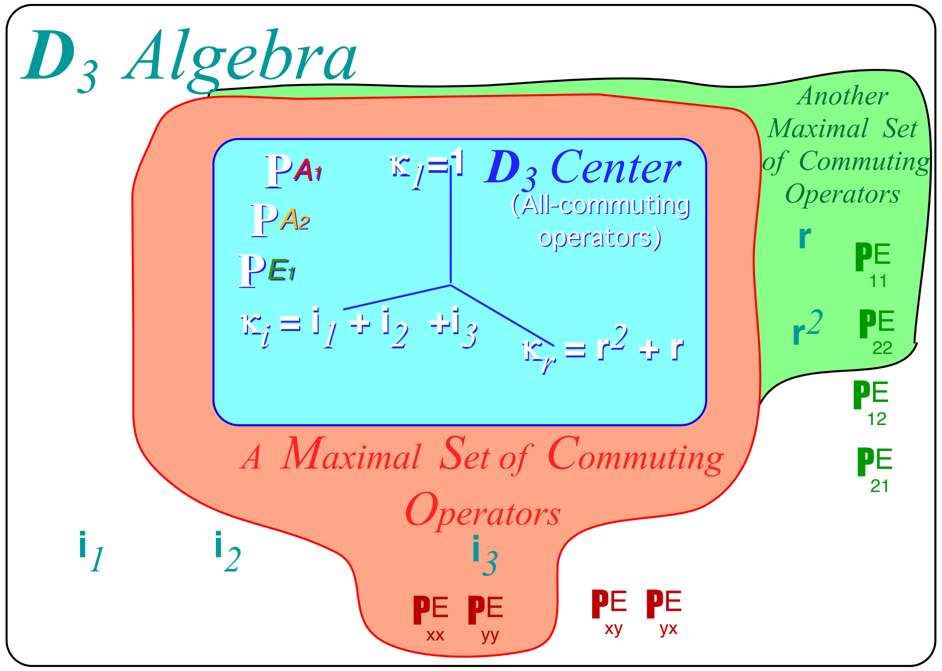
# Group Theory in Quantum Mechanics

## Lecture 14 (3.14.13)

## Spectral decomposition of groups $D_3 \sim C_{3v}$

(Int.J.Mol.Sci, 14, 714(2013) p.755-774 , QTCA Unit 5 Ch. 15 )
(PSDS - Ch. 3 )




(Fig. 15.2.1 QTCA)

Review: Spectral resolution of **$D_3$ Center** (Class algebra)
- Group theory of equivalence transformations and classes
  - Lagrange theorems
- All-commuting class projectors and $D_3$-invariant characters
  - Character ortho-completeness
- Group invariant numbers: Centrum, Rank, and Order

2nd-Stage spectral decompositions of global/local $D_3$
- Splitting class projectors using subgroup chains $D_3 \supset C_2$ and $D_3 \supset C_3$
  - Splitting classes

3rd-stage spectral resolution to **irreducible representations** (ireps) and Hamiltonian eigensolutions
- Tunneling modes and spectra for $D_3 \supset C_2$ and $D_3 \supset C_3$ local subgroup chains

*Review:Spectral resolution of **D$_3$ Center** (Class algebra)*

*Group theory of equivalence transformations and classes*

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*All-commuting class projectors and D$_3$-invariant characters*

*Character ortho-completeness*

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# *Review:* *Spectral resolution of* ***D₃ Center*** *(Class algebra)*

| | | | | | |
|---|---|---|---|---|---|
| $\mathbf{1}$ | $\mathbf{r}^2$ | $\mathbf{r}$ | $\mathbf{i}_1$ | $\mathbf{i}_2$ | $\mathbf{i}_3$ |
| $\mathbf{r}$ | $\mathbf{1}$ | $\mathbf{r}^2$ | $\mathbf{i}_3$ | $\mathbf{i}_1$ | $\mathbf{i}_2$ |
| $\mathbf{r}^2$ | $\mathbf{r}$ | $\mathbf{1}$ | $\mathbf{i}_2$ | $\mathbf{i}_3$ | $\mathbf{i}_1$ |
| $\mathbf{i}_1$ | $\mathbf{i}_3$ | $\mathbf{i}_2$ | $\mathbf{1}$ | $\mathbf{r}$ | $\mathbf{r}^2$ |
| $\mathbf{i}_2$ | $\mathbf{i}_1$ | $\mathbf{i}_3$ | $\mathbf{r}^2$ | $\mathbf{1}$ | $\mathbf{r}$ |
| $\mathbf{i}_3$ | $\mathbf{i}_2$ | $\mathbf{i}_1$ | $\mathbf{r}$ | $\mathbf{r}^2$ | $\mathbf{1}$ |

$\rightarrow$

| | $\kappa_1 = \mathbf{1}$ | $\kappa_r = \mathbf{r} + \mathbf{r}^2$ | $\kappa_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$ |
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| $\kappa_1$ | $\kappa_1$ | $\kappa_r$ | $\kappa_i$ |
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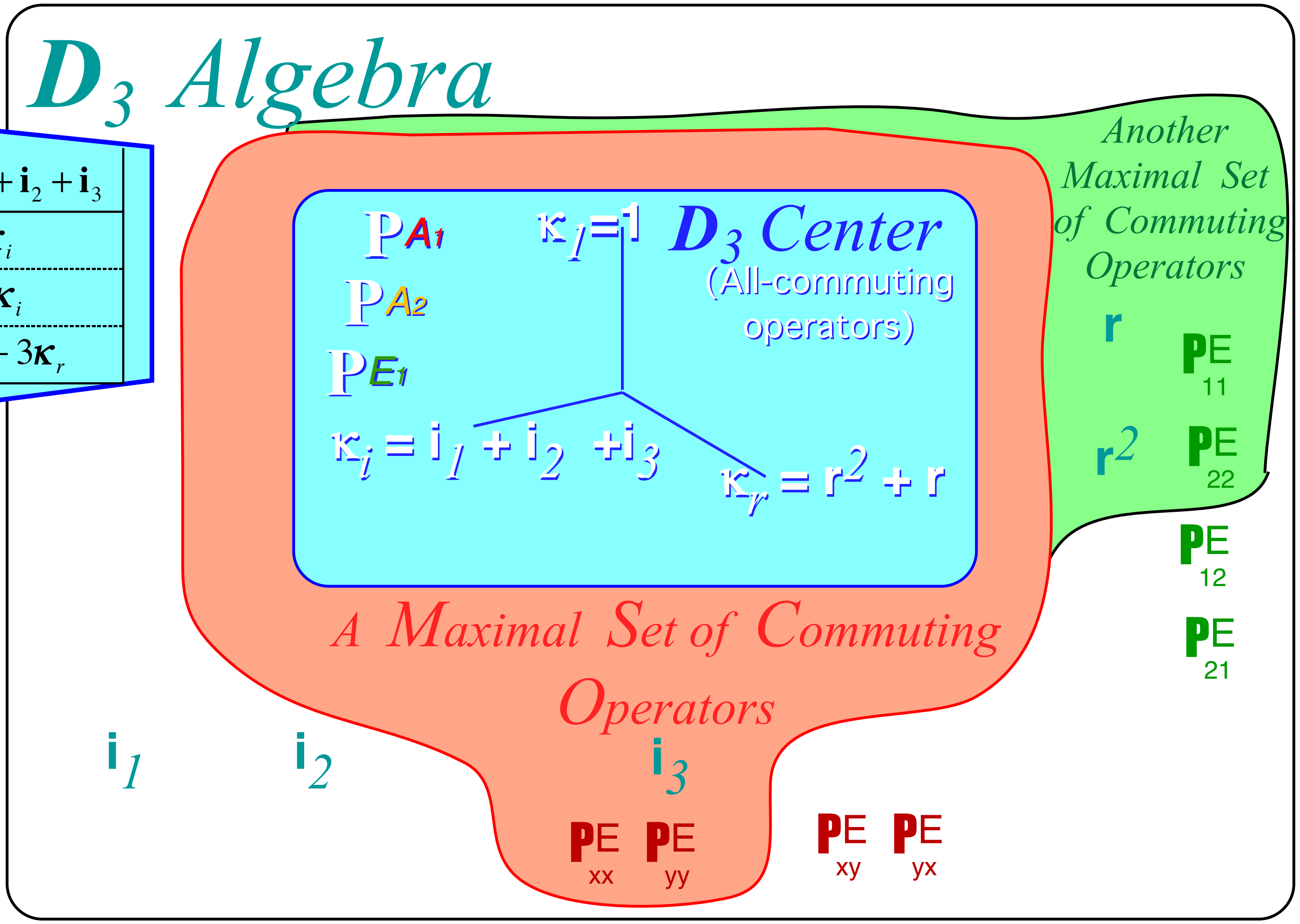
| $\kappa_i$ | $\kappa_i$ | $2\kappa_i$ | $3\kappa_1 + 3\kappa_r$ |

*Class-sum* $\boldsymbol{\kappa}_k$ *commutes with all* $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t \boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k \mathbf{g}_t$



## *Review: Spectral resolution of **$D_3$ Center** (Class algebra)*

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|---|---|---|---|---|---|
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| $\mathbf{r}$ | $\mathbf{1}$ | $\mathbf{r}^2$ | $\mathbf{i}_3$ | $\mathbf{i}_1$ | $\mathbf{i}_2$ |
| $\mathbf{r}^2$ | $\mathbf{r}$ | $\mathbf{1}$ | $\mathbf{i}_2$ | $\mathbf{i}_3$ | $\mathbf{i}_1$ |
| $\mathbf{i}_1$ | $\mathbf{i}_3$ | $\mathbf{i}_2$ | $\mathbf{1}$ | $\mathbf{r}$ | $\mathbf{r}^2$ |
| $\mathbf{i}_2$ | $\mathbf{i}_1$ | $\mathbf{i}_3$ | $\mathbf{r}^2$ | $\mathbf{1}$ | $\mathbf{r}$ |
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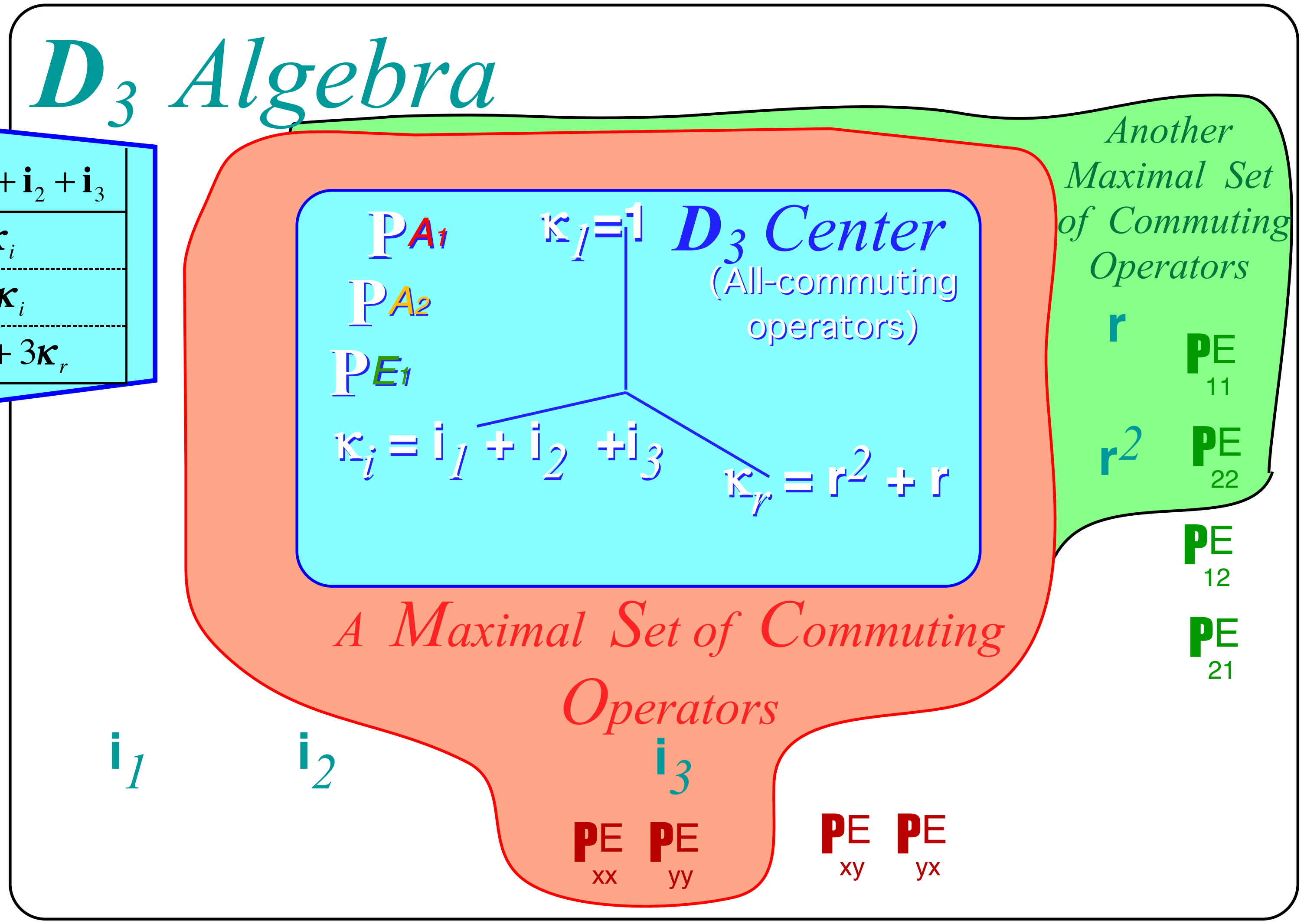

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| $\mathbf{i}_1$ | $\mathbf{i}_3$ | $\mathbf{i}_2$ | $\mathbf{1}$ | $\mathbf{r}$ | $\mathbf{r}^2$ |
| $\mathbf{i}_2$ | $\mathbf{i}_1$ | $\mathbf{i}_3$ | $\mathbf{r}^2$ | $\mathbf{1}$ | $\mathbf{r}$ |
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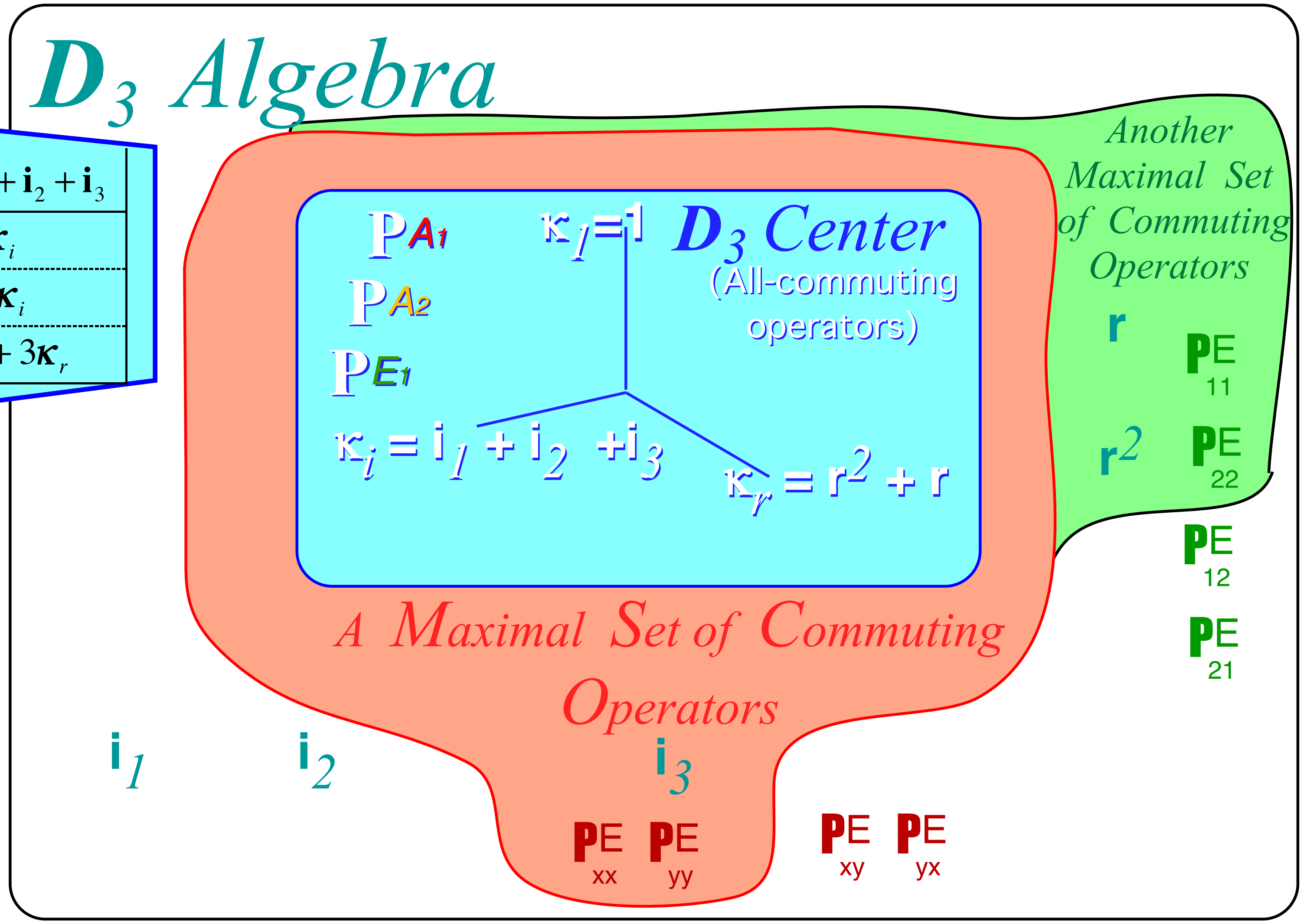

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| $\mathbf{r}^2$ | $\mathbf{r}$ | $\mathbf{1}$ | $\mathbf{i}_2$ | $\mathbf{i}_3$ | $\mathbf{i}_1$ |
| $\mathbf{i}_1$ | $\mathbf{i}_3$ | $\mathbf{i}_2$ | $\mathbf{1}$ | $\mathbf{r}$ | $\mathbf{r}^2$ |
| $\mathbf{i}_2$ | $\mathbf{i}_1$ | $\mathbf{i}_3$ | $\mathbf{r}^2$ | $\mathbf{1}$ | $\mathbf{r}$ |
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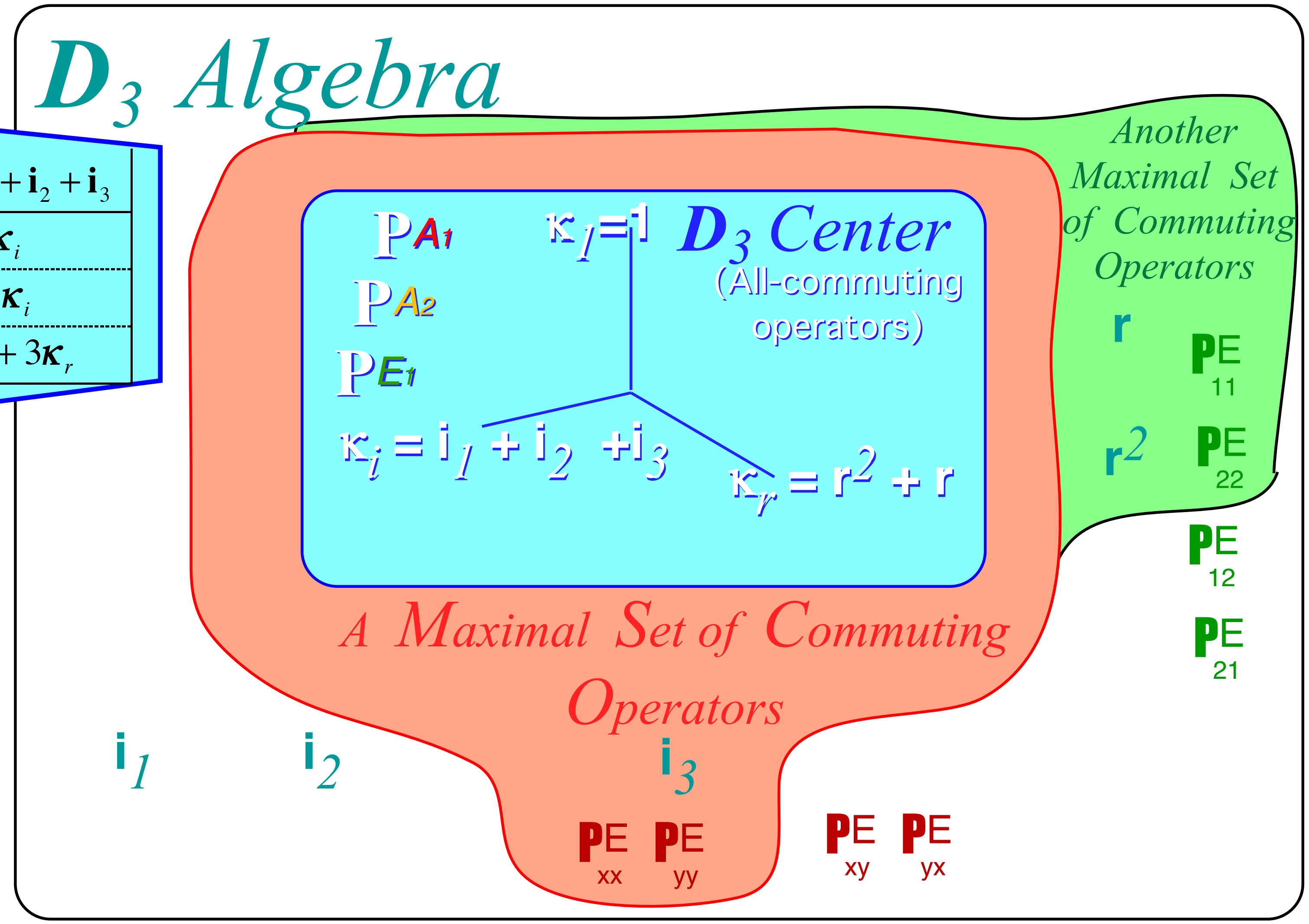

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| $\mathbf{r}^2$ | $\mathbf{r}$ | $\mathbf{1}$ | $\mathbf{i}_2$ | $\mathbf{i}_3$ | $\mathbf{i}_1$ |
| $\mathbf{i}_1$ | $\mathbf{i}_3$ | $\mathbf{i}_2$ | $\mathbf{1}$ | $\mathbf{r}$ | $\mathbf{r}^2$ |
| $\mathbf{i}_2$ | $\mathbf{i}_1$ | $\mathbf{i}_3$ | $\mathbf{r}^2$ | $\mathbf{1}$ | $\mathbf{r}$ |
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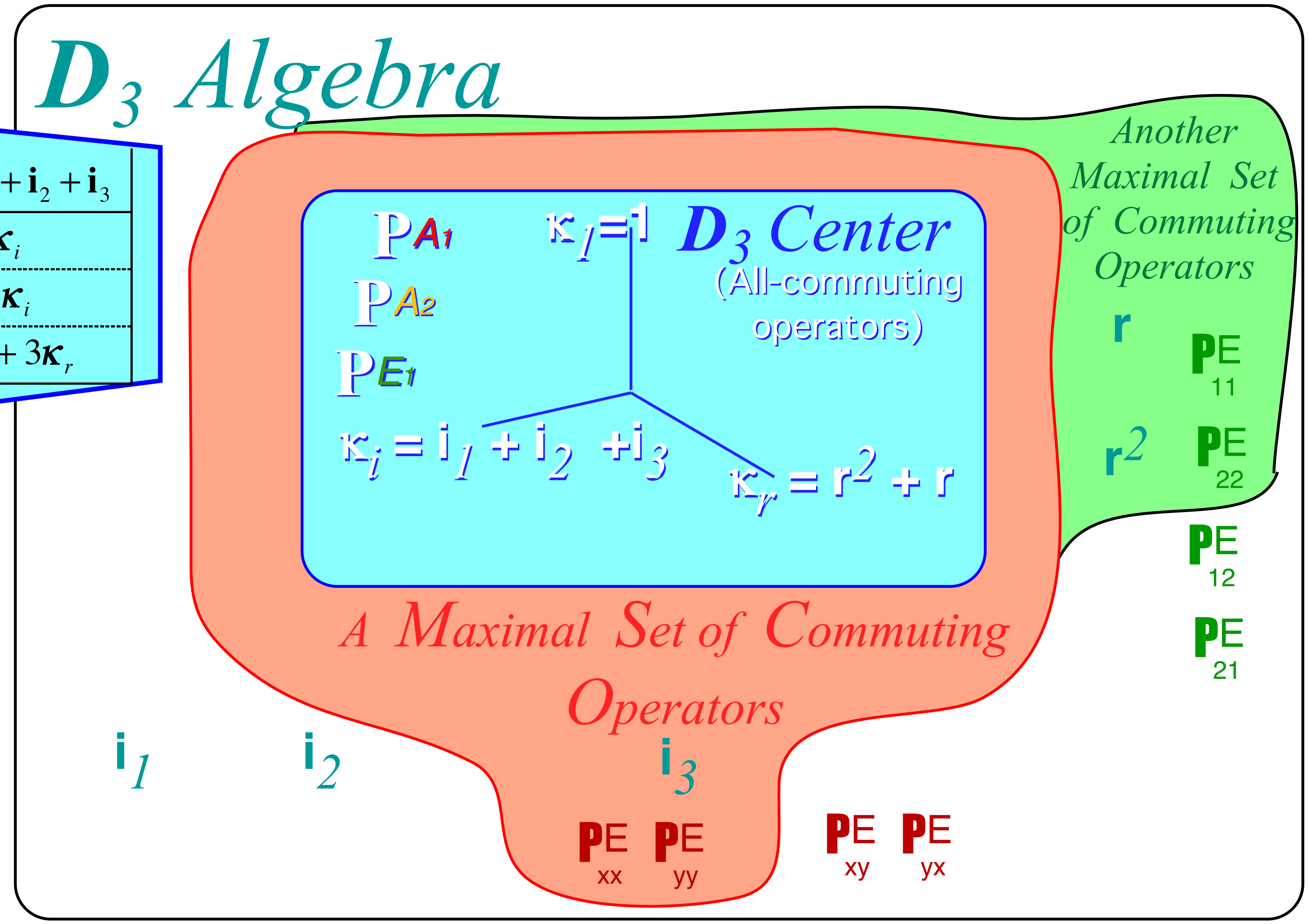

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| $\mathbf{r}^2$ | $\mathbf{r}$ | $\mathbf{1}$ | $\mathbf{i}_2$ | $\mathbf{i}_3$ | $\mathbf{i}_1$ |
| $\mathbf{i}_1$ | $\mathbf{i}_3$ | $\mathbf{i}_2$ | $\mathbf{1}$ | $\mathbf{r}$ | $\mathbf{r}^2$ |
| $\mathbf{i}_2$ | $\mathbf{i}_1$ | $\mathbf{i}_3$ | $\mathbf{r}^2$ | $\mathbf{1}$ | $\mathbf{r}$ |
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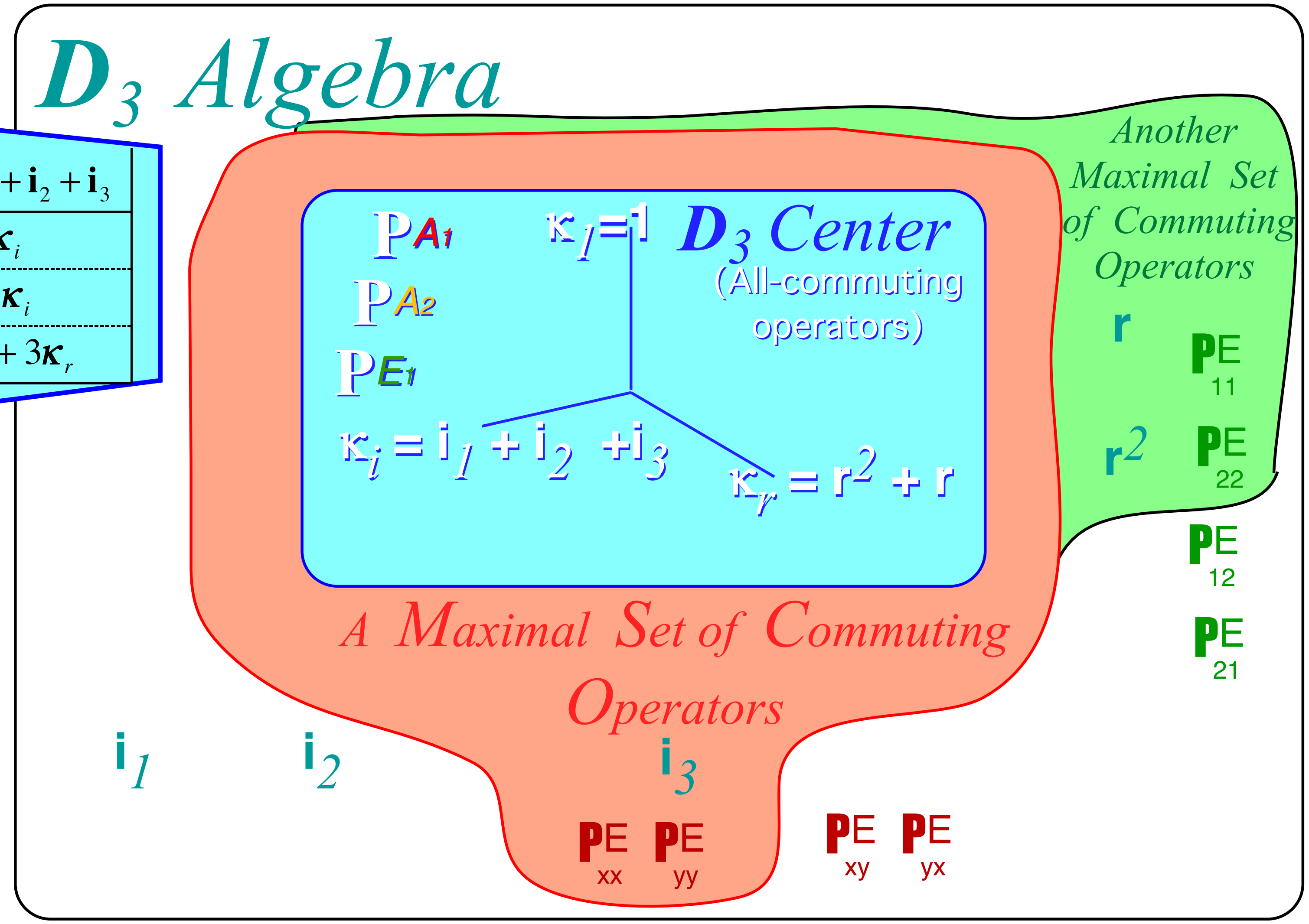


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$°s_k$=order of $\mathbf{g}_k$-self-symmetry: $(°s_1 = 6,\ °s_r = 3,\ °s_i = 2)$

$°s_k = °G / °\kappa_k$ *$°s_k$ is an integer count of $D_3$ operators $\mathbf{g}_s$ that commute with $\mathbf{g}_k$.*

*These operators $\mathbf{g}_s$ form the $\mathbf{g}_k$-self-symmetry group $s_k$. Each $\mathbf{g}_s$ transforms $\mathbf{g}_k$ into itself: $\mathbf{g}_s \mathbf{g}_k \mathbf{g}_s^{-1} = \mathbf{g}_k$*

*Review: Spectral resolution of **D3 Center** (Class algebra)*

| | | | | | |
|---|---|---|---|---|---|
| $\mathbf{1}$ | $\mathbf{r}^2$ | $\mathbf{r}$ | $\mathbf{i}_1$ | $\mathbf{i}_2$ | $\mathbf{i}_3$ |
| $\mathbf{r}$ | $\mathbf{1}$ | $\mathbf{r}^2$ | $\mathbf{i}_3$ | $\mathbf{i}_1$ | $\mathbf{i}_2$ |
| $\mathbf{r}^2$ | $\mathbf{r}$ | $\mathbf{1}$ | $\mathbf{i}_2$ | $\mathbf{i}_3$ | $\mathbf{i}_1$ |
| $\mathbf{i}_1$ | $\mathbf{i}_3$ | $\mathbf{i}_2$ | $\mathbf{1}$ | $\mathbf{r}$ | $\mathbf{r}^2$ |
| $\mathbf{i}_2$ | $\mathbf{i}_1$ | $\mathbf{i}_3$ | $\mathbf{r}^2$ | $\mathbf{1}$ | $\mathbf{r}$ |
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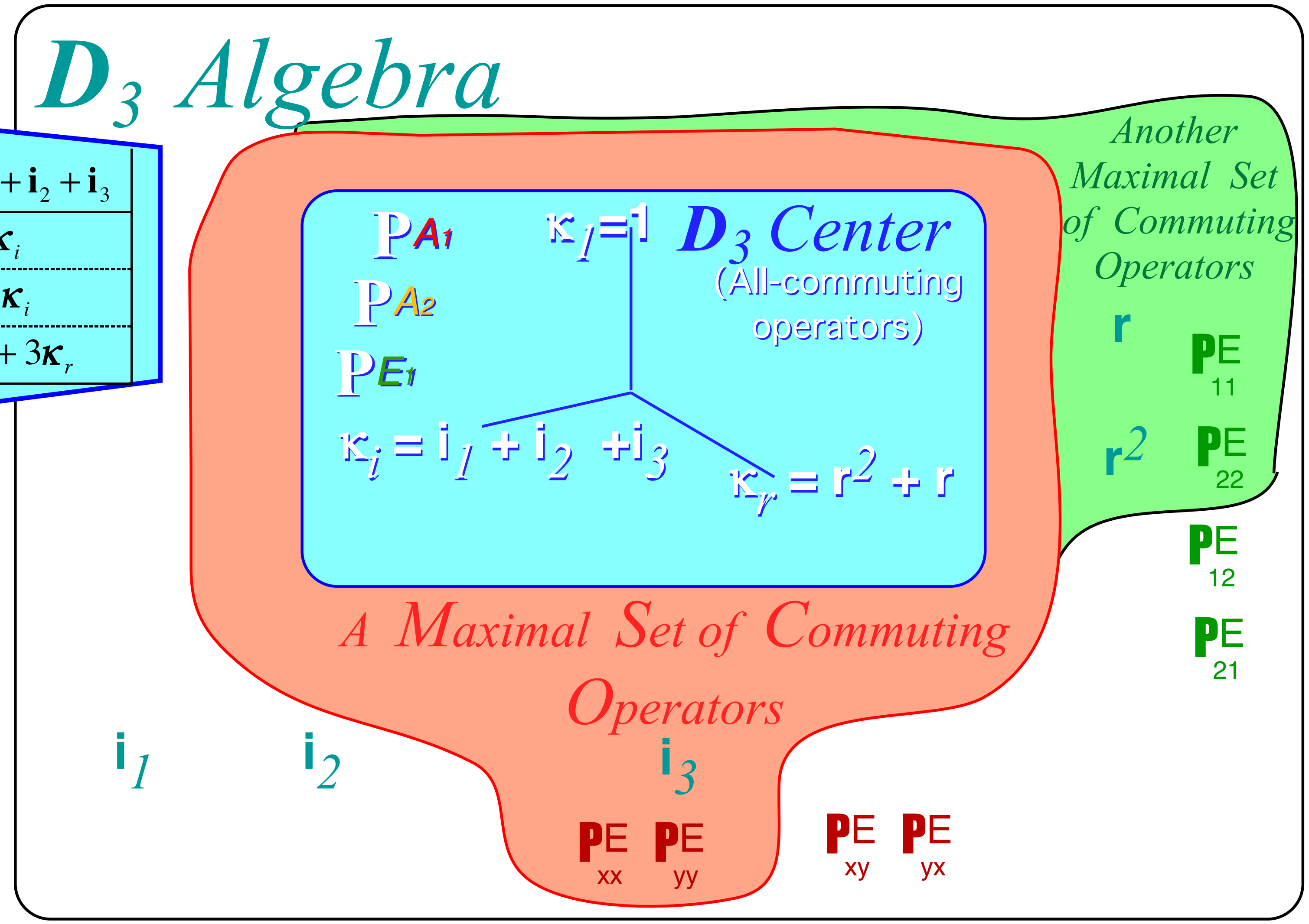


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*If an operator $\mathbf{g}_t$ transforms $\mathbf{g}_k$ into a different element $\mathbf{g}'_k$ of its class: $\mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}=\mathbf{g}'_k$, then so does $\mathbf{g}_t\mathbf{g}_s$.*
*that is: $\mathbf{g}_t\mathbf{g}_s\mathbf{g}_k(\mathbf{g}_t\mathbf{g}_s)^{-1}=\mathbf{g}_t\mathbf{g}_s\mathbf{g}_k\mathbf{g}_s^{-1}\mathbf{g}_t^{-1}=\mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}=\mathbf{g}'_k$,*

⋮

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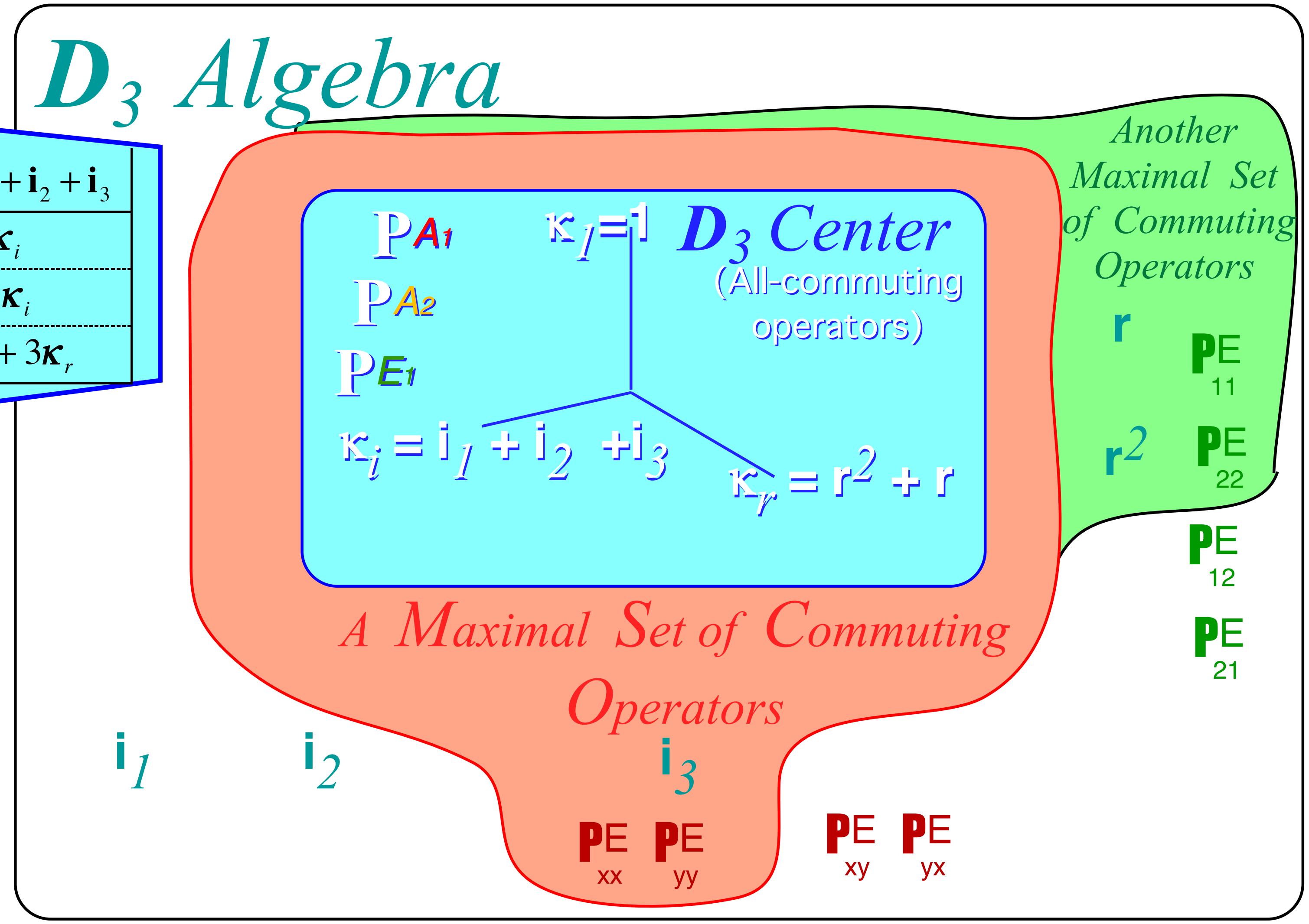


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*These operators $\mathbf{g}_s$ form the $\mathbf{g}_k$-self-symmetry group $s_k$. Each $\mathbf{g}_s$ transforms $\mathbf{g}_k$ into itself: $\mathbf{g}_s\mathbf{g}_k\mathbf{g}_s^{-1}=\mathbf{g}_k$*

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*that is:* $\mathbf{g}_t\mathbf{g}_s\mathbf{g}_k(\mathbf{g}_t\mathbf{g}_s)^{-1}=\mathbf{g}_t\mathbf{g}_s\mathbf{g}_k\mathbf{g}_s^{-1}\mathbf{g}_t^{-1}=\mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}=\mathbf{g}'_k$ ,

*Subgroup* $s_k=\{\mathbf{g}_0=\mathbf{1},\ \mathbf{g}_1=\mathbf{g}_k,\ \mathbf{g}_2,\ldots\}$ *has* $\ell=({}^\circ\kappa_k-1)$ ***Left Cosets*** (one coset for each member of class $\kappa_k$).

$\mathbf{g}_1 s_k=\mathbf{g}_1\{\mathbf{g}_0=1,\ \mathbf{g}_1=\mathbf{g}_k,\ \mathbf{g}_2,\ldots\}$,

⋮

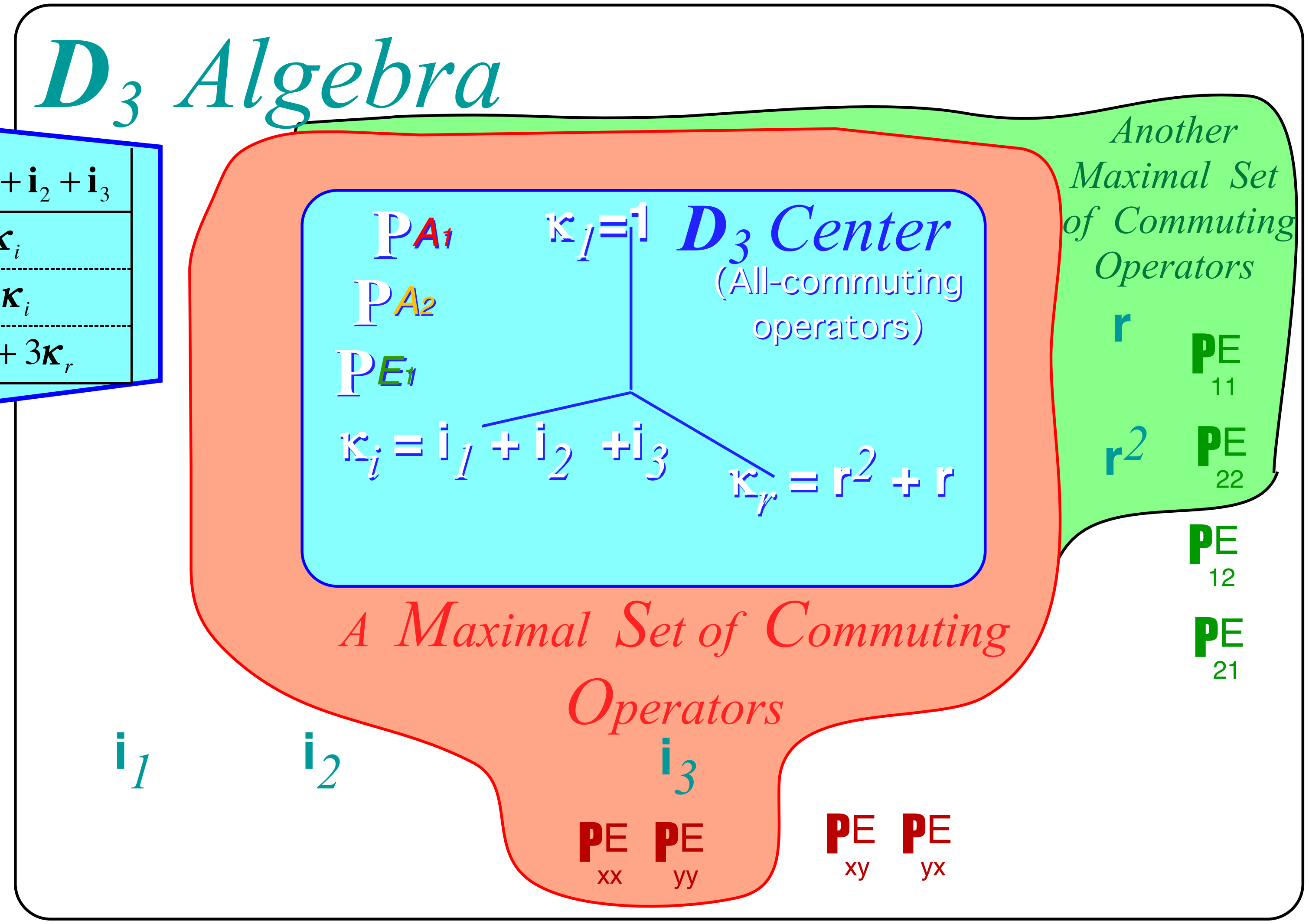
⋮

## *Review: Spectral resolution of* ***D₃ Center*** *(Class algebra)*

| | | | | | |
|---|---|---|---|---|---|
| $\mathbf{1}$ | $\mathbf{r}^2$ | $\mathbf{r}$ | $\mathbf{i}_1$ | $\mathbf{i}_2$ | $\mathbf{i}_3$ |
| $\mathbf{r}$ | $\mathbf{1}$ | $\mathbf{r}^2$ | $\mathbf{i}_3$ | $\mathbf{i}_1$ | $\mathbf{i}_2$ |
| $\mathbf{r}^2$ | $\mathbf{r}$ | $\mathbf{1}$ | $\mathbf{i}_2$ | $\mathbf{i}_3$ | $\mathbf{i}_1$ |
| $\mathbf{i}_1$ | $\mathbf{i}_3$ | $\mathbf{i}_2$ | $\mathbf{1}$ | $\mathbf{r}$ | $\mathbf{r}^2$ |
| $\mathbf{i}_2$ | $\mathbf{i}_1$ | $\mathbf{i}_3$ | $\mathbf{r}^2$ | $\mathbf{1}$ | $\mathbf{r}$ |
| $\mathbf{i}_3$ | $\mathbf{i}_2$ | $\mathbf{i}_1$ | $\mathbf{r}$ | $\mathbf{r}^2$ | $\mathbf{1}$ |

$\rightarrow$

| | $\kappa_1=\mathbf{1}$ | $\kappa_r=\mathbf{r}+\mathbf{r}^2$ | $\kappa_i=\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$ |
|---|---|---|---|
| $\kappa_1$ | $\kappa_1$ | $\kappa_r$ | $\kappa_i$ |
| $\kappa_r$ | $\kappa_r$ | $2\kappa_1+\kappa_r$ | $2\kappa_i$ |
| $\kappa_i$ | $\kappa_i$ | $2\kappa_i$ | $3\kappa_1+3\kappa_r$ |




*Class-sum* $\boldsymbol{\kappa}_k$ *commutes with all* $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t\boldsymbol{\kappa}_k=\boldsymbol{\kappa}_k\mathbf{g}_t$

$^\circ G$ = order of group: $(^\circ D_3=6)$

$^\circ\kappa_k$ = order of class $\kappa_k$: $(^\circ\kappa_1=1,\ ^\circ\kappa_r=2,\ ^\circ\kappa_i=3)$

$$\mathbf{g}_t\boldsymbol{\kappa}_k\mathbf{g}_t^{-1}=\boldsymbol{\kappa}_k \quad \text{where: } \boldsymbol{\kappa}_k=\sum_{j=1}^{j={}^\circ\kappa_k}\mathbf{g}_j=\frac{1}{^\circ s_k}\sum_{t=1}^{t={}^\circ G}\mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}$$

$^\circ s_k$=order of $\mathbf{g}_k$-self-symmetry: $(^\circ s_1=6,\ ^\circ s_r=3,\ ^\circ s_i=2)$

$^\circ s_k={}^\circ G/{}^\circ\kappa_k$ *$^\circ s_k$ is an integer count of $D_3$ operators $\mathbf{g}_s$ that commute with $\mathbf{g}_k$.*

*These operators $\mathbf{g}_s$ form the $\mathbf{g}_k$-self-symmetry group $s_k$. Each $\mathbf{g}_s$ transforms $\mathbf{g}_k$ into itself: $\mathbf{g}_s\mathbf{g}_k\mathbf{g}_s^{-1}=\mathbf{g}_k$*

*If an operator $\mathbf{g}_t$ transforms $\mathbf{g}_k$ into a different element $\mathbf{g}'_k$ of its class: $\mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}=\mathbf{g}'_k$, then so does $\mathbf{g}_t\mathbf{g}_s$.*

*that is: $\mathbf{g}_t\mathbf{g}_s\mathbf{g}_k(\mathbf{g}_t\mathbf{g}_s)^{-1}=\mathbf{g}_t\mathbf{g}_s\mathbf{g}_k\mathbf{g}_s^{-1}\mathbf{g}_t^{-1}=\mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}=\mathbf{g}'_k$,*

*Subgroup* $s_k=\{\mathbf{g}_0=\mathbf{1},\ \mathbf{g}_1=\mathbf{g}_k,\ \mathbf{g}_2,\ldots\}$ *has* $\ell=(^\circ\kappa_k-1)$ ***Left Cosets*** (one coset for each member of class $\kappa_k$).

$\mathbf{g}_1 s_k=\mathbf{g}_1\{\mathbf{g}_0=1,\ \mathbf{g}_1=\mathbf{g}_k,\ \mathbf{g}_2,\ldots\}$,

$\mathbf{g}_2 s_k=\mathbf{g}_2\{\mathbf{g}_0=1,\ \mathbf{g}_1=\mathbf{g}_k,\ \mathbf{g}_2,\ldots\}$,...

$\vdots$

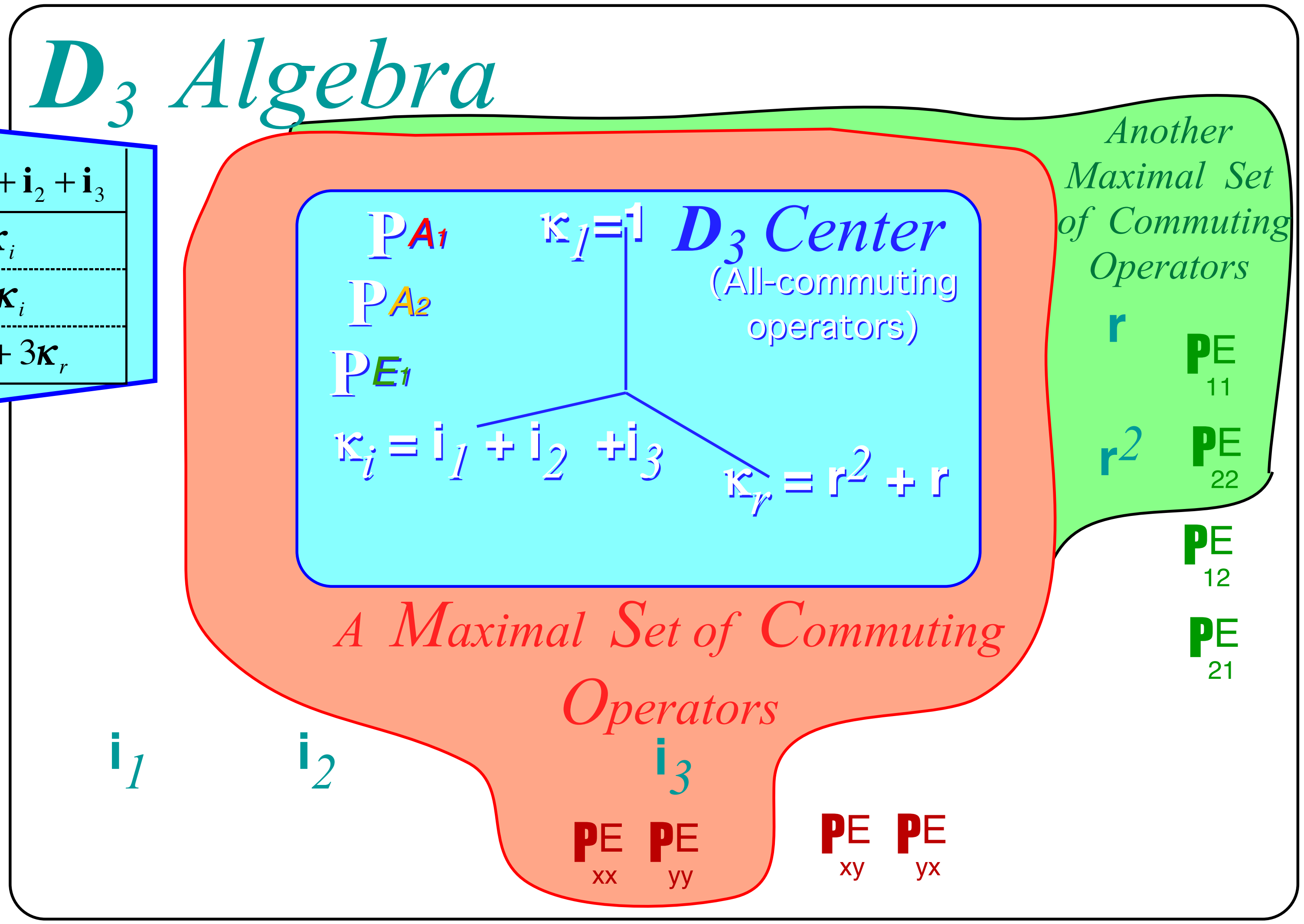
$\mathbf{g}_\ell s_k=\mathbf{g}_\ell\{\mathbf{g}_0=1,\ \mathbf{g}_1=\mathbf{g}_k,\ \mathbf{g}_2,\ldots\}$

# Review: Spectral resolution of $D_3$ Center (Class algebra)

| | | | | | |
|---|---|---|---|---|---|
| $\mathbf{1}$ | $\mathbf{r}^2$ | $\mathbf{r}$ | $\mathbf{i}_1$ | $\mathbf{i}_2$ | $\mathbf{i}_3$ |
| $\mathbf{r}$ | $\mathbf{1}$ | $\mathbf{r}^2$ | $\mathbf{i}_3$ | $\mathbf{i}_1$ | $\mathbf{i}_2$ |
| $\mathbf{r}^2$ | $\mathbf{r}$ | $\mathbf{1}$ | $\mathbf{i}_2$ | $\mathbf{i}_3$ | $\mathbf{i}_1$ |
| $\mathbf{i}_1$ | $\mathbf{i}_3$ | $\mathbf{i}_2$ | $\mathbf{1}$ | $\mathbf{r}$ | $\mathbf{r}^2$ |
| $\mathbf{i}_2$ | $\mathbf{i}_1$ | $\mathbf{i}_3$ | $\mathbf{r}^2$ | $\mathbf{1}$ | $\mathbf{r}$ |
| $\mathbf{i}_3$ | $\mathbf{i}_2$ | $\mathbf{i}_1$ | $\mathbf{r}$ | $\mathbf{r}^2$ | $\mathbf{1}$ |

$\rightarrow$

| | $\kappa_1 = \mathbf{1}$ | $\kappa_r = \mathbf{r} + \mathbf{r}^2$ | $\kappa_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$ |
|---|---|---|---|
| $\kappa_1$ | $\kappa_1$ | $\kappa_r$ | $\kappa_i$ |
| $\kappa_r$ | $\kappa_r$ | $2\kappa_1 + \kappa_r$ | $2\kappa_i$ |
| $\kappa_i$ | $\kappa_i$ | $2\kappa_i$ | $3\kappa_1 + 3\kappa_r$ |




*Class-sum* $\kappa_k$ *commutes with all* $\mathbf{g}_t$

Class-sum $\kappa_k$ invariance: $\mathbf{g}_t \kappa_k = \kappa_k \mathbf{g}_t$

$°G$ = order of group: $(°D_3 = 6)$

$°\kappa_k$ = order of class $\kappa_k$: $(°\kappa_1 = 1,\ °\kappa_r = 2,\ °\kappa_i = 3)$

$$\mathbf{g}_t \kappa_k \mathbf{g}_t^{-1} = \kappa_k \quad \text{where: } \kappa_\mathbf{k} = \sum_{j=1}^{j=°\kappa_k} \mathbf{g}_j = \frac{1}{°s_k} \sum_{t=1}^{t=°G} \mathbf{g}_t \mathbf{g}_k \mathbf{g}_t^{-1}$$

$°s_k$=order of $\mathbf{g}_k$-self-symmetry: $(°s_1 = 6,\ °s_r = 3,\ °s_i = 2)$

$°s_k = °G / °\kappa_k$ *$°s_k$ is an integer count of $D_3$ operators $\mathbf{g}_s$ that commute with $\mathbf{g}_k$.*

*These operators $\mathbf{g}_s$ form the $\mathbf{g}_k$-self-symmetry group $s_k$. Each $\mathbf{g}_s$ transforms $\mathbf{g}_k$ into itself:* $\mathbf{g}_s \mathbf{g}_k \mathbf{g}_s^{-1} = \mathbf{g}_k$

*If an operator $\mathbf{g}_t$ transforms $\mathbf{g}_k$ into a different element $\mathbf{g}'_k$ of its class:* $\mathbf{g}_t \mathbf{g}_k \mathbf{g}_t^{-1} = \mathbf{g}'_k$ *, then so does* $\mathbf{g}_t \mathbf{g}_s$ *.*

*that is:* $\mathbf{g}_t \mathbf{g}_s \mathbf{g}_k (\mathbf{g}_t \mathbf{g}_s)^{-1} = \mathbf{g}_t \mathbf{g}_s \mathbf{g}_k \mathbf{g}_s^{-1} \mathbf{g}_t^{-1} = \mathbf{g}_t \mathbf{g}_k \mathbf{g}_t^{-1} = \mathbf{g}'_k$ *,*

*Subgroup* $s_k = \{\mathbf{g}_0 = \mathbf{1},\ \mathbf{g}_1 = \mathbf{g}_k,\ \mathbf{g}_2, \ldots\}$ *has* $\ell = (°\kappa_k - 1)$ ***Left Cosets*** (one coset for each member of class $\kappa_k$).

$\mathbf{g}_1 s_k = \mathbf{g}_1 \{\mathbf{g}_0 = 1,\ \mathbf{g}_1 = \mathbf{g}_k,\ \mathbf{g}_2, \ldots\}$,

$\mathbf{g}_2 s_k = \mathbf{g}_2 \{\mathbf{g}_0 = 1,\ \mathbf{g}_1 = \mathbf{g}_k,\ \mathbf{g}_2, \ldots\}, \ldots$

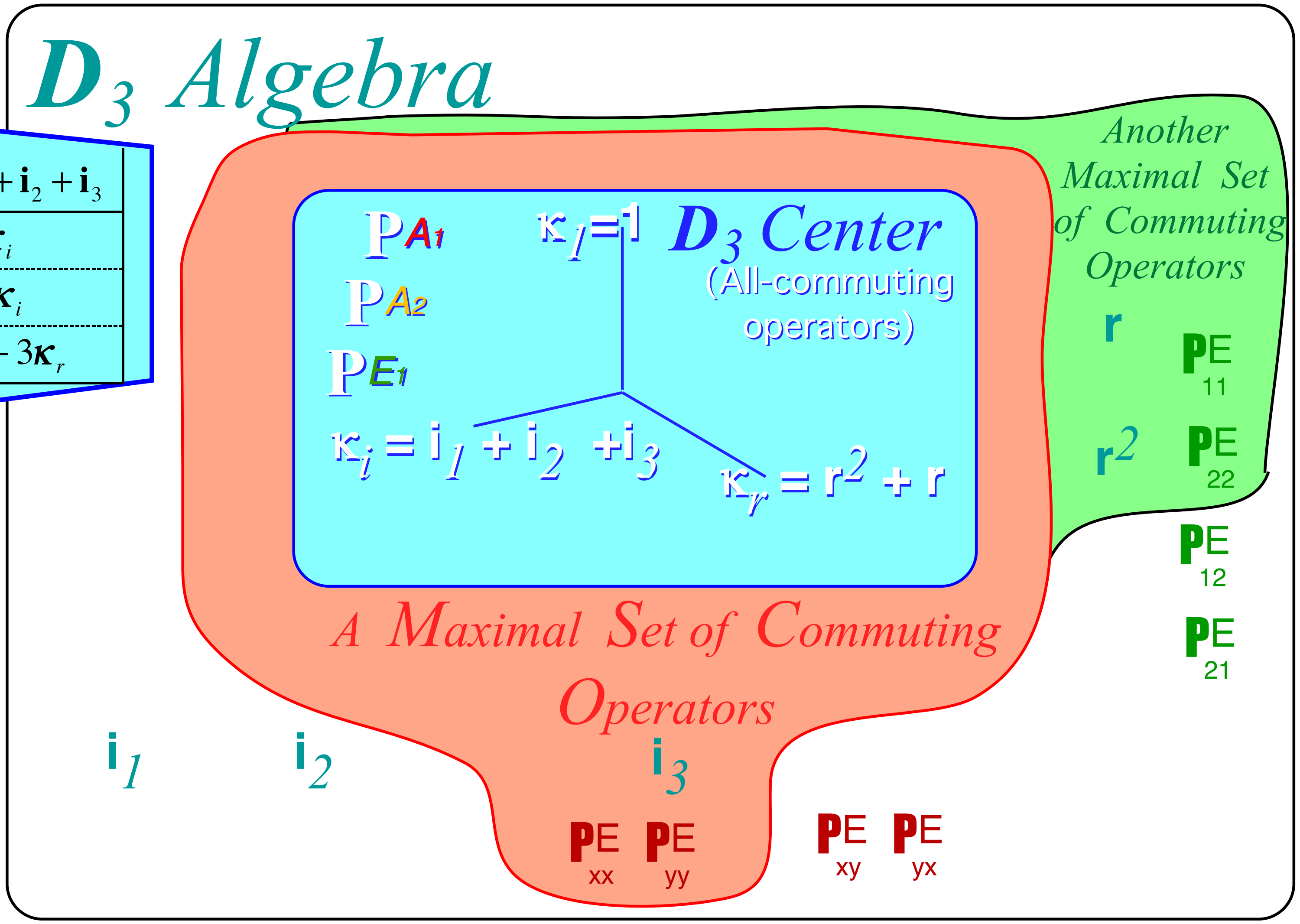
They will divide the group of order $°D_3 = °\kappa_k \cdot °s_k$ evenly into $°\kappa_k$ subsets each of order $°s_k$.

# Review: Spectral resolution of $D_3$ Center (Class algebra)

| | | | | | |
|---|---|---|---|---|---|
| $\mathbf{1}$ | $\mathbf{r}^2$ | $\mathbf{r}$ | $\mathbf{i}_1$ | $\mathbf{i}_2$ | $\mathbf{i}_3$ |
| $\mathbf{r}$ | $\mathbf{1}$ | $\mathbf{r}^2$ | $\mathbf{i}_3$ | $\mathbf{i}_1$ | $\mathbf{i}_2$ |
| $\mathbf{r}^2$ | $\mathbf{r}$ | $\mathbf{1}$ | $\mathbf{i}_2$ | $\mathbf{i}_3$ | $\mathbf{i}_1$ |
| $\mathbf{i}_1$ | $\mathbf{i}_3$ | $\mathbf{i}_2$ | $\mathbf{1}$ | $\mathbf{r}$ | $\mathbf{r}^2$ |
| $\mathbf{i}_2$ | $\mathbf{i}_1$ | $\mathbf{i}_3$ | $\mathbf{r}^2$ | $\mathbf{1}$ | $\mathbf{r}$ |
| $\mathbf{i}_3$ | $\mathbf{i}_2$ | $\mathbf{i}_1$ | $\mathbf{r}$ | $\mathbf{r}^2$ | $\mathbf{1}$ |

→

| | $\kappa_1 = \mathbf{1}$ | $\kappa_r = \mathbf{r} + \mathbf{r}^2$ | $\kappa_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$ |
|---|---|---|---|
| $\kappa_1$ | $\kappa_1$ | $\kappa_r$ | $\kappa_i$ |
| $\kappa_r$ | $\kappa_r$ | $2\kappa_1 + \kappa_r$ | $2\kappa_i$ |
| $\kappa_i$ | $\kappa_i$ | $2\kappa_i$ | $3\kappa_1 + 3\kappa_r$ |




*Class-sum* $\kappa_k$ *commutes with all* $\mathbf{g}_t$

Class-sum $\kappa_k$ invariance: $\mathbf{g}_t \kappa_k = \kappa_k \mathbf{g}_t$

$°G$ = order of group: $(°D_3 = 6)$

$°\kappa_k$ = order of class $\kappa_k$: $(°\kappa_1 = 1,\ °\kappa_r = 2,\ °\kappa_i = 3)$

$$\mathbf{g}_t \kappa_k \mathbf{g}_t^{-1} = \kappa_k \quad \text{where: } \kappa_k = \sum_{j=1}^{j=°\kappa_k} \mathbf{g}_j = \frac{1}{°s_k} \sum_{t=1}^{t=°G} \mathbf{g}_t \mathbf{g}_k \mathbf{g}_t^{-1}$$

$°s_k$=order of $\mathbf{g}_k$-self-symmetry: $(°s_1 = 6,\ °s_r = 3,\ °s_i = 2)$

$°s_k = °G / °\kappa_k$ $\quad$ *$°s_k$ is an integer count of $D_3$ operators $\mathbf{g}_s$ that commute with $\mathbf{g}_k$.*

*These operators $\mathbf{g}_s$ form the $\mathbf{g}_k$-self-symmetry group $s_k$ . Each $\mathbf{g}_s$ transforms $\mathbf{g}_k$ into itself: $\mathbf{g}_s \mathbf{g}_k \mathbf{g}_s^{-1} = \mathbf{g}_k$*

*If an operator $\mathbf{g}_t$ transforms $\mathbf{g}_k$ into a different element $\mathbf{g}'_k$ of its class: $\mathbf{g}_t \mathbf{g}_k \mathbf{g}_t^{-1} = \mathbf{g}'_k$ , then so does $\mathbf{g}_t \mathbf{g}_s$ .*

*that is: $\mathbf{g}_t \mathbf{g}_s \mathbf{g}_k (\mathbf{g}_t \mathbf{g}_s)^{-1} = \mathbf{g}_t \mathbf{g}_s \mathbf{g}_k \mathbf{g}_s^{-1} \mathbf{g}_t^{-1} = \mathbf{g}_t \mathbf{g}_k \mathbf{g}_t^{-1} = \mathbf{g}'_k$ ,*

*Subgroup $s_k = \{\mathbf{g}_0 = \mathbf{1},\ \mathbf{g}_1 = \mathbf{g}_k,\ \mathbf{g}_2, \ldots\}$ has $\ell = (°\kappa_k - 1)$* ***Left Cosets*** (one coset for each member of class $\kappa_k$).

$\mathbf{g}_1 s_k = \mathbf{g}_1 \{\mathbf{g}_0 = 1,\ \mathbf{g}_1 = \mathbf{g}_k,\ \mathbf{g}_2, \ldots\}$,

$\mathbf{g}_2 s_k = \mathbf{g}_2 \{\mathbf{g}_0 = 1,\ \mathbf{g}_1 = \mathbf{g}_k,\ \mathbf{g}_2, \ldots\}$,...

*These results are known as* ***Lagrange's Coset Theorem(s)***

They will divide the group of order $°D_3 = °\kappa_k \cdot °s_k$ evenly into $°\kappa_k$ subsets each of order $°s_k$.

*Review:Spectral resolution of **$D_3$ Center** (Class algebra)*

*Group theory of equivalence transformations and classes*

*Lagrange theorems*

*All-commuting class projectors and $D_3$-invariant characters*

*Character ortho-completeness*

*Group invariant numbers: Centrum, Rank, and Order*

*2nd-Stage spectral decompositions of global/local $D_3$*

*Splitting class projectors using subgroup chains $D_3 \supset C_2$ and $D_3 \supset C_3$*

*3rd-stage spectral resolution to **irreducible representations** (ireps) and Hamiltonian eigensolutions*

*Tunneling modes and spectra for $D_3 \supset C_2$ and $D_3 \supset C_3$ local subgroup chains*

*Review:* *Spectral resolution of **$D_3$ Center** (Class algebra)*

| | | | | | |
|---|---|---|---|---|---|
| $\mathbf{1}$ | $\mathbf{r}^2$ | $\mathbf{r}$ | $\mathbf{i}_1$ | $\mathbf{i}_2$ | $\mathbf{i}_3$ |
| $\mathbf{r}$ | $\mathbf{1}$ | $\mathbf{r}^2$ | $\mathbf{i}_3$ | $\mathbf{i}_1$ | $\mathbf{i}_2$ |
| $\mathbf{r}^2$ | $\mathbf{r}$ | $\mathbf{1}$ | $\mathbf{i}_2$ | $\mathbf{i}_3$ | $\mathbf{i}_1$ |
| $\mathbf{i}_1$ | $\mathbf{i}_3$ | $\mathbf{i}_2$ | $\mathbf{1}$ | $\mathbf{r}$ | $\mathbf{r}^2$ |
| $\mathbf{i}_2$ | $\mathbf{i}_1$ | $\mathbf{i}_3$ | $\mathbf{r}^2$ | $\mathbf{1}$ | $\mathbf{r}$ |
| $\mathbf{i}_3$ | $\mathbf{i}_2$ | $\mathbf{i}_1$ | $\mathbf{r}$ | $\mathbf{r}^2$ | $\mathbf{1}$ |

$\rightarrow$

| | $\kappa_1 = \mathbf{1}$ | $\kappa_r = \mathbf{r} + \mathbf{r}^2$ | $\kappa_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$ |
|---|---|---|---|
| $\kappa_1$ | $\kappa_1$ | $\kappa_r$ | $\kappa_i$ |
| $\kappa_r$ | $\kappa_r$ | $2\kappa_1 + \kappa_r$ | $2\kappa_i$ |
| $\kappa_i$ | $\kappa_i$ | $2\kappa_i$ | $3\kappa_1 + 3\kappa_r$ |

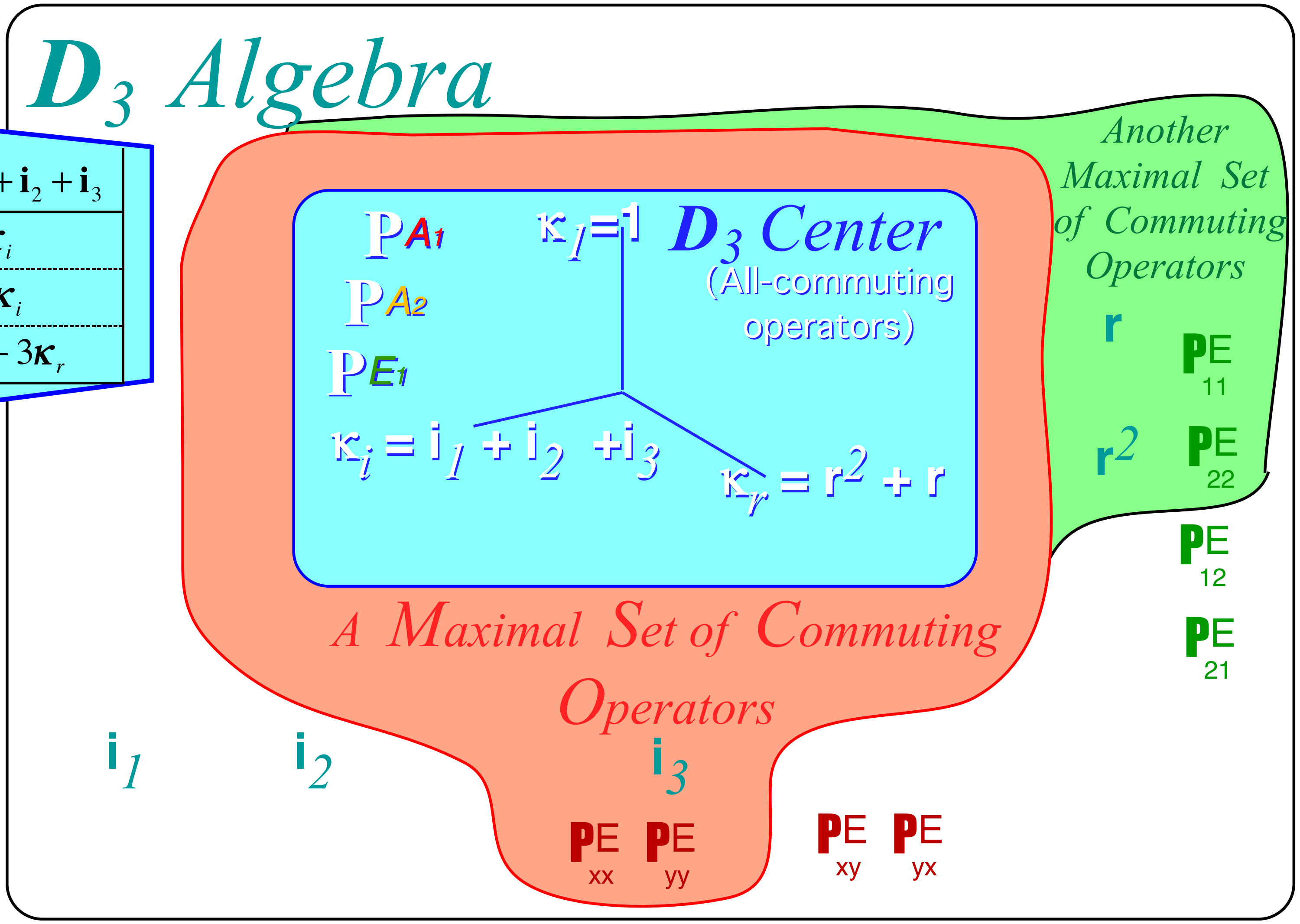
*Class-sum $\boldsymbol{\kappa}_k$ commutes with all $\mathbf{g}_t$*

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t \boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k \mathbf{g}_t$

$°G$ = order of group: $(°D_3 = 6)$

$°\kappa_k$ = order of class $\kappa_k$: $(°\kappa_1 = 1,\ °\kappa_r = 2,\ °\kappa_i = 3)$

*Class minimal equation*



# Review: Spectral resolution of $D_3$ Center (Class algebra)

| | | | | | |
|---|---|---|---|---|---|
| $\mathbf{1}$ | $\mathbf{r}^2$ | $\mathbf{r}$ | $\mathbf{i}_1$ | $\mathbf{i}_2$ | $\mathbf{i}_3$ |
| $\mathbf{r}$ | $\mathbf{1}$ | $\mathbf{r}^2$ | $\mathbf{i}_3$ | $\mathbf{i}_1$ | $\mathbf{i}_2$ |
| $\mathbf{r}^2$ | $\mathbf{r}$ | $\mathbf{1}$ | $\mathbf{i}_2$ | $\mathbf{i}_3$ | $\mathbf{i}_1$ |
| $\mathbf{i}_1$ | $\mathbf{i}_3$ | $\mathbf{i}_2$ | $\mathbf{1}$ | $\mathbf{r}$ | $\mathbf{r}^2$ |
| $\mathbf{i}_2$ | $\mathbf{i}_1$ | $\mathbf{i}_3$ | $\mathbf{r}^2$ | $\mathbf{1}$ | $\mathbf{r}$ |
| $\mathbf{i}_3$ | $\mathbf{i}_2$ | $\mathbf{i}_1$ | $\mathbf{r}$ | $\mathbf{r}^2$ | $\mathbf{1}$ |

$\rightarrow$

| | $\kappa_1 = \mathbf{1}$ | $\kappa_r = \mathbf{r} + \mathbf{r}^2$ | $\kappa_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$ |
|---|---|---|---|
| $\kappa_1$ | $\kappa_1$ | $\kappa_r$ | $\kappa_i$ |
| $\kappa_r$ | $\kappa_r$ | $2\kappa_1 + \kappa_r$ | $2\kappa_i$ |
| $\kappa_i$ | $\kappa_i$ | $2\kappa_i$ | $3\kappa_1 + 3\kappa_r$ |

*Class-sum $\boldsymbol{\kappa}_k$ commutes with all $\mathbf{g}_t$*

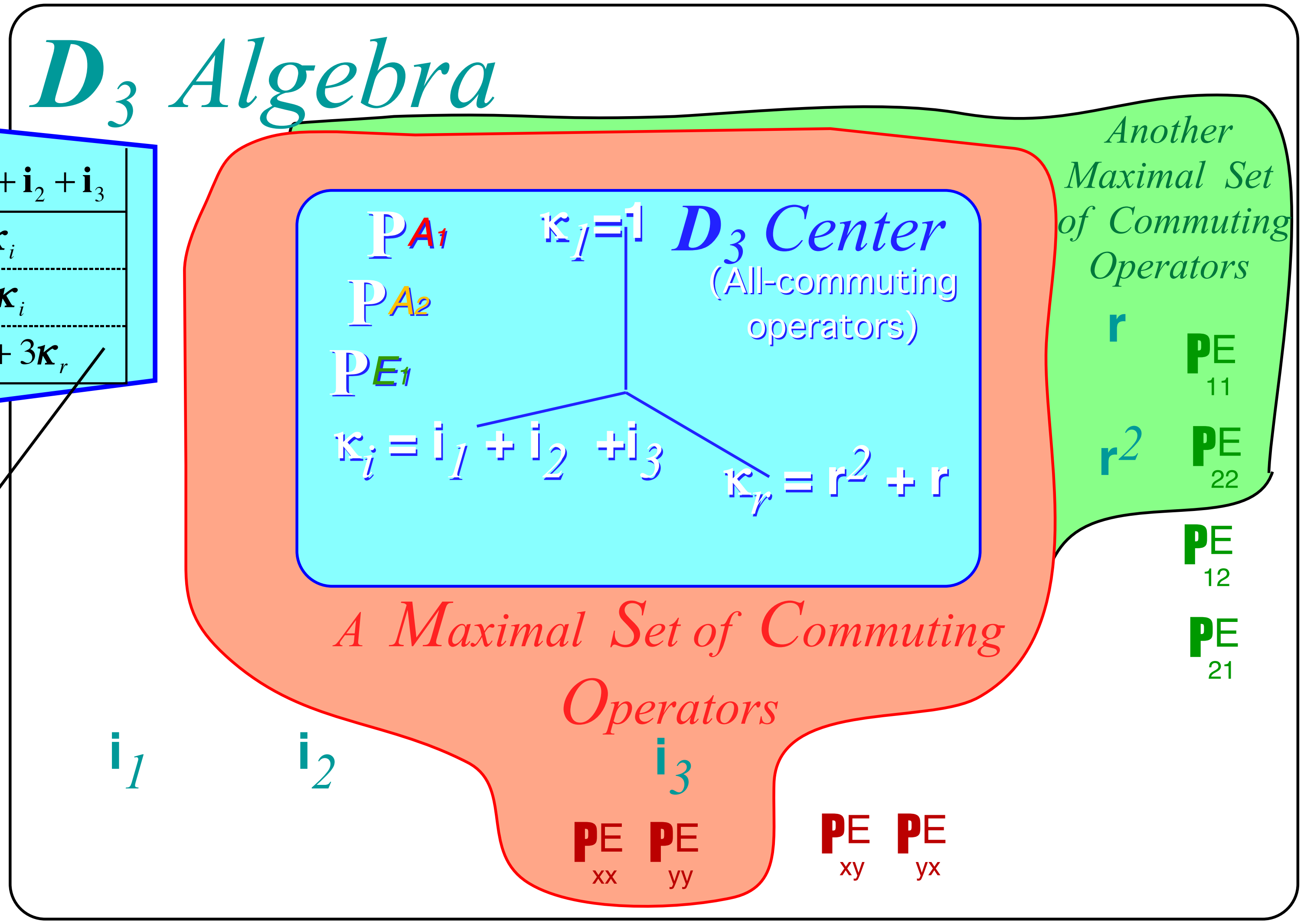
Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t \boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k \mathbf{g}_t$

$°G$ = order of group: $(°D_3 = 6)$

$°\kappa_k$ = order of class $\kappa_k$: $(°\kappa_1 = 1,\ °\kappa_r = 2,\ °\kappa_i = 3)$

*Class minimal equation*

$$\boldsymbol{\kappa}_i^2 = 3 \cdot \boldsymbol{\kappa}_r + 3 \cdot \mathbf{1}$$



## Review: *Spectral resolution of* ***D₃ Center*** *(Class algebra)*

| | | | | | |
|---|---|---|---|---|---|
| $\mathbf{1}$ | $\mathbf{r}^2$ | $\mathbf{r}$ | $\mathbf{i}_1$ | $\mathbf{i}_2$ | $\mathbf{i}_3$ |
| $\mathbf{r}$ | $\mathbf{1}$ | $\mathbf{r}^2$ | $\mathbf{i}_3$ | $\mathbf{i}_1$ | $\mathbf{i}_2$ |
| $\mathbf{r}^2$ | $\mathbf{r}$ | $\mathbf{1}$ | $\mathbf{i}_2$ | $\mathbf{i}_3$ | $\mathbf{i}_1$ |
| $\mathbf{i}_1$ | $\mathbf{i}_3$ | $\mathbf{i}_2$ | $\mathbf{1}$ | $\mathbf{r}$ | $\mathbf{r}^2$ |
| $\mathbf{i}_2$ | $\mathbf{i}_1$ | $\mathbf{i}_3$ | $\mathbf{r}^2$ | $\mathbf{1}$ | $\mathbf{r}$ |
| $\mathbf{i}_3$ | $\mathbf{i}_2$ | $\mathbf{i}_1$ | $\mathbf{r}$ | $\mathbf{r}^2$ | $\mathbf{1}$ |

→

| | $\kappa_1=\mathbf{1}$ | $\kappa_r=\mathbf{r}+\mathbf{r}^2$ | $\kappa_i=\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$ |
|---|---|---|---|
| $\kappa_1$ | $\kappa_1$ | $\kappa_r$ | $\kappa_i$ |
| $\kappa_r$ | $\kappa_r$ | $2\kappa_1+\kappa_r$ | $2\kappa_i$ |
| $\kappa_i$ | $\kappa_i$ | $2\kappa_i$ | $3\kappa_1+3\kappa_r$ |

*Class-sum $\boldsymbol{\kappa}_k$ commutes with all $\mathbf{g}_t$*

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t\boldsymbol{\kappa}_k=\boldsymbol{\kappa}_k\mathbf{g}_t$

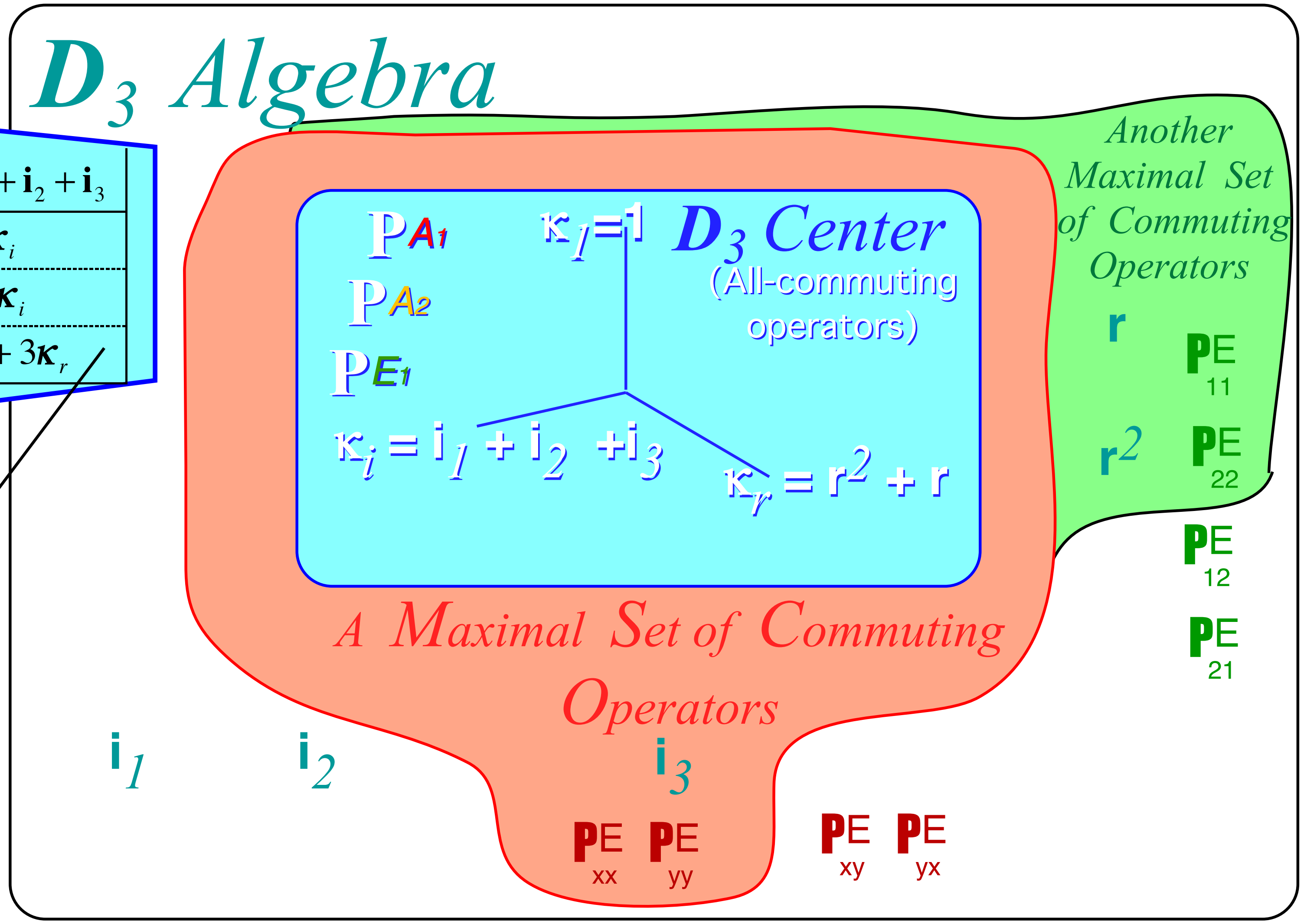
$°G$ = order of group: $(°D_3=6)$

$°\kappa_k$ = order of class $\kappa_k$: $(°\kappa_1=1,\ °\kappa_r=2,\ °\kappa_i=3)$

*Class minimal equation*

$$\boldsymbol{\kappa}_i^2=3\cdot\boldsymbol{\kappa}_r+3\cdot\mathbf{1}$$

$$\boldsymbol{\kappa}_i^3=3\cdot\boldsymbol{\kappa}_r\boldsymbol{\kappa}_i+3\cdot\boldsymbol{\kappa}_i=9\cdot\boldsymbol{\kappa}_i$$



# Review: Spectral resolution of $D_3$ Center (Class algebra)

| | | | | | |
|---|---|---|---|---|---|
| $\mathbf{1}$ | $\mathbf{r}^2$ | $\mathbf{r}$ | $\mathbf{i}_1$ | $\mathbf{i}_2$ | $\mathbf{i}_3$ |
| $\mathbf{r}$ | $\mathbf{1}$ | $\mathbf{r}^2$ | $\mathbf{i}_3$ | $\mathbf{i}_1$ | $\mathbf{i}_2$ |
| $\mathbf{r}^2$ | $\mathbf{r}$ | $\mathbf{1}$ | $\mathbf{i}_2$ | $\mathbf{i}_3$ | $\mathbf{i}_1$ |
| $\mathbf{i}_1$ | $\mathbf{i}_3$ | $\mathbf{i}_2$ | $\mathbf{1}$ | $\mathbf{r}$ | $\mathbf{r}^2$ |
| $\mathbf{i}_2$ | $\mathbf{i}_1$ | $\mathbf{i}_3$ | $\mathbf{r}^2$ | $\mathbf{1}$ | $\mathbf{r}$ |
| $\mathbf{i}_3$ | $\mathbf{i}_2$ | $\mathbf{i}_1$ | $\mathbf{r}$ | $\mathbf{r}^2$ | $\mathbf{1}$ |

$\rightarrow$

| | $\kappa_1 = \mathbf{1}$ | $\kappa_r = \mathbf{r}+\mathbf{r}^2$ | $\kappa_i = \mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$ |
|---|---|---|---|
| $\kappa_1$ | $\kappa_1$ | $\kappa_r$ | $\kappa_i$ |
| $\kappa_r$ | $\kappa_r$ | $2\kappa_1+\kappa_r$ | $2\kappa_i$ |
| $\kappa_i$ | $\kappa_i$ | $2\kappa_i$ | $3\kappa_1+3\kappa_r$ |

*Class-sum $\kappa_k$ commutes with all $\mathbf{g}_t$*

Class-sum $\kappa_k$ invariance: $\mathbf{g}_t\kappa_k = \kappa_k\mathbf{g}_t$

$°G$ = order of group: $(°D_3 = 6)$

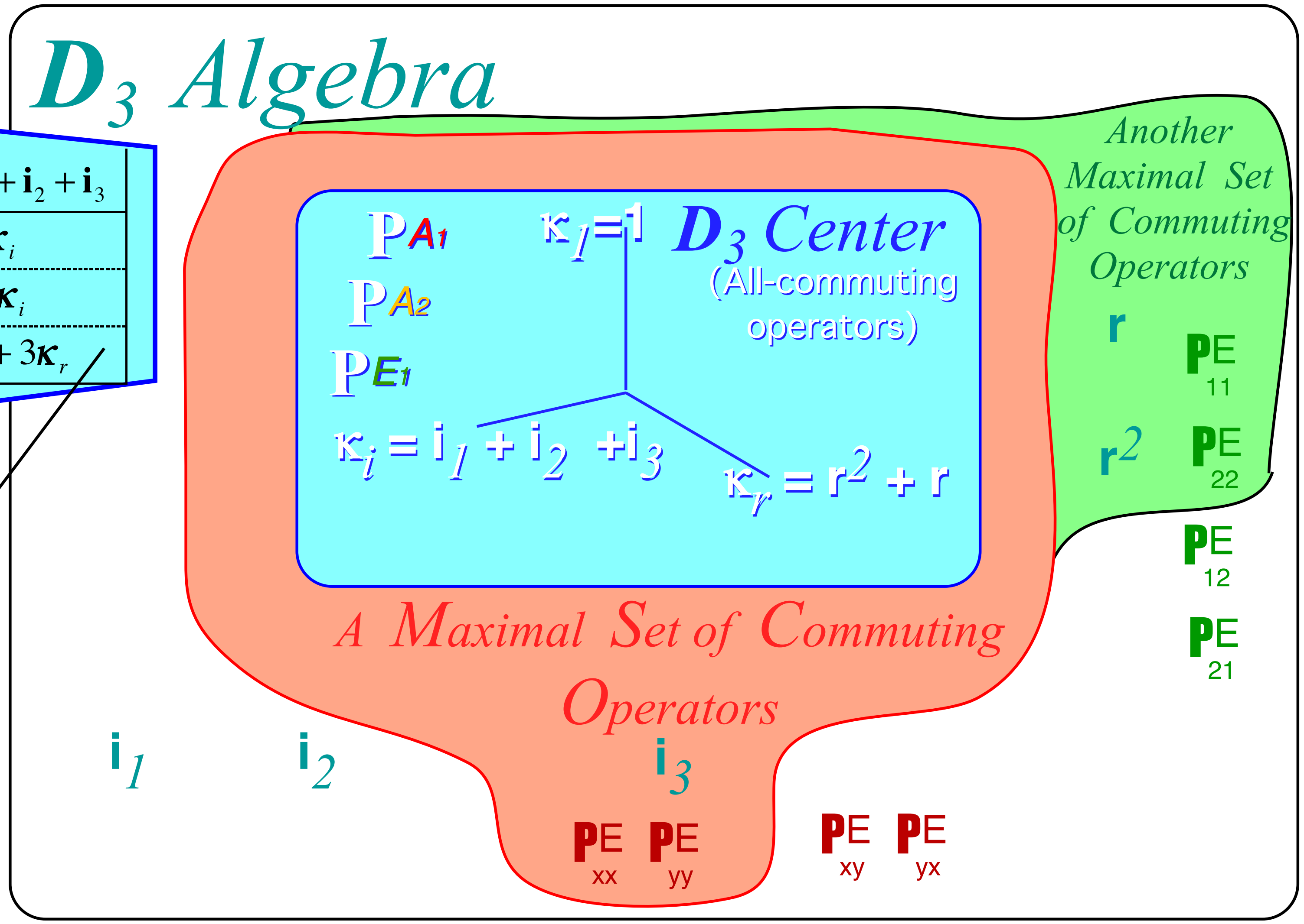
$°\kappa_k$ = order of class $\kappa_k$: $(°\kappa_1 = 1,\ °\kappa_r = 2,\ °\kappa_i = 3)$

*Class minimal equation*

$$\kappa_i^2 = 3\cdot\kappa_r + 3\cdot\mathbf{1}$$

$$\kappa_i^3 = 3\cdot\kappa_r\kappa_i + 3\cdot\kappa_i = 9\cdot\kappa_i$$

$$0 = \kappa_i^3 - 9\cdot\kappa_i = (\kappa_i - 3\cdot\mathbf{1})(\kappa_i + 3\cdot\mathbf{1})(\kappa_i - 0\cdot\mathbf{1})$$



# Review: *Spectral resolution of $D_3$ Center (Class algebra)*

| | | | | | |
|---|---|---|---|---|---|
| $\mathbf{1}$ | $\mathbf{r}^2$ | $\mathbf{r}$ | $\mathbf{i}_1$ | $\mathbf{i}_2$ | $\mathbf{i}_3$ |
| $\mathbf{r}$ | $\mathbf{1}$ | $\mathbf{r}^2$ | $\mathbf{i}_3$ | $\mathbf{i}_1$ | $\mathbf{i}_2$ |
| $\mathbf{r}^2$ | $\mathbf{r}$ | $\mathbf{1}$ | $\mathbf{i}_2$ | $\mathbf{i}_3$ | $\mathbf{i}_1$ |
| $\mathbf{i}_1$ | $\mathbf{i}_3$ | $\mathbf{i}_2$ | $\mathbf{1}$ | $\mathbf{r}$ | $\mathbf{r}^2$ |
| $\mathbf{i}_2$ | $\mathbf{i}_1$ | $\mathbf{i}_3$ | $\mathbf{r}^2$ | $\mathbf{1}$ | $\mathbf{r}$ |
| $\mathbf{i}_3$ | $\mathbf{i}_2$ | $\mathbf{i}_1$ | $\mathbf{r}$ | $\mathbf{r}^2$ | $\mathbf{1}$ |

→

| | $\kappa_1 = \mathbf{1}$ | $\kappa_r = \mathbf{r} + \mathbf{r}^2$ | $\kappa_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$ |
|---|---|---|---|
| $\kappa_1$ | $\kappa_1$ | $\kappa_r$ | $\kappa_i$ |
| $\kappa_r$ | $\kappa_r$ | $2\kappa_1 + \kappa_r$ | $2\kappa_i$ |
| $\kappa_i$ | $\kappa_i$ | $2\kappa_i$ | $3\kappa_1 + 3\kappa_r$ |

*Class-sum $\boldsymbol{\kappa}_k$ commutes with all $\mathbf{g}_t$*

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t \boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k \mathbf{g}_t$

$°G$ = order of group: $(°D_3 = 6)$

$°\kappa_k$ = order of class $\kappa_k$: $(°\kappa_1 = 1,\ °\kappa_r = 2,\ °\kappa_i = 3)$

*Class minimal equation*

$$\boldsymbol{\kappa}_i^2 = 3\cdot\boldsymbol{\kappa}_r + 3\cdot\mathbf{1}$$

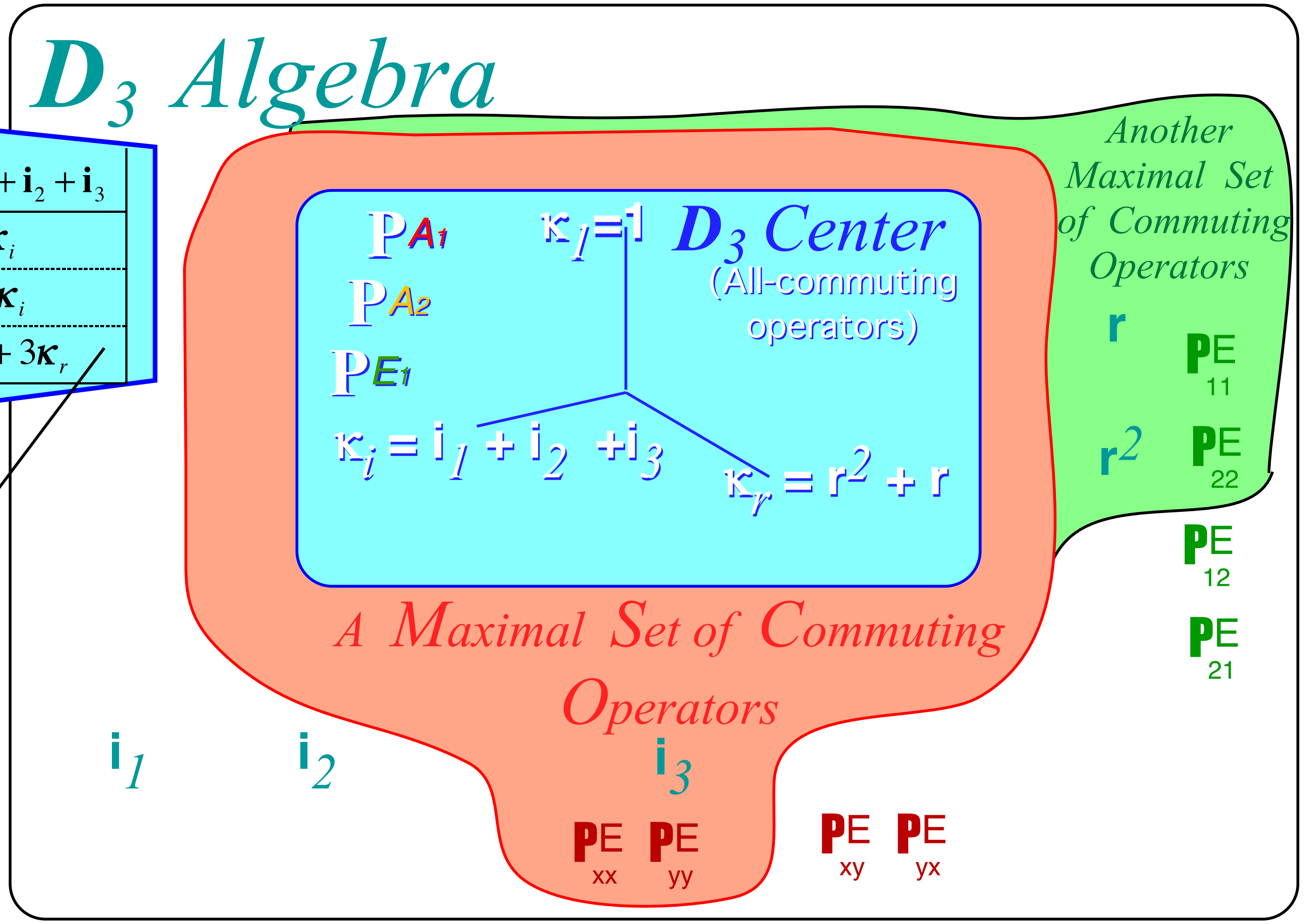
$$\boldsymbol{\kappa}_i^3 = 3\cdot\boldsymbol{\kappa}_r\boldsymbol{\kappa}_i + 3\cdot\boldsymbol{\kappa}_i = 9\cdot\boldsymbol{\kappa}_i$$

$$0 = \boldsymbol{\kappa}_i^3 - 9\cdot\boldsymbol{\kappa}_i = (\boldsymbol{\kappa}_i - 3\cdot\mathbf{1})(\boldsymbol{\kappa}_i + 3\cdot\mathbf{1})(\boldsymbol{\kappa}_i - 0\cdot\mathbf{1})$$

$$\boldsymbol{\kappa}_1 = 1\cdot\mathbf{P}^{A_1} + 1\cdot\mathbf{P}^{A_2} + 1\cdot\mathbf{P}^{E} = \mathbf{1} \quad \textit{(Completeness)}$$

$$\boldsymbol{\kappa}_r = 2\cdot\mathbf{P}^{A_1} - 2\cdot\mathbf{P}^{A_2} - 1\cdot\mathbf{P}^{E}$$

$$\boldsymbol{\kappa}_i = 3\cdot\mathbf{P}^{A_1} - 3\cdot\mathbf{P}^{A_2} + 0\cdot\mathbf{P}^{E}$$



## *Review: Spectral resolution of* ***$D_3$ Center*** *(Class algebra)*

| | | | | | |
|---|---|---|---|---|---|
| **1** | $\mathbf{r}^2$ | **r** | $\mathbf{i}_1$ | $\mathbf{i}_2$ | $\mathbf{i}_3$ |
| **r** | **1** | $\mathbf{r}^2$ | $\mathbf{i}_3$ | $\mathbf{i}_1$ | $\mathbf{i}_2$ |
| $\mathbf{r}^2$ | **r** | **1** | $\mathbf{i}_2$ | $\mathbf{i}_3$ | $\mathbf{i}_1$ |
| $\mathbf{i}_1$ | $\mathbf{i}_3$ | $\mathbf{i}_2$ | **1** | **r** | $\mathbf{r}^2$ |
| $\mathbf{i}_2$ | $\mathbf{i}_1$ | $\mathbf{i}_3$ | $\mathbf{r}^2$ | **1** | **r** |
| $\mathbf{i}_3$ | $\mathbf{i}_2$ | $\mathbf{i}_1$ | **r** | $\mathbf{r}^2$ | **1** |

→

| | $\kappa_1 = \mathbf{1}$ | $\kappa_r = \mathbf{r} + \mathbf{r}^2$ | $\kappa_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$ |
|---|---|---|---|
| $\kappa_1$ | $\kappa_1$ | $\kappa_r$ | $\kappa_i$ |
| $\kappa_r$ | $\kappa_r$ | $2\kappa_1 + \kappa_r$ | $2\kappa_i$ |
| $\kappa_i$ | $\kappa_i$ | $2\kappa_i$ | $3\kappa_1 + 3\kappa_r$ |

*Class-sum* $\boldsymbol{\kappa}_k$ *commutes with all* $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t \boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k \mathbf{g}_t$

$°G$ = order of group: $(°D_3 = 6)$

$°\kappa_k$ = order of class $\kappa_k$: $(°\kappa_1 = 1,\ °\kappa_r = 2,\ °\kappa_i = 3)$

*Class minimal equation*

$$\boldsymbol{\kappa}_i^2 = 3\cdot\boldsymbol{\kappa}_r + 3\cdot\mathbf{1}$$

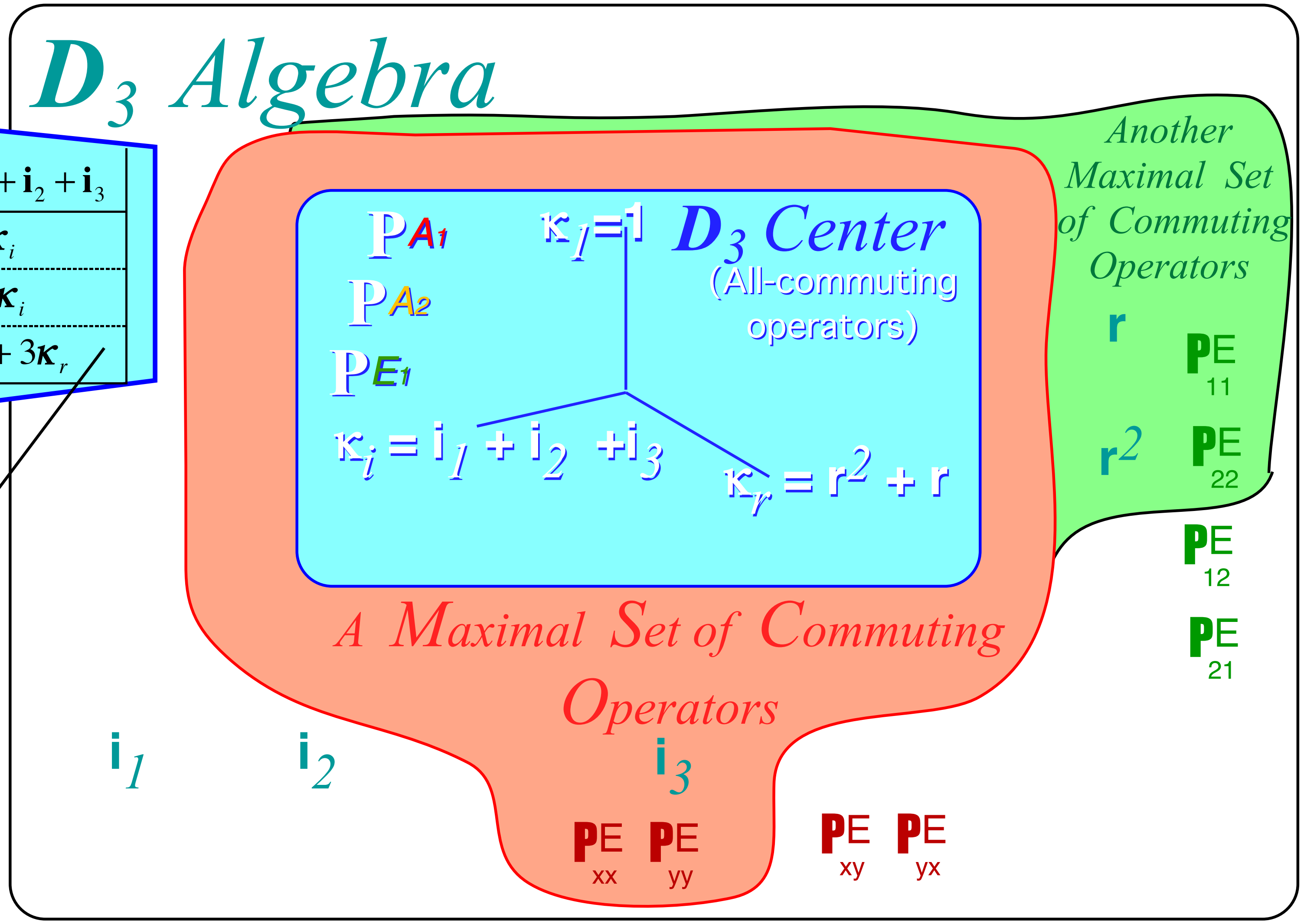
$$\boldsymbol{\kappa}_i^3 = 3\cdot\boldsymbol{\kappa}_r\boldsymbol{\kappa}_i + 3\cdot\boldsymbol{\kappa}_i = 9\cdot\boldsymbol{\kappa}_i$$

$$0 = \boldsymbol{\kappa}_i^3 - 9\cdot\boldsymbol{\kappa}_i = (\boldsymbol{\kappa}_i - 3\cdot\mathbf{1})(\boldsymbol{\kappa}_i + 3\cdot\mathbf{1})(\boldsymbol{\kappa}_i - 0\cdot\mathbf{1})$$

$$\boldsymbol{\kappa}_1 = 1\cdot\mathbf{P}^{A_1} + 1\cdot\mathbf{P}^{A_2} + 1\cdot\mathbf{P}^{E} = \mathbf{1}$$ *(Completeness)*

$$\boldsymbol{\kappa}_r = 2\cdot\mathbf{P}^{A_1} - 2\cdot\mathbf{P}^{A_2} - 1\cdot\mathbf{P}^{E}$$

$$\boldsymbol{\kappa}_i = 3\cdot\mathbf{P}^{A_1} - 3\cdot\mathbf{P}^{A_2} + 0\cdot\mathbf{P}^{E}$$




$$\mathbf{P}^{A_1} = \frac{(\boldsymbol{\kappa}_i + 3\cdot\mathbf{1})(\boldsymbol{\kappa}_i - 0\cdot\mathbf{1})}{(+3+3)(+3-0)}$$

$$\mathbf{P}^{A_2} = \frac{(\boldsymbol{\kappa}_i - 3\cdot\mathbf{1})(\boldsymbol{\kappa}_i - 0\cdot\mathbf{1})}{(-3-3)(-3-0)}$$

$$\mathbf{P}^{E} = \frac{(\boldsymbol{\kappa}_i - 3\cdot\mathbf{1})(\boldsymbol{\kappa}_i + 3\cdot\mathbf{1})}{(+0-3)(+0+3)}$$

## Review: *Spectral resolution of* $\mathbf{D_3}$ ***Center*** *(Class algebra)*

| | | | | | |
|---|---|---|---|---|---|
| $\mathbf{1}$ | $\mathbf{r}^2$ | $\mathbf{r}$ | $\mathbf{i}_1$ | $\mathbf{i}_2$ | $\mathbf{i}_3$ |
| $\mathbf{r}$ | $\mathbf{1}$ | $\mathbf{r}^2$ | $\mathbf{i}_3$ | $\mathbf{i}_1$ | $\mathbf{i}_2$ |
| $\mathbf{r}^2$ | $\mathbf{r}$ | $\mathbf{1}$ | $\mathbf{i}_2$ | $\mathbf{i}_3$ | $\mathbf{i}_1$ |
| $\mathbf{i}_1$ | $\mathbf{i}_3$ | $\mathbf{i}_2$ | $\mathbf{1}$ | $\mathbf{r}$ | $\mathbf{r}^2$ |
| $\mathbf{i}_2$ | $\mathbf{i}_1$ | $\mathbf{i}_3$ | $\mathbf{r}^2$ | $\mathbf{1}$ | $\mathbf{r}$ |
| $\mathbf{i}_3$ | $\mathbf{i}_2$ | $\mathbf{i}_1$ | $\mathbf{r}$ | $\mathbf{r}^2$ | $\mathbf{1}$ |

→

| | $\kappa_1=\mathbf{1}$ | $\kappa_r=\mathbf{r}+\mathbf{r}^2$ | $\kappa_i=\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$ |
|---|---|---|---|
| $\kappa_1$ | $\kappa_1$ | $\kappa_r$ | $\kappa_i$ |
| $\kappa_r$ | $\kappa_r$ | $2\kappa_1+\kappa_r$ | $2\kappa_i$ |
| $\kappa_i$ | $\kappa_i$ | $2\kappa_i$ | $3\kappa_1+3\kappa_r$ |

*Class-sum* $\boldsymbol{\kappa}_k$ *commutes with all* $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t\boldsymbol{\kappa}_k=\boldsymbol{\kappa}_k\mathbf{g}_t$

$°G$ = order of group: $(°D_3=6)$

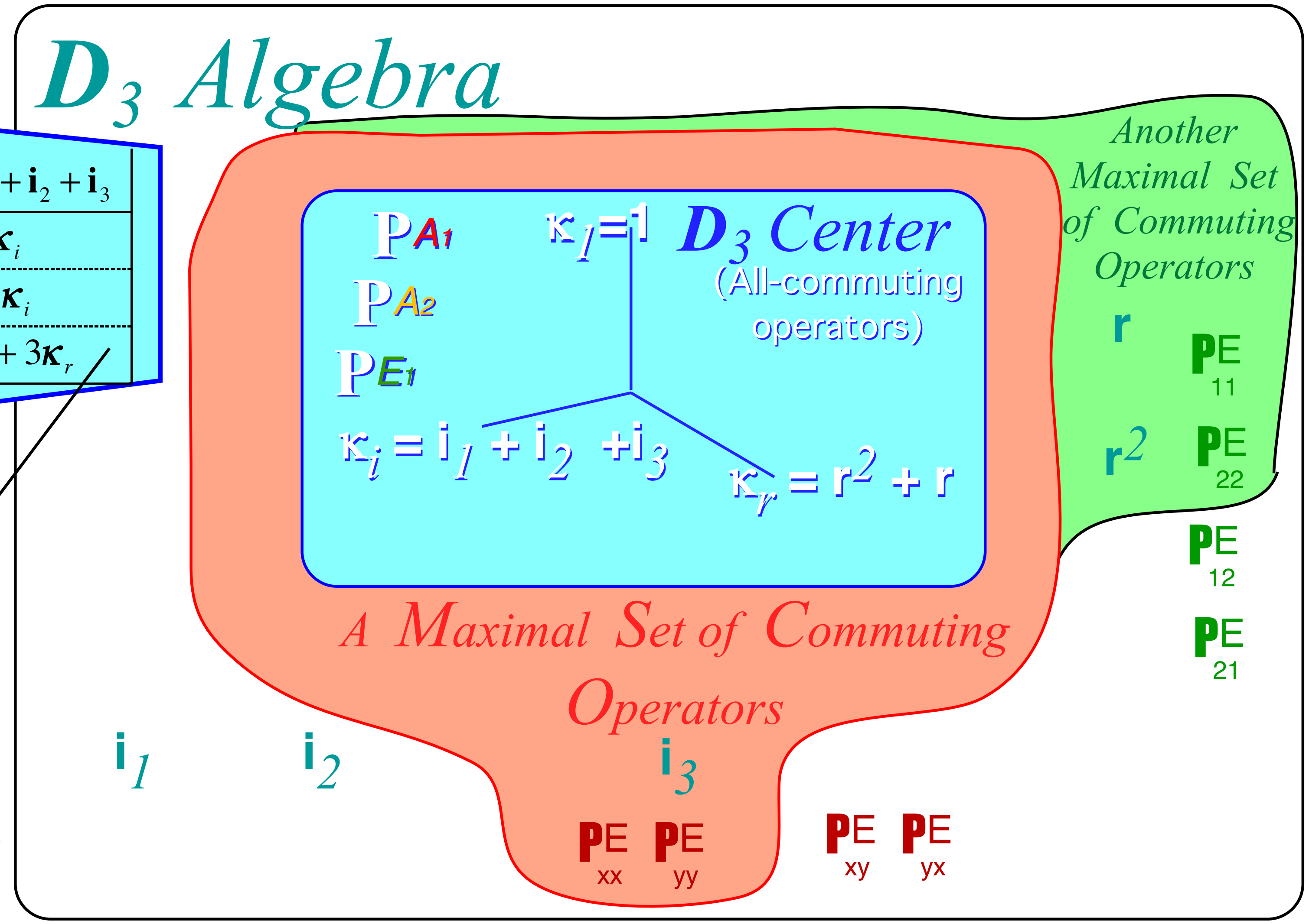
$°\kappa_k$ = order of class $\kappa_k$: $(°\kappa_1=1,\ °\kappa_r=2,\ °\kappa_i=3)$

*Class minimal equation*

$$\boldsymbol{\kappa}_i^2=3\cdot\boldsymbol{\kappa}_r+3\cdot\mathbf{1}$$

$$\boldsymbol{\kappa}_i^3=3\cdot\boldsymbol{\kappa}_r\boldsymbol{\kappa}_i+3\cdot\boldsymbol{\kappa}_i=9\cdot\boldsymbol{\kappa}_i$$

$$0=\boldsymbol{\kappa}_i^3-9\cdot\boldsymbol{\kappa}_i=(\boldsymbol{\kappa}_i-3\cdot\mathbf{1})(\boldsymbol{\kappa}_i+3\cdot\mathbf{1})(\boldsymbol{\kappa}_i-0\cdot\mathbf{1})$$




*Class ortho-complete projector relations*

$$\boldsymbol{\kappa}_1=1\cdot\mathbf{P}^{A_1}+1\cdot\mathbf{P}^{A_2}+1\cdot\mathbf{P}^{E}=\mathbf{1}\quad \textit{(Completeness)}$$

$$\boldsymbol{\kappa}_r=2\cdot\mathbf{P}^{A_1}-2\cdot\mathbf{P}^{A_2}-1\cdot\mathbf{P}^{E}$$

$$\boldsymbol{\kappa}_i=3\cdot\mathbf{P}^{A_1}-3\cdot\mathbf{P}^{A_2}+0\cdot\mathbf{P}^{E}$$

---

$$\mathbf{P}^{A_1}=(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_r+\boldsymbol{\kappa}_i)/6=(\mathbf{1}+\mathbf{r}+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2}=(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_r-\boldsymbol{\kappa}_i)/6=(\mathbf{1}+\mathbf{r}+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$$

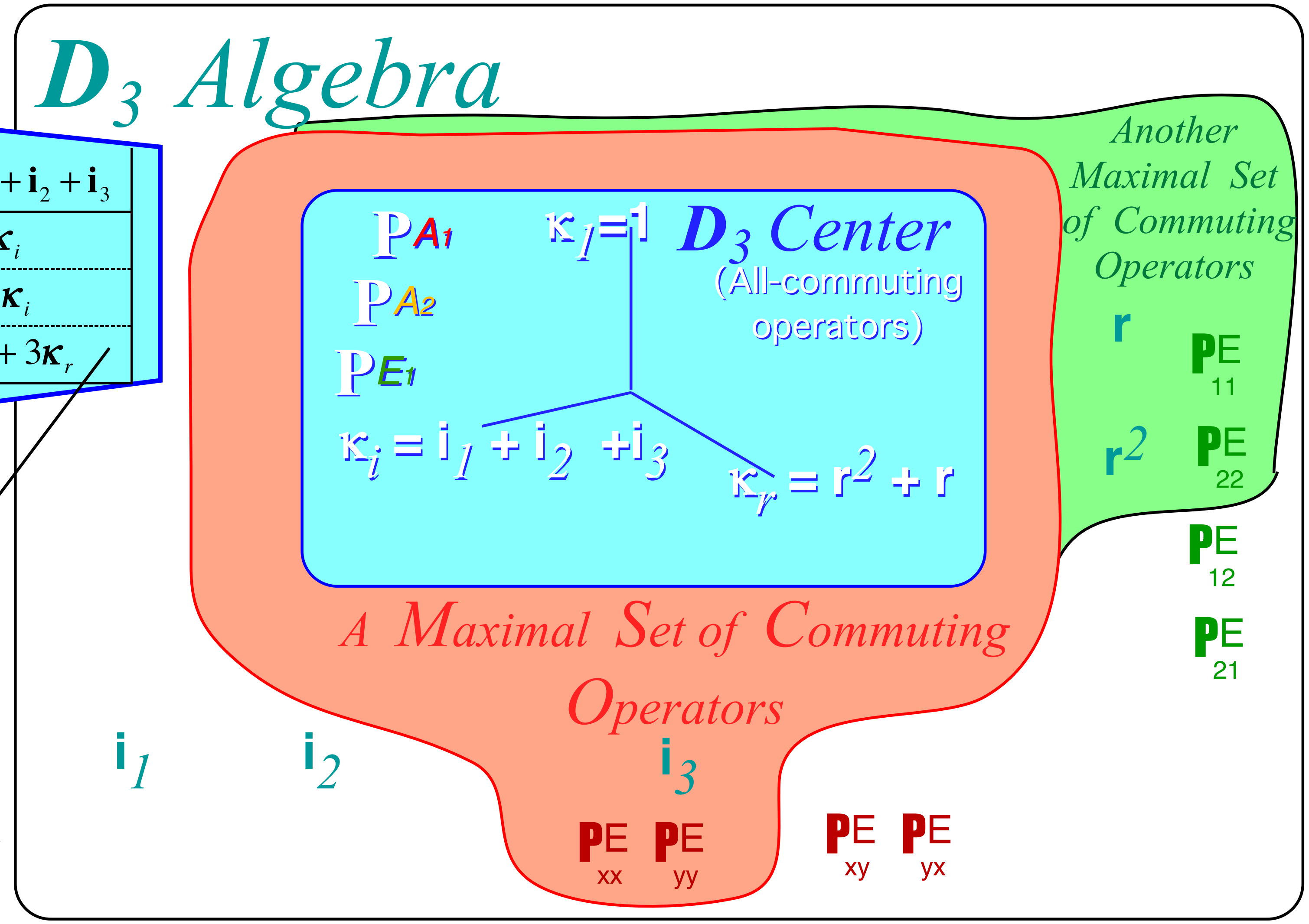
$$\mathbf{P}^{E}=(2\boldsymbol{\kappa}_1-\boldsymbol{\kappa}_r+0)/3=(2\mathbf{1}-\mathbf{r}-\mathbf{r}^2)/3$$

# *Review: Spectral resolution of* $D_3$ *Center (Class algebra)*

| | | | | | |
|---|---|---|---|---|---|
| $\mathbf{1}$ | $\mathbf{r}^2$ | $\mathbf{r}$ | $\mathbf{i}_1$ | $\mathbf{i}_2$ | $\mathbf{i}_3$ |
| $\mathbf{r}$ | $\mathbf{1}$ | $\mathbf{r}^2$ | $\mathbf{i}_3$ | $\mathbf{i}_1$ | $\mathbf{i}_2$ |
| $\mathbf{r}^2$ | $\mathbf{r}$ | $\mathbf{1}$ | $\mathbf{i}_2$ | $\mathbf{i}_3$ | $\mathbf{i}_1$ |
| $\mathbf{i}_1$ | $\mathbf{i}_3$ | $\mathbf{i}_2$ | $\mathbf{1}$ | $\mathbf{r}$ | $\mathbf{r}^2$ |
| $\mathbf{i}_2$ | $\mathbf{i}_1$ | $\mathbf{i}_3$ | $\mathbf{r}^2$ | $\mathbf{1}$ | $\mathbf{r}$ |
| $\mathbf{i}_3$ | $\mathbf{i}_2$ | $\mathbf{i}_1$ | $\mathbf{r}$ | $\mathbf{r}^2$ | $\mathbf{1}$ |

→

| | $\kappa_1=\mathbf{1}$ | $\kappa_r=\mathbf{r}+\mathbf{r}^2$ | $\kappa_i=\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$ |
|---|---|---|---|
| $\kappa_1$ | $\kappa_1$ | $\kappa_r$ | $\kappa_i$ |
| $\kappa_r$ | $\kappa_r$ | $2\kappa_1+\kappa_r$ | $2\kappa_i$ |
| $\kappa_i$ | $\kappa_i$ | $2\kappa_i$ | $3\kappa_1+3\kappa_r$ |




*Class-sum* $\kappa_k$ *commutes with all* $\mathbf{g}_t$

Class-sum $\kappa_k$ invariance: $\mathbf{g}_t\kappa_k=\kappa_k\mathbf{g}_t$

$°G$ = order of group: $(°D_3=6)$

$°\kappa_k$ = order of class $\kappa_k$: $(°\kappa_1=1,\ °\kappa_r=2,\ °\kappa_i=3)$

*Class minimal equation*

$$\kappa_i^2=3\cdot\kappa_r+3\cdot\mathbf{1}$$

$$\kappa_i^3=3\cdot\kappa_r\kappa_i+3\cdot\kappa_i=9\cdot\kappa_i$$

$$0=\kappa_i^3-9\cdot\kappa_i=(\kappa_i-3\cdot\mathbf{1})(\kappa_i+3\cdot\mathbf{1})(\kappa_i-0\cdot\mathbf{1})$$

$$\kappa_1=1\cdot\mathbf{P}^{A_1}+1\cdot\mathbf{P}^{A_2}+\ 1\cdot\mathbf{P}^{E}=\mathbf{1}\quad\textit{(Completeness)}$$

$$\kappa_r=2\cdot\mathbf{P}^{A_1}+2\cdot\mathbf{P}^{A_2}-\ 1\cdot\mathbf{P}^{E}$$

$$\kappa_i=3\cdot\mathbf{P}^{A_1}-3\cdot\mathbf{P}^{A_2}+\ 0\cdot\mathbf{P}^{E}$$

---

$$\mathbf{P}^{A_1}=(\kappa_1+\kappa_r+\kappa_i)/6=(\mathbf{1}+\mathbf{r}+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2}=(\kappa_1+\kappa_r-\kappa_i)/6=(\mathbf{1}+\mathbf{r}+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}=(2\kappa_1-\kappa_r+0)/3=(2\mathbf{1}-\mathbf{r}-\mathbf{r}^2)/3$$

*Class ortho-complete projector and character relations*

$$\kappa_k=\sum_{(\alpha)}\frac{°\kappa_k\chi_k^{(\alpha)}}{\ell^{(\alpha)}}\mathbf{P}^{(\alpha)}$$

$$\mathbf{P}^{(\alpha)}=\frac{\ell^{(\alpha)}}{°G}\sum_k\chi_k^{(\alpha)*}\kappa_k=\frac{\ell^{(\alpha)}}{°G}\sum_{g=1}^{°G}\chi_g^{(\alpha)*}\mathbf{g}$$

| $\chi_k^\alpha$ | $\chi_1^\alpha$ | $\chi_r^\alpha$ | $\chi_i^\alpha$ |
|---|---|---|---|
| $\alpha=A_1$ | 1 | 1 | 1 |
| $\alpha=A_2$ | 1 | 1 | −1 |
| $\alpha=E$ | 2 | −1 | 0 |

*Review: Spectral resolution of **$D_3$ Center** (Class algebra)*

*Group theory of equivalence transformations and classes*

*Lagrange theorems*

*All-commuting class projectors and $D_3$-invariant character ortho-completeness*

*Subgroup splitting and correlation frequency formula: $f^{(a)}(D^{(\alpha)}(G)\downarrow H)$*

*Group invariant numbers: Centrum, Rank, and Order*

*2nd-Stage spectral decompositions of global/local $D_3$*

*Splitting class projectors using subgroup chains $D_3 \supset C_2$ and $D_3 \supset C_3$*

*3rd-stage spectral resolution to **irreducible representations** (ireps) and Hamiltonian eigensolutions*

*Tunneling modes and spectra for $D_3 \supset C_2$ and $D_3 \supset C_3$ local subgroup chains*

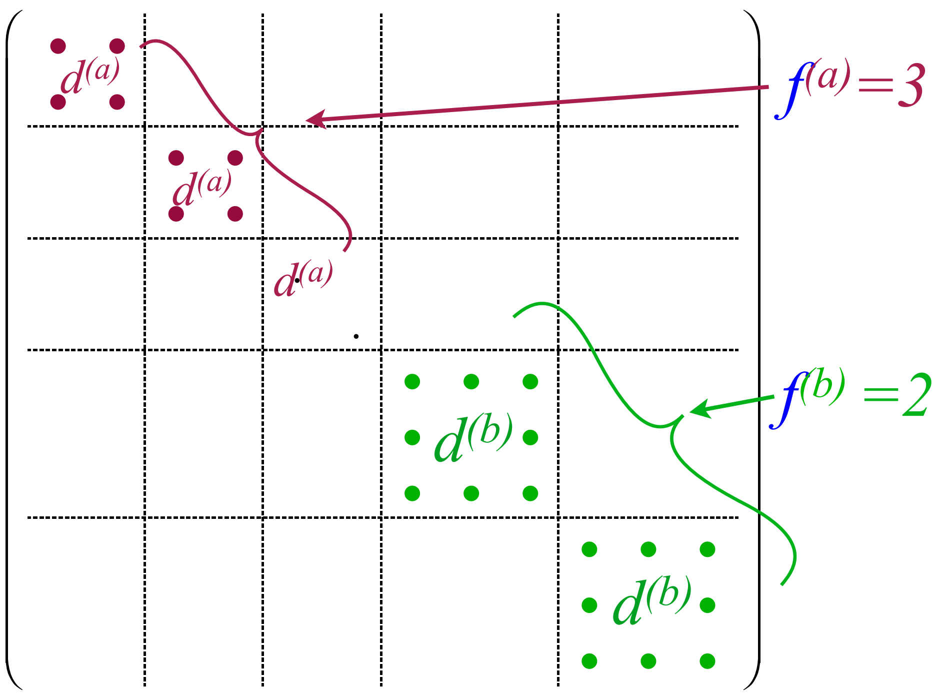
# Subgroup splitting and correlation frequency formula: $f^{(a)}(D^{(\alpha)}(G)\downarrow H)$

(*irep* ≡ *ir*reducible *rep*resentation)

*Symmetry reduction of G to H⊂G involves splitting of G-ireps $D^{(\alpha)}(G)$ into smaller H-ireps $d^{(a)}(H)$*

$D^{(\alpha)}(G)\downarrow H \equiv D^{(\alpha)}(H)$ *is reducible to:* $T_{reducer}\, D^{(\alpha)}(H) T^{\dagger}_{reducer} = f^{(a)} d^{(a)}(H) \oplus f^{(b)} d^{(b)}(H) \oplus \ldots$

$$T_{reducer} D^{(\alpha)}(h) T^{\dagger}_{reducer} =$$




*The following derives formulae for integral H⊂G correlation coefficients $f^{(a)}(D^{(\alpha)}(G)\downarrow H)$*

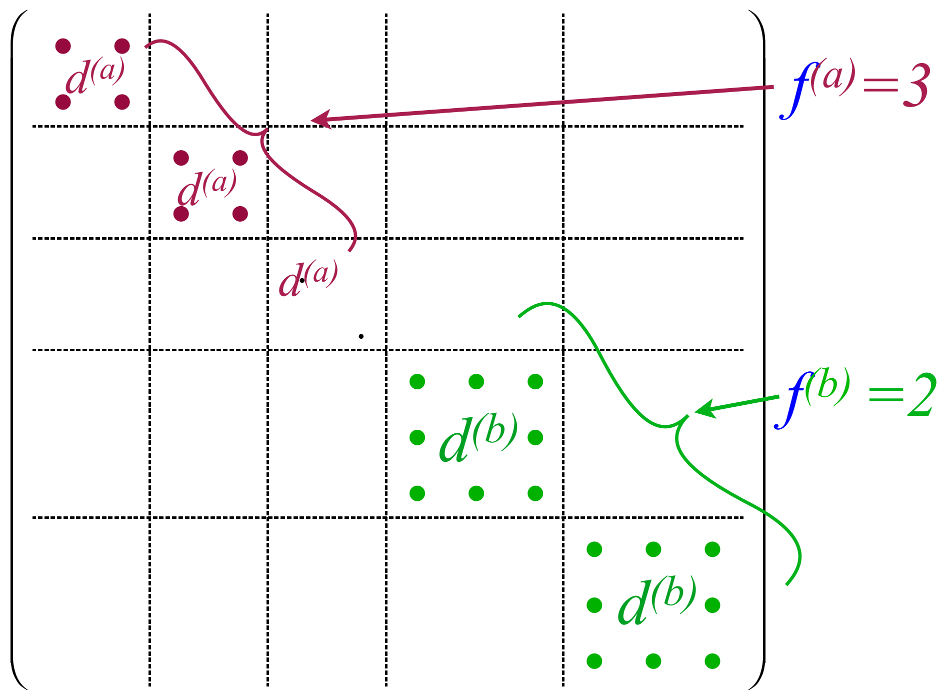
# *Subgroup splitting and correlation frequency formula:* $f^{(a)}(D^{(\alpha)}(G)\downarrow H)$

*(irep ≡ irreducible representation)*

*Symmetry reduction of G to H⊂G involves splitting of G-ireps* $D^{(\alpha)}(G)$ *into smaller H-ireps* $d^{(a)}(H)$

$D^{(\alpha)}(G)\downarrow H \equiv D^{(\alpha)}(H)$ *is reducible to:* $T_{reducer}\, D^{(\alpha)}(H) T^{\dagger}_{reducer} = f^{(a)} d^{(a)}(H) \oplus f^{(b)} d^{(b)}(H) \oplus \ldots$

$$T_{reducer} D^{(\alpha)}(h) T^{\dagger}_{reducer} =$$




*The following derives formulae for integral H⊂G correlation coefficients* $f^{(b)}(D^{(\alpha)}(G)\downarrow H)$

$$Trace D^{(\alpha)}(\mathbf{P}^{(b)}) = f^{(b)} \cdot \ell^{(b)}$$

*Since each* $d^{(b)}(\mathbf{P}^{(b)})$ *is* $\ell^{(b)}$*-by-*$\ell^{(b)}$ *unit matrix*

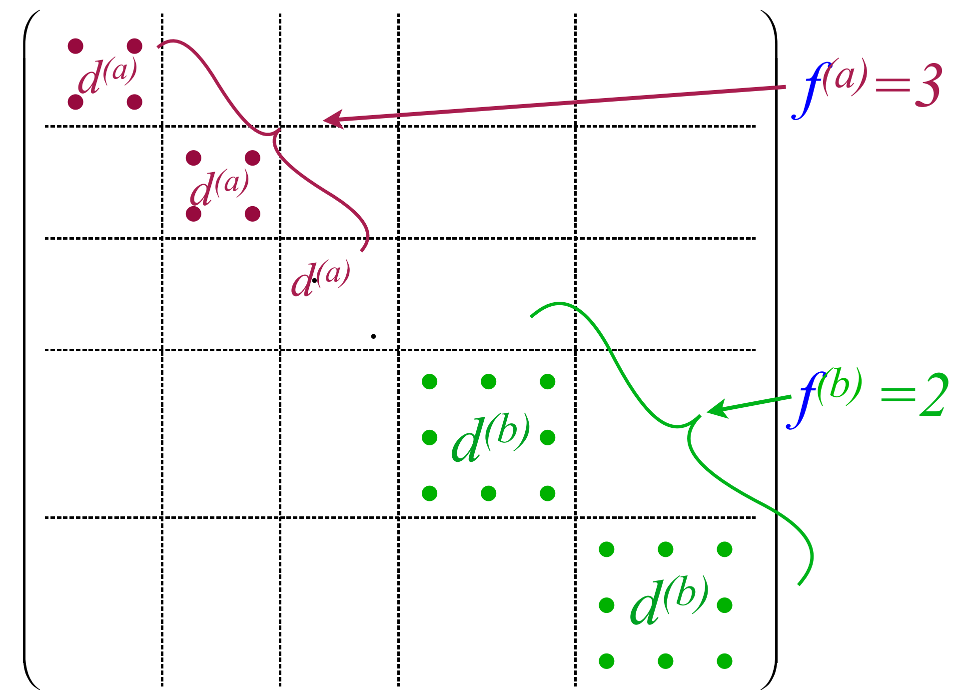
## *Subgroup splitting and correlation frequency formula:* $f^{(a)}(D^{(\alpha)}(G)\downarrow H)$

*(irep ≡ irreducible representation)*

*Symmetry reduction of G to H⊂G involves splitting of G-ireps* $D^{(\alpha)}(G)$ *into smaller H-ireps* $d^{(a)}(H)$

$$D^{(\alpha)}(G)\downarrow H \equiv D^{(\alpha)}(H) \text{ is reducible to: } T_{reducer}\, D^{(\alpha)}(H) T^{\dagger}_{reducer} = f^{(a)} d^{(a)}(H) \oplus f^{(b)} d^{(b)}(H) \oplus \ldots$$

$$T_{reducer} D^{(\alpha)}(h) T^{\dagger}_{reducer} =$$




*The following derives formulae for integral H⊂G correlation coefficients* $f^{(a)}(D^{(\alpha)}(G)\downarrow H)$

$$Trace D^{(\alpha)}(\mathbf{P}^{(b)}) = f^{(b)} \cdot \ell^{(b)}$$

$$f^{(b)} = \frac{1}{\ell^{(b)}} Trace D^{(\alpha)}(\mathbf{P}^{(b)})$$

# *Subgroup splitting and correlation frequency formula:* $f^{(a)}(D^{(\alpha)}(G)\downarrow H)$

*(irep ≡ irreducible representation)*

*Symmetry reduction of G to H⊂G involves splitting of G-ireps* $D^{(\alpha)}(G)$ *into smaller H-ireps* $d^{(a)}(H)$

$D^{(\alpha)}(G)\downarrow H \equiv D^{(\alpha)}(H)$ *is reducible to:* $T_{reducer}\, D^{(\alpha)}(H) T^{\dagger}_{reducer} = f^{(a)} d^{(a)}(H) \oplus f^{(b)} d^{(b)}(H) \oplus \ldots$

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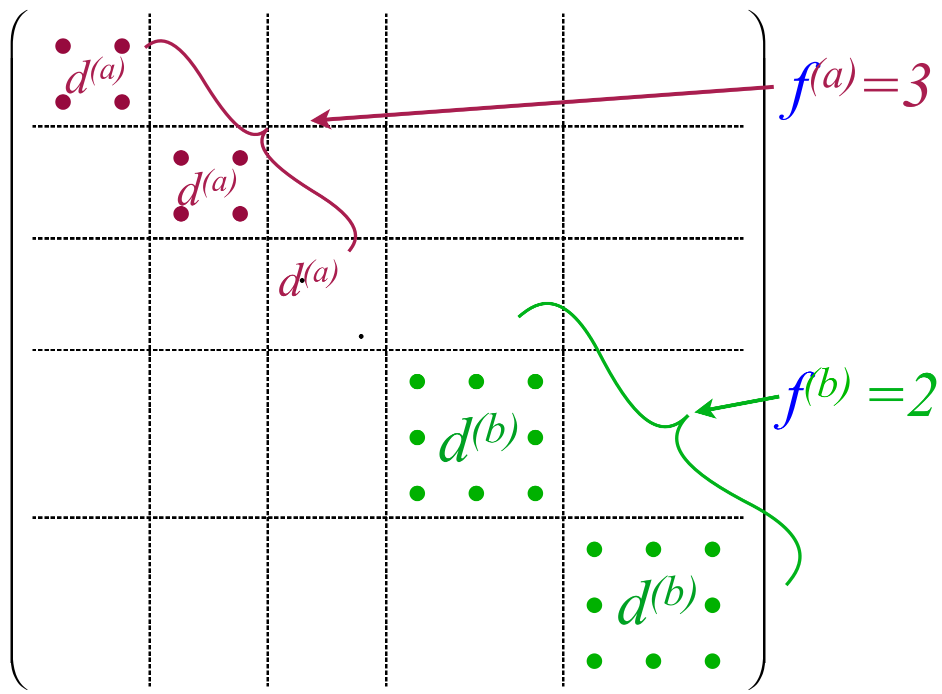



*The following derives formulae for integral H⊂G correlation coefficients* $f^{(a)}(D^{(\alpha)}(G)\downarrow H)$

$$Trace D^{(\alpha)}(\mathbf{P}^{(b)}) = f^{(b)} \cdot \ell^{(b)}$$

$$\mathbf{P}^{(\alpha)} = \frac{\ell^{(\alpha)}}{{}^{\circ}G} \sum_{k \in G} \chi_k^{(\alpha)*} \boldsymbol{\kappa}_k$$

$$\mathbf{P}^{(b)} = \frac{\ell^{(b)}}{{}^{\circ}H} \sum_{k \in H} \chi_k^{(b)*} \boldsymbol{\kappa}_k$$

$$f^{(b)} = \frac{1}{\ell^{(b)}} Trace D^{(\alpha)}(\mathbf{P}^{(b)}) = \frac{1}{\ell^{(b)}} \frac{\ell^{(b)}}{{}^{\circ}H} \sum_{\substack{classes \\ \boldsymbol{\kappa}_k \in H}} \chi_k^{(b)*} Trace D^{(\alpha)}(\boldsymbol{\kappa}_k)$$

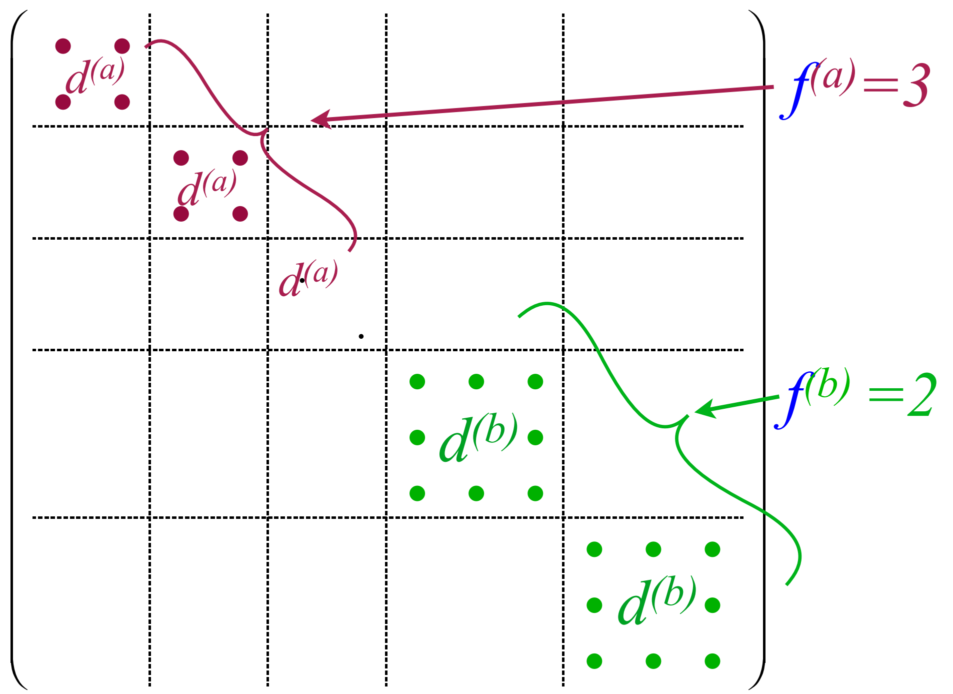
# *Subgroup splitting and correlation frequency formula:* $f^{(a)}(D^{(\alpha)}(G)\downarrow H)$

*(irep ≡ irreducible representation)*

*Symmetry reduction of G to H⊂G involves splitting of G-ireps $D^{(\alpha)}(G)$ into smaller H-ireps $d^{(a)}(H)$*

$D^{(\alpha)}(G)\downarrow H \equiv D^{(\alpha)}(H)$ *is reducible to:* $T_{reducer}\, D^{(\alpha)}(H) T^{\dagger}_{reducer} = f^{(a)} d^{(a)}(H) \oplus f^{(b)} d^{(b)}(H) \oplus \ldots$

$$T_{reducer} D^{(\alpha)}(h) T^{\dagger}_{reducer} =$$




*The following derives formulae for integral H⊂G correlation coefficients* $f^{(a)}(D^{(\alpha)}(G)\downarrow H)$

$$Trace D^{(\alpha)}(\mathbf{P}^{(b)}) = f^{(b)} \cdot \ell^{(b)}$$

$$\mathbf{P}^{(\alpha)} = \frac{\ell^{(\alpha)}}{{}^{\circ}G} \sum_{k \in G} \chi_k^{(\alpha)*} \boldsymbol{\kappa}_k$$

$$\mathbf{P}^{(b)} = \frac{\ell^{(b)}}{{}^{\circ}H} \sum_{k \in H} \chi_k^{(b)*} \boldsymbol{\kappa}_k$$

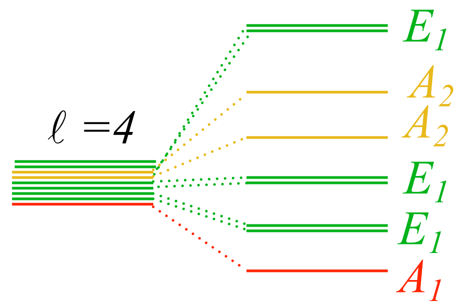
$$f^{(b)} = \frac{1}{\ell^{(b)}} Trace D^{(\alpha)}(\mathbf{P}^{(b)}) = \frac{1}{\ell^{(b)}} \frac{\ell^{(b)}}{{}^{\circ}H} \sum_{\substack{classes \\ \boldsymbol{\kappa}_k \in H}} \chi_k^{(b)*} \underbrace{Trace D^{(\alpha)}(\boldsymbol{\kappa}_k)}$$

$$\chi^{(\alpha)}(\boldsymbol{\kappa}_k) = {}^{\circ}\kappa_k \chi_k^{(\alpha)}$$

$$f^{(b)} = \frac{1}{{}^{\circ}H} \sum_{\substack{classes \\ \boldsymbol{\kappa}_k \in H}} {}^{\circ}\kappa_k \chi_k^{(b)*} \chi_k^{(\alpha)}$$

# *From end of Lecture 13*

*Example to use:* $f^{(b)} = \frac{1}{{}^{\circ}H} \sum_{\substack{classes \\ \kappa_k \in H}} {}^{\circ}\kappa_k \chi_k^{(b)*} \chi_k^{(\alpha)}$




| $\chi^\ell(\Theta)$ | $\Theta=0$ | $\frac{2\pi}{3}$ | $\pi$ |
|---|---|---|---|
| $\ell=0$ | 1 | 1 | 1 |
| 1 | 3 | 0 | –1 |
| 2 | 5 | –1 | 1 |
| 3 | 7 | 1 | –1 |
| 4 | 9 | 0 | 1 |
| 5 | 11 | –1 | –1 |
| 6 | 13 | 1 | 1 |
| 7 | 15 | 0 | –1 |

$$\chi^\ell(\Theta) = \frac{\sin(\ell+\frac{1}{2})\Theta}{\sin\frac{\Theta}{2}}$$

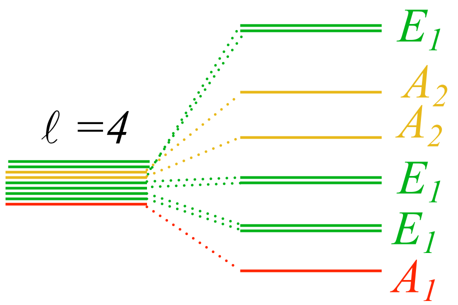
...and $D_3$ character table:

| $(\mathbf{g})=$ | $\{\mathbf{1}\}$ | $\{\mathbf{r}^1,\mathbf{r}^2\}$ | $\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$ |
|---|---|---|---|
| $\chi^{A_1}(\mathbf{g})=$ | 1 | 1 | 1 |
| $\chi^{A_2}(\mathbf{g})=$ | 1 | 1 | –1 |
| $\chi^{E_1}(\mathbf{g})=$ | 2 | –1 | 0 |

| $f^{(\alpha)}(\ell)$ | $f^{A_1}$ | $f^{A_2}$ | $f^{E_1}$ | |
|---|---|---|---|---|
| $\ell=0$ | 1 | · | · | $1A_1$ |
| 1 | · | 1 | 1 | $0A_1 \oplus A_2 \oplus E_1$ |
| 2 | 1 | · | 2 | $1A_1 \quad \oplus 2E_1$ |
| 3 | 1 | 2 | 2 | $1A_1 \oplus 2A_2 \oplus 2E_1$ |
| 4 | 1 | 2 | 3 | $1A_1 \oplus 2A_2 \oplus 3E_1$ |
| 5 | 2 | 1 | 3 | $2A_1 \oplus \; A_2 \oplus 3E_1$ |
| 6 | 3 | 2 | 4 | $3A_1 \oplus 2A_2 \oplus 4E_1$ |
| 7 | 2 | 3 | 5 | $2A_1 \oplus 3A_2 \oplus 5E_1$ |

# *From end of Lecture 13*

*Example to use:* $$f^{(b)} = \frac{1}{{}^{\circ}H}\sum_{\substack{classes\\ \kappa_k \in H}} {}^{\circ}\kappa_k \chi_k^{(b)*}\chi_k^{(\alpha)}$$




$$f^{(E_1)} = \frac{1}{{}^{\circ}D_3}\sum_{\substack{classes\\ \kappa_k \in D_3}} {}^{\circ}\kappa_k \chi_k^{(E_1)*}\chi_k^{(\ell=4)} = \frac{1}{{}^{\circ}D_3}\left({}^{\circ}\kappa_{0^{\circ}}\chi_{0^{\circ}}^{(E_1)*}\chi_{0^{\circ}}^{(\ell=4)} + {}^{\circ}\kappa_{120^{\circ}}\chi_{120^{\circ}}^{(E_1)*}\chi_{120^{\circ}}^{(\ell=4)} + {}^{\circ}\kappa_{180^{\circ}}\chi_{180^{\circ}}^{(E_1)*}\chi_{180^{\circ}}^{(\ell=4)}\right)$$

| $\chi^{\ell}(\Theta)$ | $\Theta=0$ | $\frac{2\pi}{3}$ | $\pi$ |
|---|---|---|---|
| $\ell=0$ | 1 | 1 | 1 |
| 1 | 3 | 0 | −1 |
| 2 | 5 | −1 | 1 |
| 3 | 7 | 1 | −1 |
| 4 | 9 | 0 | 1 |
| 5 | 11 | −1 | −1 |
| 6 | 13 | 1 | 1 |
| 7 | 15 | 0 | −1 |

$$\chi^{\ell}(\Theta) = \frac{\sin(\ell+\frac{1}{2})\Theta}{\sin\frac{\Theta}{2}}$$

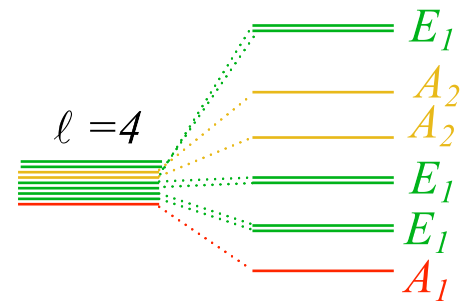
...and $D_3$ character table:

| $(\mathbf{g})=$ | $\{\mathbf{1}\}$ | $\{\mathbf{r}^1,\mathbf{r}^2\}$ | $\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$ |
|---|---|---|---|
| $\chi^{A_1}(\mathbf{g})=$ | 1 | 1 | 1 |
| $\chi^{A_2}(\mathbf{g})=$ | 1 | 1 | −1 |
| $\chi^{E_1}(\mathbf{g})=$ | 2 | −1 | 0 |

| $f^{(\alpha)}(\ell)$ | $f^{A_1}$ | $f^{A_2}$ | $f^{E_1}$ | |
|---|---|---|---|---|
| $\ell=0$ | 1 | · | · | $1A_1$ |
| 1 | · | 1 | 1 | $0A_1 \oplus A_2 \oplus E_1$ |
| 2 | 1 | · | 2 | $1A_1 \quad \oplus 2E_1$ |
| 3 | 1 | 2 | 2 | $1A_1 \oplus 2A_2 \oplus 2E_1$ |
| 4 | 1 | 2 | 3 | $1A_1 \oplus 2A_2 \oplus 3E_1$ |
| 5 | 2 | 1 | 3 | $2A_1 \oplus \quad A_2 \oplus 3E_1$ |
| 6 | 3 | 2 | 4 | $3A_1 \oplus 2A_2 \oplus 4E_1$ |
| 7 | 2 | 3 | 5 | $2A_1 \oplus 3A_2 \oplus 5E_1$ |

# *From end of Lecture 13*

*Example to use:* $f^{(b)} = \frac{1}{{}^{\circ}H} \sum_{\substack{classes \\ \kappa_k \in H}} {}^{\circ}\kappa_k \chi_k^{(b)*} \chi_k^{(\alpha)}$




$$f^{(E_1)} = \frac{1}{{}^{\circ}D_3} \sum_{\substack{classes \\ \kappa_k \in D_3}} {}^{\circ}\kappa_k \chi_k^{(E_1)*} \chi_k^{(\ell=4)} = \frac{1}{{}^{\circ}D_3}\left( {}^{\circ}\kappa_{0^\circ}\chi_{0^\circ}^{(E_1)*}\chi_{0^\circ}^{(\ell=4)} + {}^{\circ}\kappa_{120^\circ}\chi_{120^\circ}^{(E_1)*}\chi_{120^\circ}^{(\ell=4)} + {}^{\circ}\kappa_{180^\circ}\chi_{180^\circ}^{(E_1)*}\chi_{180^\circ}^{(\ell=4)} \right)$$

$$= \frac{1}{6}\left( 1 \cdot 2^* \cdot 9 \quad + \quad 2 \cdot -1^* \cdot 0 \quad + \quad 3 \cdot 0^* \cdot 1 \right)$$

| $\chi^{\ell}(\Theta)$ | $\Theta = 0$ | $\frac{2\pi}{3}$ | $\pi$ |
|---|---|---|---|
| $\ell = 0$ | 1 | 1 | 1 |
| 1 | 3 | 0 | −1 |
| 2 | 5 | −1 | 1 |
| 3 | 7 | 1 | −1 |
| 4 | 9 | 0 | 1 |
| 5 | 11 | −1 | −1 |
| 6 | 13 | 1 | 1 |
| 7 | 15 | 0 | −1 |

$$\chi^{\ell}(\Theta) = \frac{\sin(\ell + \frac{1}{2})\Theta}{\sin\frac{\Theta}{2}}$$

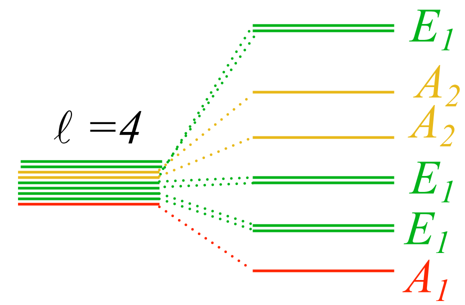
...and $D_3$ character table:

| $(\mathbf{g}) =$ | $\{\mathbf{1}\}$ | $\{\mathbf{r}^1, \mathbf{r}^2\}$ | $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$ |
|---|---|---|---|
| $\chi^{A_1}(\mathbf{g}) =$ | 1 | 1 | 1 |
| $\chi^{A_2}(\mathbf{g}) =$ | 1 | 1 | −1 |
| $\chi^{E_1}(\mathbf{g}) =$ | 2 | −1 | 0 |

| $f^{(\alpha)}(\ell)$ | $f^{A_1}$ | $f^{A_2}$ | $f^{E_1}$ | |
|---|---|---|---|---|
| $\ell = 0$ | 1 | · | · | $1A_1$ |
| 1 | · | 1 | 1 | $0A_1 \oplus A_2 \oplus E_1$ |
| 2 | 1 | · | 2 | $1A_1 \quad \oplus 2E_1$ |
| 3 | 1 | 2 | 2 | $1A_1 \oplus 2A_2 \oplus 2E_1$ |
| 4 | 1 | 2 | 3 | $1A_1 \oplus 2A_2 \oplus 3E_1$ |
| 5 | 2 | 1 | 3 | $2A_1 \oplus \ A_2 \oplus 3E_1$ |
| 6 | 3 | 2 | 4 | $3A_1 \oplus 2A_2 \oplus 4E_1$ |
| 7 | 2 | 3 | 5 | $2A_1 \oplus 3A_2 \oplus 5E_1$ |

# *From end of Lecture 13*

## *Example to use:* $f^{(b)} = \frac{1}{{}^{\circ}H} \sum_{\substack{classes \\ \kappa_k \in H}} {}^{\circ}\kappa_k \chi_k^{(b)*} \chi_k^{(\alpha)}$




$$f^{(E_1)} = \frac{1}{{}^{\circ}D_3} \sum_{\substack{classes \\ \kappa_k \in D_3}} {}^{\circ}\kappa_k \chi_k^{(E_1)*} \chi_k^{(\ell=4)} = \frac{1}{{}^{\circ}D_3}\left( {}^{\circ}\kappa_{0^\circ} \chi_{0^\circ}^{(E_1)*} \chi_{0^\circ}^{(\ell=4)} + {}^{\circ}\kappa_{120^\circ} \chi_{120^\circ}^{(E_1)*} \chi_{120^\circ}^{(\ell=4)} + {}^{\circ}\kappa_{180^\circ} \chi_{180^\circ}^{(E_1)*} \chi_{180^\circ}^{(\ell=4)} \right)$$

$$= \frac{1}{6}\left( 1 \cdot 2^* \cdot 9 \quad + \quad 2 \cdot -1^* \cdot 0 \quad + \quad 3 \cdot 0^* \cdot 1 \right)$$

$$f^{(E_1)} = 3$$

| $\chi^{\ell}(\Theta)$ | $\Theta = 0$ | $\frac{2\pi}{3}$ | $\pi$ |
|---|---|---|---|
| $\ell = 0$ | 1 | 1 | 1 |
| 1 | 3 | 0 | −1 |
| 2 | 5 | −1 | 1 |
| 3 | 7 | 1 | −1 |
| 4 | 9 | 0 | 1 |
| 5 | 11 | −1 | −1 |
| 6 | 13 | 1 | 1 |
| 7 | 15 | 0 | −1 |

$$\chi^{\ell}(\Theta) = \frac{\sin(\ell + \frac{1}{2})\Theta}{\sin\frac{\Theta}{2}}$$

...and $D_3$ character table:

| $(\mathbf{g}) =$ | $\{\mathbf{1}\}$ | $\{\mathbf{r}^1, \mathbf{r}^2\}$ | $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$ |
|---|---|---|---|
| $\chi^{A_1}(\mathbf{g}) =$ | 1 | 1 | 1 |
| $\chi^{A_2}(\mathbf{g}) =$ | 1 | 1 | −1 |
| $\chi^{E_1}(\mathbf{g}) =$ | 2 | −1 | 0 |

| $f^{(\alpha)}(\ell)$ | $f^{A_1}$ | $f^{A_2}$ | $f^{E_1}$ | |
|---|---|---|---|---|
| $\ell = 0$ | 1 | · | · | $1A_1$ |
| 1 | · | 1 | 1 | $0A_1 \oplus A_2 \oplus E_1$ |
| 2 | 1 | · | 2 | $1A_1 \quad \oplus 2E_1$ |
| 3 | 1 | 2 | 2 | $1A_1 \oplus 2A_2 \oplus 2E_1$ |
| 4 | 1 | 2 | 3 | $1A_1 \oplus 2A_2 \oplus 3E_1$ |
| 5 | 2 | 1 | 3 | $2A_1 \oplus \ A_2 \oplus 3E_1$ |
| 6 | 3 | 2 | 4 | $3A_1 \oplus 2A_2 \oplus 4E_1$ |
| 7 | 2 | 3 | 5 | $2A_1 \oplus 3A_2 \oplus 5E_1$ |

*Review:Spectral resolution of **$D_3$ Center** (Class algebra)*

*Group theory of equivalence transformations and classes*

*Lagrange theorems*

*All-commuting class projectors and $D_3$-invariant character ortho-completeness*

*Subgroup splitting and correlation frequency formula: $f^{(a)}(D^{(\alpha)}(G)\downarrow H)$*

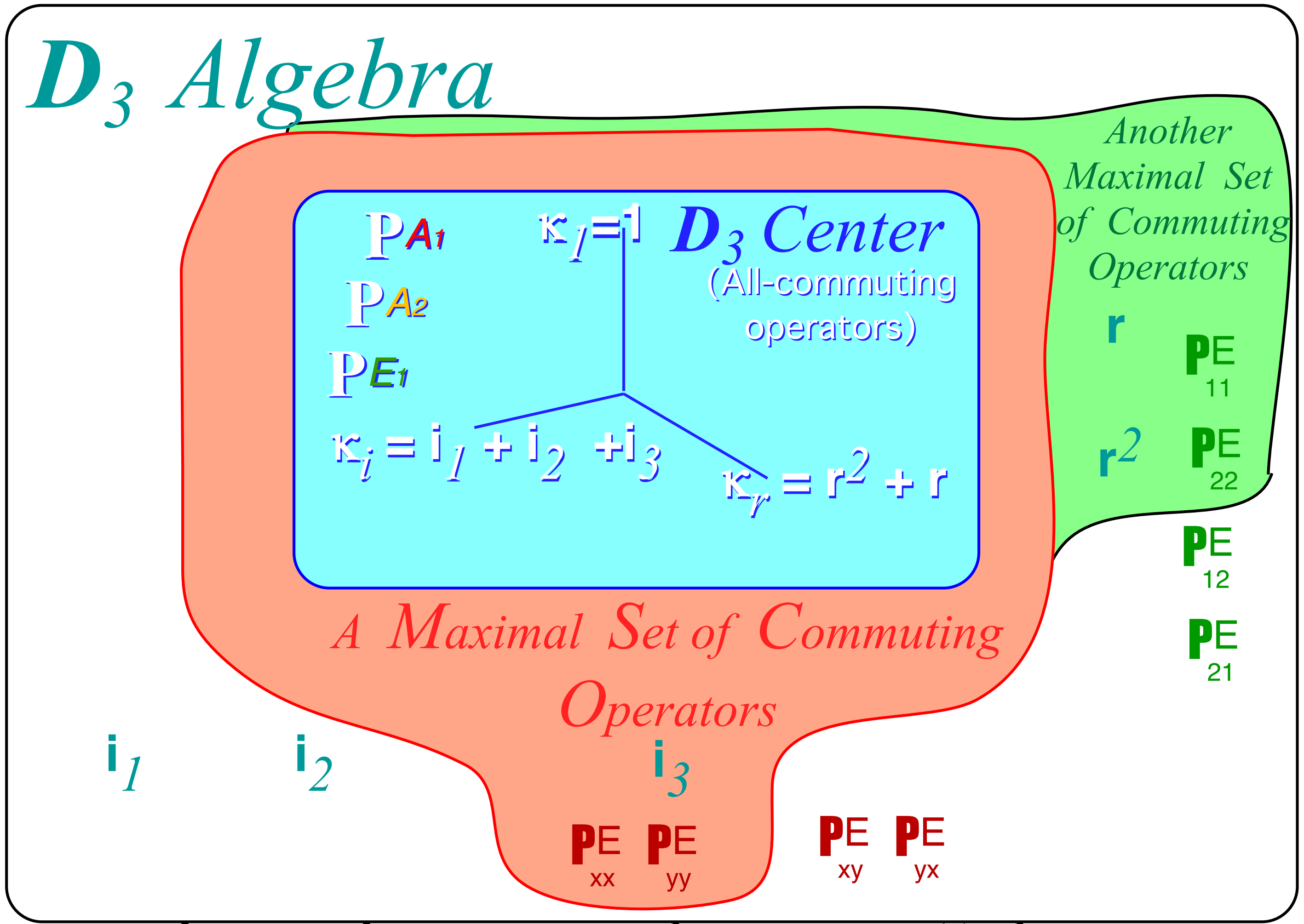
*Group invariant numbers: Centrum, Rank, and Order*

*2nd-Stage spectral decompositions of global/local $D_3$*

*Splitting class projectors using subgroup chains $D_3 \supset C_2$ and $D_3 \supset C_3$*

*3rd-stage spectral resolution to **irreducible representations** (ireps) and Hamiltonian eigensolutions*

*Tunneling modes and spectra for $D_3 \supset C_2$ and $D_3 \supset C_3$ local subgroup chains*




| $D_3$ | $\kappa = \mathbf{1}$ | $\mathbf{r}^1+\mathbf{r}^2$ | $\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$ | |
|---|---|---|---|---|
| $\mathbf{P}^{A_1}=$ | 1 | 1 | 1 | /6 |
| $\mathbf{P}^{A_2}=$ | 1 | 1 | -1 | /6 |
| $\mathbf{P}^{E}=$ | 2 | -1 | 0 | /3 |

# Important invariant numbers or "characters"

$\ell^\alpha =$ Irreducible representation (irrep) *dimension* or level *degeneracy*
*For symmetry group or algebra G*

*Centrum:* $\kappa(G)=\Sigma_{irrep(\alpha)} (\ell^\alpha)^0$ =Number of classes, invariants, irrep types, *all-commuting* ops

*Rank:* $\rho(G)=\Sigma_{irrep(\alpha)} (\ell^\alpha)^1$ =Number of irrep idempotents $\mathbf{P}^{(\alpha)}_{n,n}$, *mutually-commuting* ops

*Order:* $°(G)=\Sigma_{irrep(\alpha)} (\ell^\alpha)^2$ =*Total* number of irrep projectors $\mathbf{P}^{(\alpha)}_{m,n}$ or symmetry ops

$\kappa(D_3)=(1)^0+(1)^0+(2)^0=3$

$\rho(D_3)=(1)^1+(1)^1+(2)^1=4$

$°(D_3)=(1)^2+(1)^2+(2)^2=6$

*Review:Spectral resolution of **$D_3$ Center** (Class algebra)*

*Group theory of equivalence transformations and classes*

*Lagrange theorems*

*All-commuting class projectors and $D_3$-invariant character ortho-completeness*

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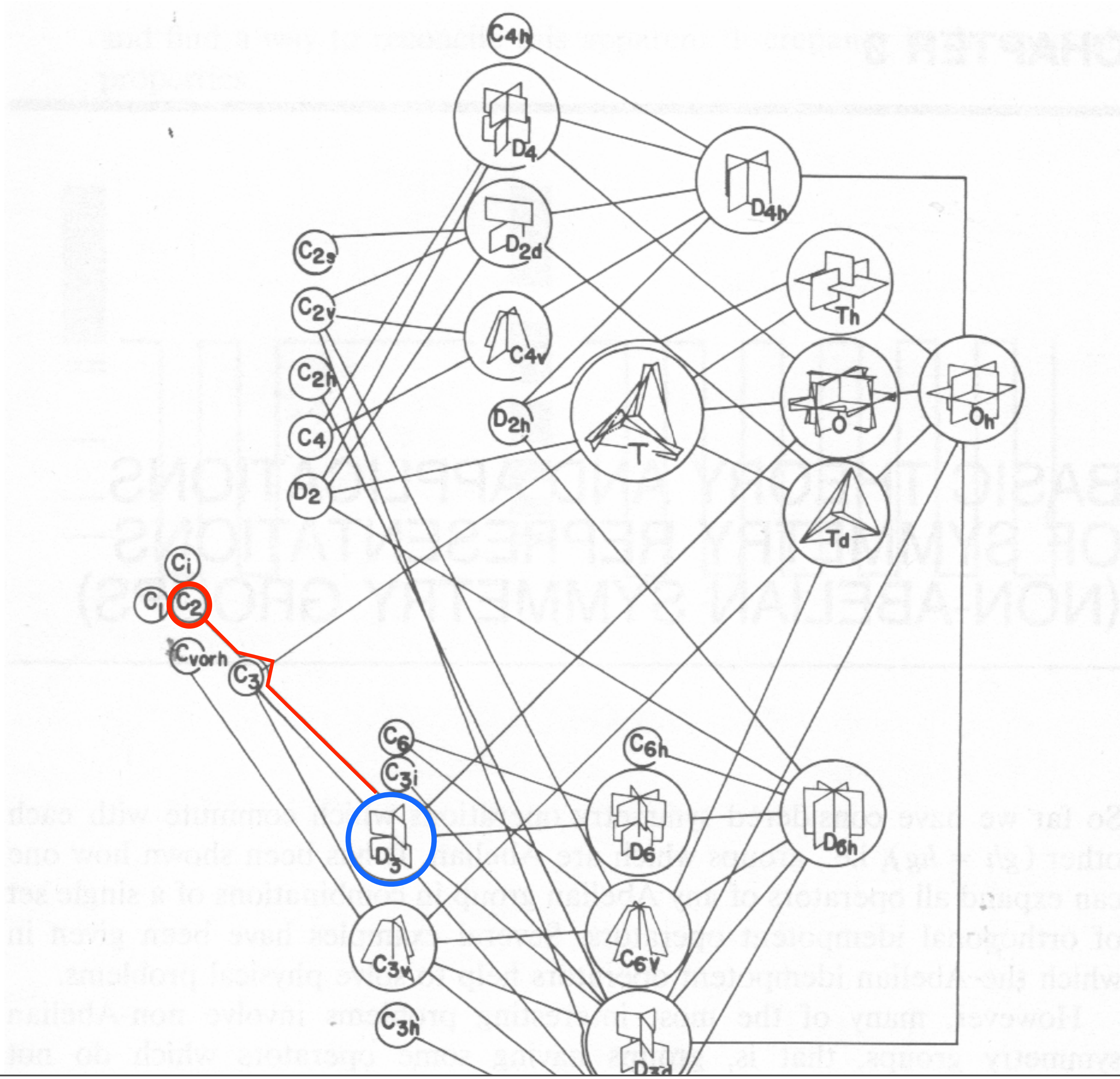
# Spectral reduction of non-commutative "Group-table Hamiltonian"

## *$D_3$ Example*

***2nd Step: Spectral resolution of Class Projector(s) of $D_3$***

***Correlate $D_3$ characters with its subgoup(s) $C_2$(i)***



# Spectral reduction of non-commutative "Group-table Hamiltonian"

## $D_3$ Example

*2nd Step: Spectral resolution of Class Projector(s) of $D_3$*

*Correlate $D_3$ characters with its subgoup(s) $C_2(\mathbf{i})$*

| $D_3$ $\kappa =$ | $\mathbf{1}$ | $\mathbf{r}^1+\mathbf{r}^2$ | $\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$ | |
|---|---|---|---|---|
| $\mathbf{P}^{A_1}=$ | 1 | 1 | 1 | /6 |
| $\mathbf{P}^{A_2}=$ | 1 | 1 | -1 | /6 |
| $\mathbf{P}^{E}=$ | 2 | -1 | 0 | /3 |

| $C_2$ $\kappa =$ | $\mathbf{1}$ | $\mathbf{i}_3$ | |
|---|---|---|---|
| $p^{0_2}=$ | 1 | 1 | /2 |
| $p^{1_2}=$ | 1 | -1 | /2 |

# Spectral reduction of non-commutative "Group-table Hamiltonian"

## $D_3$ Example

### 2nd Step: Spectral resolution of Class Projector(s) of $D_3$

**Correlate $D_3$ characters with its subgoup(s) $C_2(\mathbf{i})$**

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| $C_2$ $\kappa$ = | $\mathbf{1}$ | $\mathbf{i}_3$ | |
|---|---|---|---|
| $p^{0_2}=$ | 1 | 1 | /2 |
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*$D_3 \supset C_2$ Correlation table shows which products of class projector $\mathbf{P}^{(\alpha)}$ with $C_2$-unit $1 = p^{0_2} + p^{1_2}$ will make **IRREDUCIBLE** $\mathbf{P}^{(\alpha)}_{n,n}$ )*

| $D_3 \supset C_2$ | $0_2$ | $1_2$ |
|---|---|---|
| $n^{A_1}=$ | 1 | · |
| $n^{A_2}=$ | · | 1 |
| $n^{E}=$ | 1 | 1 |

# Spectral reduction of non-commutative "Group-table Hamiltonian"

## $D_3$ Example

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| $D_3 \supset C_2$ | $0_2$ | $1_2$ |
|---|---|---|
| $n^{A_1}=$ | 1 | · |
| $n^{A_2}=$ | · | 1 |
| $n^{E}=$ | 1 | 1 |

*Rank* $\rho(D_3)$=4 implies there will be exactly 4 "$C_2$-friendly" irep projectors

$$\mathbf{P}^{(\alpha)} 1 = \mathbf{P}^{(\alpha)}(p^{0_2} + p^{1_2})$$
$$= \mathbf{P}^{(\alpha)}_{0_2 0_2} + \mathbf{P}^{(\alpha)}_{1_2 1_2}$$

# Spectral reduction of non-commutative "Group-table Hamiltonian"

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| $C_2$ $\kappa=$ | $\mathbf{1}$ | $\mathbf{i}_3$ | |
|---|---|---|---|
| $p^{0_2}=$ | 1 | 1 | /2 |
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| $D_3 \supset C_2$ | $0_2$ | $1_2$ |
|---|---|---|
| $n^{A_1}=$ | 1 | · |
| $n^{A_2}=$ | · | 1 |
| $n^{E}=$ | 1 | 1 |

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$$\mathbf{P}^{(\alpha)} 1 = \mathbf{P}^{(\alpha)}(p^{0_2} + p^{1_2})$$
$$= \mathbf{P}^{(\alpha)}_{0_2 0_2} + \mathbf{P}^{(\alpha)}_{1_2 1_2}$$

| | $1 = p^{0_2}$ | $+\ p^{1_2}$ |
|---|---|---|
| $\mathbf{P}^{A_1}=$ | $\mathbf{P}^{A_1}_{0_2 0_2}$ | · |
| $\mathbf{P}^{A_2}=$ | · | $\mathbf{P}^{A_2}_{1_2 1_2}$ |
| $\mathbf{P}^{E}=$ | $\mathbf{P}^{E}_{0_2 0_2}$ | $\mathbf{P}^{E}_{1_2 1_2}$ |

# Spectral reduction of non-commutative "Group-table Hamiltonian"

## $D_3$ Example

### 2nd Step: Spectral resolution of Class Projector(s) of $D_3$

**Correlate $D_3$ characters with its subgoup(s) $C_2(\mathbf{i})$**

| $D_3$ $\kappa=$ | $\mathbf{1}$ | $\mathbf{r}^1+\mathbf{r}^2$ | $\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$ | |
|---|---|---|---|---|
| $\mathbf{P}^{A_1}=$ | 1 | 1 | 1 | /6 |
| $\mathbf{P}^{A_2}=$ | 1 | 1 | -1 | /6 |
| $\mathbf{P}^{E}=$ | 2 | -1 | 0 | /3 |

| $C_2$ $\kappa=$ | $\mathbf{1}$ | $\mathbf{i}_3$ | |
|---|---|---|---|
| $p^{0_2}=$ | 1 | 1 | /2 |
| $p^{1_2}=$ | 1 | -1 | /2 |

*$D_3 \supset C_2$ Correlation table shows which products of class projector $\mathbf{P}^{(\alpha)}$ with $C_2$-unit $1 = p^{0_2} + p^{1_2}$ will make **IRREDUCIBLE** $\mathbf{P}^{(\alpha)}_{n,n}$)*

| $D_3 \supset C_2$ | $0_2$ | $1_2$ |
|---|---|---|
| $n^{A_1}=$ | 1 | · |
| $n^{A_2}=$ | · | 1 |
| $n^{E}=$ | 1 | 1 |

*Rank* $\rho(D_3)=4$ implies there will be exactly 4 "$C_2$-friendly" irep projectors

$$\mathbf{P}^{(\alpha)} 1 = \mathbf{P}^{(\alpha)}(p^{0_2} + p^{1_2}) = \mathbf{P}^{(\alpha)}_{0_2 0_2} + \mathbf{P}^{(\alpha)}_{1_2 1_2}$$

| $1 =$ | $p^{0_2}$ + | $p^{1_2}$ |
|---|---|---|
| $\mathbf{P}^{A_1}=$ | $\mathbf{P}^{A_1}_{0_2 0_2}$ | · |
| $\mathbf{P}^{A_2}=$ | · | $\mathbf{P}^{A_2}_{1_2 1_2}$ |
| $\mathbf{P}^{E}=$ | $\mathbf{P}^{E}_{0_2 0_2}$ | $\mathbf{P}^{E}_{1_2 1_2}$ |

$$\mathbf{P}^{A_1} = \mathbf{P}^{A_1} p^{0_2} = \mathbf{P}^{A_1}(1+\mathbf{i}_3)/2 = (\ 1+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2} = \mathbf{P}^{A_2} p^{1_2} = \mathbf{P}^{A_2}(1-\mathbf{i}_3)/2 = (\ 1+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$$

$$\mathbf{P}^{E} = \mathbf{P}^{E} p^{0_2} = \mathbf{P}^{E}(1+\mathbf{i}_3)/2 = (21-\mathbf{r}^1-\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2+2\mathbf{i}_3)/6$$

$$\mathbf{P}^{E} = \mathbf{P}^{E} p^{1_2} = \mathbf{P}^{E}(1-\mathbf{i}_3)/2 = (21-\mathbf{r}^1-\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2-2\mathbf{i}_3)/6$$

*Review: Spectral resolution of **$D_3$ Center** (Class algebra)*

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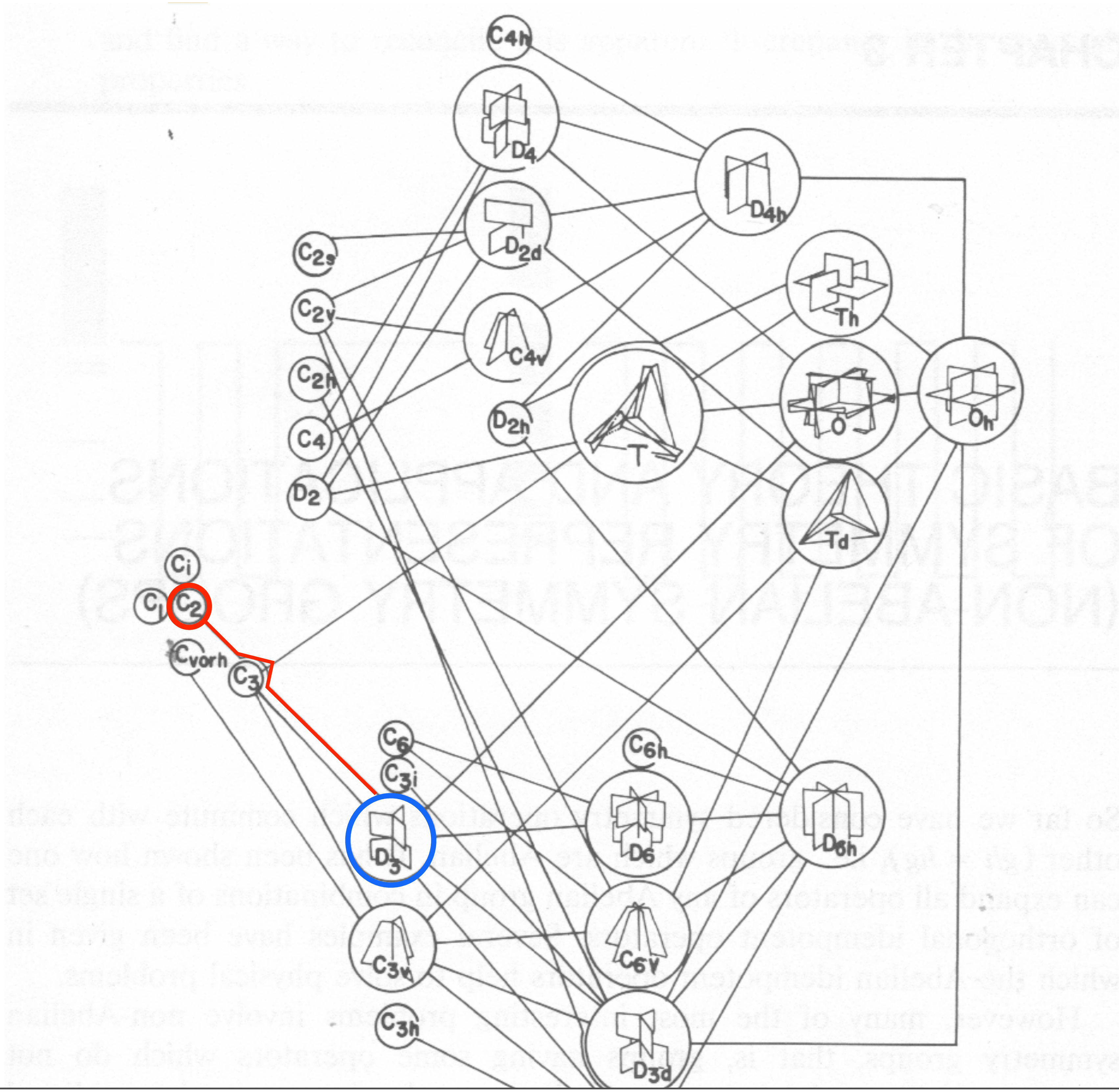
# Spectral reduction of non-commutative "Group-table Hamiltonian"

## *$D_3$ Example*

***2nd Step: Spectral resolution of Class Projector(s) of $D_3$***

***Correlate $D_3$ characters with its subgoup(s) $C_2$(i)***

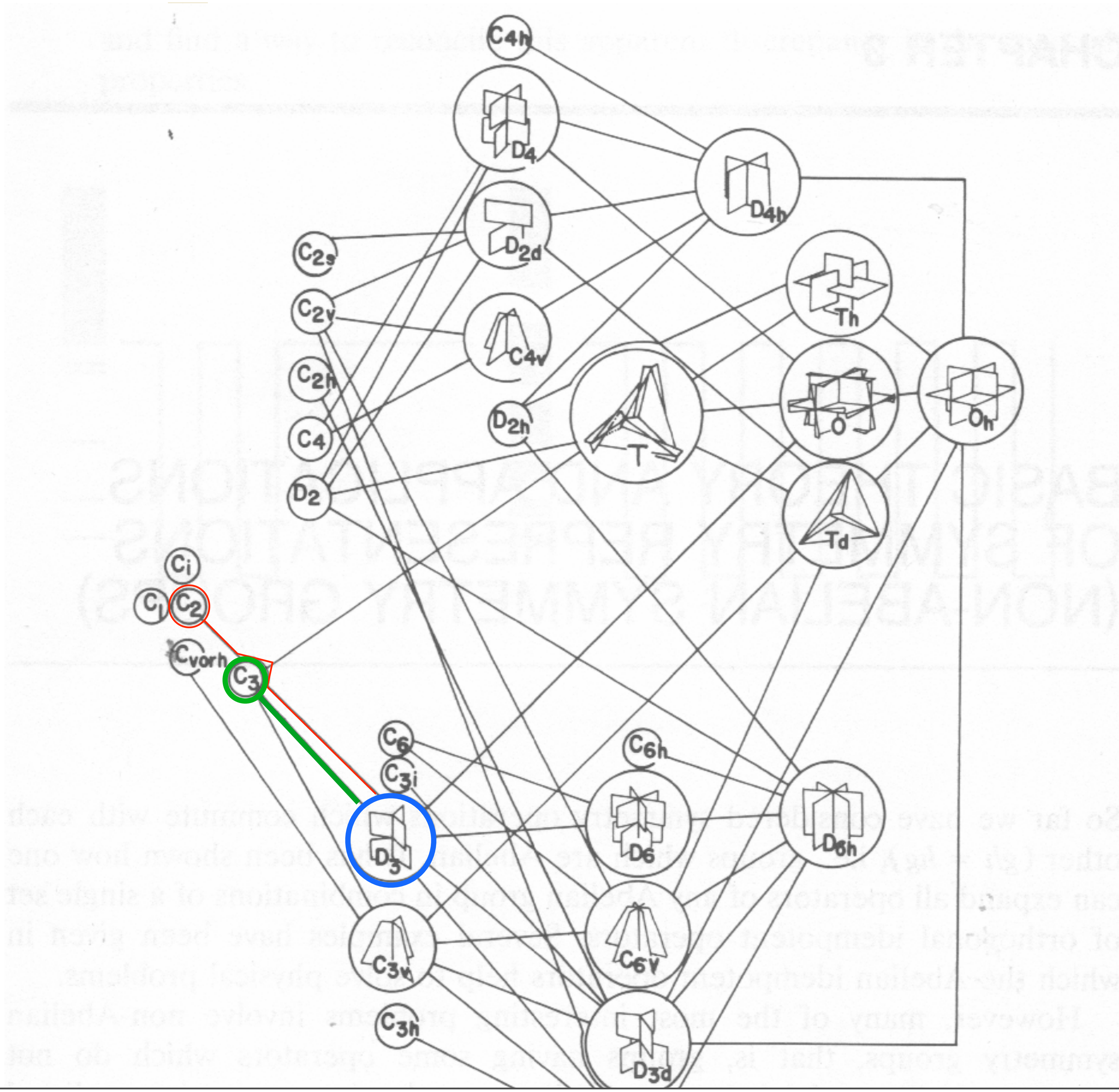


# Spectral reduction of non-commutative "Group-table Hamiltonian"

## *$D_3$ Example*

***2nd Step: Spectral resolution of Class Projector(s) of $D_3$***

***Correlate $D_3$ characters with its subgoup(s) $C_2$(i) or ELSE $C_3$(r)*** ***($C_2$ and $C_3$ don't commute)***



# Spectral reduction of non-commutative "Group-table Hamiltonian"

## $D_3$ Example

### 2nd Step: Spectral resolution of Class Projector(s) of $D_3$

*Correlate $D_3$ characters with its subgoup(s) $C_2(\mathbf{i})$ or ELSE $C_3(\mathbf{r})$ ($C_2$ and $C_3$ don't commute)*

| $D_3$ $\kappa=$ | $\mathbf{1}$ | $\mathbf{r}^1+\mathbf{r}^2$ | $\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$ | |
|---|---|---|---|---|
| $\mathbf{P}^{A_1}=$ | 1 | 1 | 1 | /6 |
| $\mathbf{P}^{A_2}=$ | 1 | 1 | -1 | /6 |
| $\mathbf{P}^{E}=$ | 2 | -1 | 0 | /3 |

| $C_2$ $\kappa=$ | $\mathbf{1}$ | $\mathbf{i}_3$ | |
|---|---|---|---|
| $p^{0_2}=$ | 1 | 1 | /2 |
| $p^{1_2}=$ | 1 | -1 | /2 |

Let:

$\varepsilon=e^{-2\pi i/3}$

| $C_3$ $\kappa=$ | $\mathbf{1}$ | $\mathbf{r}^1$ | $\mathbf{r}^2$ | |
|---|---|---|---|---|
| $p^{0_3}=$ | 1 | 1 | 1 | /3 |
| $p^{1_3}=$ | 1 | $\varepsilon$ | $\varepsilon^*$ | /3 |
| $p^{2_3}=$ | 2 | $\varepsilon^*$ | $\varepsilon$ | /3 |

*$D_3 \supset C_2$ Correlation table shows which products of class projector $\mathbf{P}^{(\alpha)}$ with $C_2$-unit $1 = p^{0_2} + p^{1_2}$ will make IRREDUCIBLE $\mathbf{P}^{(\alpha)}_{n,n}$ )*

| $D_3 \supset C_2$ | $0_2$ | $1_2$ |
|---|---|---|
| $n^{A_1}=$ | 1 | · |
| $n^{A_2}=$ | · | 1 |
| $n^{E}=$ | 1 | 1 |

*Rank* $\rho(D_3)=4$ implies there will be exactly 4 "$C_2$-friendly" irep projectors

$$\mathbf{P}^{(\alpha)}1 = \mathbf{P}^{(\alpha)}(p^{0_2} + p^{1_2})$$
$$= \mathbf{P}^{(\alpha)}_{0_2 0_2} + \mathbf{P}^{(\alpha)}_{1_2 1_2}$$

$1 = p^{0_2} + p^{1_2}$

| | | |
|---|---|---|
| $\mathbf{P}^{A_1}=$ | $\mathbf{P}^{A_1}_{0_2 0_2}$ | · |
| $\mathbf{P}^{A_2}=$ | · | $\mathbf{P}^{A_2}_{1_2 1_2}$ |
| $\mathbf{P}^{E}=$ | $\mathbf{P}^{E}_{0_2 0_2}$ | $\mathbf{P}^{E}_{1_2 1_2}$ |

$$\mathbf{P}^{A_1}=\mathbf{P}^{A_1}p^{0_2} = \mathbf{P}^{A_1}(1+\mathbf{i}_3)/2 = (\ 1+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2}=\mathbf{P}^{A_2}p^{1_2} = \mathbf{P}^{A_2}(1-\mathbf{i}_3)/2 = (\ 1+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$$

$$\mathbf{P}^{E} = \mathbf{P}^{E}p^{0_2} = \mathbf{P}^{E}(1+\mathbf{i}_3)/2 = (21-\mathbf{r}^1-\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2+2\mathbf{i}_3)/6$$

$$\mathbf{P}^{E} = \mathbf{P}^{E}p^{1_2} = \mathbf{P}^{E}(1-\mathbf{i}_3)/2 = (21-\mathbf{r}^1-\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2-2\mathbf{i}_3)/6$$

# Spectral reduction of non-commutative "Group-table Hamiltonian"

## $D_3$ Example

**2nd Step: Spectral resolution of Class Projector(s) of $D_3$**

**Correlate $D_3$ characters with its subgoup(s) $C_2(\mathbf{i})$ or ELSE $C_3(\mathbf{r})$ ($C_2$ and $C_3$ don't commute)**

| $D_3$ $\kappa=$ | $\mathbf{1}$ | $\mathbf{r}^1+\mathbf{r}^2$ | $\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$ | |
|---|---|---|---|---|
| $\mathbf{P}^{A_1}=$ | 1 | 1 | 1 | /6 |
| $\mathbf{P}^{A_2}=$ | 1 | 1 | -1 | /6 |
| $\mathbf{P}^{E}=$ | 2 | -1 | 0 | /3 |

| $C_2$ $\kappa=$ | $\mathbf{1}$ | $\mathbf{i}_3$ | |
|---|---|---|---|
| $p^{0_2}=$ | 1 | 1 | /2 |
| $p^{1_2}=$ | 1 | -1 | /2 |

Let: $\varepsilon=e^{-2\pi i/3}$

| $C_3$ $\kappa=$ | $\mathbf{1}$ | $\mathbf{r}^1$ | $\mathbf{r}^2$ | |
|---|---|---|---|---|
| $p^{0_3}=$ | 1 | 1 | 1 | /3 |
| $p^{1_3}=$ | 1 | $\varepsilon$ | $\varepsilon^*$ | /3 |
| $p^{2_3}=$ | 2 | $\varepsilon^*$ | $\varepsilon$ | /3 |

*$D_3 \supset C_2$ Correlation table shows which products of class projector $\mathbf{P}^{(\alpha)}$ with $C_2$-unit $1=p^{0_2}+p^{1_2}$ will make IRREDUCIBLE $\mathbf{P}^{(\alpha)}_{n,n}$)*

| $D_3 \supset C_2$ | $0_2$ | $1_2$ |
|---|---|---|
| $n^{A_1}=$ | 1 | · |
| $n^{A_2}=$ | · | 1 |
| $n^{E}=$ | 1 | 1 |

**Same for** *Correlation table:*

| $D_3 \supset C_3$ | $0_3$ | $1_3$ | $2_3$ |
|---|---|---|---|
| $n^{A_1}=$ | 1 | · | · |
| $n^{A_2}=$ | 1 | · | · |
| $n^{E}=$ | · | 1 | 1 |

*Rank* $\rho(D_3)=4$ implies there will be exactly 4 "$C_2$-friendly" irep projectors

$$\mathbf{P}^{(\alpha)}1=\mathbf{P}^{(\alpha)}(p^{0_2}+p^{1_2})$$
$$=\mathbf{P}^{(\alpha)}_{0_2 0_2}+\mathbf{P}^{(\alpha)}_{1_2 1_2}$$

$1=p^{0_2}+p^{1_2}$

| | | |
|---|---|---|
| $\mathbf{P}^{A_1}=$ | $\mathbf{P}^{A_1}_{0_2 0_2}$ | · |
| $\mathbf{P}^{A_2}=$ | · | $\mathbf{P}^{A_2}_{1_2 1_2}$ |
| $\mathbf{P}^{E}=$ | $\mathbf{P}^{E}_{0_2 0_2}$ | $\mathbf{P}^{E}_{1_2 1_2}$ |

$$\mathbf{P}^{A_1}=\mathbf{P}^{A_1}p^{0_2}=\mathbf{P}^{A_1}(1+\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$
$$\mathbf{P}^{A_2}=\mathbf{P}^{A_2}p^{1_2}=\mathbf{P}^{A_2}(1-\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$$
$$\mathbf{P}^{E}=\mathbf{P}^{E}p^{0_2}=\mathbf{P}^{E}(1+\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2+2\mathbf{i}_3)/6$$
$$\mathbf{P}^{E}=\mathbf{P}^{E}p^{1_2}=\mathbf{P}^{E}(1-\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2-2\mathbf{i}_3)/6$$

# Spectral reduction of non-commutative "Group-table Hamiltonian"

## $D_3$ Example

### 2nd Step: Spectral resolution of Class Projector(s) of $D_3$

**Correlate $D_3$ characters with its subgoup(s) $C_2(\mathbf{i})$ or ELSE $C_3(\mathbf{r})$ ($C_2$ and $C_3$ don't commute)**

| $D_3$ | $\kappa=\mathbf{1}$ | $\mathbf{r}^1+\mathbf{r}^2$ | $\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$ | |
|---|---|---|---|---|
| $\mathbf{P}^{A_1}=$ | 1 | 1 | 1 | /6 |
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| $\mathbf{P}^{E}=$ | 2 | -1 | 0 | /3 |

| $C_2$ | $\kappa=\mathbf{1}$ | $\mathbf{i}_3$ | |
|---|---|---|---|
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Let: $\varepsilon=e^{-2\pi i/3}$

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|---|---|---|---|---|
| $p^{0_3}=$ | 1 | 1 | 1 | /3 |
| $p^{1_3}=$ | 1 | $\varepsilon$ | $\varepsilon^*$ | /3 |
| $p^{2_3}=$ | 2 | $\varepsilon^*$ | $\varepsilon$ | /3 |

$D_3\supset C_2$ *Correlation table shows which products of class projector* $\mathbf{P}^{(\alpha)}$ *with* $C_2$*-unit* $1=p^{0_2}+p^{1_2}$ *will make* ***IRREDUCIBLE*** $\mathbf{P}^{(\alpha)}_{n,n}$ )

| $D_3\supset C_2$ | $0_2$ | $1_2$ |
|---|---|---|
| $n^{A_1}=$ | 1 | · |
| $n^{A_2}=$ | · | 1 |
| $n^{E}=$ | 1 | 1 |

***Same for*** *Correlation table:*

| $D_3\supset C_3$ | $0_3$ | $1_3$ | $2_3$ |
|---|---|---|---|
| $n^{A_1}=$ | 1 | · | · |
| $n^{A_2}=$ | 1 | · | · |
| $n^{E}=$ | · | 1 | 1 |

*Rank* $\rho(D_3)=4$ implies there will be exactly 4 "$C_2$-friendly" irep projectors

$$\mathbf{P}^{(\alpha)}1=\mathbf{P}^{(\alpha)}(p^{0_2}+p^{1_2})$$
$$=\mathbf{P}^{(\alpha)}_{0_20_2}+\mathbf{P}^{(\alpha)}_{1_21_2}$$

$$1=p^{0_2}+p^{1_2}$$

| | $p^{0_2}$ | $p^{1_2}$ |
|---|---|---|
| $\mathbf{P}^{A_1}=$ | $\mathbf{P}^{A_1}_{0_20_2}$ | · |
| $\mathbf{P}^{A_2}=$ | · | $\mathbf{P}^{A_2}_{1_21_2}$ |
| $\mathbf{P}^{E}=$ | $\mathbf{P}^{E}_{0_20_2}$ | $\mathbf{P}^{E}_{1_21_2}$ |

*Rank* $\rho(D_3)=4$ implies there will be exactly 4 "$C_3$-friendly" irreducible projectors

$$\mathbf{P}^{(\alpha)}1=\mathbf{P}^{(\alpha)}(p^{0_3}+p^{1_3}+p^{2_3})$$
$$=\mathbf{P}^{(\alpha)}_{0_20_2}+\mathbf{P}^{(\alpha)}_{1_31_3}+\mathbf{P}^{(\alpha)}_{2_32_3}$$

$$\mathbf{P}^{A_1}=\mathbf{P}^{A_1}p^{0_2}=\mathbf{P}^{A_1}(1+\mathbf{i}_3)/2=(1+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2}=\mathbf{P}^{A_2}p^{1_2}=\mathbf{P}^{A_2}(1-\mathbf{i}_3)/2=(1+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}=\mathbf{P}^{E}p^{0_2}=\mathbf{P}^{E}(1+\mathbf{i}_3)/2=(21-\mathbf{r}^1-\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2+2\mathbf{i}_3)/6$$

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# Spectral reduction of non-commutative "Group-table Hamiltonian"

## $D_3$ Example

### 2nd Step: Spectral resolution of Class Projector(s) of $D_3$

*Correlate $D_3$ characters with its subgoup(s) $C_2(\mathbf{i})$ or ELSE $C_3(\mathbf{r})$ ($C_2$ and $C_3$ don't commute)*

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|---|---|---|---|---|
| $\mathbf{P}^{A_1}=$ | 1 | 1 | 1 | /6 |
| $\mathbf{P}^{A_2}=$ | 1 | 1 | -1 | /6 |
| $\mathbf{P}^{E}=$ | 2 | -1 | 0 | /3 |

| $C_2$ $\kappa=$ | $\mathbf{1}$ | $\mathbf{i}_3$ | |
|---|---|---|---|
| $p^{0_2}=$ | 1 | 1 | /2 |
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Let: $\varepsilon=e^{-2\pi i/3}$

| $C_3$ $\kappa=$ | $\mathbf{1}$ | $\mathbf{r}^1$ | $\mathbf{r}^2$ | |
|---|---|---|---|---|
| $p^{0_3}=$ | 1 | 1 | 1 | /3 |
| $p^{1_3}=$ | 1 | $\varepsilon$ | $\varepsilon^*$ | /3 |
| $p^{2_3}=$ | 2 | $\varepsilon^*$ | $\varepsilon$ | /3 |

*$D_3\supset C_2$ Correlation table shows which products of class projector $\mathbf{P}^{(\alpha)}$ with $C_2$-unit $1=p^{0_2}+p^{1_2}$ will make IRREDUCIBLE $\mathbf{P}^{(\alpha)}_{n,n}$)*

| $D_3\supset C_2$ | $0_2$ | $1_2$ |
|---|---|---|
| $n^{A_1}=$ | 1 | · |
| $n^{A_2}=$ | · | 1 |
| $n^{E}=$ | 1 | 1 |

*Same for* *Correlation table:*

| $D_3\supset C_3$ | $0_3$ | $1_3$ | $2_3$ |
|---|---|---|---|
| $n^{A_1}=$ | 1 | · | · |
| $n^{A_2}=$ | 1 | · | · |
| $n^{E}=$ | · | 1 | 1 |

*Rank* $\rho(D_3)=4$ implies there will be exactly 4 "$C_2$-friendly" irep projectors

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$$=\mathbf{P}^{(\alpha)}_{0_20_2}+\mathbf{P}^{(\alpha)}_{1_21_2}$$

$1=p^{0_2}+p^{1_2}$

| | | |
|---|---|---|
| $\mathbf{P}^{A_1}=$ | $\mathbf{P}^{A_1}_{0_20_2}$ | · |
| $\mathbf{P}^{A_2}=$ | · | $\mathbf{P}^{A_2}_{1_21_2}$ |
| $\mathbf{P}^{E}=$ | $\mathbf{P}^{E}_{0_20_2}$ | $\mathbf{P}^{E}_{1_21_2}$ |

*Rank* $\rho(D_3)=4$ implies there will be exactly 4 "$C_3$-friendly" irreducible projectors

$$\mathbf{P}^{(\alpha)}1=\mathbf{P}^{(\alpha)}(p^{0_3}+p^{1_3}+p^{2_3})$$
$$=\mathbf{P}^{(\alpha)}_{0_20_2}+\mathbf{P}^{(\alpha)}_{1_31_3}+\mathbf{P}^{(\alpha)}_{2_32_3}$$

$1=p^{0_3}+p^{1_3}+p^{2_3}$

| | | | |
|---|---|---|---|
| $\mathbf{P}^{A_1}=$ | $\mathbf{P}^{A_1}_{0_30_3}$ | · | · |
| $\mathbf{P}^{A_2}=$ | $\mathbf{P}^{A_2}_{0_30_3}$ | · | · |
| $\mathbf{P}^{E}=$ | · | $\mathbf{P}^{E}_{1_31_3}$ | $\mathbf{P}^{E}_{2_32_3}$ |

$$\mathbf{P}^{A_1}=\mathbf{P}^{A_1}p^{0_2}=\mathbf{P}^{A_1}(1+\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$
$$\mathbf{P}^{A_2}=\mathbf{P}^{A_2}p^{1_2}=\mathbf{P}^{A_2}(1-\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$$
$$\mathbf{P}^{E}=\mathbf{P}^{E}p^{0_2}=\mathbf{P}^{E}(1+\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2+2\mathbf{i}_3)/6$$
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# Spectral reduction of non-commutative "Group-table Hamiltonian"

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| $\mathbf{P}^{E}=$ | 2 | -1 | 0 | /3 |

| $C_2$ $\kappa=$ | $\mathbf{1}$ | $\mathbf{i}_3$ | |
|---|---|---|---|
| $p^{0_2}=$ | 1 | 1 | /2 |
| $p^{1_2}=$ | 1 | -1 | /2 |

Let: $\varepsilon=e^{-2\pi i/3}$

| $C_3$ $\kappa=$ | $\mathbf{1}$ | $\mathbf{r}^1$ | $\mathbf{r}^2$ | |
|---|---|---|---|---|
| $p^{0_3}=$ | 1 | 1 | 1 | /3 |
| $p^{1_3}=$ | 1 | $\varepsilon$ | $\varepsilon^*$ | /3 |
| $p^{2_3}=$ | 2 | $\varepsilon^*$ | $\varepsilon$ | /3 |

*$D_3\supset C_2$ Correlation table shows which products of class projector $\mathbf{P}^{(\alpha)}$ with $C_2$-unit $1=p^{0_2}+p^{1_2}$ will make IRREDUCIBLE $\mathbf{P}^{(\alpha)}_{n,n}$)*

| $D_3\supset C_2$ | $0_2$ | $1_2$ |
|---|---|---|
| $n^{A_1}=$ | 1 | · |
| $n^{A_2}=$ | · | 1 |
| $n^{E}=$ | 1 | 1 |

*Same for Correlation table:*

| $D_3\supset C_3$ | $0_3$ | $1_3$ | $2_3$ |
|---|---|---|---|
| $n^{A_1}=$ | 1 | · | · |
| $n^{A_2}=$ | 1 | · | · |
| $n^{E}=$ | · | 1 | 1 |

*Rank* $\rho(D_3)=4$ implies there will be exactly 4 "$C_2$-friendly" irep projectors

$$\mathbf{P}^{(\alpha)}1=\mathbf{P}^{(\alpha)}(p^{0_2}+p^{1_2})=\mathbf{P}^{(\alpha)}_{0_20_2}+\mathbf{P}^{(\alpha)}_{1_21_2}$$

$1=p^{0_2}+p^{1_2}$

| | | |
|---|---|---|
| $\mathbf{P}^{A_1}=$ | $\mathbf{P}^{A_1}_{0_20_2}$ | · |
| $\mathbf{P}^{A_2}=$ | · | $\mathbf{P}^{A_2}_{1_21_2}$ |
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$$\mathbf{P}^{(\alpha)}1=\mathbf{P}^{(\alpha)}(p^{0_3}+p^{1_3}+p^{2_3})=\mathbf{P}^{(\alpha)}_{0_20_2}+\mathbf{P}^{(\alpha)}_{1_31_3}+\mathbf{P}^{(\alpha)}_{2_32_3}$$

$1=p^{0_3}+p^{1_3}+p^{2_3}$

| | | | |
|---|---|---|---|
| $\mathbf{P}^{A_1}=$ | $\mathbf{P}^{A_1}_{0_30_3}$ | · | · |
| $\mathbf{P}^{A_2}=$ | $\mathbf{P}^{A_2}_{0_30_3}$ | · | · |
| $\mathbf{P}^{E}=$ | · | $\mathbf{P}^{E}_{1_31_3}$ | $\mathbf{P}^{E}_{2_32_3}$ |

$$\mathbf{P}^{A_1}=\mathbf{P}^{A_1}p^{0_2}=\mathbf{P}^{A_1}(1+\mathbf{i}_3)/2=(1+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2}=\mathbf{P}^{A_2}p^{1_2}=\mathbf{P}^{A_2}(1-\mathbf{i}_3)/2=(1+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}=\mathbf{P}^{E}p^{0_2}=\mathbf{P}^{E}(1+\mathbf{i}_3)/2=(21-\mathbf{r}^1-\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2+2\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}=\mathbf{P}^{E}p^{1_2}=\mathbf{P}^{E}(1-\mathbf{i}_3)/2=(21-\mathbf{r}^1-\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2-2\mathbf{i}_3)/6$$

$$\mathbf{P}^{A_1}_{0_30_3}=\mathbf{P}^{A_1}p^{0_3}=\mathbf{P}^{A_1}(1+\mathbf{r}^1+\mathbf{r}^2)/3=(1+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2}_{0_30_3}=\mathbf{P}^{A_2}p^{0_3}=\mathbf{P}^{A_2}(1+\mathbf{r}^1+\mathbf{r}^2)/3=(1+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{1_31_3}=\mathbf{P}^{E}p^{1_3}=\mathbf{P}^{E}(1+\varepsilon^*\mathbf{r}^1+\varepsilon\mathbf{r}^2)/3=(1+\varepsilon\mathbf{r}^1+\varepsilon^*\mathbf{r}^2)/6$$

$$\mathbf{P}^{E}_{2_32_3}=\mathbf{P}^{E}p^{2_3}=\mathbf{P}^{E}(1+\varepsilon\mathbf{r}^1+\varepsilon^*\mathbf{r}^2)/3=(1+\varepsilon^*\mathbf{r}^1+\varepsilon\mathbf{r}^2)/6$$

*Review: Spectral resolution of* ***$D_3$ Center*** *(Class algebra)*

*Group theory of equivalence transformations and classes*

*Lagrange theorems*

*All-commuting class projectors and $D_3$-invariant character ortho-completeness*

*Subgroup splitting and correlation frequency formula: $f^{(a)}(D^{(\alpha)}(G)\downarrow H)$*

*Group invariant numbers: Centrum, Rank, and Order*

*2nd-Stage spectral decompositions of global/local $D_3$*

*Splitting class projectors using subgroup chains $D_3 \supset C_2$ and $D_3 \supset C_3$*

*Splitting classes*

*3rd-stage spectral resolution to* ***irreducible representations*** *(ireps) and Hamiltonian eigensolutions*

*Tunneling modes and spectra for $D_3 \supset C_2$ and $D_3 \supset C_3$ local subgroup chains*

## 2nd Step: (contd.) While some class projectors $\mathbf{P}^{(\alpha)}$ split in two, so ALSO DO some classes $\kappa_k$

*Rank* $\rho(D_3)=4$ idempotents $\mathbf{P}^{(\alpha)}$

4 *different* idempotent $\mathbf{P}^{(\alpha)}_{n,n}$

*Centrum* $\kappa(D_3)=3$ idempotents $\mathbf{P}^{(\alpha)}$

| $D_3$ $\kappa$ = | $\mathbf{1}$ | $\mathbf{r}^1+\mathbf{r}^2$ | $\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$ | |
|---|---|---|---|---|
| $\mathbf{P}^{A_1}=$ | 1 | 1 | 1 | /6 |
| $\mathbf{P}^{A_2}=$ | 1 | 1 | -1 | /6 |
| $\mathbf{P}^{E}=$ | 2 | -1 | 0 | /3 |

$$\mathbf{P}^{A_1}_{0_20_2}=\mathbf{P}^{A_1}p^{0_2}=\mathbf{P}^{A_1}(1+\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2}_{1_21_2}=\mathbf{P}^{A_2}p^{1_2}=\mathbf{P}^{A_2}(1-\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{0_20_2}=\mathbf{P}^{E}p^{0_2}=\mathbf{P}^{E}(1+\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2+2\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{1_21_2}=\mathbf{P}^{E}p^{1_2}=\mathbf{P}^{E}(1-\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2-2\mathbf{i}_3)/6$$

$\mathbf{P}^E$ *splits into* $\mathbf{P}^E=\mathbf{P}^E_{0_20_2}+\mathbf{P}^E_{1_21_2}$,

*class* $\kappa_{\mathbf{i}}$ *splits into* $\kappa_{\mathbf{i}_{12}}$ *and* $\kappa_{\mathbf{i}_3}$

$$\mathbf{P}^{A_1}_{0_30_3}=\mathbf{P}^{A_1}p^{0_3}=\mathbf{P}^{A_1}(1+\mathbf{r}^1+\mathbf{r}^2)/3=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2}_{0_30_3}=\mathbf{P}^{A_2}p^{0_3}=\mathbf{P}^{A_2}(1+\mathbf{r}^1+\mathbf{r}^2)/3=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{1_31_3}=\mathbf{P}^{E}p^{1_3}=\mathbf{P}^{E}(1+\varepsilon^*\mathbf{r}^1+\varepsilon\mathbf{r}^2)/3=(\mathbf{1}+\varepsilon\mathbf{r}^1+\varepsilon^*\mathbf{r}^2)/6$$

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$\mathbf{P}^E$ *splits into* $\mathbf{P}^E=\mathbf{P}^E_{1_31_3}+\mathbf{P}^E_{2_32_3}$,

*class* $\kappa_{\mathbf{r}}$ *splits into* $\kappa_{\mathbf{r}^1}$ *and* $\kappa_{\mathbf{r}^2}$

## 2nd Step: (contd.) While some class projectors $\mathbf{P}^{(\alpha)}$ split in two, so ALSO DO some classes $\kappa_k$

| $D_3$ $\kappa$ = | $\mathbf{1}$ | $\mathbf{r}^1+\mathbf{r}^2$ | $\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$ | |
|---|---|---|---|---|
| $\mathbf{P}^{A_1}$ = | 1 | 1 | 1 | /6 |
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| $\mathbf{P}^{E}$ = | 2 | -1 | 0 | /3 |

*Rank* $\rho(D_3)=4$ idempotents $\mathbf{P}^{(\alpha)}$

4 different idempotent $\mathbf{P}^{(\alpha)}_{n,n}$

*Centrum* $\kappa(D_3)=3$ idempotents $\mathbf{P}^{(\alpha)}$

$$\mathbf{P}^{A_1}_{0_2 0_2}=\mathbf{P}^{A_1}p^{0_2}=\mathbf{P}^{A_1}(1+\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2}_{1_2 1_2}=\mathbf{P}^{A_2}p^{1_2}=\mathbf{P}^{A_2}(1-\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{0_2 0_2}=\mathbf{P}^{E}p^{0_2}=\mathbf{P}^{E}(1+\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2+2\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{1_2 1_2}=\mathbf{P}^{E}p^{1_2}=\mathbf{P}^{E}(1-\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2-2\mathbf{i}_3)/6$$

$\mathbf{P}^E$ splits into $\mathbf{P}^E=\mathbf{P}^E_{0_2 0_2}+\mathbf{P}^E_{1_2 1_2}$,

class $\kappa_{\mathbf{i}}$ splits into $\kappa_{\mathbf{i}_{12}}$ and $\kappa_{\mathbf{i}_3}$

$$\mathbf{P}^{A_1}_{0_3 0_3}=\mathbf{P}^{A_1}p^{0_3}=\mathbf{P}^{A_1}(1+\mathbf{r}^1+\mathbf{r}^2)/3=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2}_{0_3 0_3}=\mathbf{P}^{A_2}p^{0_3}=\mathbf{P}^{A_2}(1+\mathbf{r}^1+\mathbf{r}^2)/3=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{1_3 1_3}=\mathbf{P}^{E}p^{1_3}=\mathbf{P}^{E}(1+\varepsilon^*\mathbf{r}^1+\varepsilon\mathbf{r}^2)/3=(\mathbf{1}+\varepsilon\mathbf{r}^1+\varepsilon^*\mathbf{r}^2)/6$$

$$\mathbf{P}^{E}_{2_3 2_3}=\mathbf{P}^{E}p^{2_3}=\mathbf{P}^{E}(1+\varepsilon\mathbf{r}^1+\varepsilon^*\mathbf{r}^2)/3=(\mathbf{1}+\varepsilon^*\mathbf{r}^1+\varepsilon\mathbf{r}^2)/6$$

$\mathbf{P}^E$ splits into $\mathbf{P}^E=\mathbf{P}^E_{1_3 1_3}+\mathbf{P}^E_{2_3 2_3}$,

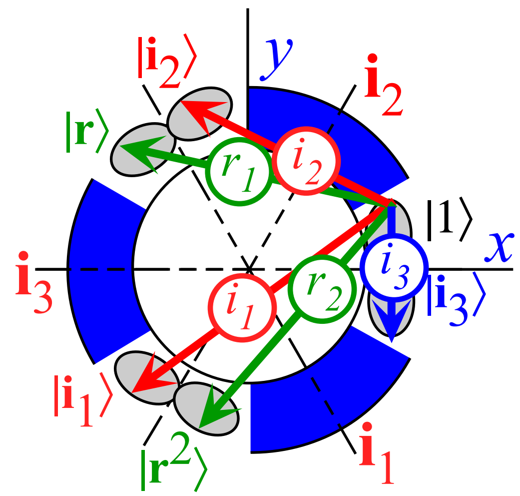
class $\kappa_{\mathbf{r}}$ splits into $\kappa_{\mathbf{r}^1}$ and $\kappa_{\mathbf{r}^2}$

*$r=r_2$ must equal $r_1$*

*$i=i_2$ must equal $i_1$*

***For Local $D_3 \supset C_2(\mathbf{i}_3)$ symmetry***

*$i_3$ is free parameter*




*Rank* $\rho(D_3)$= 4 parameters in either case

*$i=i_1=i_2=i_3$*

***For Local $D_3 \supset C_3(\mathbf{r}^p)$ symmetry***

*$r_1$ and $r_2$ are free*

*Review: Spectral resolution of **$D_3$ Center** (Class algebra)*

*Group theory of equivalence transformations and classes*

*Lagrange theorems*

*All-commuting class projectors and $D_3$-invariant character ortho-completeness*

*Subgroup splitting and correlation frequency formula: $f^{(a)}(D^{(\alpha)}(G)\downarrow H)$*

*Group invariant numbers: Centrum, Rank, and Order*

*2nd-Stage spectral decompositions of global/local $D_3$*

*Splitting class projectors using subgroup chains $D_3\supset C_2$ and $D_3\supset C_3$*

*Splitting classes*

*3rd-stage spectral resolution to **irreducible representations** (ireps) and Hamiltonian eigensolutions*

*Tunneling modes and spectra for $D_3\supset C_2$ and $D_3\supset C_3$ local subgroup chains*

*Centrum* $\kappa(D_3)=3$
idempotents
$\mathbf{P}^{(\alpha)}$

| $D_3$ $\kappa=$ | $\mathbf{1}$ | $\mathbf{r}^1+\mathbf{r}^2$ | $\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$ | |
|---|---|---|---|---|
| $\mathbf{P}^{A_1}=$ | 1 | 1 | 1 | /6 |
| $\mathbf{P}^{A_2}=$ | 1 | 1 | -1 | /6 |
| $\mathbf{P}^{E}=$ | 2 | -1 | 0 | /3 |

*Rank* $\rho(D_3)=4$
idempotents
$\mathbf{P}^{(\alpha)}_{n,n}$

$\mathbf{P}^{A_1}_{x,x}=\mathbf{P}^{A_1}_{0_2 0_2}=\mathbf{P}^{A_1}p^{0_2}=\mathbf{P}^{A_1}(\mathbf{1}+\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$

$\mathbf{P}^{A_2}_{y,y}=\mathbf{P}^{A_2}_{1_2 1_2}=\mathbf{P}^{A_2}p^{1_2}=\mathbf{P}^{A_2}(\mathbf{1}-\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$

$\mathbf{P}^{E}_{x,x}=\mathbf{P}^{E}_{0_2 0_2}=\mathbf{P}^{E}p^{0_2}=\mathbf{P}^{E}(\mathbf{1}+\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2+2\mathbf{i}_3)/6$

$\mathbf{P}^{E}_{y,y}=\mathbf{P}^{E}_{1_2 1_2}=\mathbf{P}^{E}p^{1_2}=\mathbf{P}^{E}(\mathbf{1}-\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2-2\mathbf{i}_3)/6$

# 3rd and Final Step:
## Spectral resolution of ALL 6 of D3 :

*Centrum* $\kappa(D_3)=3$ idempotents $\mathbf{P}^{(\alpha)}$

| $D_3$ | $\kappa = \mathbf{1}$ | $\mathbf{r}^1+\mathbf{r}^2$ | $\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$ | |
|---|---|---|---|---|
| $\mathbf{P}^{A_1}=$ | 1 | 1 | 1 | /6 |
| $\mathbf{P}^{A_2}=$ | 1 | 1 | -1 | /6 |
| $\mathbf{P}^{E}=$ | 2 | -1 | 0 | /3 |

*Rank* $\rho(D_3)=4$ idempotents $\mathbf{P}^{(\alpha)}_{n,n}$

$$\mathbf{P}^{A_1}_{x,x} = \mathbf{P}^{A_1}_{0_2 0_2} = \mathbf{P}^{A_1} p^{0_2} = \mathbf{P}^{A_1}(\mathbf{1}+\mathbf{i}_3)/2 = (\mathbf{1} + \mathbf{r}^1 + \mathbf{r}^2 + \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2}_{y,y} = \mathbf{P}^{A_2}_{1_2 1_2} = \mathbf{P}^{A_2} p^{1_2} = \mathbf{P}^{A_2}(\mathbf{1}-\mathbf{i}_3)/2 = (\mathbf{1} + \mathbf{r}^1 + \mathbf{r}^2 - \mathbf{i}_1 - \mathbf{i}_2 - \mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{x,x} = \mathbf{P}^{E}_{0_2 0_2} = \mathbf{P}^{E} p^{0_2} = \mathbf{P}^{E}(\mathbf{1}+\mathbf{i}_3)/2 = (2\mathbf{1} - \mathbf{r}^1 - \mathbf{r}^2 - \mathbf{i}_1 - \mathbf{i}_2 + 2\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{y,y} = \mathbf{P}^{E}_{1_2 1_2} = \mathbf{P}^{E} p^{1_2} = \mathbf{P}^{E}(\mathbf{1}-\mathbf{i}_3)/2 = (2\mathbf{1} - \mathbf{r}^1 - \mathbf{r}^2 + \mathbf{i}_1 + \mathbf{i}_2 - 2\mathbf{i}_3)/6$$

# *3rd and Final Step:*

## *Spectral resolution of ALL 6 of D3 :*

## The old '**g**-equals-**1**-times-**g**-times-**1**' Trick

$$\mathbf{g} = \mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1} = (\mathbf{P}^{A_1}_{x,x} + \mathbf{P}^{A_2}_{y,y} + \mathbf{P}^{E}_{x,x} + \mathbf{P}^{E}_{y,y})\cdot\mathbf{g}\cdot(\mathbf{P}^{A_1}_{x,x} + \mathbf{P}^{A_2}_{y,y} + \mathbf{P}^{E}_{x,x} + \mathbf{P}^{E}_{y,y})$$

*Centrum* $\kappa(D_3)=3$ idempotents $\mathbf{P}^{(\alpha)}$

$D_3$ $\kappa=$ $\mathbf{1}$ | $\mathbf{r}^1+\mathbf{r}^2$ | $\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$

| | | | | |
|---|---|---|---|---|
| $\mathbf{P}^{A_1}=$ | 1 | 1 | 1 | /6 |
| $\mathbf{P}^{A_2}=$ | 1 | 1 | -1 | /6 |
| $\mathbf{P}^{E}=$ | 2 | -1 | 0 | /3 |

*Rank* $\rho(D_3)=4$ idempotents $\mathbf{P}^{(\alpha)}_{n,n}$

$$\mathbf{P}^{A_1}_{x,x}=\mathbf{P}^{A_1}_{0_2 0_2}=\mathbf{P}^{A_1}\mathbf{p}^{0_2}=\mathbf{P}^{A_1}(\mathbf{1}+\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2}_{y,y}=\mathbf{P}^{A_2}_{1_2 1_2}=\mathbf{P}^{A_2}\mathbf{p}^{1_2}=\mathbf{P}^{A_2}(\mathbf{1}-\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{x,x}=\mathbf{P}^{E}_{0_2 0_2}=\mathbf{P}^{E}\mathbf{p}^{0_2}=\mathbf{P}^{E}(\mathbf{1}+\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2+2\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{y,y}=\mathbf{P}^{E}_{1_2 1_2}=\mathbf{P}^{E}\mathbf{p}^{1_2}=\mathbf{P}^{E}(\mathbf{1}-\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2-2\mathbf{i}_3)/6$$

# *3rd and Final Step:*

## *Spectral resolution of ALL 6 of D3 :*

The old '**g**-equals-**1**-times-**g**-times-**1**' Trick

$$\mathbf{g}=\mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1}=(\mathbf{P}^{A_1}_{x,x}+\mathbf{P}^{A_2}_{y,y}+\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y})\cdot\mathbf{g}\cdot(\mathbf{P}^{A_1}_{x,x}+\mathbf{P}^{A_2}_{y,y}+\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y})$$

$$\mathbf{g}=\mathbf{P}^{A_1}\cdot\mathbf{g}\cdot\mathbf{P}^{A_1}+\mathbf{P}^{A_2}\cdot\mathbf{g}\cdot\mathbf{P}^{A_2}+\mathbf{P}^{E}_{x,x}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{x,x}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{y,y}$$
$$+\mathbf{P}^{E}_{y,y}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{y,y}$$

*Centrum* $\kappa(D_3)=3$
idempotents
$\mathbf{P}^{(\alpha)}$

| $D_3$ $\kappa=$ | $\mathbf{1}$ | $\mathbf{r}^1+\mathbf{r}^2$ | $\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$ | |
|---|---|---|---|---|
| $\mathbf{P}^{A_1}=$ | 1 | 1 | 1 | /6 |
| $\mathbf{P}^{A_2}=$ | 1 | 1 | -1 | /6 |
| $\mathbf{P}^{E}=$ | 2 | -1 | 0 | /3 |

*Rank* $\rho(D_3)=4$
idempotents
$\mathbf{P}^{(\alpha)}_{n,n}$

$\mathbf{P}^{A_1}_{x,x}=\mathbf{P}^{A_1}_{0_2 0_2}=\mathbf{P}^{A_1}p^{0_2}=\mathbf{P}^{A_1}(\mathbf{1}+\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$

$\mathbf{P}^{A_2}_{y,y}=\mathbf{P}^{A_2}_{1_2 1_2}=\mathbf{P}^{A_2}p^{1_2}=\mathbf{P}^{A_2}(\mathbf{1}-\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$

$\mathbf{P}^{E}_{x,x}=\mathbf{P}^{E}_{0_2 0_2}=\mathbf{P}^{E}p^{0_2}=\mathbf{P}^{E}(\mathbf{1}+\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2+2\mathbf{i}_3)/6$

$\mathbf{P}^{E}_{y,y}=\mathbf{P}^{E}_{1_2 1_2}=\mathbf{P}^{E}p^{1_2}=\mathbf{P}^{E}(\mathbf{1}-\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2-2\mathbf{i}_3)/6$

# 3rd and Final Step:
## Spectral resolution of ALL 6 of D3 :

The old ‘g-equals-1-times-g-times-1’ Trick

$$\mathbf{g}=\mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1}=(\mathbf{P}^{A_1}_{x,x}+\mathbf{P}^{A_2}_{y,y}+\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y})\cdot\mathbf{g}\cdot(\mathbf{P}^{A_1}_{x,x}+\mathbf{P}^{A_2}_{y,y}+\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y})$$

$$\mathbf{g}=\mathbf{P}^{A_1}\cdot\mathbf{g}\cdot\mathbf{P}^{A_1}+\mathbf{P}^{A_2}\cdot\mathbf{g}\cdot\mathbf{P}^{A_2}+\mathbf{P}^{E}_{x,x}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{x,x}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{y,y}+\mathbf{P}^{E}_{y,y}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{y,y}$$

*Order* $^{o}(D_3)=6$
projectors
$\mathbf{P}^{(\alpha)}_{m,n}$

*Centrum* $\kappa(D_3)=3$
idempotents
$\mathbf{P}^{(\alpha)}$

| $D_3$ $\kappa=$ | $\mathbf{1}$ | $\mathbf{r}^1+\mathbf{r}^2$ | $\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$ | |
|---|---|---|---|---|
| $\mathbf{P}^{A_1}=$ | 1 | 1 | 1 | /6 |
| $\mathbf{P}^{A_2}=$ | 1 | 1 | -1 | /6 |
| $\mathbf{P}^{E}=$ | 2 | -1 | 0 | /3 |

*Rank* $\rho(D_3)=4$
idempotents
$\mathbf{P}^{(\alpha)}_{n,n}$

$\mathbf{P}^{A_1}_{x,x}=\mathbf{P}^{A_1}_{0_2 0_2}=\mathbf{P}^{A_1}p^{0_2}=\mathbf{P}^{A_1}(1+\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$

$\mathbf{P}^{A_2}_{y,y}=\mathbf{P}^{A_2}_{1_2 1_2}=\mathbf{P}^{A_2}p^{1_2}=\mathbf{P}^{A_2}(1-\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$

$\mathbf{P}^{E}_{x,x}=\mathbf{P}^{E}_{0_2 0_2}=\mathbf{P}^{E}p^{0_2}=\mathbf{P}^{E}(1+\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2+2\mathbf{i}_3)/6$

$\mathbf{P}^{E}_{y,y}=\mathbf{P}^{E}_{1_2 1_2}=\mathbf{P}^{E}p^{1_2}=\mathbf{P}^{E}(1-\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2-2\mathbf{i}_3)/6$

# *3rd and Final Step:*
## *Spectral resolution of ALL 6 of D3 :*

## The old 'g-equals-1-times-g-times-1' Trick

$$\mathbf{g}=\mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1}=(\mathbf{P}^{A_1}_{x,x}+\mathbf{P}^{A_2}_{y,y}+\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y})\cdot\mathbf{g}\cdot(\mathbf{P}^{A_1}_{x,x}+\mathbf{P}^{A_2}_{y,y}+\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y})$$

$$\mathbf{g}=\mathbf{P}^{A_1}\cdot\mathbf{g}\cdot\mathbf{P}^{A_1}+\mathbf{P}^{A_2}\cdot\mathbf{g}\cdot\mathbf{P}^{A_2}+\mathbf{P}^{E}_{x,x}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{x,x}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{y,y}+\mathbf{P}^{E}_{y,y}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{y,y}$$

*Order* $^{o}(D_3)=6$
projectors
$\mathbf{P}^{(\alpha)}_{m,n}$

*Six $D_3$ projectors: 4 idempotents + 2 nilpotents (off-diag.)*

| | $\mathbf{1}$ | $\mathbf{r}^1$ | $\mathbf{r}^2$ | $\mathbf{i}_1$ | $\mathbf{i}_2$ | $\mathbf{i}_3$ | |
|---|---|---|---|---|---|---|---|
| $\mathbf{P}^{A_1}_{x,x}=$ | (1 | 1 | 1 | 1 | 1 | 1) | /6 |
| $\mathbf{P}^{A_2}_{y,y}=$ | (1 | 1 | 1 | -1 | -1 | -1) | /6 |

| | $\mathbf{1}$ | $\mathbf{r}^1$ | $\mathbf{r}^2$ | $\mathbf{i}_1$ | $\mathbf{i}_2$ | $\mathbf{i}_3$ | |
|---|---|---|---|---|---|---|---|
| $\mathbf{P}^{E}_{x,x}=$ | (2 | -1 | -1 | -1 | -1 | +2) | /6 |
| $\mathbf{P}^{E}_{y,x}=$ | (0 | 1 | -1 | -1 | +1 | 0) | /$\sqrt{3}/2$ |

| | $\mathbf{1}$ | $\mathbf{r}^1$ | $\mathbf{r}^2$ | $\mathbf{i}_1$ | $\mathbf{i}_2$ | $\mathbf{i}_3$ | |
|---|---|---|---|---|---|---|---|
| $\mathbf{P}^{E}_{x,y}=$ | (0 | -1 | 1 | -1 | +1 | 0) | /$\sqrt{3}/2$ |
| $\mathbf{P}^{E}_{y,y}=$ | (2 | -1 | -1 | +1 | +1 | -2) | /6 |

*Centrum* $\kappa(D_3)=3$
idempotents
$\mathbf{P}^{(\alpha)}$

| $D_3$ $\kappa=$ | $\mathbf{1}$ | $\mathbf{r}^1+\mathbf{r}^2$ | $\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$ | |
|---|---|---|---|---|
| $\mathbf{P}^{A_1}=$ | 1 | 1 | 1 | /6 |
| $\mathbf{P}^{A_2}=$ | 1 | 1 | -1 | /6 |
| $\mathbf{P}^{E}=$ | 2 | -1 | 0 | /3 |

*Rank* $\rho(D_3)=4$
idempotents
$\mathbf{P}^{(\alpha)}_{n,n}$

$$\mathbf{P}^{A_1}_{x,x}=\mathbf{P}^{A_1}_{0_2 0_2}=\mathbf{P}^{A_1}p^{0_2}=\mathbf{P}^{A_1}(1+\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2}_{y,y}=\mathbf{P}^{A_2}_{1_2 1_2}=\mathbf{P}^{A_2}p^{1_2}=\mathbf{P}^{A_2}(1-\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{x,x}=\mathbf{P}^{E}_{0_2 0_2}=\mathbf{P}^{E}p^{0_2}=\mathbf{P}^{E}(1+\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2+2\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{y,y}=\mathbf{P}^{E}_{1_2 1_2}=\mathbf{P}^{E}p^{1_2}=\mathbf{P}^{E}(1-\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2-2\mathbf{i}_3)/6$$

# 3rd and Final Step:
## Spectral resolution of ALL 6 of D3 :

## The old 'g-equals-1-times-g-times-1' Trick

$$\mathbf{g}=\Sigma_m\Sigma_e\Sigma_b D^{(m)}_{eb}(\mathbf{g})\,\mathbf{P}^{(m)}_{eb}$$

$$\mathbf{P}^{(m)}_{eb}={}_{(norm)}\Sigma_{\mathbf{g}} D^{(m)*}_{eb}(\mathbf{g})\mathbf{g}$$

$$\mathbf{g}=\mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1}=(\mathbf{P}^{A_1}_{x,x}+\mathbf{P}^{A_2}_{y,y}+\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y})\cdot\mathbf{g}\cdot(\mathbf{P}^{A_1}_{x,x}+\mathbf{P}^{A_2}_{y,y}+\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y})$$

$$\mathbf{g}=\mathbf{P}^{A_1}\cdot\mathbf{g}\cdot\mathbf{P}^{A_1}+\mathbf{P}^{A_2}\cdot\mathbf{g}\cdot\mathbf{P}^{A_2}+\mathbf{P}^{E}_{x,x}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{x,x}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{y,y}+\mathbf{P}^{E}_{y,y}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{y,y}$$

*Order* $^{o}(D_3)=6$
projectors
$\mathbf{P}^{(\alpha)}_{m,n}$

*Six $D_3$ projectors: 4 idempotents + 2 nilpotents (off-diag.)*

| | 1 | $\mathbf{r}^1$ | $\mathbf{r}^2$ | $\mathbf{i}_1$ | $\mathbf{i}_2$ | $\mathbf{i}_3$ | |
|---|---|---|---|---|---|---|---|
| $\mathbf{P}^{A_1}_{x,x}=$ | (1 | 1 | 1 | 1 | 1 | 1) | /6 |
| $\mathbf{P}^{A_2}_{y,y}=$ | (1 | 1 | 1 | -1 | -1 | -1) | /6 |

| | 1 | $\mathbf{r}^1$ | $\mathbf{r}^2$ | $\mathbf{i}_1$ | $\mathbf{i}_2$ | $\mathbf{i}_3$ | |
|---|---|---|---|---|---|---|---|
| $\mathbf{P}^{E}_{x,x}=$ | (2 | -1 | -1 | -1 | -1 | +2) | /6 |
| $\mathbf{P}^{E}_{y,x}=$ | (0 | 1 | -1 | -1 | +1 | 0) | /√3/2 |

| | 1 | $\mathbf{r}^1$ | $\mathbf{r}^2$ | $\mathbf{i}_1$ | $\mathbf{i}_2$ | $\mathbf{i}_3$ | |
|---|---|---|---|---|---|---|---|
| $\mathbf{P}^{E}_{x,y}=$ | (0 | -1 | 1 | -1 | +1 | 0) | /√3/2 |
| $\mathbf{P}^{E}_{y,y}=$ | (2 | -1 | -1 | +1 | +1 | -2) | /6 |

# $D_3 > C_2\, \mathbf{i}_3$ projector states

$$\left|{}^{(m)}_{eb}\right\rangle = \mathbf{P}^{(m)}_{eb}\left|1\right\rangle$$

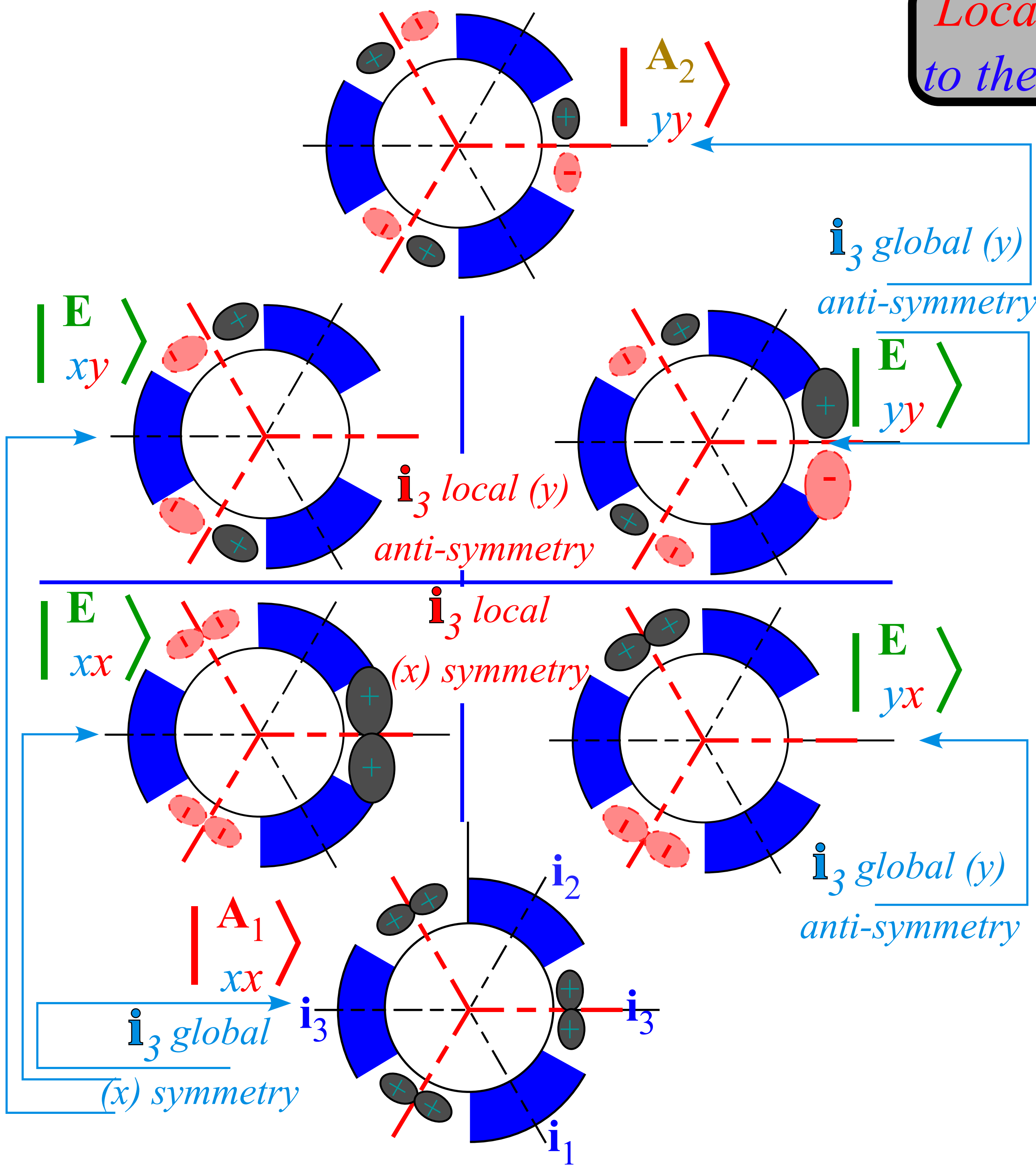
**Global (LAB) symmetry**

$$\mathbf{i}_3\left|{}^{(m)}_{eb}\right\rangle = \mathbf{i}_3\mathbf{P}^{(m)}_{eb}\left|1\right\rangle = (-1)^e\left|{}^{(m)}\right\rangle$$

**Local (BOD) symmetry**

$$\bar{\mathbf{i}}_3\left|{}^{(m)}_{eb}\right\rangle = \bar{\mathbf{i}}_3\mathbf{P}^{(m)}_{eb}\left|1\right\rangle = \mathbf{P}^{(m)}_{eb}\bar{\mathbf{i}}_3\left|1\right\rangle = \mathbf{P}^{(m)}_{eb}\mathbf{i}_3^{\dagger}\left|1\right\rangle = (-1)^b\left|{}^{(m)}\right\rangle$$

Local $\bar{\mathbf{g}}$ commute through to the "inside" to be a $\mathbf{g}^{\dagger}$




| | $\mathbf{1}$ | $\mathbf{r}^1$ | $\mathbf{r}^2$ | $\mathbf{i}_1$ | $\mathbf{i}_2$ | $\mathbf{i}_3$ |
|---|---|---|---|---|---|---|

$$\mathbf{P}^{A_2}_{y,y} = (1\ \ 1\ \ 1\ \ -1\ \ -1\ \ -1)/6$$

$$\mathbf{P}^{E}_{x,y} = (0\ \ -1\ \ 1\ \ -1\ \ +1\ \ 0)/\sqrt{3}/2$$

$$\mathbf{P}^{E}_{y,y} = (2\ \ -1\ \ -1\ \ +1\ \ +1\ \ -2)/6$$

$$\mathbf{P}^{E}_{x,x} = (2\ \ -1\ \ -1\ \ -1\ \ -1\ \ +2)/6$$

$$\mathbf{P}^{E}_{y,x} = (0\ \ 1\ \ -1\ \ -1\ \ +1\ \ 0)/\sqrt{3}/2$$

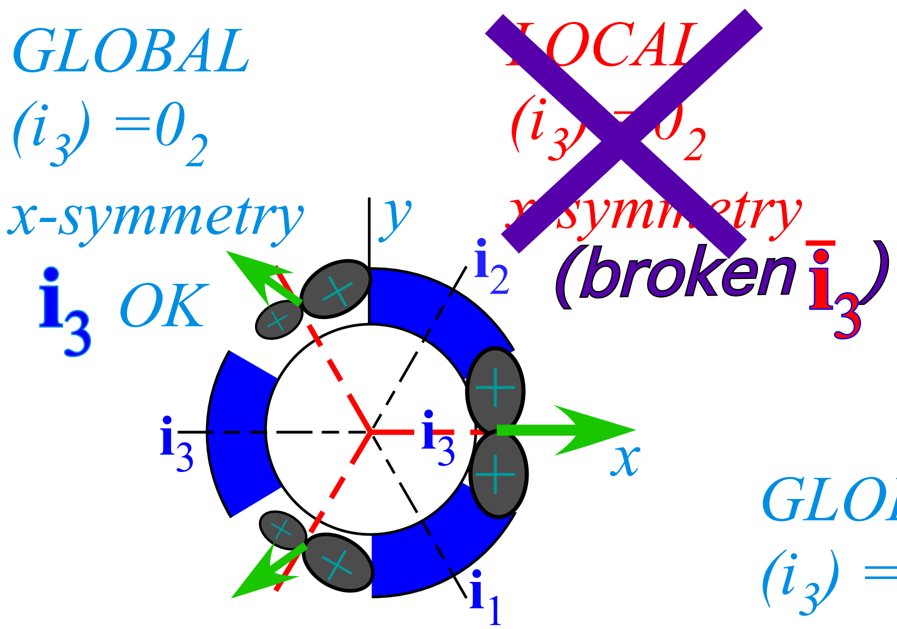
$$\mathbf{P}^{A_1}_{x,x} = (1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1)/6$$

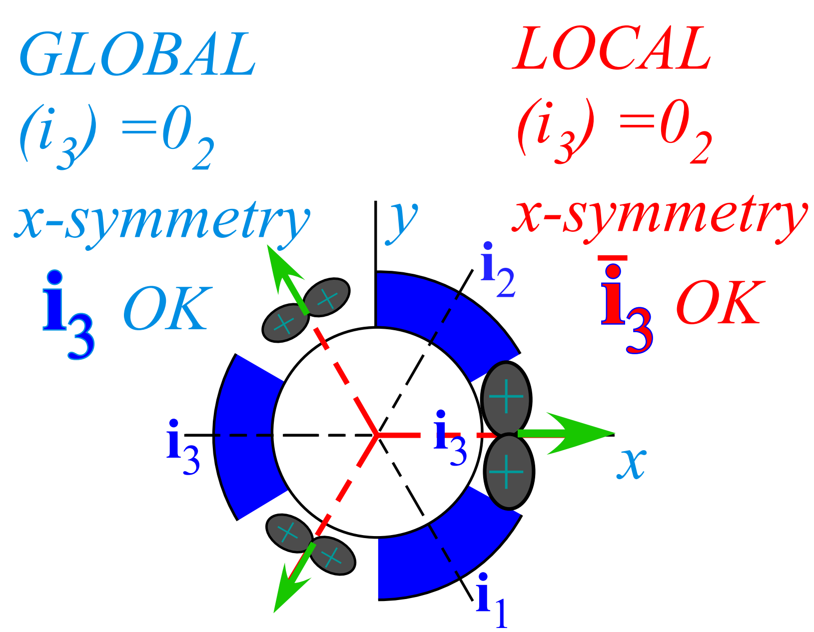
$$|{}^{(m)}_{eb}\rangle = \mathbf{P}^{(m)}_{eb}|\mathbf{1}\rangle$$

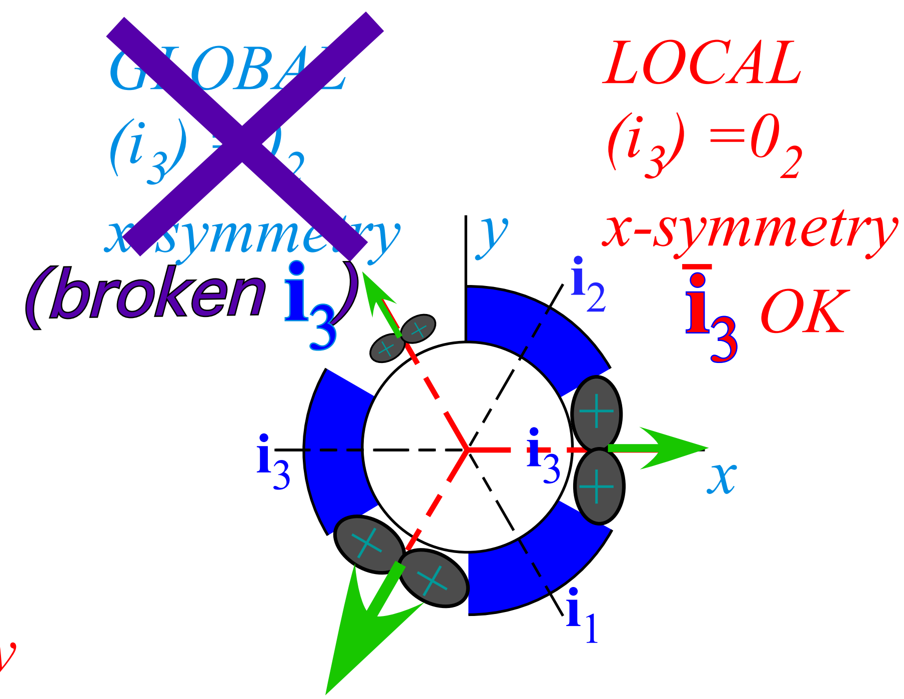
*external LAB* | *internal BOD*

*symmety label-e* | *symmety label-b*

*GLOBAL* | *LOCAL*



$$\mathbf{P}^{(\alpha)}_{mn} = \frac{\ell^{(\alpha)}}{°G} \Sigma_{\mathbf{g}} D^{(\alpha)*}_{mn}(\mathbf{g})\mathbf{g}$$

# *Spectral Efficiency: Same $D(a)_{mn}$ projectors give a lot!*

| | $\mathbf{1}$ | $\mathbf{r}^1$ | $\mathbf{r}^2$ | $\mathbf{i}_1$ | $\mathbf{i}_2$ | $\mathbf{i}_3$ | |
|---|---|---|---|---|---|---|---|
| $\mathbf{P}^{A_1}_{x,x}=$ | (1 | 1 | 1 | 1 | 1 | 1) | /6 |
| $\mathbf{P}^{A_2}_{y,y}=$ | (1 | 1 | 1 | -1 | -1 | -1) | /6 |

| | $\mathbf{1}$ | $\mathbf{r}^1$ | $\mathbf{r}^2$ | $\mathbf{i}_1$ | $\mathbf{i}_2$ | $\mathbf{i}_3$ | |
|---|---|---|---|---|---|---|---|
| $\mathbf{P}^{E}_{x,x}=$ | (2 | -1 | -1 | -1 | -1 | +2) | /6 |
| $\mathbf{P}^{E}_{y,x}=$ | (0 | 1 | -1 | -1 | +1 | 0) | /√3/2 |

| | $\mathbf{1}$ | $\mathbf{r}^1$ | $\mathbf{r}^2$ | $\mathbf{i}_1$ | $\mathbf{i}_2$ | $\mathbf{i}_3$ | |
|---|---|---|---|---|---|---|---|
| $\mathbf{P}^{E}_{x,y}=$ | (0 | -1 | 1 | -1 | +1 | 0) | /√3/2 |
| $\mathbf{P}^{E}_{y,y}=$ | (2 | -1 | -1 | +1 | +1 | -2) | /6 |

- *Eigenstates (shown before)*
- *Complete Hamiltonian*

$$\begin{pmatrix} H+r_1+r_2+i_1+i_2+i_3 & & & \\ & H+r_1+r_2-i_1-i_2-i_3 & & \\ & & H-\frac{1}{2}r_1-\frac{1}{2}r_2-\frac{1}{2}i_1-\frac{1}{2}i_2+i_3 & \frac{\sqrt{3}}{2}(-r_1+r_2-i_1+i_2) \\ & & \frac{\sqrt{3}}{2}(+r_1-r_2-i_1+i_2) & H-\frac{1}{2}r_1-\frac{1}{2}r_2+\frac{1}{2}i_1+\frac{1}{2}i_2-i_3 \end{pmatrix}$$

*$A_1$-block*   *$A_2$-block*

- *Local symmetery eigenvalue formulae* (L.S.=> off-diagonal zero.)

$r_1 = r_2 = -r_1{}^* = r,\quad i_1 = i_2 = -i_1{}^* = i$

*gives:*

| | |
|---|---|
| *$A_1$-level:* | $H + 2r + 2i + i_3$ |
| *$A_1$-level:* | $H + 2r - 2i - i_3$ |
| *$E_x$-level:* | $H - r - i + i_3$ |
| *$E_y$-level:* | $H - r + i - i_3$ |

# $D_3 > C_2\ \mathbf{i}_3$ projector states

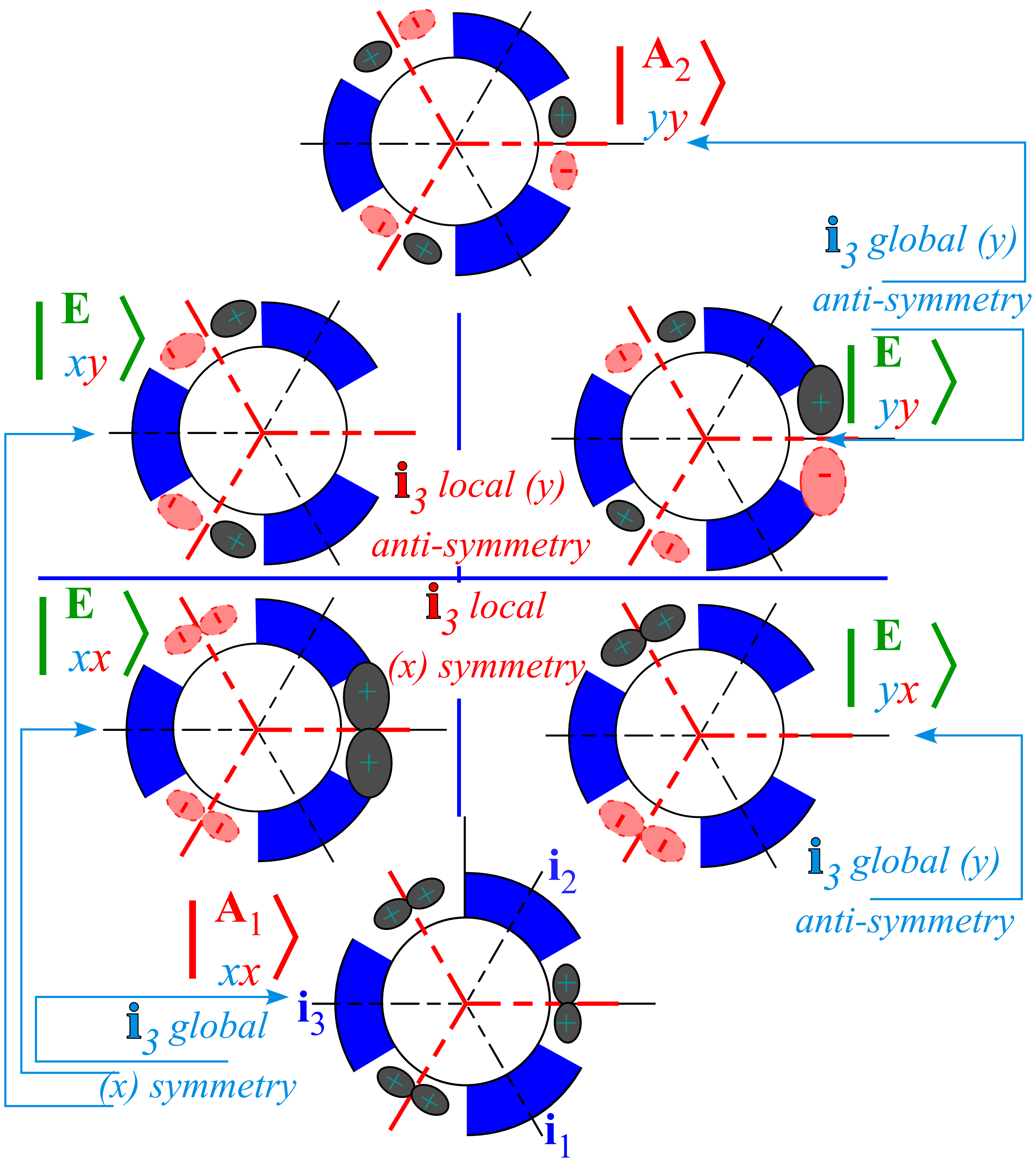
$$\left|{}^{(m)}_{eb}\right\rangle = \mathbf{P}^{(m)}_{eb}\left|1\right\rangle$$

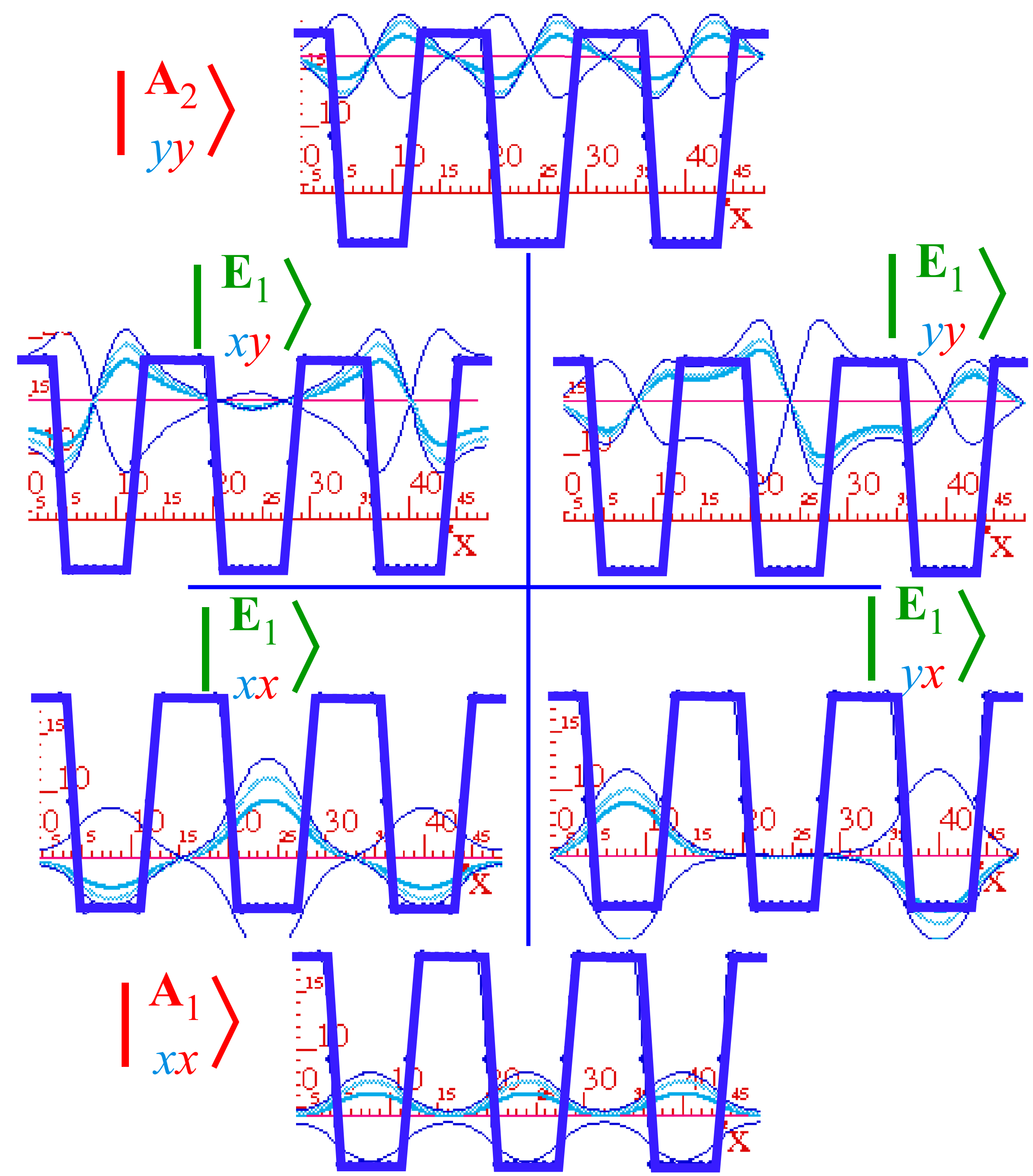
*Global (LAB) symmetry*

$$\mathbf{i}_3\left|{}^{(m)}_{eb}\right\rangle = \mathbf{i}_3\mathbf{P}^{(m)}_{eb}\left|1\right\rangle = (-1)^e\left|{}^{(m)}\right\rangle$$

*Local (BOD) symmetry*

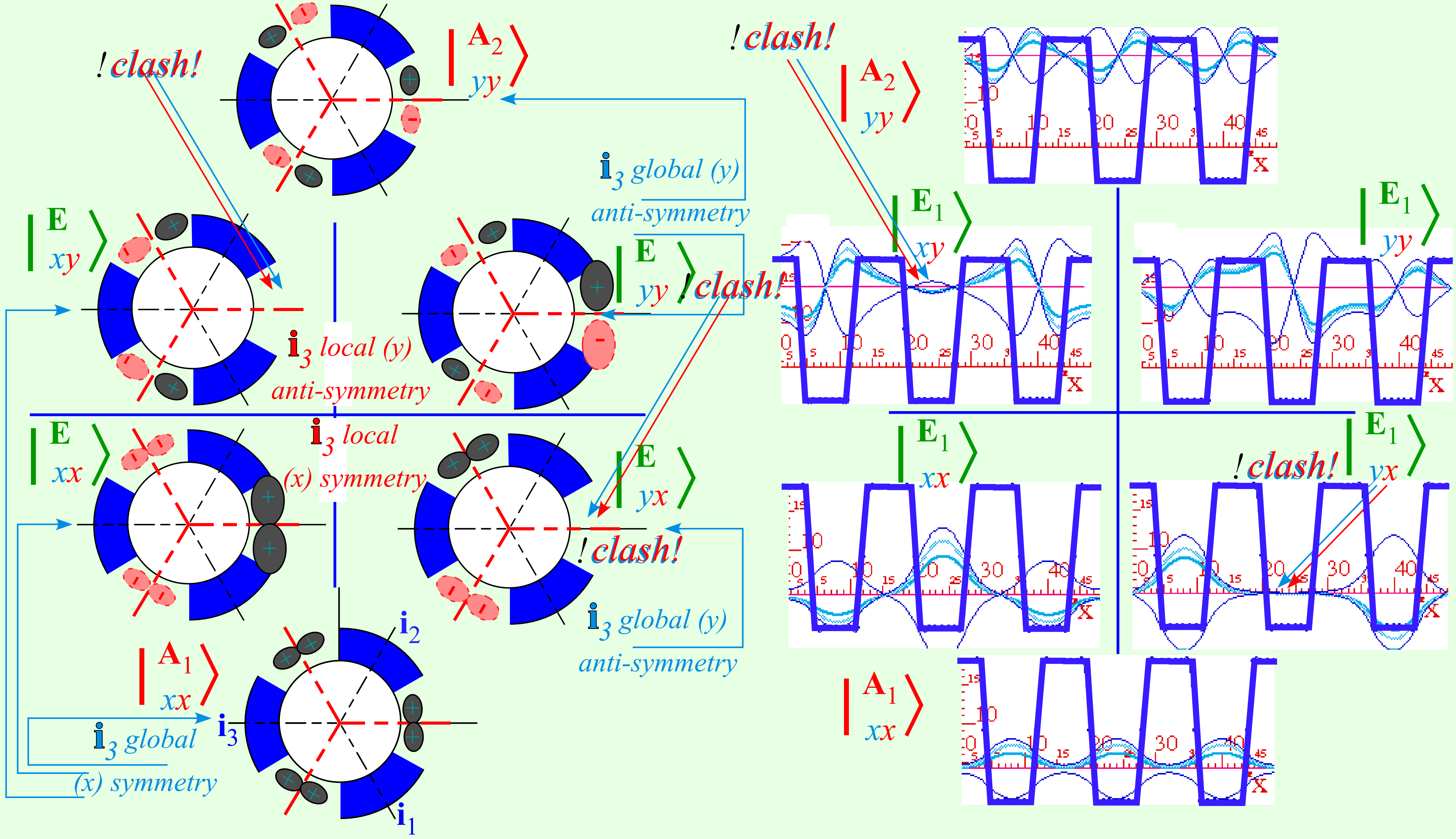
$$\bar{\mathbf{i}}_3\left|{}^{(m)}_{eb}\right\rangle = \bar{\mathbf{i}}_3\mathbf{P}^{(m)}_{eb}\left|1\right\rangle = \mathbf{P}^{(m)}_{eb}\bar{\mathbf{i}}_3\left|1\right\rangle = \mathbf{P}^{(m)}_{eb}\mathbf{i}_3^{\dagger}\left|1\right\rangle = (-1)^b\left|{}^{(m)}\right\rangle$$



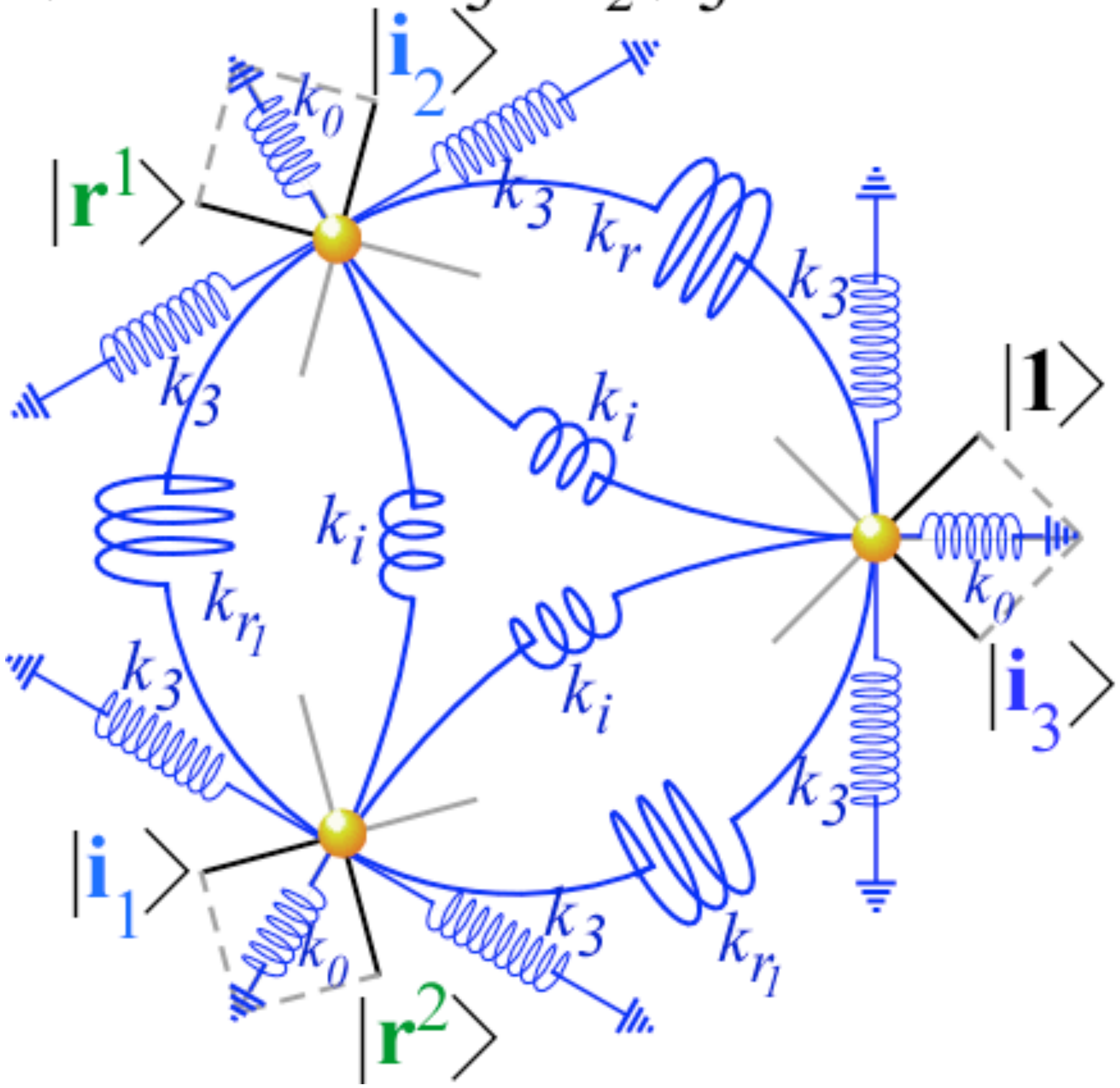



# *When there is no there, there...*

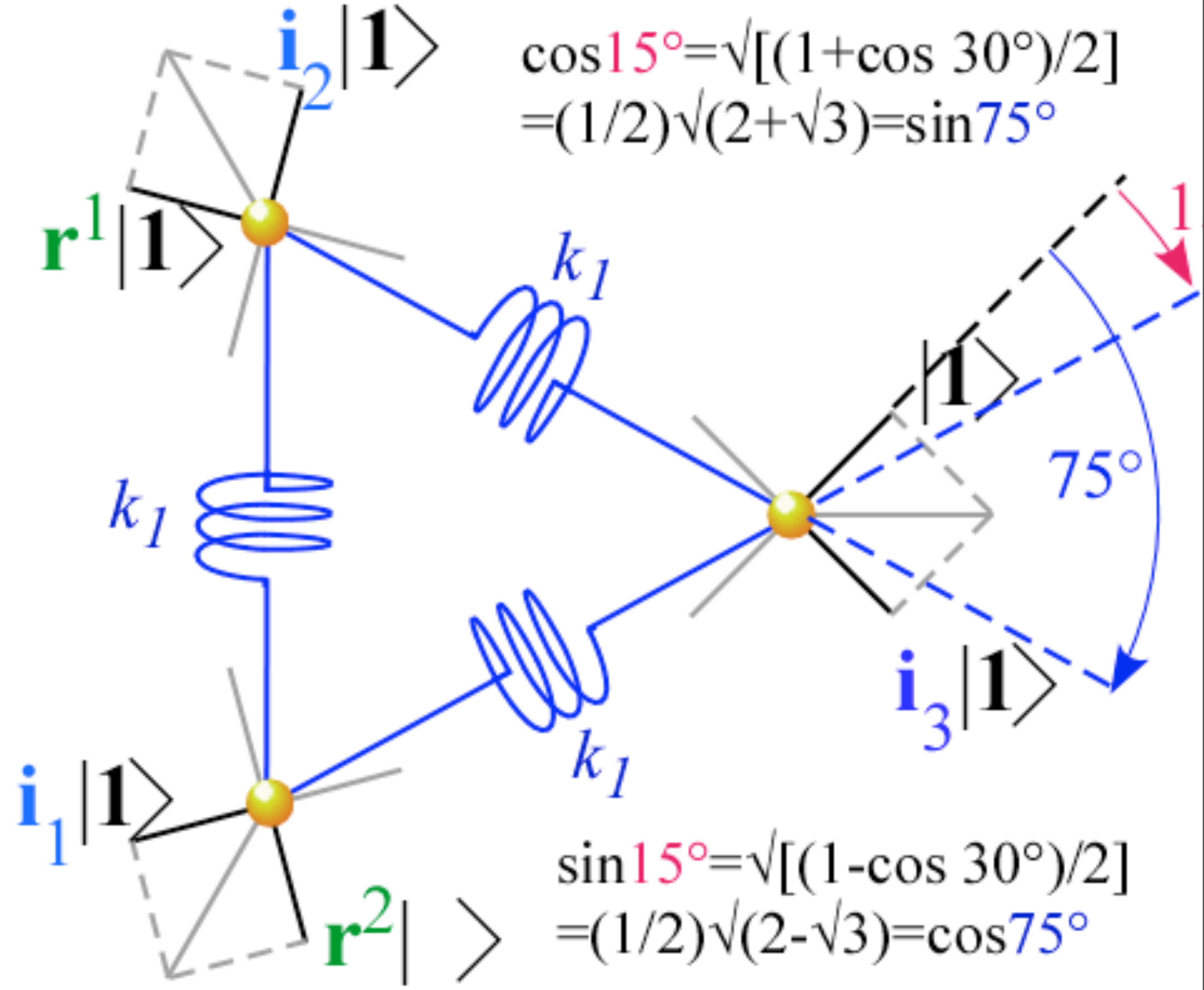
*Nobody Home*
*where LOCAL*
*and GLOBAL*

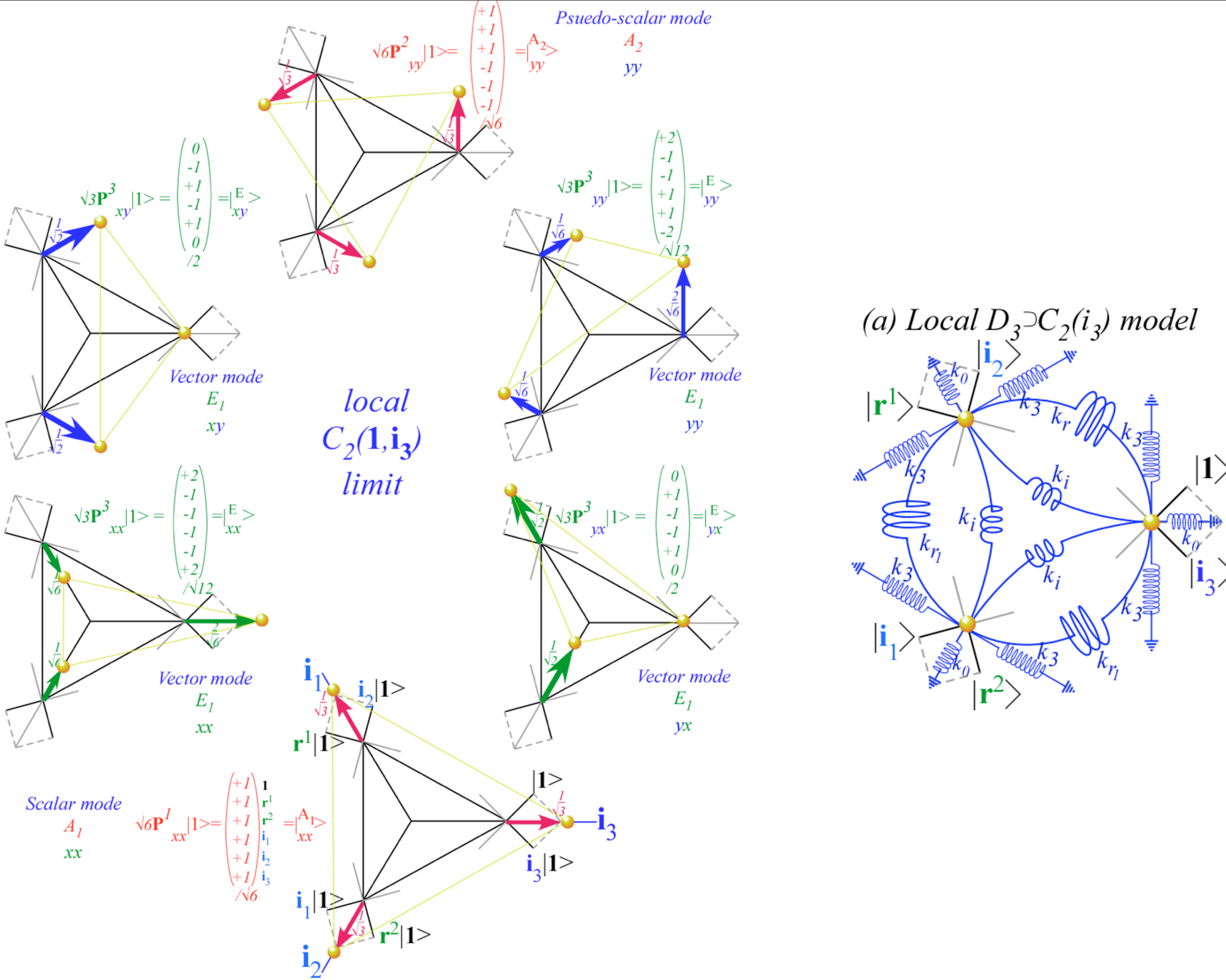


## (a) Local $D_3 \supset C_2(i_3)$ model




## (b) Mixed local symmetry $D_3$ model




Pseudo-scalar mode
$A_2$
yy
$\sqrt{6}\mathbf{P}^2_{yy}|1\rangle = (+1, +1, +1, -1, -1, -1)/\sqrt{6} = |^{A_2}_{yy}\rangle$
$\sqrt{3}\mathbf{P}^3_{xy}|1\rangle = (0, -1, +1, -1, +1, 0)/2 = |^{E}_{xy}\rangle$
Vector mode
$E_1$
xy
$\sqrt{3}\mathbf{P}^3_{yy}|1\rangle = (+2, -1, -1, +1, +1, -2)/\sqrt{12} = |^{E}_{yy}\rangle$
Vector mode
$E_1$
yy
local
$C_2(\mathbf{1},\mathbf{i}_3)$
limit
$\sqrt{3}\mathbf{P}^3_{xx}|1\rangle = (+2, -1, -1, -1, -1, +2)/\sqrt{12} = |^{E}_{xx}\rangle$
Vector mode
$E_1$
xx
$\sqrt{3}\mathbf{P}^3_{yx}|1\rangle = (0, +1, -1, -1, +1, 0)/2 = |^{E}_{yx}\rangle$
Vector mode
$E_1$
yx
Scalar mode
$A_1$
xx
$\sqrt{6}\mathbf{P}^1_{xx}|1\rangle = (+1, +1, +1, +1, +1, +1)/\sqrt{6} = |^{A_1}_{xx}\rangle$
$\mathbf{1}$, $\mathbf{r}^1$, $\mathbf{r}^2$, $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$
$\mathbf{i}_1$
$\mathbf{i}_2|\mathbf{1}\rangle$
$\mathbf{r}^1|\mathbf{1}\rangle$
$|\mathbf{1}\rangle$
$\mathbf{i}_3$
$\mathbf{i}_3|\mathbf{1}\rangle$
$\mathbf{i}_1|\mathbf{1}\rangle$
$\mathbf{r}^2|\mathbf{1}\rangle$
$\mathbf{i}_2$
(a) Local $D_3 \supset C_2(i_3)$ model
$|\mathbf{i}_2\rangle$
$|\mathbf{r}^1\rangle$
$|\mathbf{1}\rangle$
$|\mathbf{i}_3\rangle$
$|\mathbf{i}_1\rangle$
$|\mathbf{r}^2\rangle$
$k_0$
$k_3$
$k_r$
$k_i$
$k_{r_1}$