Lecture 26

Mon. 11.19.2018

Geometry and Symmetry of Coulomb Orbital Dynamics

(Ch. 2-4 of Unit 5)

Rutherford scattering and hyperbolic orbit geometry

Backward vs forward scattering angles and orbit construction example

Parabolic "kite" and orbital envelope geometry

Differential and total scattering cross-sections

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{r}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and orbital radius* $\mathbf{r}$

Review and connection to usual orbital algebra (previous lecture)

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{p}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and momentum* $\mathbf{p}=m\mathbf{v}$

General geometric orbit construction using $\boldsymbol{\varepsilon}$*-vector and* (γ,R)*-parameters*

Derivation of $\boldsymbol{\varepsilon}$*-construction by analytic geometry*

Coulomb orbit algebra of $\boldsymbol{\varepsilon}$*-vector and Kepler dynamics of momentum* $\mathbf{p}=m\mathbf{v}$

Example of complete $(\mathbf{r},\mathbf{p})$*-geometry of elliptical orbit*

Connection formulas for (γ,R)*-parameters with (a,b) and* (ε,λ)

Excerpts from Lect. 27

A running collection of links to course-relevant sites and articles

Physics Web Resources

Comprehensive Harter-Soft Resource Listing

UAF Physics YouTube channel

LearnIt Physics Web Applications

"Texts"

Classical Mechanics with a Bang!

Quantum Theory for the Computer Age

Principles of Symmetry, Dynamics, and Spectroscopy

Modern Physics and its Classical Foundations

Classes

2014 AMOP

2017 Group Theory for QM

2018 AMOP

2018 Adv Mechanics

Neat external material to start the class:
- AIP publications
- AJP article on superball dynamics
- AAPT summer reading

These *are* hot off the presses:
- Sorting ultracold atoms in a 3D optical lattice in a realization of Maxwell's demon - Kumar-Nature-Letters-2018
- Synthetic three-dimensional atomic structures assembled atom by atom - Berredo-Nature-Letters-2018

Slightly Older ones:
- Wave–particle duality of C60 molecules
- Optical vortex knots – One Photon at a Time

AMOP Ch 0 Space-Time Symmetry - 2019
- Seminar at Rochester Institute of Optics, Auxiliary slides, June 19, 2018

Springer AMO Handbook - Ch 32 - Harter-Reimer-2019

"*Relawavity*" and quantum basis of *Lagrangian* & *Hamiltonian* mechanics:
- 2-CW laser wave - BohrIt Web App
- Lagrangian vs Hamiltonian - RelaWavity Web App

Older Links from Lectures 14-20

http://thearmchaircritic.blogspot.com/2011/11/punkin-chunkin.html
http://www.sussexcountyonline.com/news/photos/punkinchunkin.html
Shooting-range-for-medieval-siege-weapons-Anybody-knows
https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html
https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html?scenario=MontezumasRevenge
https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html?scenario=SeigeOfKenilworth
The trebuchet, Chevedden, Sci Am 1995
'*Simple*' *Pendulum* Sim: https://modphys.hosted.uark.edu/markup/PendulumWeb.html
'*Cycloid*' *Pendulum*: https://modphys.hosted.uark.edu/markup/CycloidulumWeb.html
Google search on: "Satelite view of Patricia" (Images)
Physics Girl Channel - Fun with Vortex Rings in the Pool
iBall demo - Quasi-periodicity: https://youtu.be/_jntDtULxDc
https://modphys.hosted.uark.edu/markup/CoulItWeb.html?scenario=SynchrotronMotion
https://modphys.hosted.uark.edu/markup/CoulItWeb.html?scenario=SynchrotronMotion2
Mechanical Analog to EM Motion (YouTube video) - https://youtu.be/hTd5FTJ-vRk
CoulIt Web Simulation: Bound-state motion in parabolic coordinates
CoulIt Web Simulation: Bound-state motion in hyperbolic coordinates
OscillIt Web App: Simulations of various types of resonance: 18, 27, 31, 35, 38, 39
Smith Chart
http://nobelprize.org/

AnalyIt Web Application, posted 10/22/2018 in our *testing* area:
https://modphys.hosted.uark.edu/testing/markup/AnalyItBJS.html

Older Links from Lectures 21-23

Advanced Atomic and Molecular Optical Physics 2018 Class #9, pages: 5, 61
BoxIt Web Simulations
- Pure A-Type w/Cosine
- Pure B-Type w/Cosine
- Pure B-Type w/Freq ratios
- Mixed AB-Type 2:1 Freq ratio
- Pure A-Type A=4.9, B=0 ,C=0, & D=4.0
- Pure B-Type: A=4.0, B=-0.2, C=0, & D=4.0
- Pure C-Type A,D=4.055, B=0, C=0.1
- Mixed AB-Type w/Cosine
- Mixed AB Type A=4.0, BU2=0.866…, CU2=0, & D=1.0 w/Stokes & Freq rats
- Mixed AB Type A=5.086 B=-0.27 C=0 D=2.024 w/Stokes plot
- Mixed ABC Type A=4.833 B=0.2403 C=0.4162 D=4.277 w/Stokes plot
- Recent mixed ABC Type A=0.325 B=0.375 C=0.825 D=0.05 w/Stokes plot

Classical Mechanics with a Bang! 2018
- Lectures 8, 9, 23 page 93
- Text Unit 6, page=27

ColorU2 for the Web - in development
Group Theory for Quantum Mechanics - 2017 Lectures: 6, 7, 8,
and the combined 9-10
Quantum Theory for the Computer Age Unit 3 Ch.7-10, page=90
Web based 3D & XR ($x \in \{A,M,V\}$, R=Reality) https://www.babylonjs.com/
Web based 3D graphics WebGL API (Graphics Layer modeled after OpenGL)
Wiki on Pafnuty Chebyshev

continued ↘

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2018 AMOP

2018 Adv Mechanics

Older Links from Lectures 24-25

JerkIt Web App: 2-, 2+, Amp50Omega147-, Amp50Omega296, Amp50Omega602, Gap(1)
MolVibes Web App: C3vN3
WaveIt Web App:
- Dim = 3 w/Wave Components;
- Static Char Table: 6, 12, 12(b), 16, 36, 256
- Quantum Carpet with N=20: Gaussian, Boxcar

Quantum Revivals of Morse Oscillators and Farey-Ford Geometry - Li-Harter-CPL-2015
QTCA Unit_5 Ch14 2013
Lester. R. Ford, Am. Math. Monthly 45,586(1938)
John Farey, Phil. Mag.(1816) Wolfram
Harter, J. Mol. Spec. 210, 166-182 (2001)
Harter, Li IMSS (2013)
Li, Harter, Chem.Phys.Letters (2015)
OscillatorPE Web App: IHO Scenario 2, Coulomb Scenario 3
RelaWavity Web App/Simulator/Calculator: Elliptical - IHO orbits

Links to supplement Lecture 26

CoulIt Web App Simulations: p19, p32, p72, p73, p92, R=-0.375, R=+0.5, Rutherford

➤ *Rutherford scattering and hyperbolic orbit geometry*

Backward vs forward scattering angles and orbit construction example

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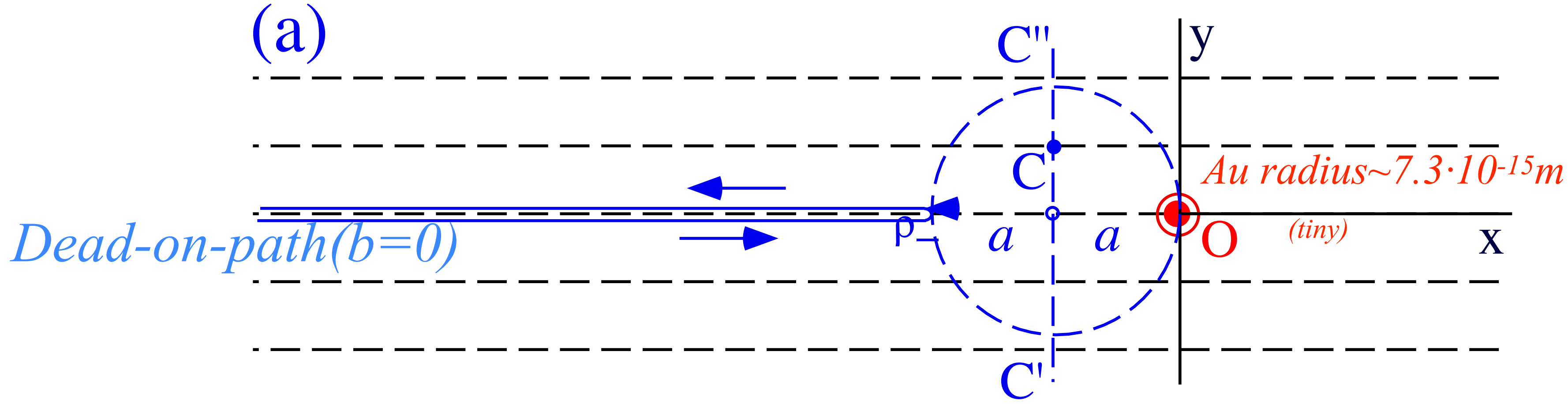
Example of complete $(\mathbf{r},\mathbf{p})$*-geometry of elliptical orbit*

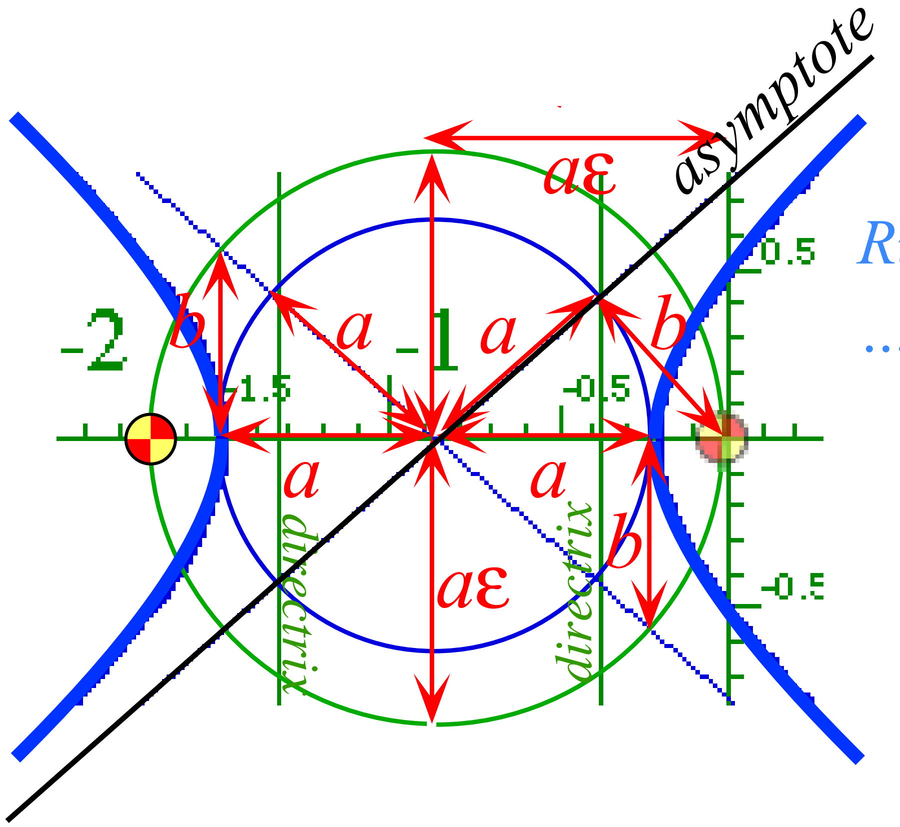
Connection formulas for (γ,R)*-parameters with* (a,b) *and* (ε,λ)



Rutherford scattering of α^{+2} particles from Au^{+79} nucleus at O

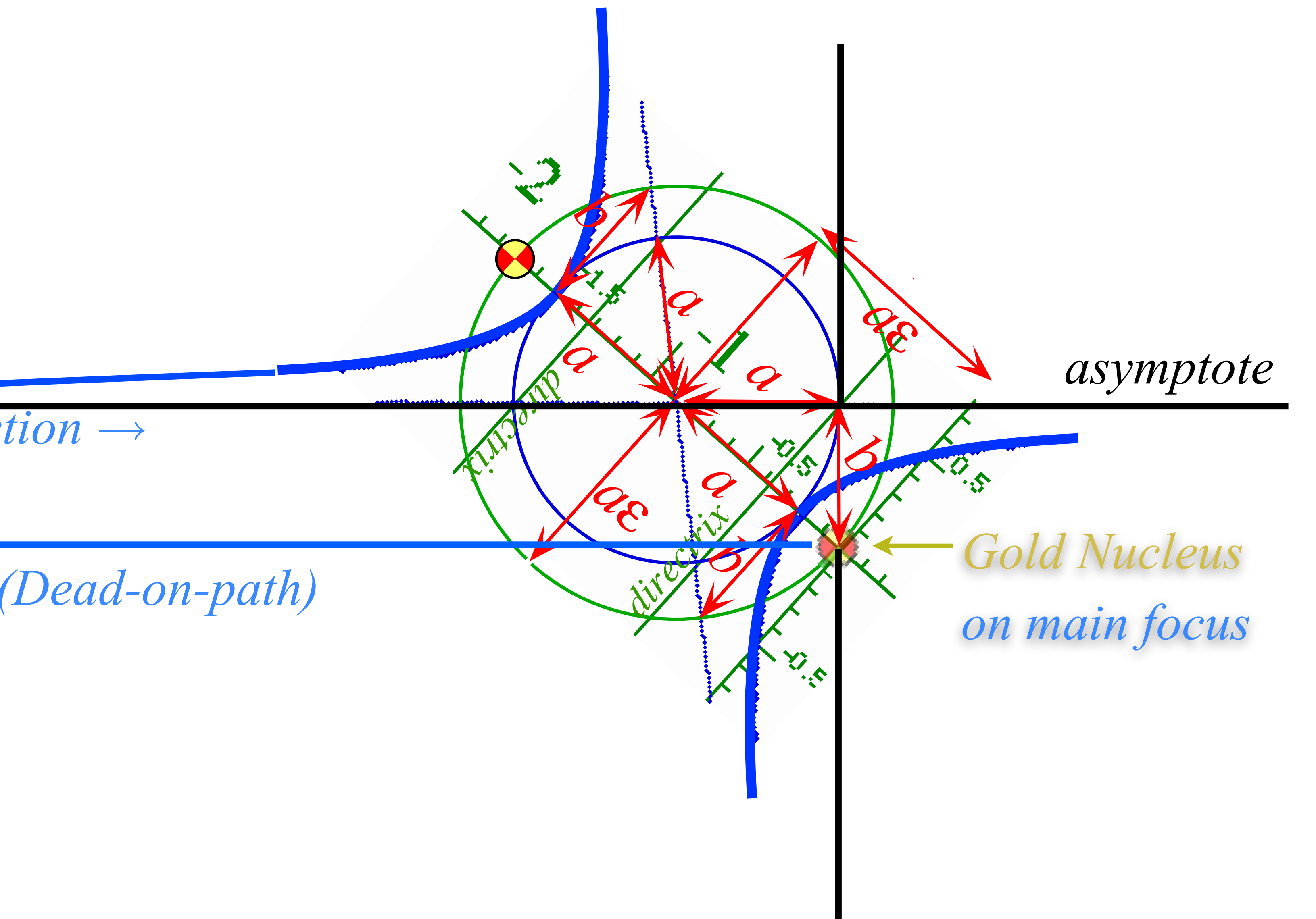
Assume "Dead-On" closest approach 2a.

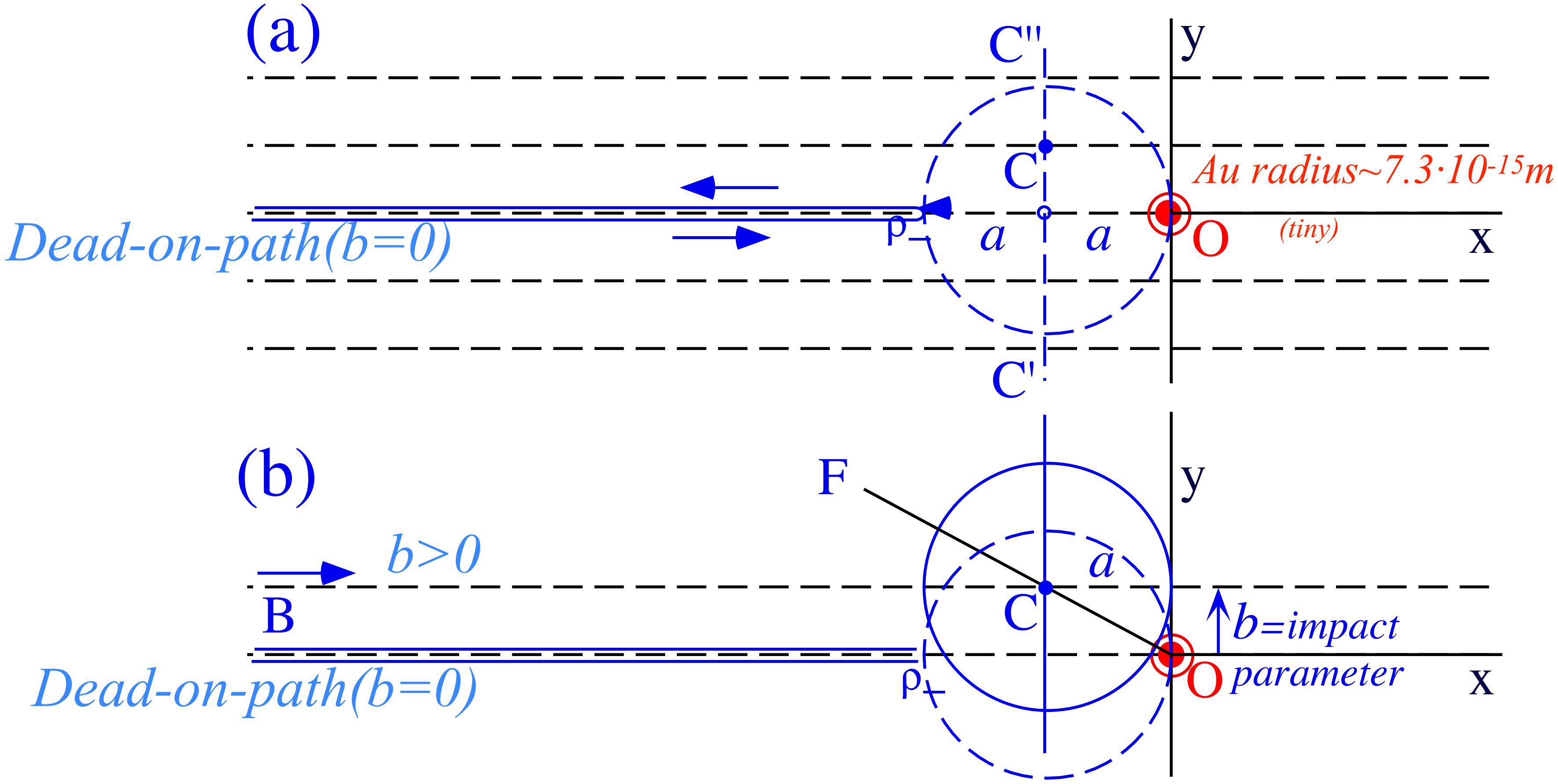
(E=k/2a) *a~10^{-11}m >>7.3·10^{-15}m*



Rutherford scattering geometry...

...(rotated so asymptote lies on an alpha path at -∞)





Rutherford scattering of α^{+2} *particles from* Au^{+79} *nucleus at* O

Assume "Dead-On" closest approach 2a.

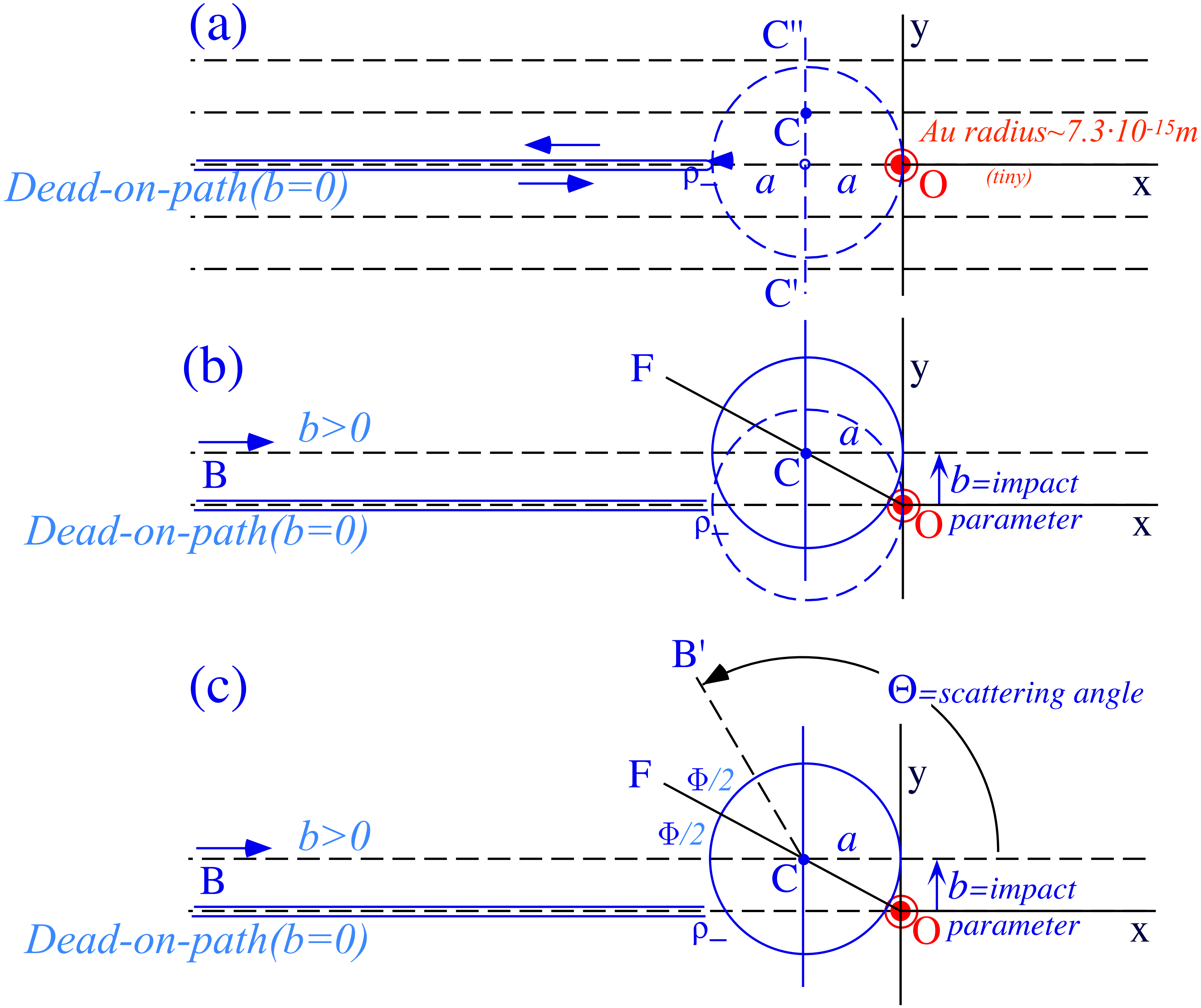
(E=k/2a) $a \sim 10^{-11}m >> 7.3 \cdot 10^{-15}m$

Pick an "impact parameter" line y =b.

Draw circle of radius ***a*** *around*

center point C*=(-a,b) tangent to y-axis.*

Draw "focus-locus" line OCF.



Rutherford scattering of α^{+2} *particles from* Au^{+79} *nucleus at* O

Assume "Dead-On" closest approach $2a$.

$(E=k/2a)$ $a \sim 10^{-11}m >> 7.3 \cdot 10^{-15}m$

Pick an "impact parameter" line $y = b$.

Draw circle of radius a *around center point* $C=(-a,b)$ *tangent to* y*-axis.*

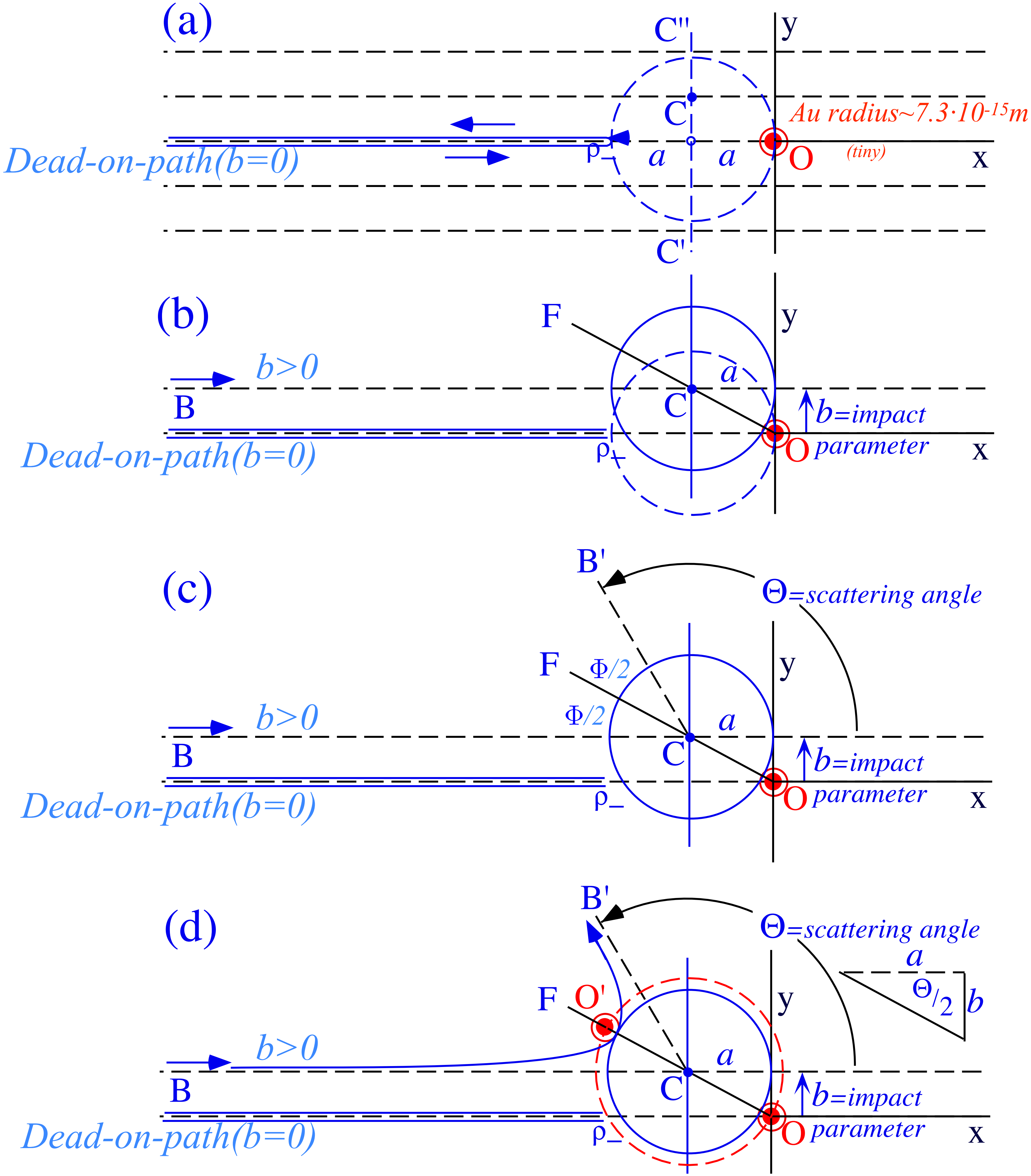
Draw "focus-locus" line OCF.

Copy angle ∠BCF (*equal to* $\Phi/2$)

to make angle ∠FCB′ (*also equal to* $\Phi/2$)

Resulting line CB′ *is outgoing asymptote*

at scattering angle Θ .



Rutherford scattering of α^{+2} particles from Au^{+79} nucleus at O

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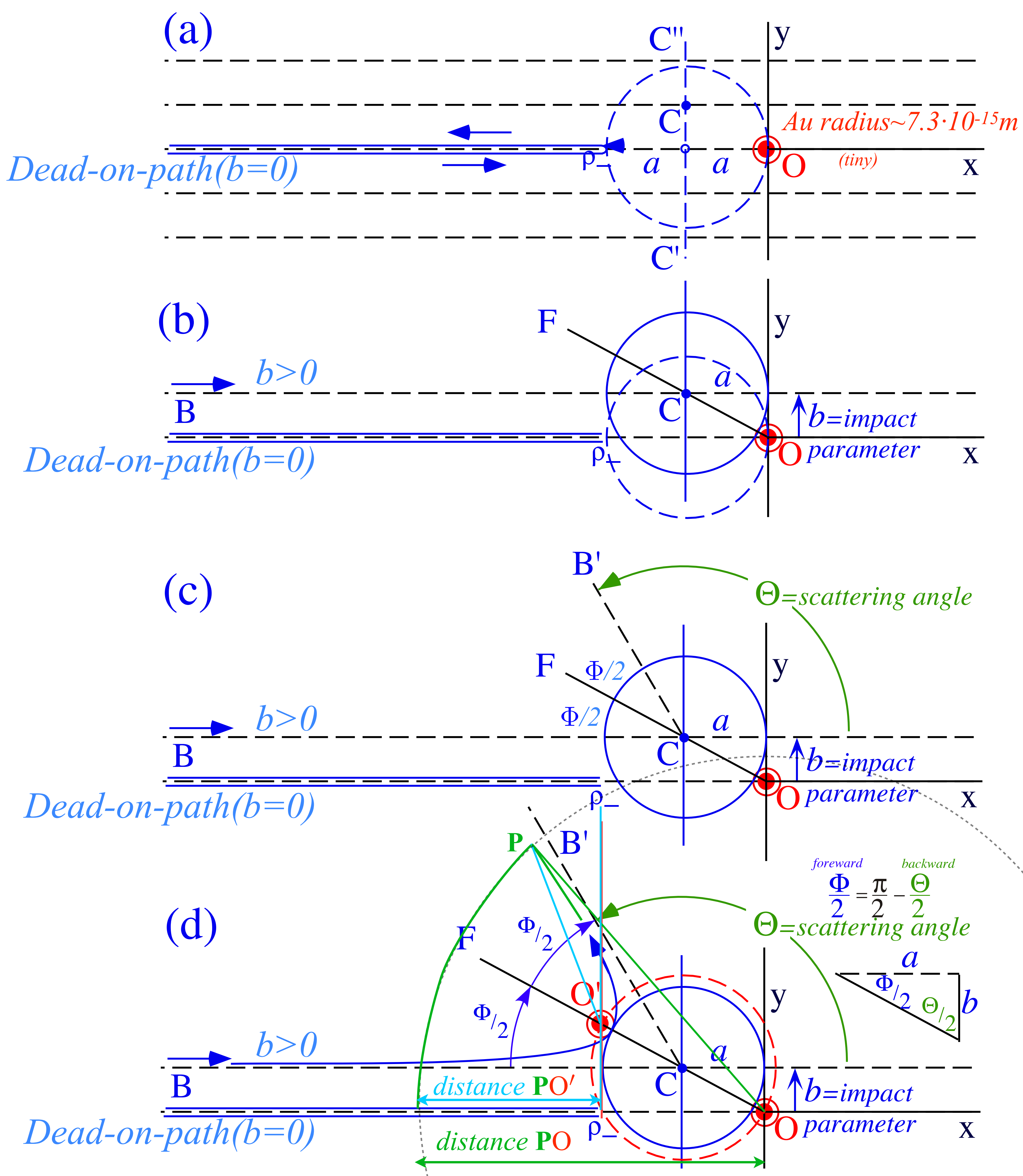
(E=k/2a) $a \sim 10^{-11}m >> 7.3\cdot 10^{-15}m$

Pick an "impact parameter" line y =b.
Draw circle of radius a around center point C=(-a,b) tangent to y-axis.
Draw "focus-locus" line OCF.

Copy angle ∠BCF (equal to Φ/2) to make angle ∠FCB′ (also equal to Φ/2)
Resulting line CB′ is outgoing asymptote at scattering angle Θ .

Locate secondary focus O′ by drawing circle around point C of diameter CO thru point O. Diameter O′CO is 2aε.
Hyperbolic orbit points **P** *now found using constant 2a=***P***O-***P***O′*



Rutherford scattering of α^{+2} *particles from* Au^{+79} *nucleus at* O

Assume "Dead-On" closest approach 2a.

$(E=k/2a)$ $\quad a\sim 10^{-11}m >> 7.3\cdot 10^{-15}m$

Pick an "impact parameter" line y = b.
Draw circle of radius a around center point C=(-a,b) *tangent to y-axis.*
Draw "focus-locus" line OCF.

Copy angle ∠BCF (*equal to* Φ/2) *to make angle* ∠FCB′ (*also equal to* Φ/2)
Resulting line CB′ *is outgoing asymptote at scattering angle* Θ .

Locate secondary focus O′ *by drawing circle around point* C *of diameter* CO *thru point* O. *Diameter* O′CO *is 2aε.*
Hyperbolic orbit points **P** *now found using constant 2a*=**P**O-**P**O′

Rutherford scattering and hyperbolic orbit geometry

➔ *Backward vs forward scattering angles and orbit construction example*

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Differential and total scattering cross-sections

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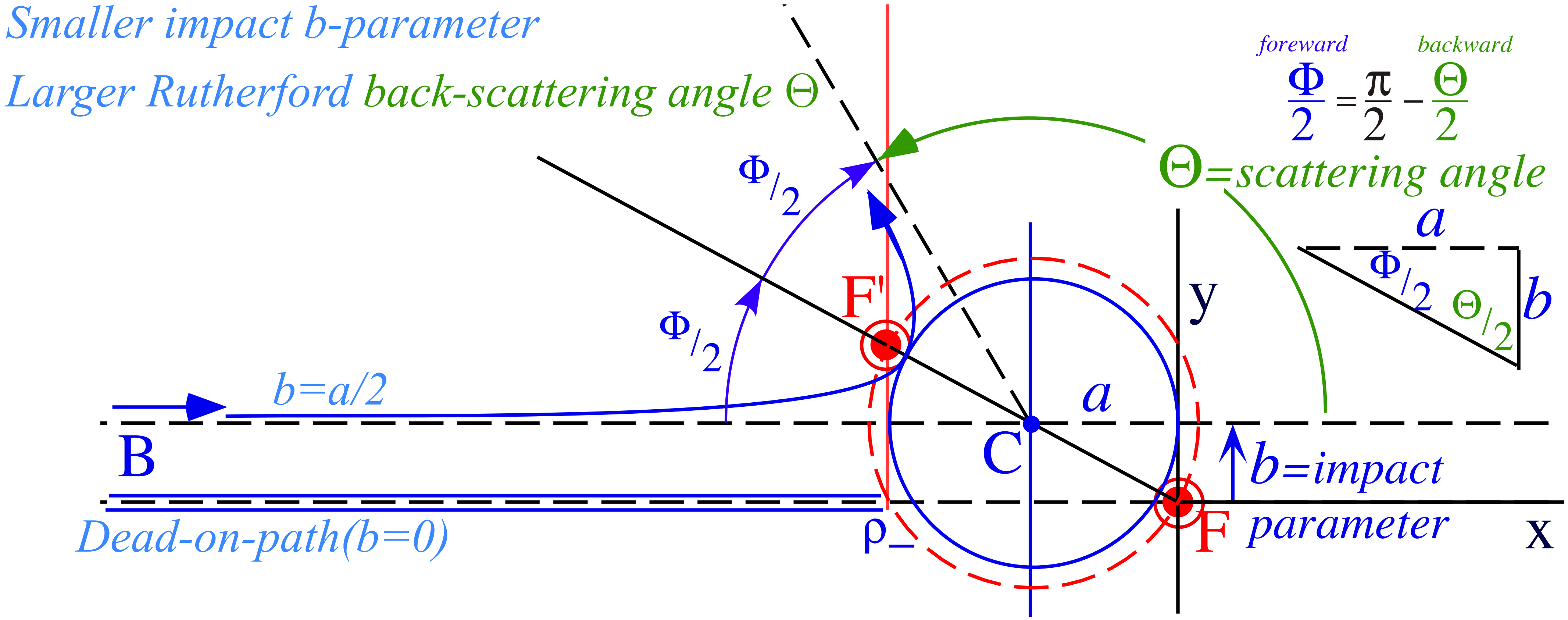
Derivation of **ε***-construction by analytic geometry*

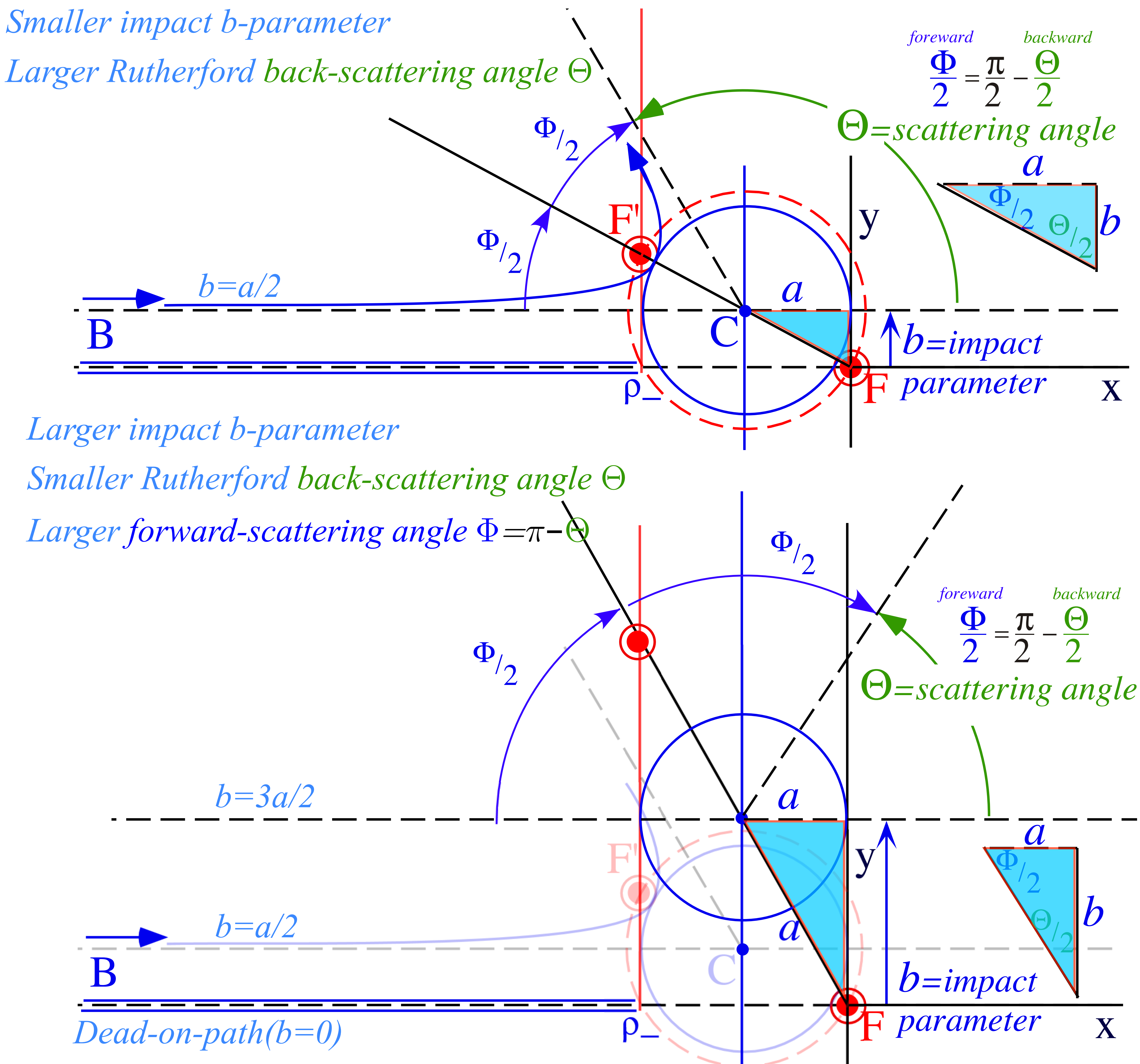
Coulomb orbit algebra of **ε***-vector and Kepler dynamics of momentum* **p**=*m***v**

*Example of complete (***r***,***p***)-geometry of elliptical orbit*

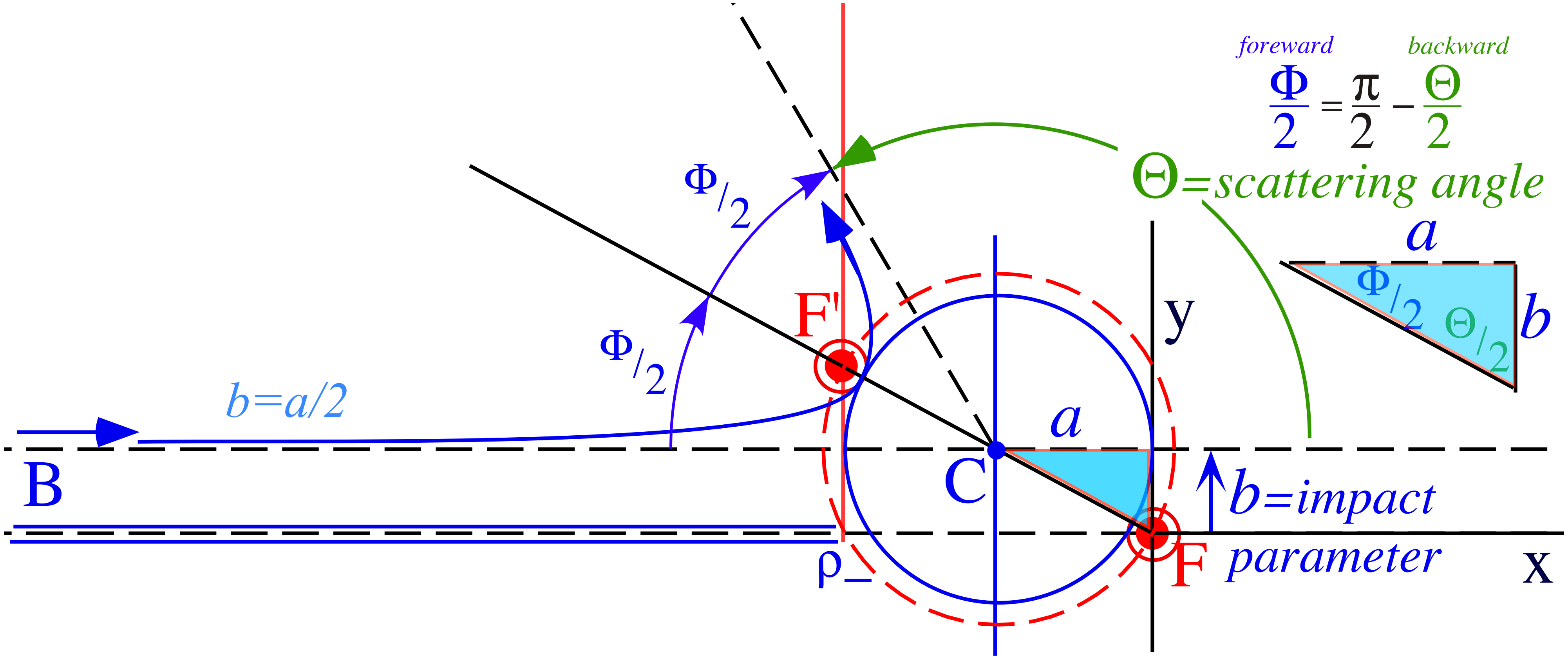
Connection formulas for (γ,R)-parameters with (a,b) and (ε,λ)



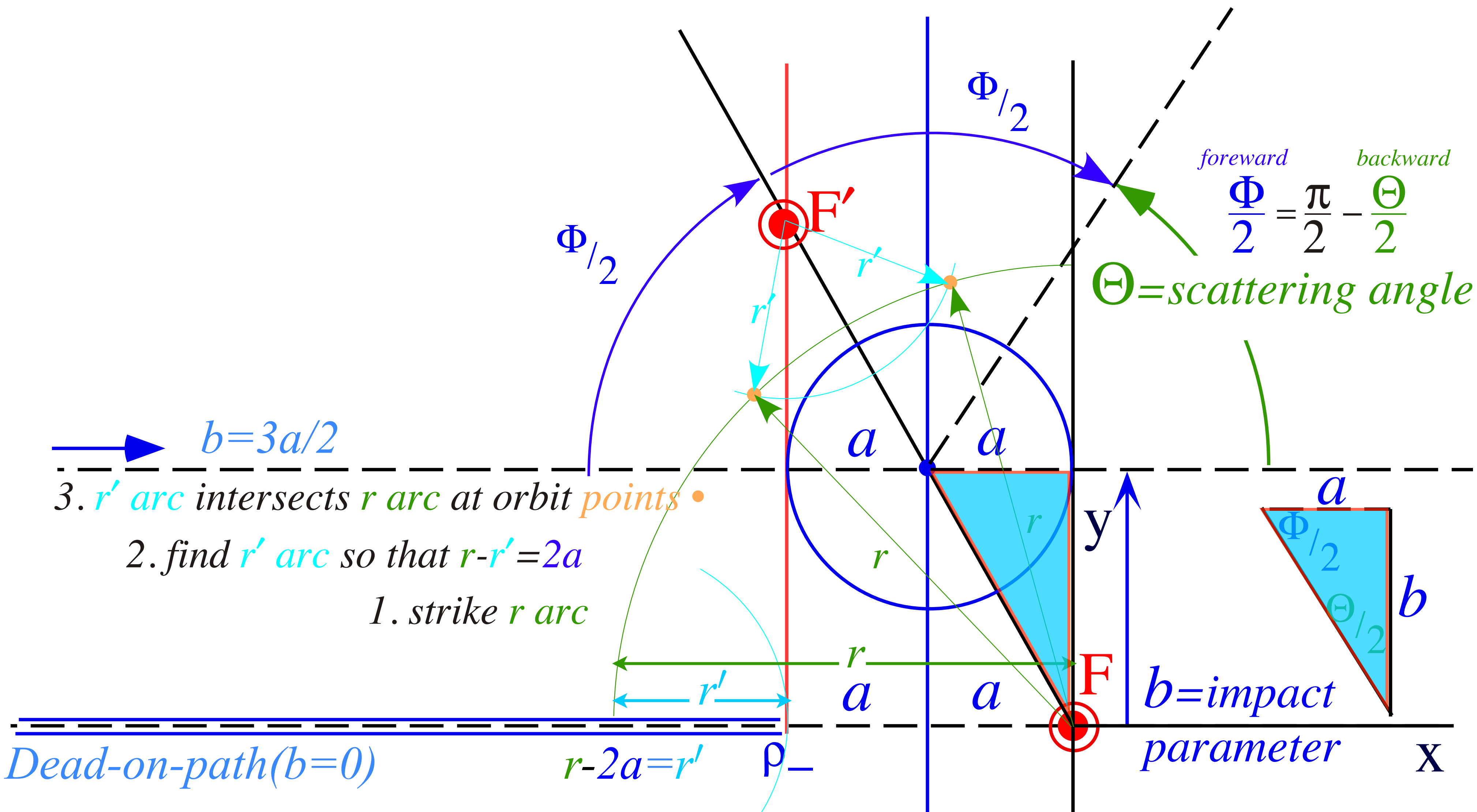
Smaller impact b-parameter
Larger Rutherford back-scattering angle Θ
foreward
backward
Φ/2 = π/2 − Θ/2
Θ=scattering angle
Φ/2
Φ/2
F'
a
Φ/2
Θ/2
b
y
b=a/2
a
B
C
b=impact
parameter
Dead-on-path(b=0)
ρ_
F
x



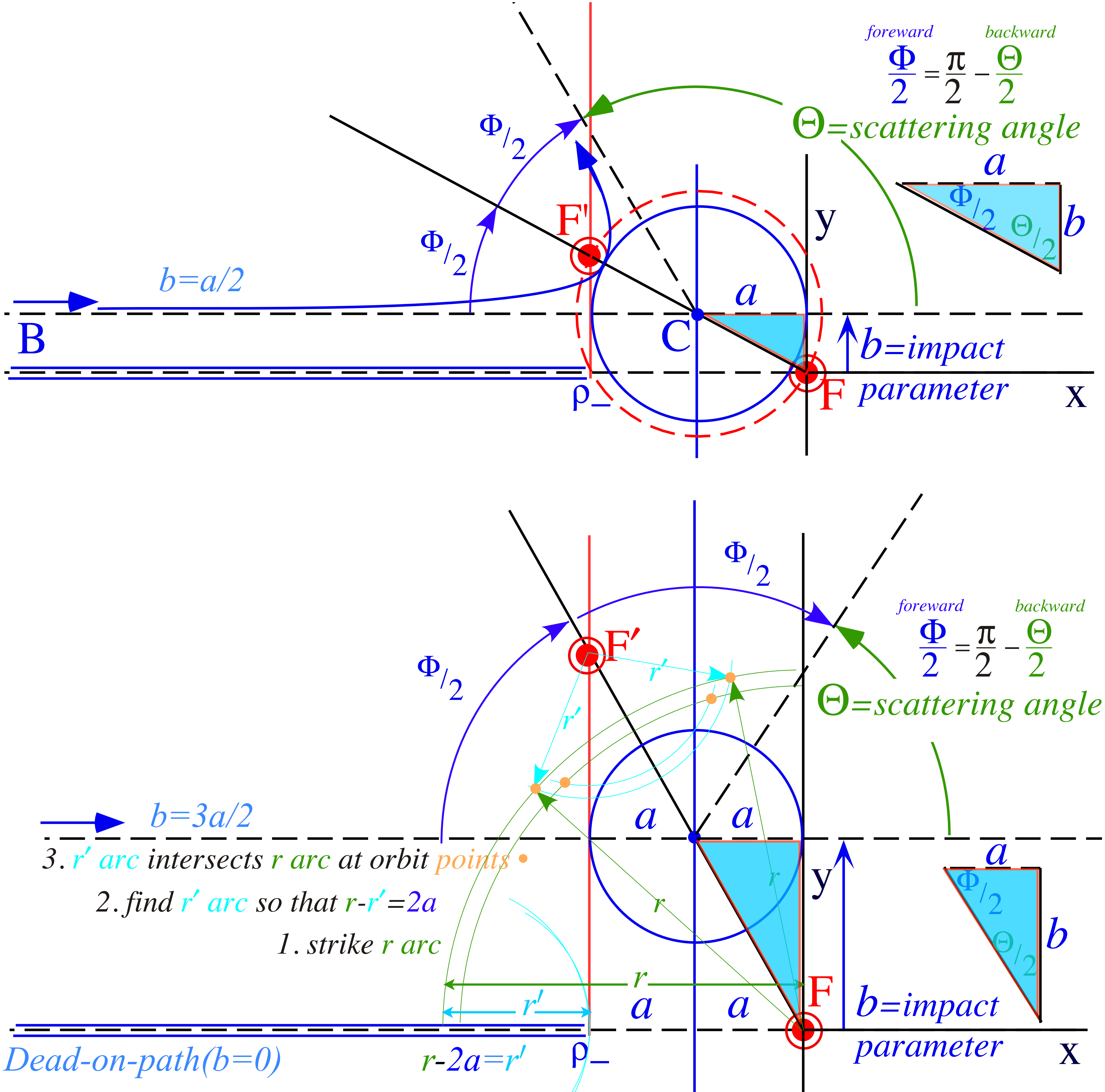
Smaller impact b-parameter
Larger Rutherford back-scattering angle Θ
foreward backward
Φ/2 = π/2 − Θ/2
Θ=scattering angle
Φ/2
Φ/2
F'
a
y
b
b=a/2
a
B
C
b=impact
parameter
ρ_
F
x
Larger impact b-parameter
Smaller Rutherford back-scattering angle Θ
Larger forward-scattering angle Φ=π−Θ
Φ/2
foreward backward
Φ/2 = π/2 − Θ/2
Φ/2
Θ=scattering angle
b=3a/2
a
F'
a
y
b
b=a/2
a
B
C
b=impact
parameter
ρ_
F
x
Dead-on-path(b=0)



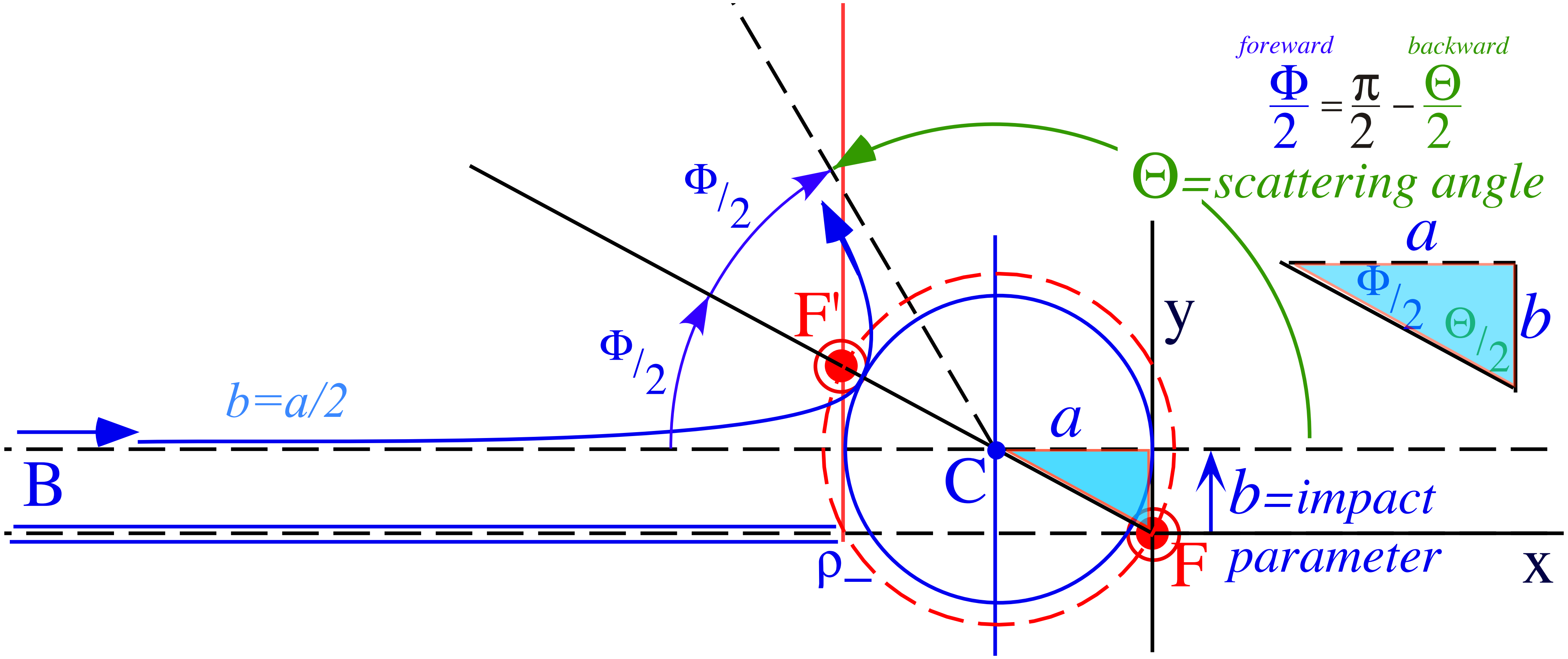
foreward
backward
Φ/2 = π/2 − Θ/2
Θ=scattering angle
Φ/2
Φ/2
F'
b=a/2
B
a
C
y
b
Θ/2
b=impact parameter
F
ρ_
x

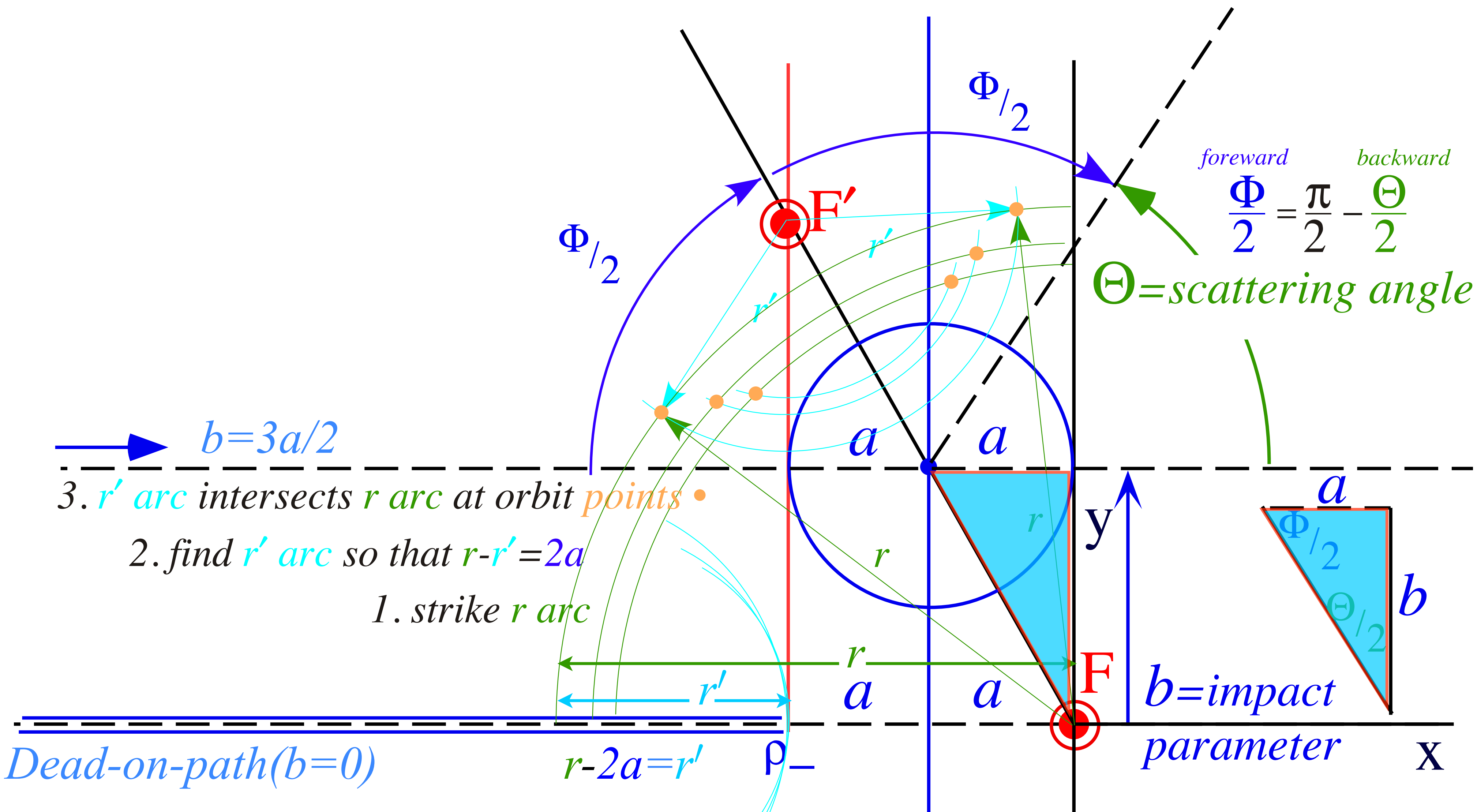


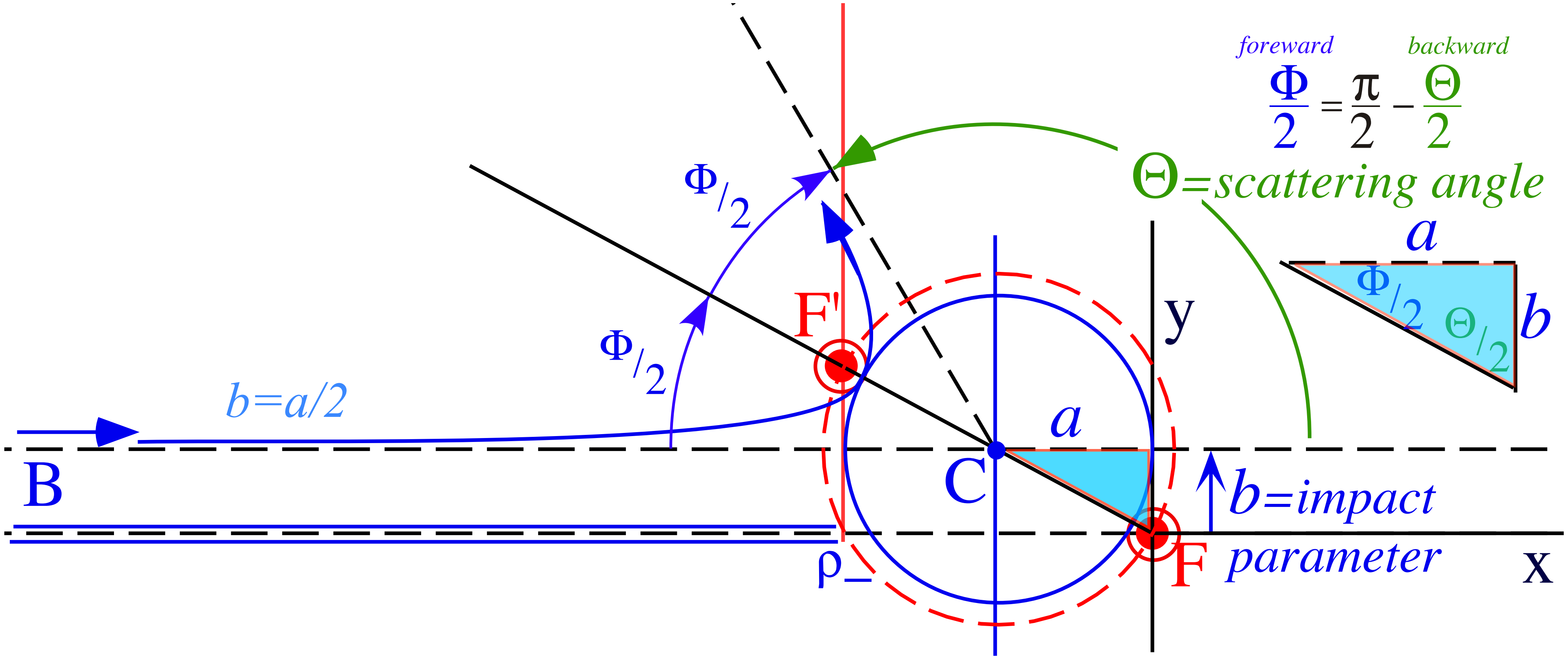
Φ/2
foreward
backward
Φ/2 = π/2 − Θ/2
F′
Φ/2
Θ=scattering angle
r′
r′
b=3a/2
a
a
3. r′ arc intersects r arc at orbit points •
2. find r′ arc so that r-r′=2a
1. strike r arc
r
r
y
a
b
Θ/2
r
F
r′
a
a
b=impact parameter
Dead-on-path(b=0)
r-2a=r′
ρ_
x

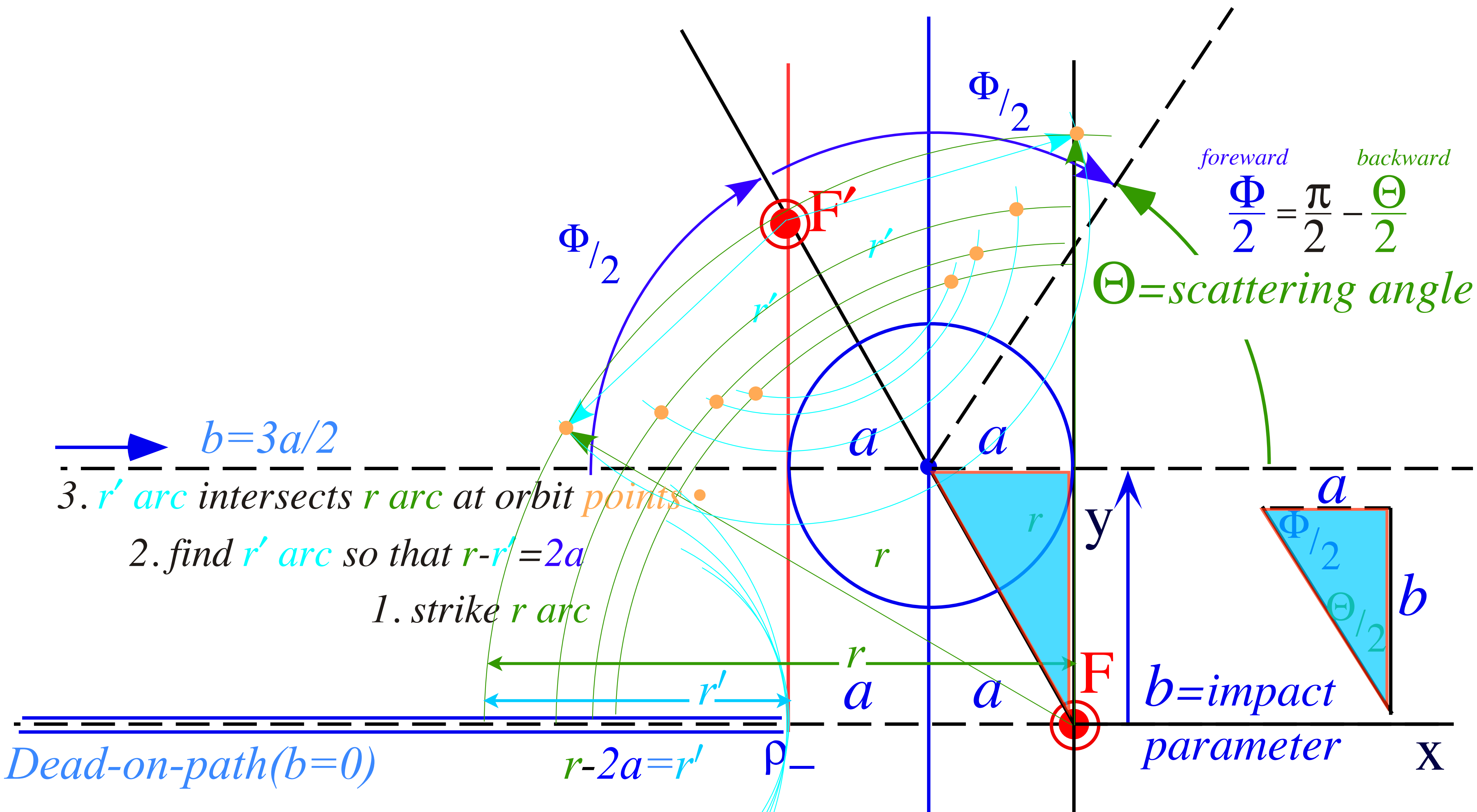


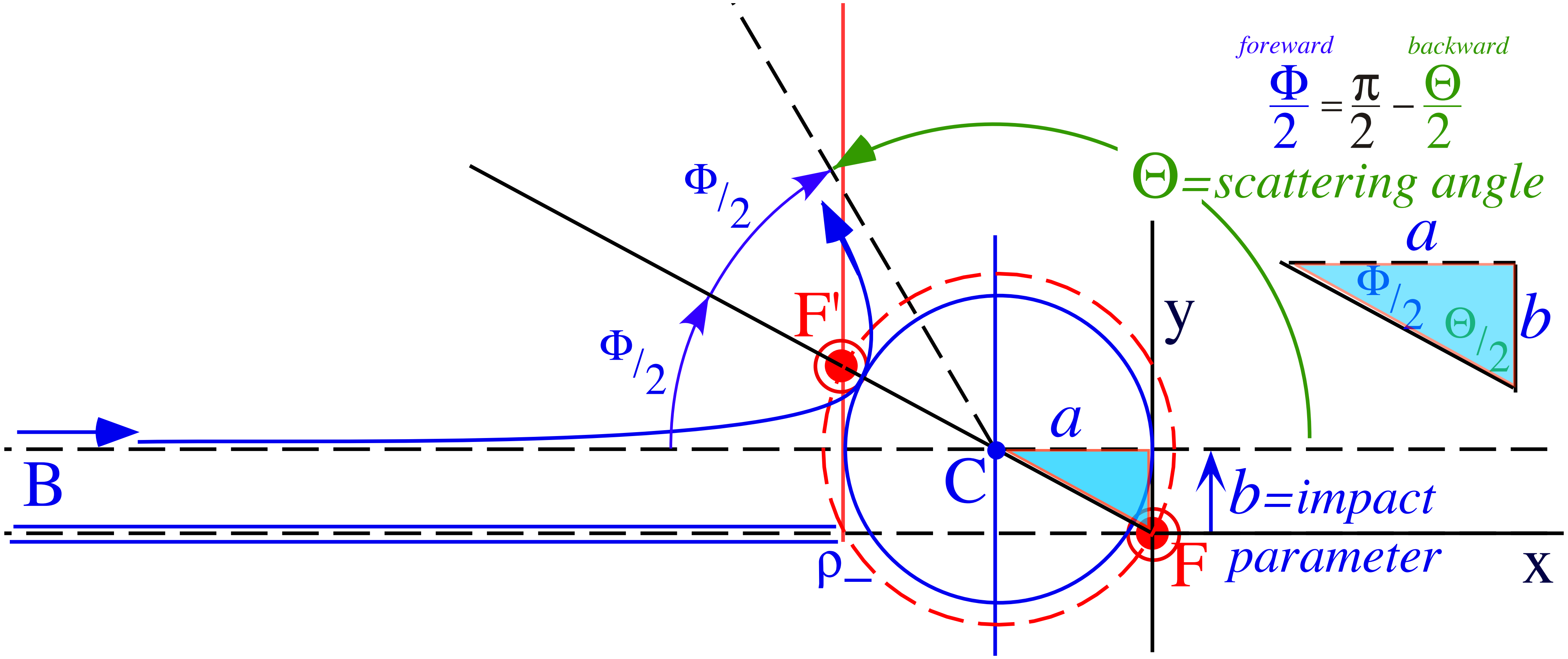
foreward
backward
Φ/2 = π/2 − Θ/2
Θ=scattering angle
a
b
F'
b=a/2
B
C
y
b=impact
parameter
F
x
ρ_
Φ/2
foreward
backward
Φ/2 = π/2 − Θ/2
Θ=scattering angle
F′
r′
b=3a/2
3. r′ arc intersects r arc at orbit points
2. find r′ arc so that r-r′=2a
1. strike r arc
r
r-2a=r′
Dead-on-path(b=0)


foreward
backward
$\frac{\Phi}{2} = \frac{\pi}{2} - \frac{\Theta}{2}$
Θ=scattering angle
Φ/2
Φ/2
F'
a
b
Φ/2
Θ/2
y
b=a/2
a
B
C
b=impact
parameter
ρ_
F
x

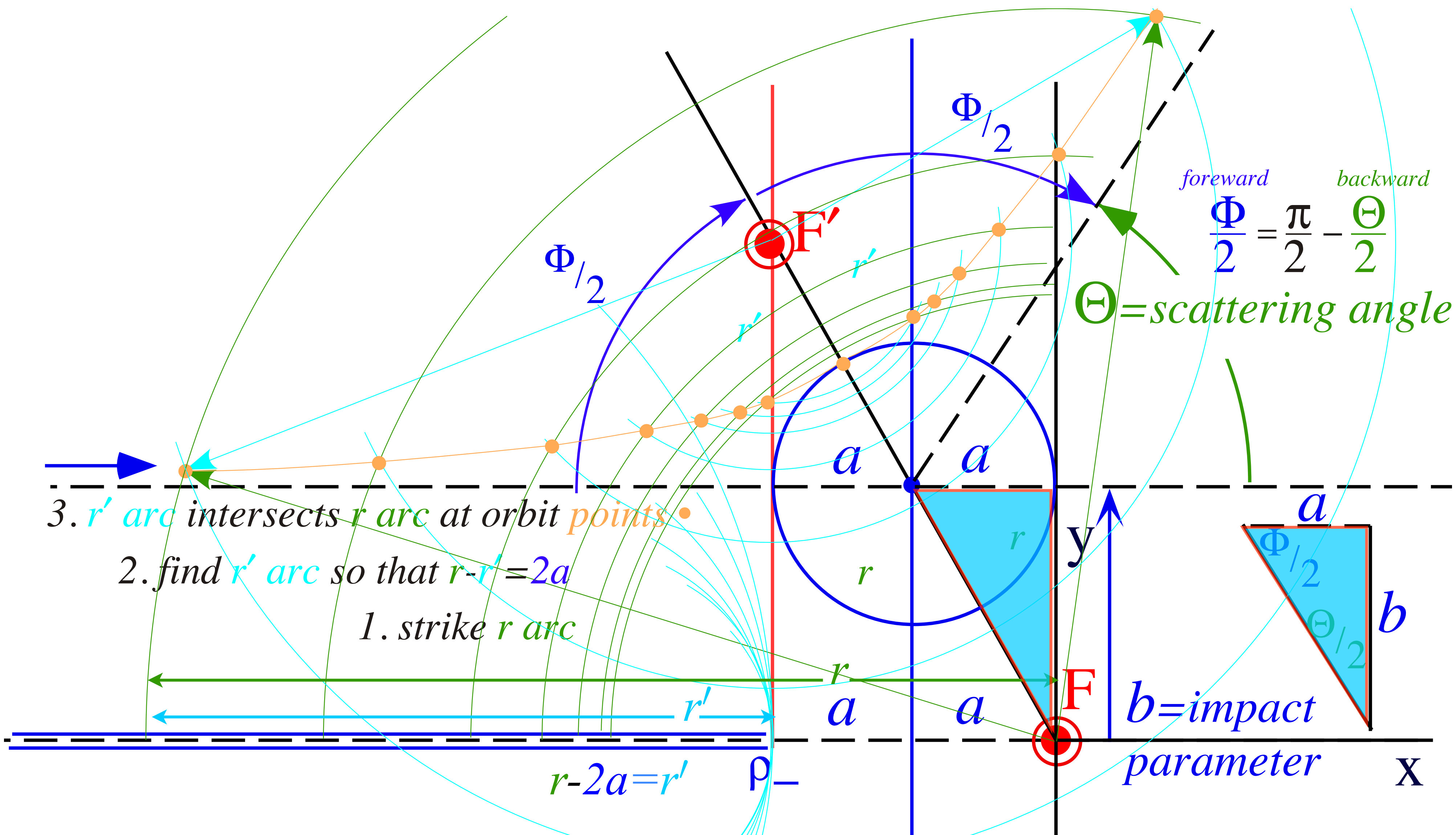

Φ/2
foreward
backward
$\frac{\Phi}{2} = \frac{\pi}{2} - \frac{\Theta}{2}$
Φ/2
F′
r′
r′
Θ=scattering angle
b=3a/2
a
a
a
3. r′ arc intersects r arc at orbit points
2. find r′ arc so that r-r′=2a
1. strike r arc
r
r
y
Φ/2
Θ/2
b
r
F
r′
a
a
b=impact
parameter
x
Dead-on-path(b=0)
r-2a=r′
ρ_


foreward
backward
$\frac{\Phi}{2} = \frac{\pi}{2} - \frac{\Theta}{2}$
Θ=scattering angle
Φ/2
Φ/2
F'
a
Φ/2
Θ/2
b
y
b=a/2
B
a
C
b=impact
parameter
ρ_
F
x

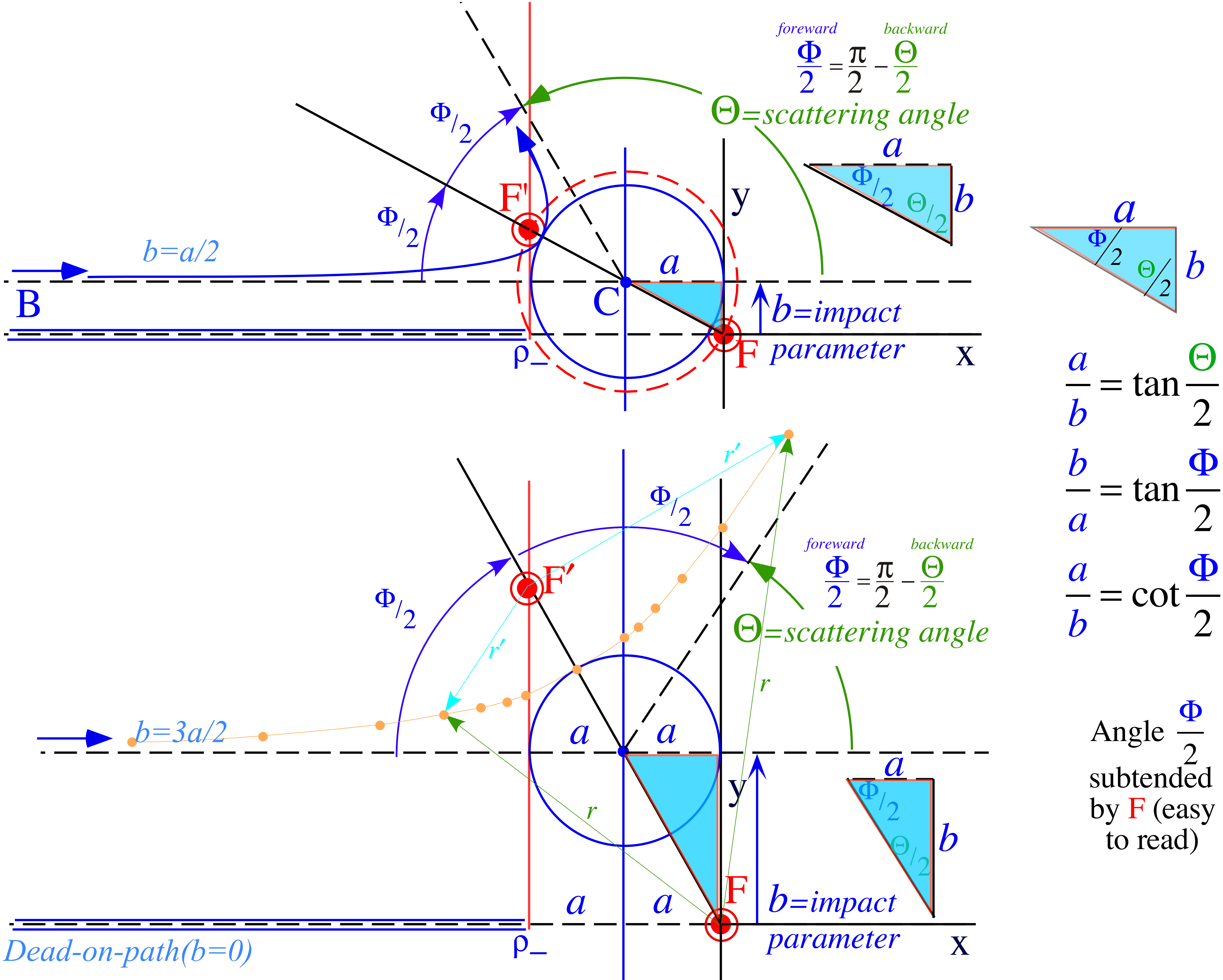

Φ/2
foreward
backward
$\frac{\Phi}{2} = \frac{\pi}{2} - \frac{\Theta}{2}$
F′
Φ/2
Θ=scattering angle
b=3a/2
a
a
3. r′ arc intersects r arc at orbit points
2. find r′ arc so that r-r′=2a
1. strike r arc
r
r
y
a
Φ/2
Θ/2
b
r
r′
a
a
F
b=impact
parameter
Dead-on-path(b=0)
r-2a=r′
ρ_
x



foreward
backward
Φ/2 = π/2 − Θ/2
Θ=scattering angle
Φ/2
Φ/2
F'
a
b
Φ/2
Θ/2
y
a
B
C
b=impact
parameter
F
ρ_
x



Φ/2
foreward
backward
Φ/2 = π/2 − Θ/2
F′
Φ/2
r′
r′
Θ=scattering angle
a
a
3. r′ arc intersects r arc at orbit points
2. find r′ arc so that r-r′=2a
1. strike r arc
r
r
a
y
Φ/2
Θ/2
b
r
r′
a
a
F
b=impact
parameter
r-2a=r′
ρ_
x


foreward
backward
Φ/2 = π/2 − Θ/2
Θ=scattering angle
Φ/2
Φ/2
F'
b=a/2
B
C
a
y
b=impact
parameter
F
ρ_
x
a
b
Φ/2
Θ/2
a/b = tan Θ/2
b/a = tan Φ/2
a/b = cot Φ/2
r'
Φ/2
F'
foreward
backward
Φ/2 = π/2 − Θ/2
Θ=scattering angle
r
b=3a/2
a
a
r
y
F
b=impact
parameter
x
ρ_
Dead-on-path(b=0)
Angle Φ/2 subtended by F (easy to read)

Rutherford scattering and hyperbolic orbit geometry

Backward vs forward scattering angles and orbit construction example

➤ *Parabolic "kite" and orbital envelope geometry*

Differential and total scattering cross-sections

Eccentricity vector **ε** *and (ε,λ)-geometry of orbital mechanics*

Projection **ε•r** *geometry of* **ε***-vector and orbital radius* **r**

Review and connection to usual orbital algebra (previous lecture)

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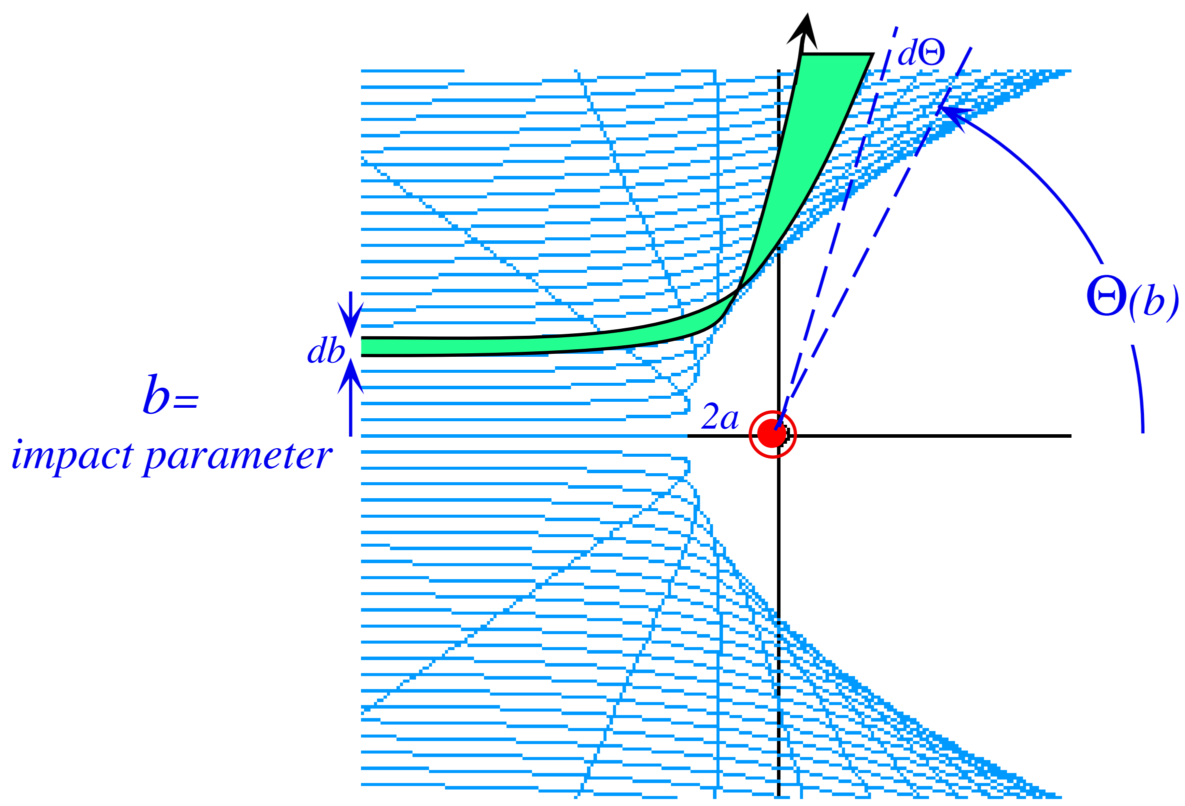
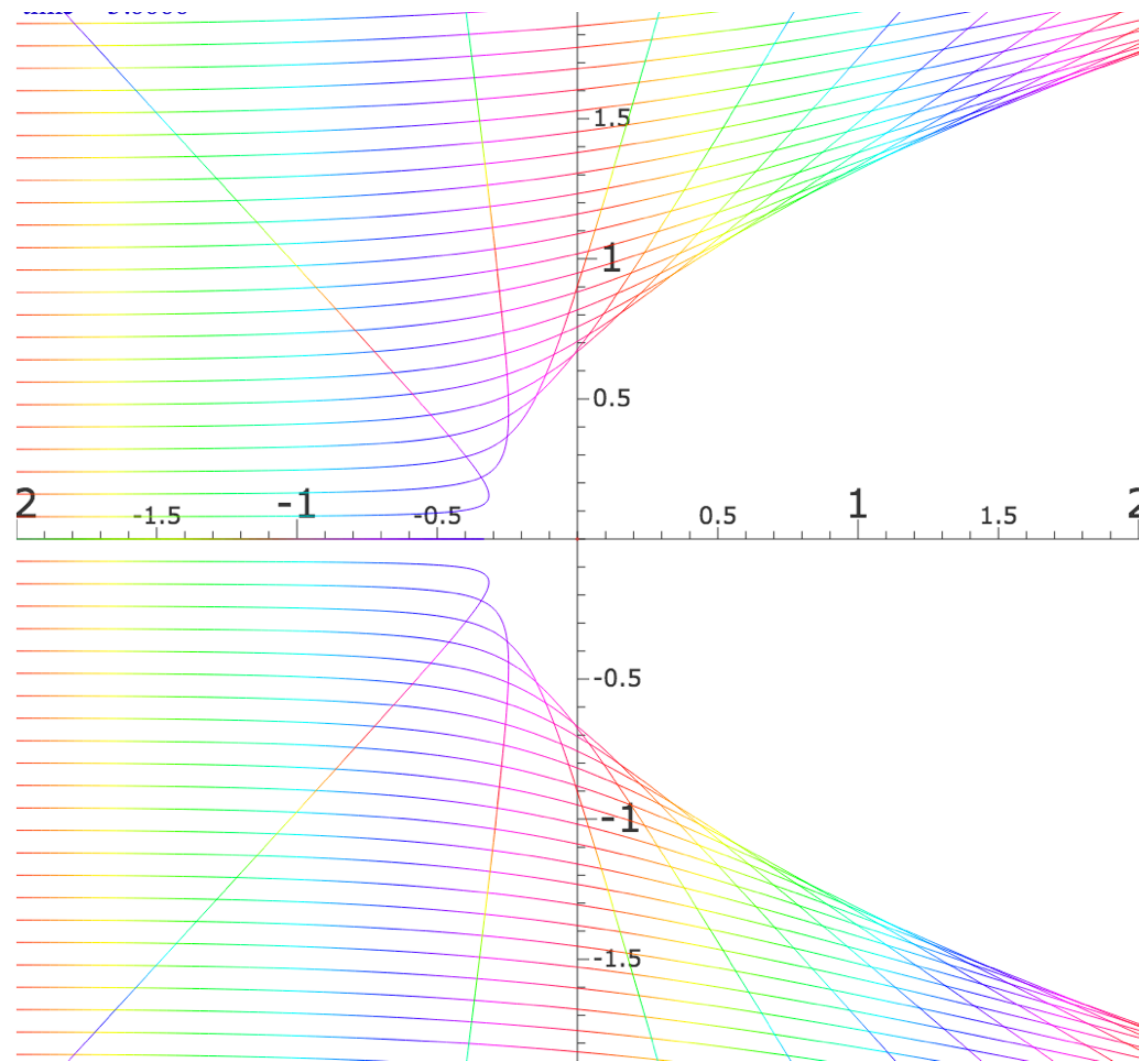
Coulomb orbit algebra of **ε***-vector and Kepler dynamics of momentum* **p**=*m***v**

*Example of complete (***r**,**p***)-geometry of elliptical orbit*

Connection formulas for (γ,R)-parameters with (a,b) and (ε,λ)

Rutherford scattering geometry





https://modphys.hosted.uark.edu/CoulItWeb.html?scenario=Rutherford

Chapter 1 Orbit Families and Action

Families of particle orbits are drawn in a varying color which represents the classical action or Hamiltoon's characteristic function SH = ∫p dq.(Sometimes SH is called 'reduced action'.) The color is chosen by first calculating c = SH modulo h-bar (You can change Planck's constant from its default value h/2π = 1.0) The chromatic value c assigns the hue by its position on the color wheel (e.g.; c=0 is red, c=0.2 is a yellow, c=0.5 is a green, etc.).

Volcanoes of Io (Paths=180, No color quant.) | Parabolic Fountain (Uniform) | Space Bomb (Coulomb) | Exploding Starlet (IHO) | Synchrotron Motion (Crossed E & B fields)

Chapter 2 Rutherford Scattering

A parallel beam of iso-energetic alpha particles undergo Rutherford scattering from a coulomb field of a nucleus as calculated in these demos. It is also the ideal pattern of paths followed by intergalactic hydrogen in perturbed by the solar wind.

Rutherford scattering | 2-Electron Orbits

Chapter 3 Coulomb Field (H atom)

Orbits in an attractive Coulomb field are calculated here. You may select the initial position (x(0),y(0)) by moving the mouse to a desired launch point, and then select the initial momentum (px(0), py(0)) by pressing the mouse button and dragging.

Atomic Orbits

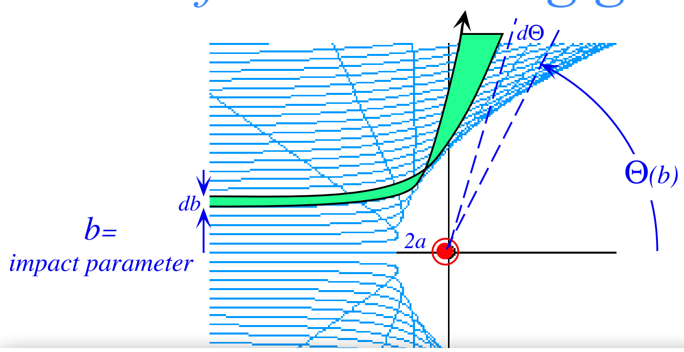
Chapter 4 Molecular Ion Orbits

Orbits around two fixed nuclei are calculated here. A set of elliptic coordinates are drawn in the background. After running a few trajectories you may notice that their caustics conform to one or two of the elliptic coordinate lines.

Molecular Ion Orbits

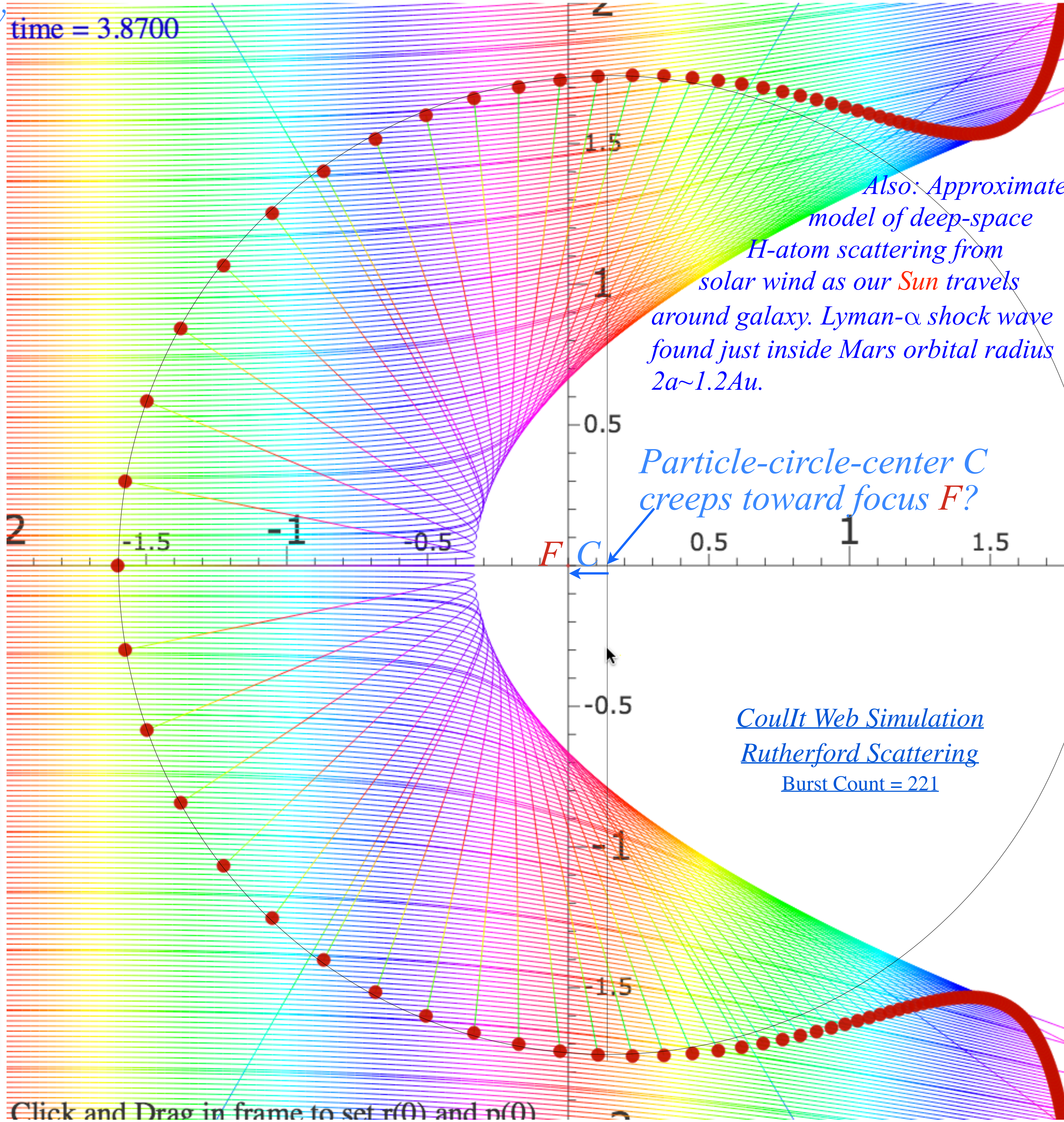
Oscillator Scattering | 2-Particle Orbits | 2-Particle Collision

Rutherford scattering geometry



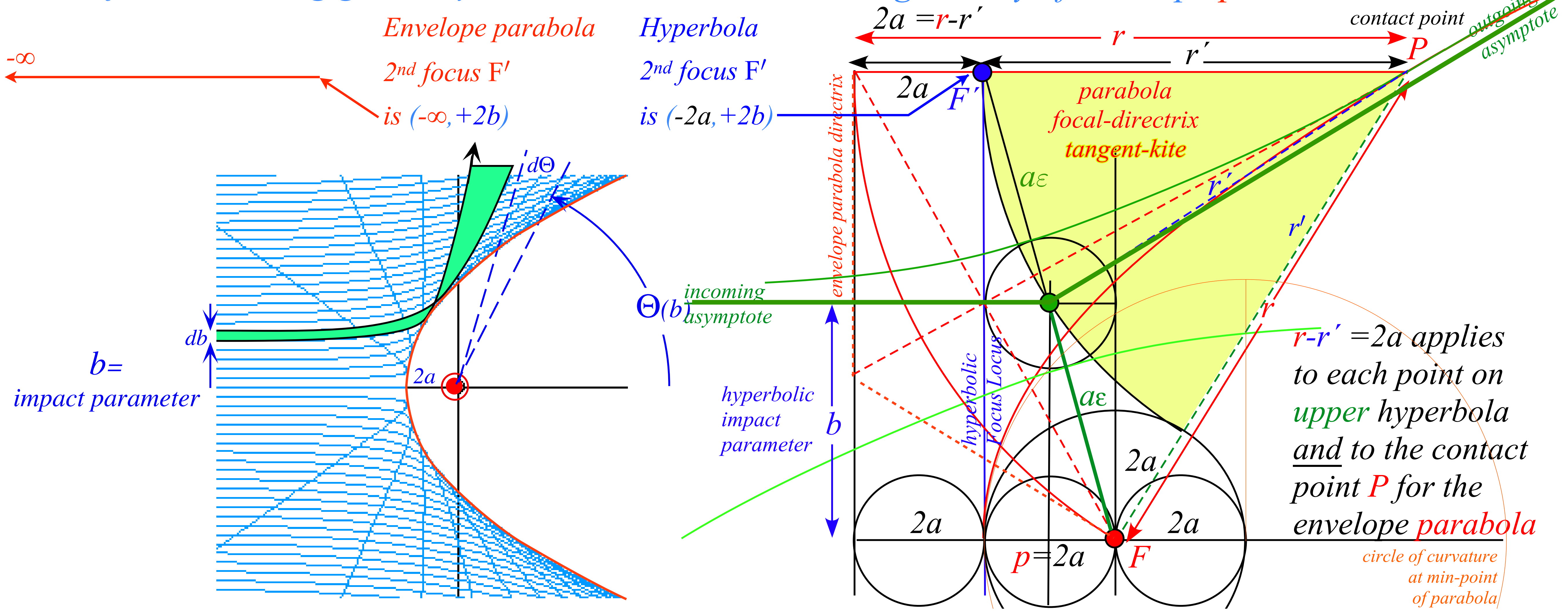
Terminal time t(off) =	5
Maximum step size dt =	0.03
Start launch angle phi1 =	-180
Start launch angle phi2 =	180
Number of burst paths =	221
Charge of Nucleus 1 =	0.2
x-Position of Nucleus 1 =	0
y-Position of Nucleus 1 =	0
Charge of Nucleus 2 =	0
Coulomb (k12) =	-1
Core thickness r =	0.000001
x-Stark field Ex =	0
y-Stark field Ey =	0
Zeeman field Bz =	0
Diamagnetic strength k =	0
Plank constant h-bar =	2
Color quantization hues =	64
Color quantization bands =	2
Fractional Error (e^{-x}), x =	8
Particle Size =	6

- Fix r(0) ☐
- Fix p(0) ☐
- Do swarm ☑
- Beam ☑
- Plot r(t) ☑
- Plot p(t) ☐
- Color action ☑
- No stops ☐
- Field vectors ☑
- Info ☑
- Draw masses ☑
- Axes ☑
- Coordinates ☐
- Lenz ☑
- Set p by ϕ ☐
- Elastic ☑
- 2 Free ☐
- Save to GIF ☐

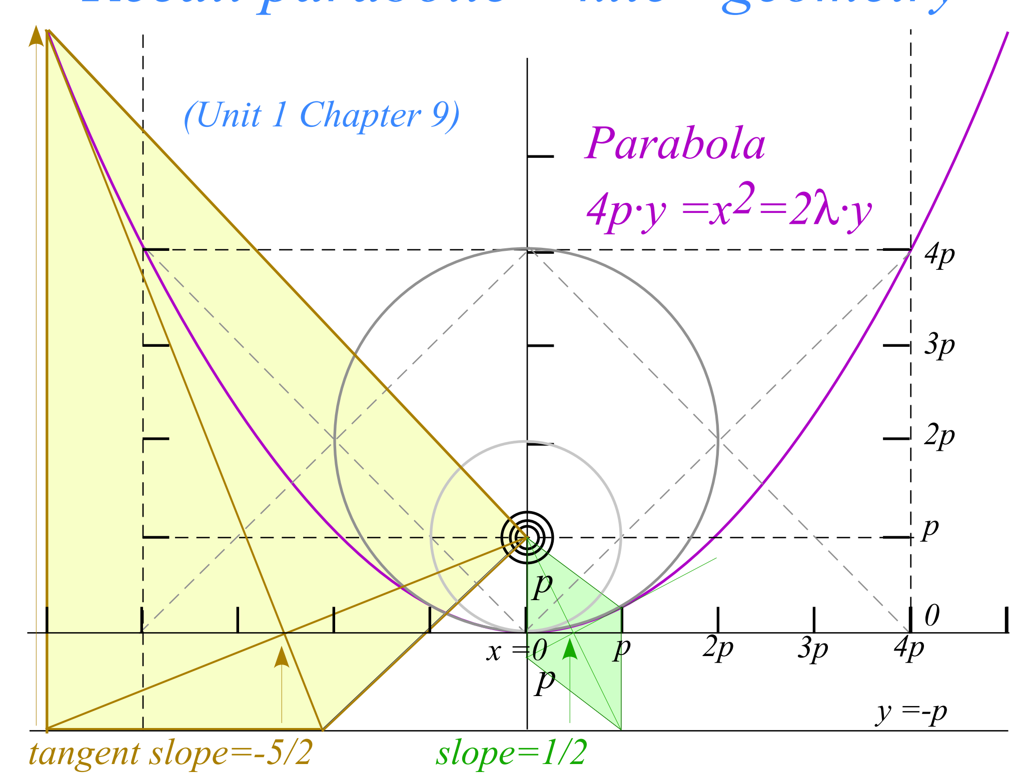


Rutherford scattering geometry

“Kite” geometry of envelope parabola

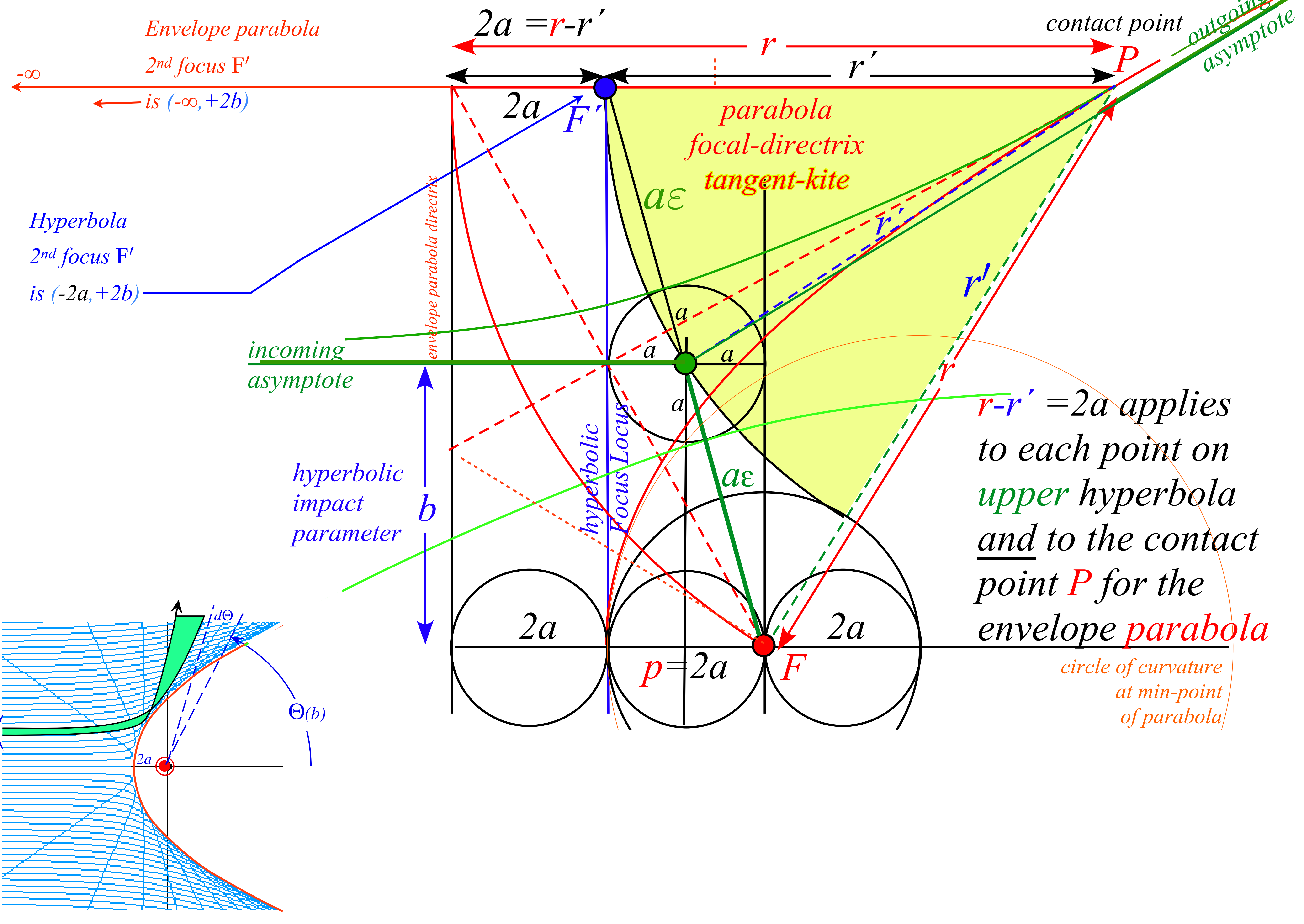


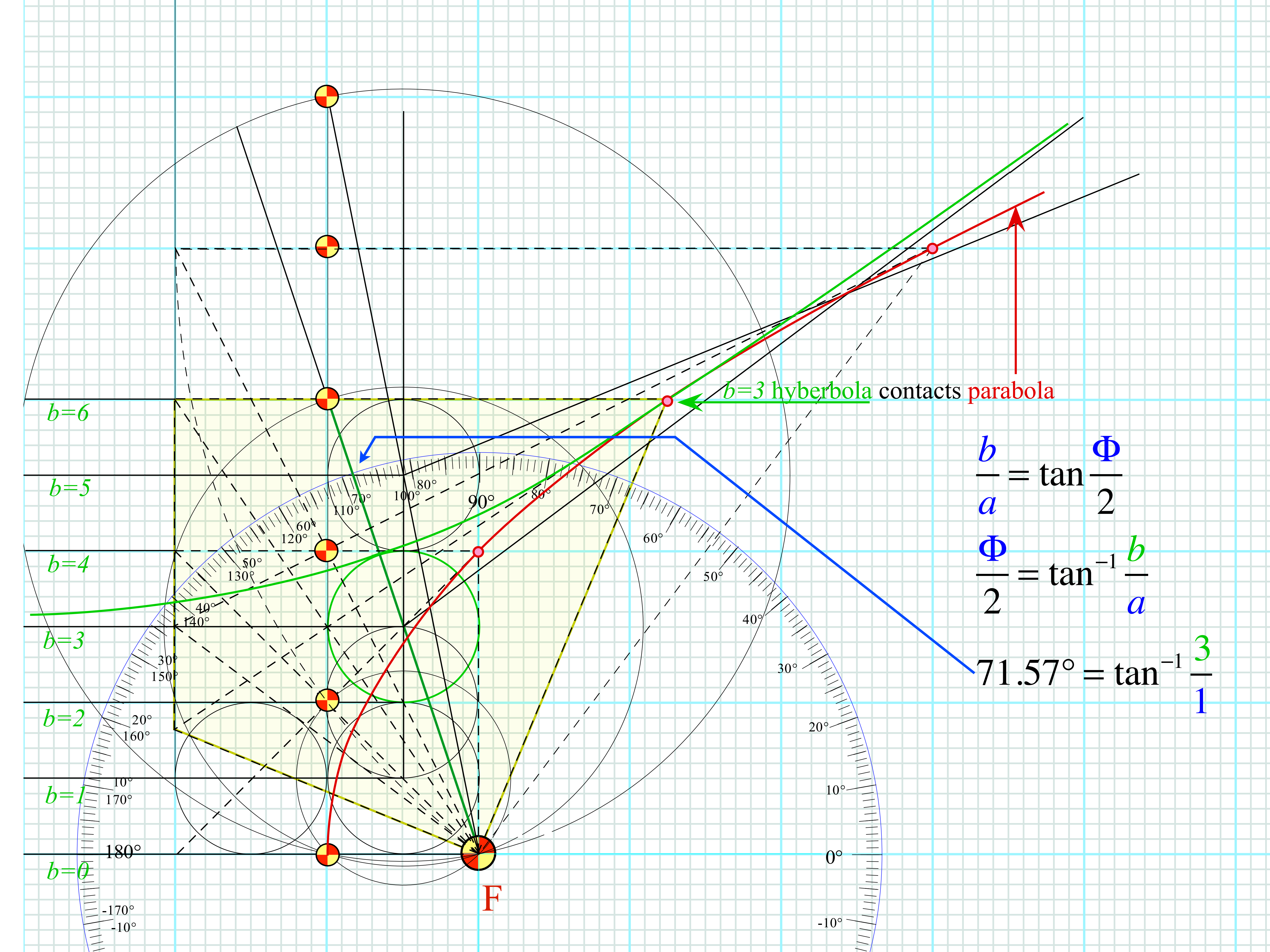
Recall parabolic “ kite” geometry



Rutherford scattering geometry

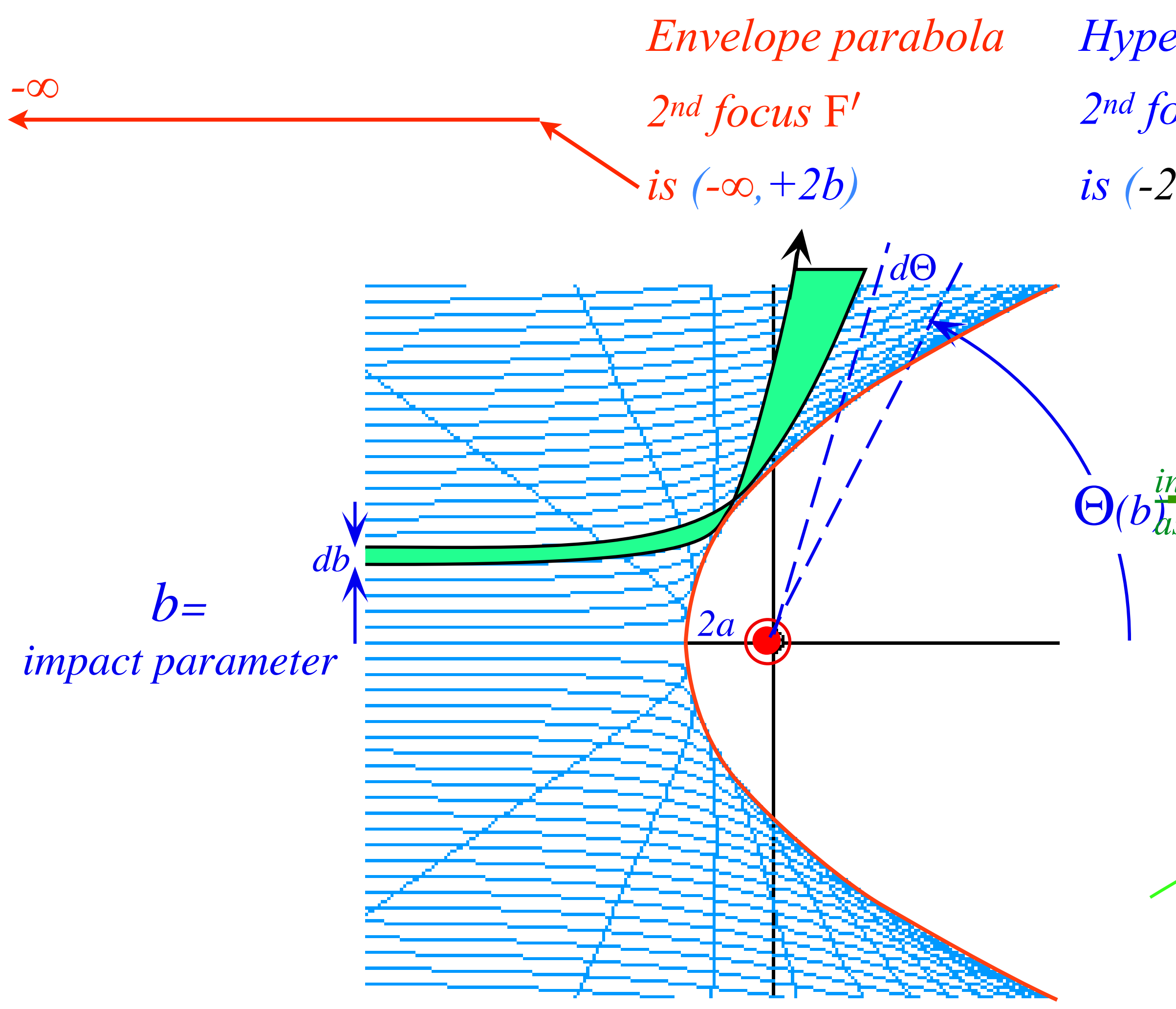
"Kite" geometry of envelope parabola





b=3 hyberbola contacts parabola
b/a = tan Φ/2
Φ/2 = tan^−1 b/a
71.57° = tan^−1 3/1
b=6
b=5
b=4
b=3
b=2
b=1
b=0
F
90°
180°
0°

Rutherford scattering geometry



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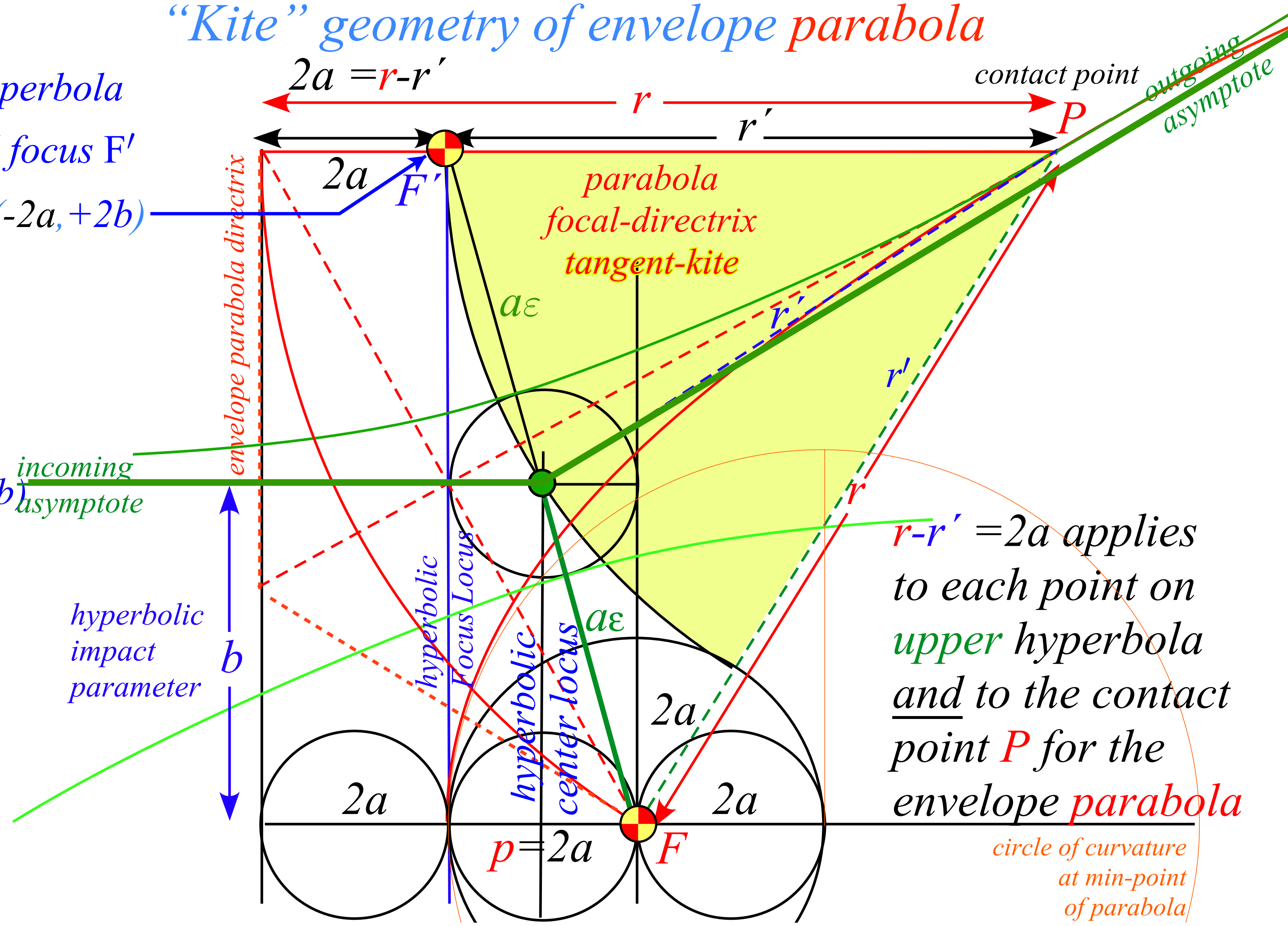

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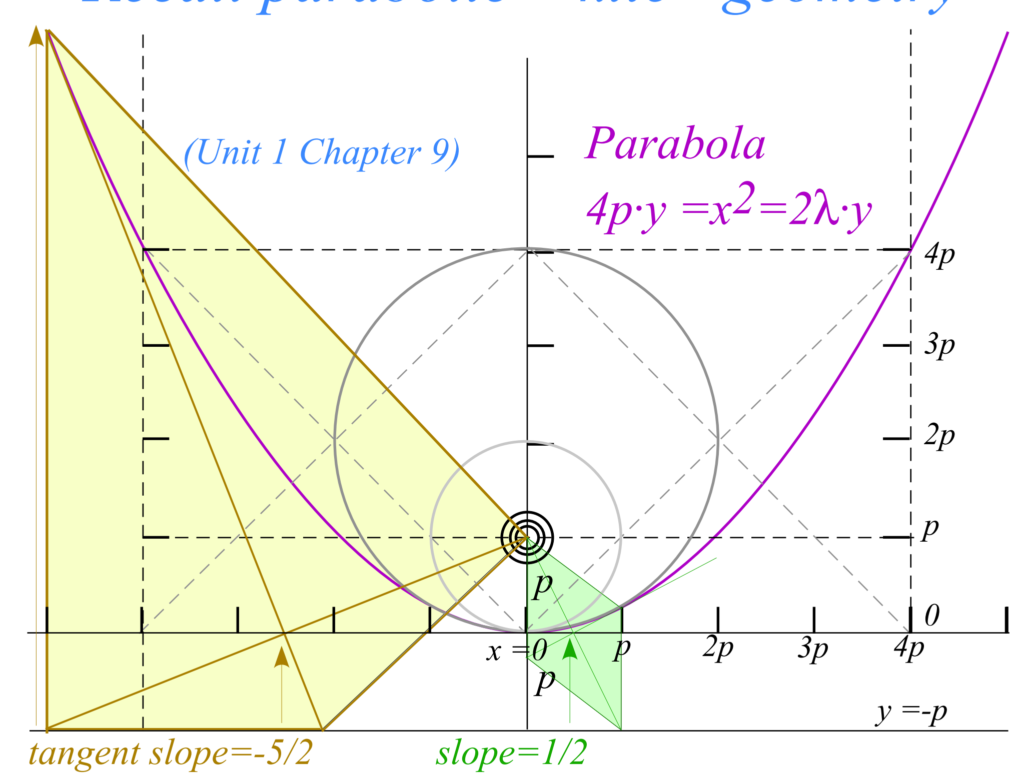
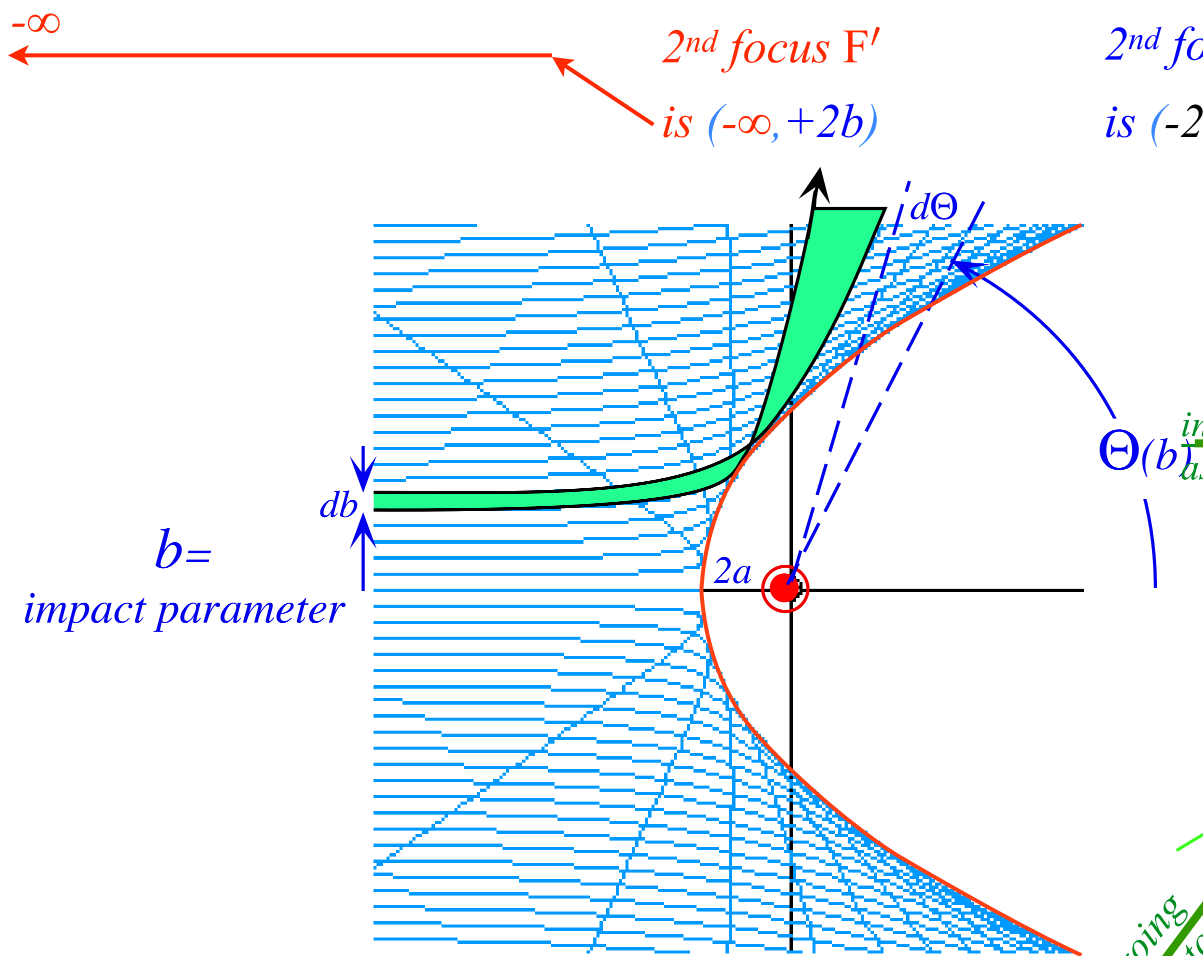
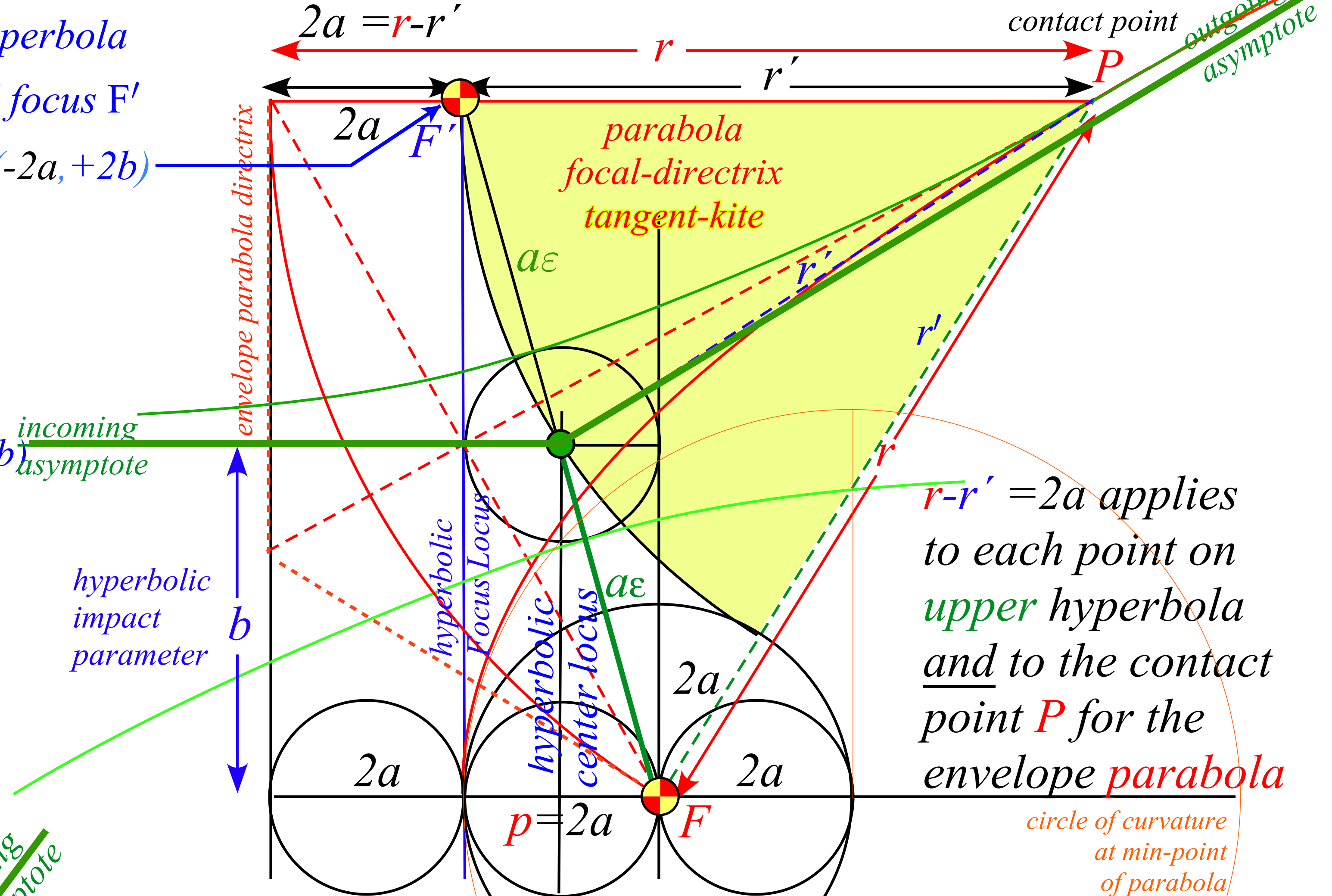


Rutherford scattering geometry



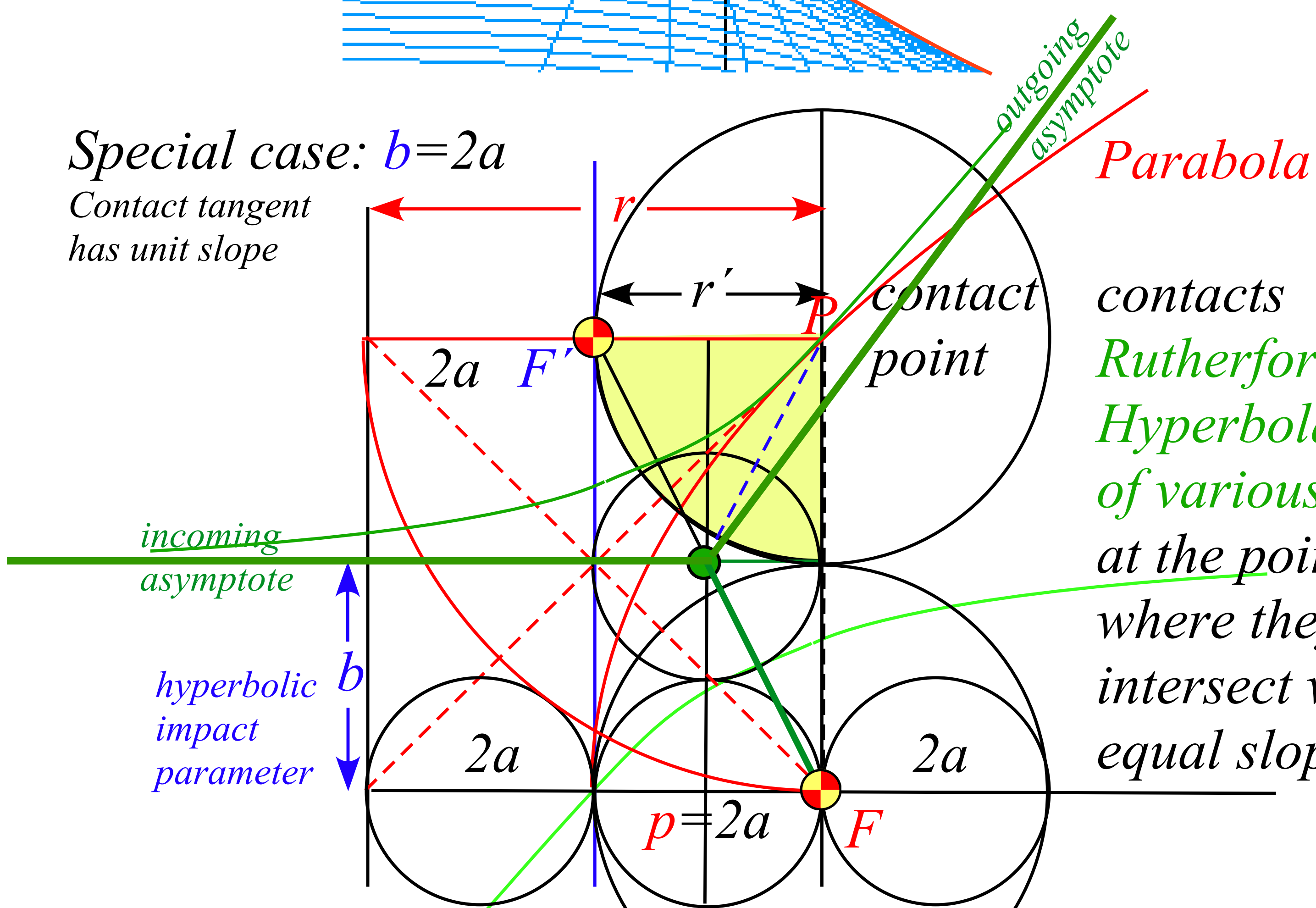
"Kite" geometry of envelope parabola



r-r′ =2a applies to each point on *upper* hyperbola and to the contact point *P* for the envelope *parabola*

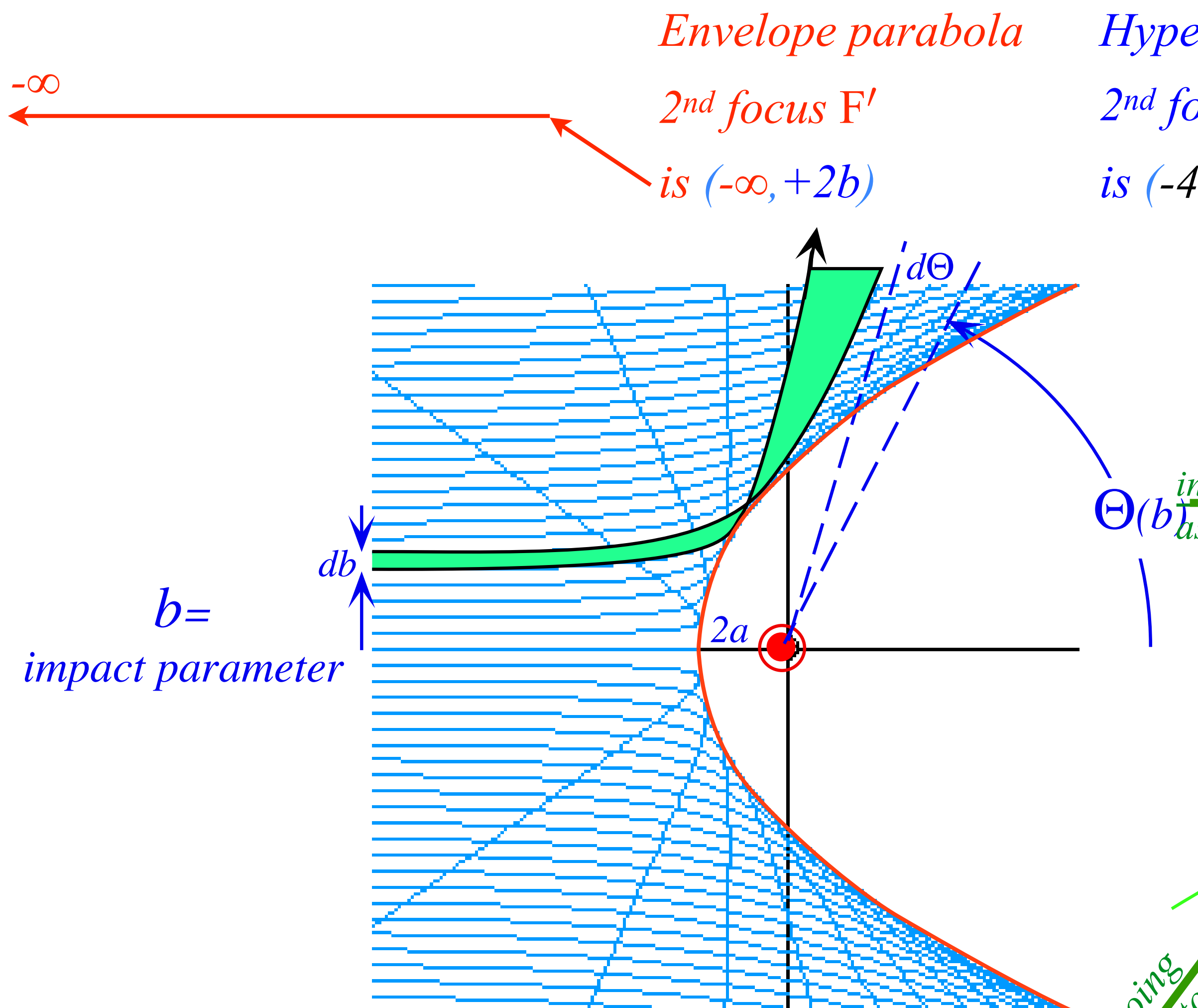
Special case: $b=2a$

Contact tangent has unit slope

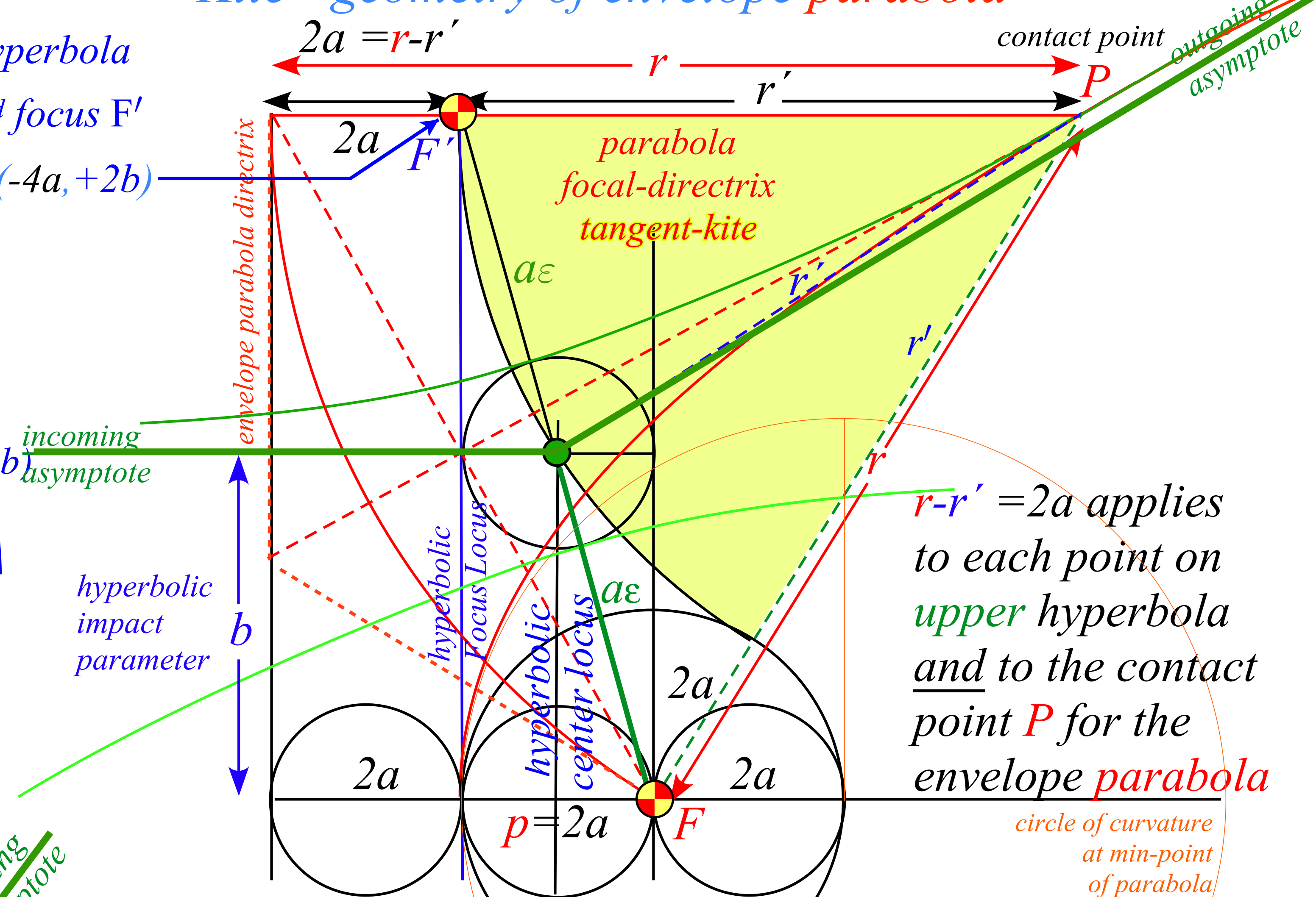


contacts Rutherford Hyperbolas of various b at the point where they intersect with equal slope

Rutherford scattering geometry



"Kite" geometry of envelope parabola



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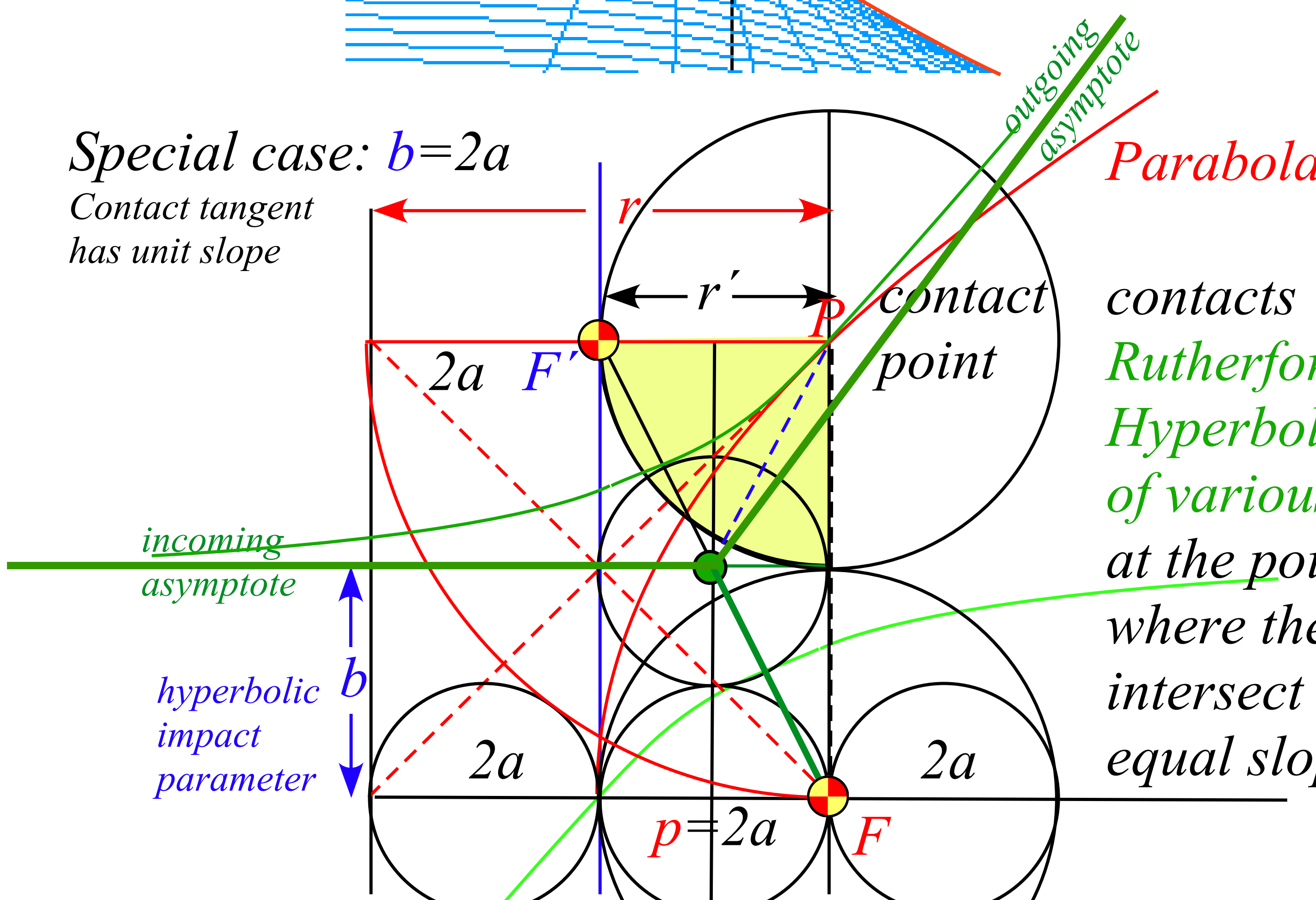
Contact tangent has unit slope



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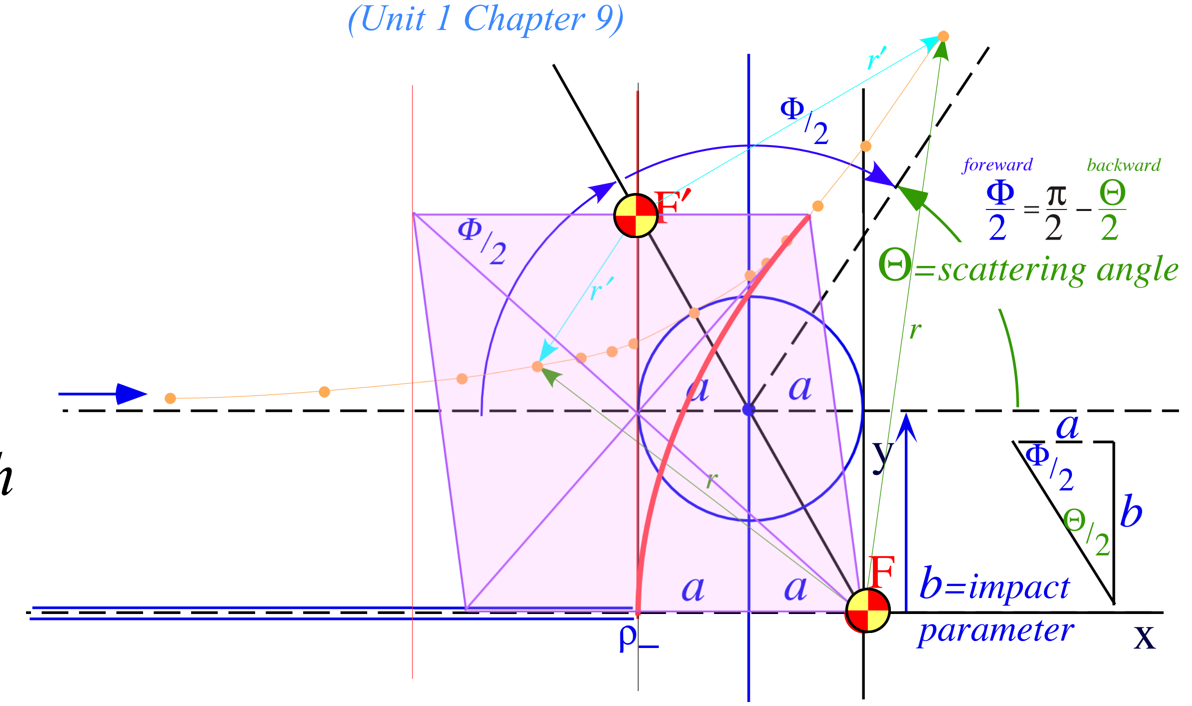
Recall parabolic " kite" geometry

(Unit 1 Chapter 9)



Rutherford scattering and hyperbolic orbit geometry

Backward vs forward scattering angles and orbit construction example

Parabolic "kite" and orbital envelope geometry

➔ *Differential and total scattering cross-sections*

Eccentricity vector **ε** *and (ε,λ)-geometry of orbital mechanics*

Projection **ε•r** *geometry of* **ε***-vector and orbital radius* **r**

Review and connection to usual orbital algebra (previous lecture)

Projection **ε•p** *geometry of* **ε***-vector and momentum* **p**=*m***v**

General geometric orbit construction using **ε***-vector and (γ,R)-parameters*

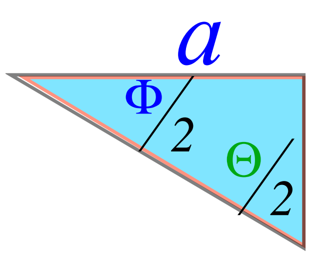
Derivation of **ε***-construction by analytic geometry*

Coulomb orbit algebra of **ε***-vector and Kepler dynamics of momentum* **p**=*m***v**

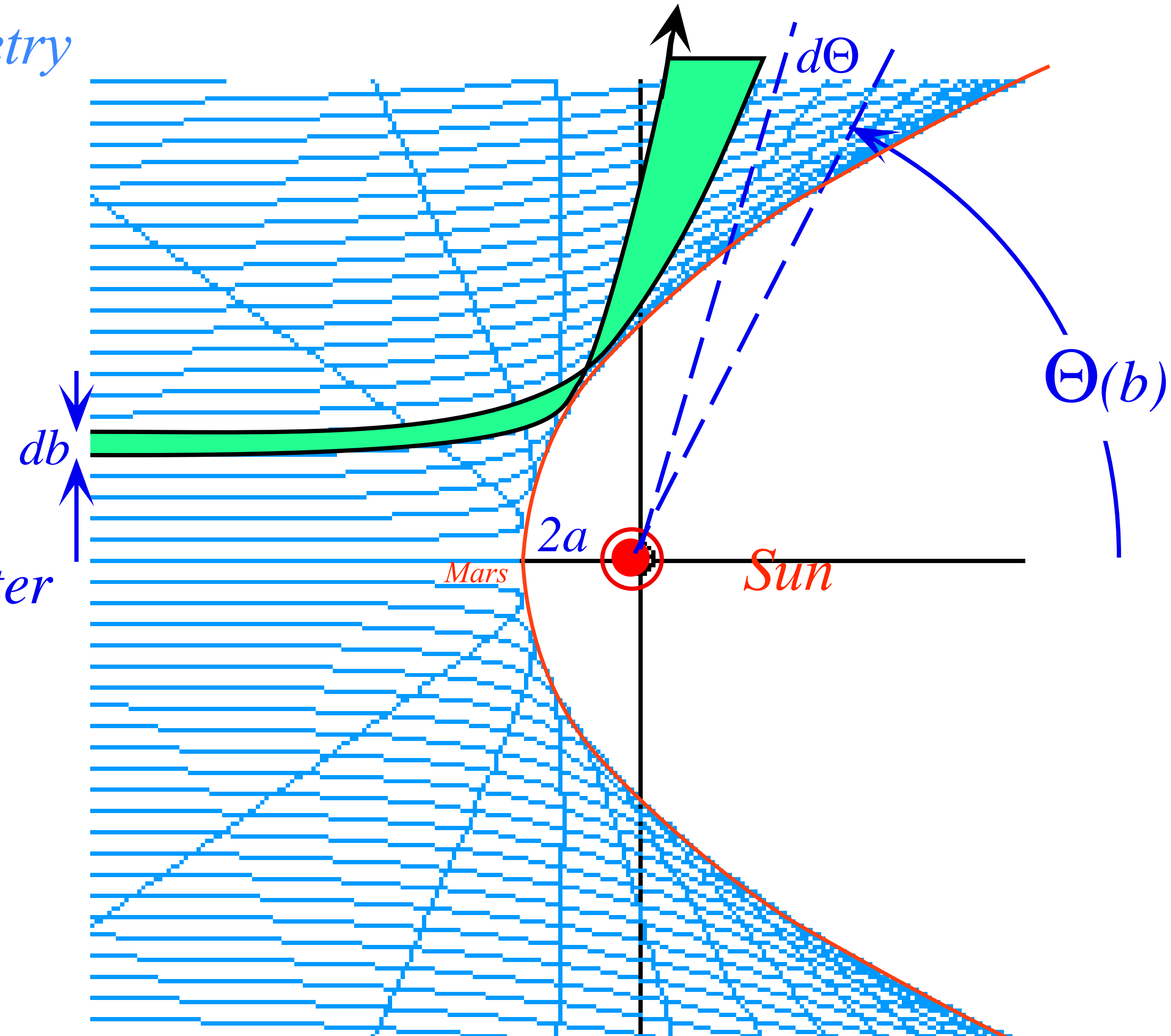
*Example of complete (***r**,**p***)-geometry of elliptical orbit*

Connection formulas for (γ,R)-parameters with (a,b) and (ε,λ)

Rutherford scattering geometry



b= impact parameter

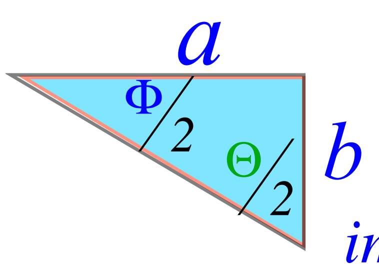


Also: Approximate model of deep-space H-atom scattering from solar wind as our Sun travels around galaxy. Lyman-α shock wave found just inside Mars orbital radius 2a~1.2Au.

Fig. 5.3.2 Family of iso-energetic Rutherford scattering orbits with varying impact parameter.

Incremental window $d\sigma = b \cdot db$ normal to beam axis at $x=-\infty$ scatters to area $dA = R^2 \sin\Theta d\Theta d\varphi = R^2 d\Omega$ onto a sphere at $R=+\infty$ where is called the *incremental solid angle* $d\Omega = \sin\Theta d\Theta d\varphi$

Rutherford scattering geometry



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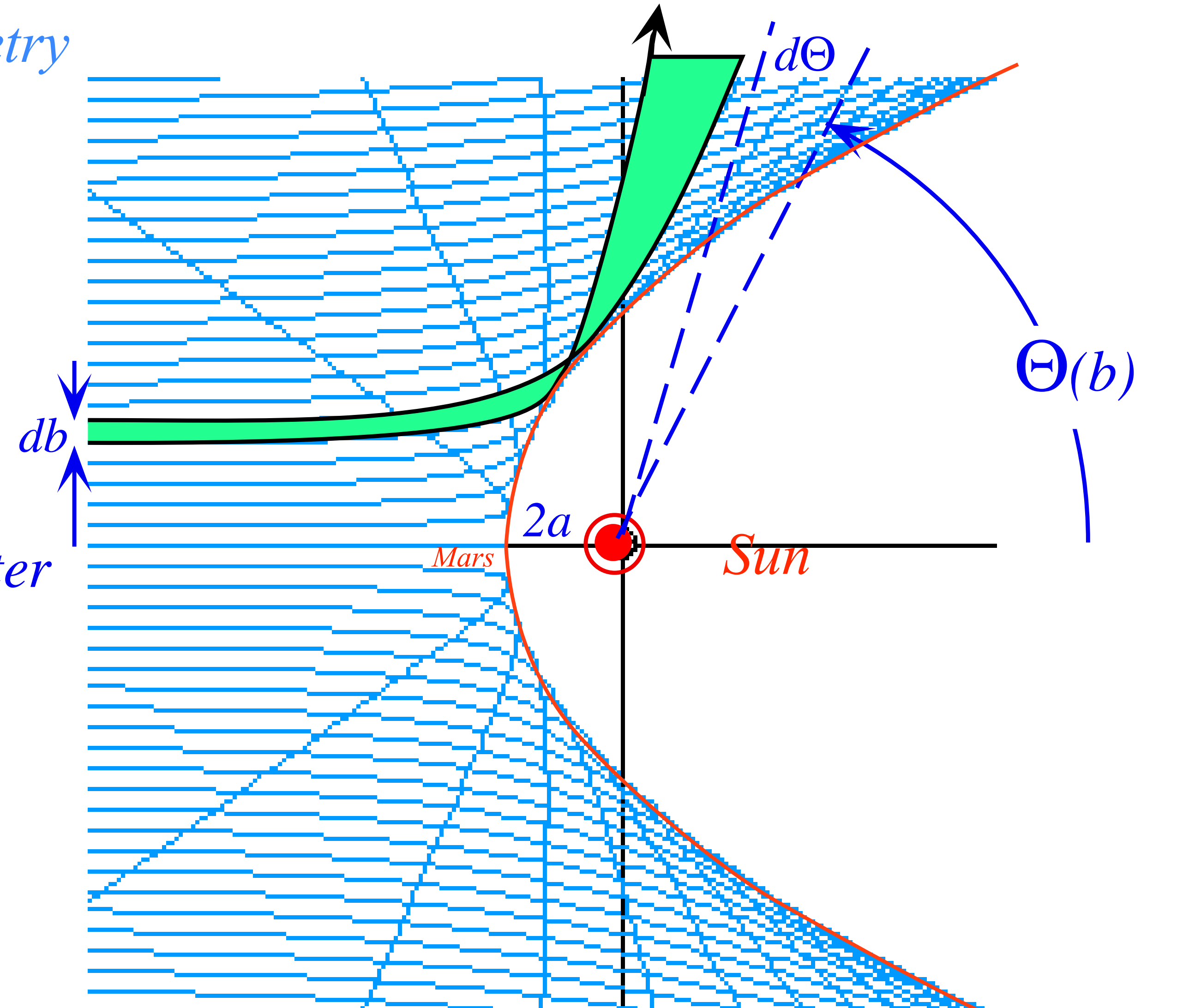

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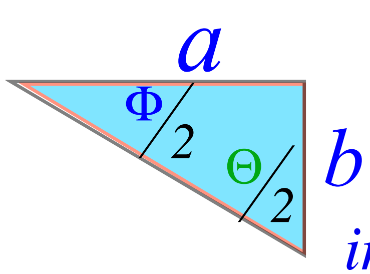
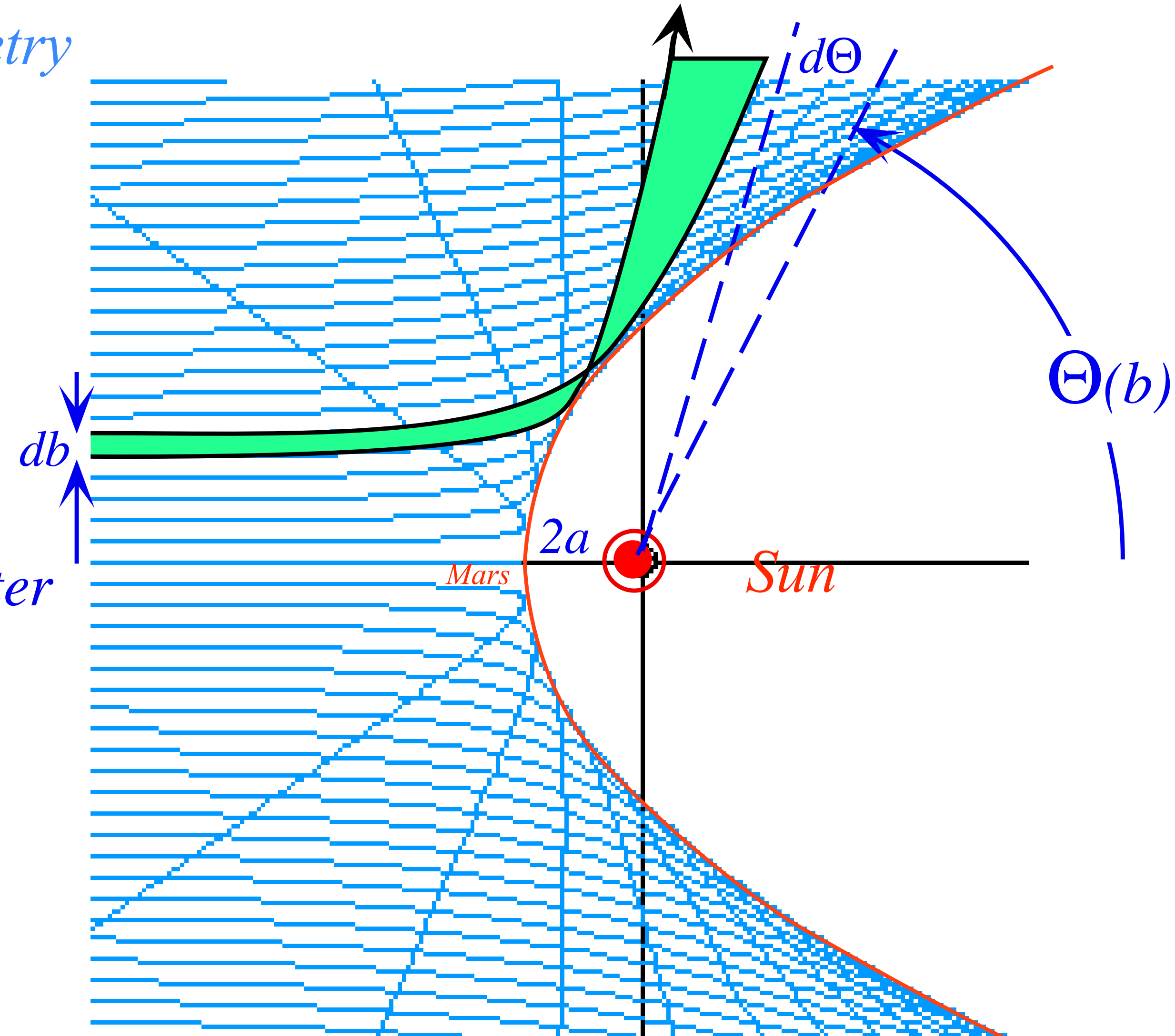
Rutherford scattering geometry



b= *impact parameter*



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$$\frac{b}{a} = \cot\frac{\Theta}{2}$$

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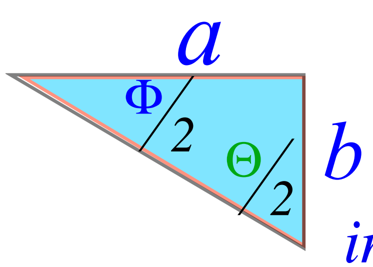
Geometry: $b = a\cot\dfrac{\Theta}{2}$

with: $\dfrac{db}{d\Theta} = \dfrac{-a}{2}\csc^2\dfrac{\Theta}{2}$

Rutherford scattering geometry



$b=$ *impact parameter*

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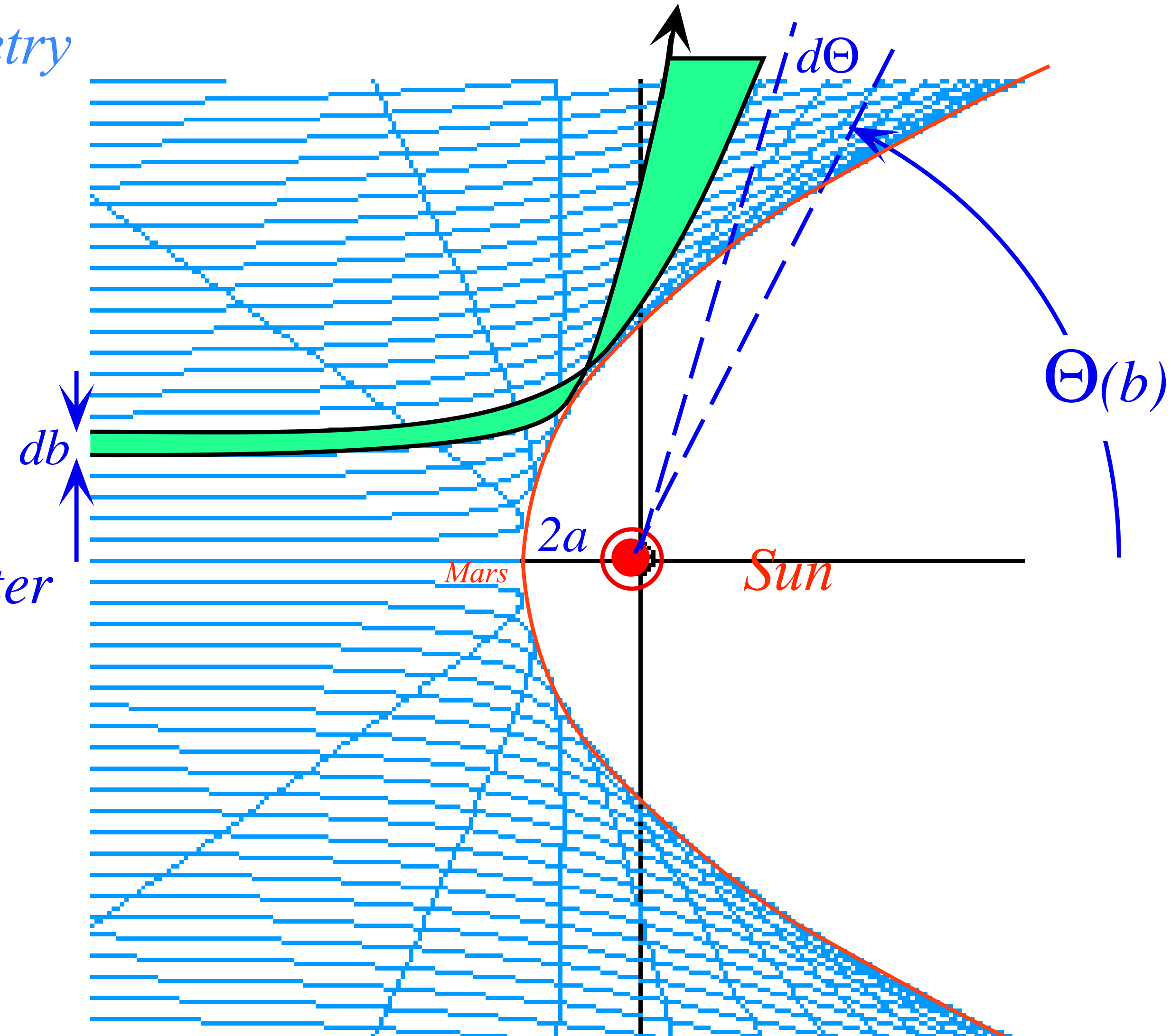

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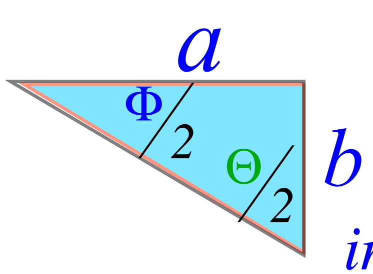
Geometry: $b = a\cot\dfrac{\Theta}{2} = \dfrac{k}{2E}\cot\dfrac{\Theta}{2}$

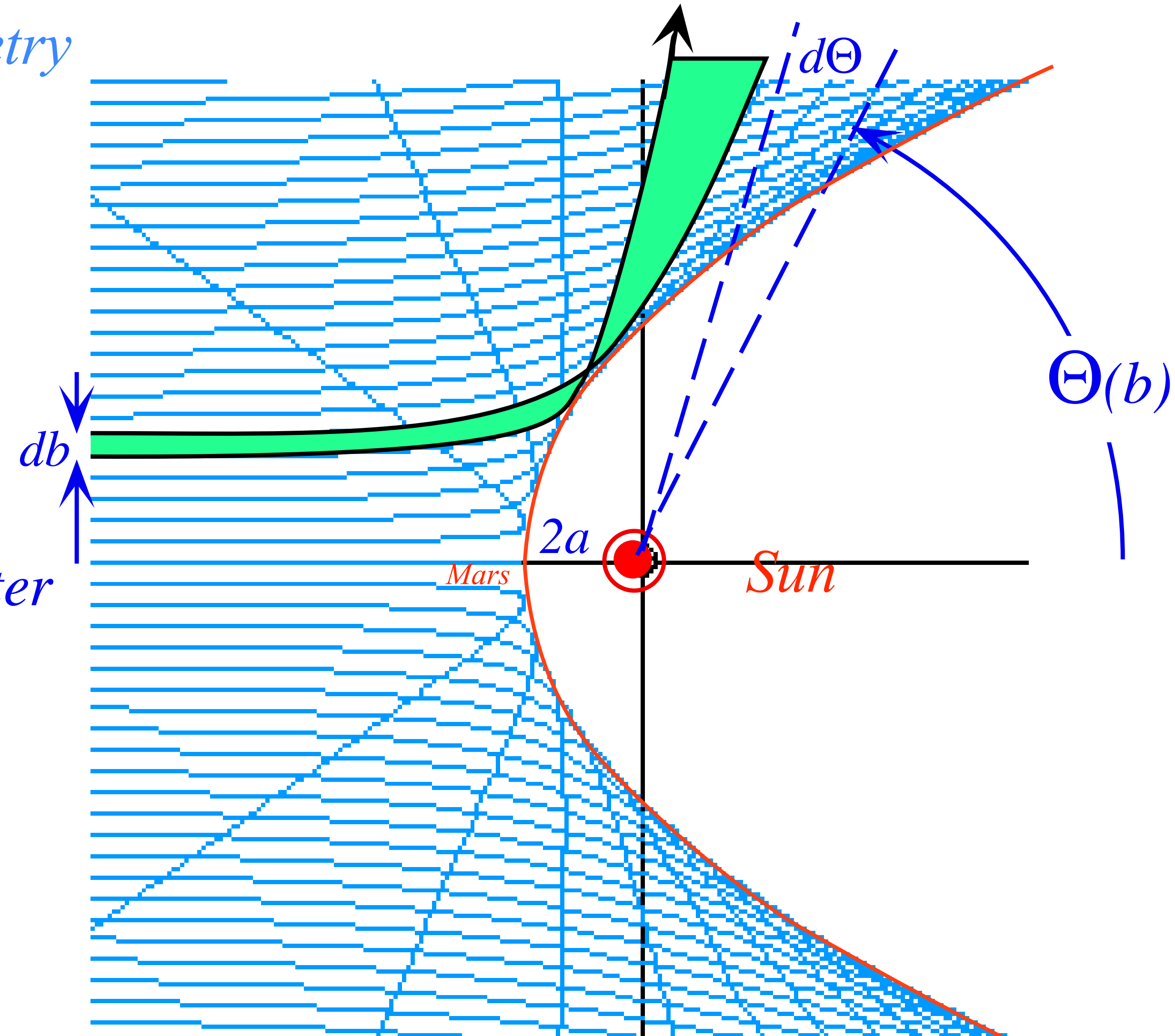
with: $\dfrac{db}{d\Theta} = \dfrac{-a}{2}\csc^2\dfrac{\Theta}{2} = \dfrac{-a}{2\sin^2\dfrac{\Theta}{2}}$

$(Never\ forget!: a = \dfrac{-k}{2E})$

Rutherford scattering geometry



$b=$ *impact parameter*



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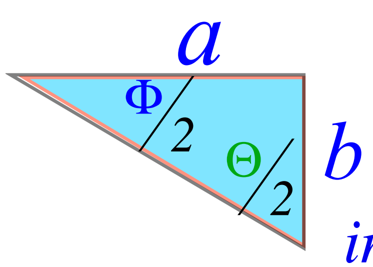
Geometry: $b = a\cot\dfrac{\Theta}{2} = \dfrac{k}{2E}\cot\dfrac{\Theta}{2}$ gives the *Rutherford DSC*. $\dfrac{d\sigma}{d\Omega} = \dfrac{-a^2\cos\frac{\Theta}{2}}{2\sin\Theta\sin^3\frac{\Theta}{2}}$

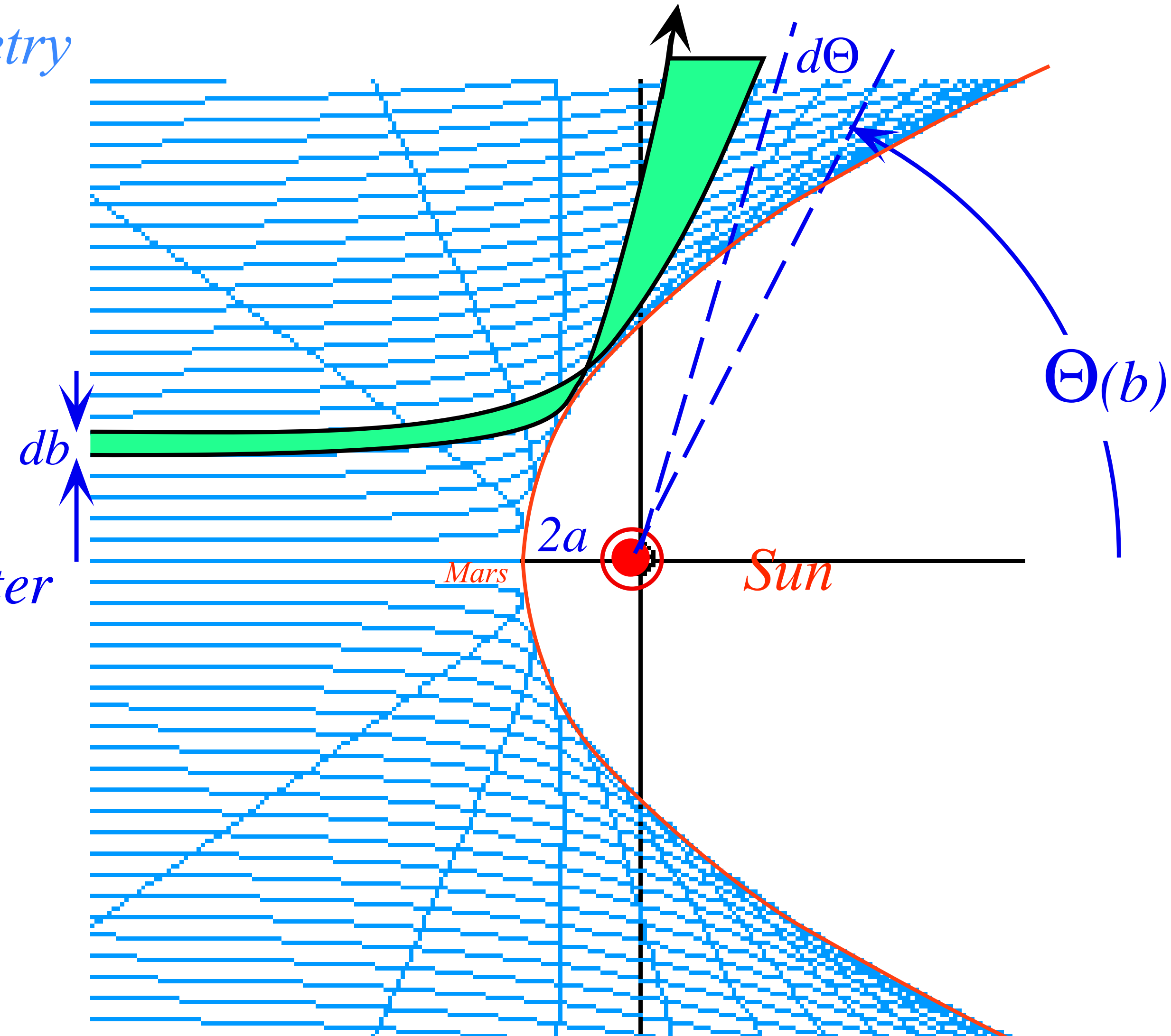
with: $\dfrac{db}{d\Theta} = \dfrac{-a}{2}\csc^2\dfrac{\Theta}{2} = \dfrac{-a}{2\sin^2\frac{\Theta}{2}}$ and: $\sin\Theta = 2\sin\dfrac{\Theta}{2}\cos\dfrac{\Theta}{2}$

(*Never forget*!: $a = \dfrac{-k}{2E}$)

Rutherford scattering geometry



b= *impact parameter*



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$$\frac{b}{a} = \cot\frac{\Theta}{2}$$

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Geometry: $b = a\cot\frac{\Theta}{2} = \frac{k}{2E}\cot\frac{\Theta}{2}$ gives the *Rutherford DSC*.

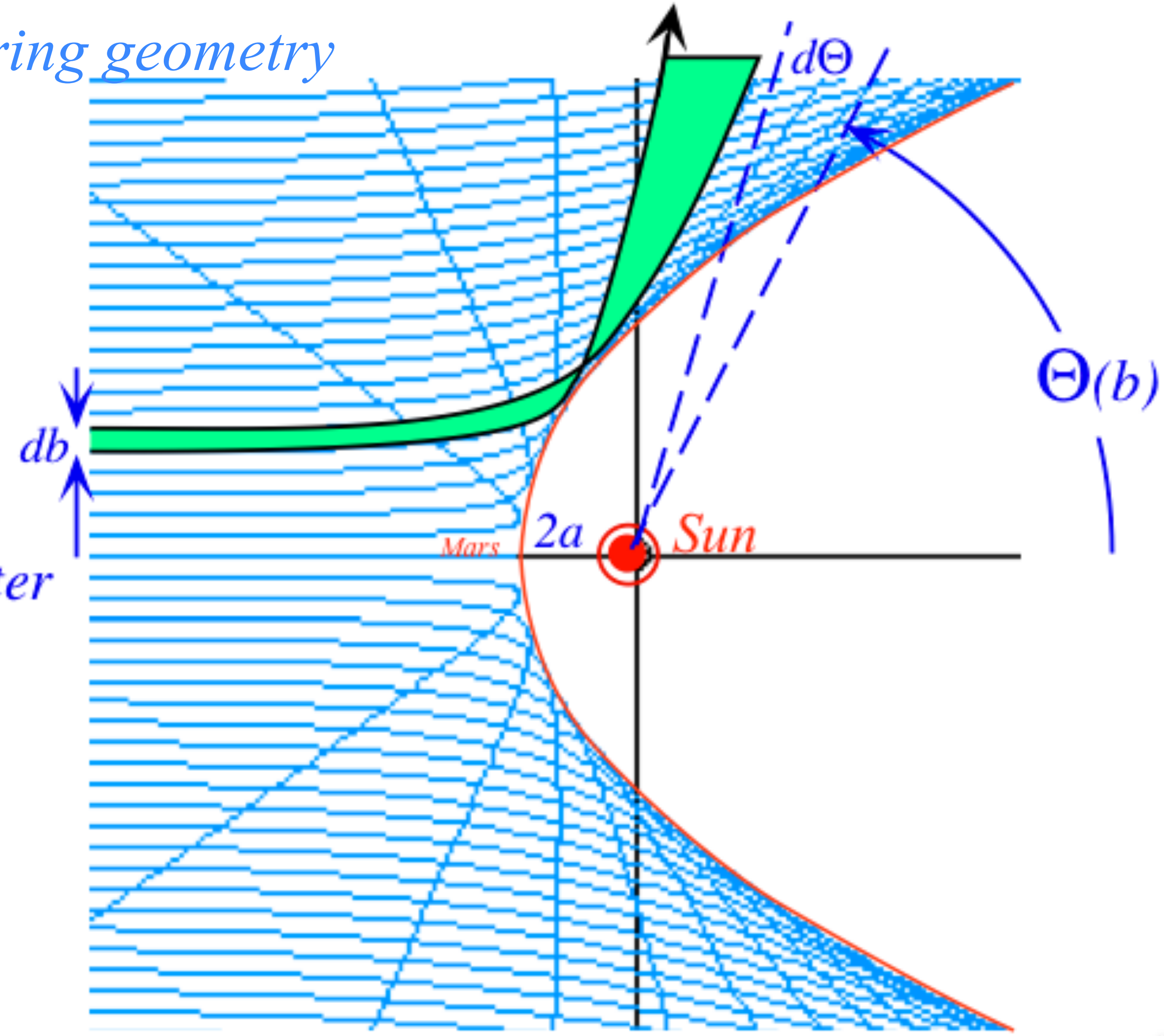
$$\frac{d\sigma}{d\Omega} = \frac{-a^2\cos\frac{\Theta}{2}}{2\sin\Theta\sin^3\frac{\Theta}{2}} = \frac{-k^4}{16E^2}\sin^{-4}\frac{\Theta}{2}$$

with: $\dfrac{db}{d\Theta} = \dfrac{-a}{2}\csc^2\dfrac{\Theta}{2} = \dfrac{-a}{2\sin^2\frac{\Theta}{2}}$ and: $\sin\Theta = 2\sin\dfrac{\Theta}{2}\cos\dfrac{\Theta}{2}$

(*Never forget*!: $a = \dfrac{-k}{2E}$)

This classical result agrees exactly with 1st Born approximation to *quantum* Coulomb *DSC*!

Rutherford scattering geometry

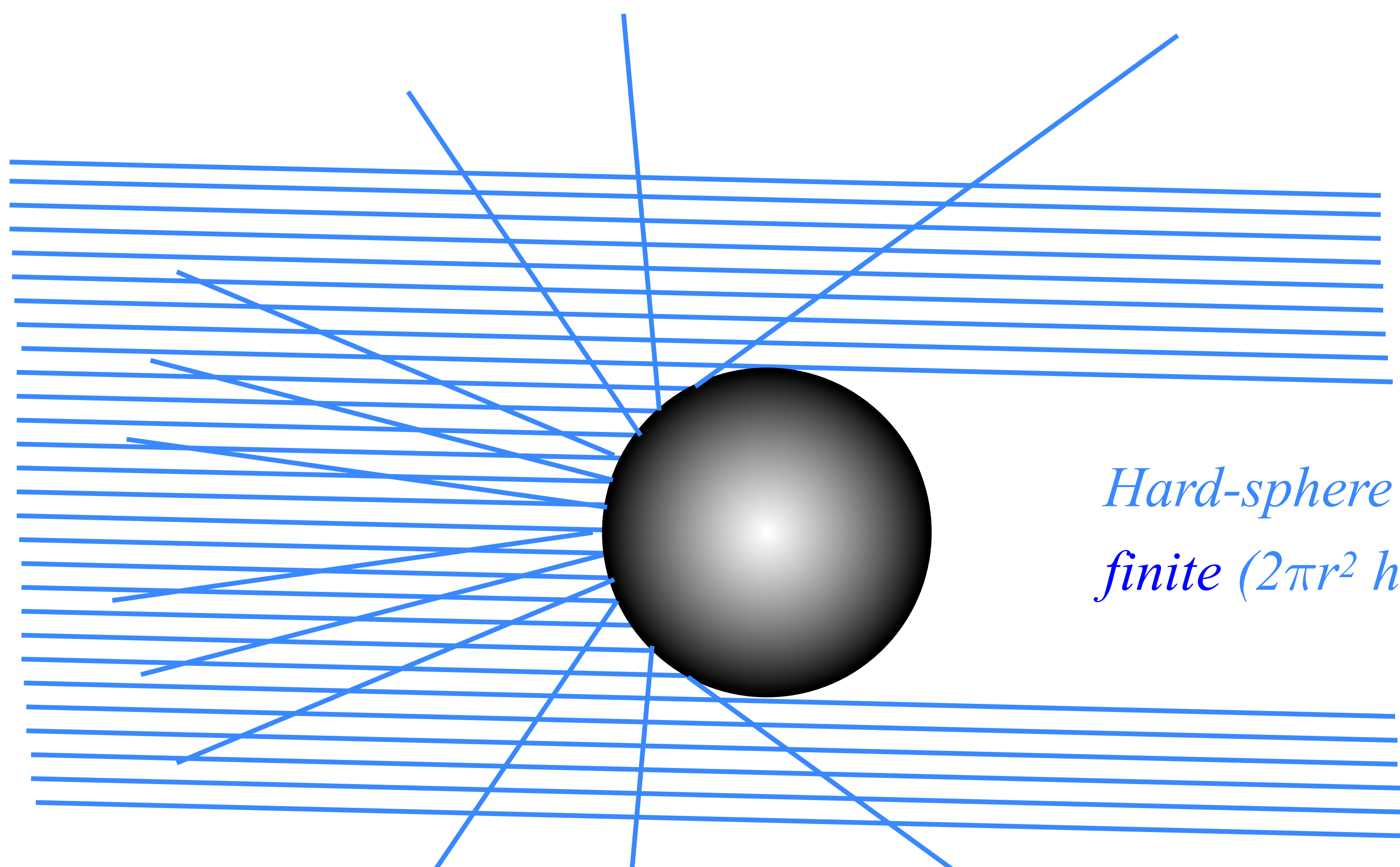


b= impact parameter

Two Extremes:

Rutherford (Coulomb) scattering has ***infinite*** *(∞) total cross section*

$$\sigma = \int d\Omega \frac{d\sigma}{d\Omega} = \int d\Omega \frac{k^4}{16E^2} \sin^{-4}\frac{\Theta}{2} = \infty$$



Hard-sphere scattering has ***finite*** *($2\pi r^2$ here) total cross section*

CoulIt Web Simulation
Hard Sphere - Elastic scattering

Rutherford scattering and hyperbolic orbit geometry

Backward vs forward scattering angles and orbit construction example

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Differential and total scattering cross-sections

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{r}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and orbital radius* $\mathbf{r}$

Review and connection to usual orbital algebra (previous lecture)

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{p}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and momentum* $\mathbf{p}=m\mathbf{v}$

General geometric orbit construction using $\boldsymbol{\varepsilon}$*-vector and* (γ,R)*-parameters*

Derivation of $\boldsymbol{\varepsilon}$*-construction by analytic geometry*

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Example of complete $(\mathbf{r},\mathbf{p})$*-geometry of elliptical orbit*

Connection formulas for (γ,R)*-parameters with (a,b) and* (ε,λ)

Eccentricity vector **ε** *and* (ε,λ) *geometry of orbital mechanics*

Isotropic field $V=V(r)$ guarantees conservation *angular momentum vector* **L**

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\,\mathbf{r} \times \dot{\mathbf{r}}$$

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ) *geometry of orbital mechanics*

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$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\,\mathbf{r} \times \dot{\mathbf{r}}$$

Coulomb *V=-k/r* also conserves *eccentricity vector* $\boldsymbol{\varepsilon}$

$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} - \frac{\mathbf{p} \times \mathbf{L}}{km} = \frac{\mathbf{r}}{r} - \frac{\mathbf{p} \times (\mathbf{r} \times \mathbf{p})}{km}$$

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ) *geometry of orbital mechanics*

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(...for sake of comparison...)

IHO $V=(k/2)r^2$ also conserves *Stokes vector* $\boldsymbol{S}$

$$S_A = \tfrac{1}{2}(x_1^2 + p_1^2 - x_2^2 - p_2^2)$$
$$S_B = x_1 p_1 + x_2 p_2$$
$$S_C = x_1 p_2 - x_2 p_1$$

$\mathbf{A} = km\cdot\boldsymbol{\varepsilon}$ is known as the *Laplace-Hamilton-Gibbs-Runge-Lenz vector*.

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ) *geometry of orbital mechanics*

Isotropic field $V=V(r)$ guarantees conservation *angular momentum vector* **L**

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Generates symmetry groups: $R(3) \subset R(3) \times R(3) \subset O(4)$

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(...for sake of comparison...)

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$$S_B = x_1 p_1 + x_2 p_2$$
$$S_C = x_1 p_2 - x_2 p_1$$

Generates symmetry groups: $U(1) \subset U(2)$

Rutherford scattering and hyperbolic orbit geometry
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- *Parabolic "kite" and orbital envelope geometry*
- *Differential and total scattering cross-sections*

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Coulomb $V=-k/r$ also conserves *eccentricity vector* $\boldsymbol{\varepsilon}$

$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} - \frac{\mathbf{p}\times\mathbf{L}}{km} = \frac{\mathbf{r}}{r} - \frac{\mathbf{p}\times(\mathbf{r}\times\mathbf{p})}{km}$$

Generates symmetry groups: $R(3) \subset R(3)\times R(3) \subset O(4)$

(...for sake of comparison...)

IHO $V=(k/2)r^2$ also conserves *Stokes vector* ***S***

$S_A = \frac{1}{2}(x_1^2+p_1^2-x_2^2-p_2^2)$

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$S_C = x_1p_2 - x_2p_1$

Generates symmetry groups: $U(1) \subset U(2)$

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Consider dot product of $\boldsymbol{\varepsilon}$ with a radial vector **r**:

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Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ) *geometry of orbital mechanics*

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Eccentricity vector **ε** *and (ε,λ) geometry of orbital mechanics*

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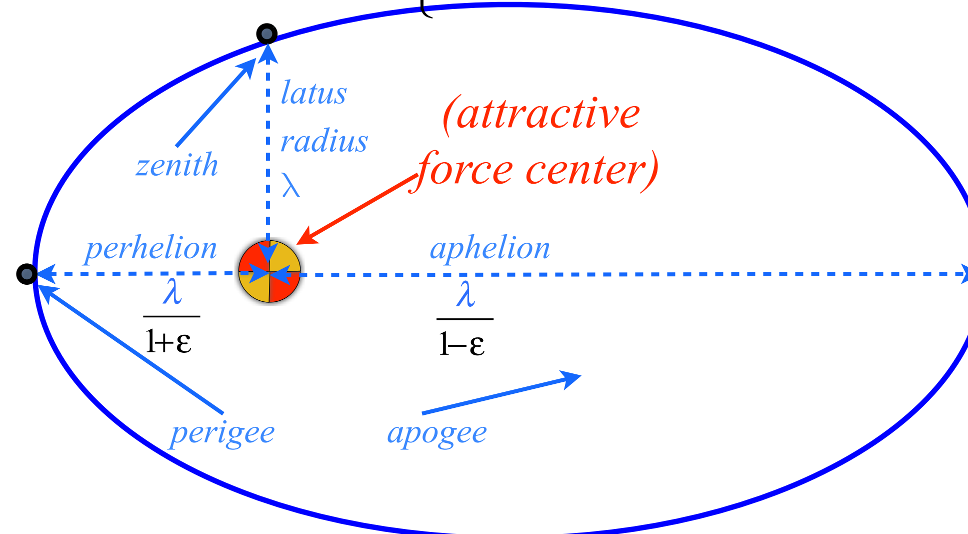

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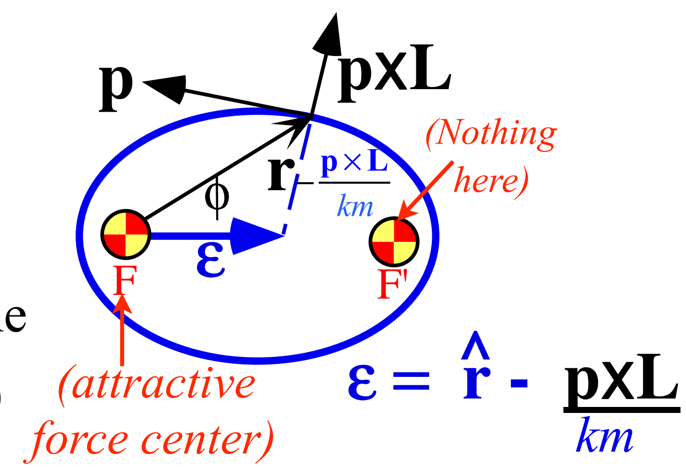
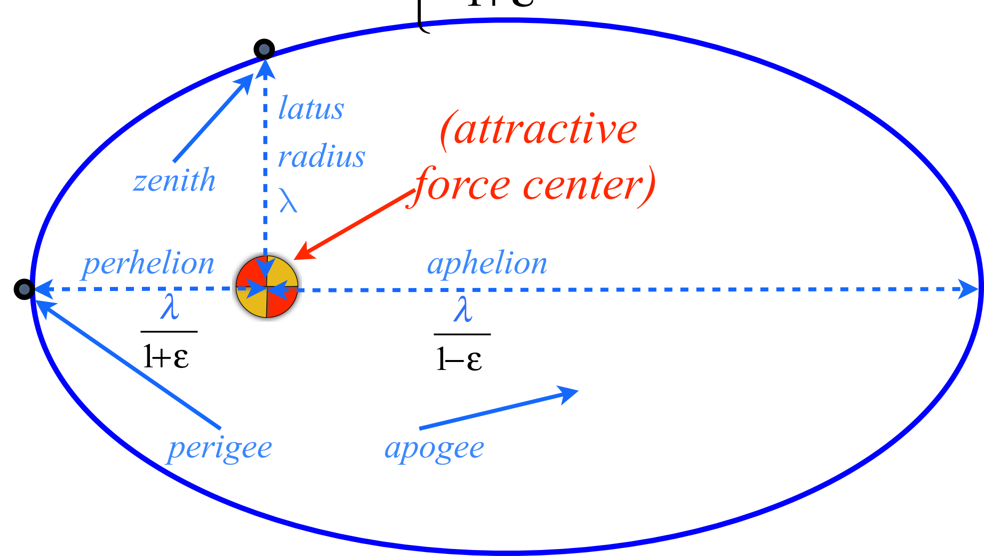
(a) Attractive (k>0)
Elliptic (E<0)

(Rotational momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is normal to the orbit plane.)





Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ) *geometry of orbital mechanics*

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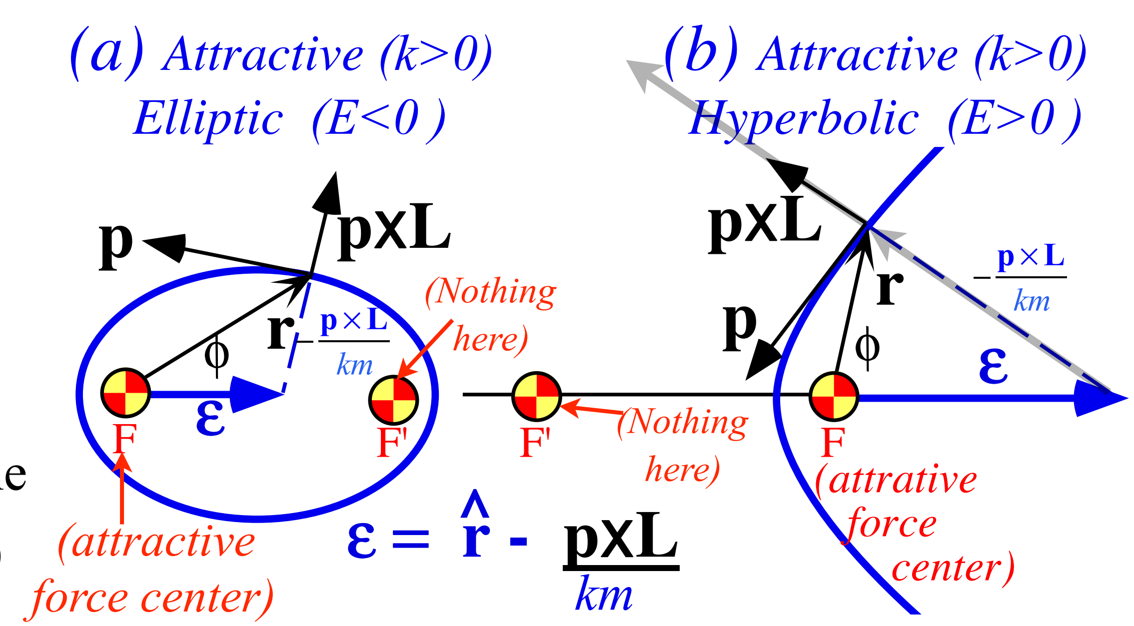

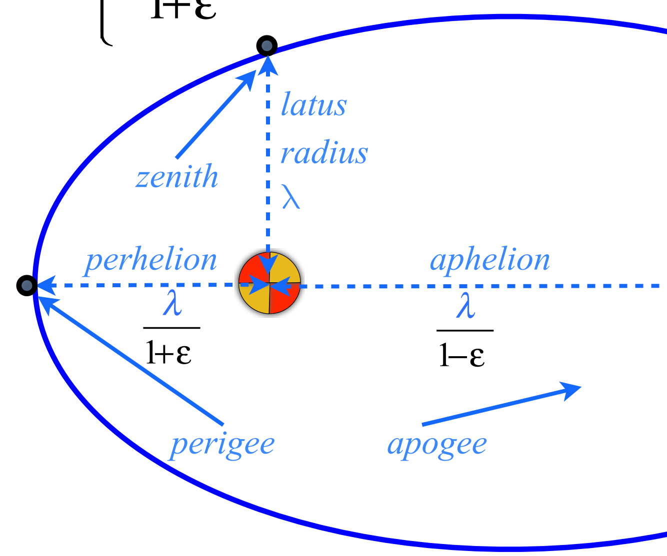

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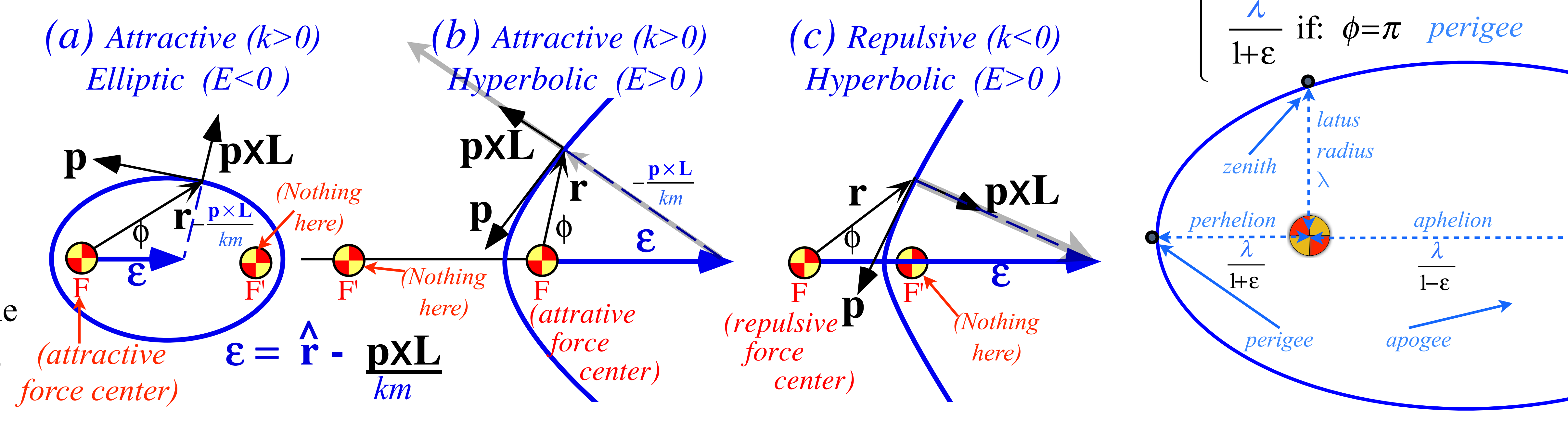


Rutherford scattering and hyperbolic orbit geometry

Backward vs forward scattering angles and orbit construction example

Parabolic "kite" and orbital envelope geometry

Differential and total scattering cross-sections

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{r}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and orbital radius* $\mathbf{r}$

➡ *Review and connection to usual orbital algebra (previous lecture)*

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{p}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and momentum* $\mathbf{p}=m\mathbf{v}$

General geometric orbit construction using $\boldsymbol{\varepsilon}$*-vector and* (γ,R)*-parameters*

Derivation of $\boldsymbol{\varepsilon}$*-construction by analytic geometry*

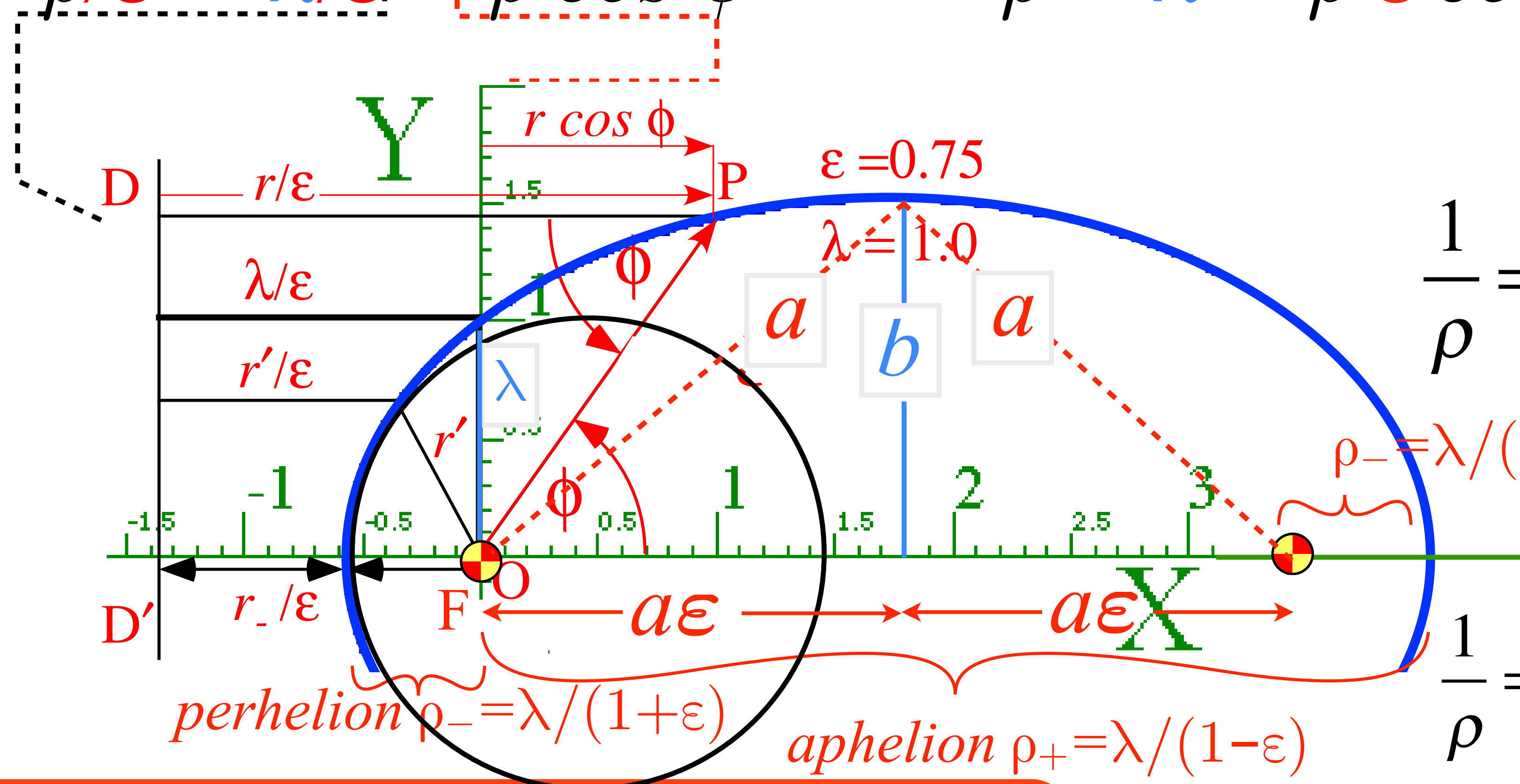
Coulomb orbit algebra of $\boldsymbol{\varepsilon}$*-vector and Kepler dynamics of momentum* $\mathbf{p}=m\mathbf{v}$

Example of complete $(\mathbf{r},\mathbf{p})$*-geometry of elliptical orbit*

Connection formulas for (γ,R)*-parameters with (a,b) and* (ε,λ)

(From Lecture 25 p. 64-74) *Geometry of Coulomb orbits (Let: $r = \rho$ here)*

$$\rho/\varepsilon = \lambda/\varepsilon + \rho\cos\phi \qquad \rho = \lambda + \rho\,\varepsilon\cos\phi \qquad \rho = \frac{\lambda}{1-\varepsilon\cos\phi}$$



$$\frac{1}{\rho} = \frac{1-\varepsilon\cos\phi}{\lambda} = \frac{1}{\lambda} - \frac{\varepsilon}{\lambda}\cos\phi$$

$$\frac{1}{\rho} = \frac{-k}{\mu^2/m} + \frac{\sqrt{k^2 + 2E\mu^2/m}}{\mu^2/m}\cos\phi$$

All conics defined by:

*Defining **eccentricity** ε*

Distance to F_{ocal}-point $= \varepsilon \cdot$ Distance to $D_{irectrix}$-line

Major axis: $\rho_+ + \rho_- = 2a$

$\rho_+ + \rho_- = [\lambda(1+\varepsilon) + \lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda/|1-\varepsilon^2|$

Focal axis: $\rho_+ - \rho_- = 2a\varepsilon$

$\rho_+ - \rho_- = [\lambda(1+\varepsilon) - \lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda\varepsilon/|1-\varepsilon^2|$

Minor radius: $b = \sqrt{(a^2 - a^2\varepsilon^2)} = \sqrt{(a\lambda)}$ *(ellipse: $\varepsilon < 1$)*

Minor radius: $b = \sqrt{(a^2\varepsilon^2 - a^2)} = \sqrt{(\lambda a)}$ *(hyperb: $\varepsilon > 1$)*

(x,y) parameters	physical constants	(r,ϕ) parameters
major radius $a = \frac{k}{2E}$	*Energy* $E = \frac{k}{2a}$	$\varepsilon = \sqrt{\frac{k^2 m + 2\mu^2 E}{k^2 m}} = \sqrt{1 \pm \frac{b^2}{a^2}}$ *eccentricity*
minor radius $b = \frac{\mu}{\sqrt{2m\lvert E\rvert}}$	*Orbital Momentum* $\mu = \sqrt{km\lambda}$	*latus radius* $\lambda = \frac{\mu^2}{km} = \frac{b^2}{a}$

$\varepsilon^2 = 1 - \frac{b^2}{a^2}$ (ellipse: $\varepsilon < 1$) $\quad \frac{b}{a} = \sqrt{1-\varepsilon^2}$

$\varepsilon^2 = 1 + \frac{b^2}{a^2}$ (hyperbola: $\varepsilon > 1$) $\quad \frac{b}{a} = \sqrt{\varepsilon^2 - 1}$

$\lambda = a(1-\varepsilon^2)$ *(ellipse: $\varepsilon < 1$)*

$\lambda = a(\varepsilon^2 - 1)$ *(hyperb: $\varepsilon > 1$)*

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Review and connection to usual orbital algebra (previous lecture)

➤ *Projection* **ε•p** *geometry of* **ε***-vector and momentum* **p**=*m***v**

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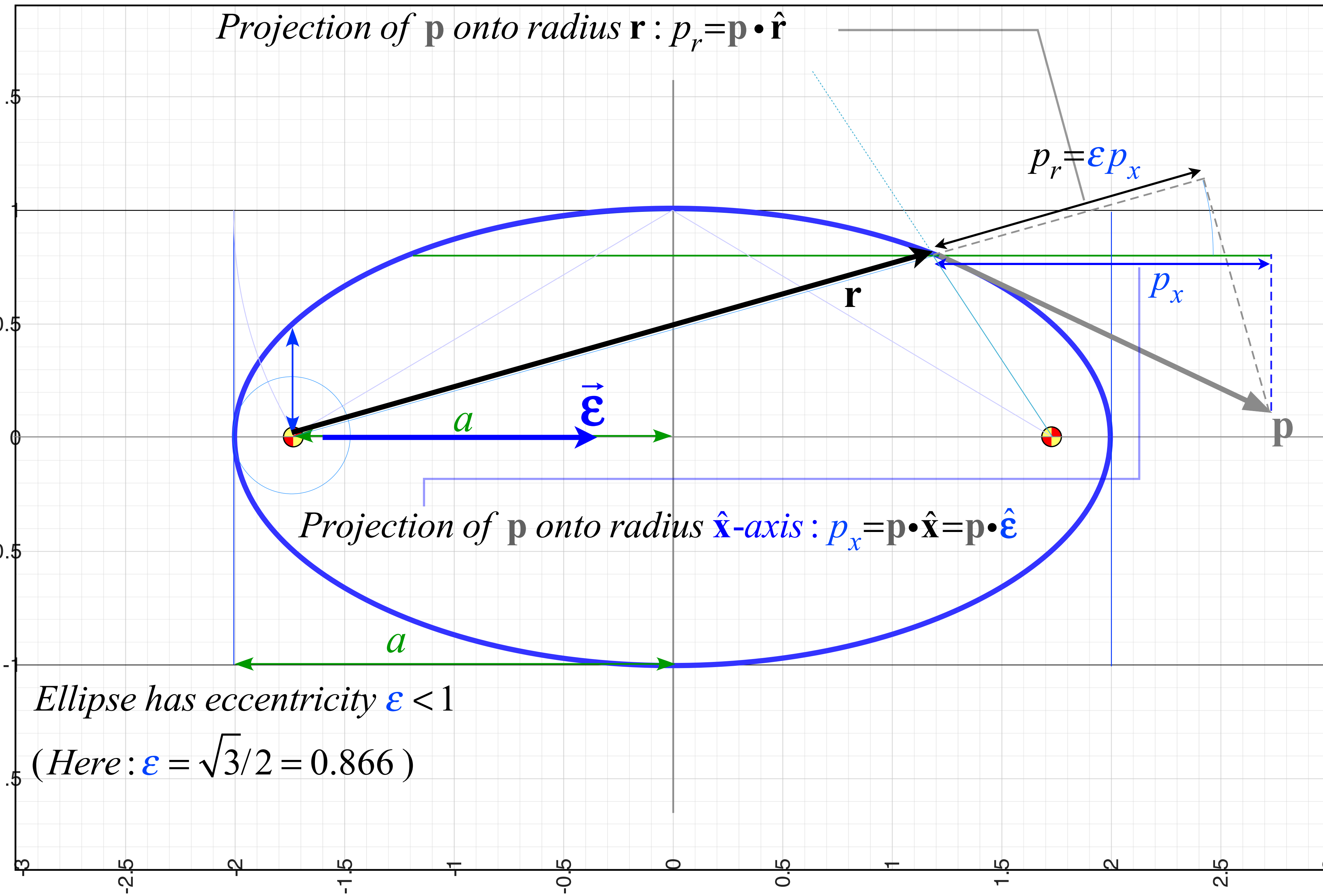
This says:
"*Projection p_r of* $\mathbf{p}$ *onto radial* $\mathbf{r}$ *or* $\mathbf{r}'$ *lines equals eccentricity* ε *times projection* p_x *of* $\mathbf{p}$ *onto orbit major axis*: $(\hat{\mathbf{x}} = \hat{\boldsymbol{\varepsilon}})$ "



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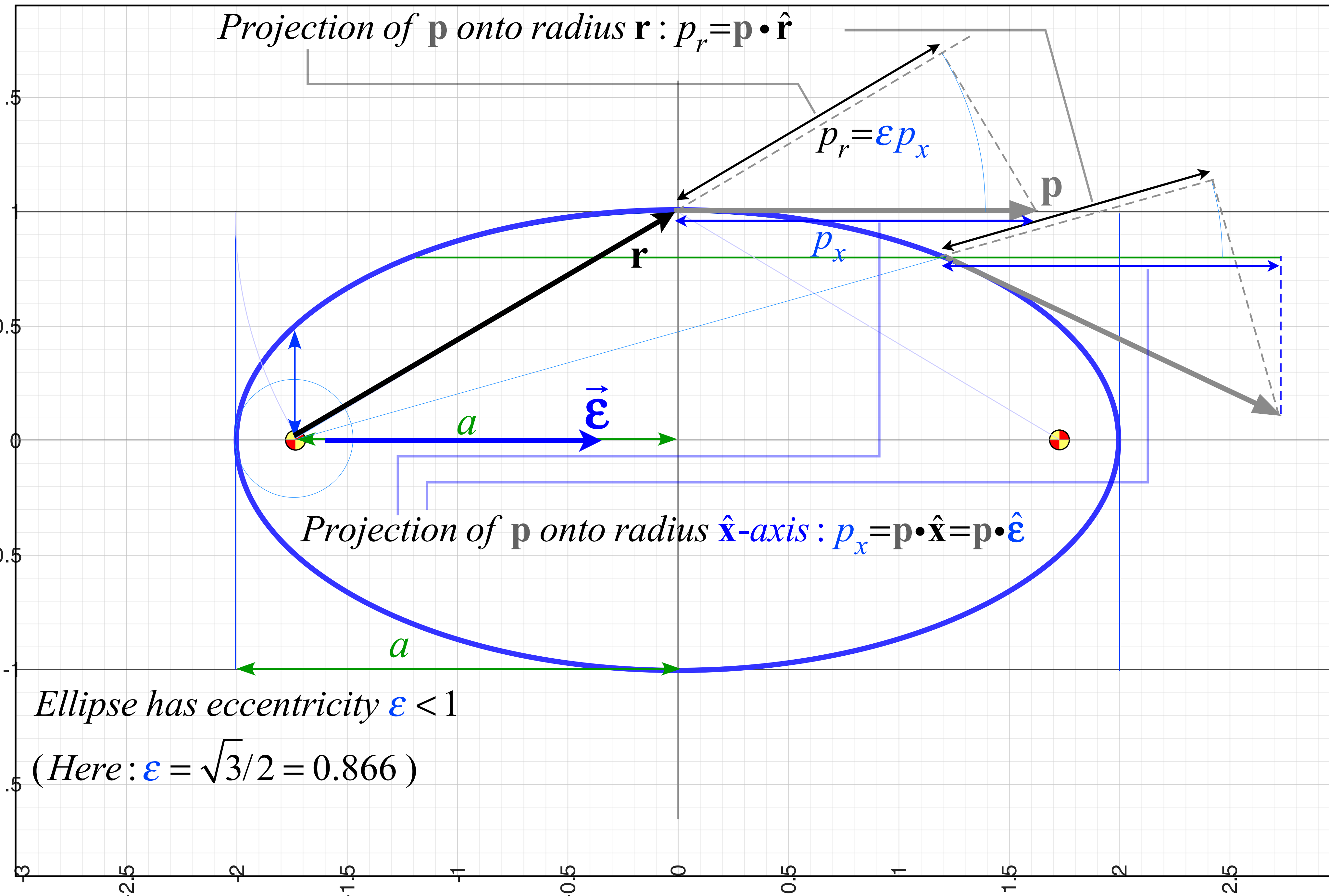

NOTE: Lengths of vectors $\mathbf{p}$ and -$\mathbf{p}$ are not drawn so they correctly show that momentum $\mathbf{p}=m\mathbf{v}$ grows as radial distance $r=|\mathbf{r}|$ falls. (To be shown on p. 92-102)

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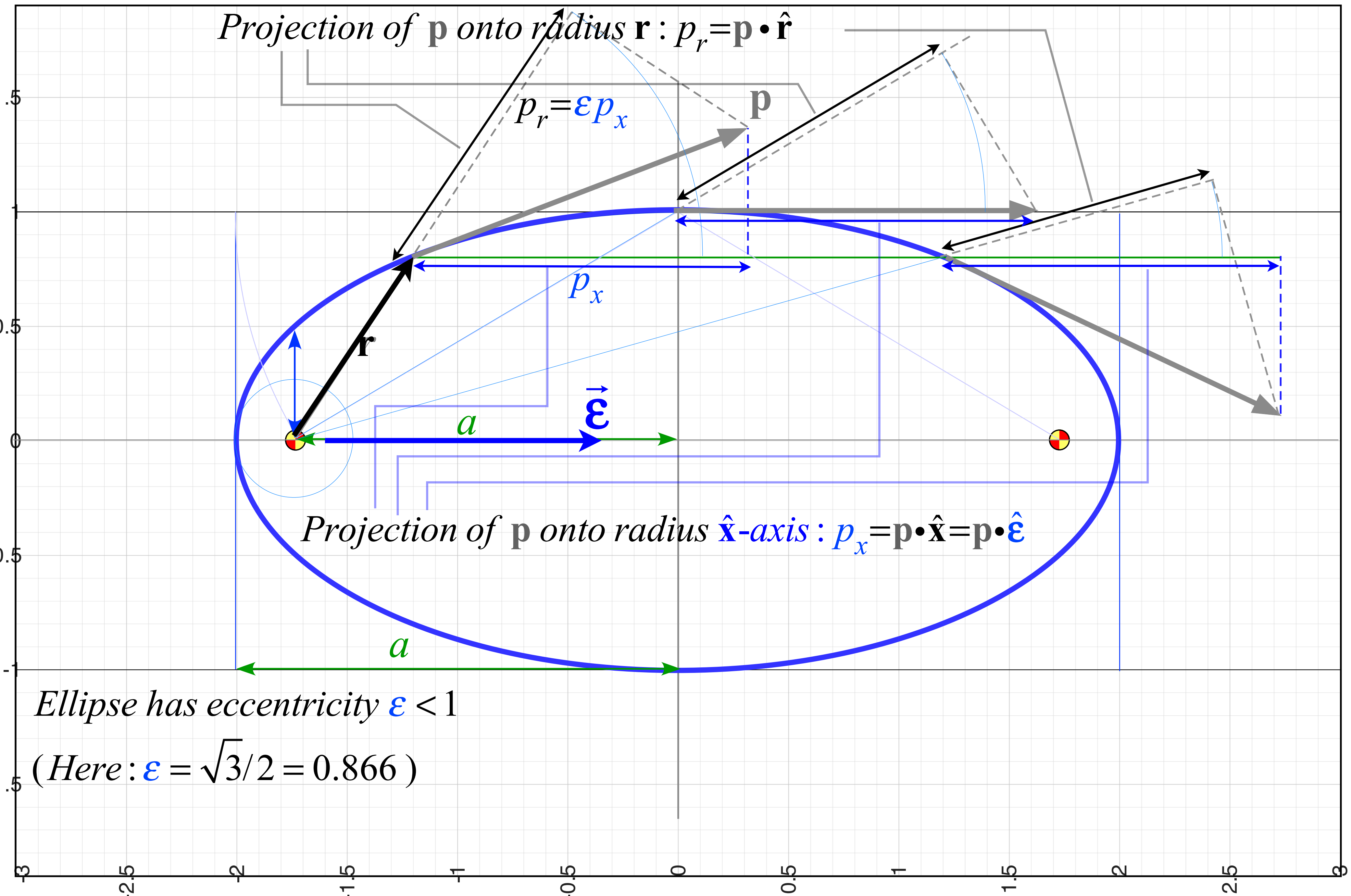

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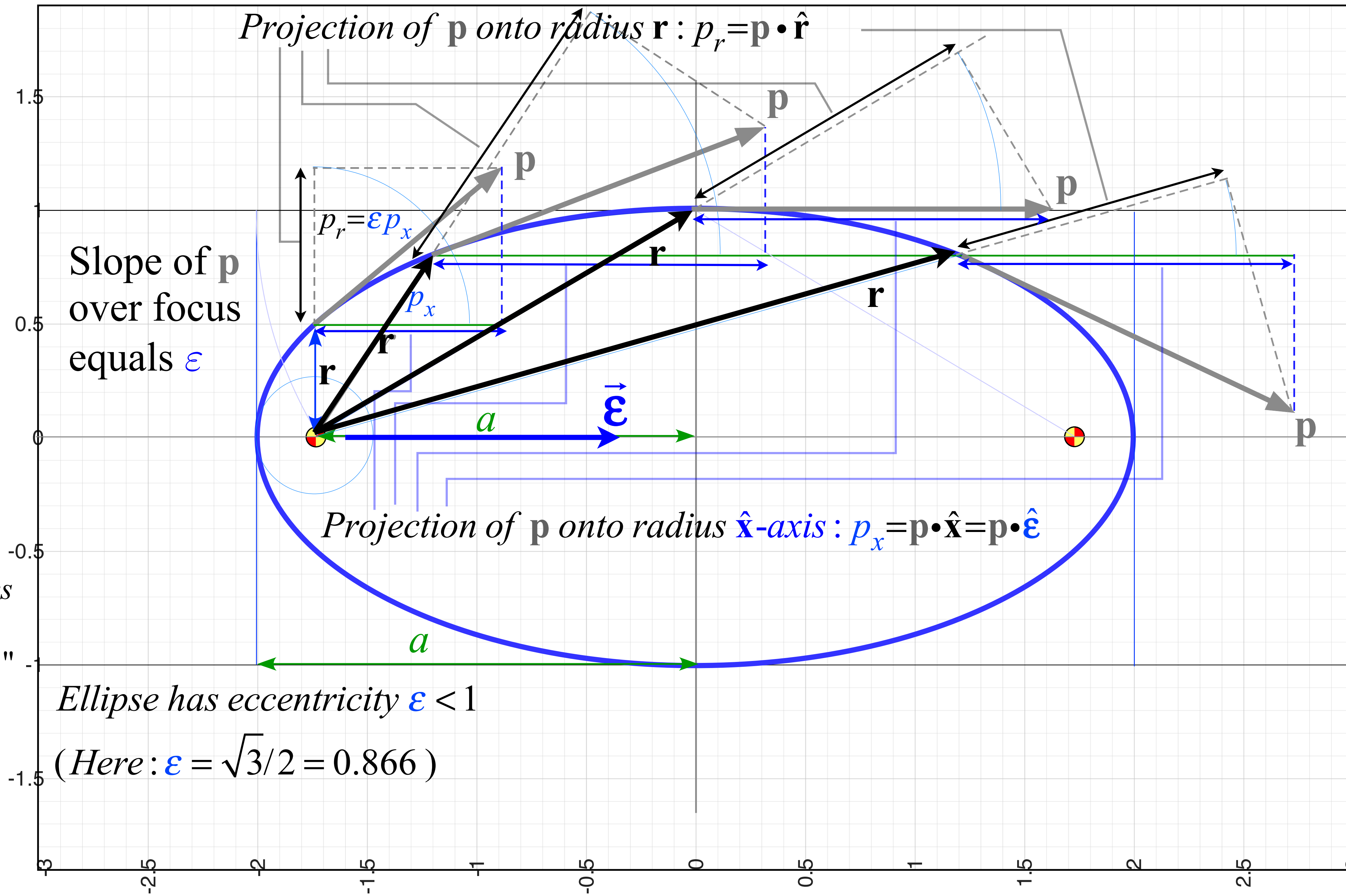
This says:
"*Projection* p_r *of* $\mathbf{p}$ *onto radial* $\mathbf{r}$ *or* $\mathbf{r}'$ *lines equals eccentricity* ε *times projection* p_x *of* $\mathbf{p}$ *onto orbit major axis*: $(\hat{\mathbf{x}} = \hat{\boldsymbol{\varepsilon}})$ "

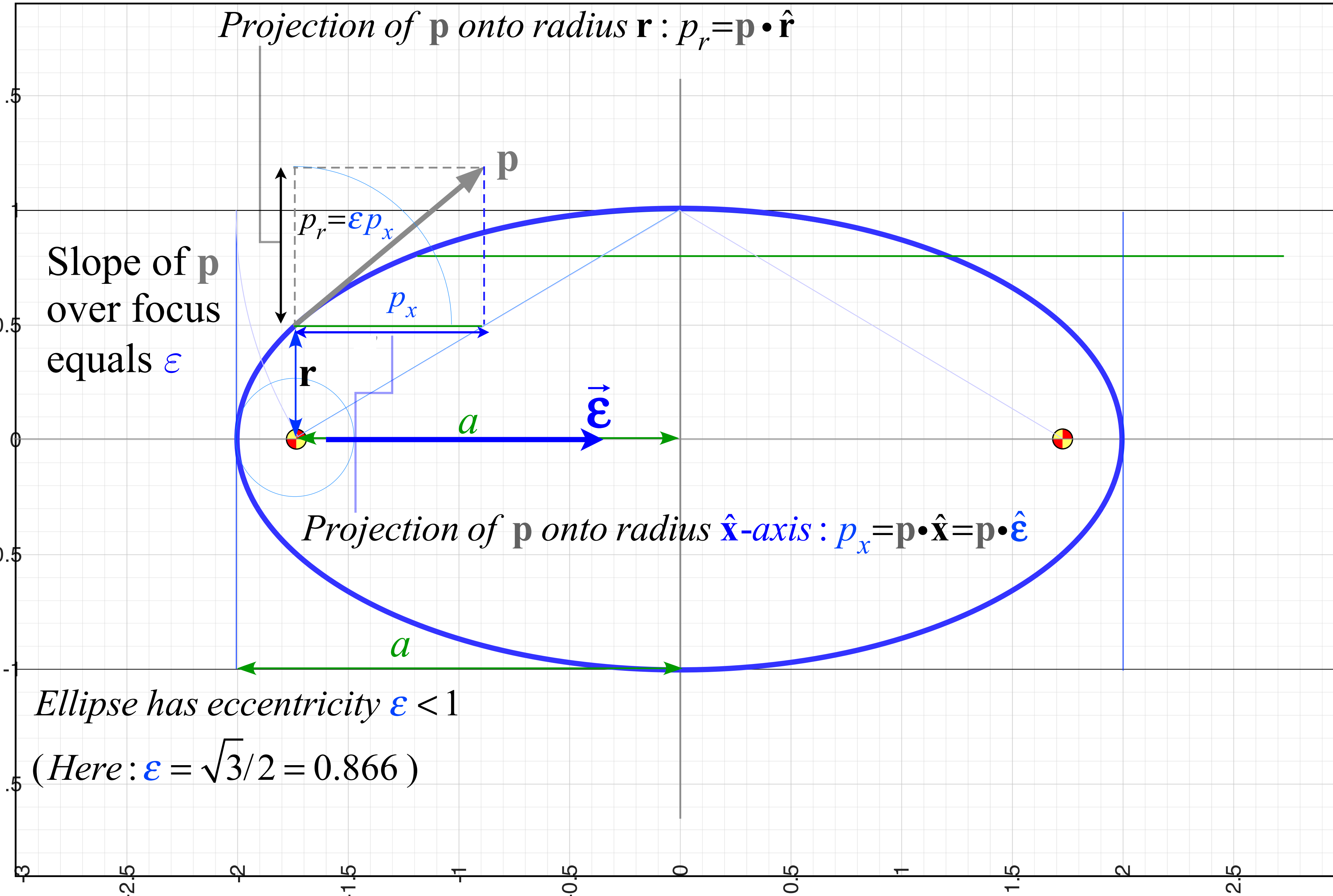


NOTE: Lengths of vectors $\mathbf{p}$ and -$\mathbf{p}$ are not drawn so they correctly show that momentum $\mathbf{p}=m\mathbf{v}$ grows as radial distance $r=|\mathbf{r}|$ falls. (To be shown on p. 92-102)

Dot product of ε with momentum vector $\mathbf{p}$:

$$\boldsymbol{\varepsilon}\bullet\mathbf{p} = \frac{\mathbf{p}\bullet\mathbf{r}}{r} - \frac{\mathbf{p}\bullet\mathbf{p}\times\mathbf{L}}{km}$$
$$= \mathbf{p}\bullet\hat{\mathbf{r}} = p_r = \varepsilon p_x$$

This says:
"*Projection* p_r *of* $\mathbf{p}$ *onto radial* $\mathbf{r}$ *or* $\mathbf{r}'$ *lines equals eccentricity* ε *times projection* p_x *of* $\mathbf{p}$ *onto orbit major axis*: $(\hat{\mathbf{x}} = \hat{\boldsymbol{\varepsilon}})$ "



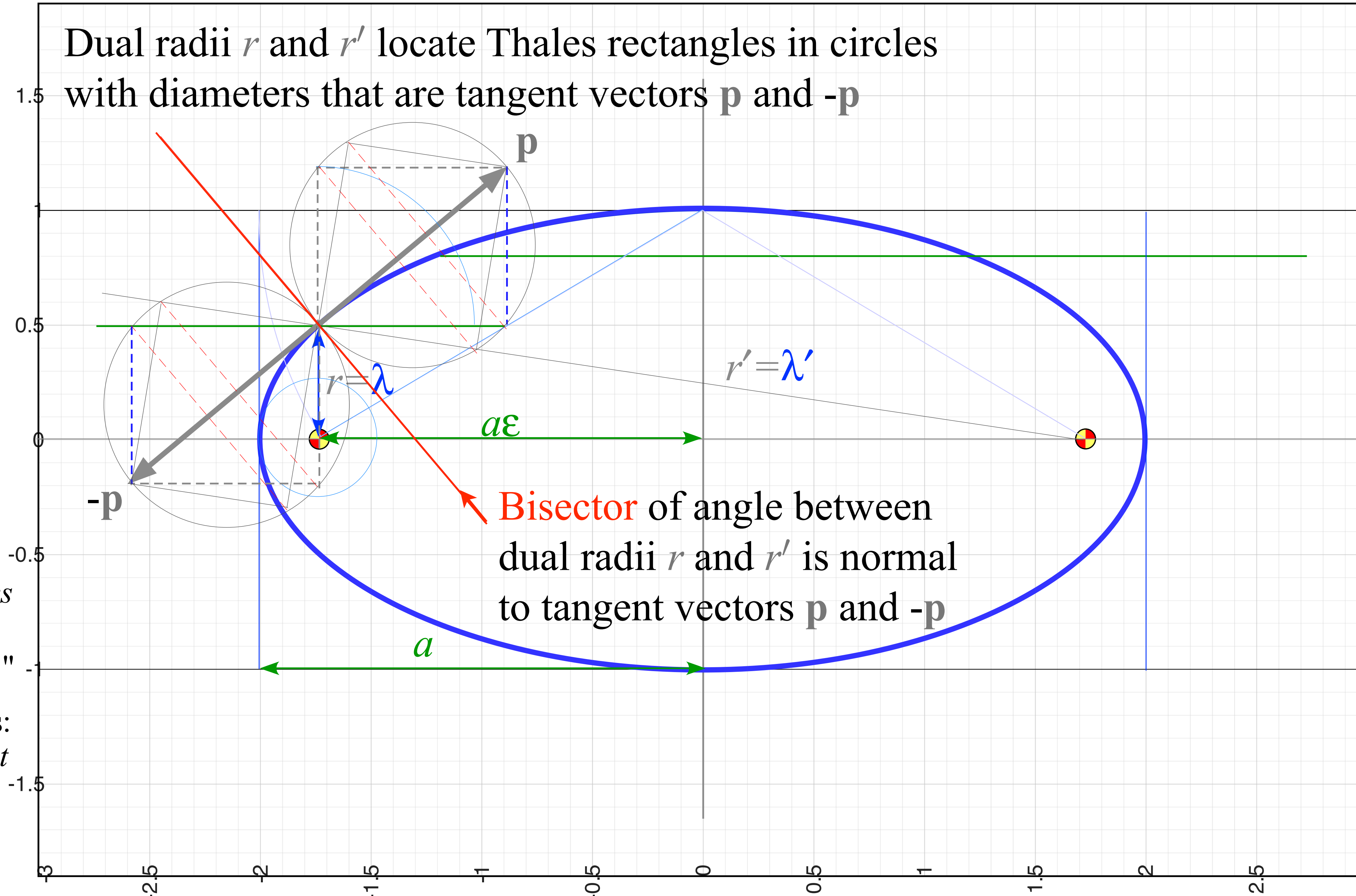
NOTE: Lengths of vectors $\mathbf{p}$ and -$\mathbf{p}$ are not drawn so they correctly show that momentum $\mathbf{p} = m\mathbf{v}$ grows as radial distance $r = |\mathbf{r}|$ falls. (To be shown on p. 92-102)

Dot product of ε with momentum vector $\mathbf{p}$:

$$\boldsymbol{\varepsilon}\bullet\mathbf{p}=\frac{\mathbf{p}\bullet\mathbf{r}}{r}-\frac{\mathbf{p}\bullet\mathbf{p}\times\mathbf{L}}{km}$$
$$=\mathbf{p}\bullet\hat{\mathbf{r}}=p_r=\varepsilon p_x$$

This says:
"*Projection* p_r *of* $\mathbf{p}$ *onto radial* $\mathbf{r}$ *or* $\mathbf{r}'$ *lines equals eccentricity* ε *times projection* p_x *of* $\mathbf{p}$ *onto orbit major axis* : $(\hat{\mathbf{x}}=\hat{\boldsymbol{\varepsilon}})$ "

Focal geometry demands:
"*Momentum* $\mathbf{p}$ *must bisect angle* $\measuredangle^{\mathbf{r}}_{\mathbf{r}'}$ *between radial* $\mathbf{r}$ *or* $\mathbf{r}'$ *lines.*"



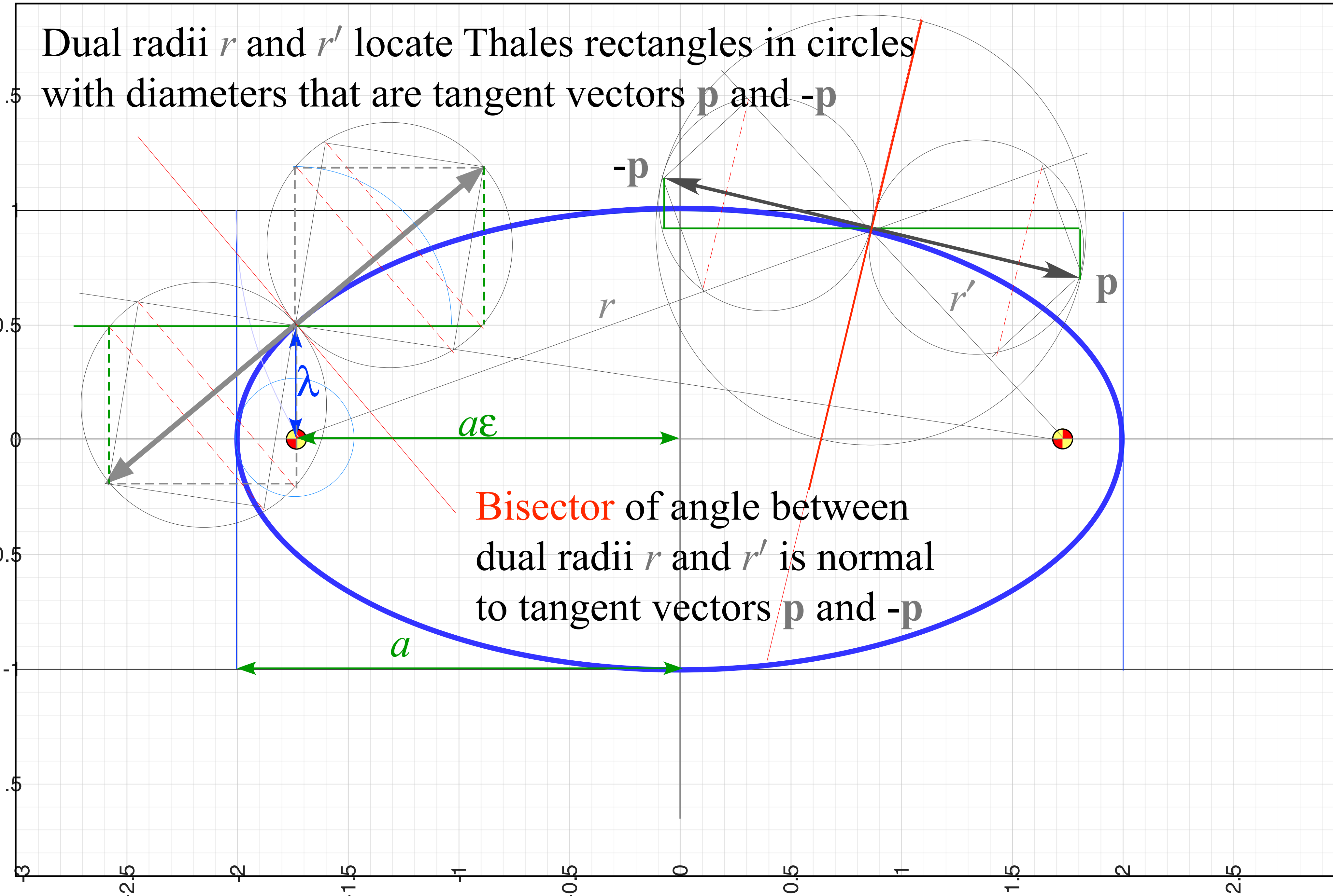
NOTE: Lengths of vectors $\mathbf{p}$ and -$\mathbf{p}$ are not drawn so they correctly show that momentum $\mathbf{p}=m\mathbf{v}$ grows as radial distance $r=|\mathbf{r}|$ falls. (To be shown on p. 92-102)

Dot product of ε with momentum vector $\mathbf{p}$:

$$\boldsymbol{\varepsilon} \bullet \mathbf{p} = \frac{\mathbf{p} \bullet \mathbf{r}}{r} - \frac{\mathbf{p} \bullet \mathbf{p} \times \mathbf{L}}{km}$$
$$= \mathbf{p} \bullet \hat{\mathbf{r}} = p_r = \varepsilon p_x$$

This says:
"*Projection p_r of* $\mathbf{p}$ *onto radial* $\mathbf{r}$ *or* $\mathbf{r}'$ *lines equals eccentricity* ε *times projection p_x of* $\mathbf{p}$ *onto orbit major axis* : $(\hat{\mathbf{x}} = \hat{\boldsymbol{\varepsilon}})$ "

Focal geometry demands:
"*Momentum* $\mathbf{p}$ *must bisect angle* $\measuredangle^{\mathbf{r}}_{\mathbf{r}'}$ *between radial* $\mathbf{r}$ *or* $\mathbf{r}'$ *lines.*"



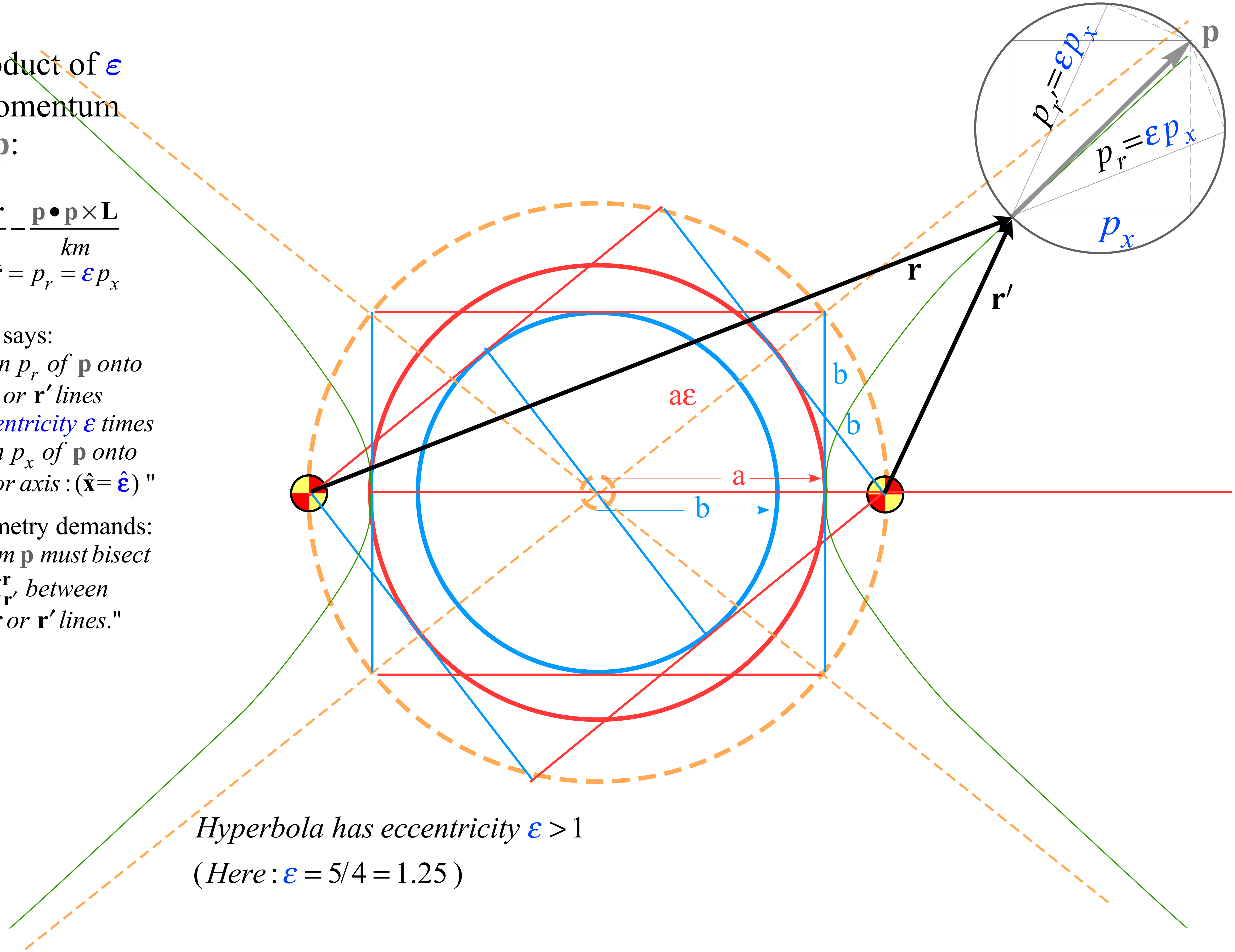
NOTE: Lengths of vectors $\mathbf{p}$ and $-\mathbf{p}$ are not drawn so they correctly show that momentum $\mathbf{p}=m\mathbf{v}$ grows as radial distance $r=|\mathbf{r}|$ falls. (To be shown on p. 92-102)

Dot product of ε with momentum vector **p**:

$$\boldsymbol{\varepsilon} \bullet \mathbf{p} = \frac{\mathbf{p}\bullet\mathbf{r}}{r} - \frac{\mathbf{p}\bullet\mathbf{p}\times\mathbf{L}}{km}$$
$$= \mathbf{p}\bullet\hat{\mathbf{r}} = p_r = \varepsilon p_x$$

This says:
"*Projection p_r of* **p** *onto radial* **r** *or* **r′** *lines equals eccentricity ε times projection p_x of* **p** *onto orbit major axis* : $(\hat{\mathbf{x}} = \hat{\boldsymbol{\varepsilon}})$ "

Focal geometry demands:
"*Momentum* **p** *must bisect angle* $\measuredangle^{\mathbf{r}}_{\mathbf{r}'}$ *between radial* **r** *or* **r′** *lines.*"



Hyperbola has eccentricity $\varepsilon > 1$

$(Here: \varepsilon = 5/4 = 1.25)$

Rutherford scattering and hyperbolic orbit geometry

Backward vs forward scattering angles and orbit construction example

Parabolic "kite" and orbital envelope geometry

Differential and total scattering cross-sections

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{r}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and orbital radius* $\mathbf{r}$

Review and connection to usual orbital algebra (previous lecture)

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{p}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and momentum* $\mathbf{p}=m\mathbf{v}$

➤ *General geometric orbit construction using* $\boldsymbol{\varepsilon}$*-vector and* (γ,R)*-parameters*

Derivation of $\boldsymbol{\varepsilon}$*-construction by analytic geometry*

Coulomb orbit algebra of $\boldsymbol{\varepsilon}$*-vector and Kepler dynamics of momentum* $\mathbf{p}=m\mathbf{v}$

Example of complete $(\mathbf{r},\mathbf{p})$*-geometry of elliptical orbit*

Connection formulas for (γ,R)*-parameters with (a,b) and* (ε,λ)

General geometric orbit construction using ***ε-vector*** *and (γ,R)-parameters*

Next several pages give step-by-step constructions of ***ε-vector*** *and Coulomb orbit and trajectory physics*

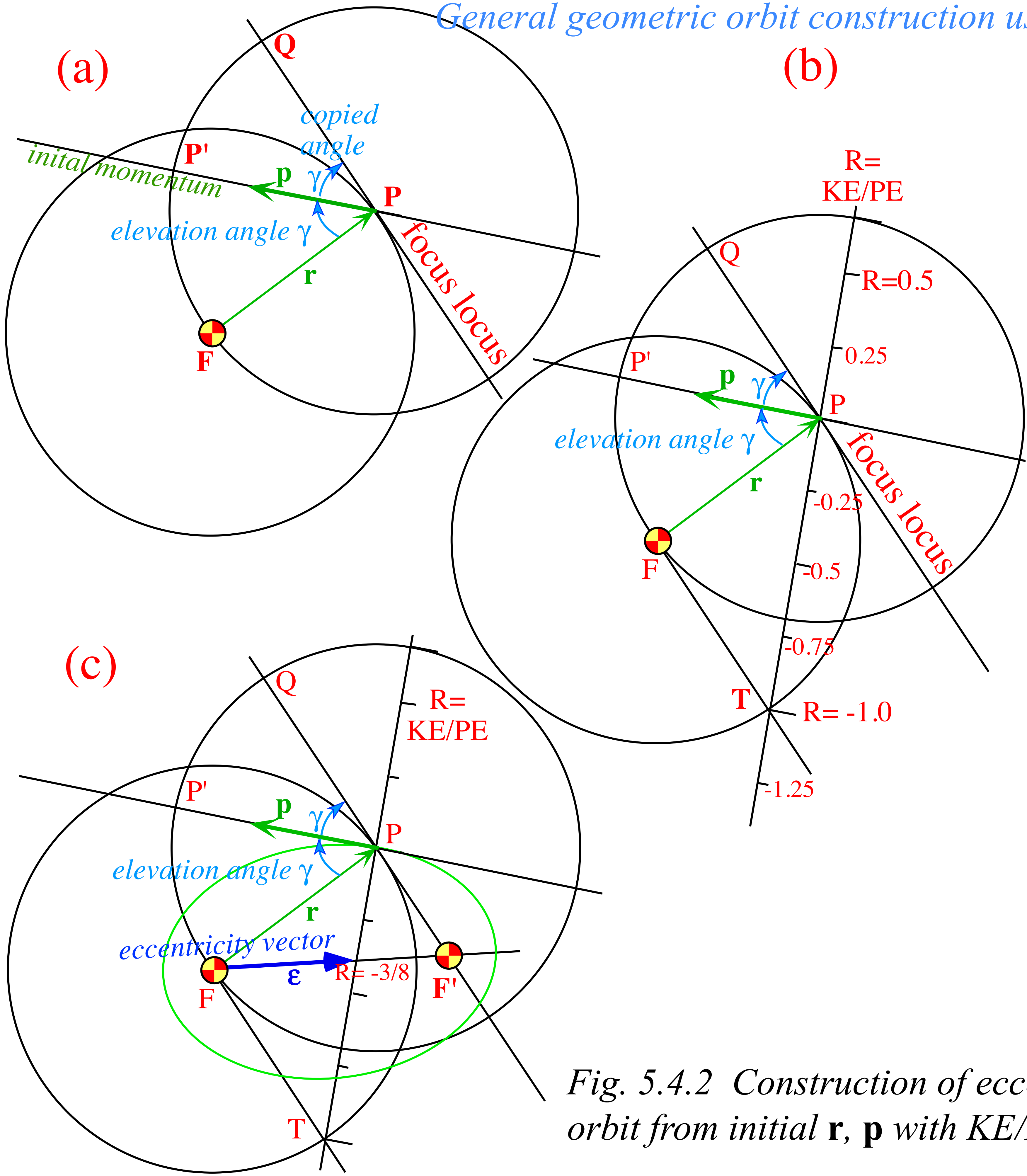


Fig. 5.4.2 Construction of eccentricity vector **ε** *and orbit from initial* **r**, **p** *with KE/PE=-3/8.*

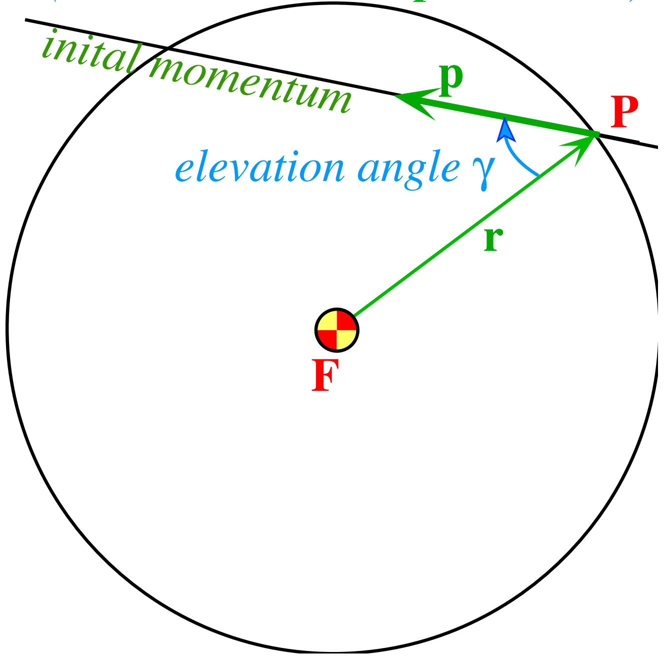
General geometric orbit construction using ε-vector and (γ,R)-parameters

Pick launch point P

(radius vector **r** *)*

and elevation angle γ *from radius*

(momentum initial **p** *direction)*



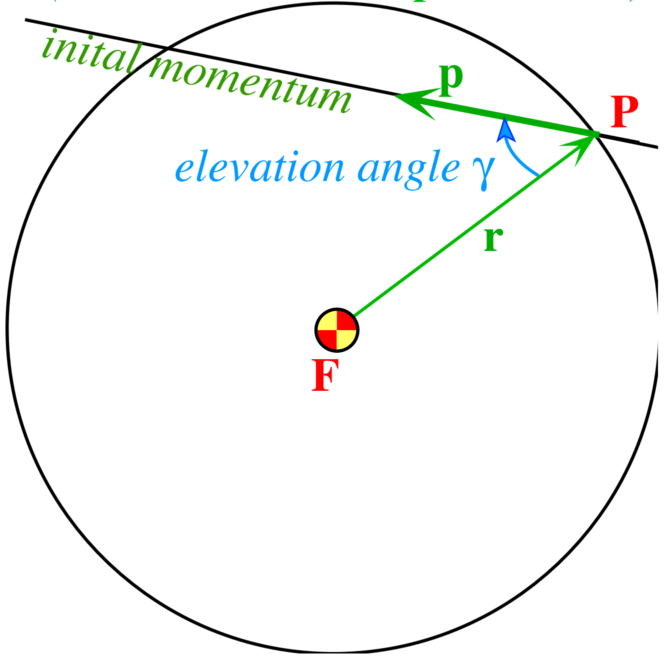
General geometric orbit construction using ε-vector and (γ,R)-parameters

Pick launch point P

(radius vector **r** *)*

and elevation angle γ *from radius*

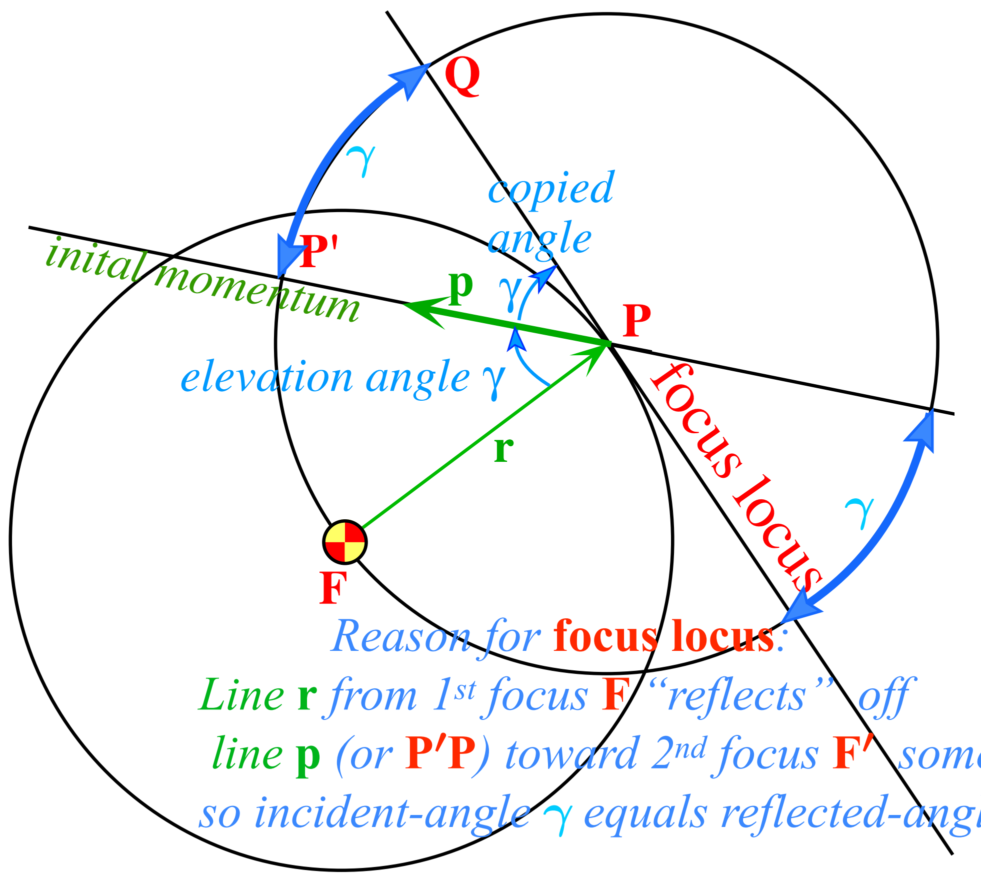
(momentum initial **p** *direction)*



Copy F-center circle around launch point P

Copy elevation angle γ *(*∠FPP′ *) onto* ∠P′PQ

Extend resulting line QPQ′ *to make* **focus locus**



Reason for **focus locus***:*

Line **r** *from 1st focus* **F** *"reflects" off*

line **p** *(or* **P′P***) toward 2nd focus* **F′** *somewhere*

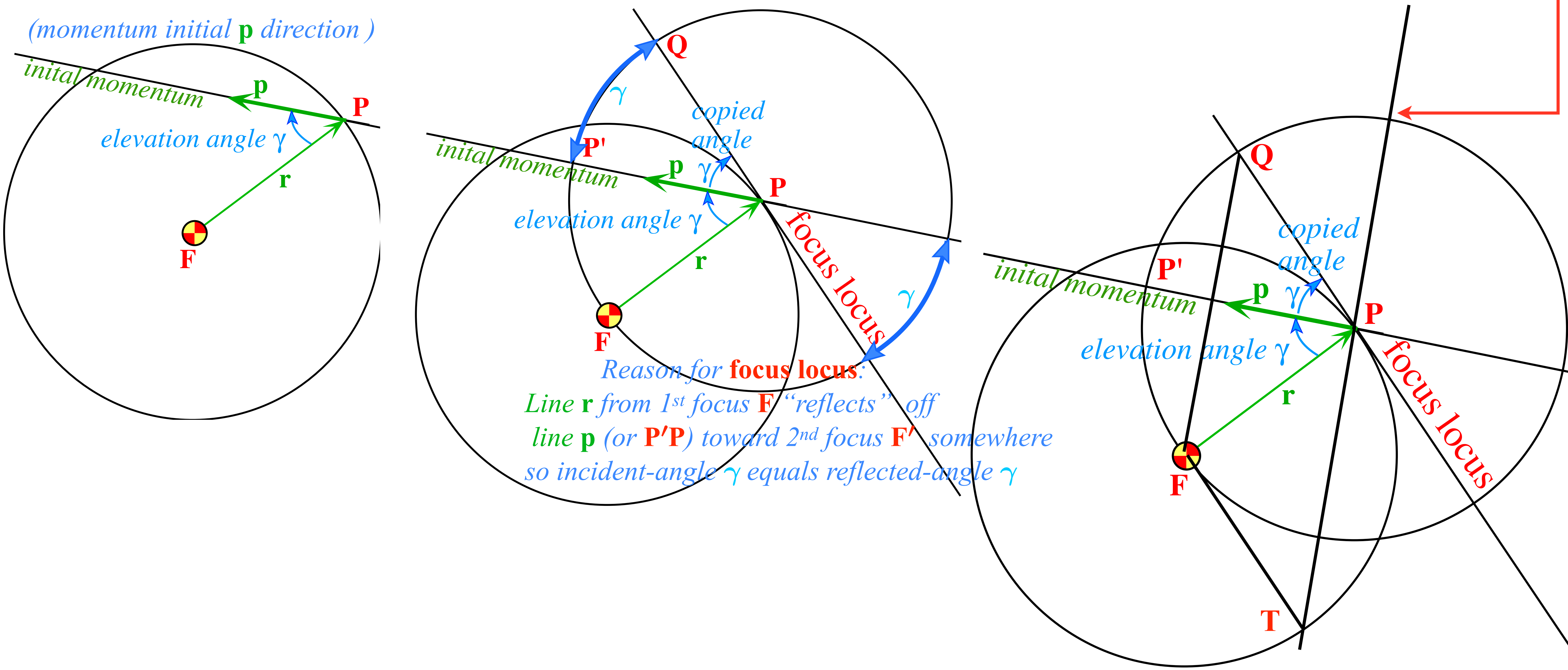
so incident-angle γ *equals reflected-angle* γ

General geometric orbit construction using ε-vector and (γ,R)-parameters

Pick launch point P
(radius vector **r** *)*
and elevation angle γ from radius
(momentum initial **p** *direction)*

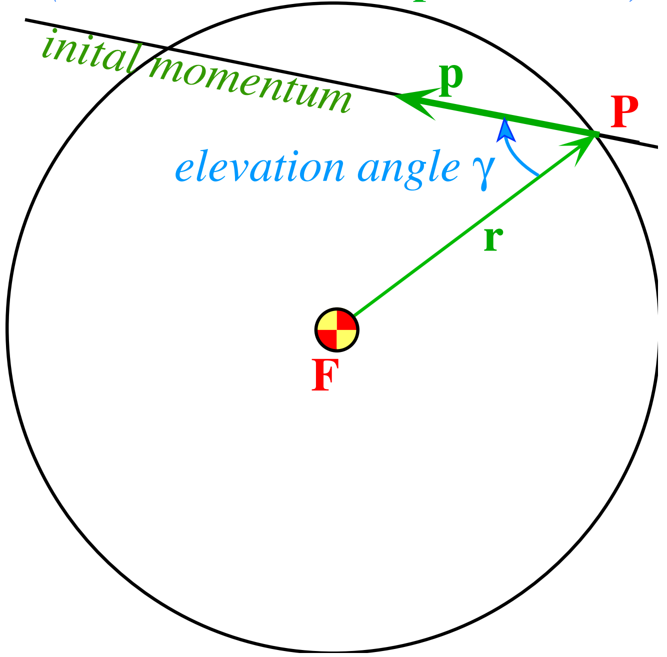
Copy F-center circle around launch point P
Copy elevation angle γ (∠FPP′) onto ∠P′PQ
Extend resulting line QPQ′ *to make* **focus locus**

Copy double angle 2γ (∠FPQ) onto ∠PFT
*Extend ∠PFT chord PT to make R-***ratio scale line**

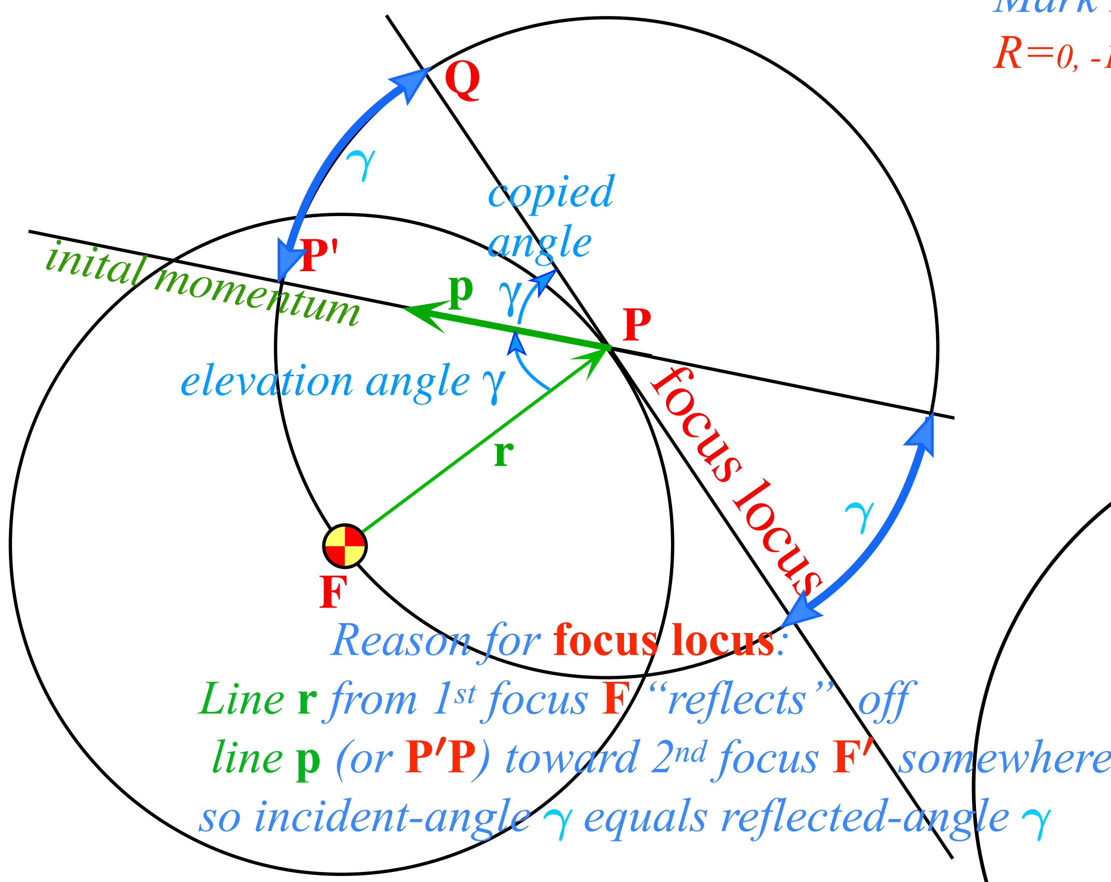


General geometric orbit construction using ε-vector and (γ,R)-parameters

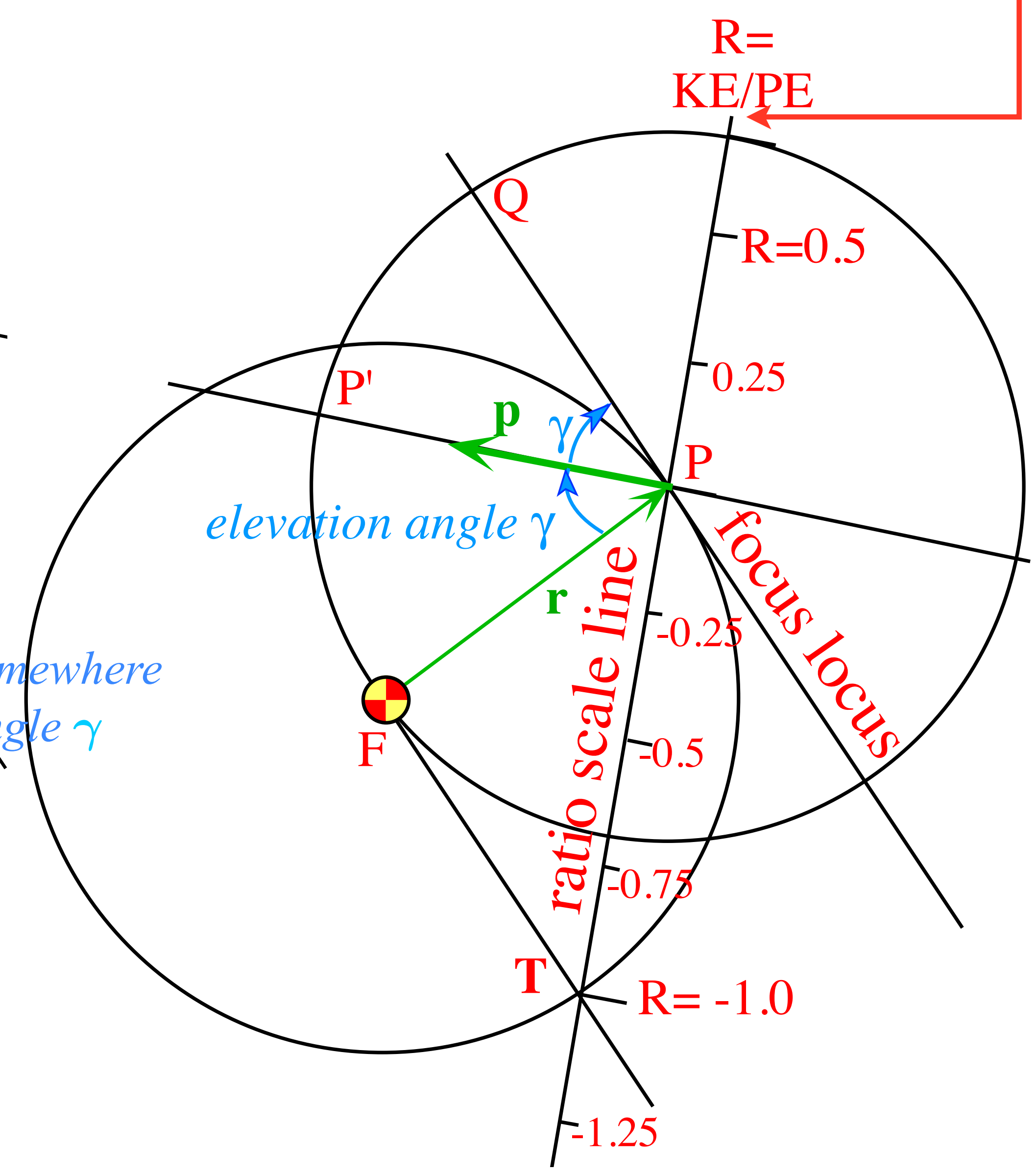
Pick launch point P
(radius vector **r** *)*
and elevation angle γ *from radius*
(momentum initial **p** *direction)*



Copy F-center circle around launch point P
Copy elevation angle γ *(∠FPP′) onto ∠P′PQ*
Extend resulting line QPQ′ *to make* **focus locus**



Copy double angle 2γ (∠FPQ) onto ∠PFT
Extend ∠PFT chord PT *to make R-***ratio scale line**
Label chord PT *with R=0 at* P *and R=-1.0 at* T.
*Mark R-***line** *fractions R=0, +1/4, +1/2,... above* P *and*
R=0, -1/8,-1/4,-1/2,...,-3/4 below P *and -5/4,-3/2,... below* T.

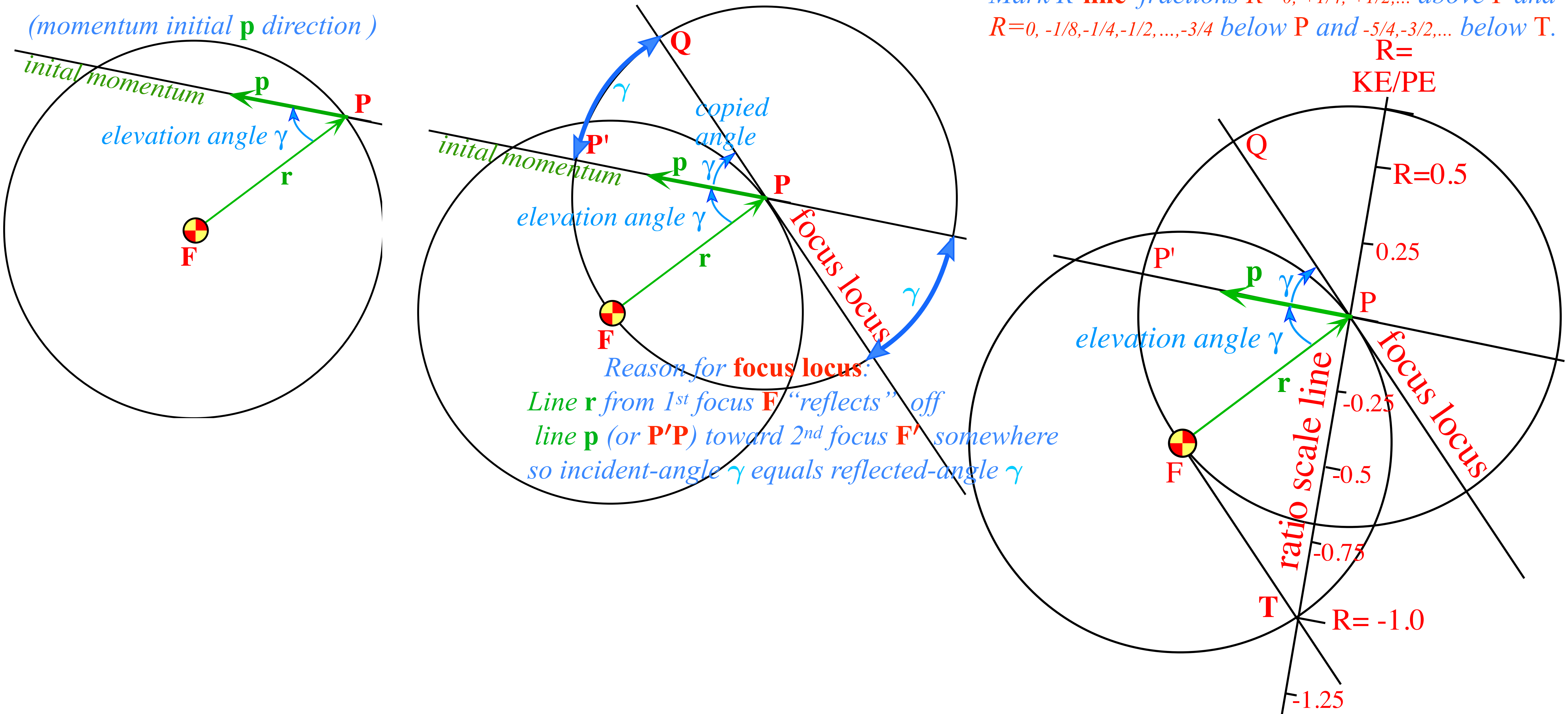


General geometric orbit construction using ε-vector and (γ,R)-parameters

Pick launch point P
(radius vector **r** *)*
and elevation angle γ from radius
(momentum initial **p** *direction)*

Copy F-center circle around launch point P
Copy elevation angle γ (∠FPP′) onto ∠P′PQ
Extend resulting line QPQ′ to make **focus locus**

Copy double angle 2γ (∠FPQ) onto ∠PFT
*Extend ∠PFT chord PT to make R-***ratio scale line**
Label chord PT with R=0 at P and R=-1.0 at T.
*Mark R-***line** *fractions R=0, +1/4, +1/2,... above P and R=0, -1/8,-1/4,-1/2,...,-3/4 below P and -5/4,-3/2,... below T.*



Reason for **focus locus**:
Line **r** *from 1st focus* **F** *"reflects" off line* **p** *(or* **P′P***) toward 2nd focus* **F′** *somewhere so incident-angle γ equals reflected-angle γ*

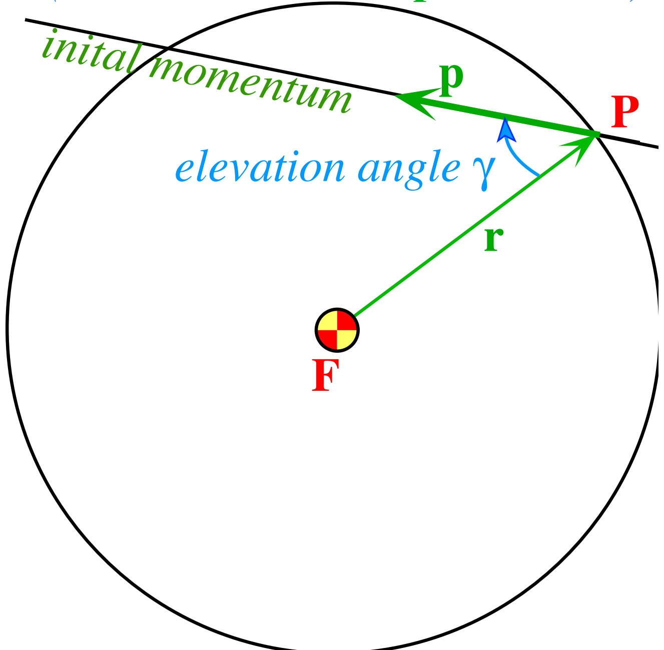
$$R = \frac{\text{Initial } KE}{\text{Initial } PE} = \frac{m\,v^2(0)/2}{-k/r(0)}$$

$$= \pm\left(\frac{\text{Initial velocity}}{\text{Escape velocity}}\right)^2 = \pm\frac{v^2(0)}{v^2(\infty)}$$

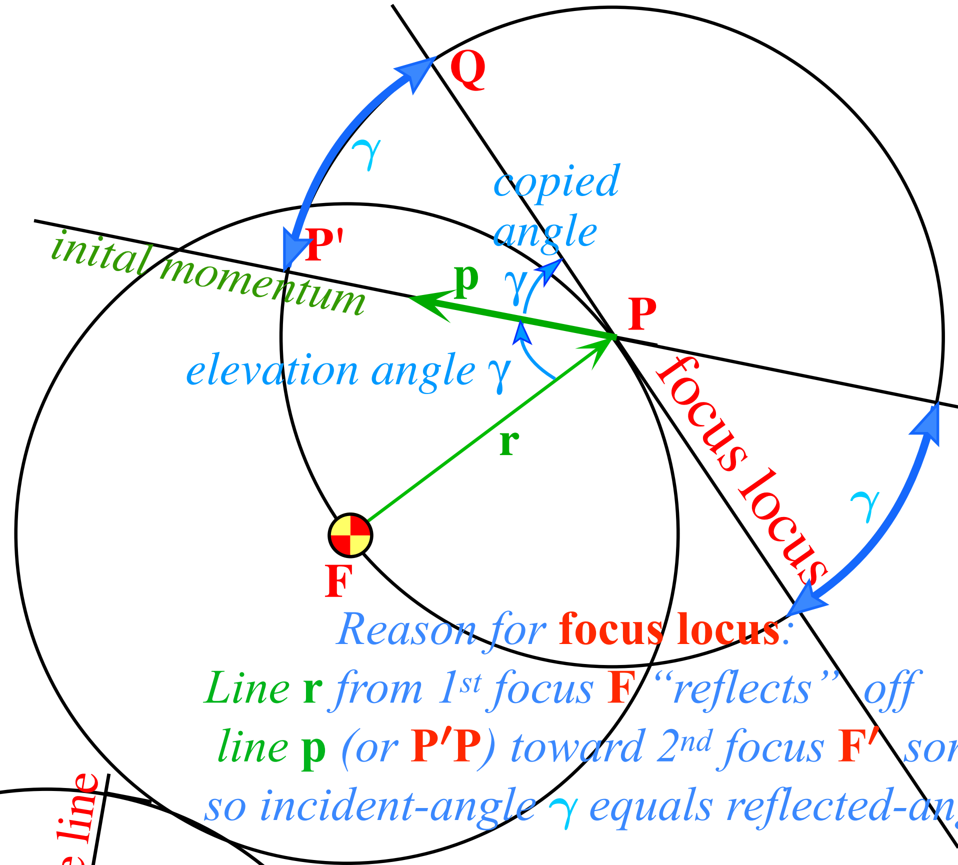
(To be proved later on p.72)

General geometric orbit construction using ε-vector and (γ,R)-parameters

Pick launch point P
(radius vector **r**)
and elevation angle γ from radius
(momentum initial **p** direction)

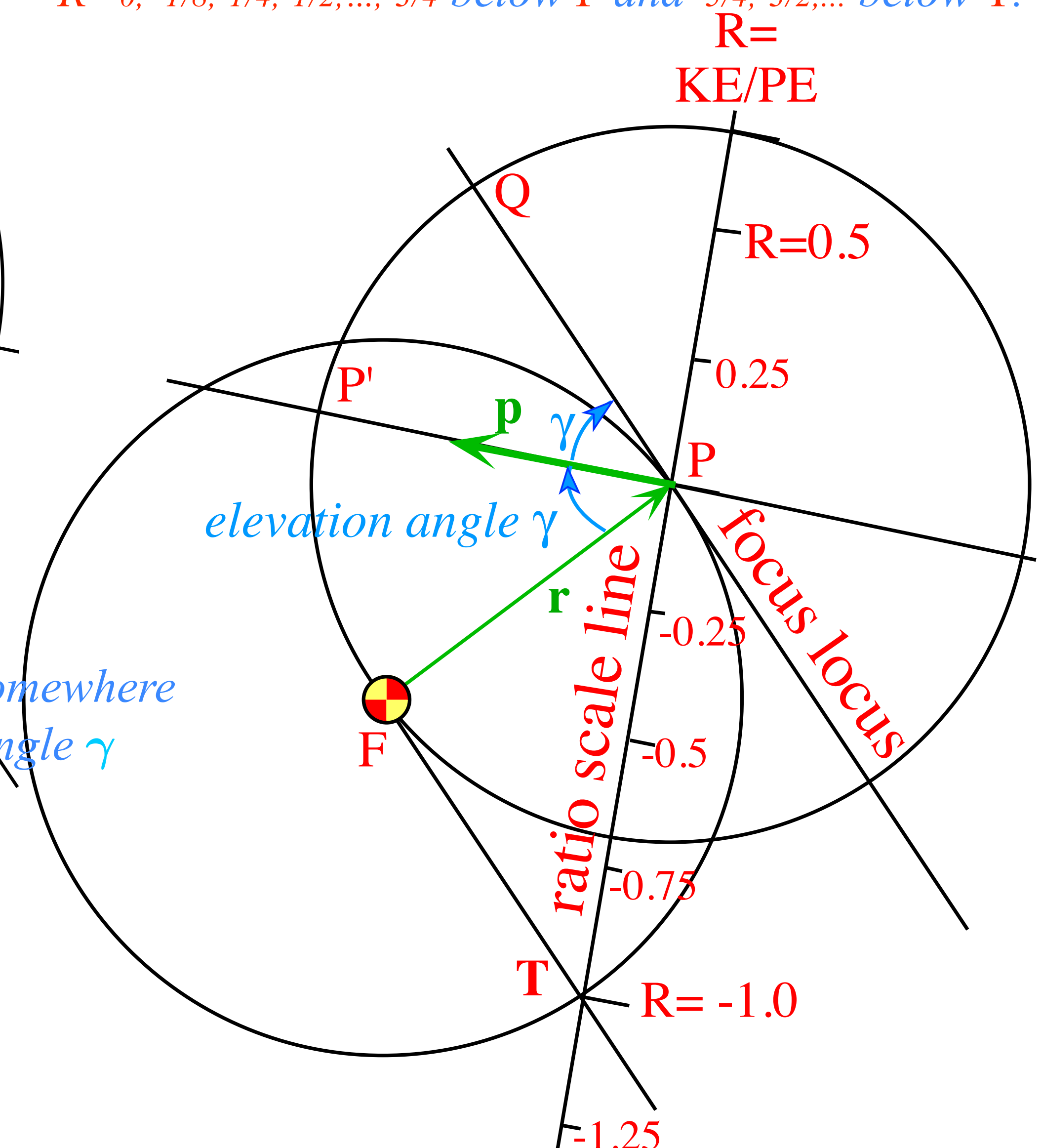


Copy F-center circle around launch point P
Copy elevation angle γ (∠FPP′) onto ∠P′PQ
Extend resulting line QPQ′ to make **focus locus**

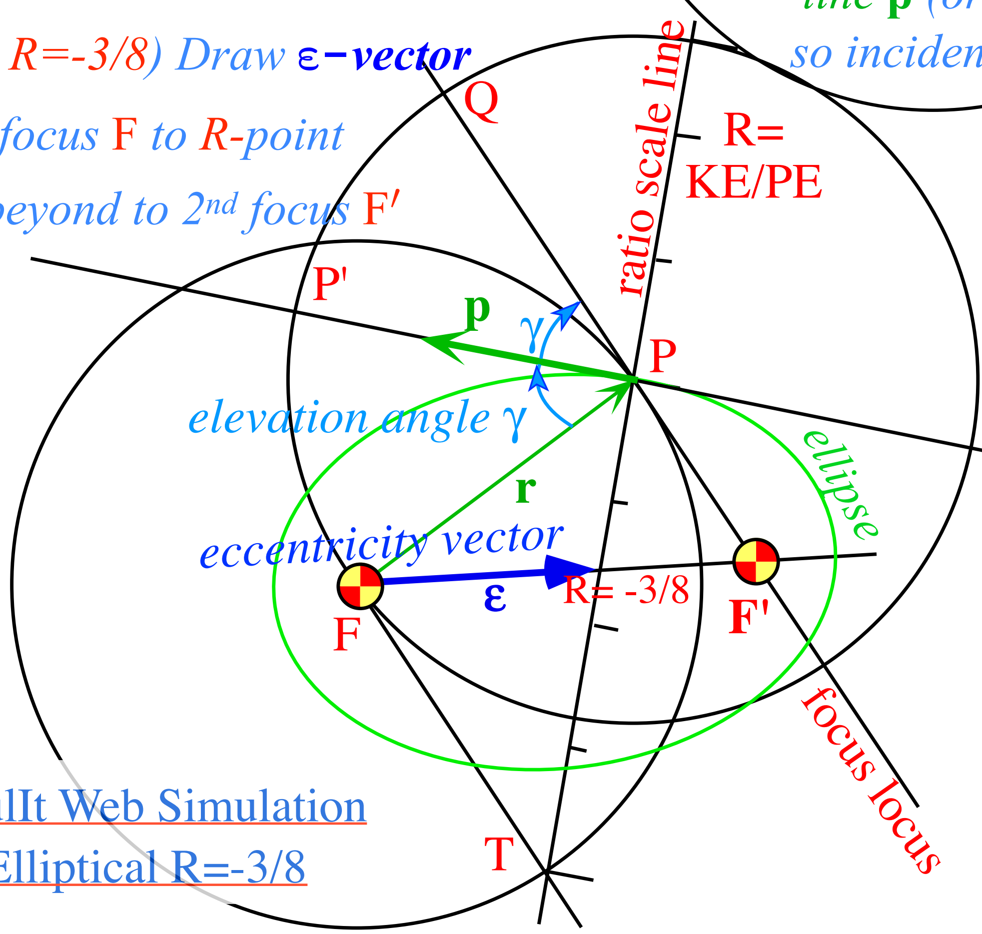


Reason for **focus locus**:
Line **r** from 1st focus **F** "reflects" off
line **p** (or **P′P**) toward 2nd focus **F′** somewhere
so incident-angle γ equals reflected-angle γ

Copy double angle 2γ (∠FPQ) onto ∠PFT
Extend ∠PFT chord PT to make R-**ratio scale line**
Label chord PT with R=0 at P and R=-1.0 at T.
Mark R-**line** fractions R=0, +1/4, +1/2,... above P and
R=0, -1/8,-1/4,-1/2,...,-3/4 below P and -5/4,-3/2,... below T.



Pick initial R=KE/PE value
(here R=-3/8) Draw **ε−vector**
from focus F to R-point
and beyond to 2nd focus F′



[CoulIt Web Simulation
Elliptical R=-3/8]

focus F and 2nd focus F′ allow final
construction of orbital trajectory.
Here it is an R=-3/8 ellipse.

$$R = \frac{\textit{Initial KE}}{\textit{Initial PE}} = \frac{m v^2(0)/2}{-k/r(0)}$$

$$= \pm \left(\frac{\textit{Initial velocity}}{\textit{Escape velocity}} \right)^2 = \pm \frac{v^2(0)}{v^2(\infty)}$$

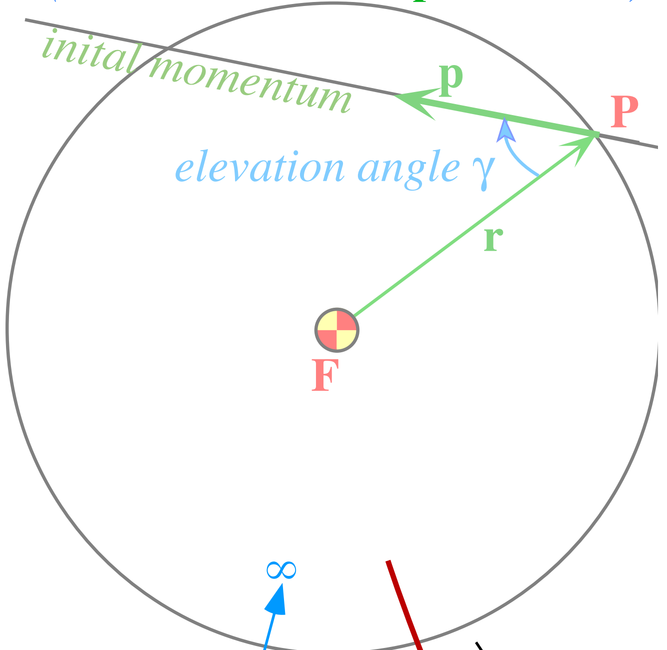
(Detailed Analytic geometry of ε-vector follows.)

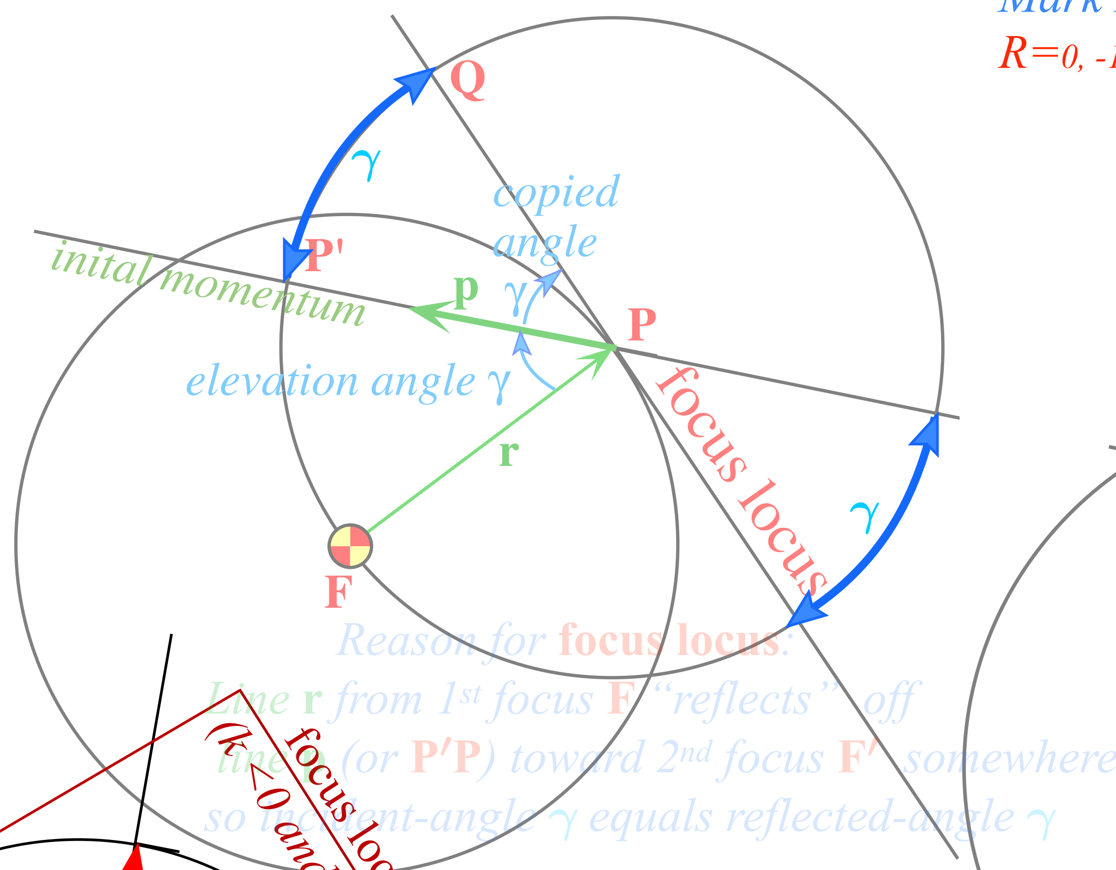
General geometric orbit construction using ε-vector and (γ,R)-parameters

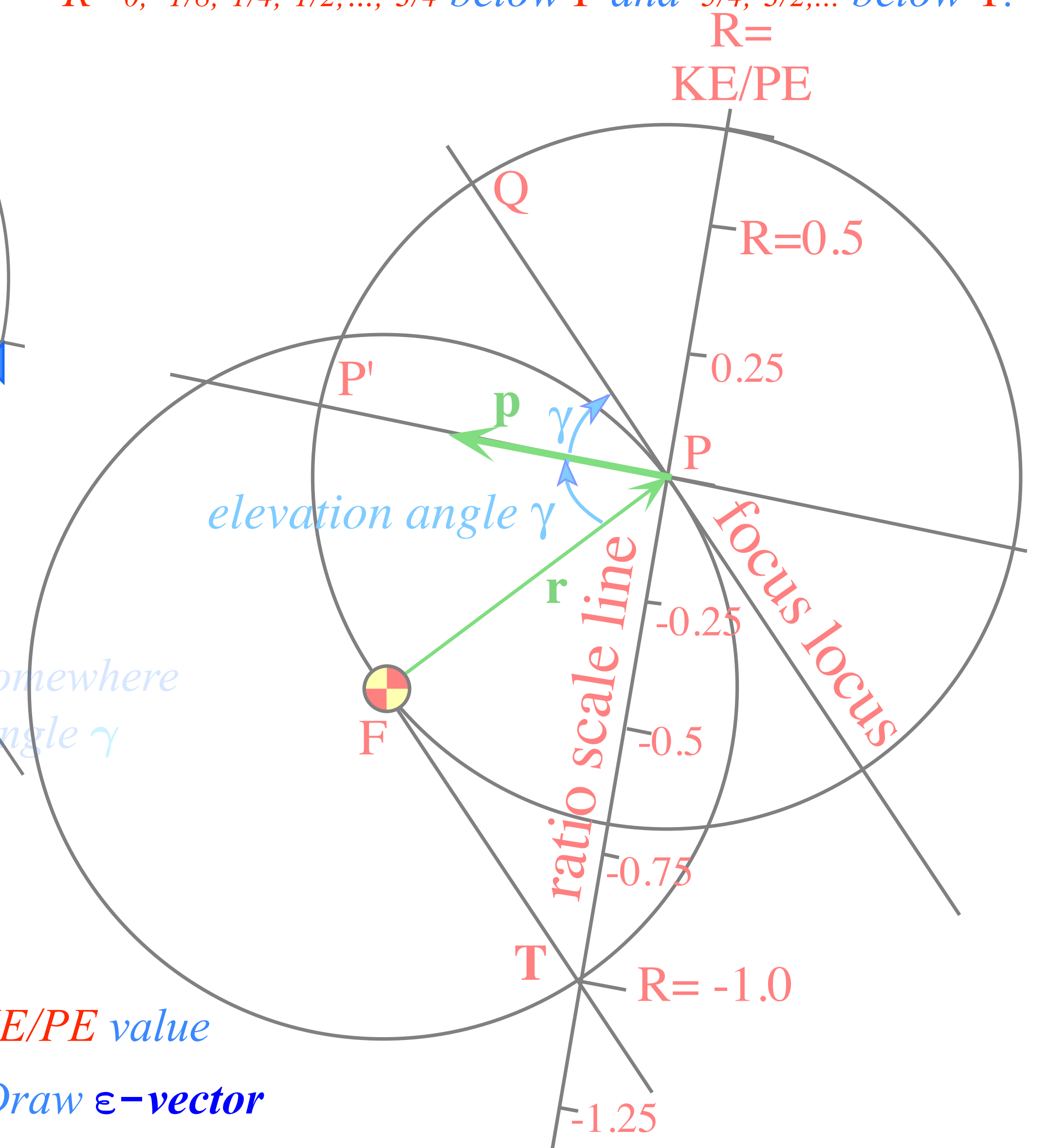
Pick launch point P
(radius vector **r** *)*
and elevation angle γ from radius
(momentum initial **p** *direction)*

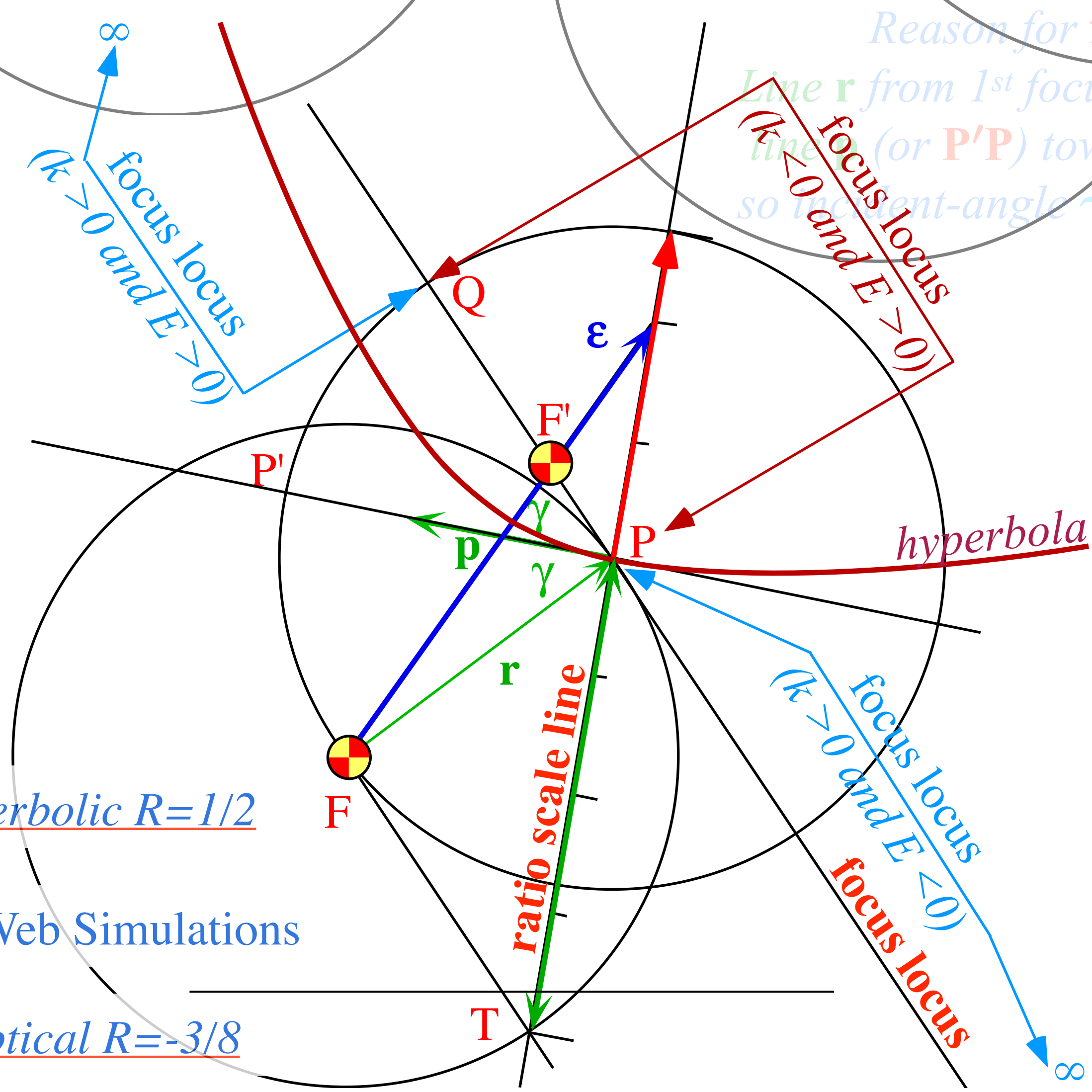
Copy F-center circle around launch point P
Copy elevation angle γ (∠FPP′) onto ∠P′PQ
Extend resulting line QPQ′ *to make* **focus locus**

Copy double angle 2γ (∠FPQ) onto ∠PFT
Extend ∠PFT chord PT *to make R-***ratio scale line**
Label chord PT *with R=0 at* P *and R=-1.0 at* T.
*Mark R-***line** *fractions R=0, +1/4, +1/2,... above* P *and R=0, -1/8,-1/4,-1/2,...,-3/4 below* P *and -5/4,-3/2,... below* T.









[*Hyperbolic R=1/2*]
CoulIt Web Simulations
[*Elliptical R=-3/8*]

Pick initial R=KE/PE value
(here R=+1/2) Draw **ε−vector**
from focus F *to R-point*
(Here it intersects 2nd focus F′

focus F *and 2nd focus* F′ *allow final construction of orbital trajectory.*
Here it is an R=+1/2 hyperbola.

(Detailed Analytic geometry of ε-vector follows.)

$$R = \frac{\textit{Initial KE}}{\textit{Initial PE}} = \frac{m v^2(0)/2}{-k/r(0)}$$

$$= \pm\left(\frac{\textit{Initial velocity}}{\textit{Escape velocity}}\right)^2 = \pm\frac{v^2(0)}{v^2(\infty)}$$

Rutherford scattering and hyperbolic orbit geometry

Backward vs forward scattering angles and orbit construction example

Parabolic "kite" and orbital envelope geometry

Differential and total scattering cross-sections

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{r}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and orbital radius* $\mathbf{r}$

Review and connection to usual orbital algebra (previous lecture)

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{p}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and momentum* $\mathbf{p}=m\mathbf{v}$

General geometric orbit construction using $\boldsymbol{\varepsilon}$*-vector and* (γ,R)*-parameters*

➤ *Derivation of* $\boldsymbol{\varepsilon}$*-construction by analytic geometry*

Coulomb orbit algebra of $\boldsymbol{\varepsilon}$*-vector and Kepler dynamics of momentum* $\mathbf{p}=m\mathbf{v}$

Example of complete $(\mathbf{r},\mathbf{p})$*-geometry of elliptical orbit*

Connection formulas for (γ,R)*-parameters with* (a,b) *and* (ε,λ)

Derivation of $\boldsymbol{\varepsilon}$*-construction by analytic geometry*

$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} - \frac{\mathbf{p}\times\mathbf{L}}{km} = \hat{\mathbf{r}} - \frac{(mv_0)(mv_0 r_0)\sin\gamma}{km}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

where: $\mathbf{L}_{\mathbf{p}\times} \equiv \mathbf{p}\times\mathbf{L}$

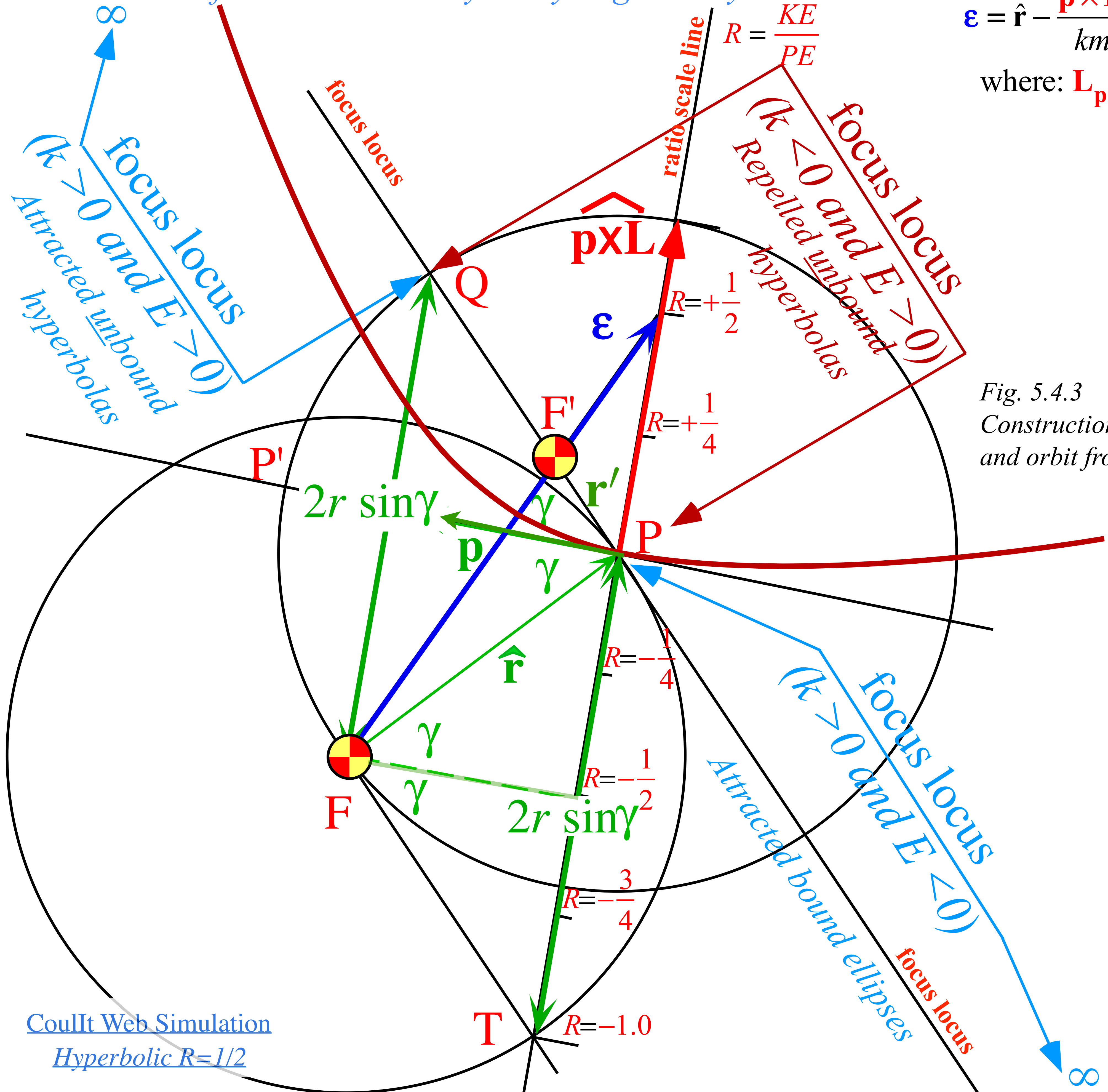


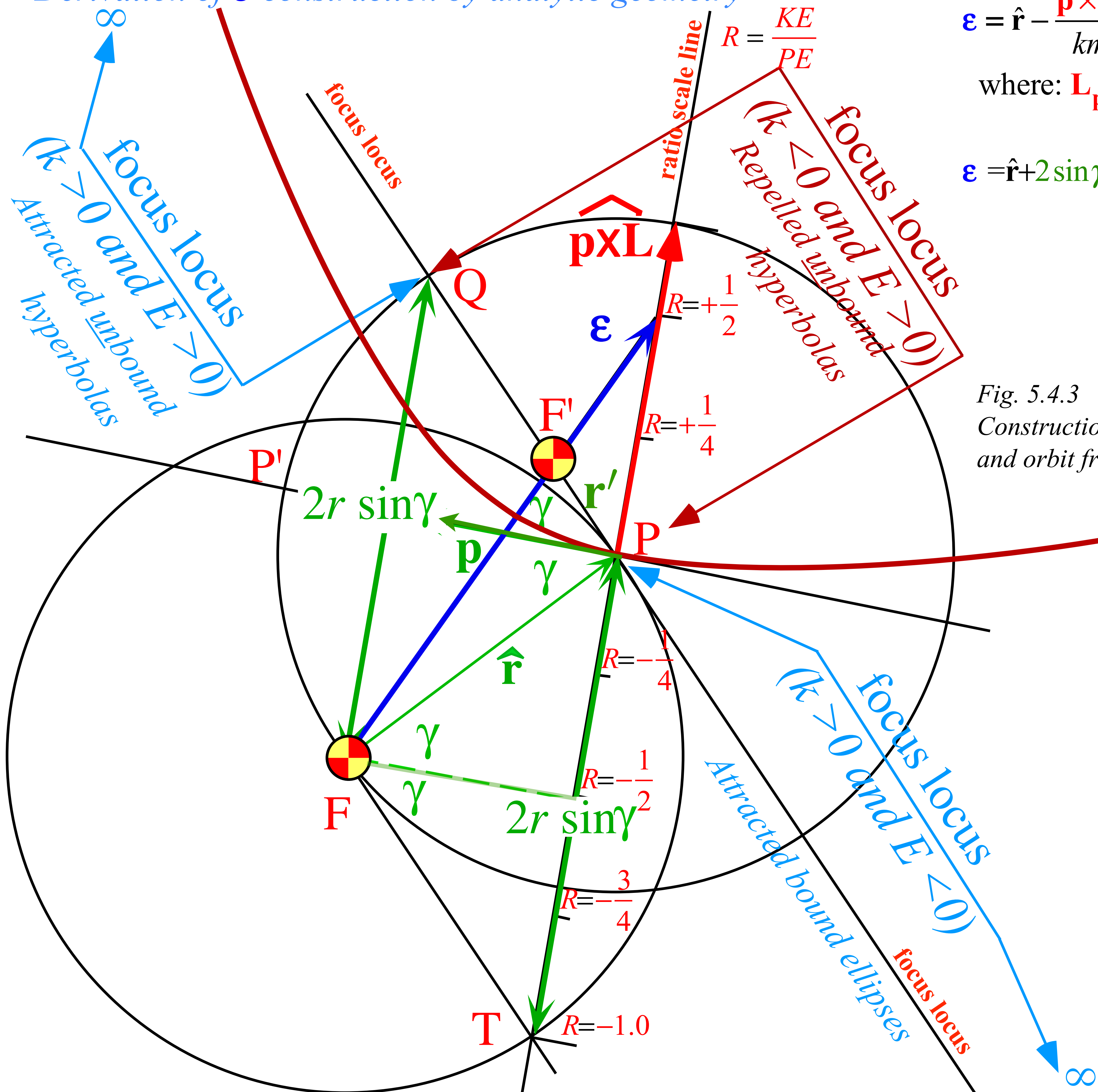
Fig. 5.4.3
Construction of eccentricity vector ε
and orbit from initial **r**, **p** *with KE/PE=+1/2.*

$$R = \frac{\textit{Initial KE}}{\textit{Initial PE}} = \frac{m\,v^2(0)/2}{-k/r(0)}$$

$$= \pm\left(\frac{\textit{Initial velocity}}{\textit{Escape velocity}}\right)^2 = \pm\frac{v^2(0)}{v^2(\infty)}$$

CoulIt Web Simulation
Hyperbolic R=1/2

Derivation of ε-construction by analytic geometry



$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} - \frac{\mathbf{p}\times\mathbf{L}}{km} = \hat{\mathbf{r}} - \frac{(mv_0)(mv_0r_0)\sin\gamma}{km}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

where: $\mathbf{L}_{\mathbf{p}\times} \equiv \mathbf{p}\times\mathbf{L}$

$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} + 2\sin\gamma \frac{mv_0^2/2}{-k/r_0}\hat{\mathbf{L}}_{\mathbf{p}\times} = \hat{\mathbf{r}} + 2\sin\gamma \frac{KE}{PE}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

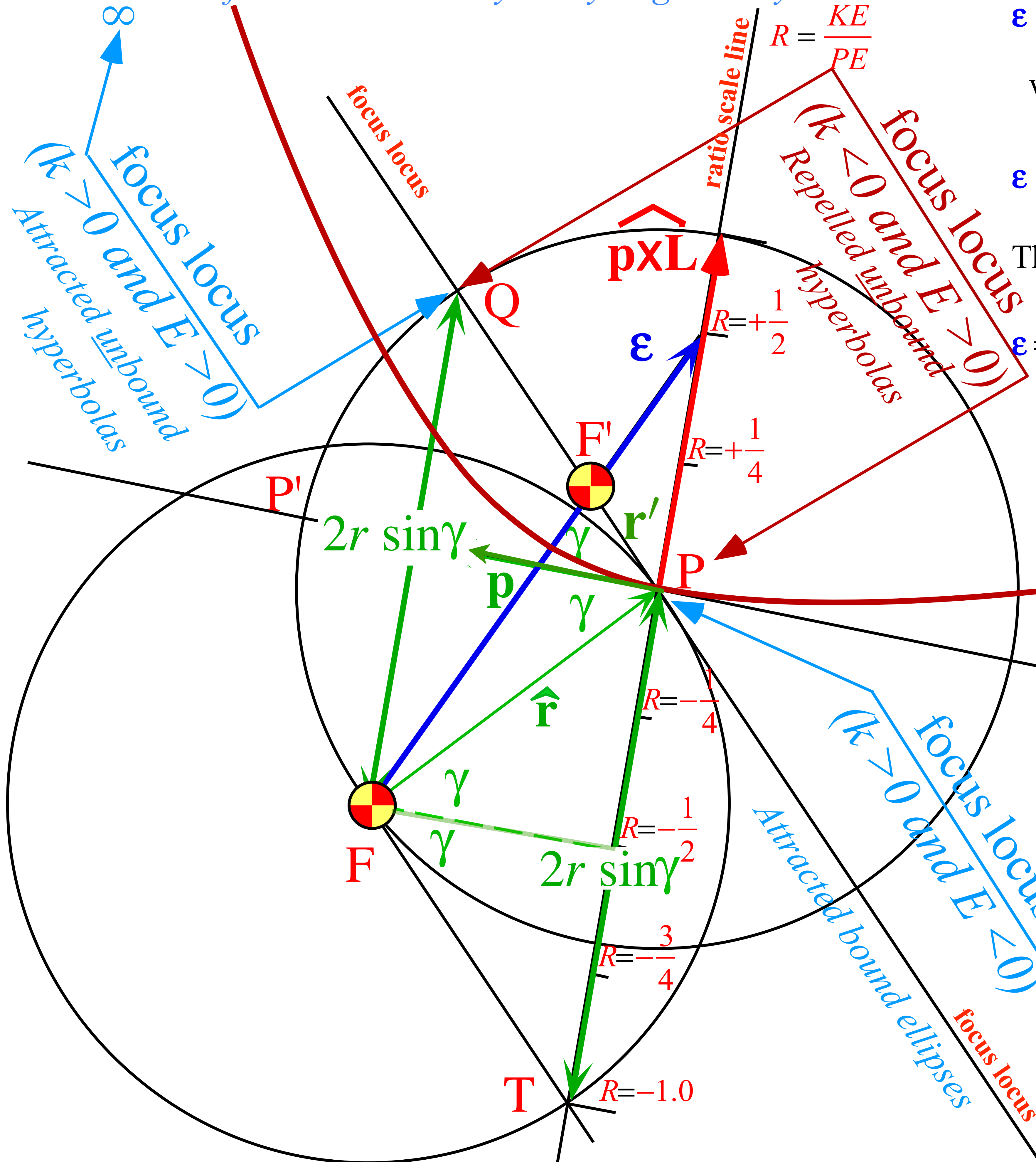
(Go back to p.67)

Fig. 5.4.3
Construction of eccentricity vector ε
and orbit from initial **r**, **p** *with KE/PE=+1/2.*

$$R = \frac{Initial\ KE}{Initial\ PE} = \frac{m v^2(0)/2}{-k/r(0)}$$

$$= \pm\left(\frac{Initial\ velocity}{Escape\ velocity}\right)^2 = \pm\frac{v^2(0)}{v^2(\infty)}$$

Derivation of ε*-construction by analytic geometry*



$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} - \frac{\mathbf{p}\times\mathbf{L}}{km} = \hat{\mathbf{r}} - \frac{(mv_0)(mv_0 r_0)\sin\gamma}{km}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

where: $\mathbf{L}_{\mathbf{p}\times} \equiv \mathbf{p}\times\mathbf{L}$

$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} + 2\sin\gamma\,\frac{mv_0^{\,2}/2}{-k/r_0}\hat{\mathbf{L}}_{\mathbf{p}\times} = \hat{\mathbf{r}} + 2\sin\gamma\,\frac{KE}{PE}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

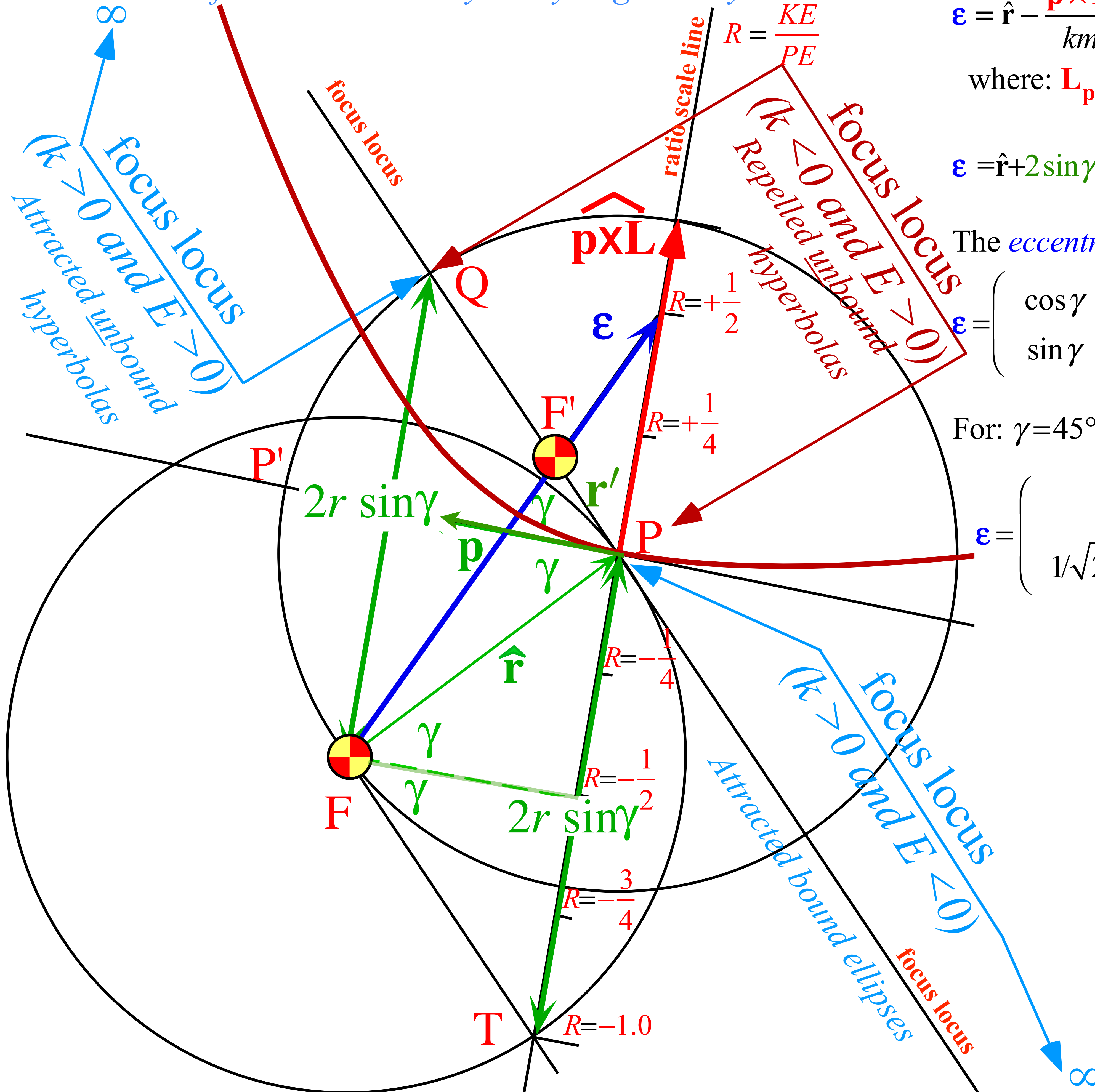
The *eccentricity* vector is:

$$\boldsymbol{\varepsilon} = \begin{pmatrix}\cos\gamma\\ \sin\gamma\end{pmatrix} + 2\sin\gamma\begin{pmatrix}0\\ 1\end{pmatrix}R = \begin{pmatrix}\cos\gamma\\ (2R+1)\sin\gamma\end{pmatrix}$$

$$R = \frac{\textit{Initial KE}}{\textit{Initial PE}} = \frac{m\,v^2(0)/2}{-k/r(0)}$$

$$= \pm\left(\frac{\textit{Initial velocity}}{\textit{Escape velocity}}\right)^2 = \pm\frac{v^2(0)}{v^2(\infty)}$$

Derivation of $\boldsymbol{\varepsilon}$*-construction by analytic geometry*



$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} - \frac{\mathbf{p}\times\mathbf{L}}{km} = \hat{\mathbf{r}} - \frac{(mv_0)(mv_0 r_0)\sin\gamma}{km}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

where: $\mathbf{L}_{\mathbf{p}\times} \equiv \mathbf{p}\times\mathbf{L}$

$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} + 2\sin\gamma\,\frac{mv_0^{\,2}/2}{-k/r_0}\hat{\mathbf{L}}_{\mathbf{p}\times} = \hat{\mathbf{r}} + 2\sin\gamma\,\frac{KE}{PE}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

The *eccentricity* vector is:

$$\boldsymbol{\varepsilon} = \begin{pmatrix}\cos\gamma\\ \sin\gamma\end{pmatrix} + 2\sin\gamma\begin{pmatrix}0\\ 1\end{pmatrix}R = \begin{pmatrix}\cos\gamma\\ (2R+1)\sin\gamma\end{pmatrix}$$

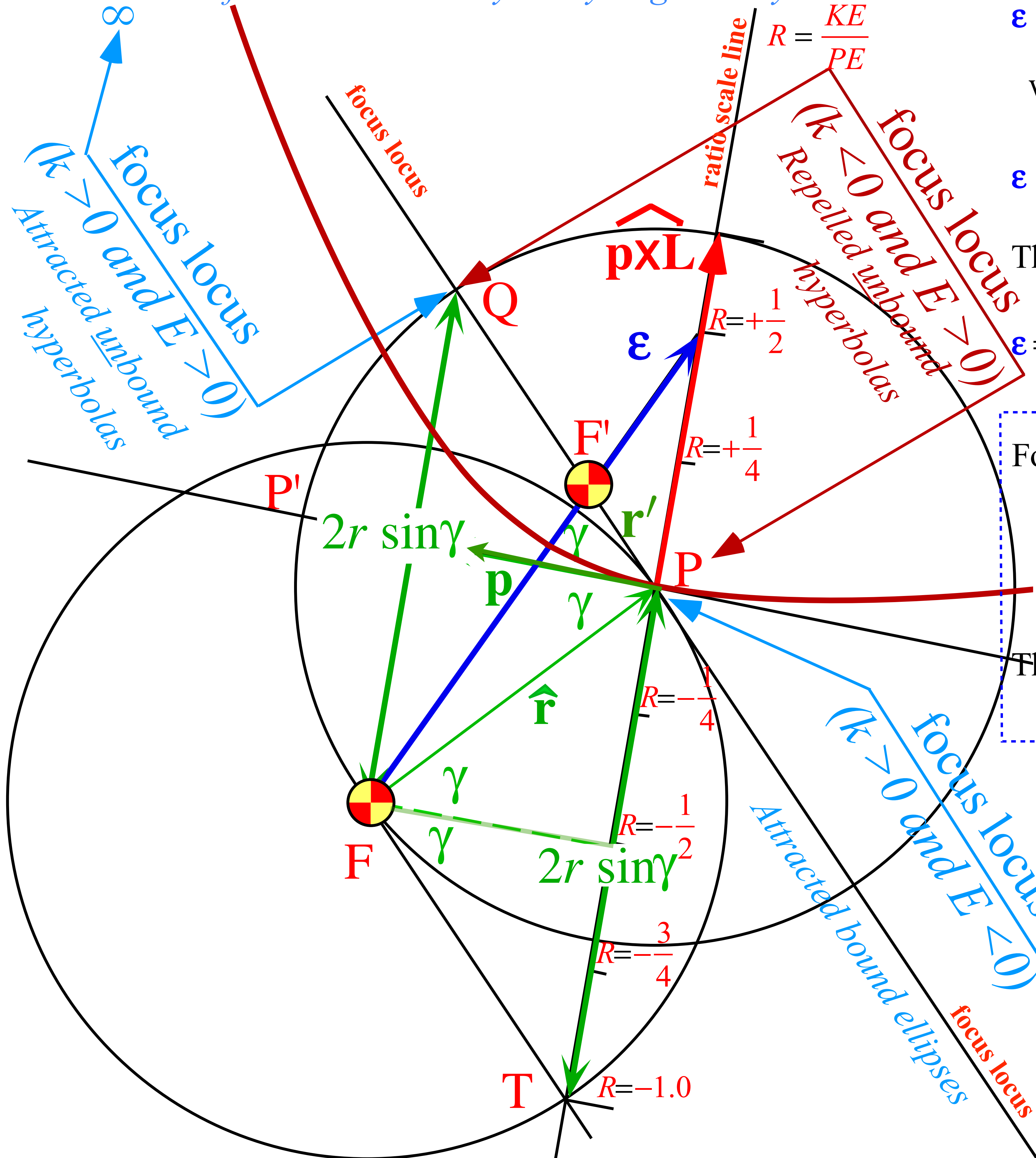
For: $\gamma = 45°$ and: $R = +\frac{1}{2}$

$$\boldsymbol{\varepsilon} = \begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}(2R+1)\end{pmatrix} = \begin{pmatrix}1/\sqrt{2}\\ 2/\sqrt{2}\end{pmatrix},$$

$$R = \frac{\textit{Initial KE}}{\textit{Initial PE}} = \frac{m\,v^2(0)/2}{-k/r(0)}$$

$$= \pm\left(\frac{\textit{Initial velocity}}{\textit{Escape velocity}}\right)^2 = \pm\frac{v^2(0)}{v^2(\infty)}$$

Derivation of ε-construction by analytic geometry



$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} - \frac{\mathbf{p}\times\mathbf{L}}{km} = \hat{\mathbf{r}} - \frac{(mv_0)(mv_0r_0)\sin\gamma}{km}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

where: $\mathbf{L}_{\mathbf{p}\times} \equiv \mathbf{p}\times\mathbf{L}$

$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} + 2\sin\gamma\,\frac{mv_0^2/2}{-k/r_0}\hat{\mathbf{L}}_{\mathbf{p}\times} = \hat{\mathbf{r}} + 2\sin\gamma\,\frac{KE}{PE}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

The *eccentricity* vector is:

$$\boldsymbol{\varepsilon} = \begin{pmatrix}\cos\gamma\\ \sin\gamma\end{pmatrix} + 2\sin\gamma\begin{pmatrix}0\\ 1\end{pmatrix}R = \begin{pmatrix}\cos\gamma\\ (2R+1)\sin\gamma\end{pmatrix}$$

For: $\gamma = 45°$ and: $R = +\frac{1}{2}$

$$\boldsymbol{\varepsilon} = \begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}(2R+1)\end{pmatrix} = \begin{pmatrix}1/\sqrt{2}\\ 2/\sqrt{2}\end{pmatrix},$$

The *eccentricity* parameter defined by:

$$\varepsilon^2 = \cos^2\gamma + (2R+1)^2\sin^2\gamma = 1 \pm \frac{a^2}{b^2}$$

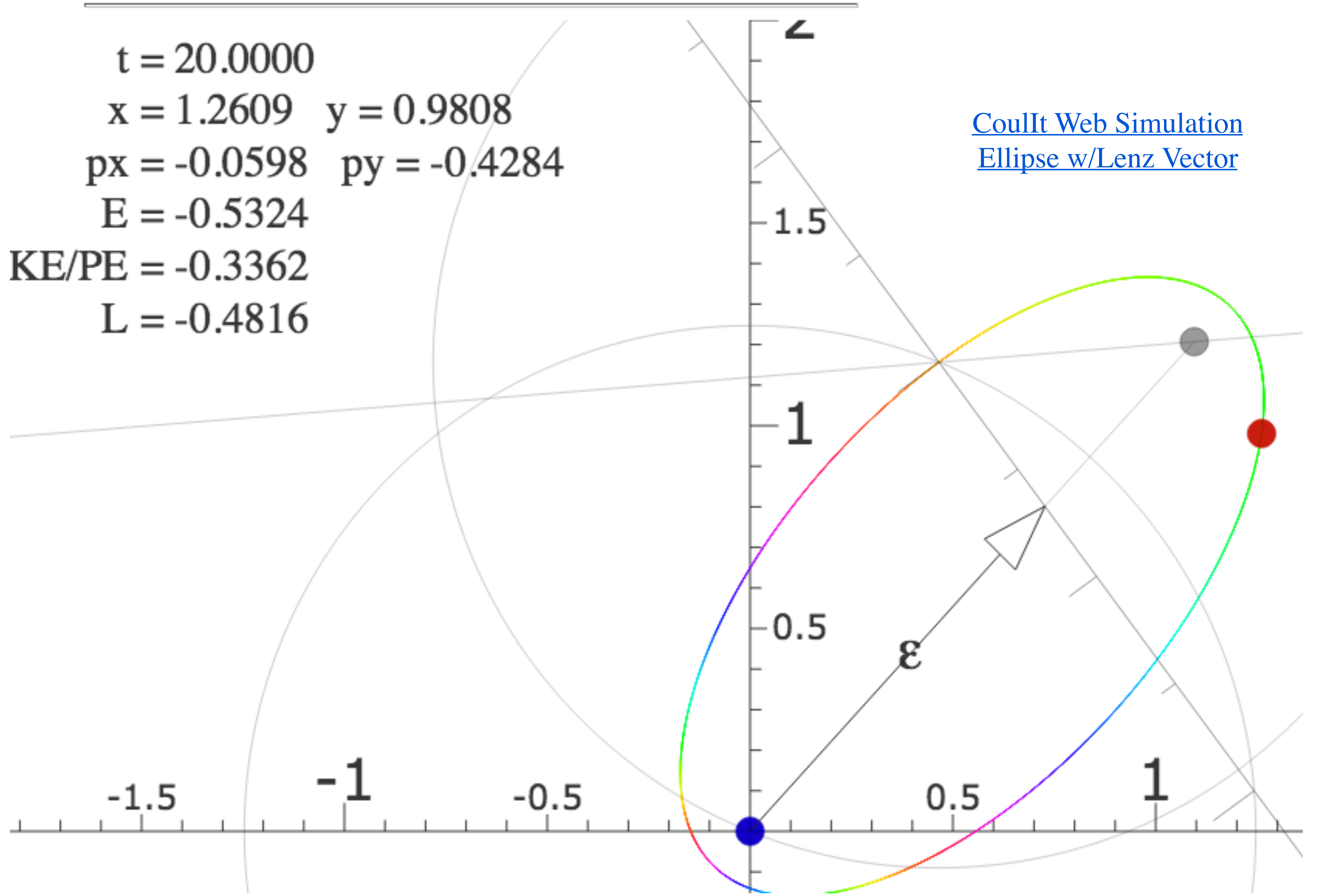
$$= 1 + 4R(R+1)\sin^2\gamma = \frac{5}{2} \quad \text{where: } \sin^2\gamma = \frac{1}{2}$$

$$R = \frac{\textit{Initial KE}}{\textit{Initial PE}} = \frac{m v^2(0)/2}{-k/r(0)}$$

$$= \pm\left(\frac{\textit{Initial velocity}}{\textit{Escape velocity}}\right)^2 = \pm\frac{v^2(0)}{v^2(\infty)}$$

Initial position x(0) =	0.465648
Initial position y(0) =	1.156488
Initial momentum px(0) =	0.591603
Initial momentum py(0) =	0.435114
Terminal time t(off) =	20
Maximum step size dt =	0.01
Charge of Nucleus 1 =	-1
x-Position of Nucleus 1 =	0
y-Position of Nucleus 1 =	0
Charge of Nucleus 2 =	0
Coulomb (k12) =	-1
Core thickness r =	0.000001
x-Stark field Ex =	0
y-Stark field Ey =	0
Zeeman field Bz =	0
Diamagnetic strength k =	0
Plank constant h-bar =	2
Color quantization hues =	64
Color quantization bands =	2
Fractional Error (e^{-x}), x =	8
Particle Size =	9

Fix r(0) ☐ Fix p(0) ☐ Do swarm ☐ Beam ☐
Plot r(t) ☑ Plot p(t) ☐
Color action ☑ No stops ☐ Field vectors ☑ Info ☑
Draw masses ☑ Axes ☑ Coordinates ☐ Lenz ☑
Set p by φ ☐ Elastic ☑ 2 Free ☐
Save to GIF ☐



Chapter 1 Orbit Families and Action

Families of particle orbits are drawn in a varying color which represents the classical action or Hamiltoon's characteristic function SH = ∫p dq.(Sometimes SH is called 'reduced action'.) The color is chosen by first calculating c = SH modulo h-bar (You can change Planck's constant from its default value h/2π = 1.0) The chromatic value c assigns the hue by its position on the color wheel (e.g.; c=0 is red, c=0.2 is a yellow, c=0.5 is a green, etc.).

Chapter 2 Rutherford Scattering

A parallel beam of iso-energetic alpha particles undergo Rutherford scattering from a coulomb field of a nucleus as calculated in these demos. It is also the ideal pattern of paths followed by intergalactic hydrogen in perturbed by the solar wind.

Chapter 3 Coulomb Field (H atom)

Orbits in an attractive Coulomb field are calculated here. You may select the initial position (x(0),y(0)) by moving the mouse to a desired launch point, and then select the initial momentum (px(0), py(0)) by pressing the mouse button and dragging.

Chapter 4 Molecular Ion Orbits

Orbits around two fixed nuclei are calculated here. A set of elliptic coordinates are drawn in the background. After running a few trajectories you may notice that their caustics conform to one or two of the elliptic coordinate lines.

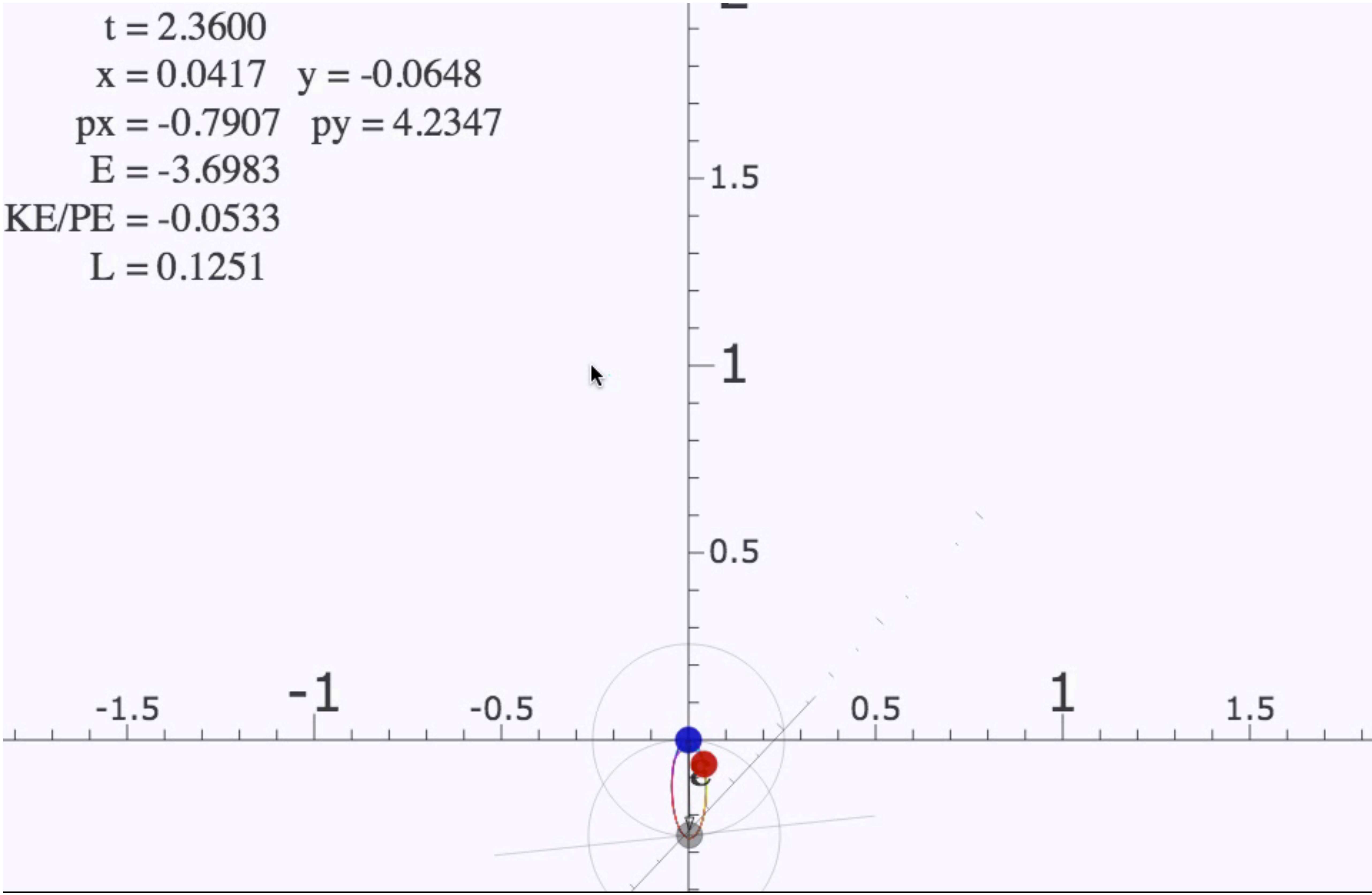
Volcanoes of Io (Paths=180, No color quant.) | Parabolic Fountain (Uniform) | Space Bomb (Coulomb) | Exploding Starlet (IHO) | Synchrotron Motion (Crossed E & B fields)

Rutherford scattering | 2-Electron Orbits

Atomic Orbits

Molecular Ion Orbits

Oscillator Scattering | 2-Particle Orbits | 2-Particle Collision



Play embedded movie with controls above.

or follow this link to try your hand at ε-construction using the CoulIt Web App

Just click and drag in main window to set new initial conditions. The Lenz vector will display as part of an overlay.

Rutherford scattering and hyperbolic orbit geometry

Backward vs forward scattering angles and orbit construction example

Parabolic "kite" and orbital envelope geometry

Differential and total scattering cross-sections

Eccentricity vector **ε** *and (ε,λ)-geometry of orbital mechanics*

Projection **ε•r** *geometry of* **ε***-vector and orbital radius* **r**

Review and connection to usual orbital algebra (previous lecture)

Projection **ε•p** *geometry of* **ε***-vector and momentum* **p**=*m***v**

General geometric orbit construction using **ε***-vector and (γ,R)-parameters*

Derivation of **ε***-construction by analytic geometry*

➔ *Coulomb orbit algebra of* **ε***-vector and Kepler dynamics of momentum* **p**=*m***v**

*Example of complete (***r***,***p***)-geometry of elliptical orbit*

Connection formulas for (γ,R)-parameters with (a,b) and (ε,λ)

Coulomb orbit algebra of ***ε*****-vector** *and Kepler dynamics of momentum* **p**$=m$**v**

Finding time derivatives of orbital coordinates r, φ, x, y, and eventually velocity **v** *or momentum* **p**$=m$**v**

Radius r:

$$r = \frac{\lambda}{1-\varepsilon\cos\phi} = \frac{L^2/km}{1-\varepsilon\cos\phi}$$

Polar angle φ using: $L = mr^2\frac{d\phi}{dt} = mr^2\dot{\phi}$

Coulomb orbit algebra of ε-vector and Kepler dynamics of momentum $\mathbf{p}=m\mathbf{v}$

Finding time derivatives of orbital coordinates r, ϕ, x, y, and eventually velocity $\mathbf{v}$ *or momentum* $\mathbf{p}=m\mathbf{v}$

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$$\dot{\phi}=\frac{L}{mr^2}=\frac{L}{m}\frac{1}{r^2}$$

Coulomb orbit algebra of **ε***-vector and Kepler dynamics of momentum* $\mathbf{p}=m\mathbf{v}$

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$$\dot{\phi}=\frac{L}{mr^2}=\frac{L}{m}\frac{1}{r^2}=\frac{L}{m}\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$$

using: $\frac{1}{r^2}=\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$

Coulomb orbit algebra of ε-vector and Kepler dynamics of momentum $\mathbf{p}=m\mathbf{v}$

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using: $\frac{1}{r^2}=\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$

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$$\dot{\phi}=\frac{L}{mr^2}=\frac{L}{m}\frac{1}{r^2}=\frac{L}{m}\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$$

$$r\dot{\phi}=\frac{L}{mr}$$

using: $\frac{1}{r^2}=\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$

Coulomb orbit algebra of ε-vector and Kepler dynamics of momentum $\mathbf{p}=m\mathbf{v}$

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$$\dot{\phi}=\frac{L}{mr^2}=\frac{L}{m}\frac{1}{r^2}=\frac{L}{m}\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$$

$$r\dot{\phi}=\frac{L}{mr}=\frac{L}{m}\frac{1}{r}=\frac{L}{m}\left(\frac{km}{L^2}\right)(1-\varepsilon\cos\phi)=\frac{k}{L}(1-\varepsilon\cos\phi)$$

using: $\frac{1}{r}=\left(\frac{km}{L^2}\right)(1-\varepsilon\cos\phi)$

Coulomb orbit algebra of ***ε-vector*** *and Kepler dynamics of momentum* **p**=*m***v**

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$$\dot r=\frac{L^2}{km}\frac{-\varepsilon\sin\phi\,\dot\phi}{(1-\varepsilon\cos\phi)^2}$$

$$\dot r=-\frac{L^2}{km}\left(\frac{km}{L^2}\right)^2 r^2\dot\phi\,\varepsilon\sin\phi$$

using: $$\frac{1}{(1-\varepsilon\cos\phi)^2}=\left(\frac{km}{L^2}\right)^2 r^2$$

Polar angle φ using: $L=mr^2\frac{d\phi}{dt}=mr^2\dot\phi$

$$\dot\phi=\frac{L}{mr^2}=\frac{L}{m}\frac{1}{r^2}=\frac{L}{m}\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$$

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Coulomb orbit algebra of ε-vector and Kepler dynamics of momentum $\mathbf{p}=m\mathbf{v}$

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$$\dot{r}=\frac{L^2}{km}\,\frac{-\varepsilon\sin\phi\,\dot{\phi}}{(1-\varepsilon\cos\phi)^2}$$

$$\dot{r}=-\frac{L^2}{km}\left(\frac{km}{L^2}\right)^2 r^2\dot{\phi}\,\varepsilon\sin\phi$$

using: $\frac{1}{(1-\varepsilon\cos\phi)^2}=\left(\frac{km}{L^2}\right)^2 r^2$

$$\dot{r}=-\frac{k}{L^2}mr^2\dot{\phi}\,\varepsilon\sin\phi=-\frac{k}{L}\varepsilon\sin\phi$$

again using: $L=mr^2\dot{\phi}$

Polar angle φ using: $L=mr^2\frac{d\phi}{dt}=mr^2\dot{\phi}$

$$\dot{\phi}=\frac{L}{mr^2}=\frac{L}{m}\frac{1}{r^2}=\frac{L}{m}\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$$

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using: $\frac{1}{r^2}=\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$

Coulomb orbit algebra of **ε**-*vector and Kepler dynamics of momentum* **p**=*m***v**

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using: $\frac{1}{(1-\varepsilon\cos\phi)^2}=\left(\frac{km}{L^2}\right)^2 r^2$

$$\dot{r}=-\frac{k}{L^2}mr^2\dot{\phi}\,\varepsilon\sin\phi=-\frac{k}{L}\varepsilon\sin\phi$$

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Polar angle φ using: $L=mr^2\frac{d\phi}{dt}=mr^2\dot{\phi}$

$$\dot{\phi}=\frac{L}{mr^2}=\frac{L}{m}\frac{1}{r^2}=\frac{L}{m}\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$$

$$r\dot{\phi}=\frac{L}{mr}=\frac{L}{m}\frac{1}{r}=\frac{L}{m}\left(\frac{km}{L^2}\right)(1-\varepsilon\cos\phi)=\frac{k}{L}(1-\varepsilon\cos\phi)$$

using: $\frac{1}{r^2}=\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$

Cartesian x = r cos φ:

$$\dot{x}=\frac{dx}{dt}=\quad \dot{r}\cos\phi-\sin\phi\; r\dot{\phi}$$

Cartesian y = r sin φ:

$$\dot{y}=\frac{dy}{dt}=\quad \dot{r}\sin\phi+\cos\phi\; r\dot{\phi}$$

Coulomb orbit algebra of ε-vector and Kepler dynamics of momentum **p**=m**v**

Finding time derivatives of orbital coordinates r, φ, x, y, and eventually velocity **v** *or momentum* **p**=m**v**

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$$\boxed{\dot{r}}=-\frac{k}{L^2}mr^2\dot{\phi}\,\varepsilon\sin\phi=\boxed{-\frac{k}{L}\varepsilon\sin\phi} \qquad \textit{again using: } L=mr^2\dot{\phi}$$

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$$\dot{\phi}=\frac{L}{mr^2}=\frac{L}{m}\frac{1}{r^2}=\frac{L}{m}\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$$

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$$\textit{using: } \frac{1}{r^2}=\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$$

Cartesian x = r cos φ:

$$\dot{x}=\frac{dx}{dt}=\boxed{\dot{r}}\cos\phi-\sin\phi\,\boxed{r\dot{\phi}}$$

$$=\boxed{-\frac{k}{L}\varepsilon\sin\phi}\cos\phi-\sin\phi\,\boxed{\frac{k}{L}(1-\varepsilon\cos\phi)}$$

Cartesian y = r sin φ:

$$\dot{y}=\frac{dy}{dt}=\boxed{\dot{r}}\sin\phi+\cos\phi\,\boxed{r\dot{\phi}}$$

$$=\boxed{-\frac{k}{L}\varepsilon\sin\phi}\sin\phi+\cos\phi\,\boxed{\frac{k}{L}(1-\varepsilon\cos\phi)}$$

Coulomb orbit algebra of ε-vector and Kepler dynamics of momentum $\mathbf{p}=m\mathbf{v}$

Finding time derivatives of orbital coordinates r, φ, x, y, and eventually velocity $\mathbf{v}$ *or momentum* $\mathbf{p}=m\mathbf{v}$

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$$\dot{r}=-\frac{L^2}{km}\left(\frac{km}{L^2}\right)^2 r^2\dot{\phi}\,\varepsilon\sin\phi$$

using: $\frac{1}{(1-\varepsilon\cos\phi)^2}=\left(\frac{km}{L^2}\right)^2 r^2$

$$\dot{r}=-\frac{k}{L^2}mr^2\dot{\phi}\,\varepsilon\sin\phi=-\frac{k}{L}\varepsilon\sin\phi$$

again using: $L=mr^2\dot{\phi}$

Polar angle φ using: $L=mr^2\frac{d\phi}{dt}=mr^2\dot{\phi}$

$$\dot{\phi}=\frac{L}{mr^2}=\frac{L}{m}\frac{1}{r^2}=\frac{L}{m}\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$$

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Cartesian x = r cos φ:

$$\dot{x}=\frac{dx}{dt}=\dot{r}\cos\phi-\sin\phi\,r\dot{\phi}$$

$$=-\frac{k}{L}\varepsilon\sin\phi\cos\phi-\sin\phi\frac{k}{L}(1-\varepsilon\cos\phi)$$

$$=-\frac{k}{L}\sin\phi$$

Cartesian y = r sin φ:

$$\dot{y}=\frac{dy}{dt}=\dot{r}\sin\phi+\cos\phi\,r\dot{\phi}$$

$$=-\frac{k}{L}\varepsilon\sin\phi\sin\phi+\cos\phi\frac{k}{L}(1-\varepsilon\cos\phi)$$

$$=\frac{k}{L}(\cos\phi-\varepsilon)$$

Coulomb orbit algebra of ε-vector and Kepler dynamics of momentum p=mv

Finding time derivatives of orbital coordinates r, φ, x, y, and eventually velocity **v** *or momentum* **p**=*m***v**

Radius r:

$$r=\frac{\lambda}{1-\varepsilon\cos\phi}=\frac{L^2/km}{1-\varepsilon\cos\phi}$$

$$\dot{r}=\frac{dr}{dt}=\frac{L^2}{km}\frac{-\frac{d}{dt}(-\varepsilon\cos\phi)}{(1-\varepsilon\cos\phi)^2}$$

$$\dot{r}=\frac{L^2}{km}\frac{-\varepsilon\sin\phi\,\dot{\phi}}{(1-\varepsilon\cos\phi)^2}$$

$$\dot{r}=-\frac{L^2}{km}\left(\frac{km}{L^2}\right)^2 r^2\dot{\phi}\,\varepsilon\sin\phi \qquad \textit{using: } \frac{1}{(1-\varepsilon\cos\phi)^2}=\left(\frac{km}{L^2}\right)^2 r^2$$

$$\boxed{\dot{r}}=-\frac{k}{L^2}mr^2\dot{\phi}\,\varepsilon\sin\phi=\boxed{-\frac{k}{L}\varepsilon\sin\phi} \qquad \textit{again using: } L=mr^2\dot{\phi}$$

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Cartesian y = r sin φ:

$$\dot{y}=\frac{dy}{dt}=\boxed{\dot{r}}\sin\phi+\cos\phi\,\boxed{r\dot{\phi}}$$

$$=\frac{k}{L}(\cos\phi-\varepsilon)$$

Velocity:

Momentum:

$$\boxed{p_x=m\dot{x}=-\frac{mk}{L}\sin\phi}$$

$$\boxed{p_y=m\dot{y}=\frac{mk}{L}(\cos\phi-\varepsilon)}$$

p *traces an off-center circle!*

Rutherford scattering and hyperbolic orbit geometry

Backward vs forward scattering angles and orbit construction example

Parabolic “kite” and orbital envelope geometry

Differential and total scattering cross-sections

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{r}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and orbital radius* $\mathbf{r}$

Review and connection to usual orbital algebra (previous lecture)

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{p}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and momentum* $\mathbf{p}=m\mathbf{v}$

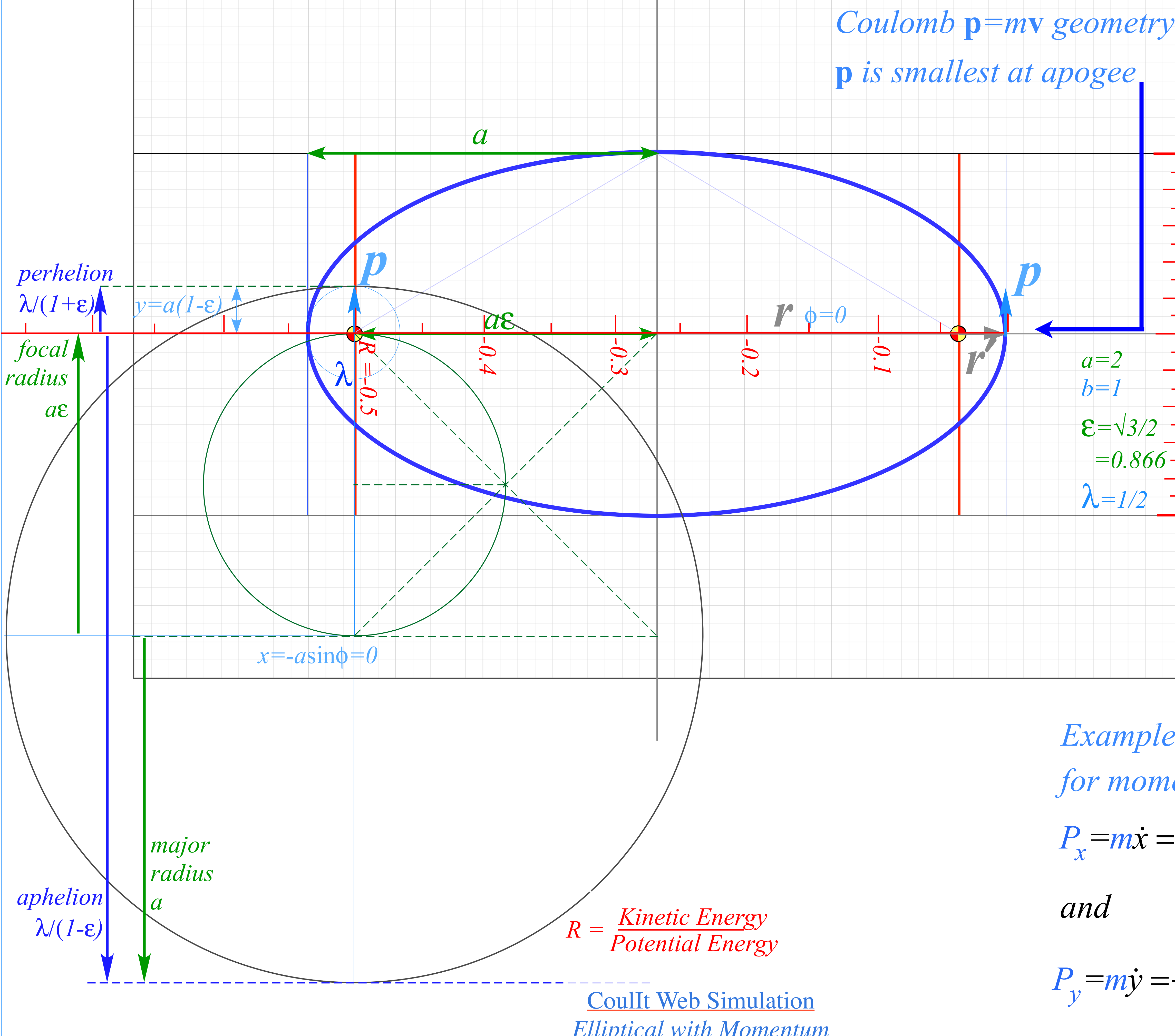
General geometric orbit construction using $\boldsymbol{\varepsilon}$*-vector and* (γ,R)*-parameters*

Derivation of $\boldsymbol{\varepsilon}$*-construction by analytic geometry*

Coulomb orbit algebra of $\boldsymbol{\varepsilon}$*-vector and Kepler dynamics of momentum* $\mathbf{p}=m\mathbf{v}$

➔ *Example of complete* $(\mathbf{r},\mathbf{p})$*-geometry of elliptical orbit*

Connection formulas for (γ,R)*-parameters with* (a,b) *and* (ε,λ)

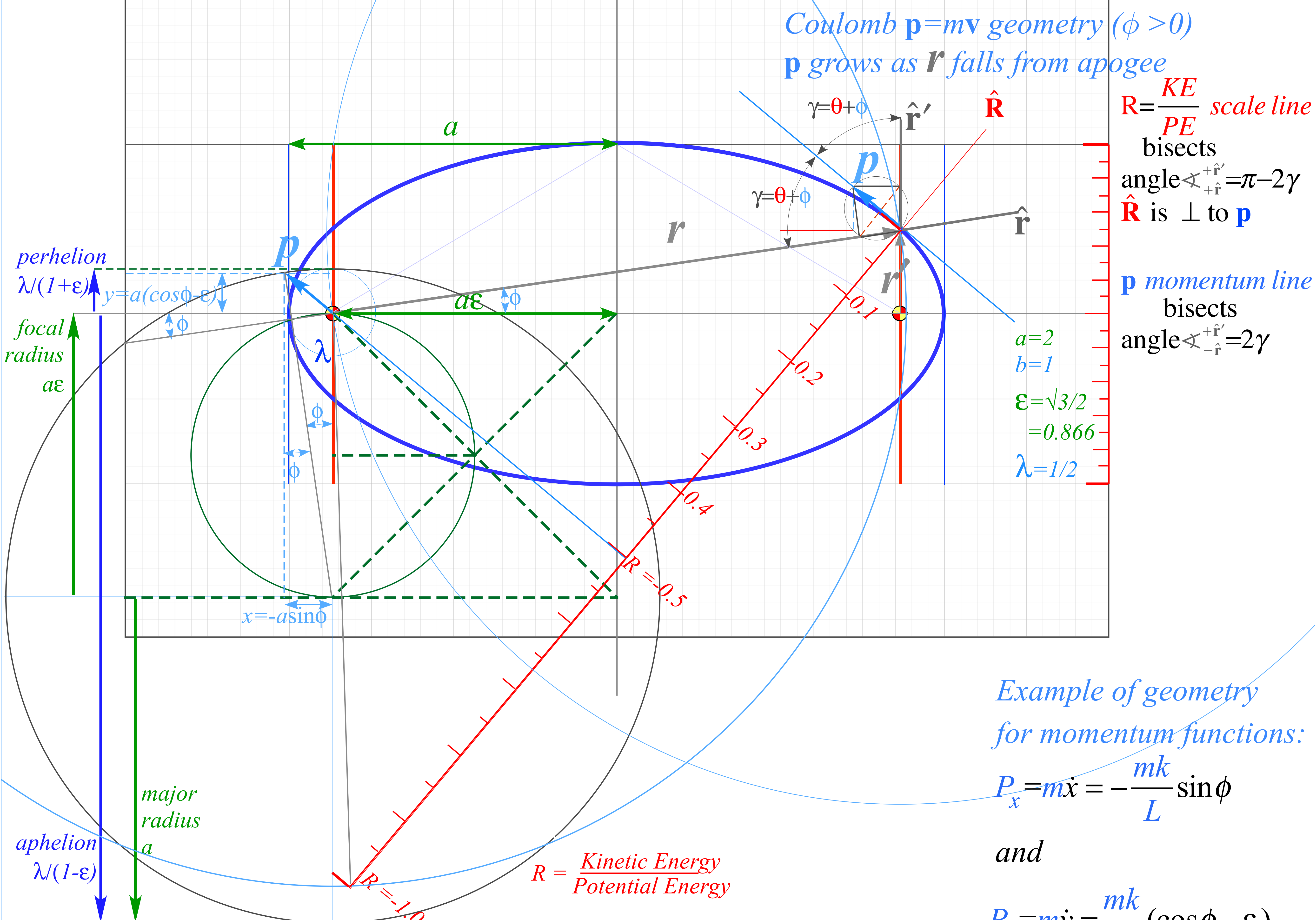


Example of geometry for momentum functions:

$$P_x = m\dot{x} = -\frac{mk}{L}\sin\phi$$

and

$$P_y = m\dot{y} = \frac{mk}{L}(\cos\phi - \varepsilon)$$



Example of geometry for momentum functions:

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and

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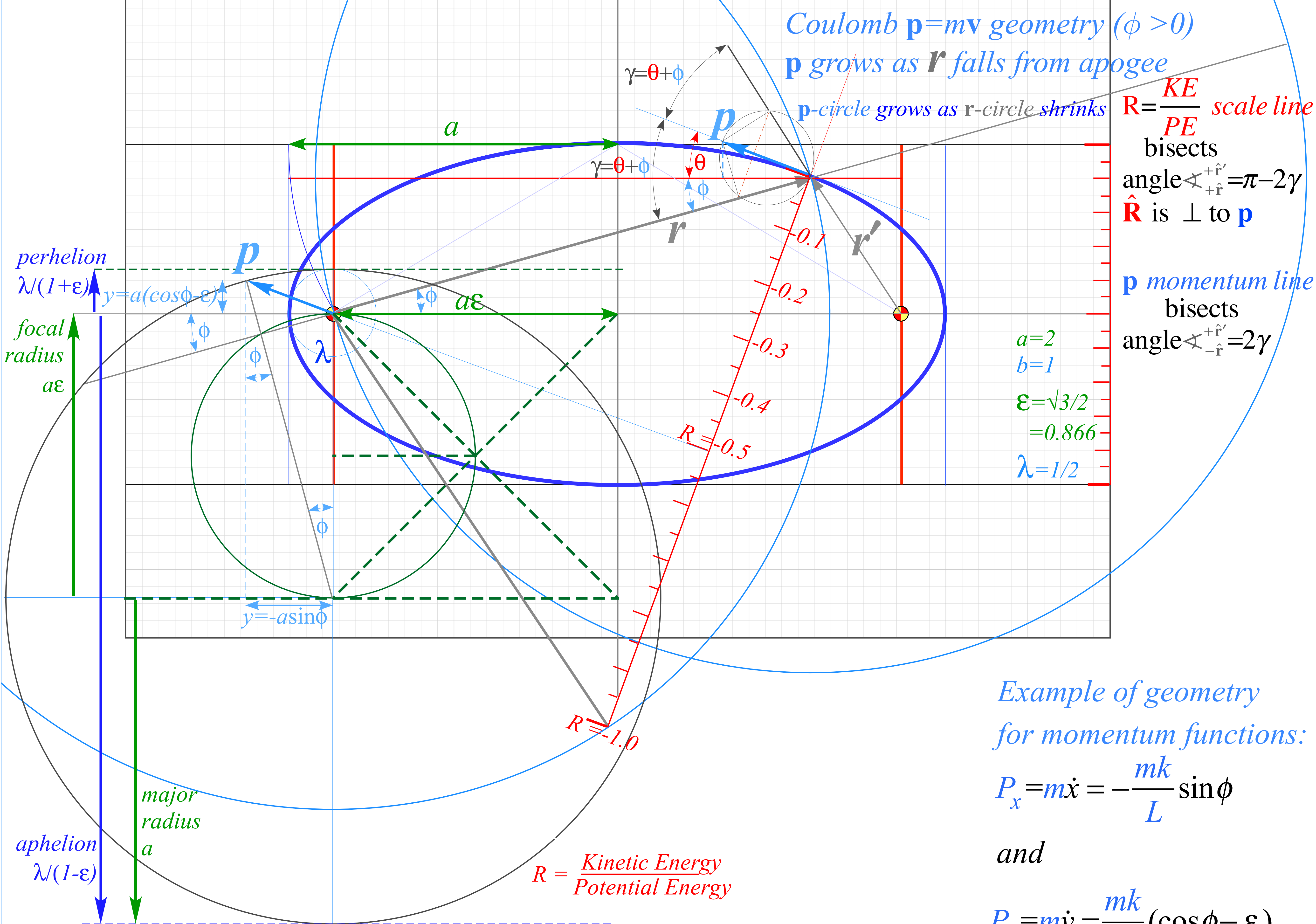
0>R=KE/PE>-1 scale subtends angle 2γ with length 2r sinγ as is derived before on p. 67-71.
*Note similarity of (**R**,**r**)-triangle in **r**-circle of radius r to that in **p**-circle of diameter p above.*



Example of geometry for momentum functions:

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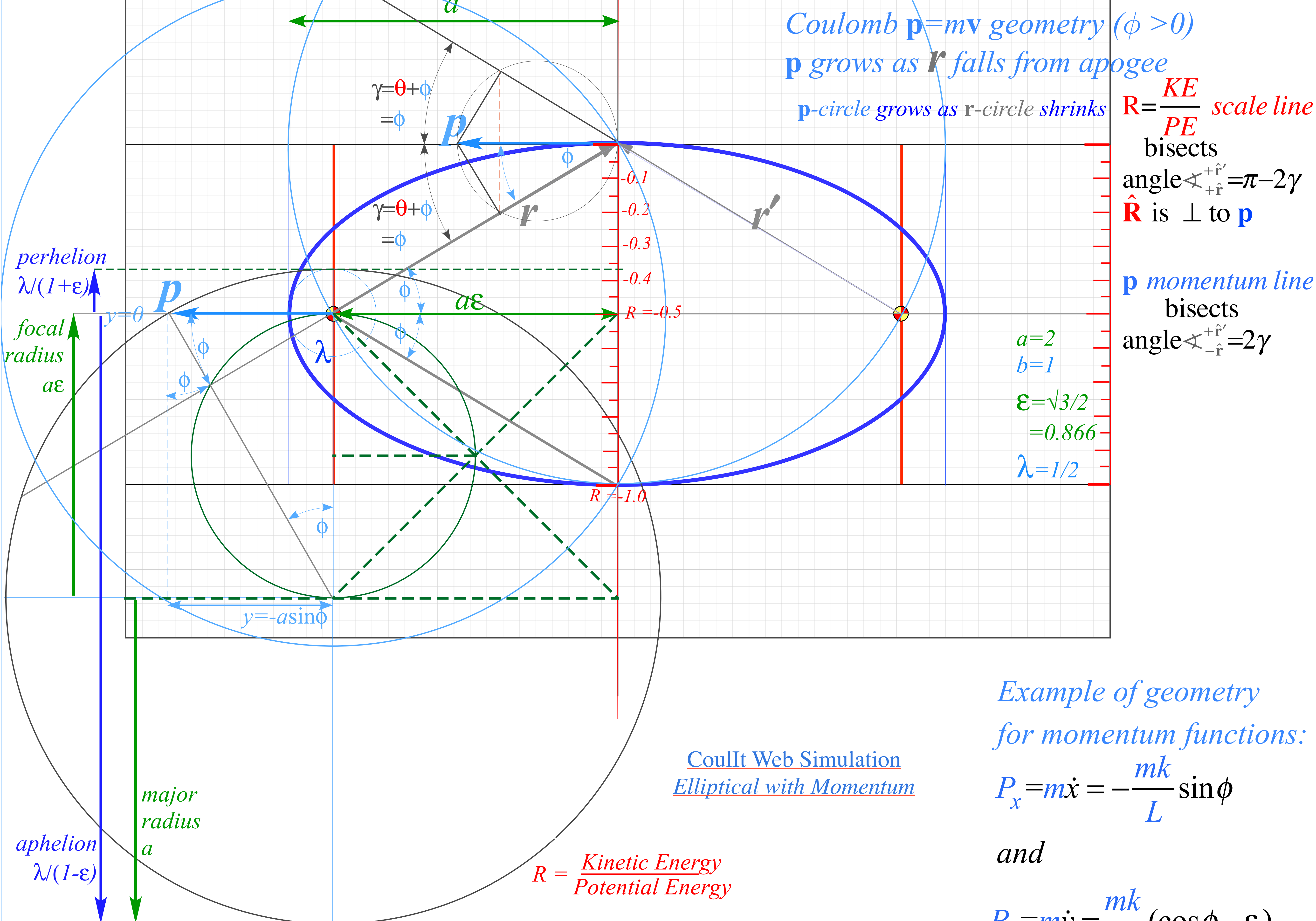
0>R=KE/PE>-1 scale subtends angle 2γ with length 2r sinγ as is derived before on p. 66-70.
*Note similarity of (**R**,**r**)-triangle in **r**-circle of radius r to that in **p**-circle of diameter p above.*



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and

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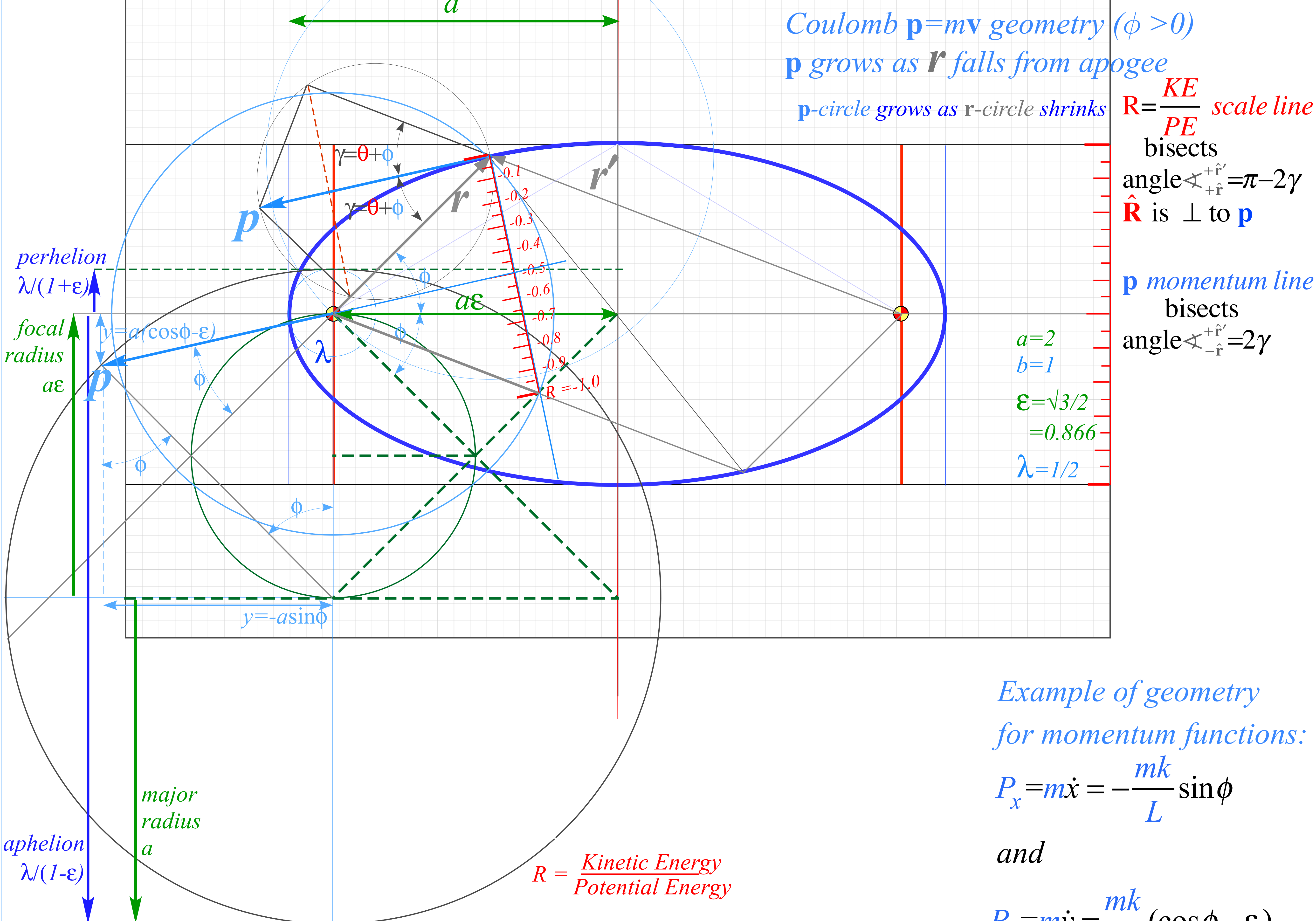

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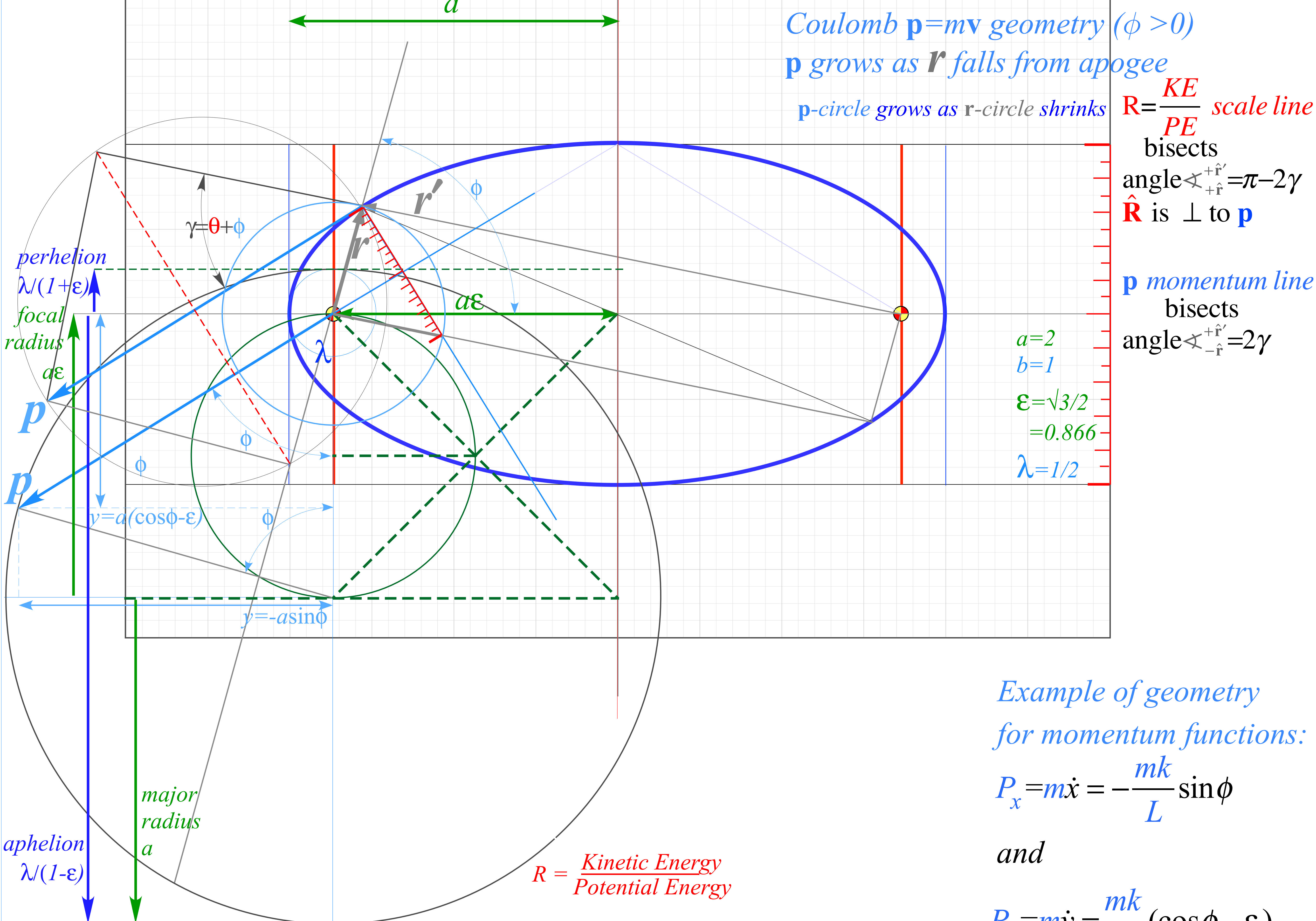

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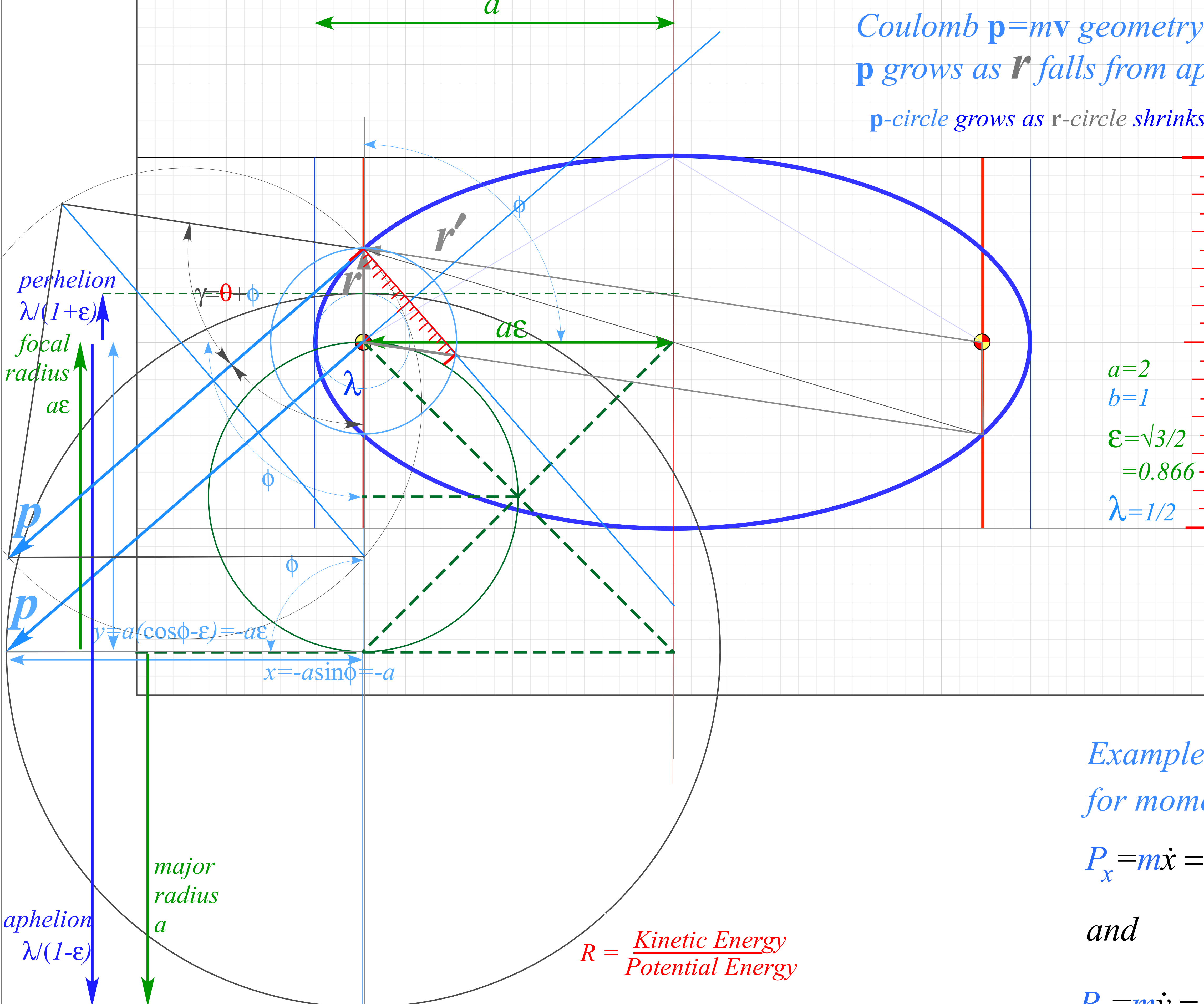

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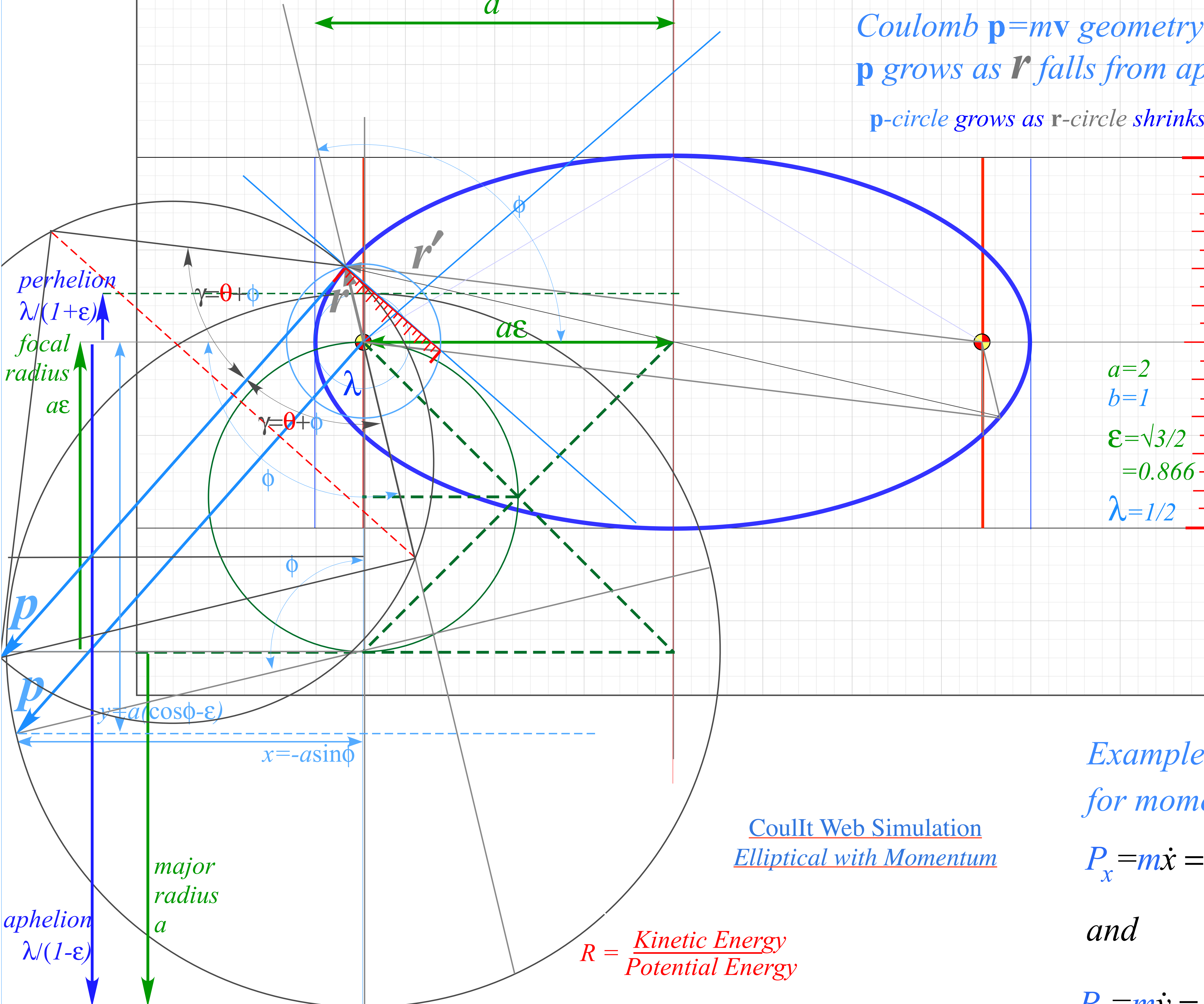

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Algebra of ε-construction geometry

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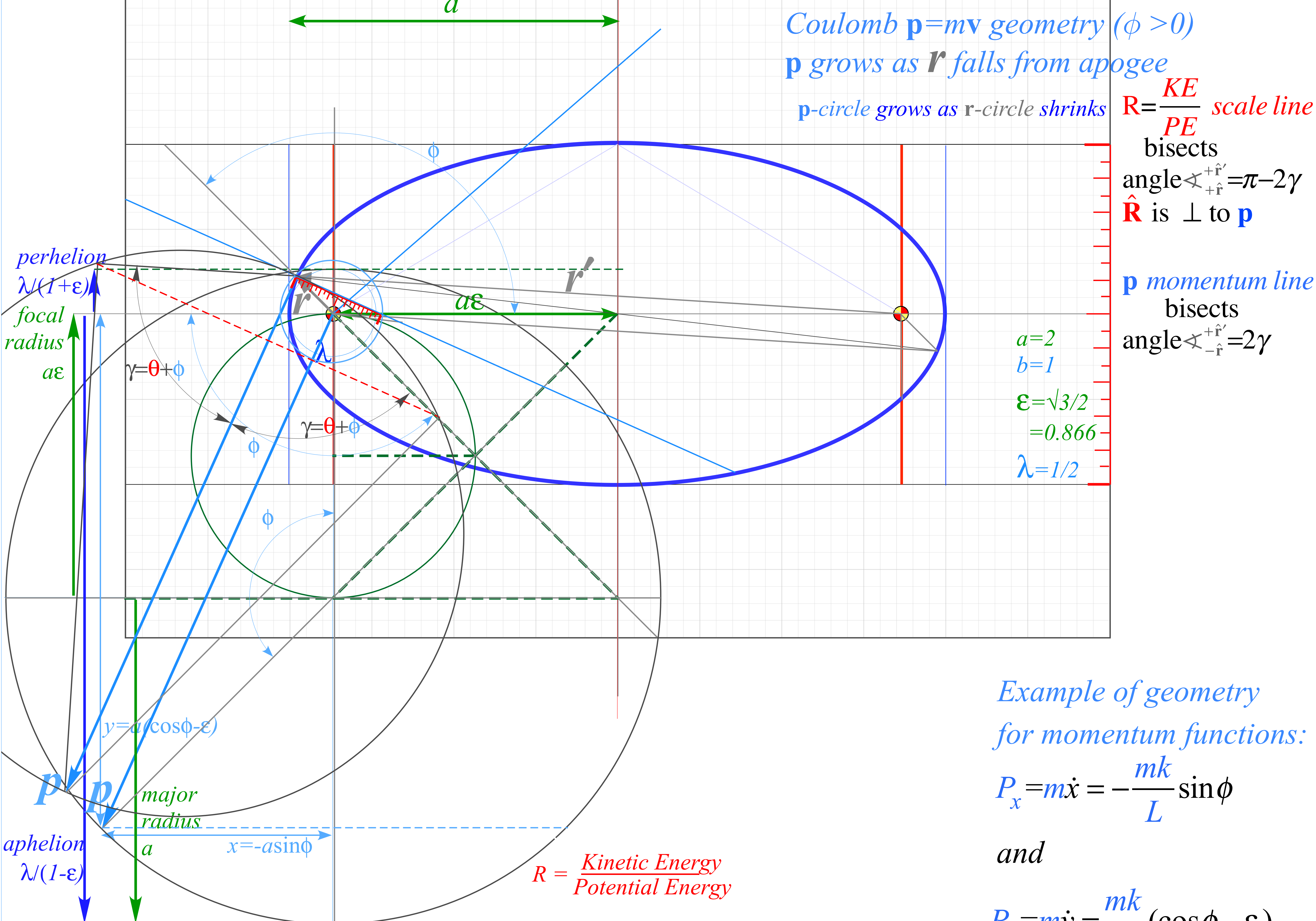

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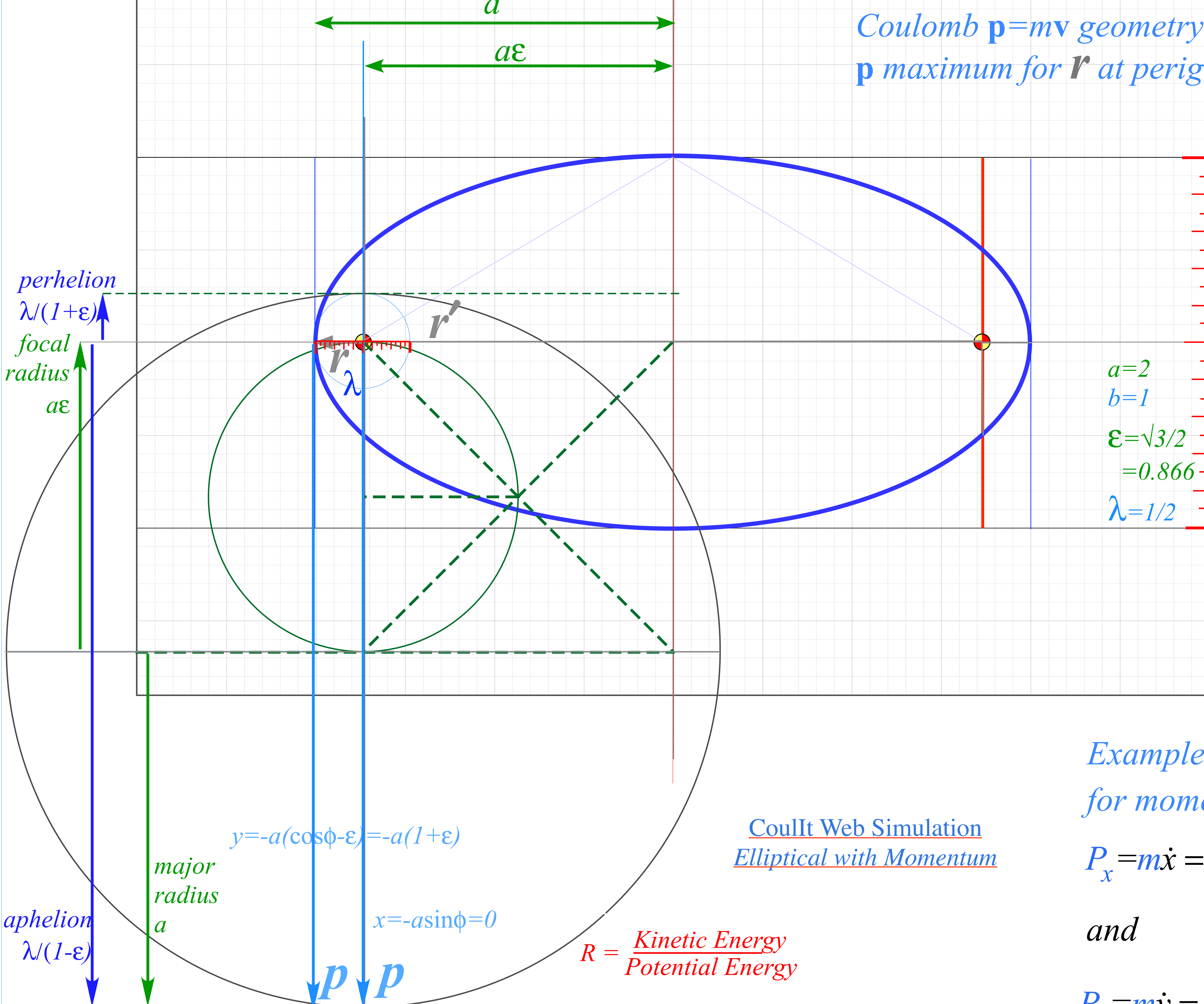
Coulomb $\mathbf{p}=m\mathbf{v}$ geometry ($\phi>0$)
$\mathbf{p}$ maximum for $\mathbf{r}$ at perigee



[CoulIt Web Simulation](#)
[Elliptical with Momentum](#)

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Rutherford scattering and hyperbolic orbit geometry

Backward vs forward scattering angles and orbit construction example

Parabolic "kite" and orbital envelope geometry

Differential and total scattering cross-sections

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{r}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and orbital radius* $\mathbf{r}$

Review and connection to usual orbital algebra (previous lecture)

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{p}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and momentum* $\mathbf{p}=m\mathbf{v}$

General geometric orbit construction using $\boldsymbol{\varepsilon}$*-vector and* (γ,R)*-parameters*

Derivation of $\boldsymbol{\varepsilon}$*-construction by analytic geometry*

Coulomb orbit algebra of $\boldsymbol{\varepsilon}$*-vector and Kepler dynamics of momentum* $\mathbf{p}=m\mathbf{v}$

Example of complete $(\mathbf{r},\mathbf{p})$*-geometry of elliptical orbit*

➔ *Connection formulas for* (γ,R)*-parameters with (a,b) and* (ε,λ)

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$$= 1 + \frac{b^2}{a^2} \quad \text{for hyperbola} \quad (\varepsilon > 1)$$

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Total $\frac{-k}{2a} = E = energy = KE + PE$ relates ratio $R = \frac{KE}{PE}$ to individual radii a, b, and λ.

$$\frac{-k}{2a} = E = KE + PE = R \cdot PE + PE = (R+1)PE = (R+1)\frac{-k}{r} \quad \text{or:} \quad \frac{1}{2a} = (R+1)\frac{1}{r} = (R+1)$$

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Latus radius is similarly related:

$$\lambda=\frac{b^2}{a}=\mp 2rR\sin^2\gamma$$

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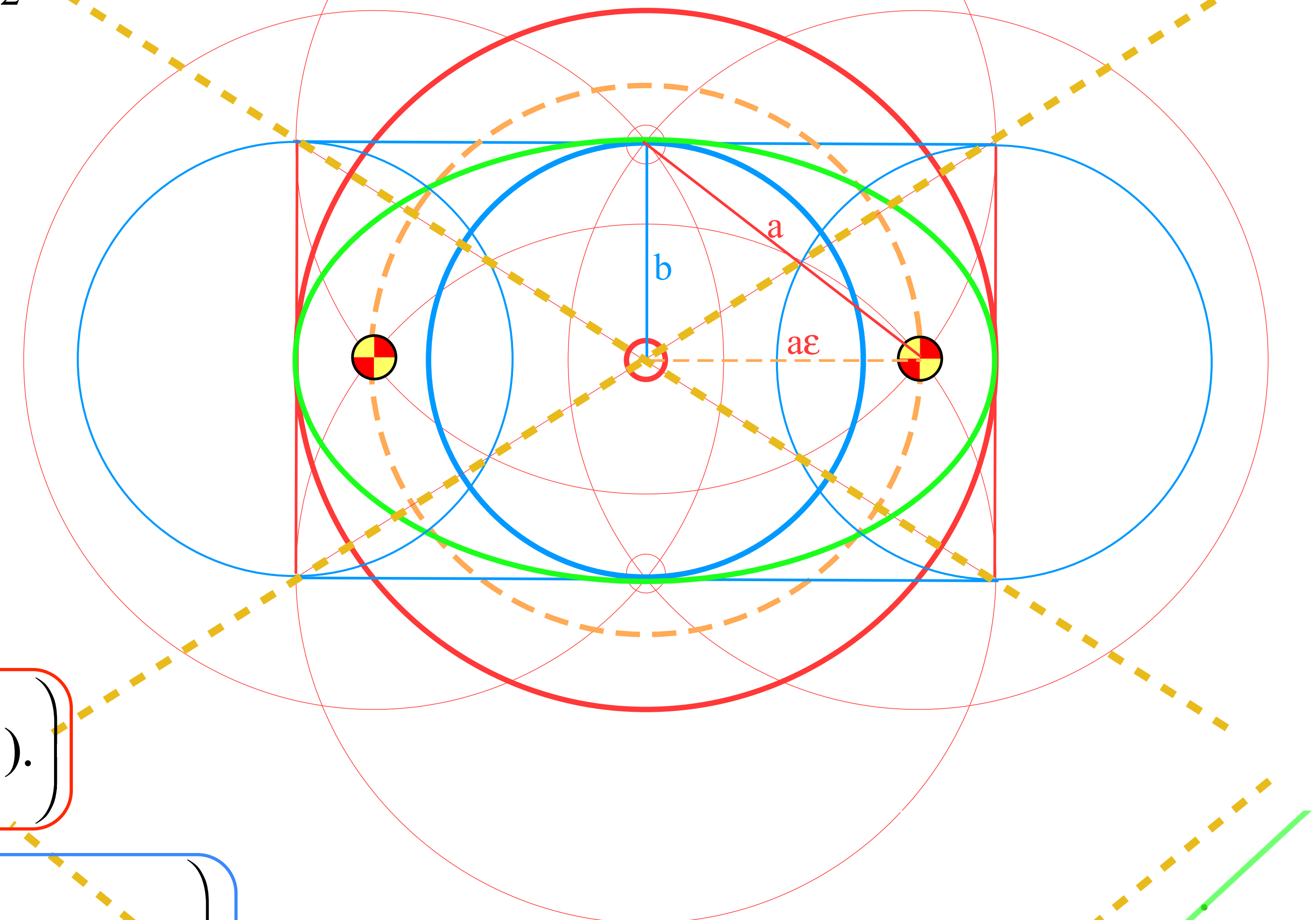

$$\varepsilon^2 = 1+4R(R+1)\sin^2\gamma$$

$$=1-\frac{b^2}{a^2}\text{ ellipse}(\varepsilon<1) \qquad 4R(R+1)\sin^2\gamma=-\frac{b^2}{a^2}$$

$$=1+\frac{b^2}{a^2}\text{ hyperbola }(\varepsilon>1) \quad 4R(R+1)\sin^2\gamma=+\frac{b^2}{a^2}$$

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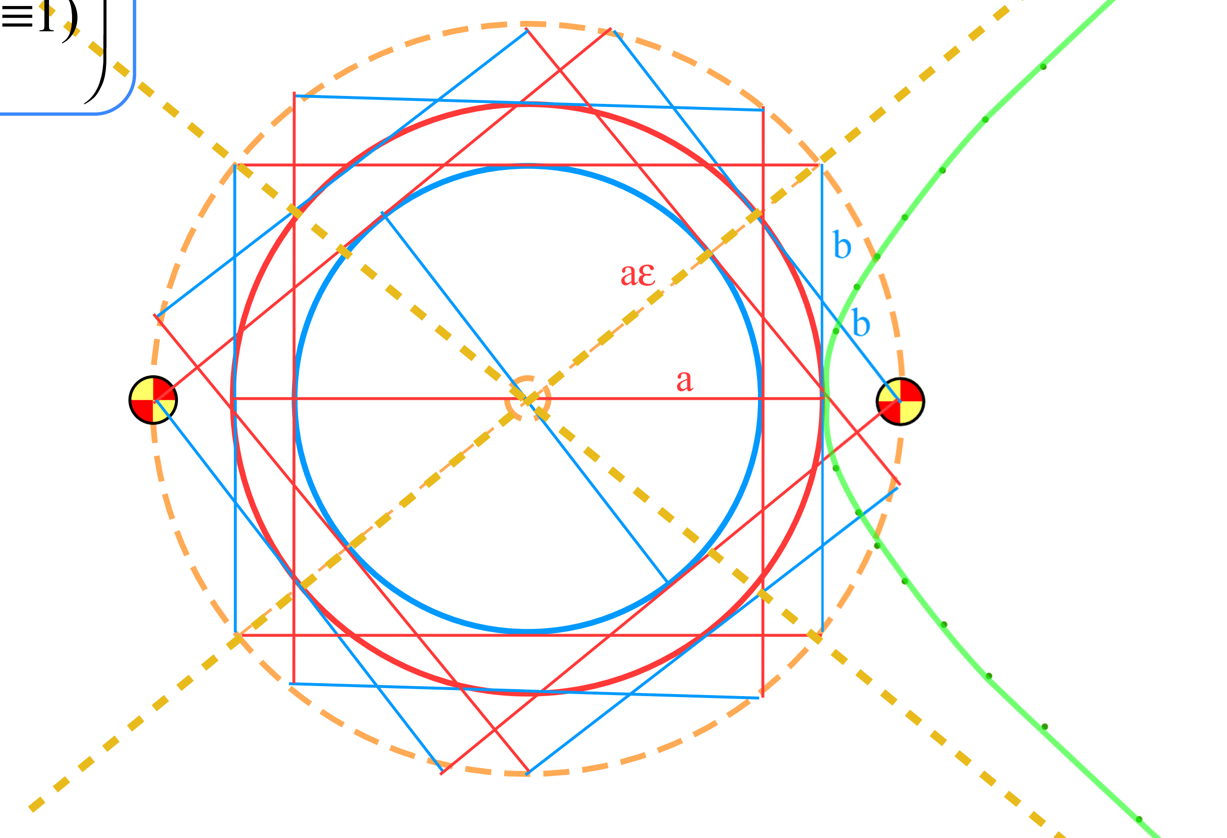
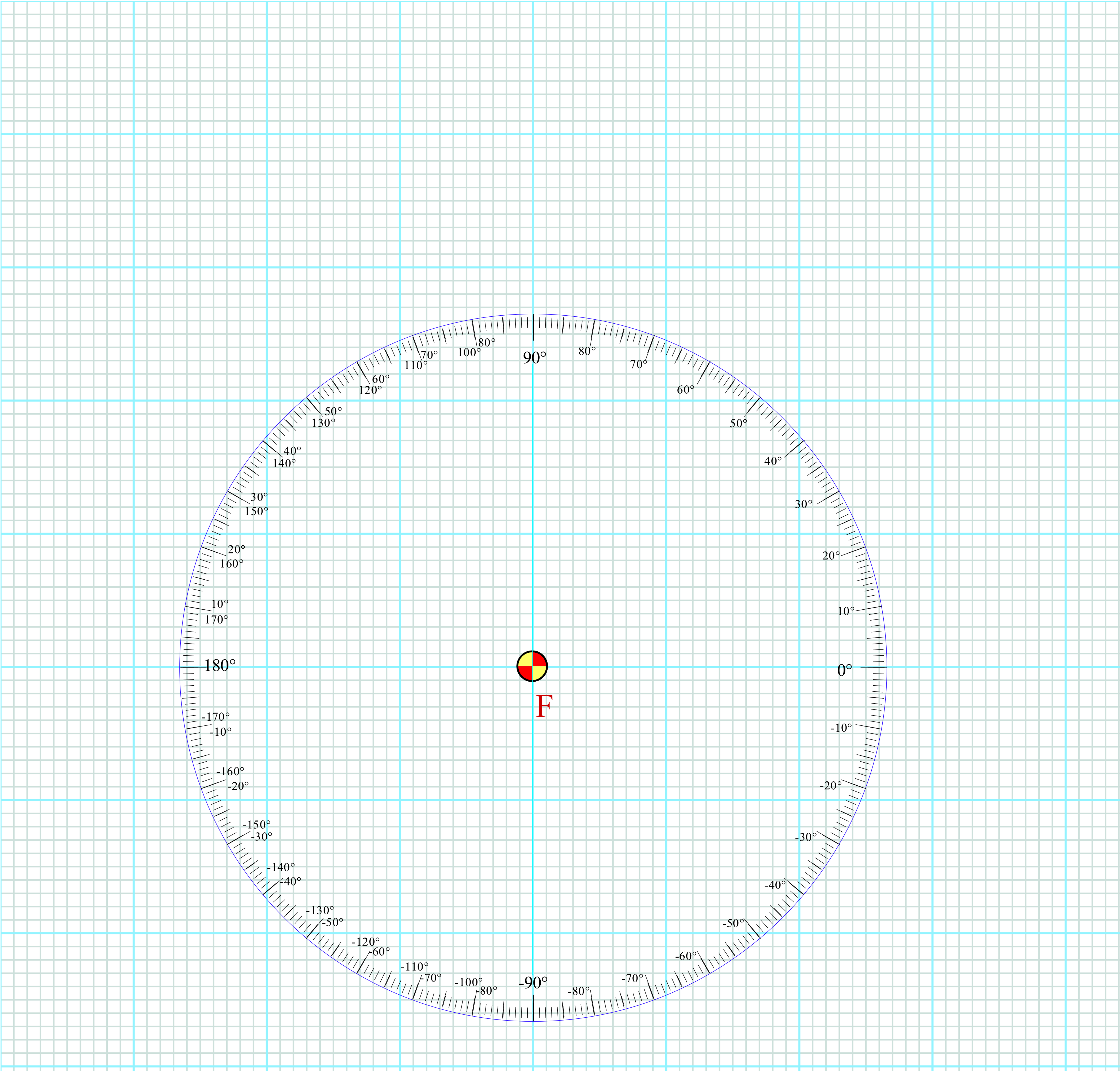
From ε^2 result (at top):

$$\frac{b}{a}=2\sqrt{\mp R(R+1)}\sin\gamma=\sqrt{\pm(1-\varepsilon^2)}$$




90°
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-100°
-110°
-120°
-130°
-140°
-150°
-160°
-170°
F

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

Analytic geometry derivation of $\boldsymbol{\varepsilon}$*-construction*

Connection formulas for (a,b) and (ε,λ) *with* (γ,R)

Detailed ruler & compass construction of $\boldsymbol{\varepsilon}$*-vector and orbits*

(R=-0.375 elliptic orbit)

(R=+0.5 hyperbolic orbit)

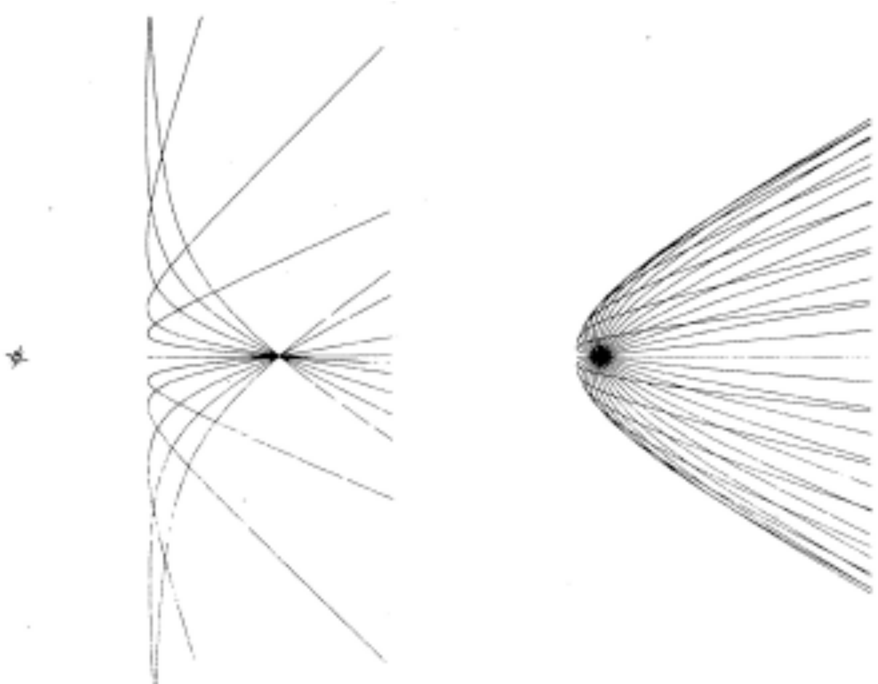
Properties of Coulomb trajectory families and envelopes

Graphical $\boldsymbol{\varepsilon}$*-development of orbits*

Launch angle fixed-Varied launch energy

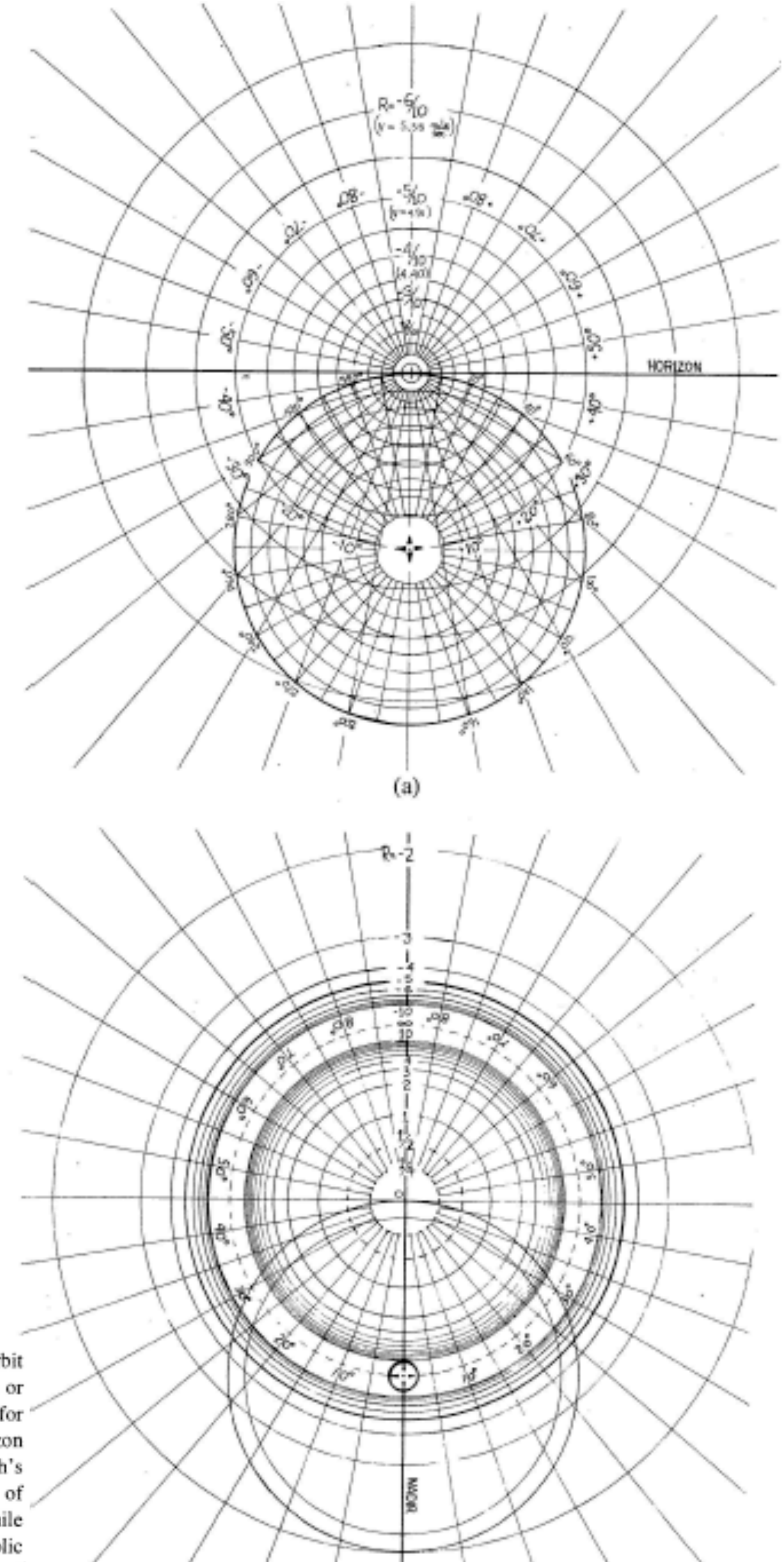
➔ *Launch energy fixed-Varied launch angle*

➔ *Launch optimization and orbit family envelopes*



(a) (b)

Fig. 11. Sample computer trajectories. (a) Family of hyperbolic orbits with $R = 1$. (b) Family of hyperbolic orbits with $R < 1$.

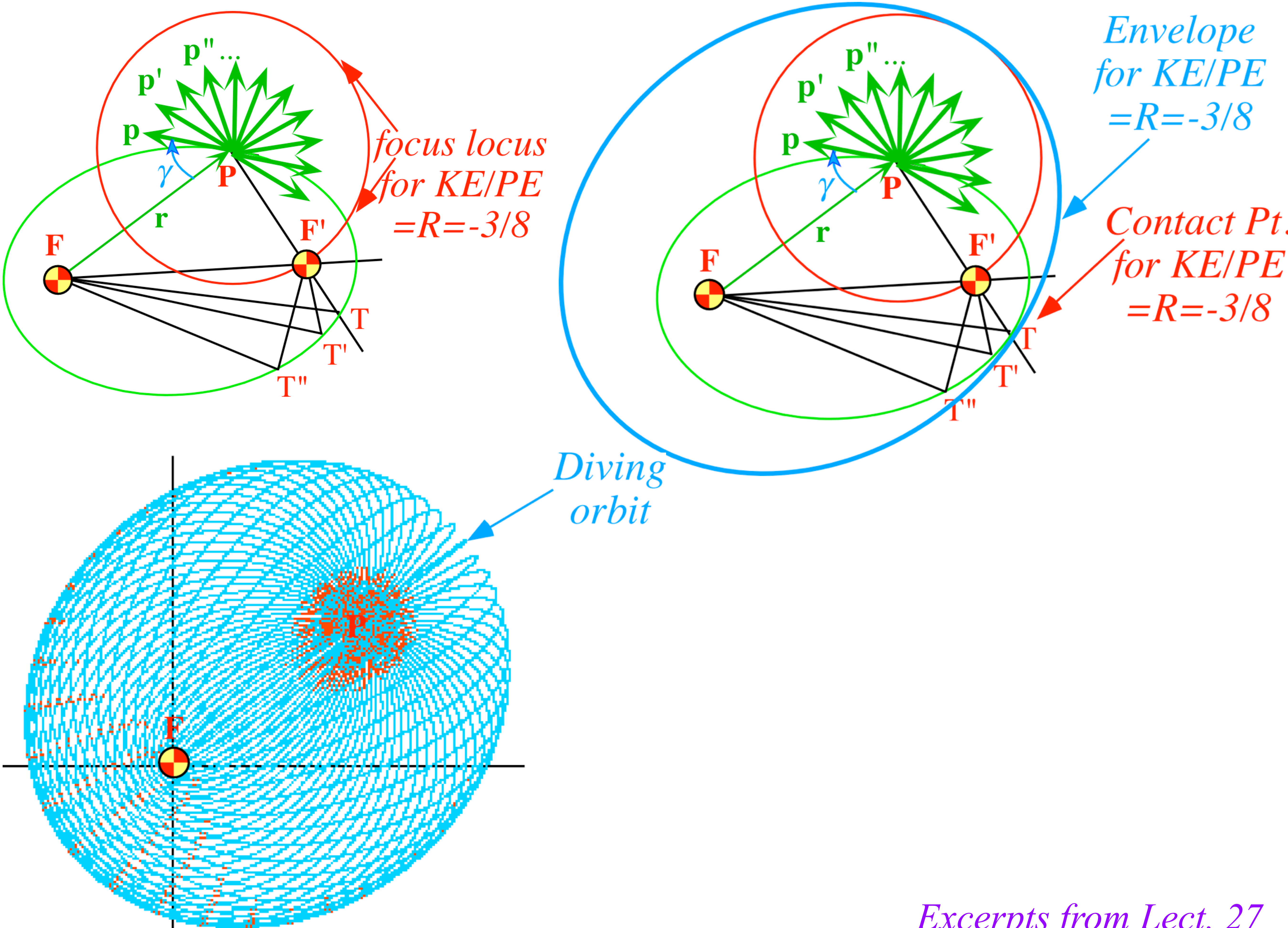


(a)

(b)

Fig. 7. Coordinate grids for orbital analog computers. (a) Elliptical orbit scale ($0 > R > -1$). This can be used with the apparatus in Figs. 8 or 9. Radial lines marked $\pm 10°$, $\pm 20°$, . . . , are each the focus locus for orbits with an initial velocity $\pm 10°$, $\pm 20°$, . . . , above the horizon line. The circle marked 20°, 40°, . . . , 340° can be taken as the Earth's surface, or any circle inside this one can be taken to be the surface of any celestial body. The R values apply correctly in either case, while the velocity values are marked for the former case only. (b) Hyperbolic orbit scale ($0 < R < \infty$) and ($-\infty < R < -1$). This can only be used with the apparatus shown in Fig. 9. Outer circles locate foci for orbits of particles attracted to the force center, while inner circles locate foci for orbits in a repulsive field. In either case a radial line marked $\pm 10°$, $\pm 20°$, . . . , is the focus locus for an orbit with the initial velocity an angle $\pm 10°$, $\pm 20°$, . . . , above the nadir line.

Excerpts from Lect. 27



p"...
p'
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γ
P
r
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F'
T
T'
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focus locus
for KE/PE
=R=-3/8
p"...
p'
p
γ
P
r
F
F'
T
T'
T"
Envelope
for KE/PE
=R=-3/8
Contact Pt.
for KE/PE
=R=-3/8
Diving
orbit
F

Coulomb envelope geometry

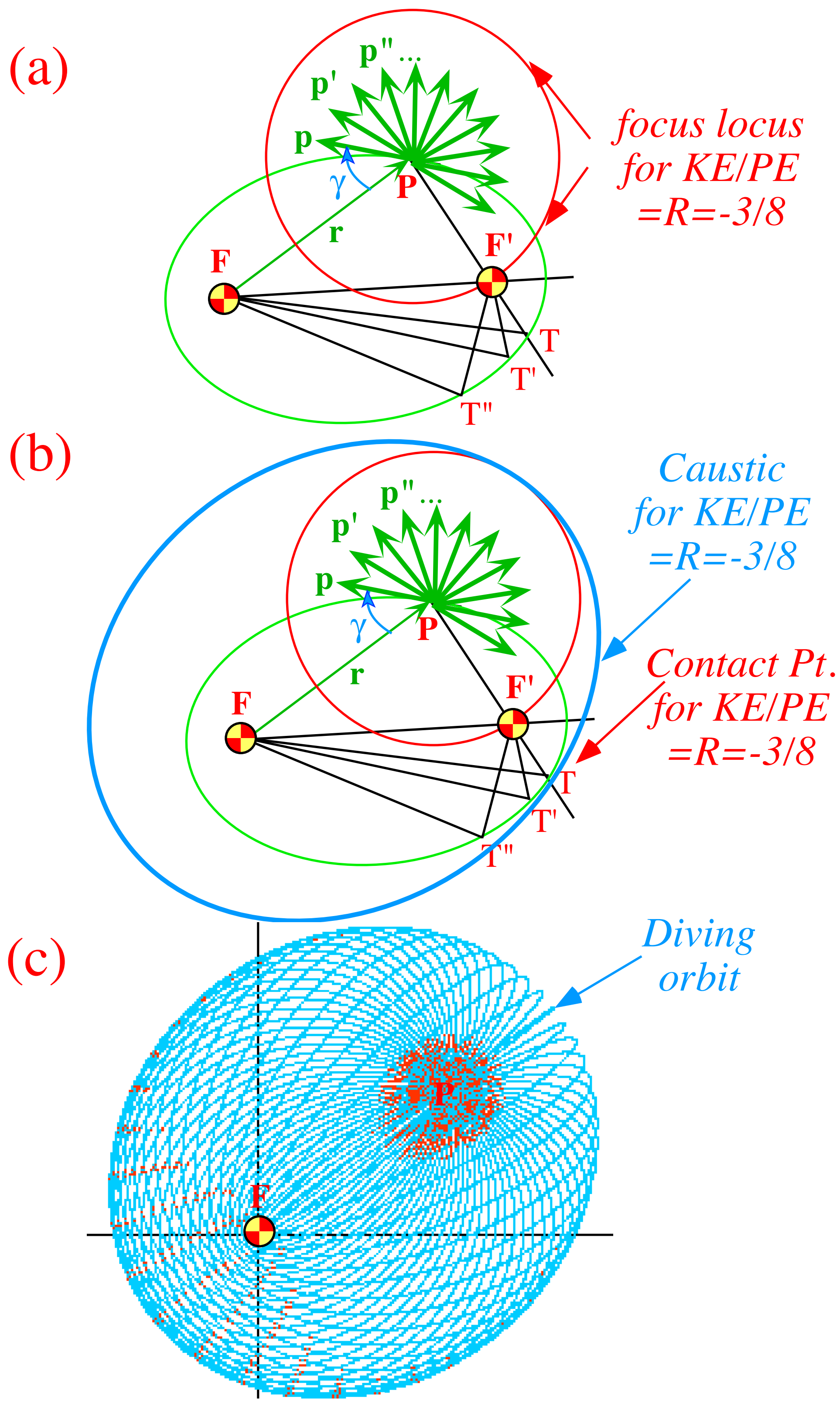
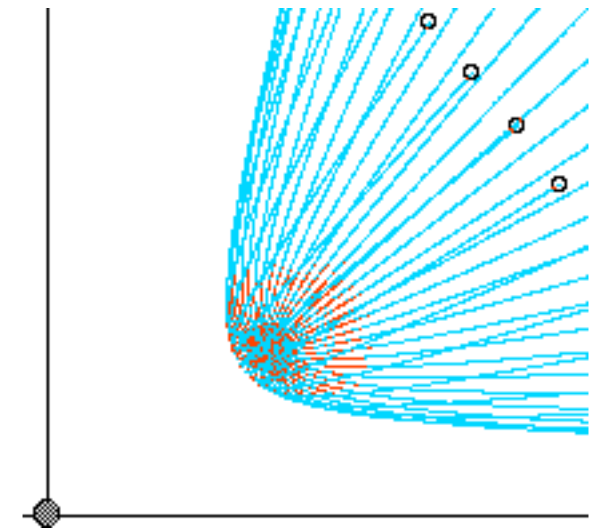


Fig. 5.4.4 in Unit 5 of CMwBANG!



Ideal comet "heads" or "tails" in solar wind

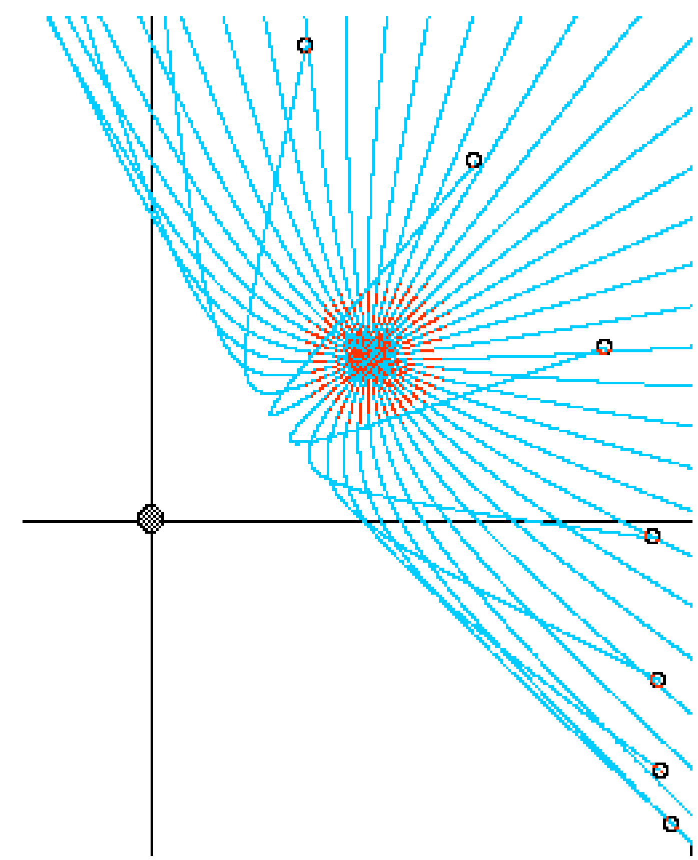
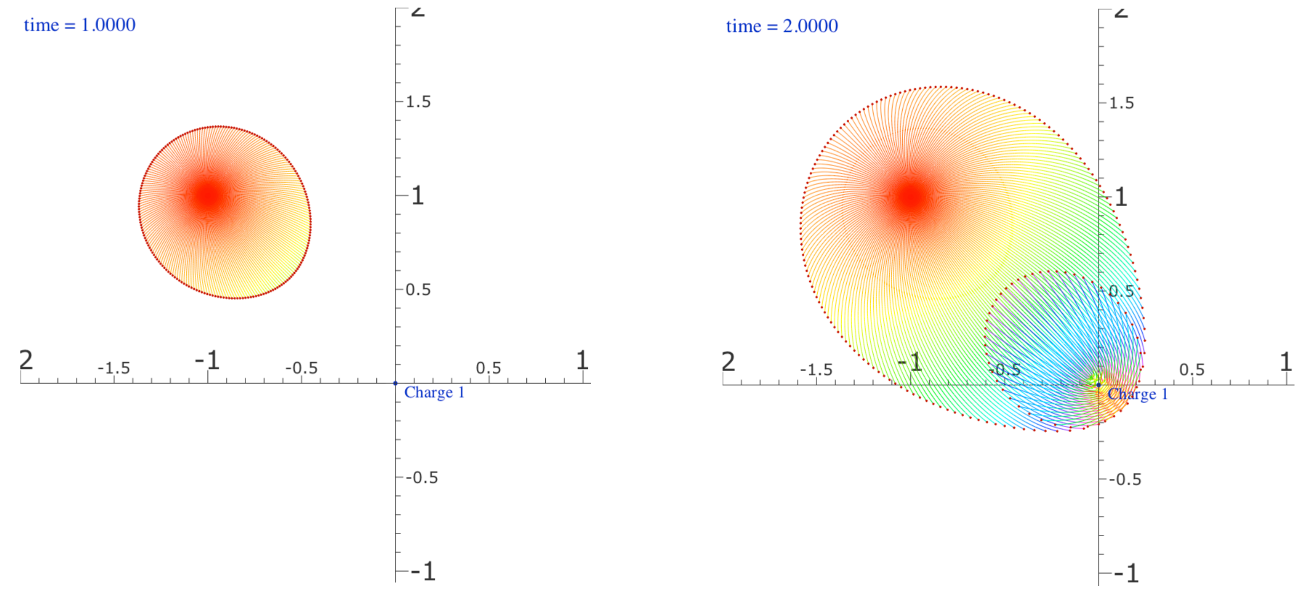
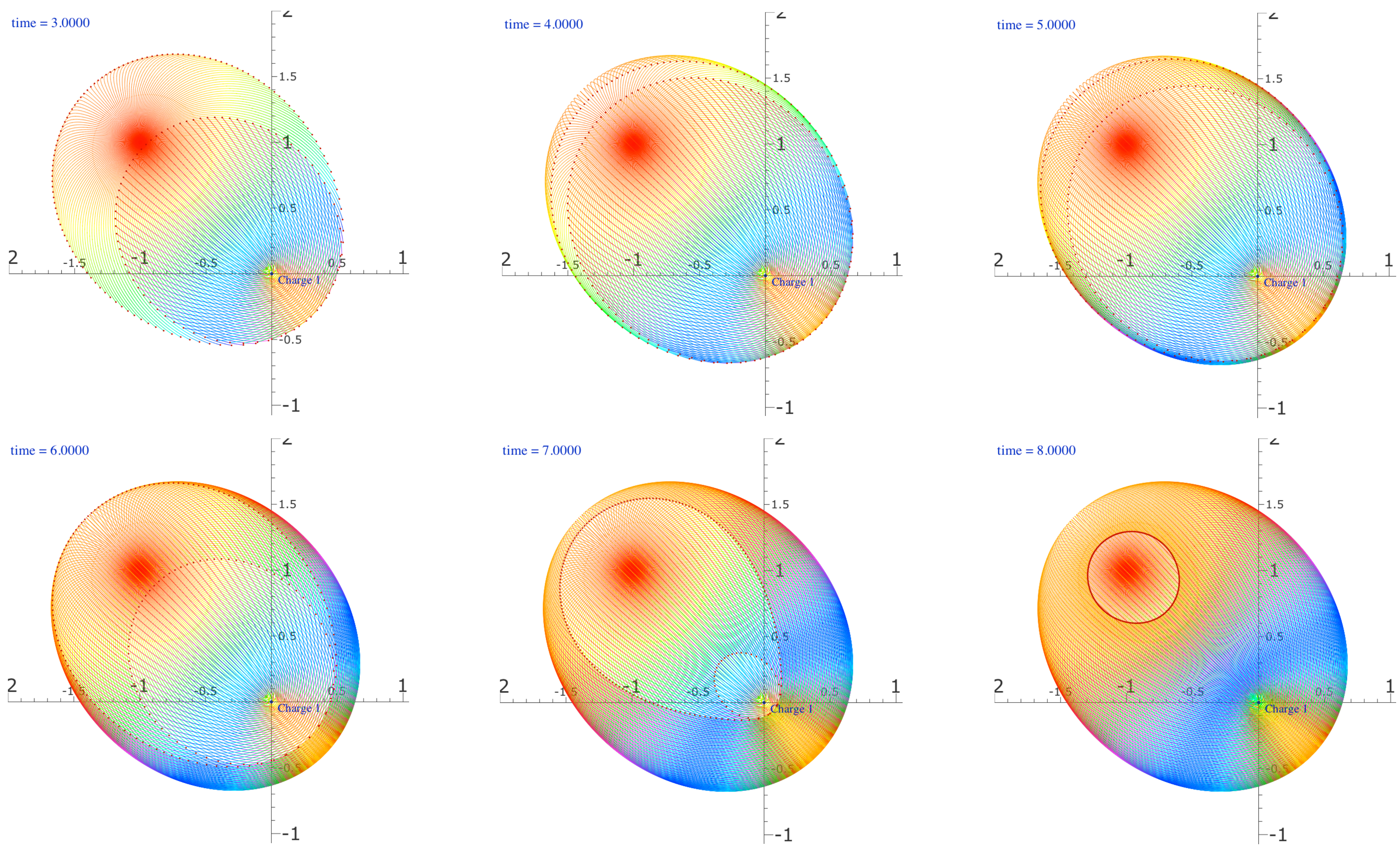


Fig. 5.4.5 in Unit 5 of CMwBANG!

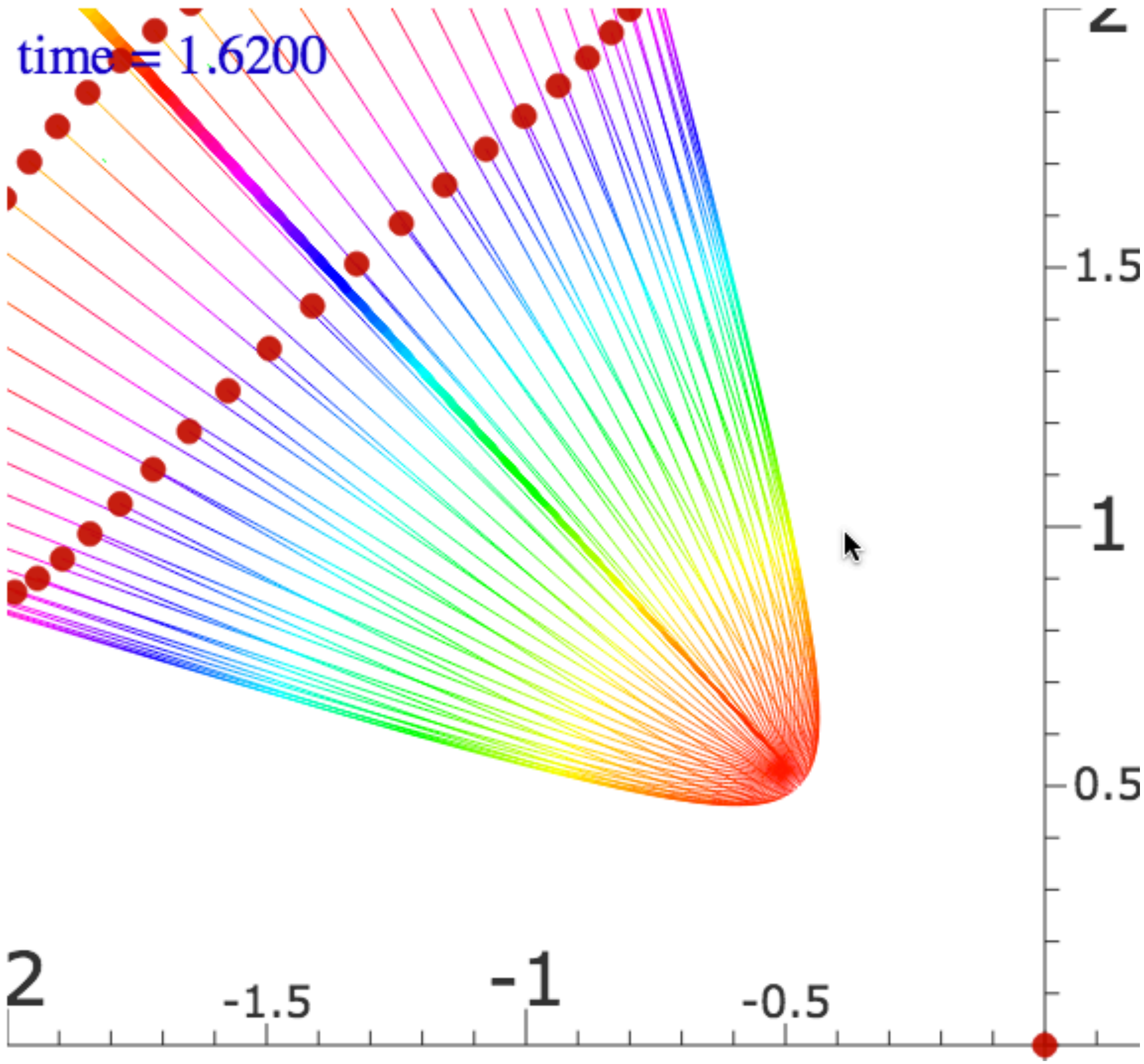
Excerpts from Lect. 27

[CoulIt Web Simulation](#)
[Attractive Coulomb Burst](#)



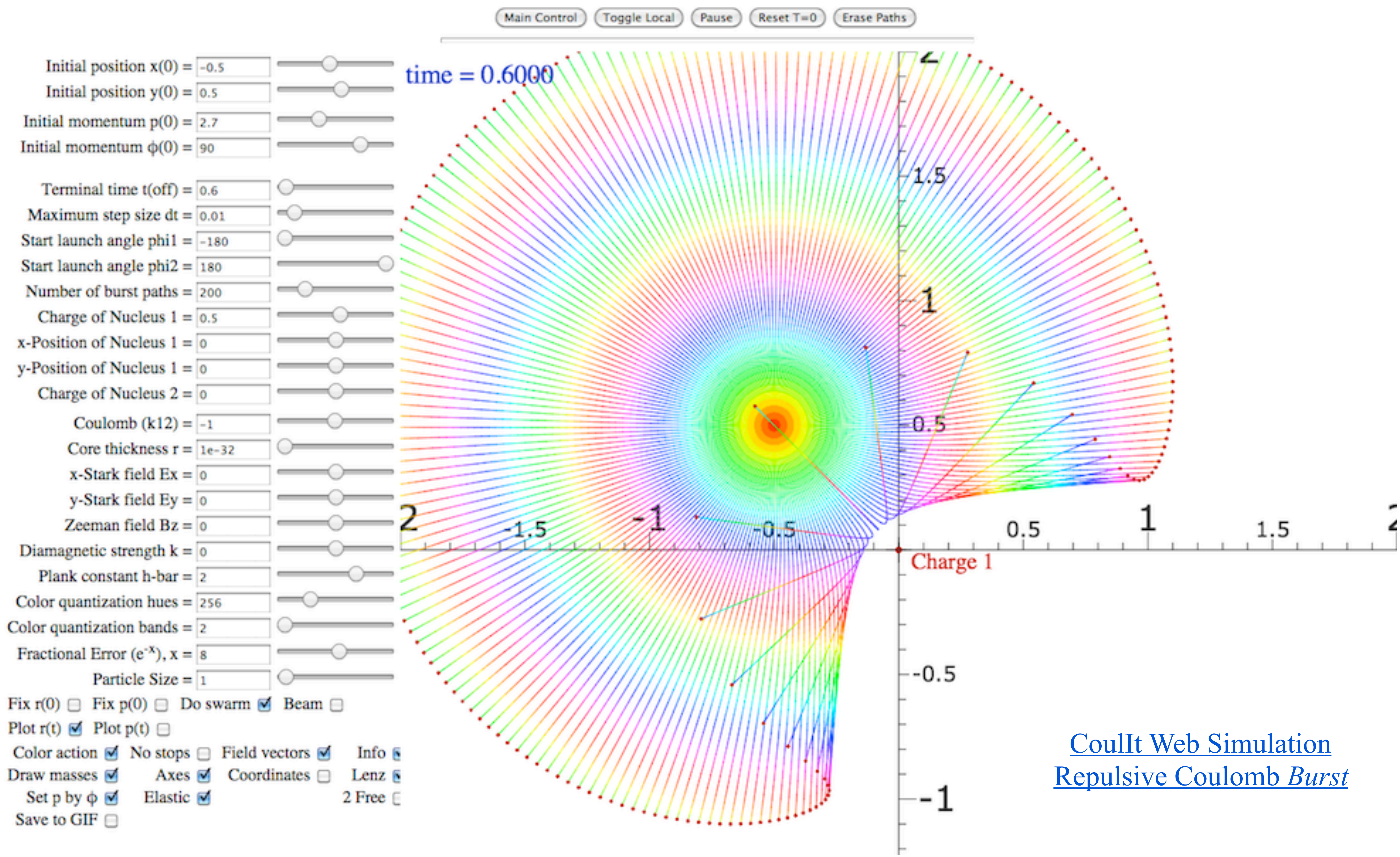


Excerpts from Lect. 27



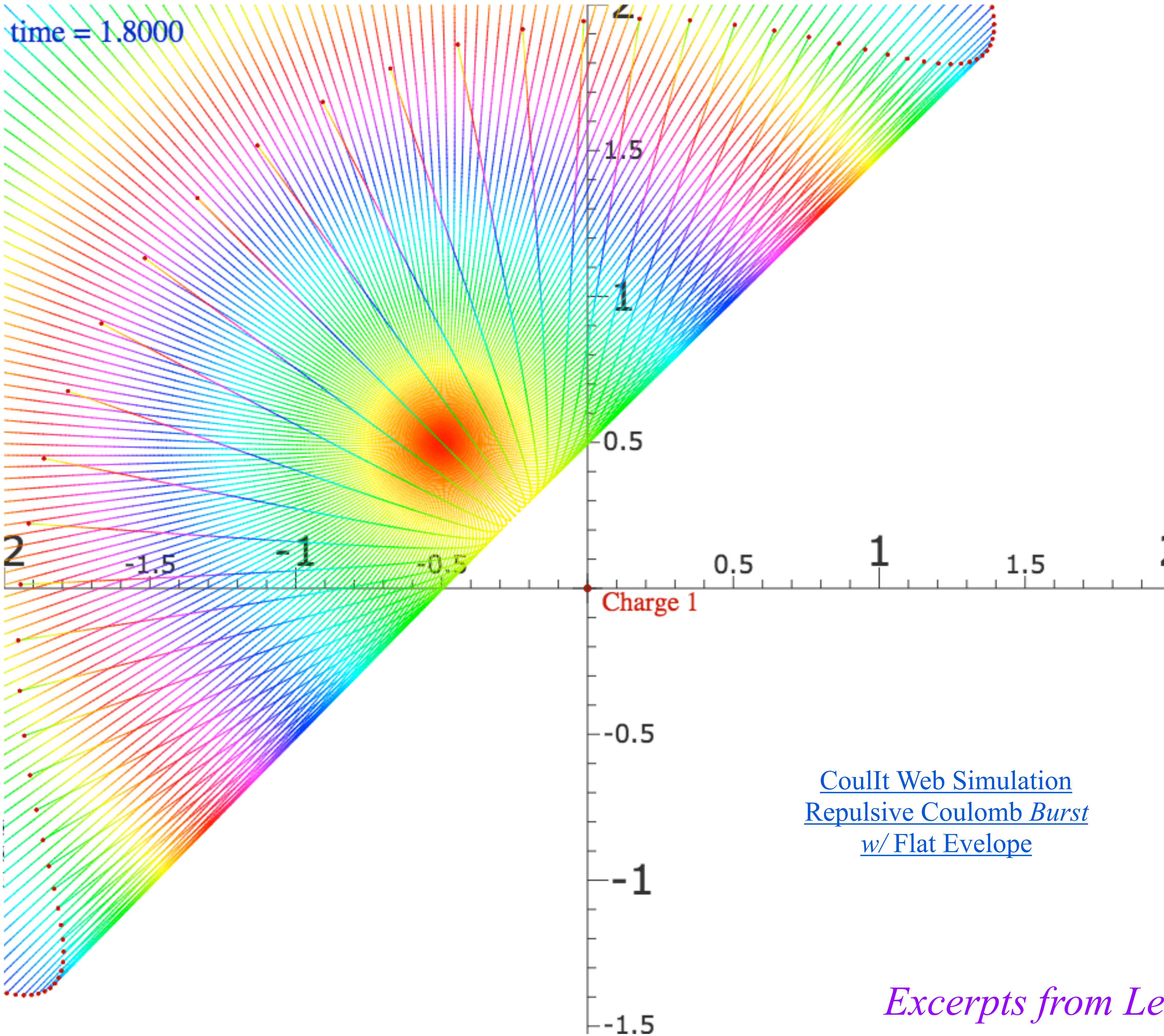
CoulIt Web Simulation Repulsive Coulomb Burst - Tight

Excerpts from Lect. 27



[CoulIt Web Simulation](#)
[Repulsive Coulomb *Burst*](#)

Excerpts from Lect. 27



[CoulIt Web Simulation
Repulsive Coulomb *Burst*
w/ Flat Evelope]()

Excerpts from Lect. 27

*Eccentricity vector **ε** and (ε,λ)-geometry of orbital mechanics*

*Analytic geometry derivation of **ε**-construction*

Connection formulas for (a,b) and (ε,λ) with (γ,R)

*Detailed ruler & compass construction of **ε-vector** and orbits*

(R=-0.375 elliptic orbit)

(R=+0.5 hyperbolic orbit)

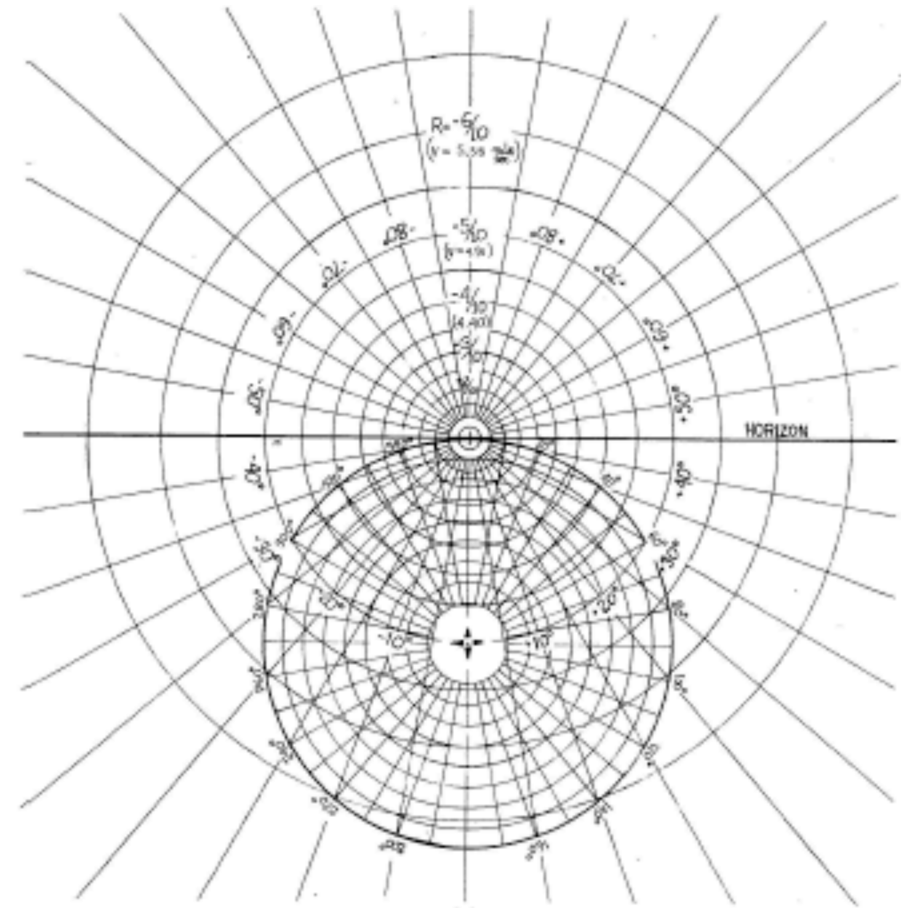
Properties of Coulomb trajectory families and envelopes

➔ *Graphical **ε**-development of orbits*

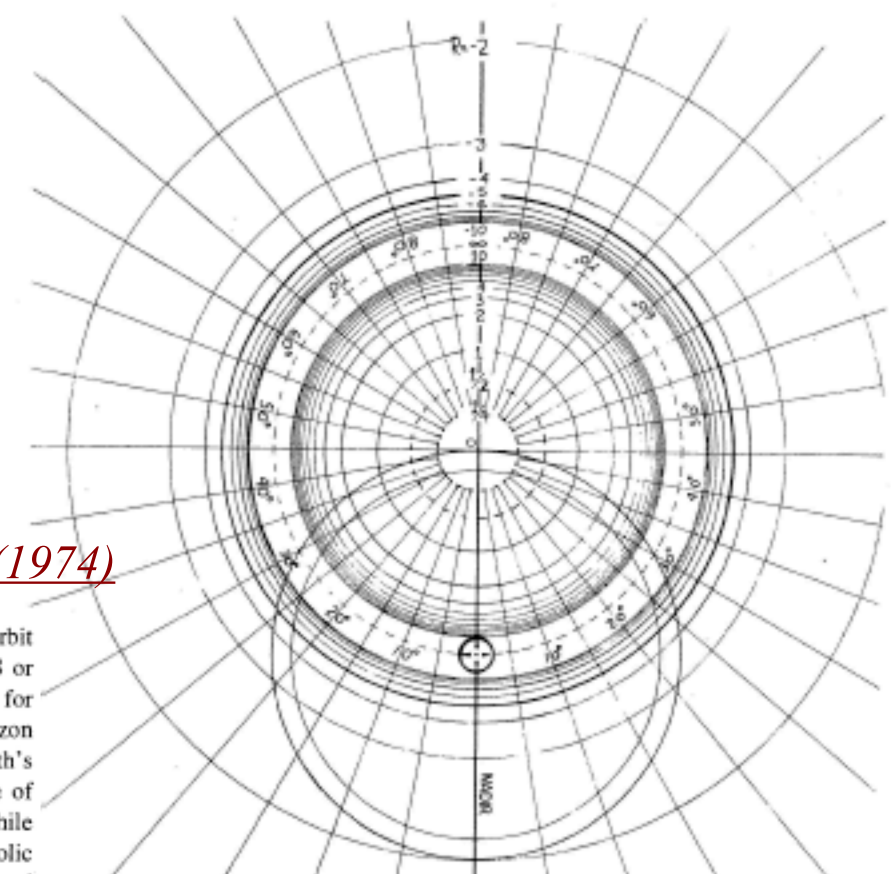
Launch angle fixed-Varied launch energy

Launch energy fixed-Varied launch angle

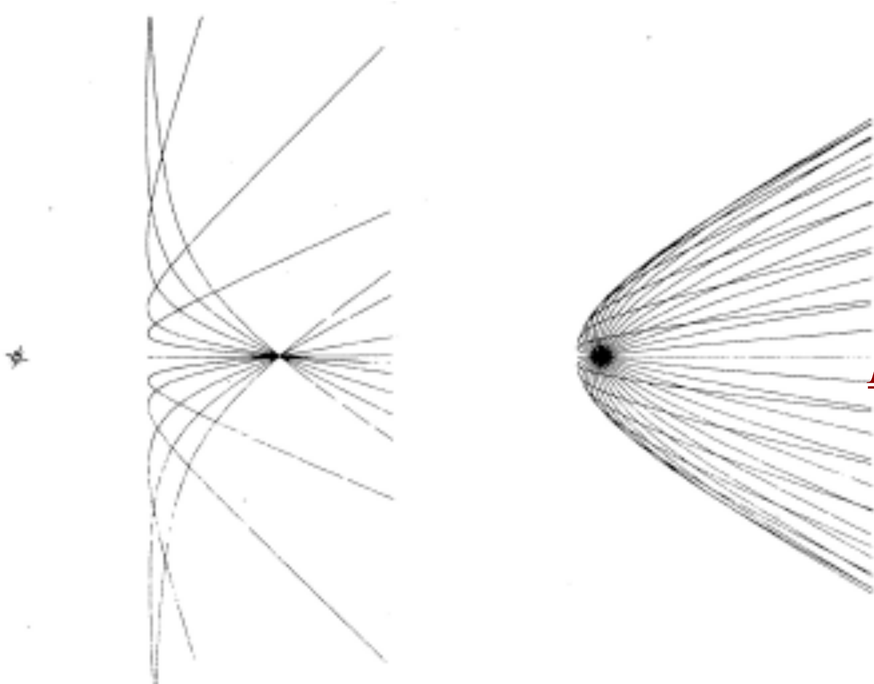
➔ *Launch optimization and orbit family envelopes*



(a)



(b)



(a) (b)

Fig. 11. Sample computer trajectories. (a) Family of hyperbolic orbits with $R = 1$. (b) Family of hyperbolic orbits with $R < 1$.

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Fig. 7. Coordinate grids for orbital analog computers. (a) Elliptical orbit scale ($0 > R > -1$). This can be used with the apparatus in Figs. 8 or 9. Radial lines marked $\pm 10°$, $\pm 20°$, . . . , are each the focus locus for orbits with an initial velocity $\pm 10°$, $\pm 20°$, . . . , above the horizon line. The circle marked 20°, 40°, . . . , 340° can be taken as the Earth's surface, or any circle inside this one can be taken to be the surface of any celestial body. The R values apply correctly in either case, while the velocity values are marked for the former case only. (b) Hyperbolic orbit scale ($0 < R < \infty$) and ($-\infty < R < -1$). This can only be used with the apparatus shown in Fig. 9. Outer circles locate foci for orbits of particles attracted to the force center, while inner circles locate foci for orbits in a repulsive field. In either case a radial line marked $\pm 10°$, $\pm 20°$, . . . , is the focus locus for an orbit with the initial velocity an angle $\pm 10°$, $\pm 20°$, . . . , above the nadir line.

The Lenz vector and orbital analog computers*

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A single geometrical diagram based on the Lenz vector shows the qualitative and quantitative features of all three types of Coulomb orbits. A simple analog computer can be made for an overhead projector by using this theory, and a number of interesting effects can be efficiently demonstrated.

I. INTRODUCTION: THE ECCENTRICITY VECTOR

Occasionally, a geometrical construction and accompanying picture is worth a great deal more to the physicist or the physics student than pages of equations and solutions, especially now that computer graphics are so available. Since Newton's time the geometrical approach has come to be regarded as more clumsy than other methods of thought, and some very pretty pictures and proofs of physical phenomena have undoubtedly been lost. An example of such a construction involving Rutherford scattering was discussed in a recent article[1] by my students and myself, and the following is an improvement of this which describes general Coulombic orbit mechanics.

The generalization we shall describe below is based partly on a more recently discovered quantity called the Lenz–Runge vector[2,3] or the "eccentricity" vector ϵ defined by Eq. (1). There $\mathbf{r}$ is the position vector of the orbiting particle, $\mathbf{L}$ is its angular momentum, $\mathbf{p}$ is its linear momentum, m is its mass, and k is the gravitational (or electrostatic) coefficient:

$$\epsilon = \mathbf{r}/r - \mathbf{L}\times\mathbf{p}/km. \tag{1}$$

Lately this quantity has received a flurry of attention in group theoretical studies of the hydrogen atom[4]; however, we shall use only its geometrical and classical properties.

In particular, the main property of ϵ is that it is a constant vector for any particle moving according to a Coulomb field. Vector ϵ points along the major axis of ellipse, parabola, or hyperbola, whichever is the appropriate orbit of the particle. Furthermore, the magnitude ϵ of this vector is the eccentricity of the orbit.

To show that this is consistent with the usual formulation, we take the dot product of this vector ϵ with the position $\mathbf{r}$ as in Eq. (2). This then reduces to the following equation (3) of a conic section in polar coordinates, which is the general orbit equation[5]:

$$\epsilon r\cos\theta = \epsilon\cdot\mathbf{r} = r - \mathbf{L}\times\mathbf{p}\cdot\mathbf{r}/km \tag{2}$$

$$= r + \mathbf{L}\cdot\mathbf{L}/km,$$

$$r = -(L^2/km)(1-\epsilon\cos\theta)^{-1}. \tag{3}$$

In Sec. II a simple geometric construction using these properties is shown to describe qualitatively and quantitatively the Coulomb orbits for all three cases: namely, the attractive case ($k < 0$) with positive energy, with negative energy, and the repulsive case ($k > 0$).

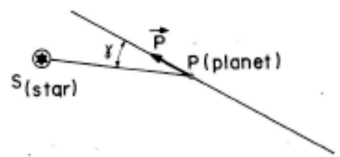

Fig. 1. Initial position and momentum must be given before construction of the resulting orbit is possible.

Finally, it is shown how this construction leads to an analog computer of orbits that can be made for a few dollars to fit onto an overhead projector. This device can be appreciated by elementary classes (even *large* elementary classes if you use the right projector) when they know only a little about conic sections, since the more tedious mathematics is built into the device.

II. COMPUTING ORBITS BY RULER AND COMPASS

We start by simply listing three steps of an orbit construction while demonstrating their application to a particular case of a satellite orbiting a star. Then a general proof of the steps will be provided along with further discussion and applications.

Suppose you are given the initial position and velocity of a satellite relative to some very massive star. If these quantities are given in a pictorial form which shows the angle γ between momentum vector vector $\mathbf{p} = m\mathbf{v}$ and the radius line PS in Fig. 1, and if the magnitude of $\mathbf{v}$ is given by the ratio $R = T/V$ of the kinetic energy ($T = mv^2/2$) to the potential energy ($V = k/r$), then the construction below proceeds immediately. Otherwise, these quantities must be calculated before proceeding. (In the potential energy of the star's gravity we have $k = -GMm$, where M is the star mass and G is the universal constant of gravitation.) Note that R is minus the squared ratio of initial velocity to the escape velocity in

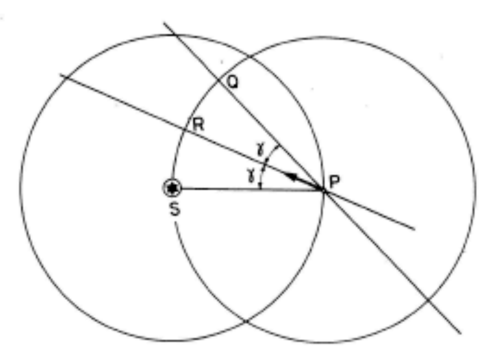

Fig. 2. Doubling the angle between the momentum and the position vectors gives a line QP which must contain the orbit focus.

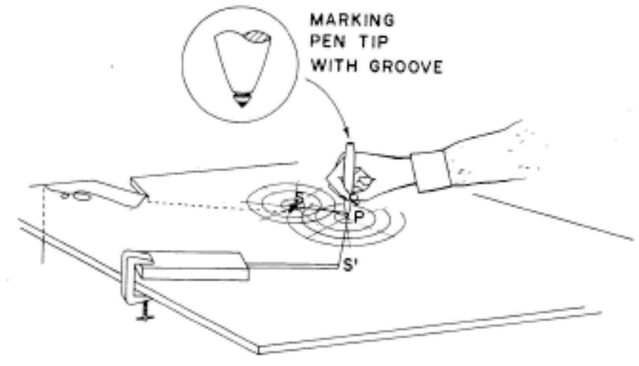

Fig. 8. Orbital computer design: cheaper model that computes elliptical orbits only using the scale of Fig. 7(a). A Plexiglas sheet that is about ⅛ in. thick has a string hole at the orbit's center. A transparency (Xerox, 3-M, etc.) of Fig. 7(a) is taped in position on the underside. (*Caution:* One should avoid marking pens that permanently mark plastic.)

8 go with Fig. 7(a) and can be assembled in a few minutes by using odds and ends. A more sophisticated apparatus is shown in Fig. 9. The simple apparatus of Fig. 8 produces the well-known elliptical orbits and trajectories of planets and satellites, but not the hyperbolic trajectories characteristic of higher-than-escape velocity meteors or of the repulsive Coulomb force problems. The second apparatus (Fig. 9) is designed to handle all cases, provided that the appropriate focal point scale is inserted.

The operation of either plotter begins with the positioning of second focal point S′ according to the scale on the plotting board. Then the marking pen is poked into a small identation at P and held while the strings to S and S′ are tightened. Finally, you slide the pen out along the board in such a way that the strings stay tight and the desired trajectory is drawn.

The apparatus in Fig. 8 will thus make an ellipse since the sum of distances SP and S′P is constant. The apparatus in Fig. 9 does the same when the spool brakes are

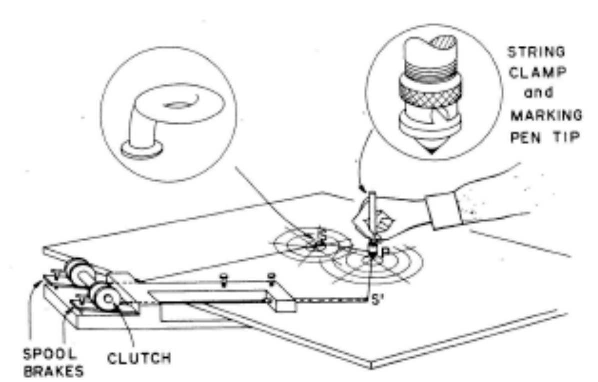

Fig. 9. Orbital computer design: a more elaborate model that computes general Coulomb orbits. The ellipse drawing mode is obtained when, first, the clamp is opened to allow the string to slide and then the spool brakes are tightened after the initial adjustment has been made with the use of the scale in Fig. 7(a). The hyperbola drawing mode is obtained when the string clamp and spool clutch are tightened but the brakes are released. The spools must turn together after the initial adjustment has been made with the use of the scale in Fig. 7(b). One hand can maintain the string tension while drawing the orbit, and the other hand can control the paying out of string. (Alternately, springs on the spool axis accomplish the same thing.)

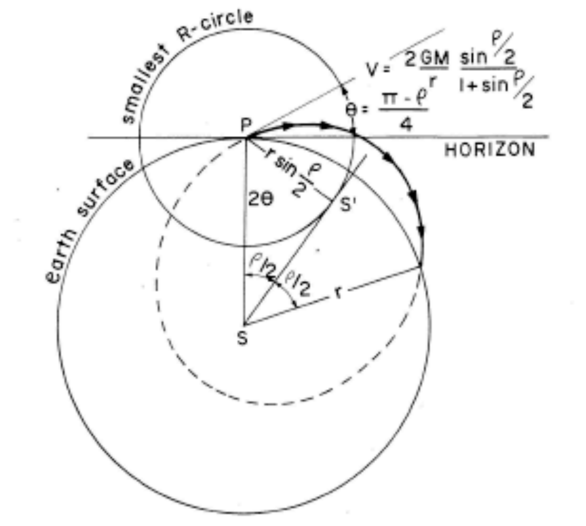

Fig. 10. Sample computer trajectory problem. One finds the minimum-energy trajectory for a given range ρ. Initial velocity v and θ follow easily from the geometry of the computer scale in Fig. 7(a).

tightened and the string clamp on the marking pen is loosened.

The apparatus in Fig. 9 will produce a hyperbola if the *difference* between distances SP and S′P is constant. This is accomplished by tightening the string clamp and the clutch so that two string spools reel equal amounts of string in or out and the constant difference in length is maintained.

IV. SOME USES FOR COMPUTERS

For the computer to be set up to draw elliptical orbits, there is one important question that can be answered immediately: To throw with *minimum* initial velocity a free falling spacecraft between two fixed points near the earth and a distance ρ apart, what initial angle θ and speed v are needed? (We imagine a fixed coordinate system here, not a rotating one.)

Measuring this range by a great circle angle ρ, we see that the focus of the orbit must be on a line through the Earth's center, making an angle $\rho/2$ with the launch point P. The smallest R circle [recall Fig. 7(a)] intersecting this line is the one tangent to it and represents the solution to the problem. Indeed, the algebraic solution to this problem follows from the diagram in Fig. 10 and is given there. Note that the angle θ approaches 45° as the range becomes small compared to the radius of the Earth.

Note that one may change the radius of the starting point P by simply reinterpreting the scale of the computer. For example, if the starting point is located at a height of, say, four times the Earth's radius, then the velocities marked on this scale are all divided by the square root of this factor, in this case, by 2.

With the computer set up to draw hyperbolic orbits and the appropriate scales available, there are a number of interesting problems to examine. For example, the attractive-field positive-energy scale allows one to exhibit the paths of meteorites. Given the impact direction and speed, one can extrapolate to find its origin.

The hyperbolic computer setup can be used to demonstrate Rutherford scattering for either the repulsive field (see Ref. 1) or the attractive field. At the same time