# Lecture 9
# Wed. 9.19.2018

# *Equations of Lagrange and Hamilton mechanics in GeneralizedCurvilinear Coordinates (GCC)*

*(Ch. 12 of Unit 1 and Ch. 1-5 of Unit 2 and Ch. 1-5 of Unit 3)*

*Quick Review of Lagrange Relations in Lectures 7-8*

*Using differential chain-rules for coordinate transformations*

*Polar coordinate example of Generalized Curvilinear Coordinates (GCC)*

*Getting the GCC ready for mechanics: Generalized velocity and Jacobian Lemma 1*

*Getting the GCC ready for mechanics: Generalized acceleration and Lemma 2*

*How to say Newton's "F=ma" in Generalized Curvilinear Coords.*

*Use Cartesian KE quadratic form* $KE=T=1/2\,\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}$ *and* $\mathbf{F}=\mathbf{M}\bullet\mathbf{a}$ *to get GCC force*

*Lagrange GCC trickery gives Lagrange force equations*

*Lagrange GCC trickery gives Lagrange potential equations (Lagrange 1 and 2)*

*GCC Cells, base vectors, and metric tensors*

*Polar coordinate examples: Covariant* $\mathbf{E}_m$ *vs. Contravariant* $\mathbf{E}^m$

*Covariant* $g_{mn}$ *vs. Invariant* $\delta_m{}^n$ *vs. Contravariant* $g^{mn}$

*Lagrange prefers Covariant* $g_{mn}$ *with Contravariant velocity*

*GCC Lagrangian definition*

*GCC "canonical" momentum* $p_m$ *definition*

*GCC "canonical" force* $F_m$ *definition*

*Coriolis "fictitious" forces (... and weather effects)*

*A running collection of links to course-relevant sites and articles*

2018 CMwBang! site

Class YouTube Channel

*You-Tube site displays related videos world-wide*

AIP publications

AJP article on superball dynamics

AAPT summer reading

These *are* hot off the presses. Out in MISC for quick reference.
https://modphys.hosted.uark.edu//ETC/MISC/Sorting_ultracold_atoms_in_a_three-dimensional_optical_lattice_in_a_realization_of_Maxwell%e2%80%99s_demon_-_Kumar-n-2018.pdf
https://modphys.hosted.uark.edu//ETC/MISC/Synthetic_three-dimensional_atomic_structures_assembled_atom_by_atom_-_Barredo-n-2018.pdf

Older ones:
https://modphys.hosted.uark.edu//ETC/MISC/Wave–particle_duality_of_C60_molecules_-_arndt-ltn-1999.pdf
https://modphys.hosted.uark.edu//ETC/MISC/Optical_Vortex_Knots_–_One_Photon__At_A_Time_-_Tempone-Wiltshire-Sr-2018.pdf

"Relawavity" and quantum basis of Lagrangian & Hamiltonian mechanics:
2-CW laser wave: https://modphys.hosted.uark.edu/markup/BohrItWeb.html?scenario=-30104&xPhasorFactor=0.5
Lagrangian vs Hamiltonian: https://modphys.hosted.uark.edu/markup/RelaWavityWeb.html?plotType=4,5&sigmaInd=0&swordLineWidth=3

***Web Resources***

Web Resources - front page

UAF Physics UTube channel

***"Texts"***

Classical Mechanics with a Bang!

Quantum Theory for the Computer Age

Principles of Symmetry, Dynamics, and Spectroscopy

Modern Physics and its Classical Foundations

***Classes***

2014 AMOP

2017 Group Theory for QM

2018 AMOP

2018 Adv Mechanics

AMOP Ch 0 Space-Time Symmetry - 2019
Seminar at Rochester Institute of Optics, Auxiliary slides, June 19, 2018

# *Quick Review of Lagrange Relations in Lectures 7-8*

→ *$0^{th}$ and $1^{st}$ equations of Lagrange and Hamilton*

# *Quick Review of Lagrange Relations in Lectures 7-8*

*0th and 1st equations of Lagrange and Hamilton*

*Starts out with simple demands for explicit-dependence, "loyalty" or "fealty to the colors"*

***Lagrangian*** *and* ***Estrangian*** *have no explicit dependence on* ***momentum*** **p**

$$\frac{\partial L}{\partial p_k} \equiv 0 \equiv \frac{\partial E}{\partial p_k}$$

***Hamiltonian*** *and* ***Estrangian*** *have no explicit dependence on* ***velocity*** **v**

$$\frac{\partial H}{\partial v_k} \equiv 0 \equiv \frac{\partial E}{\partial v_k}$$

***Lagrangian*** *and* ***Hamiltonian*** *have no explicit dependence on* ***speedinum*** **V**

$$\frac{\partial L}{\partial V_k} \equiv 0 \equiv \frac{\partial H}{\partial V_k}$$

*Such non-dependencies hold in spite of "under-the-table" matrix and partial-differential connections*

$$\nabla_v L = \frac{\partial L}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \frac{\mathbf{v} \bullet \mathbf{M} \bullet \mathbf{v}}{2}$$
$$= \mathbf{M} \bullet \mathbf{v} = \mathbf{p}$$

$$\begin{pmatrix} \frac{\partial L}{\partial v_1} \\ \frac{\partial L}{\partial v_2} \end{pmatrix} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

*Lagrange's 1st equation(s)*

$$\frac{\partial L}{\partial v_k} = p_k \quad \text{or:} \quad \frac{\partial L}{\partial \mathbf{v}} = \mathbf{p}$$

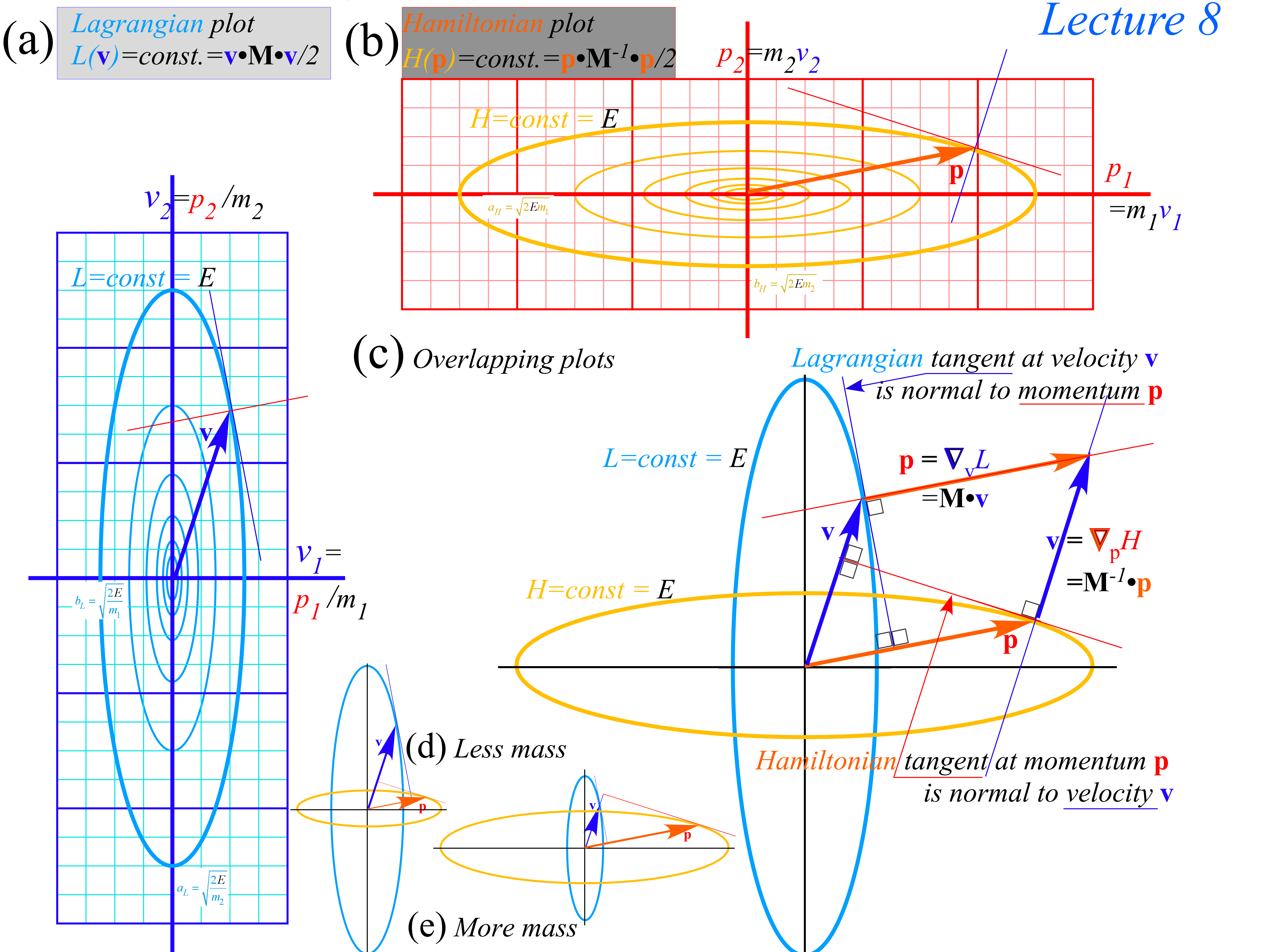
$$\nabla_p H = \mathbf{v} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\partial}{\partial \mathbf{p}} \frac{\mathbf{p} \bullet \mathbf{M}^{-1} \bullet \mathbf{p}}{2}$$
$$= \mathbf{M}^{-1} \bullet \mathbf{p} = \mathbf{v}$$

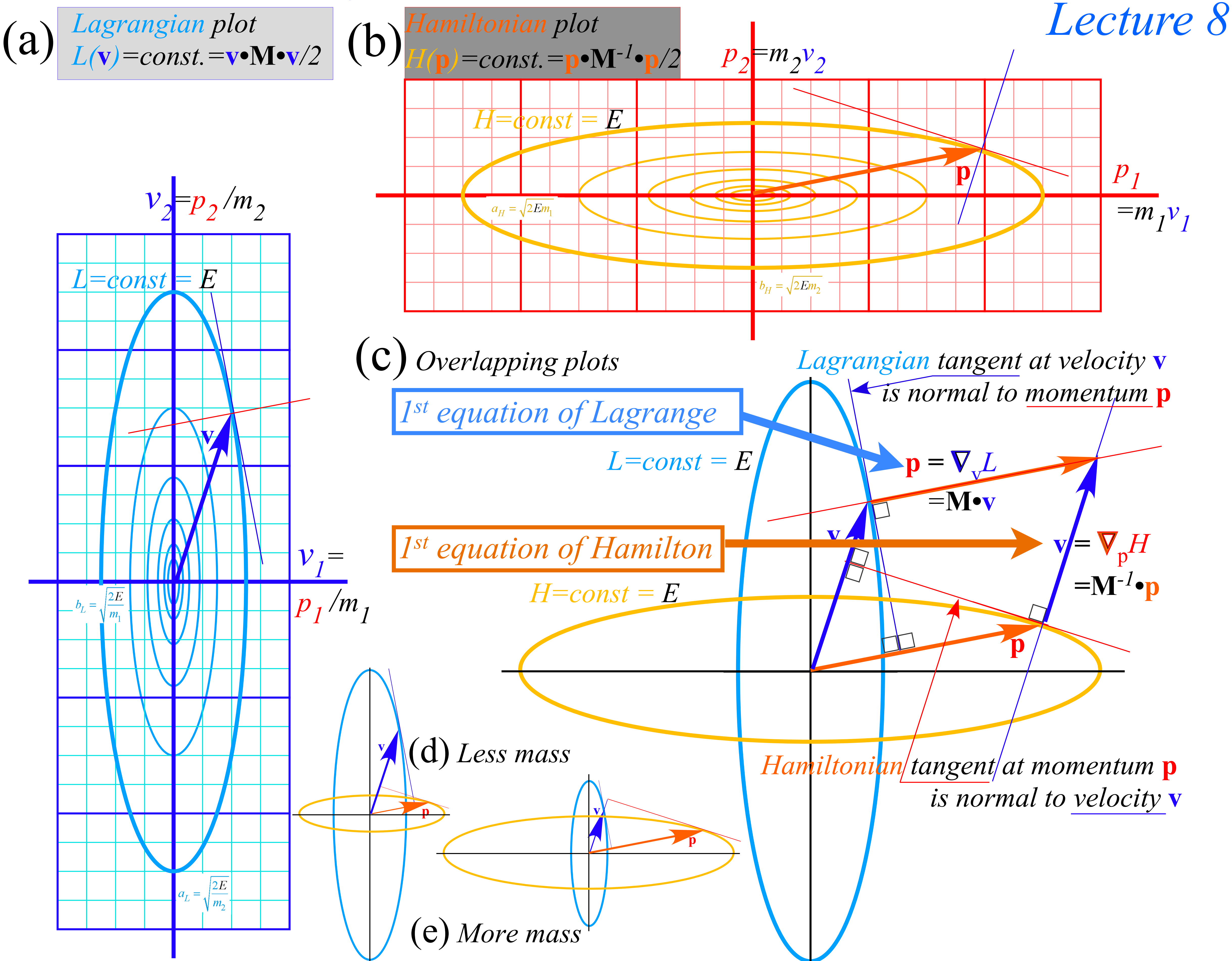
$$\begin{pmatrix} \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \end{pmatrix} = \begin{pmatrix} m_1^{-1} & 0 \\ 0 & m_2^{-1} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

*Hamilton's 1st equation(s)*

$$\frac{\partial H}{\partial p_k} = v_k \quad \text{or:} \quad \frac{\partial H}{\partial \mathbf{p}} = \mathbf{v}$$

*(Forget Estrangian for now)*


(a)
Lagrangian plot
$L(\mathbf{v})=const.=\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}/2$
(b)
Hamiltonian plot
$H(\mathbf{p})=const.=\mathbf{p}\bullet\mathbf{M}^{-1}\bullet\mathbf{p}/2$
$p_2=m_2v_2$
$H=const=E$
$p_1$
$=m_1v_1$
$\mathbf{p}$
$a_H=\sqrt{2Em_1}$
$b_H=\sqrt{2Em_2}$
$v_2=p_2/m_2$
$L=const=E$
$\mathbf{v}$
$v_1=$
$p_1/m_1$
$b_L=\sqrt{\frac{2E}{m_1}}$
$a_L=\sqrt{\frac{2E}{m_2}}$
(c) Overlapping plots
Lagrangian tangent at velocity $\mathbf{v}$
is normal to momentum $\mathbf{p}$
$L=const=E$
$\mathbf{p}=\nabla_{\mathbf{v}}L$
$=\mathbf{M}\bullet\mathbf{v}$
$\mathbf{v}$
$\mathbf{v}=\nabla_{\mathbf{p}}H$
$=\mathbf{M}^{-1}\bullet\mathbf{p}$
$H=const=E$
$\mathbf{p}$
Hamiltonian tangent at momentum $\mathbf{p}$
is normal to velocity $\mathbf{v}$
(d) Less mass
(e) More mass


(a) Lagrangian plot
L(v)=const.=v•M•v/2
(b) Hamiltonian plot
H(p)=const.=p•M^-1•p/2
p2=m2v2
H=const = E
p
p1
=m1v1
v2=p2/m2
L=const = E
v
v1=
p1/m1
(c) Overlapping plots
Lagrangian tangent at velocity v
is normal to momentum p
1st equation of Lagrange
p = ∇vL
=M•v
L=const = E
1st equation of Hamilton
v = ∇pH
=M^-1•p
H=const = E
v
p
Hamiltonian tangent at momentum p
is normal to velocity v
(d) Less mass
(e) More mass

# *Using differential chain-rules for coordinate transformations*

*Polar coordinate example of Generalized Curvilinear Coordinates (GCC)*

*Getting the GCC ready for mechanics: Generalized velocity and Jacobian Lemma 1*

*Getting the GCC ready for mechanics: Generalized acceleration and Lemma 2*

# *Using differential chain-rules† for coordinate transformations*

*A pair of 2-variable functions f(x,y) and g(x,y) can define a coordinate system on (x,y)-space*

*for example: polar coordinates*

$r^2(x,y)= x^2+y^2$ *and* $\theta(x,y)=\mathrm{atan2}(y,x)$

$$df(x,y)=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$$

$$dg(x,y)=\frac{\partial g}{\partial x}dx+\frac{\partial g}{\partial y}dy$$

*( Not in text. Recall Lecture 8 p. 16-20)†*

$$dr(x,y)=\frac{\partial r}{\partial x}dx+\frac{\partial r}{\partial y}dy$$

$$d\theta(x,y)=\frac{\partial \theta}{\partial x}dx+\frac{\partial \theta}{\partial y}dy$$

# Using differential chain-rules† for coordinate transformations

*A pair of 2-variable functions $f(x,y)$ and $g(x,y)$ can define a coordinate system on $(x,y)$-space*

*for example:* *polar coordinates*

$r^2(x,y) = x^2+y^2$ *and* $\theta(x,y) = \mathrm{atan2}(y,x)$

$$df(x,y) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \qquad dr(x,y) = \frac{\partial r}{\partial x}dx + \frac{\partial r}{\partial y}dy$$

$$dg(x,y) = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy \qquad d\theta(x,y) = \frac{\partial \theta}{\partial x}dx + \frac{\partial \theta}{\partial y}dy$$

*( Not in text. Recall Lecture 8 p. 16-20)†*

*Easy to invert differential chain relations (even if functions are not easily inverted)*

$$dx = \frac{\partial x}{\partial f}df + \frac{\partial y}{\partial g}dg$$

$$dy = \frac{\partial y}{\partial f}df + \frac{\partial y}{\partial g}dg$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta$$

$$dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta$$

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} \begin{pmatrix} dr \\ d\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} dr \\ d\theta \end{pmatrix}$$

# *Using differential chain-rules† for coordinate transformations*

*A pair of 2-variable functions f(x,y) and g(x,y) can define a coordinate system on (x,y)-space*

*for example: polar coordinates*

$r^2(x,y)= x^2+y^2$ *and* $\theta(x,y)=\mathrm{atan2}(y,x)$

$$df(x,y)=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy \qquad dr(x,y)=\frac{\partial r}{\partial x}dx+\frac{\partial r}{\partial y}dy$$

$$dg(x,y)=\frac{\partial g}{\partial x}dx+\frac{\partial g}{\partial y}dy \qquad d\theta(x,y)=\frac{\partial \theta}{\partial x}dx+\frac{\partial \theta}{\partial y}dy$$

*(Not in text. Recall Lecture 8 p. 16-20)†*

*Easy to invert differential chain relations (even if functions are not easily inverted)*

$$dx=\frac{\partial x}{\partial f}df+\frac{\partial x}{\partial g}dg \qquad x=r\cos\theta \qquad dx=\frac{\partial x}{\partial r}dr+\frac{\partial x}{\partial \theta}d\theta$$

$$dy=\frac{\partial y}{\partial f}df+\frac{\partial y}{\partial g}dg \qquad y=r\sin\theta \qquad dy=\frac{\partial y}{\partial r}dr+\frac{\partial y}{\partial \theta}d\theta$$

$$\begin{pmatrix} dx \\ dy \end{pmatrix}=\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix}\begin{pmatrix} dr \\ d\theta \end{pmatrix}=\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}\begin{pmatrix} dr \\ d\theta \end{pmatrix}$$

# *Notation for differential GCC* (Generalized Curvilinear Coordinates $\{q^1, q^2, q^3,\ldots\}$)

$$dx^j=\frac{\partial x^j}{\partial q^m}dq^m\left(\equiv\sum_{m=1}^{N}\frac{\partial x^j}{\partial q^m}dq^m \quad \left\{\begin{array}{l}\text{Defining a shorthand}\\ \text{dummy-index } m\text{-sum}\end{array}\right\}\right)$$

What does "*q*" stand for?
One guess: "*Queer*"
And they <u>do</u> get pretty queer!

These $x^j$ are plain old CC (Cartesian Coordinates $\{dx^1=dx,\ dx^2=dy,\ dx^3=dx,\ dx^4=dt\}$ )

# Using differential chain-rules† for coordinate transformations

*A pair of 2-variable functions f(x,y) and g(x,y) can define a coordinate system on (x,y)-space*

*for example: polar coordinates*

$r^2(x,y)=x^2+y^2$ and $\theta(x,y)=\mathrm{atan2}(y,x)$

$$df(x,y)=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy \qquad dr(x,y)=\frac{\partial r}{\partial x}dx+\frac{\partial r}{\partial y}dy$$

$$dg(x,y)=\frac{\partial g}{\partial x}dx+\frac{\partial g}{\partial y}dy \qquad d\theta(x,y)=\frac{\partial \theta}{\partial x}dx+\frac{\partial \theta}{\partial y}dy$$

*( Not in text. Recall Lecture 8 p. 16-20)†*

*Easy to invert differential chain relations (even if functions are not easily inverted)*

$$dx=\frac{\partial x}{\partial f}df+\frac{\partial y}{\partial g}dg \qquad x=r\cos\theta \qquad dx=\frac{\partial x}{\partial r}dr+\frac{\partial x}{\partial \theta}d\theta$$

$$dy=\frac{\partial y}{\partial f}df+\frac{\partial y}{\partial g}dg \qquad y=r\sin\theta \qquad dy=\frac{\partial y}{\partial r}dr+\frac{\partial y}{\partial \theta}d\theta$$

$$\begin{pmatrix} dx \\ dy \end{pmatrix}=\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix}\begin{pmatrix} dr \\ d\theta \end{pmatrix}=\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}\begin{pmatrix} dr \\ d\theta \end{pmatrix}$$

Index $m$ REPEATED on SAME side of $=$ is SUMMED

# Notation for differential GCC (Generalized Curvilinear Coordinates $\{q^1, q^2, q^3,\ldots\}$)

$$dx^j=\frac{\partial x^j}{\partial q^m}dq^m \left(\equiv \sum_{m=1}^{N}\frac{\partial x^j}{\partial q^m}dq^m \quad \left\{\begin{array}{l}\text{Defining a shorthand}\\ \text{dummy-index } m\text{-sum}\end{array}\right\}\right)$$

What does "$q$" stand for?
One guess: "*Queer*"
And they do get pretty queer!

Connection lines may help to indicate summation (OK on scratch paper...Difficult in text)

These $x^j$ are plain old CC (Cartesian Coordinates $\{dx^1=dx,\ dx^2=dy,\ dx^3=dx,\ dx^4=dt\}$ )

## *Using differential chain-rules for coordinate transformations*

*Polar coordinate example of Generalized Curvilinear Coordinates (GCC)*

*Getting the GCC ready for mechanics: Generalized velocity and Jacobian Lemma 1*

*Getting the GCC ready for mechanics: Generalized acceleration and Lemma 2*

# *Getting the GCC ready for mechanics:*

*Generalized velocity relation follows from GCC chain rule*

$$\boxed{dx^j = \frac{\partial x^j}{\partial q^m} dq^m}$$

Same kind of linear relation exists between CC velocity $v^j \equiv \dot{x}^j \equiv \frac{dx^j}{dt}$ and GCC velocity $\upsilon^m \equiv \dot{q}^m \equiv \frac{dq^m}{dt}$

$$\boxed{\dot{x}^j = \frac{\partial x^j}{\partial q^m} \dot{q}^m}$$

# *Getting the GCC ready for mechanics:*

*Generalized velocity relation follows from GCC chain rule*

$$dx^j = \frac{\partial x^j}{\partial q^m} dq^m$$

Same kind of linear relation exists between CC velocity $v^j \equiv \dot{x}^j \equiv \frac{dx^j}{dt}$ and GCC velocity $\upsilon^m \equiv \dot{q}^m \equiv \frac{dq^m}{dt}$

$$\dot{x}^j = \frac{\partial x^j}{\partial q^m} \dot{q}^m$$

*This is a key "lemma-1" for setting up mechanics:*

*or:*

$$\frac{\partial \dot{x}^j}{\partial \dot{q}^m} = \frac{\partial x^j}{\partial q^m}$$

*lemma-1*

# *Getting the GCC ready for mechanics:*

*Generalized velocity relation follows from GCC chain rule*

$$\boxed{dx^j = \frac{\partial x^j}{\partial q^m} dq^m}$$

Same kind of linear relation exists between CC velocity $v^j \equiv \dot{x}^j \equiv \frac{dx^j}{dt}$ and GCC velocity $\upsilon^m \equiv \dot{q}^m \equiv \frac{dq^m}{dt}$

*This is a key "lemma-1" for setting up mechanics:*

$$\boxed{\dot{x}^j = \frac{\partial x^j}{\partial q^m} \dot{q}^m}$$

*or:*

$$\boxed{\frac{\partial \dot{x}^j}{\partial \dot{q}^m} = \frac{\partial x^j}{\partial q^m} \quad \textit{lemma-1}}$$

*Jacobian* $J_m^j$ matrix gives each CCC differential $dx^j$ or velocity $\dot{x}^j$ in terms of GCC $dq^m$ or $\dot{q}^m$.

$$J_m^j \equiv \frac{\partial x^j}{\partial q^m} = \frac{\partial \dot{x}^j}{\partial \dot{q}^m} \quad \left\{ \begin{array}{l} \text{Defining } \textit{Jacobian} \\ \text{matrix component} \end{array} \right\}$$

*Recall polar coordinate transformation matrix:*

$$\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

# *Getting the GCC ready for mechanics:*

*Generalized velocity relation follows from GCC chain rule*

$$\boxed{dx^j = \frac{\partial x^j}{\partial q^m} dq^m}$$

Same kind of linear relation exists between CC velocity $v^j \equiv \dot{x}^j \equiv \frac{dx^j}{dt}$ and GCC velocity $\upsilon^m \equiv \dot{q}^m \equiv \frac{dq^m}{dt}$

*This is a key "lemma-1" for setting up mechanics:*

$$\boxed{\dot{x}^j = \frac{\partial x^j}{\partial q^m} \dot{q}^m}$$

*or:* $$\boxed{\frac{\partial \dot{x}^j}{\partial \dot{q}^m} = \frac{\partial x^j}{\partial q^m} \quad \textit{lemma-1}}$$

*Jacobian* $J_m^j$ matrix gives each CCC differential $dx^j$ or velocity $\dot{x}^j$ in terms of GCC $dq^m$ or $\dot{q}^m$.

$$J_m^j \equiv \frac{\partial x^j}{\partial q^m} = \frac{\partial \dot{x}^j}{\partial \dot{q}^m} \quad \left\{ \begin{array}{l} \text{Defining } \textit{Jacobian} \\ \text{matrix component} \end{array} \right\}$$

*Recall polar coordinate transformation matrix:*

$$\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

Inverse (so-called) *Kajobian* $K_j^m$ matrix is flipped partial derivatives of $J_m^j$.

$$K_j^m \equiv \frac{\partial q^m}{\partial x^j} = \frac{\partial \dot{q}^m}{\partial \dot{x}^j} \quad \left\{ \begin{array}{l} \text{Defining "} \textit{Kajobian} \text{"} \\ \text{(inverse to Jacobian)} \end{array} \right\}$$

*Polar coordinate inverse transformation matrix:*

$$\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{pmatrix} = \frac{\begin{pmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}{(\det J = r)} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix}$$

*Defining 2x2 matrix inverse:* *(always test inverse matrices!)*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \frac{\begin{pmatrix} D & -B \\ -C & A \end{pmatrix}}{AD - BC}$$

# Getting the GCC ready for mechanics:

*Generalized velocity relation follows from GCC chain rule*

$$\boxed{dx^j = \frac{\partial x^j}{\partial q^m} dq^m}$$

Same kind of linear relation exists between CC velocity $v^j \equiv \dot{x}^j \equiv \frac{dx^j}{dt}$ and GCC velocity $\upsilon^m \equiv \dot{q}^m \equiv \frac{dq^m}{dt}$

*This is a key "lemma-1" for setting up mechanics:*

$$\boxed{\dot{x}^j = \frac{\partial x^j}{\partial q^m} \dot{q}^m}$$

*or:*

$$\boxed{\frac{\partial \dot{x}^j}{\partial \dot{q}^m} = \frac{\partial x^j}{\partial q^m} \quad \text{lemma-1}}$$

*Jacobian* $J_m^j$ matrix gives each CCC differential $dx^j$ or velocity $\dot{x}^j$ in terms of GCC $dq^m$ or $\dot{q}^m$.

$$J_m^j \equiv \frac{\partial x^j}{\partial q^m} = \frac{\partial \dot{x}^j}{\partial \dot{q}^m} \quad \left\{ \begin{array}{l} \text{Defining } \textit{Jacobian} \\ \text{matrix component} \end{array} \right\}$$

*Recall polar coordinate transformation matrix:*

$$\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

Inverse (so-called) *Kajobian* $K_j^m$ matrix is flipped partial derivatives of $J_m^j$.

$$K_j^m \equiv \frac{\partial q^m}{\partial x^j} = \frac{\partial \dot{q}^m}{\partial \dot{x}^j} \quad \left\{ \begin{array}{l} \text{Defining } \textit{"Kajobian"} \\ \text{(inverse to Jacobian)} \end{array} \right\}$$

*Polar coordinate inverse transformation matrix:*

$$\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{pmatrix} = \frac{\begin{pmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}{(\det J = r)} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix}$$

*Defining 2x2 matrix inverse:* *(always test inverse matrices!)*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \frac{\begin{pmatrix} D & -B \\ -C & A \end{pmatrix}}{AD-BC} = \begin{pmatrix} \frac{D}{AD-BC} & \frac{-B}{AD-BC} \\ \frac{-C}{AD-BC} & \frac{A}{AD-BC} \end{pmatrix}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} D & -B \\ -C & A \end{pmatrix} = \begin{pmatrix} AD-BC & 0 \\ 0 & AD-BC \end{pmatrix}$$

# *Getting the GCC ready for mechanics:*

*Generalized velocity relation follows from GCC chain rule*

$$\boxed{dx^j = \frac{\partial x^j}{\partial q^m} dq^m}$$

Same kind of linear relation exists between CC velocity $v^j \equiv \dot{x}^j \equiv \frac{dx^j}{dt}$ and GCC velocity $\upsilon^m \equiv \dot{q}^m \equiv \frac{dq^m}{dt}$

$$\boxed{\dot{x}^j = \frac{\partial x^j}{\partial q^m} \dot{q}^m}$$

*This is a key "lemma-1" for setting up mechanics:* *or:*

$$\boxed{\frac{\partial \dot{x}^j}{\partial \dot{q}^m} = \frac{\partial x^j}{\partial q^m} \quad \text{lemma-1}}$$

*Jacobian* $J_m^j$ matrix gives each CCC differential $dx^j$ or velocity $\dot{x}^j$ in terms of GCC $dq^m$ or $\dot{q}^m$.

$$J_m^j \equiv \frac{\partial x^j}{\partial q^m} = \frac{\partial \dot{x}^j}{\partial \dot{q}^m} \quad \left\{ \begin{array}{l} \text{Defining } \textit{Jacobian} \\ \text{matrix component} \end{array} \right\}$$

*Recall polar coordinate transformation matrix:*

$$\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

Inverse (so-called) *Kajobian* $K_j^m$ matrix is flipped partial derivatives of $J_m^j$.

$$K_j^m \equiv \frac{\partial q^m}{\partial x^j} = \frac{\partial \dot{q}^m}{\partial \dot{x}^j} \quad \left\{ \begin{array}{l} \text{Defining } \textit{"Kajobian"} \\ \text{(inverse to Jacobian)} \end{array} \right\}$$

$$\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{pmatrix} = \frac{\begin{pmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}{(\det J = r)} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix}$$

Product of matrix $J_m^j$ and $K_j^m$ is a unit matrix by definition of partial derivatives. *(always test inverse matrices!)*

$$K_j^m \cdot J_n^j \equiv \frac{\partial q^m}{\partial x^j} \cdot \frac{\partial x^j}{\partial q^n} = \frac{\partial q^m}{\partial q^n} = \delta_n^m = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

$$\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

# *Using differential chain-rules for coordinate transformations*

*Polar coordinate example of Generalized Curvilinear Coordinates (GCC)*

*Getting the GCC ready for mechanics: Generalized velocity and Jacobian Lemma 1*

*Getting the GCC ready for mechanics: Generalized acceleration and Lemma 2*

# *Getting the GCC ready for mechanics (2nd part)*

*Generalized acceleration relations are a little more complicated (It's curved coords, after all!)*

First apply $\frac{d}{dt}$ to velocity $\dot{x}^j$ and use product rule: $\frac{d}{dt}(u \cdot v) = \frac{du}{dt} \cdot v + u \cdot \frac{dv}{dt}$

$$\ddot{x}^j \equiv \frac{d}{dt}\dot{x}^j = \frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\dot{q}^m\right) = \frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\right)\dot{q}^m + \frac{\partial x^j}{\partial q^m}\ddot{q}^m$$

# *Getting the GCC ready for mechanics ($2^{nd}$ part)*

*Generalized acceleration relations are a little more complicated (It's curved coords, after all!)*

First apply $\frac{d}{dt}$ to velocity $\dot{x}^j$ and use product rule: $\frac{d}{dt}(u \cdot v) = \frac{du}{dt} \cdot v + u \cdot \frac{dv}{dt}$

$$\ddot{x}^j \equiv \frac{d}{dt}\dot{x}^j = \frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\dot{q}^m\right) = \frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\right)\dot{q}^m + \frac{\partial x^j}{\partial q^m}\ddot{q}^m$$

Apply derivative chain sum to Jacobian.

$$\frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\right) = \frac{\partial}{\partial q^n}\left(\frac{\partial x^j}{\partial q^m}\right)\frac{dq^n}{dt} = \left(\frac{\partial^2 x^j}{\partial q^n \partial q^m}\right)\frac{dq^n}{dt}$$

# *Getting the GCC ready for mechanics (2nd part)*

*Generalized acceleration relations are a little more complicated (It's curved coords, after all!)*

First apply $\frac{d}{dt}$ to velocity $\dot{x}^j$ and use product rule: $\frac{d}{dt}(u \cdot v) = \frac{du}{dt} \cdot v + u \cdot \frac{dv}{dt}$

$$\ddot{x}^j \equiv \frac{d}{dt}\dot{x}^j = \frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\dot{q}^m\right) = \frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\right)\dot{q}^m + \frac{\partial x^j}{\partial q^m}\ddot{q}^m$$

*( Not in text. Recall Lecture 9 p. 15-19)†*

Apply derivative chain sum to Jacobian. Partial derivatives are reversible. $\partial_m \partial_n = \partial_n \partial_m$

$$\frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\right) = \frac{\partial}{\partial q^n}\left(\frac{\partial x^j}{\partial q^m}\right)\frac{dq^n}{dt} = \left(\frac{\partial^2 x^j}{\partial q^n \partial q^m}\right)\frac{dq^n}{dt} = \left(\frac{\partial^2 x^j}{\partial q^m \partial q^n}\right)\frac{dq^n}{dt} = \frac{\partial}{\partial q^m}\left(\frac{\partial x^j}{\partial q^n}\frac{dq^n}{dt}\right)$$

*Important thing about mechanics to recall:*
*coordinates* $q^n$
*<u>independent</u> of*
*velocities* $\frac{dq^m}{dt} = \dot{q}^m$

# *Getting the GCC ready for mechanics (2nd part)*

*Generalized acceleration relations are a little more complicated (It's curved coords, after all!)*

First apply $\frac{d}{dt}$ to velocity $\dot{x}^j$ and use product rule: $\frac{d}{dt}(u \cdot v) = \frac{du}{dt} \cdot v + u \cdot \frac{dv}{dt}$

$$\ddot{x}^j \equiv \frac{d}{dt}\dot{x}^j = \frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\dot{q}^m\right) = \frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\right)\dot{q}^m + \frac{\partial x^j}{\partial q^m}\ddot{q}^m$$

*( Not in text. Recall Lecture 9 p. 15-19)†*

Apply derivative chain sum to Jacobian. Partial derivatives are reversible. $\partial_m \partial_n = \partial_n \partial_m$

$$\frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\right) = \frac{\partial}{\partial q^n}\left(\frac{\partial x^j}{\partial q^m}\right)\frac{dq^n}{dt} = \left(\frac{\partial^2 x^j}{\partial q^n \partial q^m}\right)\frac{dq^n}{dt} = \left(\frac{\partial^2 x^j}{\partial q^m \partial q^n}\right)\frac{dq^n}{dt} = \frac{\partial}{\partial q^m}\left(\frac{\partial x^j}{\partial q^n}\frac{dq^n}{dt}\right)$$

*By chain-rule def. of CC velocity:*

$$= \frac{\partial}{\partial q^m}\left(\dot{x}^j\right)$$

*Important thing about mechanics to recall:*
*coordinates* $q^n$
*<u>independent</u> of*
*velocities* $\frac{dq^m}{dt} = \dot{q}^m$

# *Getting the GCC ready for mechanics (2nd part)*

*Generalized acceleration relations are a little more complicated (It's curved coords, after all!)*

First apply $\frac{d}{dt}$ to velocity $\dot{x}^j$ and use product rule: $\frac{d}{dt}(u \cdot v) = \frac{du}{dt} \cdot v + u \cdot \frac{dv}{dt}$

$$\ddot{x}^j \equiv \frac{d}{dt}\dot{x}^j = \frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\dot{q}^m\right) = \frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\right)\dot{q}^m + \frac{\partial x^j}{\partial q^m}\ddot{q}^m$$

*( Not in text. Recall Lecture 9 p. 15-19)*†

Apply derivative chain sum to Jacobian. Partial derivatives are reversible. $\partial_m \partial_n = \partial_n \partial_m$

$$\frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\right) = \frac{\partial}{\partial q^n}\left(\frac{\partial x^j}{\partial q^m}\right)\frac{dq^n}{dt} = \left(\frac{\partial^2 x^j}{\partial q^n \partial q^m}\right)\frac{dq^n}{dt} = \left(\frac{\partial^2 x^j}{\partial q^m \partial q^n}\right)\frac{dq^n}{dt} = \frac{\partial}{\partial q^m}\left(\frac{\partial x^j}{\partial q^n}\frac{dq^n}{dt}\right)$$

*By chain-rule def. of CC velocity:* $$= \frac{\partial}{\partial q^m}\left(\dot{x}^j\right)$$

*Important thing about mechanics to recall: coordinates* $q^n$ *<u>independent</u> of velocities* $\frac{dq^m}{dt} = \dot{q}^m$

*This is the key "lemma-2" for setting up Lagrangian mechanics .*

$$\boxed{\frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\right) = \frac{\partial \dot{x}^j}{\partial q^m} \quad \textit{lemma 2}}$$

# *Getting the GCC ready for mechanics (2nd part)*

*Generalized acceleration relations are a little more complicated (It's curved coords, after all!)*

First apply $\frac{d}{dt}$ to velocity $\dot{x}^j$ and use product rule: $\frac{d}{dt}(u \cdot v) = \frac{du}{dt} \cdot v + u \cdot \frac{dv}{dt}$

$$\ddot{x}^j \equiv \frac{d}{dt}\dot{x}^j = \frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\dot{q}^m\right) = \frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\right)\dot{q}^m + \frac{\partial x^j}{\partial q^m}\ddot{q}^m$$

*( Not in text. Recall Lecture 9 p. 15-19)*†

Apply derivative chain sum to Jacobian. Partial derivatives are reversible. $\partial_m \partial_n = \partial_n \partial_m$

$$\frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\right) = \frac{\partial}{\partial q^n}\left(\frac{\partial x^j}{\partial q^m}\right)\frac{dq^n}{dt} = \left(\frac{\partial^2 x^j}{\partial q^n \partial q^m}\right)\frac{dq^n}{dt} = \left(\frac{\partial^2 x^j}{\partial q^m \partial q^n}\right)\frac{dq^n}{dt} = \frac{\partial}{\partial q^m}\left(\frac{\partial x^j}{\partial q^n}\frac{dq^n}{dt}\right)$$

*By chain-rule def. of CC velocity:* $$= \frac{\partial}{\partial q^m}\left(\dot{x}^j\right)$$

*The "lemma-1" was in the GCC velocity analysis just before this one for acceleration.*

*This is the key "lemma-2" for setting up Lagrangian mechanics .*

$$\boxed{\frac{\partial \dot{x}^j}{\partial \dot{q}^m} = \frac{\partial x^j}{\partial q^m} \qquad \textit{lemma 1}}$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\right) = \frac{\partial \dot{x}^j}{\partial q^m} \qquad \textit{lemma 2}}$$

## *How to say Newton's "F=ma" in Generalized Curvilinear Coords.*

- *Use Cartesian KE quadratic form* $KE = T = 1/2\,\mathbf{v} \bullet \mathbf{M} \bullet \mathbf{v}$ *and* $\mathbf{F} = \mathbf{M} \bullet \mathbf{a}$ *to get GCC force*
- *Lagrange GCC trickery gives Lagrange force equations*
- *Lagrange GCC trickery gives Lagrange potential equations (Lagrange 1 and 2)*

# *Deriving GCC mechanics from Cartesian Coord. (CC) Newton I-II*

*Start with stuff we know...(sort of)*

Multidimensional CC version of kinetic energy $\frac{1}{2}\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}$

$$T = \frac{1}{2} M_{jk}\, v^j v^k = \frac{1}{2} M_{jk}\, \dot{x}^j \dot{x}^k$$

*where: $M_{jk}$ are CC inertia* *constants*

Multidimensional CC version of Newt-II (**F**=**M•a**) using $M_{jk}$ *constants*

$$f_j = M_{jk}\, a^k = M_{jk}\, \ddot{x}^k$$

# *Deriving GCC mechanics from Cartesian Coord. (CC) Newton I-II*

*Start with stuff we know...(sort of)*

Multidimensional CC version of kinetic energy $\frac{1}{2}\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}$

$$T = \frac{1}{2} M_{jk}\, v^j v^k = \frac{1}{2} M_{jk}\, \dot{x}^j \dot{x}^k$$

*where:* $M_{jk}$ *are inertia* <u>*constants*</u> *that are* <u>*symmetric*</u>*:* $M_{jk}=M_{kj}$

Multidimensional CC version of Newt-II (**F**=**M•a**) using $M_{jk}$ *constants*

$$f_j = M_{jk}\, a^k = M_{jk}\, \ddot{x}^k$$

Multidimensional CC version of work-energy differential ($dW$= **F•**$d$**x**). *Insert GCC differentials* $dq^m$

$$dW = f_j dx^j = f_j \left( \frac{\partial x^j}{\partial q^m} dq^m \right) = M_{jk}\, \ddot{x}^k \left( \frac{\partial x^j}{\partial q^m} dq^m \right)$$

*(It's time to bring in the queer* $q^m$ *!)*

# *Deriving GCC mechanics from Cartesian Coord. (CC) Newton I-II*

*Start with stuff we know...(sort of)*

Multidimensional CC version of kinetic energy $\frac{1}{2}\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}$

$$T = \frac{1}{2} M_{jk}\, v^j v^k = \frac{1}{2} M_{jk}\, \dot{x}^j \dot{x}^k$$

*where: $M_{jk}$ are inertia constants that are symmetric:* $M_{jk}=M_{kj}$

Multidimensional CC version of Newt-II (**F**=**M•a**) using $M_{jk}$ *constants*

$$f_j = M_{jk}\, a^k = M_{jk}\, \ddot{x}^k$$

Multidimensional CC version of work-energy differential ($dW$= **F•**$d$**x**). *Insert GCC differentials $dq^m$*

*(It's time to bring in the queer $q^m$ !)*

$$dW = f_j dx^j = f_j \left( \frac{\partial x^j}{\partial q^m} dq^m \right) = M_{jk}\, \ddot{x}^k \left( \frac{\partial x^j}{\partial q^m} dq^m \right)$$

$dq^m$ are independent so $dq^m$-sum is true term-by-term.

$$dW = f_j dx^j = F_m dq^m = f_j \frac{\partial x^j}{\partial q^m} dq^m = M_{jk}\, \ddot{x}^k \frac{\partial x^j}{\partial q^m} dq^m$$

# *Deriving GCC mechanics from Cartesian Coord. (CC) Newton I-II*

*Start with stuff we know...(sort of)*

Multidimensional CC version of kinetic energy $\frac{1}{2}\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}$

$$T = \frac{1}{2} M_{jk}\, v^j v^k = \frac{1}{2} M_{jk}\, \dot{x}^j \dot{x}^k$$

*where: $M_{jk}$ are inertia constants*

Multidimensional CC version of Newt-II ($\mathbf{F}=\mathbf{M}\bullet\mathbf{a}$) using $M_{jk}$ *constants*

$$f_j = M_{jk}\, a^k = M_{jk}\, \ddot{x}^k$$

Multidimensional CC version of work-energy differential ($dW = \mathbf{F}\bullet d\mathbf{x}$). *Insert GCC differentials $dq^m$*

$$dW = f_j dx^j = f_j \left( \frac{\partial x^j}{\partial q^m} dq^m \right) = M_{jk}\, \ddot{x}^k \left( \frac{\partial x^j}{\partial q^m} dq^m \right)$$

*(It's time to bring in the queer $q^m$ !)*

$dq^m$ are independent so $dq^m$-sum is true term-by-term. (Still holds if all $dq^m$ are zero but one.)

$$dW = f_j dx^j = F_m dq^m = f_j \frac{\partial x^j}{\partial q^m} dq^m = M_{jk}\, \ddot{x}^k \frac{\partial x^j}{\partial q^m} dq^m \quad \Rightarrow \quad F_m = f_j \frac{\partial x^j}{\partial q^m} = M_{jk}\, \ddot{x}^k \frac{\partial x^j}{\partial q^m}$$

# *Deriving GCC mechanics from Cartesian Coord. (CC) Newton I-II*

*Start with stuff we know...(sort of)*

Multidimensional CC version of kinetic energy $\frac{1}{2}\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}$

$$T=\frac{1}{2}M_{jk}\,v^j v^k=\frac{1}{2}M_{jk}\,\dot{x}^j\dot{x}^k$$

*where: $M_{jk}$ are inertia constants*

Multidimensional CC version of Newt-II (**F**=**M•a**) using $M_{jk}$ *constants*

$$f_j=M_{jk}\,a^k=M_{jk}\,\ddot{x}^k$$

Multidimensional CC version of work-energy differential ($dW$= **F•**$d$**x**). *Insert GCC differentials $dq^m$*

$$dW=f_j dx^j=f_j\left(\frac{\partial x^j}{\partial q^m}dq^m\right)=M_{jk}\,\ddot{x}^k\left(\frac{\partial x^j}{\partial q^m}dq^m\right)$$

*(It's time to bring in the queer $q^m$ !)*

$dq^m$ are independent so $dq^m$-sum is true term-by-term. (Still holds if all $dq^m$ are zero but one.)

$$dW=f_j dx^j=F_m dq^m=f_j\frac{\partial x^j}{\partial q^m}dq^m=M_{jk}\,\ddot{x}^k\frac{\partial x^j}{\partial q^m}dq^m \quad\Rightarrow\quad F_m=f_j\frac{\partial x^j}{\partial q^m}=M_{jk}\,\ddot{x}^k\frac{\partial x^j}{\partial q^m}$$

Here *generalized GCC force component $F_m$* is defined:

$$where:\quad F_m=f_j\frac{\partial x^j}{\partial q^m}=M_{jk}\,\ddot{x}^k\frac{\partial x^j}{\partial q^m}$$

## *How to say Newton's "F=ma" in Generalized Curvilinear Coords.*

*Use Cartesian KE quadratic form* $KE=T=1/2\,\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}$ *and* $\mathbf{F}=\mathbf{M}\bullet\mathbf{a}$ *to get GCC force*

→ *Lagrange GCC trickery gives Lagrange force equations*

*Lagrange GCC trickery gives Lagrange potential equations (Lagrange 1 and 2)*

# *Now Lagrange GCC trickery begins*

*Obvious stuff...(sort of, if you've looked at it for a century!)*

*Lagrange's clever end game:* First set $A = M_{jk}\,\ddot{x}^k$ and $B = \dfrac{\partial x^j}{\partial q^m}$ with calc. formula: $\left[\ddot{A}B = \dfrac{d}{dt}\left(\dot{A}B\right) - \dot{A}\dot{B}\right]$

$$\ddot{A}B \qquad\qquad \left(\dot{A}B\right) \qquad\qquad \dot{A}\dot{B}$$

$$F_m = f_j \frac{\partial x^j}{\partial q^m} = M_{jk}\,\ddot{x}^k \frac{\partial x^j}{\partial q^m} = \frac{d}{dt}\left( M_{jk}\dot{x}^k \frac{\partial x^j}{\partial q^m} \right) - M_{jk}\dot{x}^k \frac{d}{dt}\left( \frac{\partial x^j}{\partial q^m} \right)$$

# *Now Lagrange GCC trickery begins*

*Obvious stuff...(sort of, if you've looked at it for a century!)*

*Lagrange's clever end game:* First set $A = M_{jk}\,\ddot{x}^k$ and $B = \frac{\partial x^j}{\partial q^m}$ with calc. formula: $\left[\ddot{A}B = \frac{d}{dt}\left(\dot{A}B\right) - \dot{A}\dot{B}\right]$

$$F_m = f_j \frac{\partial x^j}{\partial q^m} = \underbrace{M_{jk}\,\ddot{x}^k \frac{\partial x^j}{\partial q^m}}_{\ddot{A}B} = \frac{d}{dt}\underbrace{\left( M_{jk}\dot{x}^k \frac{\partial x^j}{\partial q^m}\right)}_{(\dot{A}B)} - \underbrace{M_{jk}\dot{x}^k \frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\right)}_{\dot{A}\dot{B}}$$

*Cartesian* $M_{jk}$ *must be constant for this to work*

*(Bye, Bye relativistic mechanics or QM!)*

# *Now Lagrange GCC trickery begins*

*Obvious stuff...(sort of, if you've looked at it for a century!)*

*Lagrange's clever end game:* First set $A = M_{jk}\ddot{x}^k$ and $B = \frac{\partial x^j}{\partial q^m}$ with calc. formula: $\left[\ddot{A}B = \frac{d}{dt}\left(\dot{A}B\right) - \dot{A}\dot{B}\right]$

$$F_m = f_j \frac{\partial x^j}{\partial q^m} = \overbrace{M_{jk}\ddot{x}^k \frac{\partial x^j}{\partial q^m}}^{\ddot{A}B} = \frac{d}{dt}\overbrace{\left(M_{jk}\dot{x}^k \frac{\partial x^j}{\partial q^m}\right)}^{(\dot{A}B)} - \overbrace{M_{jk}\dot{x}^k \frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\right)}^{\dot{A}\dot{B}}$$

Then convert $\partial x^j$ to $\partial \dot{x}^j$ by *Lemma 1* and *Lemma 2* on 2nd term.

*Cartesian $M_{jk}$ must be constant for this to work*

*(Bye, Bye relativistic mechanics or QM!)*

$$F_m = \frac{d}{dt}\left(M_{jk}\dot{x}^k \frac{\partial \dot{x}^j}{\partial \dot{q}^m}\right) - M_{jk}\dot{x}^k\left(\frac{\partial \dot{x}^j}{\partial q^m}\right)$$

$$\boxed{\frac{\partial \dot{x}^j}{\partial \dot{q}^m} = \frac{\partial x^j}{\partial q^m} \qquad \textit{lemma 1}}$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\right) = \frac{\partial \dot{x}^j}{\partial q^m} \qquad \textit{lemma 2}}$$

# *Now Lagrange GCC trickery begins*

*Obvious stuff...(sort of, if you've looked at it for a century!)*

*Lagrange's clever end game:* First set $A = M_{jk}\,\ddot{x}^k$ and $B = \dfrac{\partial x^j}{\partial q^m}$ with calc. formula: $\left[\ddot{A}B = \dfrac{d}{dt}\left(\dot{A}B\right) - \dot{A}\dot{B}\right]$

$$F_m = f_j \frac{\partial x^j}{\partial q^m} = \overbrace{M_{jk}\,\ddot{x}^k \frac{\partial x^j}{\partial q^m}}^{\ddot{A}B} = \frac{d}{dt}\left(\overbrace{M_{jk}\dot{x}^k \frac{\partial x^j}{\partial q^m}}^{\left(\dot{A}B\right)}\right) - \overbrace{M_{jk}\dot{x}^k \frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\right)}^{\dot{A}\dot{B}}$$

*Cartesian $M_{jk}$ must be constant for this to work*

*(Bye, Bye relativistic mechanics or QM!)*

Then convert $\partial x^j$ to $\partial \dot{x}^j$ by *Lemma 1* and *Lemma 2* on 2[nd] term.

$$F_m = \frac{d}{dt}\left(M_{jk}\dot{x}^k \frac{\partial \dot{x}^j}{\partial \dot{q}^m}\right) - M_{jk}\dot{x}^k \left(\frac{\partial \dot{x}^j}{\partial q^m}\right)$$

Simplify using: $\left[M_{ij} v^i \dfrac{\partial v^j}{\partial q} = M_{ij} \dfrac{\partial}{\partial q} \dfrac{v^i v^j}{2}\right]$ where $q$ may be $\dot{q}^m$ or $q^m$

$$F_m = \frac{d}{dt}\frac{\partial}{\partial \dot{q}^m}\left(\frac{M_{jk}\dot{x}^k \dot{x}^j}{2}\right) - \frac{\partial}{\partial q^m}\left(\frac{M_{jk}\dot{x}^k \dot{x}^j}{2}\right)$$

$$\boxed{\frac{\partial \dot{x}^j}{\partial \dot{q}^m} = \frac{\partial x^j}{\partial q^m} \quad \textit{lemma 1}}$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\right) = \frac{\partial \dot{x}^j}{\partial q^m} \quad \textit{lemma 2}}$$

# *Now Lagrange GCC trickery begins*

*Obvious stuff...(sort of, if you've looked at it for a century!)*

*Lagrange's clever end game:* First set $A = M_{jk}\ddot{x}^k$ and $B = \frac{\partial x^j}{\partial q^m}$ with calc. formula: $\left[\ddot{A}B = \frac{d}{dt}\left(\dot{A}B\right) - \dot{A}\dot{B}\right]$

$$F_m = f_j \frac{\partial x^j}{\partial q^m} = \overbrace{M_{jk}\ddot{x}^k \frac{\partial x^j}{\partial q^m}}^{\ddot{A}B} = \frac{d}{dt}\overbrace{\left(M_{jk}\dot{x}^k \frac{\partial x^j}{\partial q^m}\right)}^{\left(\dot{A}B\right)} - \overbrace{M_{jk}\dot{x}^k \frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\right)}^{\dot{A}\dot{B}}$$

Then convert $\partial x^j$ to $\partial \dot{x}^j$ by *Lemma 1* and *Lemma 2* on 2[nd] term.

$$F_m = \frac{d}{dt}\left(M_{jk}\dot{x}^k \frac{\partial \dot{x}^j}{\partial \dot{q}^m}\right) - M_{jk}\dot{x}^k\left(\frac{\partial \dot{x}^j}{\partial q^m}\right)$$

Simplify using: $\left[M_{ij}v^i \frac{\partial v^j}{\partial q} = M_{ij}\frac{\partial}{\partial q}\frac{v^i v^j}{2}\right]$ where $q$ may be $\dot{q}^m$ or $q^m$

$$F_m = \frac{d}{dt}\frac{\partial}{\partial \dot{q}^m}\left(\frac{M_{jk}\dot{x}^k\dot{x}^j}{2}\right) - \frac{\partial}{\partial q^m}\left(\frac{M_{jk}\dot{x}^k\dot{x}^j}{2}\right)$$

The result is *Lagrange's GCC force equation* in terms of *kinetic energy* $T = \frac{1}{2}M_{jk}\dot{x}^j\dot{x}^k$

$$\boxed{F_m = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^m} - \frac{\partial T}{\partial q^m}}$$

*or:* $\mathbf{F} = \frac{d}{dt}\frac{\partial T}{\partial \mathbf{v}} - \frac{\partial T}{\partial \mathbf{r}}$

# *How to say Newton's "F=ma" in Generalized Curvilinear Coords.*

*Use Cartesian KE quadratic form* $KE=T=1/2\,\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}$ *and* $\mathbf{F}=\mathbf{M}\bullet\mathbf{a}$ *to get GCC force*

*Lagrange GCC trickery gives Lagrange force equations*

→ *Lagrange GCC trickery gives Lagrange potential equations (Lagrange 1 and 2)*

# *But, Lagrange GCC trickery is not yet done...*

*(Still another trick-up-the-sleeve!)*

If the force is conservative it's a gradient $\mathbf{F} = -\nabla U$ In GCC: $F_m = -\dfrac{\partial U}{\partial q^m}$

$$F_m = -\frac{\partial U}{\partial q^m} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^m} - \frac{\partial T}{\partial q^m}$$

# *But, Lagrange GCC trickery is not yet done...*

*(Still another trick-up-the-sleeve!)*

If the force is conservative it's a gradient $\mathbf{F} = -\nabla U$ In GCC: $F_m = -\frac{\partial U}{\partial q^m}$

$$F_m = -\frac{\partial U}{\partial q^m} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^m} - \frac{\partial T}{\partial q^m}$$

Becomes *Lagrange's GCC potential equation* with a new definition for the *Lagrangian: L=T-U*.

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^m} - \frac{\partial L}{\partial q^m} \qquad L(\dot{q}^m, q^m) = T(\dot{q}^m, q^m) - U(q^m)$$

This trick requires: $\frac{\partial U}{\partial \dot{q}^m} \equiv 0$

*U(r) has*
*NO explicit*
*velocity*
*dependence!*

# *But, Lagrange GCC trickery is not yet done...*

*(Still another trick-up-the-sleeve!)*

If the force is conservative it's a gradient $\mathbf{F} = -\nabla U$ In GCC: $F_m = -\frac{\partial U}{\partial q^m}$

$$F_m = -\frac{\partial U}{\partial q^m} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^m} - \frac{\partial T}{\partial q^m}$$

Becomes *Lagrange's GCC potential equation* with a new definition for the *Lagrangian: L=T-U*.

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^m} - \frac{\partial L}{\partial q^m} \qquad L(\dot{q}^m, q^m) = T(\dot{q}^m, q^m) - U(q^m)$$

This trick requires: $\frac{\partial U}{\partial \dot{q}^m} \equiv 0$ *U(r) has NO explicit velocity dependence!*

*Lagrange's 1st GCC equation*
*(Defining GCC momentum)*

$$p_m = \frac{\partial L}{\partial \dot{q}^m}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^m} = \frac{\partial L}{\partial q^m}$$

*Recall :*

$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}}$

*Lagrange's 2nd GCC equation*
*(Change of GCC momentum)*

$$\frac{dp_m}{dt} \equiv \dot{p}_m = \frac{\partial L}{\partial q^m}$$

# But, Lagrange GCC trickery is not yet done...

*(Still another trick-up-the-sleeve!)*

If the force is conservative it's a gradient $\mathbf{F} = -\nabla U$ In GCC: $F_m = -\frac{\partial U}{\partial q^m}$

$$F_m = -\frac{\partial U}{\partial q^m} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^m} - \frac{\partial T}{\partial q^m}$$

Becomes *Lagrange's GCC potential equation* with a new definition for the *Lagrangian: L=T-U*.

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^m} - \frac{\partial L}{\partial q^m} \qquad L(\dot{q}^m, q^m) = T(\dot{q}^m, q^m) - U(q^m)$$

This trick requires: $\frac{\partial U}{\partial \dot{q}^m} \equiv 0$

*U(r) has NO explicit velocity dependence!*

*If L has no explicit $q^m$ dependence then:*
$\dot{p}_m = 0$
*or :*
$p_m = const.$

*Lagrange's 1st GCC equation (Defining GCC momentum)*

$$p_m = \frac{\partial L}{\partial \dot{q}^m}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^m} = \frac{\partial L}{\partial q^m}$$

*Recall :*

$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}}$

*Lagrange's 2nd GCC equation (Change of GCC momentum)*

$$\frac{dp_m}{dt} \equiv \dot{p}_m = \frac{\partial L}{\partial q^m}$$

*GCC Cells, base vectors, and metric tensors*

*Polar coordinate examples:* Co*variant* $\mathbf{E}_m$ *vs.* Contra*variant* $\mathbf{E}^m$

Co*variant* $g_{mn}$ *vs.* In*variant* $\delta_m{}^n$ *vs.* Contra*variant* $g^{mn}$

A dual set of *quasi-unit vectors* show up in Jacobian J and Kajobian K.

J-Columns are *covariant vectors* $\{\mathbf{E}_1=\mathbf{E}_r\ \ \mathbf{E}_2=\mathbf{E}_\phi\}$ K-Rows are *contravariant vectors* $\{\mathbf{E}^1=\mathbf{E}^r\ \ \mathbf{E}^2=\mathbf{E}^\phi\}$

$$\langle J\rangle = \begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r}=\cos\phi & \frac{\partial x}{\partial \phi}=-r\sin\phi \\ \frac{\partial y}{\partial r}=\sin\phi & \frac{\partial y}{\partial \phi}=r\cos\phi \end{pmatrix}$$

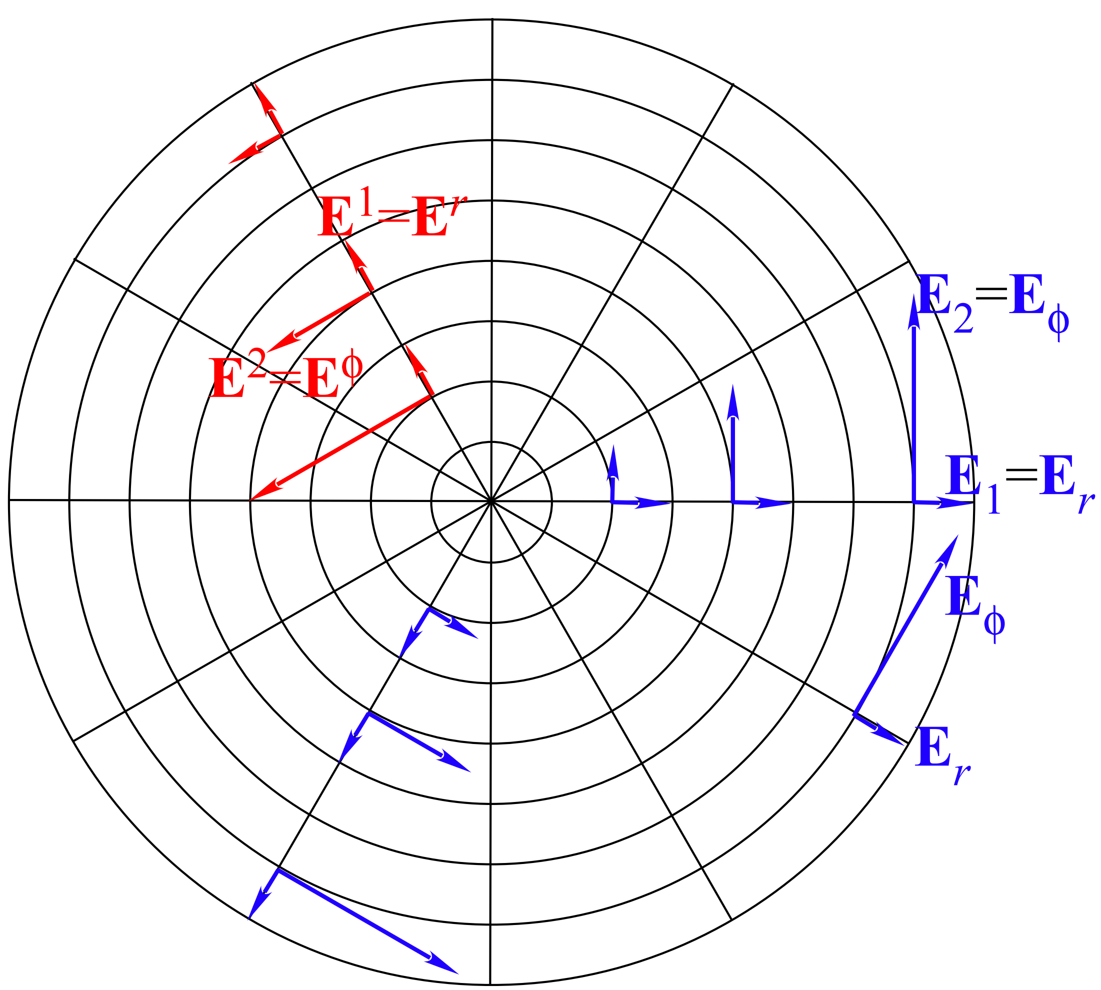
$$\uparrow\mathbf{E}_1 \quad \uparrow\mathbf{E}_2 \qquad\qquad \uparrow\mathbf{E}_r \quad \uparrow\mathbf{E}_\phi$$

$$\langle K\rangle = \langle J^{-1}\rangle = \begin{pmatrix} \frac{\partial r}{\partial x}=\cos\phi & \frac{\partial r}{\partial y}=\sin\phi \\ \frac{\partial \phi}{\partial x}=\frac{-\sin\phi}{r} & \frac{\partial \phi}{\partial y}=\frac{\cos\phi}{r} \end{pmatrix} \begin{array}{l} \leftarrow \mathbf{E}^r=\mathbf{E}^1 \\ \leftarrow \mathbf{E}^\phi=\mathbf{E}^2 \end{array}$$

*Derived from polar definition:* $x=r\cos\phi$ *and* $y=r\sin\phi$

*Inverse polar definition:*
$r^2=x^2+y^2$ *and* $\phi=\mathrm{atan2}(y,x)$

(a) Polar coordinate bases




Unit 1
Fig. 12.10

A dual set of *quasi-unit vectors* show up in Jacobian J and Kajobian K.

J-Columns are *covariant vectors* $\{\mathbf{E}_1 = \mathbf{E}_r \;\; \mathbf{E}_2 = \mathbf{E}_\phi\}$ K-Rows are *contravariant vectors* $\{\mathbf{E}^1 = \mathbf{E}^r \;\; \mathbf{E}^2 = \mathbf{E}^\phi\}$

$$\langle J \rangle = \begin{array}{c} \begin{pmatrix} \dfrac{\partial x^1}{\partial q^1} & \dfrac{\partial x^1}{\partial q^2} \\ \dfrac{\partial x^2}{\partial q^1} & \dfrac{\partial x^2}{\partial q^2} \end{pmatrix} \\ \uparrow \mathbf{E}_1 \;\; \uparrow \mathbf{E}_2 \end{array} = \begin{array}{c} \begin{pmatrix} \dfrac{\partial x}{\partial r} = \cos\phi & \dfrac{\partial x}{\partial \phi} = -r\sin\phi \\ \dfrac{\partial y}{\partial r} = \sin\phi & \dfrac{\partial y}{\partial \phi} = r\cos\phi \end{pmatrix} \\ \uparrow \mathbf{E}_r \qquad\qquad \uparrow \mathbf{E}_\phi \end{array}$$

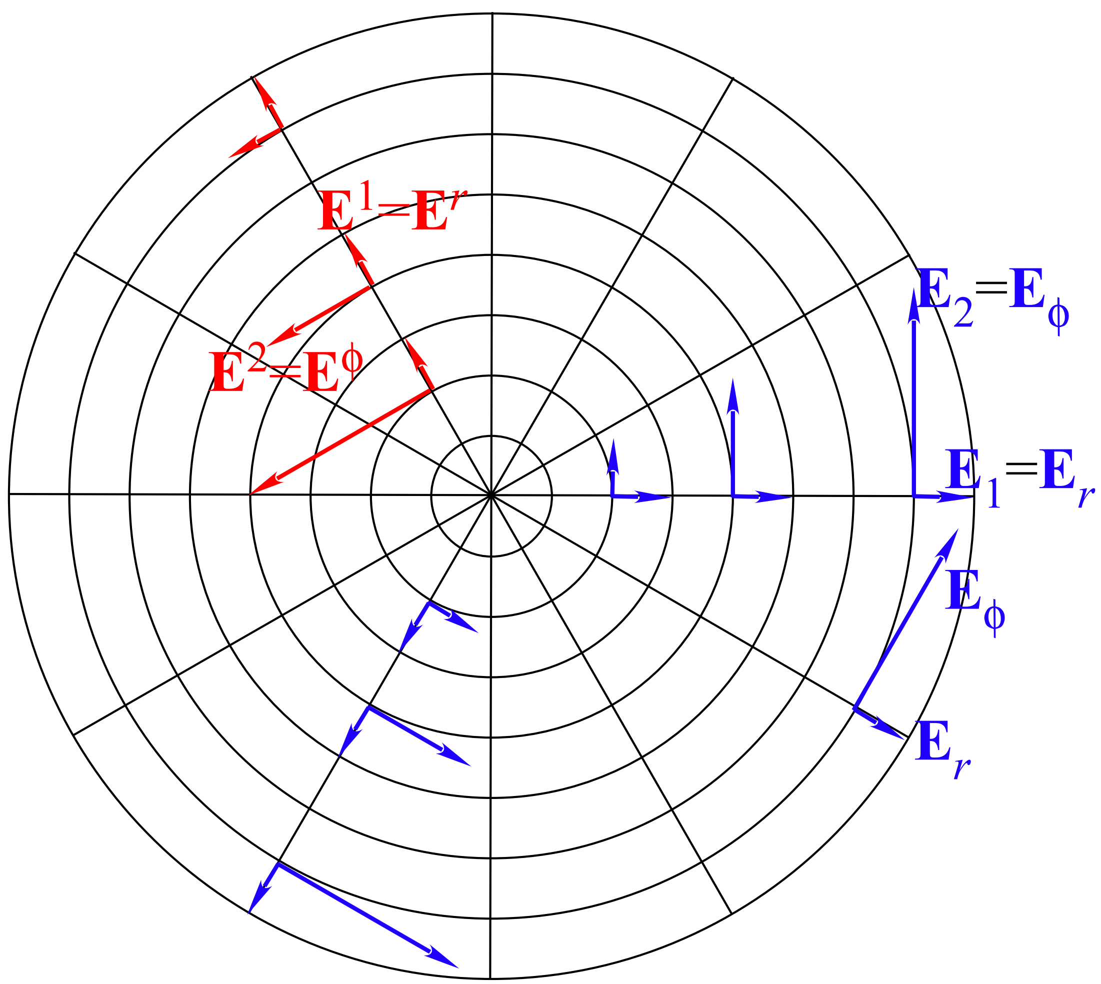
*Derived from polar definition:* $x = r\cos\phi$ *and* $y = r\sin\phi$

$$\langle K \rangle = \langle J^{-1} \rangle = \begin{pmatrix} \dfrac{\partial r}{\partial x} = \cos\phi & \dfrac{\partial r}{\partial y} = \sin\phi \\ \dfrac{\partial \phi}{\partial x} = \dfrac{-\sin\phi}{r} & \dfrac{\partial \phi}{\partial y} = \dfrac{\cos\phi}{r} \end{pmatrix} \begin{array}{l} \leftarrow \mathbf{E}^r = \mathbf{E}^1 \\ \\ \leftarrow \mathbf{E}^\phi = \mathbf{E}^2 \end{array}$$

*Inverse polar definition:*

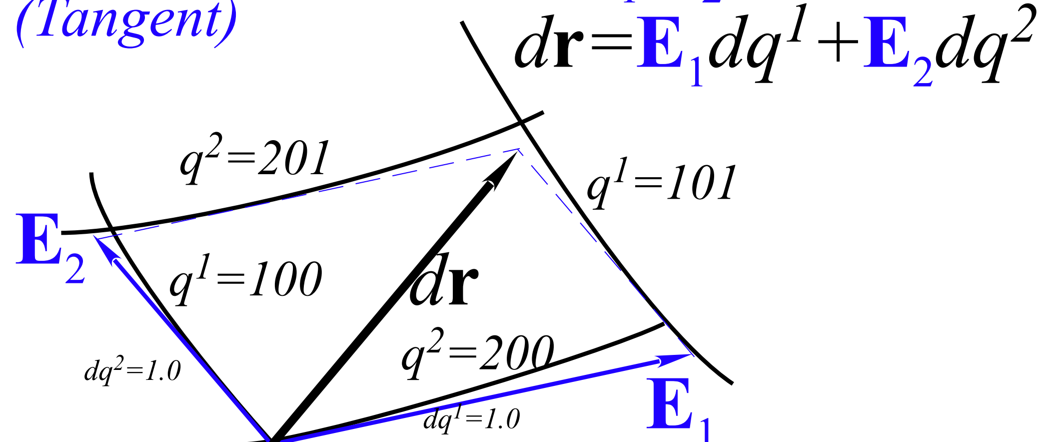
$r^2 = x^2 + y^2$ *and* $\phi = \mathrm{atan2}(y,x)$

(a) Polar coordinate bases




(b) Covariant bases $\{\mathbf{E}_1 \; \mathbf{E}_2\}$

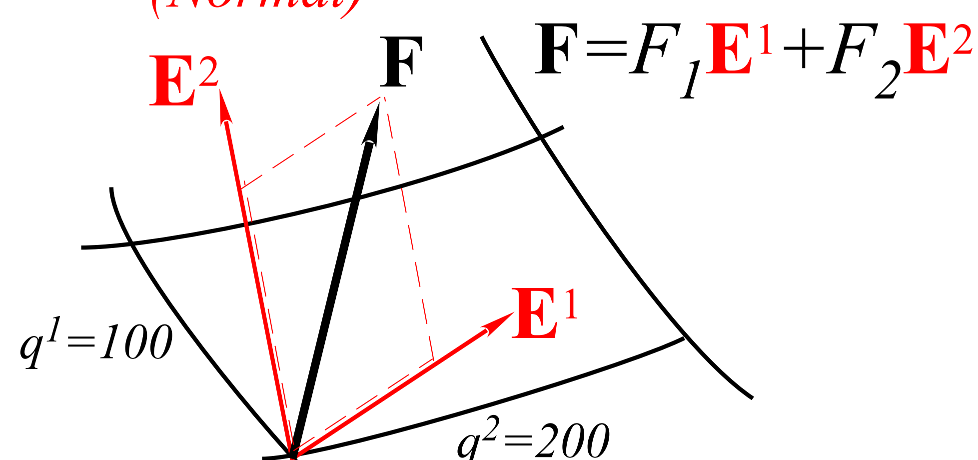
*(Tangent)*




NOTE:These are 2D drawings! No 3D perspective

(c) Contravariant bases $\{\mathbf{E}^1 \; \mathbf{E}^2\}$

*(Normal)*




Unit 1
Fig. 12.10

# *Comparison: Covariant* $\mathbf{E}_m = \frac{\partial \mathbf{r}}{\partial q^m}$ *vs. Contravariant* $\mathbf{E}^m = \frac{\partial q^m}{\partial \mathbf{r}} = \nabla q^m$

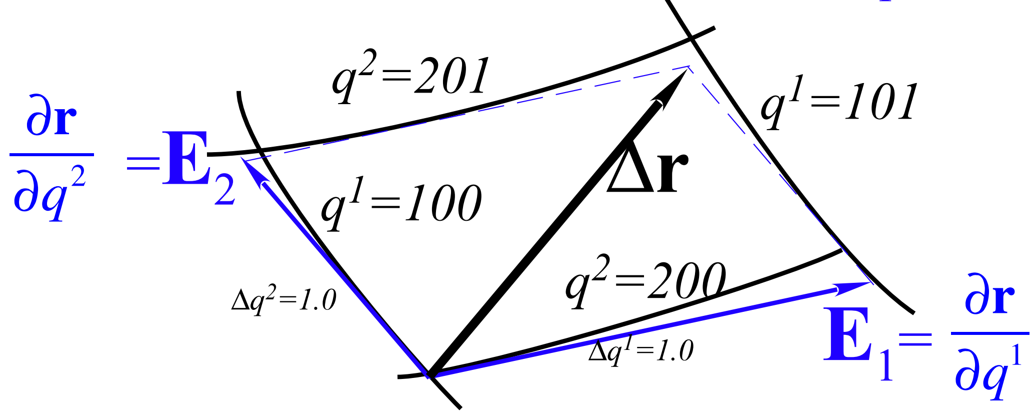
*geometric unit*

*Co*variant bases $\{\mathbf{E}_1\ \mathbf{E}_2\}$ match $\wedge$ cell walls

*(Tangent)*

$$\Delta\mathbf{r} = \mathbf{E}_1 \Delta q^1 + \mathbf{E}_2 \Delta q^2$$

*is based on chain rule:* $d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial q^1} dq^1 + \frac{\partial \mathbf{r}}{\partial q^2} dq^2 = \mathbf{E}_1 dq^1 + \mathbf{E}_2 dq^2$




NOTE:These
are 2D drawings!
*No* 3D *perspective*

# *Comparison: Covariant* $\mathbf{E}_m = \frac{\partial \mathbf{r}}{\partial q^m}$ *vs. Contravariant* $\mathbf{E}^m = \frac{\partial q^m}{\partial \mathbf{r}} = \nabla q^m$

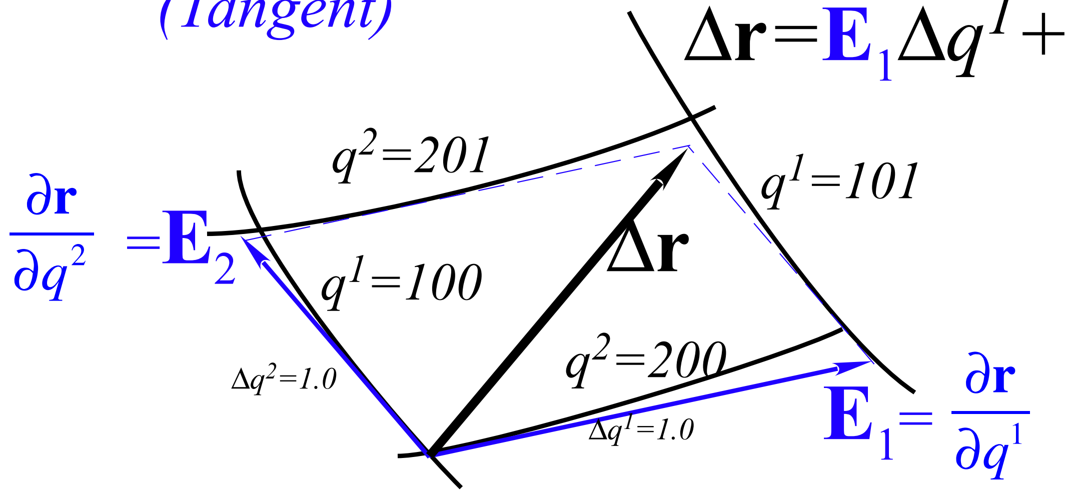
*geometric unit*

*Co*variant bases $\{\mathbf{E}_1\ \mathbf{E}_2\}$ match cell walls

*(Tangent)*

$$\Delta\mathbf{r} = \mathbf{E}_1 \Delta q^1 + \mathbf{E}_2 \Delta q^2$$

*is based on chain rule:* $d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial q^1} dq^1 + \frac{\partial \mathbf{r}}{\partial q^2} dq^2 = \mathbf{E}_1 dq^1 + \mathbf{E}_2 dq^2$




$\mathbf{E}_1$ *follows tangent to* $q^2$*=const. ...*
*since only* $q^1$ *varies in* $\frac{\partial \mathbf{r}}{\partial q^1}$
*while* $q^2$, $q^3$,*... remain constant*

NOTE:These
are 2D drawings!
*No* 3D *perspective*

# Comparison: Covariant $\mathbf{E}_m = \frac{\partial \mathbf{r}}{\partial q^m}$ vs. Contravariant $\mathbf{E}^m = \frac{\partial q^m}{\partial \mathbf{r}} = \nabla q^m$

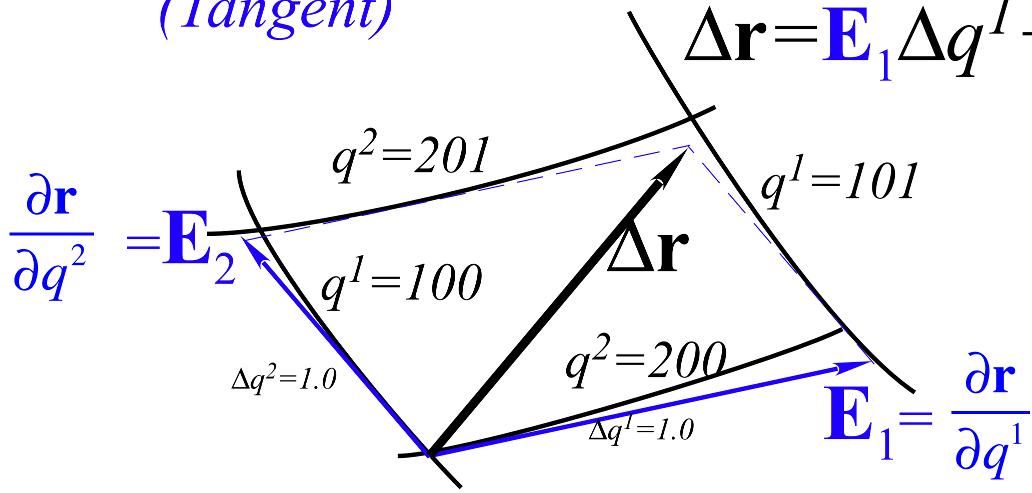
geometric unit

Covariant bases $\{\mathbf{E}_1\ \mathbf{E}_2\}$ match $\wedge$ cell walls

(Tangent)

$$\Delta\mathbf{r} = \mathbf{E}_1 \Delta q^1 + \mathbf{E}_2 \Delta q^2$$

*is based on chain rule:* $d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial q^1} dq^1 + \frac{\partial \mathbf{r}}{\partial q^2} dq^2 = \mathbf{E}_1 dq^1 + \mathbf{E}_2 dq^2$




$\mathbf{E}_1$ *follows* *tangent* *to* $q^2$=*const. ...*
*since only* $q^1$ *varies in* $\frac{\partial \mathbf{r}}{\partial q^1}$
*while* $q^2$, $q^3$,*... remain constant*

$\mathbf{E}_m$ are convenient bases for *ex*tensive quantities like distance and velocity.

$$\mathbf{V} = V^1 \mathbf{E}_1 + V^2 \mathbf{E}_2 = V^1 \frac{\partial \mathbf{r}}{\partial q^1} + V^2 \frac{\partial \mathbf{r}}{\partial q^2}$$

NOTE:These
are 2D drawings!
*No* 3D *perspective*

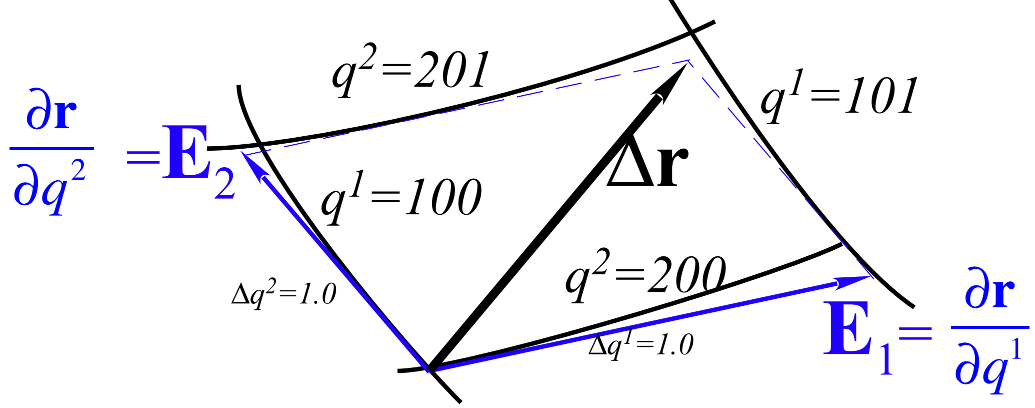
# *Comparison: Covariant* $\mathbf{E}_m = \frac{\partial \mathbf{r}}{\partial q^m}$ *vs.* *Contravariant* $\mathbf{E}^m = \frac{\partial q^m}{\partial \mathbf{r}} = \nabla q^m$

## *Co*variant bases $\{\mathbf{E}_1\ \mathbf{E}_2\}$ match *geometric unit* cell walls

*(Tangent)*

$$\Delta\mathbf{r} = \mathbf{E}_1 \Delta q^1 + \mathbf{E}_2 \Delta q^2$$

*is based on chain rule:* $d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial q^1} dq^1 + \frac{\partial \mathbf{r}}{\partial q^2} dq^2 = \mathbf{E}_1 dq^1 + \mathbf{E}_2 dq^2$




$\mathbf{E}_1$ *follows tangent to* $q^2$*=const. ...*
*since only* $q^1$ *varies in* $\frac{\partial \mathbf{r}}{\partial q^1}$
*while* $q^2$, $q^3$*,... remain constant*

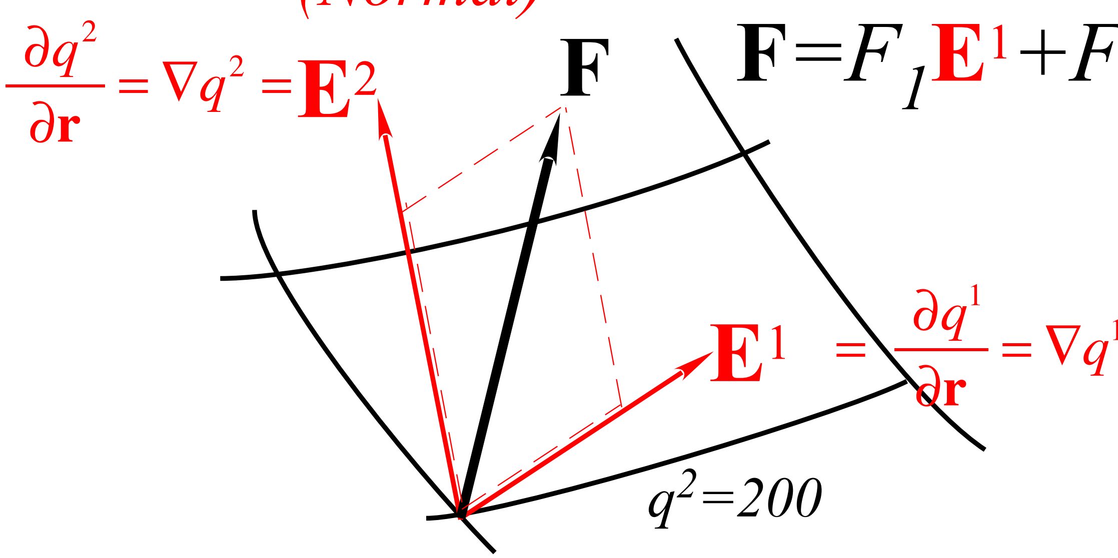
$\mathbf{E}_m$ are convenient bases for *ex*tensive quantities like distance and velocity.

$$\mathbf{V} = V^1\mathbf{E}_1 + V^2\mathbf{E}_2 = V^1\frac{\partial \mathbf{r}}{\partial q^1} + V^2\frac{\partial \mathbf{r}}{\partial q^2}$$

## *Contra*variant $\{\mathbf{E}^1\ \mathbf{E}^2\}$ match reciprocal cells

*(Normal)*

$$\mathbf{F} = F_1\mathbf{E}^1 + F_2\mathbf{E}^2$$




NOTE:These
are 2D drawings!
No 3D perspective

$\mathbf{E}^1$ *is normal to* $q^1$*=const. since*
**gradient** *of* $q^1$ *is vector sum* $\nabla q^1 = \begin{pmatrix} \frac{\partial q^1}{\partial x} \\ \frac{\partial q^1}{\partial y} \end{pmatrix}$
*of all its partial derivatives*

# *Comparison: Covariant* $\mathbf{E}_m = \frac{\partial \mathbf{r}}{\partial q^m}$ *vs.* *Contravariant* $\mathbf{E}^m = \frac{\partial q^m}{\partial \mathbf{r}} = \nabla q^m$

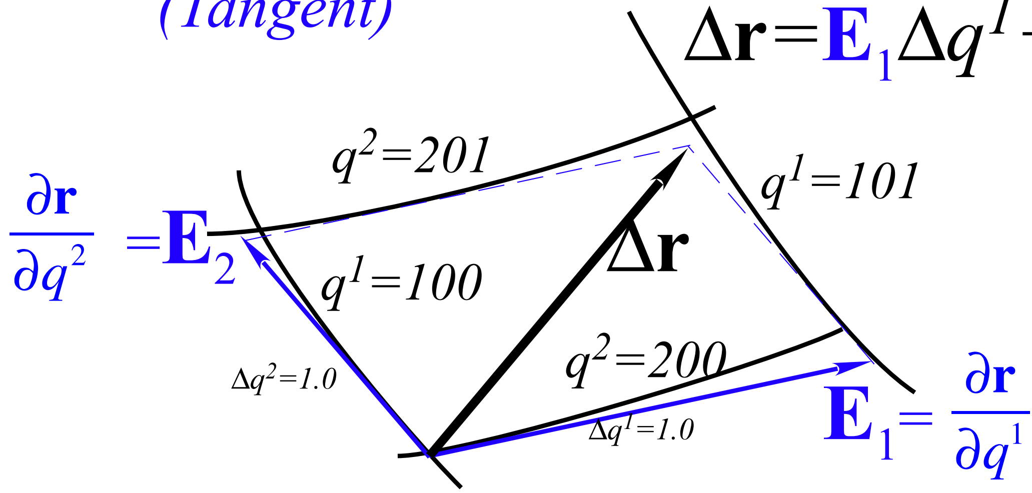
*geometric unit*

## *Co*variant bases $\{\mathbf{E}_1\ \mathbf{E}_2\}$ match cell walls

*(Tangent)*

$$\Delta\mathbf{r} = \mathbf{E}_1 \Delta q^1 + \mathbf{E}_2 \Delta q^2$$

*is based on chain rule:* $d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial q^1} dq^1 + \frac{\partial \mathbf{r}}{\partial q^2} dq^2 = \mathbf{E}_1 dq^1 + \mathbf{E}_2 dq^2$




$\mathbf{E}_1$ *follows tangent to* $q^2$*=const. ...*
*since only* $q^1$ *varies in* $\frac{\partial \mathbf{r}}{\partial q^1}$
*while* $q^2$*,* $q^3$*,... remain constant*

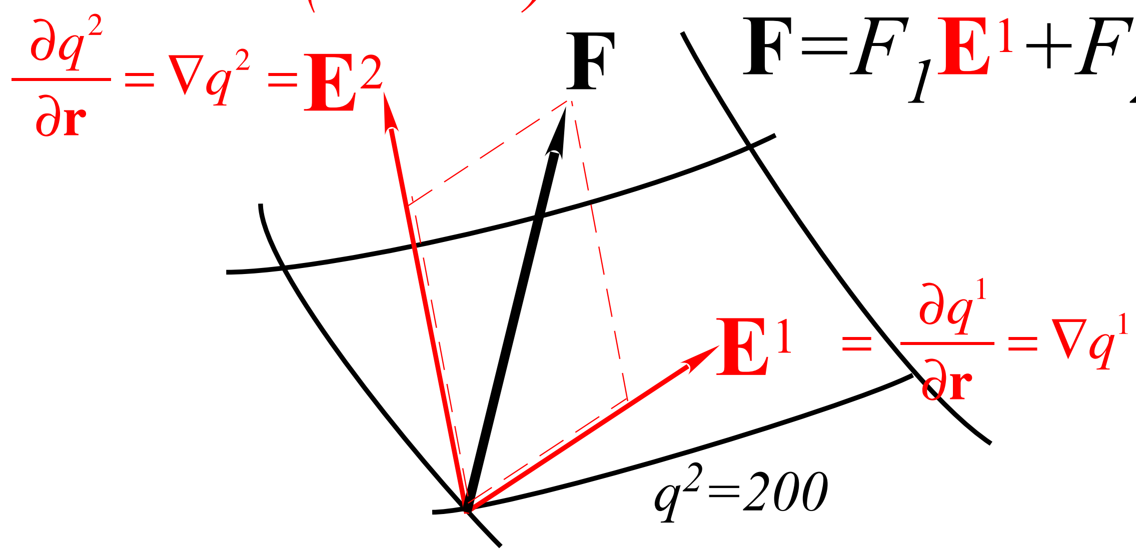
$\mathbf{E}_m$ are convenient bases for *ex*tensive quantities like distance and velocity.

$$\mathbf{V} = V^1\mathbf{E}_1 + V^2\mathbf{E}_2 = V^1\frac{\partial \mathbf{r}}{\partial q^1} + V^2\frac{\partial \mathbf{r}}{\partial q^2}$$

## *Contra*variant $\{\mathbf{E}^1\ \mathbf{E}^2\}$ match reciprocal cells

*(Normal)*

$$\mathbf{F} = F_1\mathbf{E}^1 + F_2\mathbf{E}^2$$




NOTE:These
are 2D drawings!
No 3D perspective

$\mathbf{E}^1$ *is normal to* $q^1$*=const. since* **gradient** *of* $q^1$ *is vector sum* $\nabla q^1 = \begin{pmatrix} \frac{\partial q^1}{\partial x} \\ \frac{\partial q^1}{\partial y} \end{pmatrix}$
*of all its partial derivatives*

$\mathbf{E}^m$ are convenient bases for *in*tensive quantities like force and momentum.

$$\mathbf{F} = F_1\mathbf{E}^1 + F_2\mathbf{E}^2 = F_1\frac{\partial q^1}{\partial \mathbf{r}} + F_2\frac{\partial q^2}{\partial \mathbf{r}} = F_1\nabla q^1 + F_2\nabla q^2$$

# *Comparison: Covariant* $\mathbf{E}_m = \frac{\partial \mathbf{r}}{\partial q^m}$ *vs. Contravariant* $\mathbf{E}^n = \frac{\partial q^n}{\partial \mathbf{r}} = \nabla q^n$

## *Co*variant bases $\{\mathbf{E}_1\ \mathbf{E}_2\}$ match cell walls

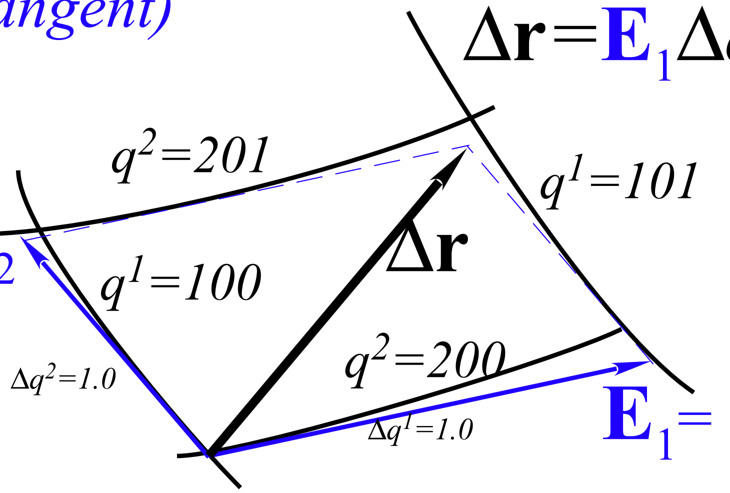
*geometric unit*

*(Tangent)*

$$\Delta\mathbf{r} = \mathbf{E}_1 \Delta q^1 + \mathbf{E}_2 \Delta q^2$$

*is based on chain rule:* $d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial q^1} dq^1 + \frac{\partial \mathbf{r}}{\partial q^2} dq^2 = \mathbf{E}_1 dq^1 + \mathbf{E}_2 dq^2$

$\frac{\partial \mathbf{r}}{\partial q^2} = \mathbf{E}_2$




$\mathbf{E}_1 = \frac{\partial \mathbf{r}}{\partial q^1}$

$\mathbf{E}_1$ *follows tangent to* $q^2$*=const. ...*
*since only* $q^1$ *varies in* $\frac{\partial \mathbf{r}}{\partial q^1}$
*while* $q^2$, $q^3$*,... remain constant*

$\mathbf{E}_m$ are convenient bases for *ex*tensive quantities like distance and velocity.

$$\mathbf{V} = V^1 \mathbf{E}_1 + V^2 \mathbf{E}_2 = V^1 \frac{\partial \mathbf{r}}{\partial q^1} + V^2 \frac{\partial \mathbf{r}}{\partial q^2}$$

## *Co-Contra dot products* $\mathbf{E}_m \cdot \mathbf{E}^n$ *are orthonormal:*

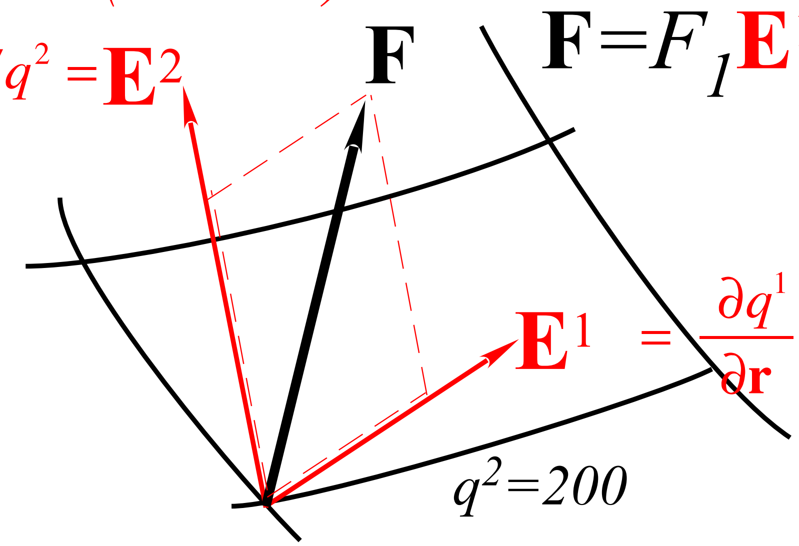
$$\mathbf{E}_m \bullet \mathbf{E}^n = \frac{\partial \mathbf{r}}{\partial q^m} \bullet \frac{\partial q^n}{\partial \mathbf{r}} = \delta_m^n$$

## *Contra*variant $\{\mathbf{E}^1\ \mathbf{E}^2\}$ match reciprocal cells

*(Normal)*

$\frac{\partial q^2}{\partial \mathbf{r}} = \nabla q^2 = \mathbf{E}^2$

$$\mathbf{F} = F_1 \mathbf{E}^1 + F_2 \mathbf{E}^2$$




$\mathbf{E}^1 = \frac{\partial q^1}{\partial \mathbf{r}} = \nabla q^1$

$\mathbf{E}^1$ *is normal to* $q^1$*=const. since* **gradient** *of* $q^1$ *is vector sum* $\nabla q^1 = \begin{pmatrix} \frac{\partial q^1}{\partial x} \\ \frac{\partial q^1}{\partial y} \end{pmatrix}$ *of all its partial derivatives*

$\mathbf{E}^m$ are convenient bases for *in*tensive quantities like force and momentum.

$$\mathbf{F} = F_1 \mathbf{E}^1 + F_2 \mathbf{E}^2 = F_1 \frac{\partial q^1}{\partial \mathbf{r}} + F_2 \frac{\partial q^2}{\partial \mathbf{r}} = F_1 \nabla q^1 + F_2 \nabla q^2$$

# *GCC Cells, base vectors, and metric tensors*

*Polar coordinate examples:* Co*variant* $\mathbf{E}_m$ *vs.* Contra*variant* $\mathbf{E}^m$

→ Co*variant* $g_{mn}$ *vs.* In*variant* $\delta_m{}^n$ *vs.* Contra*variant* $g^{mn}$

## Covariant $g_{mn}$ *vs.* Invariant $\delta_m{}^n$ *vs.* Contravariant $g^{mn}$

$$\mathbf{E}_m \bullet \mathbf{E}_n = \frac{\partial \mathbf{r}}{\partial q^m} \bullet \frac{\partial \mathbf{r}}{\partial q^n} \equiv g_{mn}$$

$$\mathbf{E}_m \bullet \mathbf{E}^n = \frac{\partial \mathbf{r}}{\partial q^m} \bullet \frac{\partial q^n}{\partial \mathbf{r}} = \delta_m^n$$

$$\mathbf{E}^m \bullet \mathbf{E}^n = \frac{\partial q^m}{\partial \mathbf{r}} \bullet \frac{\partial q^n}{\partial \mathbf{r}} \equiv g^{mn}$$

*Covariant metric tensor* $g_{mn}$

*Invariant Kroneker unit tensor*

$$\delta_m^n \equiv \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

*Contravariant metric tensor* $g^{mn}$

# Covariant $g_{mn}$ vs. Invariant $\delta_m^n$ vs. Contravariant $g^{mn}$

$$\mathbf{E}_m \bullet \mathbf{E}_n = \frac{\partial \mathbf{r}}{\partial q^m} \bullet \frac{\partial \mathbf{r}}{\partial q^n} \equiv g_{mn}$$

$$\mathbf{E}_m \bullet \mathbf{E}^n = \frac{\partial \mathbf{r}}{\partial q^m} \bullet \frac{\partial q^n}{\partial \mathbf{r}} = \delta_m^n$$

$$\mathbf{E}^m \bullet \mathbf{E}^n = \frac{\partial q^m}{\partial \mathbf{r}} \bullet \frac{\partial q^n}{\partial \mathbf{r}} \equiv g^{mn}$$

*Covariant metric tensor* $g_{mn}$

*Invariant Kroneker unit tensor*

$$\delta_m^n \equiv \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

*Contravariant metric tensor* $g^{mn}$

*Polar coordinate examples (again):*

$$\langle J \rangle = \begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} = \cos\phi & \frac{\partial x}{\partial \phi} = -r\sin\phi \\ \frac{\partial y}{\partial r} = \sin\phi & \frac{\partial y}{\partial \phi} = r\cos\phi \end{pmatrix}$$

$$\uparrow \mathbf{E}_1 \quad \uparrow \mathbf{E}_2 \qquad\qquad \uparrow \mathbf{E}_r \qquad\qquad \uparrow \mathbf{E}_\phi$$

$$\langle K \rangle = \langle J^{-1} \rangle = \begin{pmatrix} \frac{\partial r}{\partial x} = \cos\phi & \frac{\partial r}{\partial y} = \sin\phi \\ \frac{\partial \phi}{\partial x} = \frac{-\sin\phi}{r} & \frac{\partial \phi}{\partial y} = \frac{\cos\phi}{r} \end{pmatrix} \begin{array}{l} \leftarrow \mathbf{E}^r = \mathbf{E}^1 \\ \leftarrow \mathbf{E}^\phi = \mathbf{E}^2 \end{array}$$

# Covariant $g_{mn}$ vs. Invariant $\delta_m^n$ vs. Contravariant $g^{mn}$

$$\mathbf{E}_m \bullet \mathbf{E}_n = \frac{\partial \mathbf{r}}{\partial q^m} \bullet \frac{\partial \mathbf{r}}{\partial q^n} \equiv g_{mn}$$

$$\mathbf{E}_m \bullet \mathbf{E}^n = \frac{\partial \mathbf{r}}{\partial q^m} \bullet \frac{\partial q^n}{\partial \mathbf{r}} = \delta_m^n$$

$$\mathbf{E}^m \bullet \mathbf{E}^n = \frac{\partial q^m}{\partial \mathbf{r}} \bullet \frac{\partial q^n}{\partial \mathbf{r}} \equiv g^{mn}$$

Covariant metric tensor $g_{mn}$

Invariant Kroneker unit tensor

$$\delta_m^n \equiv \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

Contravariant metric tensor $g^{mn}$

*Polar coordinate examples (again):*

$$\langle J \rangle = \begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} = \cos\phi & \frac{\partial x}{\partial \phi} = -r\sin\phi \\ \frac{\partial y}{\partial r} = \sin\phi & \frac{\partial y}{\partial \phi} = r\cos\phi \end{pmatrix}$$

$$\begin{array}{cccc} \uparrow \mathbf{E}_1 & \uparrow \mathbf{E}_2 & \uparrow \mathbf{E}_r & \uparrow \mathbf{E}_\phi \end{array}$$

$$\langle K \rangle = \langle J^{-1} \rangle = \begin{pmatrix} \frac{\partial r}{\partial x} = \cos\phi & \frac{\partial r}{\partial y} = \sin\phi \\ \frac{\partial \phi}{\partial x} = \frac{-\sin\phi}{r} & \frac{\partial \phi}{\partial y} = \frac{\cos\phi}{r} \end{pmatrix} \begin{array}{l} \leftarrow \mathbf{E}^r = \mathbf{E}^1 \\ \leftarrow \mathbf{E}^\phi = \mathbf{E}^2 \end{array}$$

Covariant $g_{mn}$

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r \bullet \mathbf{E}_r & \mathbf{E}_r \bullet \mathbf{E}_\phi \\ \mathbf{E}_\phi \bullet \mathbf{E}_r & \mathbf{E}_\phi \bullet \mathbf{E}_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

Invariant $\delta_m^n$

$$\begin{pmatrix} \delta_r^r & \delta_r^\phi \\ \delta_\phi^r & \delta_\phi^\phi \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r \bullet \mathbf{E}^r & \mathbf{E}_r \bullet \mathbf{E}^\phi \\ \mathbf{E}_\phi \bullet \mathbf{E}^r & \mathbf{E}_\phi \bullet \mathbf{E}^\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Contravariant $g^{mn}$

$$\begin{pmatrix} g^{rr} & g^{r\phi} \\ g^{\phi r} & g^{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}^r \bullet \mathbf{E}^r & \mathbf{E}^r \bullet \mathbf{E}^\phi \\ \mathbf{E}^\phi \bullet \mathbf{E}^r & \mathbf{E}^\phi \bullet \mathbf{E}^\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}$$

*Lagrange prefers* <u>Co</u>*variant* $g_{mn}$ *with* <u>Contra</u>*variant velocity* $\dot{q}^m$

*GCC Lagrangian definition*

*GCC "canonical" momentum* $p_m$ *definition*

*GCC "canonical" force* $F_m$ *definition*

*Coriolis "fictitious" forces (... and weather effects)*

# *Lagrange prefers Covariant $g_{mn}$ with Contravariant velocity*

*Lagrangian L=KE-U is supposed to be explicit function of velocity.*

$$L(\mathbf{v}) = \tfrac{1}{2} M\mathbf{v}\bullet\mathbf{v} - U = \tfrac{1}{2} M\dot{\mathbf{r}}\bullet\dot{\mathbf{r}} - U = \tfrac{1}{2} M(\mathbf{E}_m \dot{q}^m)\bullet(\mathbf{E}_n \dot{q}^n) - U = \tfrac{1}{2} M(g_{mn}\dot{q}^m \dot{q}^n) - U = L(\dot{q})$$

# *Lagrange prefers Covariant $g_{mn}$ with Contravariant velocity*

*Lagrangian KE-U is supposed to be explicit function of velocity.*

$$L(\mathbf{v}) = \tfrac{1}{2}M\mathbf{v}\bullet\mathbf{v} - U = \tfrac{1}{2}M\dot{\mathbf{r}}\bullet\dot{\mathbf{r}} - U = \tfrac{1}{2}M(\mathbf{E}_m\dot{q}^m)\bullet(\mathbf{E}_n\dot{q}^n) - U = \tfrac{1}{2}M(g_{mn}\dot{q}^m\dot{q}^n) - U = L(\dot{q})$$

*Use polar coordinate Covariant $g_{mn}$ metric (page 53)*

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r\bullet\mathbf{E}_r & \mathbf{E}_r\bullet\mathbf{E}_\phi \\ \mathbf{E}_\phi\bullet\mathbf{E}_r & \mathbf{E}_\phi\bullet\mathbf{E}_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

# *Lagrange prefers Covariant $g_{mn}$ with Contravariant velocity*

*Lagrangian KE-U is supposed to be explicit function of velocity.*

$$L(\mathbf{v}) = \tfrac{1}{2} M \mathbf{v} \bullet \mathbf{v} - U = \tfrac{1}{2} M \dot{\mathbf{r}} \bullet \dot{\mathbf{r}} - U = \tfrac{1}{2} M (\mathbf{E}_m \dot{q}^m) \bullet (\mathbf{E}_n \dot{q}^n) - U = \tfrac{1}{2} M (g_{mn} \dot{q}^m \dot{q}^n) - U = L(\dot{q})$$

*Use polar coordinate Covariant $g_{mn}$ metric (page 53)*

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r \bullet \mathbf{E}_r & \mathbf{E}_r \bullet \mathbf{E}_\phi \\ \mathbf{E}_\phi \bullet \mathbf{E}_r & \mathbf{E}_\phi \bullet \mathbf{E}_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

*This gives polar GCC form (Actually it's an OCC or Orthogonal Curvilinear Coordinate form)*

$$L(\dot{r}, \dot{\phi}) = \tfrac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2) - U(r,\phi) = \tfrac{1}{2} M (1 \cdot \dot{r}^2 + r^2 \cdot \dot{\phi}^2) - U(r,\phi)$$

*Lagrange prefers* <u>Co</u>*variant* $g_{mn}$ *with* <u>Contra</u>*variant velocity* $\dot{q}^m$

*GCC Lagrangian definition*

*GCC "canonical" momentum* $p_m$ *definition*

*GCC "canonical" force* $F_m$ *definition*

*Coriolis "fictitious" forces (... and weather effects)*

# *Lagrange prefers Covariant $g_{mn}$ with Contravariant velocity*

*Lagrangian KE-U is supposed to be explicit function of velocity.*

$$L(\mathbf{v}) = \tfrac{1}{2}M\mathbf{v}\bullet\mathbf{v} - U = \tfrac{1}{2}M\dot{\mathbf{r}}\bullet\dot{\mathbf{r}} - U = \tfrac{1}{2}M(\mathbf{E}_m\dot{q}^m)\bullet(\mathbf{E}_n\dot{q}^n) - U = \tfrac{1}{2}M(g_{mn}\dot{q}^m\dot{q}^n) - U = L(\dot{q})$$

*Use polar coordinate Covariant $g_{mn}$ metric (page 53)*

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r\bullet\mathbf{E}_r & \mathbf{E}_r\bullet\mathbf{E}_\phi \\ \mathbf{E}_\phi\bullet\mathbf{E}_r & \mathbf{E}_\phi\bullet\mathbf{E}_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

*This gives polar GCC form (Actually it's an OCC or Orthogonal Curvilinear Coordinate form)*

$$L(\dot{r},\dot{\phi}) = \tfrac{1}{2}M(g_{rr}\dot{r}^2 + g_{\phi\phi}\dot{\phi}^2) - U(r,\phi) = \tfrac{1}{2}M(1\cdot\dot{r}^2 + r^2\cdot\dot{\phi}^2) - U(r,\phi)$$

*(From preceding page)*

# *Lagrange prefers Covariant $g_{mn}$ with Contravariant velocity*

*Lagrangian KE-U is supposed to be explicit function of velocity.*

$$L(\mathbf{v}) = \tfrac{1}{2}M\mathbf{v}\bullet\mathbf{v} - U = \tfrac{1}{2}M\dot{\mathbf{r}}\bullet\dot{\mathbf{r}} - U = \tfrac{1}{2}M(\mathbf{E}_m\dot{q}^m)\bullet(\mathbf{E}_n\dot{q}^n) - U = \tfrac{1}{2}M(g_{mn}\dot{q}^m\dot{q}^n) - U = L(\dot{q})$$

*Use polar coordinate Covariant $g_{mn}$ metric (page 53)*

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r\bullet\mathbf{E}_r & \mathbf{E}_r\bullet\mathbf{E}_\phi \\ \mathbf{E}_\phi\bullet\mathbf{E}_r & \mathbf{E}_\phi\bullet\mathbf{E}_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

*This gives polar GCC form (Actually it's an OCC or Orthogonal Curvilinear Coordinate form)*

$$L(\dot{r},\dot{\phi}) = \tfrac{1}{2}M(g_{rr}\dot{r}^2 + g_{\phi\phi}\dot{\phi}^2) - U(r,\phi) = \tfrac{1}{2}M(1\cdot\dot{r}^2 + r^2\cdot\dot{\phi}^2) - U(r,\phi)$$

*GCC Lagrange equations follow. 1st L-equation is momentum $p_m$ definition for each coordinate $q^m$:*

$$p_r = \frac{\partial L}{\partial \dot{r}} = M\, g_{rr}\dot{r} = M\,\dot{r}$$

*Nothing too surprising;*
*radial momentum $p_r$ has the*
*usual linear $M\cdot v$ form*

# *Lagrange prefers Covariant $g_{mn}$ with Contravariant velocity*

*Lagrangian KE-U is supposed to be explicit function of velocity.*

$$L(\mathbf{v}) = \tfrac{1}{2}M\mathbf{v}\bullet\mathbf{v} - U = \tfrac{1}{2}M\dot{\mathbf{r}}\bullet\dot{\mathbf{r}} - U = \tfrac{1}{2}M(\mathbf{E}_m\dot{q}^m)\bullet(\mathbf{E}_n\dot{q}^n) - U = \tfrac{1}{2}M(g_{mn}\dot{q}^m\dot{q}^n) - U = L(\dot{q})$$

*Use polar coordinate Covariant $g_{mn}$ metric (page 53)*

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r\bullet\mathbf{E}_r & \mathbf{E}_r\bullet\mathbf{E}_\phi \\ \mathbf{E}_\phi\bullet\mathbf{E}_r & \mathbf{E}_\phi\bullet\mathbf{E}_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

*This gives polar GCC form (Actually it's an OCC or Orthogonal Curvilinear Coordinate form)*

$$L(\dot{r},\dot{\phi}) = \tfrac{1}{2}M(g_{rr}\dot{r}^2 + g_{\phi\phi}\dot{\phi}^2) - U(r,\phi) = \tfrac{1}{2}M(1\cdot\dot{r}^2 + r^2\cdot\dot{\phi}^2) - U(r,\phi)$$

*GCC Lagrange equations follow. 1st L-equation is momentum $p_m$ definition for each coordinate $q^m$:*

$$p_r = \frac{\partial L}{\partial \dot{r}} = M\, g_{rr}\dot{r} = M\,\dot{r}$$

*Nothing too surprising; radial momentum $p_r$ has the usual linear $M\cdot v$ form*

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = Mg_{\phi\phi}\dot{\phi} = Mr^2\dot{\phi}$$

*Wow! $g_{\phi\phi}$ gives moment-of-inertia factor $Mr^2$ automatically for the angular momentum $p_\phi = Mr^2\omega$.*

**Lagrange's 1st GCC equation (Defining GCC momentum)**

$$p_m = \frac{\partial L}{\partial \dot{q}^m}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^m} = \frac{\partial L}{\partial q^m}$$

*Recall :*

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}}$$

**Lagrange's 2 (Change of GCC momentum)**

$$\frac{dp_m}{dt} \equiv \dot{p}_m = \frac{\partial L}{\partial q^m}$$

*Lagrange prefers* <u>Co</u>*variant* $g_{mn}$ *with* <u>Contra</u>*variant velocity* $\dot{q}^m$

*GCC Lagrangian definition*

*GCC "canonical" momentum* $p_m$ *definition*

*GCC "canonical" force* $F_m$ *definition*

*Coriolis "fictitious" forces (... and weather effects)*

# *Lagrange prefers Covariant $g_{mn}$ with Contravariant velocity*

*Lagrangian KE-U is supposed to be explicit function of velocity.*

$$L(\mathbf{v}) = \tfrac{1}{2}M\mathbf{v}\bullet\mathbf{v} - U = \tfrac{1}{2}M\dot{\mathbf{r}}\bullet\dot{\mathbf{r}} - U = \tfrac{1}{2}M(\mathbf{E}_m\dot{q}^m)\bullet(\mathbf{E}_n\dot{q}^n) - U = \tfrac{1}{2}M(g_{mn}\dot{q}^m\dot{q}^n) - U = L(\dot{q})$$

*Use polar coordinate Covariant $g_{mn}$ metric (page 53)*

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r\bullet\mathbf{E}_r & \mathbf{E}_r\bullet\mathbf{E}_\phi \\ \mathbf{E}_\phi\bullet\mathbf{E}_r & \mathbf{E}_\phi\bullet\mathbf{E}_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

*This gives polar GCC form (Actually it's an OCC or Orthogonal Curvilinear Coordinate form)*

$$L(\dot{r},\dot{\phi}) = \tfrac{1}{2}M(g_{rr}\dot{r}^2 + g_{\phi\phi}\dot{\phi}^2) - U(r,\phi) = \tfrac{1}{2}M(1\cdot\dot{r}^2 + r^2\cdot\dot{\phi}^2) - U(r,\phi)$$

*GCC Lagrange equations follow. 1st L-equation is momentum $p_m$ definition for each coordinate $q^m$:*

$$p_r = \frac{\partial L}{\partial \dot{r}} = M\, g_{rr}\dot{r} = M\,\dot{r}$$

*Nothing too surprising; radial momentum $p_r$ has the usual linear $M\cdot v$ form*

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = Mg_{\phi\phi}\dot{\phi} = Mr^2\dot{\phi}$$

*Wow! $g_{\phi\phi}$ gives moment-of-inertia factor $Mr^2$ automatically for the angular momentum $p_\phi = Mr^2\omega$.*

*(From preceding page)*

# *Lagrange prefers Covariant $g_{mn}$ with Contravariant velocity*

*Lagrangian KE-U is supposed to be explicit function of velocity.*

$$L(\mathbf{v}) = \tfrac{1}{2}M\mathbf{v}\bullet\mathbf{v} - U = \tfrac{1}{2}M\dot{\mathbf{r}}\bullet\dot{\mathbf{r}} - U = \tfrac{1}{2}M(\mathbf{E}_m\dot{q}^m)\bullet(\mathbf{E}_n\dot{q}^n) - U = \tfrac{1}{2}M(g_{mn}\dot{q}^m\dot{q}^n) - U = L(\dot{q})$$

*Use polar coordinate Covariant $g_{mn}$ metric (page 53)*

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r\bullet\mathbf{E}_r & \mathbf{E}_r\bullet\mathbf{E}_\phi \\ \mathbf{E}_\phi\bullet\mathbf{E}_r & \mathbf{E}_\phi\bullet\mathbf{E}_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

*This gives polar GCC form (Actually it's an OCC or Orthogonal Curvilinear Coordinate form)*

$$L(\dot{r},\dot{\phi}) = \tfrac{1}{2}M(g_{rr}\dot{r}^2 + g_{\phi\phi}\dot{\phi}^2) - U(r,\phi) = \tfrac{1}{2}M(1\cdot\dot{r}^2 + r^2\cdot\dot{\phi}^2) - U(r,\phi)$$

*GCC Lagrange equations follow. 1st L-equation is momentum $p_m$ definition for each coordinate $q^m$:*

$$p_r = \frac{\partial L}{\partial \dot{r}} = M\, g_{rr}\dot{r} = M\,\dot{r}$$

*Nothing too surprising; radial momentum $p_r$ has the usual linear $M\cdot v$ form*

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = M g_{\phi\phi}\dot{\phi} = M r^2\dot{\phi}$$

*Wow! $g_{\phi\phi}$ gives moment-of-inertia factor $Mr^2$ automatically for the angular momentum $p_\phi = Mr^2\omega$.*

*2nd L-equation involves total time derivative $\dot{p}_m$ for each momentum $p_m$:*

$$\dot{p}_r = \frac{\partial L}{\partial r} = \frac{M}{2}\frac{\partial g_{\phi\phi}}{\partial r}\dot{\phi}^2 - \frac{\partial U}{\partial r} = M\,r\,\dot{\phi}^2 - \frac{\partial U}{\partial r}$$

*Centrifugal force $Mr\omega^2$*

$$\dot{p}_\phi = \frac{\partial L}{\partial \phi} = 0 - \frac{\partial U}{\partial \phi}$$

*Angular momentum $p_\phi$ is conserved if potential $U$ has no explicit $\phi$-dependence*

*Lagrange's 1*
*(Defining GCC momentum)*

$$p_m = \frac{\partial L}{\partial \dot{q}^m}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^m} = \frac{\partial L}{\partial q^m}$$

*Recall :*

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}}$$

*Lagrange's 2nd GCC equation*
*(Change of GCC momentum)*

$$\frac{dp_m}{dt} \equiv \dot{p}_m = \frac{\partial L}{\partial q^m}$$

# *Lagrange prefers Covariant $g_{mn}$ with Contravariant velocity*

*Lagrangian KE-U is supposed to be explicit function of velocity.*

$$L(\mathbf{v}) = \tfrac{1}{2}M\mathbf{v}\bullet\mathbf{v} - U = \tfrac{1}{2}M\dot{\mathbf{r}}\bullet\dot{\mathbf{r}} - U = \tfrac{1}{2}M(\mathbf{E}_m\dot{q}^m)\bullet(\mathbf{E}_n\dot{q}^n) - U = \tfrac{1}{2}M(g_{mn}\dot{q}^m\dot{q}^n) - U = L(\dot{q})$$

*Use polar coordinate Covariant $g_{mn}$ metric (page 53)*

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r\bullet\mathbf{E}_r & \mathbf{E}_r\bullet\mathbf{E}_\phi \\ \mathbf{E}_\phi\bullet\mathbf{E}_r & \mathbf{E}_\phi\bullet\mathbf{E}_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

*This gives polar GCC form (Actually it's an OCC or Orthogonal Curvilinear Coordinate form)*

$$L(\dot{r},\dot{\phi}) = \tfrac{1}{2}M(g_{rr}\dot{r}^2 + g_{\phi\phi}\dot{\phi}^2) - U(r,\phi) = \tfrac{1}{2}M(1\cdot\dot{r}^2 + r^2\cdot\dot{\phi}^2) - U(r,\phi)$$

*GCC Lagrange equations follow. 1st L-equation is momentum $p_m$ definition for each coordinate $q^m$:*

$$p_r = \frac{\partial L}{\partial \dot{r}} = M\, g_{rr}\dot{r} = M\,\dot{r}$$

*Nothing too surprising; radial momentum $p_r$ has the usual linear $M\cdot v$ form*

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = M g_{\phi\phi}\dot{\phi} = Mr^2\dot{\phi}$$

*Wow! $g_{\phi\phi}$ gives moment-of-inertia factor $Mr^2$ automatically for the angular momentum $p_\phi = Mr^2\omega$.*

*2nd L-equation involves total time derivative $\dot{p}_m$ for each momentum $p_m$:*

$$\dot{p}_r = \frac{\partial L}{\partial r} = \frac{M}{2}\frac{\partial g_{\phi\phi}}{\partial r}\dot{\phi}^2 - \frac{\partial U}{\partial r} = M\,r\,\dot{\phi}^2 - \frac{\partial U}{\partial r}$$

*Centrifugal force $Mr\omega^2$*

$$\dot{p}_\phi = \frac{\partial L}{\partial \phi} = 0 - \frac{\partial U}{\partial \phi}$$

*Angular momentum $p_\phi$ is conserved if potential $U$ has no explicit $\phi$-dependence*

*Find $\dot{p}_m$ directly from 1st L-equation:* $\dot{p}_m \equiv \frac{dp_m}{dt} = \frac{d}{dt}M(g_{mn}\dot{q}^n) = M(\dot{g}_{mn}\dot{q}^n + g_{mn}\ddot{q}^n)$ *Equate it to $\dot{p}_m$ in 2nd L-equation:*

*Lagrange prefers <u>Co</u>variant $g_{mn}$ with <u>Contra</u>variant velocity $\dot{q}^m$*

*GCC Lagrangian definition*

*GCC "canonical" momentum $p_m$ definition*

→ *GCC "canonical" force $F_m$ definition*

*Coriolis "fictitious" forces (... and weather effects)*

# *Lagrange prefers Covariant $g_{mn}$ with Contravariant velocity*

*Lagrangian KE-U is supposed to be explicit function of velocity.*

$$L(\mathbf{v}) = \tfrac{1}{2}M\mathbf{v}\bullet\mathbf{v} - U = \tfrac{1}{2}M\dot{\mathbf{r}}\bullet\dot{\mathbf{r}} - U = \tfrac{1}{2}M(\mathbf{E}_m\dot{q}^m)\bullet(\mathbf{E}_n\dot{q}^n) - U = \tfrac{1}{2}M(g_{mn}\dot{q}^m\dot{q}^n) - U = L(\dot{q})$$

*Use polar coordinate Covariant $g_{mn}$ metric (page 53)*

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r\bullet\mathbf{E}_r & \mathbf{E}_r\bullet\mathbf{E}_\phi \\ \mathbf{E}_\phi\bullet\mathbf{E}_r & \mathbf{E}_\phi\bullet\mathbf{E}_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

*This gives polar GCC form (Actually it's an OCC or Orthogonal Curvilinear Coordinate form)*

$$L(\dot{r},\dot{\phi}) = \tfrac{1}{2}M(g_{rr}\dot{r}^2 + g_{\phi\phi}\dot{\phi}^2) - U(r,\phi) = \tfrac{1}{2}M(1\cdot\dot{r}^2 + r^2\cdot\dot{\phi}^2) - U(r,\phi)$$

*GCC Lagrange equations follow. 1st L-equation is momentum $p_m$ definition for each coordinate $q^m$:*

$$p_r = \frac{\partial L}{\partial \dot{r}} = M\, g_{rr}\dot{r} = M\,\dot{r}$$

*Nothing too surprising; radial momentum $p_r$ has the usual linear $M\cdot v$ form*

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = M g_{\phi\phi}\dot{\phi} = Mr^2\dot{\phi}$$

*Wow! $g_{\phi\phi}$ gives moment-of-inertia factor $Mr^2$ automatically for the angular momentum $p_\phi = Mr^2\omega$.*

*2nd L-equation involves total time derivative $\dot{p}_m$ for each momentum $p_m$:*

$$\dot{p}_r = \frac{\partial L}{\partial r} = \frac{M}{2}\frac{\partial g_{\phi\phi}}{\partial r}\dot{\phi}^2 - \frac{\partial U}{\partial r} = M\,r\,\dot{\phi}^2 - \frac{\partial U}{\partial r}$$

*Centrifugal force $Mr\omega^2$*

$$\dot{p}_\phi = \frac{\partial L}{\partial \phi} = 0 - \frac{\partial U}{\partial \phi}$$

*Angular momentum $p_\phi$ is conserved if potential $U$ has no explicit $\phi$-dependence*

*Find $\dot{p}_m$ directly from 1st L-equation:* $\dot{p}_m \equiv \frac{dp_m}{dt} = \frac{d}{dt}M(g_{mn}\dot{q}^n) = M(\dot{g}_{mn}\dot{q}^n + g_{mn}\ddot{q}^n)$ *Equate it to $\dot{p}_m$ in 2nd L-equation:*

*(From preceding page)*

# *Lagrange prefers Covariant $g_{mn}$ with Contravariant velocity*

*Lagrangian KE-U is supposed to be explicit function of velocity.*

$$L(\mathbf{v})=\tfrac{1}{2}M\mathbf{v}\bullet\mathbf{v}-U=\tfrac{1}{2}M\dot{\mathbf{r}}\bullet\dot{\mathbf{r}}-U=\tfrac{1}{2}M(\mathbf{E}_m\dot{q}^m)\bullet(\mathbf{E}_n\dot{q}^n)-U=\tfrac{1}{2}M(g_{mn}\dot{q}^m\dot{q}^n)-U=L(\dot{q})$$

*Use polar coordinate Covariant $g_{mn}$ metric (page 53)*

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix}=\begin{pmatrix} \mathbf{E}_r\bullet\mathbf{E}_r & \mathbf{E}_r\bullet\mathbf{E}_\phi \\ \mathbf{E}_\phi\bullet\mathbf{E}_r & \mathbf{E}_\phi\bullet\mathbf{E}_\phi \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

*This gives polar GCC form (Actually it's an OCC or Orthogonal Curvilinear Coordinate form)*

$$L(\dot{r},\dot{\phi})=\tfrac{1}{2}M(g_{rr}\dot{r}^2+g_{\phi\phi}\dot{\phi}^2)-U(r,\phi)=\tfrac{1}{2}M(1\cdot\dot{r}^2+r^2\cdot\dot{\phi}^2)-U(r,\phi)$$

*GCC Lagrange equations follow. 1st L-equation is momentum $p_m$ definition for each coordinate $q^m$:*

$$p_r=\frac{\partial L}{\partial\dot{r}}=M\,g_{rr}\dot{r}=M\,\dot{r}$$

*Nothing too surprising; radial momentum $p_r$ has the usual linear $M\cdot v$ form*

$$p_\phi=\frac{\partial L}{\partial\dot{\phi}}=Mg_{\phi\phi}\dot{\phi}=Mr^2\dot{\phi}$$

*Wow! $g_{\phi\phi}$ gives moment-of-inertia factor $Mr^2$ automatically for the angular momentum $p_\phi=Mr^2\omega$.*

*2nd L-equation involves total time derivative $\dot{p}_m$ for each momentum $p_m$:*

$$\dot{p}_r=\frac{\partial L}{\partial r}=\frac{M}{2}\frac{\partial g_{\phi\phi}}{\partial r}\dot{\phi}^2-\frac{\partial U}{\partial r}=M\,r\,\dot{\phi}^2-\frac{\partial U}{\partial r}$$

*Centrifugal force $Mr\omega^2$*

$$\dot{p}_\phi=\frac{\partial L}{\partial\phi}=0-\frac{\partial U}{\partial\phi}$$

*Angular momentum $p_\phi$ is conserved if potential $U$ has no explicit $\phi$-dependence*

*Find $\dot{p}_m$ directly from 1st L-equation:* $\dot{p}_m\equiv\frac{dp_m}{dt}=\frac{d}{dt}M(g_{mn}\dot{q}^n)=M(\dot{g}_{mn}\dot{q}^n+g_{mn}\ddot{q}^n)$ *Equate it to $\dot{p}_m$ in 2nd L-equation:*

$$\dot{p}_r\equiv\frac{dp_r}{dt}=M\,\ddot{r}$$

$$=M\,r\,\dot{\phi}^2-\frac{\partial U}{\partial r}$$

*Centrifugal (center-fleeing) force equals total Centripetal (center-pulling) force*

# *Lagrange prefers Covariant $g_{mn}$ with Contravariant velocity*

*Lagrangian KE-U is supposed to be explicit function of velocity.*

$$L(\mathbf{v}) = \tfrac{1}{2}M\mathbf{v}\bullet\mathbf{v} - U = \tfrac{1}{2}M\dot{\mathbf{r}}\bullet\dot{\mathbf{r}} - U = \tfrac{1}{2}M(\mathbf{E}_m\dot{q}^m)\bullet(\mathbf{E}_n\dot{q}^n) - U = \tfrac{1}{2}M(g_{mn}\dot{q}^m\dot{q}^n) - U = L(\dot{q})$$

*Use polar coordinate Covariant $g_{mn}$ metric (page 53)*

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r\bullet\mathbf{E}_r & \mathbf{E}_r\bullet\mathbf{E}_\phi \\ \mathbf{E}_\phi\bullet\mathbf{E}_r & \mathbf{E}_\phi\bullet\mathbf{E}_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

*This gives polar GCC form (Actually it's an OCC or Orthogonal Curvilinear Coordinate form)*

$$L(\dot{r},\dot{\phi}) = \tfrac{1}{2}M(g_{rr}\dot{r}^2 + g_{\phi\phi}\dot{\phi}^2) - U(r,\phi) = \tfrac{1}{2}M(1\cdot\dot{r}^2 + r^2\cdot\dot{\phi}^2) - U(r,\phi)$$

*GCC Lagrange equations follow. 1st L-equation is momentum $p_m$ definition for each coordinate $q^m$:*

$$p_r = \frac{\partial L}{\partial \dot{r}} = M\, g_{rr}\dot{r} = M\,\dot{r}$$

*Nothing too surprising; radial momentum $p_r$ has the usual linear $M\cdot v$ form*

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = M g_{\phi\phi}\dot{\phi} = Mr^2\dot{\phi}$$

*Wow! $g_{\phi\phi}$ gives moment-of-inertia factor $Mr^2$ automatically for the angular momentum $p_\phi = Mr^2\omega$.*

*2nd L-equation involves total time derivative $\dot{p}_m$ for each momentum $p_m$:*

$$\dot{p}_r = \frac{\partial L}{\partial r} = \frac{M}{2}\frac{\partial g_{\phi\phi}}{\partial r}\dot{\phi}^2 - \frac{\partial U}{\partial r} = M\,r\,\dot{\phi}^2 - \frac{\partial U}{\partial r}$$

*Centrifugal force $Mr\omega^2$*

$$\dot{p}_\phi = \frac{\partial L}{\partial \phi} = 0 - \frac{\partial U}{\partial \phi}$$

*Angular momentum $p_\phi$ is conserved if potential $U$ has no explicit $\phi$-dependence*

*Find $\dot{p}_m$ directly from 1st L-equation:* $\dot{p}_m \equiv \frac{dp_m}{dt} = \frac{d}{dt}M(g_{mn}\dot{q}^n) = M(\dot{g}_{mn}\dot{q}^n + g_{mn}\ddot{q}^n)$ *Equate it to $\dot{p}_m$ in 2nd L-equation:*

$$\dot{p}_r \equiv \frac{dp_r}{dt} = M\,\ddot{r} = M\,r\,\dot{\phi}^2 - \frac{\partial U}{\partial r}$$

*Centrifugal (center-fleeing) force equals total Centripetal (center-pulling) force*

$$\dot{p}_\phi \equiv \frac{dp_\phi}{dt} = 2Mr\dot{r}\dot{\phi} + Mr^2\ddot{\phi} = 0 - \frac{\partial U}{\partial \phi}$$

*Torque relates to two distinct parts: Coriolis and angular acceleration*

*Angular momentum $p_\phi$ is conserved if potential $U$ has no explicit $\phi$-dependence*

*Rewriting GCC Lagrange equations :*

$$\dot{p}_r \equiv \frac{dp_r}{dt} = M\ddot{r} = Mr\dot{\phi}^2 - \frac{\partial U}{\partial r}$$

*Centrifugal (center-fleeing) force equals total Centripetal (center-pulling) force*

$$\dot{p}_\phi \equiv \frac{dp_\phi}{dt} = 2Mr\dot{r}\dot{\phi} + Mr^2\ddot{\phi} = 0 - \frac{\partial U}{\partial \phi}$$

*Torque relates to two distinct parts: Coriolis and angular acceleration*

*Angular momentum $p_\phi$ is conserved if potential $U$ has no explicit $\phi$-dependence*

*Conventional forms*

*radial force:* $M\ddot{r} = Mr\dot{\phi}^2 - \frac{\partial U}{\partial r}$

*angular force or torque:* $Mr^2\ddot{\phi} = -2Mr\dot{r}\dot{\phi} - \frac{\partial U}{\partial \phi}$

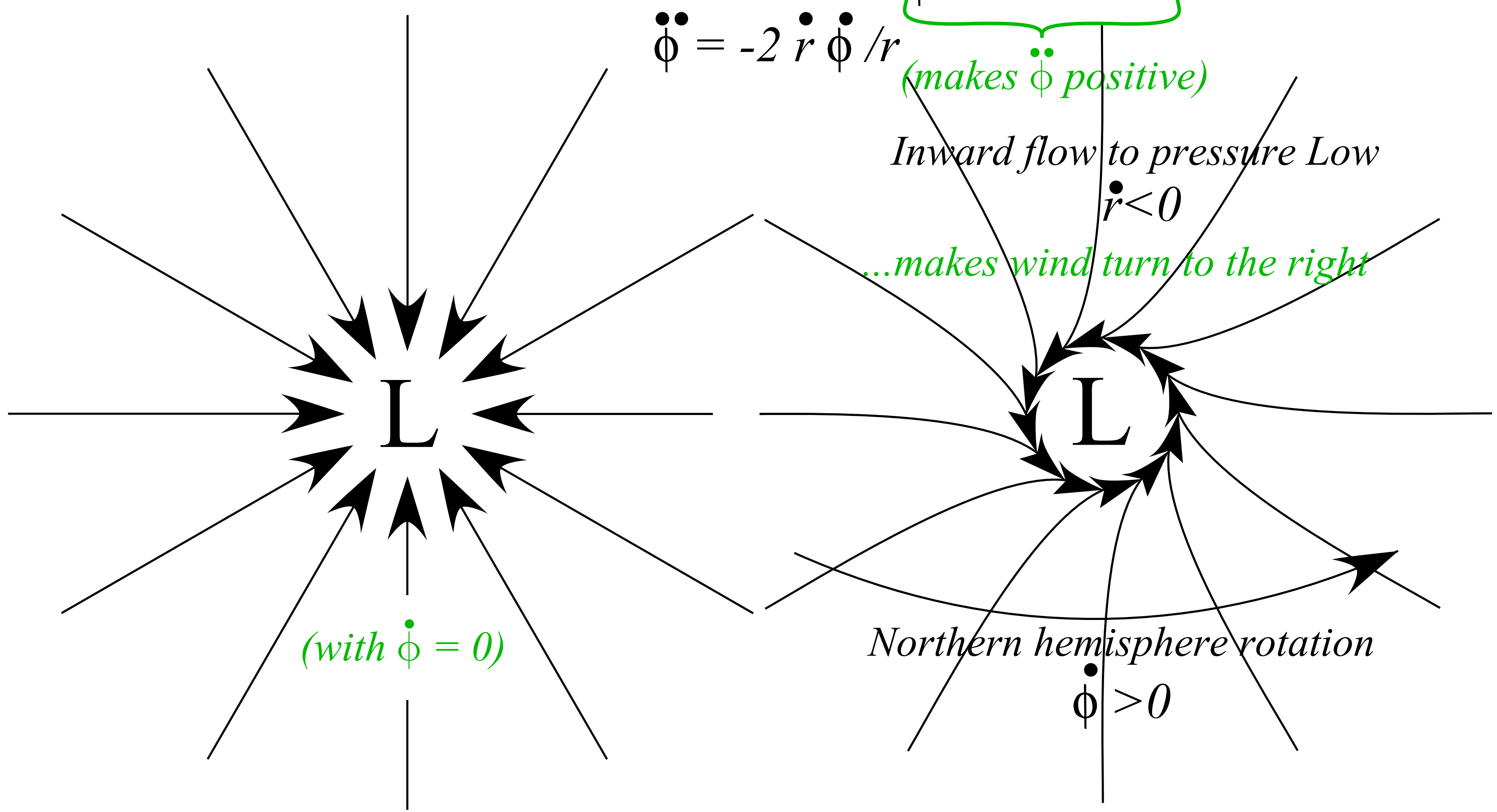
*Field-free (U=0)*

*radial acceleration:* $\ddot{r} = r\dot{\phi}^2$

*angular acceleration:* $\ddot{\phi} = -2\frac{\dot{r}\dot{\phi}}{r}$

Coriolis acceleration with $\underbrace{\dot{\phi} > 0 \text{ and } \dot{r} < 0}_{\text{(makes } \ddot{\phi} \text{ positive)}}$

$\ddot{\phi} = -2\,\dot{r}\,\dot{\phi}\,/r$




*Effect on Northern Hemisphere local weather*

*Cyclonic flow around lows*

## Rewriting GCC Lagrange equations :

$$\dot{p}_r \equiv \frac{dp_r}{dt} = M\,\ddot{r} = M\,r\,\dot{\phi}^2 - \frac{\partial U}{\partial r}$$

*Centrifugal (center-fleeing) force equals total Centripetal (center-pulling) force*

$$\dot{p}_\phi \equiv \frac{dp_\phi}{dt} = 2Mr\dot{r}\dot{\phi} + Mr^2\ddot{\phi} = 0 - \frac{\partial U}{\partial \phi}$$

*Torque relates to two distinct parts: Coriolis and angular acceleration*

*Angular momentum $p_\phi$ is conserved if potential $U$ has no explicit $\phi$-dependence*

### Conventional forms

*radial force:* $M\,\ddot{r} = M\,r\,\dot{\phi}^2 - \dfrac{\partial U}{\partial r}$

*angular force or torque:* $Mr^2\ddot{\phi} = -2Mr\dot{r}\dot{\phi} - \dfrac{\partial U}{\partial \phi}$

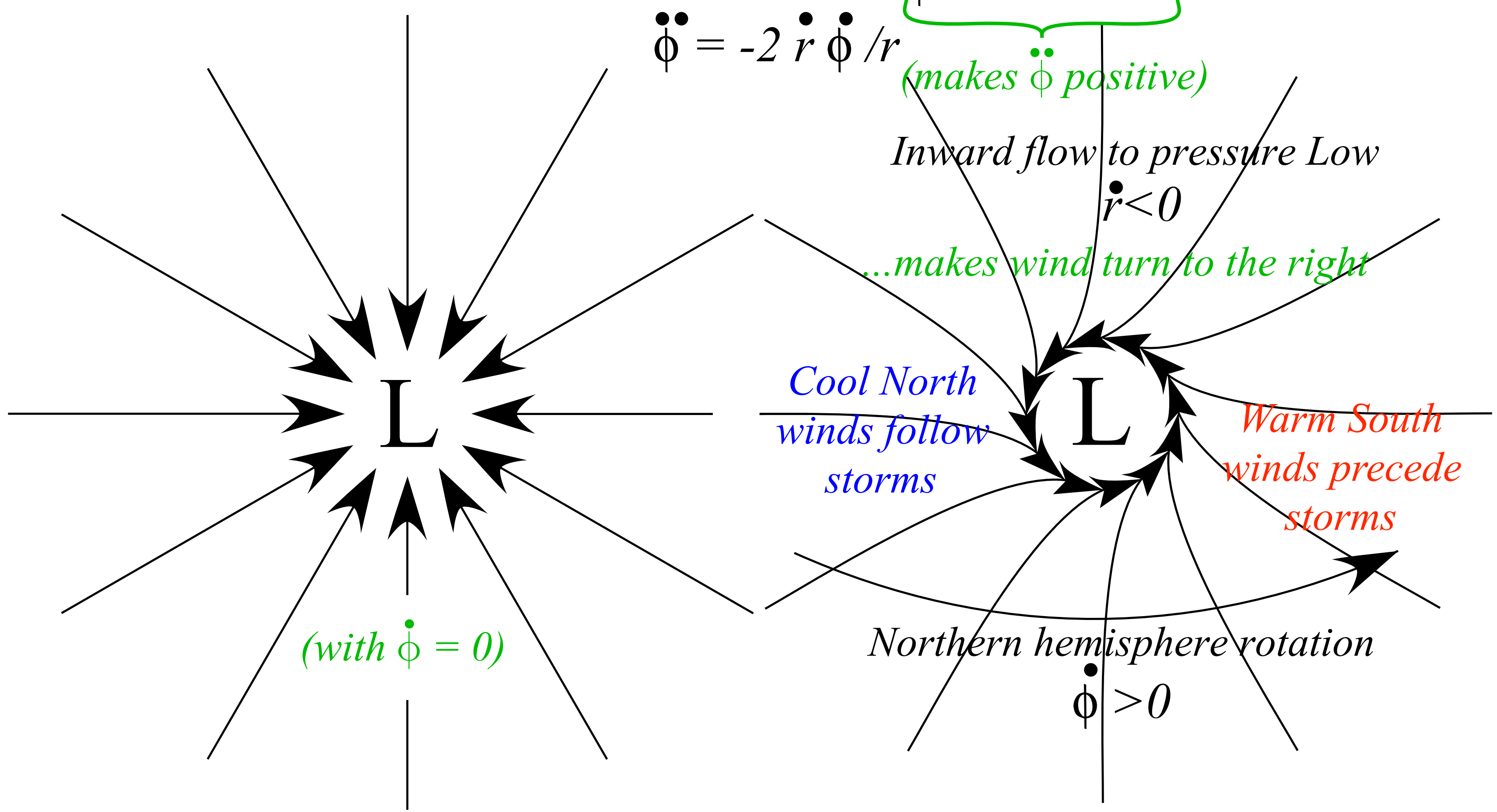
### Field-free (U=0)

*radial acceleration:* $\ddot{r} = r\,\dot{\phi}^2$

*angular acceleration:* $\ddot{\phi} = -2\dfrac{\dot{r}\dot{\phi}}{r}$

Coriolis acceleration with $\dot{\phi} > 0$ and $\dot{r} < 0$

$\ddot{\phi} = -2\,\dot{r}\,\dot{\phi}\,/r$




*Effect on Northern Hemisphere local weather*

*Cyclonic flow around lows*

## Rewriting GCC Lagrange equations :

$$\dot{p}_r \equiv \frac{dp_r}{dt} = M\ddot{r} = Mr\dot{\phi}^2 - \frac{\partial U}{\partial r}$$

Centrifugal (center-fleeing) force equals total Centripetal (center-pulling) force

$$\dot{p}_\phi \equiv \frac{dp_\phi}{dt} = 2Mr\dot{r}\dot{\phi} + Mr^2\ddot{\phi} = 0 - \frac{\partial U}{\partial \phi}$$

Torque relates to two distinct parts: Coriolis and angular acceleration

Angular momentum $p_\phi$ is conserved if potential $U$ has no explicit $\phi$-dependence

### Conventional forms

radial force: $M\ddot{r} = Mr\dot{\phi}^2 - \frac{\partial U}{\partial r}$

angular force or torque: $Mr^2\ddot{\phi} = -2Mr\dot{r}\dot{\phi} - \frac{\partial U}{\partial \phi}$

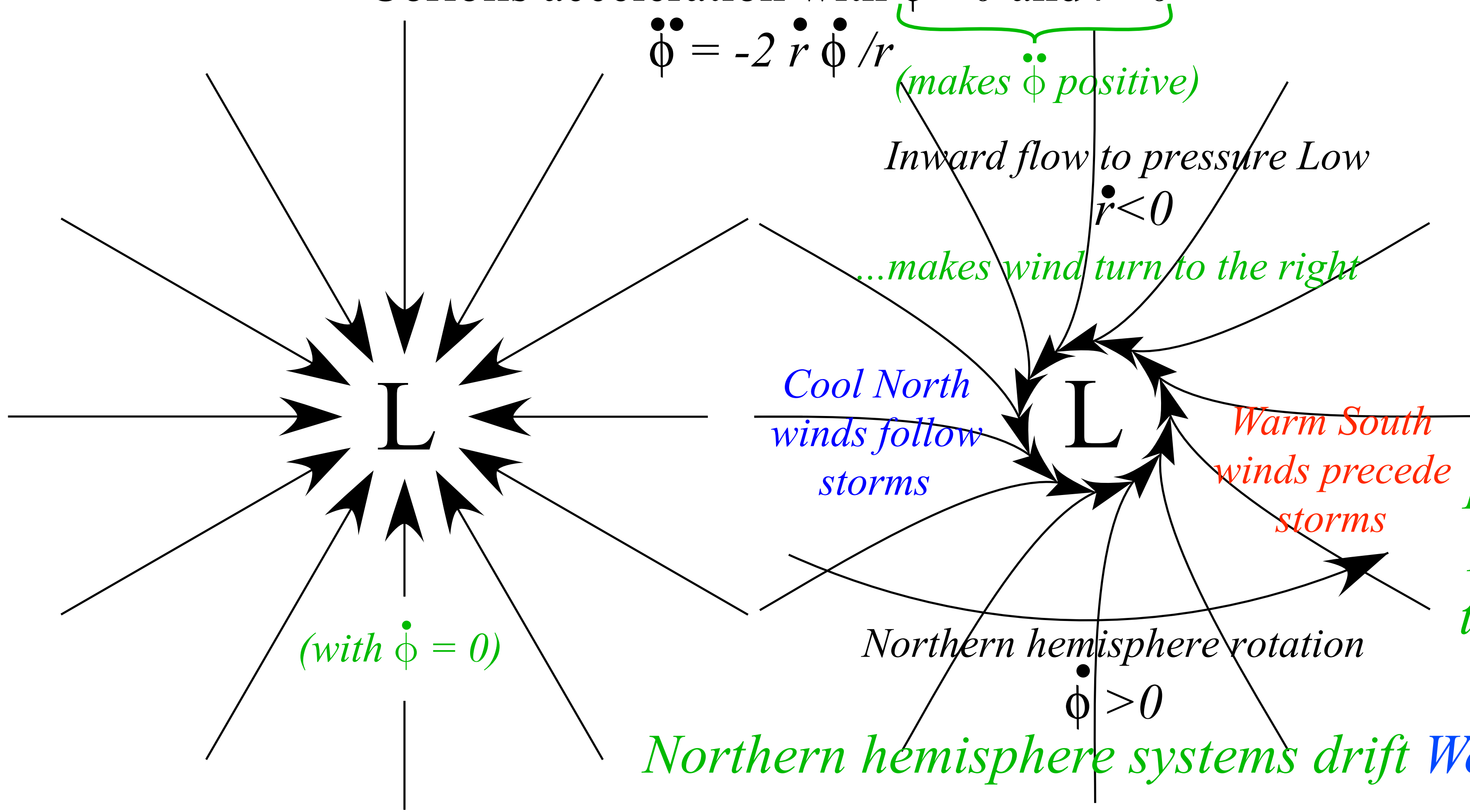
### Field-free (U=0)

radial acceleration: $\ddot{r} = r\dot{\phi}^2$

angular acceleration: $\ddot{\phi} = -2\frac{\dot{r}\dot{\phi}}{r}$

Coriolis acceleration with $\underbrace{\dot{\phi} > 0 \text{ and } \dot{r} < 0}_{\text{(makes } \ddot{\phi} \text{ positive)}}$

$\ddot{\phi} = -2\,\dot{r}\,\dot{\phi}/r$



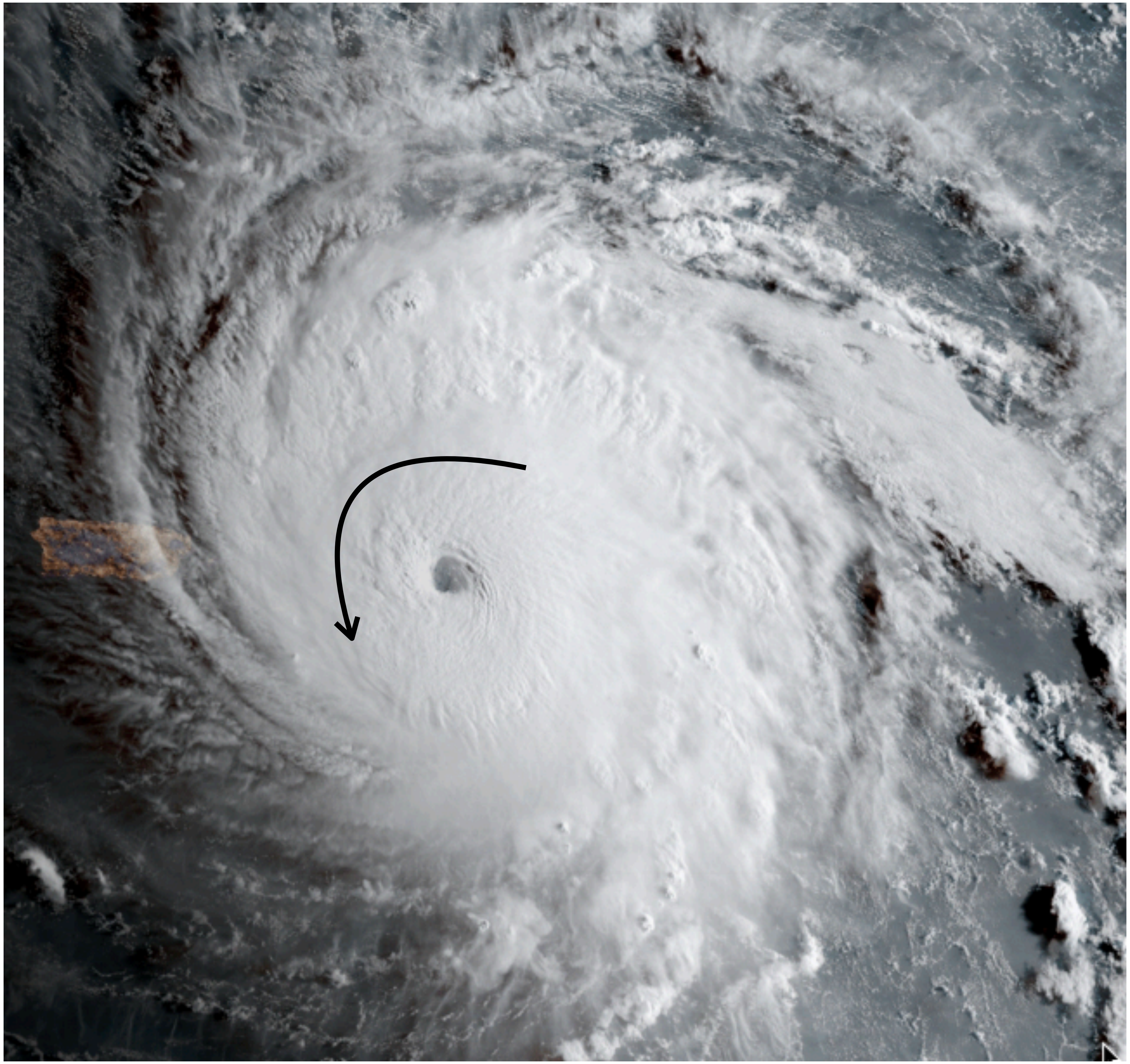

Northern hemisphere systems drift West to East

Effect on Northern Hemisphere local weather

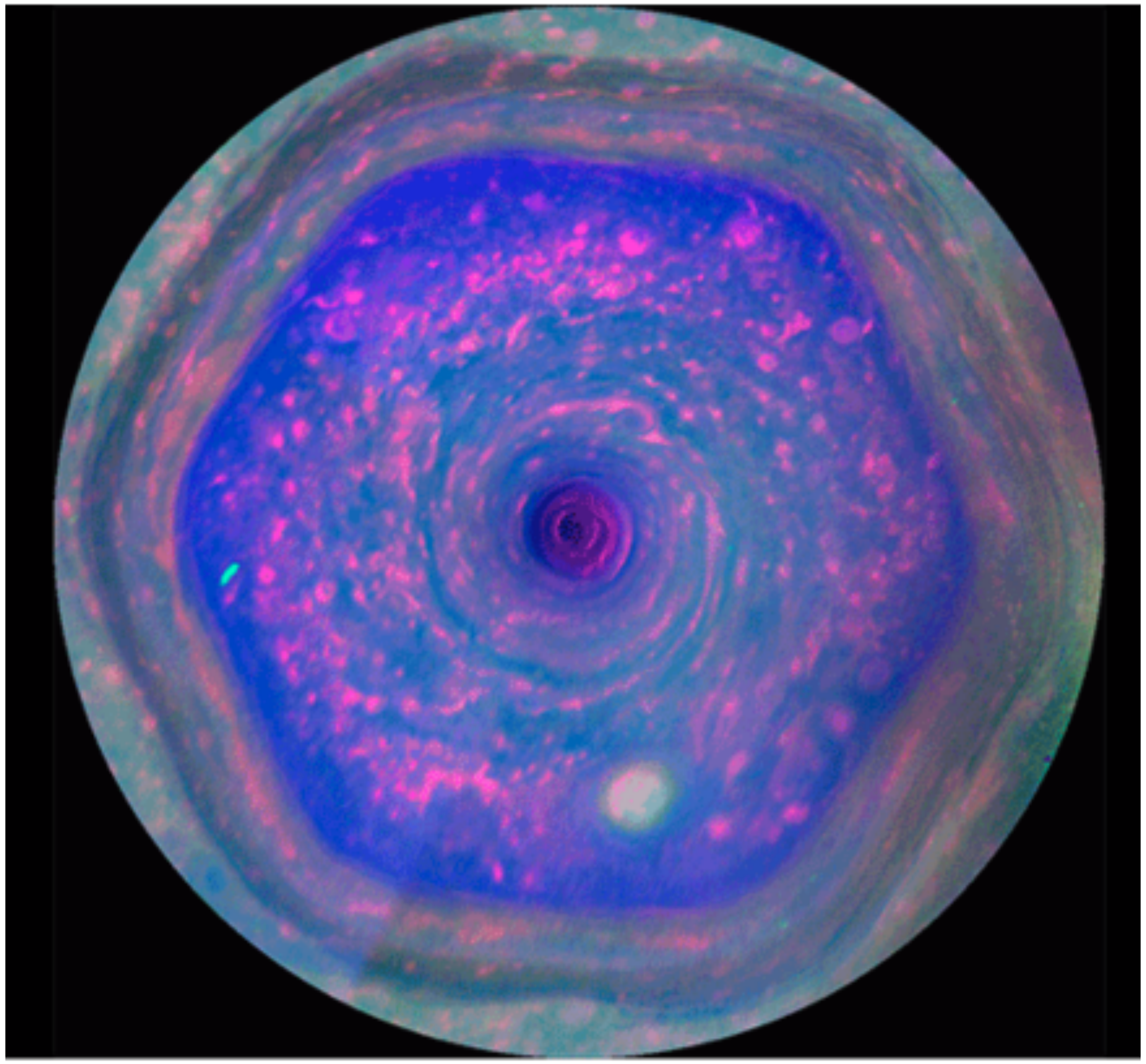
Cyclonic flow around lows

Deep quantum rule: Flow tries to mimic the external rotation (least relative v)

GOES-16 captured this geocolor image of Hurricane Irma approaching Anguilla at about 7:15 am (eastern), September 6, 2017. Irma's maximum sustained winds remain near 185 mph with higher gusts, making it a category 5 hurricane on the Saffir-Simpson Hurricane Wind Scale. According to the latest information from NOAA's National Hurricane Center (issued at 8:00 am eastern), Irma was located about 15 miles west-southwest of Anguilla and moving toward the west-northwest near 16 miles per hour.



# Science News <link>



Saturn's north pole was dark when Cassini arrived in 2004. But as the seasons changed, light illuminated a bizarre six-sided swirl of gases at the pole (shown here in false color). The hexagon has been known since the 1980s. It is about 30,000 kilometers (18,600 miles) wide with a massive hurricane centered on the north pole.
JPL-CALTECH/NASA, SPACE SCIENCE INSTITUTE

Lecture 9 ends here
Wed. 9.19.2018

https://www.sciencenewsforstudents.org/article/cassini-spacecraft-takes-its-final-bow