Group Theory in Quantum Mechanics

Lecture 11 *(2.19-3.5.13)*

Symmetry and Dynamics of C_N cyclic systems

(Geometry of U(2) characters - Ch. 6-9 of Unit 3)

(Principles of Symmetry, Dynamics, and Spectroscopy - Sec. 3-7 of Ch. 2)

Polygonal geometry of $U(2) \supset C_N$ character spectral function
Algebra
Geometry

Introduction to wave dynamics of phase, mean phase, and group velocity
Expo-Cosine identity
Relating space-time and per-space-time
- *Wave coordinates*
- *Pulse-waves (PW) vs Continuous -waves (CW)*

Introduction to C_N beat dynamics and "Revivals" due to Bohr-dispersion
- *∞-Square well PE versus Bohr rotor*
- *SinNx/x wavepackets bandwidth and uncertainty*
  - *SinNx/x explosion and revivals*
- *Bohr-rotor dynamics*
  - *Gaussian wave-packet bandwidth and uncertainty*
  - *Gaussian Bohr-rotor revivals*
- *Farey-Sums and Ford-products*
- *Phase dynamics*

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Polygonal geometry of U(2)⊃C_N character spectral function

Trace-character $\chi^j(\Theta)$ of U(2) rotation by C_n angle $\Theta=2\pi/n$

is an ($\ell^j=2j+1$)-term sum of $e^{-im\Theta}$ over allowed *m*-quanta $m=\{-j, -j+1, ..., j-1, j\}$.

$$\chi^{1/2}(\Theta) = trace D^{1/2}(\Theta) = trace\begin{pmatrix} e^{-i\theta/2} & \cdot \\ \cdot & e^{+i\theta/2} \end{pmatrix}$$

(spinor-j=1/2)

$$\chi^{1}(\Theta) = trace D^{1}(\Theta) = trace\begin{pmatrix} e^{-i\theta} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{-i\theta} \end{pmatrix}$$

(vector-j=1)

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(vector-j=1)

$\chi^j(\Theta)$ involves a sum of $2\cos(m\,\Theta/2)$ for $m\geq 0$ up to $m=j$.

$$\chi^{1/2}(\Theta)=e^{-i\frac{\Theta}{2}}+e^{i\frac{\Theta}{2}} = 2\cos\frac{\Theta}{2}$$ *(spinor-j=1/2)*

$$\chi^{3/2}(\Theta)=e^{-i\frac{3\Theta}{2}}+\ldots\quad +e^{i\frac{3\Theta}{2}}=2\cos\frac{\Theta}{2}+2\cos\frac{3\Theta}{2}$$

$$\chi^{5/2}(\Theta)=e^{-i\frac{5\Theta}{2}}+\ldots\quad +e^{i\frac{5\Theta}{2}}=2\cos\frac{\Theta}{2}+2\cos\frac{3\Theta}{2}+2\cos\frac{5\Theta}{2}$$

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$$\chi^{1/2}(\Theta) = e^{-i\frac{\Theta}{2}} + e^{i\frac{\Theta}{2}} = 2\cos\frac{\Theta}{2}$$
(spinor-j=1/2)

$$\chi^{3/2}(\Theta) = e^{-i\frac{3\Theta}{2}} + \ldots \quad + e^{i\frac{3\Theta}{2}} = 2\cos\frac{\Theta}{2} + 2\cos\frac{3\Theta}{2}$$

$$\chi^{5/2}(\Theta) = e^{-i\frac{5\Theta}{2}} + \ldots \quad + e^{i\frac{5\Theta}{2}} = 2\cos\frac{\Theta}{2} + 2\cos\frac{3\Theta}{2} + 2\cos\frac{5\Theta}{2}$$

$$\chi^{0}(\Theta) = e^{-i\Theta\cdot 0} = 1$$
(scalar-j=0)

$$\chi^{1}(\Theta) = e^{-i\Theta} + 1 + e^{i\Theta} = 1 + 2\cos\Theta$$
(vector-j=1)

$$\chi^{2}(\Theta) = e^{-i2\Theta} + \ldots e^{i2\Theta} = 1 + 2\cos\Theta + 2\cos 2\Theta$$
(tensor-j=2)

Polygonal geometry of $U(2) \supset C_N$ character spectral function
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$$\chi^{5/2}(\Theta) = e^{-i\frac{5\Theta}{2}} + \ldots \quad + e^{i\frac{5\Theta}{2}} = 2\cos\frac{\Theta}{2} + 2\cos\frac{3\Theta}{2} + 2\cos\frac{5\Theta}{2}$$

$$\chi^{0}(\Theta) = e^{-i\Theta\cdot 0} = 1$$

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$$\chi^{1}(\Theta) = e^{-i\Theta} + 1 + e^{i\Theta} = 1 + 2\cos\Theta$$

(vector-j=1)

$$\chi^{2}(\Theta) = e^{-i2\Theta} + \ldots e^{i2\Theta} = 1 + 2\cos\Theta + 2\cos 2\Theta$$

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$\chi^j(\Theta)$ is a geometric series with ratio $e^{i\Theta}$ between each successive term.

$$\chi^{j}(\Theta) = TraceD^{(j)}(\Theta) = e^{-i\Theta j} + e^{-i\Theta(j-1)} + e^{-i\Theta(j-2)} + \ldots + e^{+i\Theta(j-2)} + e^{+i\Theta(j-1)} + e^{+i\Theta j}$$

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$$\chi^{1/2}(\Theta)=e^{-i\frac{\Theta}{2}}+e^{i\frac{\Theta}{2}} \quad =2\cos\frac{\Theta}{2}$$ *(spinor-j=1/2)*

$$\chi^{3/2}(\Theta)=e^{-i\frac{3\Theta}{2}}+\ldots \quad +e^{i\frac{3\Theta}{2}}=2\cos\frac{\Theta}{2}+2\cos\frac{3\Theta}{2}$$

$$\chi^{5/2}(\Theta)=e^{-i\frac{5\Theta}{2}}+\ldots \quad +e^{i\frac{5\Theta}{2}}=2\cos\frac{\Theta}{2}+2\cos\frac{3\Theta}{2}+2\cos\frac{5\Theta}{2}$$

$$\chi^{0}(\Theta)=e^{-i\Theta\cdot 0} \quad =1$$ *(scalar-j=0)*

$$\chi^{1}(\Theta)=e^{-i\Theta}+1+e^{i\Theta} \quad =1+2\cos\Theta$$ *(vector-j=1)*

$$\chi^{2}(\Theta)=e^{-i2\Theta}+\ldots e^{i2\Theta}=1+2\cos\Theta+2\cos 2\Theta$$ *(tensor-j=2)*

$\chi^j(\Theta)$ is a geometric series with ratio $e^{i\Theta}$ between each successive term.

$$\chi^j(\Theta)=TraceD^{(j)}(\Theta)=e^{-i\Theta j}+e^{-i\Theta(j-1)}+e^{-i\Theta(j-2)}+\ldots+e^{+i\Theta(j-2)}+e^{+i\Theta(j-1)}+e^{+i\Theta j}$$

$$\chi^j(\Theta)e^{-i\Theta}=e^{-i\Theta(j+1)}+e^{-i\Theta j}+e^{-i\Theta(j-1)}+e^{-i\Theta(j-2)}+\ldots+e^{+i\Theta(j-2)}+e^{+i\Theta(j-1)}$$

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$$\chi^{1}(\Theta) = e^{-i\Theta} + 1 + e^{i\Theta} \quad = 1 + 2\cos\Theta$$ *(vector-j=1)*

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$$\chi^{j}(\Theta) = Trace D^{(j)}(\Theta) = e^{-i\Theta j} + e^{-i\Theta(j-1)} + e^{-i\Theta(j-2)} + \ldots + e^{+i\Theta(j-2)} + e^{+i\Theta(j-1)} + e^{+i\Theta j}$$

$$\chi^{j}(\Theta)e^{-i\Theta} = e^{-i\Theta(j+1)} + e^{-i\Theta j} + e^{-i\Theta(j-1)} + e^{-i\Theta(j-2)} + \ldots + e^{+i\Theta(j-2)} + e^{+i\Theta(j-1)}$$

Subtracting gives:

$$\chi^{j}(\Theta)(1-e^{-i\Theta}) = -e^{-i\Theta(j+1)} \qquad + \qquad e^{+i\Theta j}$$

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Subtracting/dividing gives $\chi^j(\Theta)$ formula.

$$\chi^{j}(\Theta)=\frac{e^{+i\Theta j}-e^{-i\Theta(j+1)}}{1-e^{-i\Theta}}=\frac{e^{+i\Theta(j+\frac{1}{2})}-e^{-i\Theta(j+\frac{1}{2})}}{e^{+i\frac{\Theta}{2}}-e^{-i\frac{\Theta}{2}}}=\frac{\sin\Theta(j+\frac{1}{2})}{\sin\frac{\Theta}{2}}$$

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Subtracting/dividing gives $\chi^j(\Theta)$ formula.

$$\chi^{j}(\Theta)=\frac{e^{+i\Theta j}-e^{-i\Theta(j+1)}}{1-e^{-i\Theta}}=\frac{e^{+i\Theta(j+\frac{1}{2})}-e^{-i\Theta(j+\frac{1}{2})}}{e^{+i\frac{\Theta}{2}}-e^{-i\frac{\Theta}{2}}}=\frac{\sin\Theta(j+\frac{1}{2})}{\sin\frac{\Theta}{2}}$$

For C_n angle $\Theta=2\pi/n$ this χ^j has a lot of geometric significance.

$$\chi^{j}(\frac{2\pi}{n})=\frac{\sin\frac{\pi}{n}(2j+1)}{\sin\frac{\pi}{n}}=\frac{\sin\frac{\pi\ell^{j}}{n}}{\sin\frac{\pi}{n}}$$

Character Spectral Function where: $\ell^j=2j+1$ is U(2) irrep dimension

Polygonal geometry of U(2)⊃C_N character spectral function
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Character Spectral Function
where: $\ell^j=2j+1$
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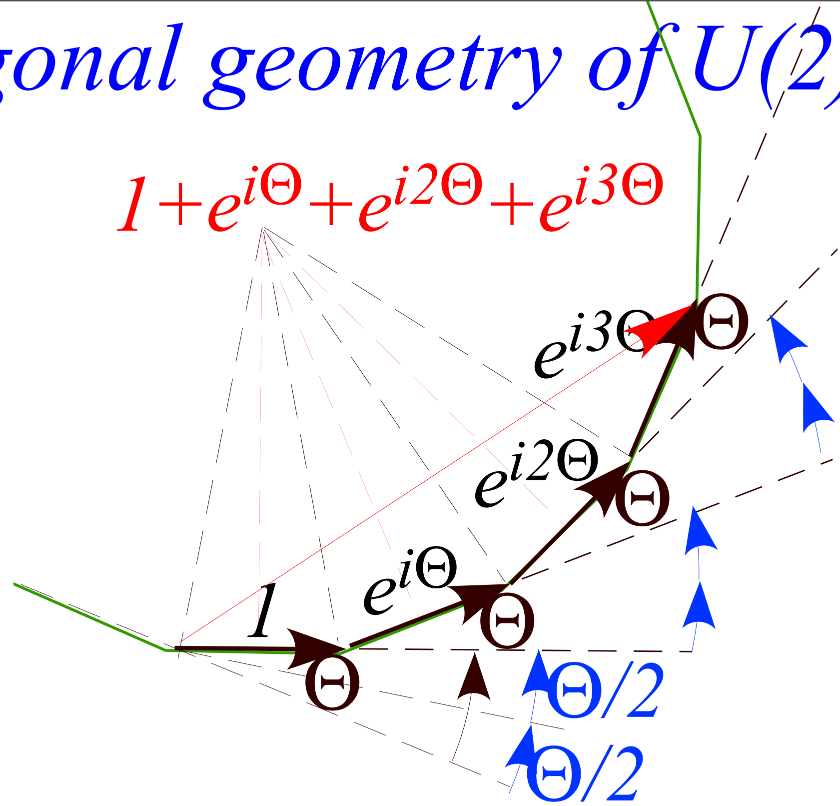
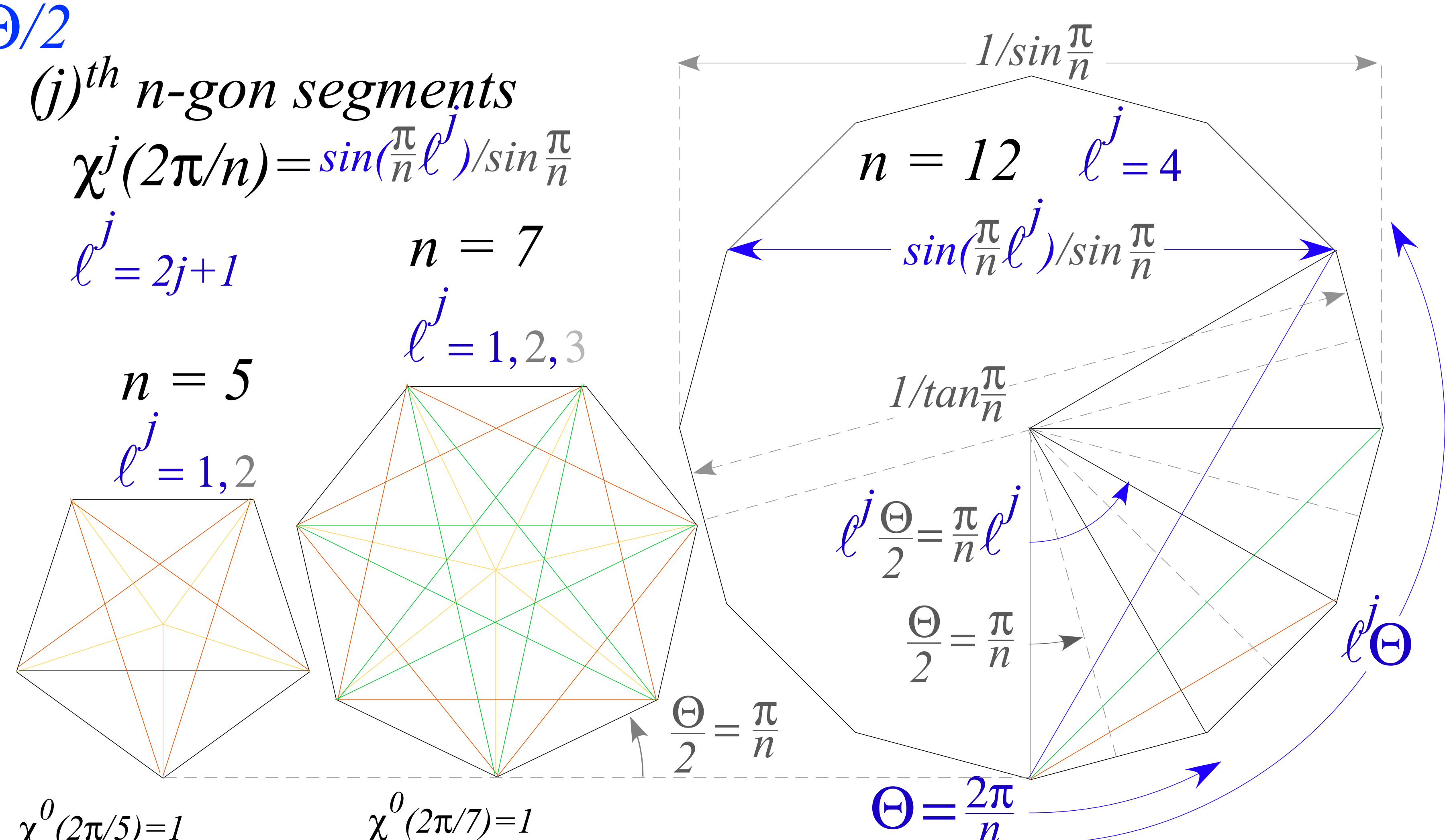


(j)th n-gon segments

$\chi^j(2\pi/n)=\sin(\frac{\pi}{n}\ell^j)/\sin\frac{\pi}{n}$

$\ell^j=2j+1$



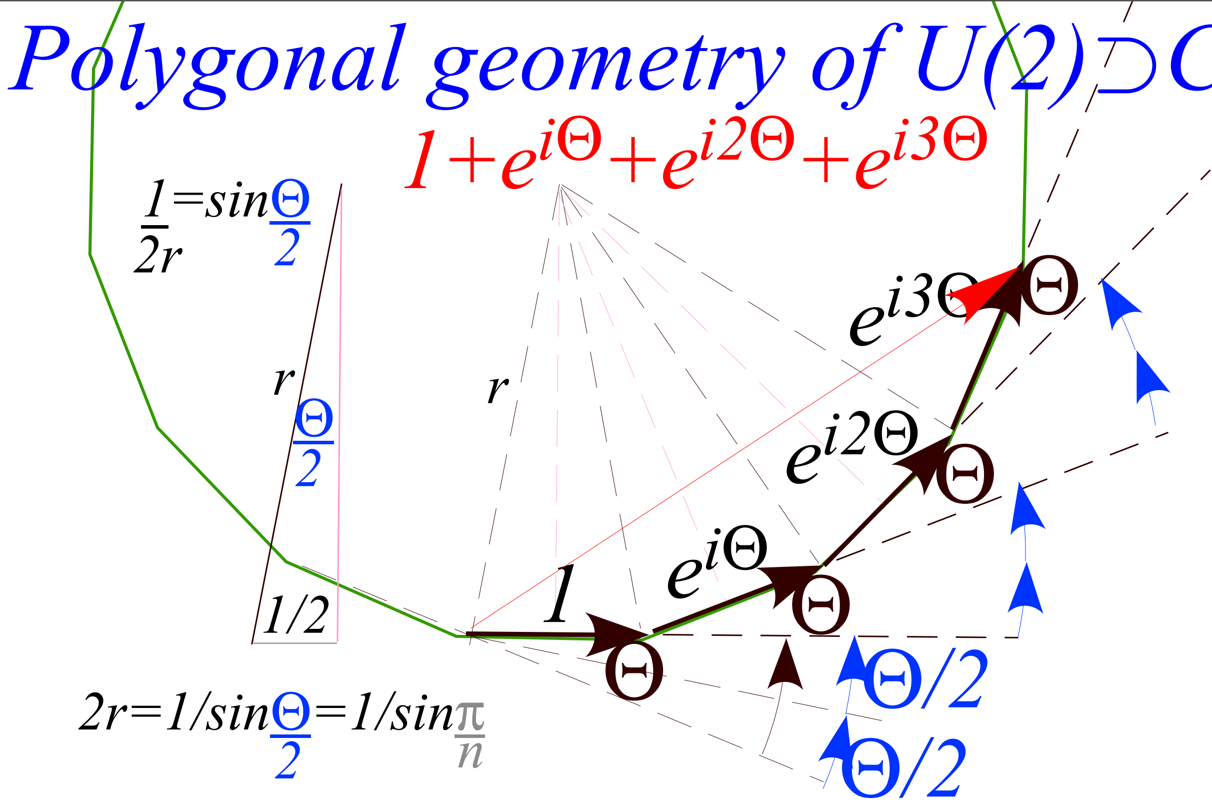
$\chi^0(2\pi/5)=1$
$\chi^{1/2}(2\pi/5)=1.618...$
$=(1+\sqrt{5})/2=$

$\chi^0(2\pi/7)=1$
$\chi^{1/2}(2\pi/7)=1.802...$
$\chi^1(2\pi/7)=2.247...$
$\chi^{3/2}(2\pi/7)=2.247...$

$\chi^{1/2}(2\pi/12)=1.932...$
$\chi^1(2\pi/12)=2.732...$
$\chi^{3/2}(2\pi/12)=3.346...$

$\chi^2(2\pi/12)=3.732...$
$\chi^{5/2}(2\pi/12)=3.864...$
$\chi^3(2\pi/12)=3.732...$

Polygonal geometry of $U(2)\supset C_N$ character spectral function



Character Spectral Function

$$\chi^j\left(\frac{2\pi}{n}\right)=\frac{\sin\frac{\pi}{n}(2j+1)}{\sin\frac{\pi}{n}}=\frac{\sin\frac{\pi\ell^J}{n}}{\sin\frac{\pi}{n}}$$

where: $\ell^j=2j+1$ is $U(2)$ irrep dimension

$(j)^{th}$ n-gon segments

$$\chi^j(2\pi/n)=\sin(\tfrac{\pi}{n}\ell^j)/\sin\tfrac{\pi}{n}$$

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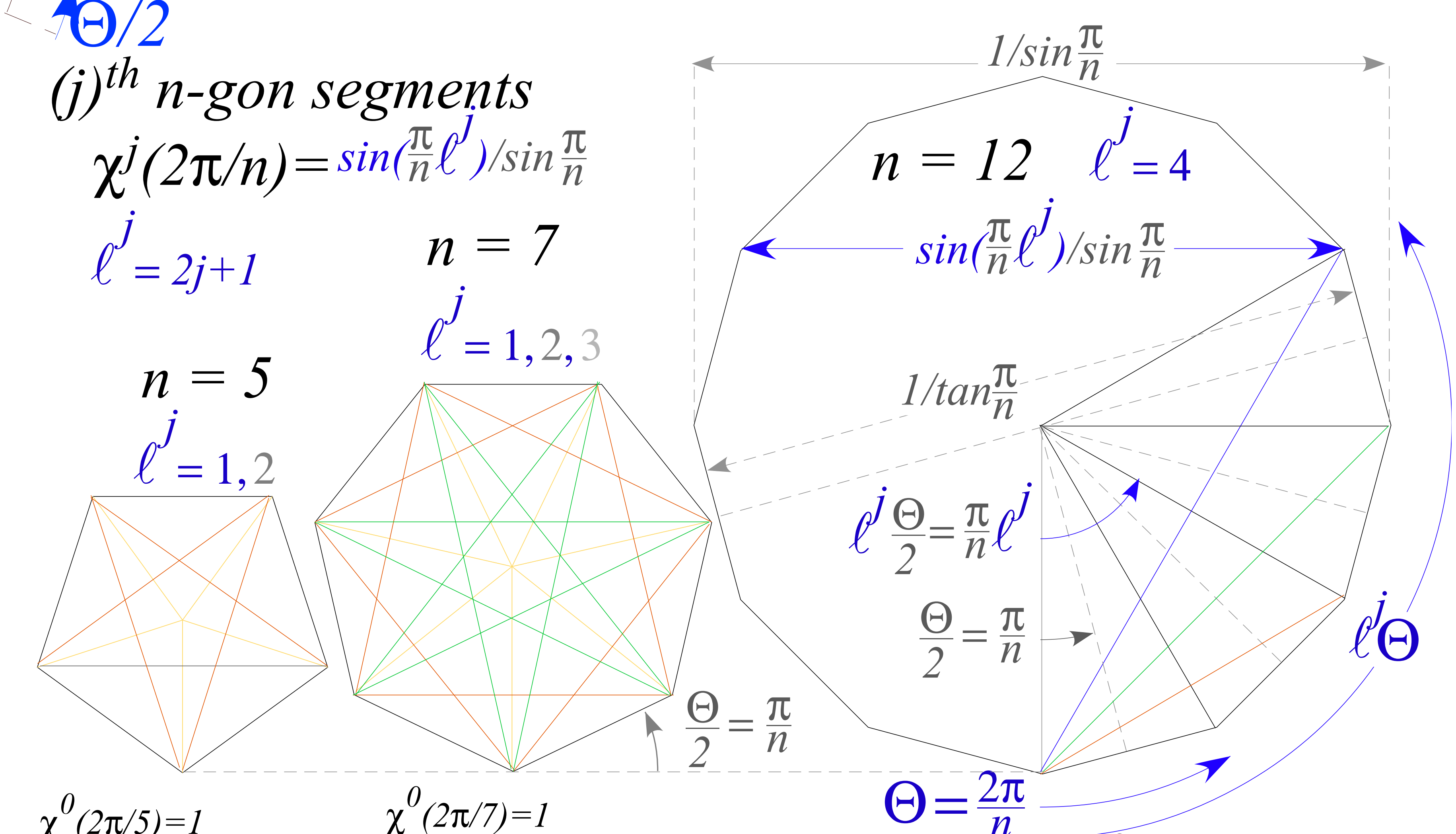

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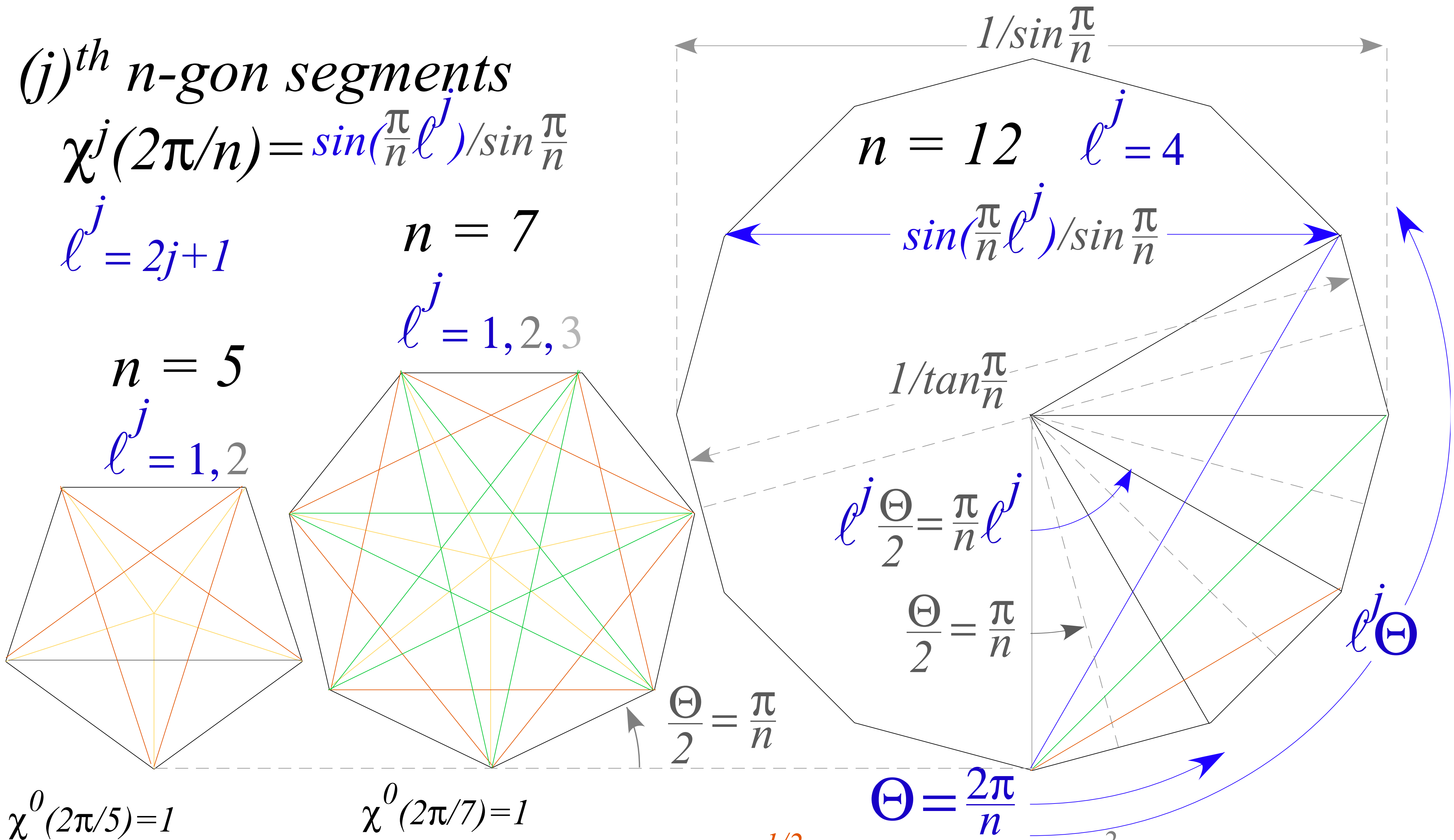
Polygonal geometry of $U(2) \supset C_N$ character spectral function

$$\chi^j\left(\frac{2\pi}{n}\right) = \frac{\sin\frac{\pi}{n}(2j+1)}{\sin\frac{\pi}{n}} = \frac{\sin\frac{\pi \ell^J}{n}}{\sin\frac{\pi}{n}}$$

Character Spectral Function where: $\ell^j = 2j+1$ is U(2) irrep dimension



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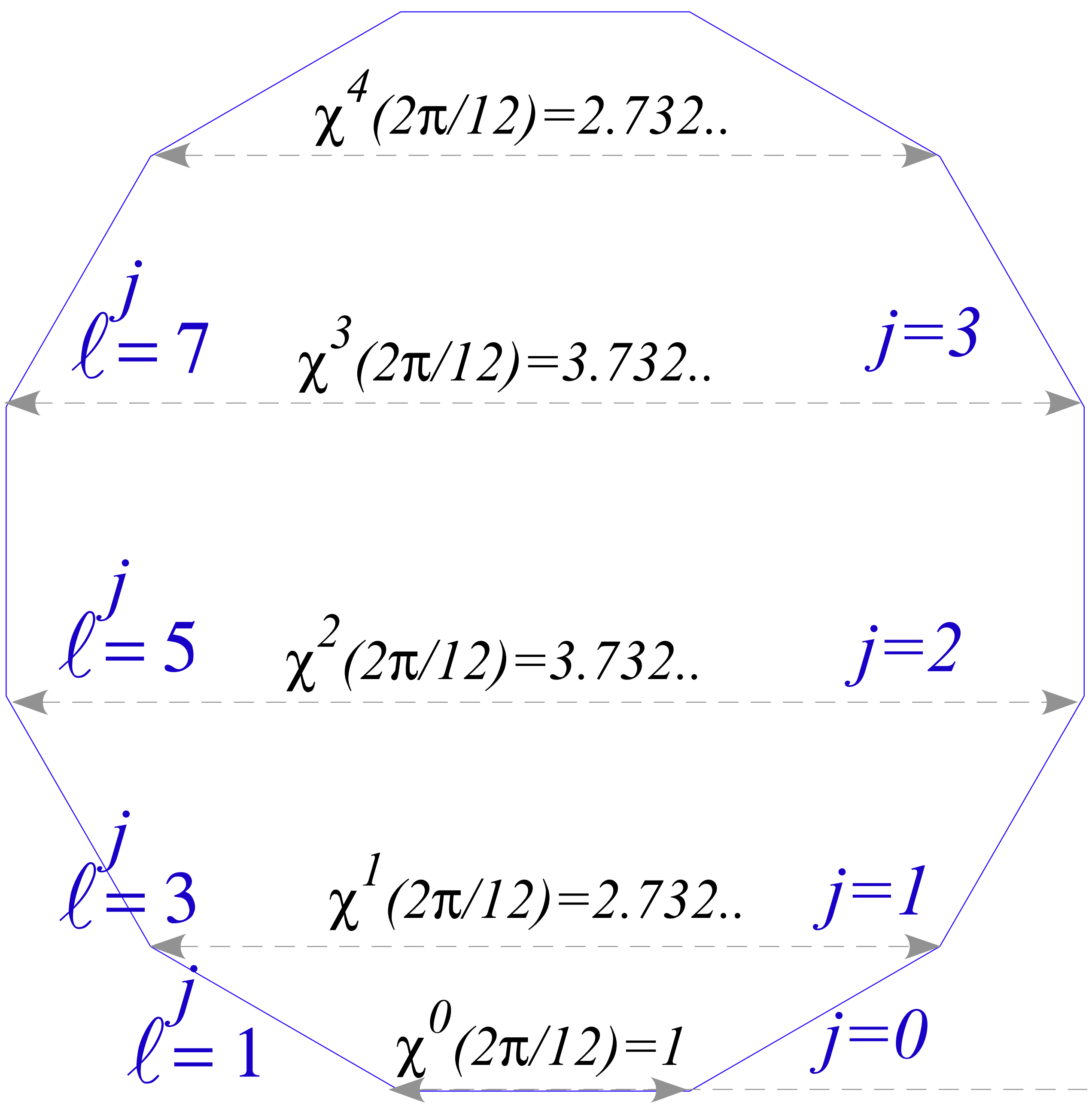
Integer j for n=12



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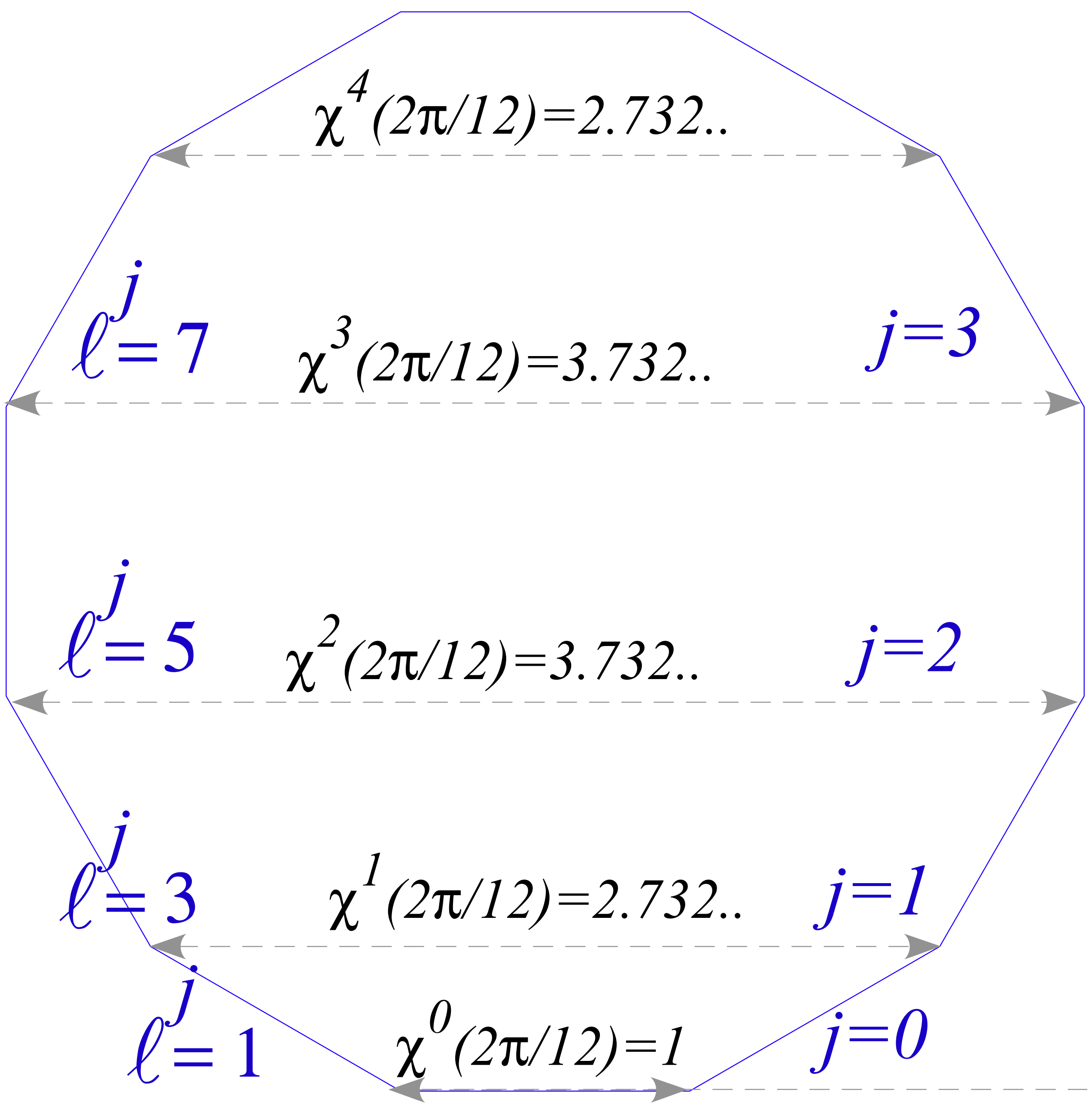
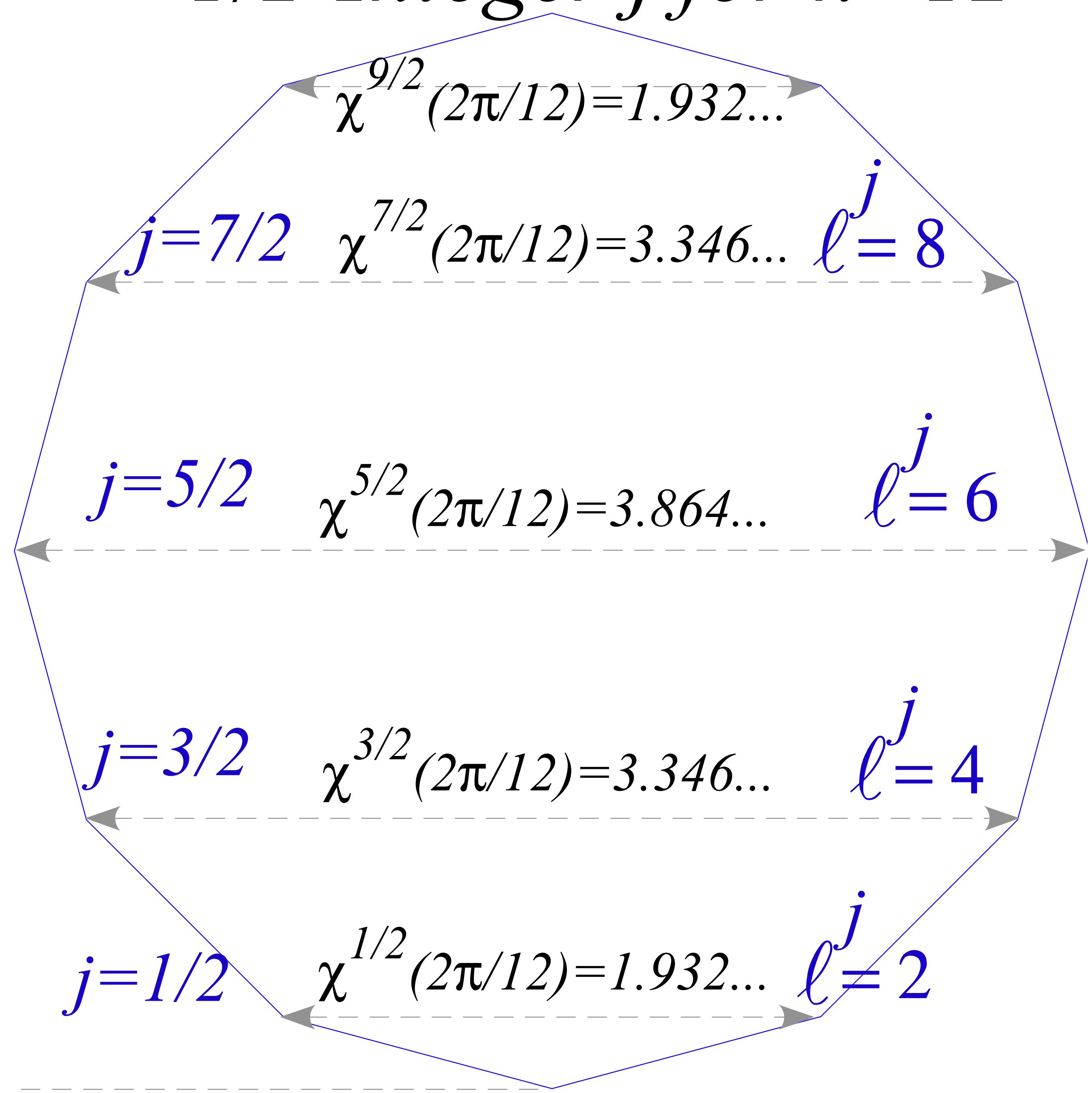
Character Spectral Function where: $\ell^j=2j+1$ is U(2) irrep dimension

Integer j for n=12



1/2-Integer j for n=12



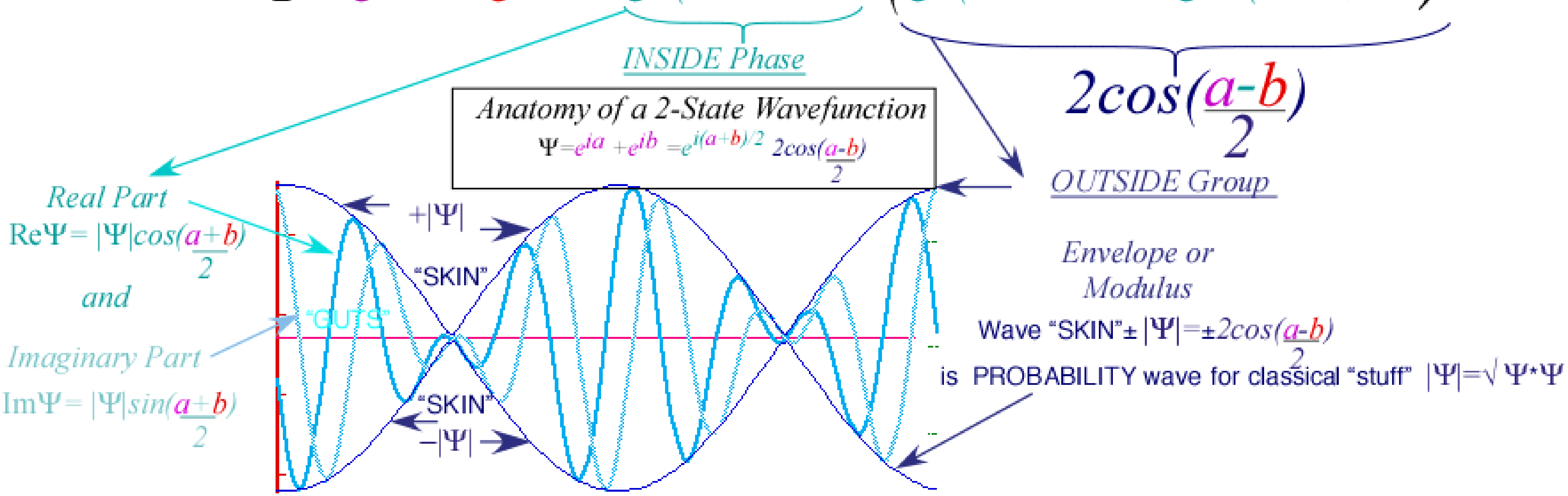
Polygonal geometry of U(2)⊃C_N character spectral function
Algebra
Geometry

Introduction to wave dynamics of phase, mean phase, and group velocity
Expo-Cosine identity
Relating space-time and per-space-time
- *Wave coordinates*
- *Pulse-waves (PW) vs Continuous -waves (CW)*

Introduction to C_N beat dynamics and "Revivals"
- *Farey-Sums and Ford-products*
- *Phase dynamics*

Interfering Plane Waves: The Expo-Cosine Identity

$$\Psi = e^{ia} + e^{ib} = e^{i(a+b)/2}\left(e^{i(a-b)/2} + e^{-i(a-b)/2}\right)$$



Fundamental wave dynamics based on Euler Expo-cosine Identity

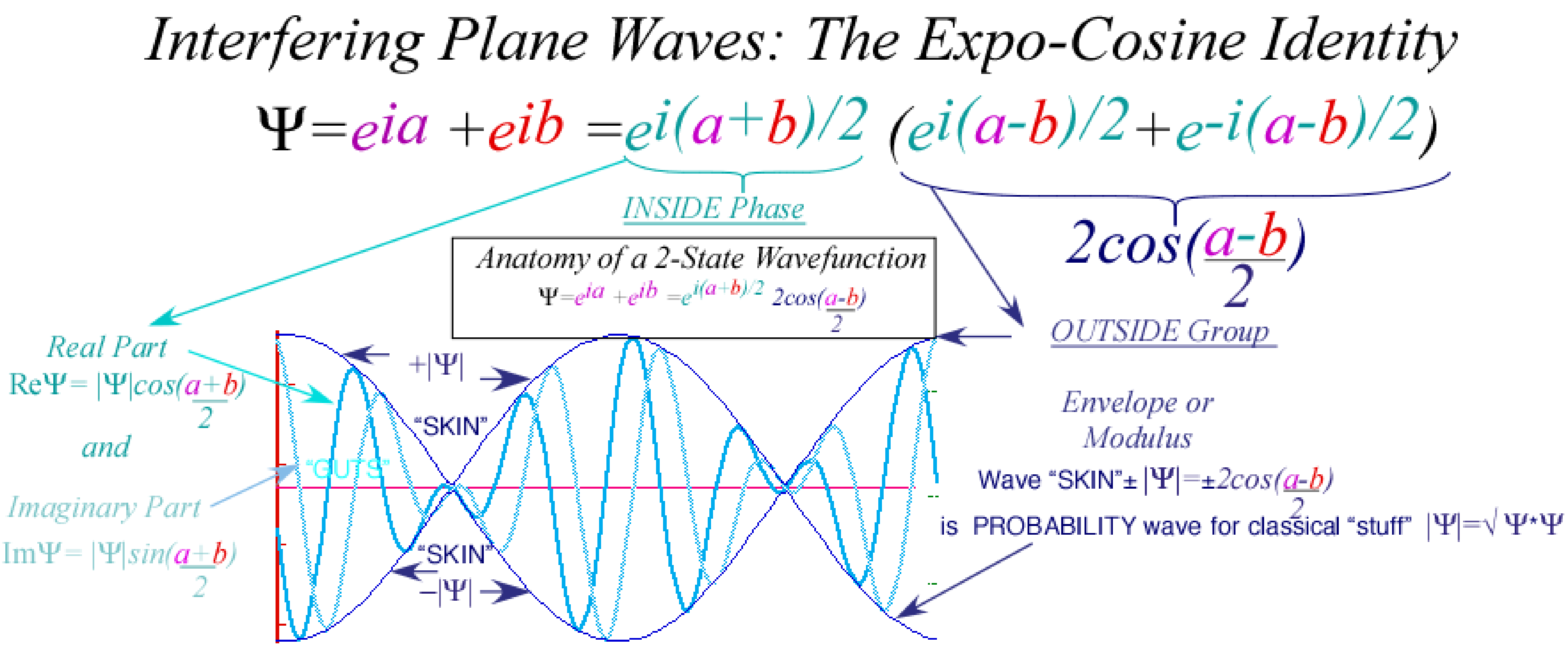
$$(e^{ia} + e^{ib})/2 = e^{i(a+b)/2}(e^{i(a-b)/2}+e^{-i(a-b)/2})/2 = e^{i(a+b)/2} \bullet \cos(a-b)/2$$

$a=k_1x-\omega_1t \qquad b= k_2x-\omega_2t$

Balanced (50-50) plane wave combination:

$$\Psi_{50_1\text{-}50_2}(x,t) = (1/2)\psi_{k_1}(x,t) + (1/2)\psi_{k_2}(x,t)$$

$$(1/2)e^{i(k_1x-\omega_1t)} + (1/2)e^{i(k_2x-\omega_2t)} = e^{i(k_px-\omega_pt)} \bullet \cos(k_gx-\omega_gt)$$



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$(e^{ia} + e^{ib})/2 = e^{i(a+b)/2}(e^{i(a-b)/2}+e^{-i(a-b)/2})/2 \;=\; e^{i(a+b)/2} \bullet \cos(a-b)/2$

$a=k_1x-\omega_1t \qquad b= k_2x-\omega_2t$

$\omega_p=(\omega_1+ \omega_2)/2 \qquad \omega_g=(\omega_1- \omega_2)/2$

$k_p=(k_1+ k_2)/2 \qquad k_g=(k_1- k_2)/2$

Overall or Mean phase — Relative or Group phase

Balanced (50-50) plane wave combination:

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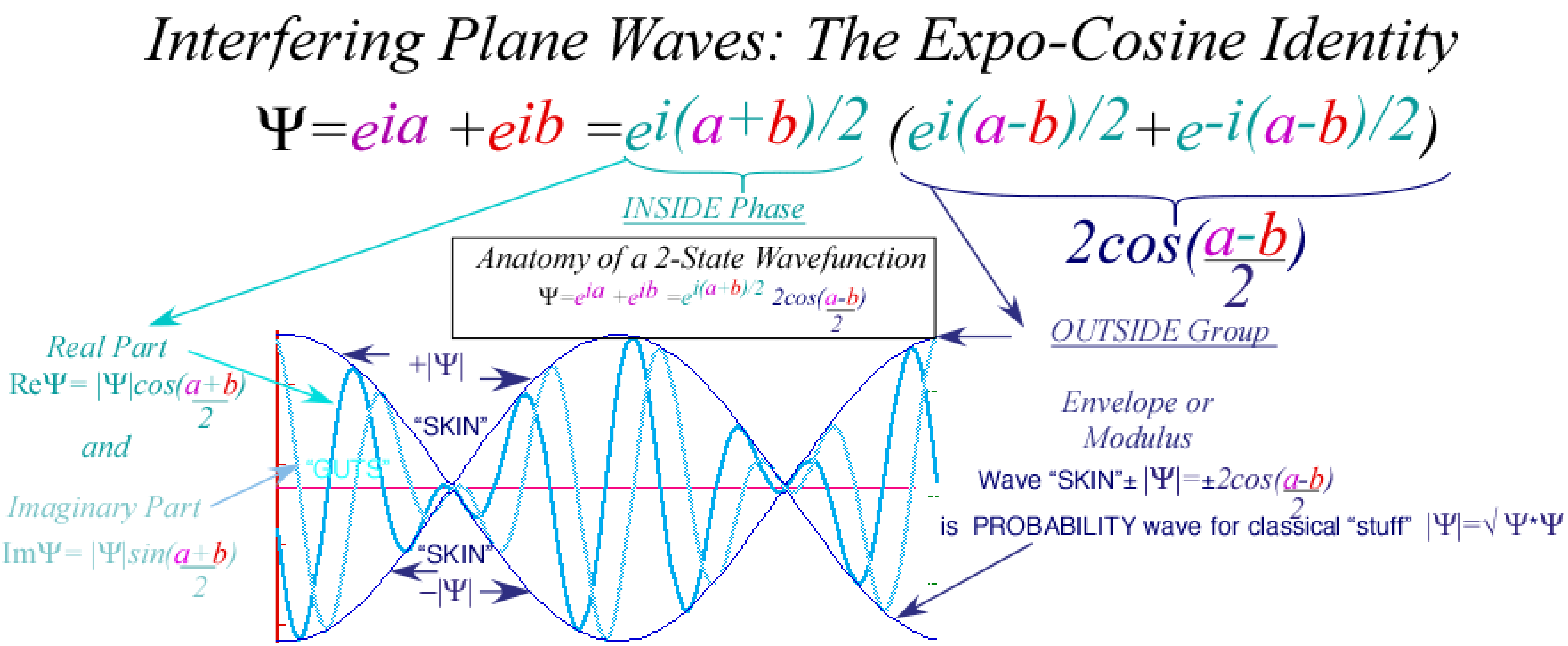

Fundamental wave dynamics based on Euler Expo-cosine Identity

$(e^{ia} + e^{ib})/2 = e^{i(a+b)/2}(e^{i(a-b)/2} + e^{-i(a-b)/2})/2 = 2e^{i(a+b)/2} \bullet \cos(a-b)/2$

$a = k_1x - \omega_1 t \qquad b = k_2 x - \omega_2 t$

$\omega_p = (\omega_1 + \omega_2)/2$

$k_p = (k_1 + k_2)/2$

Overall or Mean phase

$\omega_g = (\omega_1 - \omega_2)/2$

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1st plane phase velocity	*2nd plane phase velocity*	*Phase or Carrier velocity*	*Group or Envelope velocity*
$V_1 = \frac{\omega_1}{k_1}$	$V_2 = \frac{\omega_2}{k_2}$	$V_p = \frac{\omega_p}{k_p} = \frac{\omega_1+\omega_2}{k_1+k_2}$	$V_g = \frac{\omega_g}{k_g} = \frac{\omega_1-\omega_2}{k_1-k_2}$

Define **K***-vectors in per-spacetime*

$\mathbf{K}_1 = (\omega_1, k_1)$ $\quad \mathbf{K}_2 = (\omega_2, k_2)$ $\quad \mathbf{K}_{phase} = (\omega_p, k_p) = (\mathbf{K}_1 + \mathbf{K}_2)/2$ $\quad \mathbf{K}_{group} = (\omega_g, k_g) = (\mathbf{K}_1 - \mathbf{K}_2)/2$

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1[st] plane phase velocity	2[nd] plane phase velocity	Phase or Carrier velocity	Group or Envelope velocity
$V_1=\frac{\omega_1}{k_1}$	$V_2=\frac{\omega_2}{k_2}$	$V_p=\frac{\omega_p}{k_p}=\frac{\omega_1+\omega_2}{k_1+k_2}$	$V_g=\frac{\omega_g}{k_g}=\frac{\omega_1-\omega_2}{k_1-k_2}$

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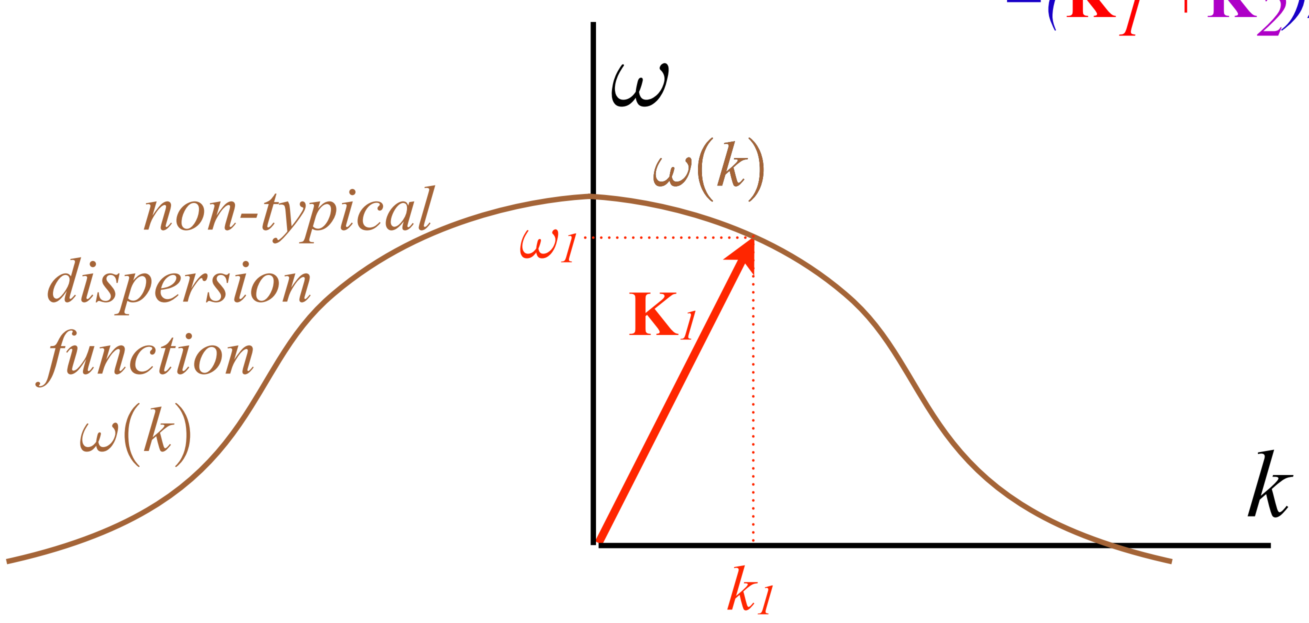

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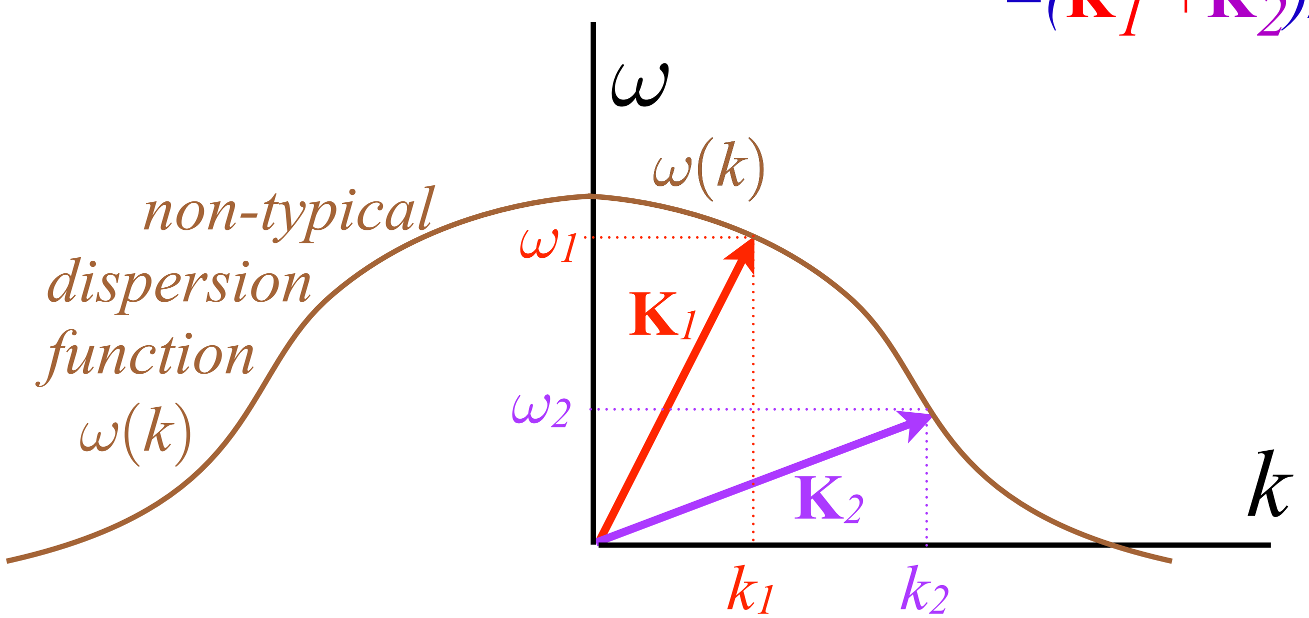

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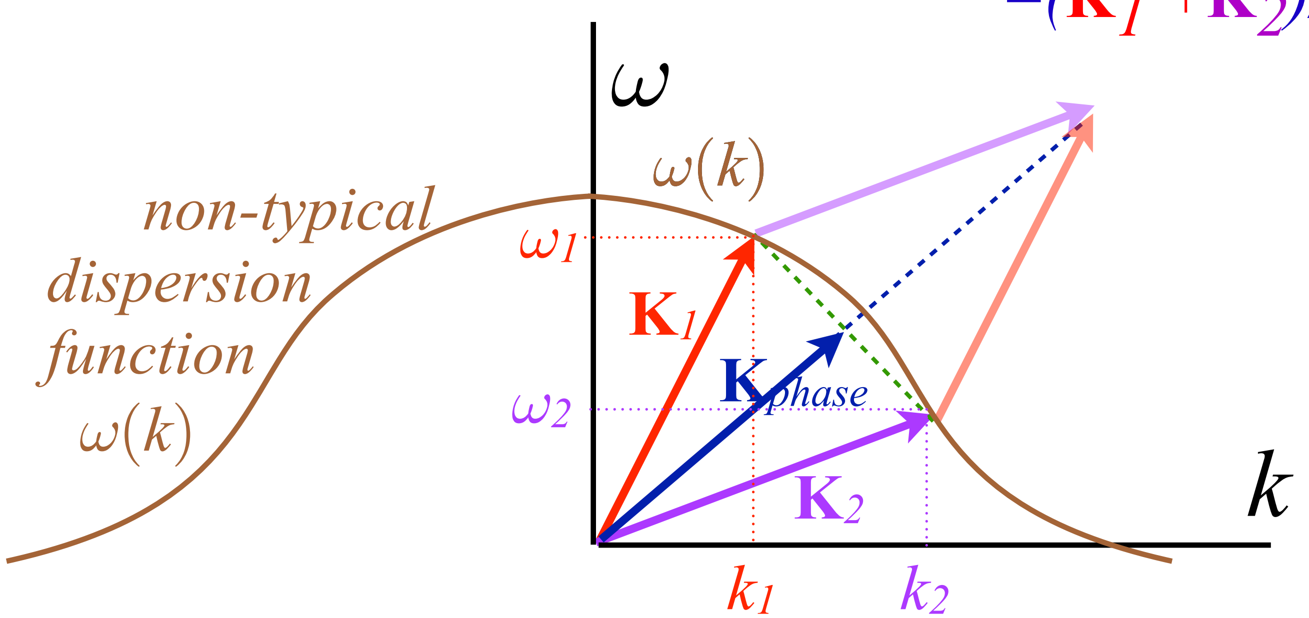

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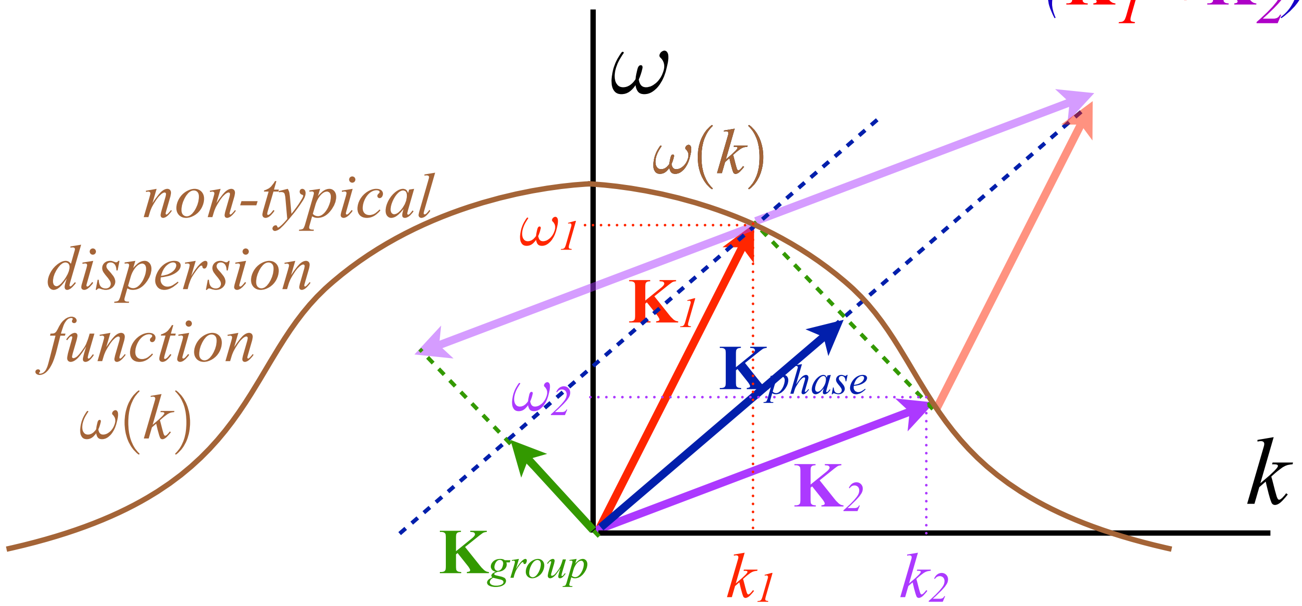

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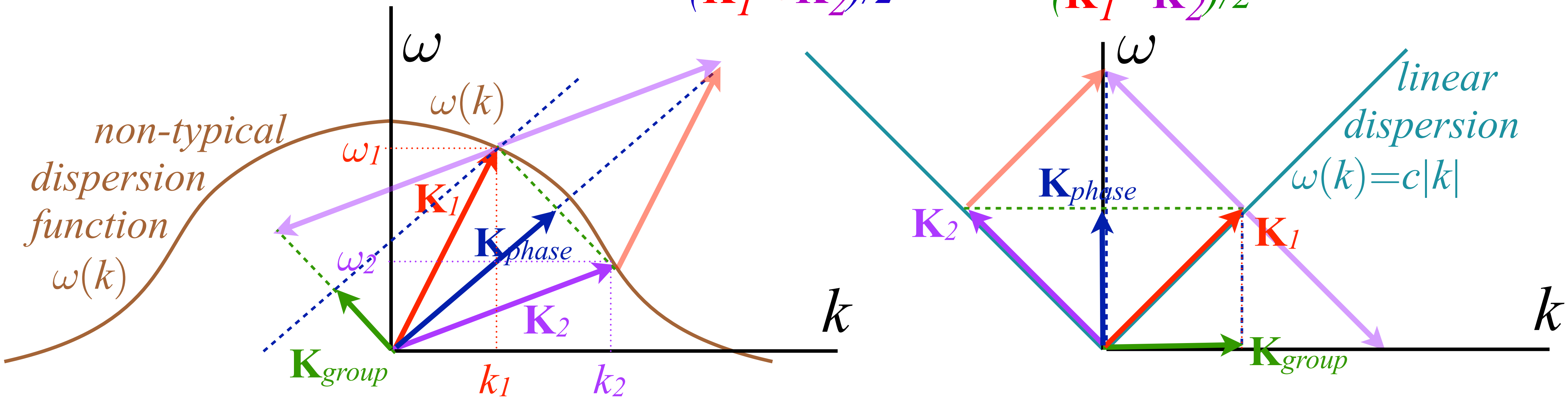
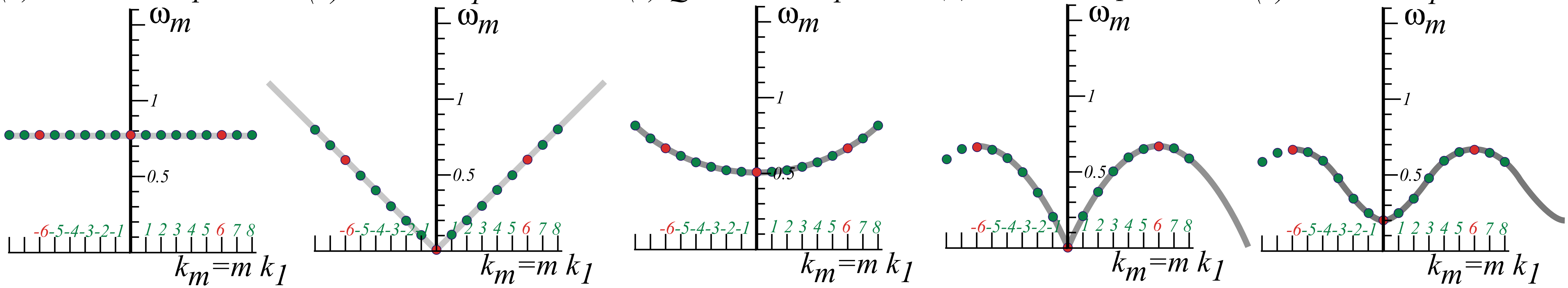


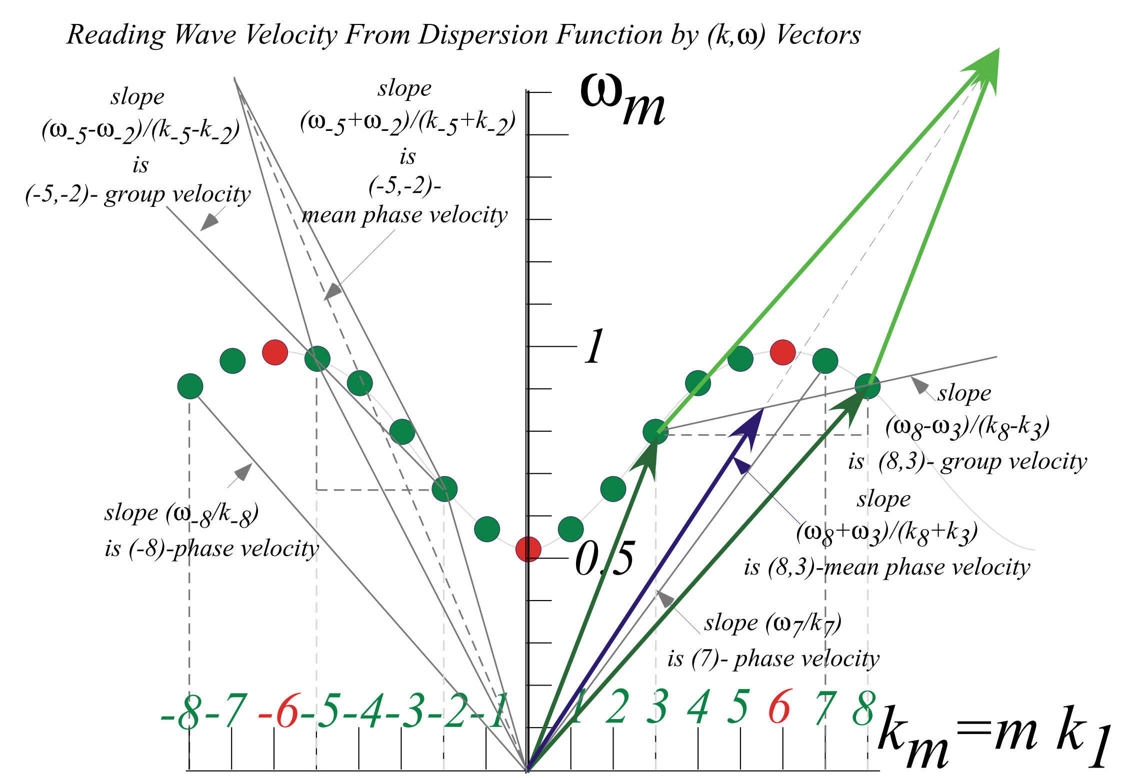
Archetypical Examples of Dispersion Functions

(a) Constant dispersion (b) Linear dispersion (c) Quadratic dispersion (d) Phonon dispersion (e) Exciton dispersion

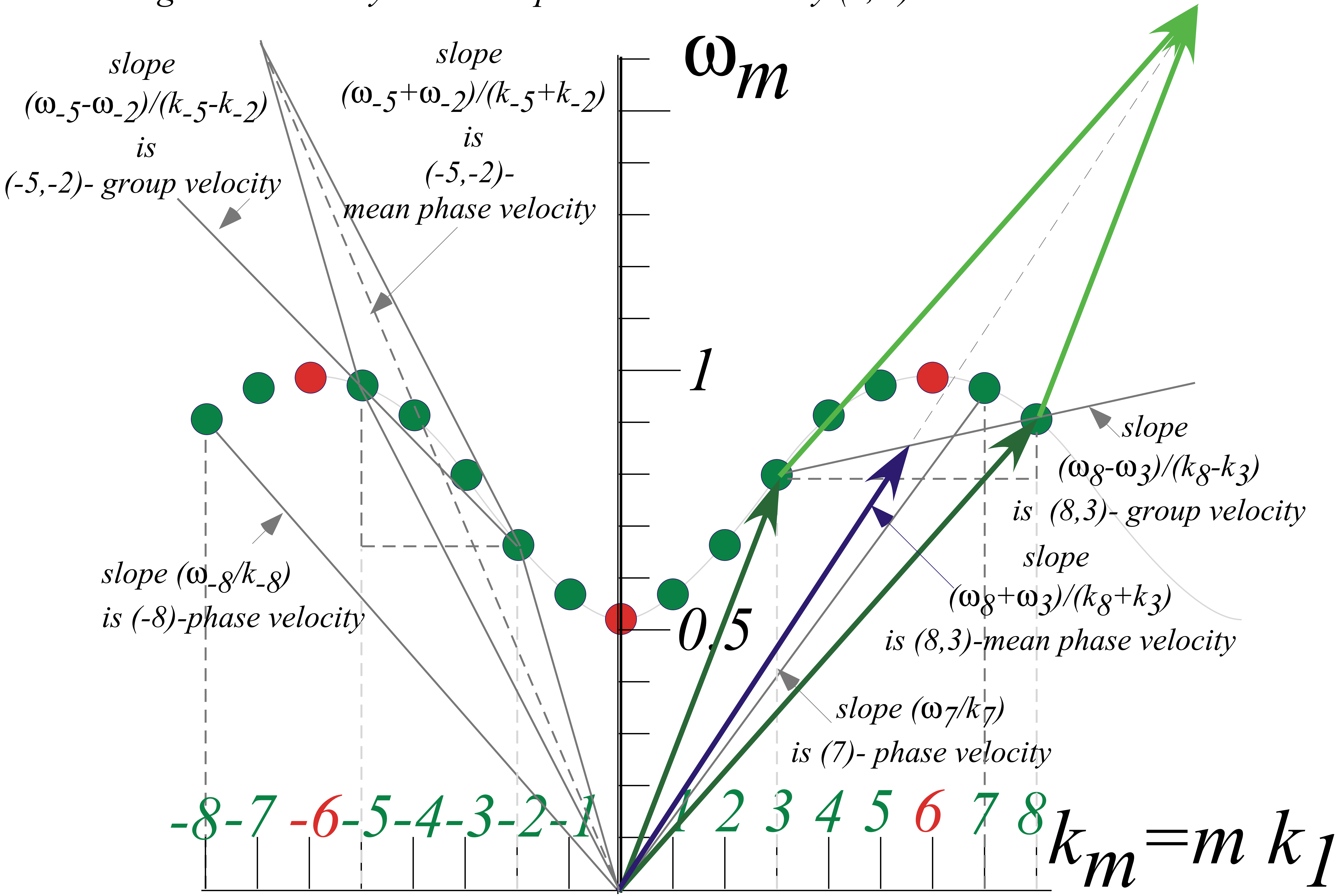


Applications:

(a)	(b)	(c)	(d)	(e)
Uncoupled pendulums	Weakly coupled pendulums (No gravity)	Weakly coupled pendulums (With gravity)	Strongly coupled pendulums (No gravity)	Strongly coupled pendulums (With gravity)
Movie marquis Xmas lights	Light in vacuum (Exactly) Sound (Approximately)	Light in fiber (Approx) Non-relativistic Schrodinger matter wave	Acoustic mode in solids	Optical mode in solids Relativistic matter (If exact hyperbola)



Reading Wave Velocity From Dispersion Function by (k,ω) Vectors



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Relating

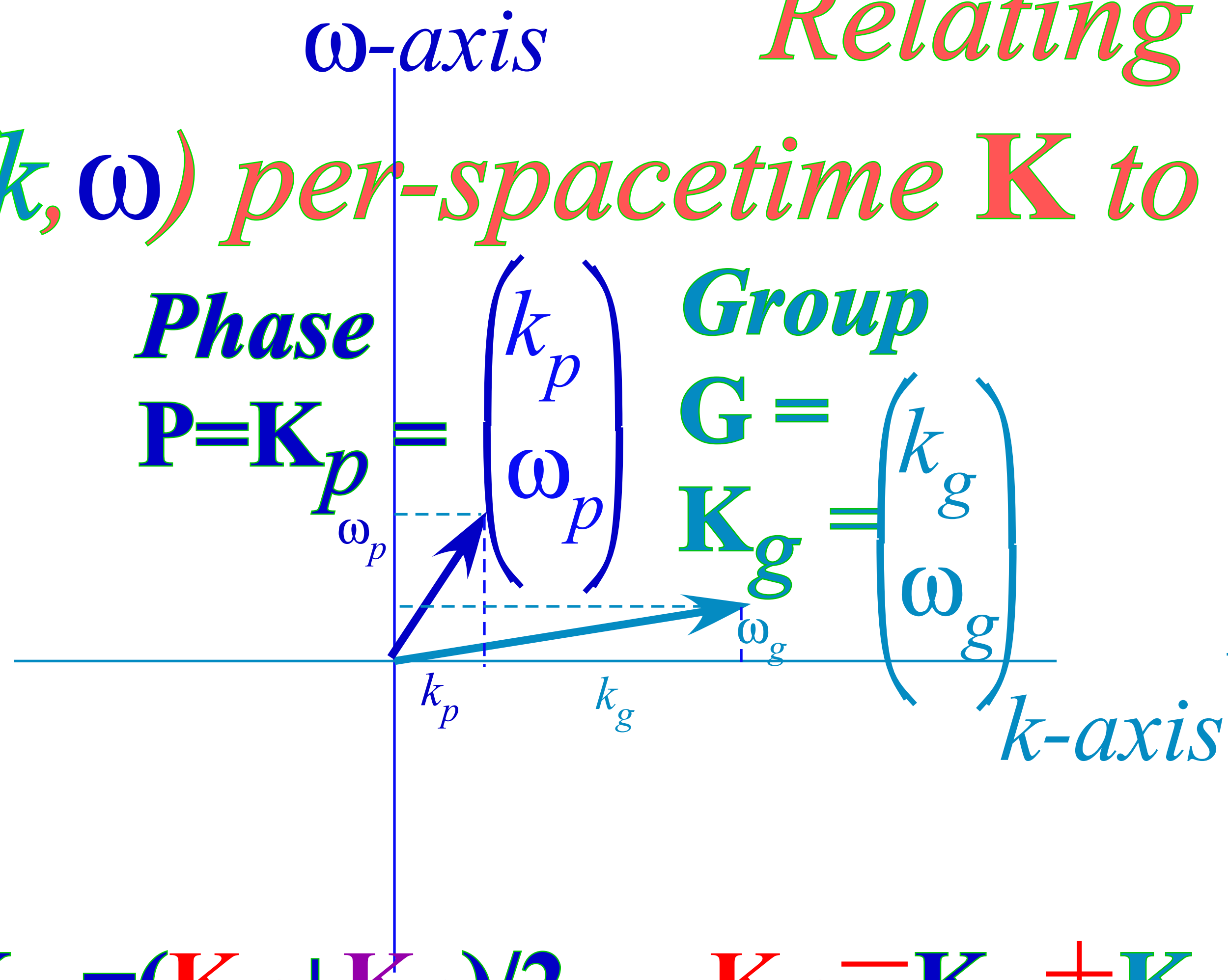
(k,ω) *per-spacetime* **K** *to* (x,t) *spacetime*

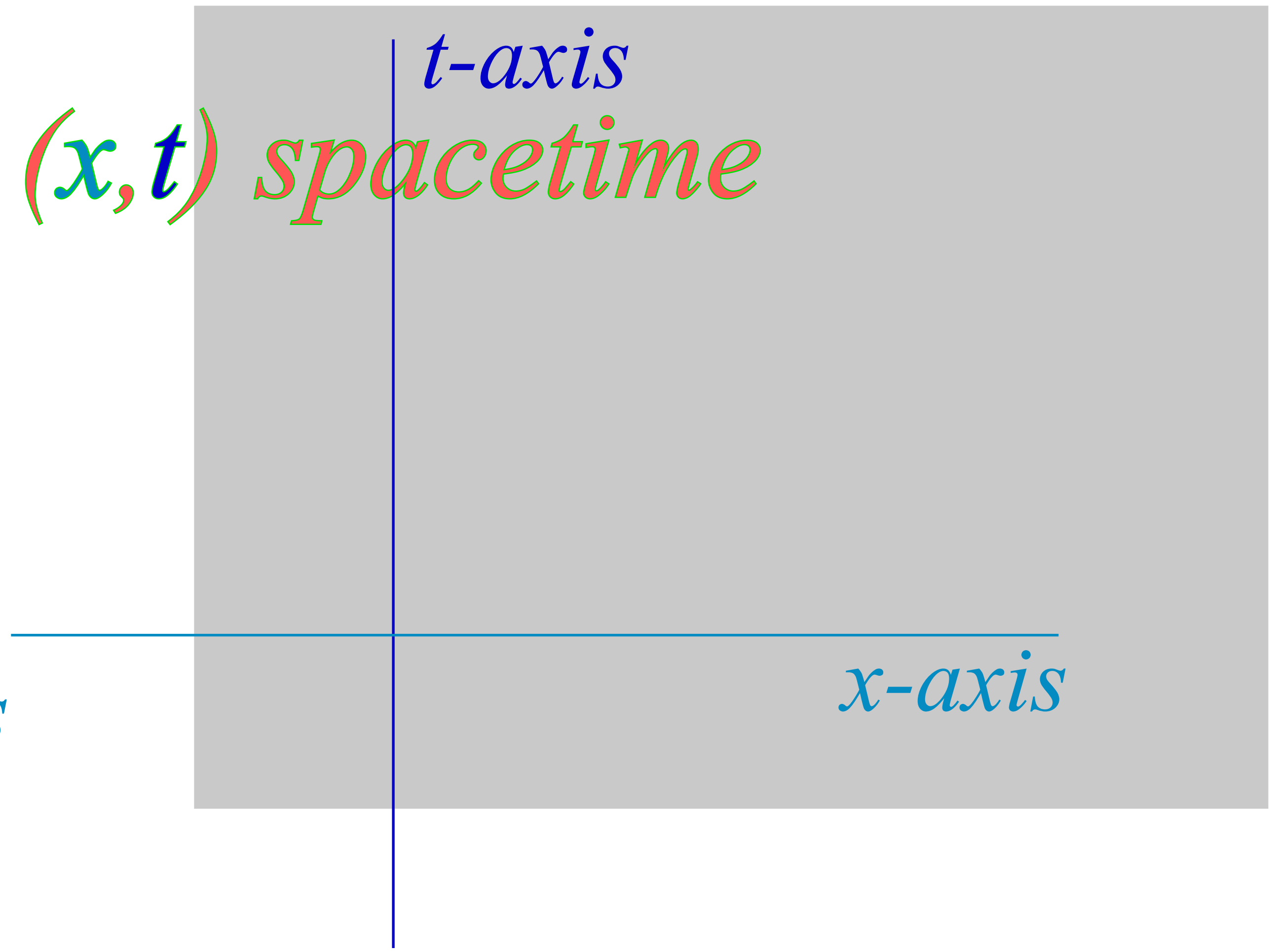
Phase

$\mathbf{P}=\mathbf{K}_p=\begin{pmatrix} k_p \\ \omega_p \end{pmatrix}$

Group

$\mathbf{G}=\mathbf{K}_g=\begin{pmatrix} k_g \\ \omega_g \end{pmatrix}$





$\mathbf{K}_p=(\mathbf{K}_1+\mathbf{K}_2)/2 \qquad \mathbf{K}_1=\mathbf{K}_p+\mathbf{K}_g$

$\mathbf{K}_g=(\mathbf{K}_1-\mathbf{K}_2)/2 \qquad \mathbf{K}_2=\mathbf{K}_p-\mathbf{K}_g$

$\omega_p=(\omega_1+\omega_2)/2$ $\qquad$ $\omega_g=(\omega_1-\omega_2)/2$

$k_p=(k_1+k_2)/2$ $\qquad$ $k_g=(k_1-k_2)/2$

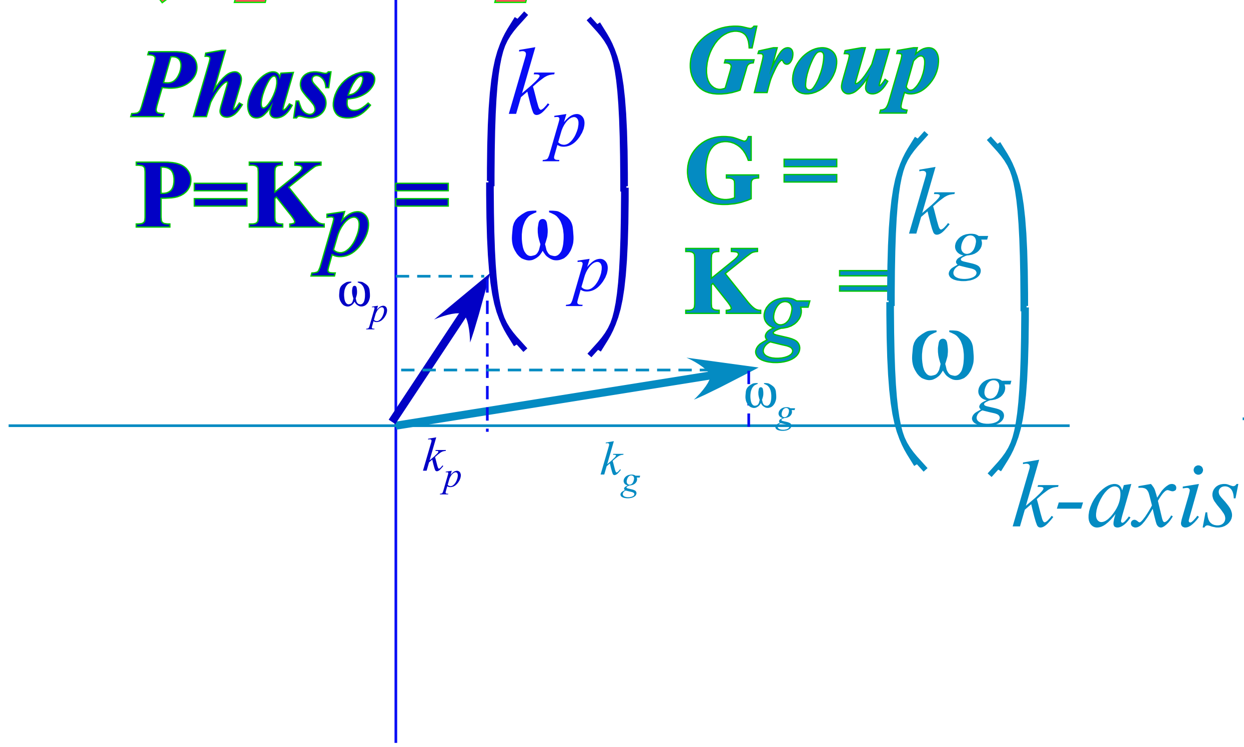
Overall or Mean phase $\qquad$ *Relative or Group phase*

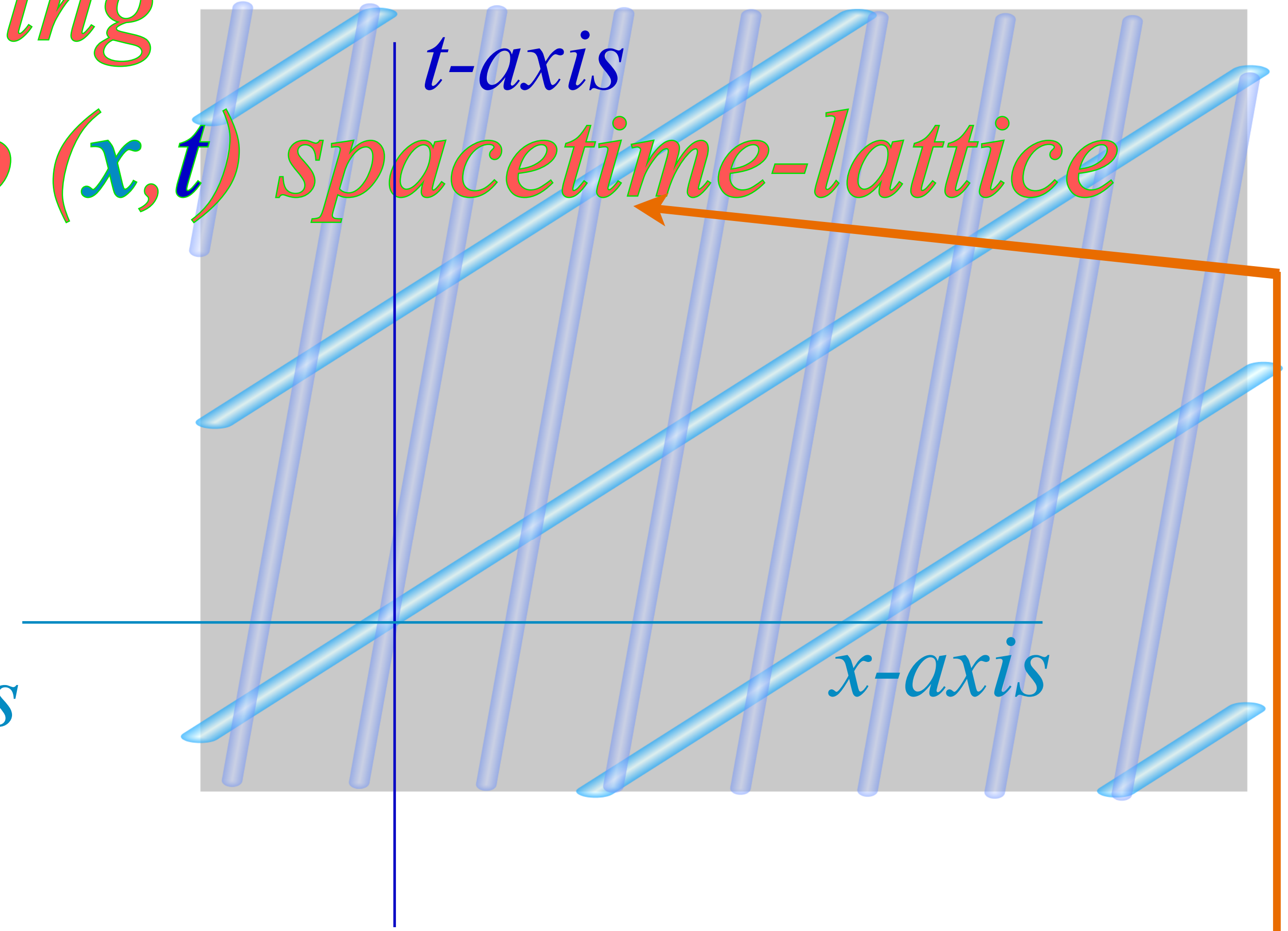
Find tracks in space-time of a balanced (50-50) plane wave combination:

$$\Psi_{50_1\text{-}50_2}(x,t)=1/2e^{i(k_1x-\omega_1t)}+1/2e^{i(k_2x-\omega_2t)}=e^{i(k_px-\omega_pt)}\bullet\cos(k_gx-\omega_gt)$$

Relating

(k,ω) per-spacetime **K** to (x,t) spacetime-lattice





$\omega_p=(\omega_1+\omega_2)/2$ $\quad$ $\omega_g=(\omega_1-\omega_2)/2$

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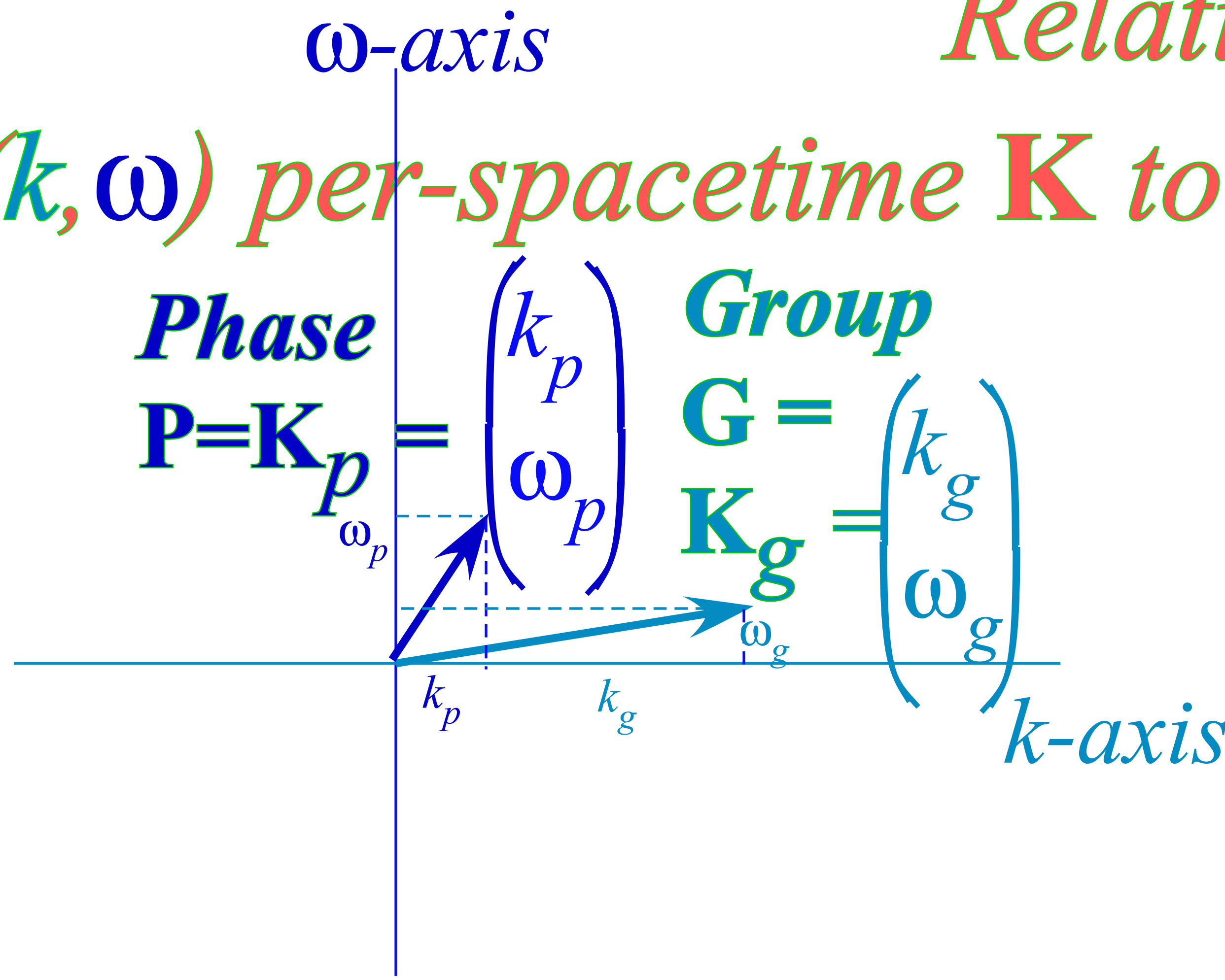
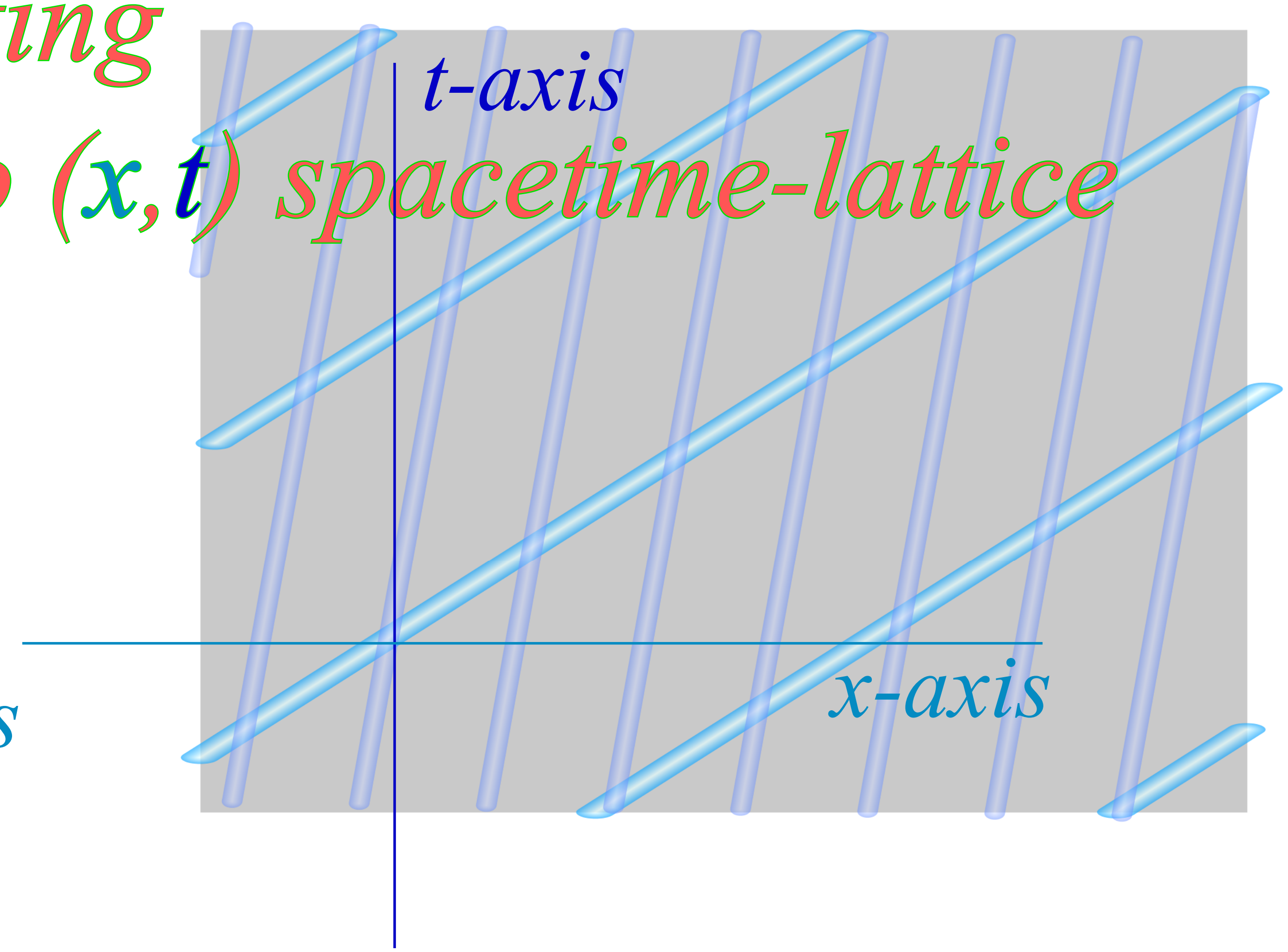
$$Re[\Psi_{50_1\text{-}50_2}(x,t)]==\cos(k_px-\omega_pt)\bullet\cos(k_gx-\omega_gt)=0$$

Real part has ZEROS that make: (x,t) *spacetime-lattice*

Relating
(k,ω) per-spacetime K to (x,t) spacetime-lattice





Find tracks in space-time of a
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$$\Psi_{50_1\text{-}50_2}(x,t) = \tfrac{1}{2}e^{i(k_1x-\omega_1t)} + \tfrac{1}{2}e^{i(k_2x-\omega_2t)} = e^{i(k_px-\omega_pt)} \bullet \cos(k_gx-\omega_gt)$$

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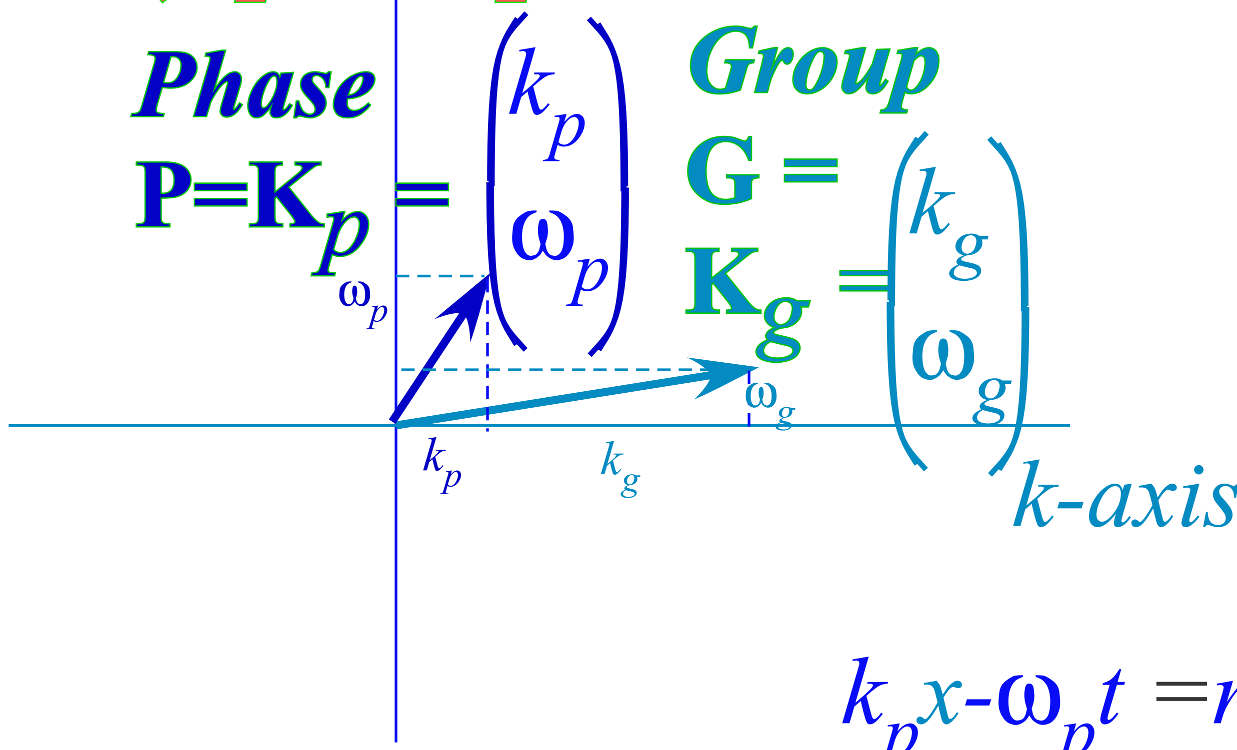
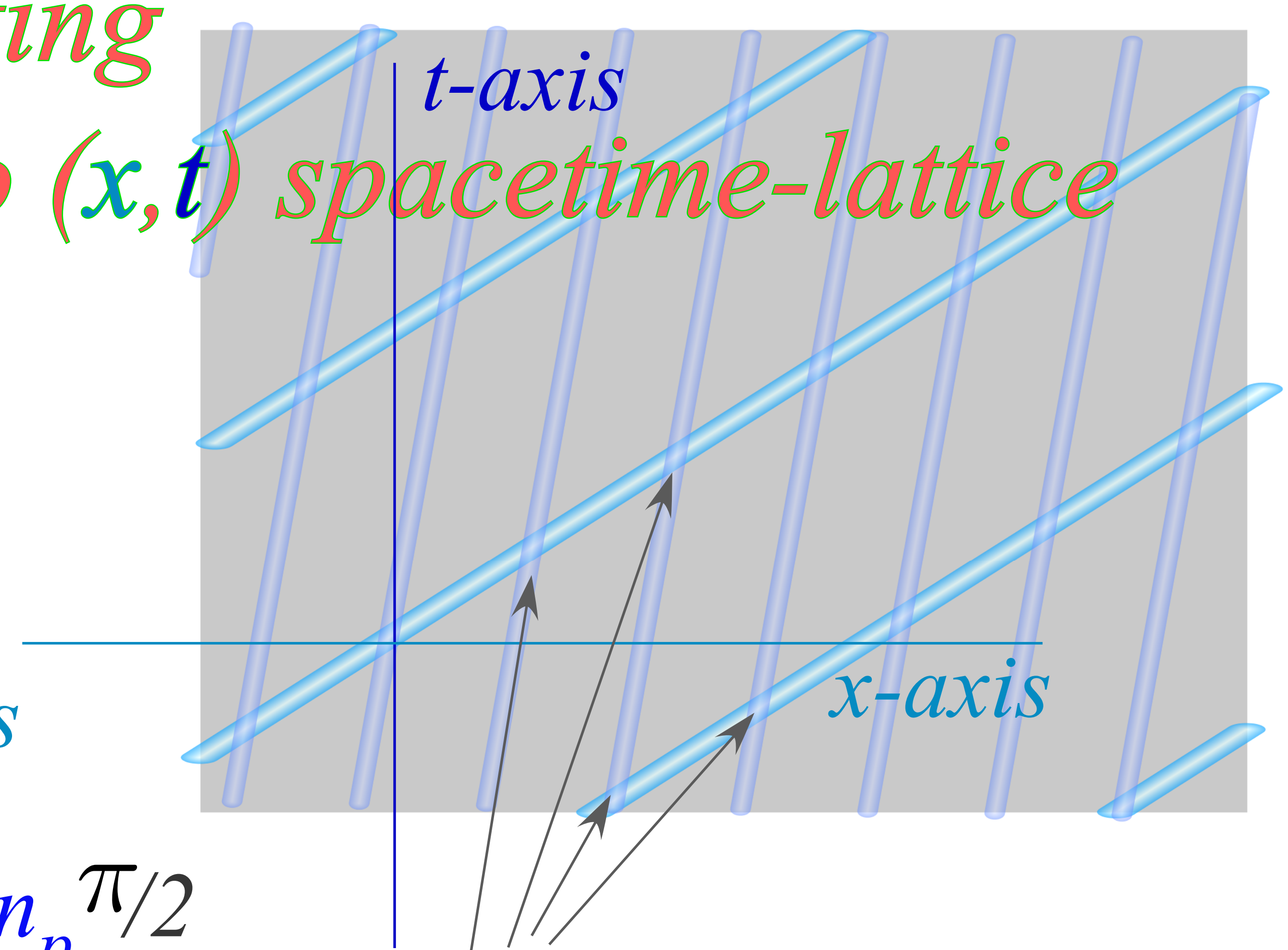
Real part has ZEROS that make: ***(x,t) CW spacetime-lattice***

Relating
(k,ω) per-spacetime **K** *to (x,t) spacetime-lattice*





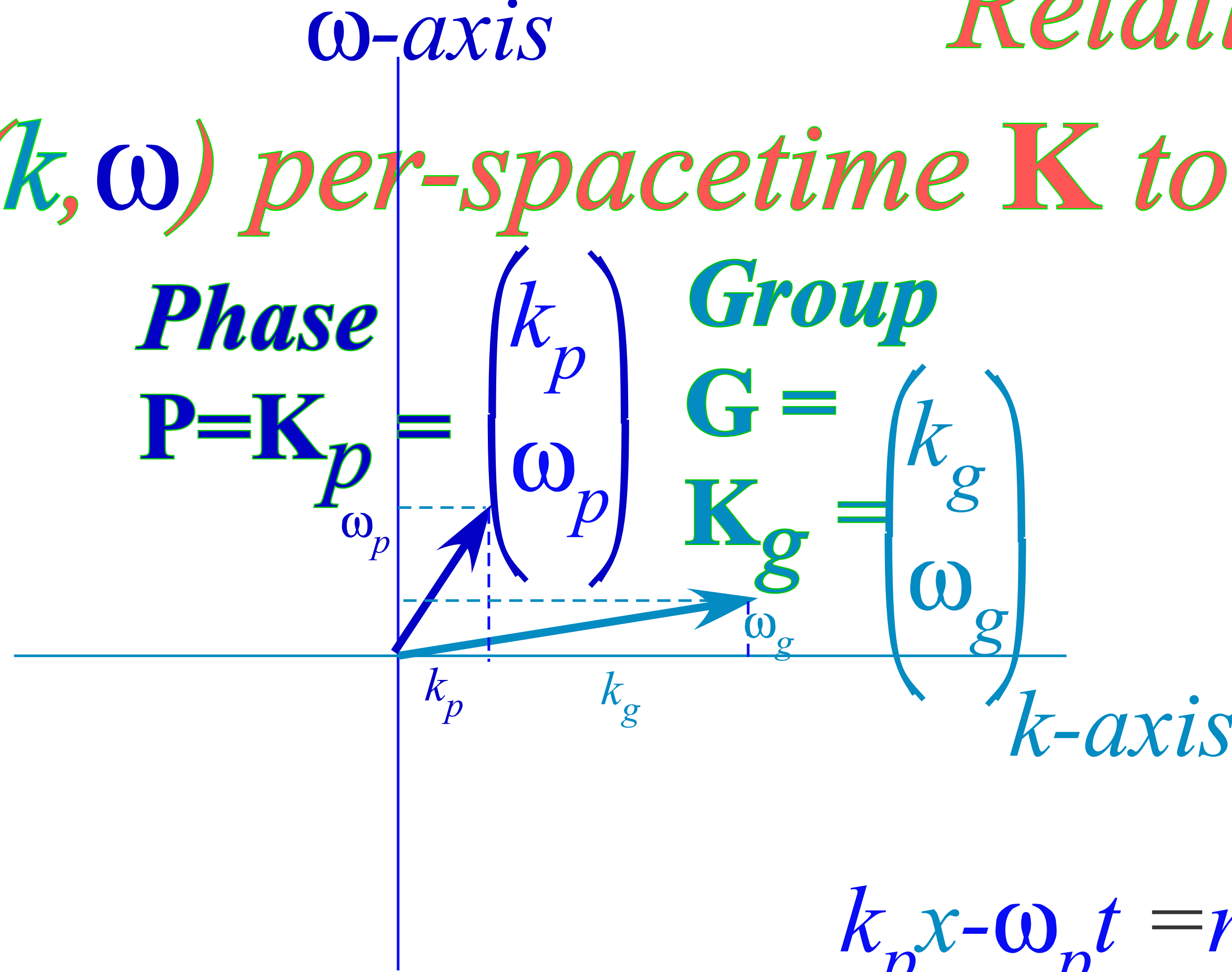
$$k_p x - \omega_p t = n_p \pi/2$$
$$k_g x - \omega_g t = n_g \pi/2$$

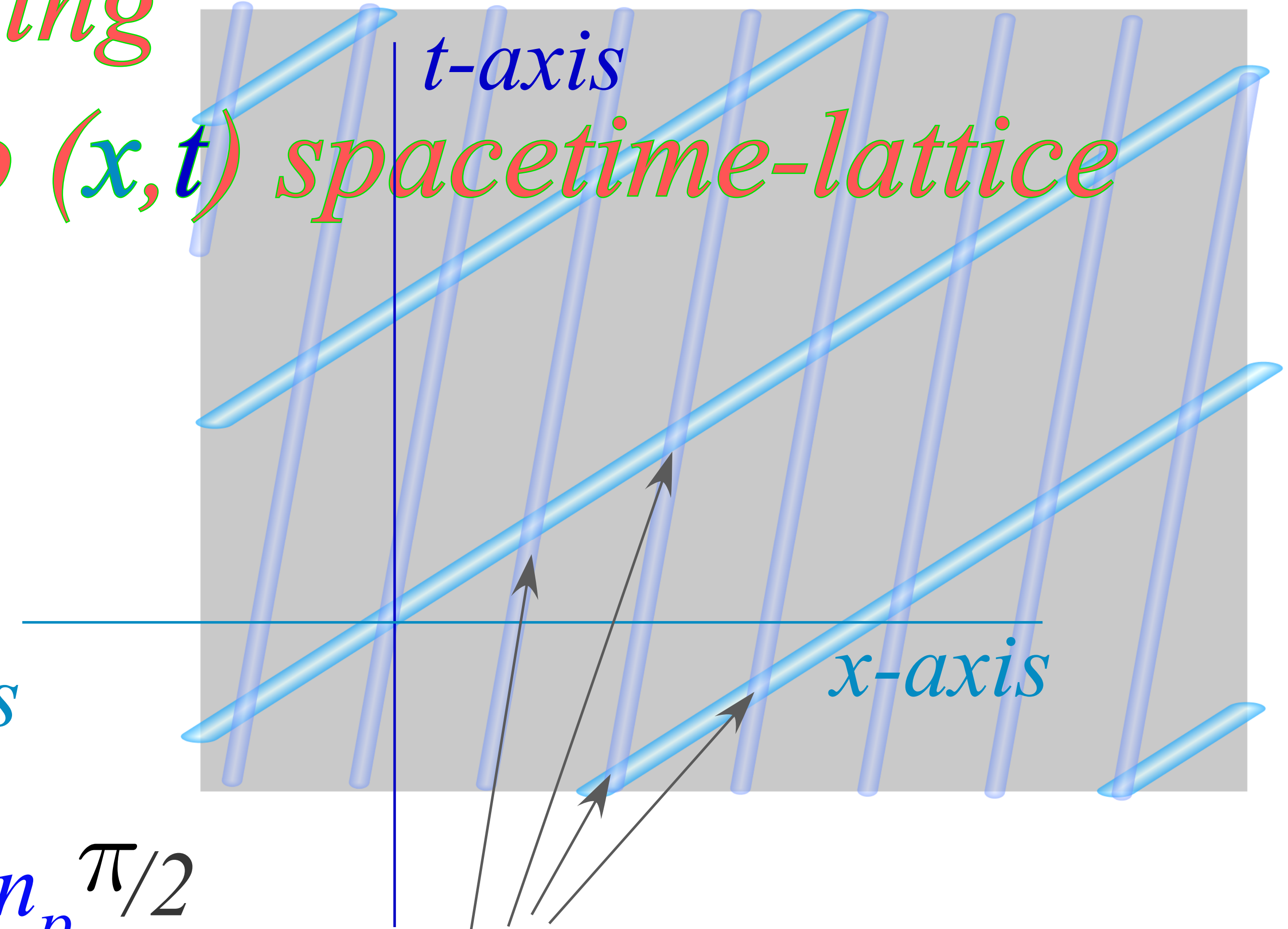
lattice point equations for:
$n_p = \pm 1, \pm 2 \ldots$ and $n_g = \pm 1, \pm 2 \ldots$

$Re[\Psi_{50_1\text{-}50_2}(x,t)] = \quad = \cos(k_p x - \omega_p t) \bullet \cos(k_g x - \omega_g t) = 0$

Real part has ZEROS that make: *(x,t) CW spacetime-lattice*

Relating *(k,ω) per-spacetime* **K** *to (x,t) spacetime-lattice*





lattice point equations for:
$n_p = \pm 1, \pm 2 \ldots$ and $n_g = \pm 1, \pm 2 \ldots$

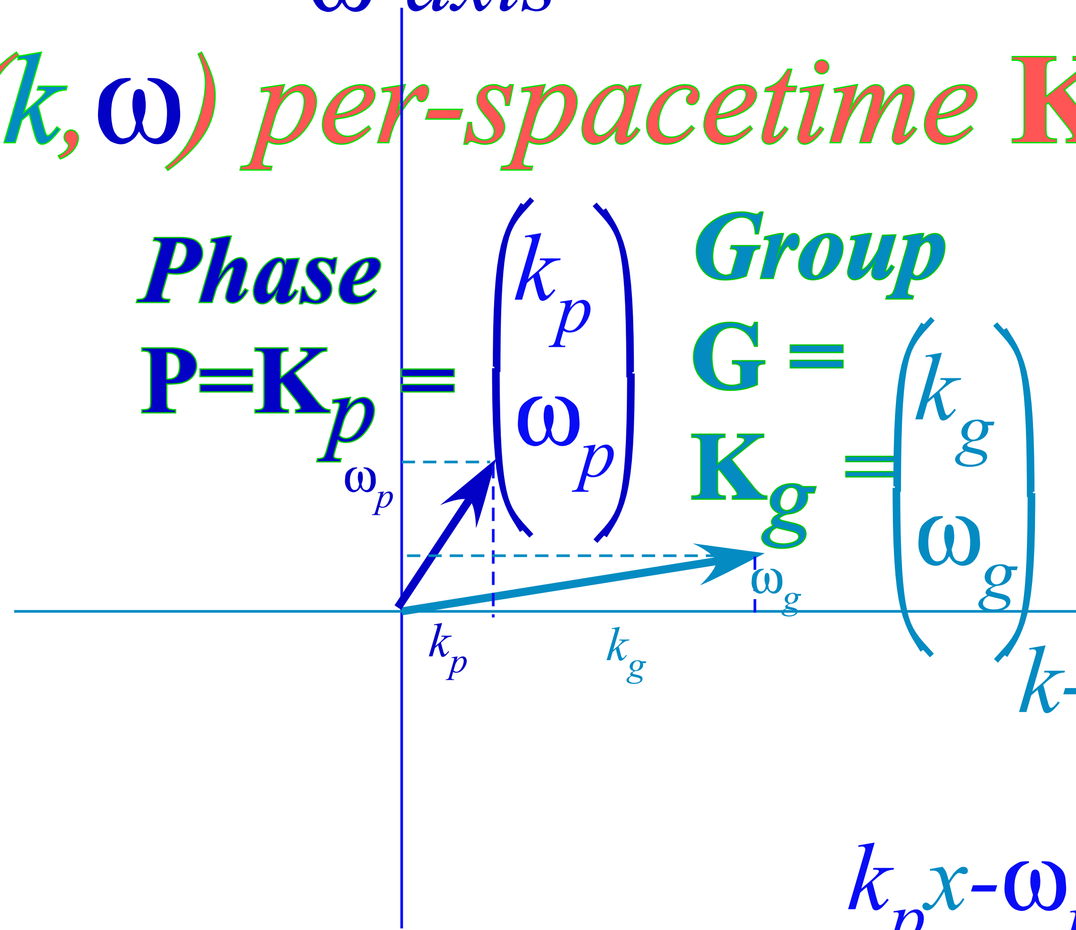
$$k_p x - \omega_p t = n_p \pi/2$$
$$k_g x - \omega_g t = n_g \pi/2$$

$$\begin{pmatrix} k_p & -\omega_p \\ k_g & -\omega_g \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} n_p \\ n_g \end{pmatrix} \pi/2$$

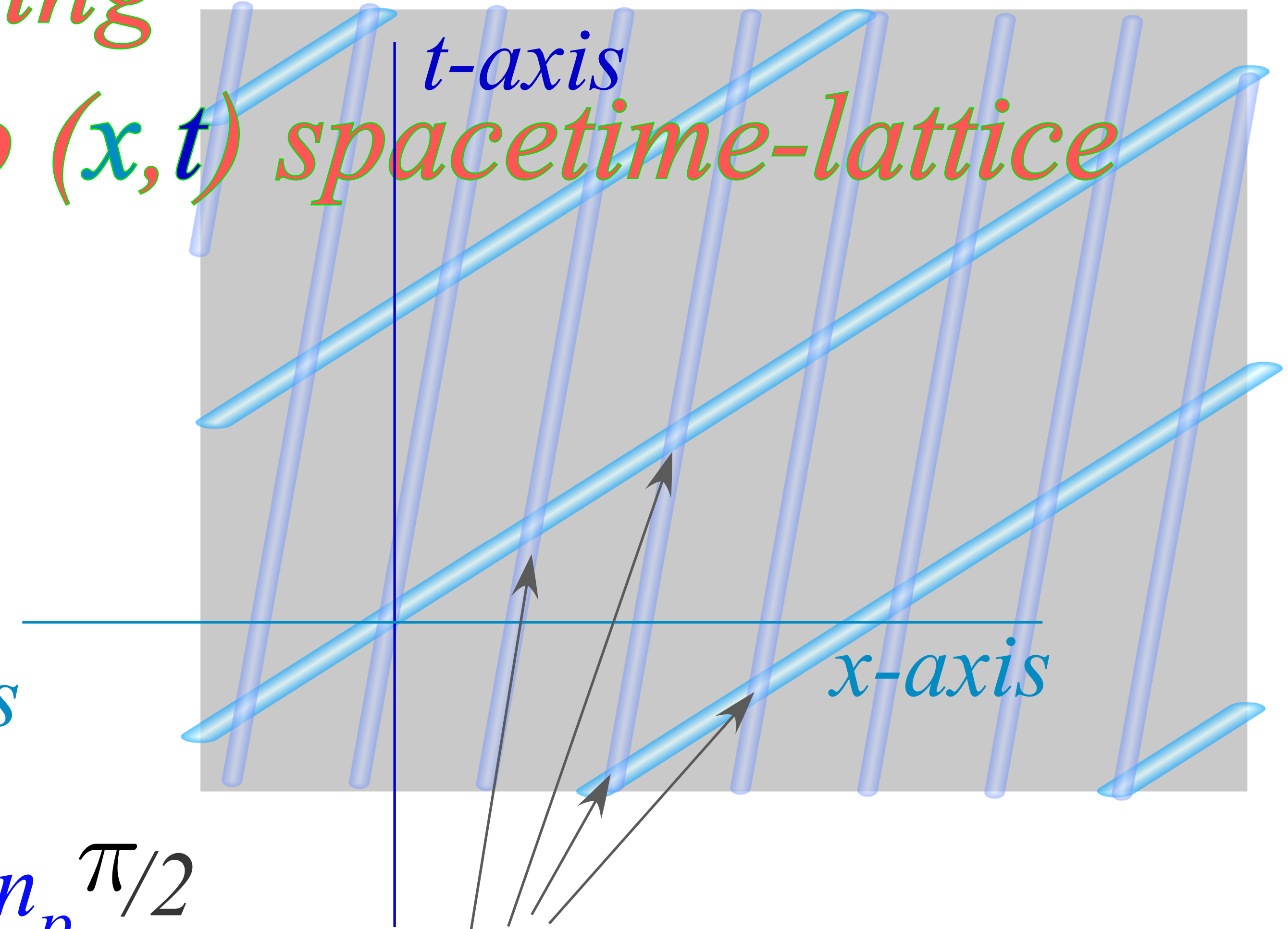
$Re[\Psi_{50_1 - 50_2}(x,t)] = \quad = \cos(k_p x - \omega_p t) \bullet \cos(k_g x - \omega_g t) = 0$

Real part has ZEROS that make: *(x,t) CW spacetime-lattice*

Relating
(k,ω) per-spacetime K to (x,t) spacetime-lattice



Phase $\mathbf{P}=\mathbf{K}_p=\begin{pmatrix} k_p \\ \omega_p \end{pmatrix}$ **Group** $\mathbf{G}=\mathbf{K}_g=\begin{pmatrix} k_g \\ \omega_g \end{pmatrix}$



$$k_p x-\omega_p t = n_p \pi/2$$
$$k_g x-\omega_g t = n_g \pi/2$$

lattice point equations for:
$n_p=\pm 1,\pm 2...$ and $n_g=\pm 1,\pm 2...$

$$\begin{pmatrix} k_p & -\omega_p \\ k_g & -\omega_g \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix}=\begin{pmatrix} n_p \\ n_g \end{pmatrix}\pi/2$$

inverted

$$\begin{pmatrix} x \\ t \end{pmatrix}=\frac{\begin{pmatrix} -\omega_g & \omega_p \\ -k_g & k_p \end{pmatrix}\begin{pmatrix} n_p \\ n_g \end{pmatrix}\pi/2}{det\left|\mathbf{K}_g \times \mathbf{K}_p\right|}=\frac{-n_p\begin{pmatrix} \omega_g \\ k_g \end{pmatrix}+n_g\begin{pmatrix} \omega_p \\ k_p \end{pmatrix}}{det\left|\mathbf{K}_g \times \mathbf{K}_p\right| 2/\pi}$$

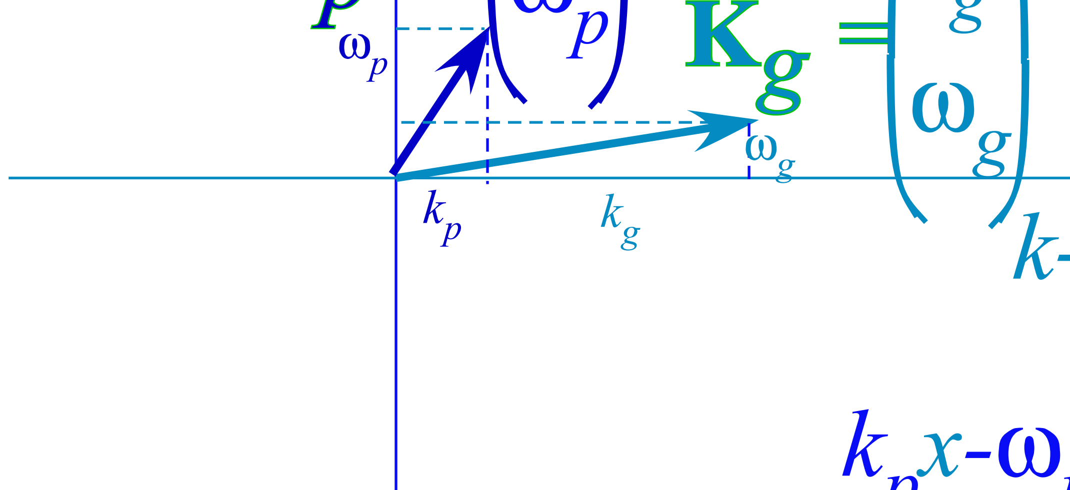
$Re[\Psi_{50_1-50_2}(x,t)]= \quad = \cos(k_p x-\omega_p t) \bullet \cos(k_g x-\omega_g t)=0$

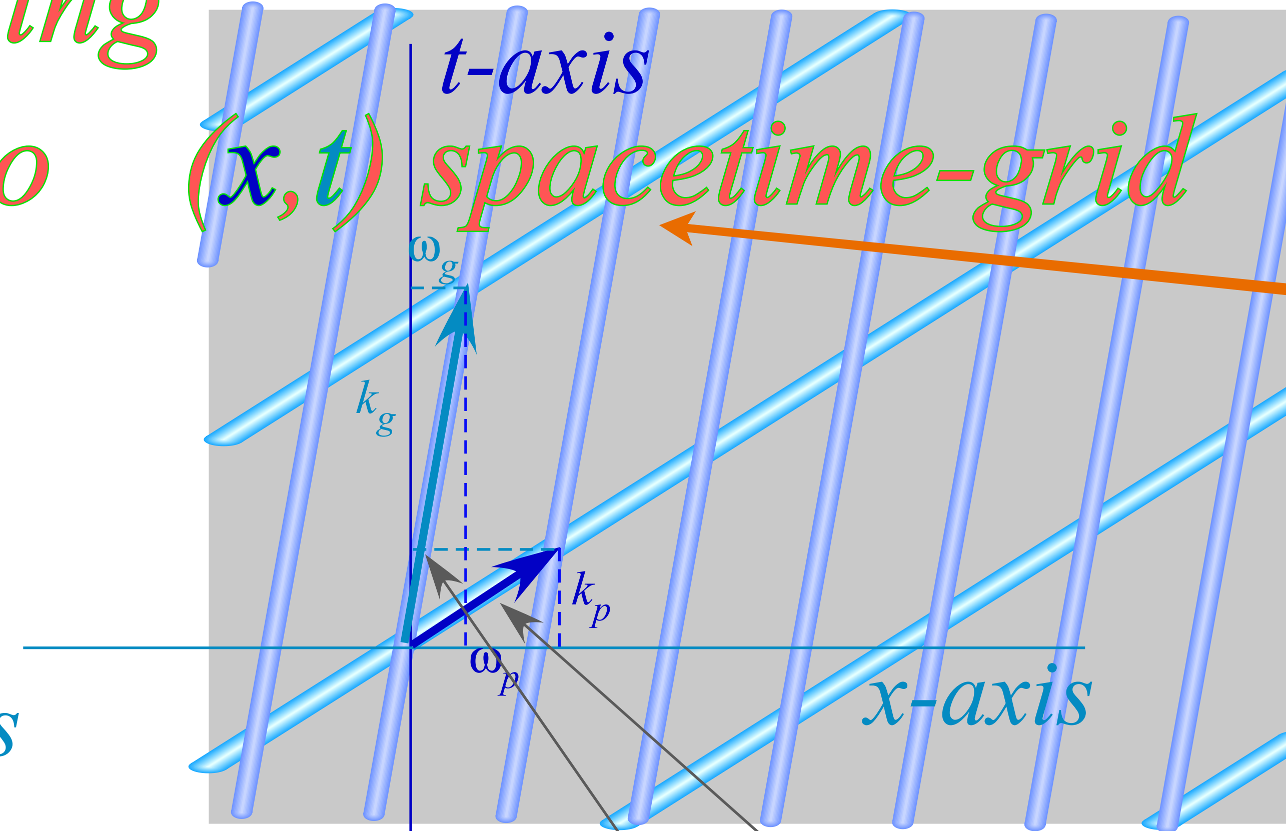
Real part has ZEROS that make: (x,t) CW spacetime-lattice

Relating (k,ω) *per-spacetime* **K** *to* (x,t) *spacetime-grid*

Phase $\mathbf{P}=\mathbf{K}_p = \begin{pmatrix} k_p \\ \omega_p \end{pmatrix}$

Group $\mathbf{G} = \mathbf{K}_g = \begin{pmatrix} k_g \\ \omega_g \end{pmatrix}$





$$k_p x - \omega_p t = n_p \pi/2$$
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inverted

$$\begin{pmatrix} x \\ t \end{pmatrix} = \frac{\begin{pmatrix} -\omega_g & \omega_p \\ -k_g & k_p \end{pmatrix}}{det\left|\mathbf{K}_g \times \mathbf{K}_p\right|}\begin{pmatrix} n_p \\ n_g \end{pmatrix}\pi/2 = \frac{-n_p\begin{pmatrix} \omega_g \\ k_g \end{pmatrix} + n_g\begin{pmatrix} \omega_p \\ k_p \end{pmatrix}}{det\left|\mathbf{K}_g \times \mathbf{K}_p\right| 2/\pi}$$

$Re[\Psi_{50_1\text{-}50_2}(x,t)] = \quad = \cos(k_p x-\omega_p t) \bullet \cos(k_g x-\omega_g t)=0$

Real part has ZEROS that make: (x,t) *CW spacetime-lattice*

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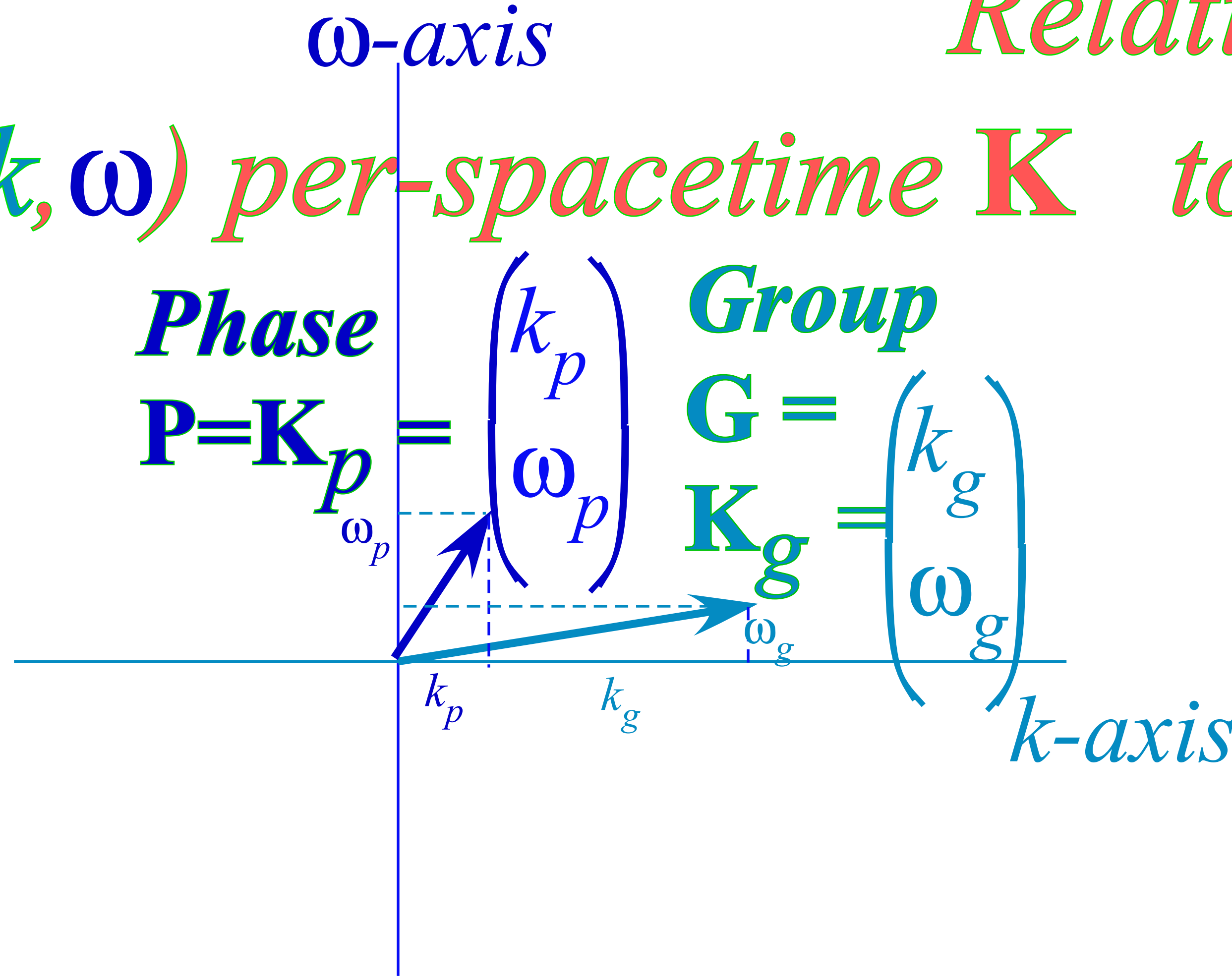
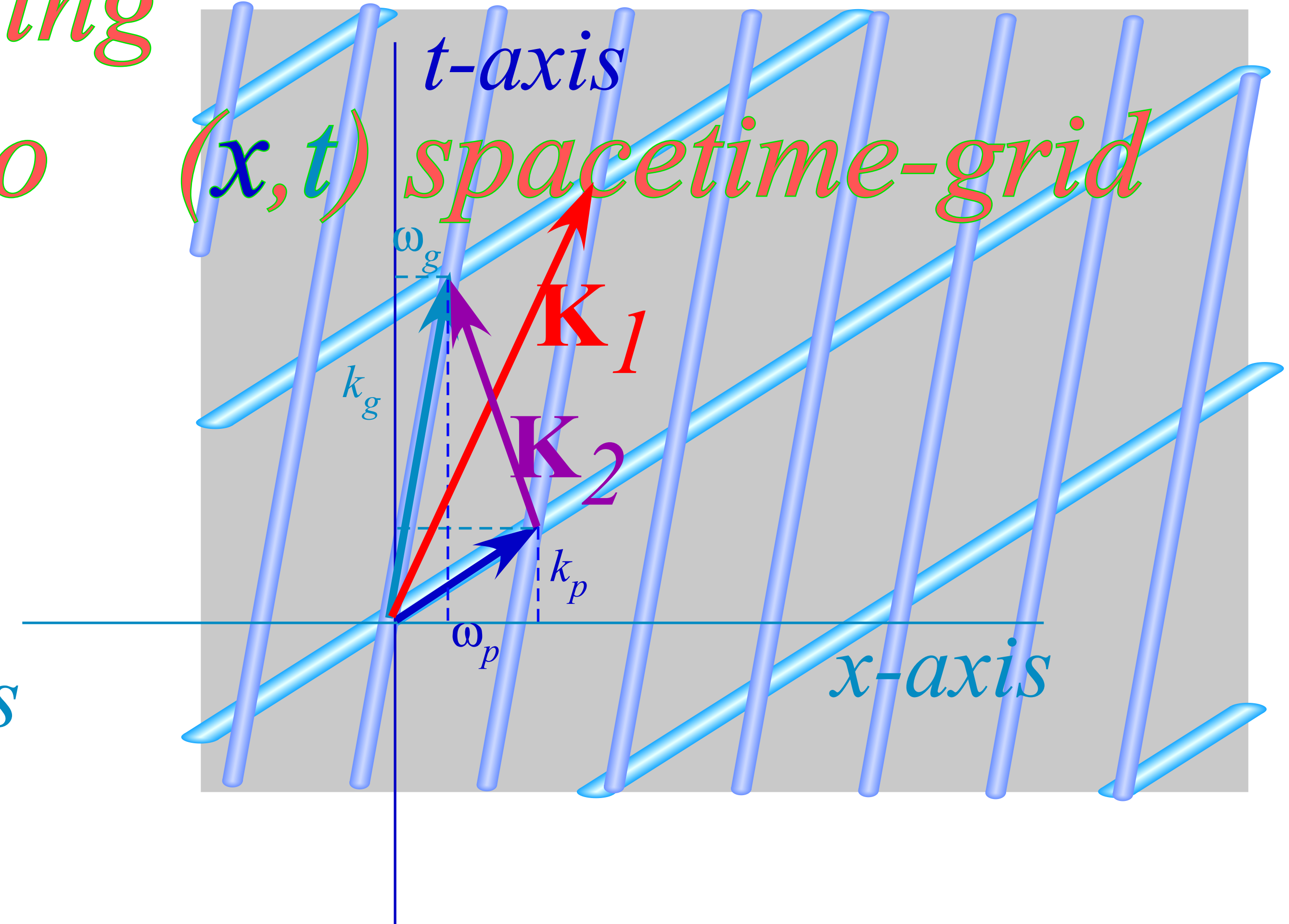
Relating (k,ω) per-spacetime **K** to (x,t) spacetime-grid





while primitive $\mathbf{K}_1$ *and* $\mathbf{K}_2$ *make:* *(x,t) PW spacetime-lattice*

$$\mathbf{K}_p = (\mathbf{K}_1 + \mathbf{K}_2)/2 \qquad \mathbf{K}_1 = \mathbf{K}_p + \mathbf{K}_g$$

$$\mathbf{K}_g = (\mathbf{K}_1 - \mathbf{K}_2)/2 \qquad \mathbf{K}_2 = \mathbf{K}_p - \mathbf{K}_g$$

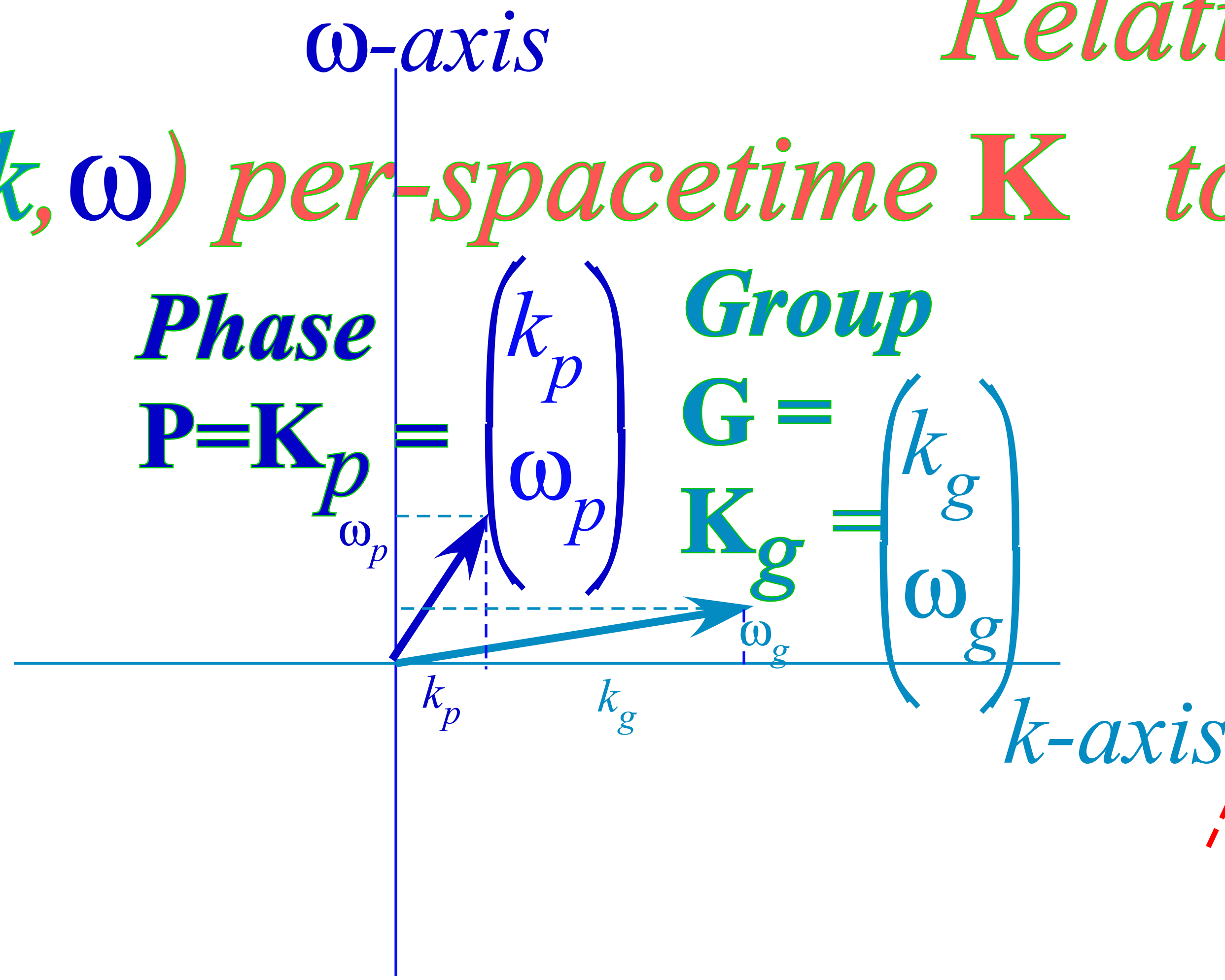
Find tracks in space-time of a
balanced (50-50) plane wave combination:

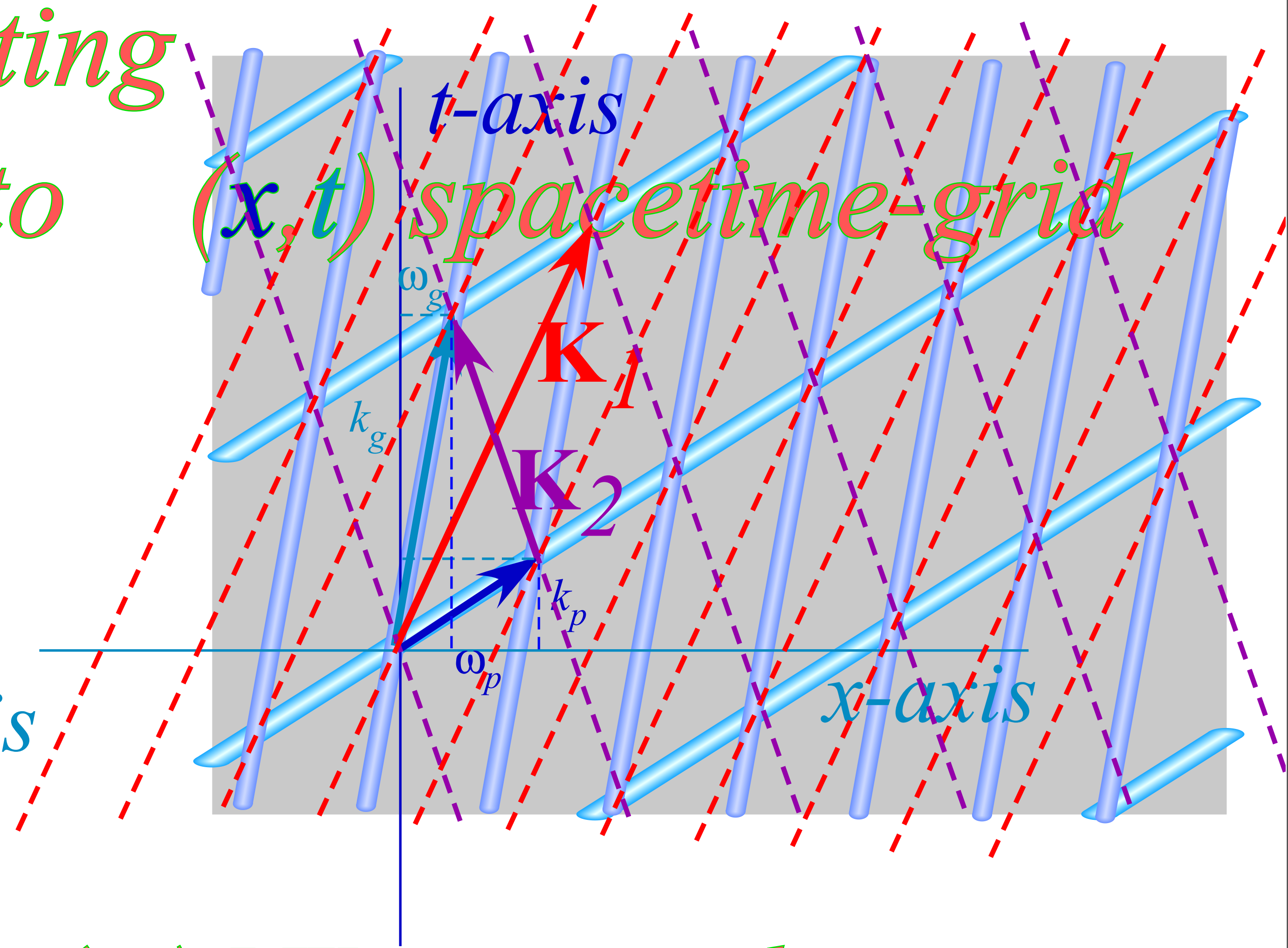
$$\Psi_{50_1\text{-}50_2}(x,t) = 1/2 e^{i(k_1 x - \omega_1 t)} + 1/2 e^{i(k_2 x - \omega_2 t)} = e^{i(k_p x - \omega_p t)} \bullet \cos(k_g x - \omega_g t)$$

$$Re[\Psi_{50_1\text{-}50_2}(x,t)] = \qquad = \cos(k_p x - \omega_p t) \bullet \cos(k_g x - \omega_g t) = 0$$

Real part has ZEROS that make: *(x,t) CW spacetime-lattice*

Relating (k,ω) per-spacetime $\mathbf{K}$ to (x,t) spacetime-grid





while primitive $\mathbf{K}_1$ *and* $\mathbf{K}_2$ *make:* *(x,t) PW spacetime-lattice*

$$\mathbf{K}_p=(\mathbf{K}_1+\mathbf{K}_2)/2 \qquad \mathbf{K}_1=\mathbf{K}_p+\mathbf{K}_g$$

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Find tracks in space-time of a balanced (50-50) plane wave combination:

$$\Psi_{50_1\text{-}50_2}(x,t)=1/2e^{i(k_1x-\omega_1t)}+1/2e^{i(k_2x-\omega_2t)}=e^{i(k_px-\omega_pt)}\bullet\cos(k_gx-\omega_gt)$$

$$Re[\Psi_{50_1\text{-}50_2}(x,t)]==\cos(k_px-\omega_pt)\bullet\cos(k_gx-\omega_gt)=0$$

Real part has ZEROS that make: *(x,t) CW spacetime-lattice*

Interfering Plane Waves: The Expo-Cosine Identity

$$\Psi = e^{ia} + e^{ib} = e^{i(a+b)/2}\left(e^{i(a-b)/2} + e^{-i(a-b)/2}\right)$$

INSIDE Phase: $e^{i(a+b)/2}$

OUTSIDE Group: $2\cos\left(\frac{a-b}{2}\right)$

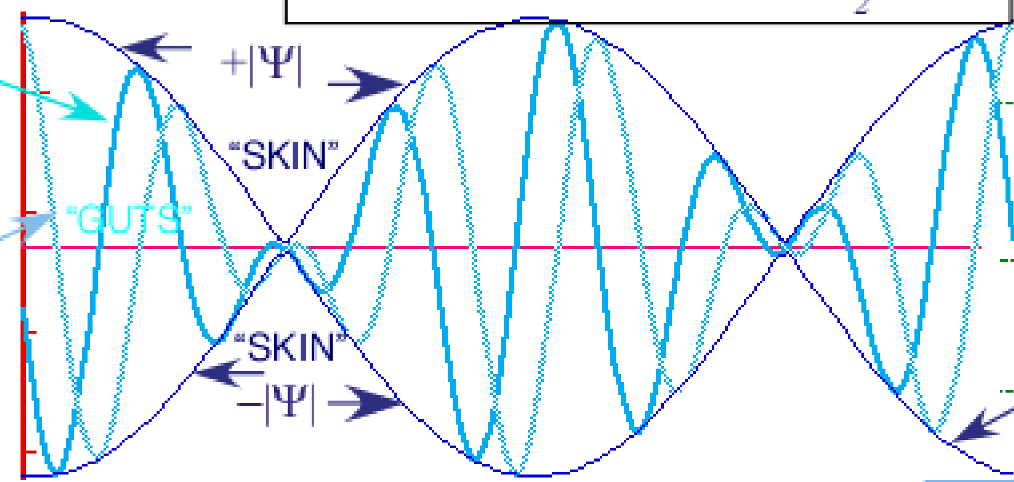
Anatomy of a 2-State Wavefunction

$\Psi = e^{ia} + e^{ib} = e^{i(a+b)/2}\, 2\cos\left(\frac{a-b}{2}\right)$

Real Part $\mathrm{Re}\Psi = |\Psi|\cos\left(\frac{a+b}{2}\right)$

and

Imaginary Part $\mathrm{Im}\Psi = |\Psi|\sin\left(\frac{a+b}{2}\right)$



Envelope or Modulus

Wave "SKIN" $\pm|\Psi| = \pm 2\cos\left(\frac{a-b}{2}\right)$ is PROBABILITY wave for classical "stuff" $|\Psi| = \sqrt{\Psi^*\Psi}$

Input phases

$a = k_a x - \omega_a t$

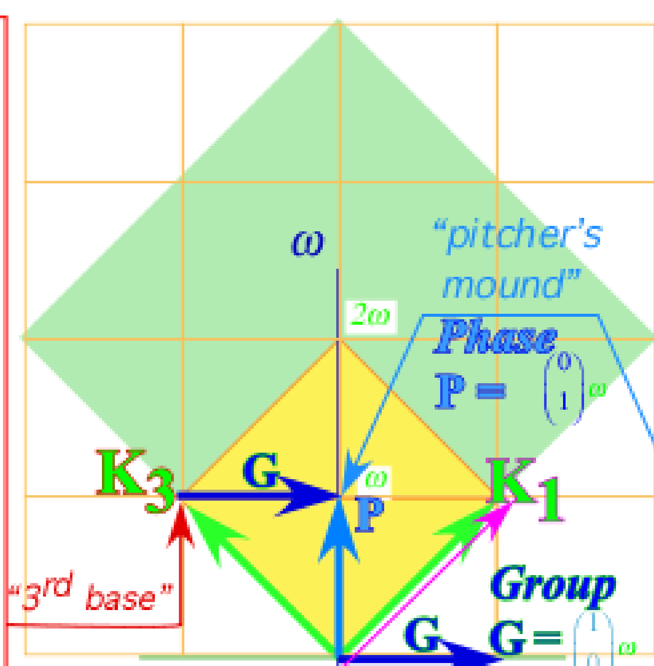
1st base vector

$$K_1 = \begin{pmatrix} ck_a \\ \omega_a \end{pmatrix} = \omega_a \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$b = k_b x - \omega_b t$

3rd base vector

$$K_3 = \begin{pmatrix} ck_b \\ \omega_b \end{pmatrix} = \omega_b \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$



(Here: $\omega_a = \omega = \omega_b$)

$\frac{1}{2}$-Sum

Phase vector

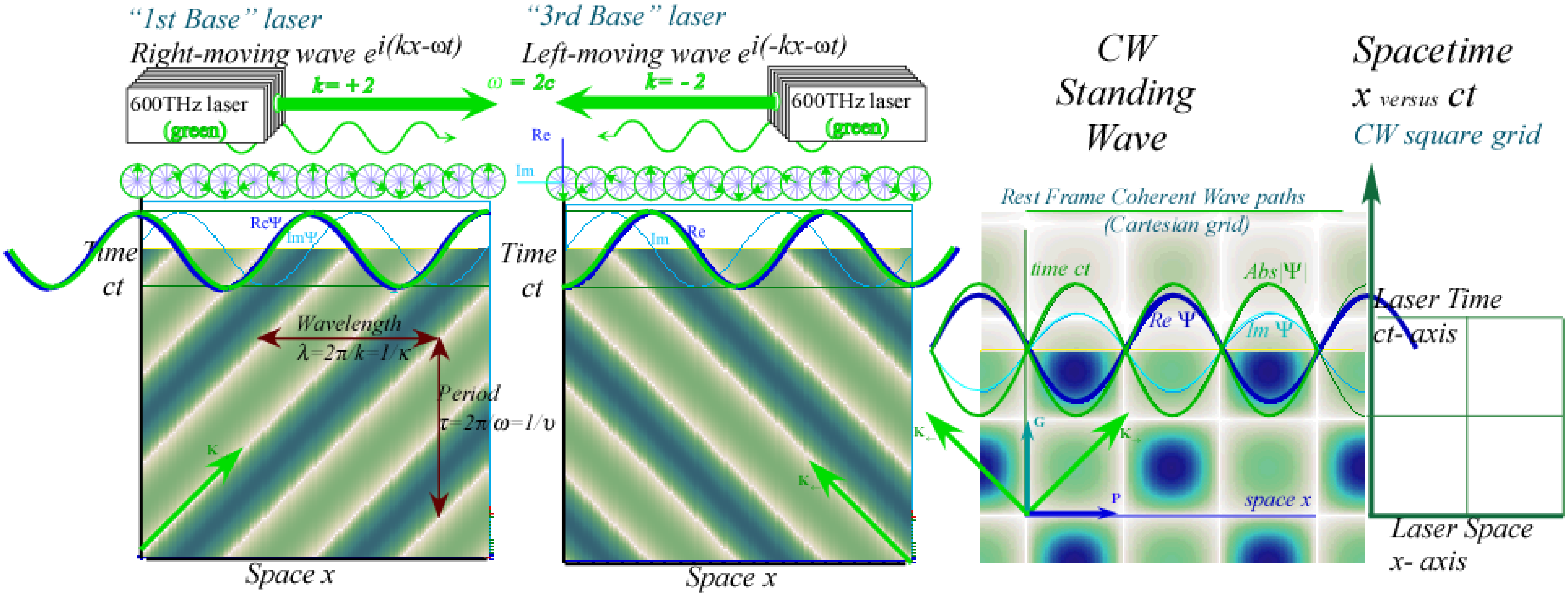
$$\mathbf{P} = \frac{1}{2}\begin{pmatrix} ck_a + ck_b \\ \omega_a + \omega_b \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \omega_a - \omega_b \\ \omega_a + \omega_b \end{pmatrix} = \omega \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\frac{1}{2}$-Difference

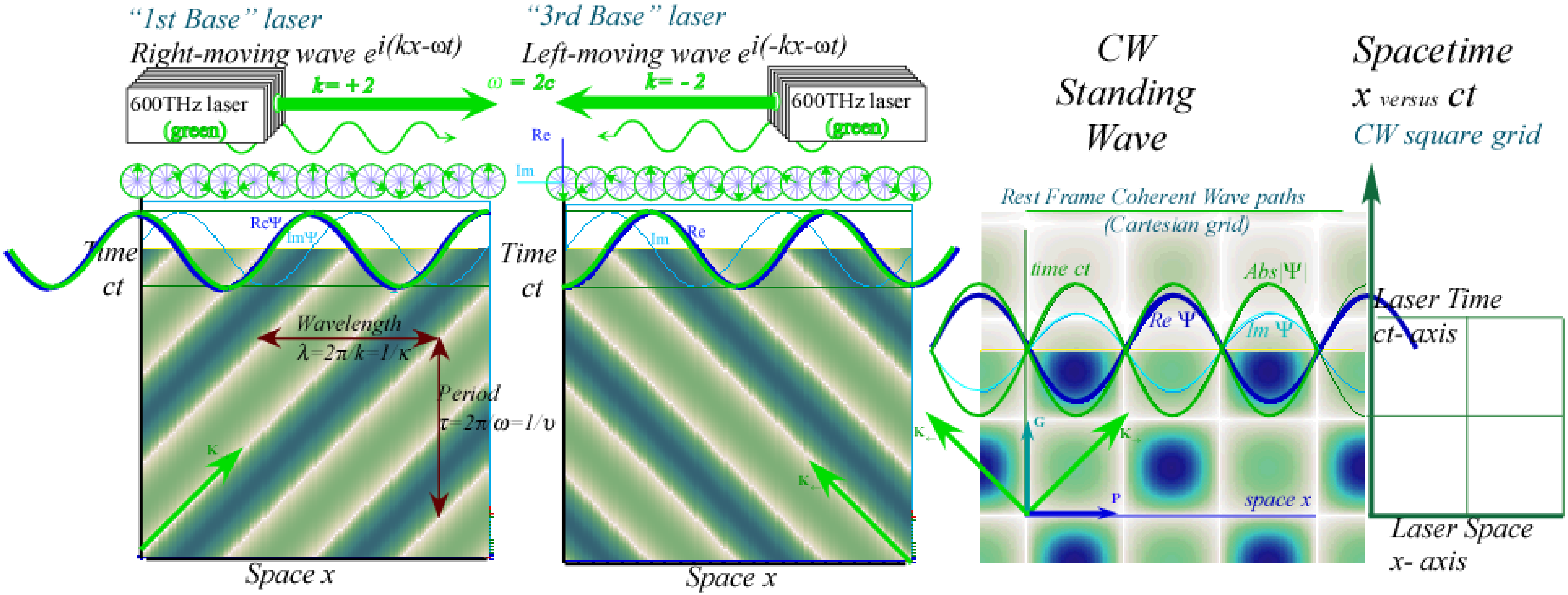
Group vector

$$\mathbf{G} = \frac{1}{2}\begin{pmatrix} ck_a - ck_b \\ \omega_a - \omega_b \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \omega_a + \omega_b \\ \omega_a - \omega_b \end{pmatrix} = \omega \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Zeros of head-on CW sum gives (x,ct)-grid



Zeros of head-on CW sum gives (x,ct)-grid



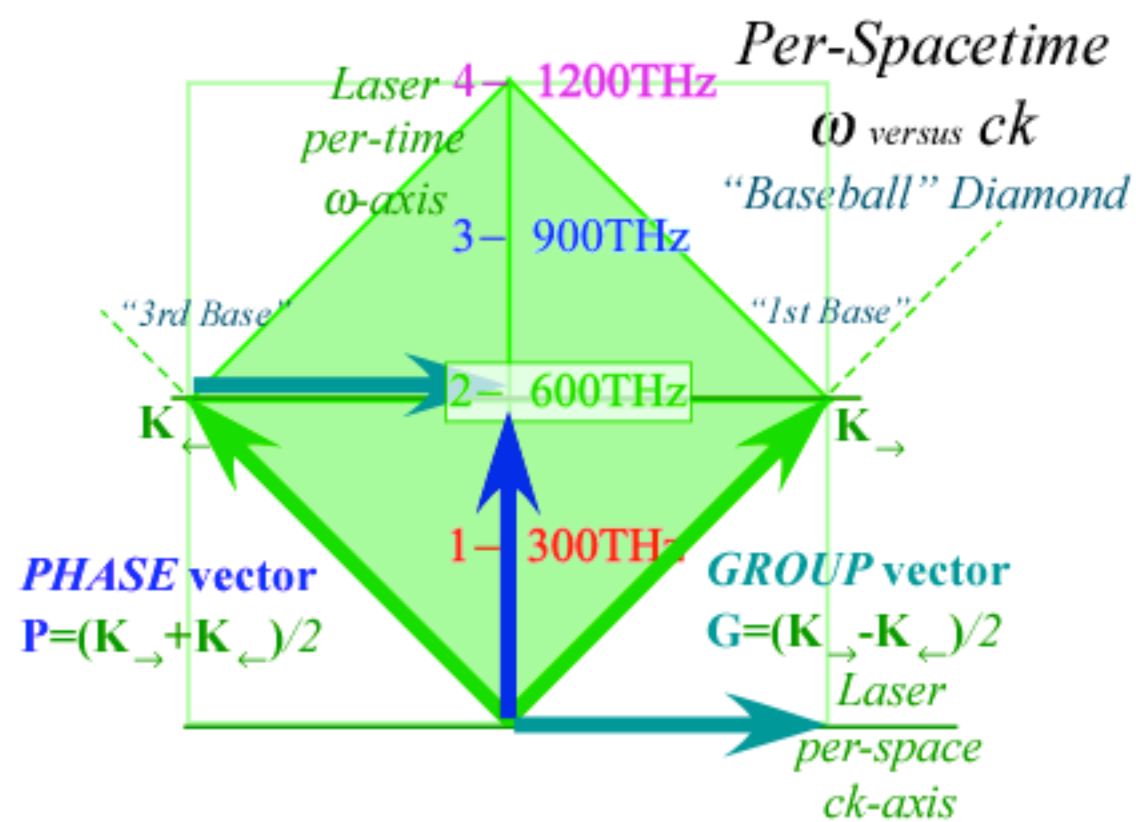
Find zeros by factoring sum:

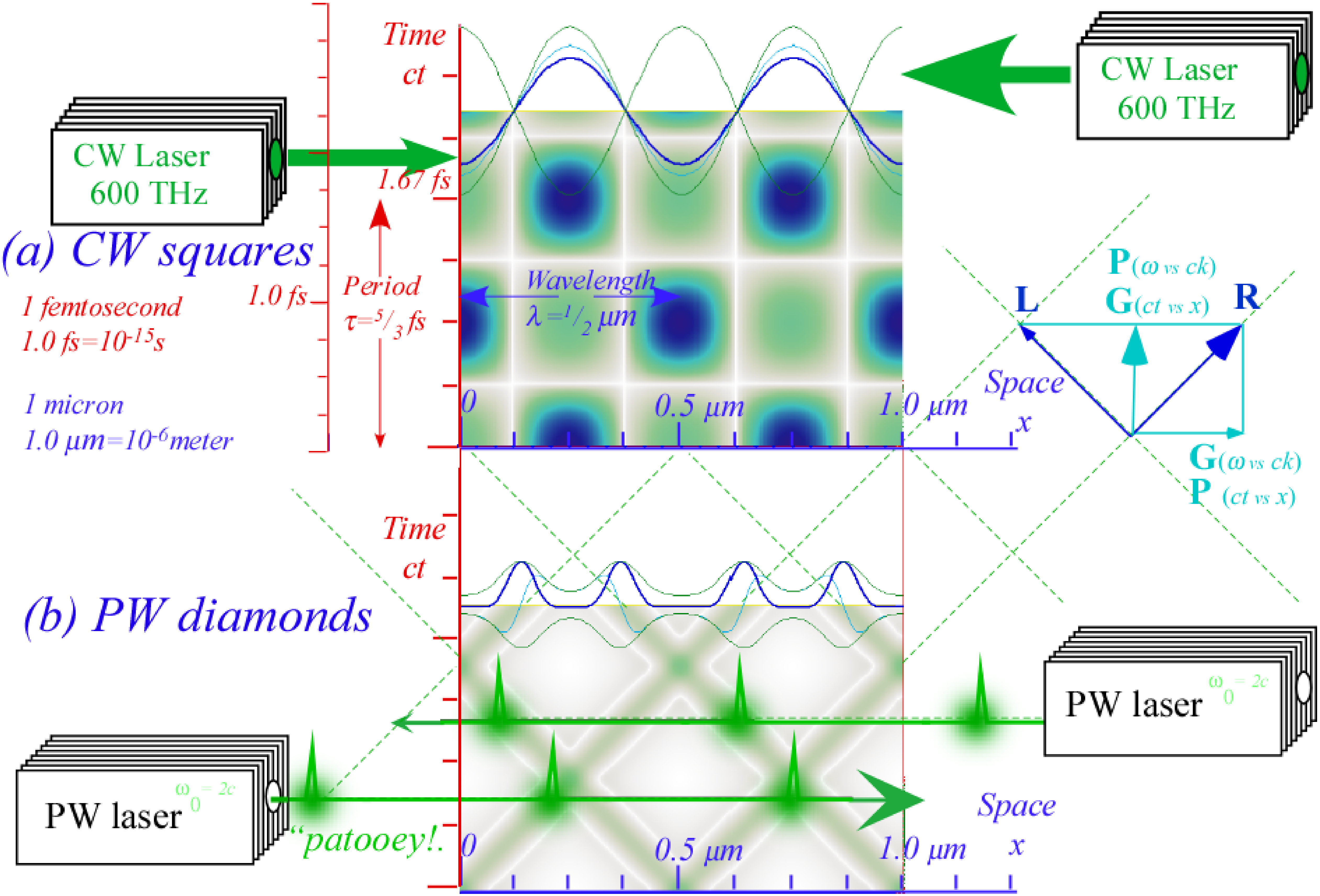
$$\Psi = \overbrace{e^{ia}}^{(kx-\omega t)} + \overbrace{e^{ib}}^{(-kx-\omega t)}$$

$$= \underbrace{e^{i(a+b)/2}}_{\text{Phase factor:}} \underbrace{\left(e^{i(a-b)/2} + e^{-i(a-b)/2}\right)}_{\text{Group factor:}}$$

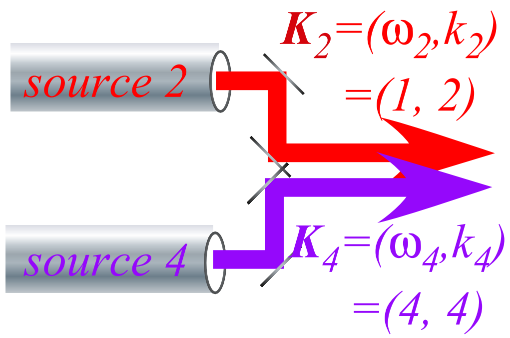
Phase factor: $\exp i\left(\frac{a+b}{2}\right) = e^{-i\omega t}$

Group factor: $2\cos\left(\frac{a-b}{2}\right) = 2\cos(kx)$

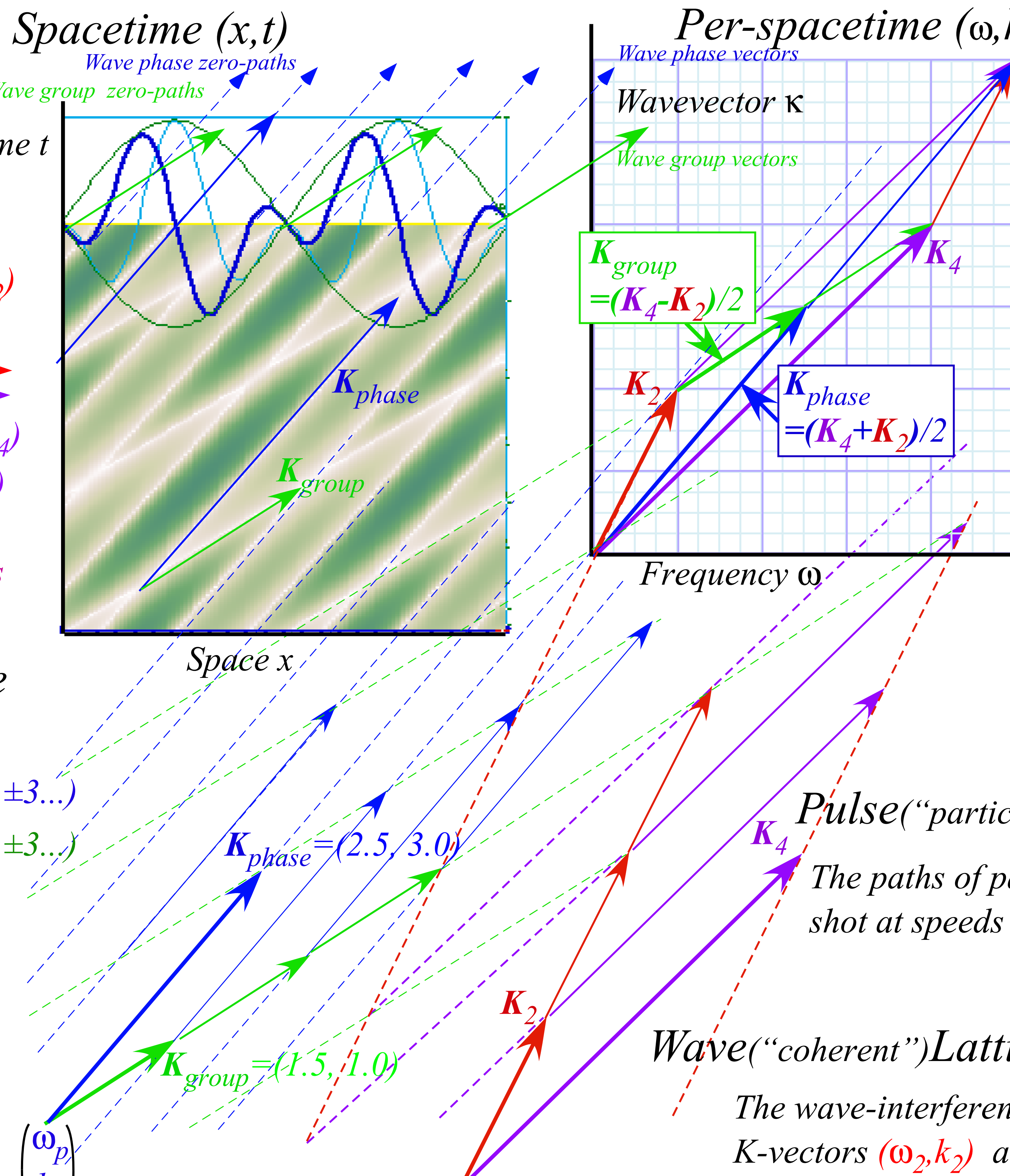



CW Laser
600 THz
CW Laser
600 THz
Time
ct
1.67 fs
(a) CW squares
1 femtosecond
1.0 fs=10-15s
1 micron
1.0 μm=10-6meter
1.0 fs
Period
τ=5/3 fs
Wavelength
λ=1/2 μm
0
0.5 μm
1.0 μm
Space
x
L
P(ω vs ck)
G(ct vs x)
R
G(ω vs ck)
P (ct vs x)
Time
ct
(b) PW diamonds
PW laser
PW laser
"patooey!.
0
0.5 μm
1.0 μm
Space
x

Suppose we are given two "mystery† sources"



†Shrodinger matter waves



$$V_4=\frac{\omega_4}{k_4}=\frac{4}{4}=1.0 \qquad V_2=\frac{\omega_2}{k_2}=\frac{1}{2}=0.5$$

$$K_{phase}=(K_4+K_2)/2=\begin{pmatrix}\omega_4+\omega_2\\k_4+k_4\end{pmatrix}/2=\begin{pmatrix}\omega_p\\k_p\end{pmatrix}=\frac{1}{2}\begin{pmatrix}4+1\\4+2\end{pmatrix}=\begin{pmatrix}2.5\\3.0\end{pmatrix}$$

$$V_{phase}=\frac{\omega_4+\omega_2}{k_4+k_4}=\frac{2.5}{3.0}=0.83$$

$$K_{group}=(K_4-K_2)/2=\begin{pmatrix}\omega_4-\omega_2\\k_4-k_4\end{pmatrix}/2=\begin{pmatrix}\omega_g\\k_g\end{pmatrix}=\frac{1}{2}\begin{pmatrix}4-1\\4-2\end{pmatrix}=\begin{pmatrix}1.5\\1.0\end{pmatrix}$$

$$V_{phase}=\frac{\omega_4-\omega_2}{k_4-k_4}=\frac{1.5}{1.0}=1.5$$

Wave("coherent")Lattice

Bases: K_{group} and K_{phase}

$k_px-\omega_pt = n_p= N_p\ /2 \quad (N_p=\pm1,\pm3...)$

$k_gx-\omega_gt = n_g= N_g\ /2 \quad (N_g=\pm1,\pm3...)$

$$\begin{pmatrix}k_p & -\omega_p\\k_g & -\omega_g\end{pmatrix}\begin{pmatrix}x\\t\end{pmatrix}=\begin{pmatrix}n_p\\n_g\end{pmatrix}$$

$$\begin{pmatrix}x\\t\end{pmatrix}=\frac{\begin{pmatrix}-\omega_g & \omega_p\\-k_g & k_p\end{pmatrix}\begin{pmatrix}n_p\\n_g\end{pmatrix}}{\omega_pk_g-\omega_gk_p}=\frac{-n_p\begin{pmatrix}\omega_g\\k_g\end{pmatrix}+n_g\begin{pmatrix}\omega_p\\k_p\end{pmatrix}}{\omega_pk_g-\omega_gk_p}=\frac{n_p}{D}\ K_{group}+\frac{n_g}{D}K_{phase}$$

Pulse("particle")Lattice (Bases: K_2 and K_4)

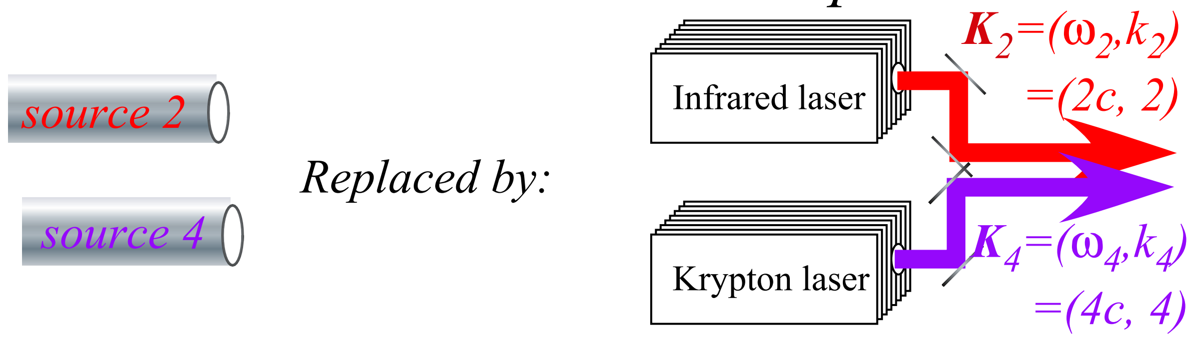
The paths of packets or Newtonian "corpuscles" shot at speeds V_2 and V_4 and rates ω_2 and ω_2

Wave("coherent")Lattice (Bases: K_{group} and K_{phase})

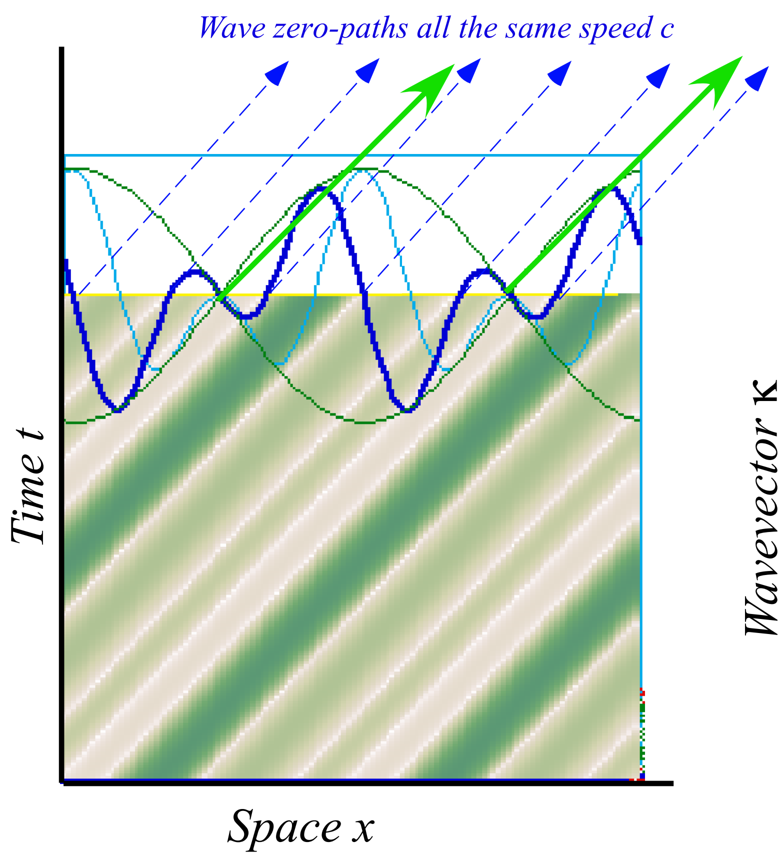
The wave-interference-zero paths given K-vectors (ω_2,k_2) and (ω_4,k_4) .

*For co-propagating laser * sources...*
...the wave-coordinate lattice collapses to lines..

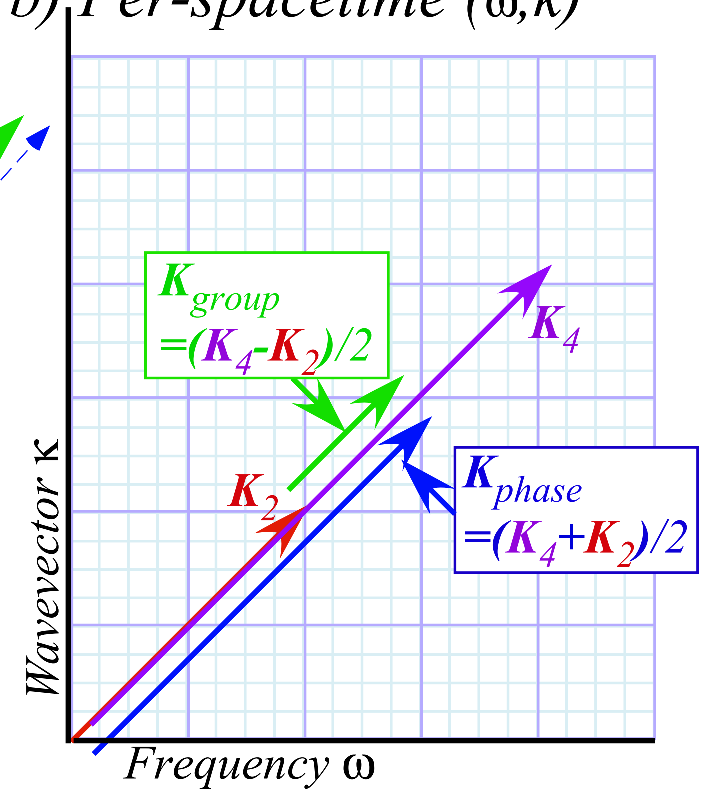
**simple linear*
ω=ck dispersion



(a) Spacetime (x,t)



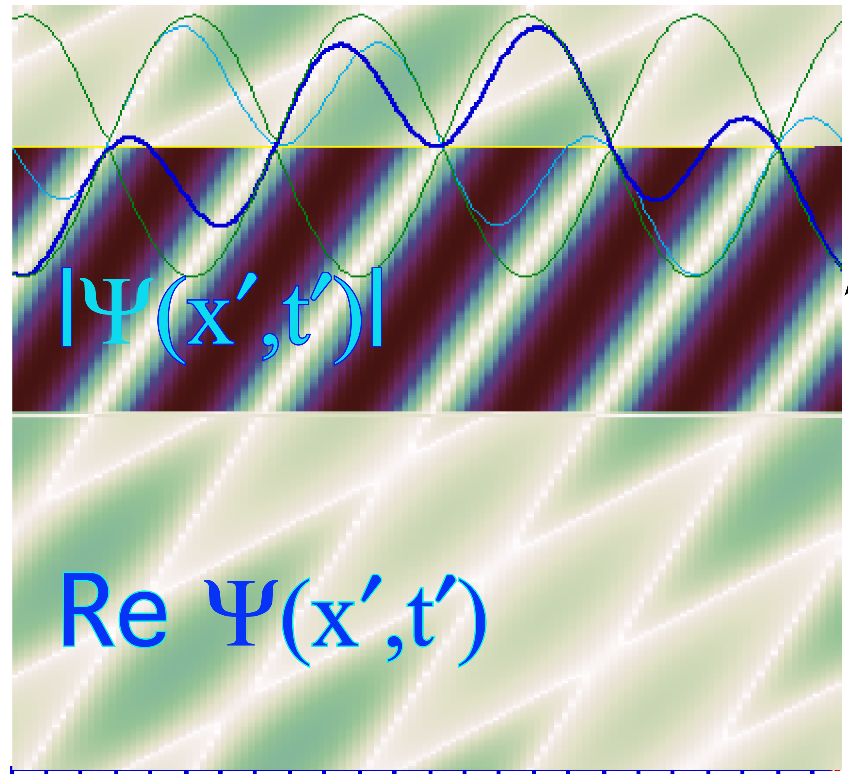
(b) Per-spacetime (ω,k)



But, for counter-propagating laser sources...
...the wave coordinate lattice is the Lorentz-Einstein-Minkowski frame!!

Phase lines may not show up in Magnitude ($|\Psi(x',t')|$) orProbability ($\Psi(x',t')^*\Psi(x',t')$) plots.

Unbiased $\Psi=\psi_{-1}+\psi_{+4}$

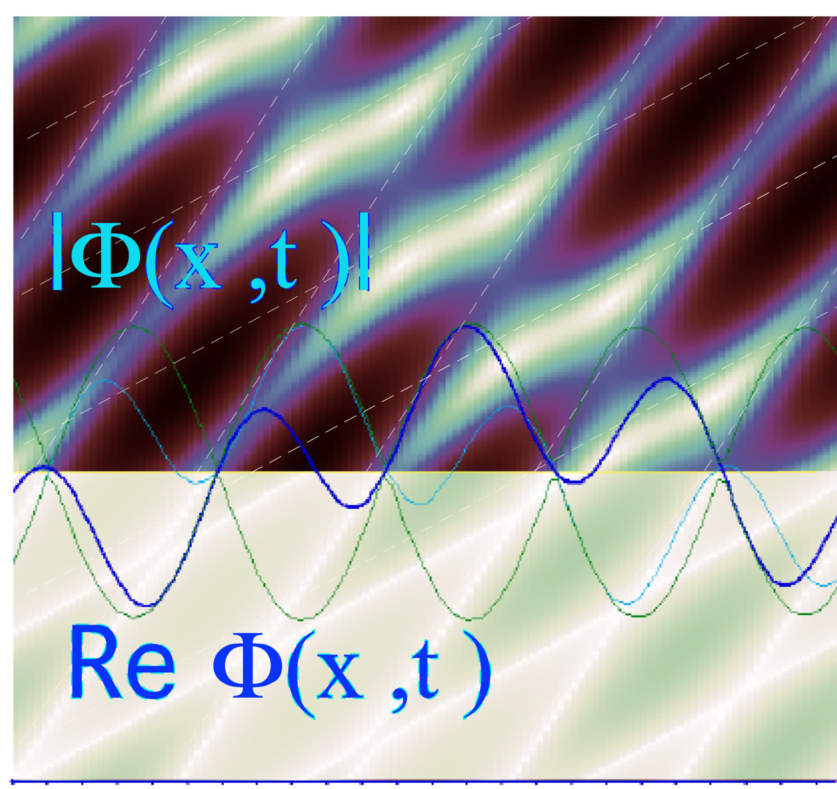


Only the group wave paths appear

The "inside phase" $e^{i[\,]}$ *gets killed in* ($\Psi(x',t')^*\Psi(x',t')$) *because* $(e^{i[\,]})^* = e^{-i[\,]}$ *and* $(e^{i[\,]})^* \bullet e^{i[\,]} = 1$

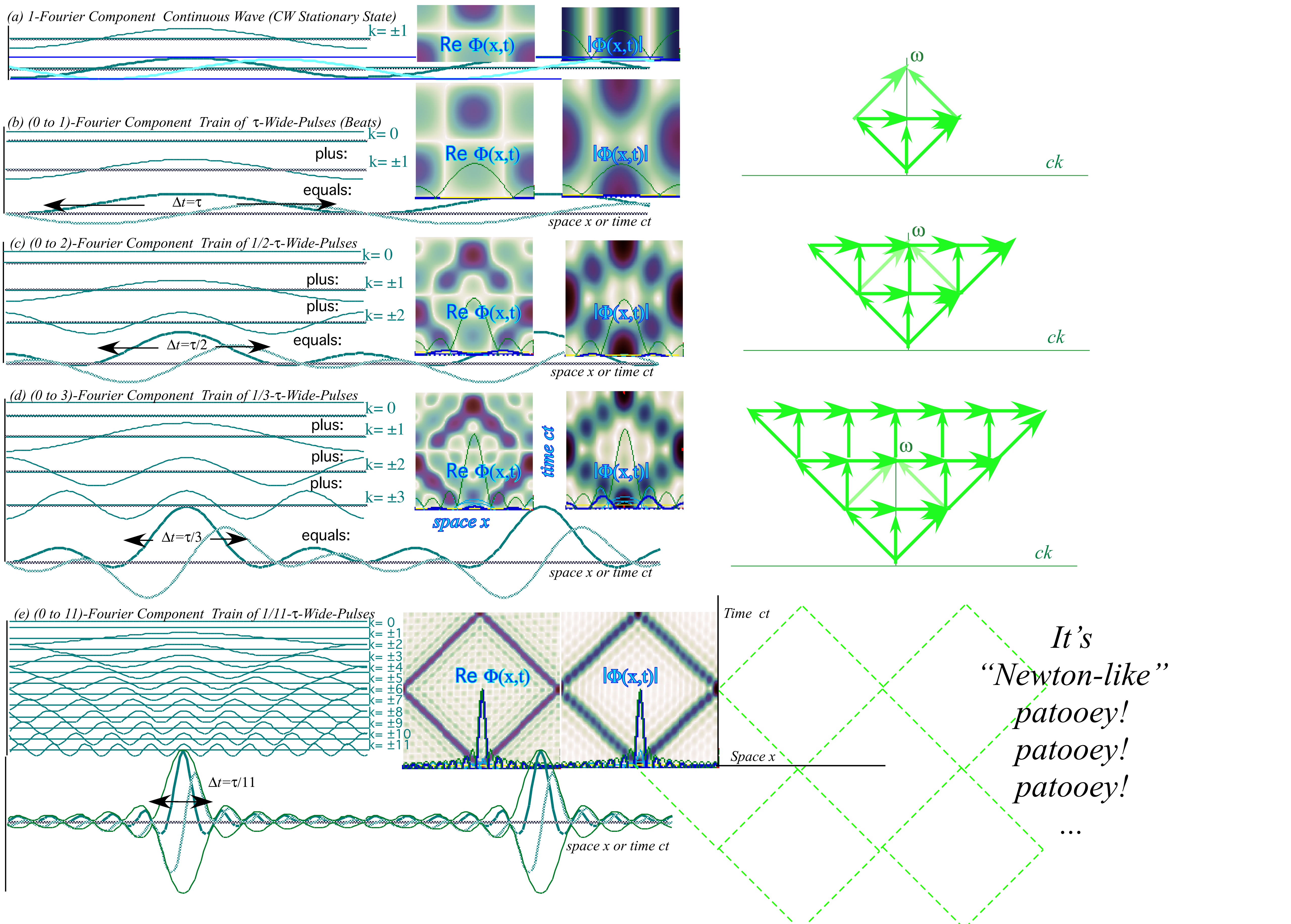
Phase structure begins to show up if ground-state (k=0) component is added.

DC biased $\Phi=\psi_0+\Psi$



Group and phase paths begin to appear

Each counter-propagating pair of beams makes a wave-interference-lattice.
“Packets” or pulses made by adding more pairs. Finally, pulse lattice apppears.



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- *SinNx/x wavepackets bandwidth and uncertainty*
  - *SinNx/x revivals*
- *Gaussian wave-packet bandwidth and uncertainty*
  - *Gaussian revivals*
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- *Phase dynamics*

∞-Square well PE versus Bohr rotor

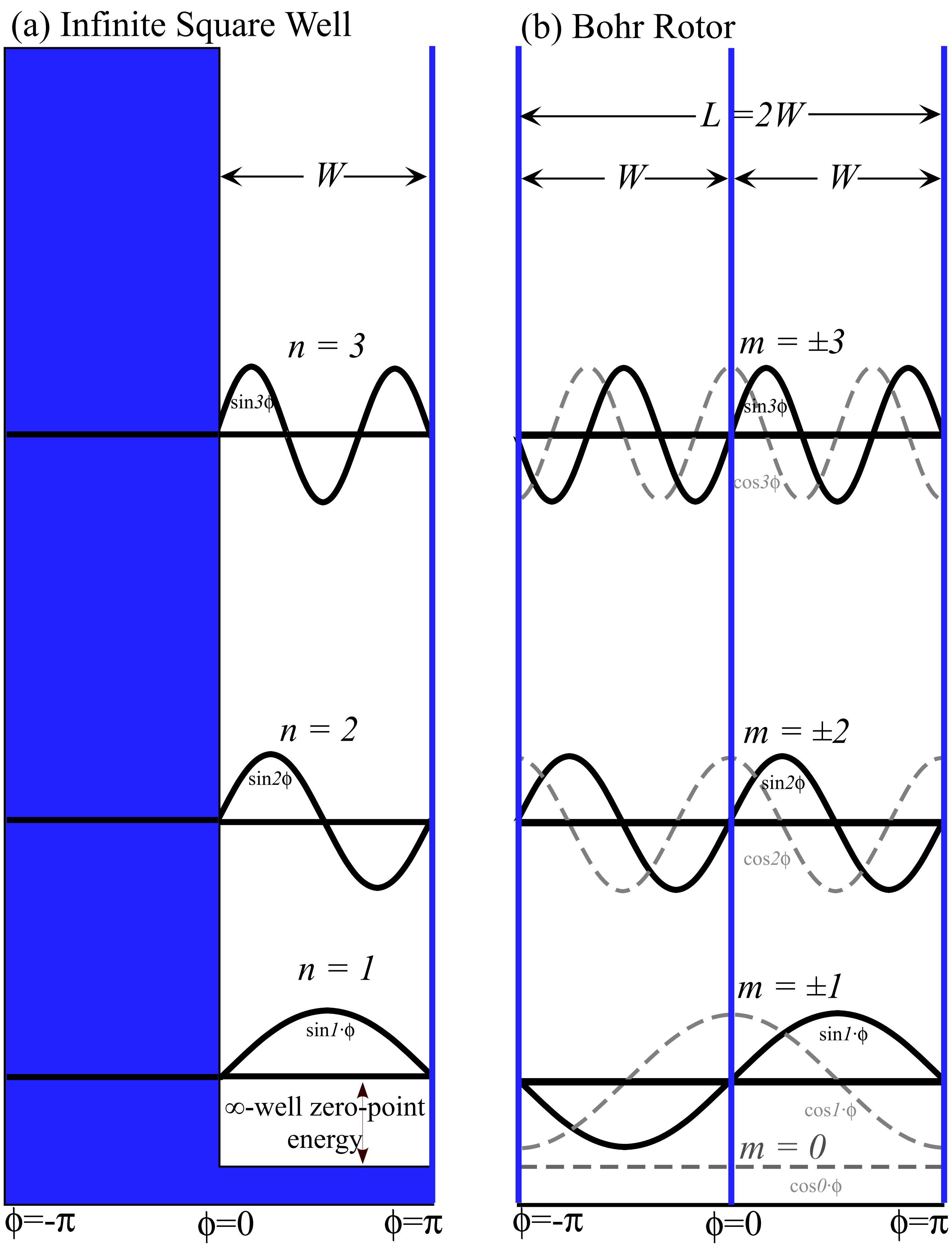
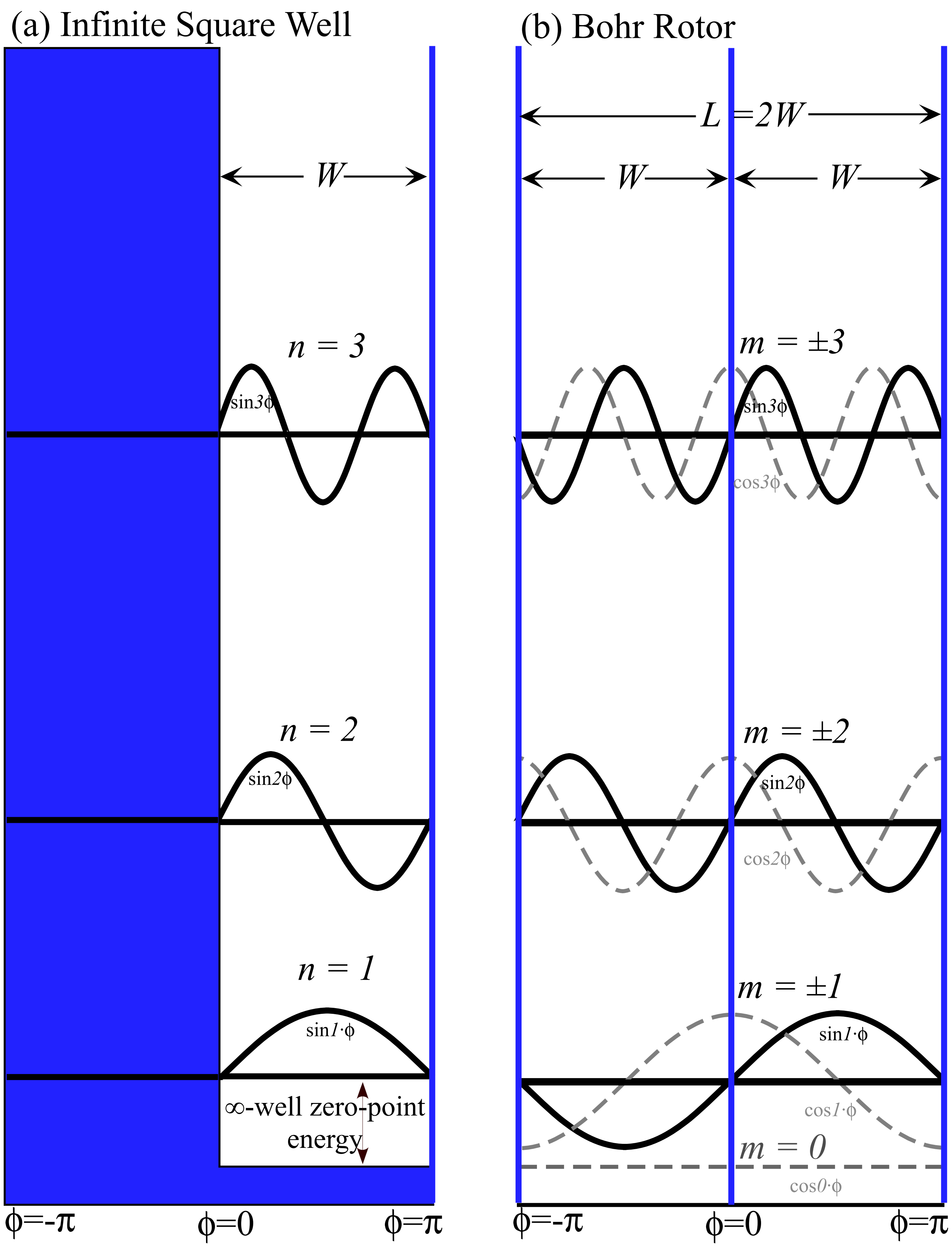


Fig. 12.2.6 Comparison of eigensolutions for (a) Infinite square well, and (b) Bohr rotor.

$m=0, \pm 1, \pm 2, \pm 3,$...are momentum quanta in wavevector formula: $k_m=2\pi m/L$

$(k_m=m \quad if: L=2\pi)$

∞-Square well PE versus Bohr rotor



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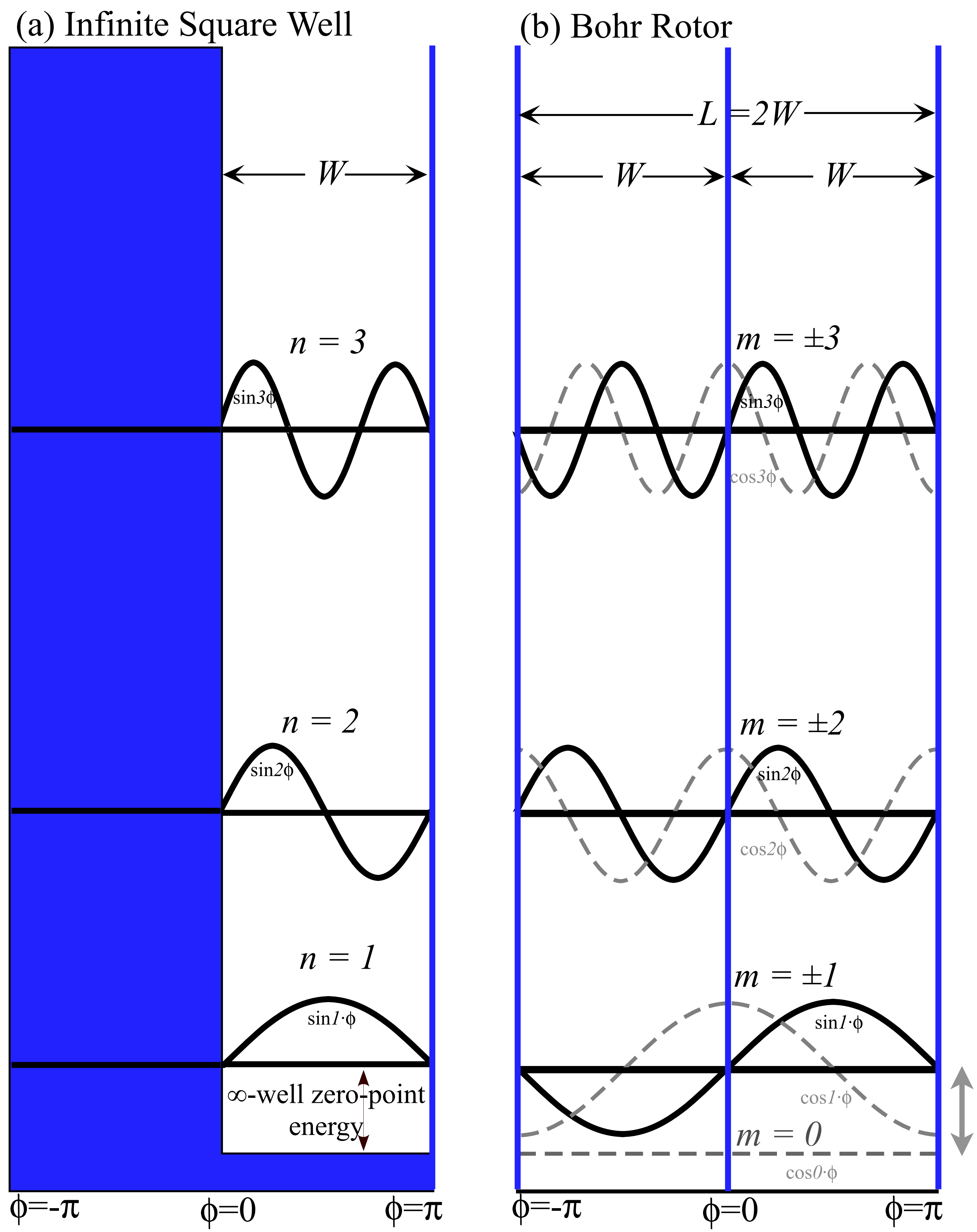
$(k_m=m \quad if{:}\ L=2\pi)$

$$E_m= (\hbar k_m)^2/2M = m^2[h^2/2ML^2]$$
$$= m^2\, h\upsilon_1 = m^2\, \hbar\omega_1$$

∞-Square well PE versus Bohr rotor



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($k_m=m$ if: $L=2\pi$)

$$E_m = (\hbar k_m)^2/2M = m^2[h^2/2ML^2]$$
$$= m^2 h\upsilon_1 = m^2 \hbar\omega_1$$

fundamental Bohr ∠-frequency

$\omega_1=2\pi\upsilon_1$

lowest *transition (beat) frequency*

$\upsilon_1 = (E_1-E_0)/h$ *(E_0 is defined as zero)*

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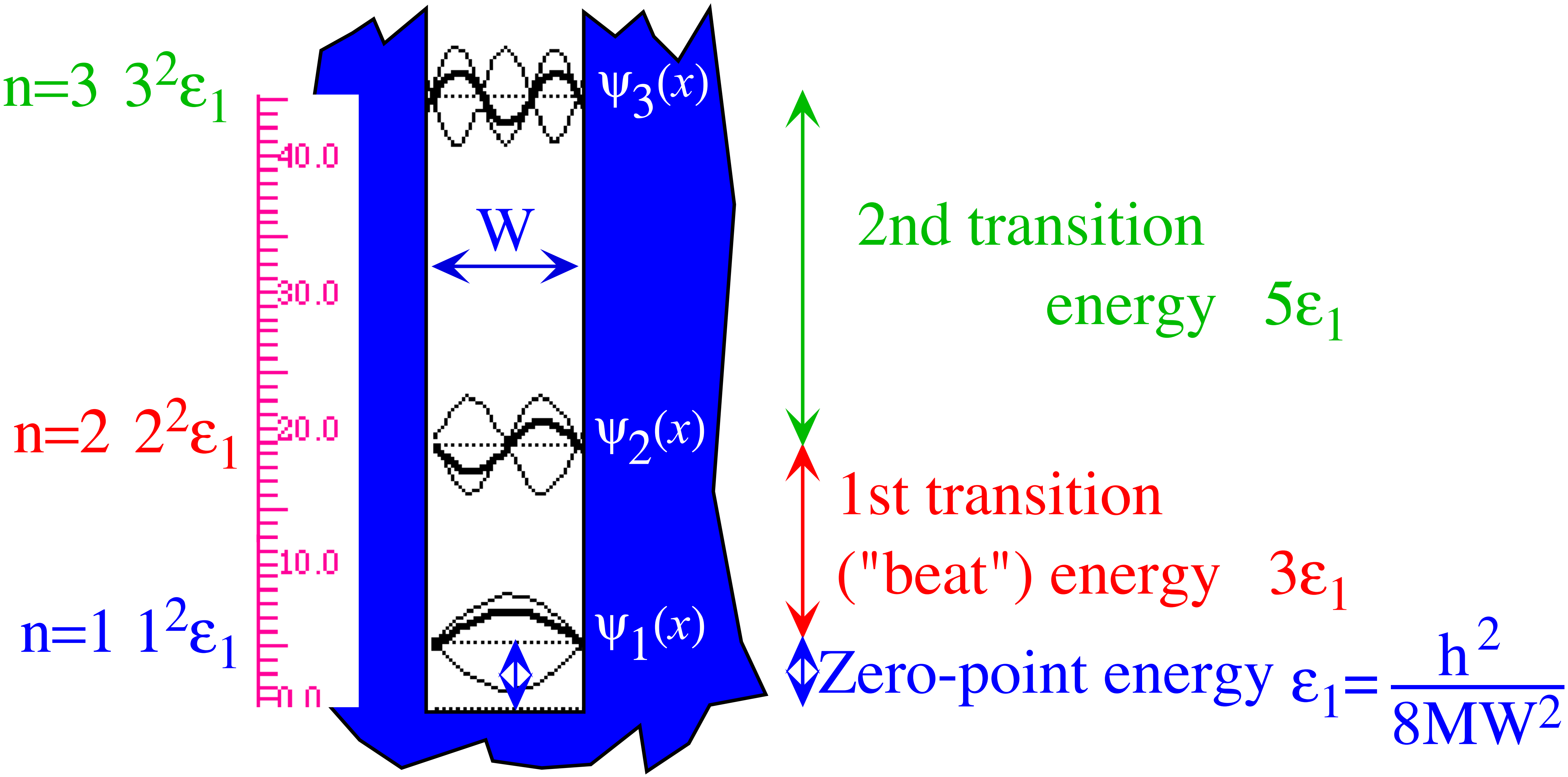
- *Phase dynamics*

∞-Square well PE versus Bohr rotor

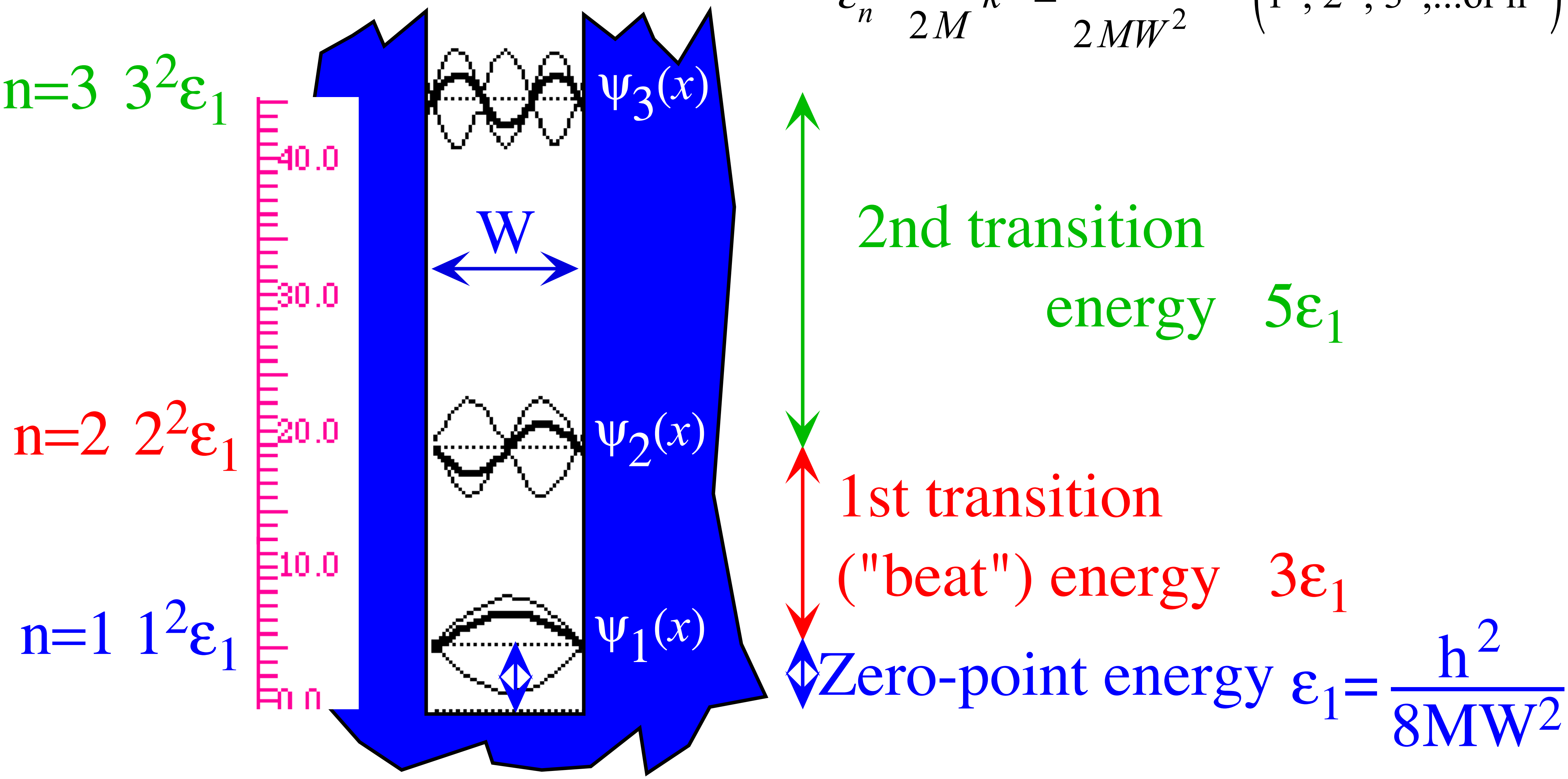


∞-Square well PE versus Bohr rotor

$$kW = n\pi \quad or: \; k = n\pi/W$$

$$\langle x|\varepsilon_n\rangle = \psi_n(x) = A\sin(k_n x) = A\sin\left(\frac{n\pi x}{W}\right) \quad (n=1,2,3,...\infty)$$

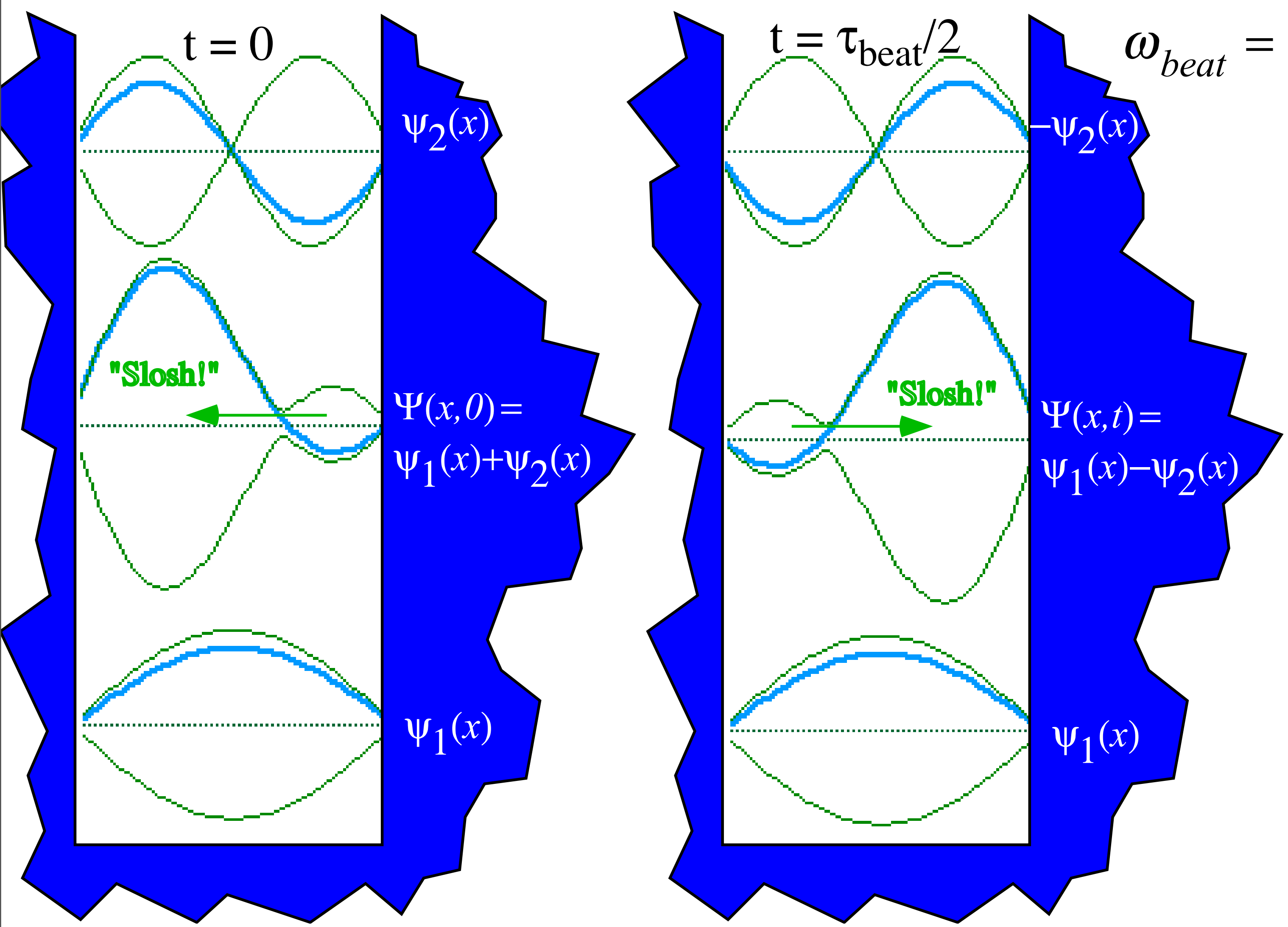
$$\varepsilon_n = \frac{\hbar^2}{2M}k^2 = \frac{\hbar^2 n^2 \pi^2}{2MW^2} = \left(1^2, 2^2, 3^2, ...\text{or } n^2\right)\frac{h^2}{8MW^2}$$



∞-Square well PE versus Bohr rotor

$kW = n\pi \quad or: \; k = n\pi/W$

$$\langle x|\varepsilon_n\rangle = \psi_n(x) = A\sin(k_n x) = A\sin\left(\frac{n\pi x}{W}\right) \quad (n=1,2,3,\ldots\infty)$$



$$\omega_{beat} = \omega_2 - \omega_1 = \frac{\varepsilon_2 - \varepsilon_1}{\hbar} = \frac{2^2 - 1^2}{\hbar}\frac{h^2}{8MW^2} = 3\frac{2\pi h}{8MW^2} = 3\omega_1$$

Fig. 12.1.2 Exercise in prison. Infinite square well eigensolution combination "sloshes" back and forth.

SinNx/x wavepackets bandwidth and uncertainty

$$\delta\left(x-a\right)=\left\langle x\middle|a\right\rangle=\sum_{n=1}^{\infty}\left\langle x\middle|\varepsilon_n\right\rangle\left\langle\varepsilon_n\middle|a\right\rangle$$

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$$\delta(x-a) = \langle x|a\rangle = \sum_{n=1}^{\infty} \langle x|\varepsilon_n\rangle\langle\varepsilon_n|a\rangle = \sum_{n=1}^{\infty} a_n \sin k_n x$$

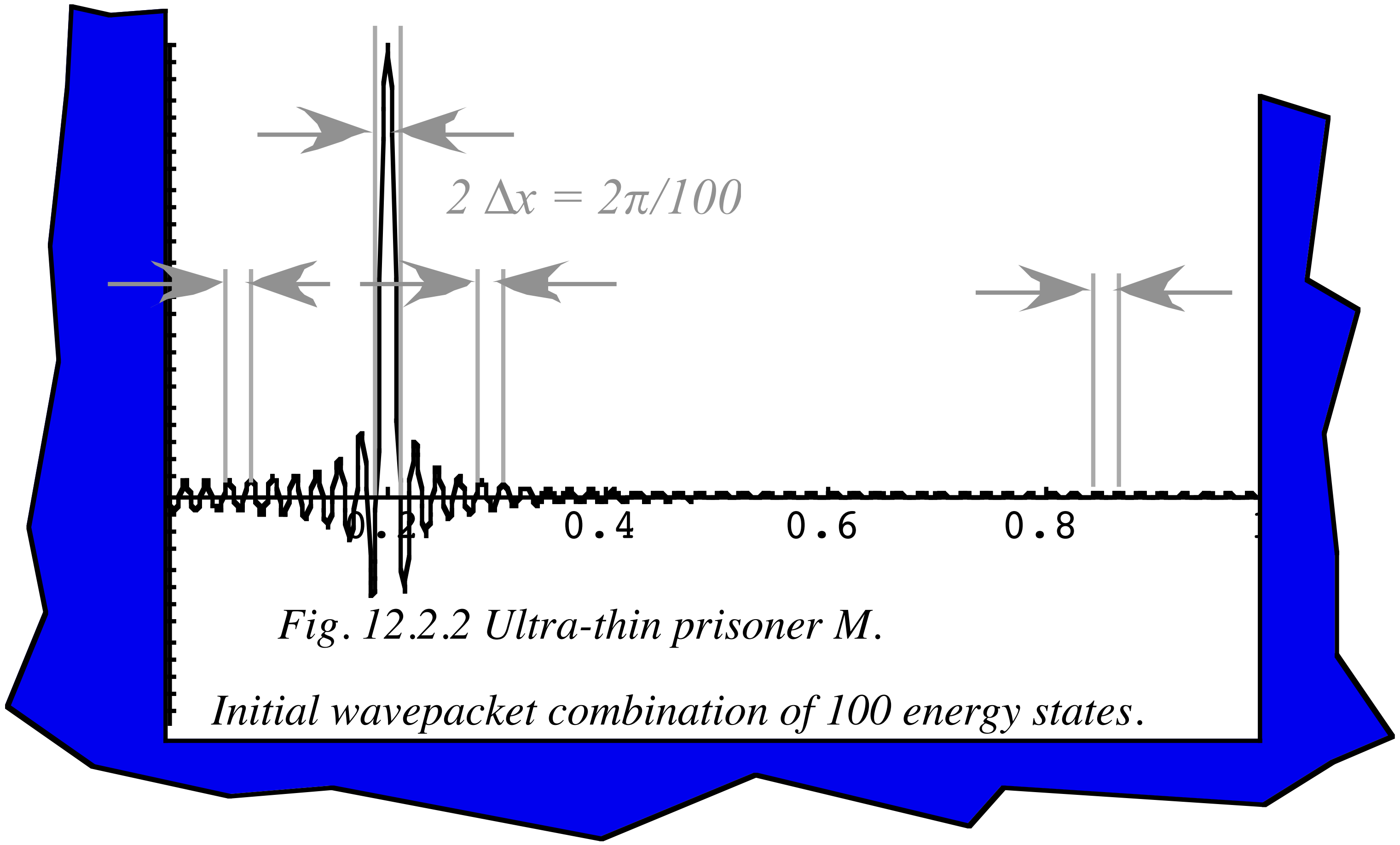


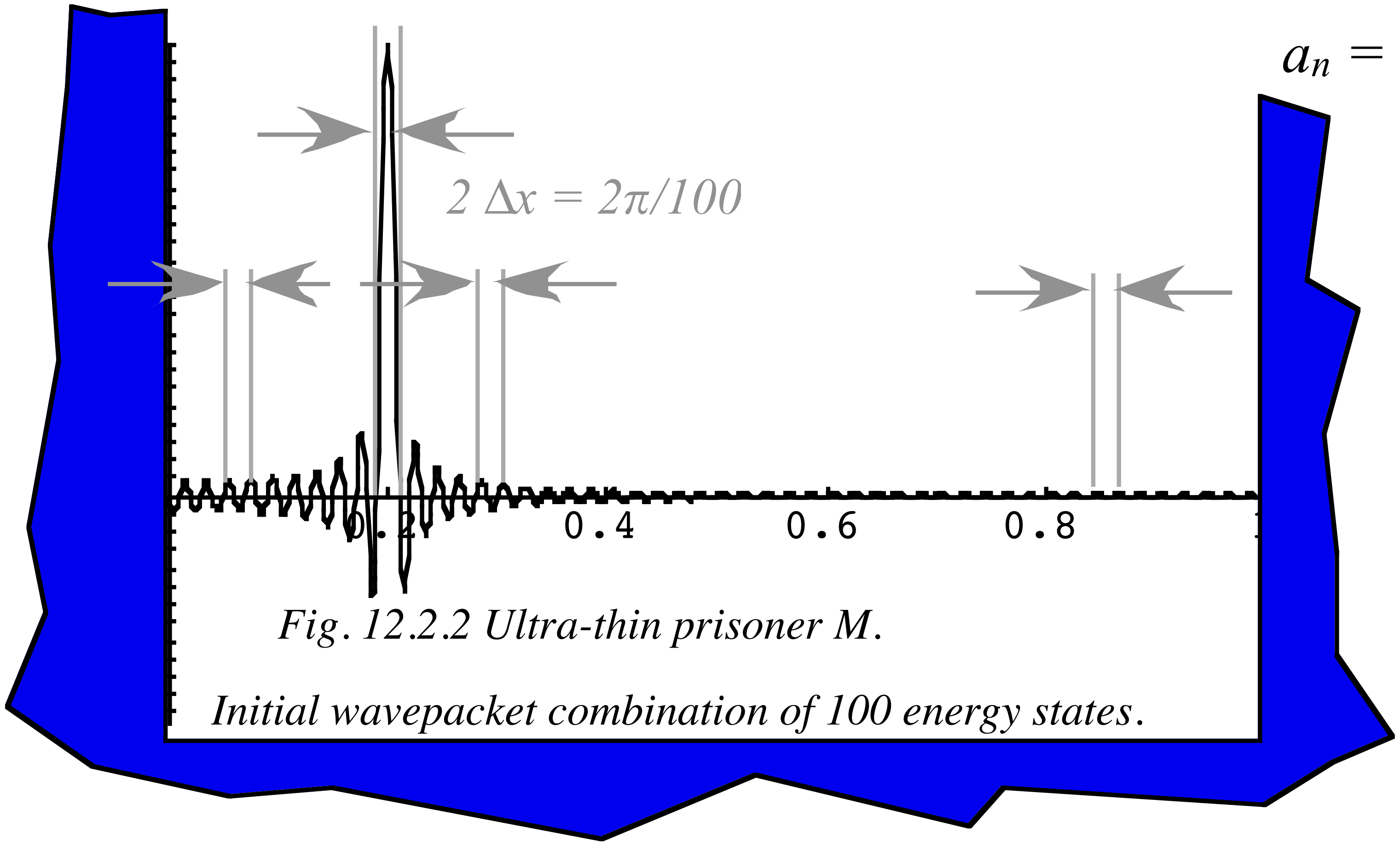
Fig. 12.2.2 Ultra-thin prisoner M.

Initial wavepacket combination of 100 energy states.

SinNx/x wavepackets bandwidth and uncertainty

$$\delta(x-a) = \langle x | a \rangle = \sum_{n=1}^{\infty} \langle x | \varepsilon_n \rangle \langle \varepsilon_n | a \rangle = \sum_{n=1}^{\infty} a_n \sin k_n x$$

$a_n = \langle \varepsilon_n | a \rangle = (2/W) \sin k_n a \quad (k_n = n\pi/W)$



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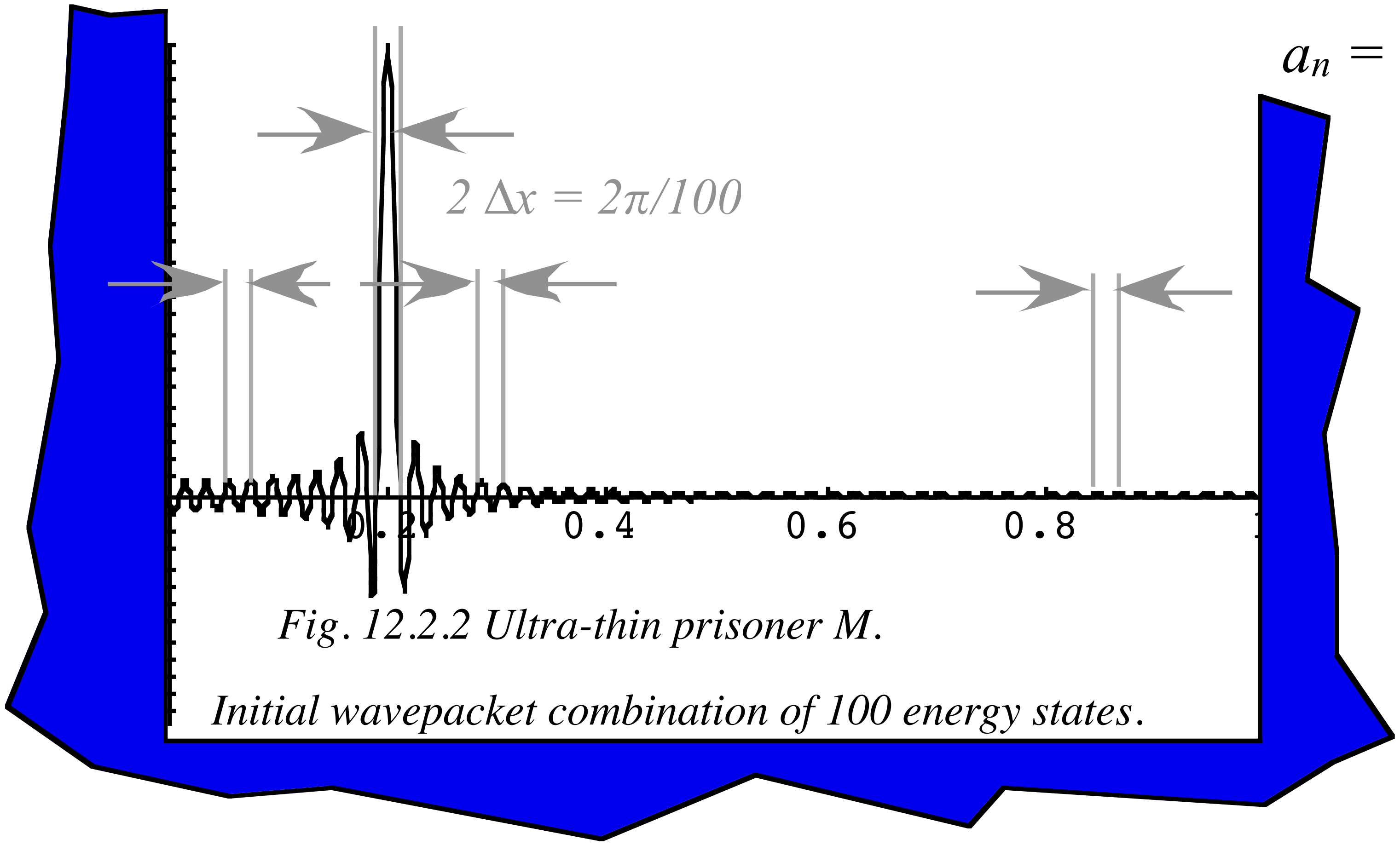

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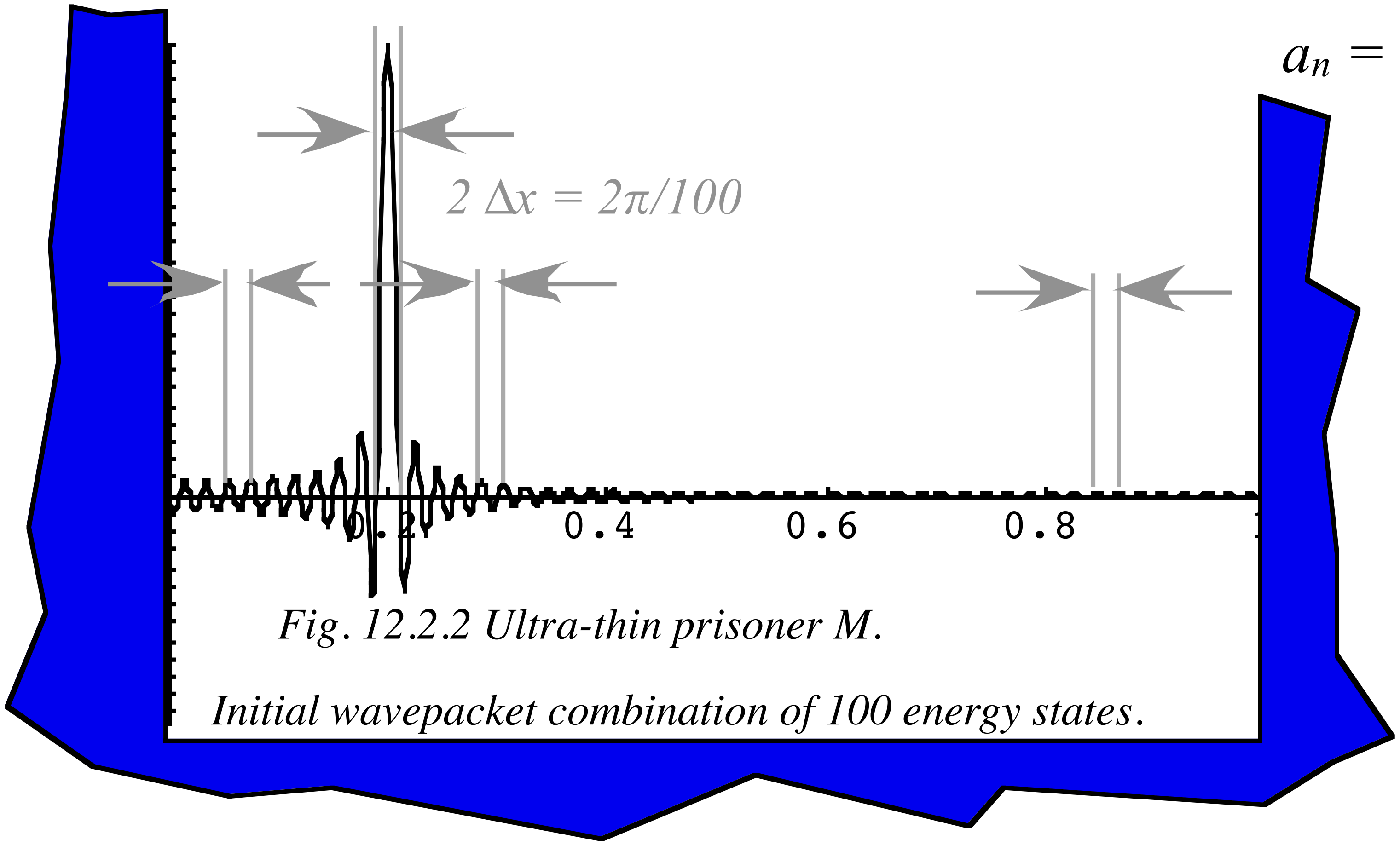

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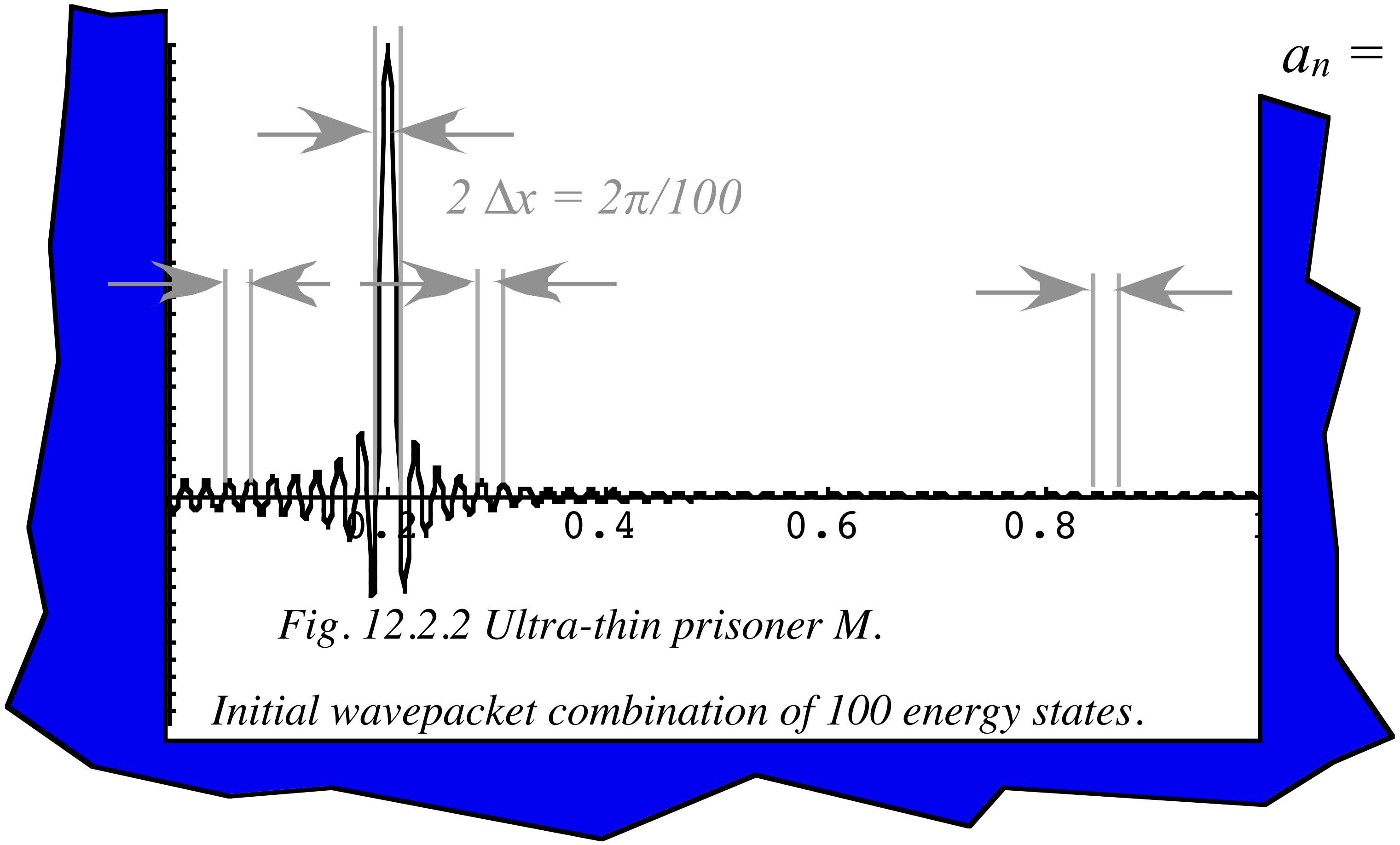

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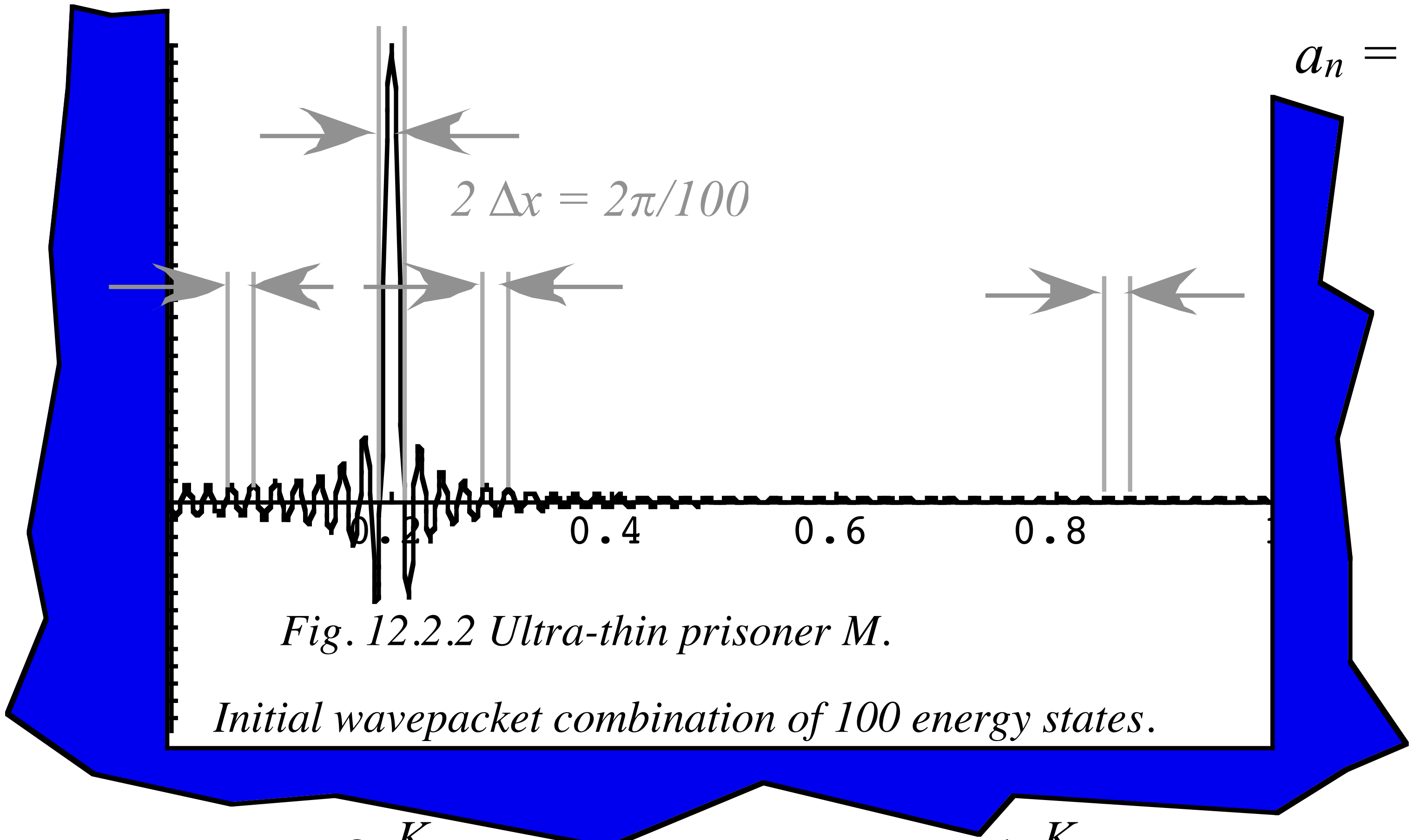

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$$\Psi(x) \cong \frac{2}{\pi}\int_0^{K_{max}} dk \sin ka \sin kx = \frac{1}{\pi}\int_0^{K_{max}} dk\left(\cos k(x-a) - \cos k(x+a)\right)$$

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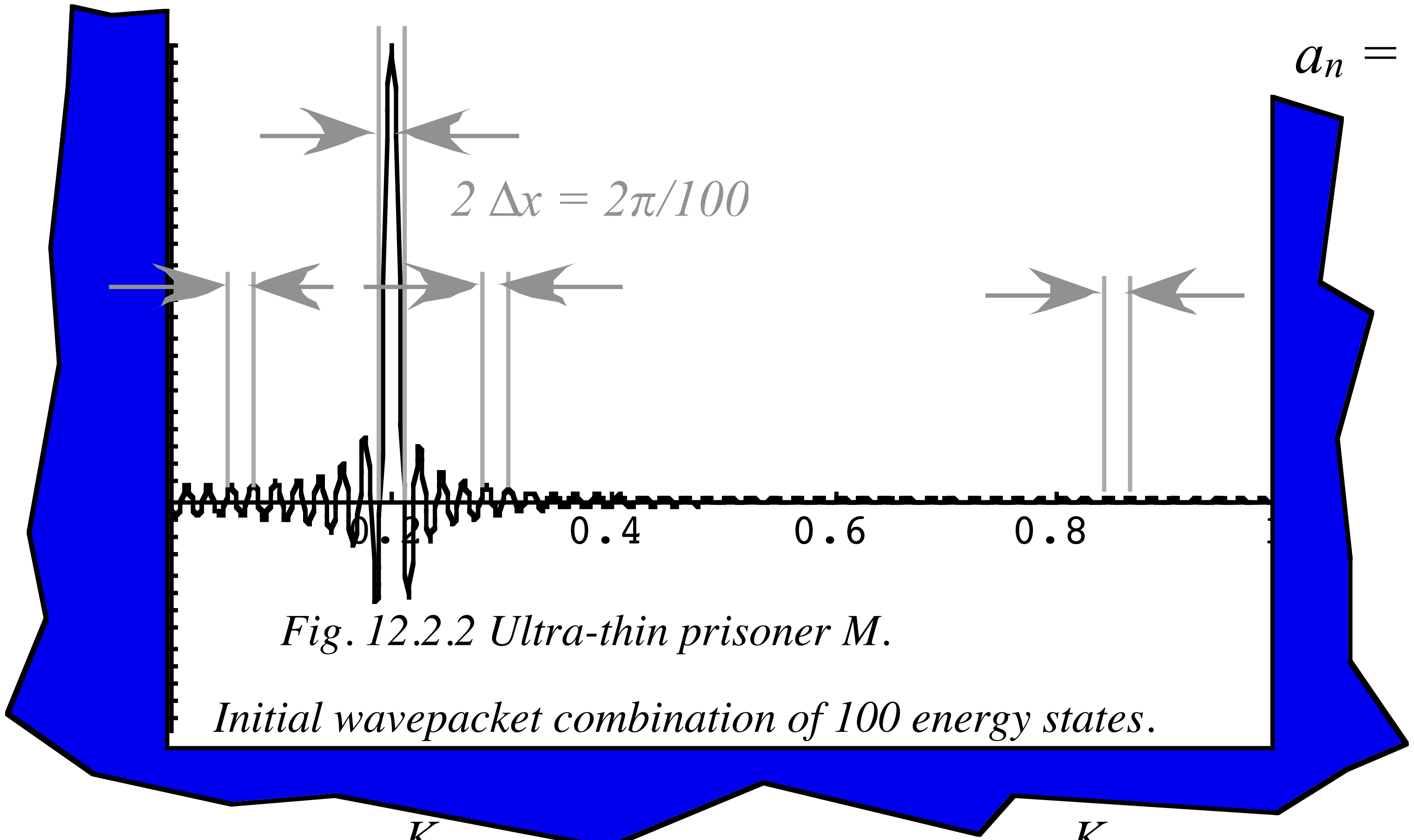

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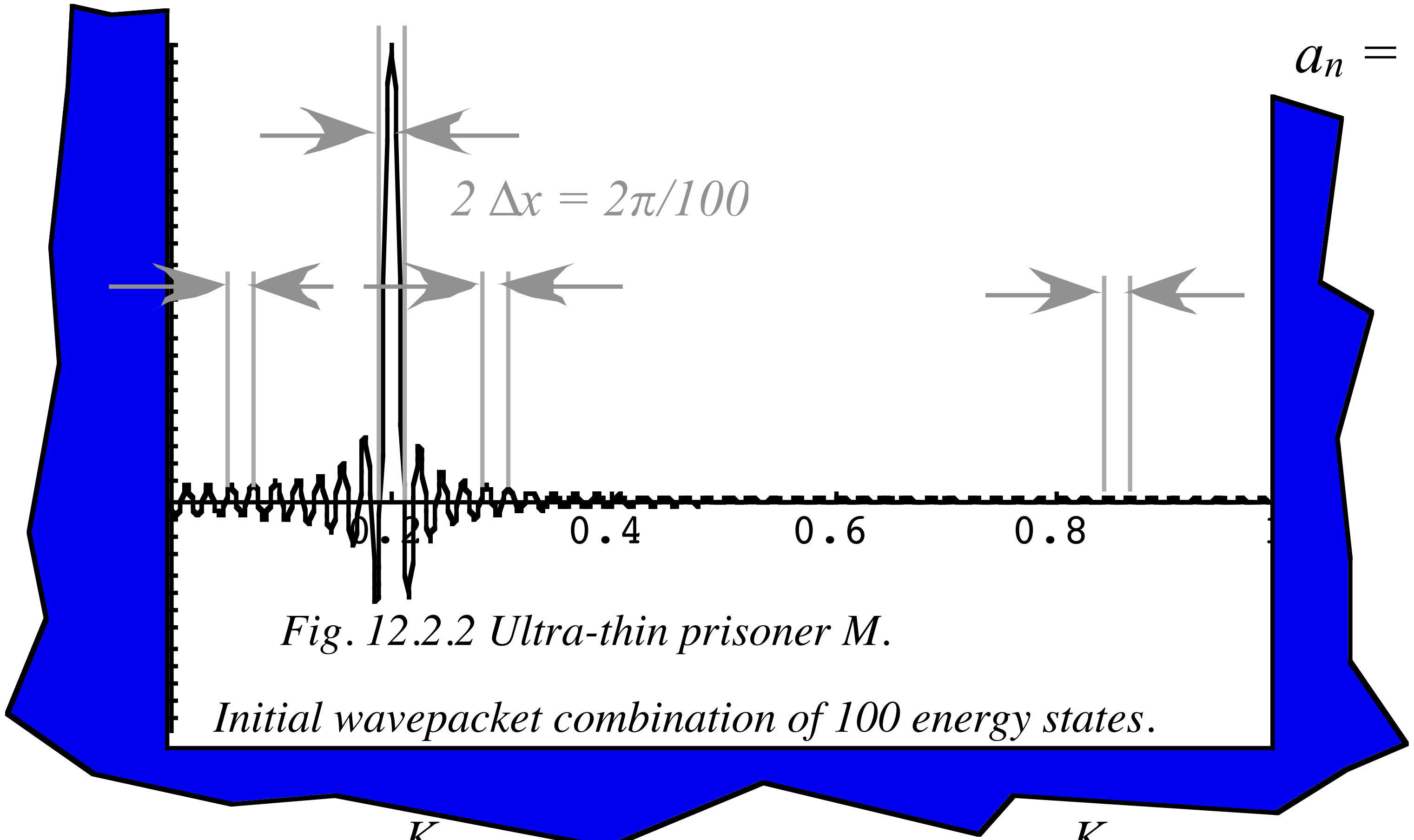

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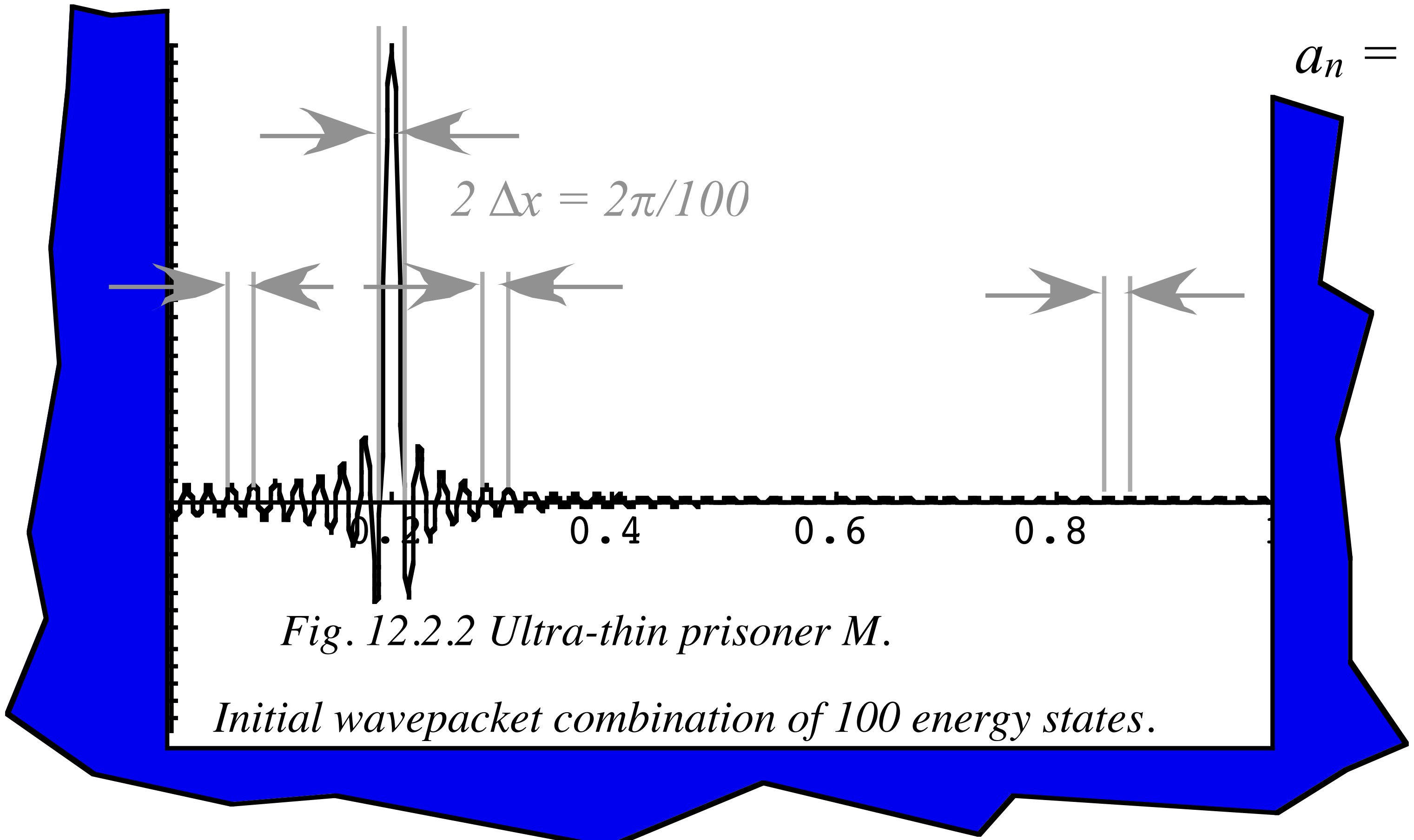

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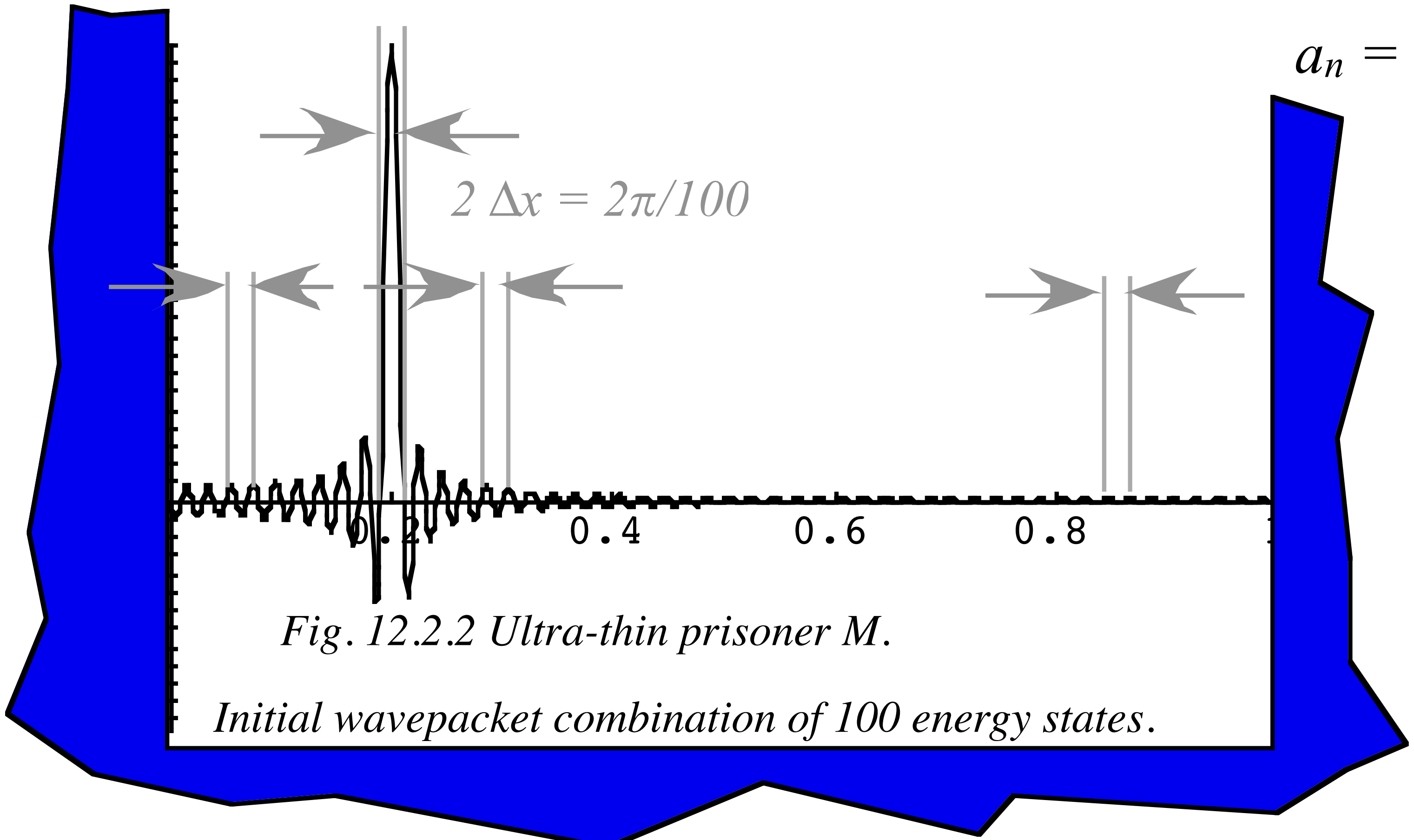

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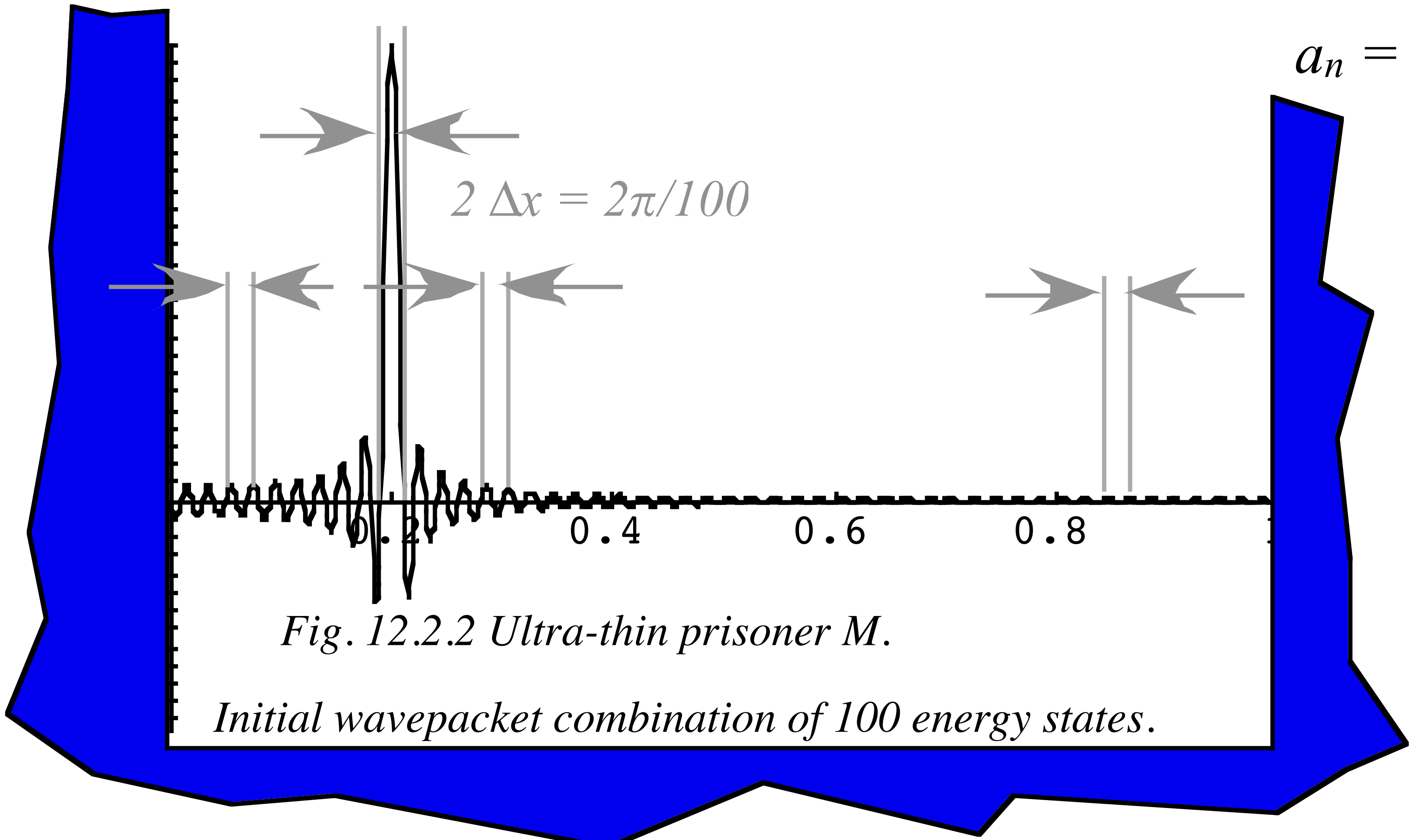

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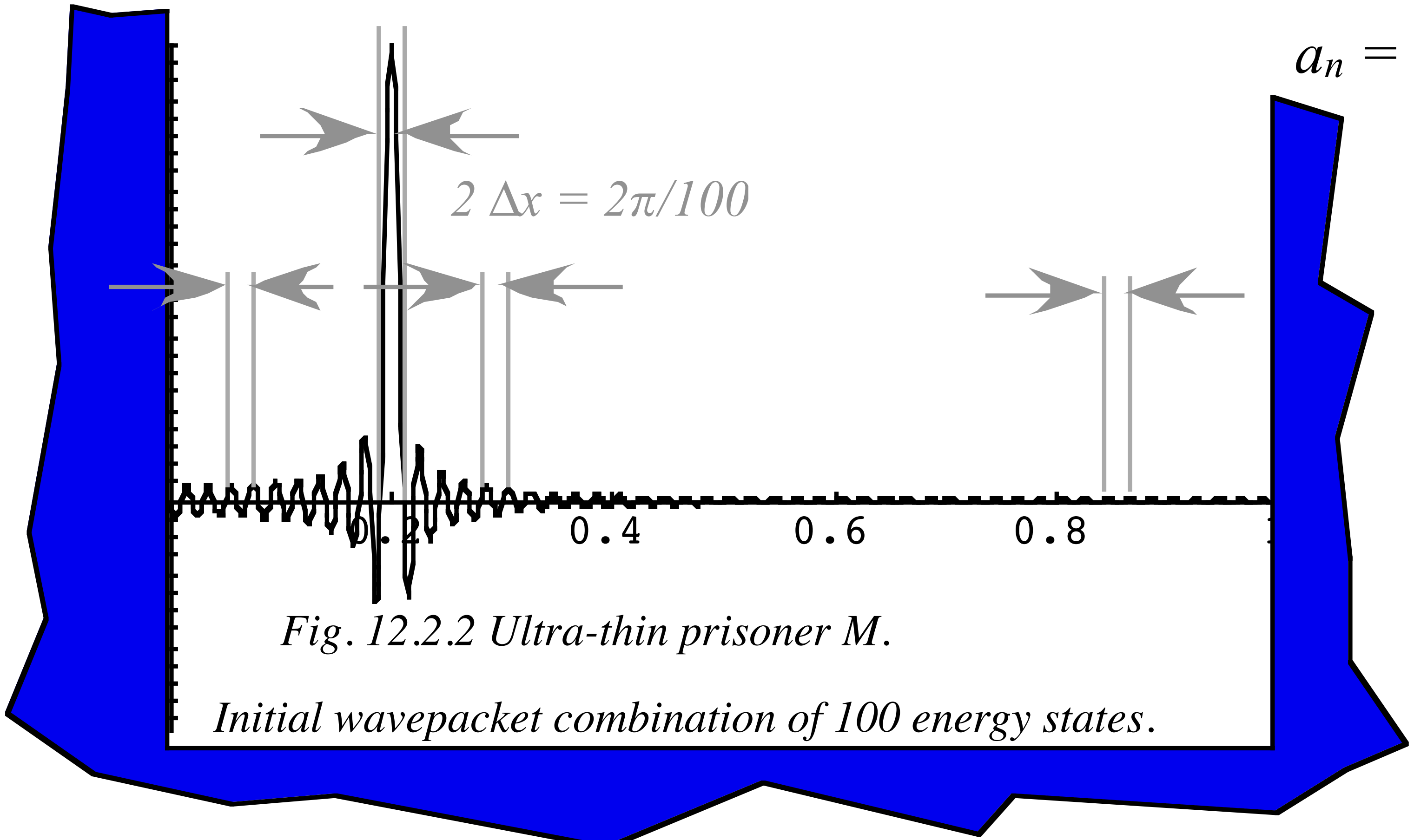

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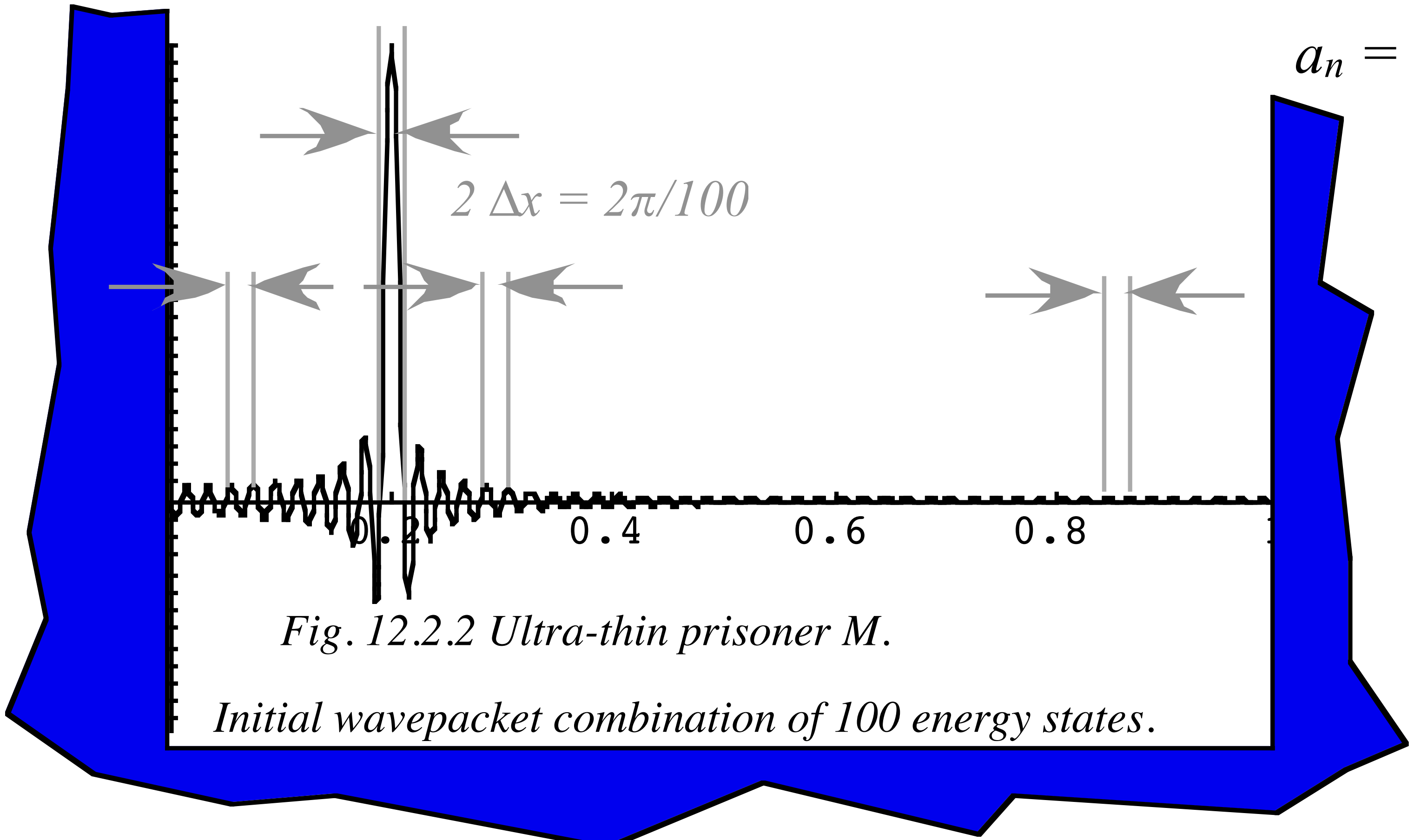

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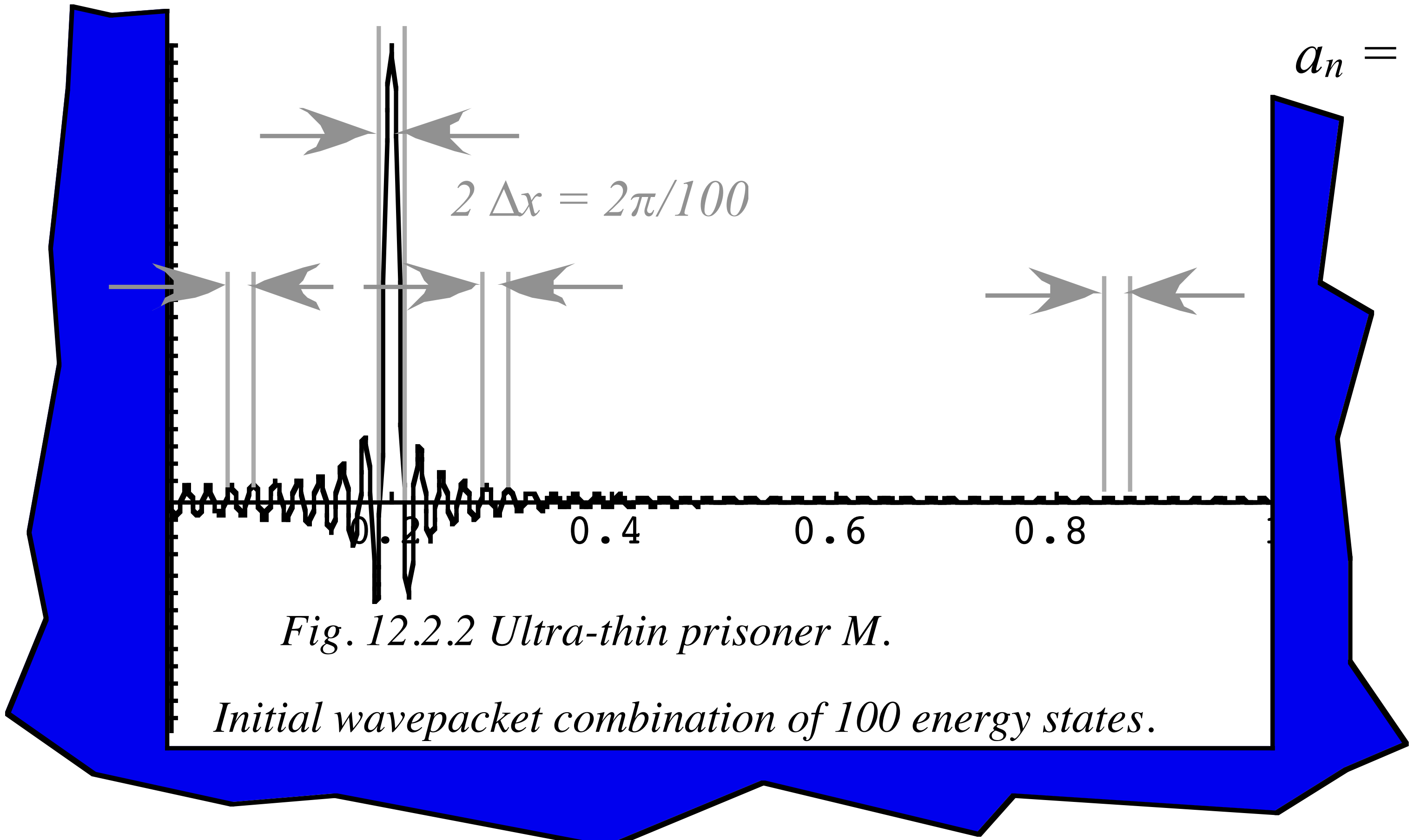

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"Last-in-first-out" effect. Last K_{max}-value dominates and "inside" K get "smothered" by interference with neighbors.

Polygonal geometry of $U(2)\supset C_N$ character spectral function
Algebra
Geometry

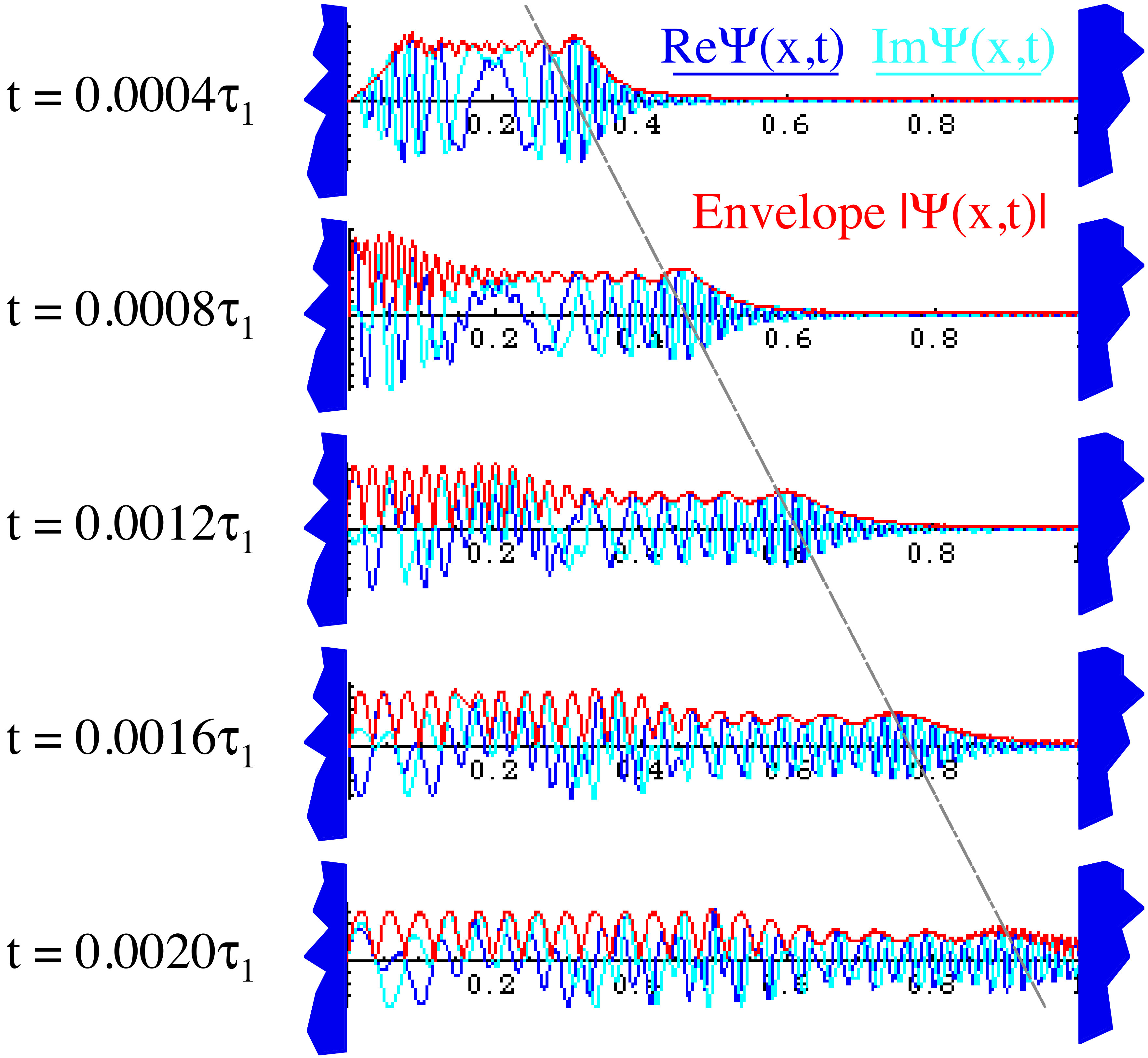
Introduction to wave dynamics of phase, mean phase, and group velocity
Expo-Cosine identity
Relating space-time and per-space-time
    Wave coordinates
    Pulse-waves (PW) vs Continuous -waves (CW)

Introduction to C_N beat dynamics and "Revivals" due to Bohr-dispersion
    ∞-Square well PE versus Bohr rotor
    SinNx/x wavepackets bandwidth and uncertainty
        → *SinNx/x explosion and revivals*
    Gaussian wave-packet bandwidth and uncertainty
        Gaussian revivals
    Farey-Sums and Ford-products
    Phase dynamics

Wavepacket explodes!

Time given in units of period τ_1 (slowest phasor of ground level).

fundamental zero-point period $\tau_1 = 1/\nu_1$



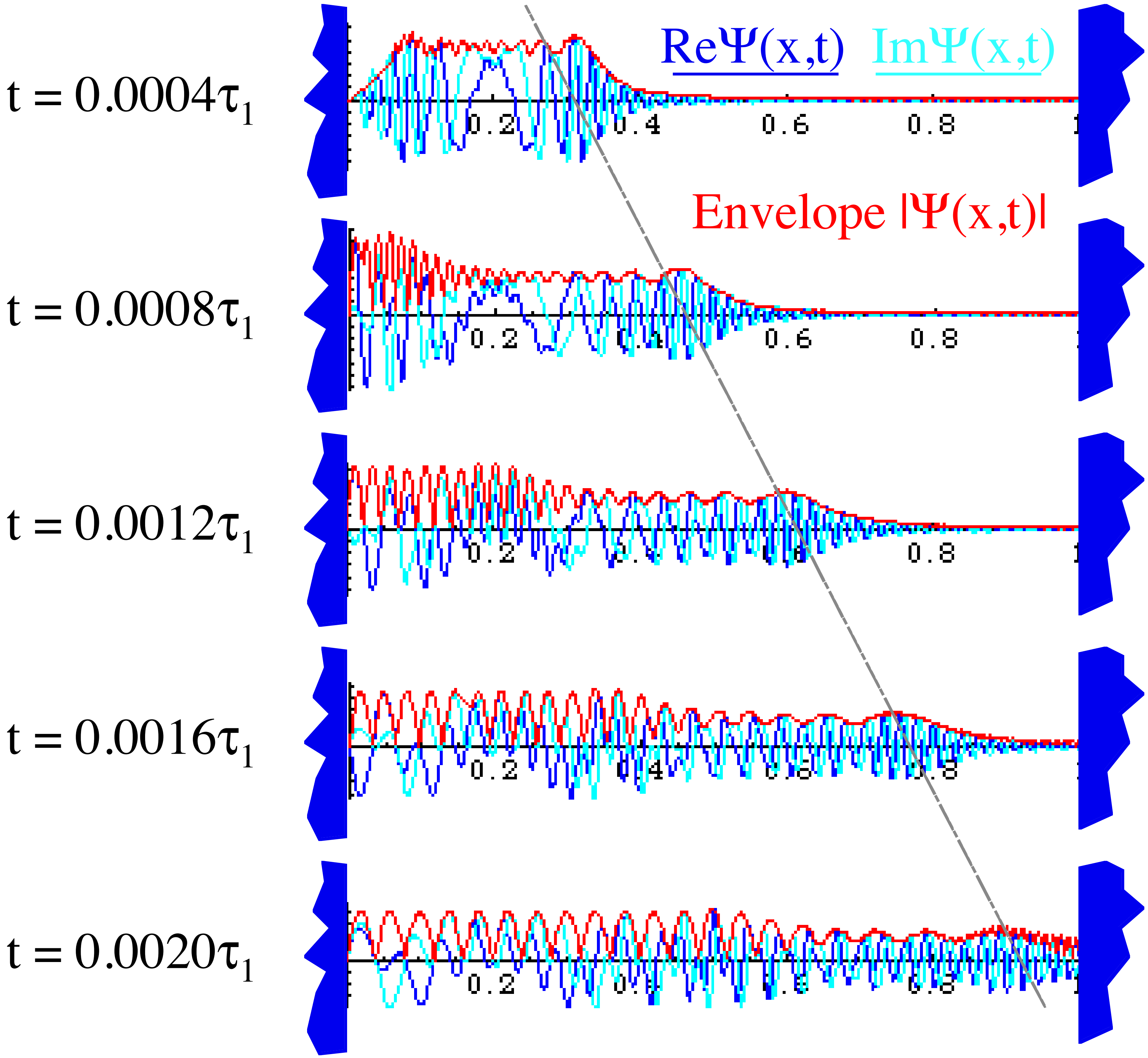
Wavepacket explodes!

Time given in units of period τ₁ (slowest phasor of ground level).

fundamental zero-point period $\tau_1 = 1/\nu_1$ is

$$\tau_1 = \frac{2\pi}{\omega_1} = \frac{2\pi\hbar}{\varepsilon_1}$$

$$= \frac{h}{h^2/8MW^2} = \frac{8MW^2}{h}$$



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ε_n-*level classical velocity*:

$$V_n = \frac{d\omega_n}{dk} = \frac{1}{\hbar}\frac{d\varepsilon_n}{dk} = \frac{1}{\hbar}\frac{\hbar^2}{2M}\frac{dk^2}{dk} = \frac{\hbar 2k_n}{2M} = \frac{\hbar n\pi}{MW} = \frac{hn}{2MW}$$

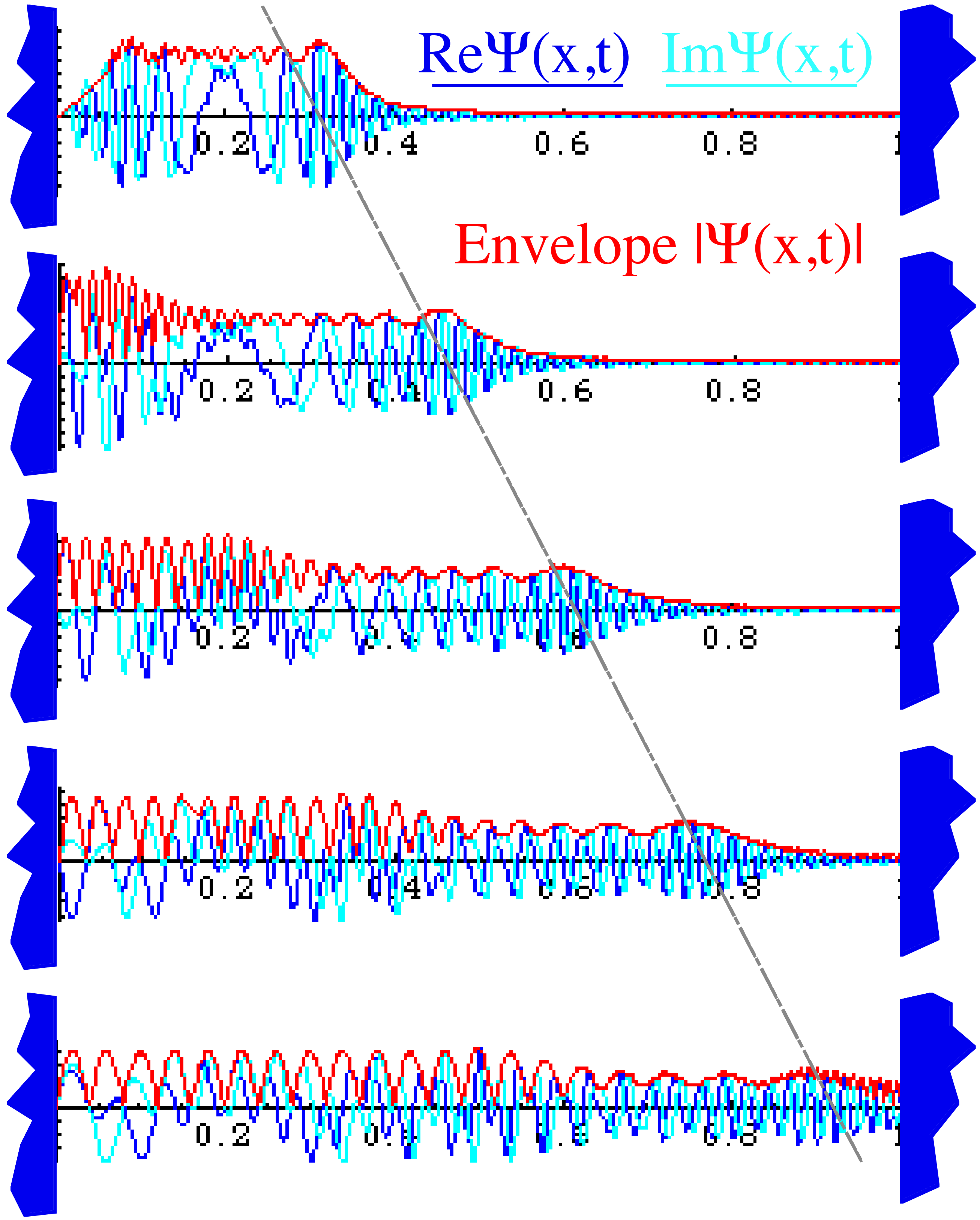
t = 0.0004τ_1

t = 0.0008τ_1

t = 0.0012τ_1

t = 0.0016τ_1

t = 0.0020τ_1



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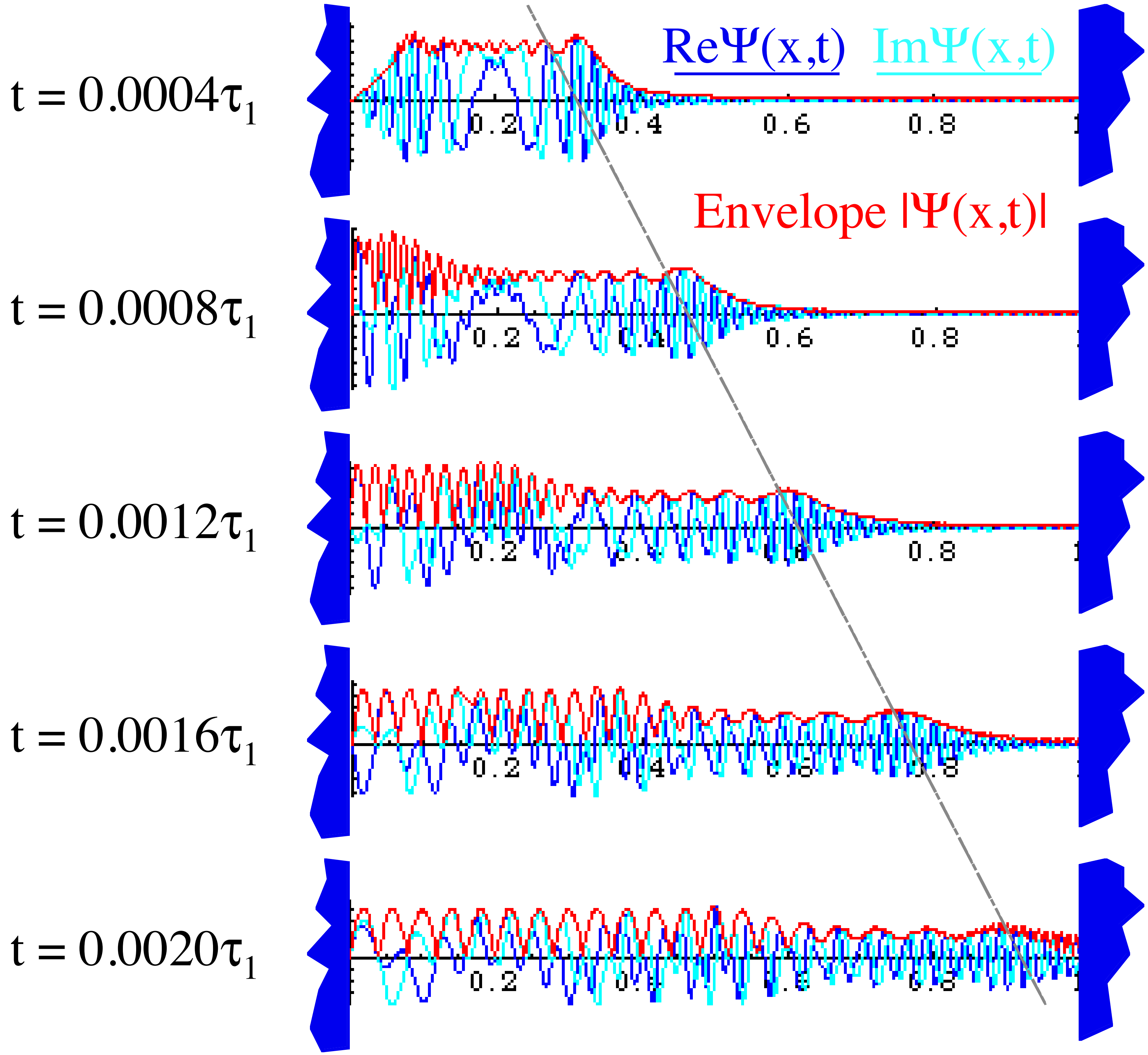


ε_n-level classical velocity:

$$V_n = \frac{d\omega_n}{dk} = \frac{1}{\hbar}\frac{d\varepsilon_n}{dk}$$
$$= \frac{1}{\hbar}\frac{\hbar^2}{2M}\frac{dk^2}{dk}$$
$$= \frac{\hbar 2k_n}{2M} = \frac{\hbar n\pi}{MW} = \frac{hn}{2MW}$$

ε_n-level classical round trip time $T_n(2W)$

$$T_n(2W) = \frac{2W}{V_n} = 2W\frac{2MW}{hn} = \frac{4MW^2}{hn}$$
$$= \frac{1}{2n}\frac{8MW^2}{h} = \frac{\tau_1}{2n}$$

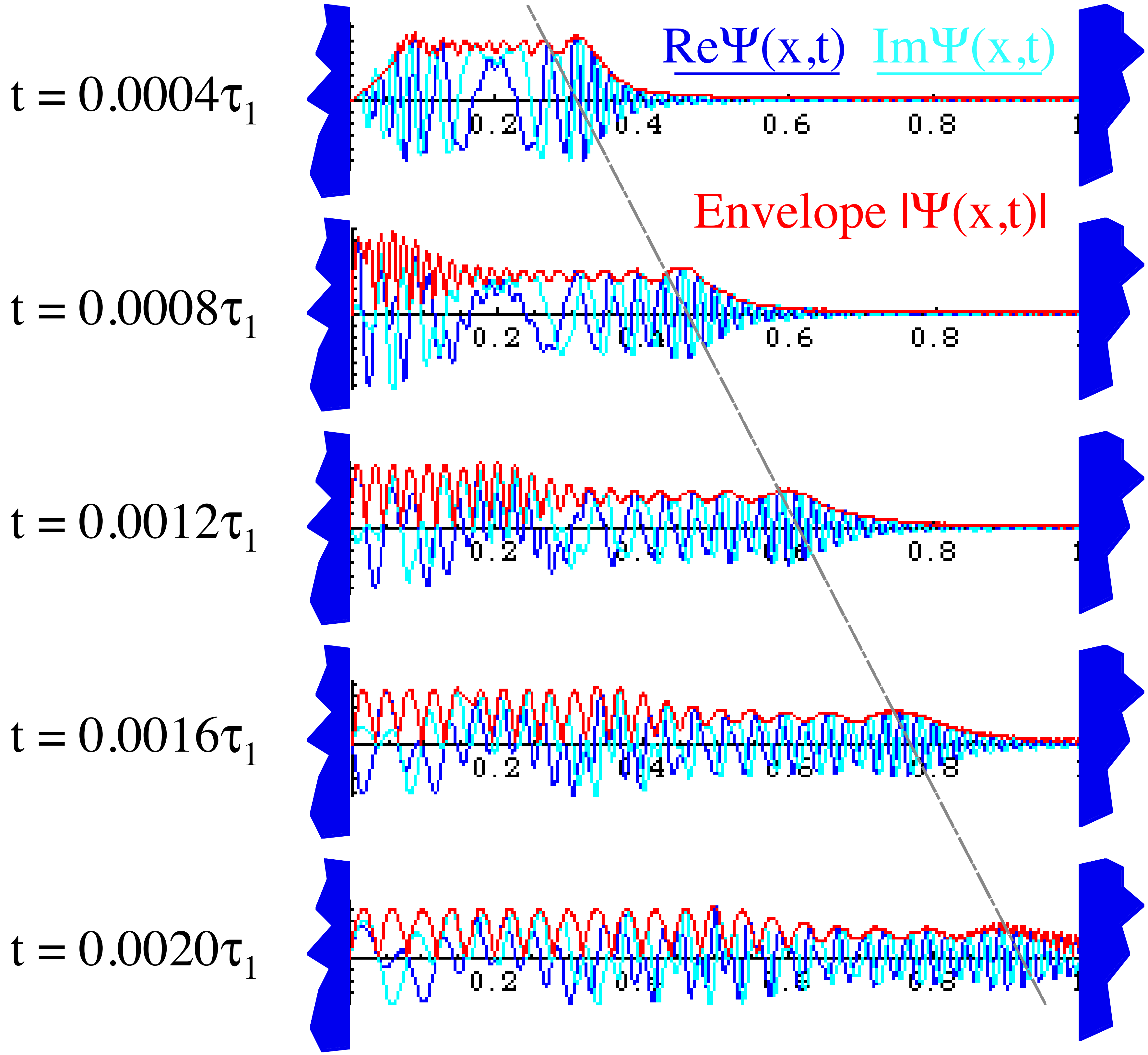
Wavepacket explodes!

Time given in units of period τ_1 (slowest phasor of ground level).

fundamental zero-point period $\tau_1 = 1/\nu_1$ is

$$\tau_1 = \frac{2\pi}{\omega_1} = \frac{2\pi\hbar}{\varepsilon_1}$$

$$= \frac{h}{h^2/8MW^2} = \frac{8MW^2}{h}$$



ε_n-level classical velocity:

$$V_n = \frac{d\omega_n}{dk} = \frac{1}{\hbar}\frac{d\varepsilon_n}{dk}$$

$$= \frac{1}{\hbar}\frac{\hbar^2}{2M}\frac{dk^2}{dk}$$

$$= \frac{\hbar 2k_n}{2M} = \frac{\hbar n\pi}{MW} = \frac{hn}{2MW}$$

ε_n-level classical round trip time $T_n(2W)$

$$T_n(2W) = \frac{2W}{V_n} = 2W\frac{2MW}{hn} = \frac{4MW^2}{hn}$$

$$= \frac{1}{2n}\frac{8MW^2}{h} = \frac{\tau_1}{2n}$$

ε_n-level 1-way time $T_n(W)$

$$T_n(W) = T_n(2W)/2 = \frac{\tau_1}{4n}$$

$$\left(= 0.0025\,\tau_1 \quad \text{for: } n=100\right)$$

Wavepacket explodes!

Time given in units of period τ_1 (slowest phasor of ground level).

fundamental zero-point period $\tau_1 = 1/\nu_1$ is

$$\tau_1 = \frac{2\pi}{\omega_1} = \frac{2\pi\hbar}{\varepsilon_1} = \frac{h}{h^2/8MW^2} = \frac{8MW^2}{h}$$

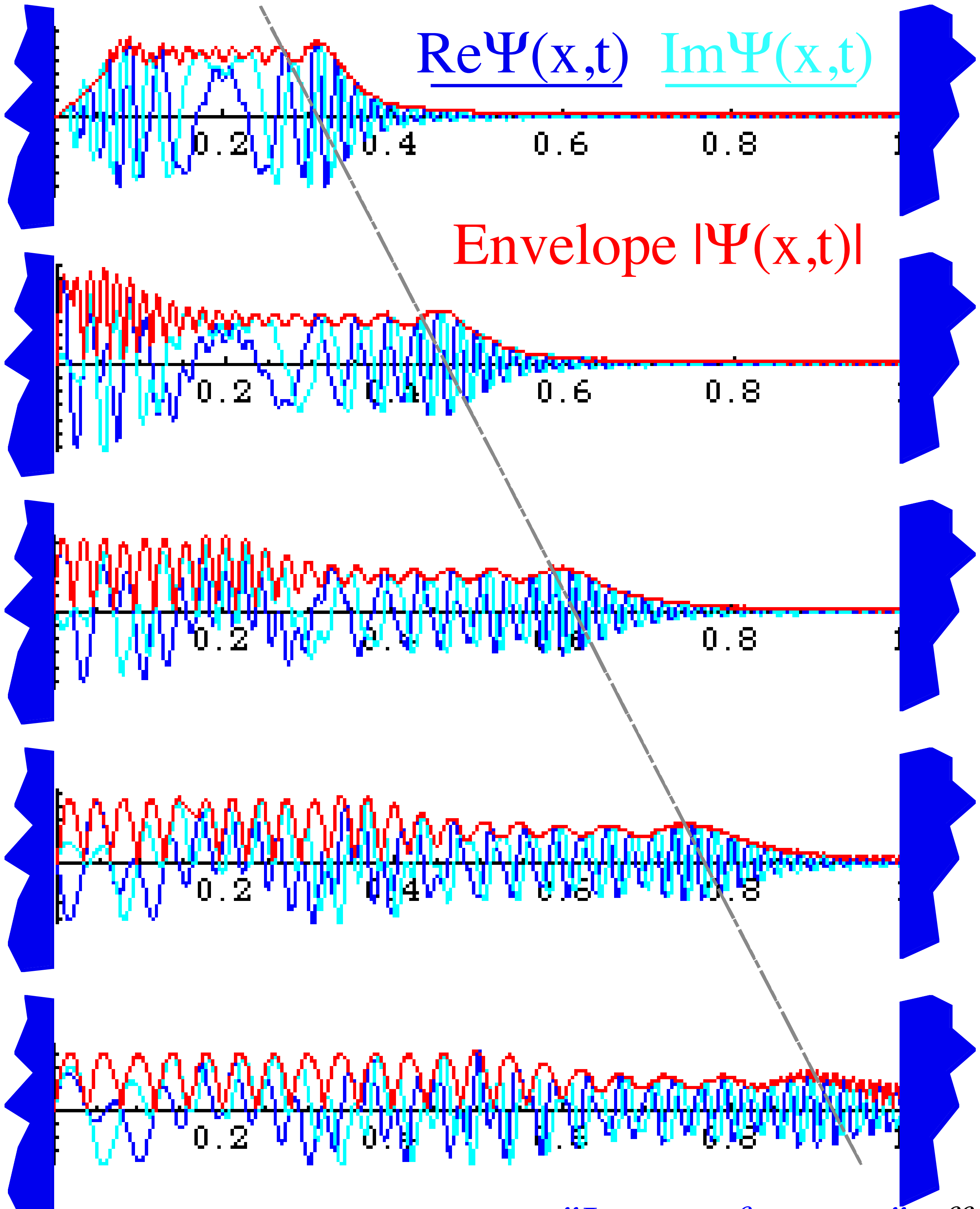
t = $0.0004\tau_1$

t = $0.0008\tau_1$

t = $0.0012\tau_1$

t = $0.0016\tau_1$

t = $0.0020\tau_1$



"Last-in-first-out" effect

ε_n-level classical velocity:

$$V_n = \frac{d\omega_n}{dk} = \frac{1}{\hbar}\frac{d\varepsilon_n}{dk} = \frac{1}{\hbar}\frac{\hbar^2}{2M}\frac{dk^2}{dk} = \frac{\hbar 2k_n}{2M} = \frac{\hbar n\pi}{MW} = \frac{hn}{2MW}$$

ε_n-level classical round trip time $T_n(2W)$

$$T_n(2W) = \frac{2W}{V_n} = 2W\frac{2MW}{hn} = \frac{4MW^2}{hn} = \frac{1}{2n}\frac{8MW^2}{h} = \frac{\tau_1}{2n}$$

ε_n-level 1-way time $T_n(W)$

$$T_n(W) = T_n(2W)/2 = \frac{\tau_1}{4n}$$

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Polygonal geometry of $U(2) \supset C_N$ character spectral function
Algebra
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Introduction to wave dynamics of phase, mean phase, and group velocity
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  Gaussian wave-packet bandwidth and uncertainty
    Gaussian revivals
  Farey-Sums and Ford-products
  Phase dynamics

Wavepacket explodes! (Then revives)

Zero-point period τ_1 is just enough time for "particle" in ε_n-level to make *2n* round trips.

$$\tau_1 = 2n\, T_n(2W) = \frac{8ML^2}{h}$$

In time τ_1 ground ε_1-level particle does 2 round trips,

ε_2-level particle makes 4 round trips,

ε_3-level particle makes 6 round trips,..,

At time τ_1, *M* undergoes a *full revival* and "unexplodes" into his original spike at $x=0.2W$,

Wavepacket explodes! (Then revives)

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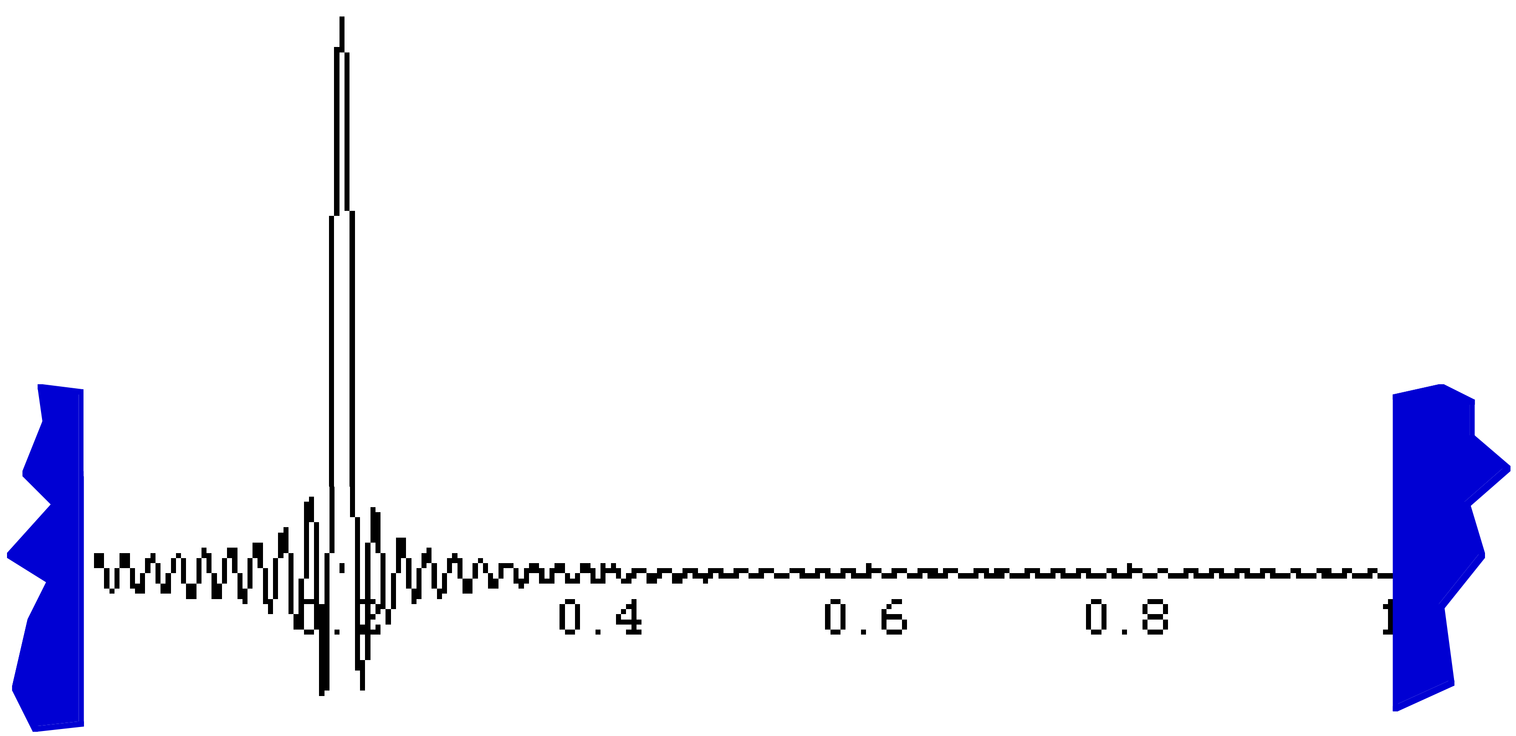
In time τ_1 ground ε_1-level particle does 2 round trips,
ε_2-level particle makes 4 round trips,
ε_3-level particle makes 6 round trips,..,

At time τ_1, *M* undergoes a *full revival* and "unexplodes" into his original spike at *x=0.2W*,

$t = 1.0000\tau_1$
$= 3.0\tau_{beat}$



Wavepacket explodes! (Then revives)

Zero-point period τ_1 is just enough time for "particle" in ε_n-level to make *2n* round trips.

$$\tau_1 = 2n\, T_n(2W) = \frac{8ML^2}{h}$$

In time τ_1 ground ε_1-level particle does 2 round trips,

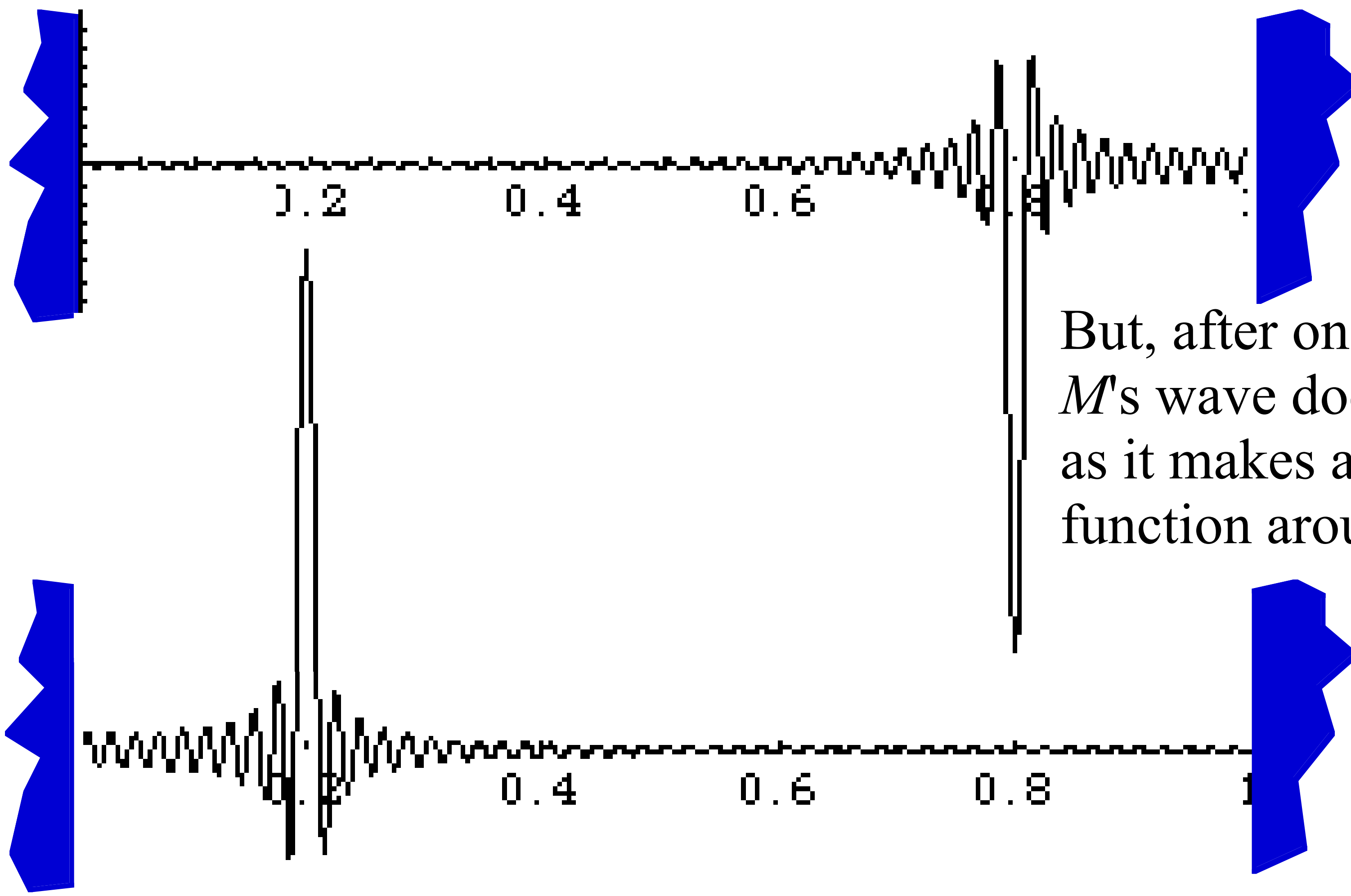
ε_2-level particle makes 4 round trips,

ε_3-level particle makes 6 round trips,..,

At time τ_1, *M* undergoes a *full revival* and "unexplodes" into his original spike at $x=0.2W$,

$t = 0.5000\tau_1$

$= 1.5\tau_{beat}$



But, after only 50 round-trips *M*'s wave does a *partial revival* as it makes an upside down-delta function around $x=0.8W$.

$t = 1.0000\tau_1$

$= 3.0\tau_{beat}$

At fractional times τ_1/n *M* undergoes a number of *fractional revivals*

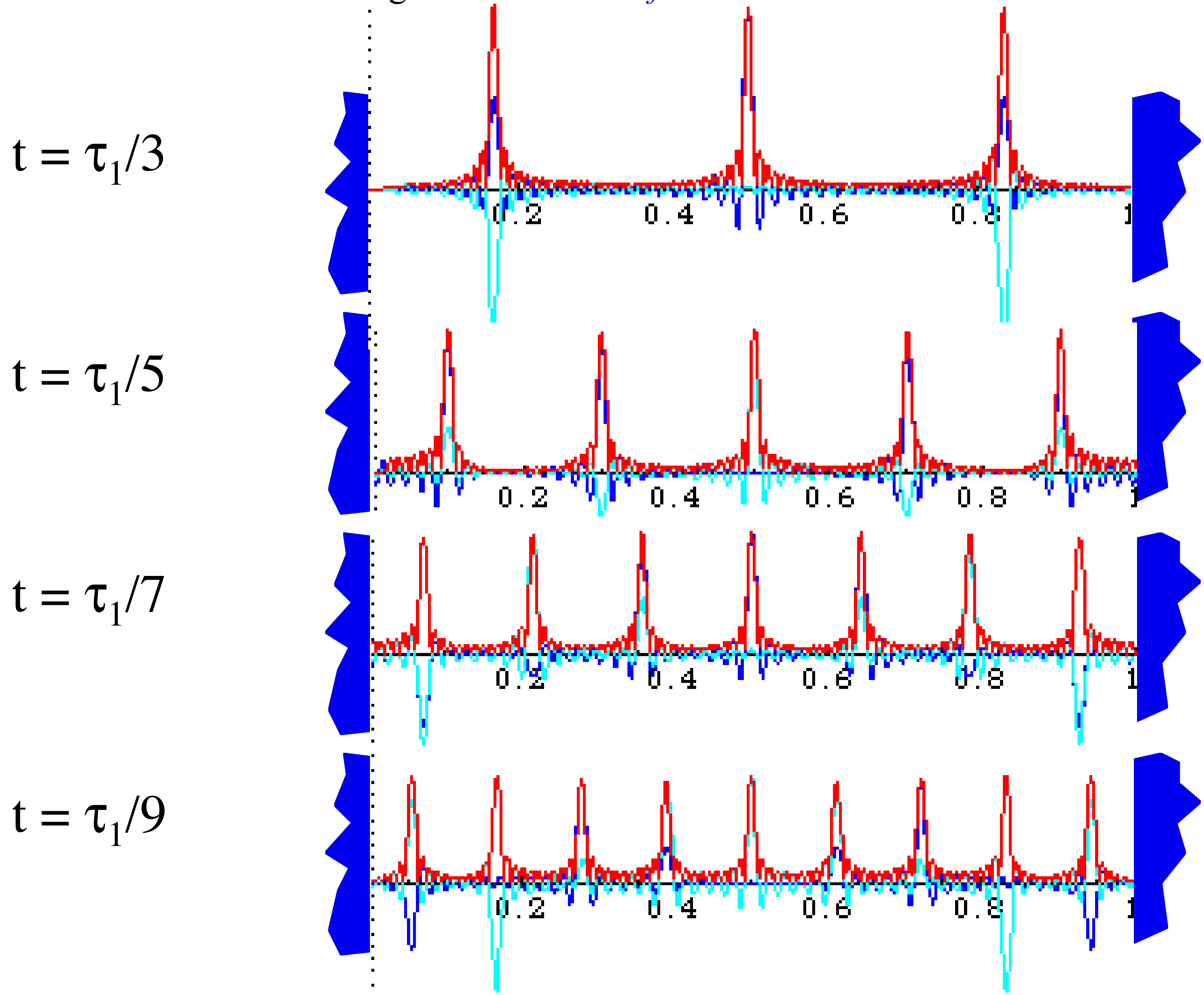
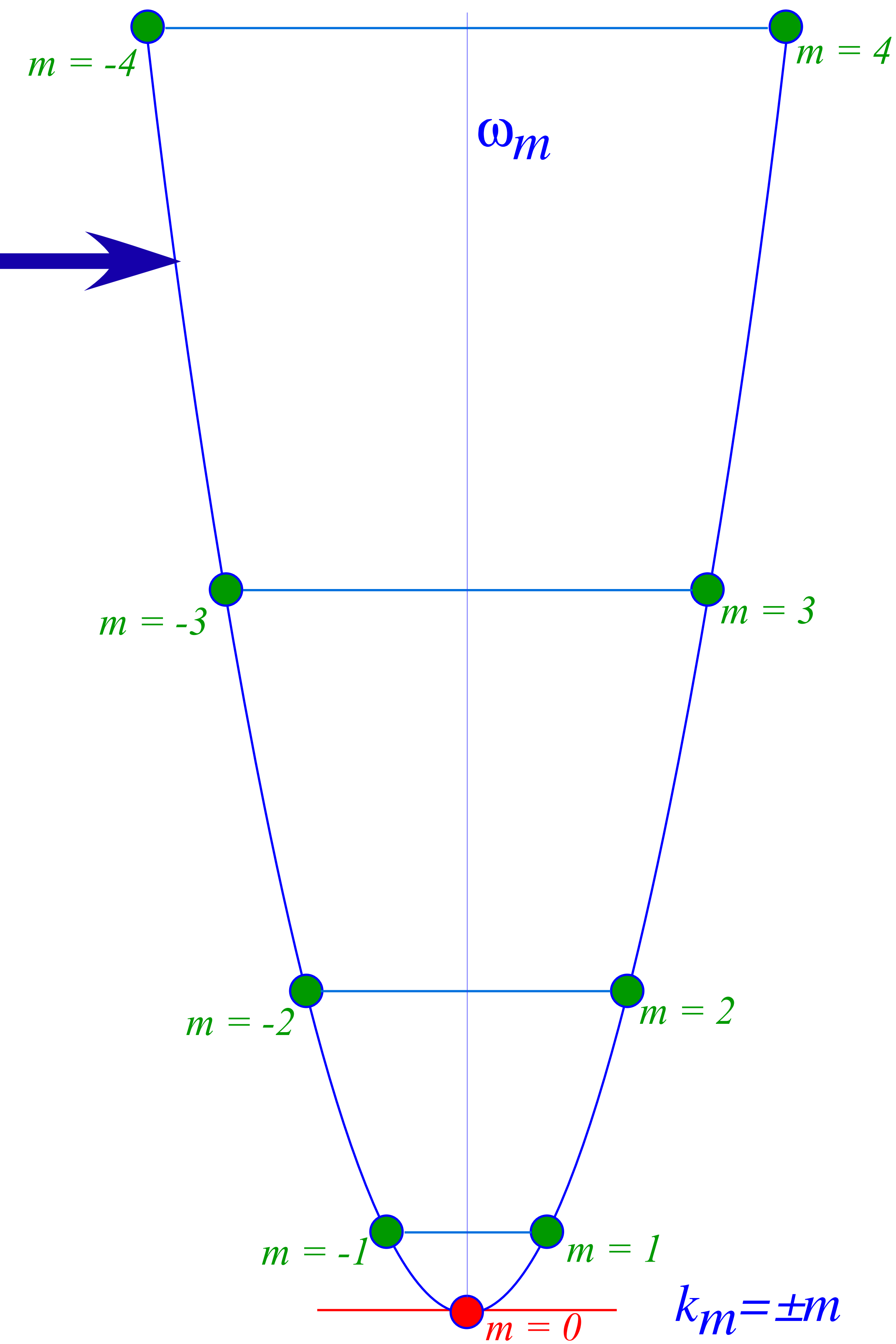


Fig. 12.2.5 The "Dance of the deltas." Mini-Revivals for prisoner M's wavepacket envelope function.

Levels
for
Quadratic (Bohr-Rotor) Spectrum

$\omega_m = Bm^2$

$k_m = \pm m$



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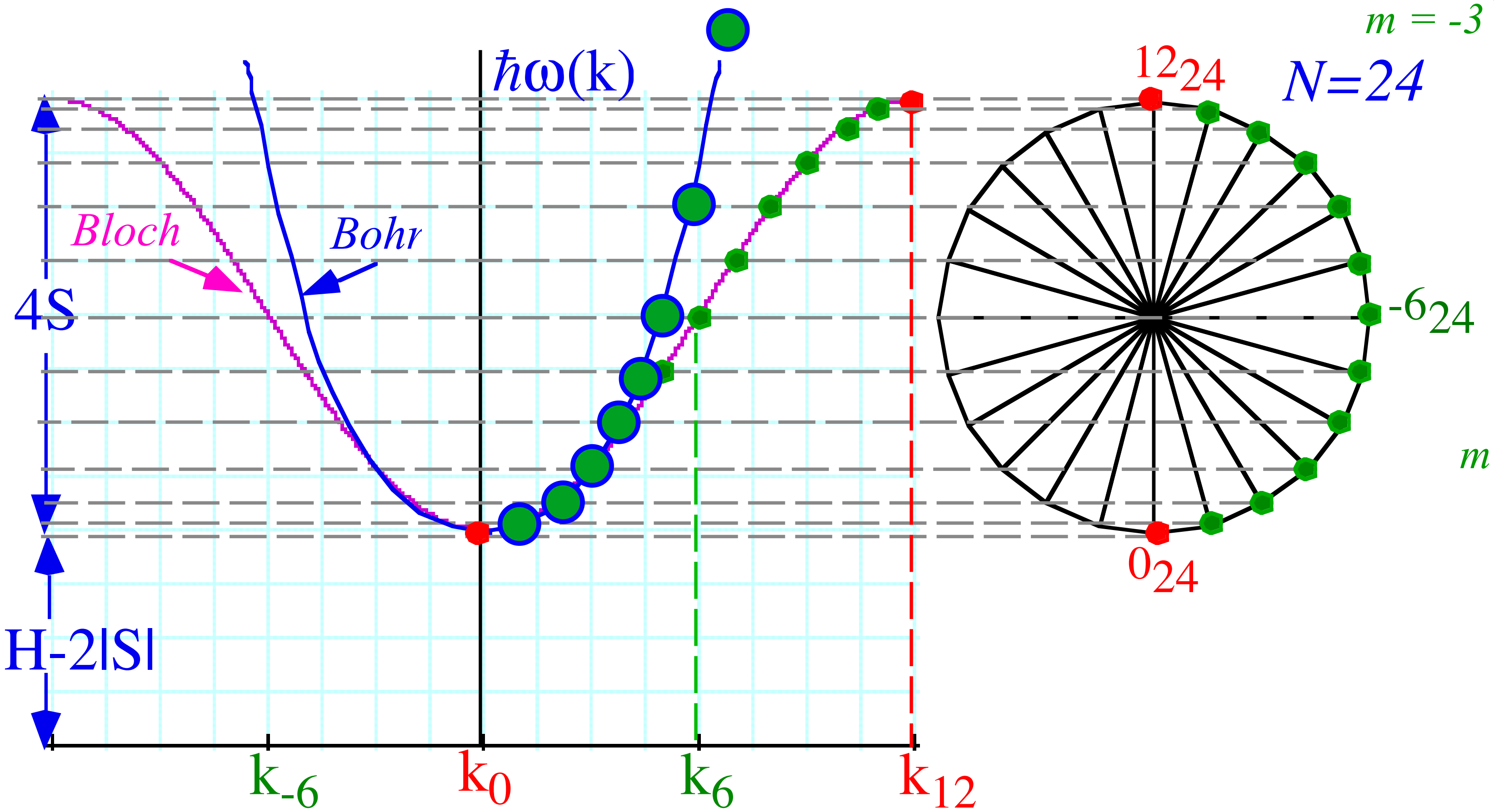
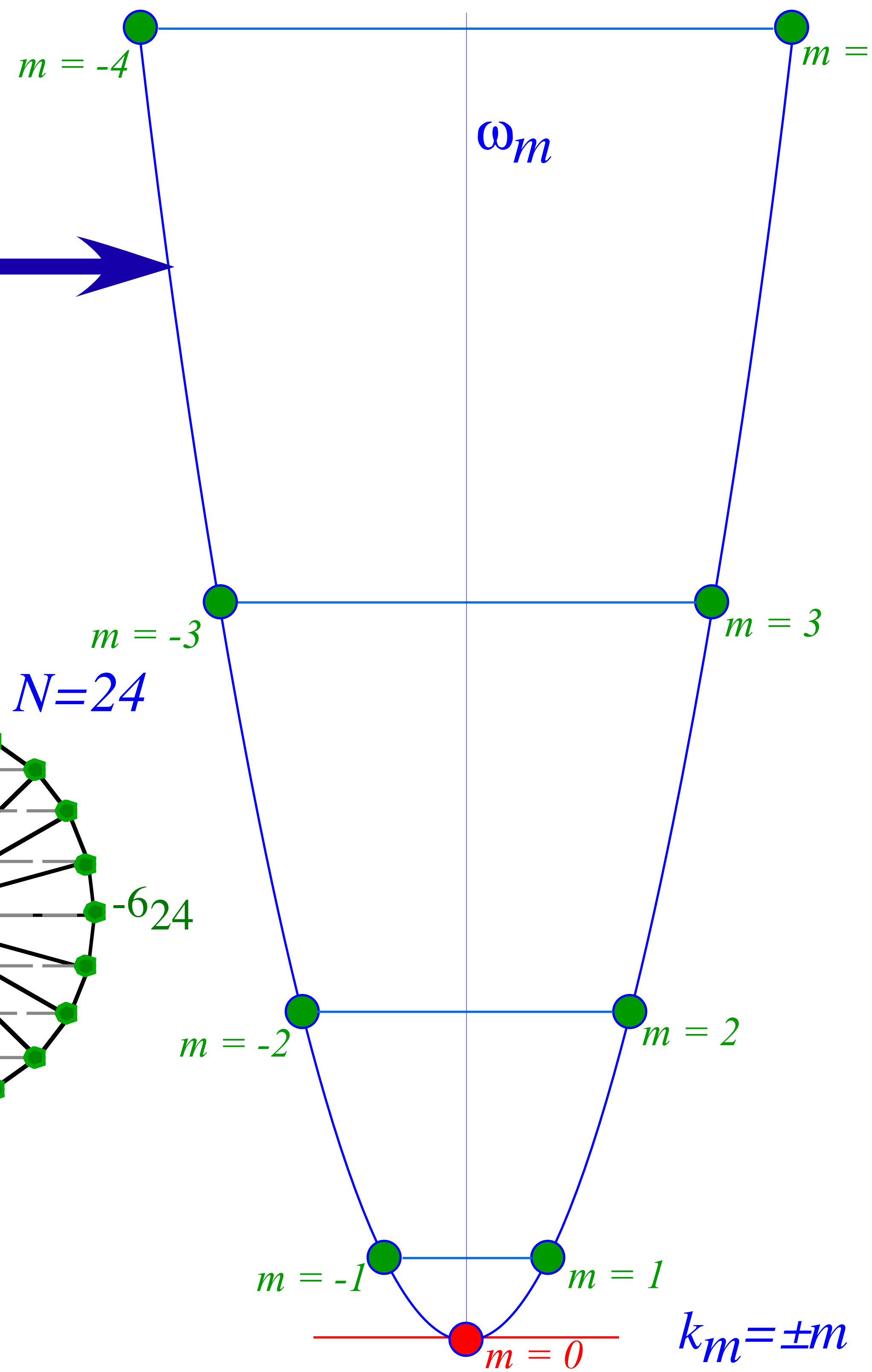


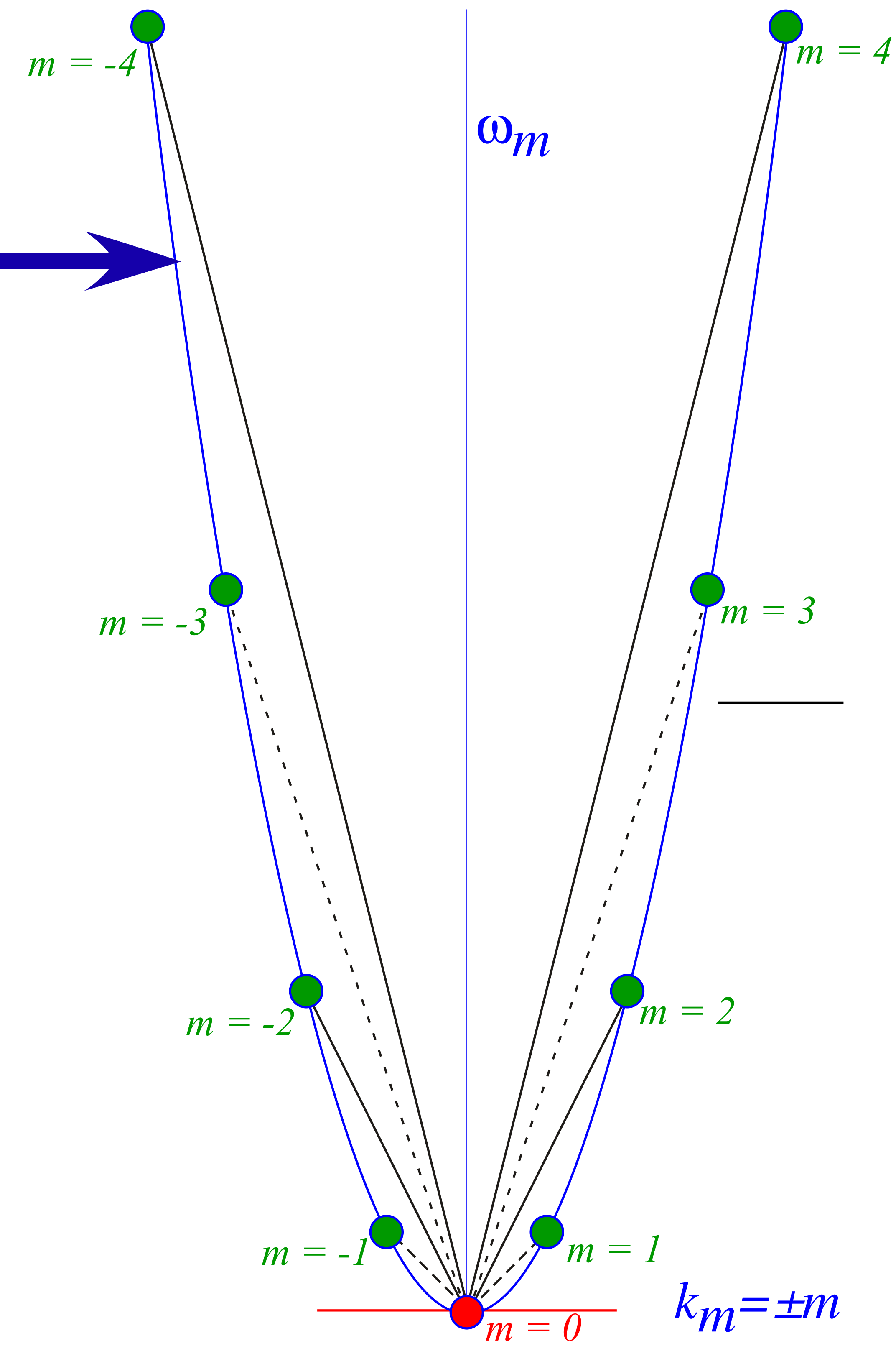


Possible wave velocities
for
Quadratic (Bohr-Rotor) Spectrum

$\omega_m = Bm^2$

$k_m = \pm m$

$$V_{phase} = \frac{\omega_m}{k_m} = \frac{Bm^2}{m} = mB$$



Possible wave velocities
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Quadratic (Bohr-Rotor) Spectrum

$\omega_m = Bm^2$

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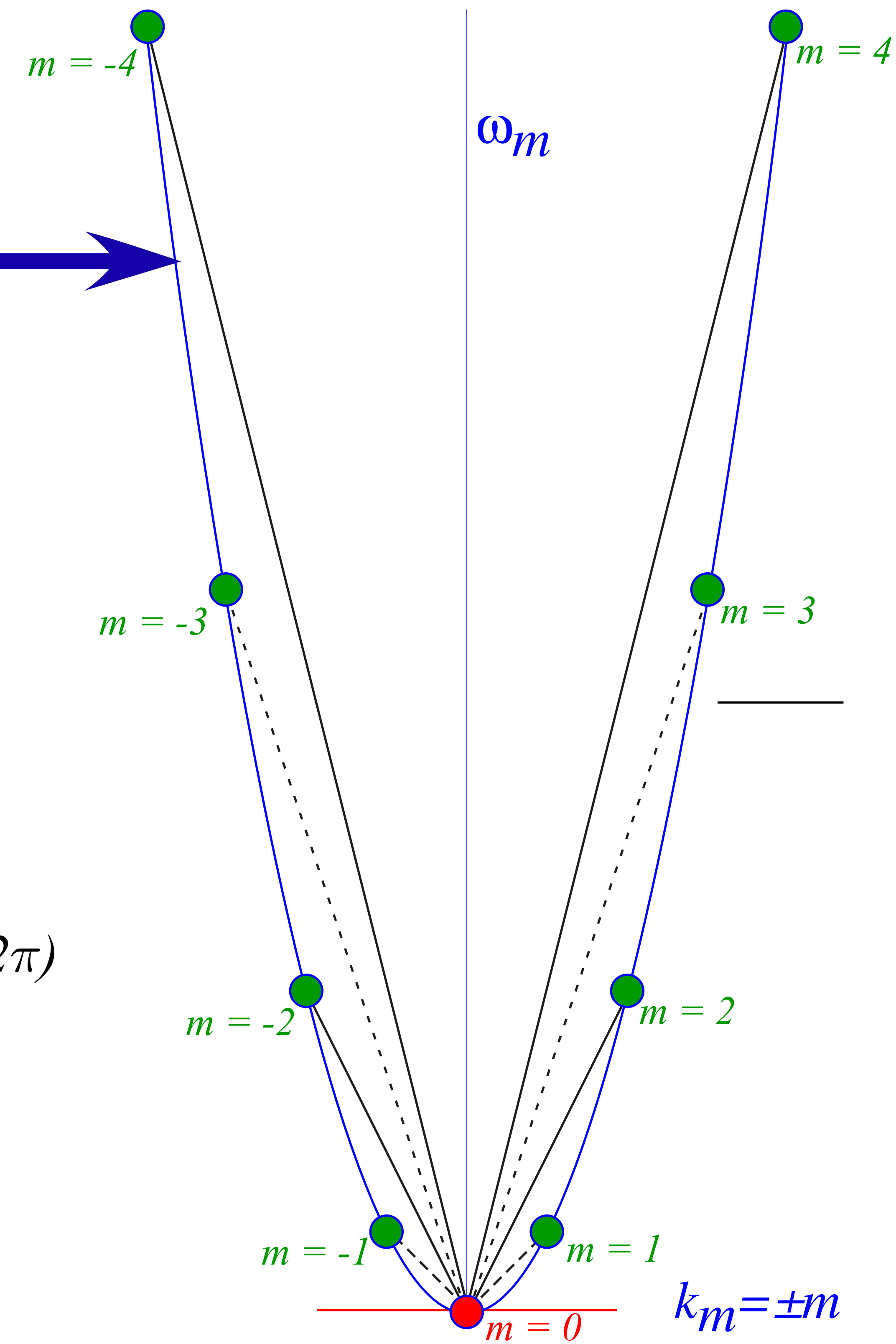

$m=0, \pm1, \pm2, \pm3,$...are momentum quanta
in wavevector formula: $k_m=2\pi m/L$ $\quad$ ($k_m=m$ $\quad$ if: $L=2\pi$)

Possible wave velocities
for
Quadratic (Bohr-Rotor) Spectrum
$\omega_m = Bm^2$
$k_m = \pm m$

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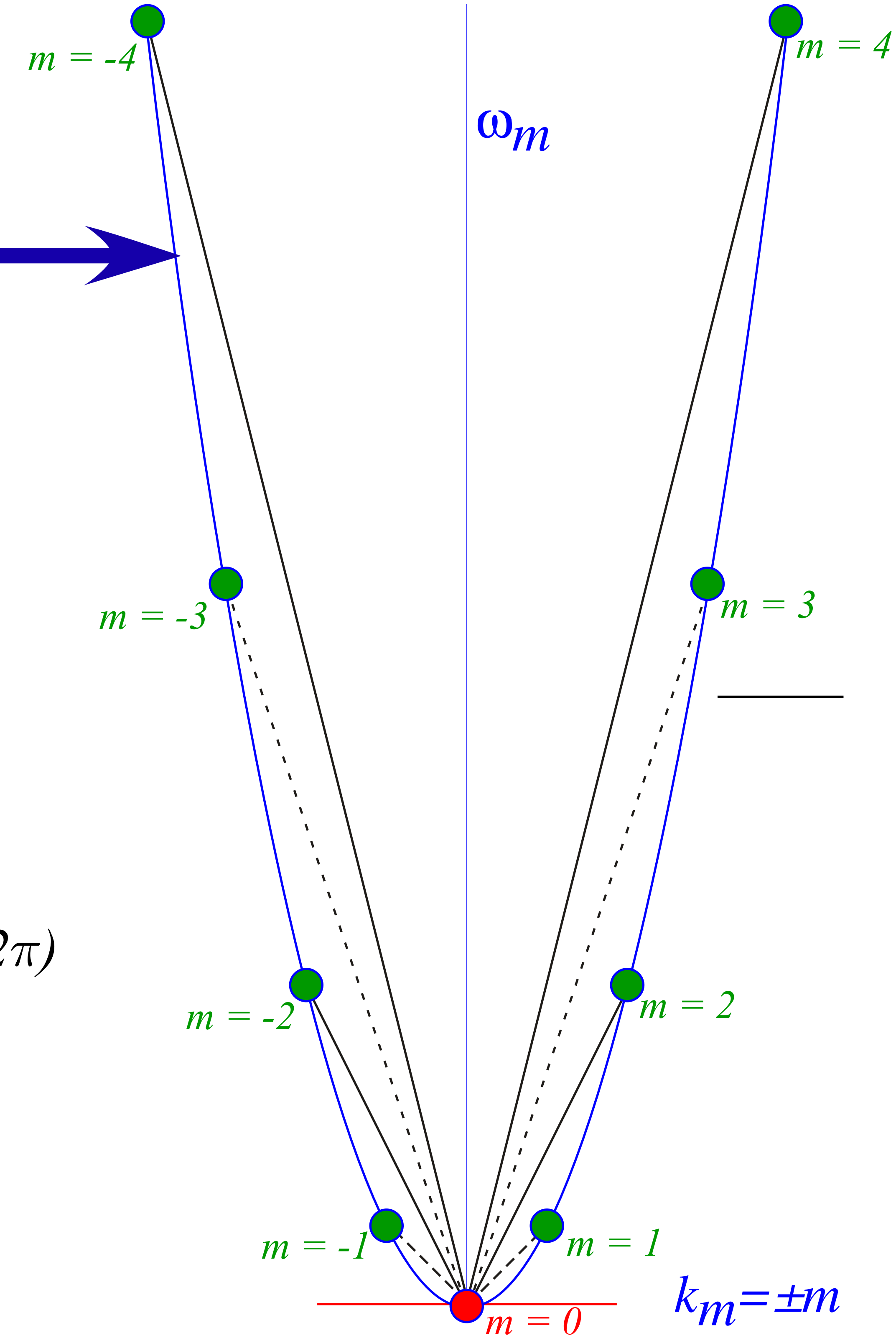


$m=0, \pm 1, \pm 2, \pm 3$,...are momentum quanta
in wavevector formula: $k_m = 2\pi m/L$ $\quad (k_m = m$ $\quad$ *if:* $L = 2\pi)$

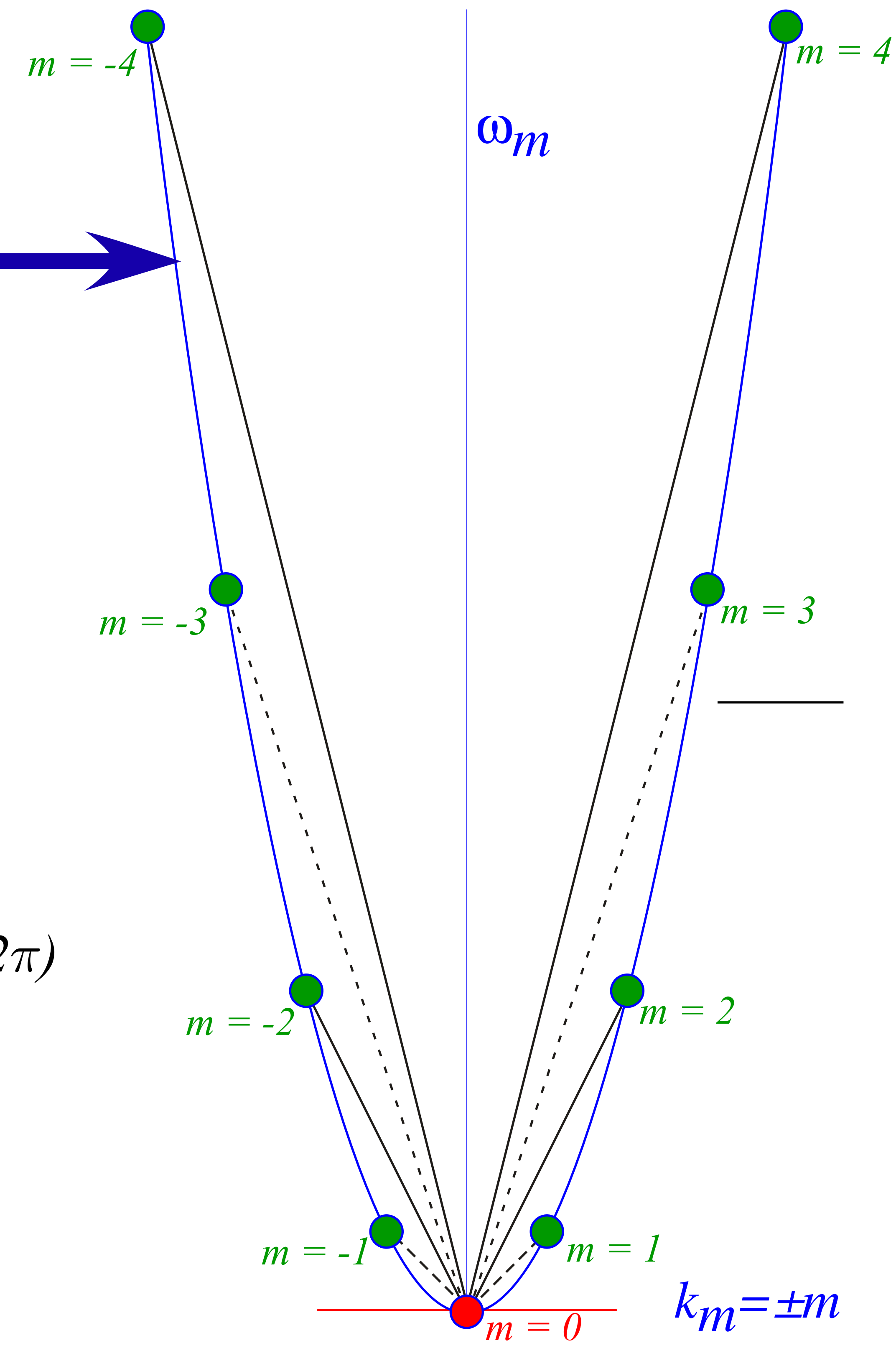
$$E_m = (\hbar k_m)^2/2M = m^2[h^2/2ML^2] = m^2\, h\upsilon_1 = m^2\, \hbar\omega_1$$

Possible wave velocities
for
Quadratic (Bohr-Rotor) Spectrum

$\omega_m = Bm^2$

$k_m = \pm m$

$$V_{phase} = \frac{\omega_m}{k_m} = \frac{Bm^2}{m} = mB$$



$m = 0, \pm 1, \pm 2, \pm 3, \ldots$ are momentum quanta
in wavevector formula: $k_m = 2\pi m/L$ $\quad$ ($k_m = m$ if: $L = 2\pi$)

$$E_m = (\hbar k_m)^2/2M = m^2[h^2/2ML^2] = m^2\, h\upsilon_1 = m^2\, \hbar\omega_1$$

fundamental Bohr ∠−frequency $\omega_1 = 2\pi\upsilon_1$
and lowest *transition (beat) frequency* $\upsilon_1 = (E_1 - E_0)/h$

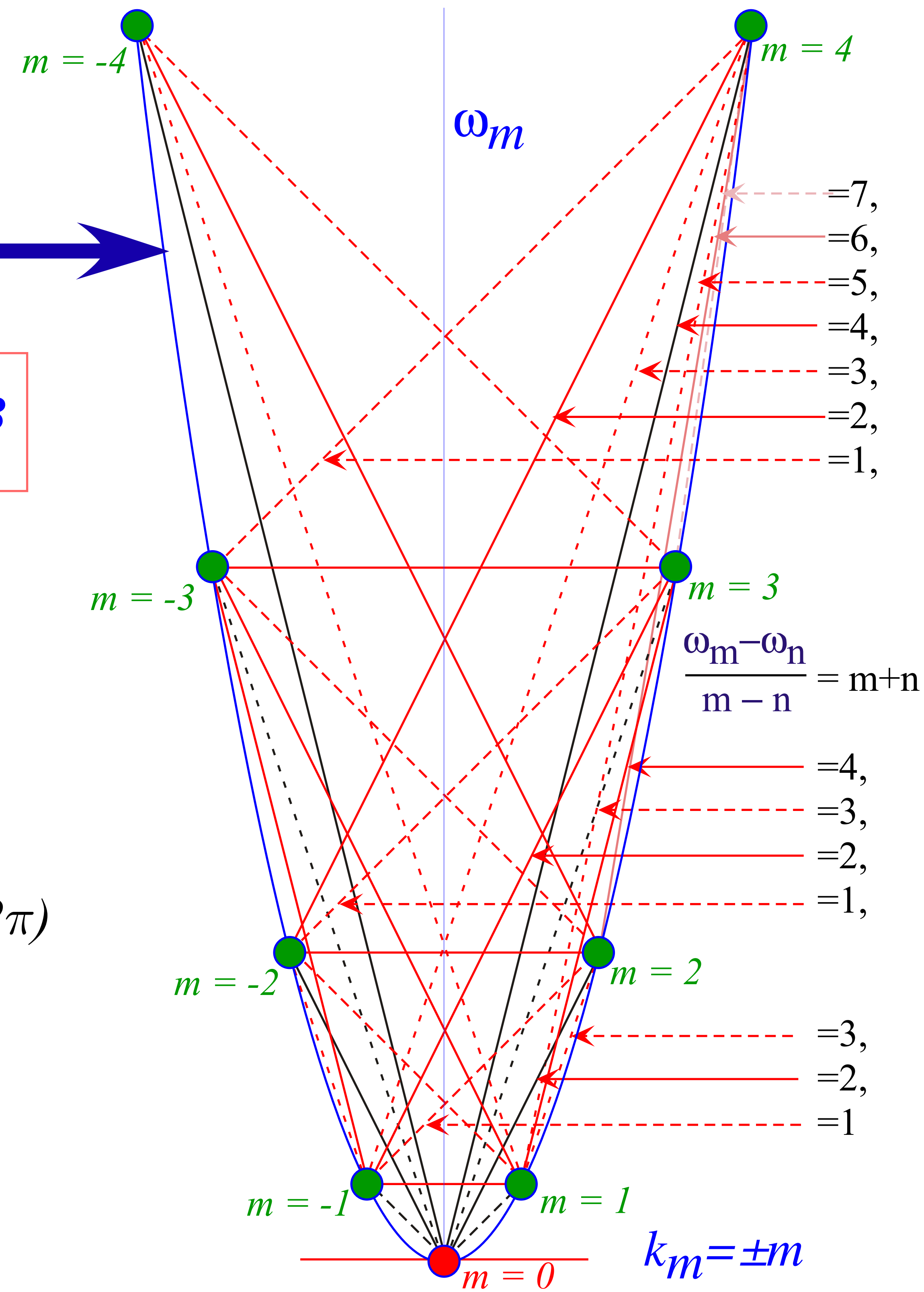
Possible wave velocities
for
Quadratic (Bohr-Rotor) Spectrum

$\omega_m = Bm^2$

$k_m = \pm m$

$$V_{phase} = \frac{\quad}{k_m} = \frac{Bm^2}{m} = mB$$

$$V_{group} = \frac{\omega_m - \omega_n}{k_m - k_n} = \frac{m^2 - n^2}{m \pm n} B = (m \pm n)B$$



$m = 0, \pm 1, \pm 2, \pm 3$,...are momentum quanta
in wavevector formula: $k_m = 2\pi m/L$ $\quad (k_m = m \quad if: L = 2\pi)$

$$E_m = (\hbar k_m)^2/2M = m^2[h^2/2ML^2] = m^2 h\upsilon_1 = m^2 \hbar\omega_1$$

fundamental Bohr ∠−frequency $\omega_1 = 2\pi\upsilon_1$
and lowest *transition (beat) frequency* $\upsilon_1 = (E_1 - E_0)/h$

Possible wave velocities
for
Quadratic (Bohr-Rotor) Spectrum

$$\omega_m = Bm^2$$
$$k_m = \pm m$$

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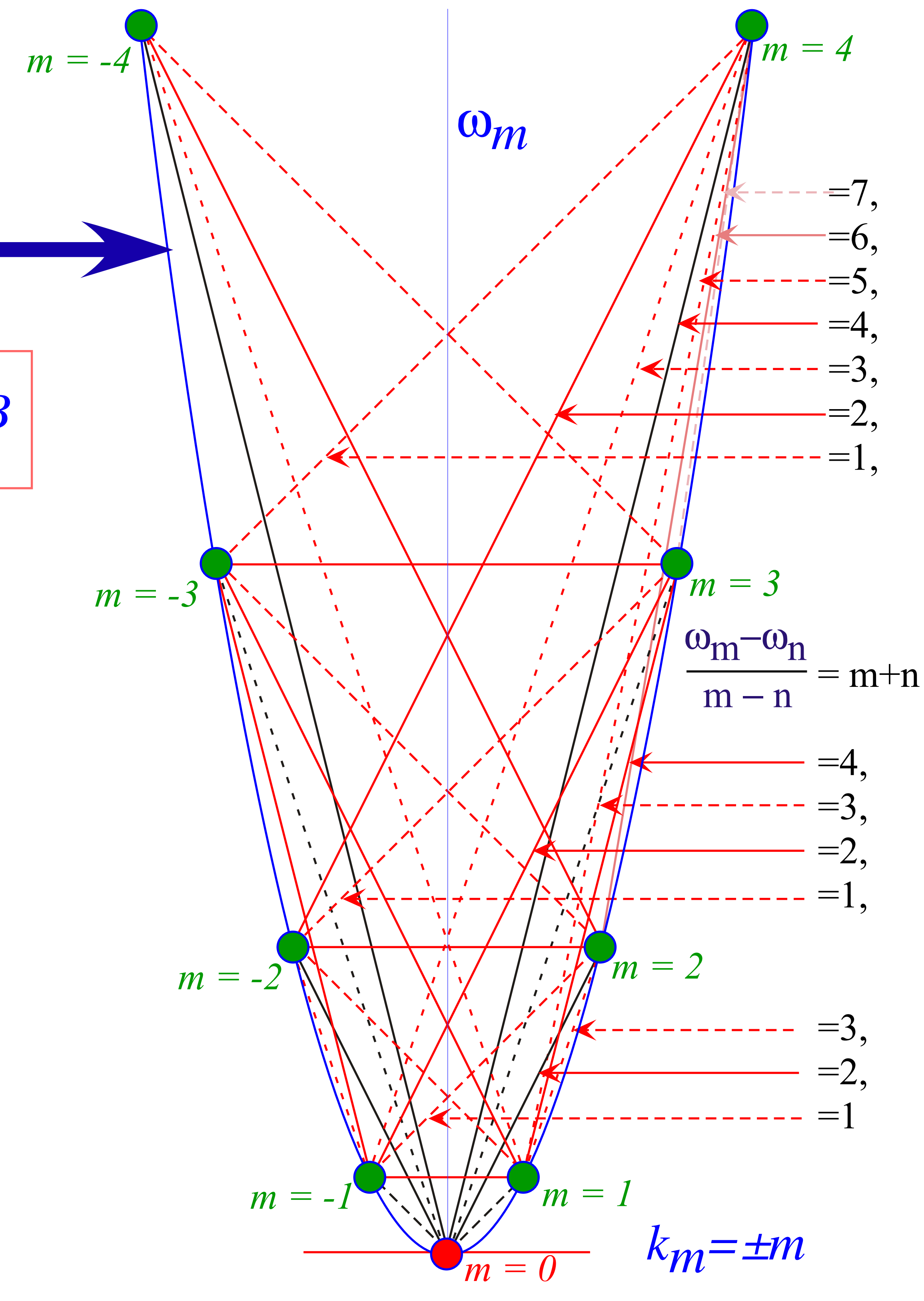
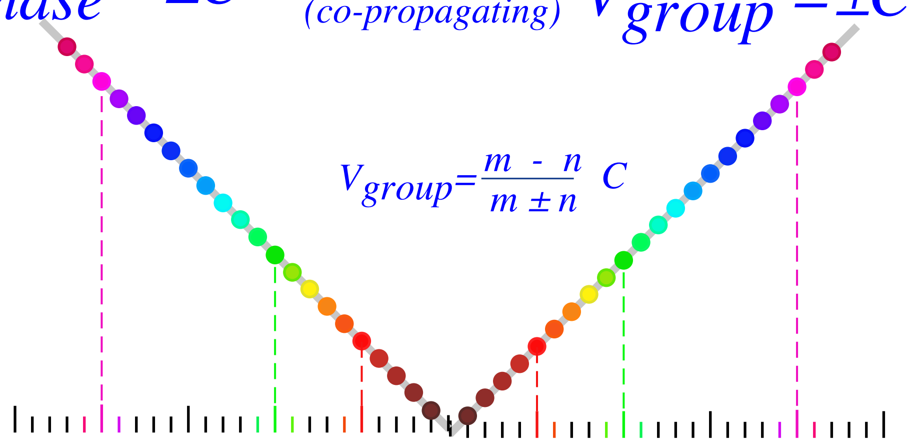


Possible wave velocities
for
Linear (Optical) Spectrum

$$\omega_m = C|m|^1$$
$$k_m = m$$

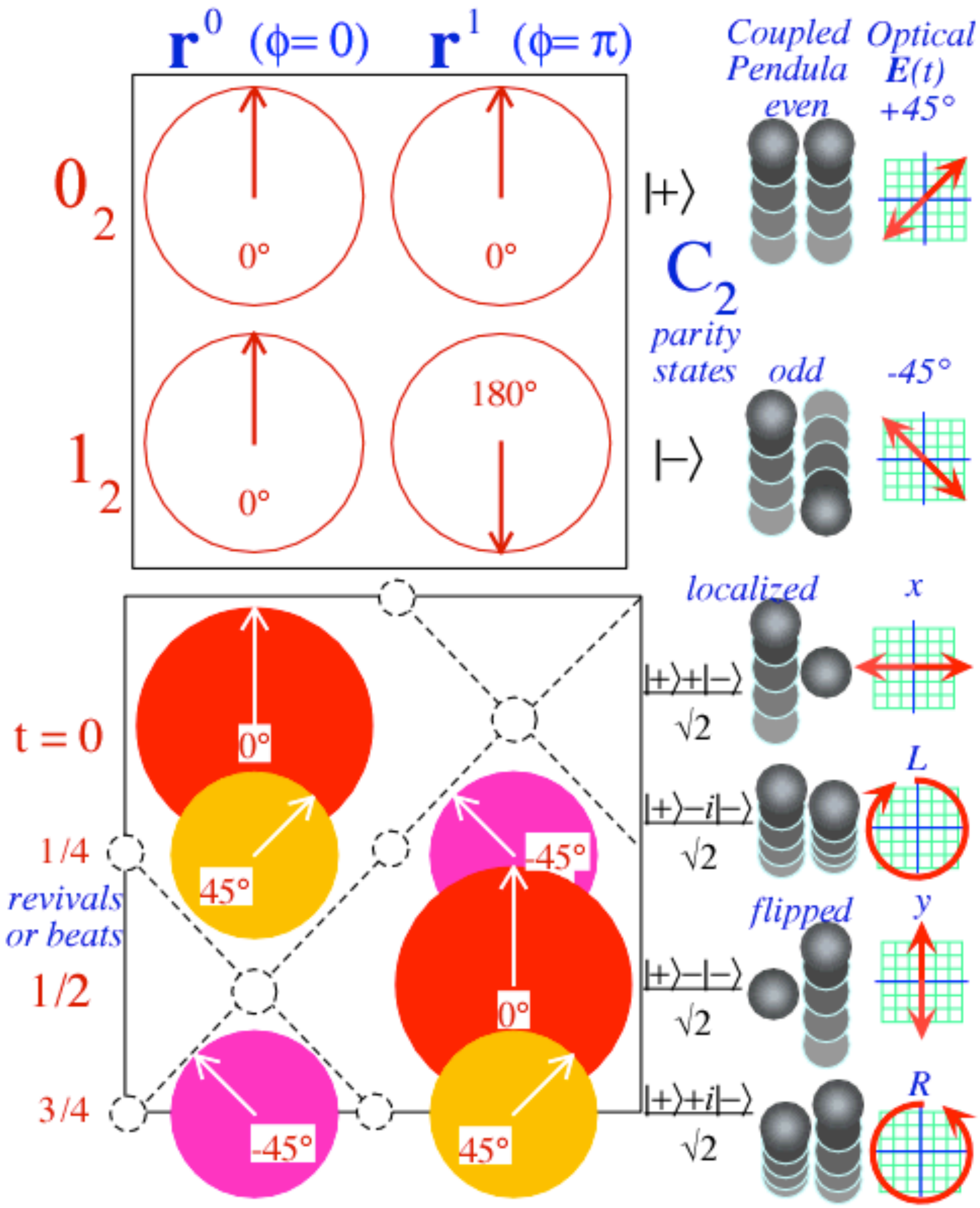
$V_{phase} = \pm C$ *(co-propagating)* $V_{group} = \pm C$



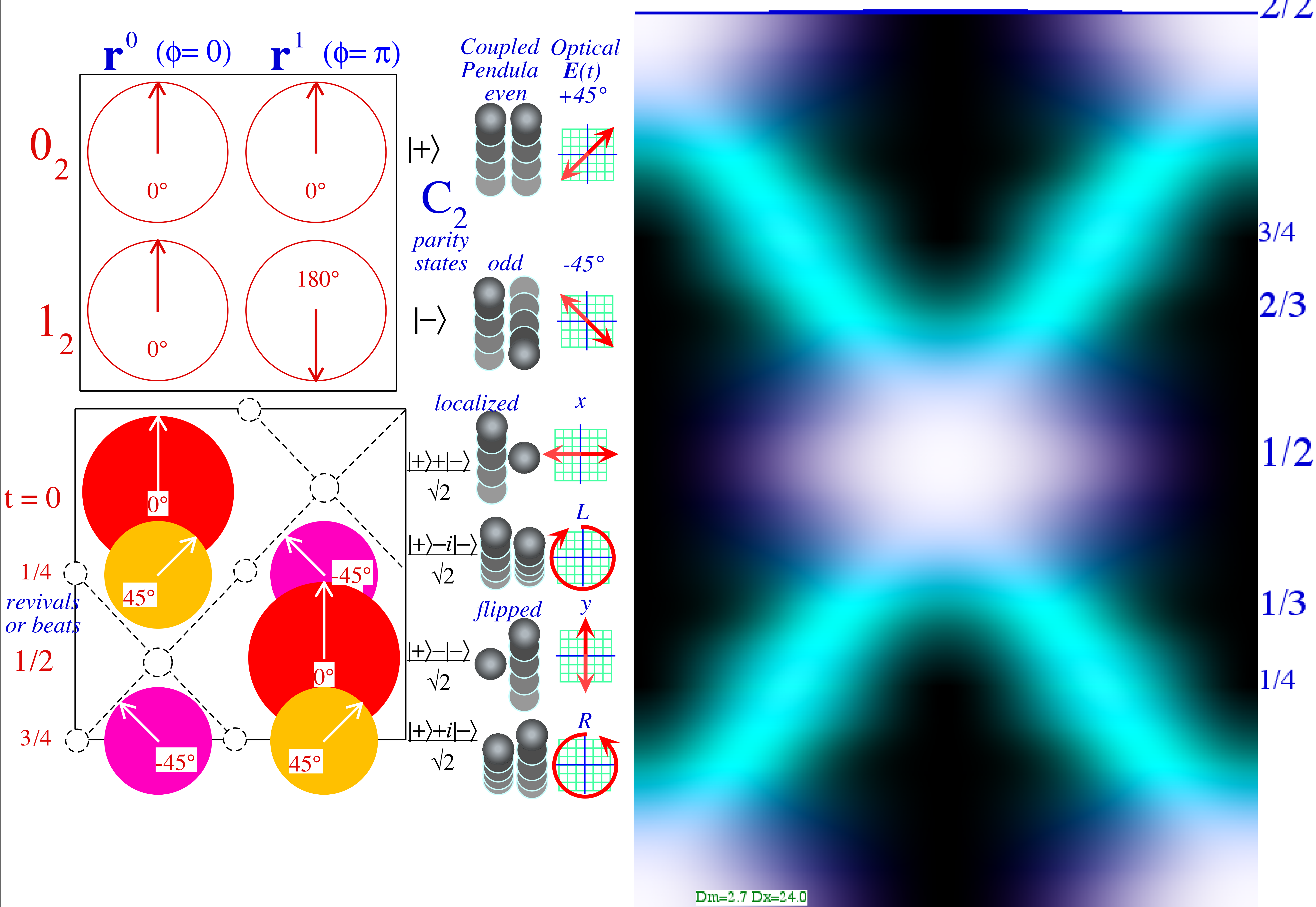
C_2 Fourier transformation matrix

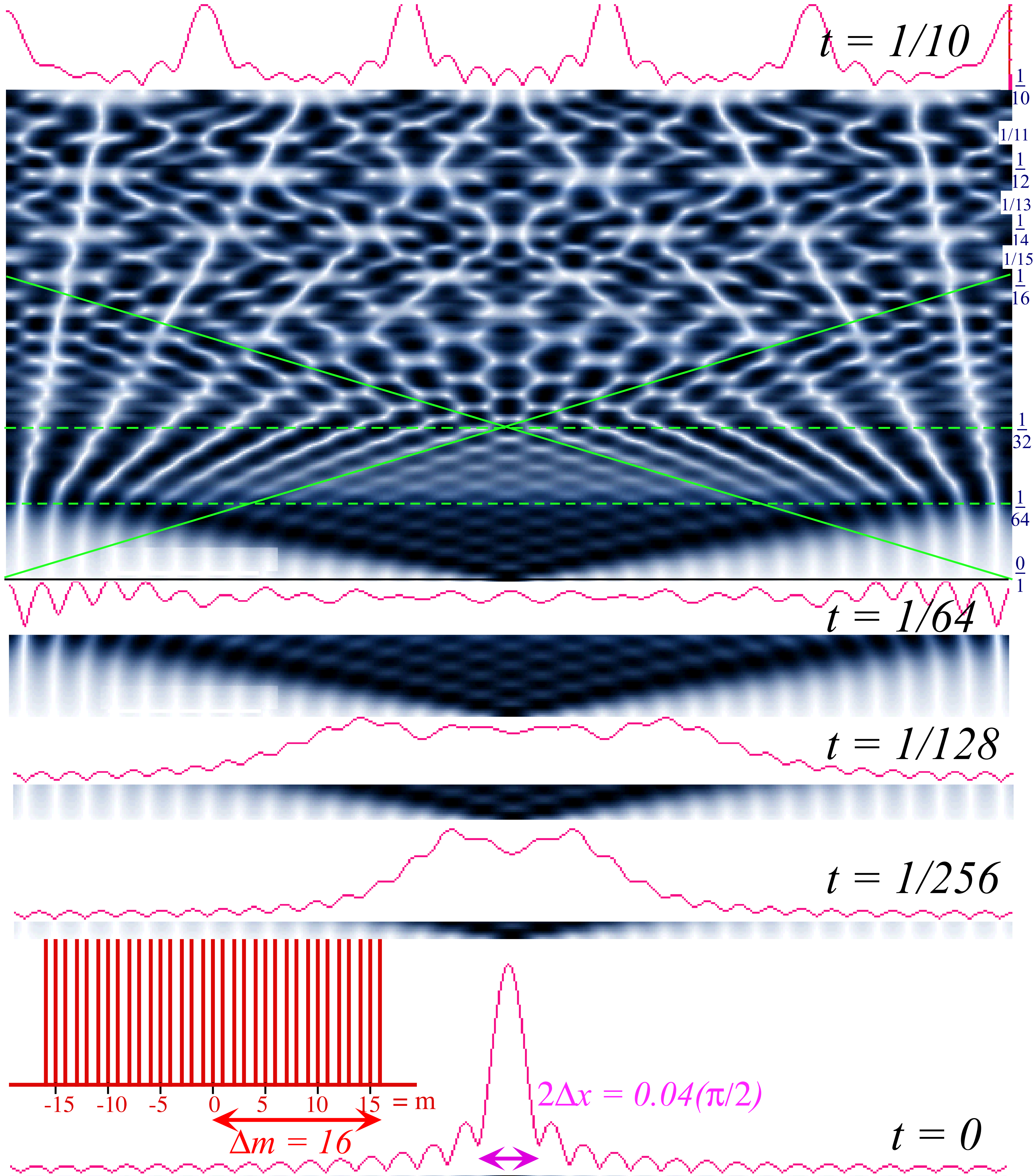
and

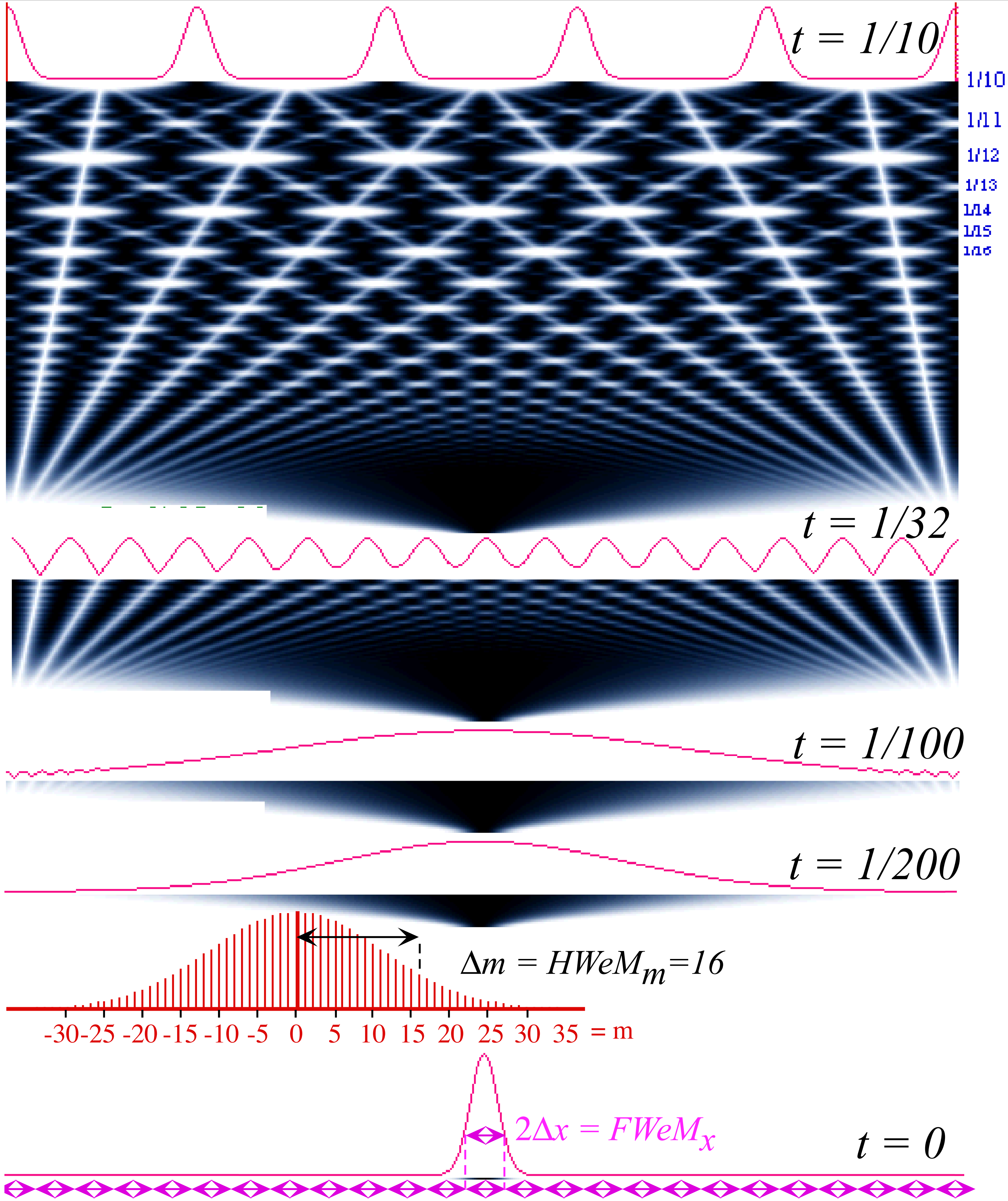
dynamics



Fundamental Beats and 2-Level Transitions: The "Mother of all symmetry" is C_2




t = 1/10
1/10
1/11
1/12
1/13
1/14
1/15
1/16
1/32
1/64
0/1
t = 1/64
t = 1/128
t = 1/256
-15 -10 -5 0 5 10 15 = m
Δm = 16
2Δx = 0.04(π/2)
t = 0



t = 1/10
1/10
1/11
1/12
1/13
1/14
1/15
1/16
t = 1/32
t = 1/100
t = 1/200
Δm = HWeM_m=16
-30 -25 -20 -15 -10 -5 0 5 10 15 20 25 30 35 = m
2Δx = FWeM_x
t = 0

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  Bohr-rotor dynamics
    → *Gaussian wave-packet bandwidth and uncertainty* ←
    Gaussian Bohr-rotor revivals
  Farey-Sums and Ford-products
  Phase dynamics

Gaussian wave-packet bandwidth and uncertainty

Suppose we excite a Gaussian combination of Bohr momentum-*m* plane waves:

$$\Psi(\phi,t{=}0) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}\right)^2} e^{im\phi}$$

Gaussian wave-packet bandwidth and uncertainty

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Complete the square in exponent to simplify ϕ-angle wavefunction.

Gaussian wave-packet bandwidth and uncertainty

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$$=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}\right)^2+im\phi+\left(\frac{\Delta_m}{2}\phi\right)^2-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

Complete the square in exponent to simplify ϕ-angle wavefunction.

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Suppose we excite a Gaussian combination of Bohr momentum-m plane waves:

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$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}\right)^2 + im\phi + \left(\frac{\Delta_m}{2}\phi\right)^2 - \left(\frac{\Delta_m}{2}\phi\right)^2}$$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m} - i\frac{\Delta_m}{2}\phi\right)^2} e^{-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

Complete the square in exponent to simplify ϕ-angle wavefunction.

Gaussian wave-packet bandwidth and uncertainty

Suppose we excite a Gaussian combination of Bohr momentum-m plane waves:

Complete the square in exponent to simplify ϕ*-angle wavefunction.*

$$\Psi(\phi,t{=}0) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}\right)^2} e^{im\phi}$$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}\right)^2 + im\phi + \left(\frac{\Delta_m}{2}\phi\right)^2 - \left(\frac{\Delta_m}{2}\phi\right)^2}$$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m} - i\frac{\Delta_m}{2}\phi\right)^2} e^{-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

$$= \frac{A(\Delta_m,\phi)}{2\pi} e^{-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

$$A(\Delta_m,\phi) = \sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m} - i\frac{\Delta_m}{2}\phi\right)^2}$$

Gaussian wave-packet bandwidth and uncertainty

Suppose we excite a Gaussian combination of Bohr momentum-*m* plane waves:

Complete the square in exponent to simplify ϕ-angle wavefunction.

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$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}\right)^2 + im\phi + \left(\frac{\Delta_m}{2}\phi\right)^2 - \left(\frac{\Delta_m}{2}\phi\right)^2}$$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m} - i\frac{\Delta_m}{2}\phi\right)^2} \quad e^{-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

$$= \frac{A\left(\Delta_m,\phi\right)}{2\pi} \quad e^{-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

$$A\left(\Delta_m,\phi\right) = \sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m} - i\frac{\Delta_m}{2}\phi\right)^2} \xrightarrow[\Delta_m >> 1]{} \int_{-\infty}^{\infty} dk \; e^{-\left(\frac{k}{\Delta_m} - i\frac{\Delta_m}{2}\phi\right)^2}$$

m=0, ±1, ±2, ±3,...are momentum quanta in wavevector formula: $k_m = 2\pi m/L$ $\quad (k_m = m$ *if:* $L = 2\pi)$

Gaussian wave-packet bandwidth and uncertainty

Suppose we excite a Gaussian combination of Bohr momentum-m plane waves:

Complete the square in exponent to simplify ϕ-angle wavefunction.

$$\Psi(\phi,t{=}0) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}\right)^2} e^{im\phi}$$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}\right)^2 + im\phi + \left(\frac{\Delta_m}{2}\phi\right)^2 - \left(\frac{\Delta_m}{2}\phi\right)^2}$$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m} - i\frac{\Delta_m}{2}\phi\right)^2} e^{-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

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$$\left[\text{let: } K = \frac{k}{\Delta_m} - i\frac{\Delta_m}{2}\phi \quad \text{so: } dk = \Delta_m dK\right]$$

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$$=\frac{A(\Delta_m,\phi)}{2\pi} e^{-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

Gaussian integral:

$$\sqrt{\int_{-\infty}^{\infty} e^{-x^2}dx}\sqrt{\int_{-\infty}^{\infty} e^{-y^2}dy}=\sqrt{\iint e^{-(x^2+y^2)}dx\,dy}$$

$$=\sqrt{\int_0^{2\pi}\int_0^{\infty} e^{-r^2}r\,dr\,d\theta}=\sqrt{2\pi\int_0^{\infty} e^{-r^2}\frac{dr^2}{2}}=\sqrt{\pi}$$

$$A(\Delta_m,\phi)=\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}-i\frac{\Delta_m}{2}\phi\right)^2} \xrightarrow[\Delta_m>>1]{} \int_{-\infty}^{\infty} dk\, e^{-\left(\frac{k}{\Delta_m}-i\frac{\Delta_m}{2}\phi\right)^2}$$

$$\left[\text{let: } K=\frac{k}{\Delta_m}-i\frac{\Delta_m}{2}\phi \quad \text{so: } dk=\Delta_m dK\right] \quad \text{then: } A(\Delta_m,\phi)\simeq\Delta_m\int_{-\infty}^{\infty} dK\, e^{-(K)^2}=\Delta_m\sqrt{\pi}$$

$m=0,\pm1,\pm2,\pm3,$...are momentum quanta in wavevector formula: $k_m=2\pi m/L$ $\quad$ ($k_m=m$ *if:* $L=2\pi$)

Gaussian wave-packet bandwidth and uncertainty

Suppose we excite a Gaussian combination of Bohr momentum-m plane waves:

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$$=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}\right)^2+im\phi+\left(\frac{\Delta_m}{2}\phi\right)^2-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

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$$=\frac{A(\Delta_m,\phi)}{2\pi} e^{-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

Complete the square in exponent to simplify ϕ-angle wavefunction.

$$\Psi(\phi,t{=}0)=\frac{\Delta_m}{2\sqrt{\pi}} e^{-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

It is a Gaussian distribution, too

$$A(\Delta_m,\phi)=\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}-i\frac{\Delta_m}{2}\phi\right)^2} \xrightarrow[\Delta_m>>1]{} \int_{-\infty}^{\infty} dk\, e^{-\left(\frac{k}{\Delta_m}-i\frac{\Delta_m}{2}\phi\right)^2}$$

$\left[\text{let: } K=\frac{k}{\Delta_m}-i\frac{\Delta_m}{2}\phi \quad \text{so: } dk=\Delta_m dK\right]$ then: $A(\Delta_m,\phi)\simeq\Delta_m\int_{-\infty}^{\infty} dK\, e^{-(K)^2}=\Delta_m\sqrt{\pi}$

$m=0, \pm1, \pm2, \pm3,\ldots$ are momentum quanta in wavevector formula: $k_m=2\pi m/L$ $\quad (k_m=m$ if: $L=2\pi)$

Gaussian wave-packet bandwidth and uncertainty

Suppose we excite a Gaussian combination of Bohr momentum-m plane waves:

$$\Psi(\phi,t{=}0) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}\right)^2} e^{im\phi}$$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}\right)^2 + im\phi + \left(\frac{\Delta_m}{2}\phi\right)^2 - \left(\frac{\Delta_m}{2}\phi\right)^2}$$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m} - i\frac{\Delta_m}{2}\phi\right)^2} e^{-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

$$= \frac{A(\Delta_m,\phi)}{2\pi} e^{-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

Complete the square in exponent to simplify ϕ-angle wavefunction.

$$\Psi(\phi,t{=}0) = \frac{\Delta_m}{2\sqrt{\pi}} e^{-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

It is a Gaussian distribution, too

$$\Psi(\phi,t{=}0) \simeq \frac{\Delta_m}{2\sqrt{\pi}} e^{-\left(\frac{\phi}{\Delta_\phi}\right)^2}$$

where: $\Delta_\phi = \frac{2}{\Delta_m}$ or: $\Delta_\phi \Delta_m = 2$

$$A(\Delta_m,\phi) = \sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m} - i\frac{\Delta_m}{2}\phi\right)^2} \xrightarrow[\Delta_m >> 1]{} \int_{-\infty}^{\infty} dk\, e^{-\left(\frac{k}{\Delta_m} - i\frac{\Delta_m}{2}\phi\right)^2}$$

$\left[\text{let: } K = \frac{k}{\Delta_m} - i\frac{\Delta_m}{2}\phi \text{ so: } dk = \Delta_m dK\right]$ then: $A(\Delta_m,\phi) \simeq \Delta_m \int_{-\infty}^{\infty} dK\, e^{-(K)^2} = \Delta_m\sqrt{\pi}$

m=0, ±1, ±2, ±3,...are momentum quanta in wavevector formula: $k_m = 2\pi m/L$ $\quad$ ($k_m = m$ *if:* $L = 2\pi$)

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$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m} - i\frac{\Delta_m}{2}\phi\right)^2} e^{-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

$$= \frac{A(\Delta_m,\phi)}{2\pi} e^{-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

Complete the square in exponent to simplify φ-angle wavefunction.

$$\Psi(\phi,t{=}0) = \frac{\Delta_m}{2\sqrt{\pi}} e^{-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

It is a Gaussian distribution, too

$$\Psi(\phi,t{=}0) \simeq \frac{\Delta_m}{2\sqrt{\pi}} e^{-\left(\frac{\phi}{\Delta_\phi}\right)^2}$$

where: $\Delta_\phi = \frac{2}{\Delta_m}$ or: $\Delta_\phi \Delta_m = 2$

Gaussian uncertainty relation

(Compare to $\Delta x \cdot \Delta k = \pi$ *for ∞-Well)*

$$A(\Delta_m,\phi) = \sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m} - i\frac{\Delta_m}{2}\phi\right)^2} \xrightarrow[\Delta_m >> 1]{} \int_{-\infty}^{\infty} dk\, e^{-\left(\frac{k}{\Delta_m} - i\frac{\Delta_m}{2}\phi\right)^2}$$

$\left[\text{let: } K = \frac{k}{\Delta_m} - i\frac{\Delta_m}{2}\phi \quad \text{so: } dk = \Delta_m dK\right]$ then: $A(\Delta_m,\phi) \simeq \Delta_m \int_{-\infty}^{\infty} dK\, e^{-(K)^2} = \Delta_m\sqrt{\pi}$

$m=0, \pm1, \pm2, \pm3,\ldots$ are momentum quanta in wavevector formula: $k_m = 2\pi m/L$ ($k_m = m$ *if:* $L = 2\pi$)

$$E_m = (\hbar k_m)^2/2M = m^2[h^2/2ML^2] = m^2 h\upsilon_1 = m^2 \hbar\omega_1$$

fundamental Bohr ∠−frequency $\omega_1 = 2\pi\upsilon_1$ and lowest *transition (beat) frequency* $\upsilon_1 = (E_1 - E_0)/h$

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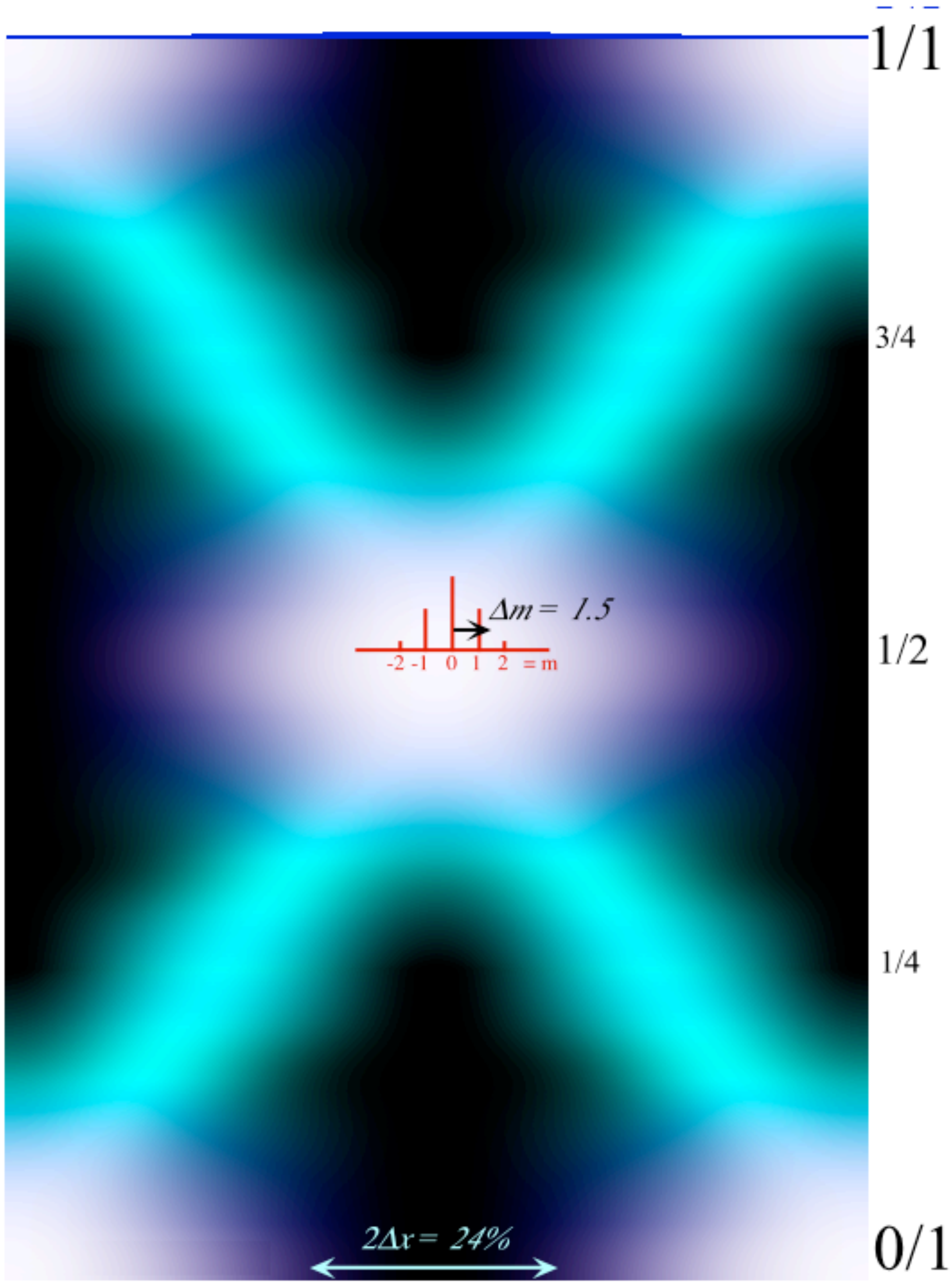
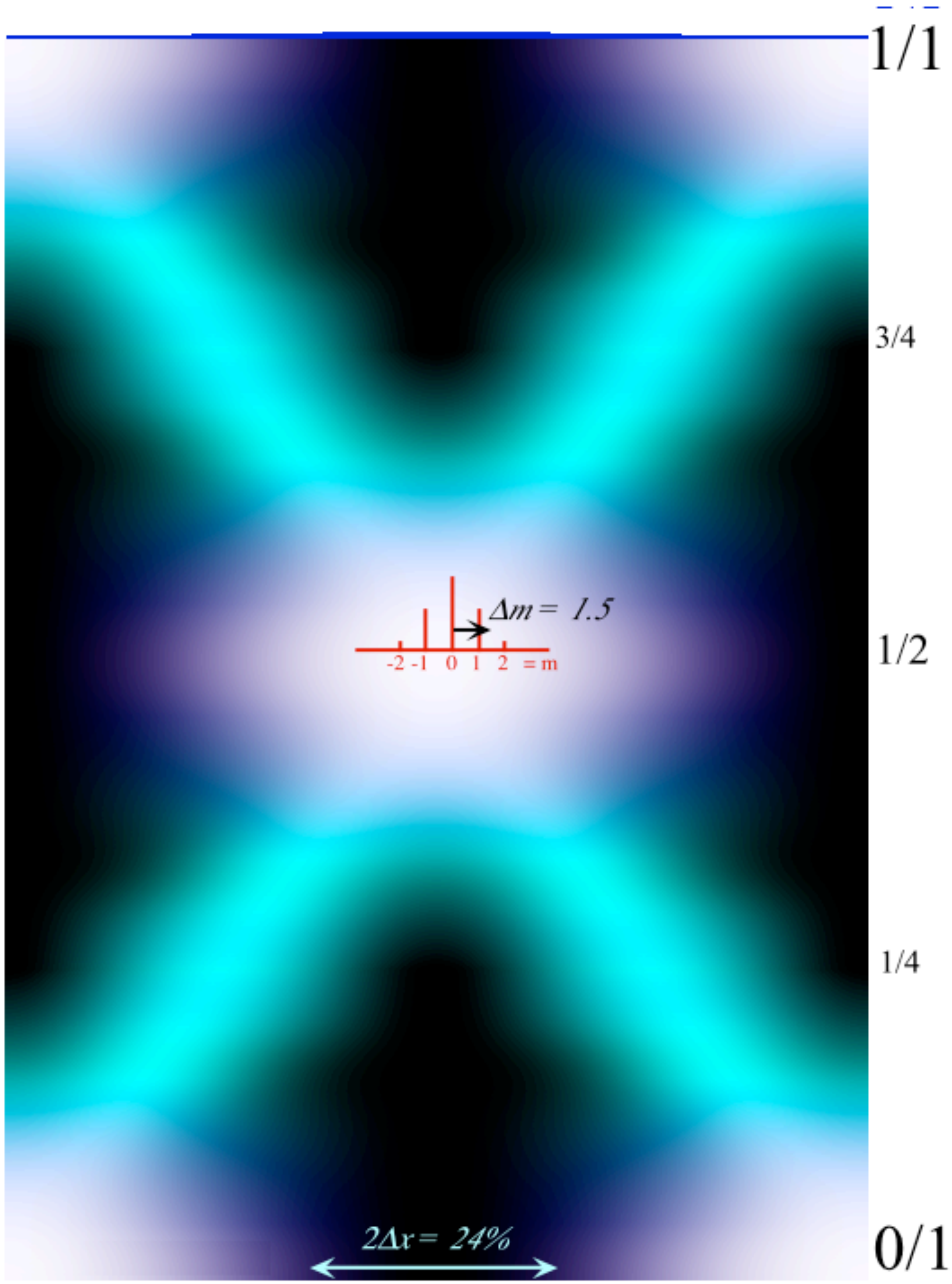
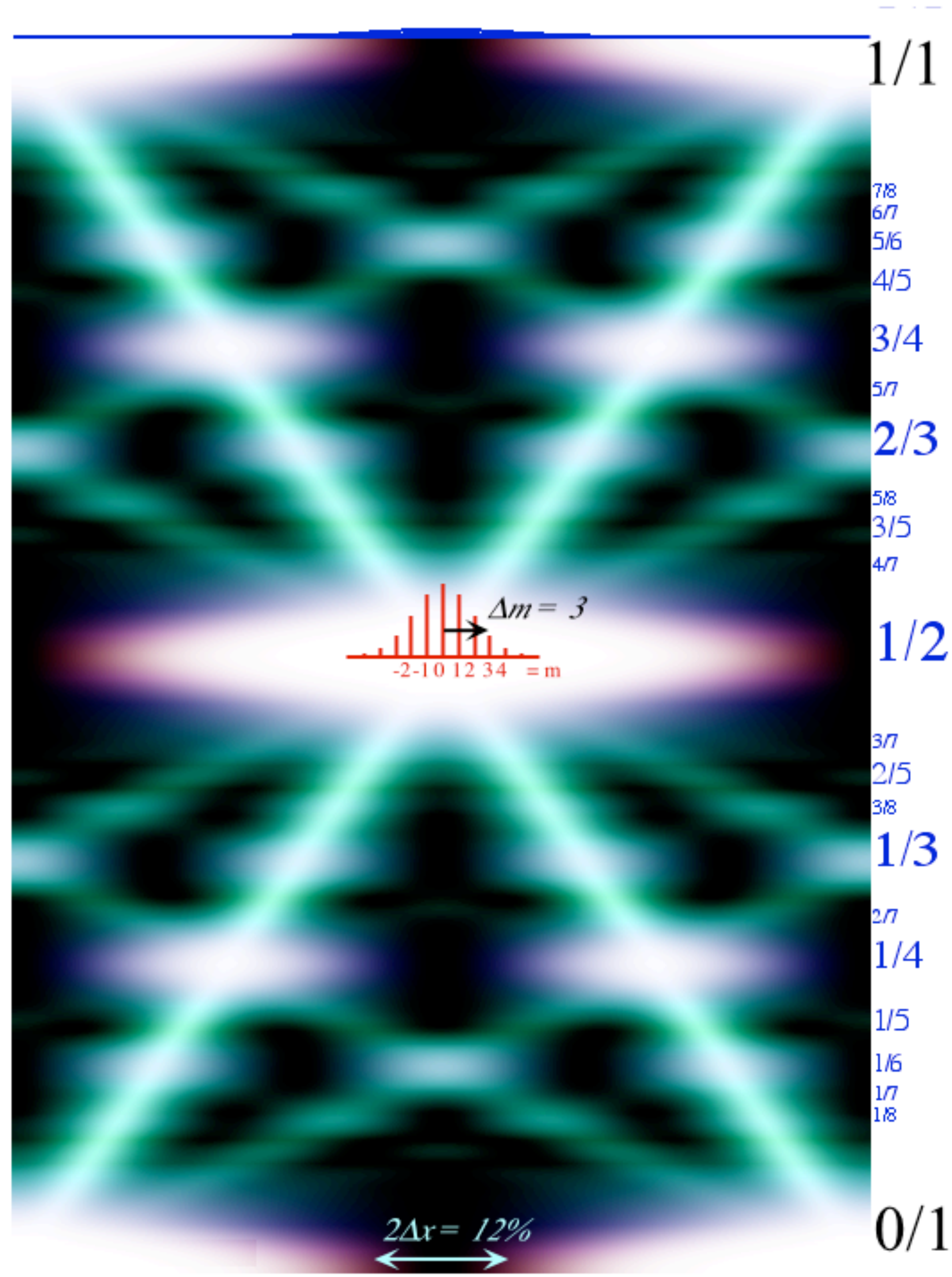
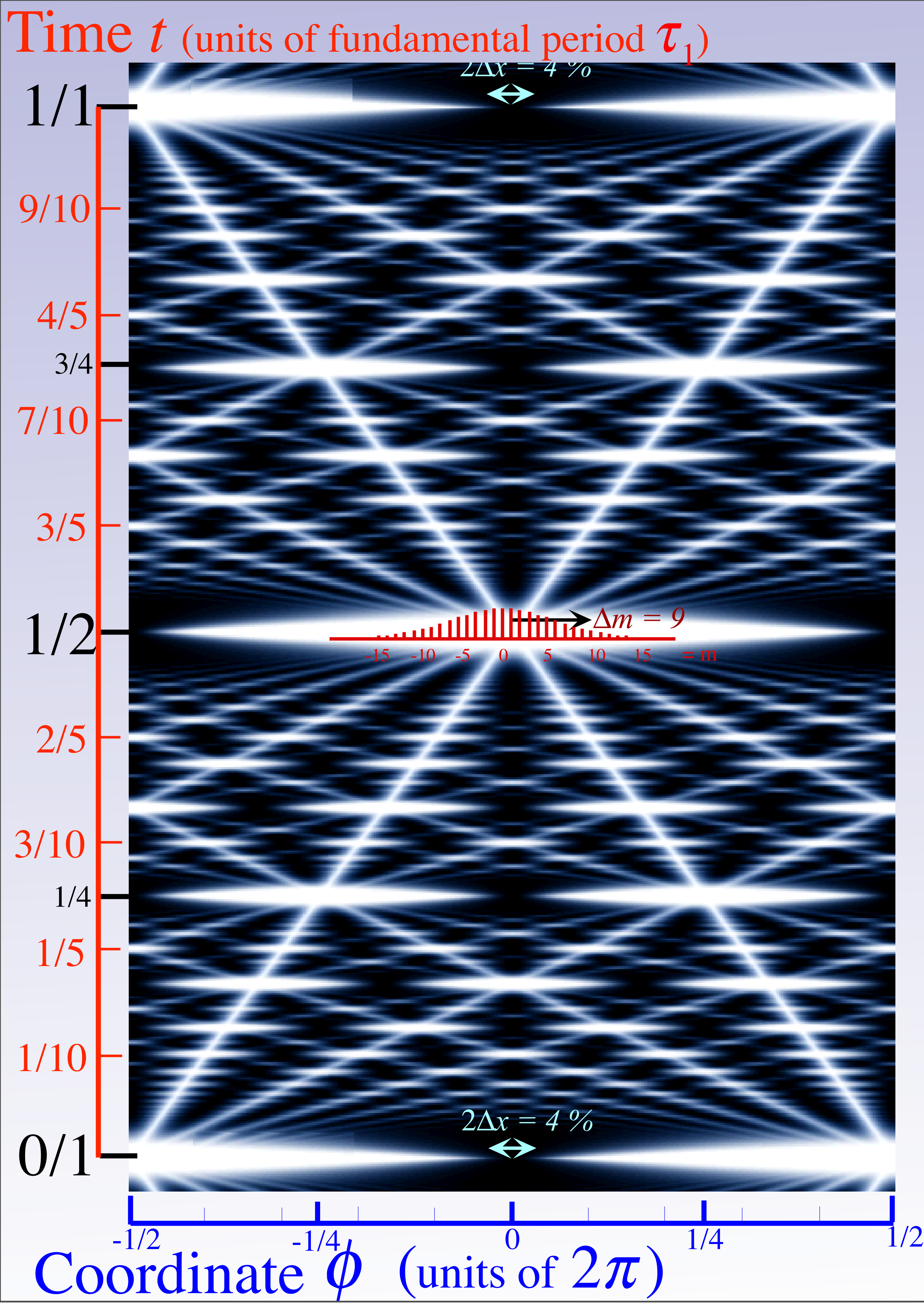
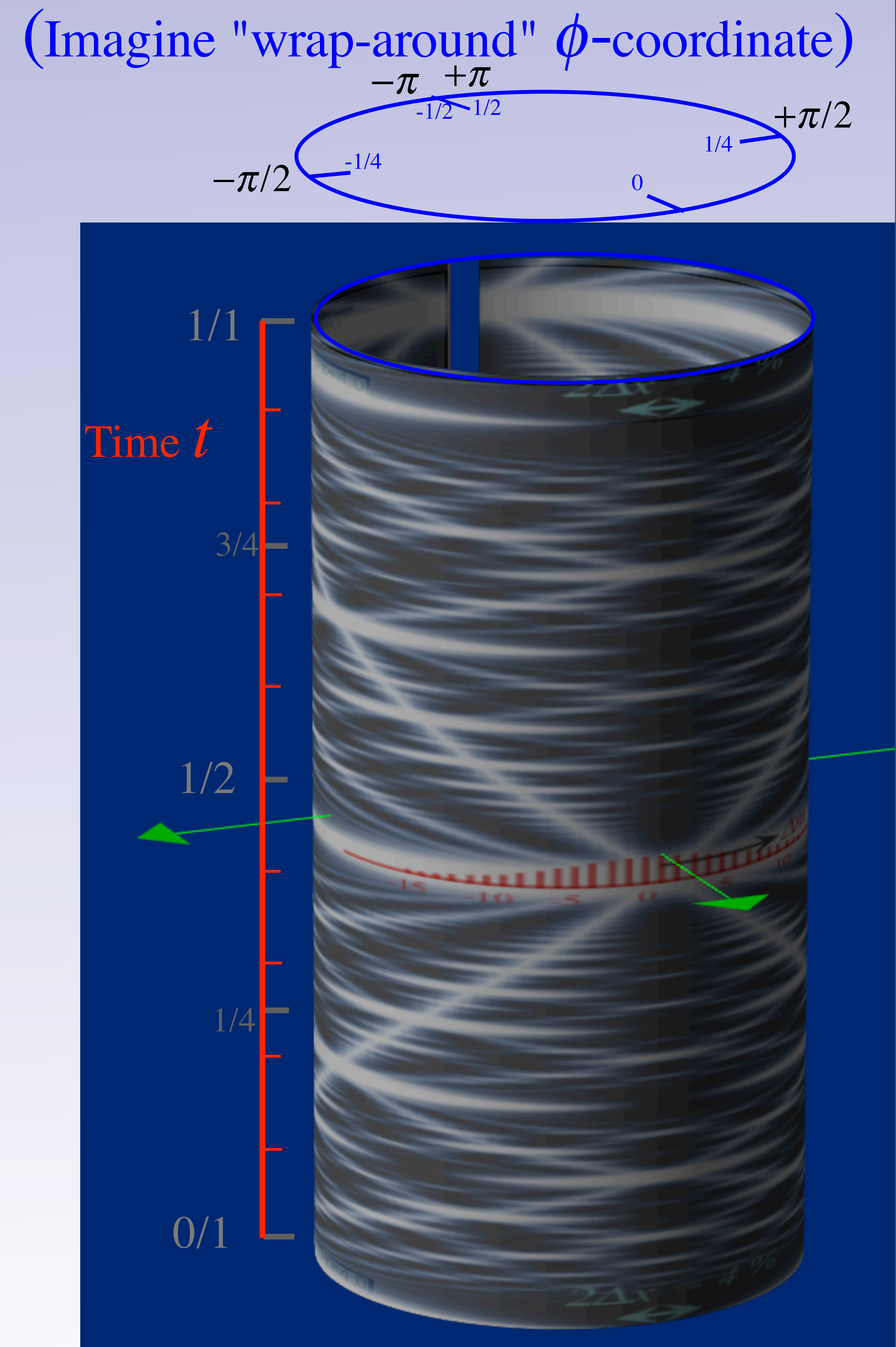
➡ *Gaussian Bohr-rotor revivals* ⬅
Farey-Sums and Ford-products
Phase dynamics


1/1
3/4
$\Delta m = 1.5$
-2 -1 0 1 2 = m
1/2
1/4
$2\Delta x = 24\%$
0/1


1/1
3/4
1/2
1/4
0/1
Δm = 1.5
-2 -1 0 1 2 = m
2Δx = 24%


1/1
7/8
6/7
5/6
4/5
3/4
5/7
2/3
5/8
3/5
4/7
1/2
3/7
2/5
3/8
1/3
2/7
1/4
1/5
1/6
1/7
1/8
0/1
Δm = 3
-2-1 0 1 2 3 4 = m
2Δx = 12%



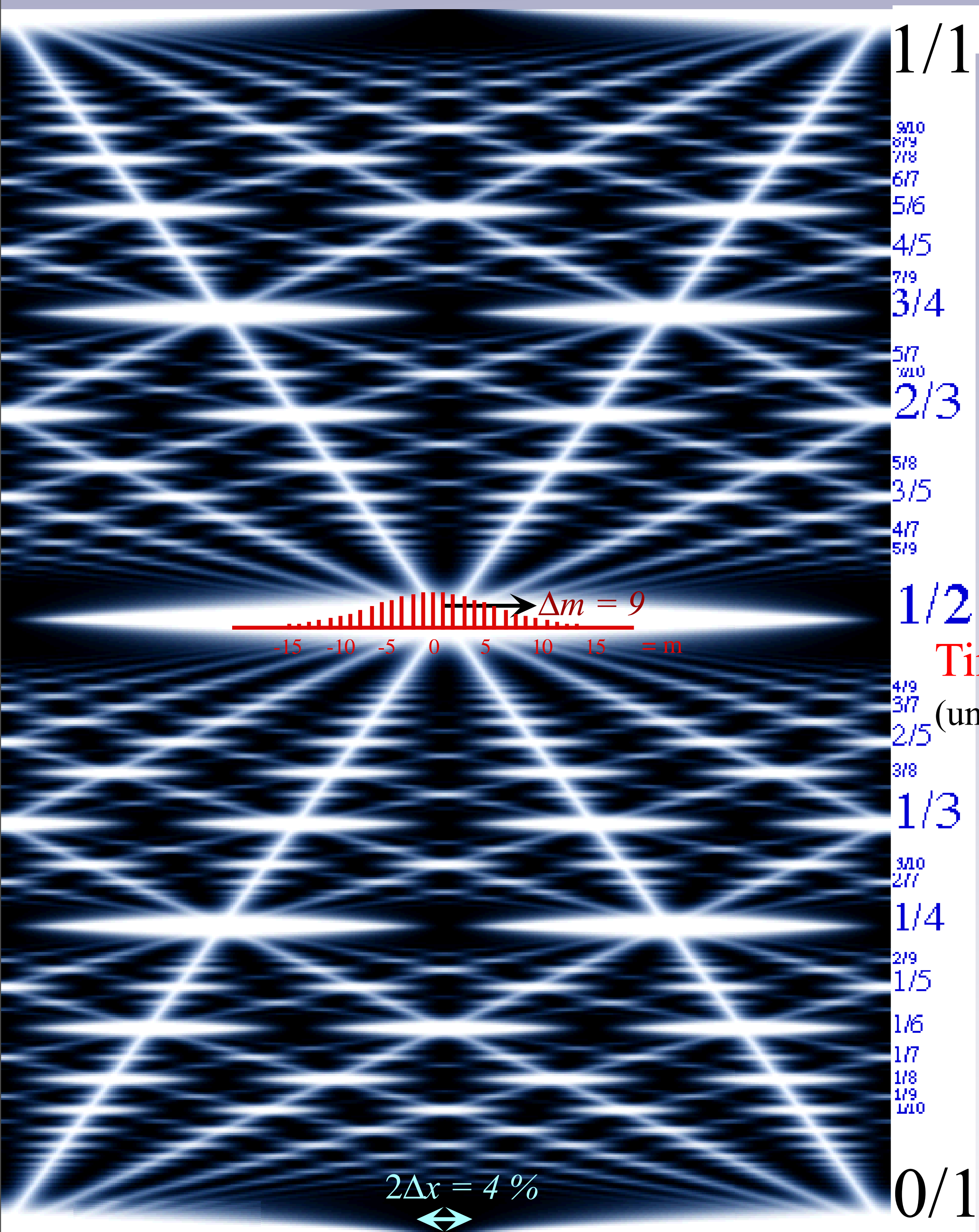


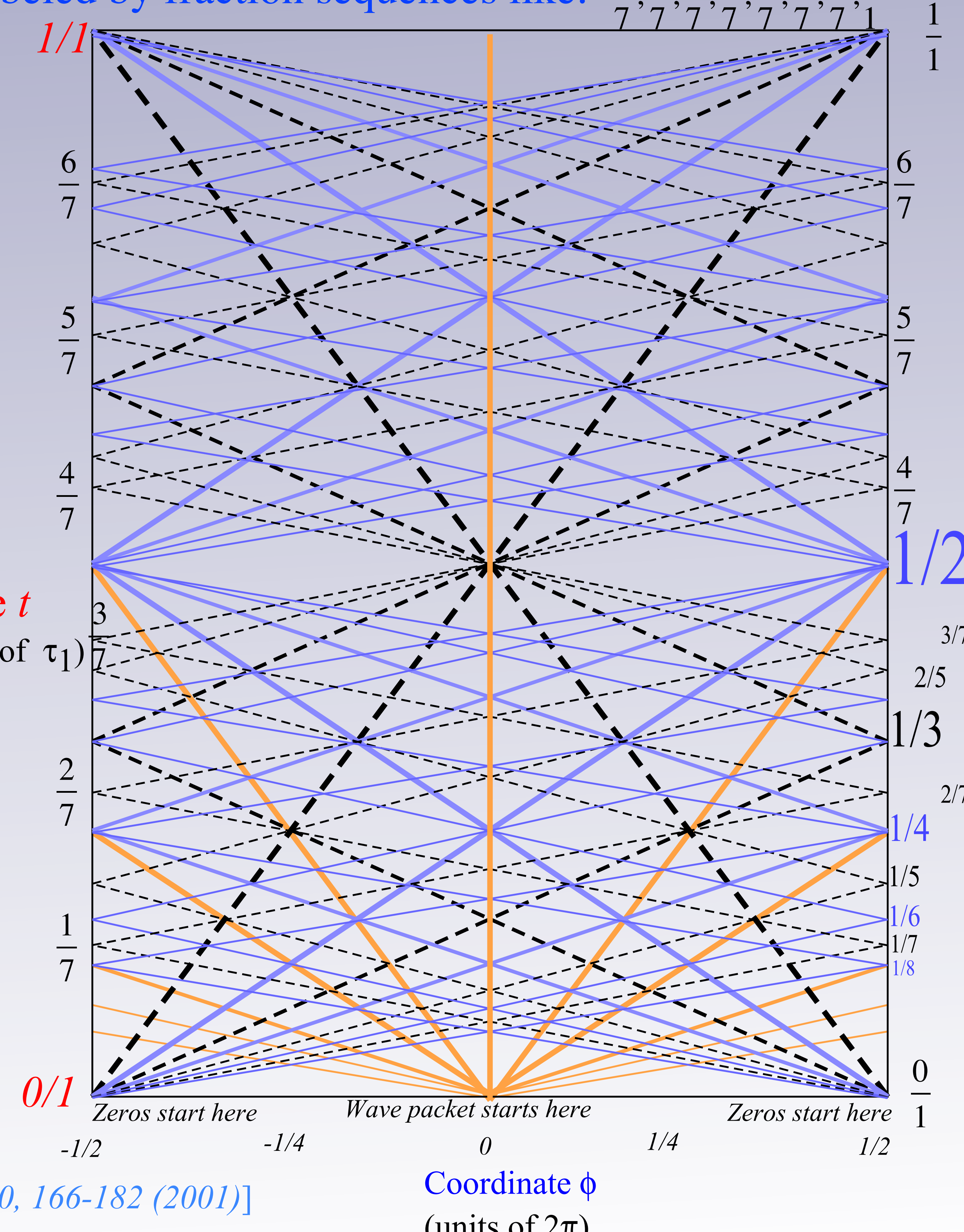
[Harter, J. Mol. Spec. 210, 166-182 (2001)]

N-level-system and revival-beat wave dynamics

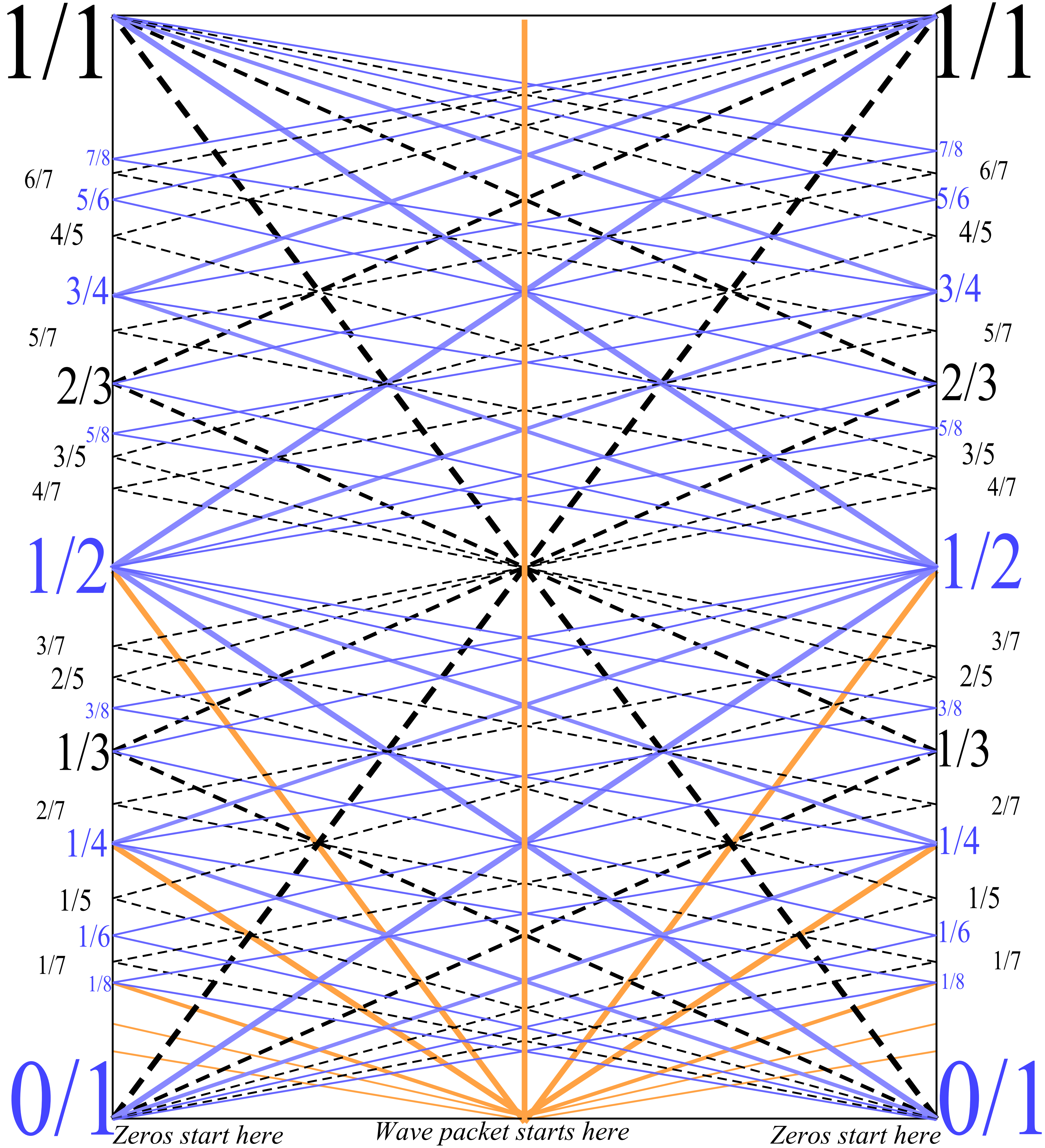
(9 or10-levels $(0, \pm1, \pm2, \pm3, \pm4, ..., \pm9, \pm10, \pm11...)$ excited)

Zeros (clearly) and "particle-packets" (faintly) have paths labeled by fraction sequences like: $\frac{0}{7}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, \frac{1}{1}$





[*Harter, J. Mol. Spec. 210, 166-182 (2001)*]


1/1
1/1
7/8
6/7
5/6
4/5
3/4
5/7
2/3
5/8
3/5
4/7
1/2
3/7
2/5
3/8
1/3
2/7
1/4
1/5
1/6
1/7
1/8
0/1
Zeros start here
Wave packet starts here
Zeros start here
7/8
6/7
5/6
4/5
3/4
5/7
2/3
5/8
3/5
4/7
1/2
3/7
2/5
3/8
1/3
2/7
1/4
1/5
1/6
1/7
1/8
0/1

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  - *SinNx/x explosion and revivals*
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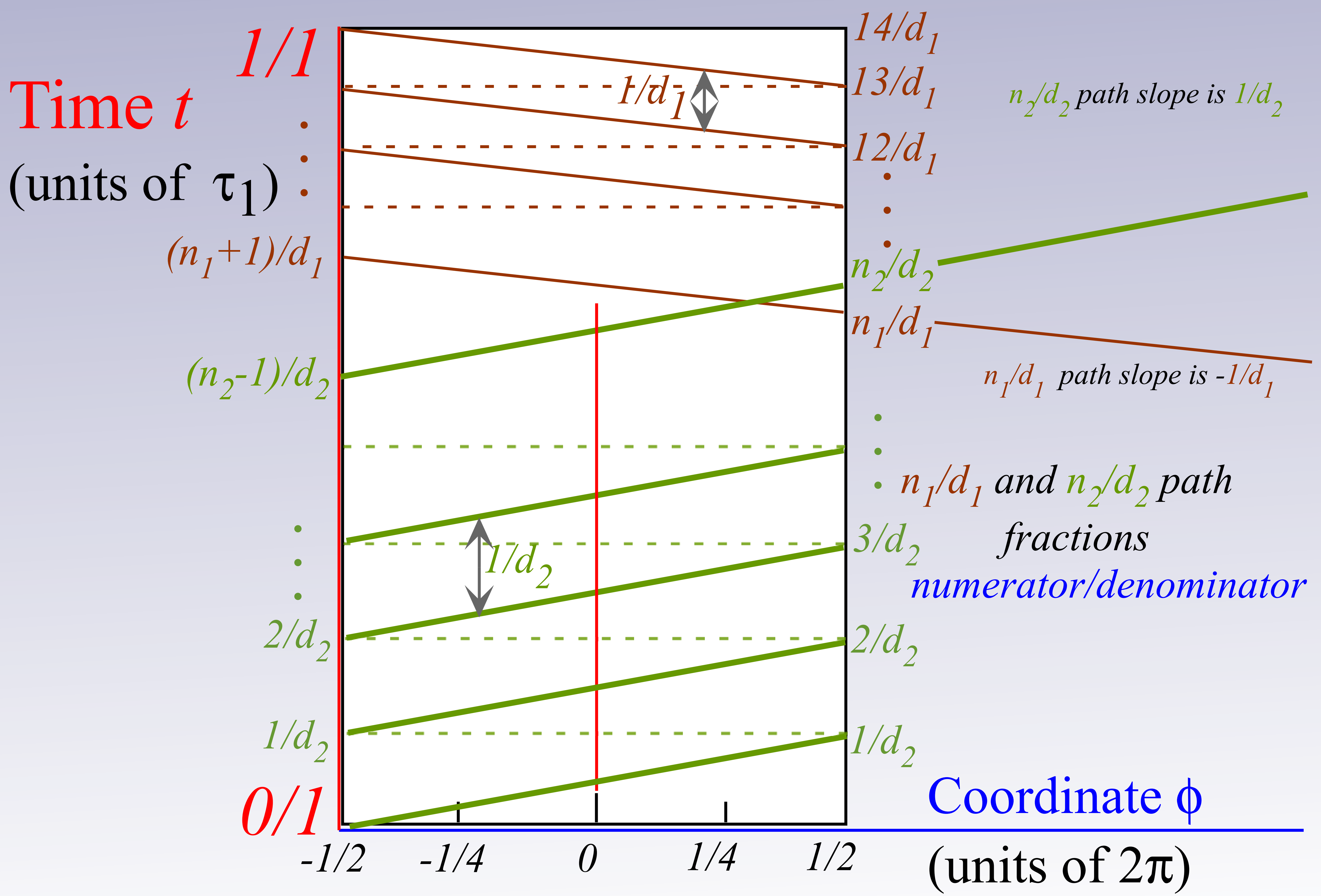
  - *Gaussian wave-packet bandwidth and uncertainty*
  - *Gaussian Bohr-rotor revivals*
- *Farey-Sums and Ford-products*
- *Phase dynamics*

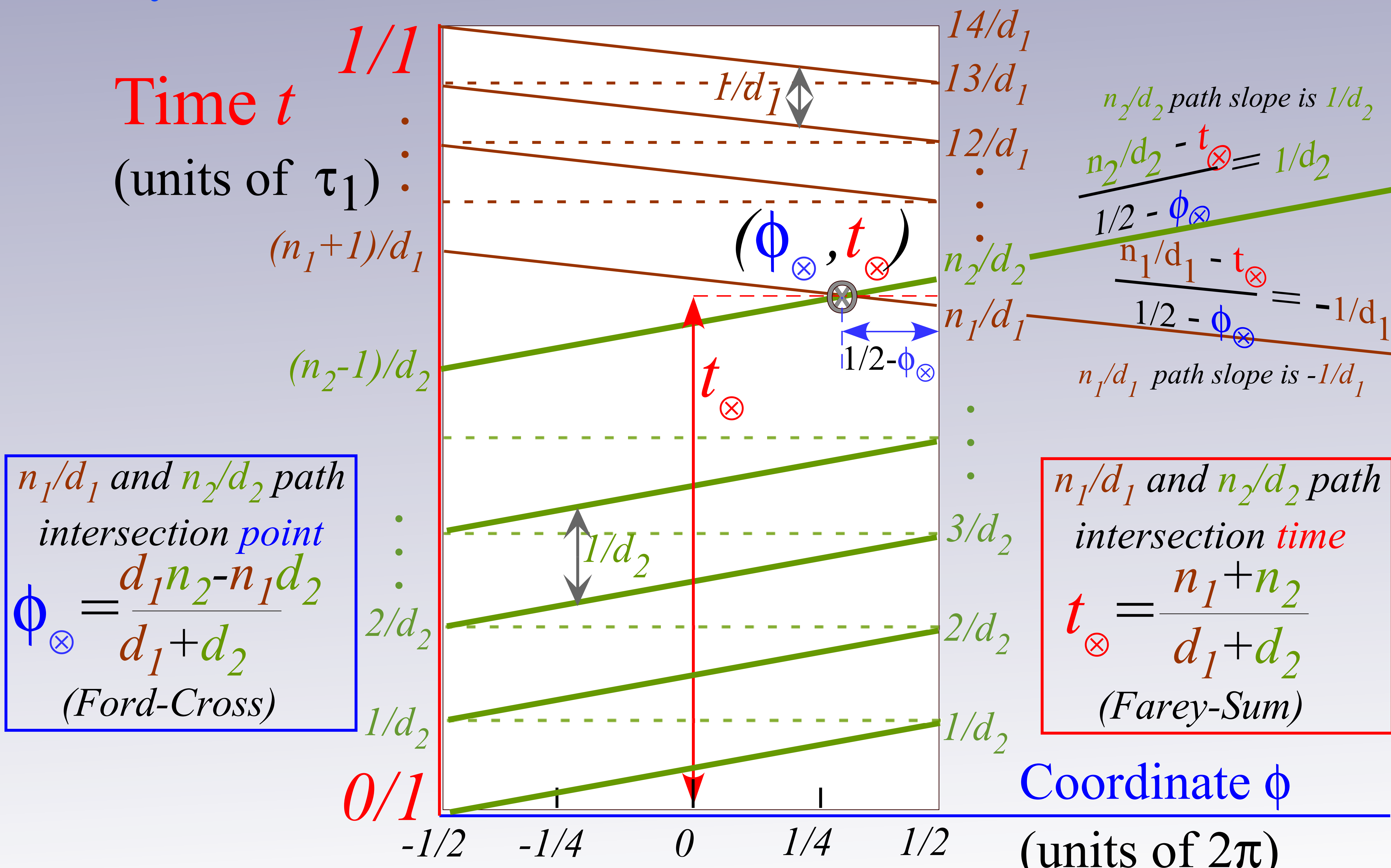
Farey Sum algebra of revival-beat wave dynamics

Label by *numerators N* and *denominators D* of rational fractions *N/D*



Farey Sum algebra of revival-beat wave dynamics

Label by *numerators N* and *denominators D* of rational fractions *N/D*

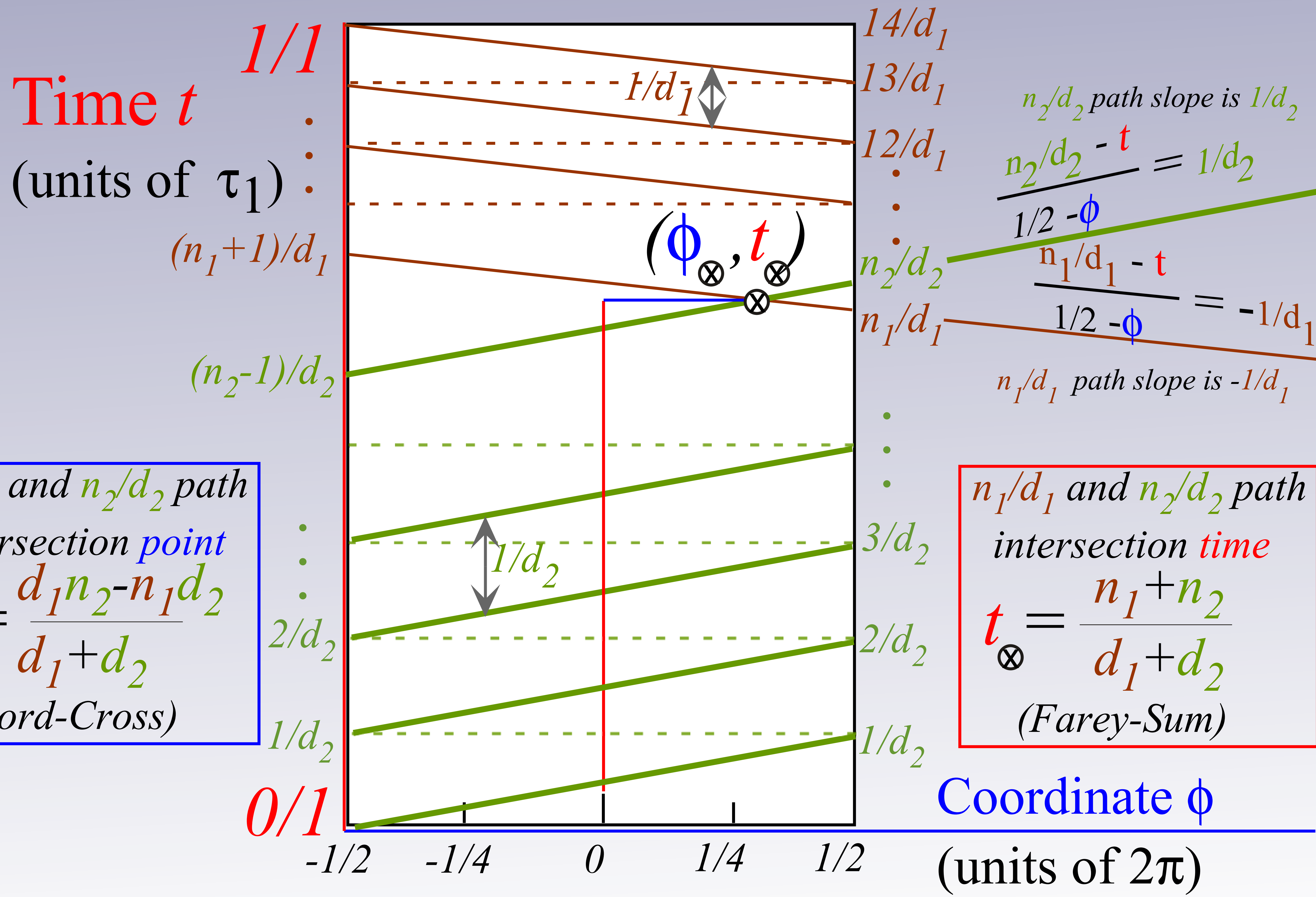


[Lester. R. Ford, Am. Math. Monthly 45,586(1938)]

[John Farey, Phil. Mag.(1816)]

Farey Sum algebra of revival-beat wave dynamics

Label by *numerators N* and *denominators D* of rational fractions *N/D*



n_1/d_1 and n_2/d_2 path intersection point

$$\phi_\otimes = \frac{d_1 n_2 - n_1 d_2}{d_1 + d_2}$$

(Ford-Cross)

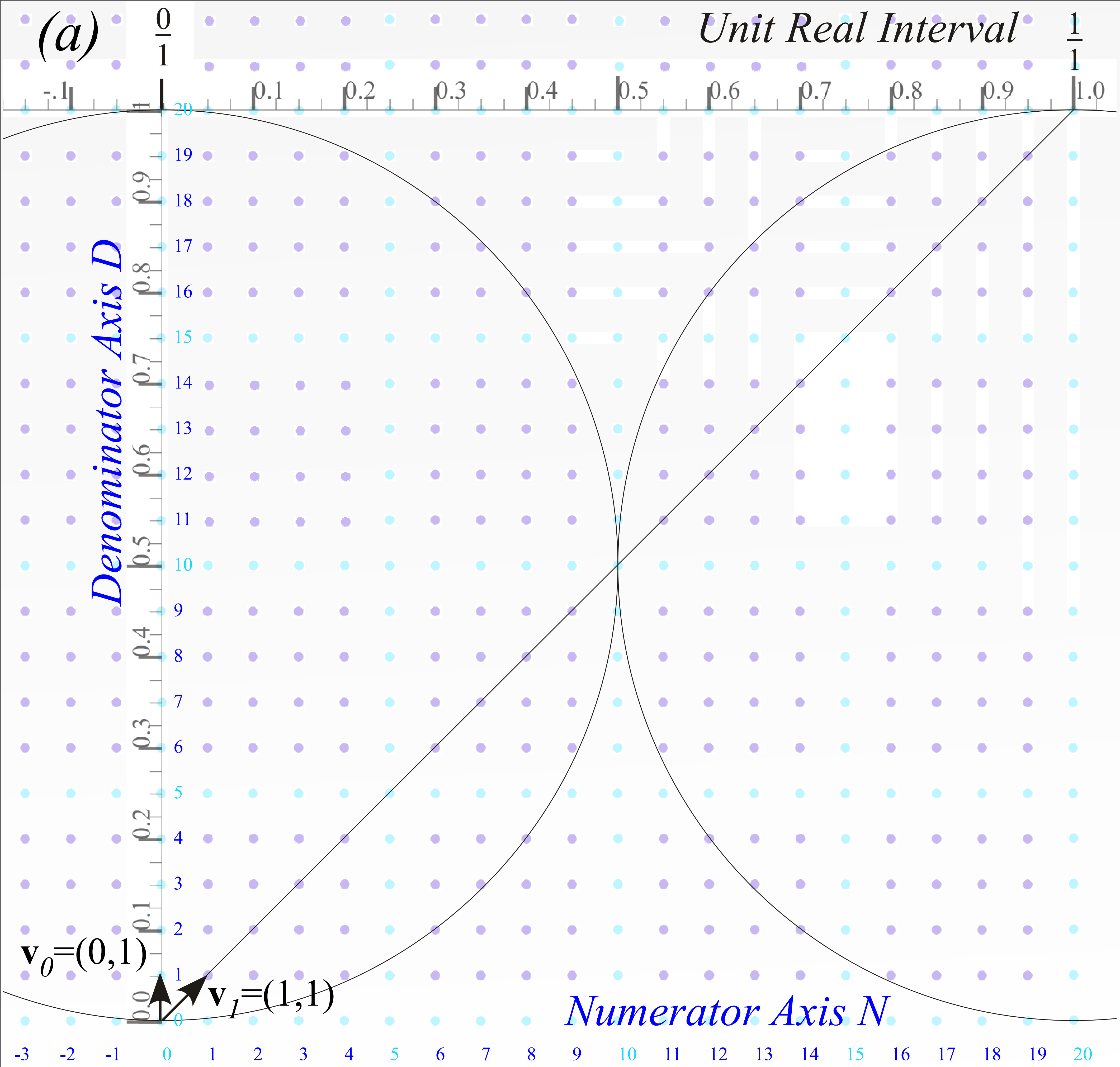
n_1/d_1 and n_2/d_2 path intersection time

$$t_\otimes = \frac{n_1 + n_2}{d_1 + d_2}$$

(Farey-Sum)

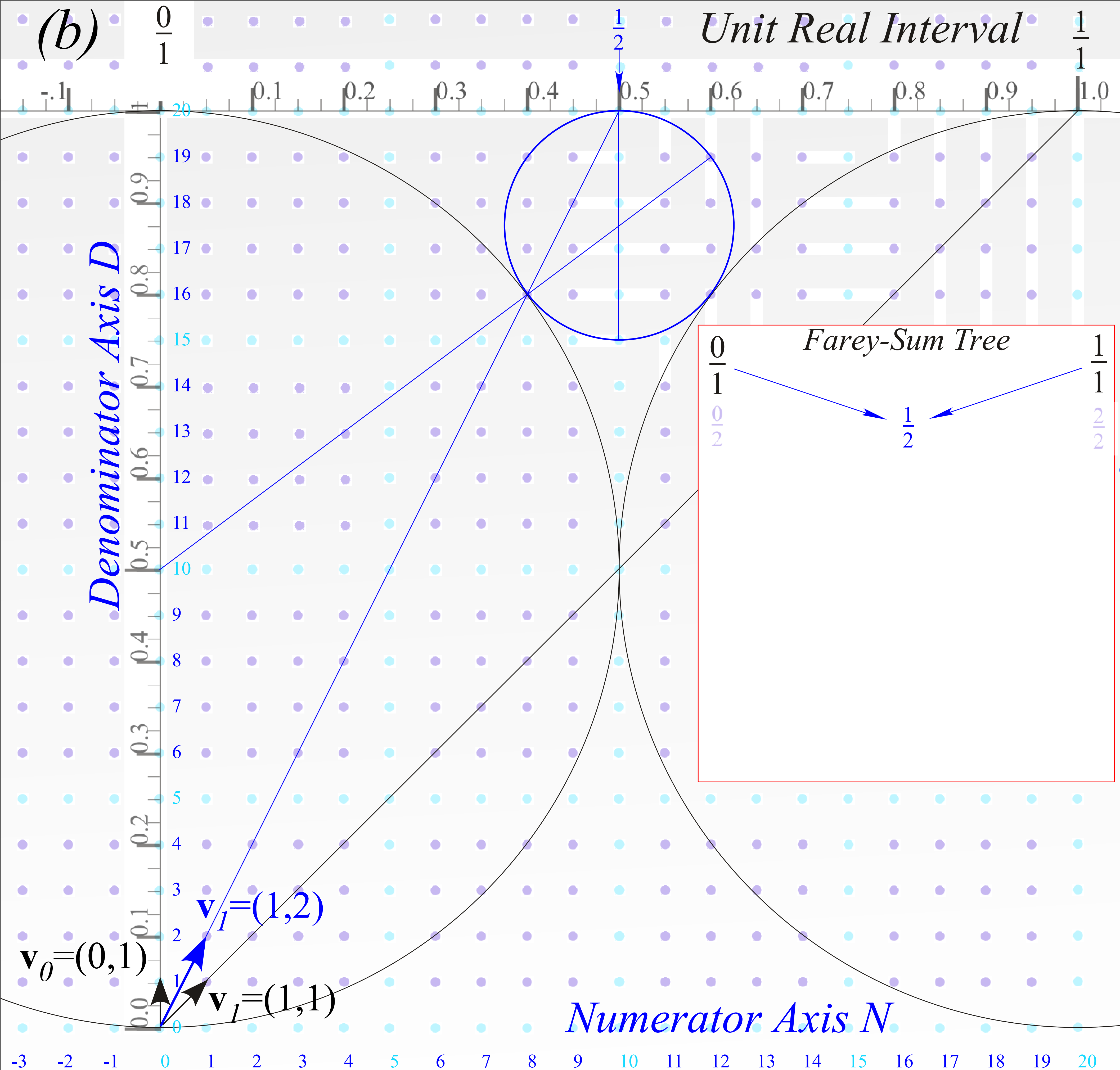
[Lester. R. Ford, Am. Math. Monthly 45,586(1938)]

[John Farey, Phil. Mag.(1816)]



Farey Sum related to vector sum and *Ford Circles*

1/1-circle has diameter *1*

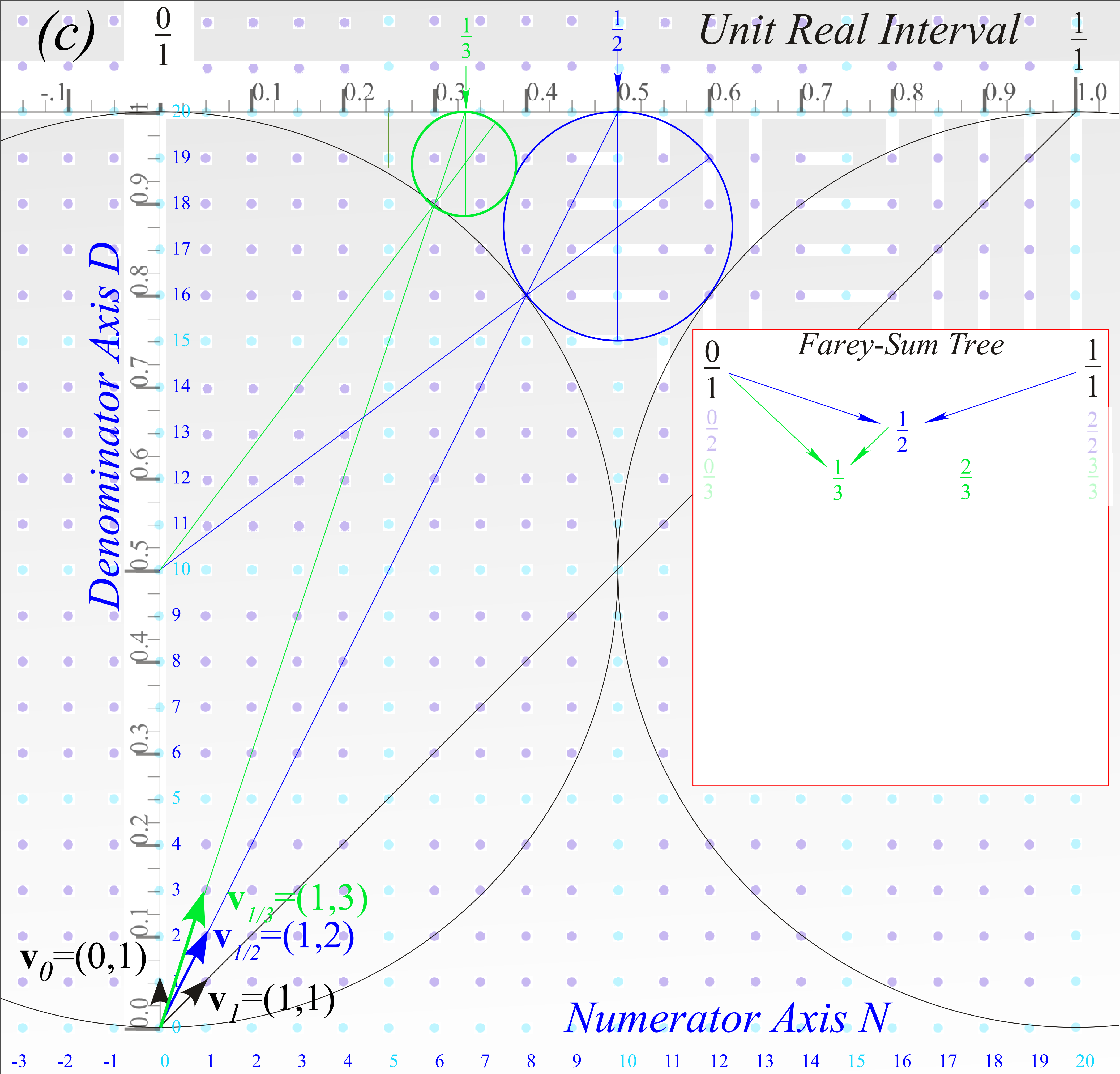


Farey Sum related to vector sum and *Ford Circles*

1/1-circle has diameter *1*

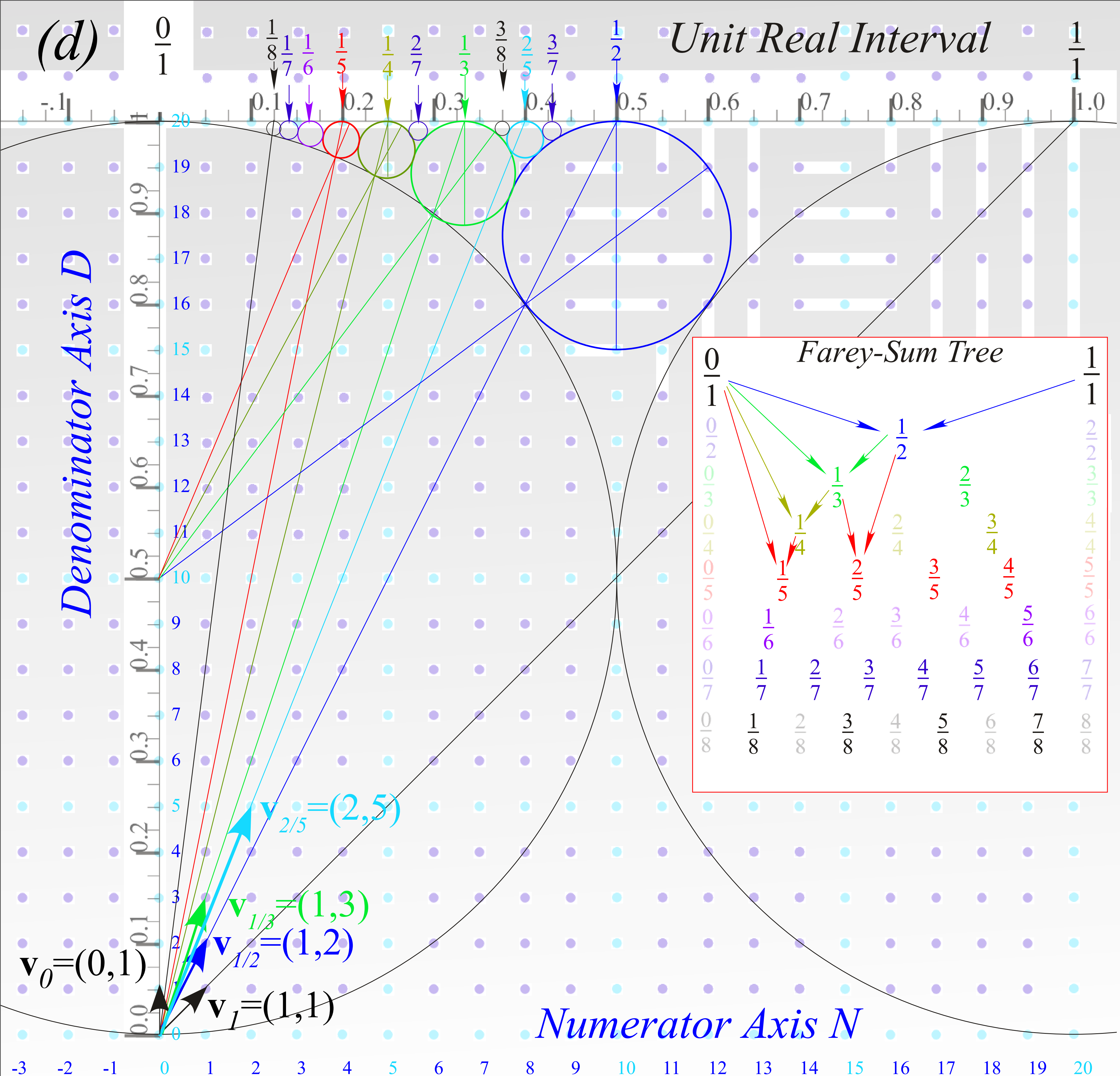
1/2-circle has diameter *1/2²=1/4*

Farey Sum related to vector sum and *Ford Circles*



1/2-circle has diameter $1/2^2=1/4$

1/3-circles have diameter $1/3^2=1/9$



Farey Sum related to vector sum and *Ford Circles*

1/2-circle has diameter $1/2^2=1/4$

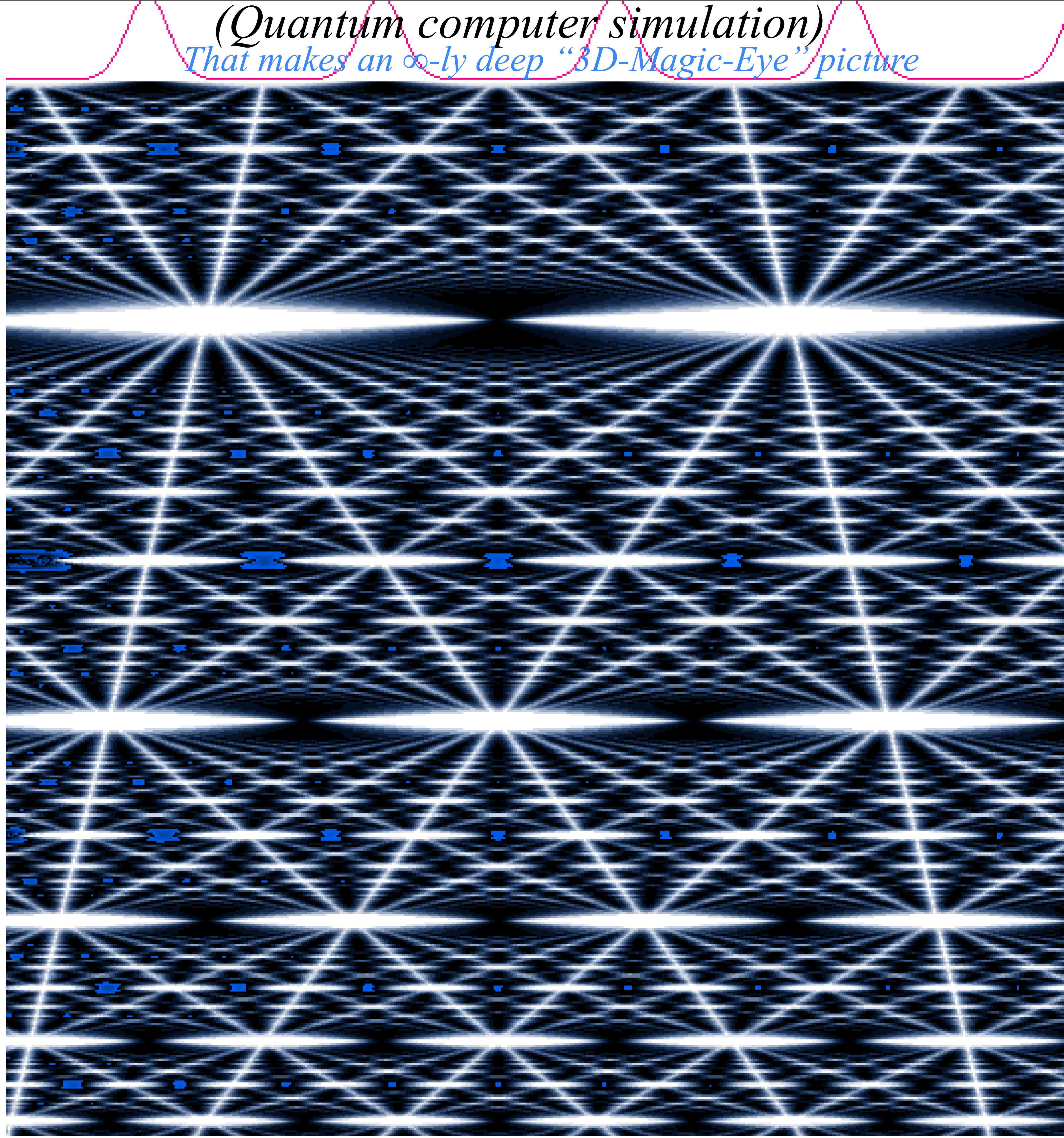
1/3-circles have diameter $1/3^2=1/9$

n/d-circles have diameter $1/d^2$

$D \leq 1$	$\mathbf{\frac{0}{1}}$																						$\mathbf{\frac{1}{1}}$
$D \leq 2$	$\frac{0}{1}$											$\mathbf{\frac{1}{2}}$											$\frac{1}{1}$
$D \leq 3$	$\frac{0}{1}$							$\mathbf{\frac{1}{3}}$				$\frac{1}{2}$				$\mathbf{\frac{2}{3}}$							$\frac{1}{1}$
$D \leq 4$	$\frac{0}{1}$					$\mathbf{\frac{1}{4}}$		$\frac{1}{3}$				$\frac{1}{2}$				$\frac{2}{3}$		$\mathbf{\frac{3}{4}}$					$\frac{1}{1}$
$D \leq 5$	$\frac{0}{1}$				$\mathbf{\frac{1}{5}}$	$\frac{1}{4}$		$\frac{1}{3}$		$\mathbf{\frac{2}{5}}$		$\frac{1}{2}$		$\mathbf{\frac{3}{5}}$		$\frac{2}{3}$		$\frac{3}{4}$	$\mathbf{\frac{4}{5}}$				$\frac{1}{1}$
$D \leq 6$	$\frac{0}{1}$			$\mathbf{\frac{1}{6}}$	$\frac{1}{5}$	$\frac{1}{4}$		$\frac{1}{3}$		$\frac{2}{5}$		$\frac{1}{2}$		$\frac{3}{5}$		$\frac{2}{3}$		$\frac{3}{4}$	$\frac{4}{5}$	$\mathbf{\frac{5}{6}}$			$\frac{1}{1}$
$D \leq 7$	$\frac{0}{1}$		$\mathbf{\frac{1}{7}}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\mathbf{\frac{2}{7}}$	$\frac{1}{3}$		$\frac{2}{5}$	$\mathbf{\frac{3}{7}}$	$\frac{1}{2}$	$\mathbf{\frac{4}{7}}$	$\frac{3}{5}$		$\frac{2}{3}$	$\mathbf{\frac{5}{7}}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\mathbf{\frac{6}{7}}$		$\frac{1}{1}$
$D \leq 8$	$\frac{0}{1}$	$\mathbf{\frac{1}{8}}$	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{2}{7}$	$\frac{1}{3}$	$\mathbf{\frac{3}{8}}$	$\frac{2}{5}$	$\frac{3}{7}$	$\frac{1}{2}$	$\frac{4}{7}$	$\frac{3}{5}$	$\mathbf{\frac{5}{8}}$	$\frac{2}{3}$	$\frac{5}{7}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$	$\mathbf{\frac{7}{8}}$	$\frac{1}{1}$

(Quantum computer simulation)

That makes an ∞-ly deep "3D-Magic-Eye" picture



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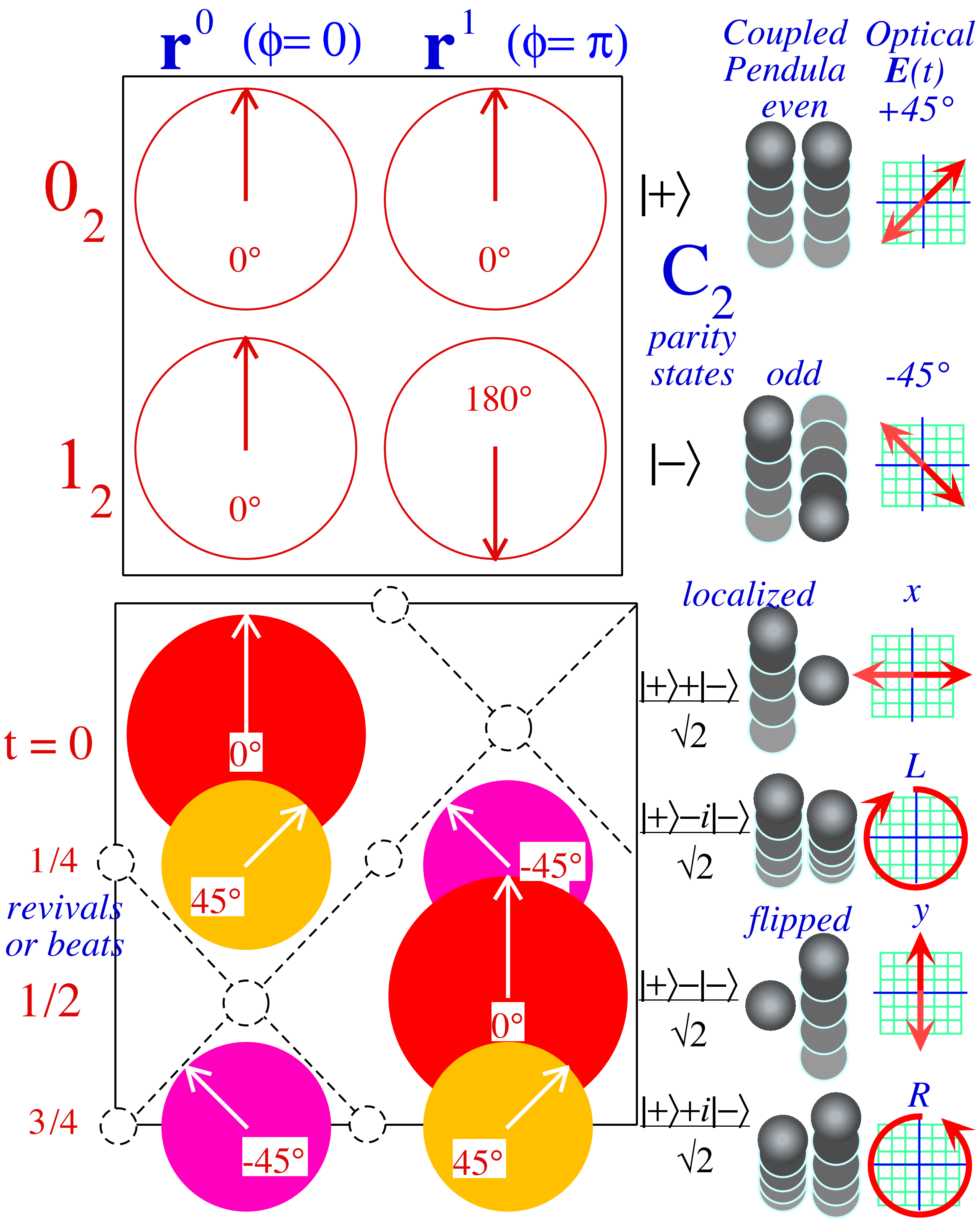
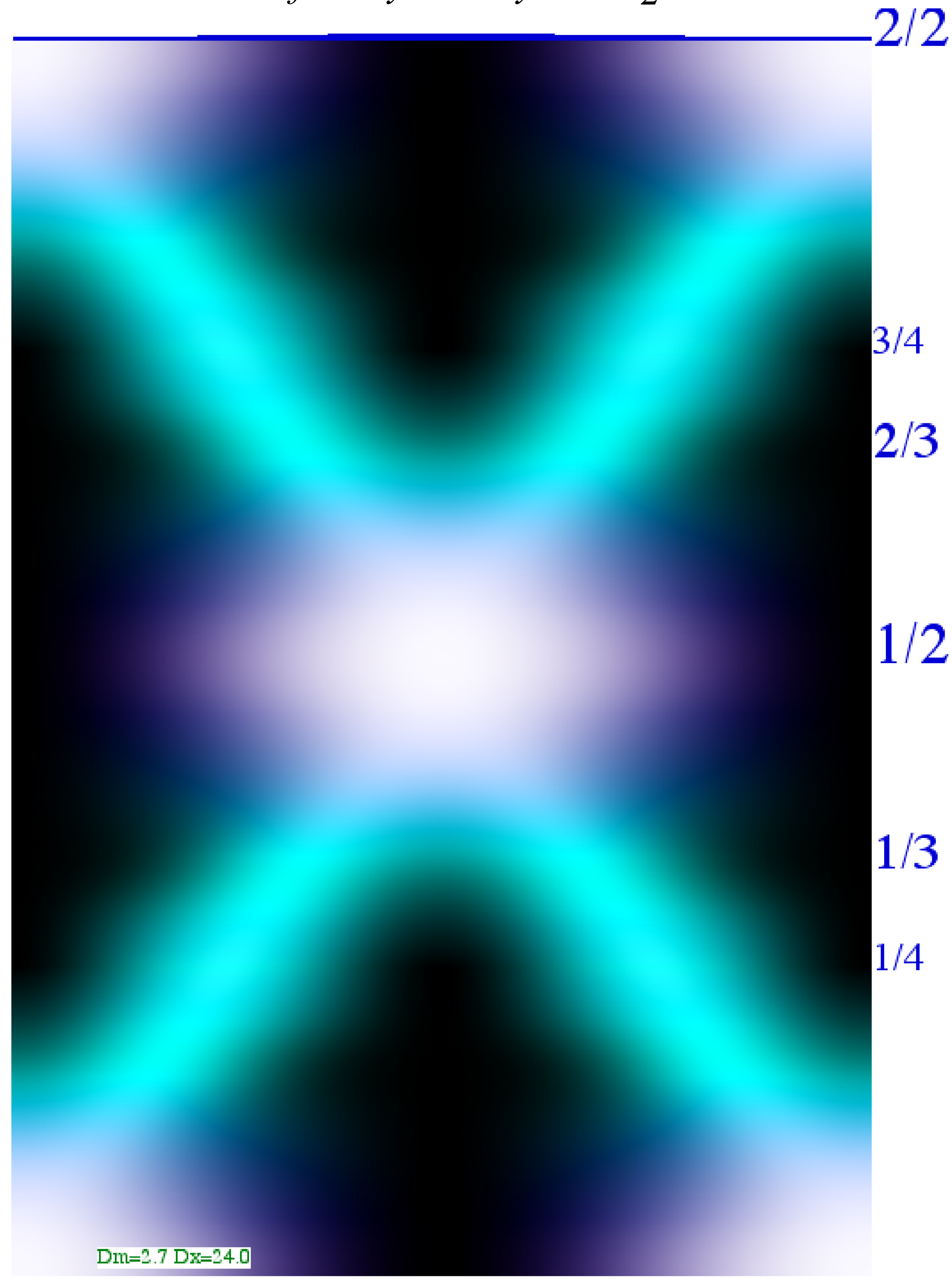
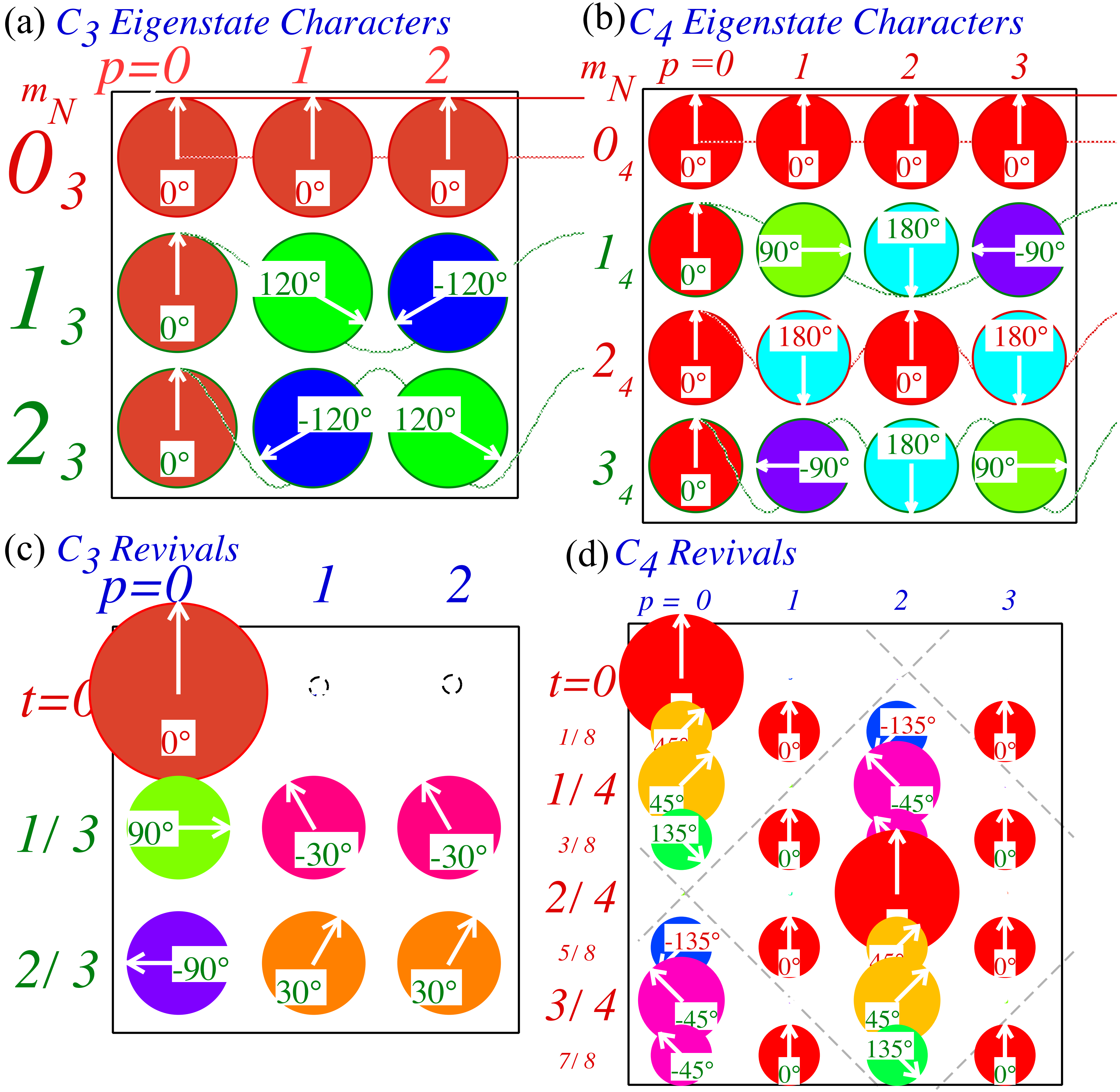
    Gaussian wave-packet bandwidth and uncertainty
    Gaussian Bohr-rotor revivals
  Farey-Sums and Ford-products
  → *Phase dynamics* ←

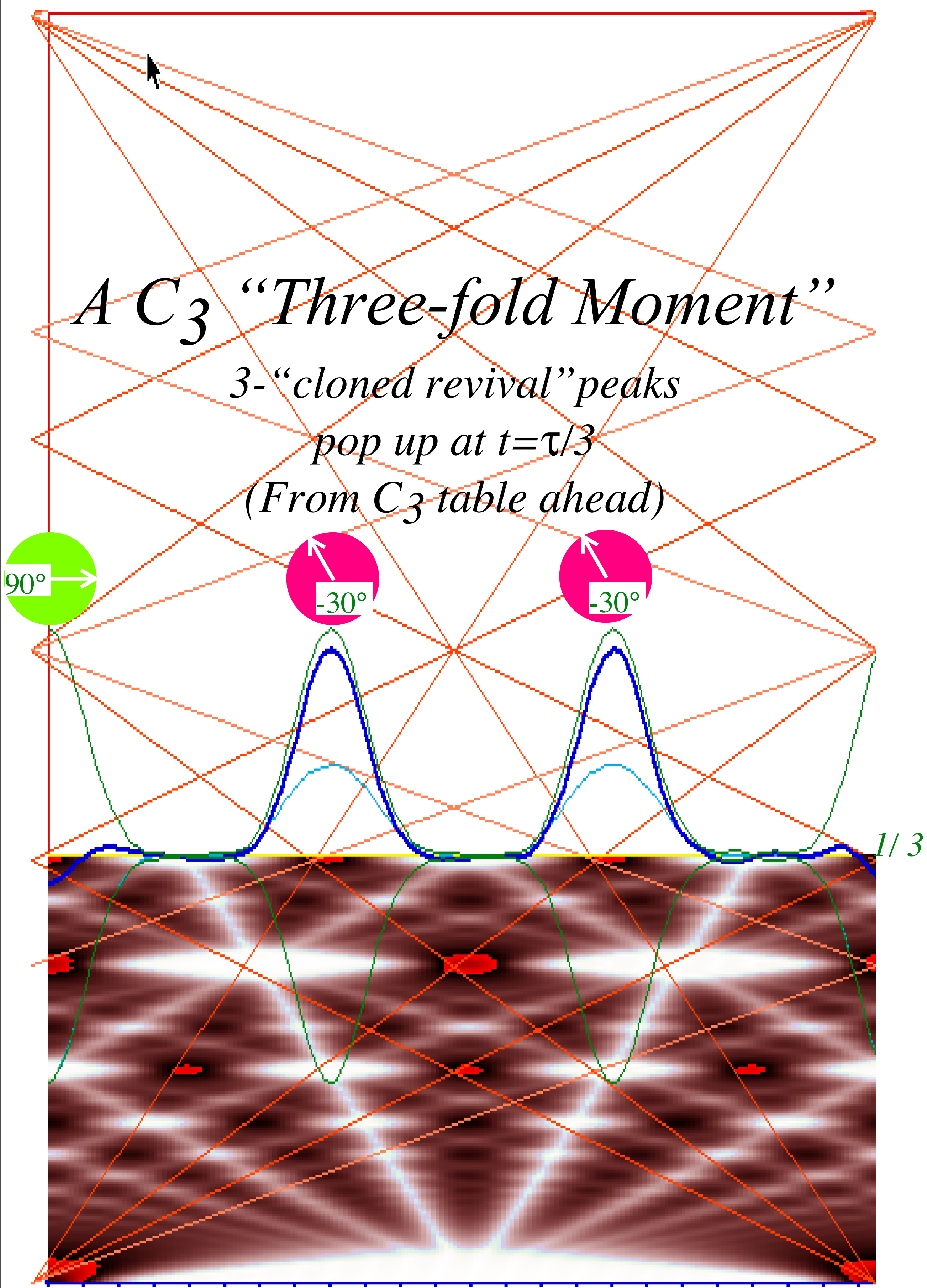
Fundamental Beats and 2-Level Transitions: The "Mother of all symmetry" is C_2

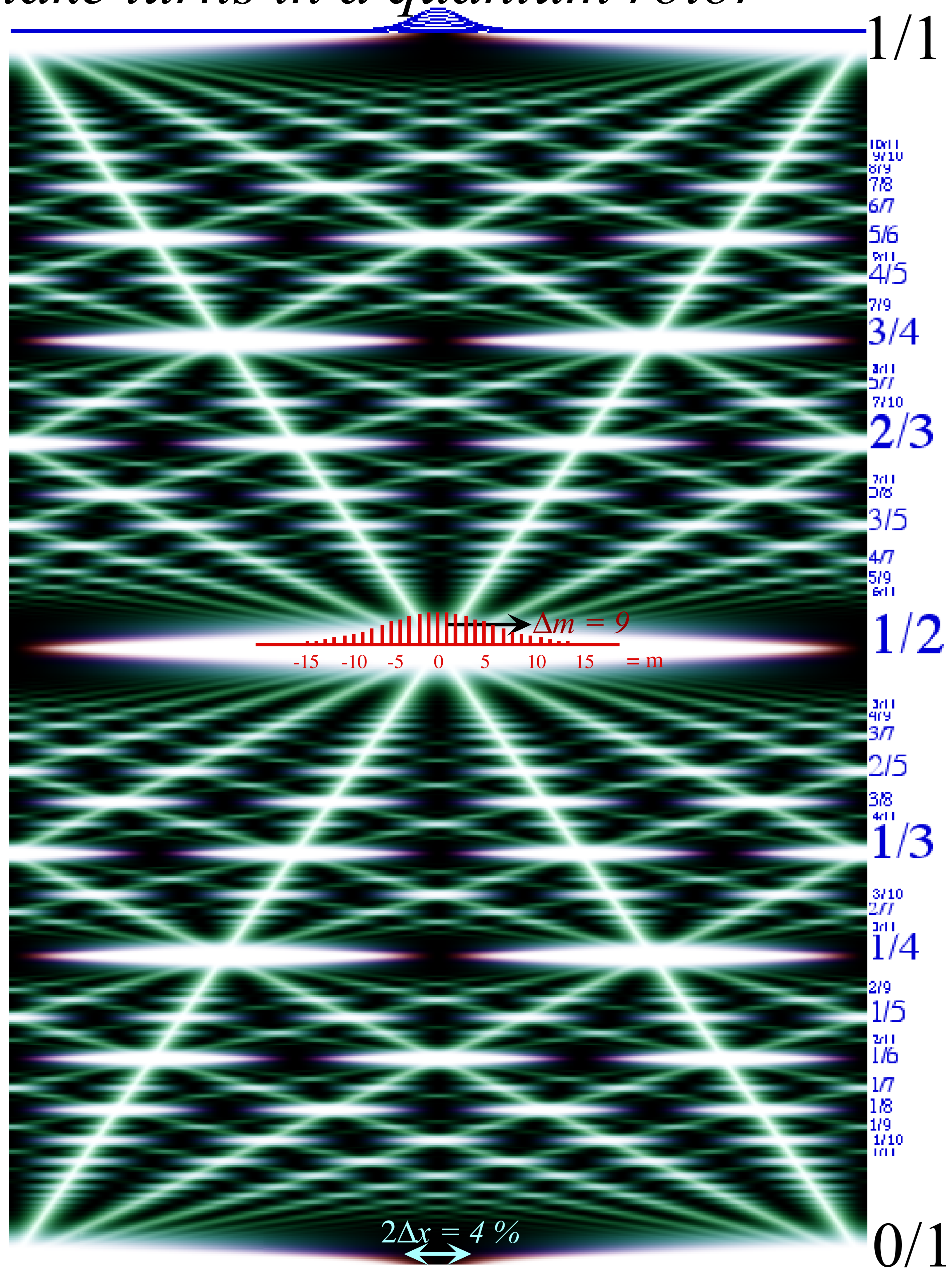


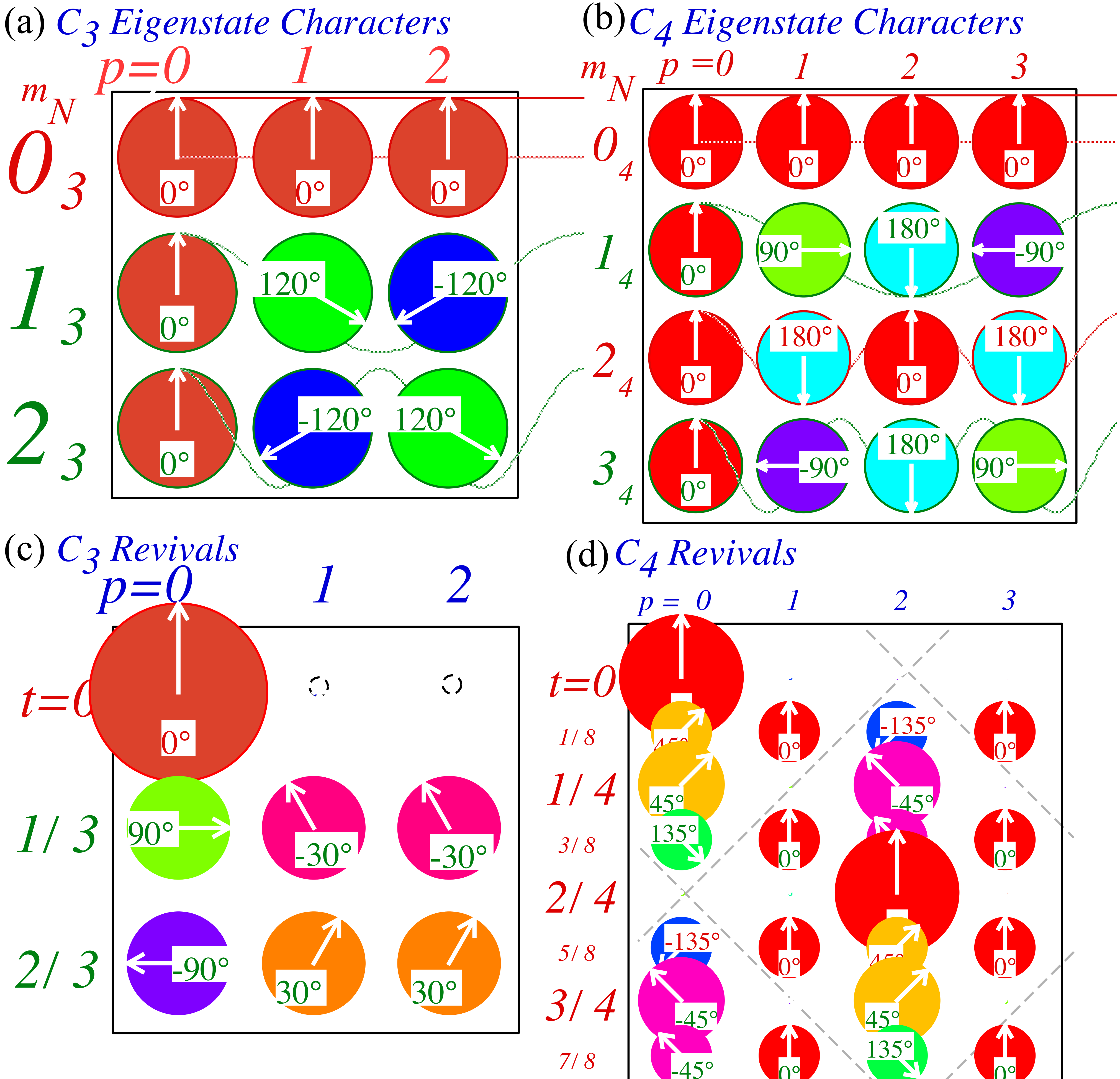



(a) C3 Eigenstate Characters
p=0 1 2
mN
03
13
23
0°
120°
-120°
(b) C4 Eigenstate Characters
mN p =0 1 2 3
04
14
24
34
0°
90°
180°
-90°
(c) C3 Revivals
p=0 1 2
t=0
1/ 3
2/ 3
90°
-30°
-90°
30°
(d) C4 Revivals
p = 0 1 2 3
t=0
1/ 8
1/ 4
3/ 8
2/ 4
5/ 8
3/ 4
7/ 8
-135°
45°
-45°
135°

Revivals: All excited transitions take turns in a quantum rotor






(a) C3 Eigenstate Characters
(b) C4 Eigenstate Characters
(c) C3 Revivals
(d) C4 Revivals
mN
p=0
1
2
3
0°
120°
-120°
90°
180°
-90°
t=0
1/ 3
2/ 3
1/ 4
2/ 4
3/ 4
1/ 8
3/ 8
5/ 8
7/ 8
-30°
30°
-135°
-45°
45°
135°

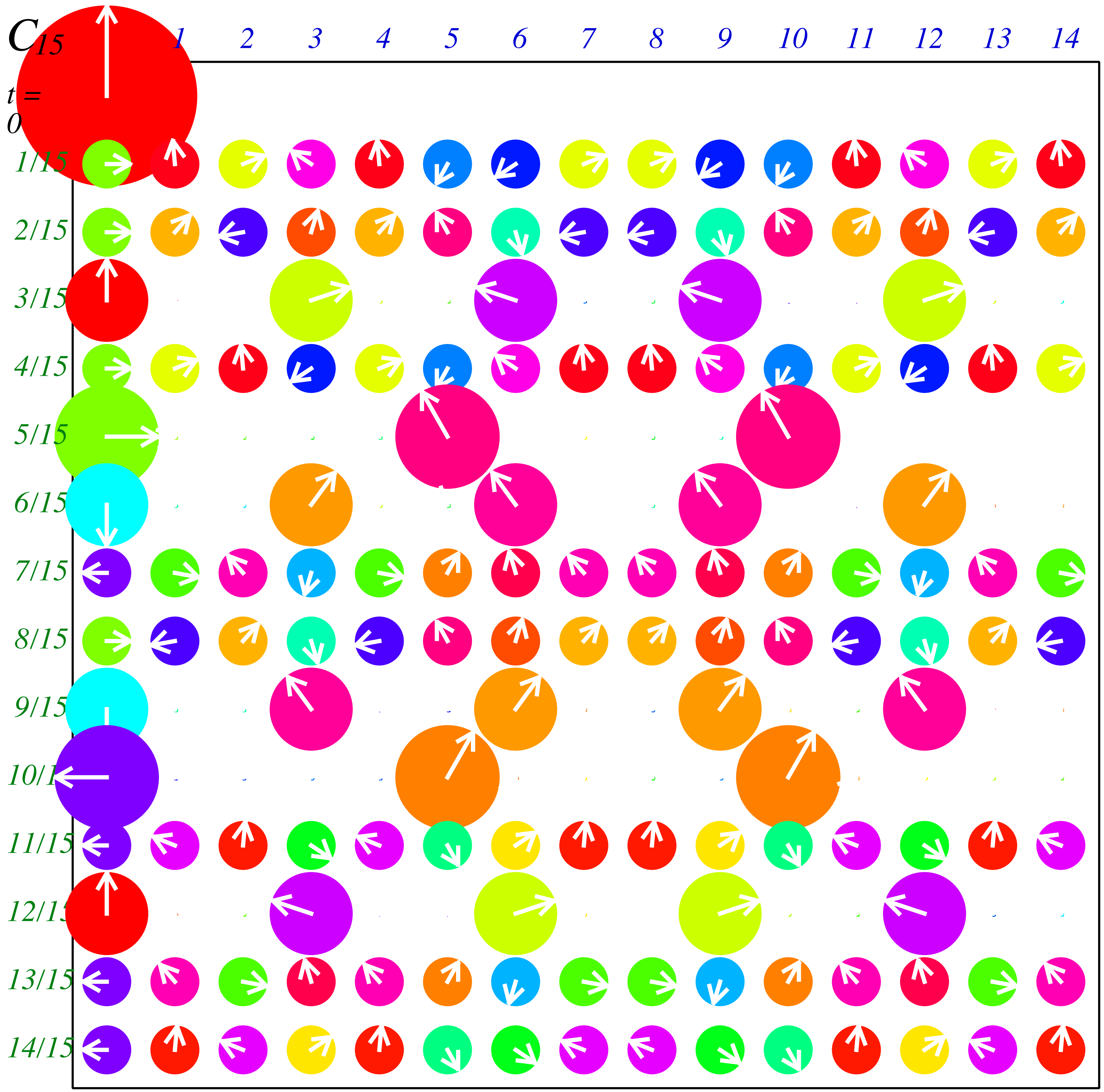
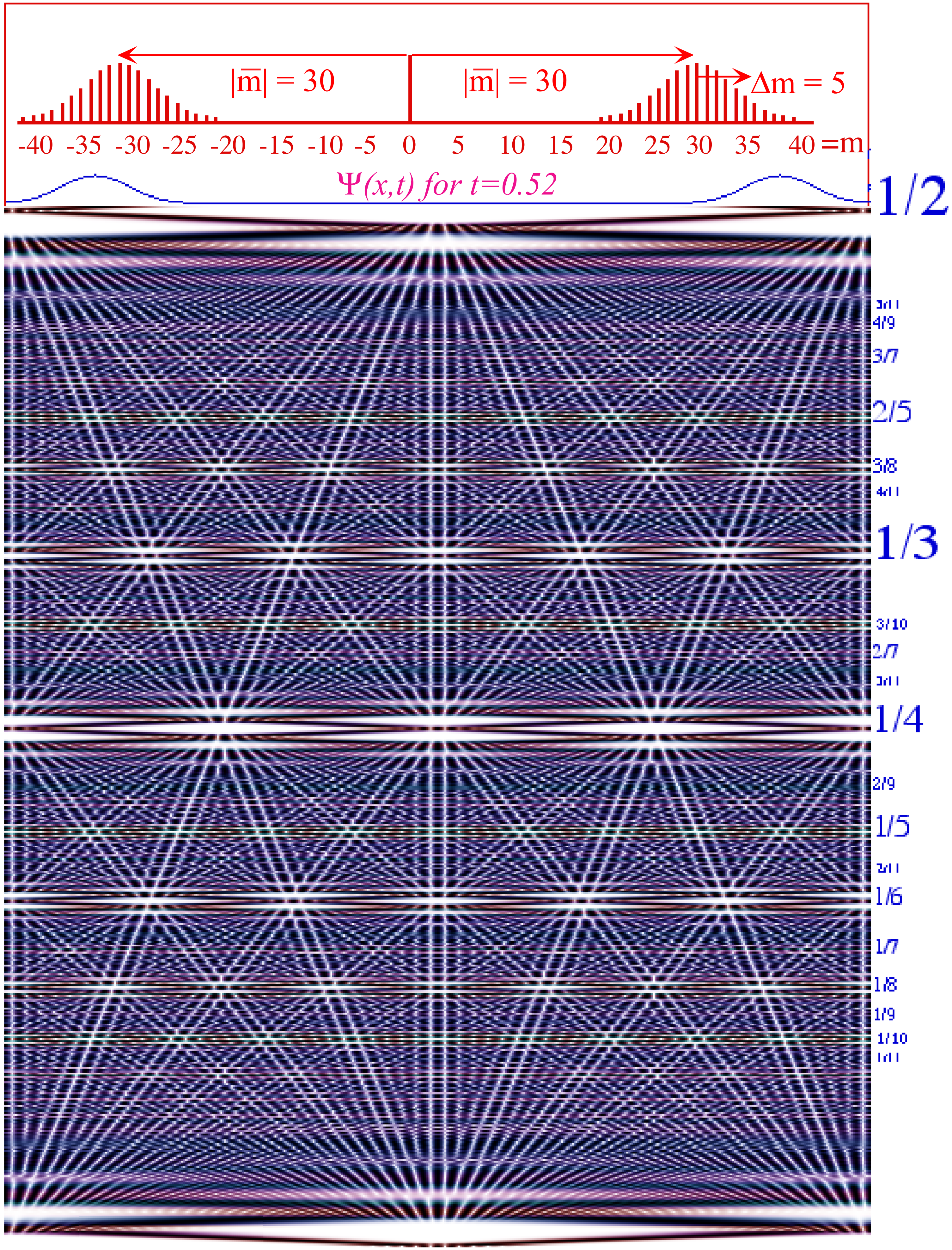
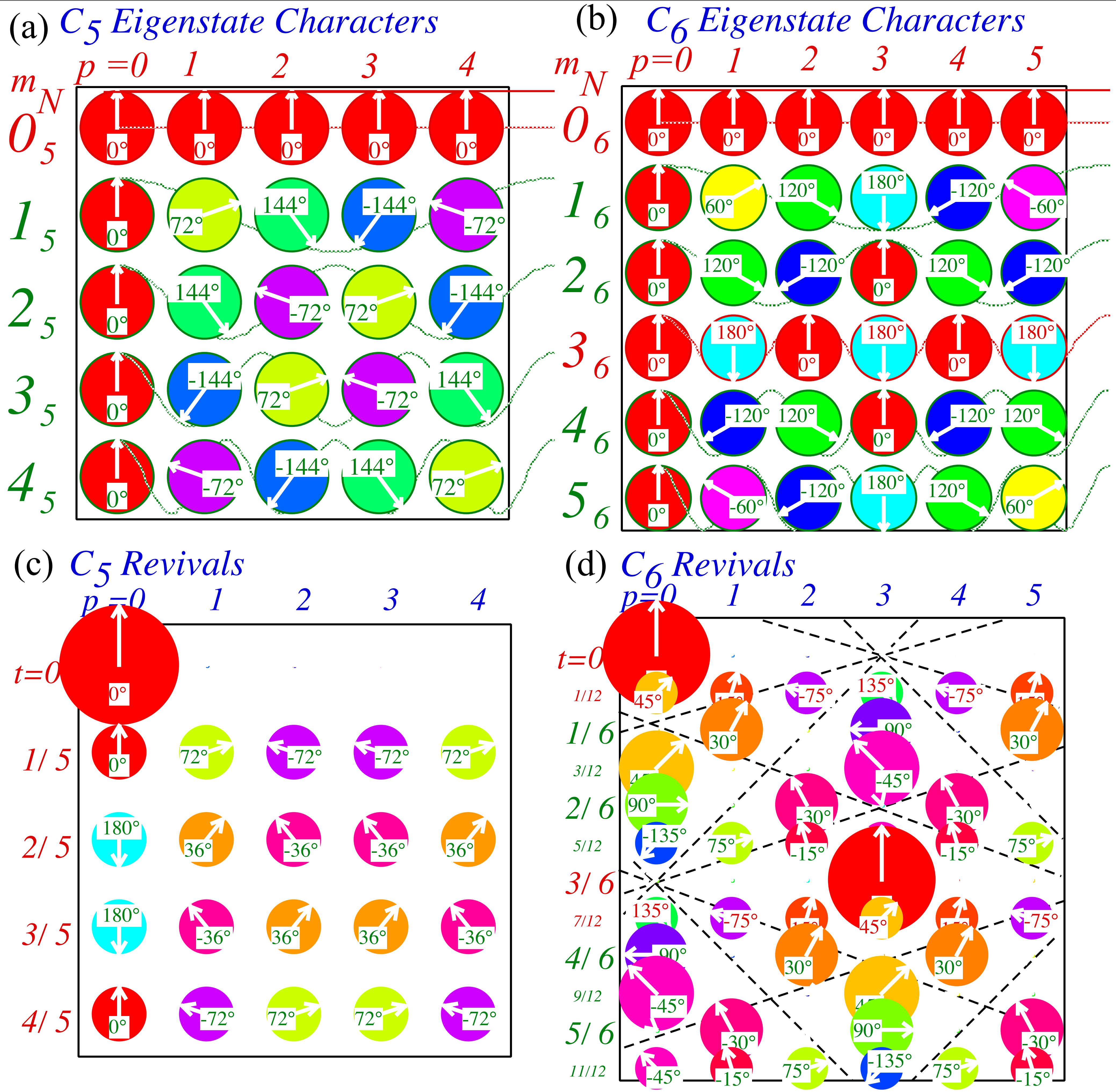


Fig. 9.4.4 Bohr space-time revival pattern for C_{15} Bohr system.


|m̄| = 30
|m̄| = 30
Δm = 5
-40 -35 -30 -25 -20 -15 -10 -5 0 5 10 15 20 25 30 35 40 =m
Ψ(x,t) for t=0.52
1/2
4/9
3/7
2/5
3/8
1/3
3/10
2/7
1/4
2/9
1/5
1/6
1/7
1/8
1/9
1/10


(a) C5 Eigenstate Characters
(b) C6 Eigenstate Characters
(c) C5 Revivals
(d) C6 Revivals
m N
p =0 1 2 3 4
p=0 1 2 3 4 5
0 5
1 5
2 5
3 5
4 5
0 6
1 6
2 6
3 6
4 6
5 6
t=0
1/ 5
2/ 5
3/ 5
4/ 5
1/12
1/ 6
3/12
2/ 6
5/12
3/ 6
7/12
4/ 6
9/12
5/ 6
11/12

Simulating Complex Systems With Simpler Ones

Discrete 3-State or Trigonal System (Tesla's 3-Phase AC)

Discrete 6-State or Hexagonal System (6-Phase AC)