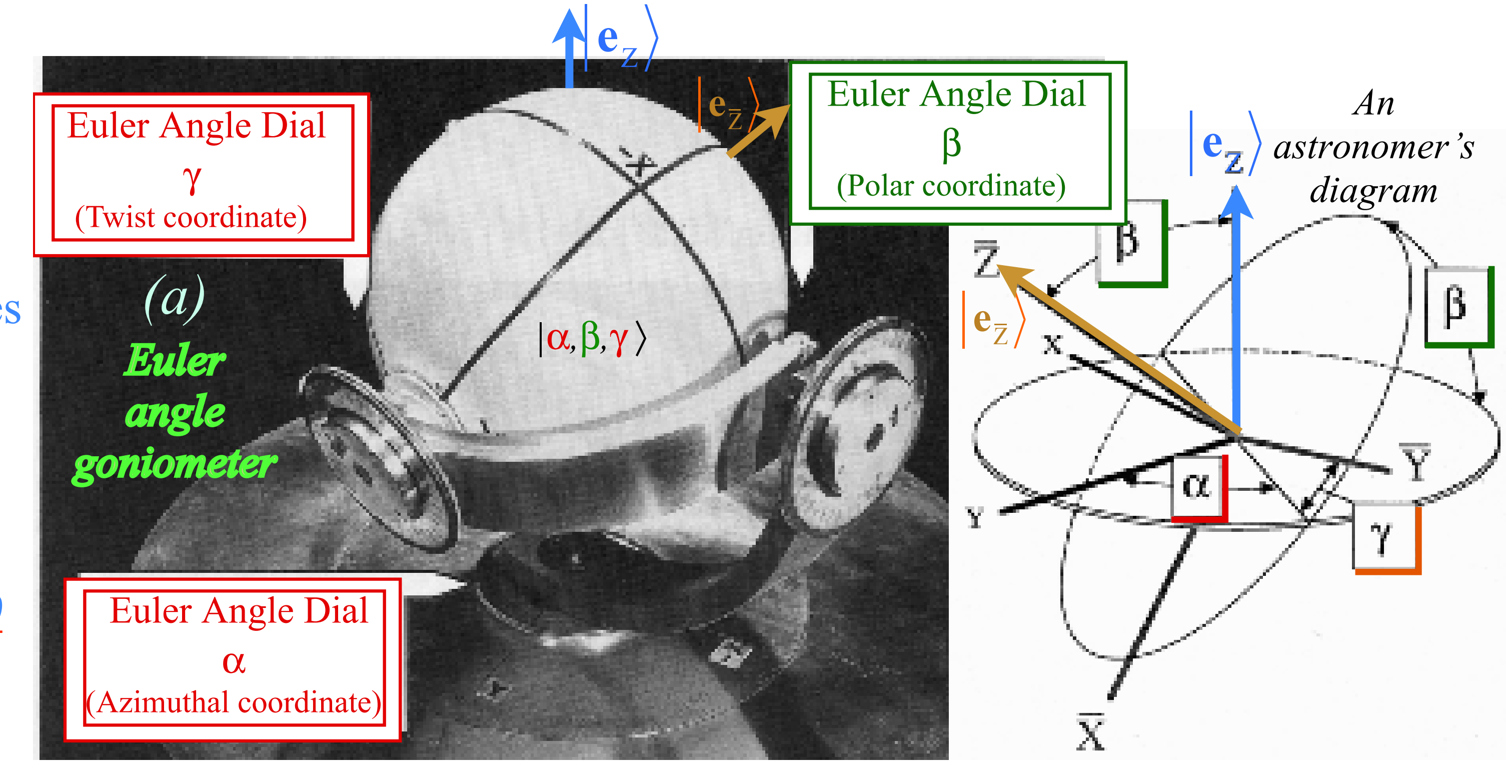
See also Alternate treatments & Supplemental references

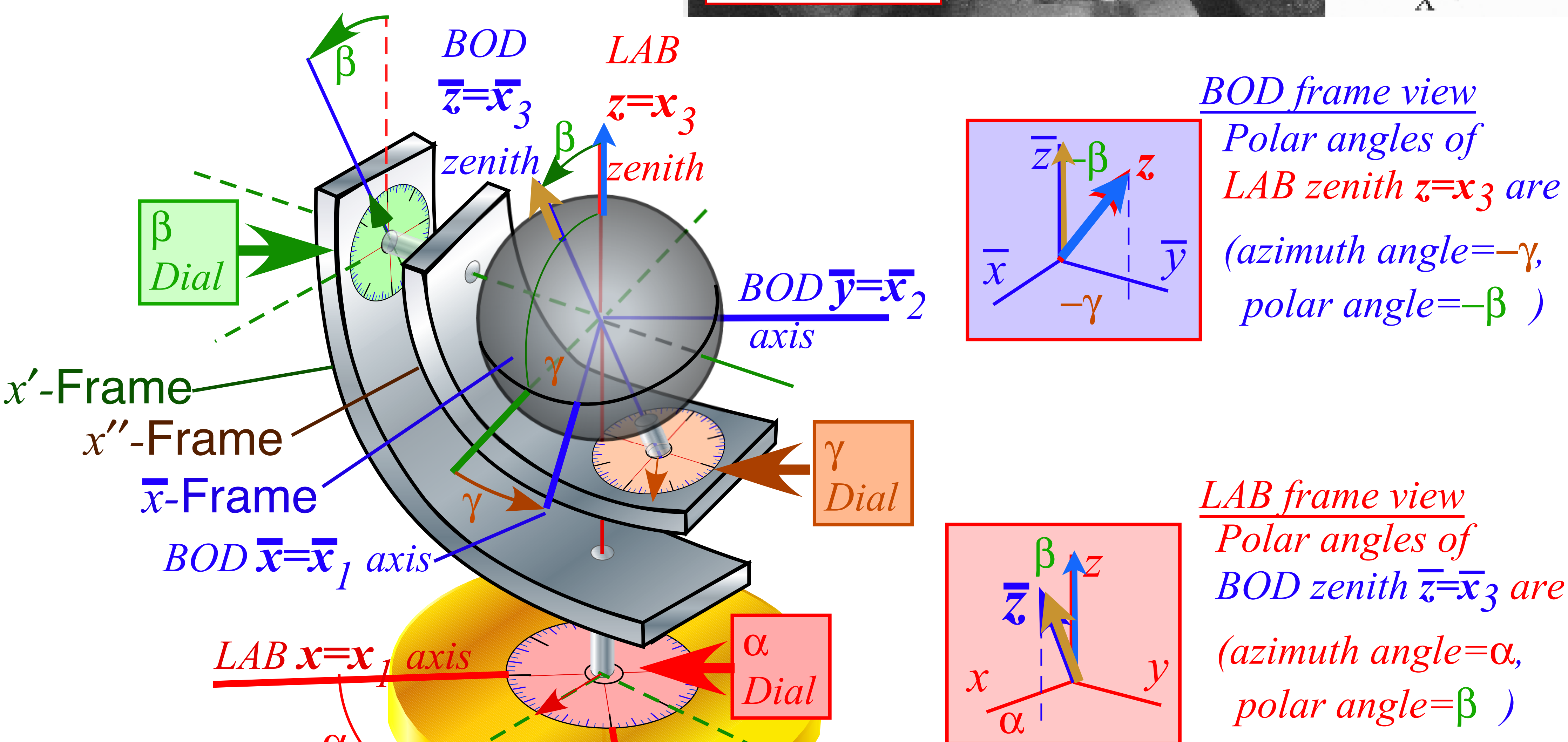
Euler Angle machine: *CMwB Unit 6, pg 23*

Previously in our own Lectures. *8* & *9*

QTofCA Unit 3 Ch. 10A-B

Group Theory in QM 5093 *Lectures 6, 7, 8, and 9-10*





$$\mathbf{x}'_1=\mathbf{x}_1\cos\alpha+\mathbf{x}_2\sin\alpha$$

$$\mathbf{x}'_2=-\mathbf{x}_1\sin\alpha+\mathbf{x}_2\cos\alpha$$

$$=\bar{\mathbf{x}}_1\sin\gamma+\bar{\mathbf{x}}_2\cos\gamma$$

Euler's state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case

➤ *Spin-1/2 (2D-complex spinor) case*

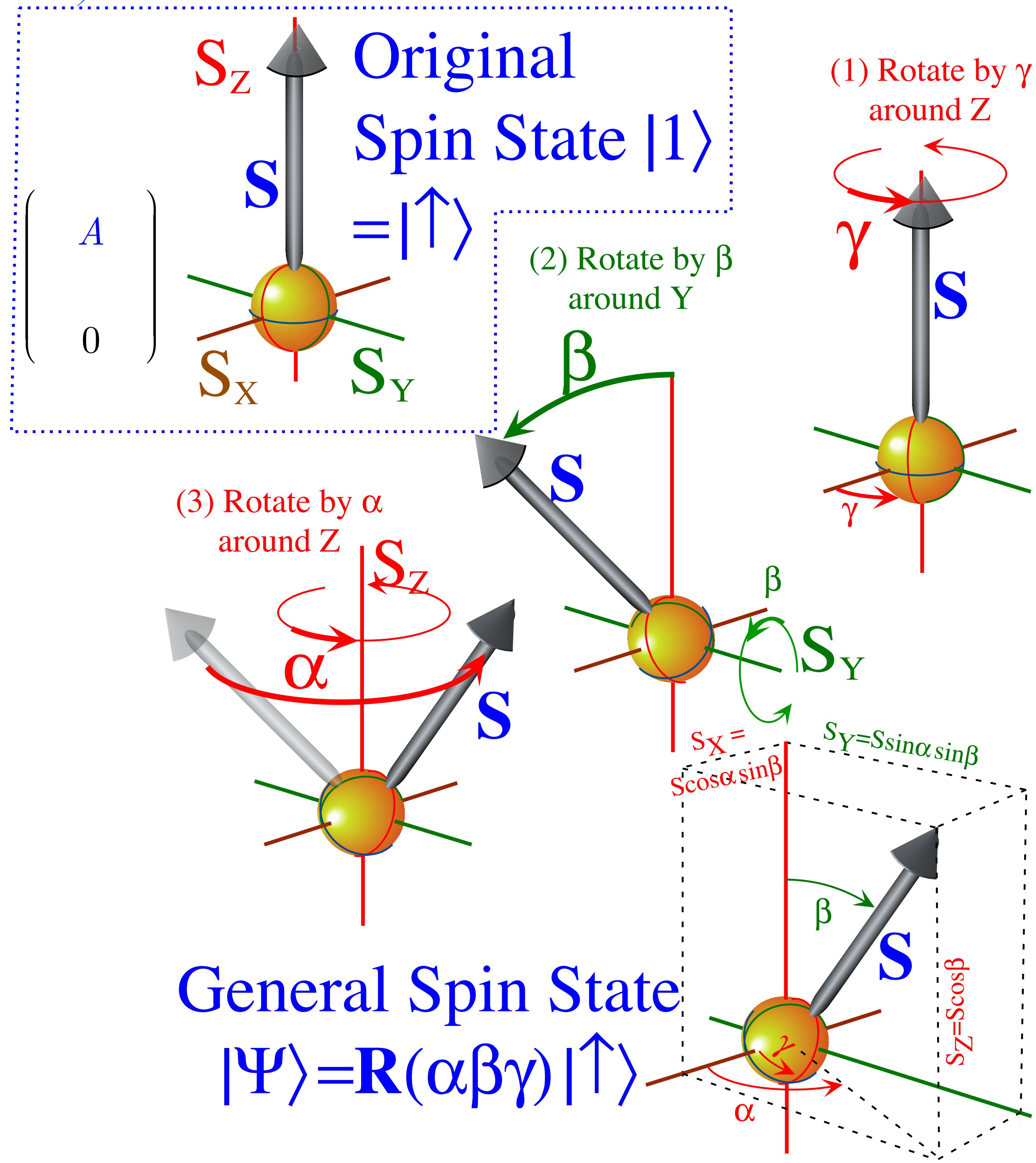
Euler's rotation state definition using rotations **R**$(\alpha,0,0)$, **R**$(0,\beta,0)$, *and* **R**$(0,0,\gamma)$

Spin-1/2 (2D-complex spinor) case

$$\left|a\right\rangle = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow\right\rangle$$

$$= \mathbf{R}[\alpha \text{ about } Z]\cdot\mathbf{R}[\beta \text{ about } Y]\cdot\mathbf{R}[\gamma \text{ about } Z]\left|\uparrow\right\rangle$$

$$= \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix} \begin{pmatrix} A \\ 0 \end{pmatrix}$$



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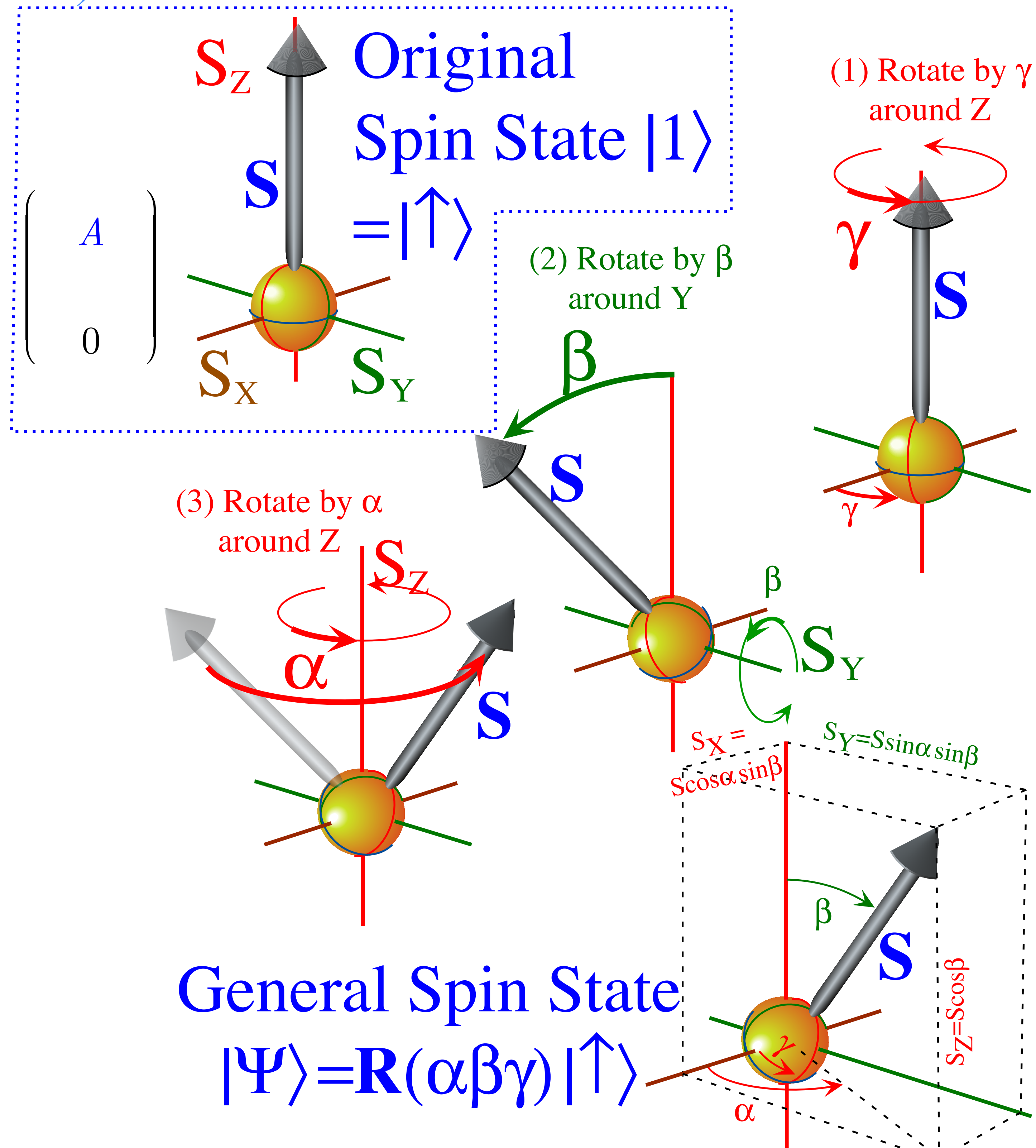
$$= \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} A \\ 0 \end{pmatrix}$$

$$= A \begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$



3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

➤ *Asymmetry* $S_A = S_Z$, *Balance* $S_B = S_X$, *and Chirality* $S_C = S_Y$

Polarization ellipse and spinor state dynamics

The "Great Spectral Avoided-Crossing" and A-to-B-to-A symmetry breaking

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This defines real *3D spin vector* (S_A, S_B, S_C) "pointing" to a polarization ellipse or state.

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$$\textit{Chirality}\quad S_C = \frac{1}{2}\left(a\middle|\sigma_C\middle|a\right) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{-i}{2}\left[a_1^* a_2 - a_2^* a_1\right] = \left[x_1 p_2 - x_2 p_1\right]$$

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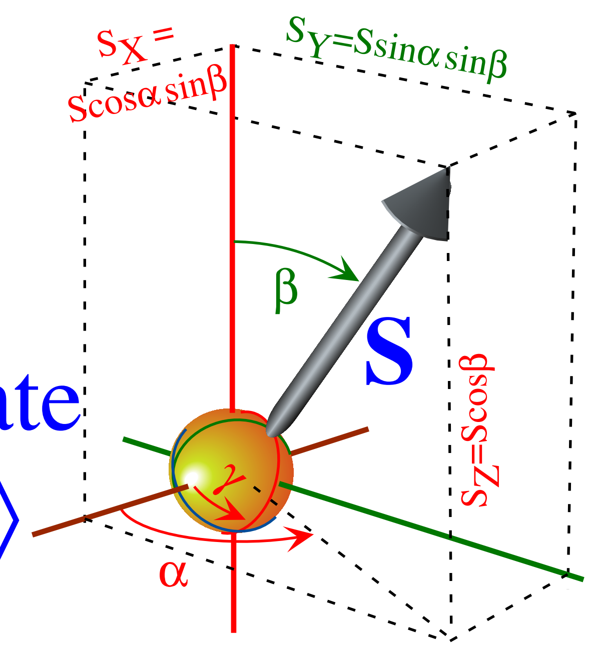
General Spin State

$|\Psi\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\uparrow\rangle$



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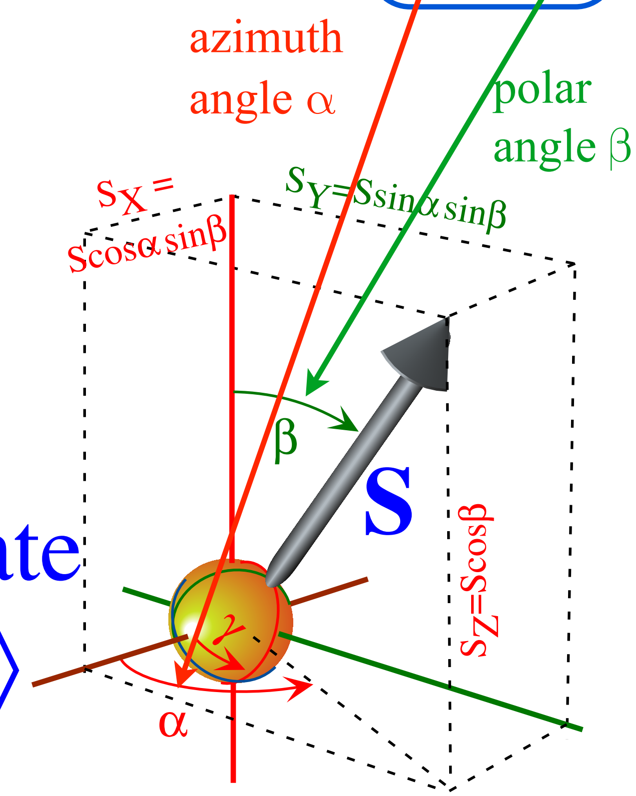

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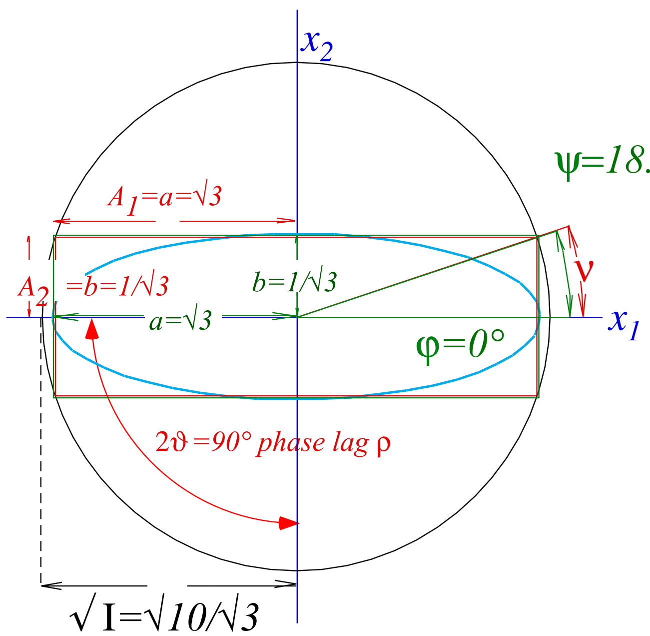

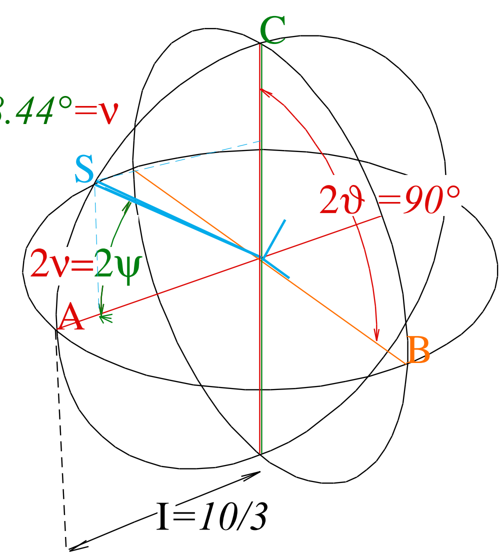

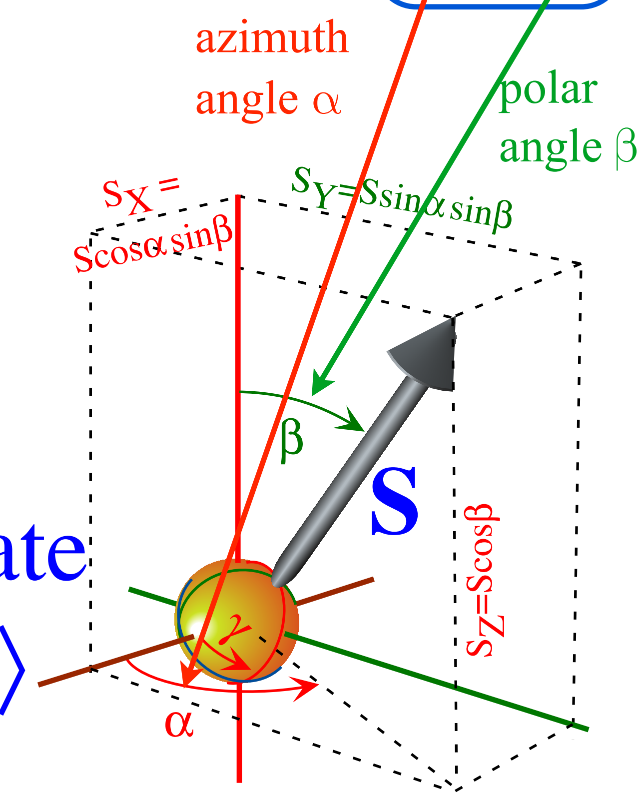

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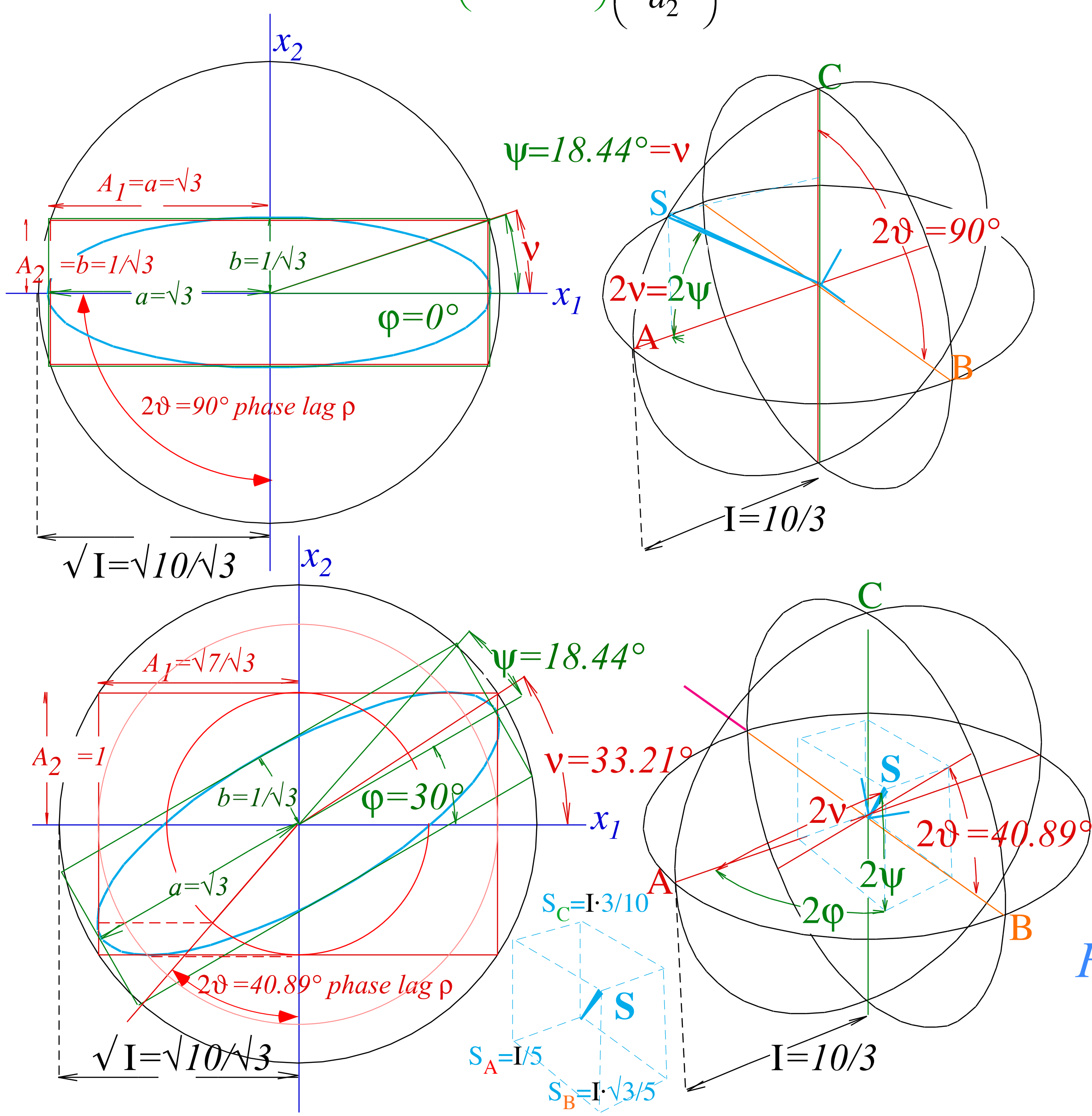

General Spin State
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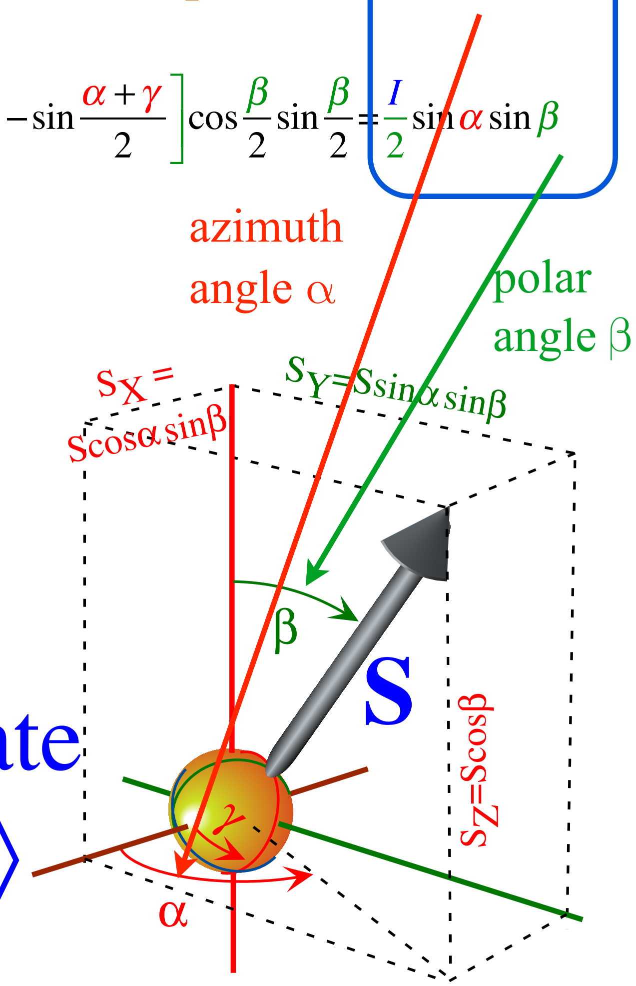
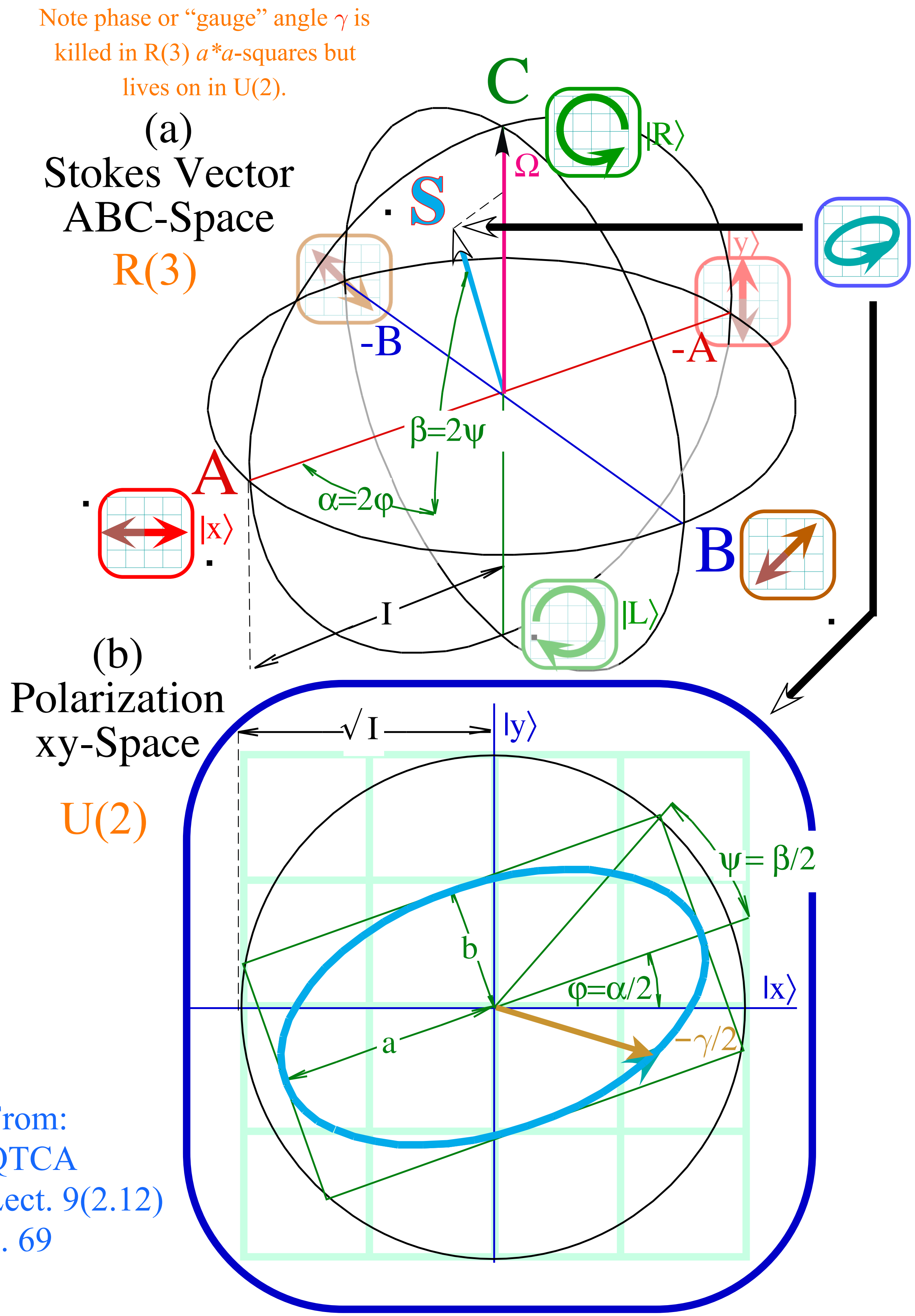


Further explanation of polarization geometry given in Lecture 23 p. 93 to 125

Note phase or "gauge" angle γ is killed in R(3) *a*a*-squares but lives on in U(2).

Polarization ellipse and spinor state dynamics



From:
QTCA
Lect. 9(2.12)
p. 69

Fig. 3.4.5 Polarization variables (a) Stokes real-vector space (ABC) (b) Complex xy-spinor-space (x_1,x_2).

Further explanation of polarization geometry given in Lecture 23 p. 93 to 125

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Polarization ellipse and spinor state dynamics

The "Great Spectral Avoided-Crossing" and A-to-B-to-A symmetry breaking

Polarization ellipse and spinor state dynamics (A-Type motion)

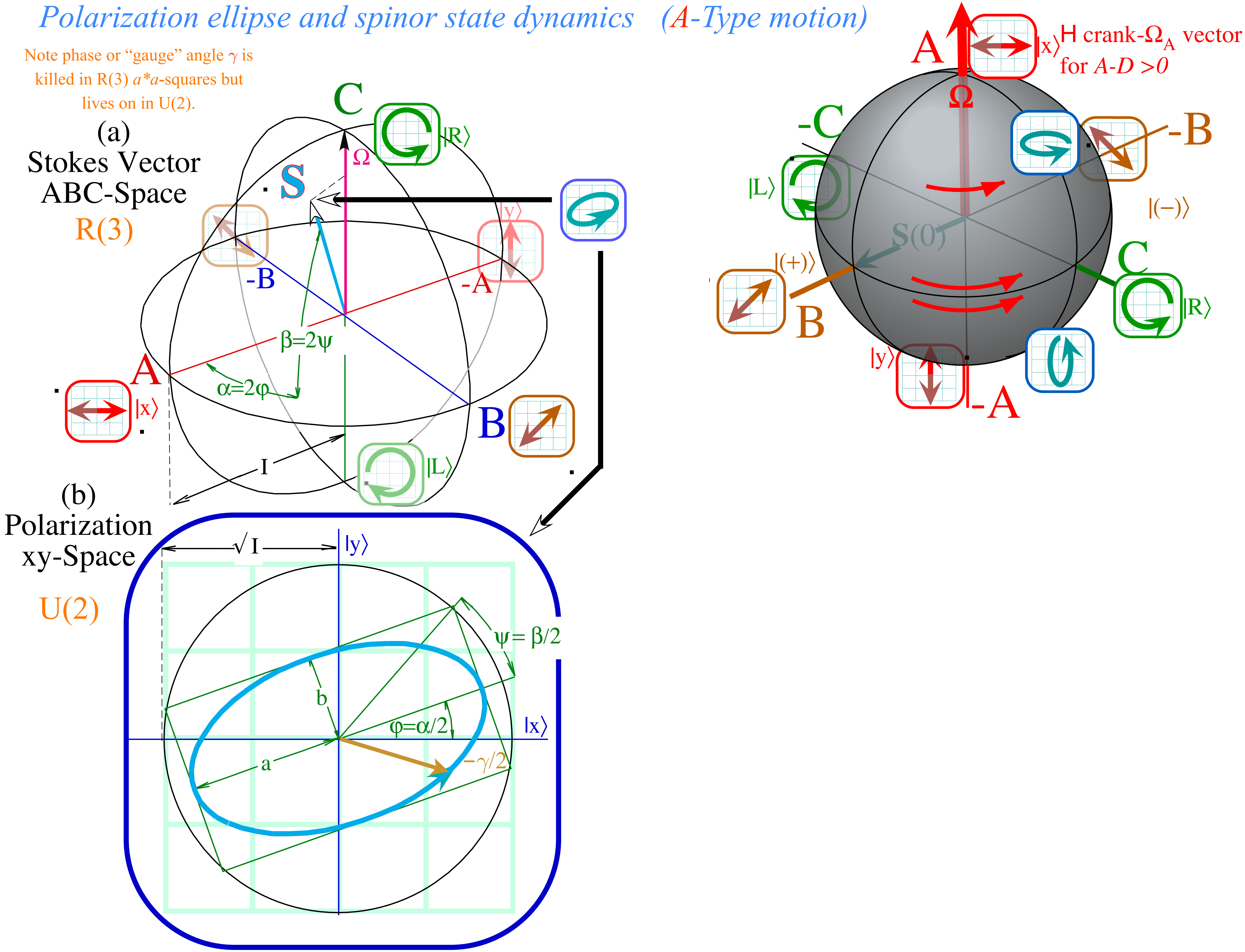


Fig. 3.4.5 Polarization variables (a) Stokes real-vector space (ABC) (b) Complex xy-spinor-space (x_1,x_2).

The ABC's of U(2) dynamics (A-Type motion)

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

From:
QTCA
Lect. 9(2.12)
p.49

$$= \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

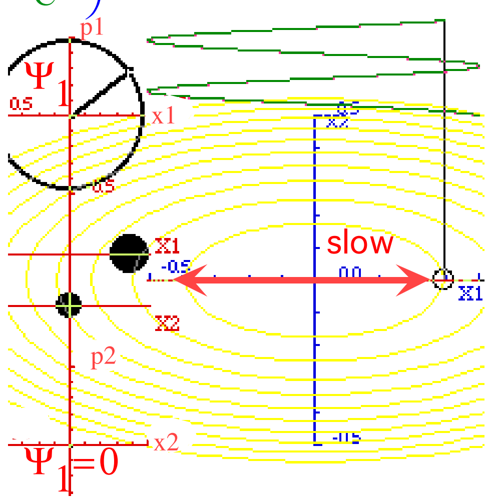
$$= \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

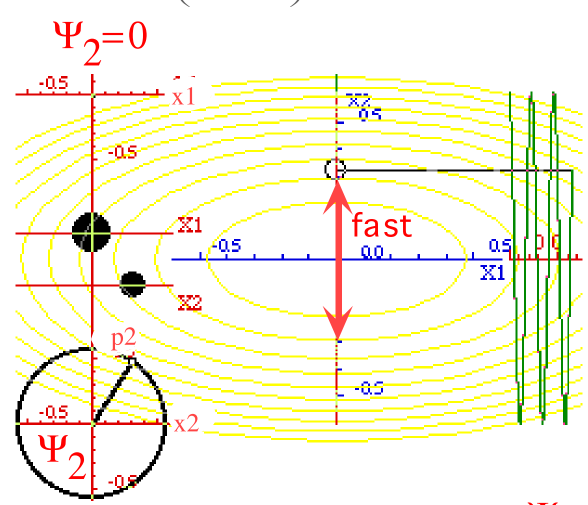
$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

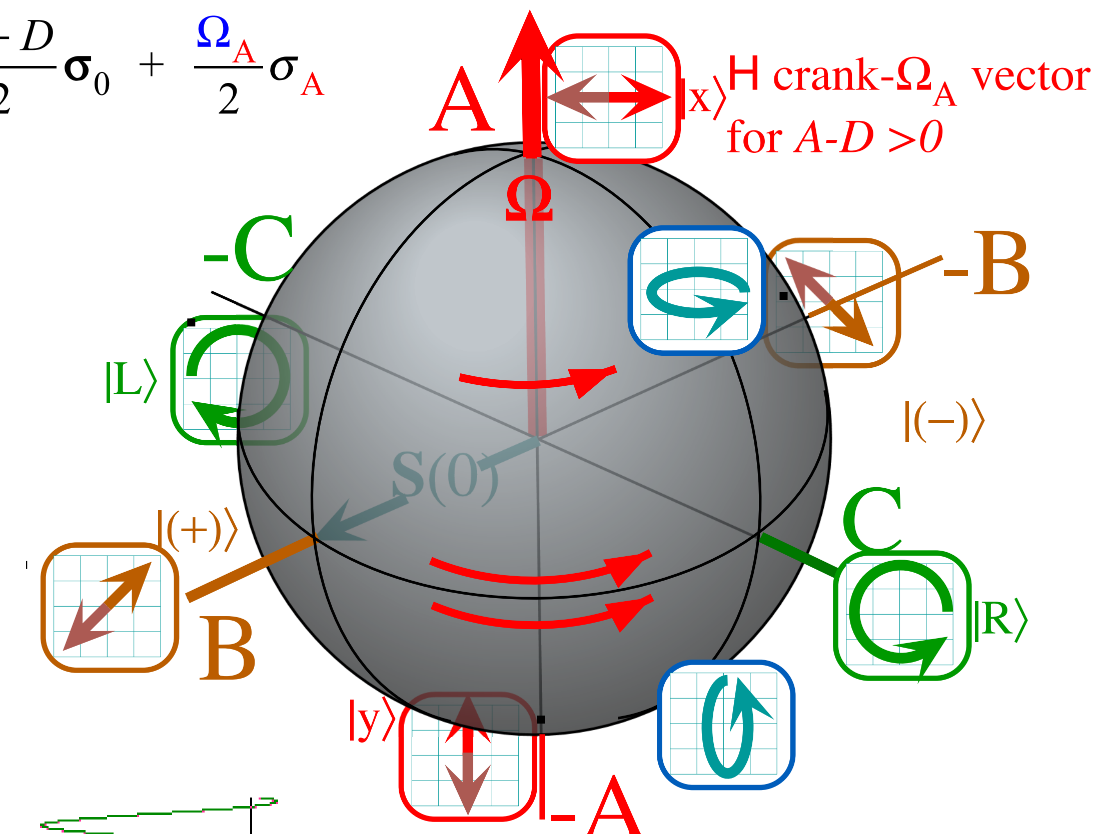
Asymmetric Diagonal A-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^A|1\rangle & \langle 1|\mathbf{H}^A|2\rangle \\ \langle 2|\mathbf{H}^A|1\rangle & \langle 2|\mathbf{H}^A|2\rangle \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_A}{2}\sigma_A$$

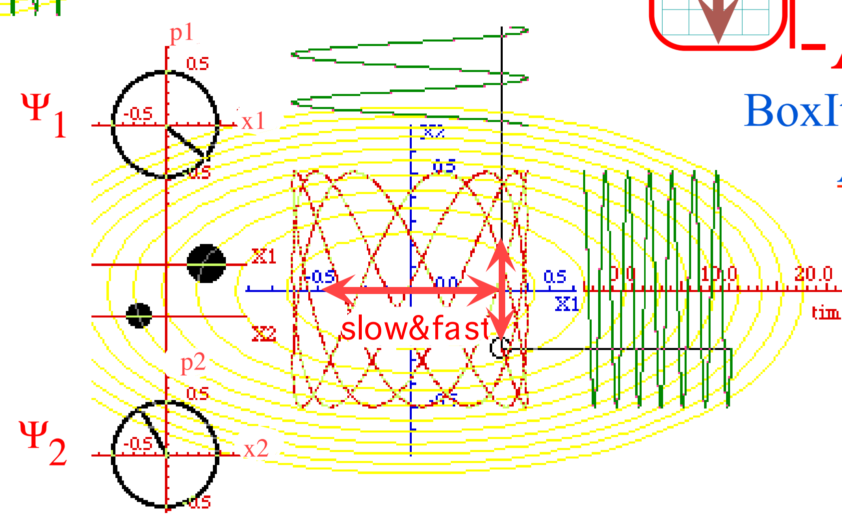
$$Crank:\ \vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 0 \\ 0 \end{pmatrix} \qquad Eigen-Spin:\ \vec{\mathbf{S}} = \begin{pmatrix} S_A \\ S_B \\ S_C \end{pmatrix} = \begin{pmatrix} \pm S \\ 0 \\ 0 \end{pmatrix}$$



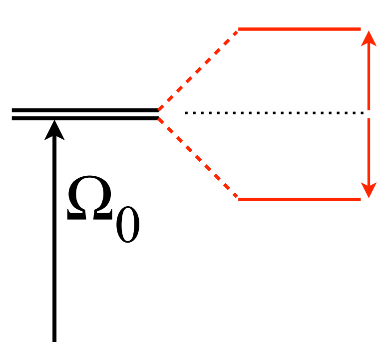




BoxIt (A-Type) Web Simulation:
A=4.9, B=C=0, D=4.0



Beat dynamics:



$\frac{\Omega_A}{2}$

$\frac{\Omega_A}{2}$

Total beat frequency Ω_A

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

Asymmetry $S_A = S_Z$, Balance $S_B = S_X$, and Chirality $S_C = S_Y$

Polarization ellipse and spinor state dynamics

The "Great Spectral Avoided-Crossing" and A-to-B-to-A symmetry breaking

Polarization ellipse and spinor state dynamics (B-Type motion)

Note phase or “gauge” angle γ is killed in R(3) a^*a-squares but lives on in U(2).

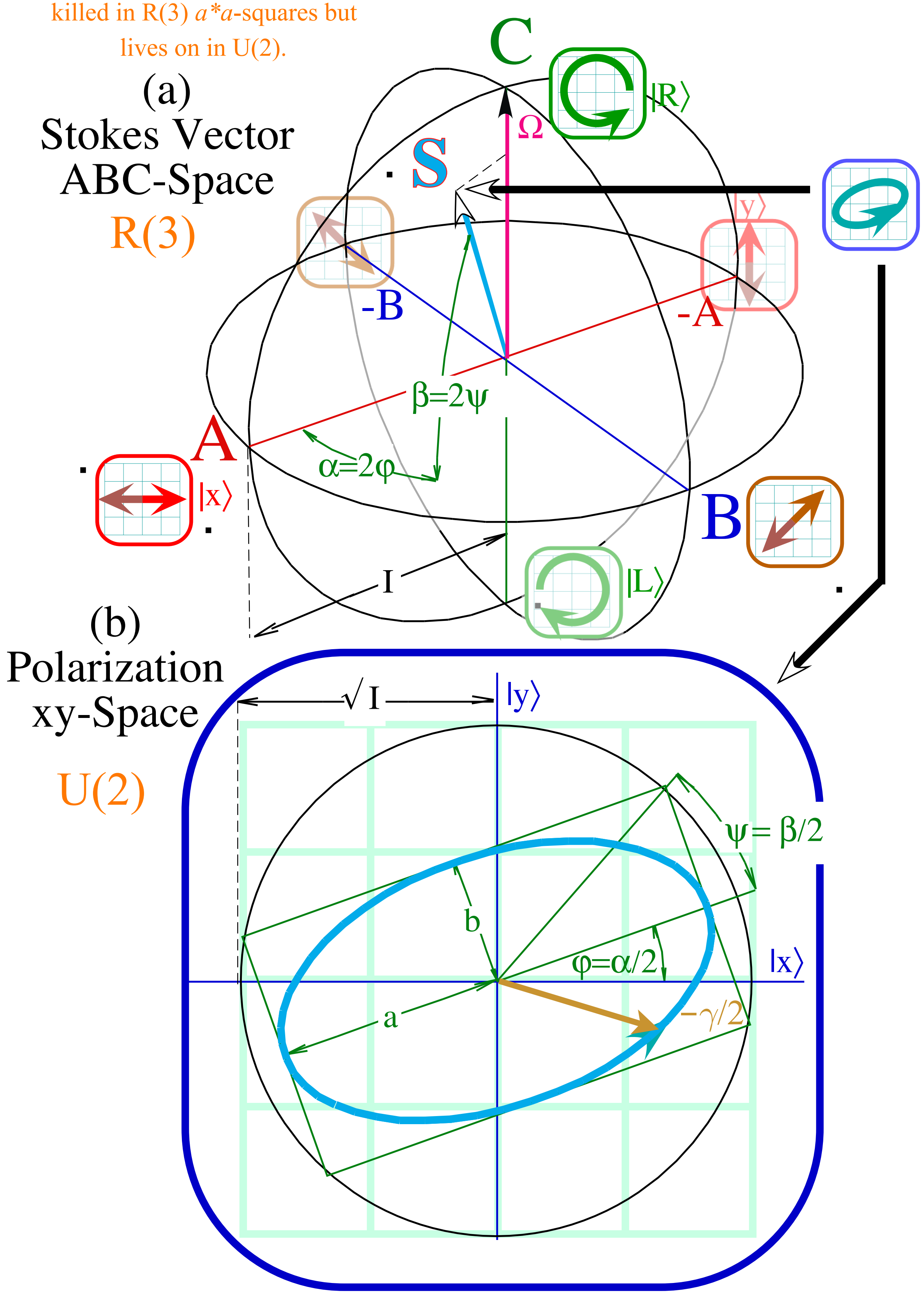


Fig. 3.4.5 Polarization variables (a) Stokes real-vector space (ABC) (b) Complex xy-spinor-space (x_1,x_2).

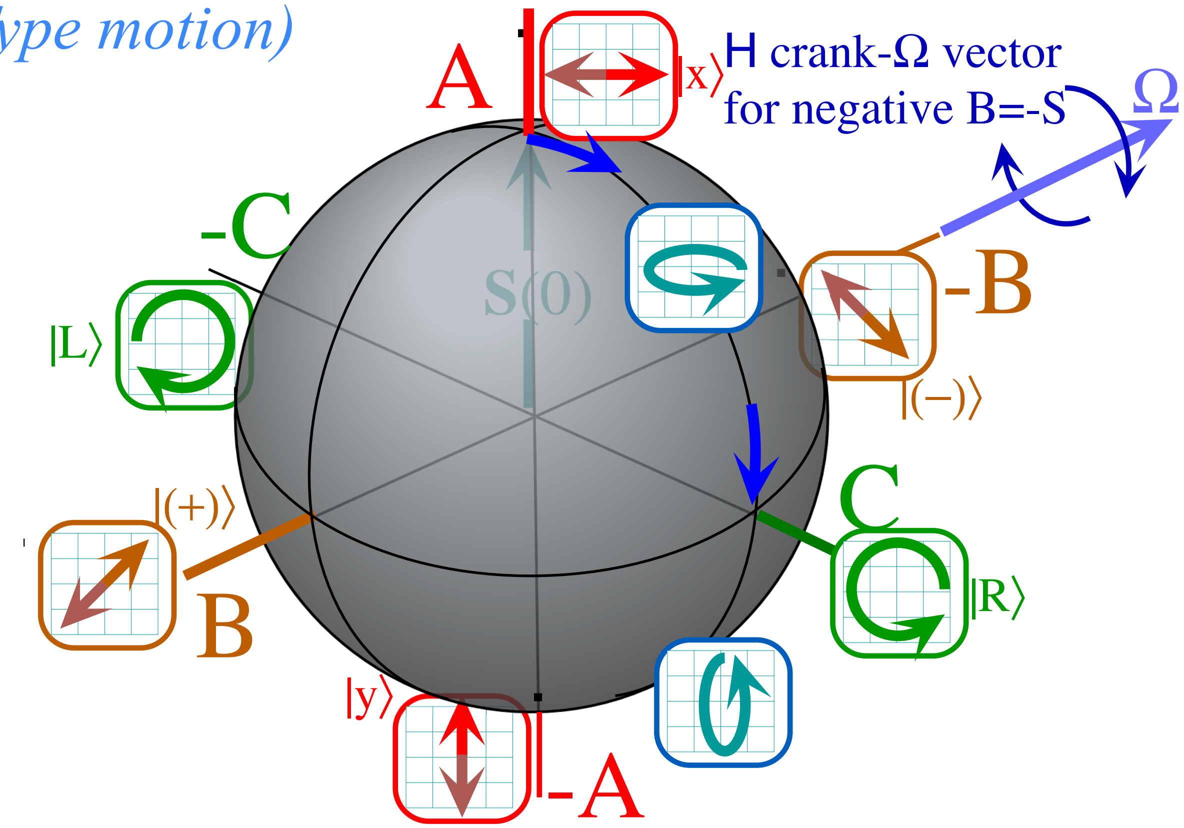


Fig. 3.4.6 Time evolution of a B-type beat. **S***-vector rotates from A to C to -A to -C and back to A.*

BoxIt (*B-Type*) *Simulation*
A=4.0, B=-0.2, C=0, D=4.0

The ABC's of U(2) dynamics (B-Type motion)

$$\rho = \frac{1}{2}\left(N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)$$
$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix}\langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle\\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle\end{pmatrix} = \begin{pmatrix}A & B-iC\\ B+iC & D\end{pmatrix} = \frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix} + B\begin{pmatrix}0&1\\1&0\end{pmatrix} + C\begin{pmatrix}0&-i\\i&0\end{pmatrix} + \frac{A-D}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$= \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

$$= \frac{A+D}{2}\sigma_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

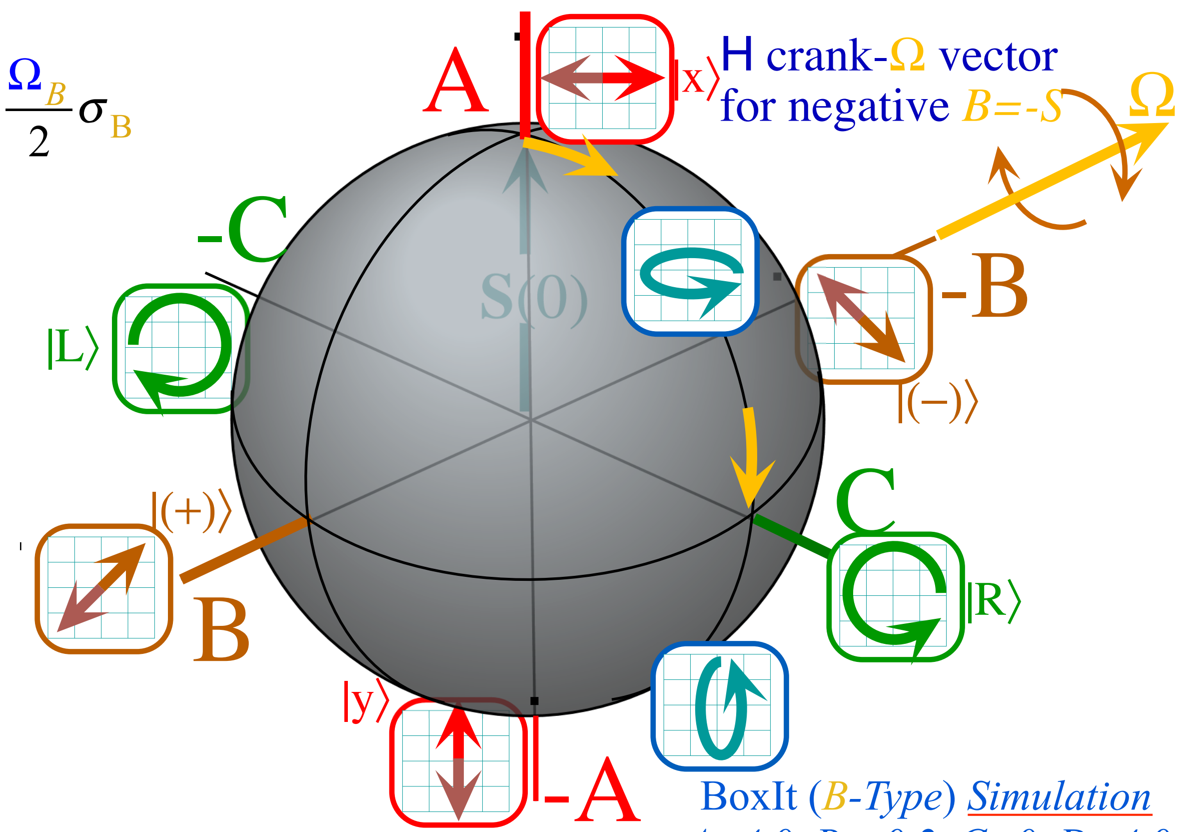
$$\vec{\Omega} = \begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix} = \begin{pmatrix}A-D\\ 2B\\ 2C\end{pmatrix}$$

From:
QTCA
Lect. 9(2.12)
p. 54

Bilateral-Balanced B-Type motion

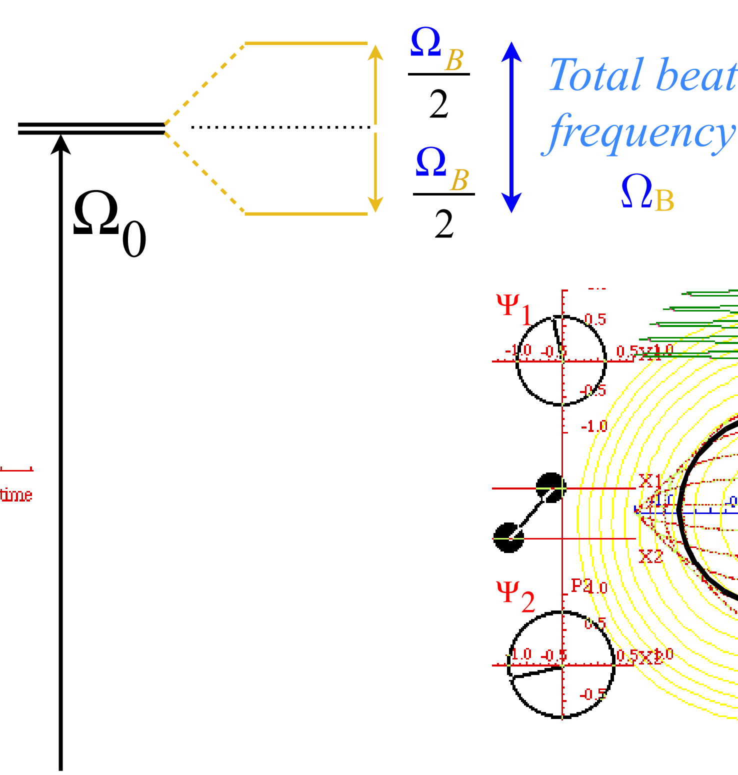
$$\begin{pmatrix}\langle 1|\mathbf{H}^B|1\rangle & \langle 1|\mathbf{H}^B|2\rangle\\ \langle 2|\mathbf{H}^B|1\rangle & \langle 2|\mathbf{H}^B|2\rangle\end{pmatrix} = \begin{pmatrix}\Omega_0 & B\\ B & \Omega_0\end{pmatrix} = \Omega_0\begin{pmatrix}1&0\\0&1\end{pmatrix} + B\begin{pmatrix}0&1\\1&0\end{pmatrix} = \Omega_0\sigma_0 + \frac{\Omega_B}{2}\sigma_B$$

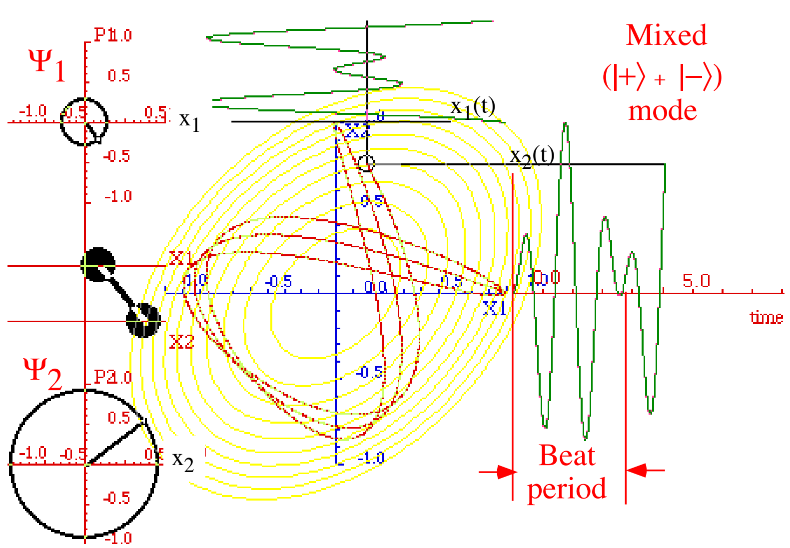
$$\textit{Crank}:\ \vec{\Omega} = \begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix} = \begin{pmatrix}0\\ 2B\\ 0\end{pmatrix} \qquad \textit{Eigen-Spin}:\ \vec{\mathbf{S}} = \begin{pmatrix}S_A\\ S_B\\ S_C\end{pmatrix} = \begin{pmatrix}0\\ \pm S\\ 0\end{pmatrix}$$

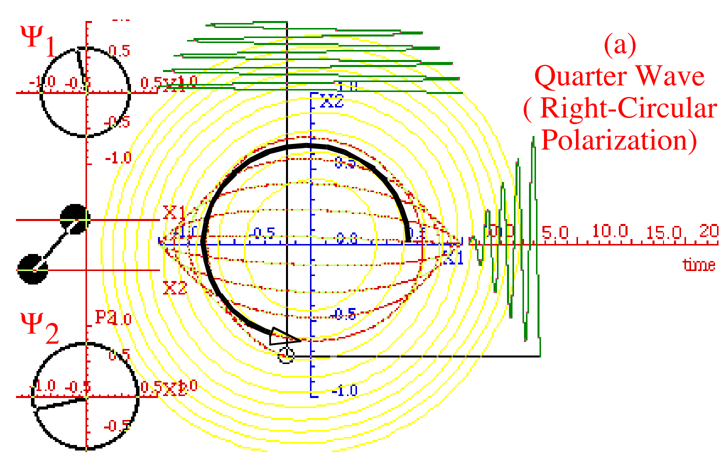


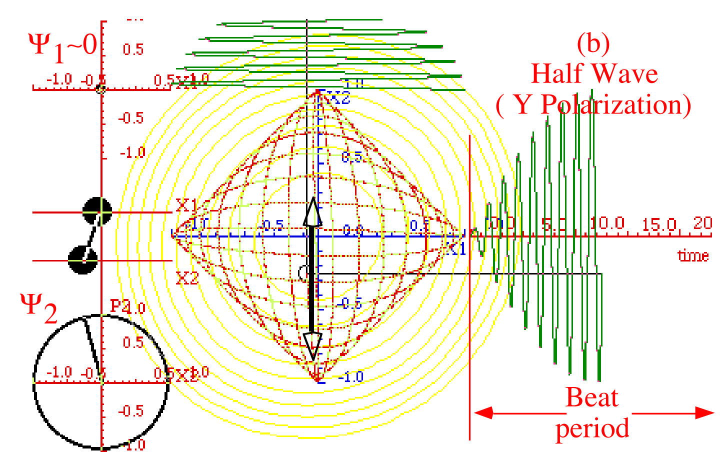
BoxIt (B-Type) Simulation
A=4.0, B=-0.2, C=0, D=4.0

Beat dynamics:









3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

Asymmetry $S_A = S_Z$, Balance $S_B = S_X$, and Chirality $S_C = S_Y$

Polarization ellipse and spinor state dynamics

➡ The "Great Spectral Avoided-Crossing" and A-to-B-to-A symmetry breaking

Polarization ellipse and spinor state dynamics (C-Type motion)

Note phase or “gauge” angle γ is killed in R(3) a^*a-squares but lives on in U(2).

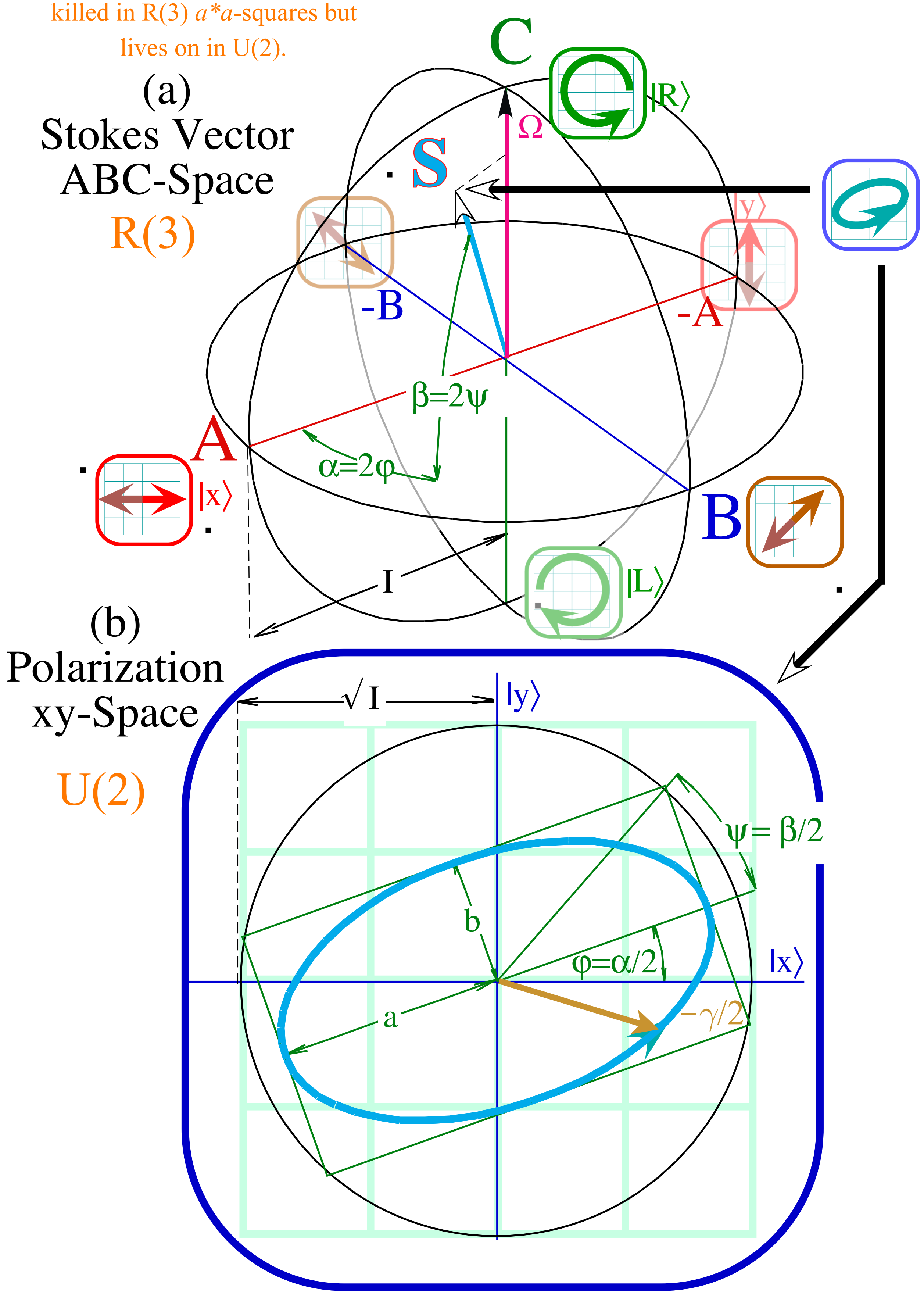


Fig. 3.4.5 Polarization variables (a) Stokes real-vector space (ABC) (b) Complex xy-spinor-space (x_1,x_2).

C (Chiral-circular-complex-Coriolis-cyclotron-curly...current-carrier...)

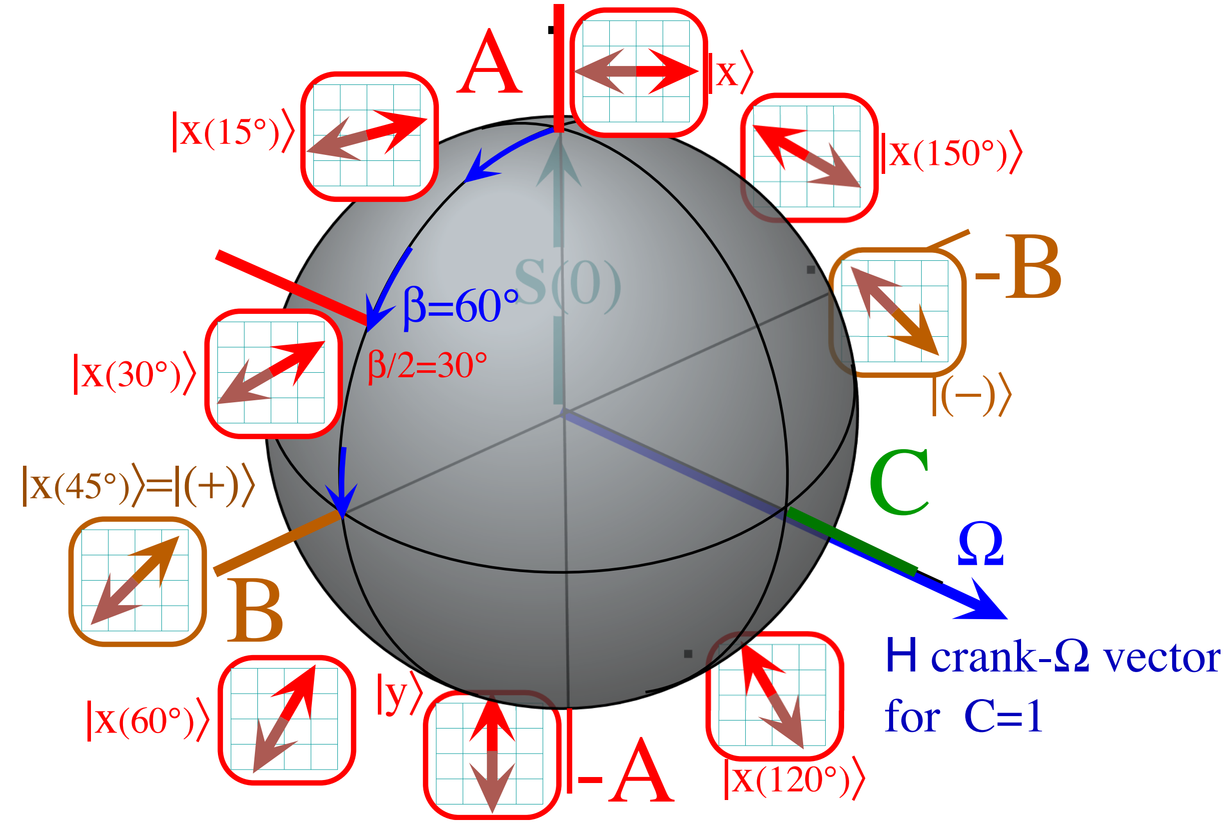


Fig. 3.4.7 Time evolution of a C-type beat. **S**-*vector rotates from A to B to -A to -B and back to A.*

The ABC's of U(2) dynamics (C-Type motion)

$$\rho = \frac{1}{2}\left(N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)$$
$$\mathbf{H} = \Omega_0 \mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

From:
QTCA
Lect. 9(2.12)
p. 58

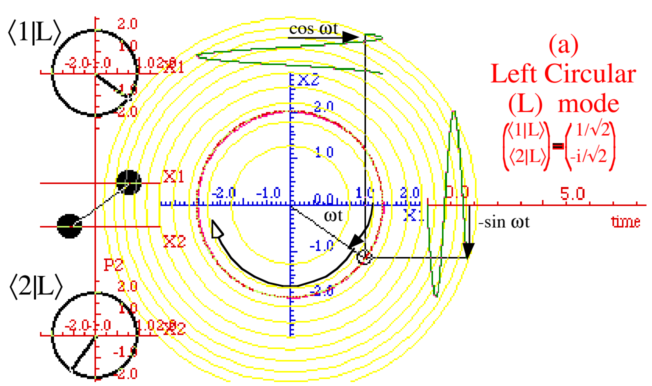
$$= \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$
$$= \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

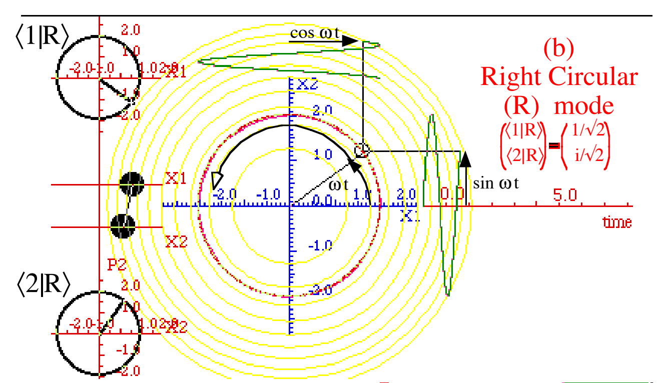
Circular-Coriolis... C-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^C|1\rangle & \langle 1|\mathbf{H}^C|2\rangle \\ \langle 2|\mathbf{H}^C|1\rangle & \langle 2|\mathbf{H}^C|2\rangle \end{pmatrix} = \begin{pmatrix} \Omega_0 & -iC \\ iC & \Omega_0 \end{pmatrix} = \Omega_0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \Omega_0\boldsymbol{\sigma}_0 + \frac{\Omega_C}{2}\sigma_C$$

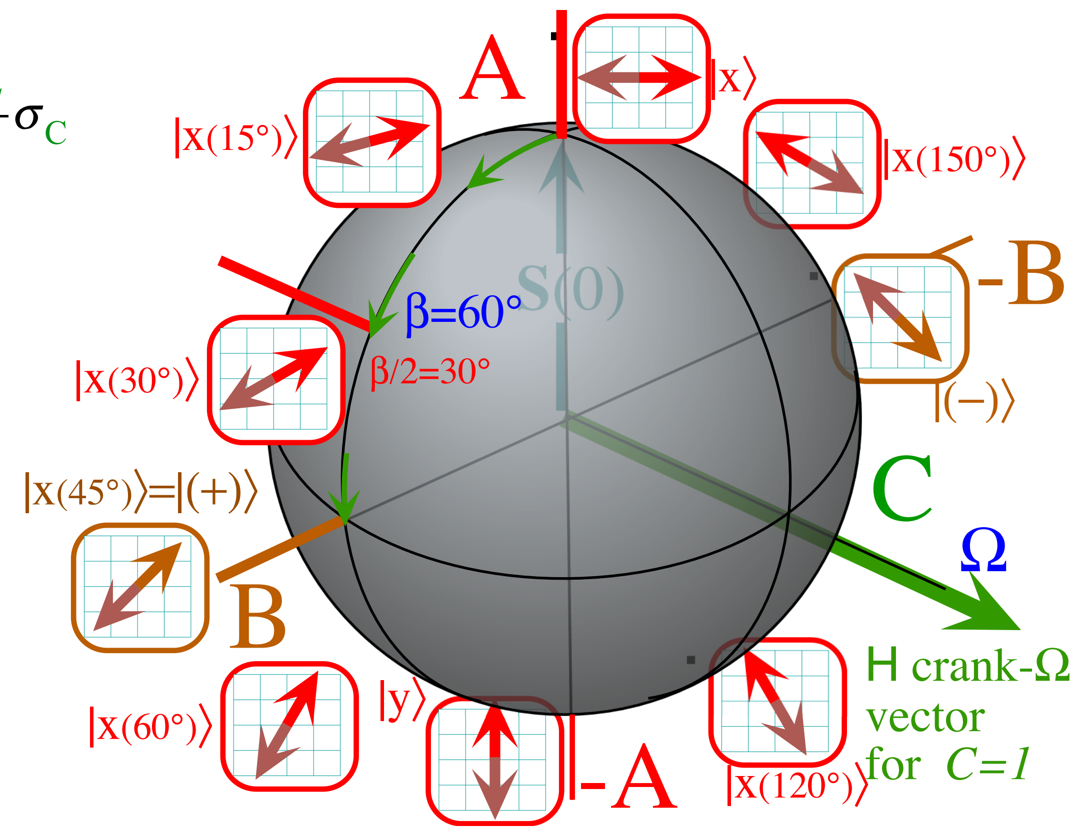
$$\textit{Crank}: \vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2C \end{pmatrix} \qquad \textit{Eigen-Spin}: \vec{\mathbf{S}} = \begin{pmatrix} S_A \\ S_B \\ S_C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \pm S \end{pmatrix}$$



(a) Left Circular (L) mode $\begin{pmatrix}\langle 1|L\rangle \\ \langle 2|L\rangle\end{pmatrix} = \begin{pmatrix}1/\sqrt{2} \\ -i/\sqrt{2}\end{pmatrix}$

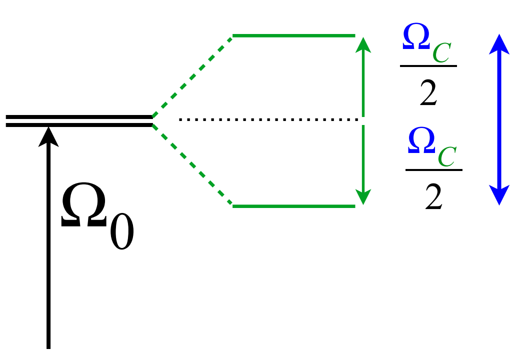


(b) Right Circular (R) mode $\begin{pmatrix}\langle 1|R\rangle \\ \langle 2|R\rangle\end{pmatrix} = \begin{pmatrix}1/\sqrt{2} \\ i/\sqrt{2}\end{pmatrix}$

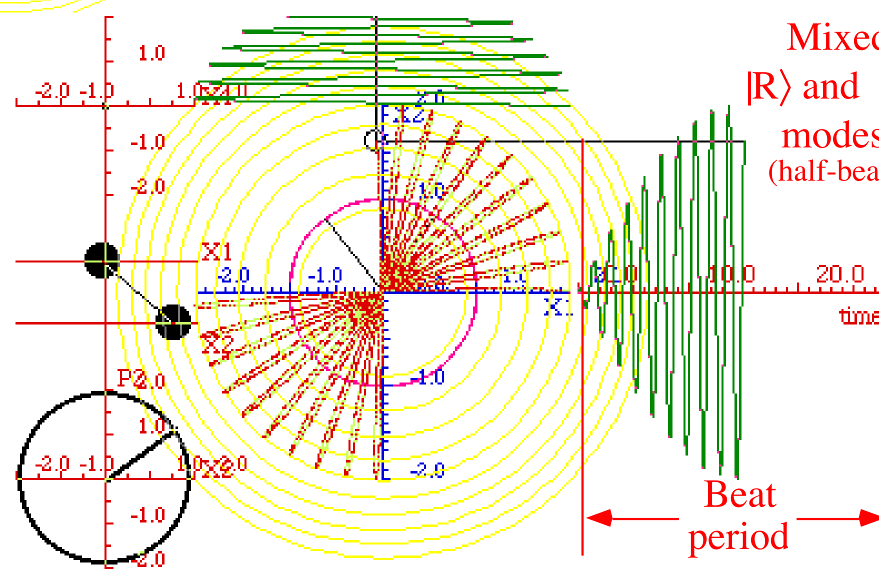


BoxIt (C-Type) Simulation
A=4.055, B=0, C=0.1, D=4.055

Beat dynamics:



Total beat frequency Ω_C



Mixed |R⟩ and |L⟩ modes (half-beat)

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

Asymmetry $S_A = S_Z$, Balance $S_B = S_X$, and Chirality $S_C = S_Y$

Polarization ellipse and spinor state dynamics

➤ *The "Great Spectral Avoided-Crossing" and A-to-B-to-A symmetry breaking*

The ABC's of U(2) dynamics-Mixed modes (AB-Type motion)

$$\rho = \frac{1}{2} N \mathbf{1} + \vec{\mathbf{S}} \bullet \boldsymbol{\sigma}$$
$$\mathbf{H} = \Omega_0 \mathbf{1} + \frac{\vec{\Omega}}{2} \bullet \boldsymbol{\sigma}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

From:
QTCA
Lect. 9(2.12)
p.60

$$= \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$
$$= \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

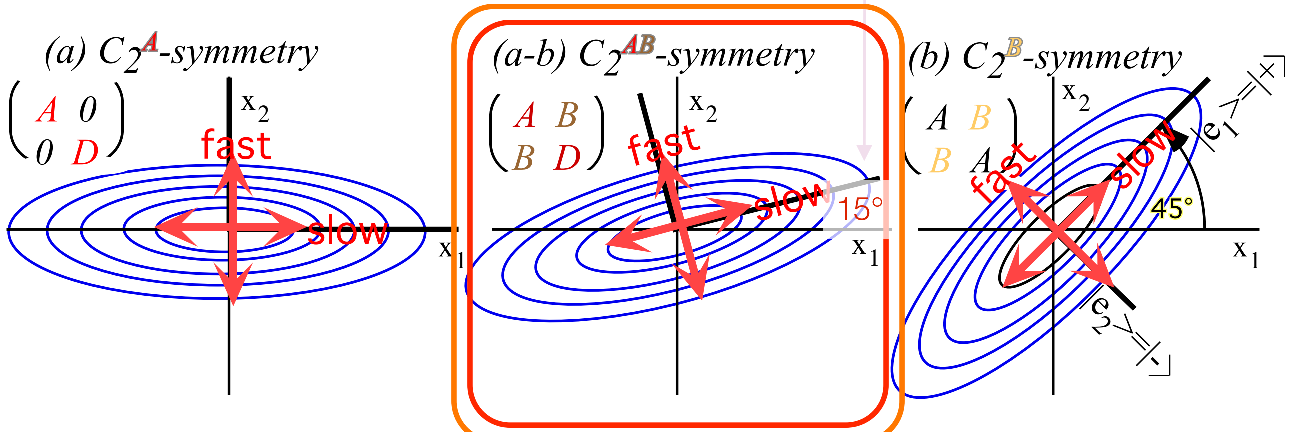
$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

Tilted-plane polarization AB-Type motion

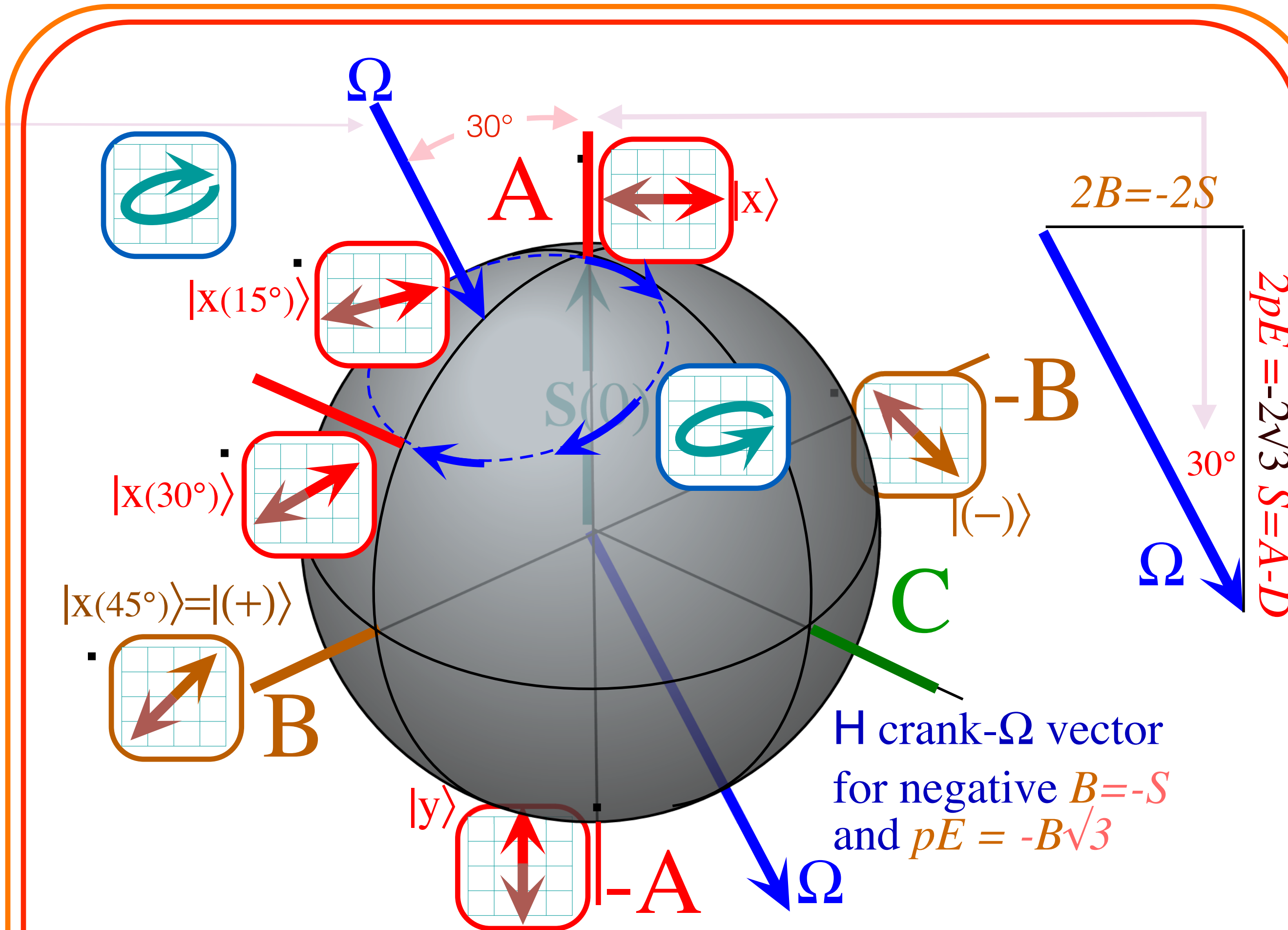
$$\begin{pmatrix} \langle 1|\mathbf{H}^{AB}|1\rangle & \langle 1|\mathbf{H}^{AB}|2\rangle \\ \langle 2|\mathbf{H}^{AB}|1\rangle & \langle 2|\mathbf{H}^{AB}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B \\ B & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \Omega_0\boldsymbol{\sigma}_0 + \frac{\Omega_A}{2}\sigma_A + \frac{\Omega_B}{2}\sigma_B$$

Crank : $\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 0 \end{pmatrix}$ *Eigen-Spin :* $\vec{\mathbf{S}} = \pm S \cdot \hat{\Omega}$

Note the *relative* factor of 1/2



Beat dynamics:



BoxIt web simulation: AB-Type motion w/Stoke's plot & frequency ratios
A=4.0, B=0.866, C=0.0, D=1.0
Note: *In the Web App* the C axis is vertical

The Great Spectral "Avoided-Crossing"

A to B to A Symmetry breaking described by hyperbolic eigenvalues of $A\boldsymbol{\sigma}_A + B\boldsymbol{\sigma}_B = \mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$

$\mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$ Secular equation: $\varepsilon^2 - 0 \cdot \varepsilon - (A^2 + B^2)$ gives *hyperbolic* energy levels: $\varepsilon = \pm\sqrt{A^2 + B^2}$

The Great Spectral "Avoided-Crossing"

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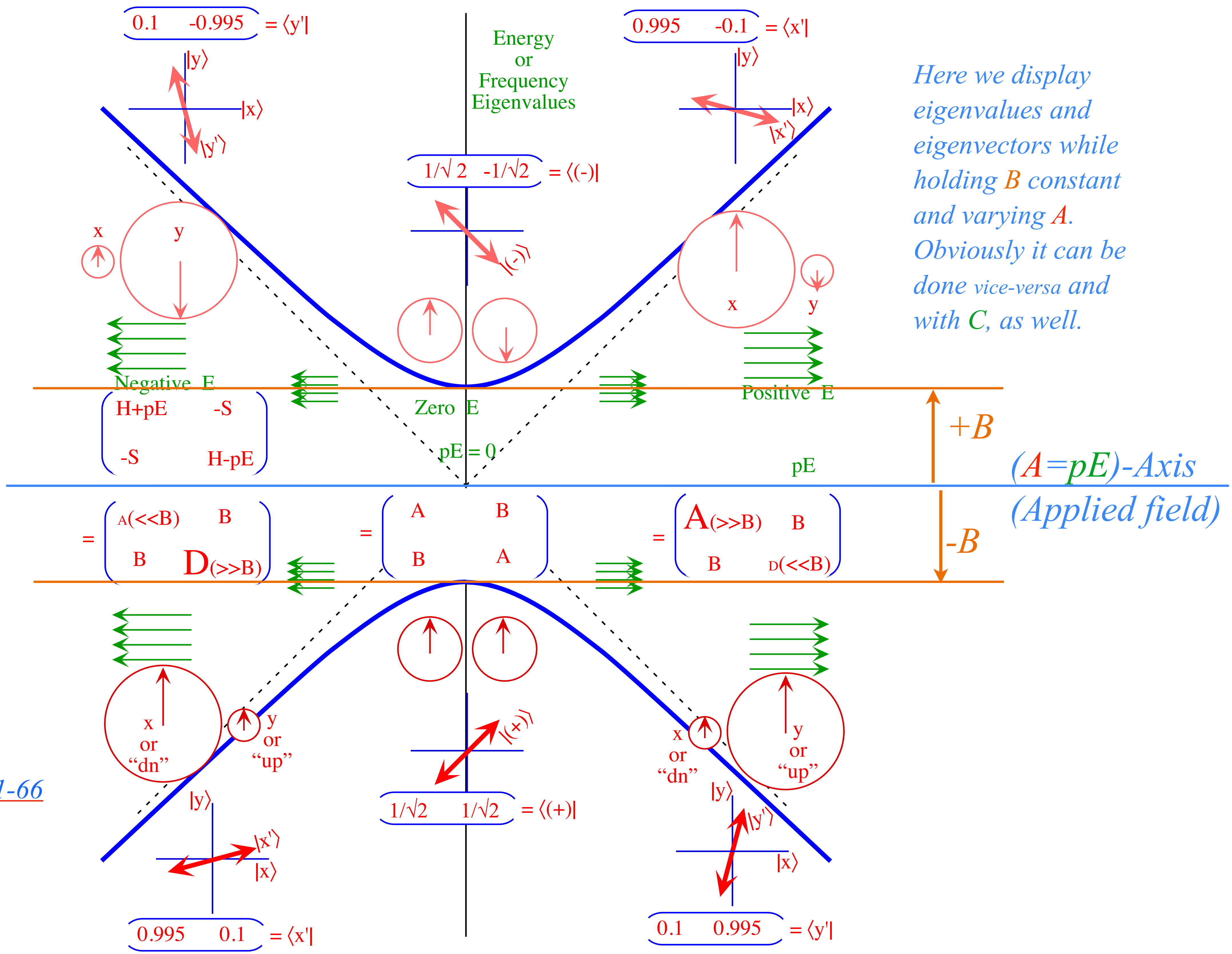
$\mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$ Secular equation: $\varepsilon^2 - 0 \cdot \varepsilon - (A^2 + B^2)$ gives *hyperbolic* energy levels: $\varepsilon = \pm\sqrt{A^2 + B^2}$



Here we display eigenvalues and eigenvectors while holding B constant and varying A. Obviously it can be done vice-versa and with C, as well.

See also:
QTCA Lect. 9 (2.12) p.61-66

The Great Spectral "Avoided-Crossing"

A to B to A Symmetry breaking described by hyperbolic eigenvalues of $A\boldsymbol{\sigma}_A + B\boldsymbol{\sigma}_B = \mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$

$\mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$ Secular equation: $\varepsilon^2 - 0 \cdot \varepsilon - (A^2 + B^2)$ gives *hyperbolic* energy levels: $\varepsilon = \pm\sqrt{A^2 + B^2}$

These on-and-off resonance effects are key to:

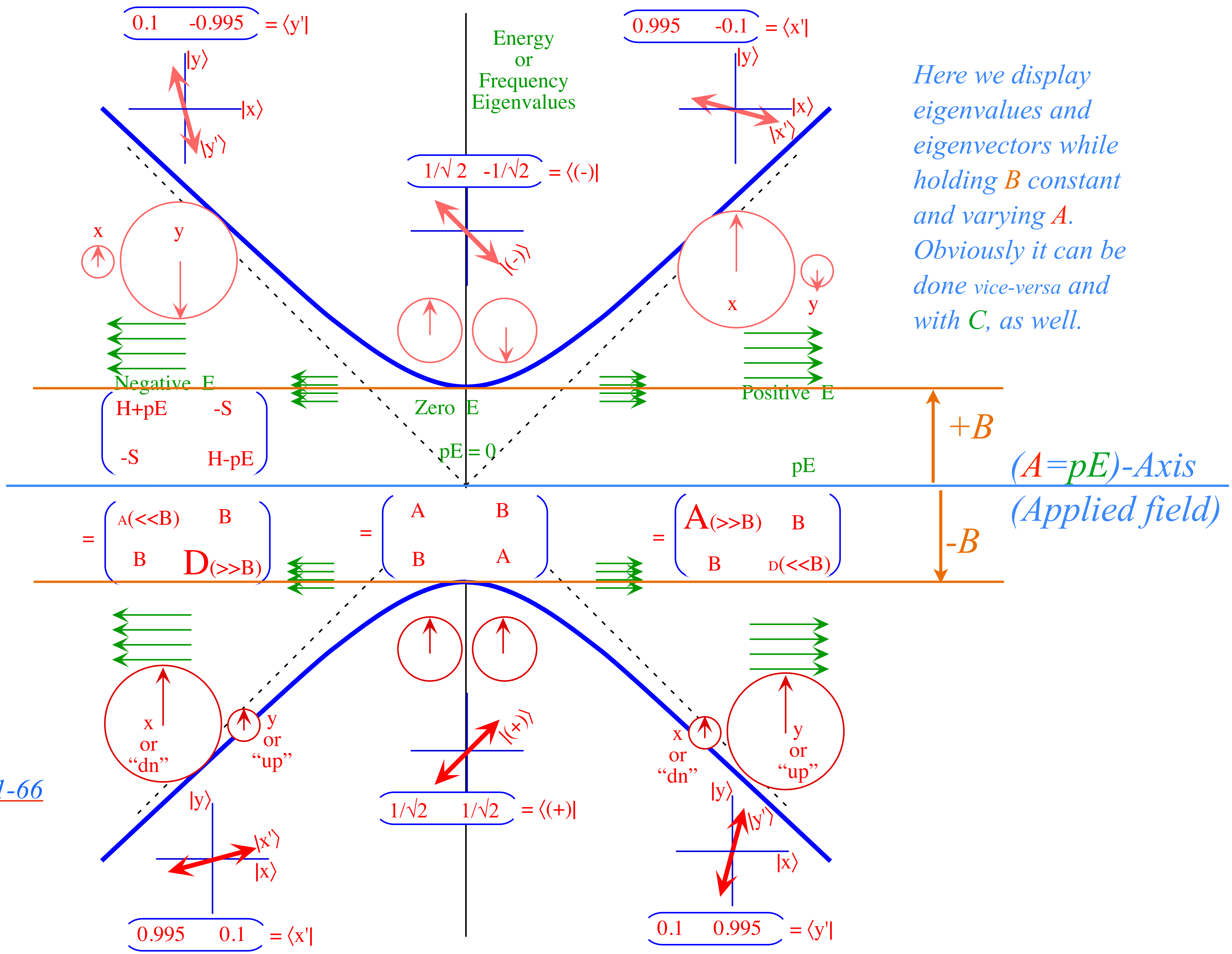
Laser QCD

Relativistic QED

Quantum computing

Photosynthesis

and a whole lot of other things...



Here we display eigenvalues and eigenvectors while holding B constant and varying A. Obviously it can be done vice-versa and with C, as well.

See also:

[QTCA Lect. 9 (2.12) p.61-66](#)

OBJECTIVE: Evaluate and (*most* important!) *visualize* matrix-exponent solutions.

ABCD Time evolution operator

$$\left|\Psi(t)\right\rangle = e^{-i\mathbf{H}\cdot t}\left|\Psi(0)\right\rangle$$

Hamilton generalized Euler's expansion $e^{-i\Omega t} = \cos\Omega t - i\sin\Omega t$ so matrix exponential becomes powerful.

$$e^{-i\mathbf{H}\cdot t} = e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D\end{pmatrix}\cdot t} = e^{-i\frac{A-D}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}t - iB\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}t - iC\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}t - i\frac{A+D}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}t} = e^{-i(\omega_0\sigma_0 + \vec{\omega}\bullet\vec{\sigma})\cdot t} = e^{-i\omega_0\cdot t}\left(\mathbf{1}\cos\omega\cdot t - i\boldsymbol{\sigma}_\omega \sin\omega\cdot t\right)$$

$\sigma_A = \sigma_Z$ $\quad$ $\sigma_B = \sigma_X$ $\quad$ $\sigma_C = \sigma_Y$

where: $\vec{\varphi} = \vec{\omega}\cdot t = \begin{pmatrix}\omega_A\\ \omega_B\\ \omega_C\end{pmatrix}\cdot t = \begin{pmatrix}\frac{A-D}{2}\\ B\\ C\end{pmatrix}\cdot t$ and: $\omega_0 = \frac{A+D}{2}$ $\qquad$ and: $\vec{\Theta} = \vec{\Omega}\cdot t = \begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix}\cdot t = \begin{pmatrix}A-D\\ 2B\\ 2C\end{pmatrix}\cdot t$ and: $\Omega_0 = \frac{A+D}{2}$

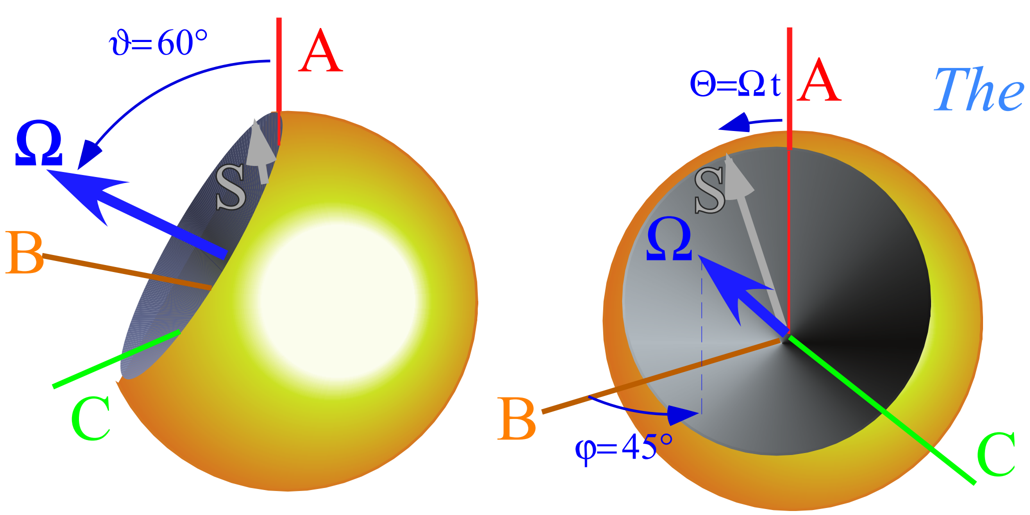
Symmetry relations make spinors σ_X, σ_Y, and σ_Z or quaternions $\mathbf{i} = -i\sigma_X$, $\mathbf{j} = -i\sigma_Y$, and $\mathbf{k} = -i\sigma_Z$ powerful.

3D crank vector $\vec{\Theta} = \vec{\Omega}\cdot t$ and *spin operator* $\mathbf{S}$ defines *3D ABC*-rotation with ratio $\frac{1}{2}$ or 2 between Θ_a and $\varphi_a = \frac{1}{2}\Theta_a$ or between $\mathbf{S}$ and $\boldsymbol{\sigma} = 2\mathbf{S}$.

$$e^{-i\boldsymbol{\sigma}\bullet\vec{\varphi}} = e^{-i\boldsymbol{\sigma}\bullet\vec{\Theta}/2} = e^{-i\mathbf{S}\bullet\vec{\Theta}} = \mathbf{1}\cos\frac{\Theta}{2} - i\,(\boldsymbol{\sigma}\bullet\hat{\Theta})\sin\frac{\Theta}{2} = \begin{pmatrix}\cos\frac{\Theta}{2} - i\hat{\Theta}_A\sin\frac{\Theta}{2} & (-i\hat{\Theta}_B - \hat{\Theta}_C)\sin\frac{\Theta}{2}\\ (-i\hat{\Theta}_B + \hat{\Theta}_C)\sin\frac{\Theta}{2} & \cos\frac{\Theta}{2} + i\hat{\Theta}_A\sin\frac{\Theta}{2}\end{pmatrix}$$

Example 3: Any $\boldsymbol{\Theta} = \boldsymbol{\Omega}t$-axial rotation

2D angle : $\varphi = \frac{1}{2}\Theta$ $\qquad$ *3D Crank vector* : $\vec{\Theta} = \Theta\hat{\Theta} = 2\varphi_a\hat{\mathbf{a}} = 2\vec{\varphi}$ $\qquad$ *2D spin matrix* : $\mathbf{S} = \frac{1}{2}\boldsymbol{\sigma}$

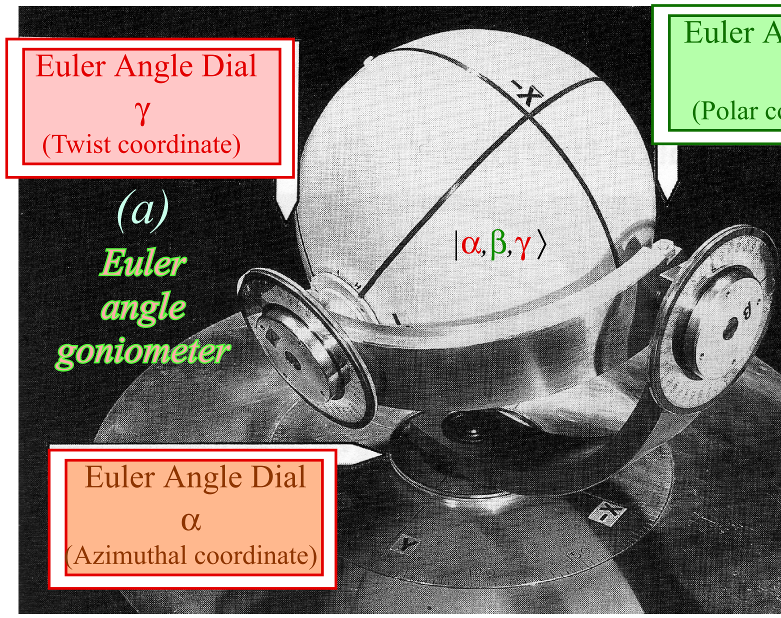


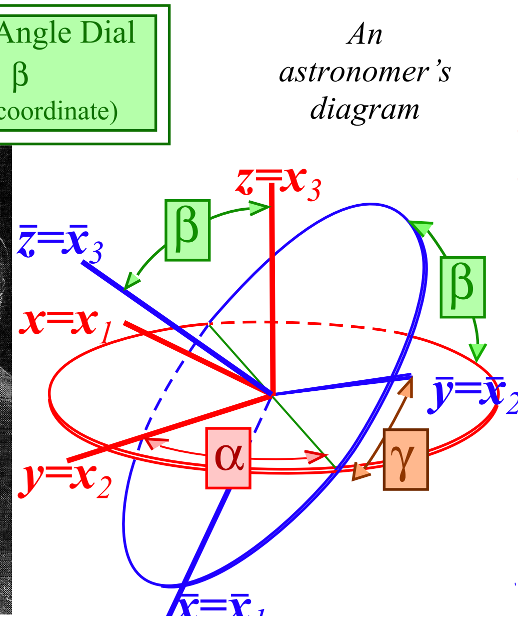
The driving $\boldsymbol{\Theta} = \boldsymbol{\Omega}t$ vector is defined by the ABCD of Hamiltonian **H** .

The driven spin vector **S** *defines the state. But, how?*

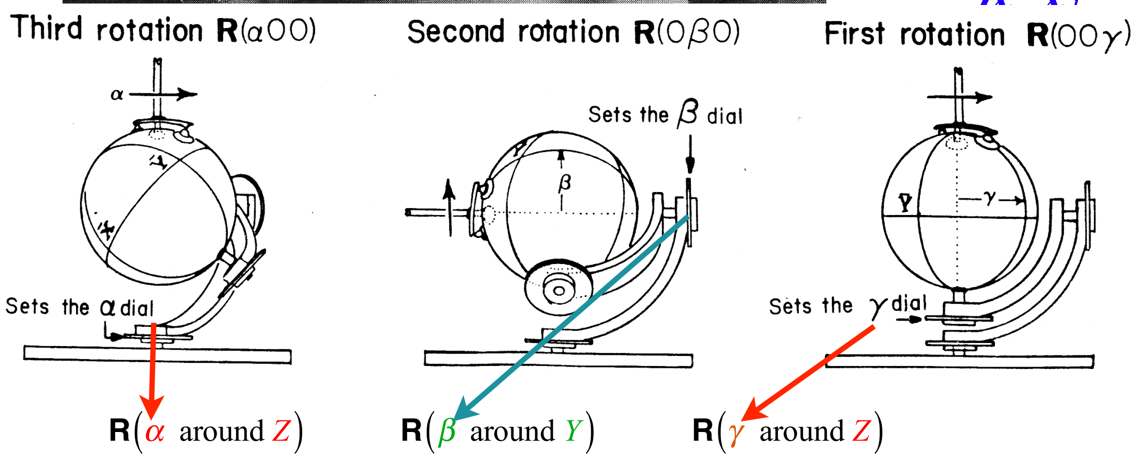
Fig. 3.4.2 Two views of Hamilton crank vector Ω(φ,ϑ) whirling Stokes state vector S in ABC-space.

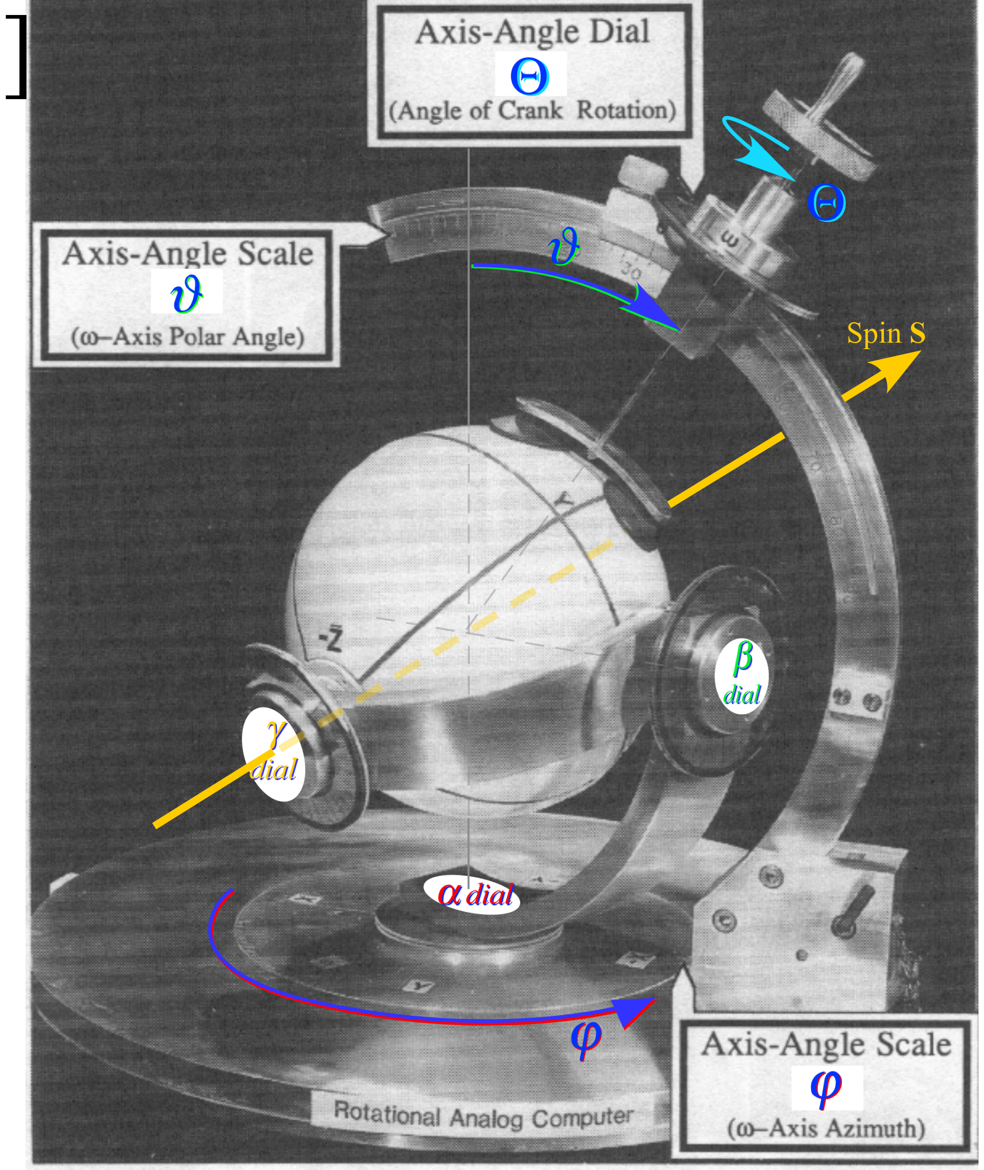
Euler R(αβγ) versus *Darboux* R[φϑΘ]





From: QTCA Lect. 9 (2.12) (See p.5-23 there)





$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}}\end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}}\end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}=$$

$$=\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

$$\mathbf{R}[\vec{\Theta}]=\begin{pmatrix}\cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}(\hat{\Theta}_X-i\hat{\Theta}_Y) \\ -i\sin\frac{\Theta}{2}(\hat{\Theta}_X+i\hat{\Theta}_Y) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2}\end{pmatrix}=\mathbf{R}[\varphi\vartheta\Theta]=e^{-i\mathbf{H}t}$$

$$=\cos\frac{\Theta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\hat{\Theta}_X\sin\frac{\Theta}{2}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\hat{\Theta}_Y\sin\frac{\Theta}{2}-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\hat{\Theta}_Z\sin\frac{\Theta}{2}$$

$\hat{\Theta}_X$: cosφ sinϑ $\hat{\Theta}_Y$: sinφ sinϑ $\hat{\Theta}_Z$: cosϑ

$$=\begin{pmatrix}\cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta) \\ \sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2}\end{pmatrix}$$

Euler R(αβγ) is simpler to form than Θ-axis *Darboux* R[φϑΘ].

Euler *state definition* lets us relate R(αβγ) to R[φϑΘ] ...

|αβγ⟩= R(αβγ)|000⟩   *(αβγ make better coordinates)*

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}x_1+ip_1\\x_2+ip_2\end{pmatrix}$$

$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$

$-p_2 = \sin[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_X \sin\Theta/2 = \cos\varphi\ \sin\vartheta\ \sin\Theta/2$

$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y \sin\Theta/2 = \sin\varphi\ \sin\vartheta\ \sin\Theta/2$

$-p_1 = \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z \sin\Theta/2 = \cos\vartheta\ \sin\Theta/2$

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates related to Euler Angles $(\alpha\beta\gamma)$

2D elliptic frequency ω orbit has amplitudes A_1 and A_2, and phase shifts ρ_1 and $\rho_2=-\rho_1$.

$$\begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$x_1 = A_1 \cos(\omega t+\rho_1)$$
$$-p_1 = A_1 \sin(\omega t+\rho_1)$$
$$x_2 = A_2 \cos(\omega t-\rho_1)$$
$$-p_2 = A_2 \sin(\omega t-\rho_1)$$

Let: $A_1 = A\cos\beta/2$

$A_2 = A\sin\beta/2$

$$\tan\beta/2 = A_2/A_1 \qquad A^2 = A_1^2 + A_2^2$$

Real x_k and imaginary p_k parts of phasor amplitudes $a_k = x_k + ip_k$ depend on Euler angles $(\alpha\beta\gamma)$ and A.

$$x_1 = A\cos\beta/2\cos[(\gamma+\alpha)/2]$$
$$-p_1 = A\cos\beta/2\sin[(\gamma+\alpha)/2]$$
$$x_2 = A\sin\beta/2\cos[(\gamma-\alpha)/2]$$
$$-p_2 = A\sin\beta/2\sin[(\gamma-\alpha)/2]$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

Let: $\omega t+\rho_1 = (\gamma+\alpha)/2$

$\omega t-\rho_1 = (\gamma-\alpha)/2$

$$\alpha = 2\rho_1 \qquad \gamma = 2\omega\cdot t$$

Euler parameters (α,β,γ, A) in terms of *amp-phase parameters* $(A_1, A_2, \omega t, \rho_1)$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$