Lecture 25
Wed. 11.14.2018

Introduction to Orbital Dynamics

(Ch. 2-4 of Unit 5)

Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

- *Effective potentials for IHO and Coulomb orbits*
  - *Stable equilibrium radii and radial/angular frequency ratios*
  - *Classical turning radii and apogee/perigee parameters*
  - *Polar coordinate differential equations*
  - *Quadrature integration techniques*
    - *Detailed orbital functions*
  - *Relating orbital energy-momentum to conic-sectional orbital geometry*
    - *Kepler equation of time and phase geometry*

Review: "3 steps from Hell"
(Lect. 7 Ch. 9 Unit 1)

(A mystery similarity appears)

Geometry and Symmetry of Coulomb orbits

- *Detailed elliptic geometry*
- *Detailed hyperbolic geometry*

A running collection of links to course-relevant sites and articles

Physics Web Resources

Comprehensive Harter-Soft Resource Listing

UAF Physics YouTube channel

LearnIt Physics Web Applications

"Texts"

Classical Mechanics with a Bang!

Quantum Theory for the Computer Age

Principles of Symmetry, Dynamics, and Spectroscopy

Modern Physics and its Classical Foundations

Classes

2014 AMOP

2017 Group Theory for QM

2018 AMOP

2018 Adv Mechanics

Neat external material to start the class:
- AIP publications
- AJP article on superball dynamics
- AAPT summer reading

AMOP Ch 0 Space-Time Symmetry - 2019

Seminar at Rochester Institute of Optics, Auxiliary slides, June 19, 2018

Springer AMO Handbook - Ch 32 - Harter-Reimer-2019

These *are* hot off the presses:
- Sorting ultracold atoms in a 3D optical lattice in a realization of Maxwell's demon - Kumar-Nature-Letters-2018
- Synthetic three-dimensional atomic structures assembled atom by atom - Berredo-Nature-Letters-2018

Slightly Older ones:
- Wave–particle duality of C60 molecules
- Optical vortex knots – One Photon at a Time

"*Relawavity*" and quantum basis of *Lagrangian* & *Hamiltonian* mechanics:
- 2-CW laser wave - BohrIt Web App
- Lagrangian vs Hamiltonian - RelaWavity Web App

Older Links from Lectures 14-20

http://thearmchaircritic.blogspot.com/2011/11/punkin-chunkin.html
http://www.sussexcountyonline.com/news/photos/punkinchunkin.html
Shooting-range-for-medieval-siege-weapons-Anybody-knows
https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html
https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html?scenario=MontezumasRevenge
https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html?scenario=SeigeOfKenilworth
The trebuchet, Chevedden, Sci Am 1995
'*Simple*' *Pendulum* Sim: https://modphys.hosted.uark.edu/markup/PendulumWeb.html
'*Cycloid*' *Pendulum*: https://modphys.hosted.uark.edu/markup/CycloidulumWeb.html
Google search on: "Satelite view of Patricia" (Images)
Physics Girl Channel - Fun with Vortex Rings in the Pool
iBall demo - Quasi-periodicity: https://youtu.be/_jntDtULxDc
https://modphys.hosted.uark.edu/markup/CoulItWeb.html?scenario=SynchrotronMotion
https://modphys.hosted.uark.edu/markup/CoulItWeb.html?scenario=SynchrotronMotion2
Mechanical Analog to EM Motion (YouTube video) - https://youtu.be/hTd5FTJ-vRk
CoulIt Web Simulation: Bound-state motion in parabolic coordinates
CoulIt Web Simulation: Bound-state motion in hyperbolic coordinates
OscillIt Web App: Simulations of various types of resonance: 18, 27, 31, 35, 38, 39
Smith Chart
http://nobelprize.org/

AnalyIt Web Application, posted 10/22/2018 in our *testing* area:
https://modphys.hosted.uark.edu/testing/markup/AnalyItBJS.html

Older Links from Lectures 21-23

Advanced Atomic and Molecular Optical Physics 2018 Class #9, pages: 5, 61
BoxIt Web Simulations
- Pure A-Type w/Cosine
- Pure B-Type w/Cosine
- Pure B-Type w/Freq ratios
- Mixed AB-Type 2:1 Freq ratio
- Pure A-Type A=4.9, B=0 ,C=0, & D=4.0
- Pure B-Type: A=4.0, B=-0.2, C=0, & D=4.0
- Pure C-Type A,D=4.055, B=0, C=0.1
- Mixed AB-Type w/Cosine
- Mixed AB Type A=4.0, BU2=0.866…, CU2=0, & D=1.0 w/Stokes & Freq rats
- Mixed AB Type A=5.086 B=-0.27 C=0 D=2.024 w/Stokes plot
- Mixed ABC Type A=4.833 B=0.2403 C=0.4162 D=4.277 w/Stokes plot
- Recent mixed ABC Type A=0.325 B=0.375 C=0.825 D=0.05 w/Stokes plot

Classical Mechanics with a Bang! 2018
- Lectures 8, 9, 23 page 93
- Text Unit 6, page=27

ColorU2 for the Web - in development
Group Theory for Quantum Mechanics - 2017 Lectures: 6, 7, 8,
and the combined 9-10
Quantum Theory for the Computer Age Unit 3 Ch.7-10, page=90
Web based 3D & XR ($x \in \{A,M,V\}$, R=Reality) https://www.babylonjs.com/
Web based 3D graphics WebGL API (Graphics Layer modeled after OpenGL)
Wiki on Pafnuty Chebyshev

continued ↘

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Classes

2014 AMOP

2017 Group Theory for QM

2018 AMOP

2018 Adv Mechanics

Older Links from Lectures 24

JerkIt Web App: 2-, 2+, Amp50Omega147-, Amp50Omega296, Amp50Omega602, Gap(1)
MolVibes Web App: C3vN3
WaveIt Web App:
  Dim = 3 w/Wave Components;
  Static Char Table: 6, 12, 12(b), 16, 36, 256
  Quantum Carpet with N=20: Gaussian, Boxcar
Quantum Revivals of Morse Oscillators and Farey-Ford Geometry - Li-Harter-CPL-2015
QTCA Unit_5 Ch14 2013
Lester. R. Ford, Am. Math. Monthly 45,586(1938)
John Farey, Phil. Mag.(1816) Wolfram
Harter, J. Mol. Spec. 210, 166-182 (2001)
Harter, Li IMSS (2013)
Li, Harter, Chem.Phys.Letters (2015)

Links to supplement Lecture 25

OscillatorPE Web App: IHO Scenario 2, Coulomb Scenario 3
RelaWavity Web App/Simulator/Calculator: Elliptical - IHO orbits

Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

➤ *Effective potentials for IHO and Coulomb orbits*

Stable equilibrium radii and radial/angular frequency ratios

Classical turning radii and apogee/perigee parameters

Polar coordinate differential equations

Quadrature integration techniques

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Relating orbital energy-momentum to conic-sectional orbital geometry

Kepler equation of time and phase geometry

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T = \frac{m}{2} g_{\rho\rho}\dot{\rho}^2 + \frac{m}{2} g_{\phi\phi}\dot{\phi}^2 = \frac{m}{2}\dot{\rho}^2 + \frac{m}{2}\rho^2\dot{\phi}^2 = \frac{m}{2}\dot{\rho}^2 + \frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi = \frac{\partial T}{\partial \dot{\phi}} = m\rho^2\dot{\phi} = const = \mu$$

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where: $p_\phi = \frac{\partial T}{\partial \dot{\phi}} = m\rho^2\dot{\phi} = const = \mu$

For ALL central forces

$$\dot{\phi} = \frac{\mu}{m\rho^2}$$

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For ALL central forces

$$\dot{\phi} = \frac{\mu}{m\rho^2}$$

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Orbits in Isotropic Oscillator and Coulomb Potentials

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Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

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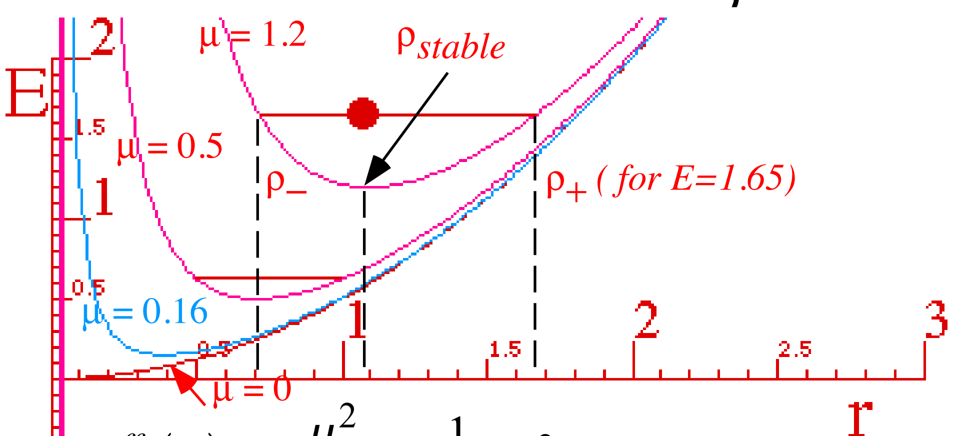

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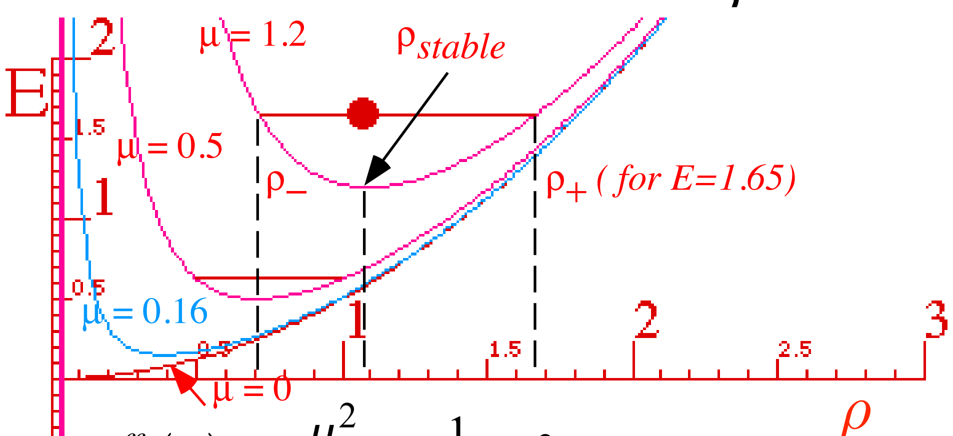

$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

Effective potential for Coulomb $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

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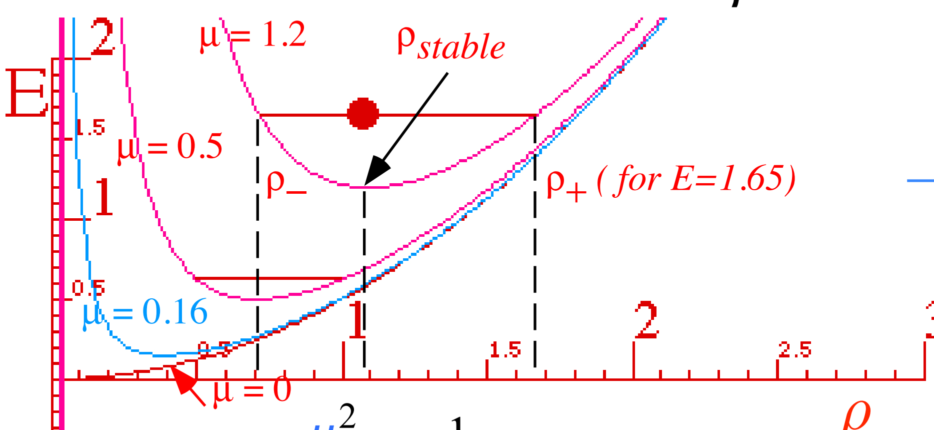

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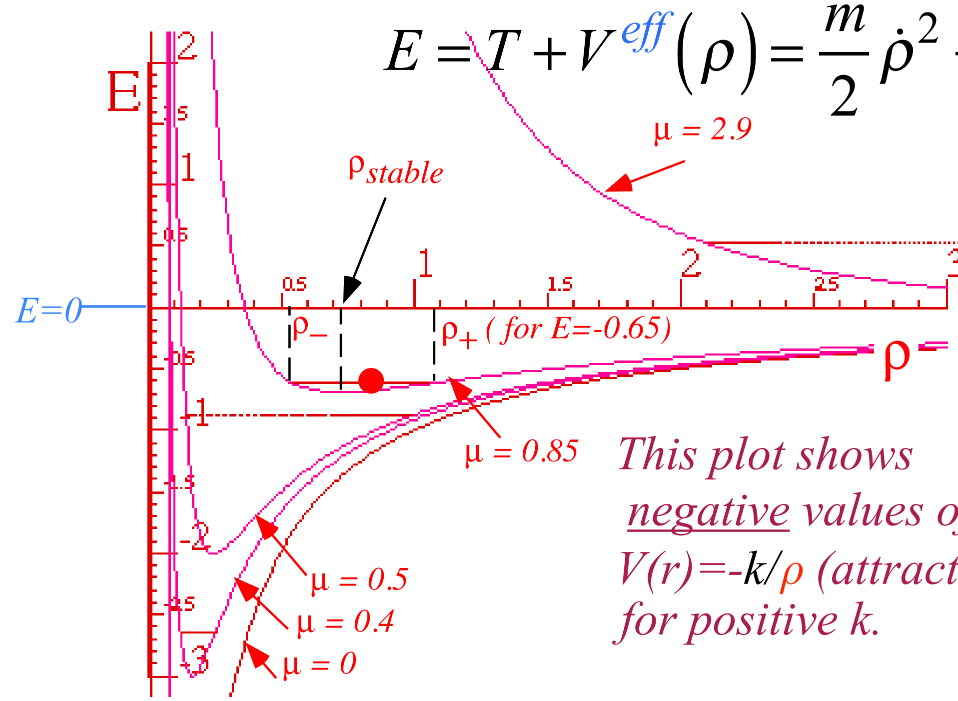

$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

This plot shows <u>negative</u> values of $V(r)=-k/\rho$ (attractive) for positive k.

(Web Simulation: OscillatorPE - IHO)

Web Simulation: OscillatorPE - Coulomb

Orbits in Isotropic Oscillator and Coulomb Potentials

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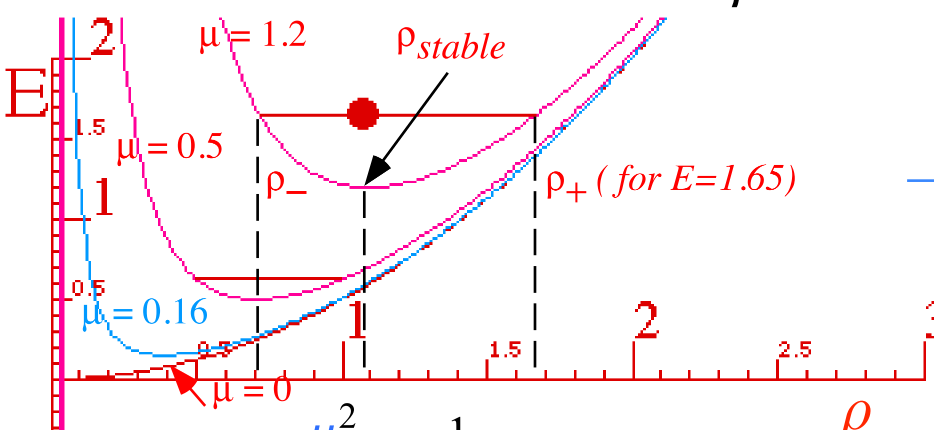

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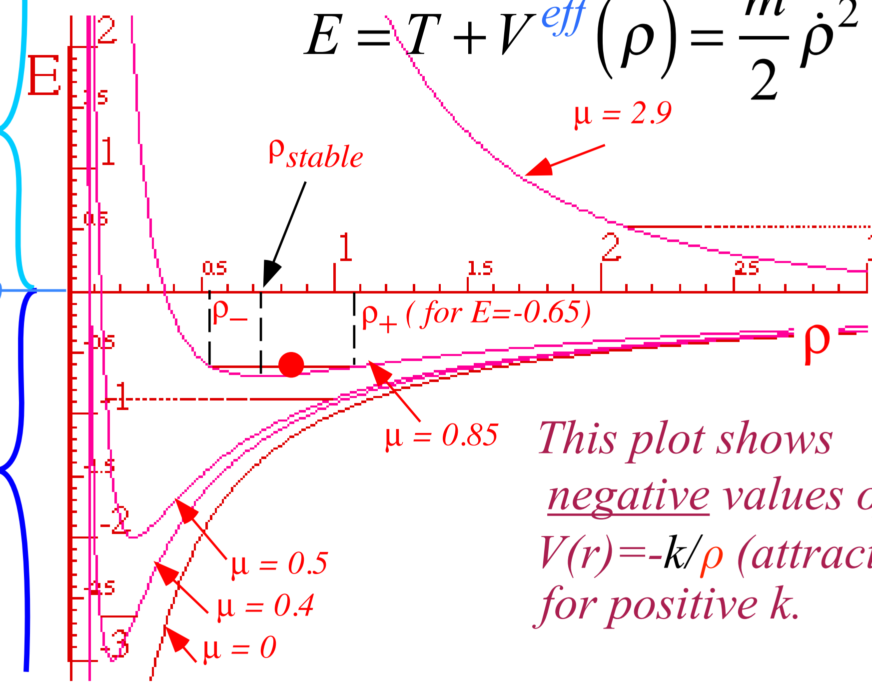
E > 0 (unbound orbits)

E=0

E < 0 (bound orbits)



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

This plot shows negative values of $V(r)=-k/\rho$ (attractive) for positive k.

(Web Simulation: OscillatorPE - IHO)

Web Simulation: OscillatorPE - Coulomb

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

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Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

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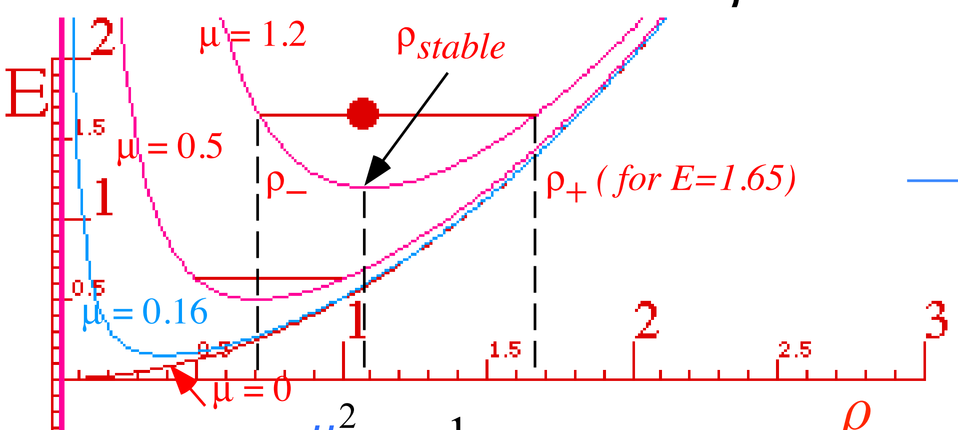

$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

Effective potential for Coulomb $V(\rho)=-k/\rho$

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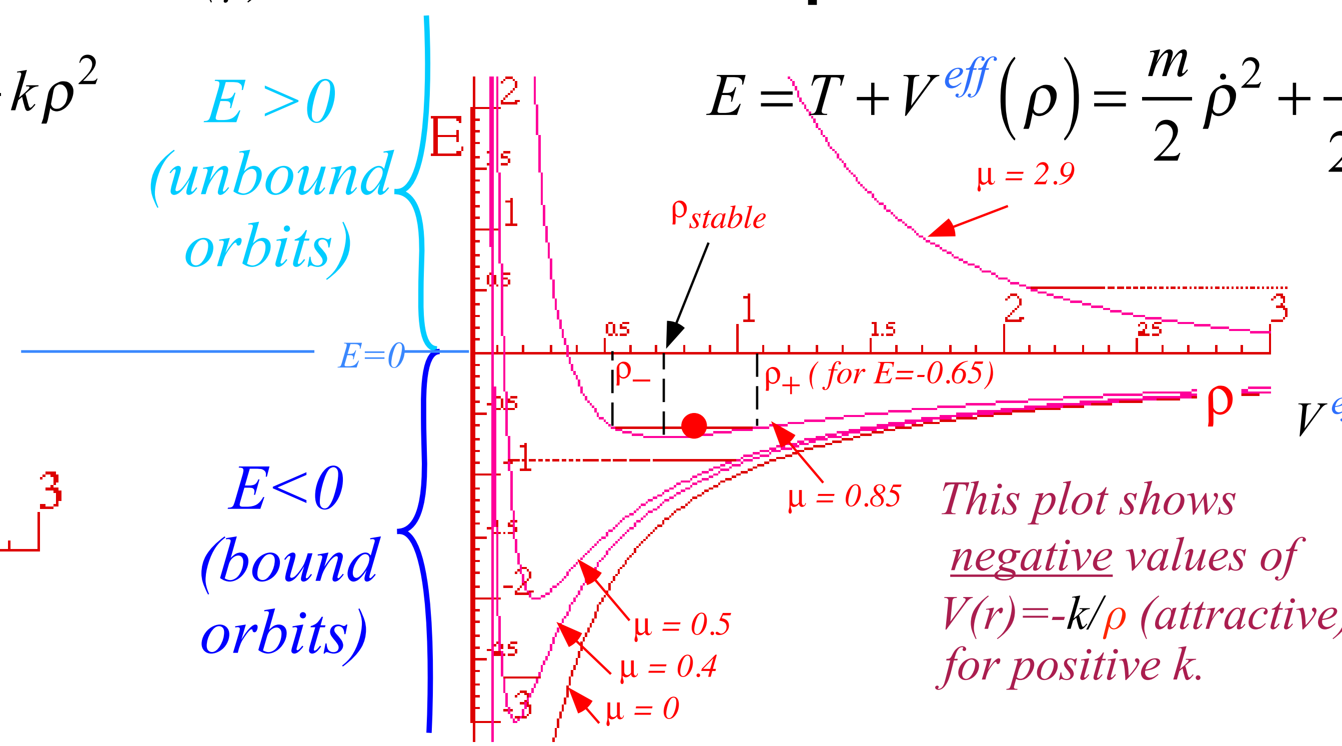

$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

This plot shows <u>negative</u> values of $V(r)=-k/\rho$ (attractive) for positive k.

In either case: *IHO or Coulomb orbit blows up if* k *is negative.*

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

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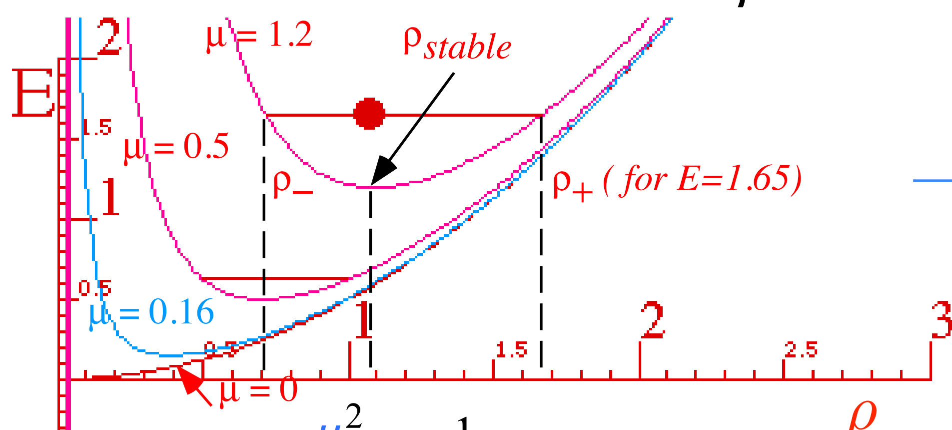

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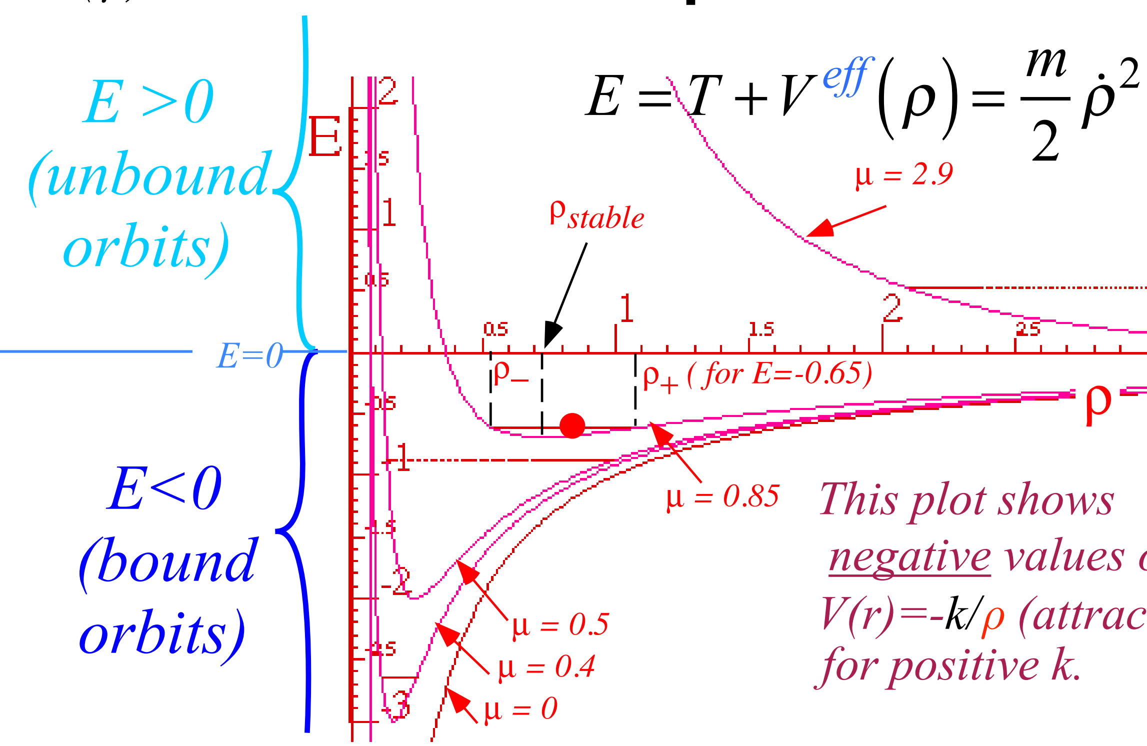

$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

This plot shows negative values of $V(r)=-k/\rho$ (attractive) for positive k.

In either case: *IHO or Coulomb orbit blows up if k is negative.*

NOTE: Our Coulomb field is attractive if k is positive

That is, if $-k/\rho$ is negative.

Coulomb $V(\rho)=-k/\rho$

(Explicit minus (-) convention)

Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

Effective potentials for IHO and Coulomb orbits ➔ *Review: "3steps from Hell" (Lect. 7 Ch. 9 Unit 1)*

Stable equilibrium radii and radial/angular frequency ratios

Classical turning radii and apogee/perigee parameters

Polar coordinate differential equations

Quadrature integration techniques

Detailed orbital functions

Relating orbital energy-momentum to conic-sectional orbital geometry

Kepler equation of time and phase geometry

Review: "Three (equal) steps from Hell" (Lect. 7 Ch. 9 Unit 1)

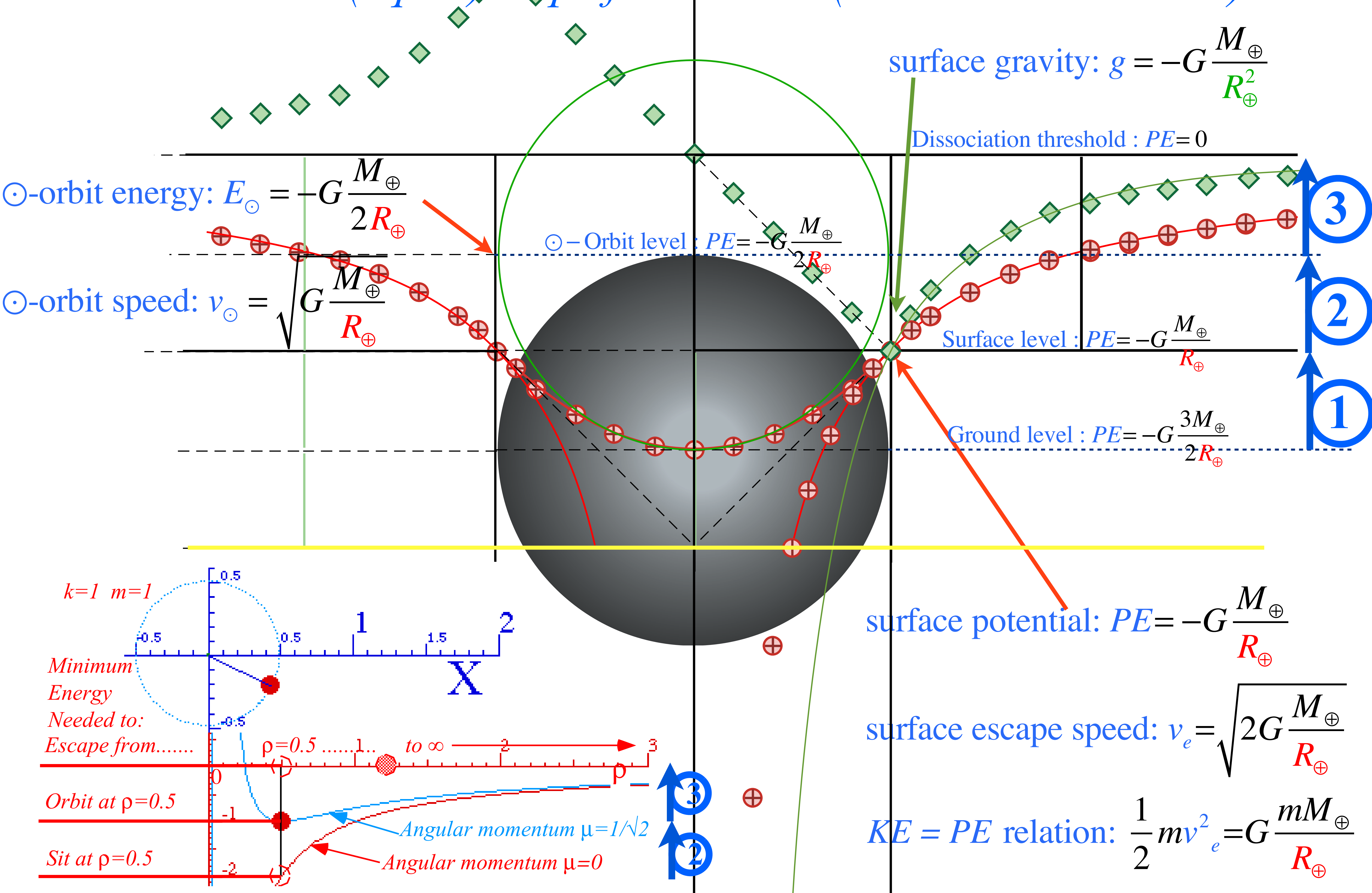

Fig. 5.2.10 Three Coulomb threshold energies: to sit (E=-2), to orbit (E=-1), and to escape (E=0) from R.
A 4th energy (E=-3) is for sitting at the center (ρ=0) of a uniform mass planet of radius ρ=R.

$G = 0.67 \cdot 10^{-10}$

Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

- *Effective potentials for IHO and Coulomb orbits*
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Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

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For ALL central forces

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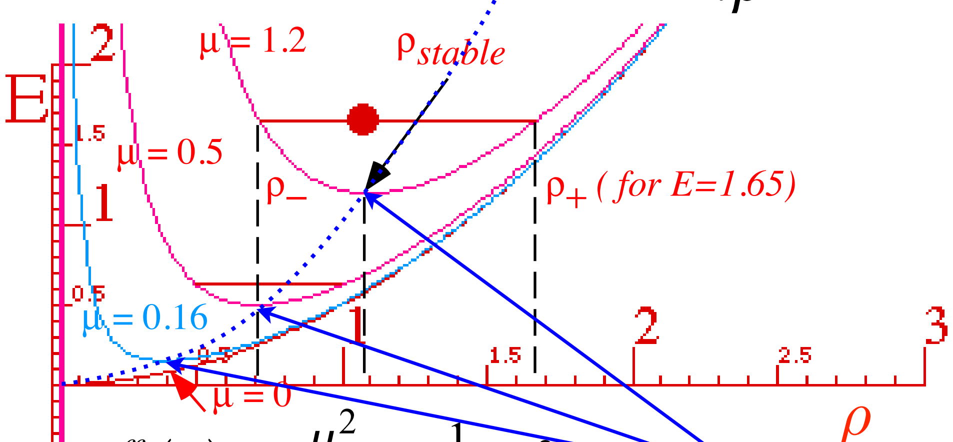

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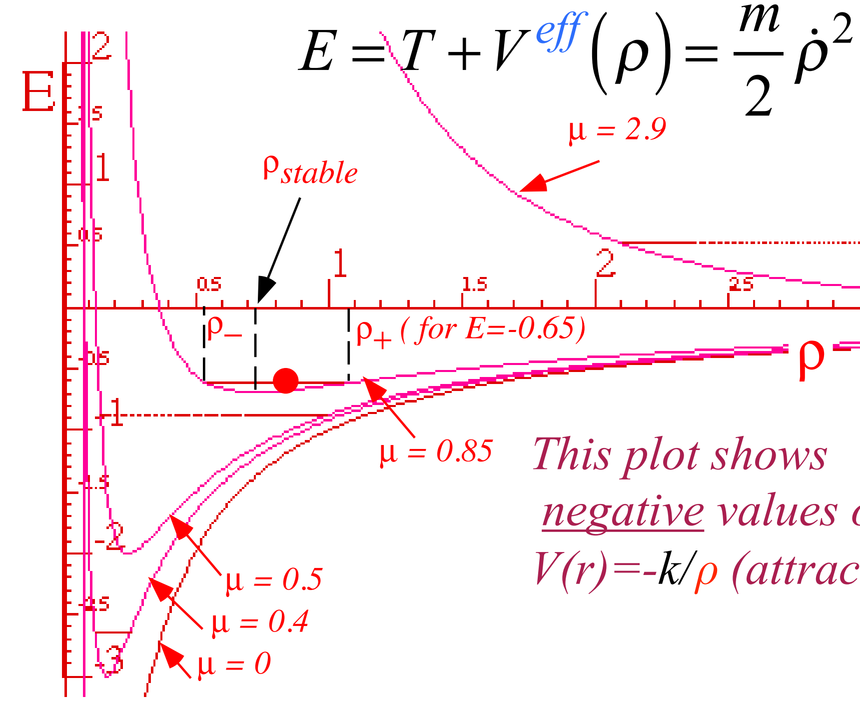


$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

This plot shows negative values of $V(r)=-k/\rho$ (attractive)

Stability radius ρ_{stable} for circular orbits: force or V^{eff} derivative is zero.

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}}=0=\frac{-\mu^2}{m\rho^3}+k\rho \qquad \text{or: } \rho_{stable}=\sqrt{\frac{\mu}{\sqrt{mk}}}$$

$$\frac{\mu^2}{m}=+k\rho^4$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

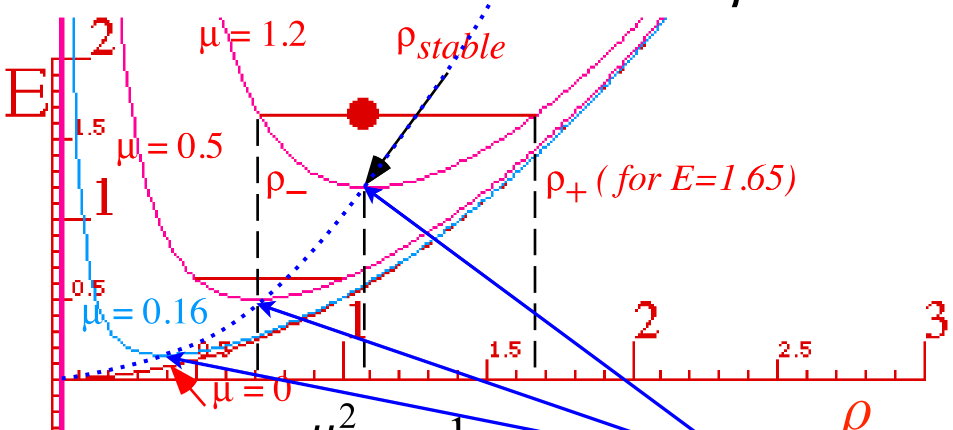
$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

Effective potential for Coulomb $V(\rho)=-k/\rho$

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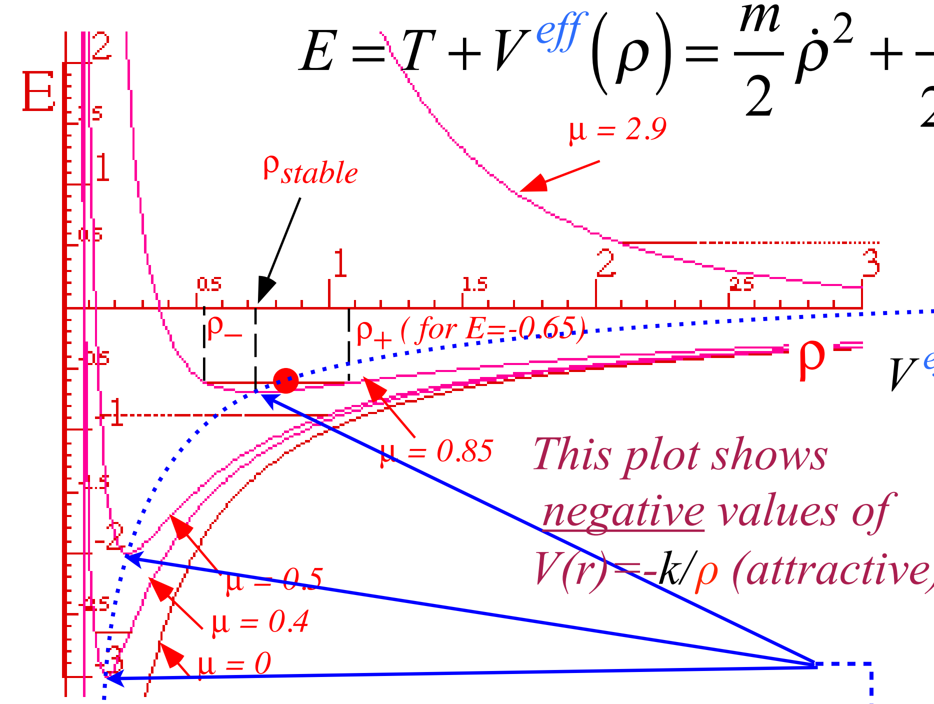

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Stability radius ρ_{stable} for circular orbits: force or V^{eff} derivative is zero.

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}}=0=\frac{-\mu^2}{m\rho^3}+k\rho \qquad \text{or: } \rho_{stable}=\sqrt{\frac{\mu}{\sqrt{mk}}}$$

$$\frac{\mu^2}{m}=+k\rho^4$$

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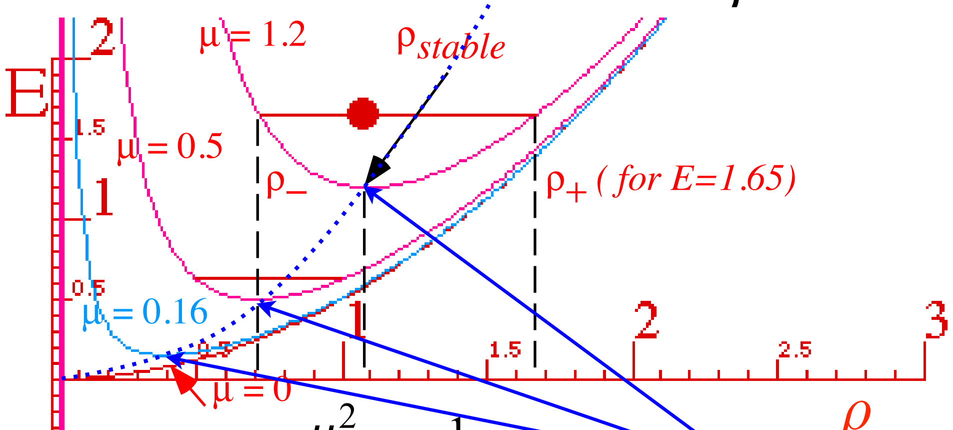

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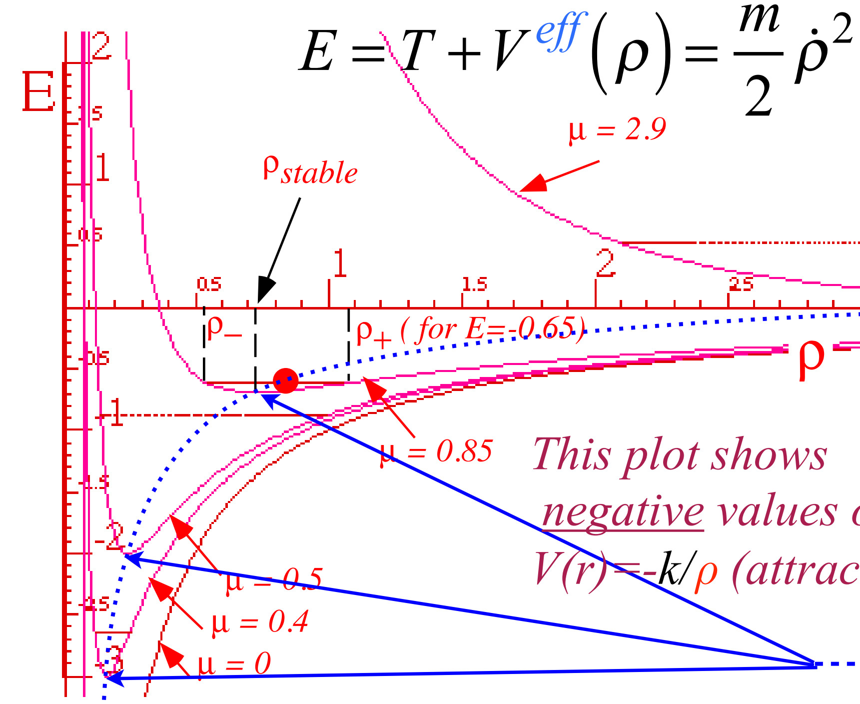

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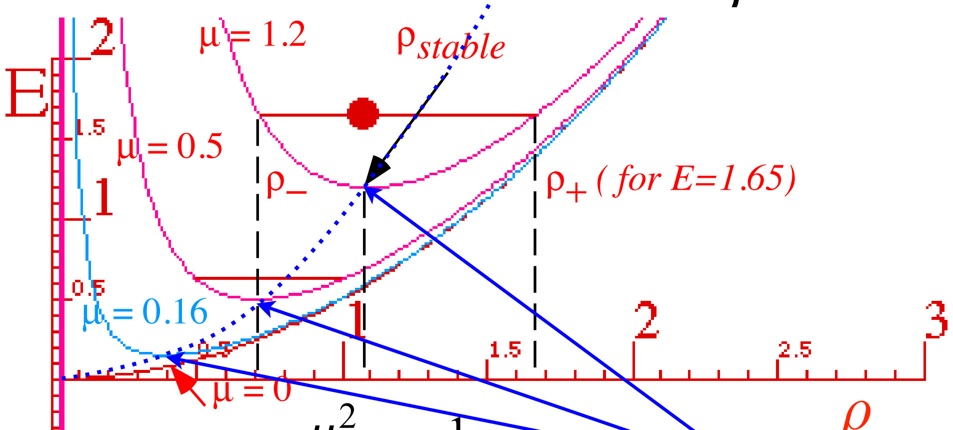

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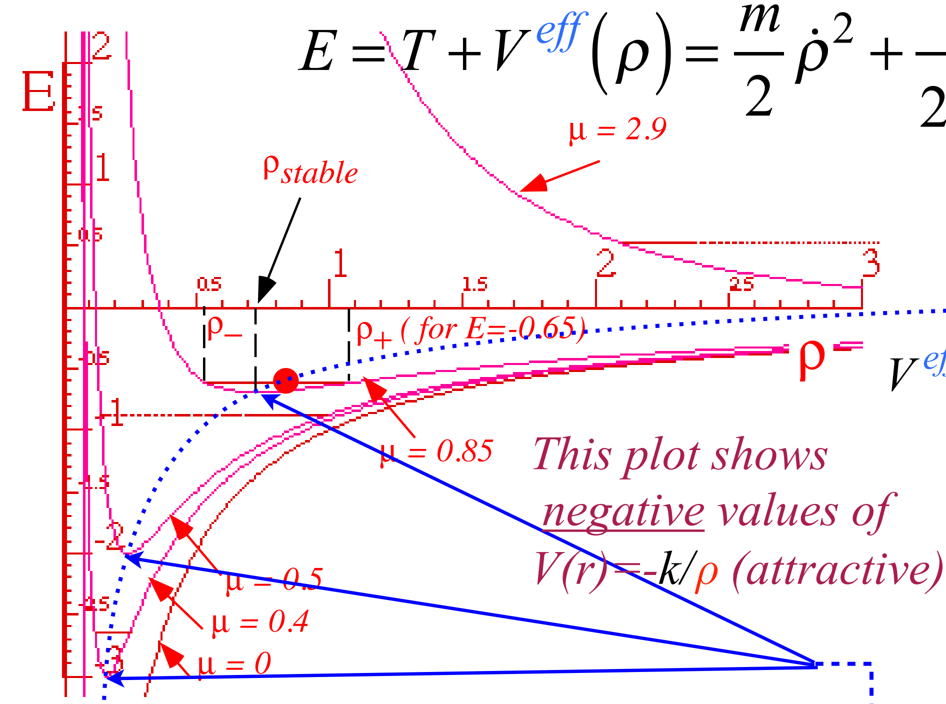

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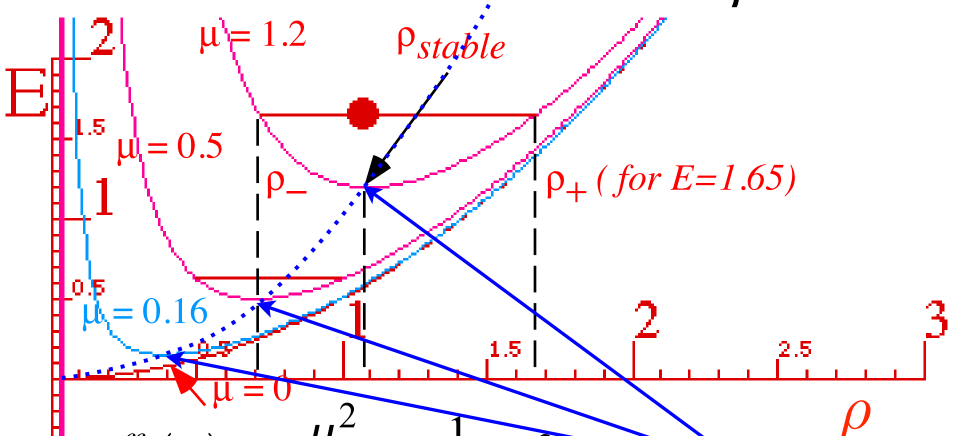

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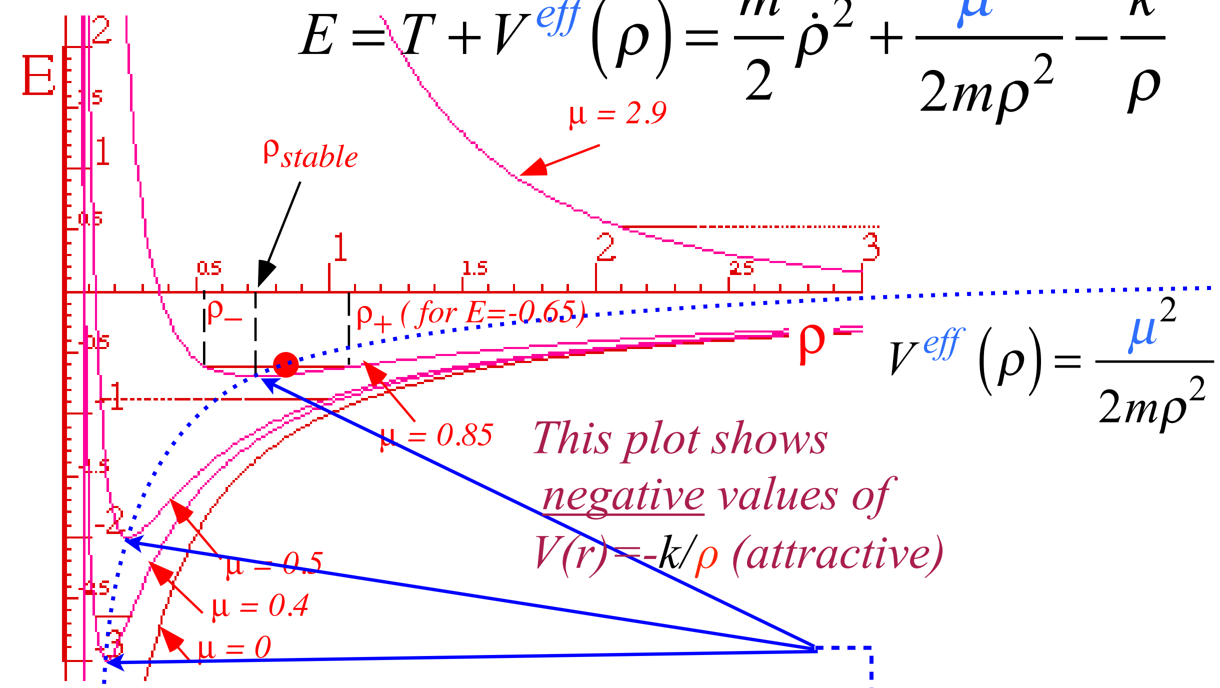

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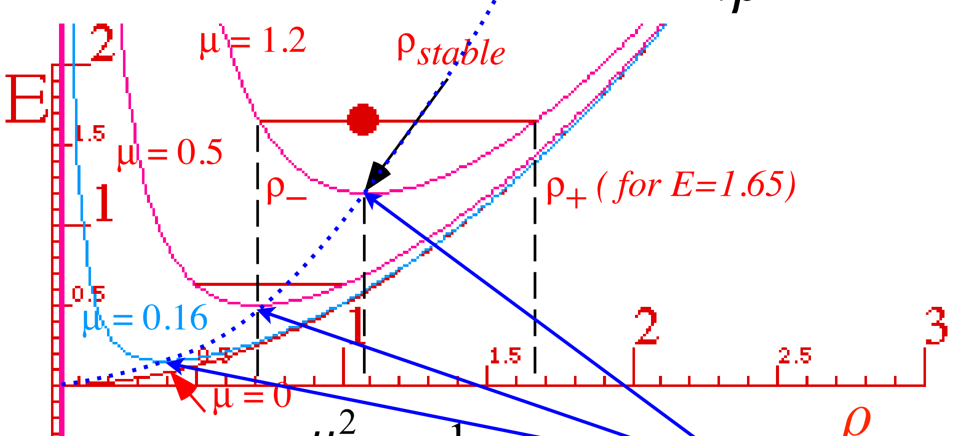

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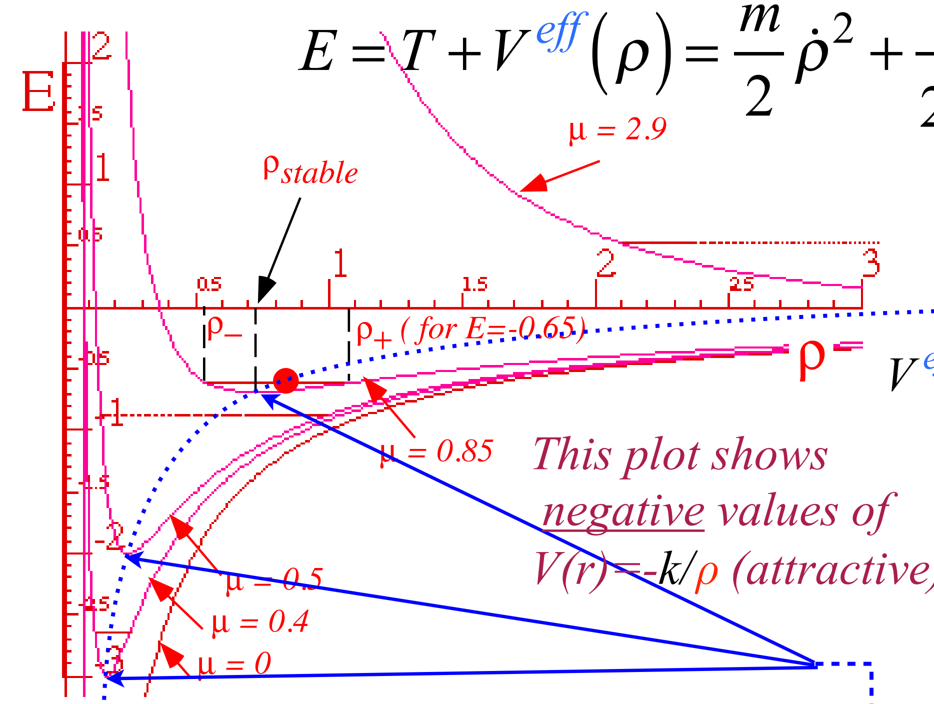

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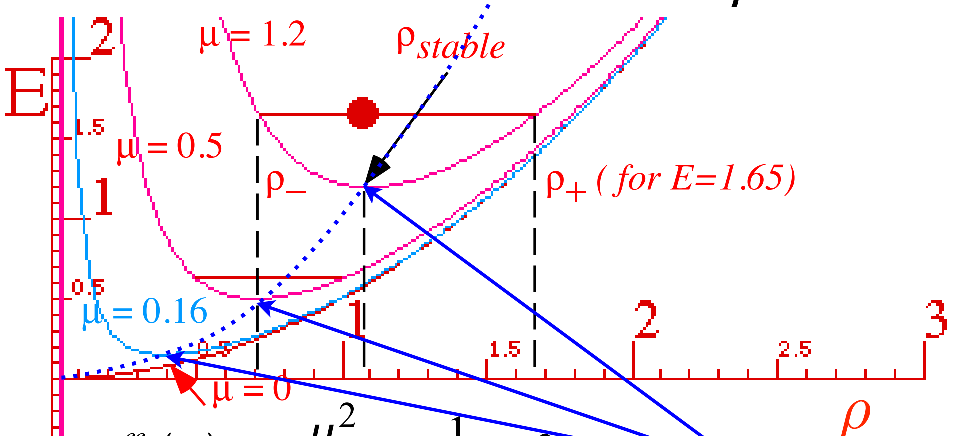

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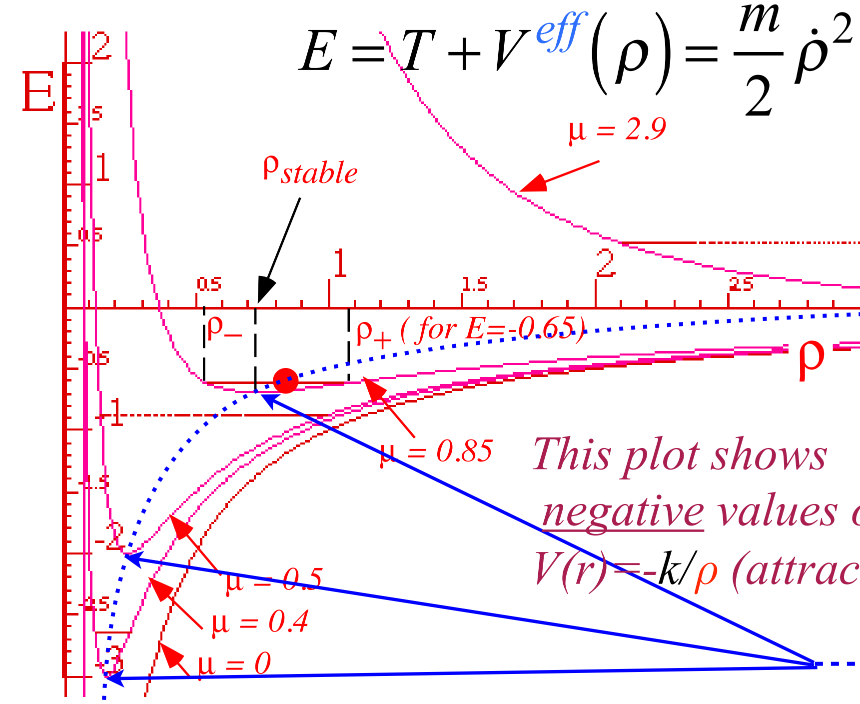

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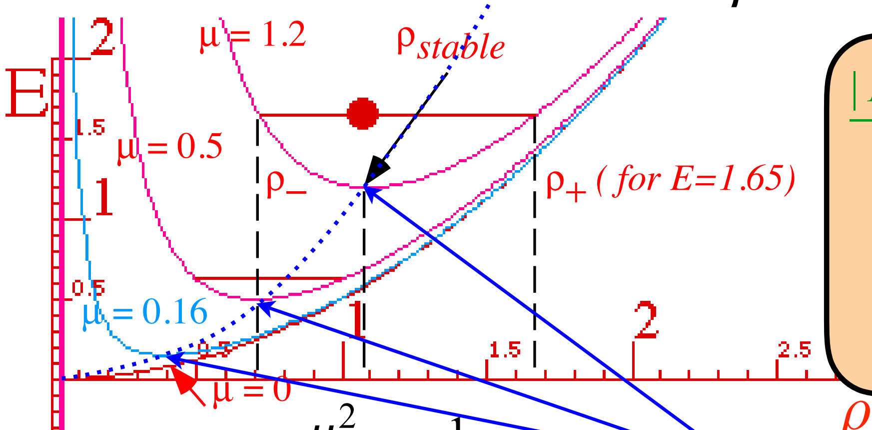

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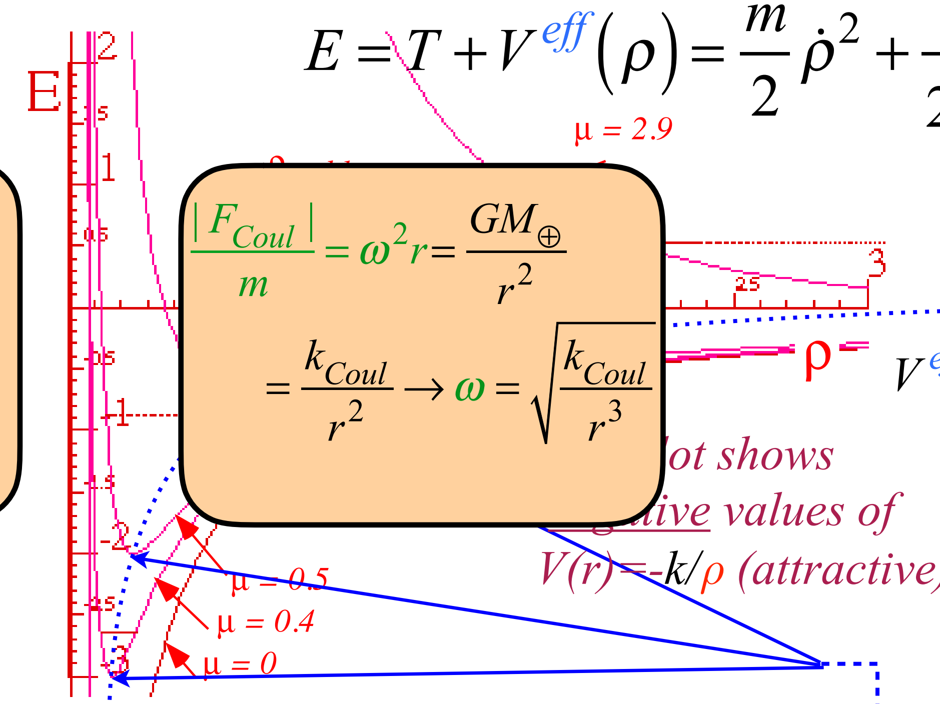

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Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

- *Effective potentials for IHO and Coulomb orbits*
  - *Stable equilibrium radii and radial/angular frequency ratios*
  - ➤ *Classical turning radii and apogee/perigee parameters*
  - *Polar coordinate differential equations*
  - *Quadrature integration techniques*
    - *Detailed orbital functions*
  - *Relating orbital energy-momentum to conic-sectional orbital geometry*
    - *Kepler equation of time and phase geometry*

(A mystery similarity appears)

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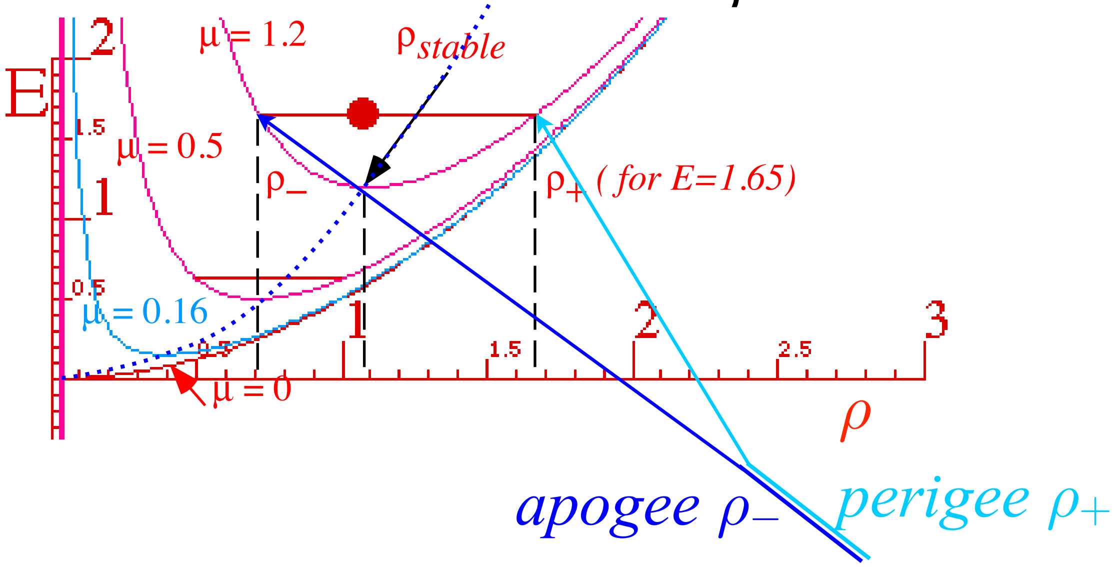

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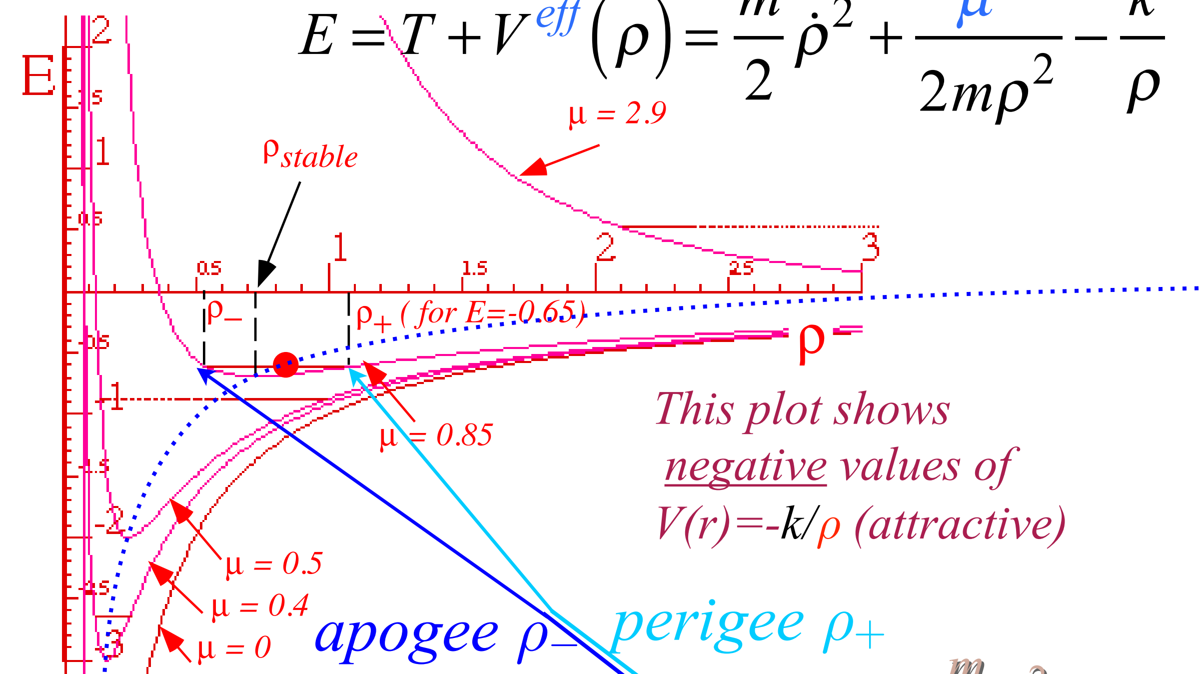


Classical turning radii $\rho_\pm$ for bound orbits are where radial kinetic energy $\frac{m}{2}\dot{\rho}^2$ is zero.

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

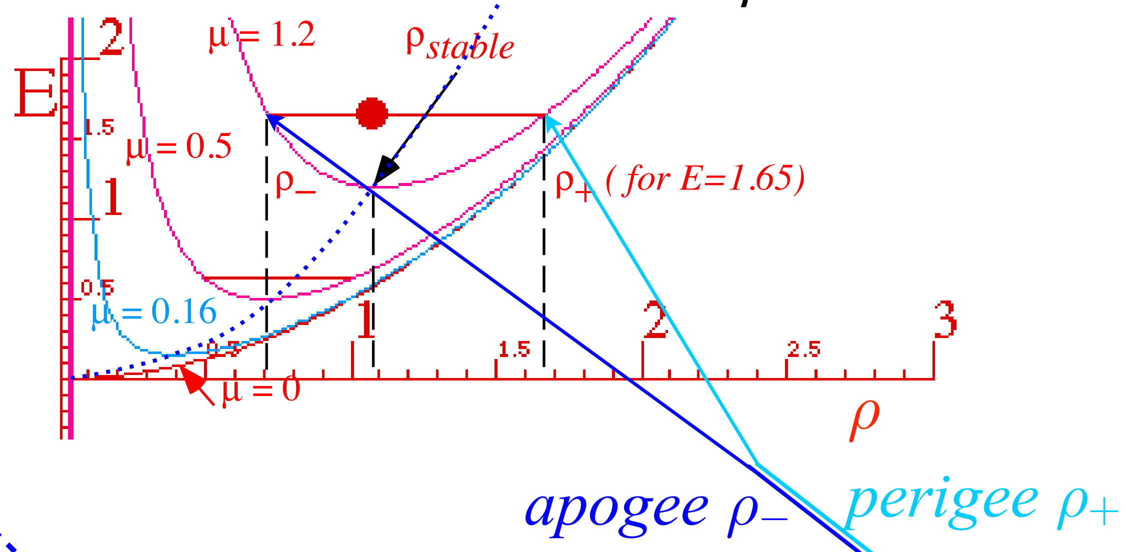
$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial \dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



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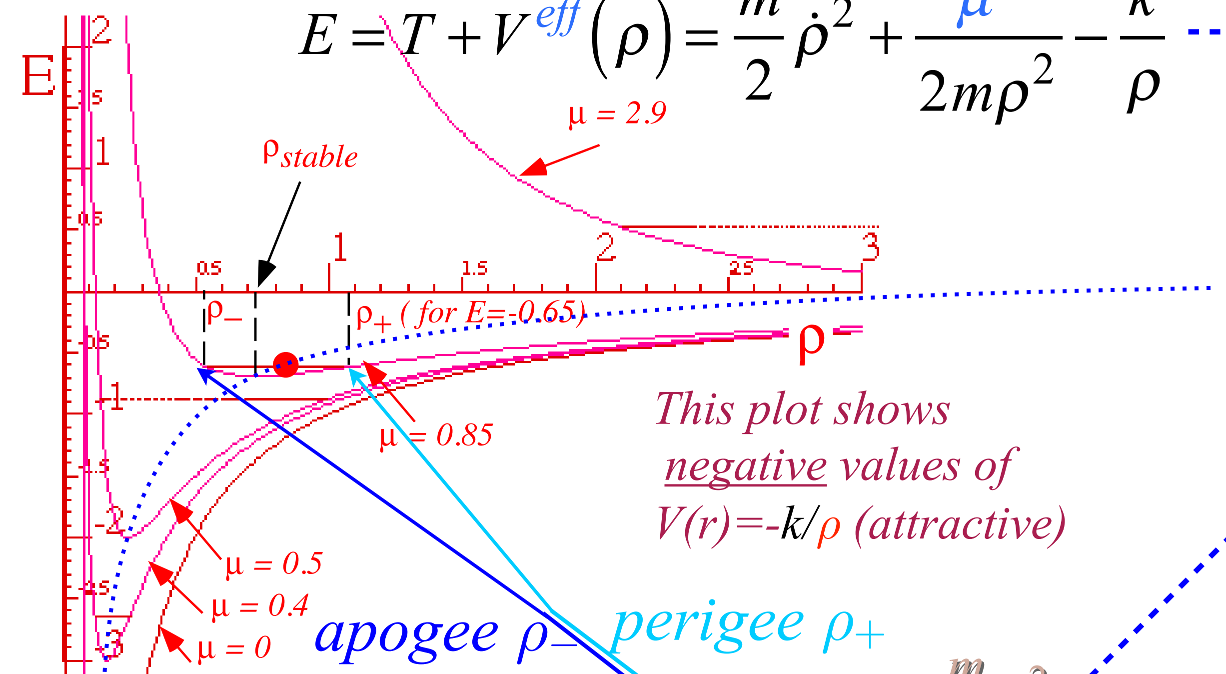

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Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

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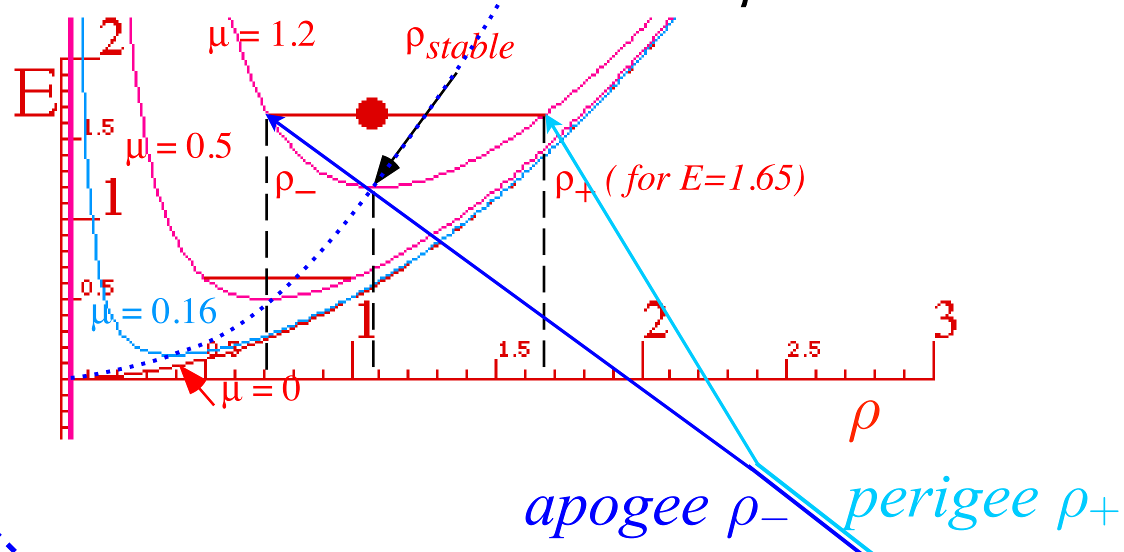

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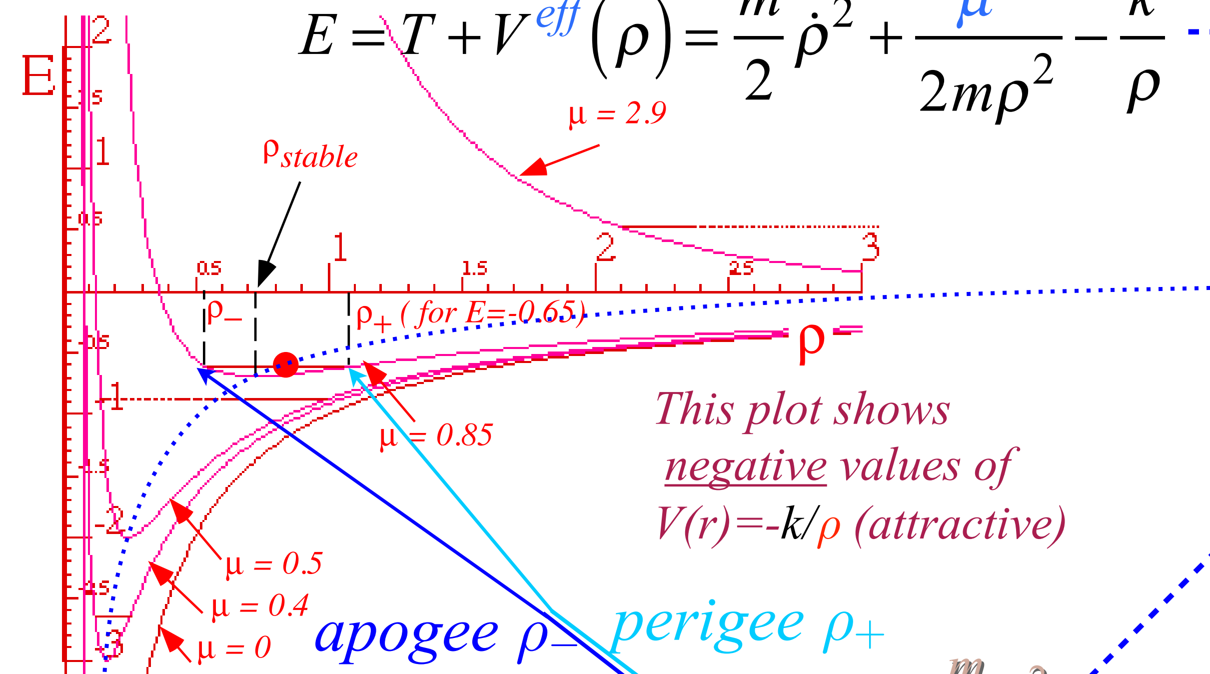

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$$0=\frac{\mu^2}{2m}-E\rho^2+\frac{k}{2}\rho^4 \qquad \text{or else: } 0=\frac{\mu^2}{2m}\frac{1}{\rho^4}-E\frac{1}{\rho^2}+\frac{k}{2}$$

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Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

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For ALL central forces

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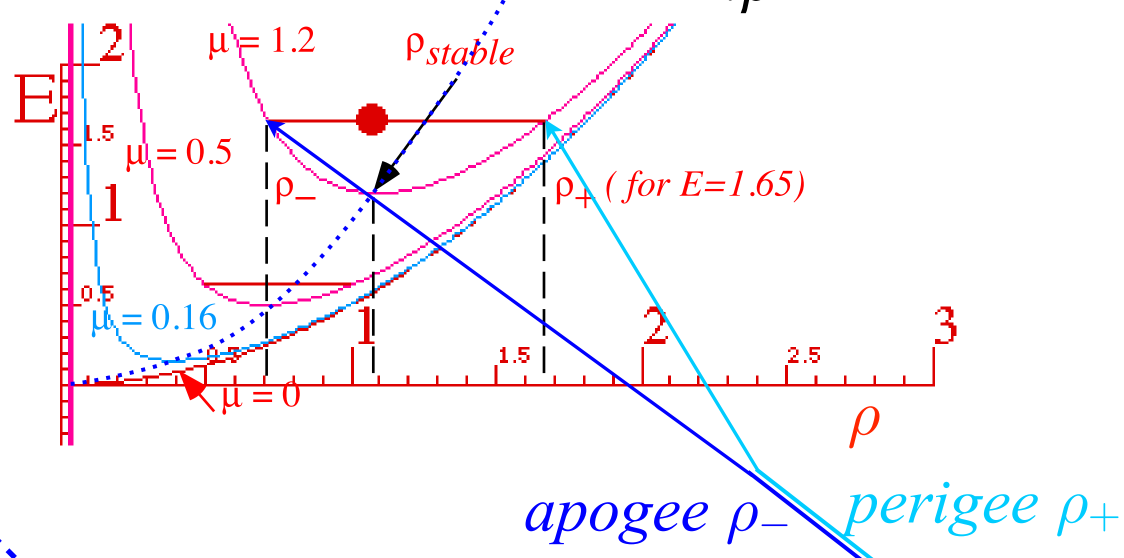

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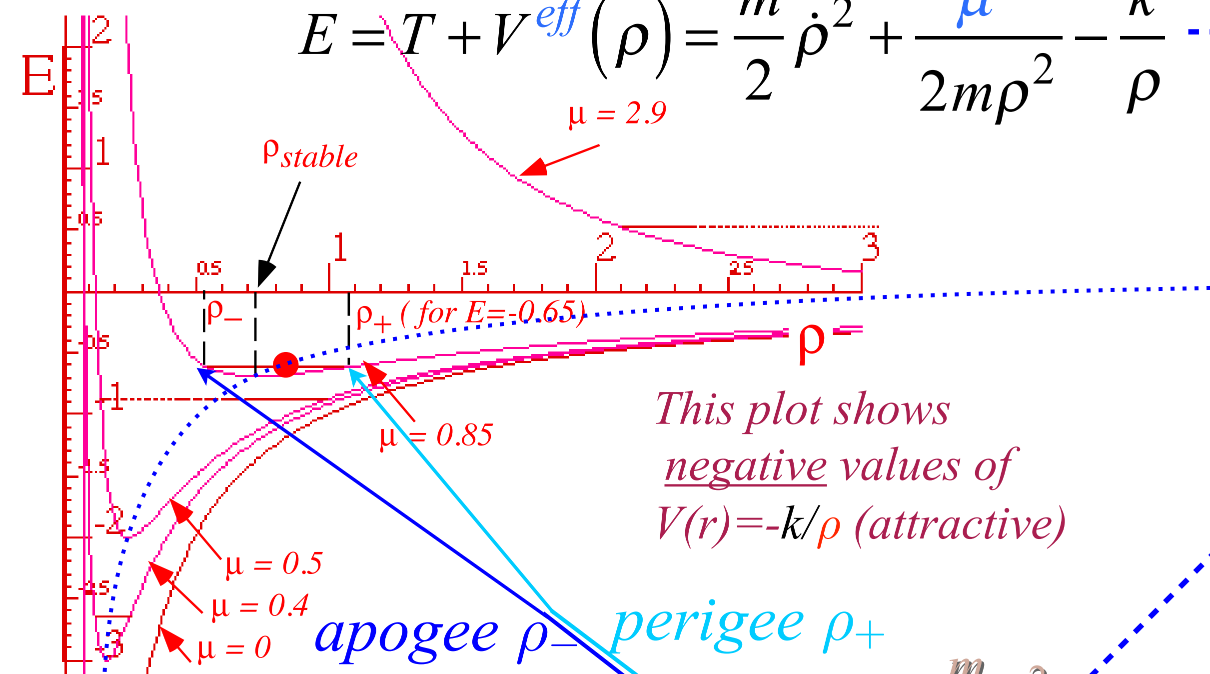

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$$0=\frac{-\mu^2}{2m}+k\rho+E\rho^2 0\quad\text{or else: }0=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}-E$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T = \frac{m}{2} g_{\rho\rho}\dot{\rho}^2 + \frac{m}{2} g_{\phi\phi}\dot{\phi}^2 = \frac{m}{2}\dot{\rho}^2 + \frac{m}{2}\rho^2\dot{\phi}^2 = \frac{m}{2}\dot{\rho}^2 + \frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi = \frac{\partial T}{\partial \dot{\phi}} = m\rho^2\dot{\phi} = const = \mu \qquad \dot{\phi} = \frac{\mu}{m\rho^2}$$

For ALL central forces

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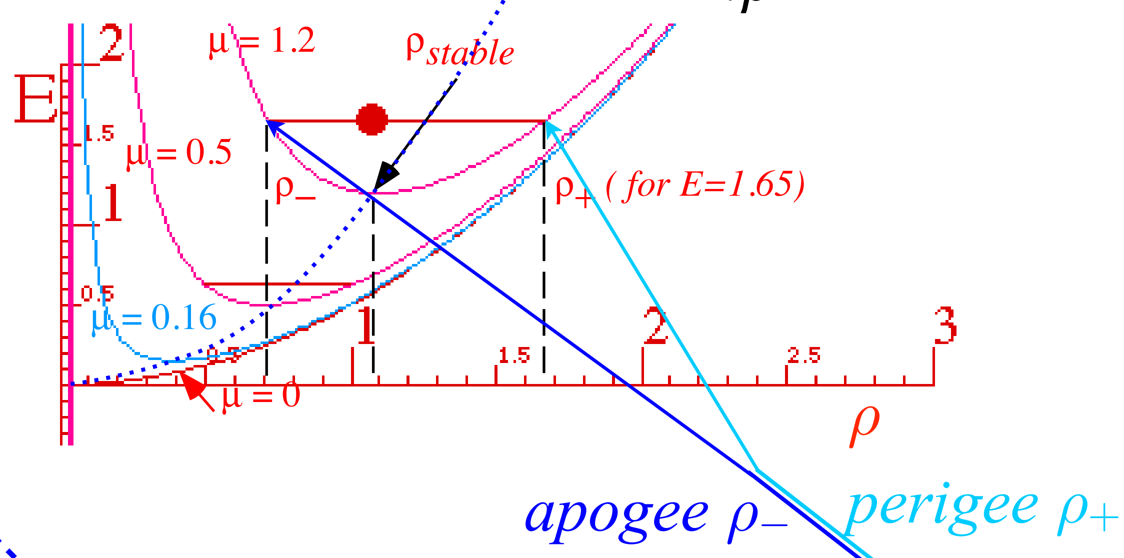
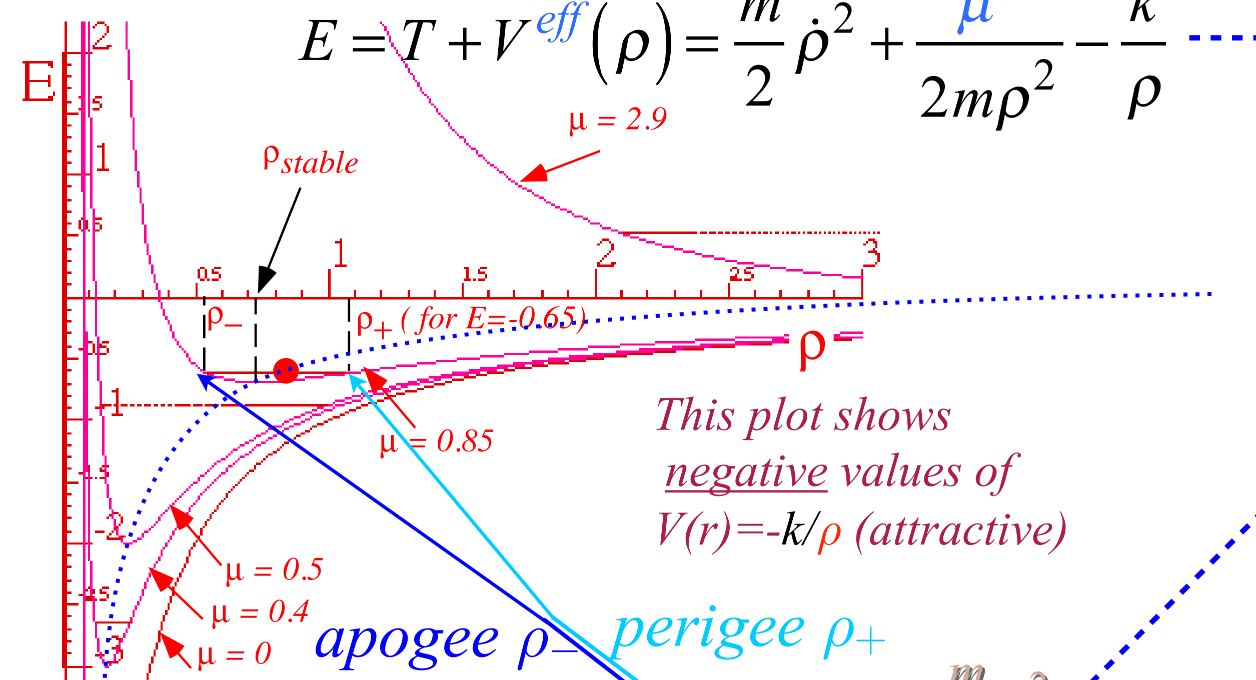


Effective potential for Coulomb $V(\rho)=-k/\rho$

$$E = T + V^{eff}(\rho) = \frac{m}{2}\dot{\rho}^2 + \frac{\mu^2}{2m\rho^2} - \frac{k}{\rho}$$



Classical turning radii $\rho_\pm$ for bound orbits are where radial kinetic energy $\frac{m}{2}\dot{\rho}^2$ is zero.

$$0 = -E + \frac{m}{2}\dot{\rho}^2 + \frac{\mu^2}{2m\rho^2} + \frac{1}{2}k\rho^2$$

$$0 = \frac{\mu^2}{2m} - E\rho^2 + \frac{k}{2}\rho^4 \quad \text{or else: } 0 = \frac{\mu^2}{2m}\frac{1}{\rho^4} - E\frac{1}{\rho^2} + \frac{k}{2}$$

$$\rho_\pm{}^2 = \frac{E \pm \sqrt{E^2 - k\mu^2/m}}{k} \quad \text{or else: } \frac{1}{\rho_\pm{}^2} = \frac{E \mp \sqrt{E^2 - k\mu^2/m}}{\mu^2/m}$$

$$0 = -E + \frac{m}{2}\dot{\rho}^2 + \frac{\mu^2}{2m\rho^2} - \frac{k}{\rho}$$

$$0 = \frac{-\mu^2}{2m} + k\rho + E\rho^2 0 \quad \text{or else: } 0 = \frac{\mu^2}{2m\rho^2} - \frac{k}{\rho} - E$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

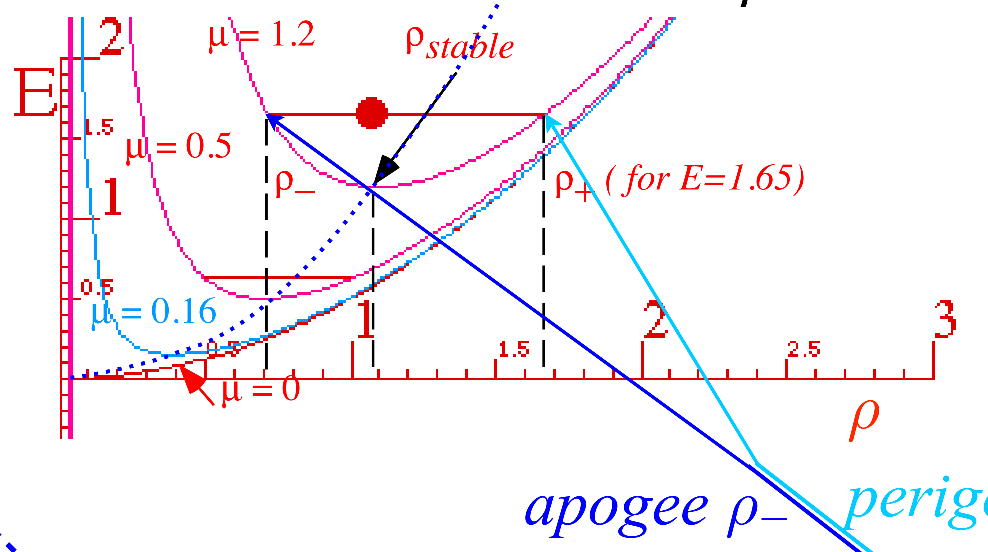
$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

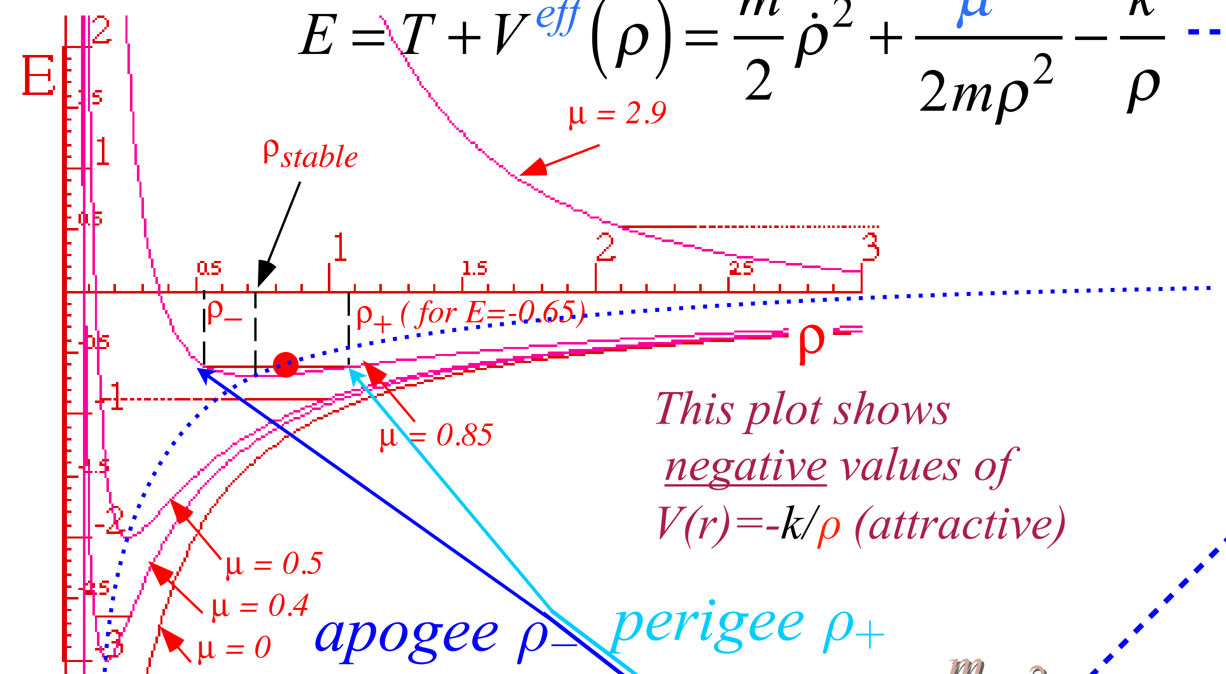
For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$ **Effective potential for Coulomb** $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2 \qquad\qquad E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$





Classical turning radii $\rho_\pm$ for bound orbits are where radial kinetic energy $\frac{m}{2}\dot{\rho}^2$ is zero.

$$0=-E+\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2 \qquad\qquad 0=-E+\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

$$0=\frac{\mu^2}{2m}-E\rho^2+\frac{k}{2}\rho^4 \quad \text{or else: } 0=\frac{\mu^2}{2m}\frac{1}{\rho^4}-E\frac{1}{\rho^2}+\frac{k}{2} \qquad 0=\frac{-\mu^2}{2m}+k\rho+E\rho^2 0 \quad \text{or else: } 0=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}-E$$

$$\rho_\pm{}^2=\frac{E\pm\sqrt{E^2-k\mu^2/m}}{k} \quad \text{or else: } \frac{1}{\rho_\pm{}^2}=\frac{E\mp\sqrt{E^2-k\mu^2/m}}{\mu^2/m} \qquad \rho_\pm=\frac{-k\pm\sqrt{k^2+2E\mu^2/m}}{2E} \quad \text{or else: } \frac{1}{\rho_\pm}=\frac{k\pm\sqrt{k^2+2E\mu^2/m}}{\mu^2/m}$$

Note: $\rho^2\rightarrow\rho$ similarity: $E\rightarrow k$ and $k\rightarrow 2E$

(See p.60)

Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

Effective potentials for IHO and Coulomb orbits

Stable equilibrium radii and radial/angular frequency ratios

Classical turning radii and apogee/perigee parameters

➜ *Polar coordinate differential equations*

Quadrature integration techniques

Detailed orbital functions

Relating orbital energy-momentum to conic-sectional orbital geometry

Kepler equation of time and phase geometry

(A mystery similarity appears)

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T = \frac{m}{2} g_{\rho\rho}\dot{\rho}^2 + \frac{m}{2} g_{\phi\phi}\dot{\phi}^2 = \frac{m}{2}\dot{\rho}^2 + \frac{m}{2}\rho^2\dot{\phi}^2 = \frac{m}{2}\dot{\rho}^2 + \frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi = \frac{\partial T}{\partial \dot{\phi}} = m\rho^2\dot{\phi} = const = \mu \qquad \boxed{\dot{\phi} = \frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) **equations for IHOscillator** $V(\rho)=k\rho^2/2$ (ρ,ϕ) **equations for Coulomb** $V(\rho)=-k/\rho$

$$\frac{m}{2}\dot{\rho}^2 = E - \frac{\mu^2}{2m\rho^2} - \frac{1}{2}k\rho^2 \qquad \dot{\phi} = \frac{\mu}{m\rho^2} \equiv \frac{d\phi}{dt} \qquad \frac{m}{2}\dot{\rho}^2 = E - \frac{\mu^2}{2m\rho^2} - k/\rho$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

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$$\frac{d\phi}{dt}\frac{dt}{d\rho}=\frac{d\phi}{d\rho}=\frac{\dot{\phi}}{\dot{\rho}}=\frac{\mu}{m\rho^2\dot{\rho}}$$

(Finding $\rho=\rho(\phi)$ trajectory equations)

Parameter table on p.77

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

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(ρ,ϕ) equations for IHOscillator $V(\rho)=k\rho^2/2$

(ρ,ϕ) equations for Coulomb $V(\rho)=-k/\rho$

$$\frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-\frac{1}{2}k\rho^2$$

$$\dot{\phi}=\frac{\mu}{m\rho^2}\equiv\frac{d\phi}{dt}$$

$$\frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-k/\rho$$

$$\dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}}$$

$$\frac{d\phi}{dt}\frac{dt}{d\rho}=\frac{d\phi}{d\rho}=\frac{\dot{\phi}}{\dot{\rho}}=\frac{\mu}{m\rho^2\dot{\rho}}$$

$$\dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}$$

(Time solution begins: p. 79)

(Time solution ends: p. 88)

(Finding $\rho=\rho(\phi)$ trajectory equations)

Parameter table on p.77

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

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$$d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot{\rho}}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}}}$$

$$\dot{\phi}=\frac{\mu}{m\rho^2}\equiv\frac{d\phi}{dt}$$

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(ρ,ϕ) equations for Coulomb $V(\rho)=-k/\rho$

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(Finding $\rho=\rho(\phi)$ trajectory equations)

Parameter table on p.77

Orbits in Isotropic Oscillator and Coulomb Potentials

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$$d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot{\rho}}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}}}$$

Let: $\frac{1}{\rho}=u$ so: $\begin{cases}-\frac{d\rho}{\rho^2}=du\\ d\rho=-\frac{du}{u^2}\end{cases}$

$$d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot{\rho}}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}}$$

(Finding $\rho=\rho(\phi)$ trajectory equations)

Parameter table on p.77

Orbits in Isotropic Oscillator and Coulomb Potentials

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$$\dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}} \qquad \frac{d\phi}{d\rho}=\frac{\dot{\phi}}{\dot{\rho}}=\frac{\mu}{m\rho^2\dot{\rho}} \qquad \dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}$$

$$d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot{\rho}}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}}}$$

Let: $\frac{1}{\rho}=u$ so: $\begin{cases}-\frac{d\rho}{\rho^2}=du\\ d\rho=-\frac{du}{u^2}\end{cases}$

$$d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot{\rho}}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}}$$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{\frac{2E}{m}-\frac{\mu^2u^2}{m^2}-\frac{k}{mu^2}}} \qquad d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{\frac{2E}{m}-\frac{\mu^2u^2}{m^2}-\frac{2ku}{m}}}$$

(Finding $\rho=\rho(\phi)$ trajectory equations)

Parameter table on p.77

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \dot{\phi}=\frac{\mu}{m\rho^2}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) equations for IHOscillator $V(\rho)=k\rho^2/2$

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$$\frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-\frac{1}{2}k\rho^2 \qquad \dot{\phi}=\frac{\mu}{m\rho^2}\equiv\frac{d\phi}{dt} \qquad \frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-k/\rho$$

IHOscillator:

$$\dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}}$$

$$d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot{\rho}}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}}}$$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{\frac{2E}{m}-\frac{\mu^2u^2}{m^2}-\frac{k}{mu^2}}}$$

Common:

$$\frac{d\phi}{d\rho}=\frac{\dot{\phi}}{\dot{\rho}}=\frac{\mu}{m\rho^2\dot{\rho}}$$

Let: $\frac{1}{\rho}=u$ so: $\begin{cases}-\frac{d\rho}{\rho^2}=du\\ d\rho=-\frac{du}{u^2}\end{cases}$

Let: $x=u^2=\frac{1}{\rho^2}$ so: $\begin{cases}dx=2udu\\ du=\frac{dx}{2\sqrt{x}}\end{cases}$

Coulomb:

$$\dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}$$

$$d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot{\rho}}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}}$$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{\frac{2E}{m}-\frac{\mu^2u^2}{m^2}-\frac{2ku}{m}}}$$

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$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{x}\sqrt{\frac{2E}{m}-\frac{\mu^2x}{m^2}-\frac{k}{mx}}}$$

Common:

$$\frac{d\phi}{d\rho}=\frac{\dot{\phi}}{\dot{\rho}}=\frac{\mu}{m\rho^2\dot{\rho}}$$

Let: $\frac{1}{\rho}=u$ so: $-\frac{d\rho}{\rho^2}=du$, $d\rho=-\frac{du}{u^2}$

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Coulomb:

$$\dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}$$

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where: $p_\phi=\frac{\partial T}{\partial\dot\phi}=m\rho^2\dot\phi=const=\mu$

$$\boxed{\dot\phi=\frac{\mu}{m\rho^2}}$$

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$$\dot\rho=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}}$$

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$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}$$

$$\dot\phi=\frac{\mu}{m\rho^2}\equiv\frac{d\phi}{dt}$$

$$\frac{d\phi}{d\rho}=\frac{\dot\phi}{\dot\rho}=\frac{\mu}{m\rho^2\dot\rho}$$

Let: $\frac{1}{\rho}=u$ so: $-\frac{d\rho}{\rho^2}=du$, $\quad d\rho=-\frac{du}{u^2}$

Let: $x=u^2=\frac{1}{\rho^2}$ so: $dx=2u\,du$, $\quad du=\frac{dx}{2\sqrt{x}}$

(Finding $\rho=\rho(\phi)$ trajectory equations)

Parameter table on p.77

Note: $\rho^2\rightarrow\rho$ similarity: $E\rightarrow k$ and $k\rightarrow 2E$

(ρ,ϕ) equations for Coulomb $V(\rho)=-k/\rho$

$$\frac{m}{2}\dot\rho^2=E-\frac{\mu^2}{2m\rho^2}-k/\rho$$

$$\dot\rho=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}$$

$$d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot\rho}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}}$$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{\frac{2E}{m}-\frac{\mu^2u^2}{m^2}-\frac{2ku}{m}}}$$

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$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}}$$

Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

- *Effective potentials for IHO and Coulomb orbits*
  - *Stable equilibrium radii and radial/angular frequency ratios*
  - *Classical turning radii and apogee/perigee parameters*
  - *Polar coordinate differential equations*
  - ➤ *Quadrature integration techniques*
    - *Detailed orbital functions*
  - *Relating orbital energy-momentum to conic-sectional orbital geometry*
    - *Kepler equation of time and phase geometry*

(A mystery similarity appears)

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \dot{\phi}=\frac{\mu}{m\rho^2}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) equations for IHOscillator $V(\rho)=k\rho^2/2$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}$$

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$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}}$$

Some radial $V(\rho)=k\rho^n$ repeatedly enjoy the integral $\phi(z)$ below. *(Introduced briefly in Unit 3)*

$$\phi(z)=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}$$

(Finding $\rho=\rho(\phi)$ trajectory solutions)

Orbits in Isotropic Oscillator and Coulomb Potentials

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$$\phi(z)=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{-\left(z-z_+\right)\left(x-x_-\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{\left(z_+-z\right)\left(z-z_-\right)}}$$

(Finding $\rho=\rho(\phi)$ trajectory solutions)

Orbits in Isotropic Oscillator and Coulomb Potentials

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Roots $z_\pm$ are *classical turning points* (*perigee* $z_-=\alpha-\beta$, *apogee* $z_+=\alpha+\beta$). Solve integral $\phi(z)$ for $z(\phi)$.

Defining α and β:

$$z_\pm=\alpha\pm\beta$$

(Finding $\rho=\rho(\phi)$ trajectory solutions)

Orbits in Isotropic Oscillator and Coulomb Potentials

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Solution based on quadratic roots of $Az^2+Bz+C=0$.

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$$\frac{\sqrt{A}}{D}\phi(z)=\int\frac{dz}{\sqrt{\beta^2-(z-\alpha)^2}}=\sin^{-1}\frac{z-\alpha}{\beta}$$

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Solution based on quadratic roots of $Az^2+Bz+C=0$. Variable z may be ρ or $u=1/\rho$ or ρ^2 or $x=1/\rho^2$...

$$\frac{\sqrt{A}}{D}\phi(z)=\int\frac{dz}{\sqrt{\beta^2-(z-\alpha)^2}}=\sin^{-1}\frac{z-\alpha}{\beta}\qquad z(\phi)=\frac{\sqrt{B^2-4AC}}{2A}\sin\frac{\sqrt{A}}{D}\phi-\frac{B}{2A}$$

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Defining α and β:

$$z_\pm=\alpha\pm\beta \text{ , where: } \alpha=\frac{z_++z_-}{2}=\frac{-B}{2A} \text{ , and: } \beta=\frac{z_+-z_-}{2}=\frac{\sqrt{B^2-4AC}}{2A}$$

Solution based on quadratic roots of $Az^2+Bz+C=0$. Variable z may be ρ or $u=1/\rho$ or ρ^2 or $x=1/\rho^2$...

$$\frac{\sqrt{A}}{D}\phi(z)=\int\frac{dz}{\sqrt{\beta^2-(z-\alpha)^2}}=\sin^{-1}\frac{z-\alpha}{\beta}$$

$$z(\phi)=\frac{\sqrt{B^2-4AC}}{2A}\sin\frac{\sqrt{A}}{D}\phi-\frac{B}{2A}$$

$$z(\phi)=\beta\cdot\sin\frac{\sqrt{A}}{D}\phi-\alpha$$

(Finding $\rho=\rho(\phi)$ trajectory solutions)

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) equations for IHOscillator $V(\rho)=k\rho^2/2$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}$$

Let: $x=u^2=\frac{1}{\rho^2}$

(ρ,ϕ) equations for Coulomb $V(\rho)=-k/\rho$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}}$$

Some radial $V(\rho)=k\rho^n$ repeatedly enjoy the integral $\phi(z)$ below. *(Introduced briefly in Unit 3)*

$$\phi(z)=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{-(z-z_+)(x-x_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{(z_+-z)(z-z_-)}}$$

Roots $z_\pm$ are *classical turning points* (*perigee* $z_-=\alpha-\beta$, *apogee* $z_+=\alpha+\beta$). Solve integral $\phi(z)$ for $z(\phi)$.

$$z_\pm=\alpha\pm\beta \text{ , where: } \alpha=\frac{z_++z_-}{2}=\frac{-B}{2A} \text{ , and: } \beta=\frac{z_+-z_-}{2}=\frac{\sqrt{B^2-4AC}}{2A}$$

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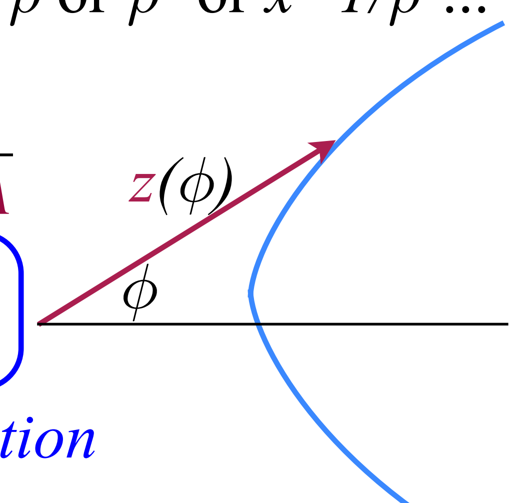
$$\boxed{z(\phi)=\beta\cdot\sin\frac{\sqrt{A}}{D}\phi-\alpha}$$

radial-polar-coordinate orbit function



Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

Effective potentials for IHO and Coulomb orbits

Stable equilibrium radii and radial/angular frequency ratios

Classical turning radii and apogee/perigee parameters

Polar coordinate differential equations

Quadrature integration techniques

Detailed orbital functions

Relating orbital energy-momentum to conic-sectional orbital geometry

Kepler equation of time and phase geometry

(A mystery similarity appears)

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Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \dot{\phi}=\frac{\mu}{m\rho^2}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) orbits for IHOscillator $V(\rho)=k\rho^2/2$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}$$

Let: $x=u^2=\frac{1}{\rho^2}$

(ρ,ϕ) orbits for Coulomb $V(\rho)=-k/\rho$

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Let: $u=\frac{1}{\rho}$

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where: $p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu$ $\qquad \dot{\phi}=\frac{\mu}{m\rho^2}$

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$$A=\frac{\mu^2}{m^2}\quad B=-\frac{2E}{m}\quad C=\frac{k}{m}\quad D=-\frac{\mu}{2m}$$

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$\alpha=\frac{E}{\mu^2/m}$

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$$\alpha=\frac{-B}{2A},\quad \beta=\frac{\sqrt{B^2-4AC}}{2A}\qquad z=\alpha+\beta\sin\frac{\phi\sqrt{A}}{D}$$

Oscillator:

$$A=\frac{\mu^2}{m^2}\quad B=-\frac{2E}{m}\quad C=\frac{k}{m}\quad D=-\frac{\mu}{2m}$$

$$\alpha=\frac{E}{\mu^2/m}$$

$$\beta=\frac{\sqrt{E^2-k\mu^2/m}}{\mu^2/m}$$

Coulomb:

$$A=\frac{\mu^2}{m^2}\quad B=\frac{2k}{m}\quad C=-\frac{2E}{m}\quad D=-\frac{\mu}{m}$$

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Algebra details on following pages

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Let: $x=u^2=\frac{1}{\rho^2}$

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Let: $u=\frac{1}{\rho}$

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IHO: $A=\frac{\mu^2}{m^2}$ $\quad B=-\frac{2E}{m}$ $\quad C=\frac{k}{m}$ $\quad D=-\frac{\mu}{2m}$

$$\alpha=\frac{E}{\mu^2/m}$$

$$\beta=\frac{\sqrt{E^2-k\mu^2/m}}{\mu^2/m}$$

$$x=\frac{1}{\rho^2}=\frac{E}{\mu^2/m}+\frac{\sqrt{E^2-k\mu^2/m}}{\mu^2/m}\sin(-2\phi)$$

Coulomb: $A=\frac{\mu^2}{m^2}$ $\quad B=\frac{2k}{m}$ $\quad C=-\frac{2E}{m}$ $\quad D=-\frac{\mu}{m}$

$$\alpha=\frac{-k}{\mu^2/m}\ ,$$

$$\beta=\frac{\sqrt{k^2+2E\mu^2/m}}{\mu^2/m}$$

$$u=\frac{1}{\rho}=\frac{-k}{\mu^2/m}+\frac{\sqrt{k^2+2E\mu^2/m}}{\mu^2/m}\sin(-\phi)$$

Algebra details on following pages

Algebra details and checks

$$\alpha = \frac{-B}{2A}\ ,\ \beta = \frac{\sqrt{B^2-4AC}}{2A}$$

(ρ,ϕ) orbits for IHOscillator $V(\rho)=k\rho^2/2$

$$A=\frac{\mu^2}{m^2}\quad B=-\frac{2E}{m}\quad C=\frac{k}{m}\quad D=-\frac{\mu}{2m}$$

$$\alpha = \frac{\frac{2E}{m}}{2\frac{\mu^2}{m^2}} = \frac{E}{\mu^2/m} = \frac{z_+ + z_-}{2}$$

$$\beta = \frac{\sqrt{\left(\frac{2E}{m}\right)^2 - 4\frac{\mu^2}{m^2}\frac{k}{m}}}{2\frac{\mu^2}{m^2}}$$

$$\beta = \frac{\sqrt{\left(\frac{2Em}{m\cdot m}\right)^2 - 4\frac{\mu^2}{m^2}\frac{km}{m^2}}}{2\frac{\mu^2}{m^2}} = \frac{\sqrt{E^2-k\mu^2/m}}{\mu^2/m} = \frac{z_+ - z_-}{2}$$

(ρ,ϕ) equations for Coulomb $V(\rho)=-k/\rho$

$$A=\frac{\mu^2}{m^2}\quad B=\frac{2k}{m}\quad C=-\frac{2E}{m}\quad D=-\frac{\mu}{m}$$

$$\alpha = \frac{-\frac{2k}{m}}{2\frac{\mu^2}{m^2}} = \frac{-k}{\mu^2/m} = \frac{z_+ + z_-}{2}$$

$$\beta = \frac{\sqrt{\left(\frac{2k}{m}\right)^2 + 4\frac{\mu^2}{m^2}\frac{2E}{m}}}{2\frac{\mu^2}{m^2}} = \frac{\sqrt{k^2+2E\mu^2/m}}{\mu^2/m} = \frac{z_+ - z_-}{2}$$

Parameter table on p.77

Checking that roots $z_\pm$ are *classical turning points* (*perigee* $z_-=\alpha-\beta$, *apogee* $z_+=\alpha+\beta$) (See p.32)

$$\rho_\pm^{\ 2} = \frac{E\pm\sqrt{E^2-k\mu^2/m}}{k} \quad \text{or else:} \quad \frac{1}{\rho_\pm^{\ 2}} = \frac{E\mp\sqrt{E^2-k\mu^2/m}}{\mu^2/m}$$

$$\rho_\pm = \frac{-k\pm\sqrt{k^2+2E\mu^2/m}}{2E} \quad \text{or else:} \quad \frac{1}{\rho_\pm} = \frac{k\pm\sqrt{k^2+2E\mu^2/m}}{\mu^2/m}$$

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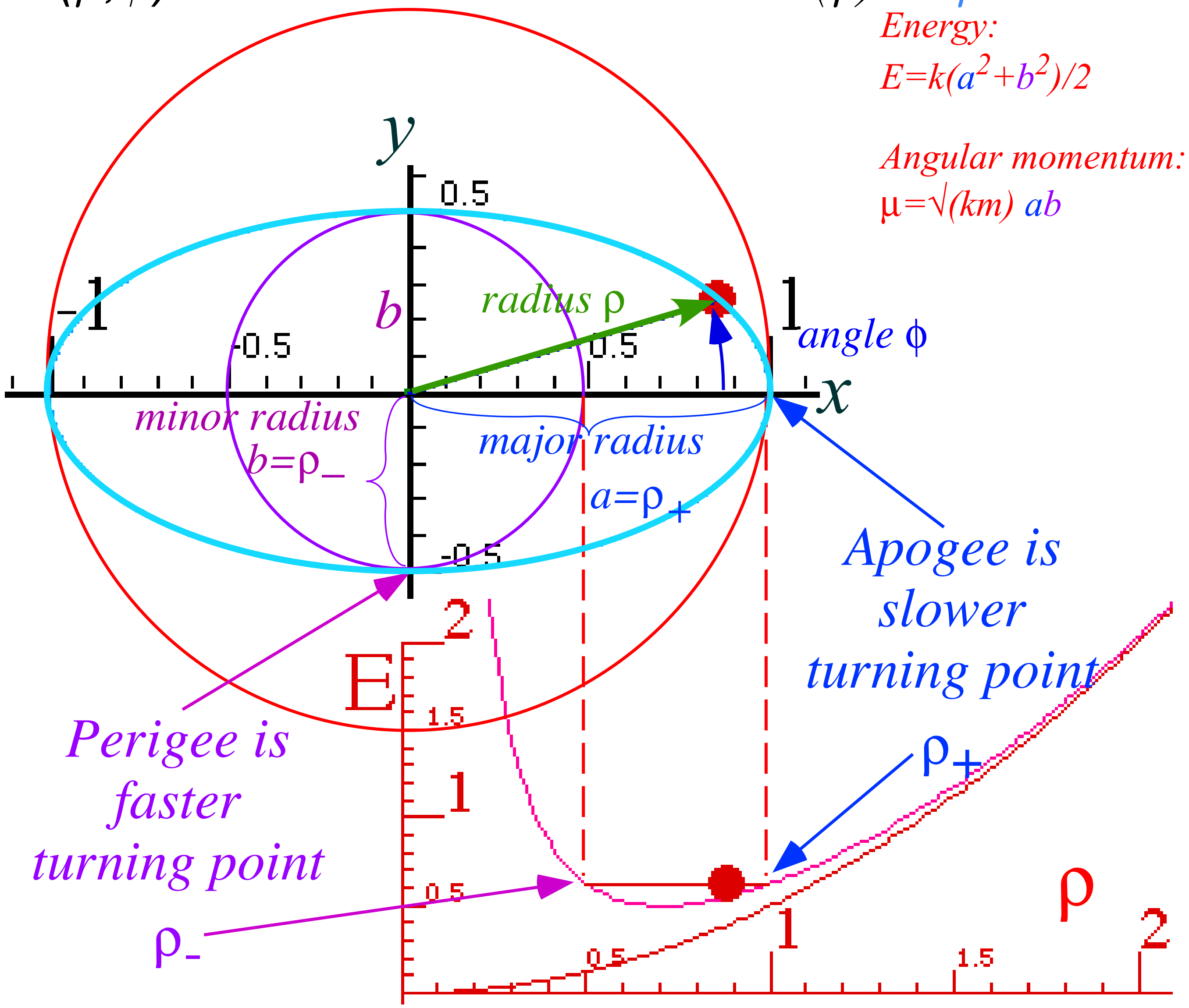
➔ *Detailed orbital functions*

Relating orbital energy-momentum to conic-sectional orbital geometry

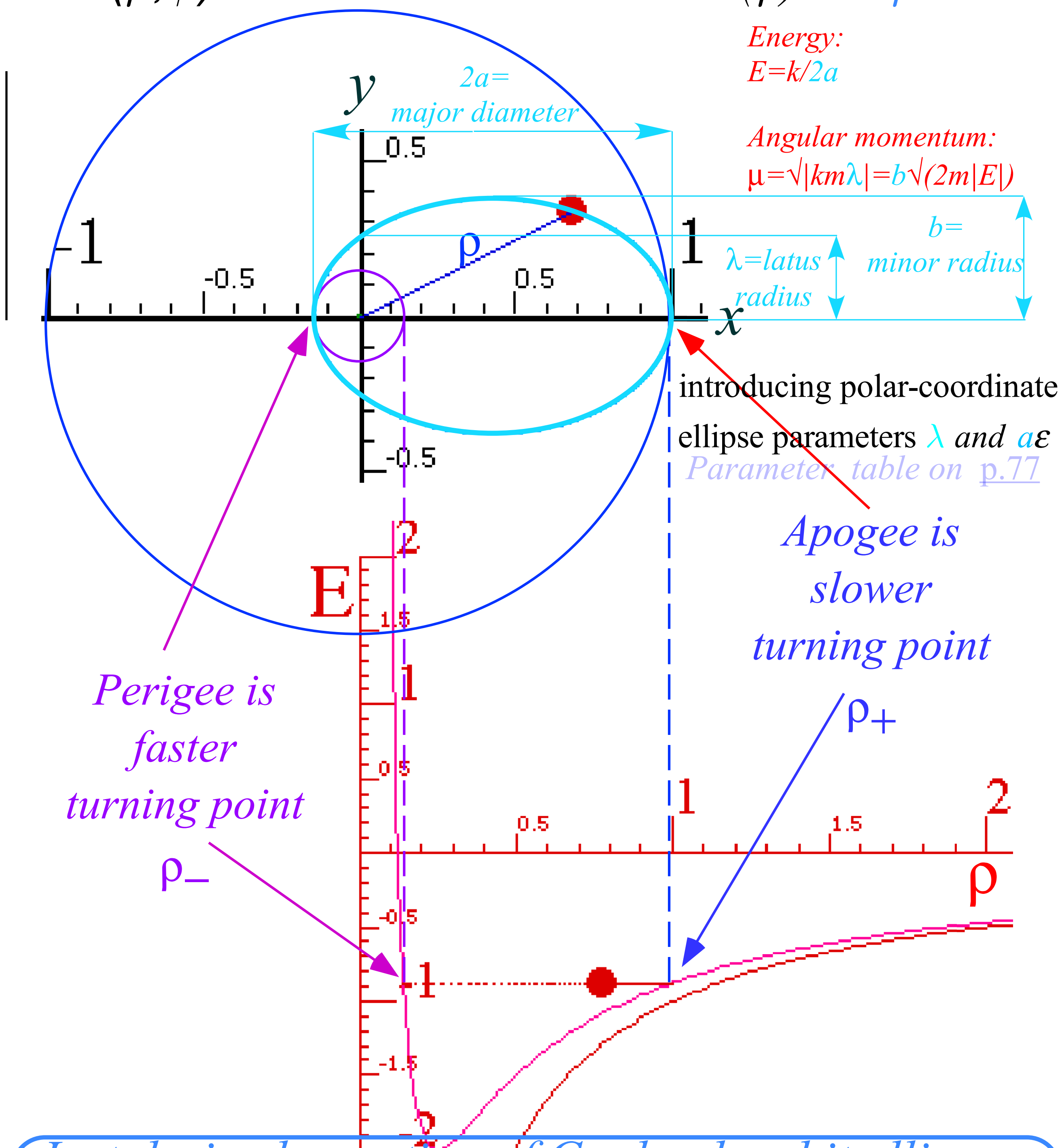
Kepler equation of time and phase geometry

Orbits in Isotropic Oscillator and Coulomb Potentials

(ρ,ϕ) **orbits for IHOscillator** $V(\rho) = k\rho^2/2$



(ρ,ϕ) **orbits for Coulomb** $V(\rho) = -k/\rho$



Just derived equation of IHO orbit ellipse

$$x = \frac{1}{\rho^2} = \frac{E}{\mu^2/m} + \frac{\sqrt{E^2 - k\mu^2/m}}{\mu^2/m}\sin(-2\phi)$$

$$\frac{1}{\rho^2} = +\frac{1}{2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)+\frac{1}{2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\cos(2\phi)$$

One of many equations of center-centered ellipse

Just derived equation of Coulomb orbit ellipse

$$u = \frac{1}{\rho} = \frac{-k}{\mu^2/m} + \frac{\sqrt{k^2 + 2E\mu^2/m}}{\mu^2/m}\sin(-\phi)$$

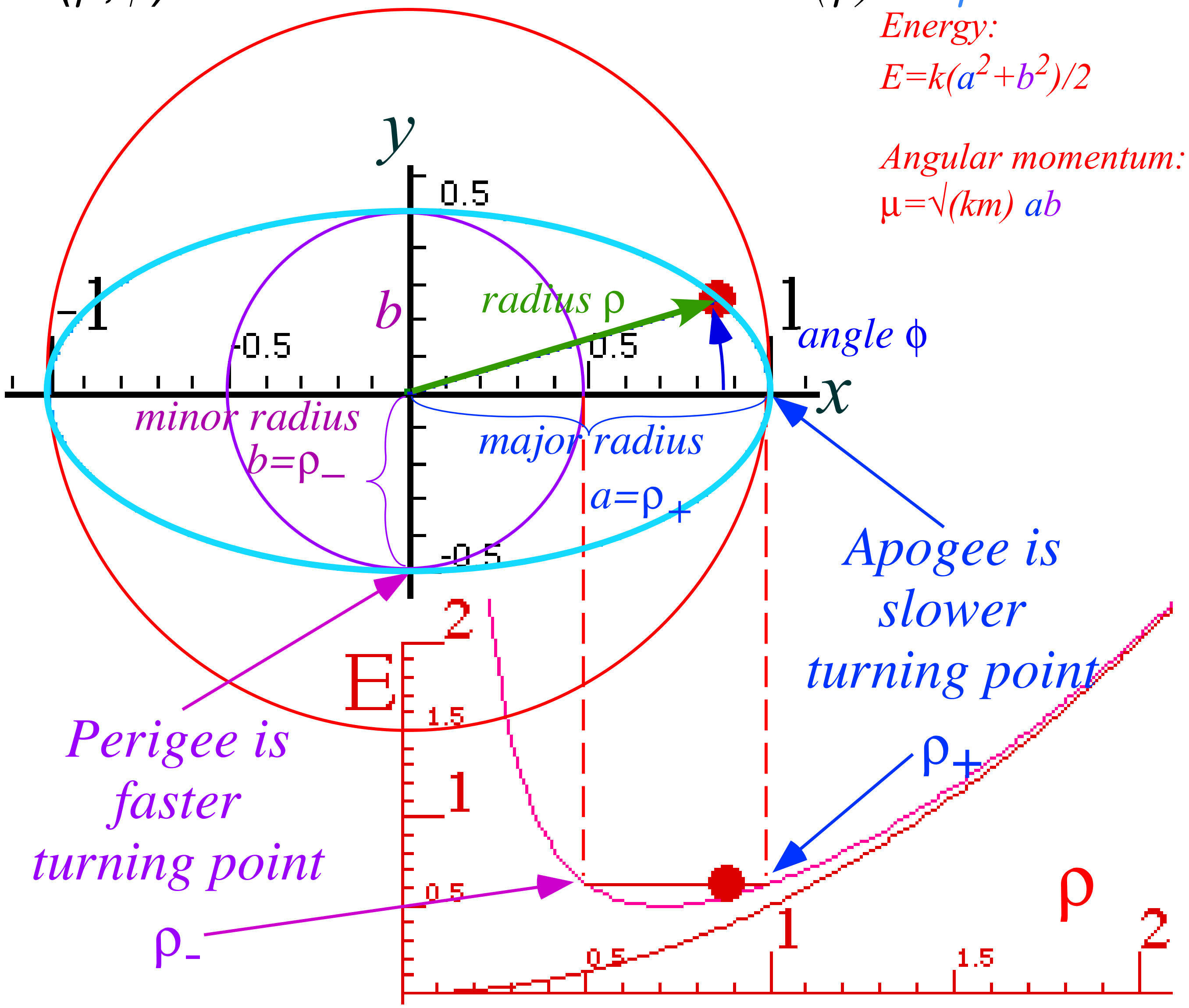
(to be discussed shortly)

$$\frac{1}{\rho} = \frac{a}{b^2} - \frac{\sqrt{a^2 - b^2}}{b^2}\cos\phi$$

One of many equations of focus-centered ellipse

Orbits in Isotropic Oscillator and Coulomb Potentials

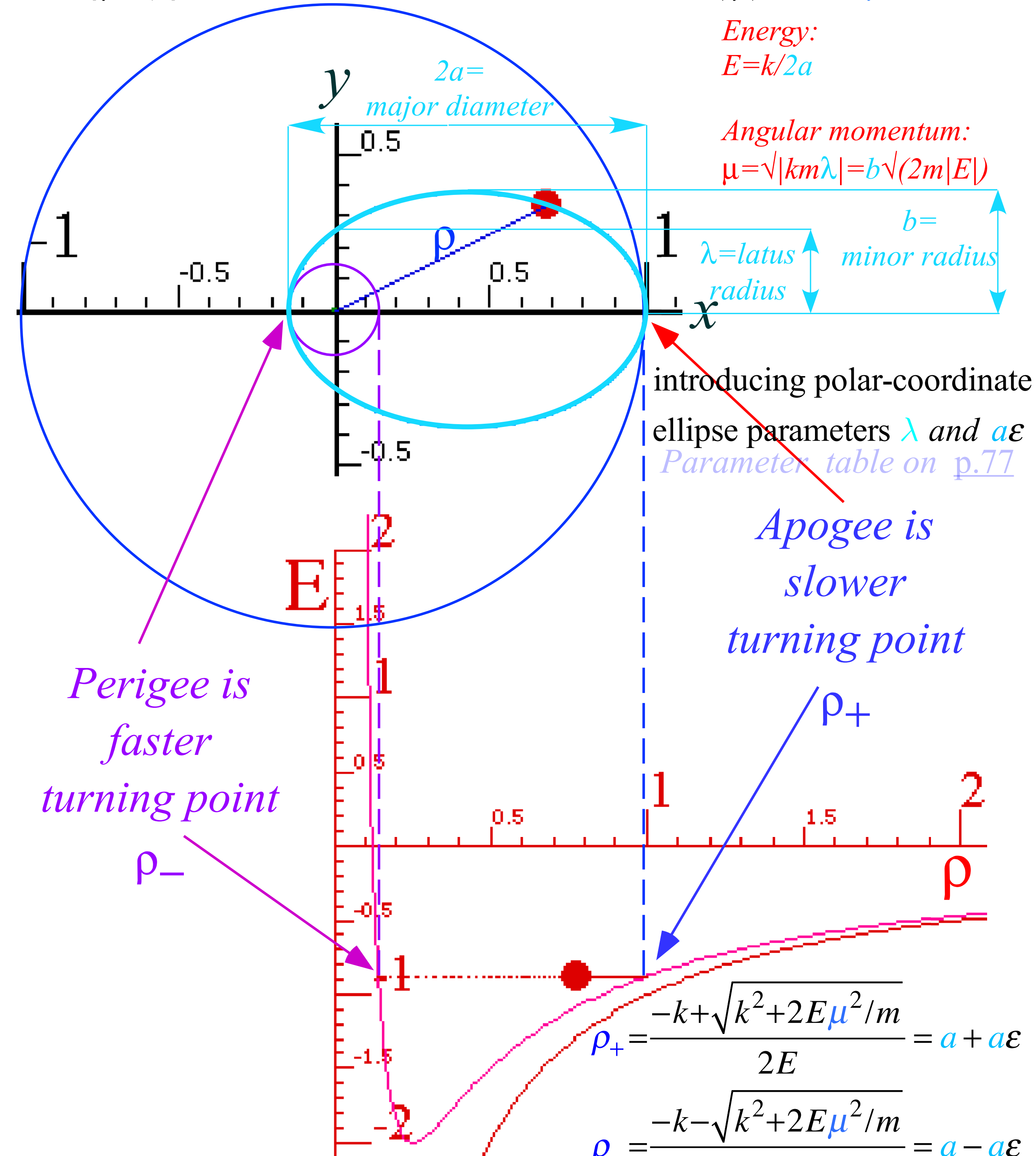
(ρ,ϕ) orbits for IHOscillator $V(\rho)=k\rho^2/2$



$$\rho_+^{\,2}=\frac{E+\sqrt{E^2-k\mu^2/m}}{k}=a^2$$

$$\rho_-^{\,2}=\frac{E-\sqrt{E^2-k\mu^2/m}}{k}=b^2$$

(ρ,ϕ) orbits for Coulomb $V(\rho)=-k/\rho$



$$\rho_+=\frac{-k+\sqrt{k^2+2E\mu^2/m}}{2E}=a+a\varepsilon$$

$$\rho_-=\frac{-k-\sqrt{k^2+2E\mu^2/m}}{2E}=a-a\varepsilon$$

(from p.29 or p.60)

(to be discussed first: turning point relations)

Just derived equation of IHO orbit ellipse

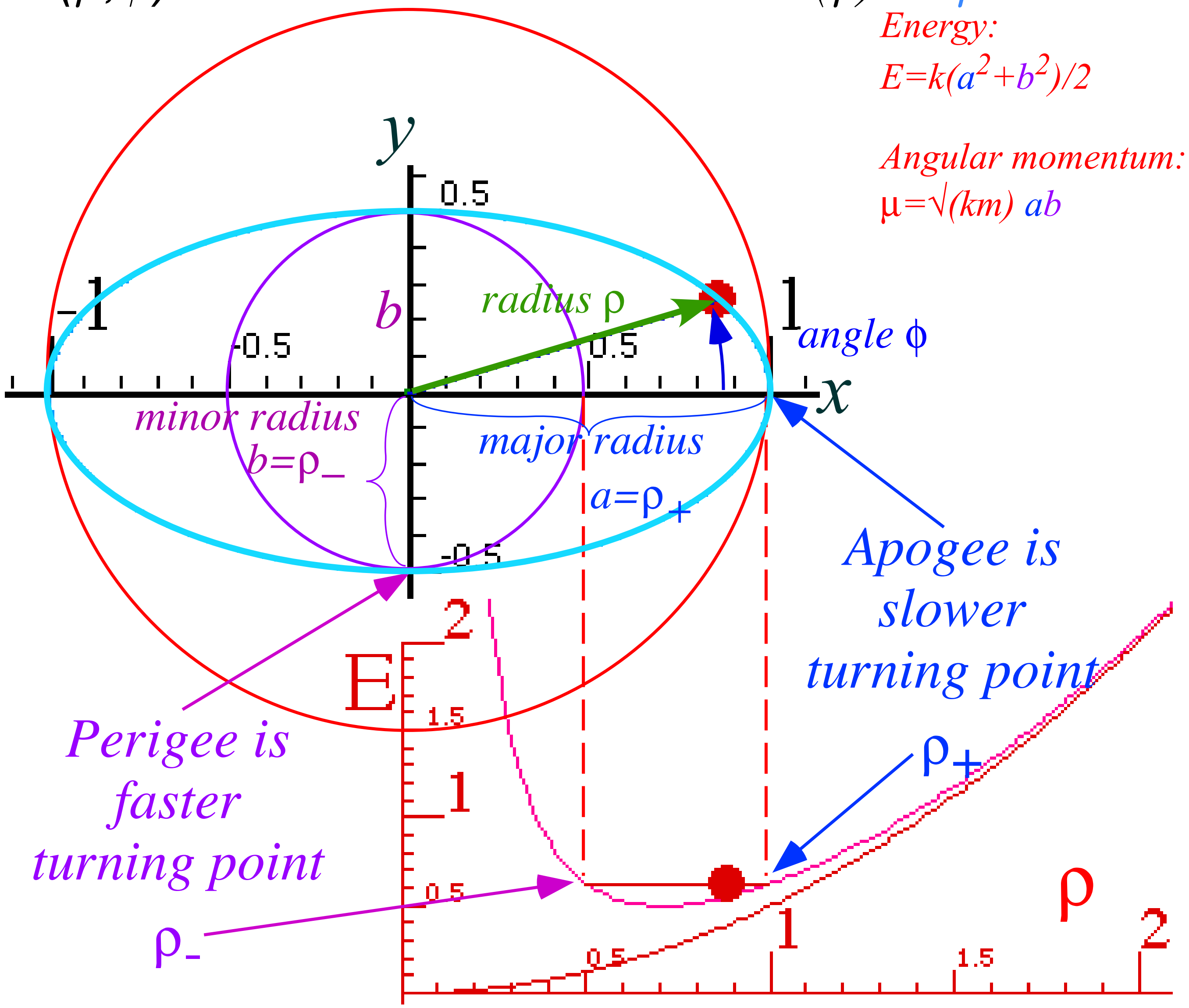
$$x=\frac{1}{\rho^2}=\frac{E}{\mu^2/m}+\frac{\sqrt{E^2-k\mu^2/m}}{\mu^2/m}\sin(-2\phi)$$

Just derived equation of Coulomb orbit ellipse

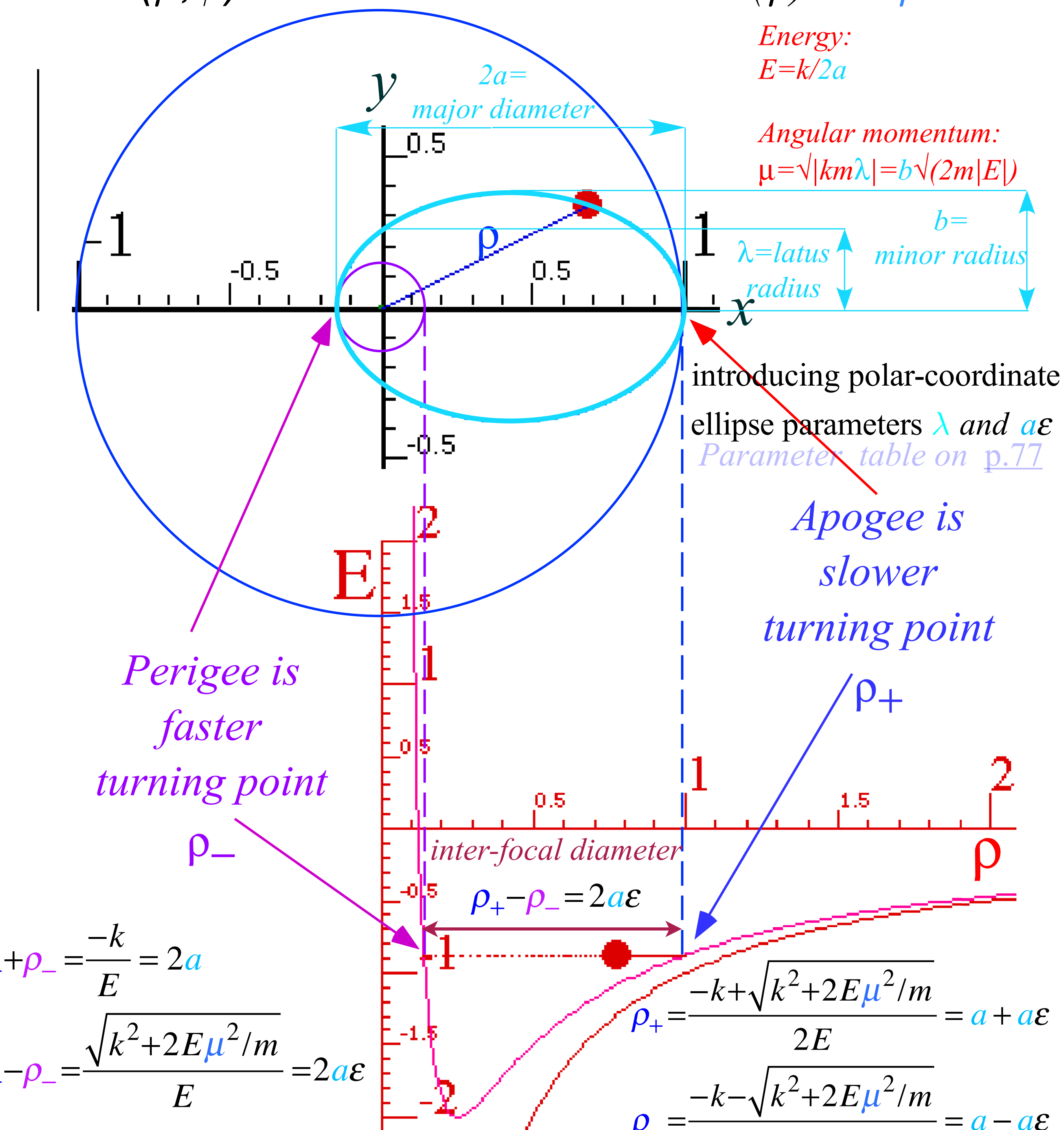
$$u=\frac{1}{\rho}=\frac{-k}{\mu^2/m}+\frac{\sqrt{k^2+2E\mu^2/m}}{\mu^2/m}\sin(-\phi)$$

Orbits in Isotropic Oscillator and Coulomb Potentials

(ρ,ϕ) orbits for IHOscillator $V(\rho)=k\rho^2/2$



(ρ,ϕ) orbits for Coulomb $V(\rho)=-k/\rho$



$$\rho_+^2 = \frac{E+\sqrt{E^2-k\mu^2/m}}{k} = a^2 \qquad \rho_+^2+\rho_-^2 = \frac{2E}{k} = a^2+b^2$$

$$\rho_-^2 = \frac{E-\sqrt{E^2-k\mu^2/m}}{k} = b^2 \qquad \rho_+^2-\rho_-^2 = \frac{2\sqrt{E^2-k\mu^2/m}}{k} = a^2-b^2 = a\varepsilon$$

$$\rho_+^2\rho_-^2 = \frac{E^2 - E^2-k\mu^2/m}{k^2} = a^2b^2 = \frac{-\mu^2}{km}$$

$$\rho_+ + \rho_- = \frac{-k}{E} = 2a$$

$$\rho_+ - \rho_- = \frac{\sqrt{k^2+2E\mu^2/m}}{E} = 2a\varepsilon$$

$$\rho_+ = \frac{-k+\sqrt{k^2+2E\mu^2/m}}{2E} = a + a\varepsilon$$

$$\rho_- = \frac{-k-\sqrt{k^2+2E\mu^2/m}}{2E} = a - a\varepsilon$$

$$\rho_+\rho_- = \frac{k^2-k^2-2E\mu^2/m}{(2E)^2} = a^2 - a^2\varepsilon^2 = \frac{-\mu^2}{2Em} = b^2$$

(Turning points $\rho_\pm$ on p.60 or p.32)

(to be discussed first: turning point relations)

Just derived equation of IHO orbit ellipse

$$x = \frac{1}{\rho^2} = \frac{E}{\mu^2/m} + \frac{\sqrt{E^2-k\mu^2/m}}{\mu^2/m}\sin(-2\phi)$$

Just derived equation of Coulomb orbit ellipse

(See p.67)

$$u = \frac{1}{\rho} = \frac{-k}{\mu^2/m} + \frac{\sqrt{k^2+2E\mu^2/m}}{\mu^2/m}\sin(-\phi)$$

Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

Effective potentials for IHO and Coulomb orbits

Stable equilibrium radii and radial/angular frequency ratios

Classical turning radii and apogee/perigee parameters

Polar coordinate differential equations

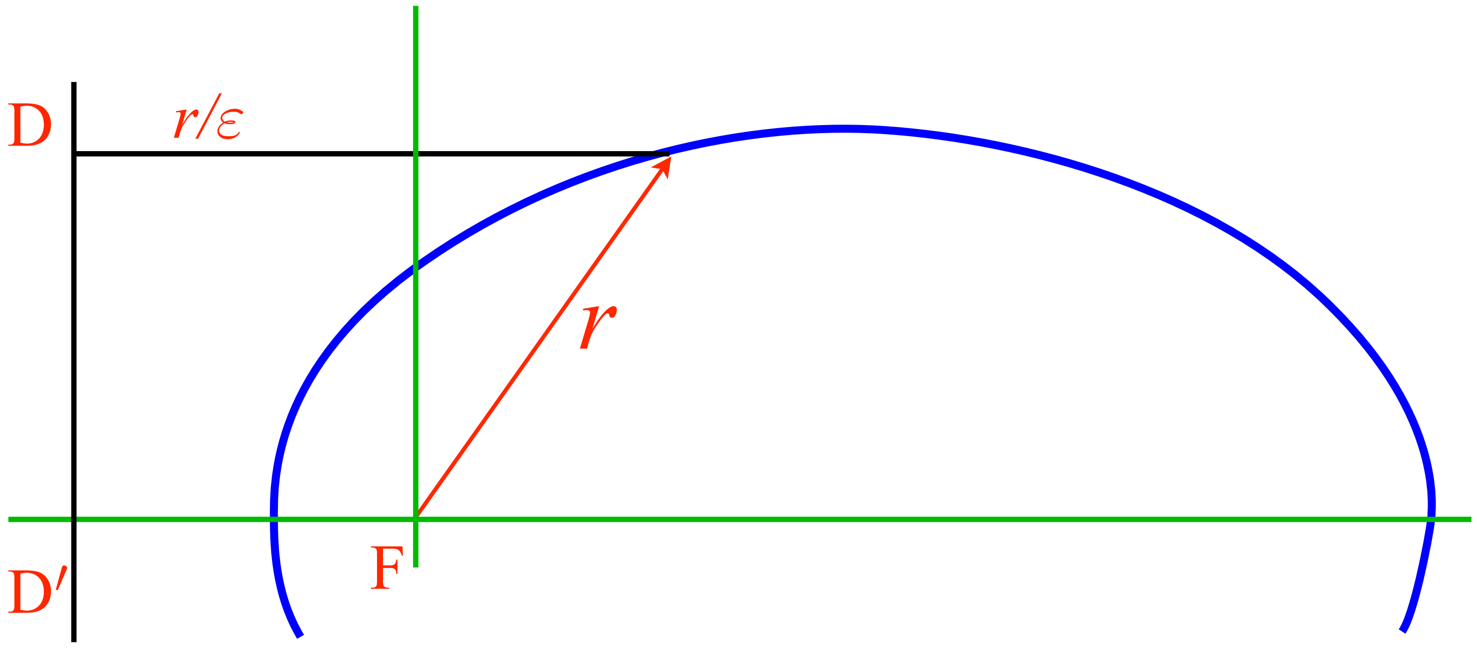
Quadrature integration techniques

Detailed orbital functions

➤ *Relating orbital energy-momentum to conic-sectional orbital geometry*

Kepler equation of time and phase geometry

Geometry of ALL Coulomb conic section orbits (Let: $r = \rho$ here)



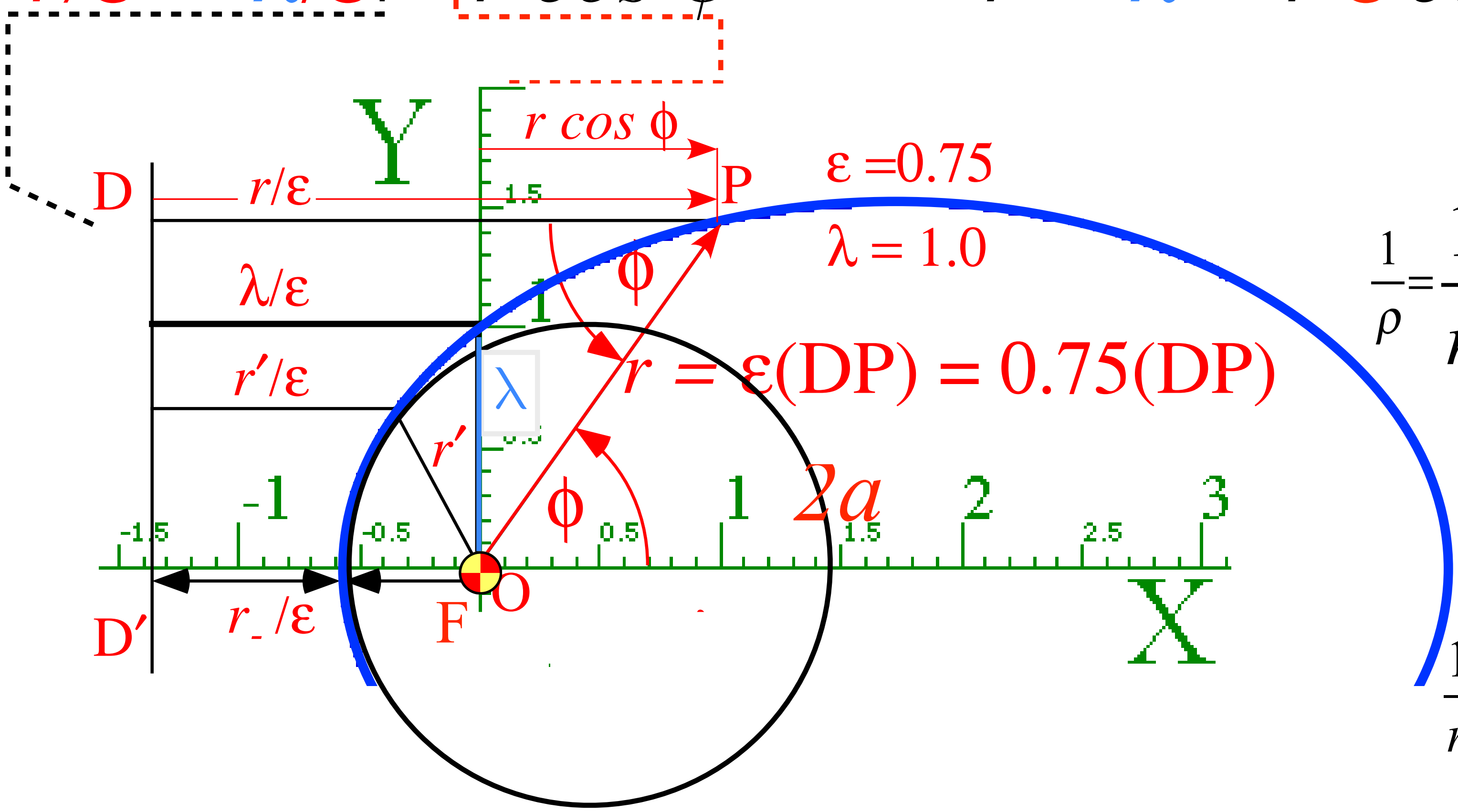
Parameter table on p.77

All conics defined by: ***Eccentricity*** ε

Distance to *Focus* **F** = ε·Distance to *Directrix* **DD′**

Geometry of ALL Coulomb conic section orbits (Let: $r \equiv \rho$ here)

$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi \qquad r = \lambda + r\,\varepsilon\cos\phi \qquad r = \frac{\lambda}{1-\varepsilon\cos\phi}$$



By geometry:

$$\frac{1}{\rho} = \frac{1}{r} = \frac{1-\varepsilon\cos\phi}{\lambda} = \frac{1}{\lambda} - \frac{\varepsilon}{\lambda}\cos\phi$$

Recall p.64 formula:

$$\frac{1}{r} = \frac{1}{\rho} = \frac{-k}{\mu^2/m} + \frac{\sqrt{k^2 + 2E\mu^2/m}}{\mu^2/m}\cos\phi$$

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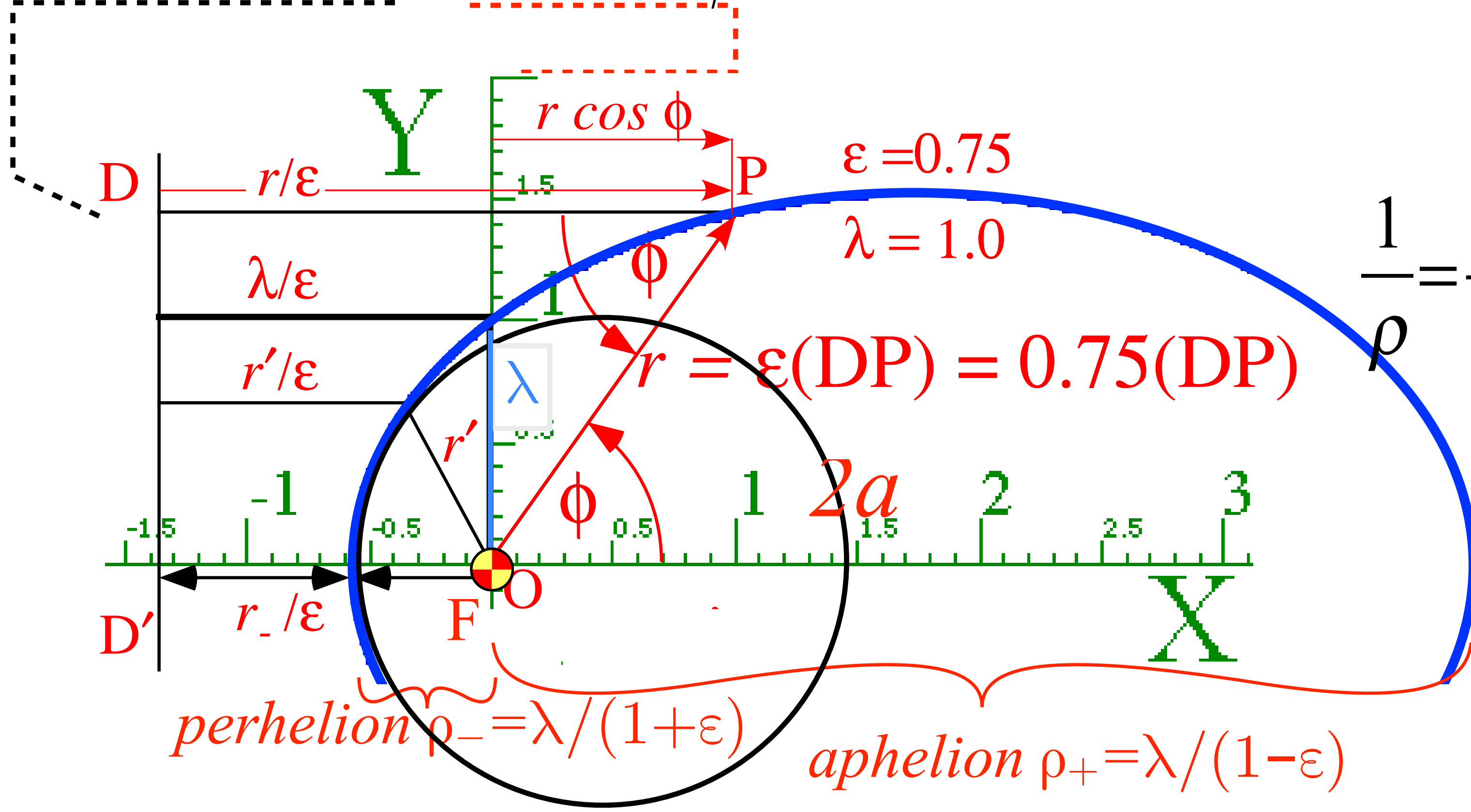

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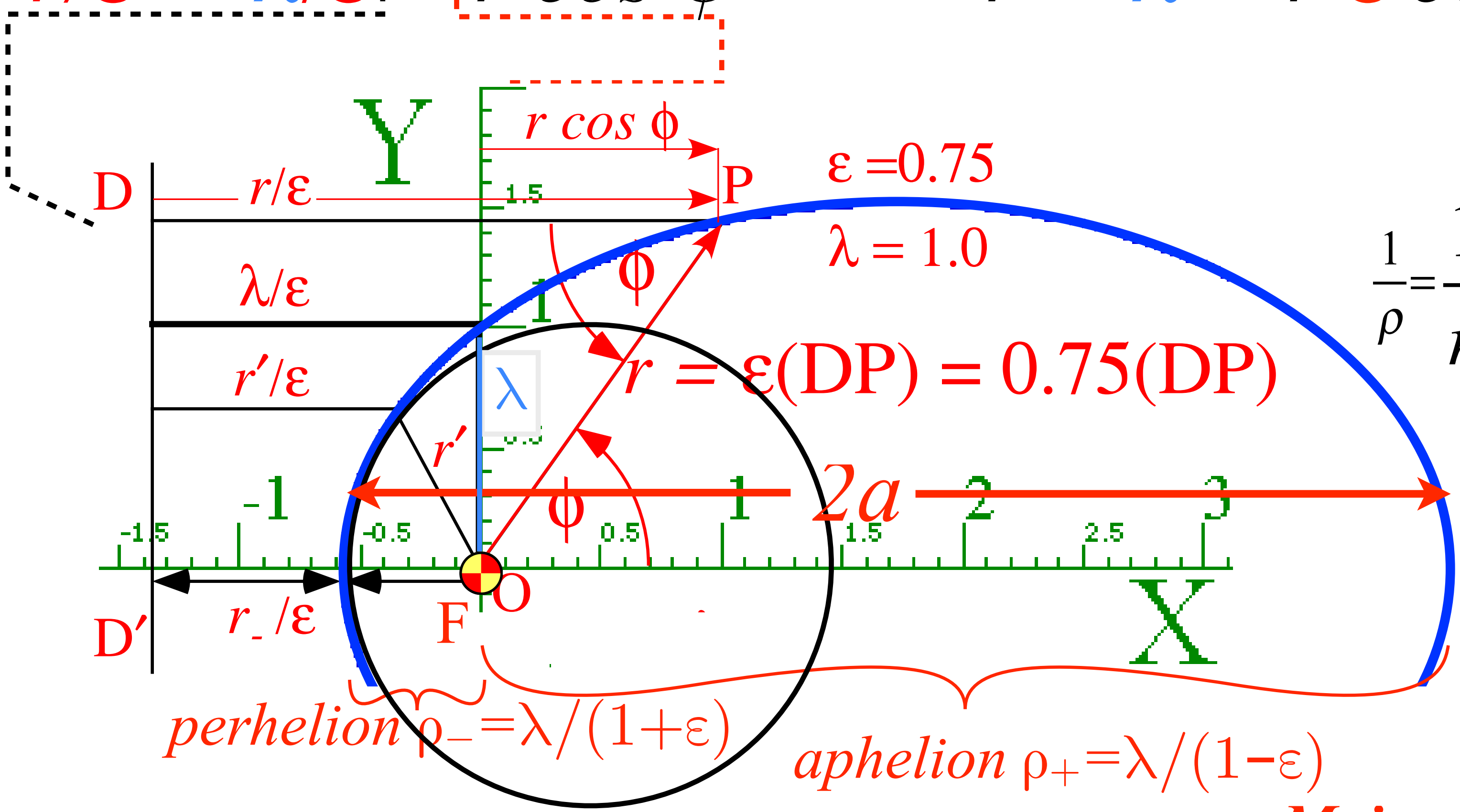

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$$r_{\pm} = \rho_{\pm} = \frac{-k \pm \sqrt{k^2 + 2E\mu^2/m}}{2E}$$

***Major axis**:* $\rho_+ + \rho_- = 2a$

$$\rho_+ + \rho_- = [\lambda(1+\varepsilon) + \lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda/(1-\varepsilon^2)$$

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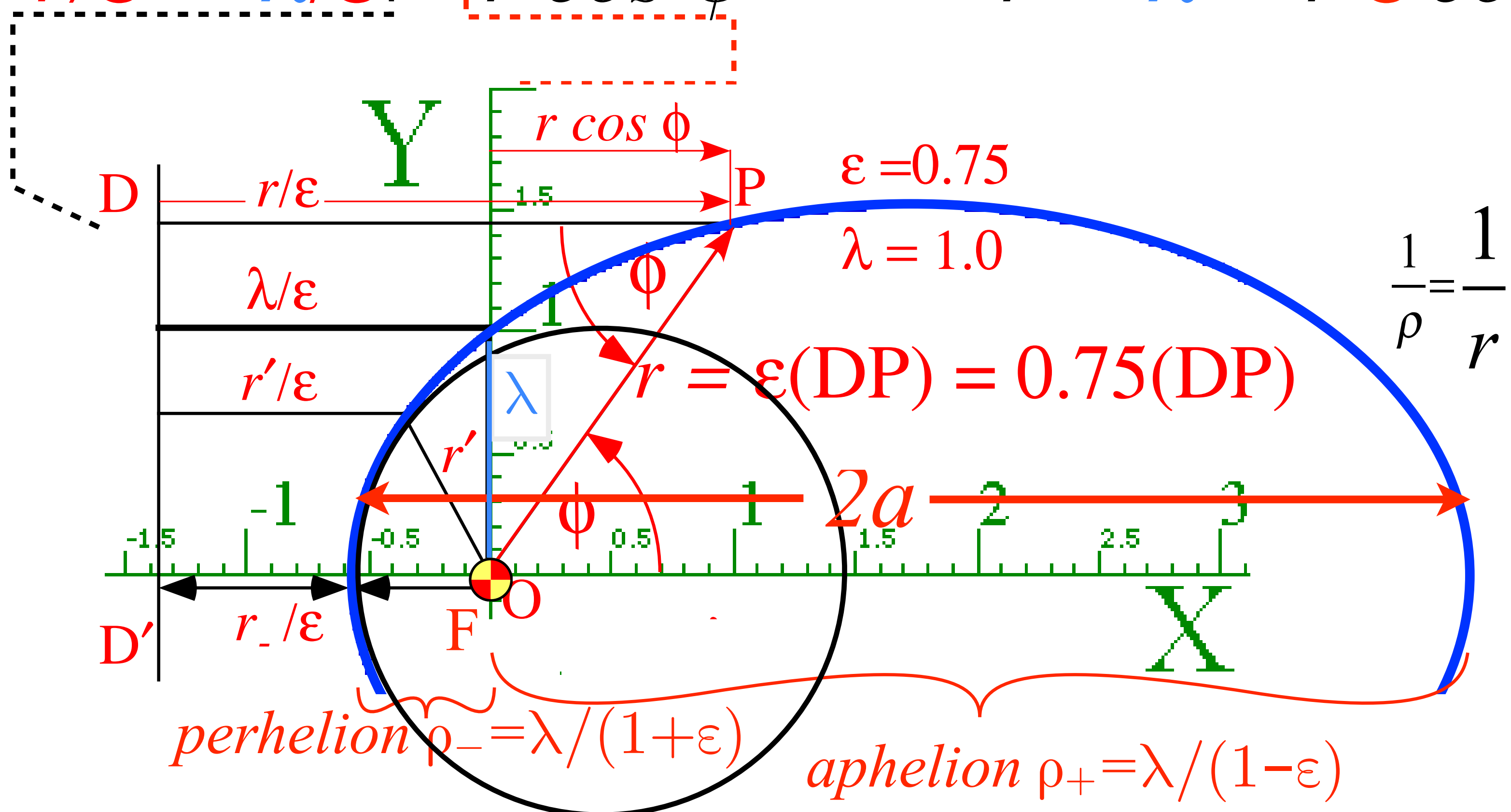

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$$\rho_+ + \rho_- = \frac{-k}{E} = 2a \quad \text{implies:} \quad \boxed{E = \frac{-k}{2a}}$$

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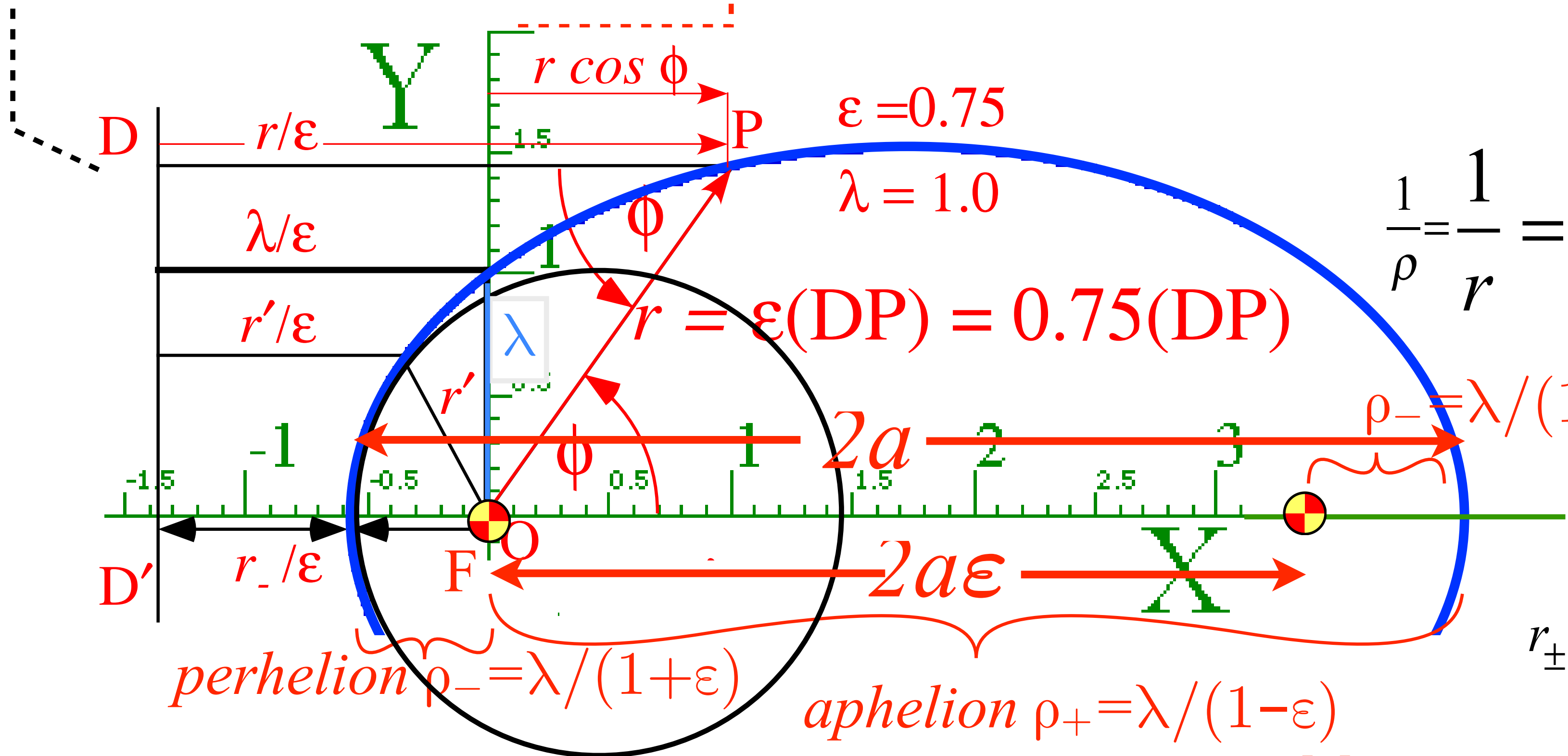

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$$\rho_+ - \rho_- = \sqrt{\left(\frac{k}{E}\right)^2 + \frac{2\mu^2}{Em}} = \left|\frac{k}{E}\right|\sqrt{1 + \frac{2\mu^2 E}{k^2 m}} = 2a\varepsilon = \frac{2\lambda\varepsilon}{|1-\varepsilon^2|}$$

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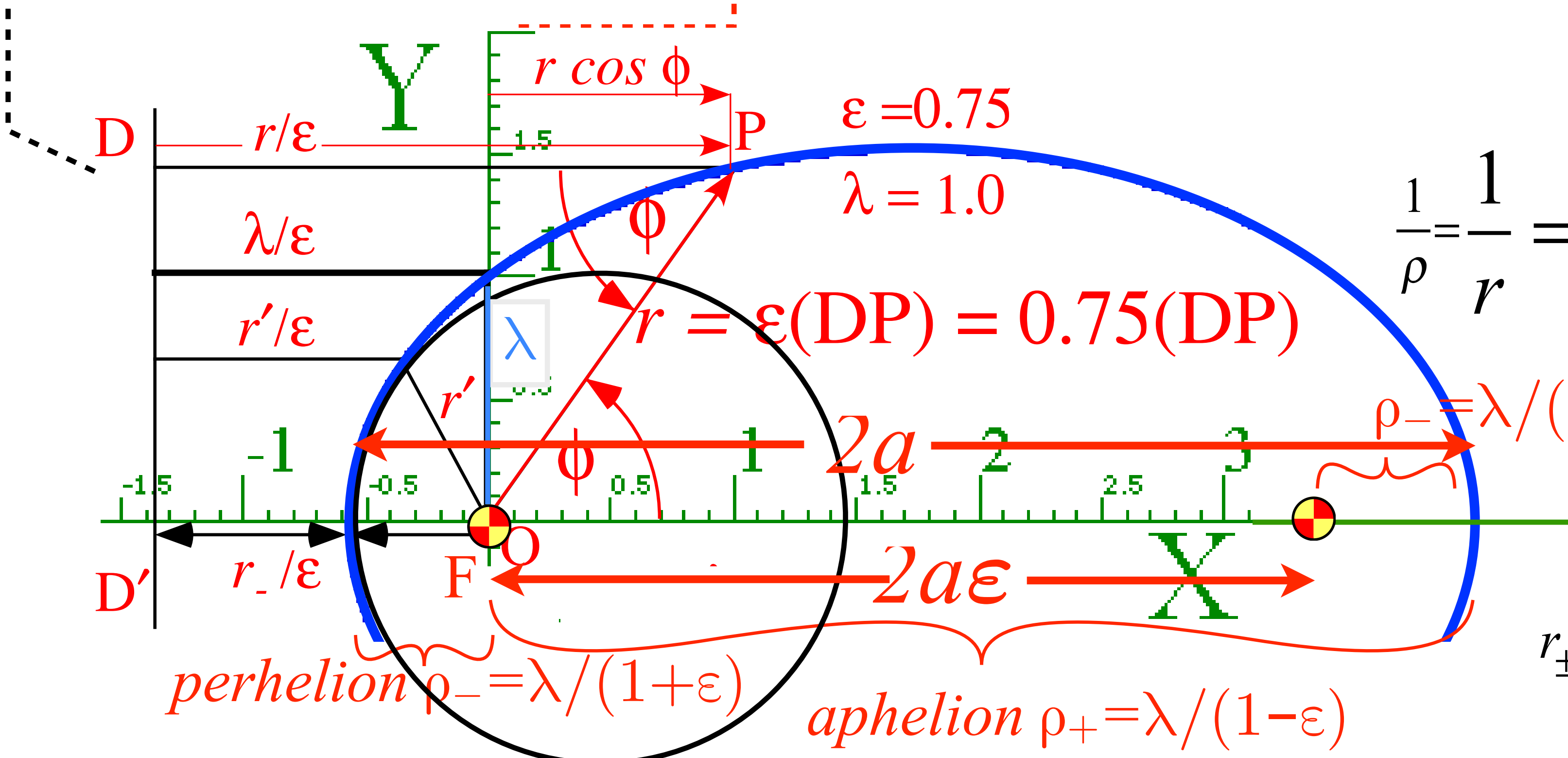

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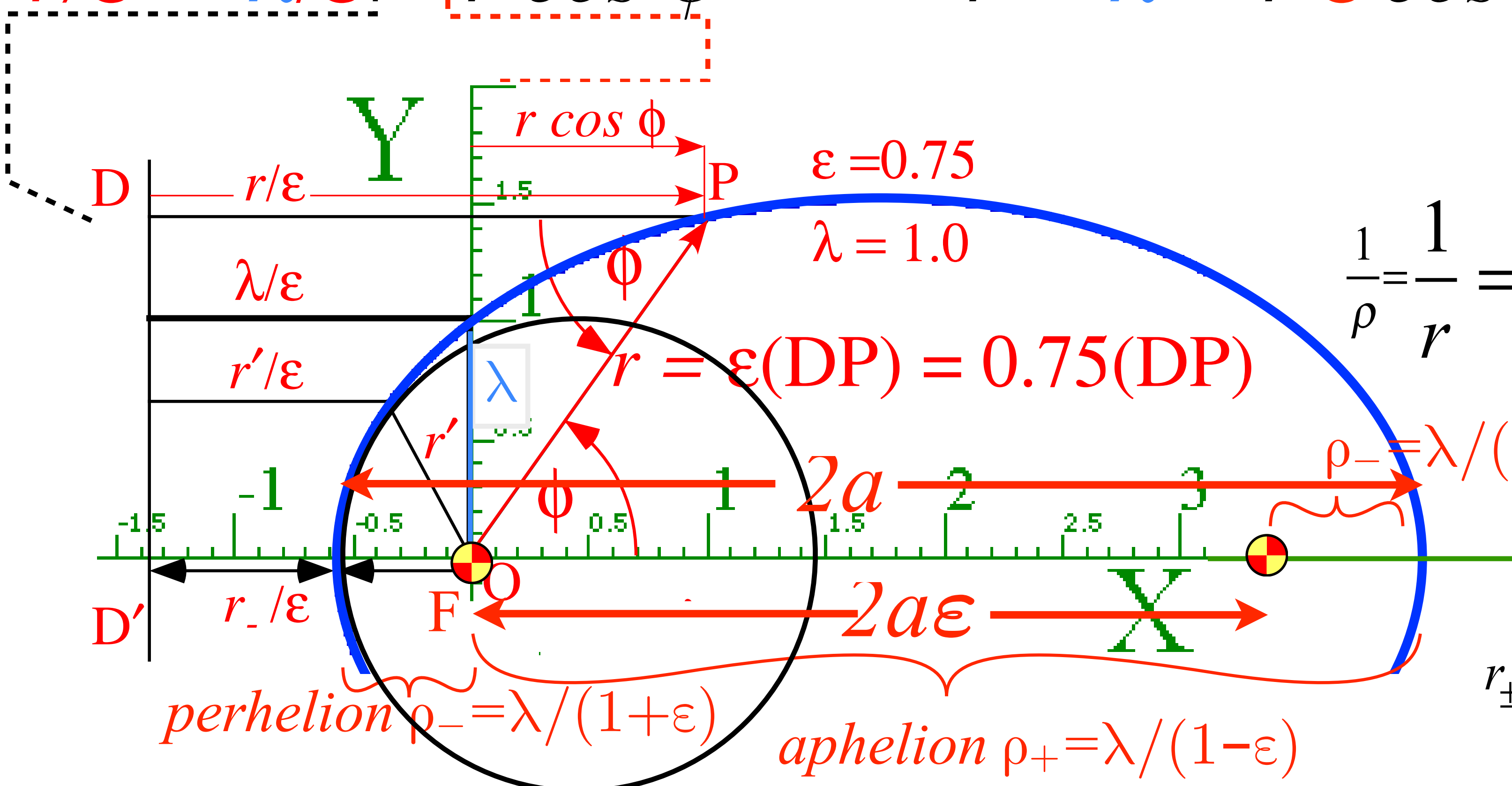

$\rho_-=\lambda/(1+\varepsilon)$ *perhelion*

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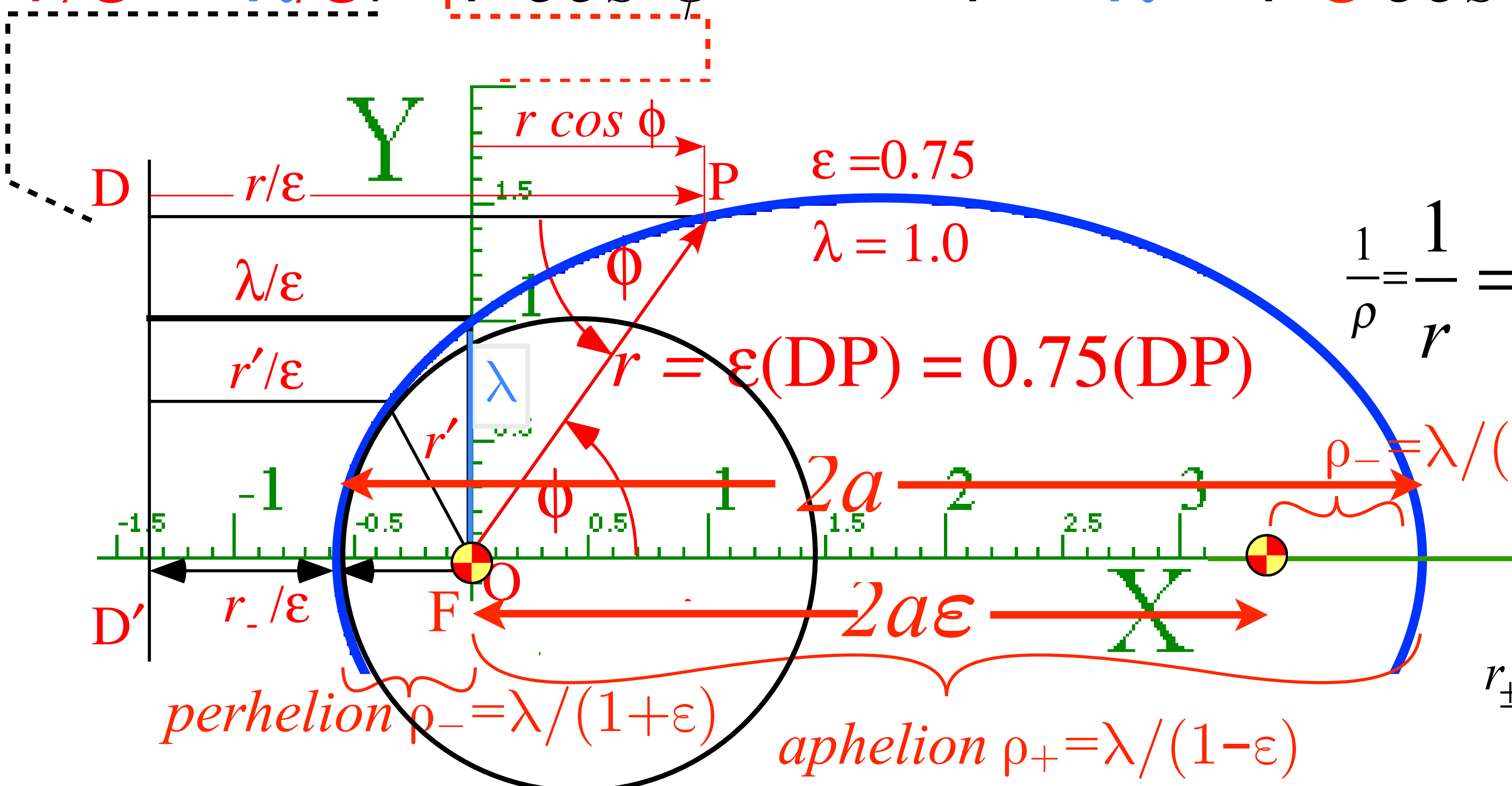

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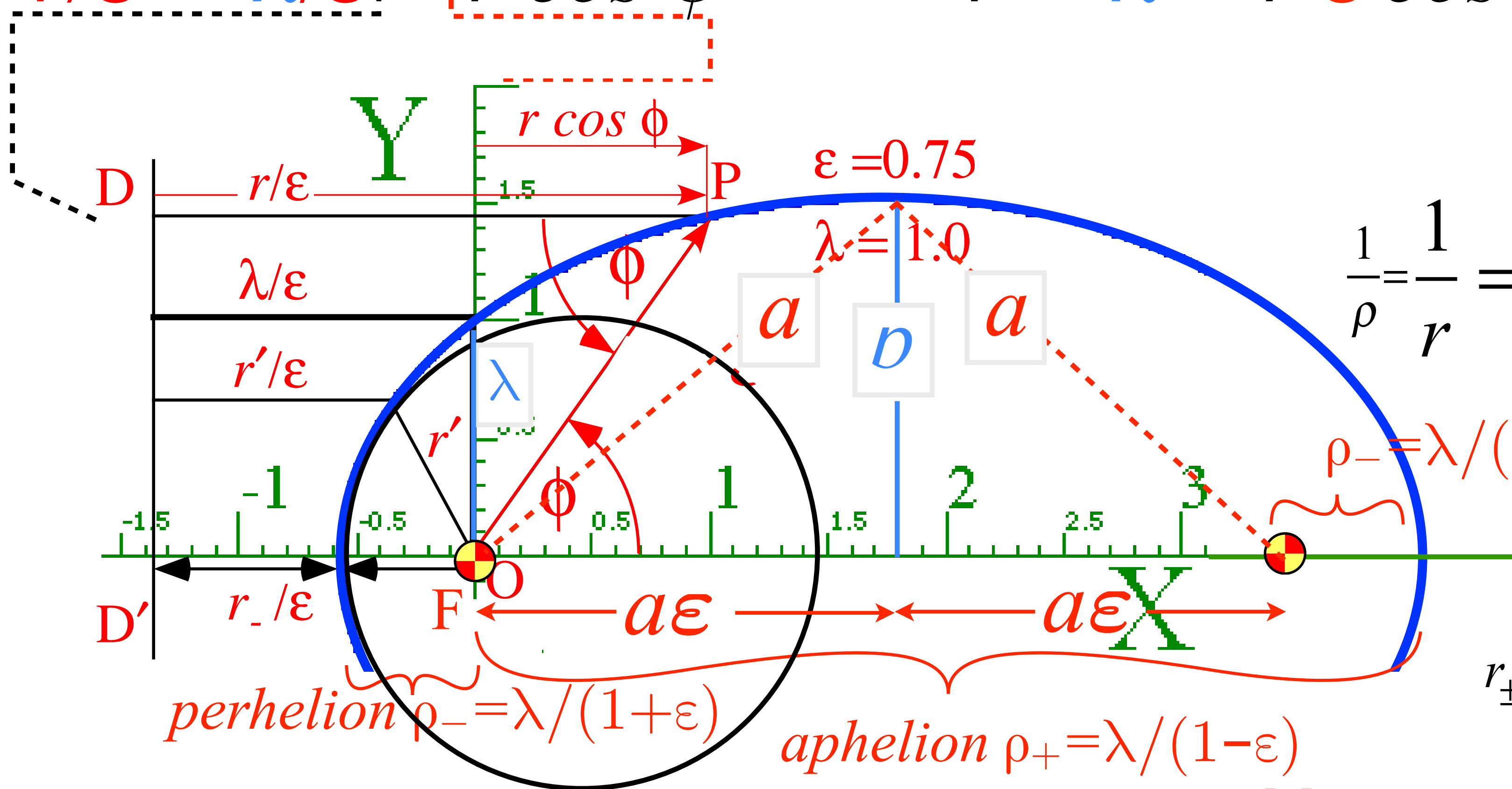
Also important! $\mu = \sqrt{km\lambda}$

Geometry of ALL Coulomb conic section orbits (Let: $r \equiv \rho$ here)

$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi \qquad r = \lambda + r\,\varepsilon\cos\phi \qquad r = \frac{\lambda}{1-\varepsilon\cos\phi}$$



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Recall p.64 formula:

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Latus radius: $\lambda = a(1-\varepsilon^2)$

Minor radius:

$b = \sqrt{(a^2 - a^2\varepsilon^2)} = \sqrt{(a\lambda)}$ *(ellipse: $\varepsilon < 1$)*

$b = \sqrt{(a^2\varepsilon^2 - a^2)} = \sqrt{(\lambda a)}$ *(hyperb: $\varepsilon > 1$)*

All conics defined by: *Eccentricity* ε

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$$\rho_+ + \rho_- = \frac{-k}{E} = 2a \quad \text{implies:} \quad \boxed{E = \frac{-k}{2a}}$$

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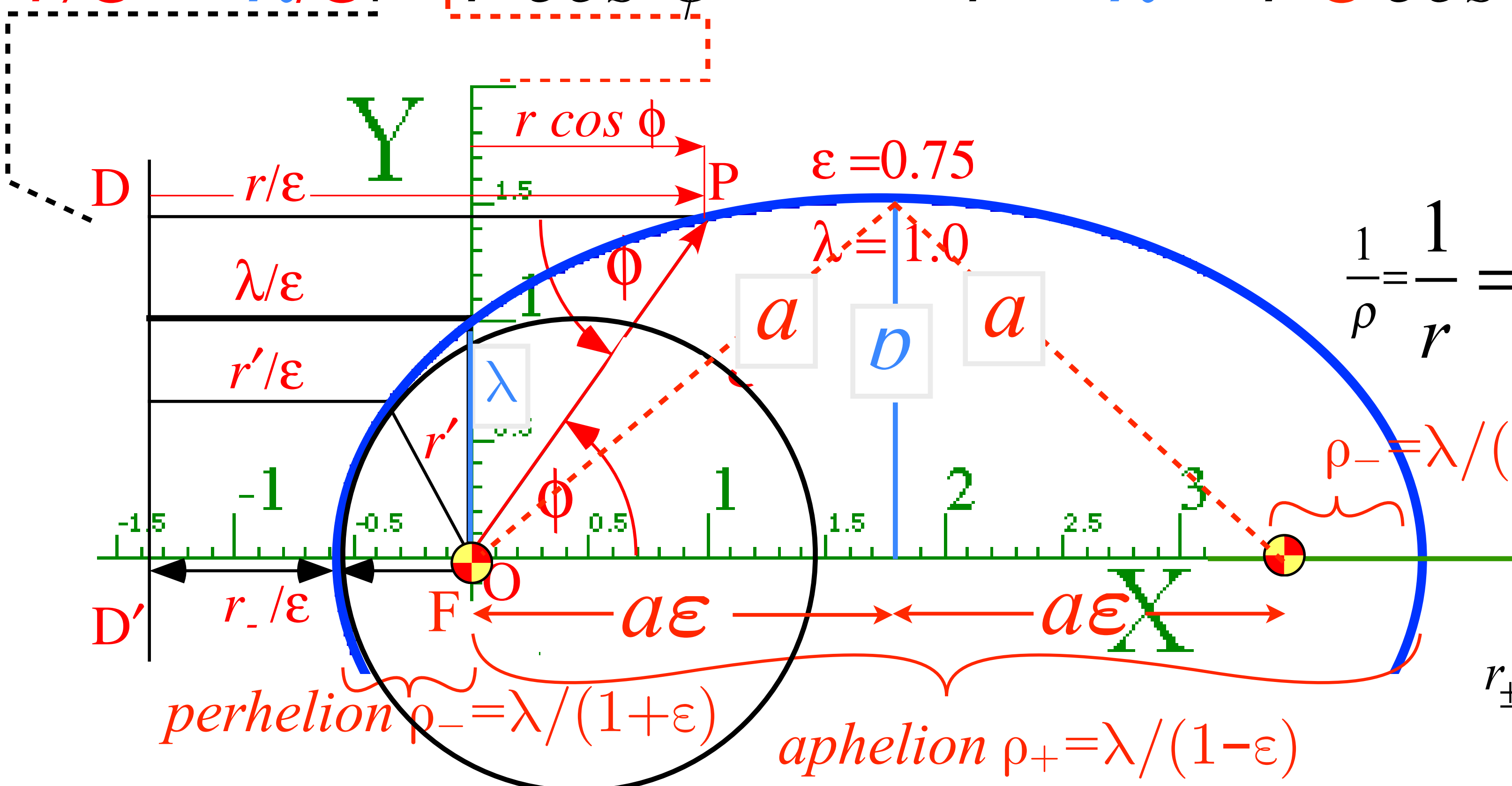

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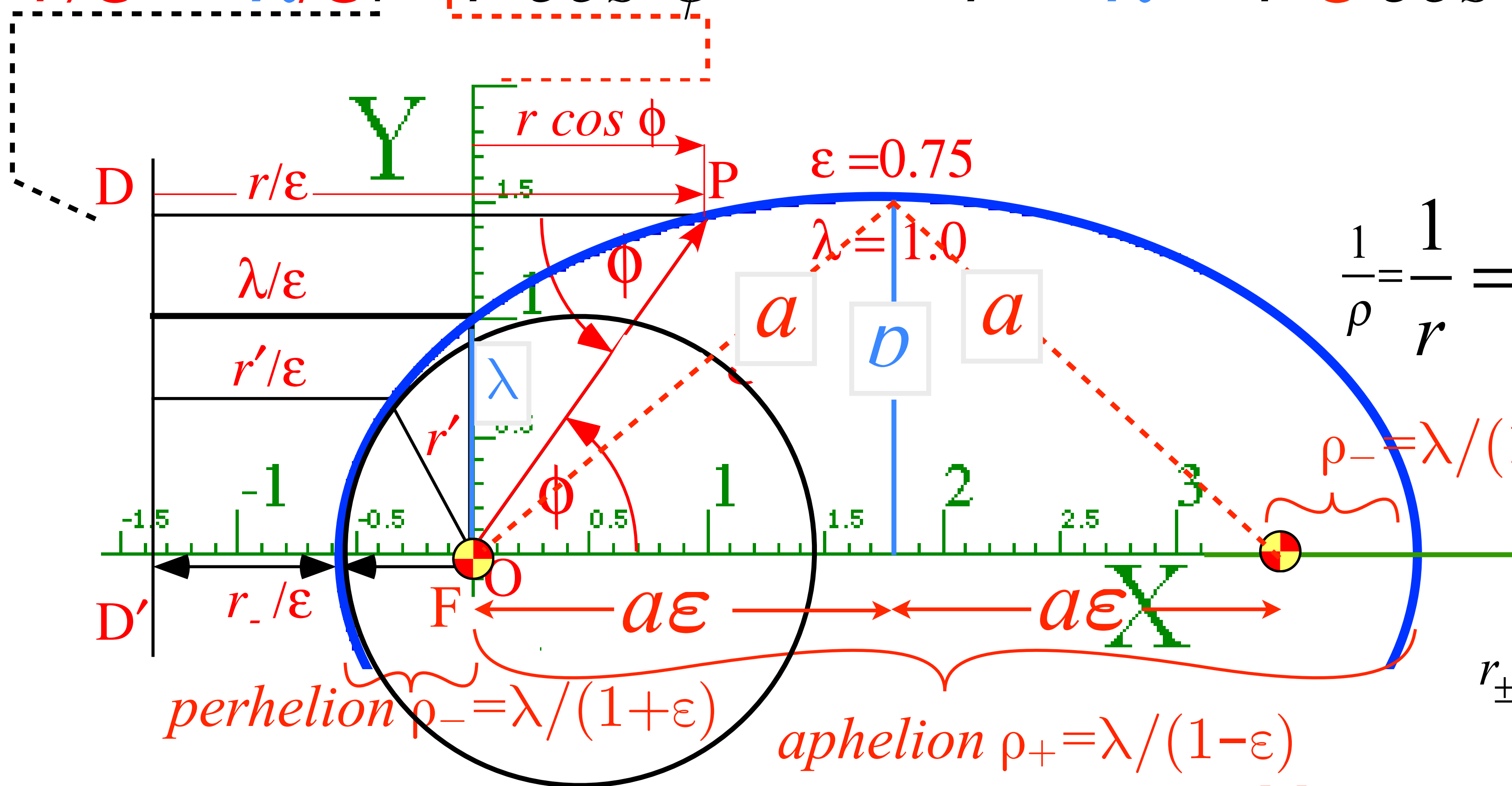

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Parameter table on *p.77*

$\rho_- = \lambda/(1+\varepsilon)$ *perhelion*

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(x,y) parameters	physical parameters	(r,ϕ) parameters
major radius $a = \frac{k}{2E}$	*Energy* $E = \frac{k}{2a}$	*eccentricity* $\varepsilon = \sqrt{\frac{k^2 m + 2L^2 E}{k^2 m}}$
minor radius $b = \frac{L}{\sqrt{2m\lvert E\rvert}}$	*∠-momentum* $L = \sqrt{km\lambda} \equiv \mu$	*latus radius* $\lambda = \frac{L^2}{km}$

Minor radius: $b = \sqrt{(a^2 - a^2\varepsilon^2)} = \sqrt{(a\lambda)}$ *(ellipse: $\varepsilon < 1$)*

Minor radius: $b = \sqrt{(a^2\varepsilon^2 - a^2)} = \sqrt{(\lambda a)}$ *(hyperb: $\varepsilon > 1$)*

$b/a = \sqrt{(1-\varepsilon^2)}$ *(ellipse: $\varepsilon < 1$)* $\quad \varepsilon^2 = 1 - b^2/a^2$

$b/a = \sqrt{(\varepsilon^2 - 1)}$ *(hyperb: $\varepsilon > 1$)* $\quad \varepsilon^2 = 1 + b^2/a^2$

$\lambda = a(1-\varepsilon^2)$ *(ellipse: $\varepsilon < 1$)* $\quad a\varepsilon^2 = a - \lambda$

$\lambda = a(\varepsilon^2 - 1)$ *(hyperb: $\varepsilon > 1$)* $\quad a\varepsilon^2 = a + \lambda$

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Effective potentials for IHO and Coulomb orbits

Stable equilibrium radii and radial/angular frequency ratios

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Relating orbital energy-momentum to conic-sectional orbital geometry

➔ *Kepler equation of time and phase geometry*

Starting with KE-eff.-PE results: $\frac{m}{2}\dot{\rho}^2 = E - \frac{\mu^2}{2m\rho^2} - k/\rho$ *or* *p.36**:* $\dot{\rho} = \frac{d\rho}{dt} = \sqrt{\frac{2E}{m} - \frac{\mu^2}{m^2\rho^2} - \frac{2k}{m\rho}}$

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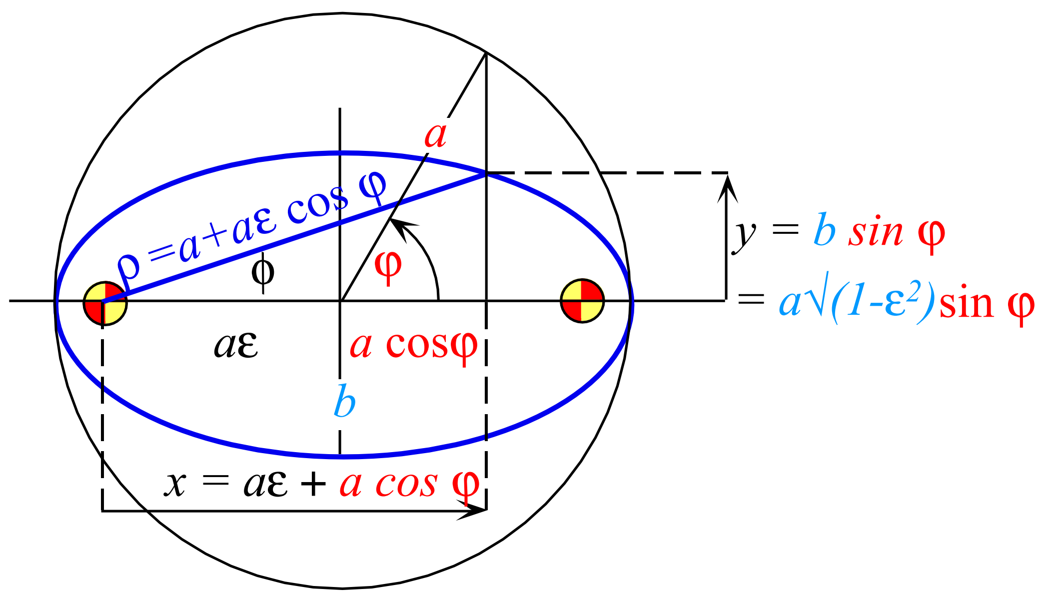
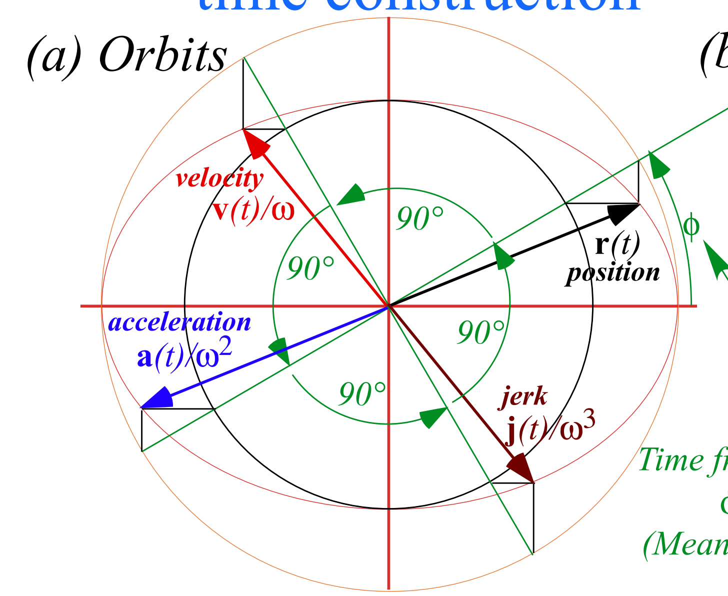


Unit 1 Ch. 9
Recall IHO orbit
time construction



[RelaWavity Web Simulation: IHO time rates](#)

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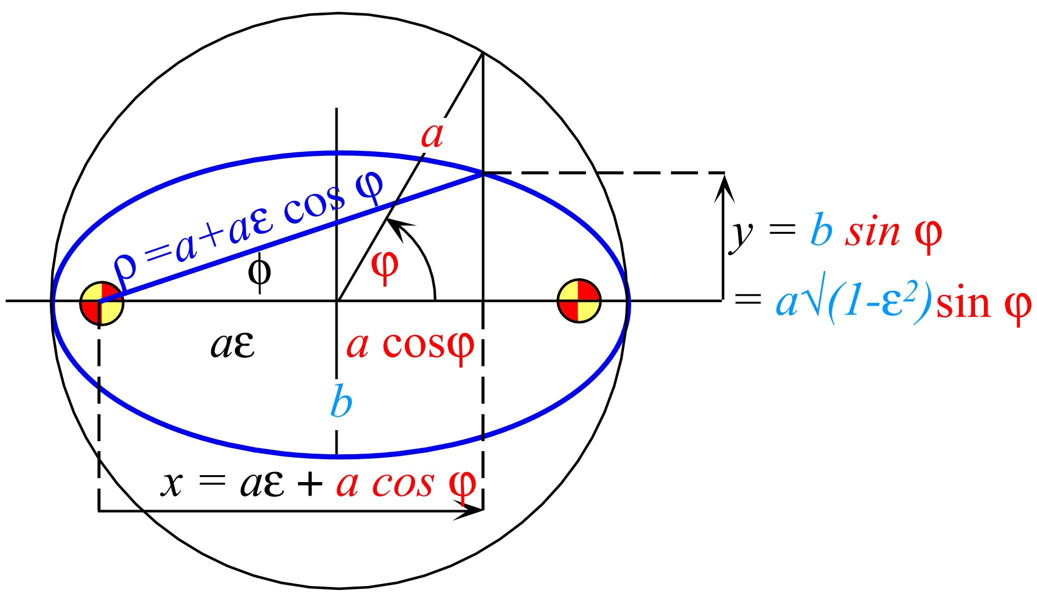

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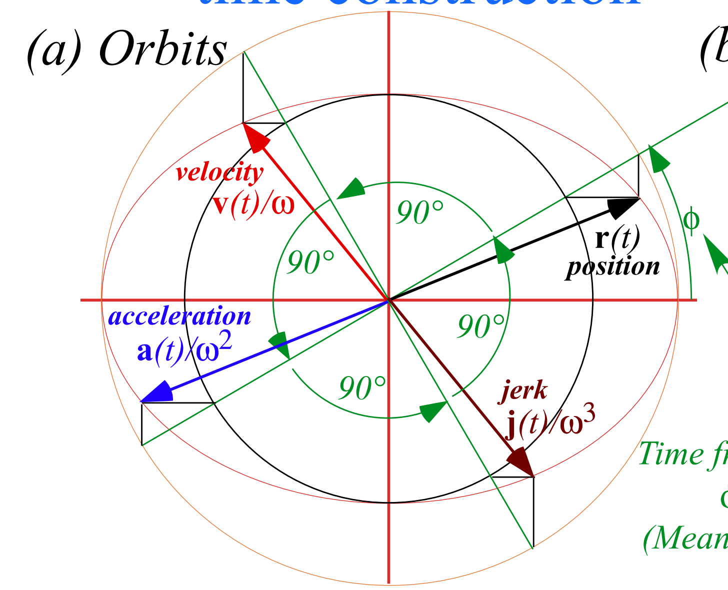

RelaWavity Web Simulation:
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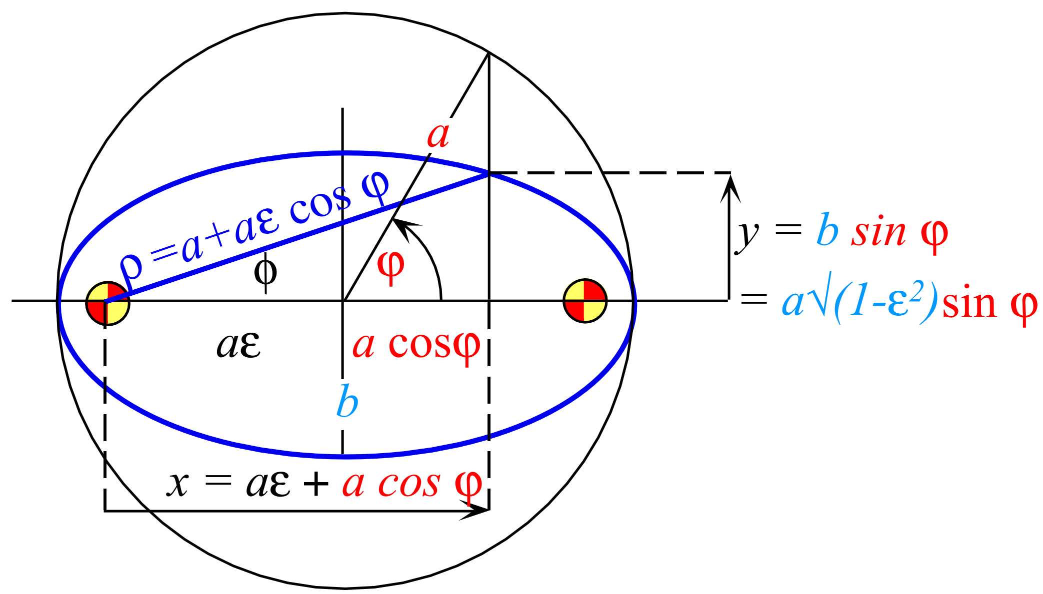
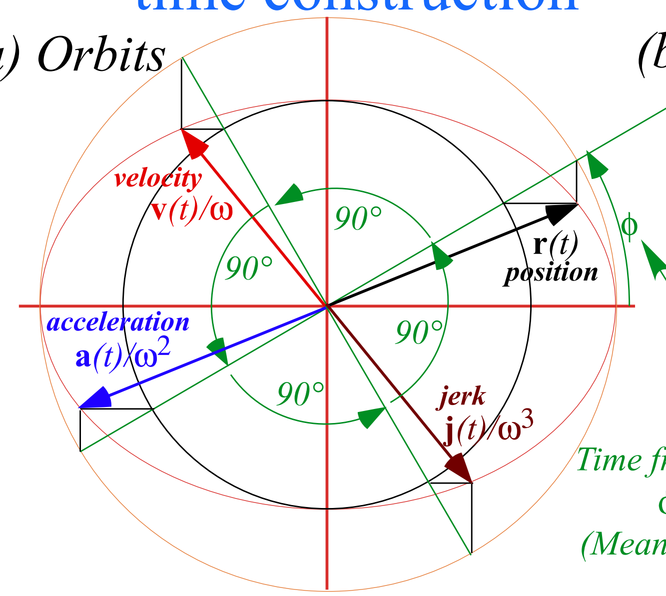


Unit 1 Ch. 9
Recall IHO orbit time construction

(a) Orbits



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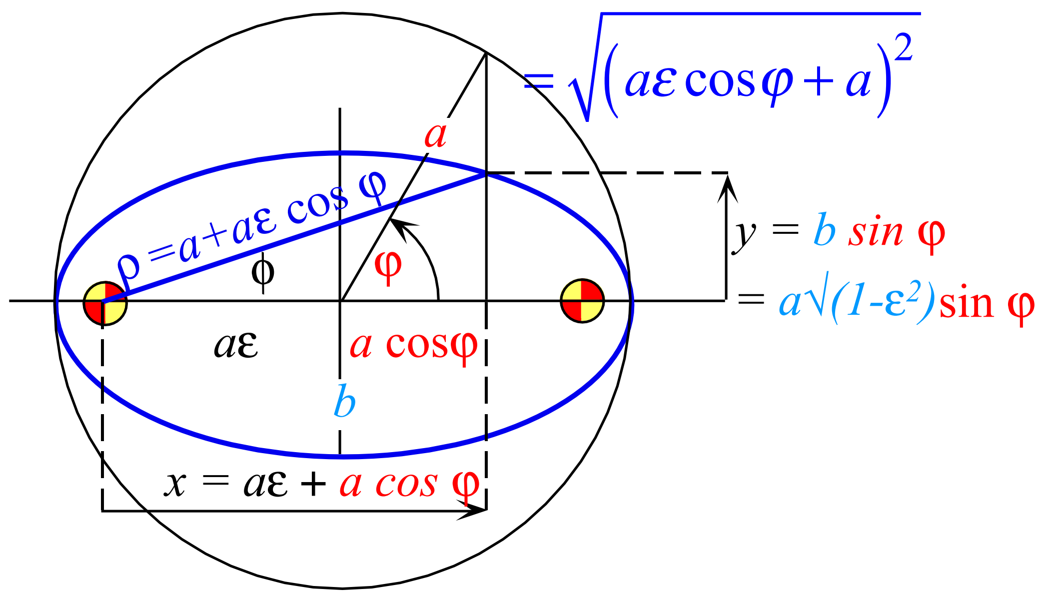

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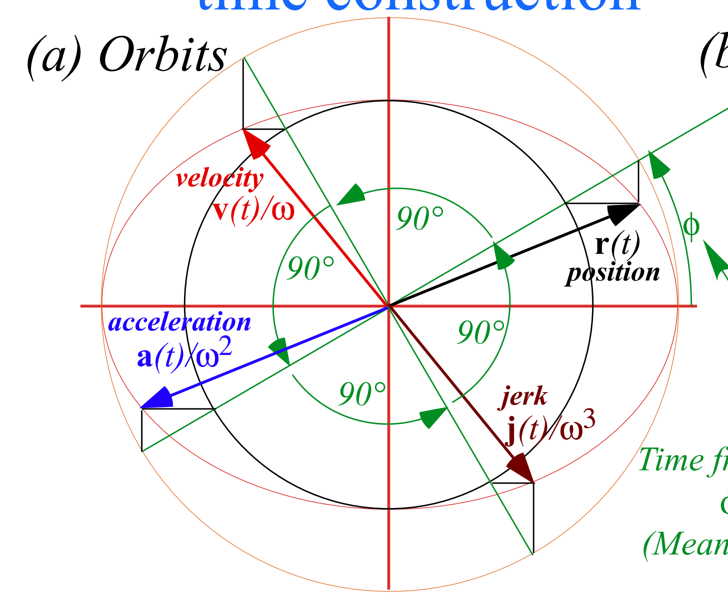

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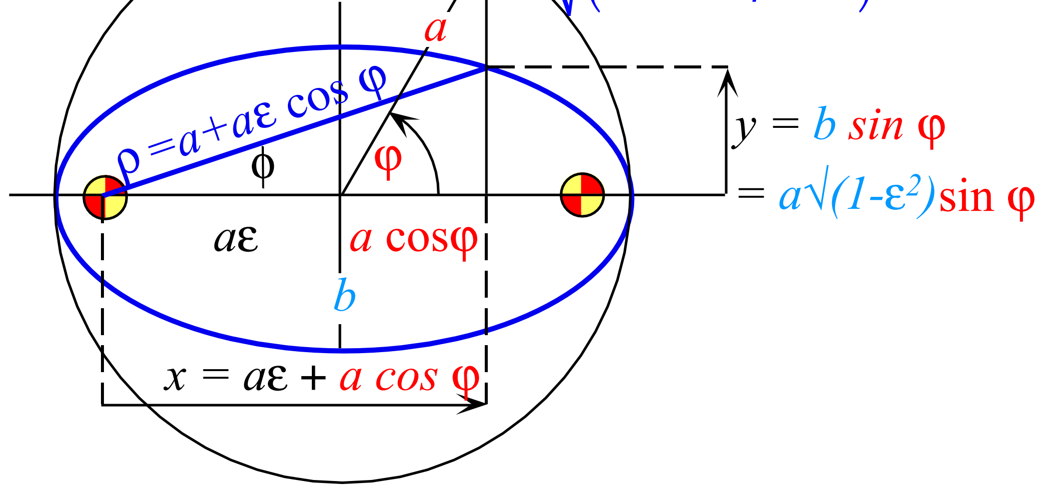

Unit 1 Ch. 9
Recall IHO orbit
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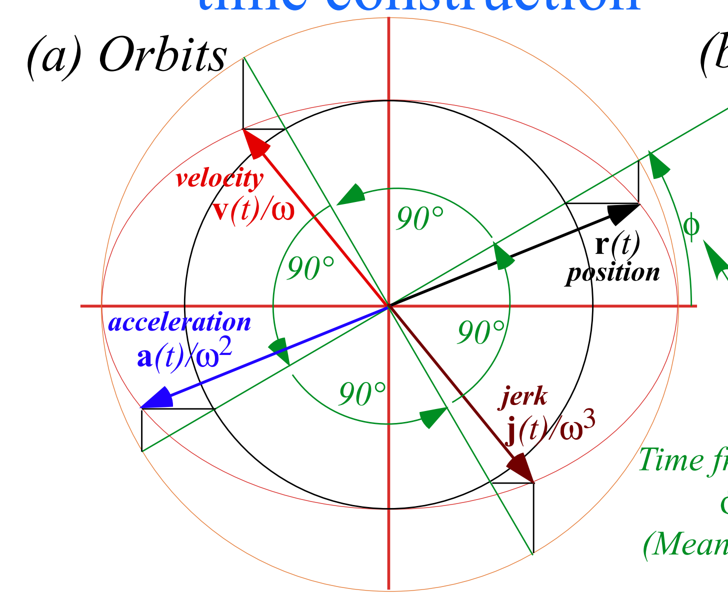

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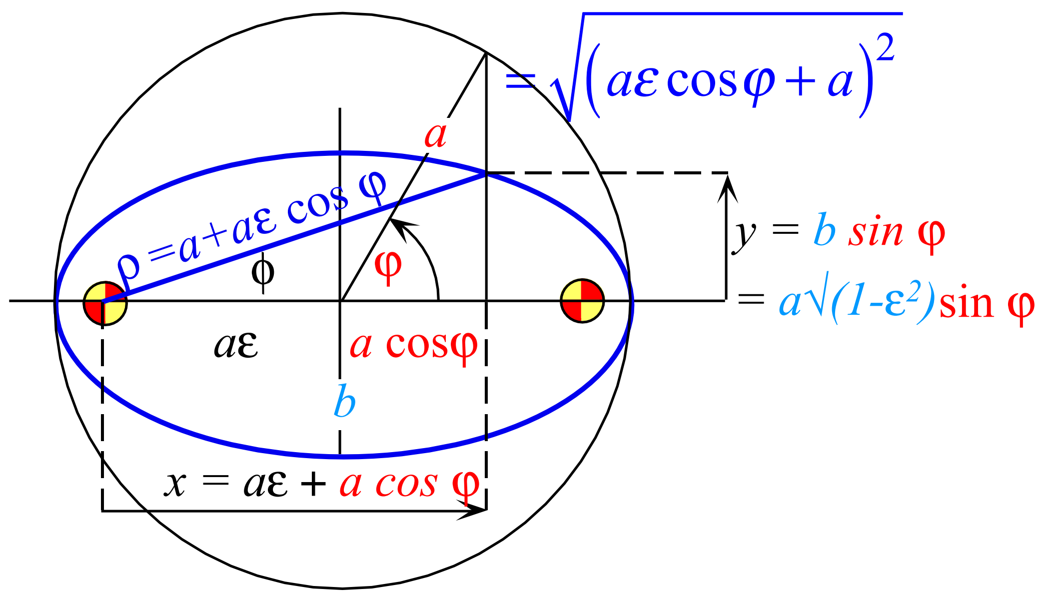

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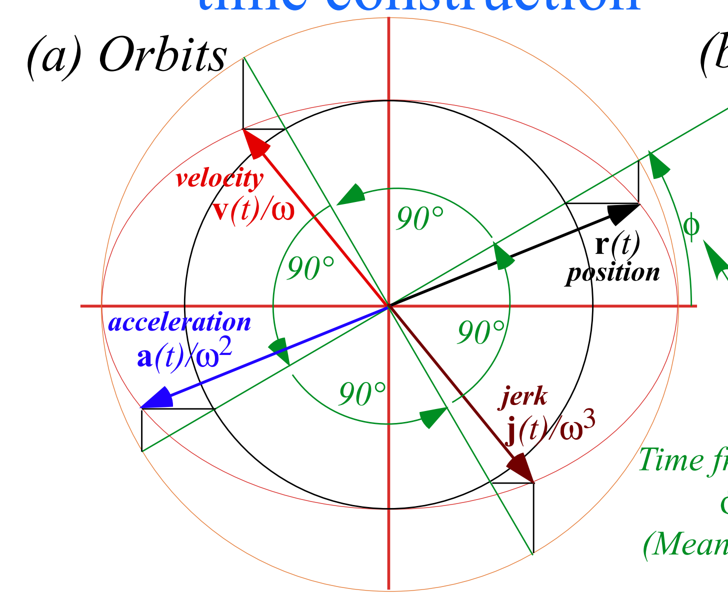

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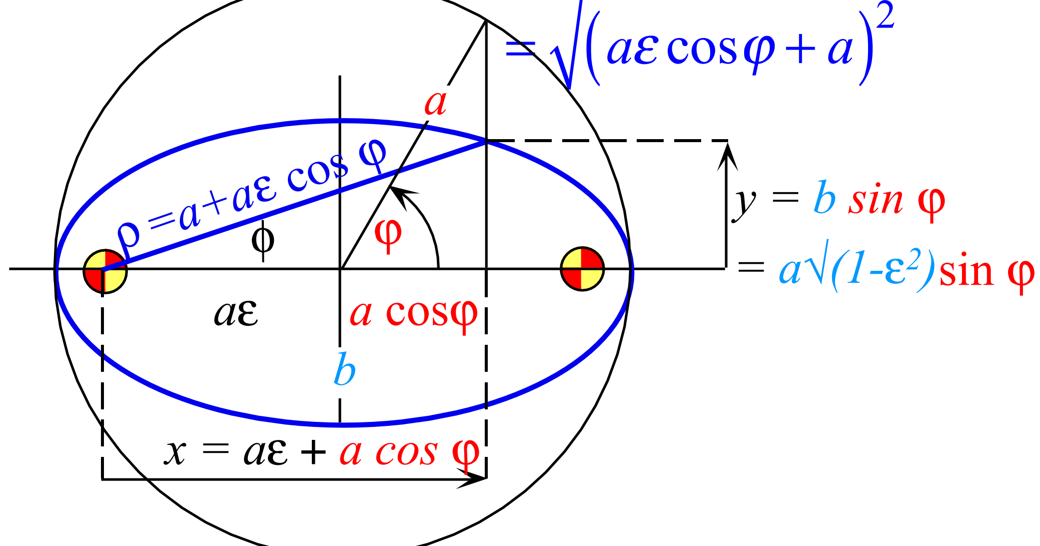

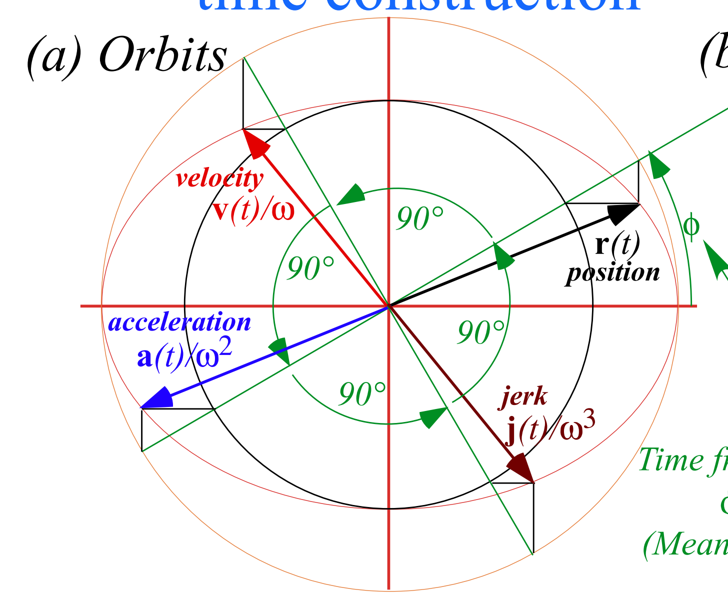

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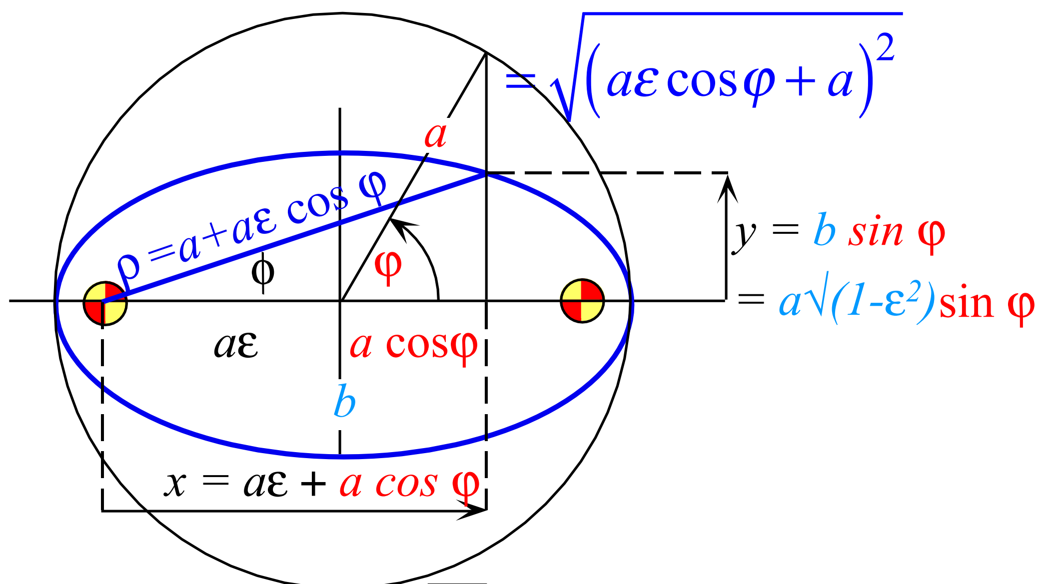

Unit 1 Ch. 9
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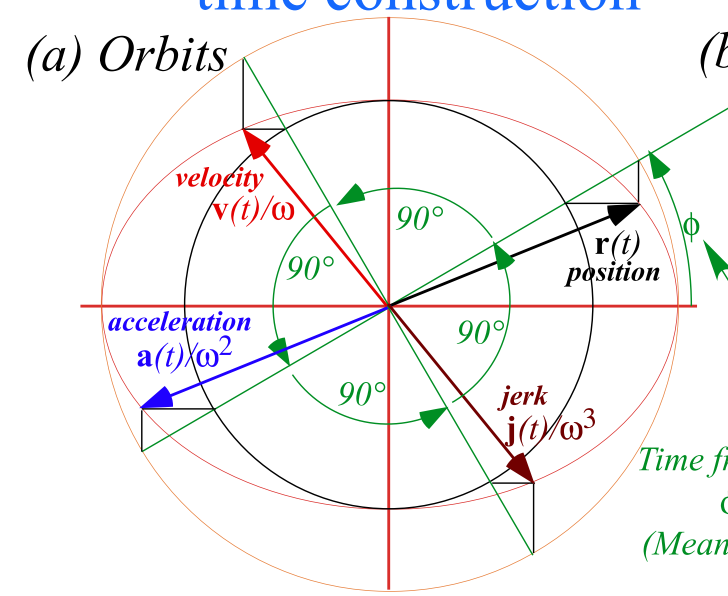

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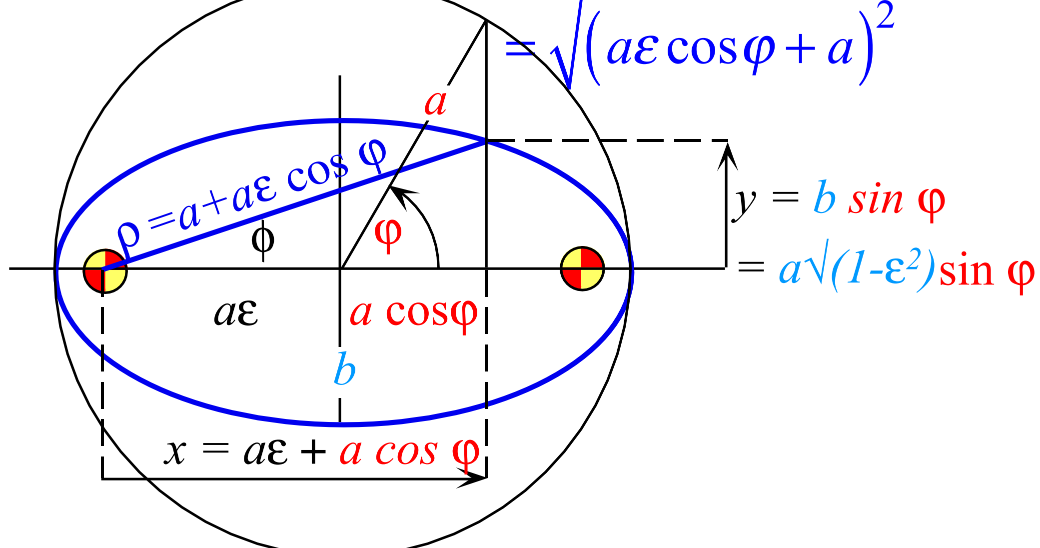

Unit 1 Ch. 9
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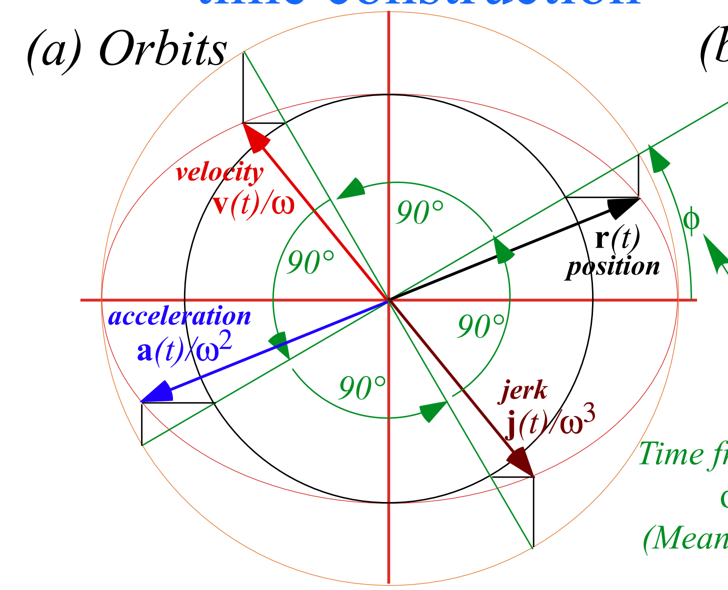

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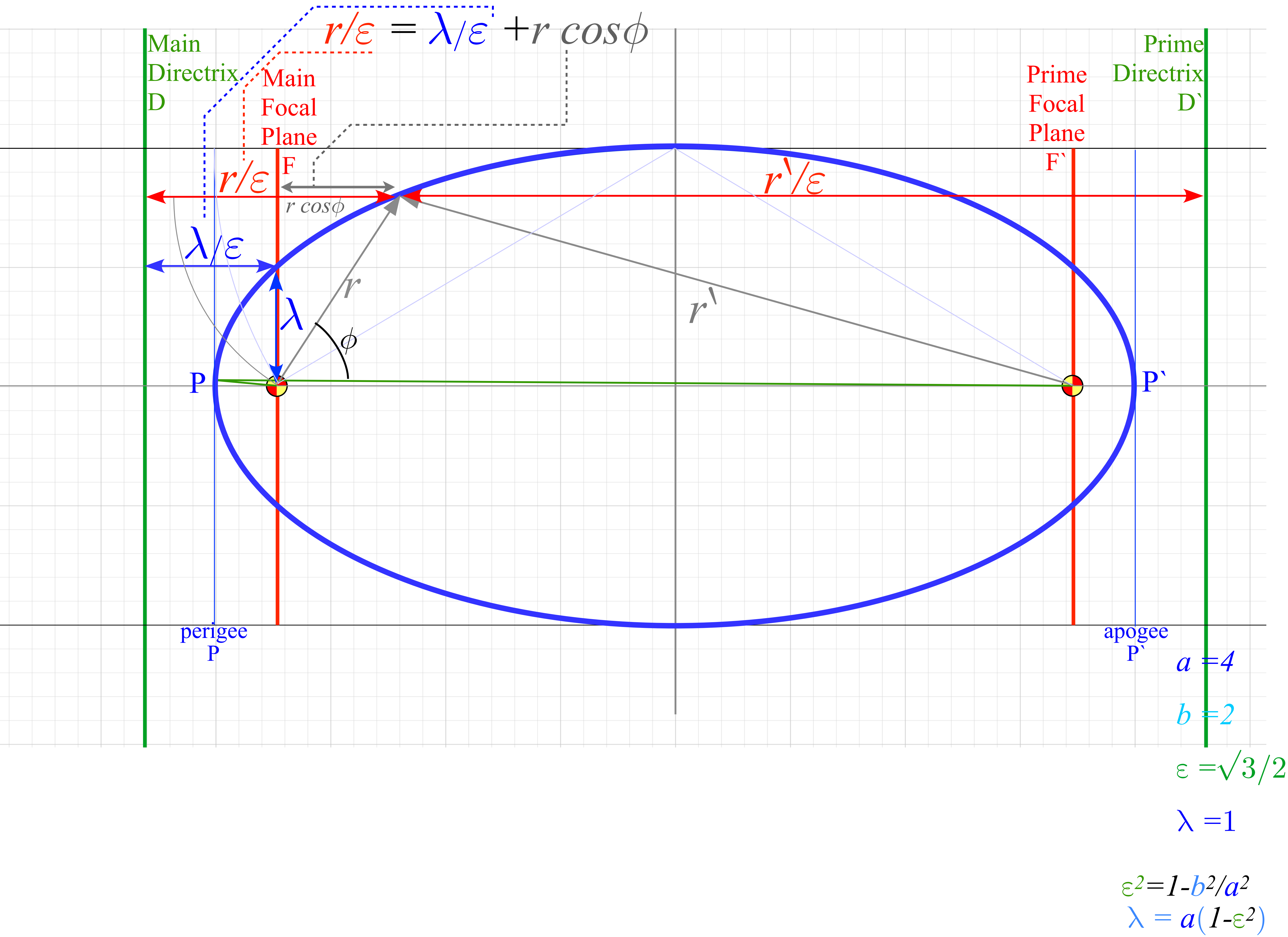
Kepler's equations
of orbital time

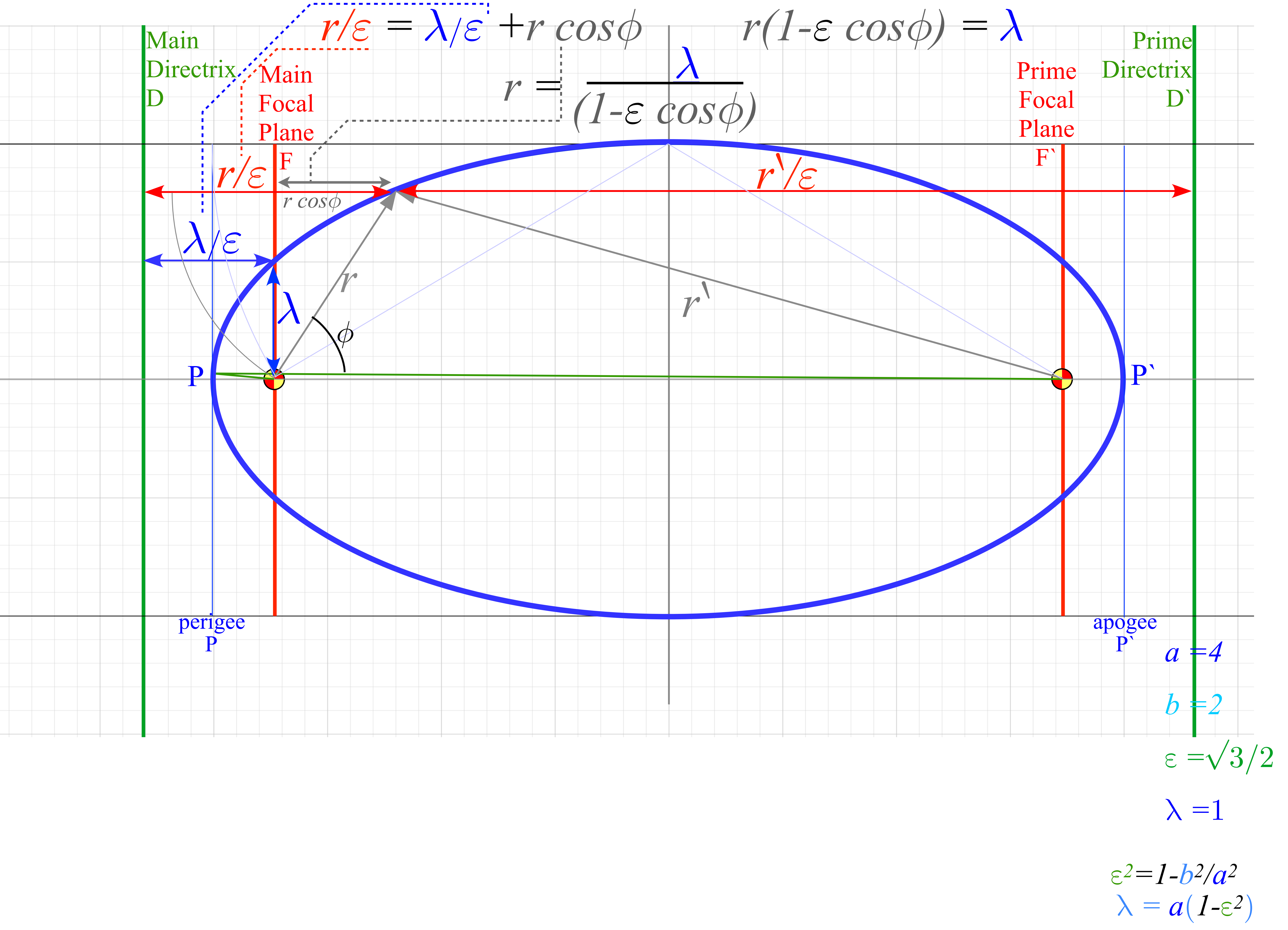
Orbit Period $= T = \frac{2\pi}{\omega_\varphi} = 2\pi\sqrt{\frac{ma^3}{k}}$

Geometry and Symmetry of Coulomb orbits

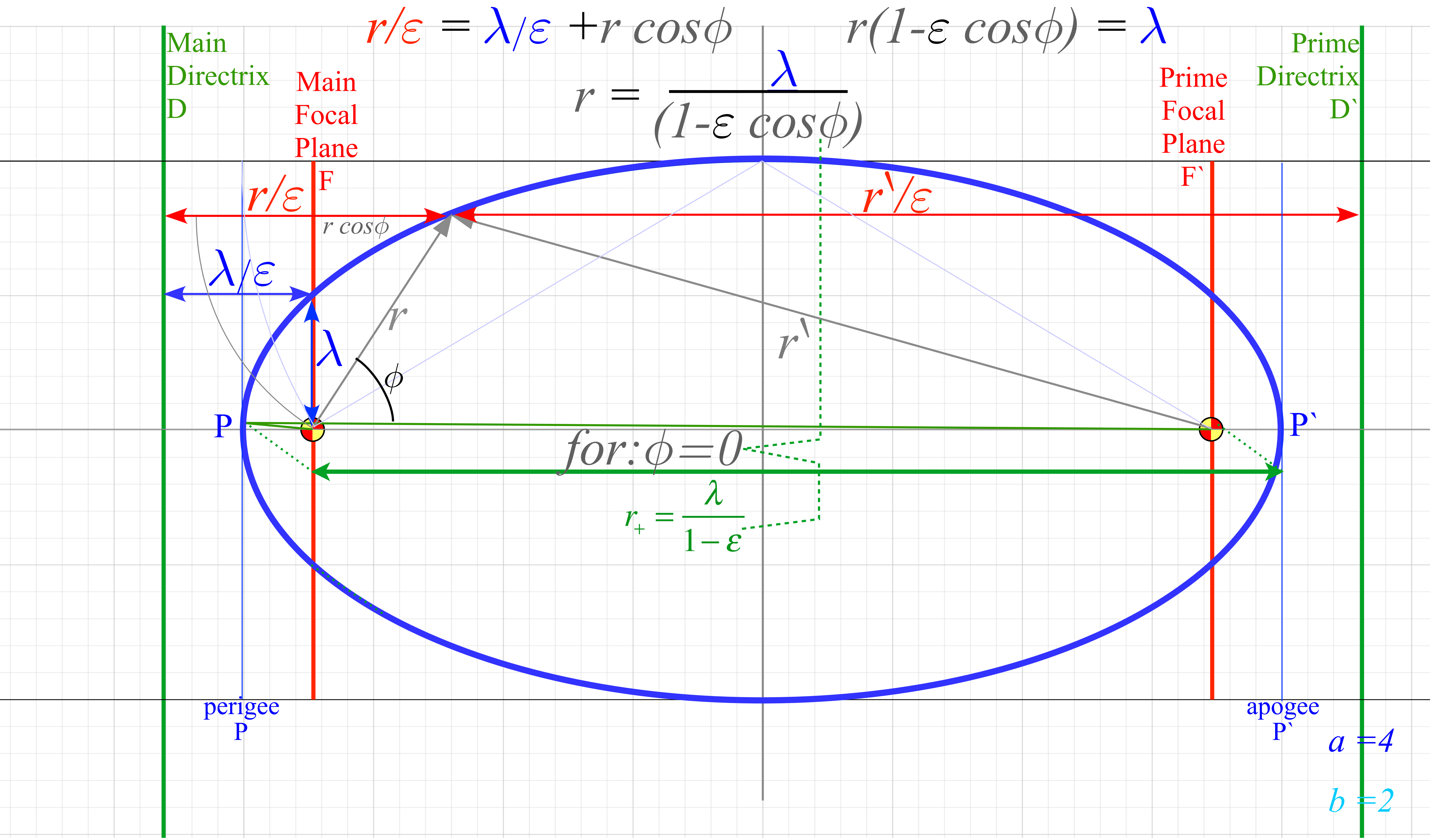
- ➤ *Detailed elliptic geometry*
- *Detailed hyperbolic geometry*



$r/\varepsilon = \lambda/\varepsilon + r\cos\phi$
Main Directrix D
Main Focal Plane
F
Prime Focal Plane
F`
Prime Directrix D`
r/ε
r'/ε
$r\cos\phi$
λ/ε
λ
r
r'
ϕ
P
P`
perigee P
apogee P`
$a = 4$
$b = 2$
$\varepsilon = \sqrt{3}/2$
$\lambda = 1$
$\varepsilon^2 = 1 - b^2/a^2$
$\lambda = a(1-\varepsilon^2)$



r/ε = λ/ε + r cosφ
r(1-ε cosφ) = λ
r = λ / (1-ε cosφ)
Main Directrix D
Main Focal Plane
F
Prime Focal Plane
F`
Prime Directrix D`
r/ε
r cosφ
r`/ε
λ/ε
λ
r
r`
φ
P
P`
perigee P
apogee P`
a =4
b =2
ε =√3/2
λ =1
ε2=1-b2/a2
λ = a(1-ε2)



$a = 4$

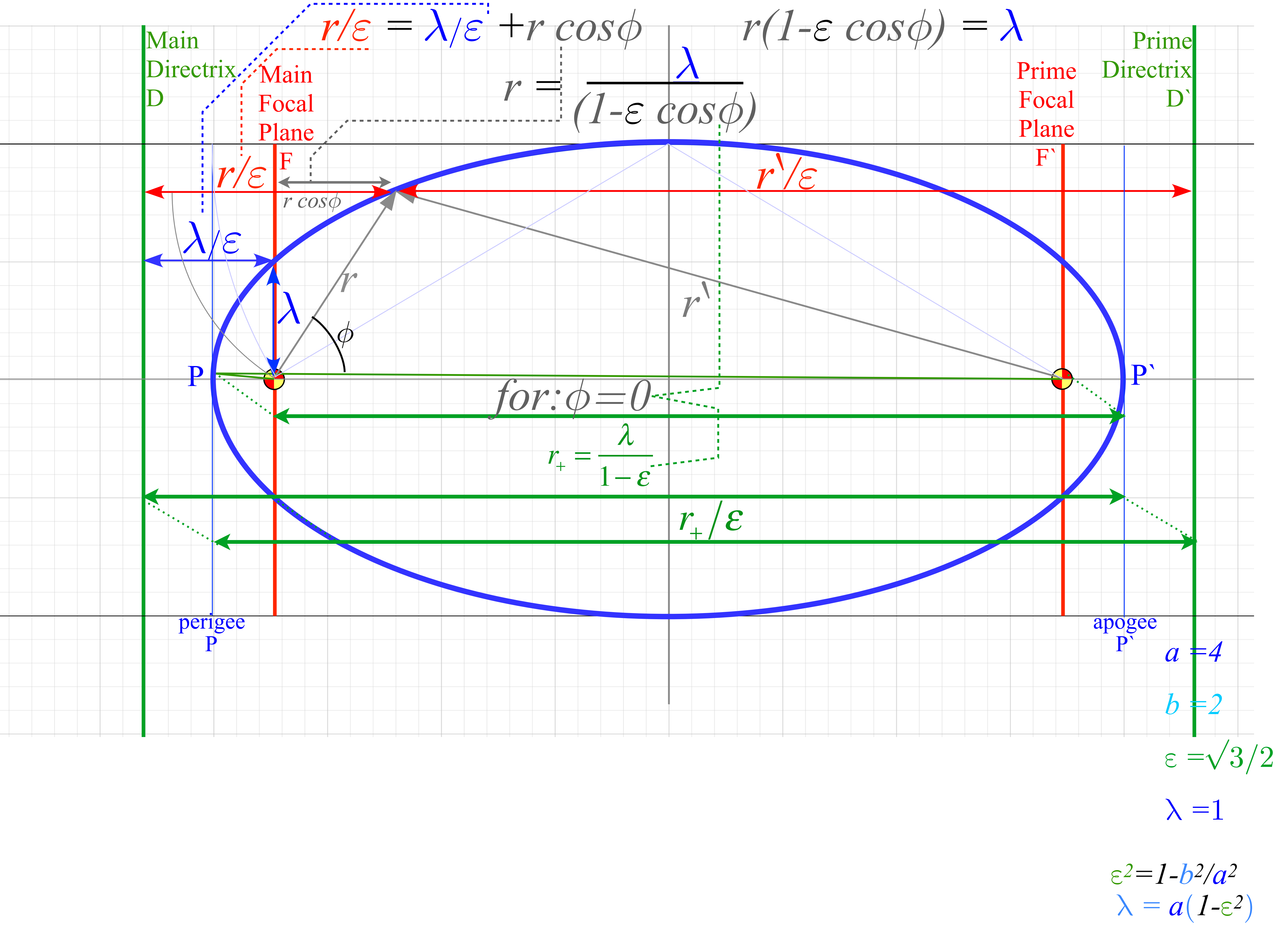
$b = 2$

$\varepsilon = \sqrt{3}/2$

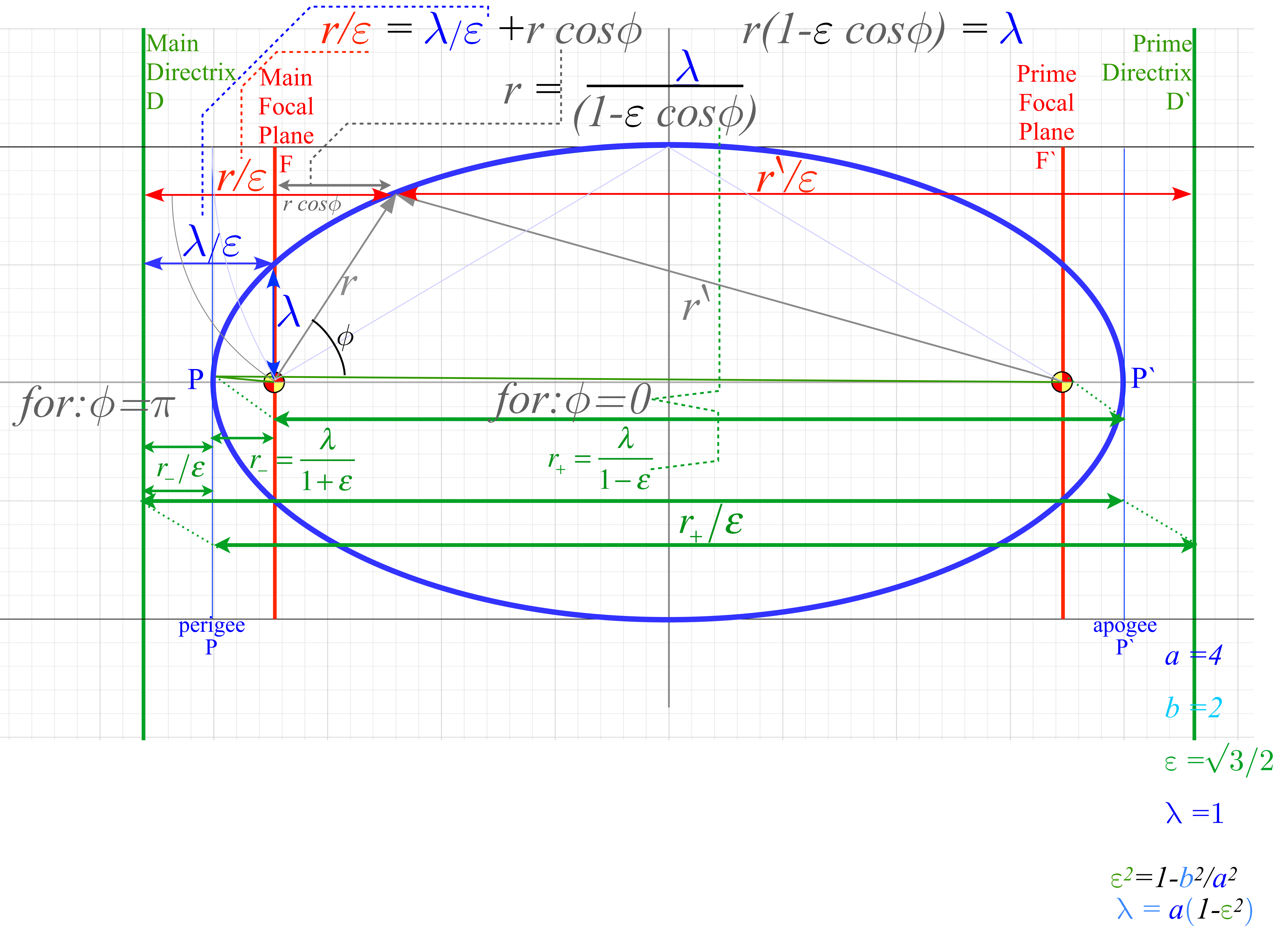
$\lambda = 1$

$\varepsilon^2 = 1 - b^2/a^2$

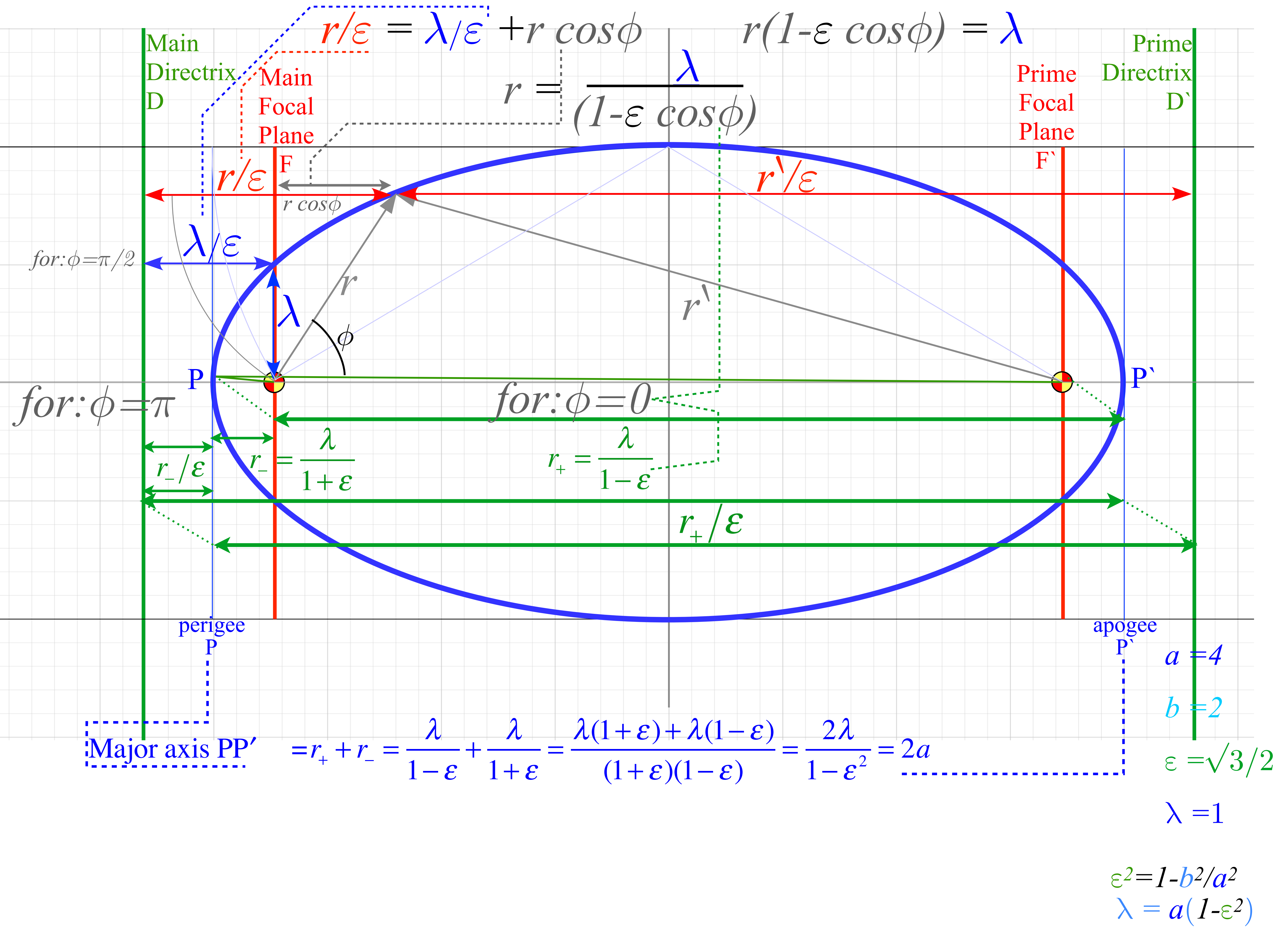
$\lambda = a(1 - \varepsilon^2)$



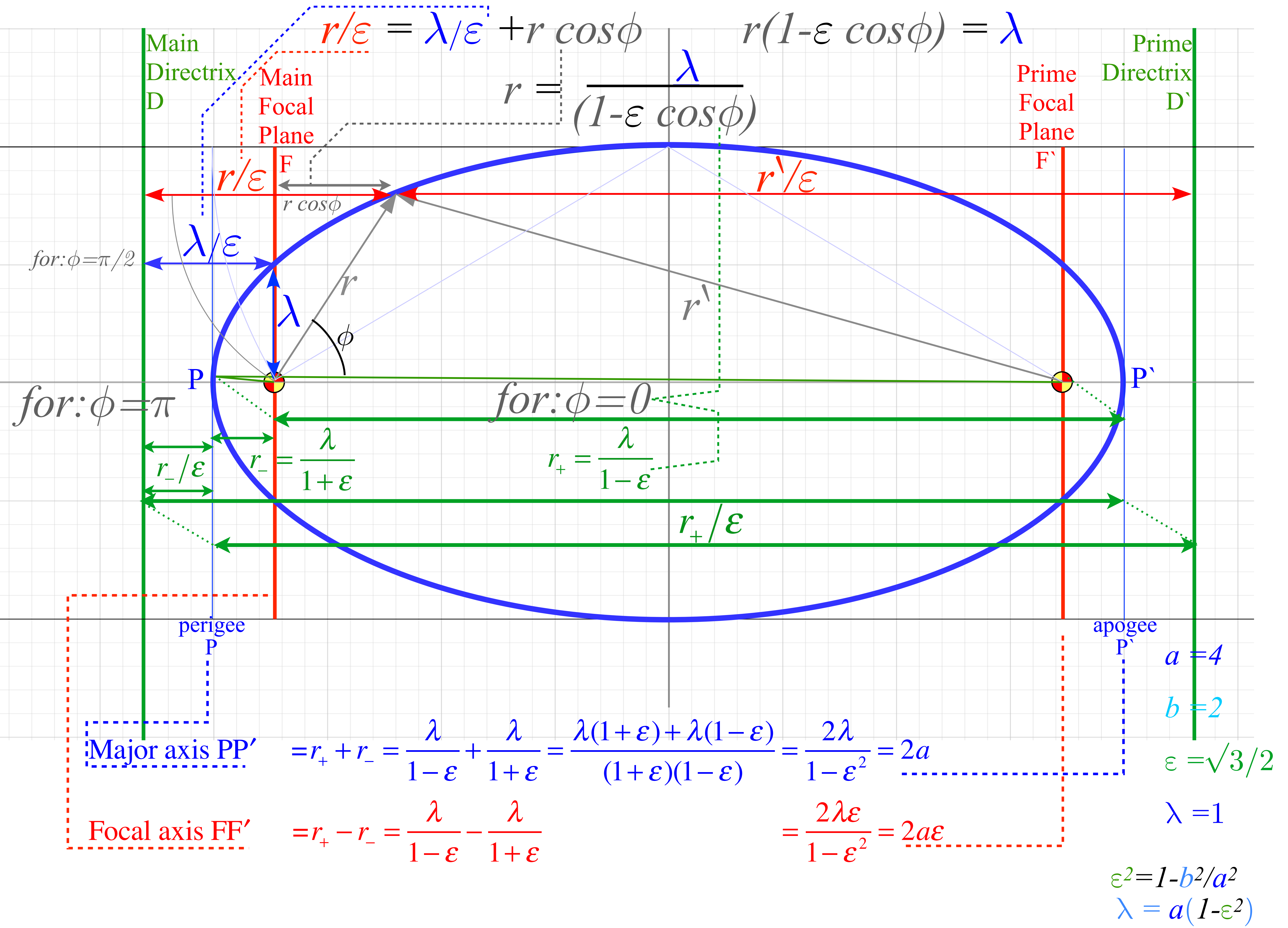
r/ε = λ/ε + r cosφ
r(1-ε cosφ) = λ
r = λ / (1-ε cosφ)
Main Directrix D
Main Focal Plane
F
Prime Focal Plane
F`
Prime Directrix D`
r/ε
r cosφ
r`/ε
λ/ε
λ
r
φ
r`
P
P`
for:φ=0
r₊ = λ / (1−ε)
r₊/ε
perigee P
apogee P`
a =4
b =2
ε =√3/2
λ =1
ε²=1-b²/a²
λ = a(1-ε²)



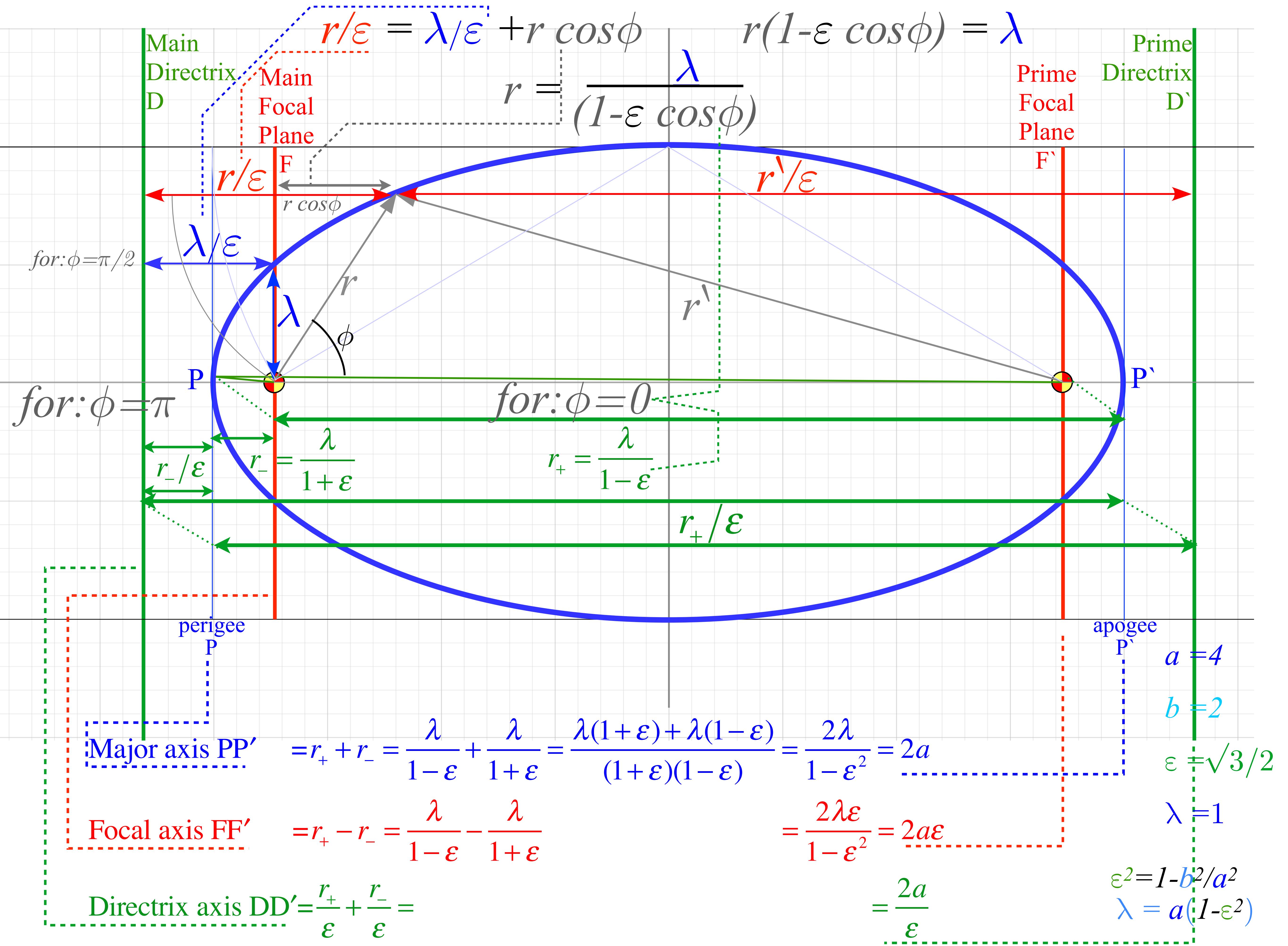
r/ε = λ/ε +r cosφ
r(1-ε cosφ) = λ
r = λ / (1-ε cosφ)
Main Directrix D
Main Focal Plane F
Prime Focal Plane F`
Prime Directrix D`
r/ε
r cosφ
r`/ε
λ/ε
λ
r
φ
r`
P
P`
for:φ=π
for:φ=0
r_- = λ/(1+ε)
r_+ = λ/(1−ε)
r_-/ε
r_+/ε
perigee P
apogee P`
a =4
b =2
ε =√3/2
λ =1
ε2=1-b2/a2
λ = a(1-ε2)



$r/\varepsilon = \lambda/\varepsilon + r\cos\phi$
$r(1-\varepsilon\cos\phi) = \lambda$
$r = \dfrac{\lambda}{(1-\varepsilon\cos\phi)}$
Main Directrix D
Main Focal Plane F
Prime Focal Plane F`
Prime Directrix D`
r/ε
r cosφ
r`/ε
λ/ε
for:φ=π/2
λ
r
φ
r`
P
P`
for:φ=π
for:φ=0
$r_- = \dfrac{\lambda}{1+\varepsilon}$
$r_+ = \dfrac{\lambda}{1-\varepsilon}$
r_-/ε
r_+/ε
perigee P
apogee P`
Major axis PP′
$= r_+ + r_- = \dfrac{\lambda}{1-\varepsilon} + \dfrac{\lambda}{1+\varepsilon} = \dfrac{\lambda(1+\varepsilon)+\lambda(1-\varepsilon)}{(1+\varepsilon)(1-\varepsilon)} = \dfrac{2\lambda}{1-\varepsilon^2} = 2a$
a =4
b =2
ε =√3/2
λ =1
$\varepsilon^2 = 1 - b^2/a^2$
$\lambda = a(1-\varepsilon^2)$

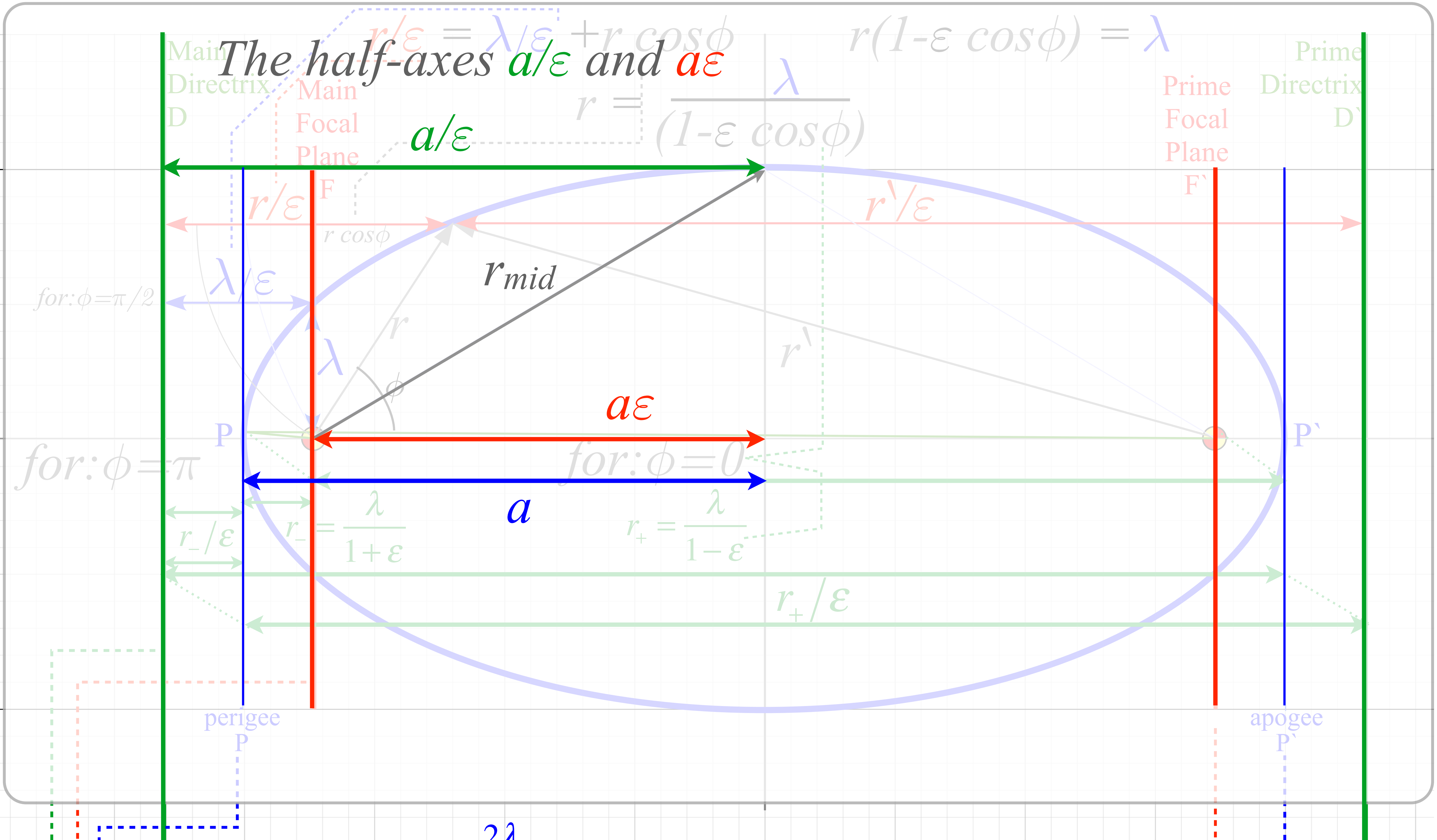


r/ε = λ/ε +r cosφ
r(1-ε cosφ) = λ
r = λ / (1-ε cosφ)
Main Directrix D
Main Focal Plane
F
Prime Focal Plane
F`
Prime Directrix D`
r/ε
r cosφ
r`/ε
for:φ=π/2
λ/ε
λ
r
φ
r`
P
P`
for:φ=π
for:φ=0
r_ = λ/(1+ε)
r+ = λ/(1−ε)
r_/ε
r+/ε
perigee P
apogee P`
a =4
b =2
ε =√3/2
λ =1
Major axis PP′ =r+ + r_ = λ/(1−ε) + λ/(1+ε) = (λ(1+ε)+λ(1−ε))/((1+ε)(1−ε)) = 2λ/(1−ε²) = 2a
Focal axis FF′ =r+ − r_ = λ/(1−ε) − λ/(1+ε) = 2λε/(1−ε²) = 2aε
ε²=1-b²/a²
λ = a(1-ε²)



$r/\varepsilon = \lambda/\varepsilon + r\cos\phi$
$r(1-\varepsilon\cos\phi) = \lambda$
$r = \dfrac{\lambda}{(1-\varepsilon\cos\phi)}$
Main Directrix D
Main Focal Plane F
Prime Focal Plane F`
Prime Directrix D`
r/ε
$r\cos\phi$
r'/ε
λ/ε
for: $\phi=\pi/2$
λ
r
r'
ϕ
P
P`
for: $\phi=\pi$
for: $\phi=0$
r_-/ε
$r_- = \dfrac{\lambda}{1+\varepsilon}$
$r_+ = \dfrac{\lambda}{1-\varepsilon}$
r_+/ε
perigee P
apogee P`
$a = 4$
$b = 2$
$\varepsilon = \sqrt{3}/2$
$\lambda = 1$
Major axis PP′ $= r_+ + r_- = \dfrac{\lambda}{1-\varepsilon} + \dfrac{\lambda}{1+\varepsilon} = \dfrac{\lambda(1+\varepsilon)+\lambda(1-\varepsilon)}{(1+\varepsilon)(1-\varepsilon)} = \dfrac{2\lambda}{1-\varepsilon^2} = 2a$
Focal axis FF′ $= r_+ - r_- = \dfrac{\lambda}{1-\varepsilon} - \dfrac{\lambda}{1+\varepsilon} = \dfrac{2\lambda\varepsilon}{1-\varepsilon^2} = 2a\varepsilon$
Directrix axis DD′ $= \dfrac{r_+}{\varepsilon} + \dfrac{r_-}{\varepsilon} = \dfrac{2a}{\varepsilon}$
$\varepsilon^2 = 1 - b^2/a^2$
$\lambda = a(1-\varepsilon^2)$

The half-axes a/ε and $a\varepsilon$

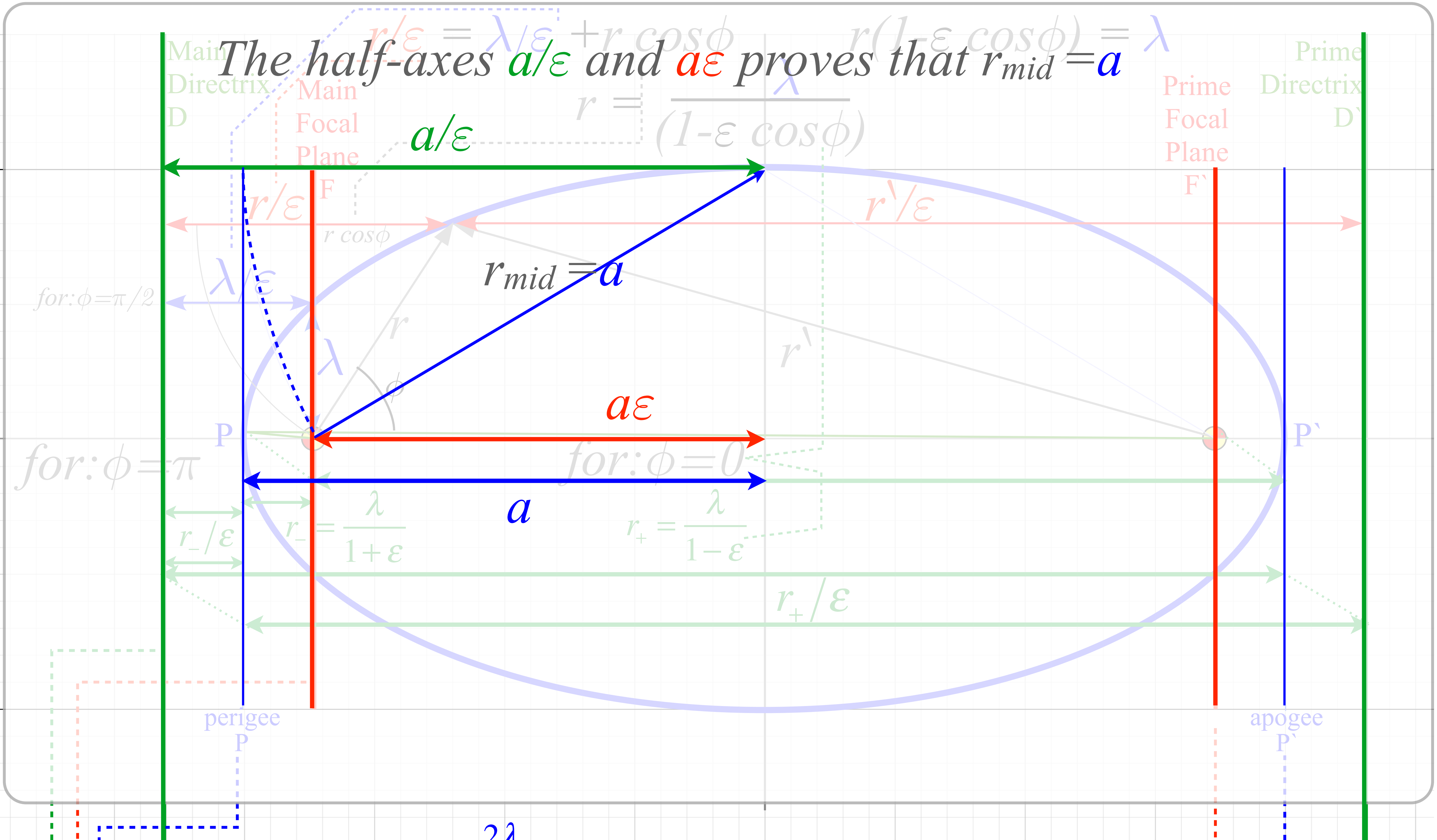


Major axis PP′ $= r_+ + r_- = \dfrac{2\lambda}{1-\varepsilon^2} = 2a$

Focal axis FF′ $= r_+ - r_- = \dfrac{2\lambda\varepsilon}{1-\varepsilon^2} = 2a\varepsilon$

Directrix axis DD′ $= \dfrac{r_+}{\varepsilon} + \dfrac{r_-}{\varepsilon} = \dfrac{2a}{\varepsilon}$

The half-axes a/ε and $a\varepsilon$ proves that $r_{mid} = a$

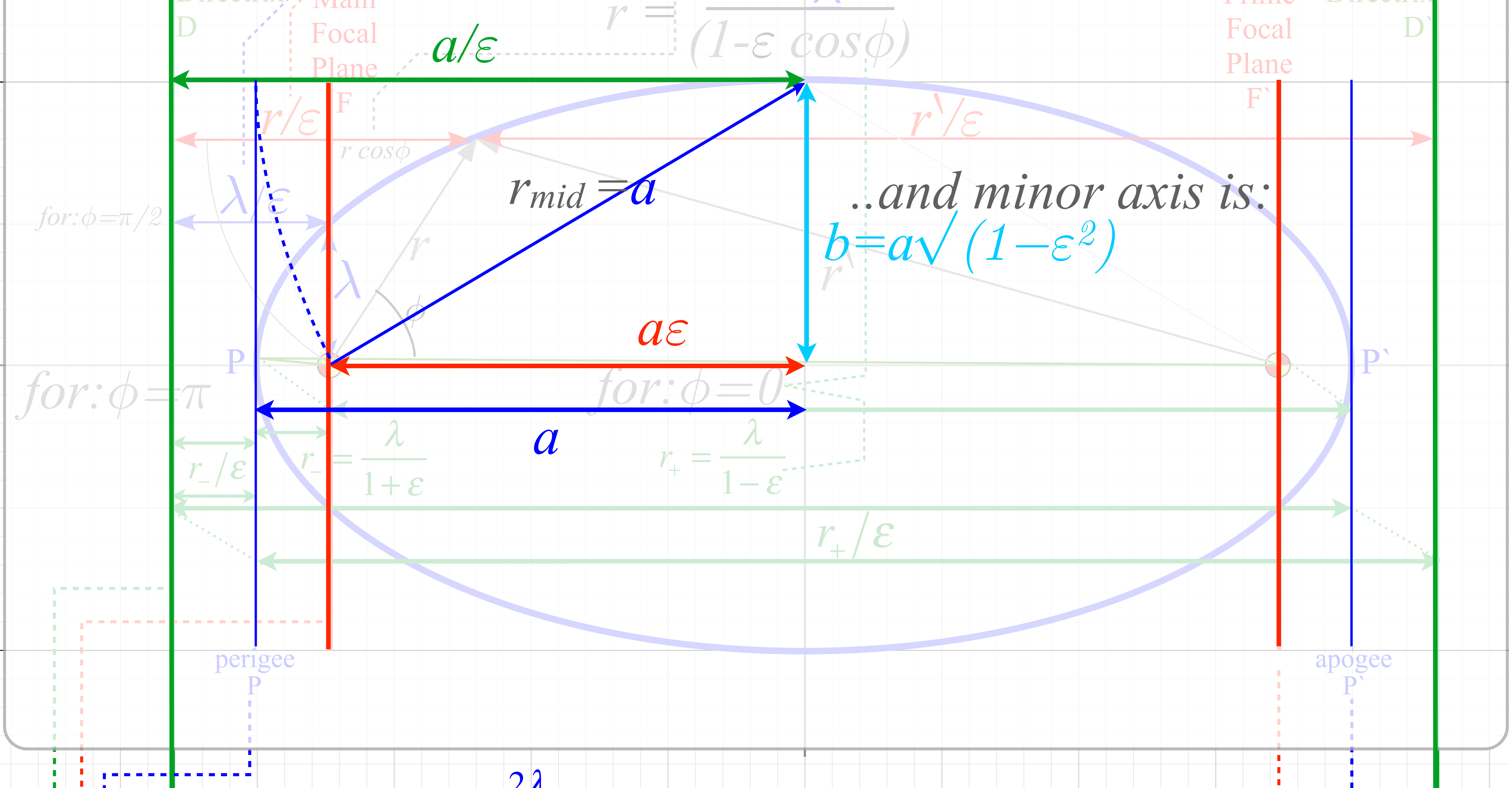


Major axis PP′ $= r_+ + r_- = \dfrac{2\lambda}{1-\varepsilon^2} = 2a$

Focal axis FF′ $= r_+ - r_- = \dfrac{2\lambda\varepsilon}{1-\varepsilon^2} = 2a\varepsilon$

Directrix axis DD′ $= \dfrac{r_+}{\varepsilon} + \dfrac{r_-}{\varepsilon} = \dfrac{2a}{\varepsilon}$

The half-axes a/ε and $a\varepsilon$ proves that $r_{mid} = a$



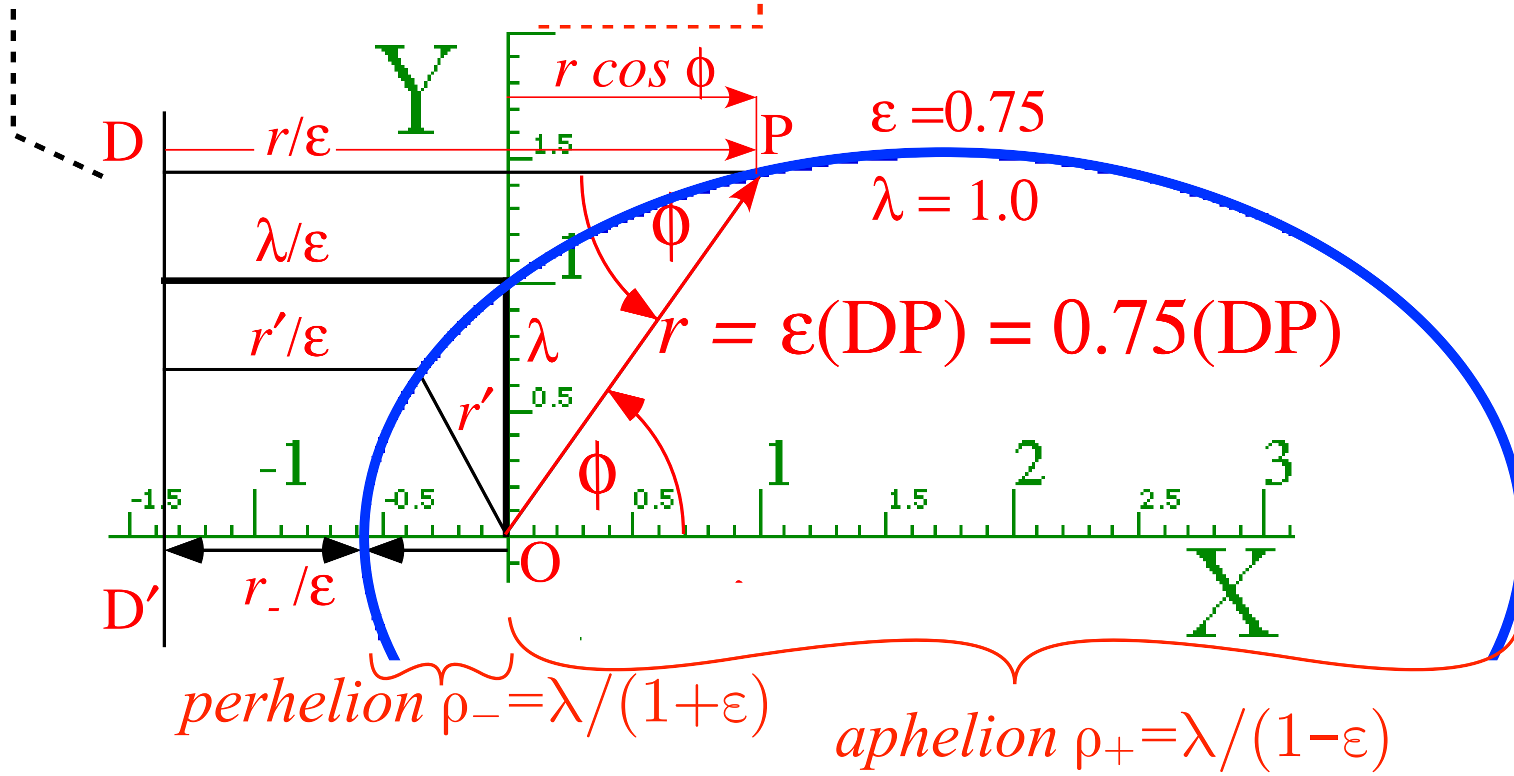
Major axis PP′ $= r_+ + r_- = \dfrac{2\lambda}{1-\varepsilon^2} = 2a$

Focal axis FF′ $= r_+ - r_- = \dfrac{2\lambda\varepsilon}{1-\varepsilon^2} = 2a\varepsilon$

Directrix axis DD′ $= \dfrac{r_+}{\varepsilon} + \dfrac{r_-}{\varepsilon} = \dfrac{2a}{\varepsilon}$

Geometry of Coulomb conic section orbits (Let: $r = \rho$ here)

$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi \qquad r = \lambda + r\,\varepsilon\cos\phi \qquad r = \frac{\lambda}{1-\varepsilon\cos\phi}$$



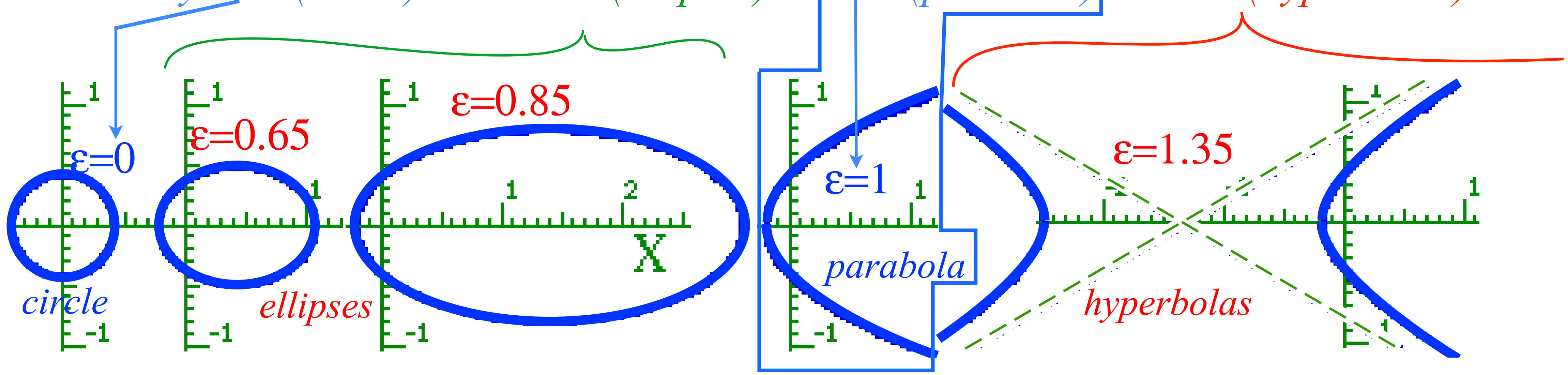
$$\frac{1}{r} = \frac{1-\varepsilon\cos\phi}{\lambda} = \frac{1}{\lambda} - \frac{\varepsilon}{\lambda}\cos\phi$$

$$\frac{1}{\rho} = \frac{-k}{\mu^2/m} + \frac{\sqrt{k^2 + 2E\mu^2/m}}{\mu^2/m}\cos\phi$$

Becoming more and more eccentric...

Singular Case!

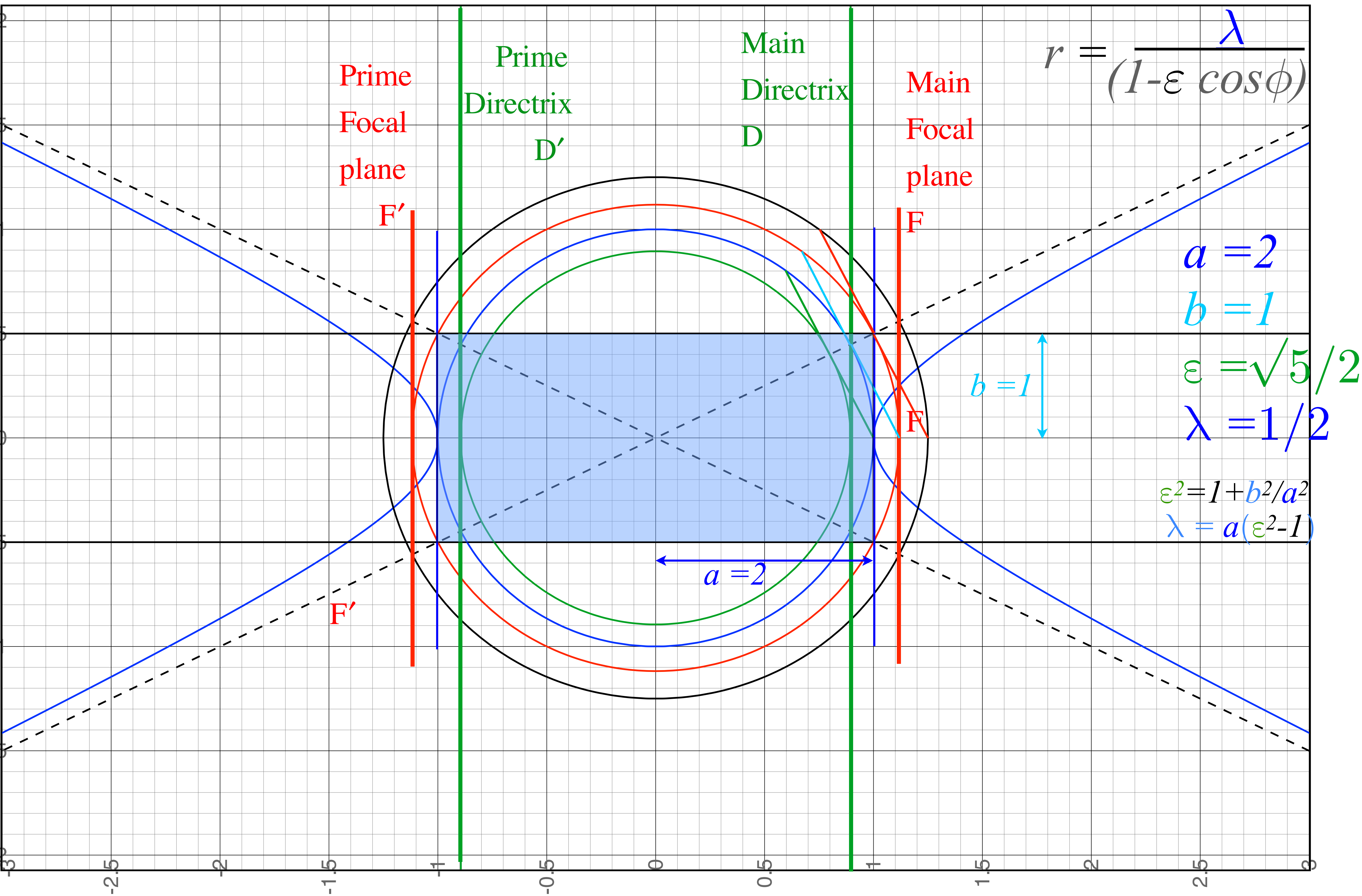
Eccentricity ε=0 (circle) to 0<ε<1 (ellipses) to ε=1 (parabola) to ε >1 (hyperbolas)



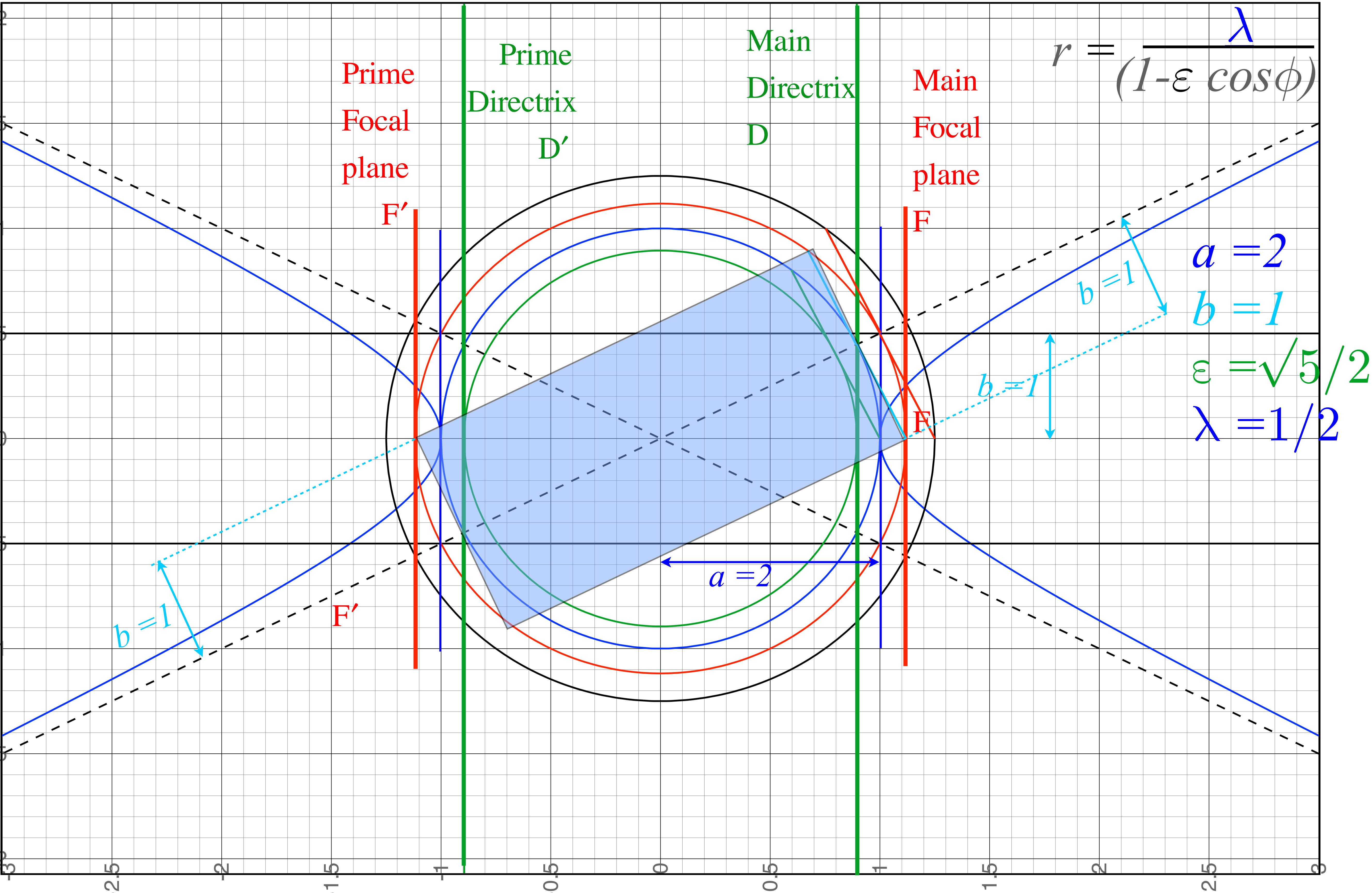
Geometry and Symmetry of Coulomb orbits

Detailed elliptic geometry

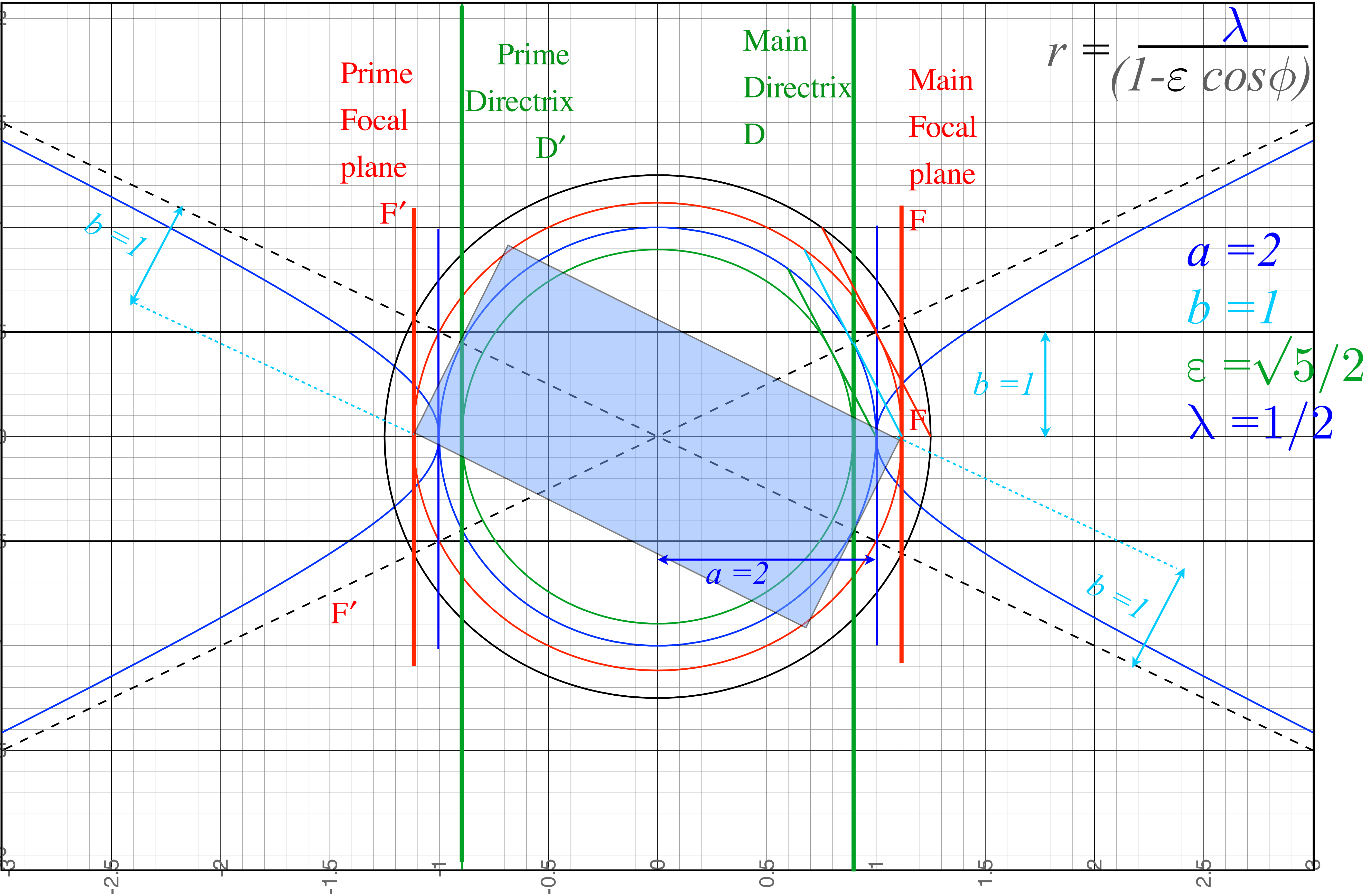
➤ *Detailed hyperbolic geometry*



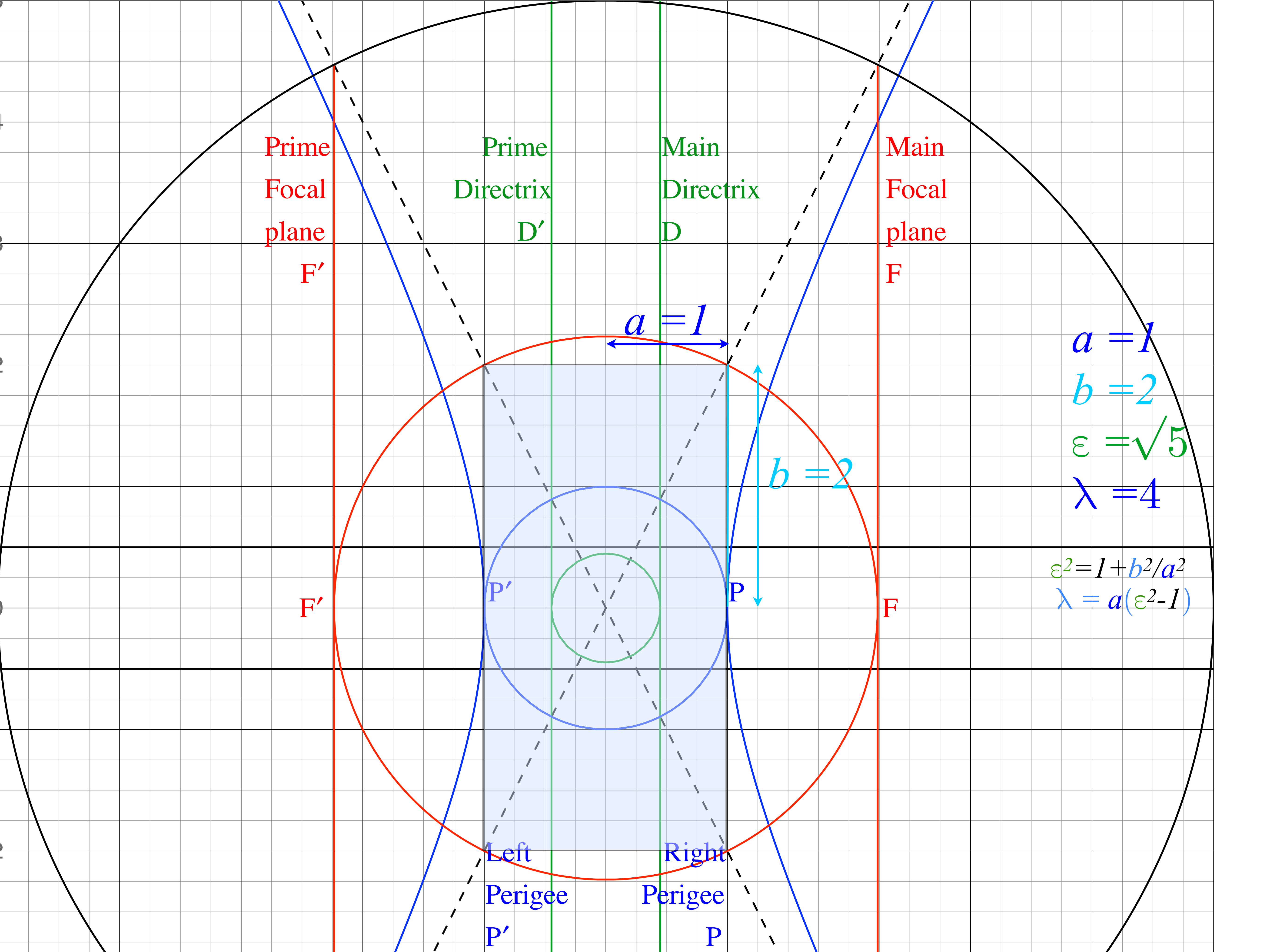
Prime Focal plane
F′
Prime Directrix D′
Main Directrix D
Main Focal plane
F
$r = \frac{\lambda}{(1-\varepsilon \cos\phi)}$
$a = 2$
$b = 1$
$\varepsilon = \sqrt{5}/2$
$\lambda = 1/2$
$b = 1$
F
$\varepsilon^2 = 1 + b^2/a^2$
$\lambda = a(\varepsilon^2 - 1)$
$a = 2$
F′
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3



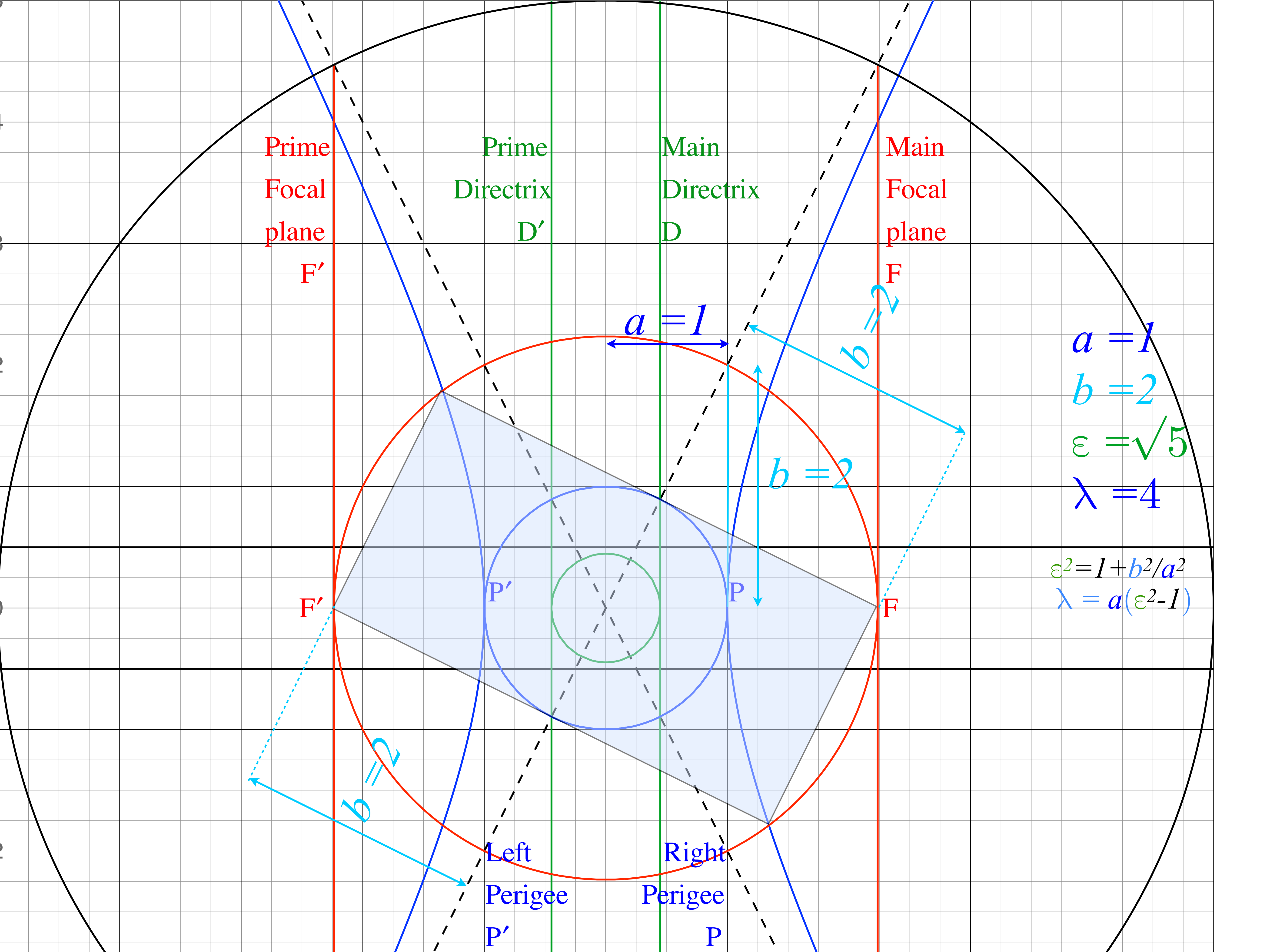
Prime
Focal
plane
F′
Prime
Directrix
D′
Main
Directrix
D
Main
Focal
plane
F
$r = \frac{\lambda}{(1-\varepsilon \cos\phi)}$
a =2
b =1
ε =√5/2
λ =1/2
b =1
b =1
b =1
F
F′
a =2
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3



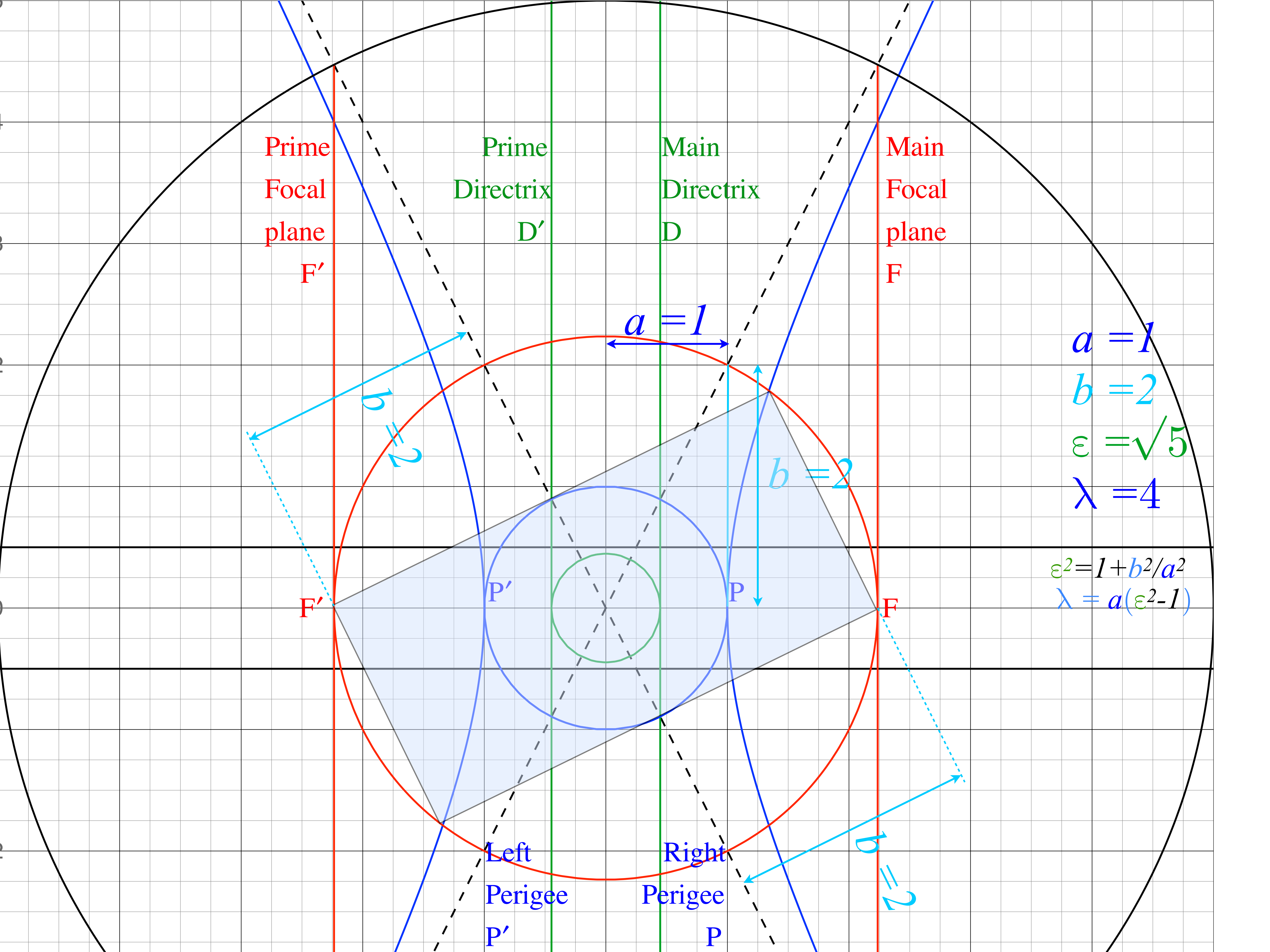
Prime Focal plane
F′
Prime Directrix
D′
Main Directrix
D
Main Focal plane
F
r = λ / (1-ε cosφ)
a =2
b =1
ε =√5/2
λ =1/2
b =1
F
b =1
a =2
b =1
F′



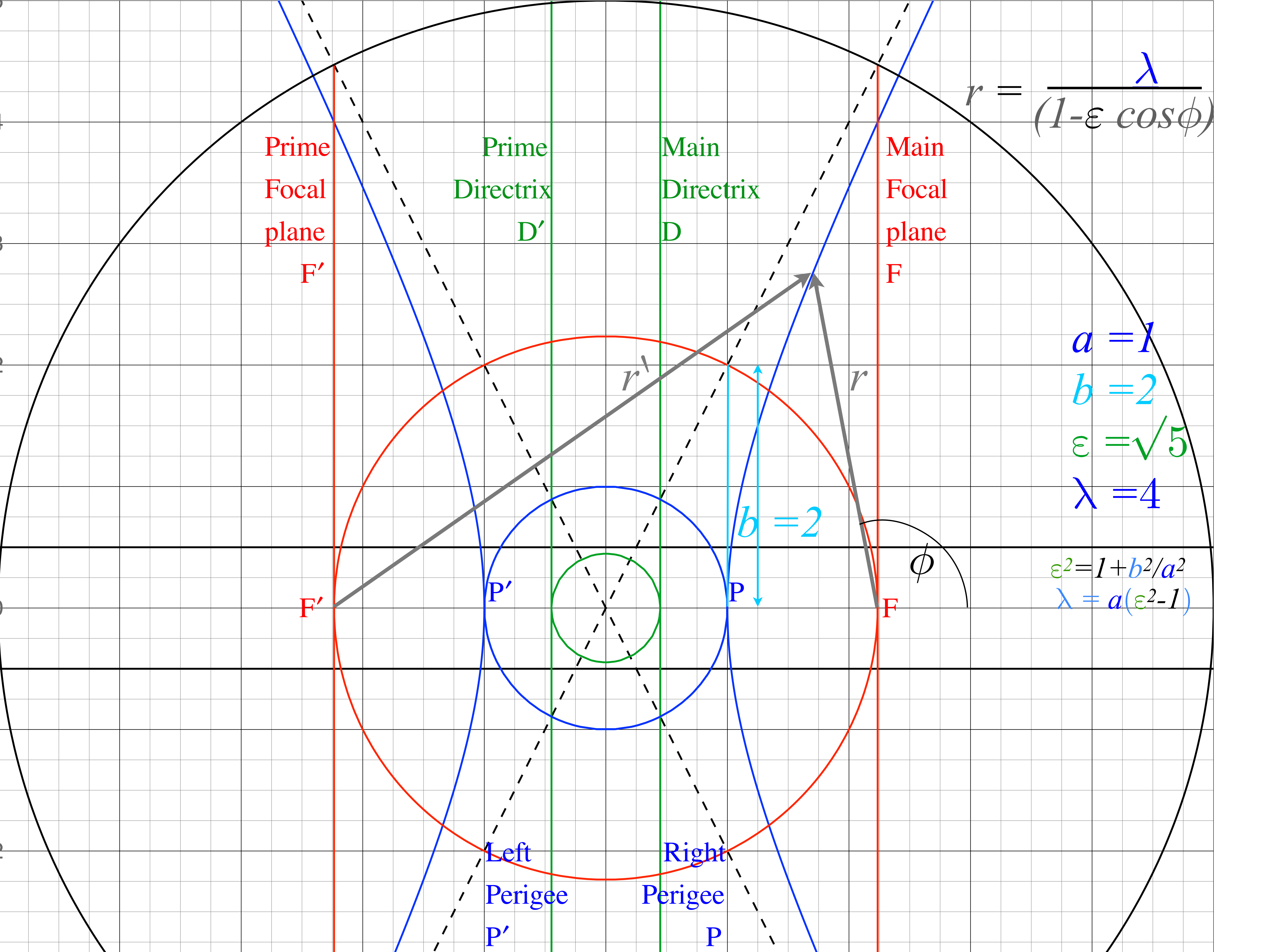
Prime Focal plane F′
Prime Directrix D′
Main Directrix D
Main Focal plane F
a =1
b =2
a =1
b =2
ε =√5
λ =4
ε2=1+b2/a2
λ = a(ε2-1)
P′
P
F′
F
Left Perigee P′
Right Perigee P



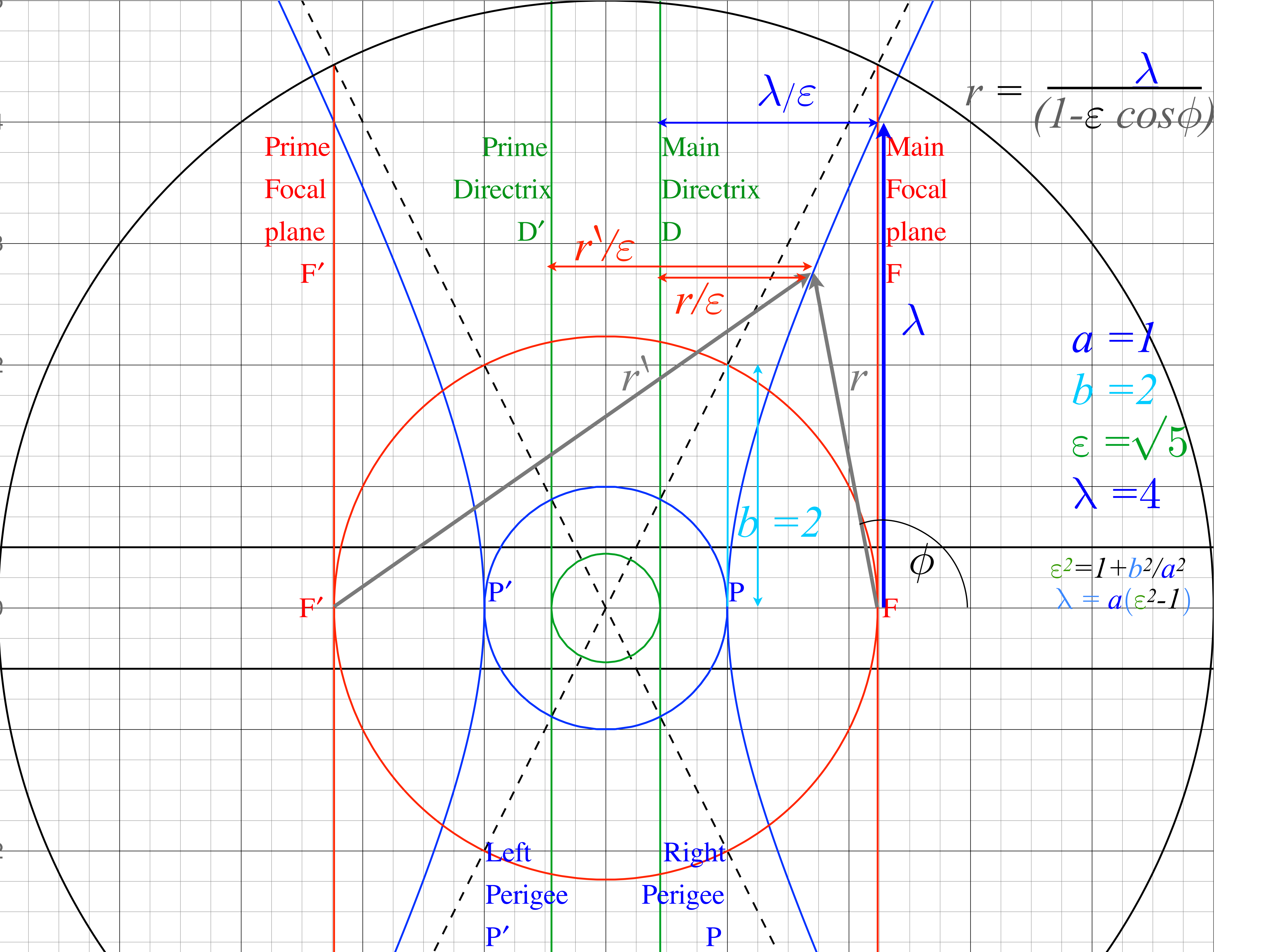
Prime Focal plane F′
Prime Directrix D′
Main Directrix D
Main Focal plane F
a =1
b =2
b =2
b =2
b =2
a =1
b =2
ε =√5
λ =4
ε²=1+b²/a²
λ = a(ε²-1)
P′
P
F′
F
Left Perigee P′
Right Perigee P



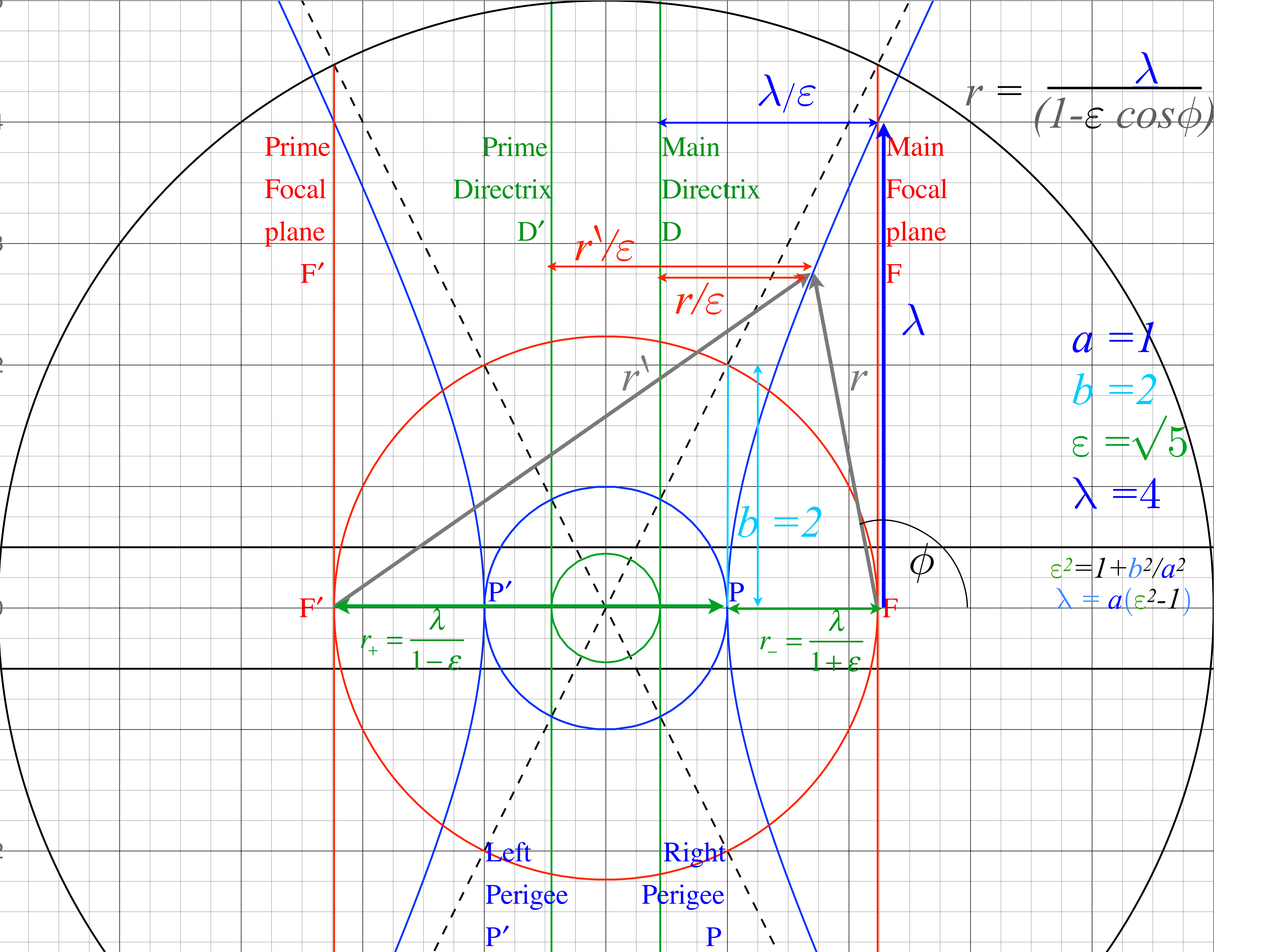
Prime Focal plane F′
Prime Directrix D′
Main Directrix D
Main Focal plane F
a = 1
b = 2
b = 2
b = 2
b = 2
a = 1
b = 2
ε = √5
λ = 4
ε² = 1 + b²/a²
λ = a(ε²-1)
F′
P′
P
F
Left Perigee P′
Right Perigee P



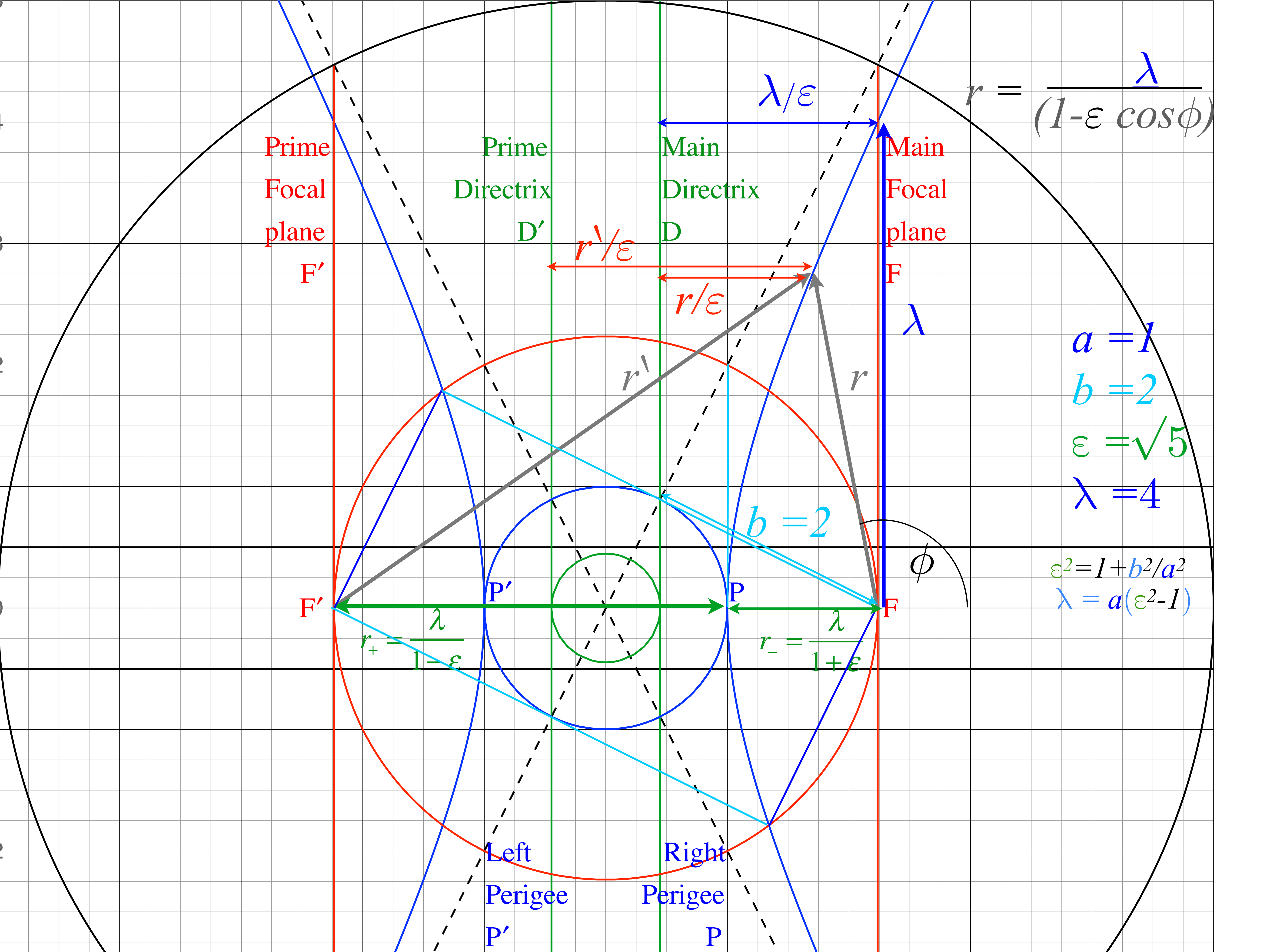
$r = \frac{\lambda}{(1-\varepsilon \cos\phi)}$
Prime Focal plane F′
Prime Directrix D′
Main Directrix D
Main Focal plane F
$a = 1$
$b = 2$
$\varepsilon = \sqrt{5}$
$\lambda = 4$
r′
r
b = 2
ϕ
$\varepsilon^2 = 1 + b^2/a^2$
$\lambda = a(\varepsilon^2 - 1)$
F′
P′
P
F
Left Perigee P′
Right Perigee P



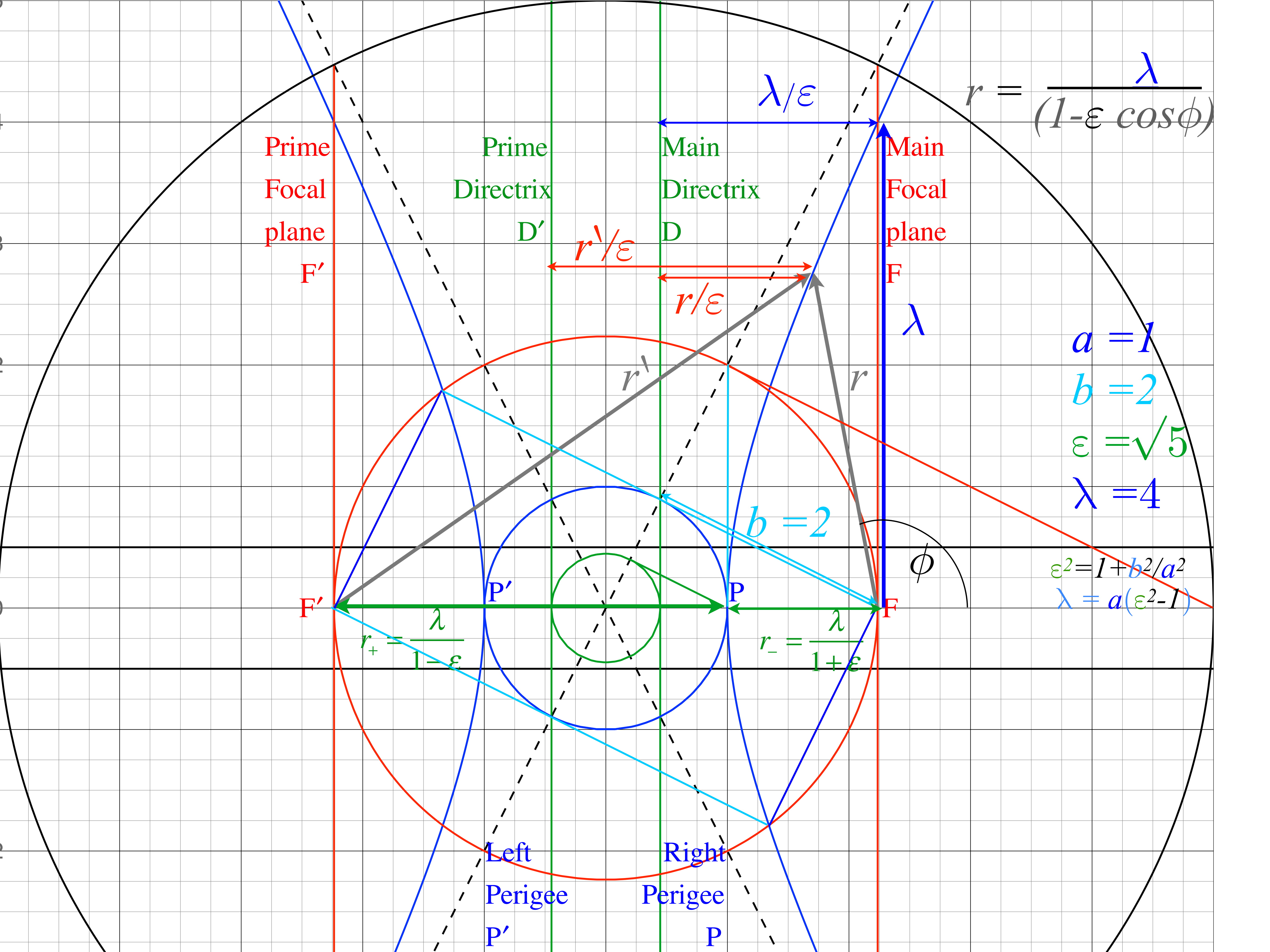
$r = \frac{\lambda}{(1-\varepsilon \cos\phi)}$
λ/ε
Prime Focal plane F′
Prime Directrix D′
Main Directrix D
Main Focal plane F
r′/ε
r/ε
λ
a =1
b =2
ε =√5
λ =4
r′
r
b =2
ϕ
F′
P′
P
F
ε²=1+b²/a²
λ = a(ε²-1)
Left Perigee P′
Right Perigee P



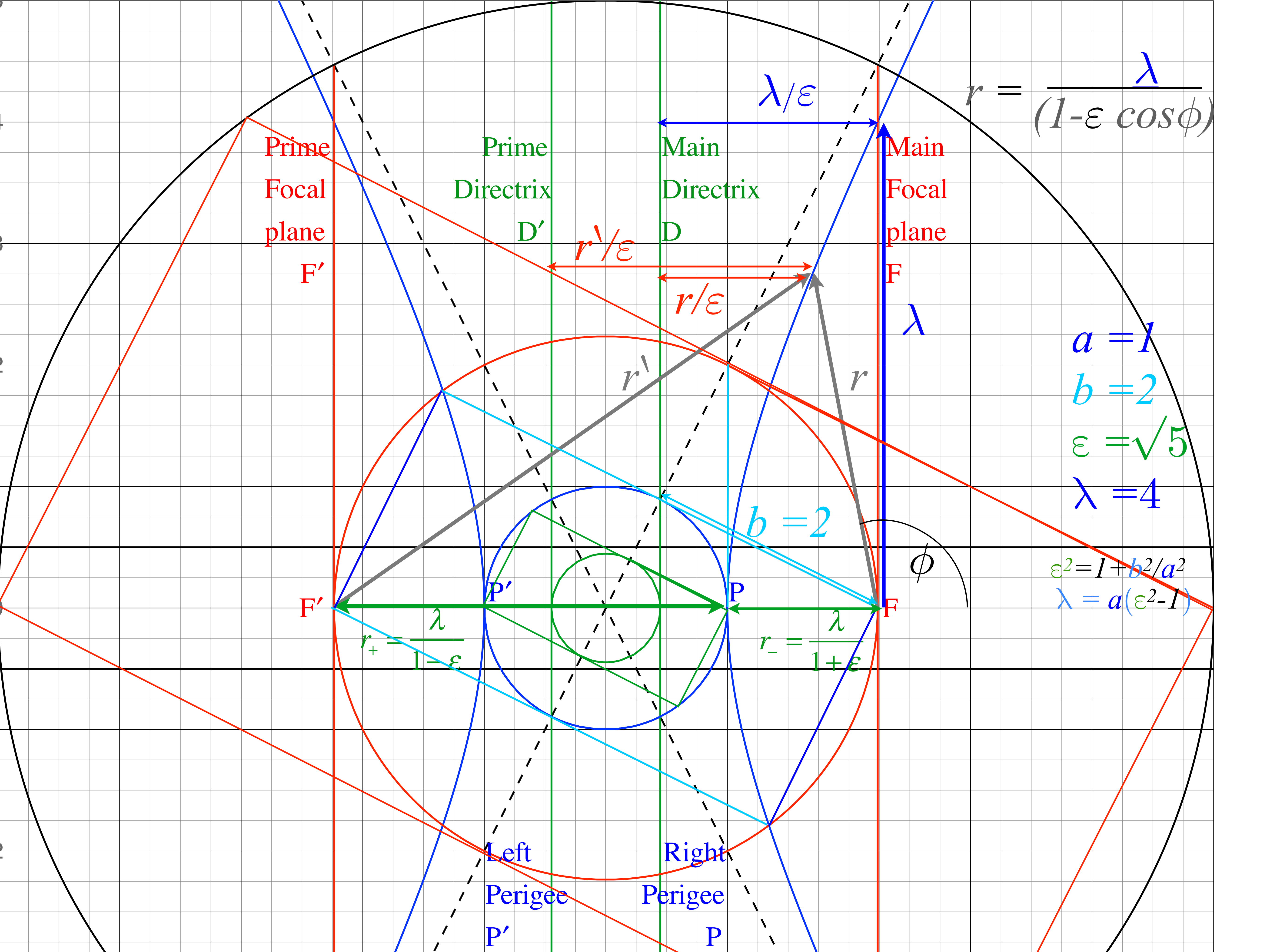
$r = \frac{\lambda}{(1-\varepsilon\cos\phi)}$
λ/ε
Prime Focal plane F′
Prime Directrix D′
Main Directrix D
Main Focal plane F
r'/ε
r/ε
λ
$a = 1$
$b = 2$
$\varepsilon = \sqrt{5}$
$\lambda = 4$
r'
r
$b = 2$
ϕ
$\varepsilon^2 = 1 + b^2/a^2$
$\lambda = a(\varepsilon^2 - 1)$
F′
P′
P
F
$r_+ = \frac{\lambda}{1-\varepsilon}$
$r_- = \frac{\lambda}{1+\varepsilon}$
Left Perigee P′
Right Perigee P



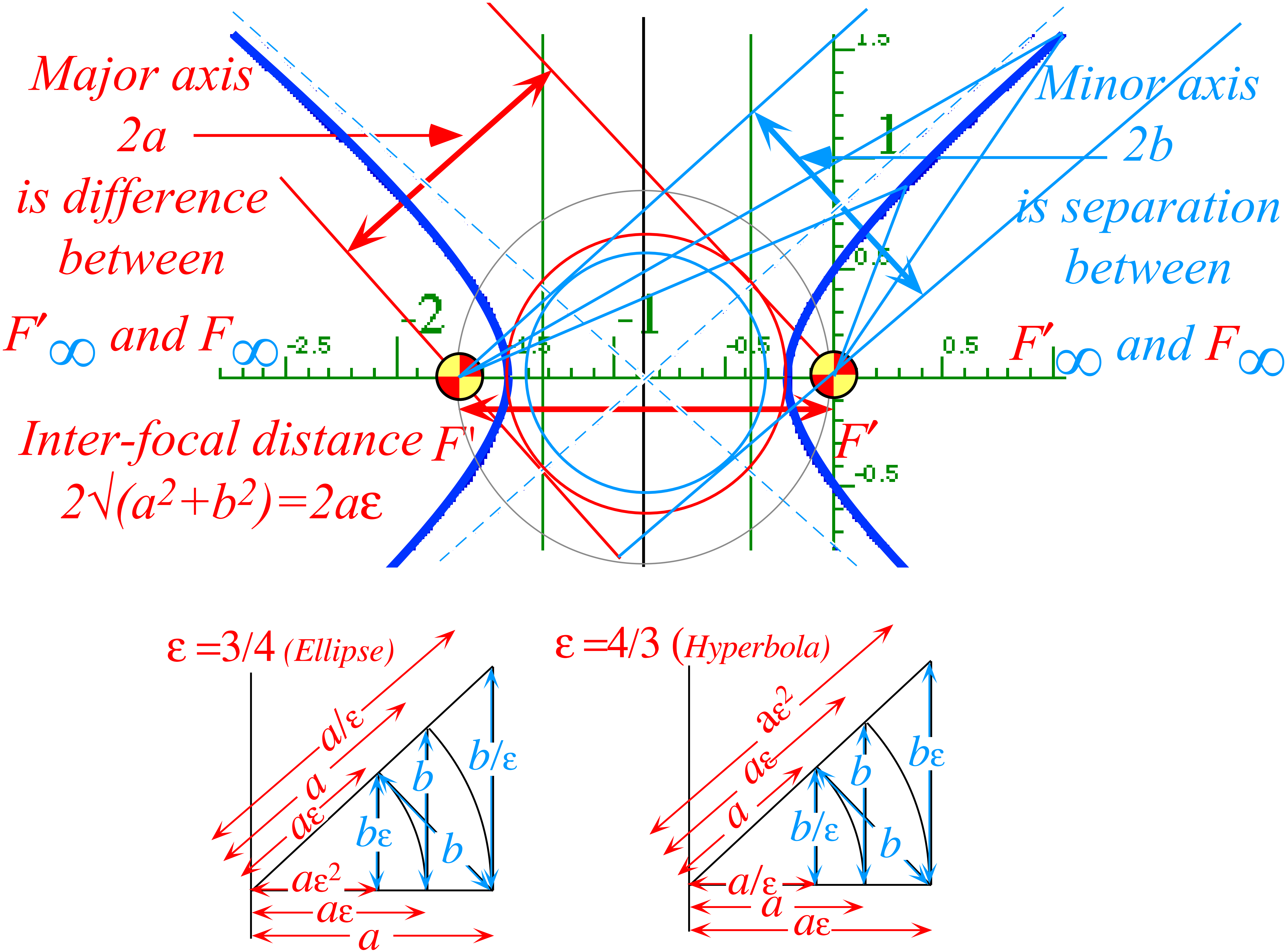
$r = \frac{\lambda}{(1-\varepsilon \cos\phi)}$
λ/ε
Prime
Focal
plane
F′
Prime
Directrix
D′
Main
Directrix
D
Main
Focal
plane
F
r'/ε
r/ε
λ
$a = 1$
$b = 2$
$\varepsilon = \sqrt{5}$
$\lambda = 4$
r'
r
$b = 2$
ϕ
$\varepsilon^2 = 1 + b^2/a^2$
$\lambda = a(\varepsilon^2 - 1)$
F′
P′
P
F
$r_+ = \frac{\lambda}{1-\varepsilon}$
$r_- = \frac{\lambda}{1+\varepsilon}$
Left
Perigee
P′
Right
Perigee
P

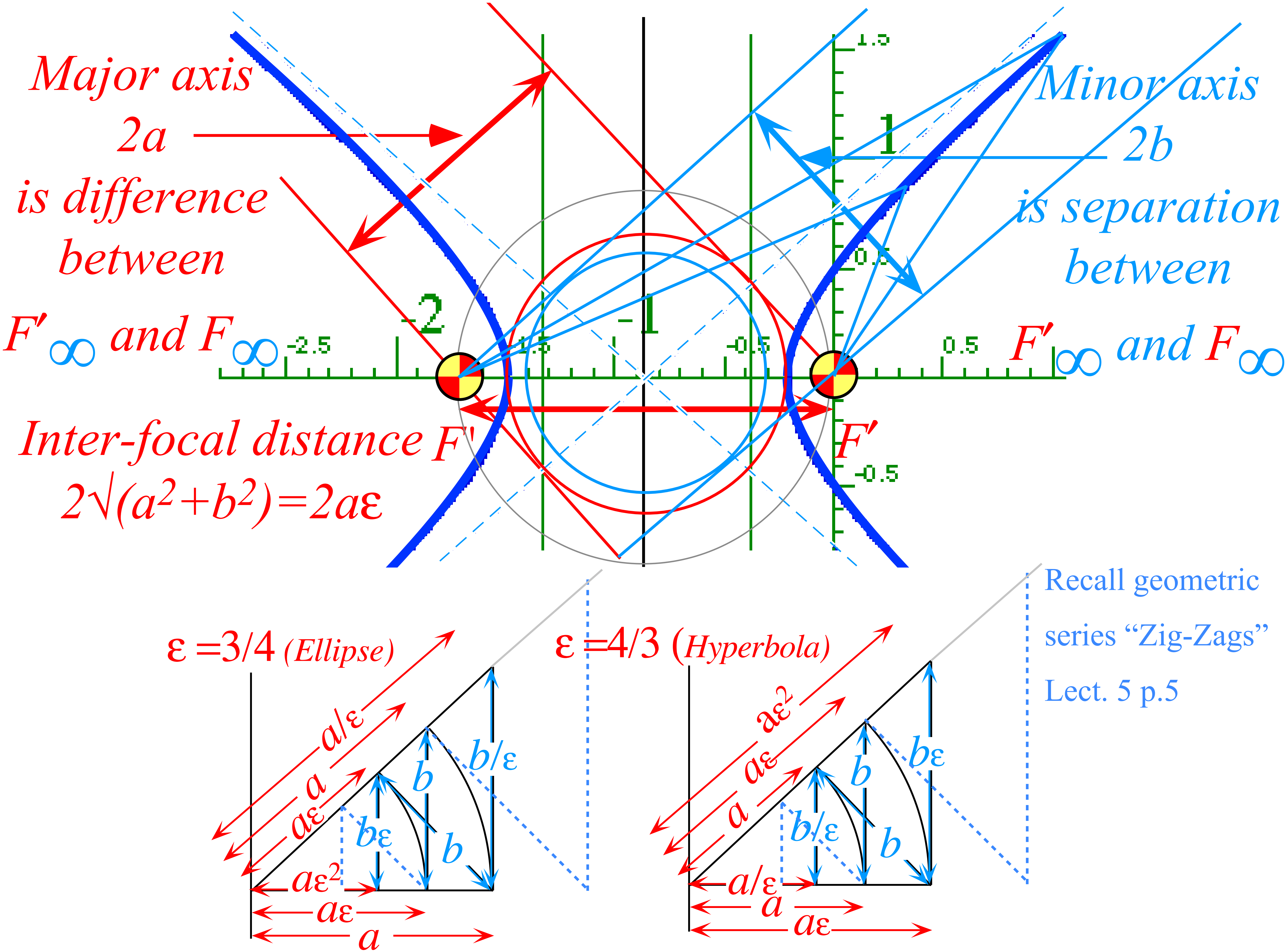


$r = \frac{\lambda}{(1-\varepsilon \cos\phi)}$
λ/ε
Prime Focal plane F′
Prime Directrix D′
Main Directrix D
Main Focal plane F
r′/ε
r/ε
λ
r′
r
a =1
b =2
ε =√5
λ =4
b =2
ϕ
ε²=1+b²/a²
λ = a(ε²-1)
F′
P′
P
F
$r_+ = \frac{\lambda}{1-\varepsilon}$
$r_- = \frac{\lambda}{1+\varepsilon}$
Left Perigee P′
Right Perigee P

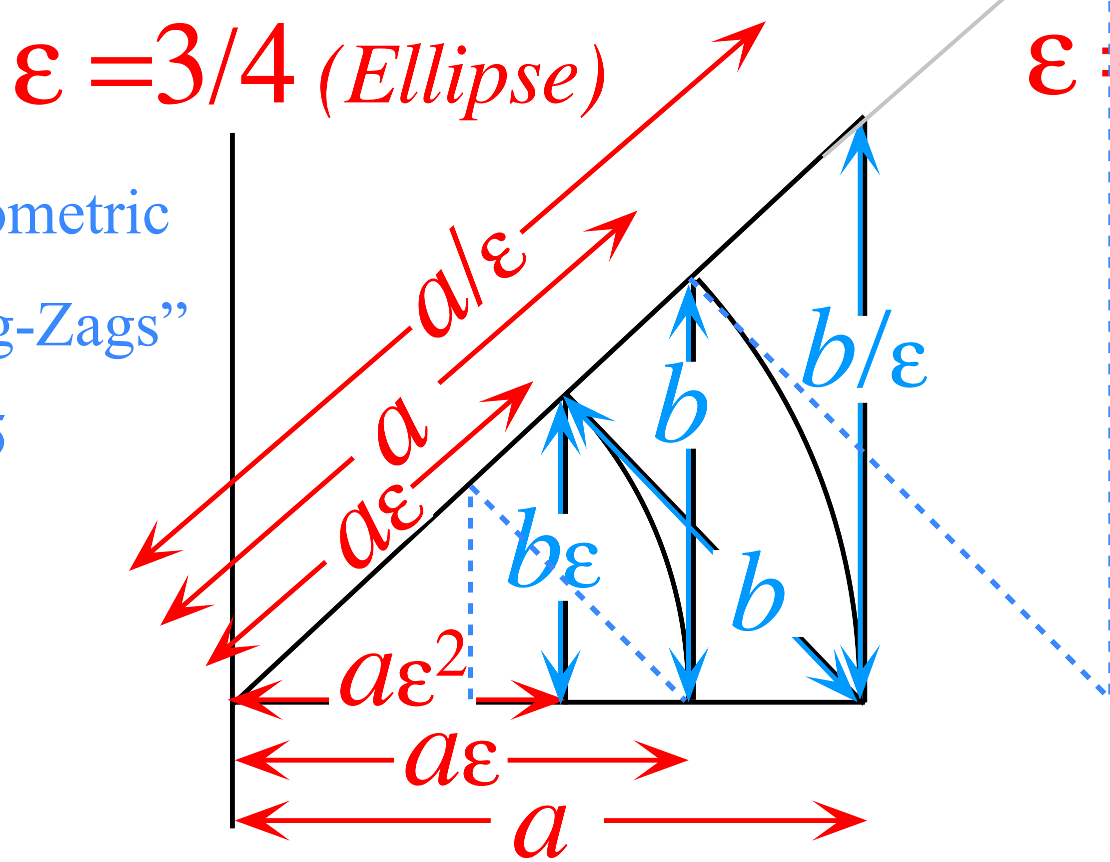


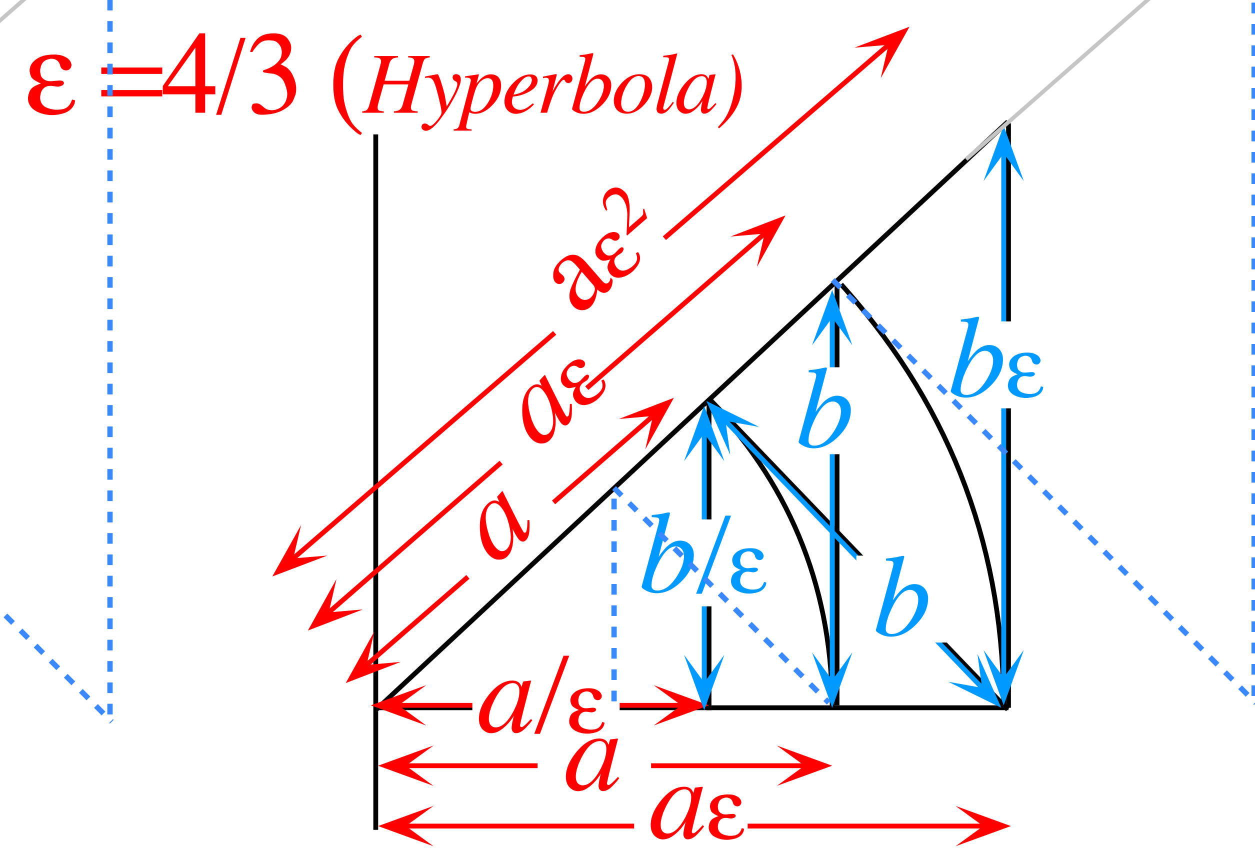
$r = \frac{\lambda}{(1-\varepsilon \cos\phi)}$
λ/ε
Prime Focal plane F′
Prime Directrix D′
Main Directrix D
Main Focal plane F
r'/ε
r/ε
λ
$a = 1$
$b = 2$
$\varepsilon = \sqrt{5}$
$\lambda = 4$
r'
r
$b = 2$
ϕ
$\varepsilon^2 = 1 + b^2/a^2$
$\lambda = a(\varepsilon^2 - 1)$
F′
P′
P
F
$r_+ = \frac{\lambda}{1-\varepsilon}$
$r_- = \frac{\lambda}{1+\varepsilon}$
Left Perigee P′
Right Perigee P


Major axis
2a
is difference
between
F′∞ and F∞
Minor axis
2b
is separation
between
F′∞ and F∞
Inter-focal distance F′
2√(a²+b²)=2aε
F′
-2
-1
1
-2.5
-1.5
-0.5
0.5
1.5
0.5
-0.5
ε =3/4 (Ellipse)
a/ε
a
aε
aε²
aε
a
bε
b
b
b/ε
ε =4/3 (Hyperbola)
aε²
aε
a
a/ε
a
aε
b/ε
b
b
bε


Major axis
2a
is difference
between
F′∞ and F∞
Minor axis
2b
is separation
between
F′∞ and F∞
Inter-focal distance F′F
2√(a²+b²)=2aε
ε =3/4 (Ellipse)
ε =4/3 (Hyperbola)
Recall geometric
series "Zig-Zags"
Lect. 5 p.5
a/ε
a
aε
aε²
b/ε
b
bε
aε²
aε
a
a/ε
b/ε
b
bε
-2.5
-2
-1.5
-1
-0.5
0.5
1.5
1
0.5
-0.5

Recall geometric series "Zig-Zags" Lect. 5 p.5





For the elliptic geometry *(ε<1)*:

$b^2 = a^2 - a^2\varepsilon^2 = a\lambda$,

$b = a\sqrt{(1-\varepsilon^2)} = \sqrt{(a\lambda)}$,

For hyperbolic geometry *(ε>1)*:

$b^2 = a^2\varepsilon^2 - a^2 = a\lambda$,

$b = a\sqrt{(\varepsilon^2-1)} = \sqrt{(a\lambda)}$.

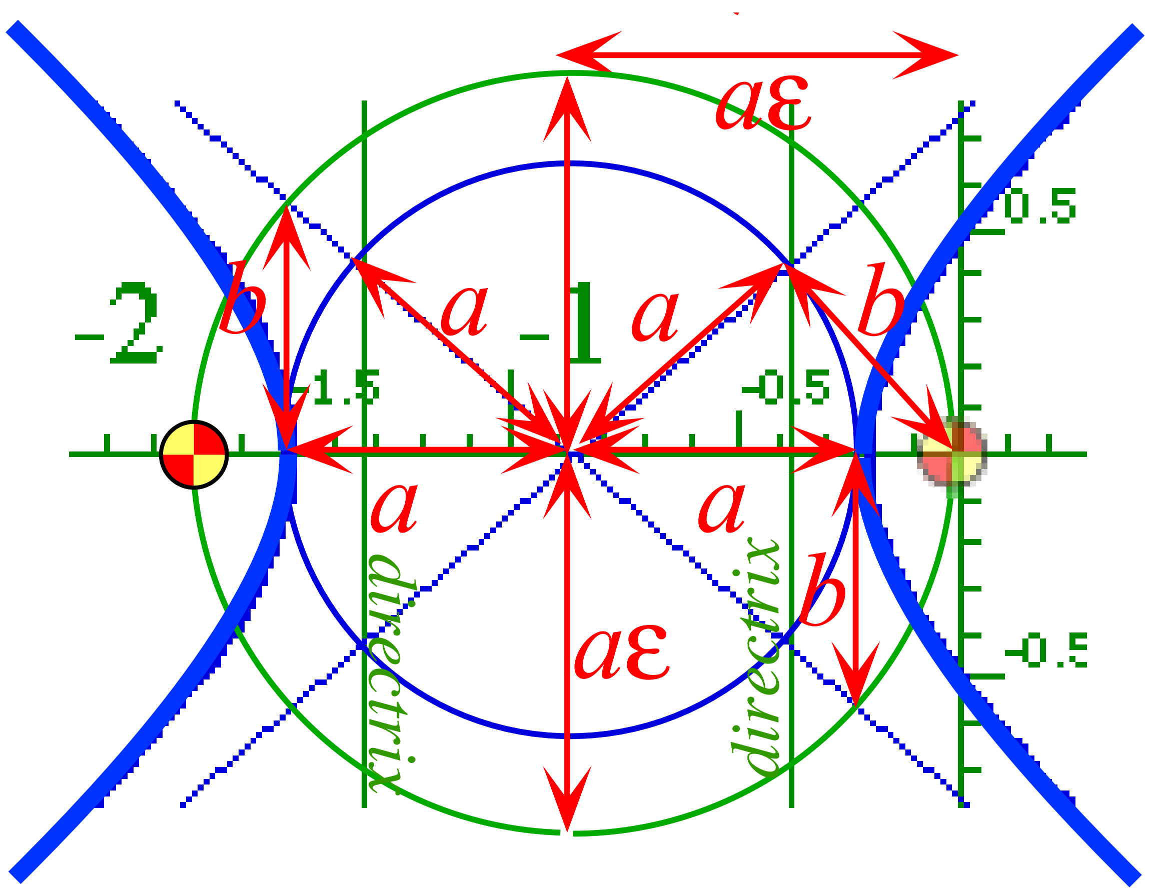
(λ,ε)-(a,b) expressions and their inverses follow.

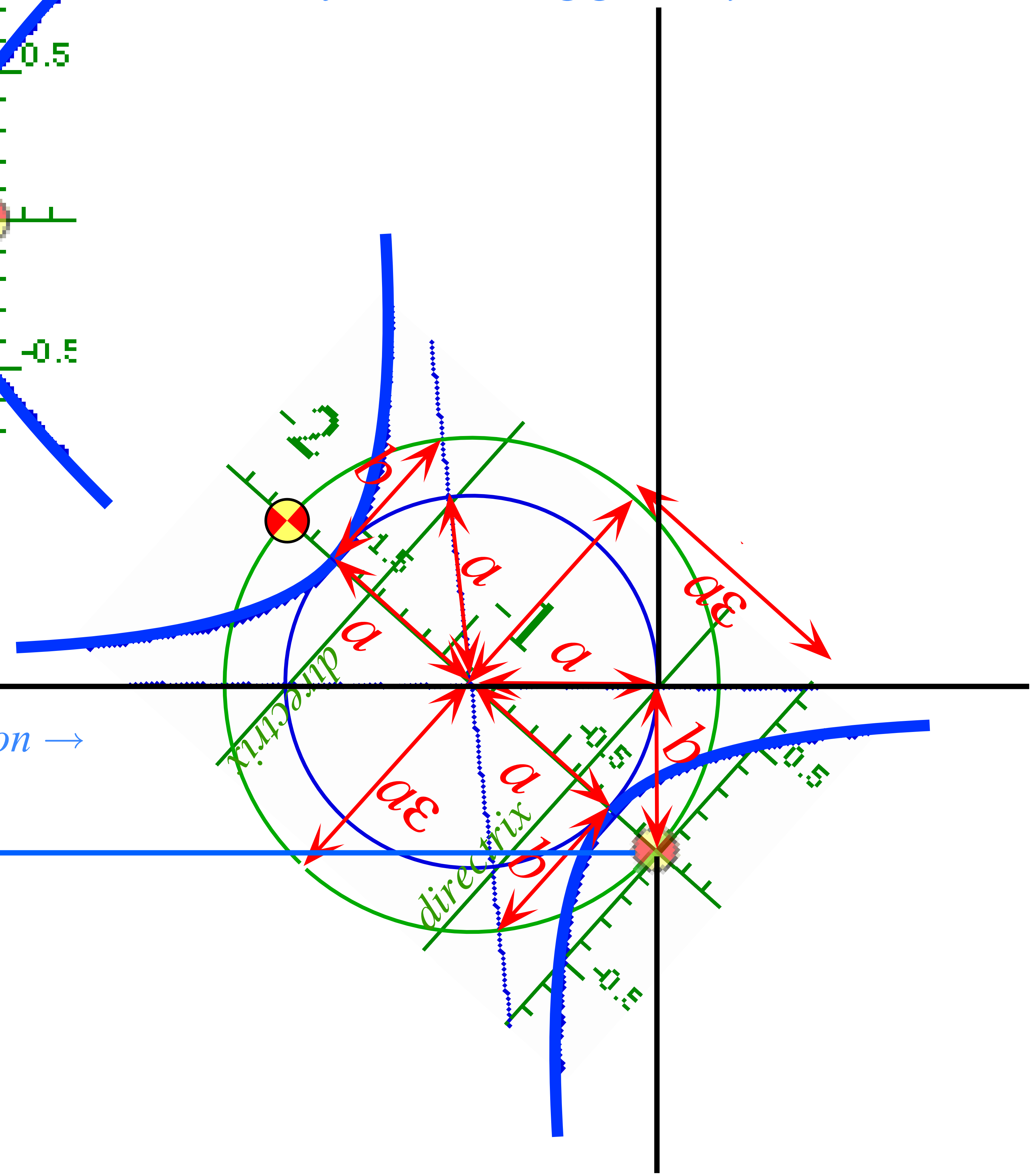
$a = \lambda/(1-\varepsilon^2)$	$a = \lambda/(\varepsilon^2-1)$
$b^2 = \lambda^2/(1-\varepsilon^2)$	$b^2 = \lambda^2/(\varepsilon^2-1)$
$\lambda = a(1-\varepsilon^2) = b^2/a$	$\lambda = a(\varepsilon^2-1) = b^2/a$
$\varepsilon^2 = 1 - b^2/a^2$	$\varepsilon^2 = 1 + b^2/a^2$

To be discussed
In next Lecture….

Cartesian Parameters		Physics		Polar Parameters
Semi-major axis $a=k/\|2E\|$	←	Energy E	→	Eccentricity $\varepsilon=\sqrt{(1+2\mu^2E/(k^2m))}$
Semi-minor axis $b=\mu/\sqrt{\|2mE\|}$	←	Angular momentum $\mu=\ell$	→	Latus radius $\lambda=\mu^2/(km)$

Rutherford scattering geometry...





To be discussed
In next Lecture….