

Group Theory in Quantum Mechanics

Lecture 19 (4.9.13)

Octahedral-tetrahedral $O \sim T_d$ symmetries

(Int.J.Mol.Sci, 14, 714(2013) p.755-774 , QTCA Unit 5 Ch. 15)

(PSDS - Ch. 4)

Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$

Octahedral-cubic O symmetry and group operations

Tetrahedral symmetry becomes Icosahedral

Octahedral groups $O_h \supset O \sim T_d \supset T$

Octahedral O and spin- $O \subset U(2)$

Tetrahedral T class algebra

Tetrahedral T class minimal equations

Tetrahedral T class projectors and characters

Octahedral O class algebra

Octahedral O class minimal equations

Octahedral O class projectors and characters

Octahedral $O_h \supset O$ subgroup correlations

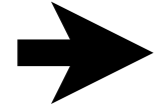
Octahedral $O_h \supset O$ subgroup correlations

$O_h \supset O \supset D_4$ subgroup correlations

$O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

Applications

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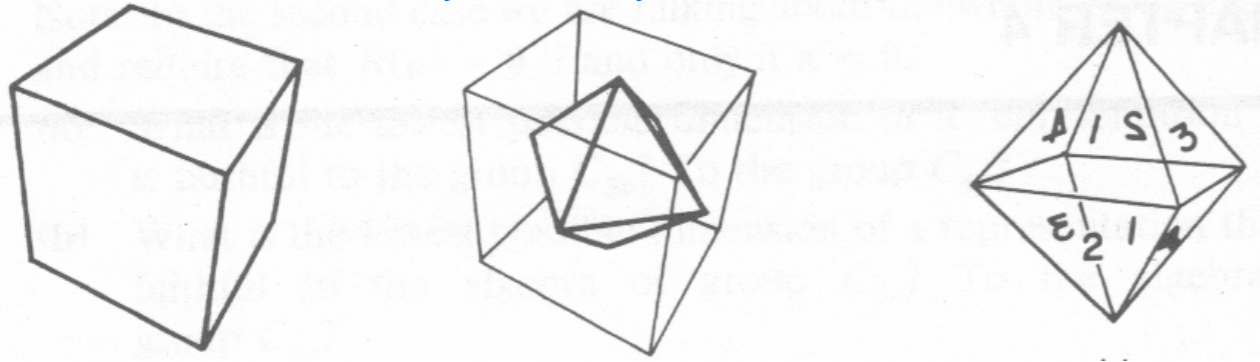
$O_h \supset O \supset D_4$ subgroup correlations

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Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$

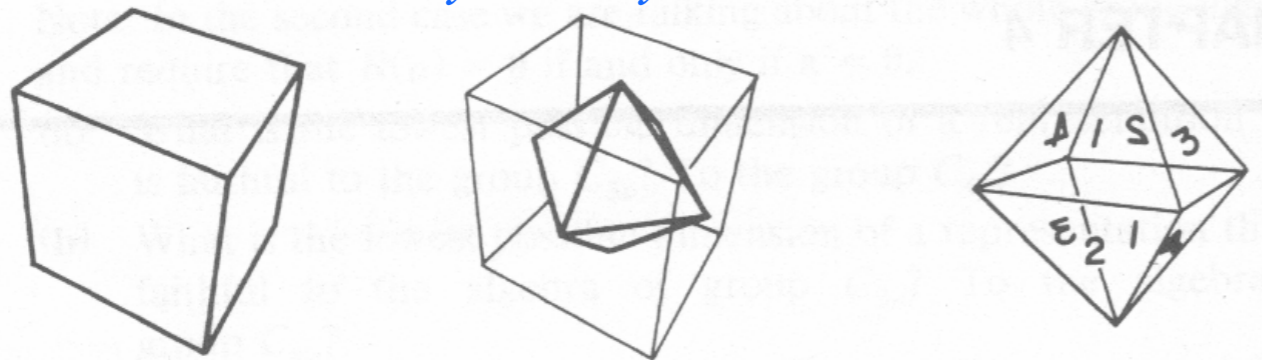
Octahedral-cubic O symmetry



*Order $^{\circ}O = 6$ hexahedron squares $\cdot 4$ pts = 24
= 8 octahedron triangles $\cdot 3$ pts = 24
= 12 lines $\cdot 2$ pts = 24 positions*

Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$

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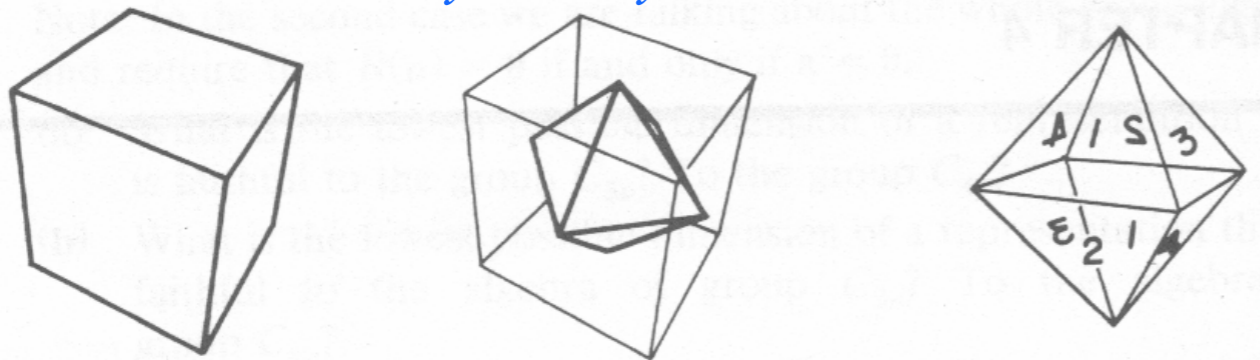
Octahedral group O operations

*Class of 1: **1***



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Octahedral-cubic O symmetry



Order $^{\circ}O = 6 \text{ hexahedron squares} \cdot 4 \text{ pts} = 24$
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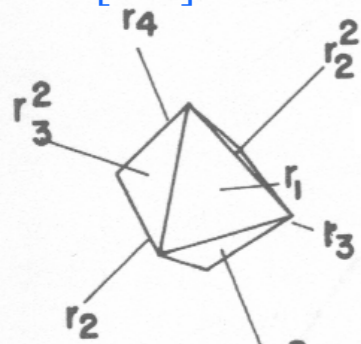
Octahedral group O operations

Class of 1: $\mathbf{1}$

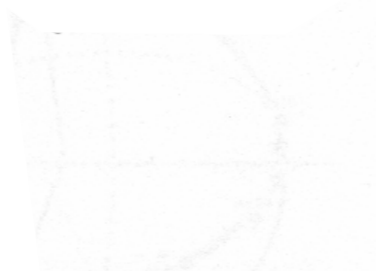
$$\mathbf{r}_k = \mathbf{r}_k$$

Class of 8:

120° rotations
 on $[111]$ axes

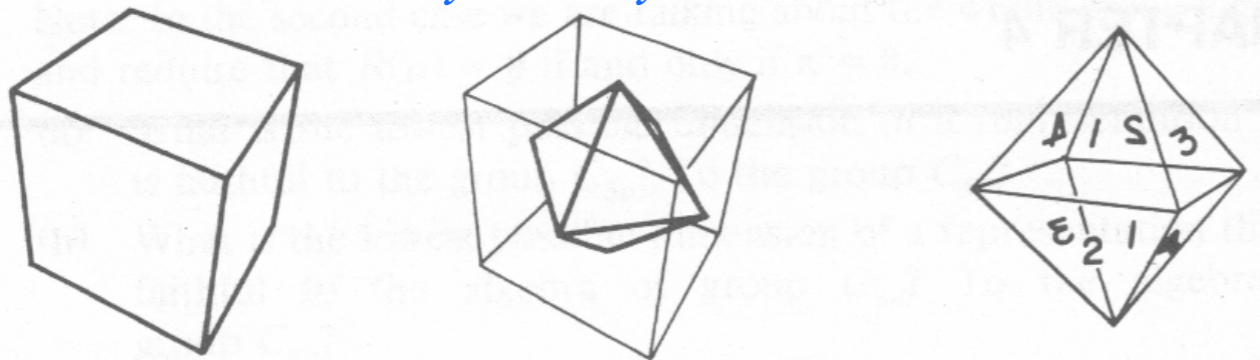


$$\tilde{\mathbf{r}}_k = \mathbf{r}_k^2 = \mathbf{r}_k^{-1}$$



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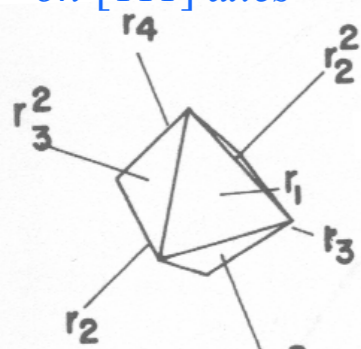
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Octahedral group O operations

Class of 1: $\mathbf{1}$

$\mathbf{r}_k = \mathbf{r}_k$

Class of 8:
 120° rotations
 on $[111]$ axes



Class of 3:
 180° rotations
 on $[100]$ axes

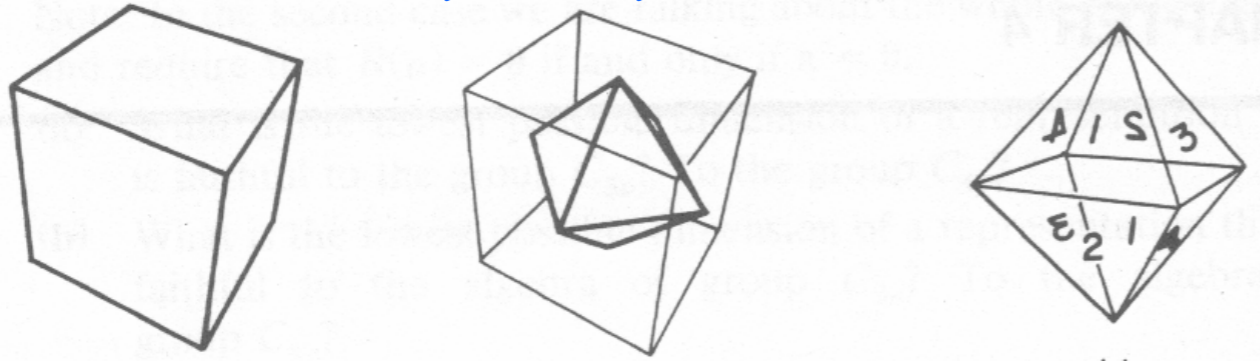
$\rho_{x,y,z} = \mathbf{R}_{1,2,3}^2$

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Octahedral-cubic O symmetry



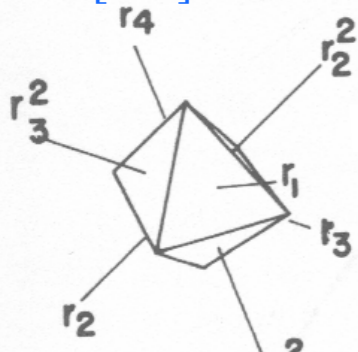
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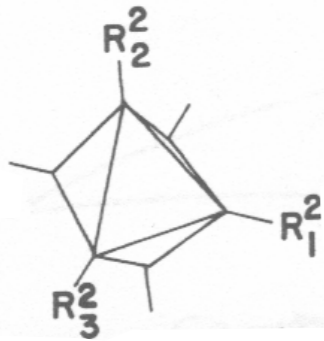
Class of 1: $\mathbf{1}$

$\mathbf{r}_k = \mathbf{r}_k$

Class of 8:
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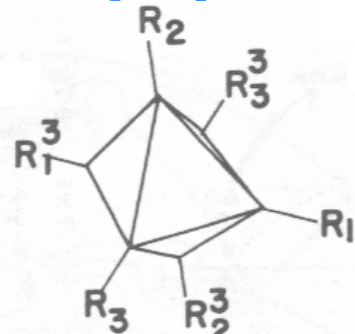


Class of 3:
 180° rotations
 on $[100]$ axes

$\mathbf{p}_{x,y,z} = \mathbf{R}_{1,2,3}^2$

$\mathbf{R}_{x,y,z} = \mathbf{R}_{1,2,3}$

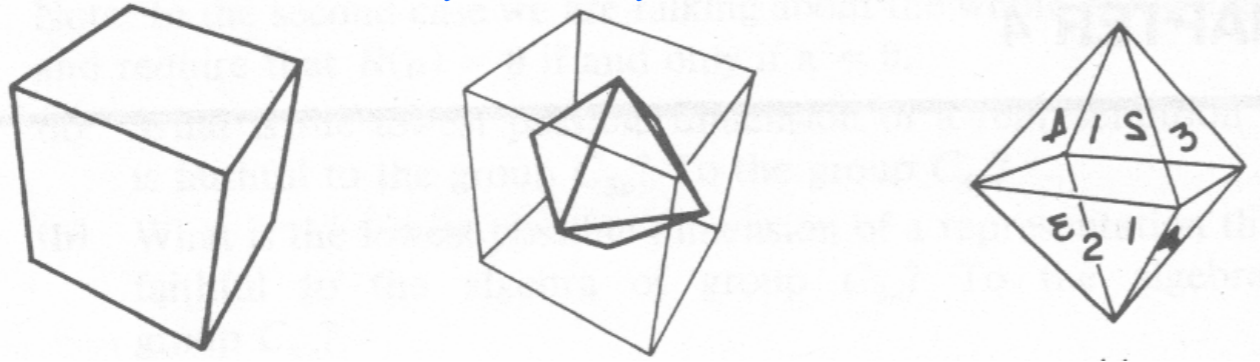
Class of 6:
 $\pm 90^\circ$ rotations
 on $[100]$ axes



$\tilde{\mathbf{R}}_{x,y,z} = \mathbf{R}_{1,2,3}^3 = \mathbf{R}_{1,2,3}^{-1}$

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Octahedral-cubic O symmetry



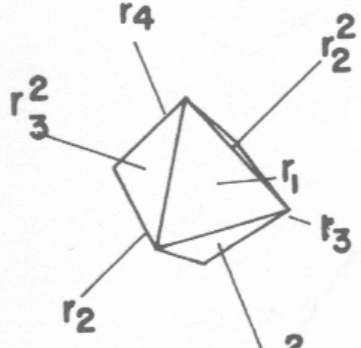
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Octahedral group O operations

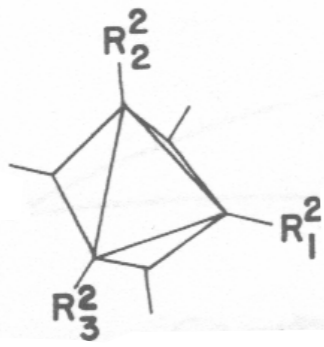
Class of 1: $\mathbf{1}$

$$\mathbf{r}_k = \mathbf{r}_k$$

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$$\tilde{\mathbf{r}}_k = \mathbf{r}_k^2 = \mathbf{r}_k^{-1}$$

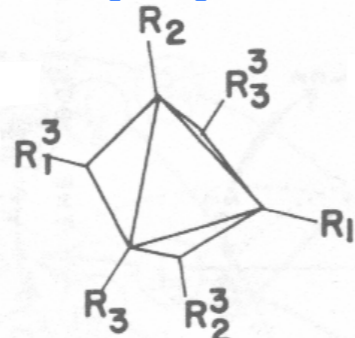


Class of 3:
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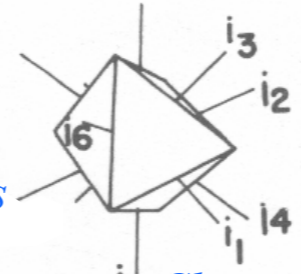
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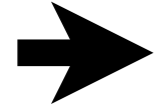
$$\tilde{\mathbf{R}}_{x,y,z} = \mathbf{R}_{1,2,3}^3 = \mathbf{R}_{1,2,3}^{-1}$$



Class of 6:
 180° rotations
 on $[110]$ diagonals

$$\mathbf{i}_k = \mathbf{i}_k$$

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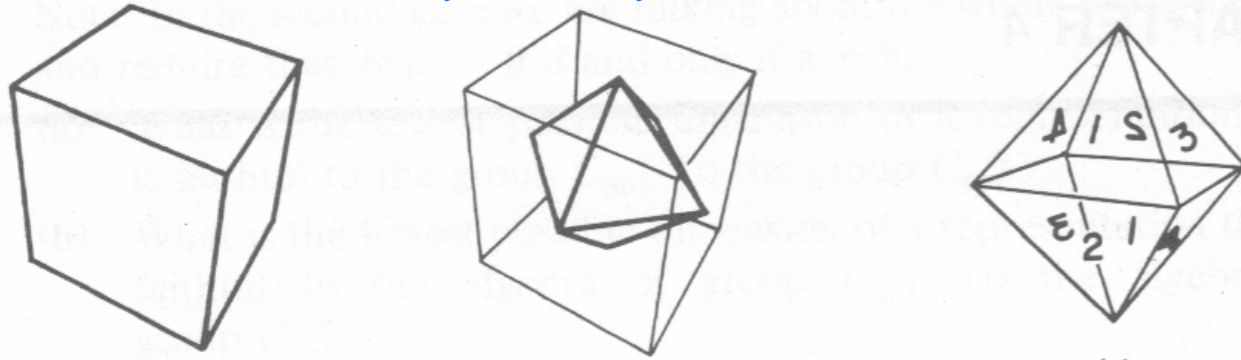
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Octahedral-cubic O symmetry



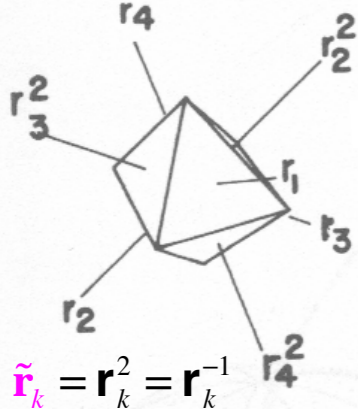
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Octahedral group O operations

Class of 1: $\mathbf{1}$

$$\mathbf{r}_k = \mathbf{r}_k$$

Class of 8:
 $\pm 120^\circ$ rotations
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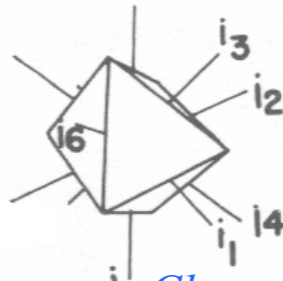


Class of 3:
 180° rotations
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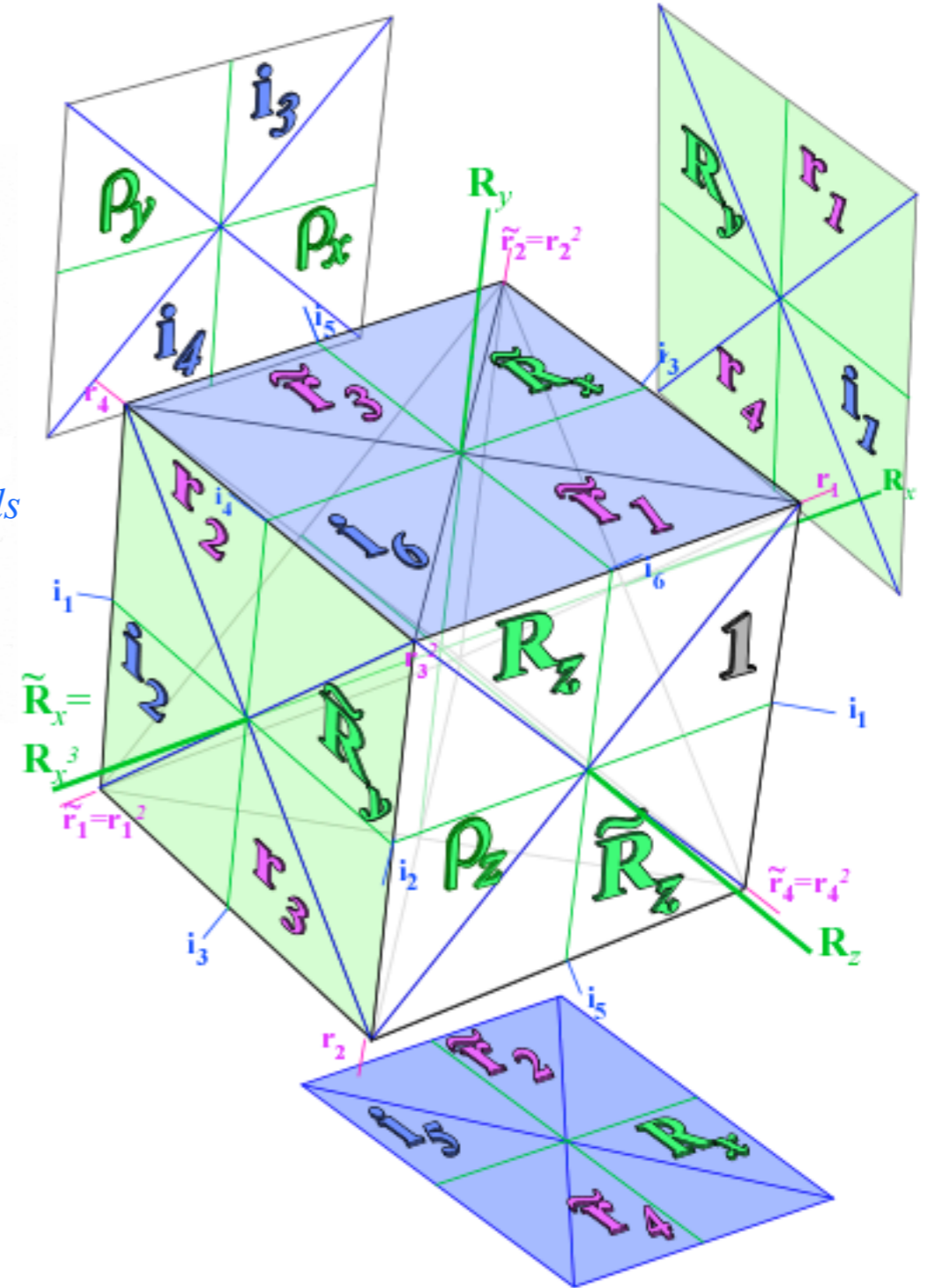
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Class of 6:
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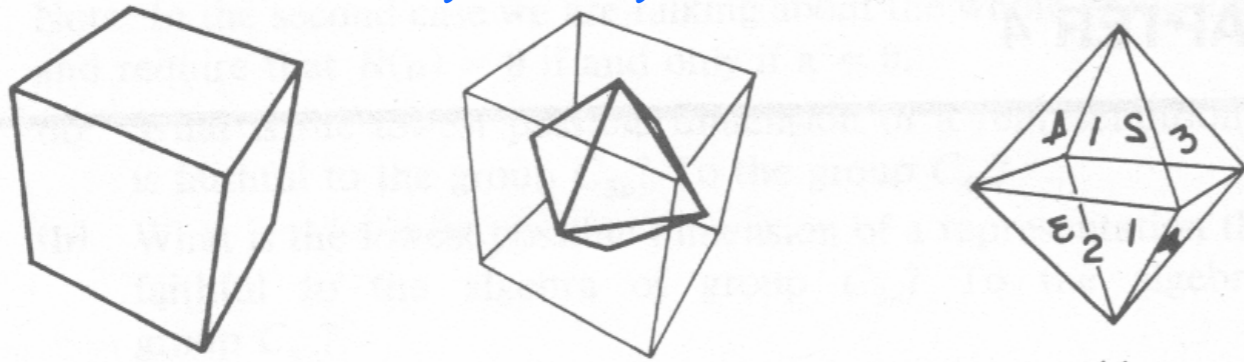
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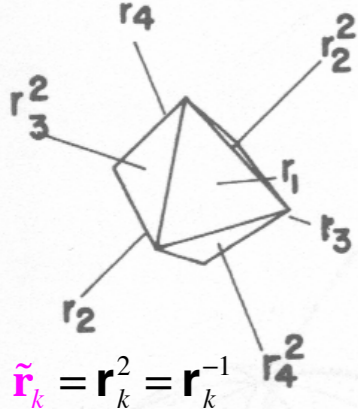
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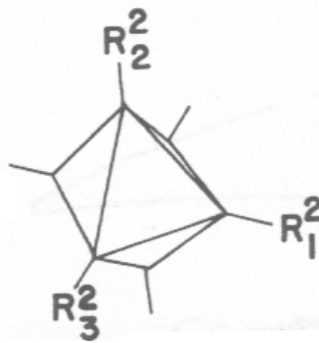
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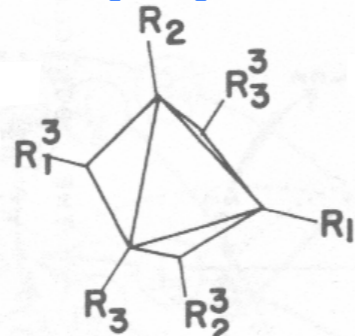


Class of 3:
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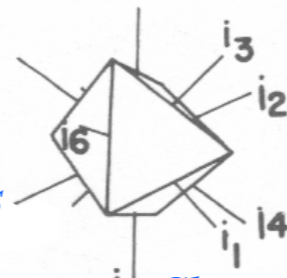
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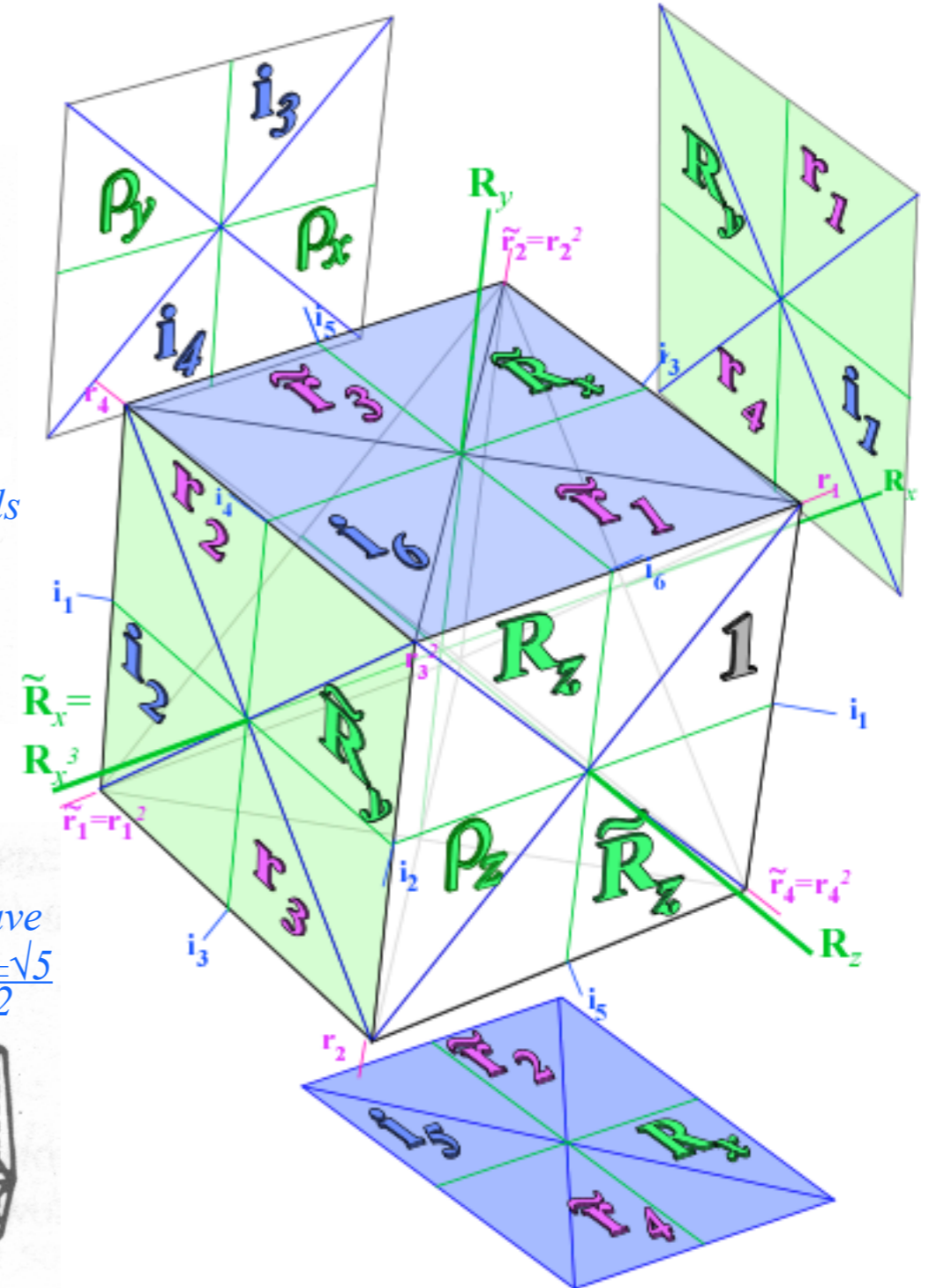
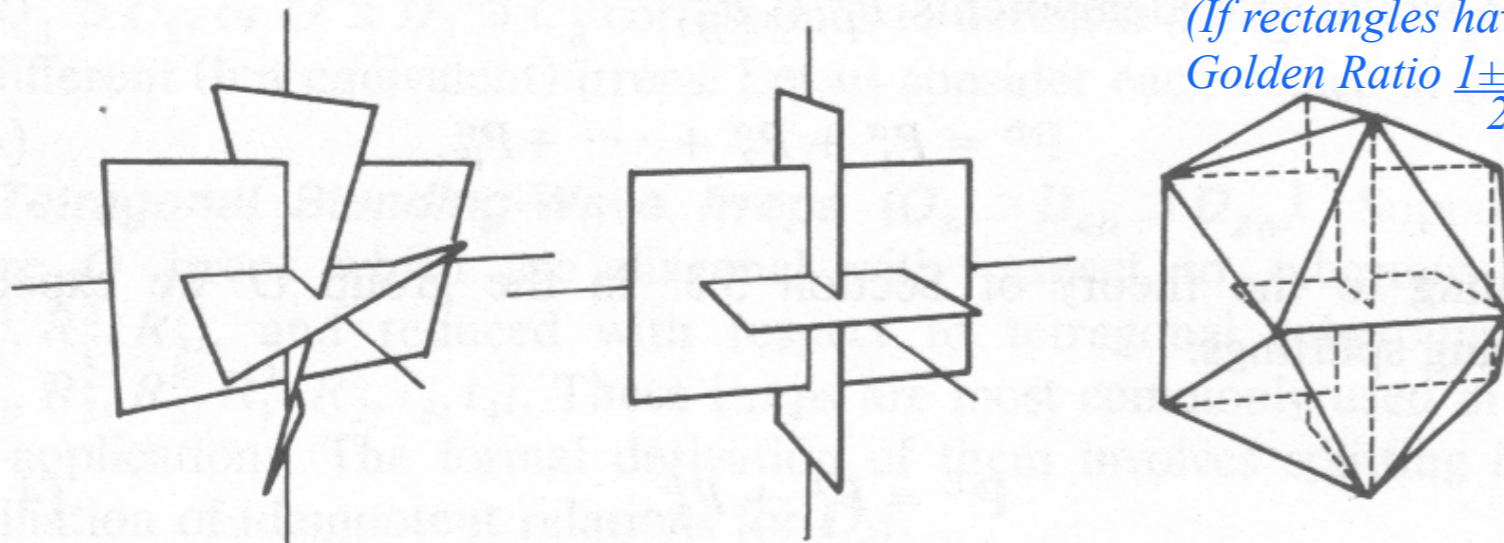
Tetrahedral symmetry becomes Icosahedral

T symmetry

T_h symmetry

I_h symmetry

(If rectangles have
 Golden Ratio $\frac{1 \pm \sqrt{5}}{2}$)



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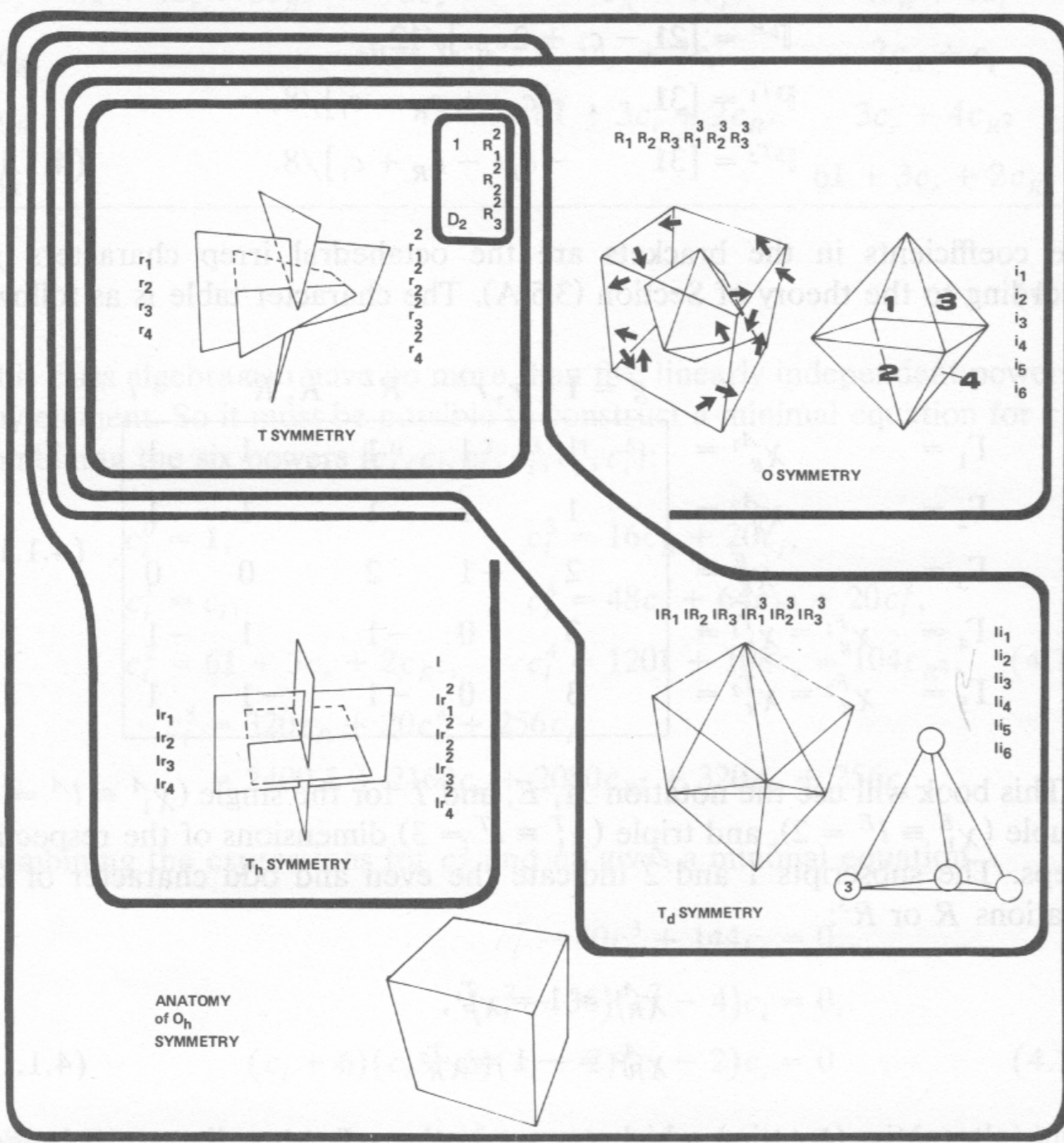


Figure 4.1.5 The full octahedral group (O_h) and four non-Abelian subgroups T , T_h , T_d , and O . The Abelian D_2 subgroup of T is indicated also.

Octahedral groups $O_h \supset O \sim T_d \supset T$

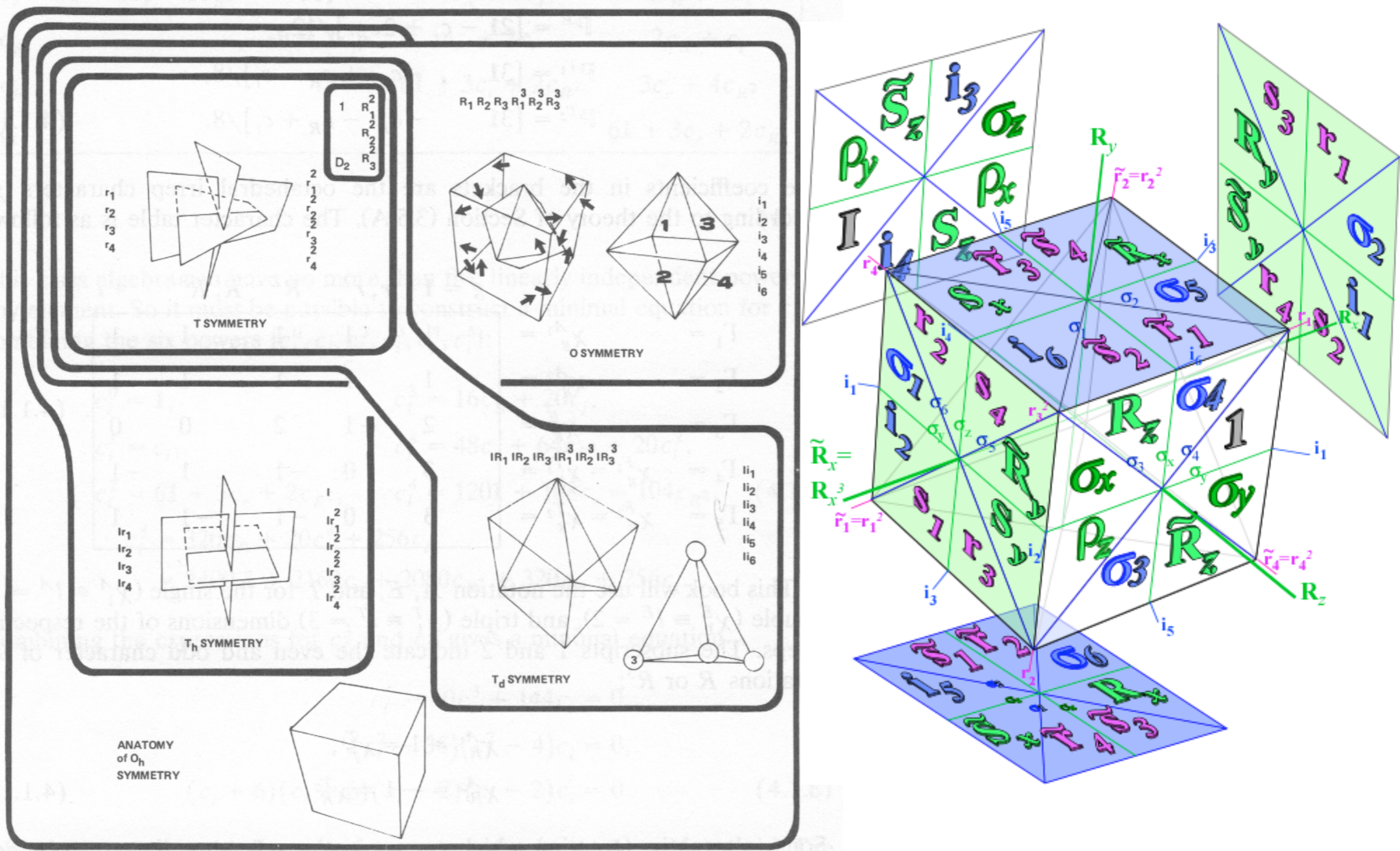


Figure 4.1.5 The full octahedral group (O_h) and four non-Abelian subgroups T , T_h , T_d , and O . The Abelian D_2 subgroup of T is indicated also.

Introduction to octahedral tetrahedral symmetry $O_h \supset O \sim T_d \supset T$

Octahedral groups $O_h \supset O \sim T_d$ and $O_h \supset T_h \supset T$

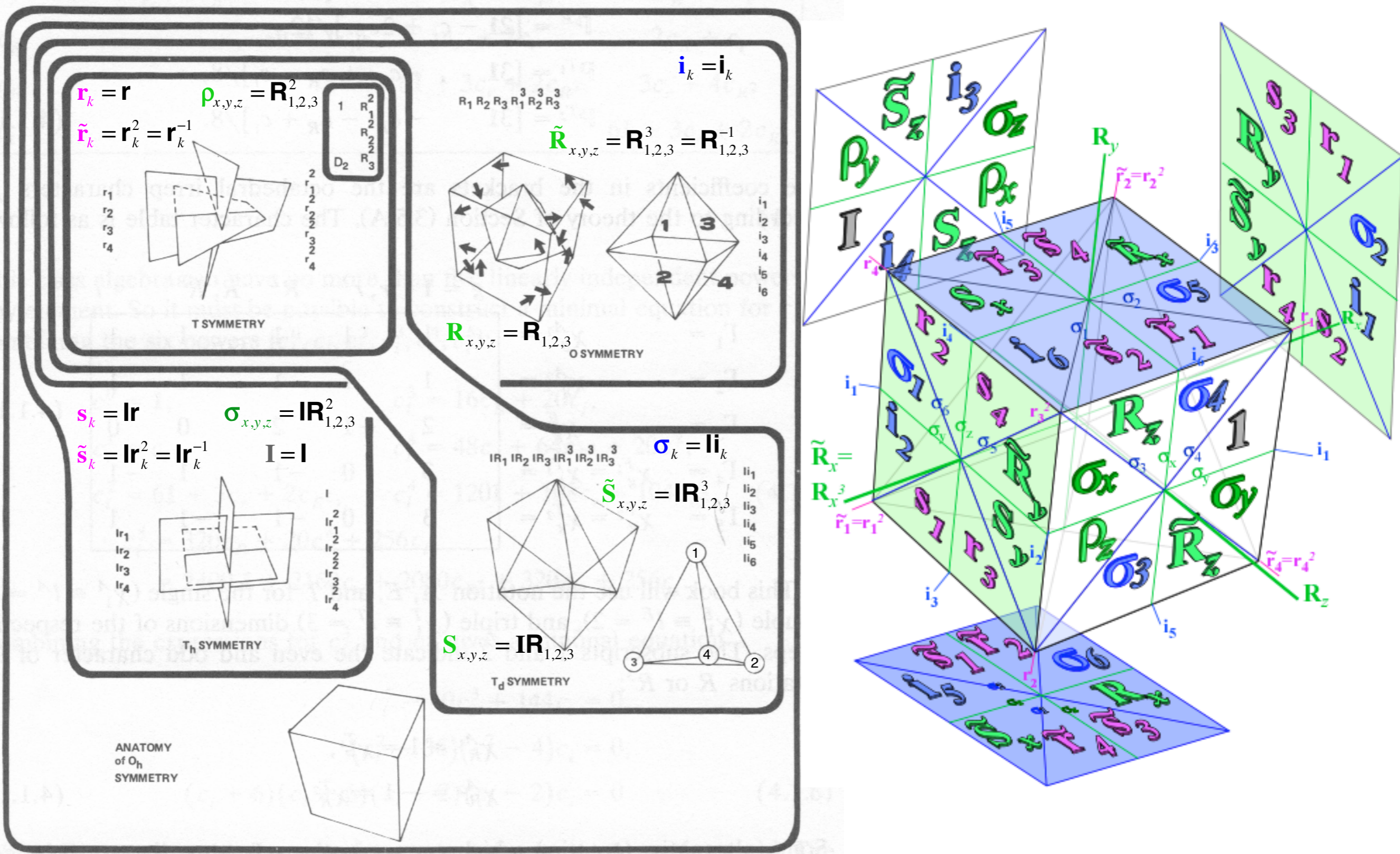


Figure 4.1.5 The full octahedral group (O_h) and four non-Abelian subgroups T , T_h , T_d , and O . The Abelian D_2 subgroup of T is indicated also.

Fig. 4.1.5 from *Principles of Symmetry, Dynamics and Spectroscopy*

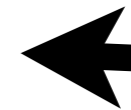
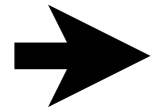
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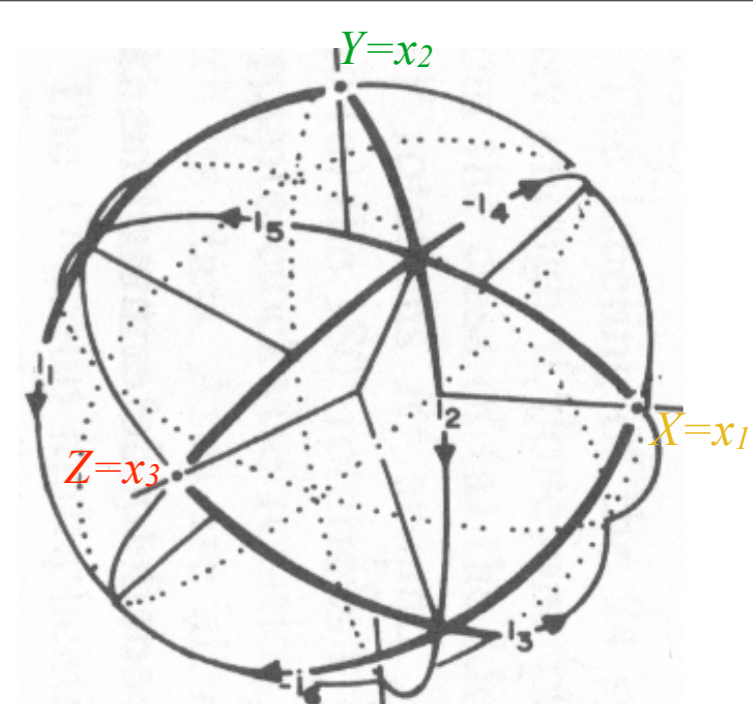
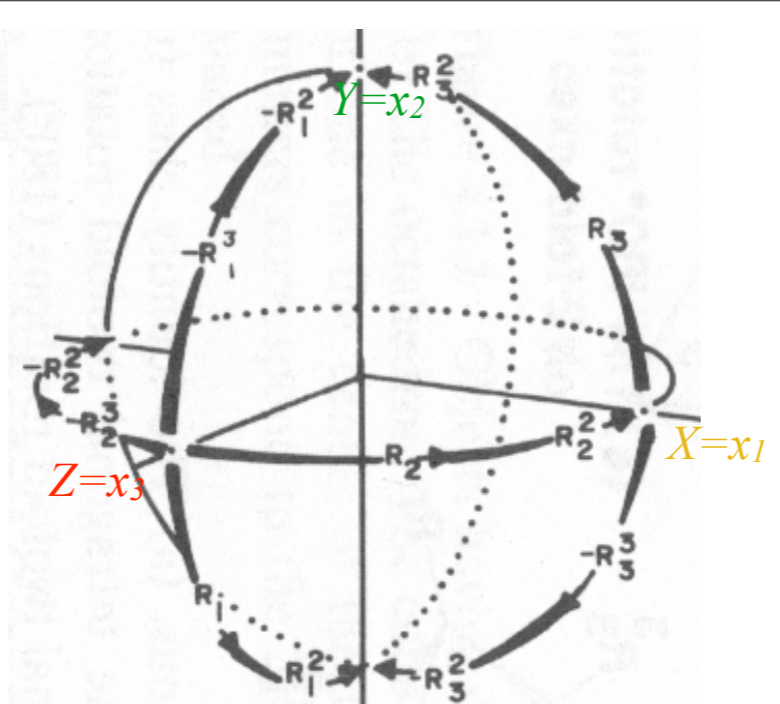
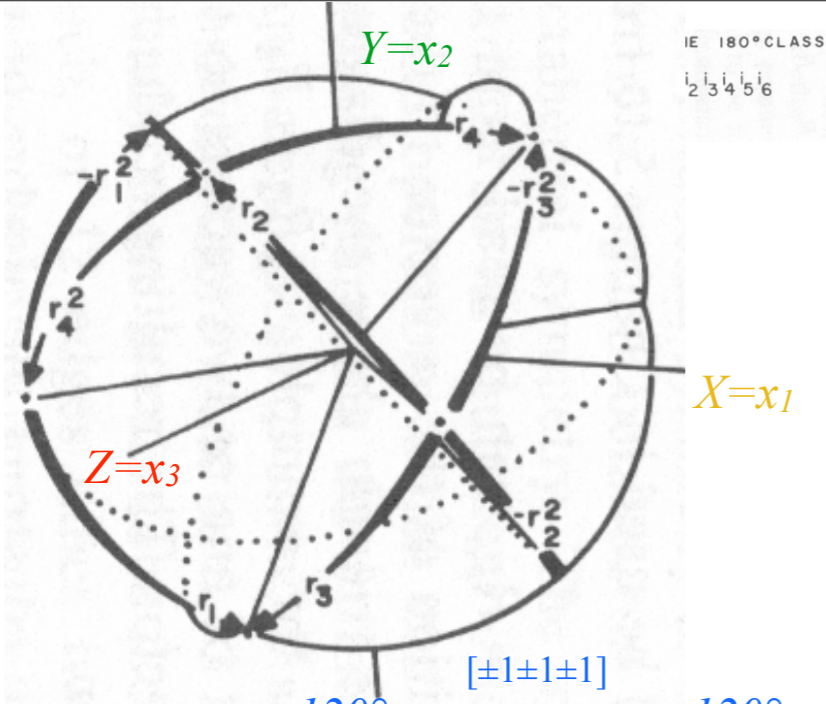
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$O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

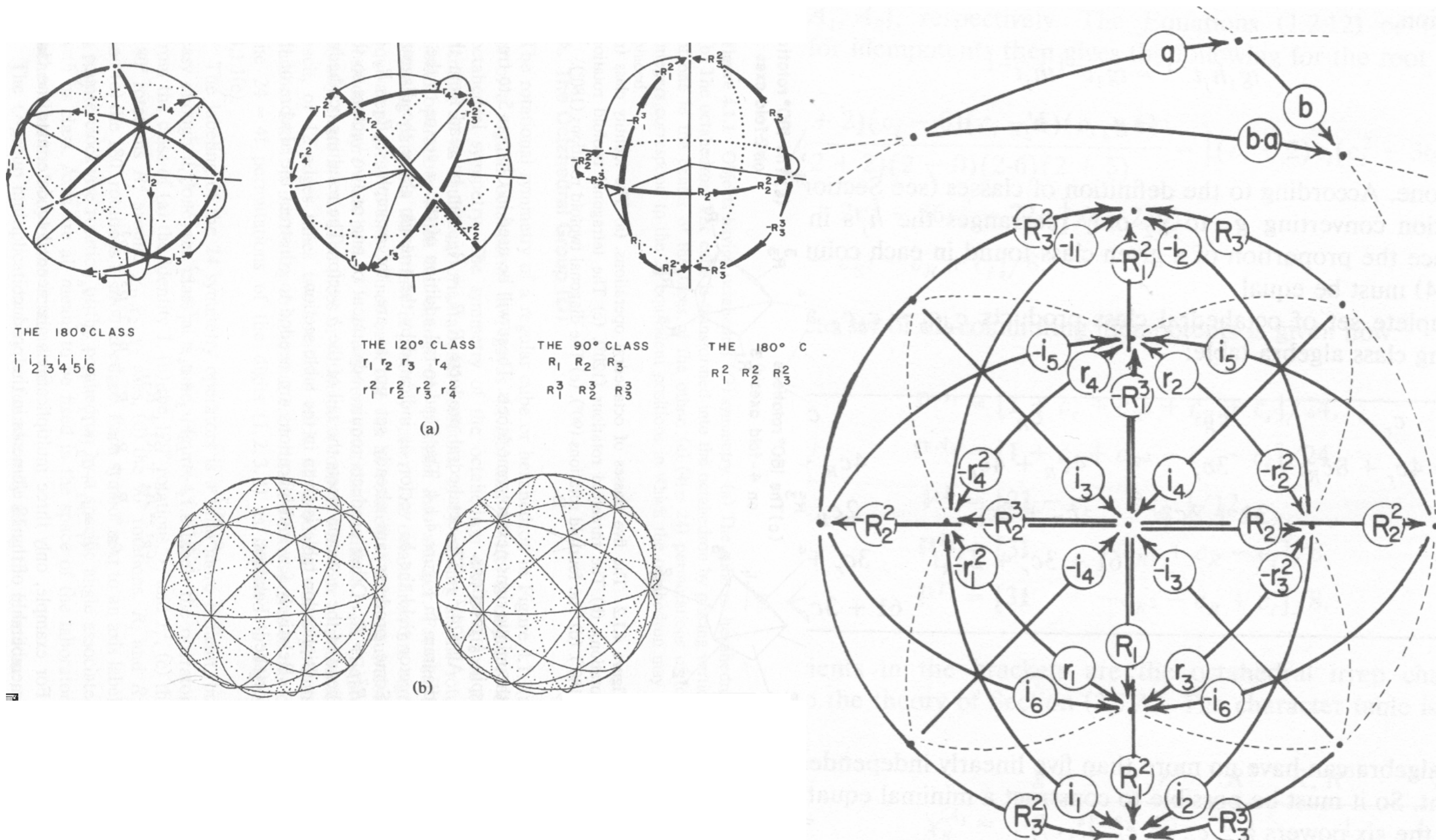
Applications



$[111]$ $[\bar{1}\bar{1}\bar{1}]$ $[1\bar{1}\bar{1}]$ $[\bar{1}1\bar{1}]$ $[\bar{1}\bar{1}1]$ $[11\bar{1}]$ $[\bar{1}11]$ $[1\bar{1}1]$ $[11\bar{1}]$ $[\bar{1}00]$ $[010]$ $[001]$ $[100]$ $[010]$ $[001]$ $[\bar{1}00]$ $[0\bar{1}0]$ $[00\bar{1}]$ $[101]$ $[10\bar{1}]$ $[110]$ $[\bar{1}\bar{1}0]$ $[01\bar{1}]$ $[011]$

1	r_1	r_2	r_3	r_4	r_1^2	r_2^2	r_3^2	r_4^2	R_1^2	R_2^2	R_3^2	R_1	R_2	R_3	R_1^3	R_2^3	R_3^3	i_1	i_2	i_3	i_4	i_5	i_6
r_1	r_1^2	$-r_4^2$	$-r_2^2$	$-r_3^2$	-1	$-R_2^2$	$-R_3^2$	$-R_1^2$	$-r_2$	$-r_3$	$-r_4$	i_3	i_6	i_1	$-R_3$	$-R_1$	$-R_2$	R_1^3	i_5	R_2^3	i_2	$-i_4$	R_3^3
r_2	$-r_3^2$	r_2^2	$-r_4^2$	$-r_1^2$	R_2^2	-1	R_1^2	$-R_3^2$	r_1	r_4	$-r_3$	R_3	$-R_1^3$	i_2	i_3	$-i_5$	R_2^3	i_6	$-R_1$	R_2	$-i_1$	R_3^3	i_4
r_3	$-r_4^2$	$-r_1^2$	r_3^2	$-r_2^2$	R_3^2	$-R_1^2$	-1	R_2^2	$-r_4$	r_1	r_2	$-i_4$	R_1	$-R_2^3$	R_3^3	i_6	i_2	i_5	$-R_1^3$	i_1	R_2	$-i_3$	R_3
r_4	$-r_2^2$	$-r_3^2$	$-r_1^2$	r_4^2	R_1^2	R_3^2	$-R_2^2$	-1	r_3	$-r_2$	r_1	$-R_3^3$	$-i_5$	R_2	$-i_4$	R_1^3	i_1	R_1	i_6	$-i_2$	R_2^3	R_3	i_3
r_1^2	-1	R_1^2	R_2^2	R_3^2	$-r_1$	r_3	r_4	r_2	r_4^2	r_2^2	r_3^2	R_2^3	R_3^3	R_1^3	$-i_1$	$-i_3$	$-i_6$	$-R_3$	$-i_4$	$-R_1$	i_5	$-i_2$	$-R_2$
r_2^2	$-R_1^2$	-1	R_3^2	$-R_2^2$	r_4	$-r_2$	r_1	r_3	$-r_3^2$	$-r_1^2$	r_4^2	i_2	$-i_3$	$-R_1$	R_2	$-R_3^3$	$-i_5$	i_4	$-R_3$	$-R_1^3$	$-i_6$	R_2^3	$-i_1$
r_3^2	$-R_2^2$	$-R_3^2$	-1	R_1^2	r_2	r_4	$-r_3$	r_1	r_2^2	$-r_4^2$	$-r_1^2$	$-R_2$	$-i_4$	$-i_6$	i_2	R_3	$-R_1^3$	$-i_3$	$-R_3^3$	i_5	R_1	$-i_1$	$-R_2^3$
r_4^2	$-R_3^2$	R_2^2	$-R_1^2$	-1	r_3	r_1	r_2	$-r_4$	$-r_1^2$	r_3^2	$-r_2^2$	$-i_1$	$-R_3$	$-i_5$	$-R_2^3$	$-i_4$	R_1	$-R_3^3$	i_3	$-i_6$	R_1^3	R_2	$-i_2$
R_1^2	$-r_4$	r_3	$-r_2$	r_1	r_2^2	$-r_1^2$	r_4^2	$-r_3^2$	-1	R_3^2	$-R_2^2$	R_1^3	i_1	$-i_4$	$-R_1$	i_2	$-i_3$	$-R_2$	$-R_3^3$	R_3^3	R_3	$-i_6$	i_5
R_2^2	$-r_2$	r_1	r_4	$-r_3$	r_3^2	$-r_4^2$	$-r_1^2$	r_2^2	$-R_3^2$	-1	R_1^2	$-i_5$	R_2^3	i_3	$-i_6$	$-R_2$	$-i_4$	$-i_2$	i_1	$-R_3$	R_3^3	R_1	R_1^3
R_3^2	$-r_3$	$-r_4$	r_1	r_2	r_4^2	r_3^2	$-r_2^2$	$-r_1^2$	R_2^2	$-R_1^2$	-1	i_6	i_2	R_3^3	$-i_5$	$-i_1$	$-R_3$	R_2^3	$-R_2$	i_4	$-i_3$	R_1^3	$-R_1$
R_1	i_1	$-R_2^3$	$-i_2$	R_2	R_3^3	$-i_3$	$-R_3$	i_4	R_1^3	i_6	i_5	R_1^2	r_1	$-r_4^2$	-1	$-r_3$	r_2^2	$-r_4$	r_2	r_1^2	$-r_3^2$	$-R_2^2$	R_3^2
R_2	i_3	R_3	$-R_3^3$	i_4	R_1^3	i_5	$-i_6$	$-R_1$	$-i_2$	R_2^3	i_1	$-r_2^2$	R_2^2	r_1	r_3^2	-1	$-r_4$	R_1^2	R_3^3	$-r_2$	$-r_3$	$-r_4^2$	r_1^2
R_3	i_6	i_5	R_1	$-R_1^3$	R_2^3	$-R_2$	$-i_2$	$-i_1$	i_3	i_4	R_3^3	r_1	$-r_3^2$	R_2^2	$-r_2$	r_4^2	-1	r_1^2	r_2^2	R_2^2	$-R_1^2$	$-r_4$	$-r_3$
R_1^3	$-R_2$	$-i_2$	R_2^3	i_1	$-i_3$	$-R_3^3$	i_4	R_3	$-R_1$	i_5	$-i_6$	-1	$-r_4$	r_3^2	$-R_1^2$	r_2	$-r_1^2$	$-r_1$	r_3	r_2^2	$-r_4^2$	$-R_2^3$	$-R_2^2$
R_2^3	$-R_3$	i_3	i_4	R_3^3	$-i_6$	R_1	$-R_1^3$	i_5	$-i_1$	$-R_2$	$-i_2$	r_4^2	-1	$-r_2$	$-r_1^2$	$-R_2^2$	r_3	$-R_3^2$	R_1^2	$-r_1$	$-r_4$	$-r_2^2$	r_3^2
R_3^3	$-R_1$	R_1^3	i_6	i_5	$-i_1$	$-i_2$	R_2	$-R_2^3$	i_4	$-i_3$	$-R_3$	$-r_3$	r_2^2	-1	r_4	$-r_1^2$	$-R_3^2$	r_4^2	r_3^2	$-R_1^2$	$-R_2^2$	$-r_2$	$-r_1$
i_1	R_3^3	$-i_4$	i_3	R_3	$-R_1$	$-i_6$	$-i_5$	$-R_1^3$	R_2^3	i_2	$-R_2$	r_1^2	R_3^2	$-r_4$	r_4^2	$-R_1^2$	$-r_1$	-1	$-R_2^2$	$-r_3$	r_2	r_3^2	r_2^2
i_2	i_4	R_3^3	R_3	$-i_3$	$-i_5$	R_1^3	R_1	$-i_6$	R_2	$-i_1$	R_2^2	$-r_3^2$	$-R_1^2$	$-r_3$	$-r_2^2$	$-R_3^2$	$-r_2$	R_2^2	-1	r_4	$-r_1$	r_1^2	r_4^2
i_3	R_1^3	R_1	$-i_5$	i_6	$-R_2$	$-R_2^3$	$-i_1$	i_2	$-R_3$	R_3^3	$-i_4$	$-r_2$	r_1^2	R_1^2	$-r_1$	r_2^2	$-R_2^2$	r_3^2	$-r_4^2$	-1	R_2^3	r_3	$-r_4$
i_4	$-i_5$	i_6	$-R_1^3$	$-R_1$	$-i_2$	i_1	$-R_2^2$	$-R_2$	$-R_3^3$	$-R_3$	i_3	r_4	r_4^2	R_2^2	r_3	r_3^2	R_1^2	$-r_2^2$	r_1^2	$-R_3^3$	-1	r_1	$-r_2$
i_5	i_2	$-R_2$	i_1	$-R_2^3$	i_4	$-R_3$	i_3	$-R_3^3$	i_6	$-R_1^3$	$-R_1$	R_3^3	r_2	r_2^2	R_2^2	r_4	r_4^2	$-r_3$	$-r_1$	$-r_3^2$	$-r_1^2$	-1	$-R_1^2$
i_6	R_2^3	i_1	R_2	i_2	$-R_3$	$-i_4$	$-R_3^3$	$-i_3$	$-i_5$	$-R_1$	R_1^3	R_2^2	$-r_3$	r_1^2	$-R_3^2$	$-r_1$	r_3^2	$-r_2$	$-r_4$	r_4^2	r_2^2	R_1^2	-1

Octahedral O and spin-OCU(2) rotation product Table F.2.1 from Principles of Symmetry, Dynamics and Spectroscopy



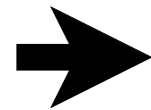
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Tetrahedral symmetry becomes Icosahedral

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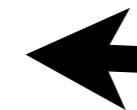
Octahedral O and spin- $O \subset U(2)$



Tetrahedral T class algebra

Tetrahedral T class minimal equations

Tetrahedral T class projectors and characters



Octahedral O class algebra

Octahedral O class minimal equations

Octahedral O class projectors and characters

Octahedral $O_h \supset O$ subgroup correlations

Octahedral $O_h \supset O$ subgroup correlations

$O_h \supset O \supset D_4$ subgroup correlations

$O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

Applications

Tetrahedral T class algebra

$$\mathbf{c}_I = \mathbf{1}, \quad \mathbf{c}_r = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4, \quad \mathbf{c}_r^\dagger = \mathbf{r}_1^2 + \mathbf{r}_2^2 + \mathbf{r}_3^2 + \mathbf{r}_4^2, \quad \mathbf{c}_\rho = \mathbf{R}_1^2 + \mathbf{R}_2^2 + \mathbf{R}_3^2,$$

T group products

	$+120^\circ$				-120°				$\pm 180^\circ$ XYZ			
	$[111]$	$[\bar{1}\bar{1}1]$	$[1\bar{1}\bar{1}]$	$[\bar{1}1\bar{1}]$	$[\bar{1}\bar{1}\bar{1}]$	$[11\bar{1}]$	$[\bar{1}11]$	$[1\bar{1}1]$	$[\bar{1}\bar{1}1]$	$[100]$	$[010]$	$[001]$
1	r_1	r_2	r_3	r_4	\tilde{r}_1^2	\tilde{r}_2^2	\tilde{r}_3^2	\tilde{r}_4^2	ρ_x	ρ_y	ρ_z	
	r_1^2	r_2^2	r_3^2	r_4^2	R_1^2	R_2^2	R_3^2	R_4^2	R_1^2	R_2^2	R_3^2	
r_1	r_1^2	$-r_4^2$	$-r_2^2$	$-r_3^2$	-1	$-R_2^2$	$-R_3^2$	$-R_1^2$	$-r_2$	$-r_3$	$-r_4$	
r_2	$-r_3^2$	r_2^2	$-r_4^2$	$-r_1^2$	R_2^2	-1	R_1^2	$-R_3^2$	r_1	r_4	$-r_3$	
r_3	$-r_4^2$	$-r_1^2$	r_3^2	$-r_2^2$	R_3^2	$-R_1^2$	-1	R_2^2	$-r_4$	r_1	r_2	
r_4	$-r_2^2$	$-r_3^2$	$-r_1^2$	r_4^2	R_1^2	R_3^2	$-R_2^2$	-1	r_3	$-r_2$	r_1	
r_1^2	-1	R_1^2	R_2^2	R_3^2	$-r_1$	r_3	r_4	r_2	r_4^2	r_2^2	r_3^2	
r_2^2	$-R_1^2$	-1	R_3^2	$-R_2^2$	r_4	$-r_2$	r_1	r_3	$-r_3^2$	$-r_1^2$	r_4^2	
r_3^2	$-R_2^2$	$-R_3^2$	-1	R_1^2	r_2	r_4	$-r_3$	r_1	r_2^2	$-r_4^2$	$-r_1^2$	
r_4^2	$-R_3^2$	R_2^2	$-R_1^2$	-1	r_3	r_1	r_2	$-r_4$	$-r_1^2$	r_3^2	$-r_2^2$	
R_1^2	$-r_4$	r_3	$-r_2$	r_1	r_2^2	$-r_1^2$	r_4^2	$-r_3^2$	-1	R_3^2	$-R_2^2$	
R_2^2	$-r_2$	r_1	r_4	$-r_3$	r_3^2	$-r_4^2$	$-r_1^2$	r_2^2	$-R_3^2$	-1	R_1^2	
R_3^2	$-r_3$	$-r_4$	r_1	r_2	r_4^2	r_3^2	$-r_2^2$	$-r_1^2$	R_2^2	$-R_1^2$	-1	

Tetrahedral T class algebra

$$\mathbf{c}_I = \mathbf{1}, \quad \mathbf{c}_r = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4, \quad \mathbf{c}_r^\dagger = \mathbf{r}_1^2 + \mathbf{r}_2^2 + \mathbf{r}_3^2 + \mathbf{r}_4^2,$$

$$\mathbf{c}_\rho = \mathbf{R}_1^2 + \mathbf{R}_2^2 + \mathbf{R}_3^2,$$

T group products

	$+120^\circ$				-120°				$\pm 180^\circ$ XYZ			
	$[111]$	$[\bar{1}\bar{1}1]$	$[1\bar{1}\bar{1}]$	$[\bar{1}1\bar{1}]$	$[\bar{1}\bar{1}\bar{1}]$	$[11\bar{1}]$	$[\bar{1}11]$	$[1\bar{1}1]$	$[\bar{1}\bar{1}1]$	$[100]$	$[010]$	$[001]$
1	r_1	r_2	r_3	r_4	\tilde{r}_1^2	\tilde{r}_2^2	\tilde{r}_3^2	\tilde{r}_4^2	ρ_x	ρ_y	ρ_z	
	r_1^2	r_2^2	r_3^2	r_4^2	R_1^2	R_2^2	R_3^2	R_4^2	R_1^2	R_2^2	R_3^2	
r_1	r_1^2	$-r_4^2$	$-r_2^2$	$-r_3^2$	-1	$-R_2^2$	$-R_3^2$	$-R_1^2$	$-r_2$	$-r_3$	$-r_4$	
r_2	$-r_3^2$	r_2^2	$-r_4^2$	$-r_1^2$	R_2^2	-1	R_1^2	$-R_3^2$	r_1	r_4	$-r_3$	
r_3	$-r_4^2$	$-r_1^2$	r_3^2	$-r_2^2$	R_3^2	$-R_1^2$	-1	R_2^2	$-r_4$	r_1	r_2	
r_4	$-r_2^2$	$-r_3^2$	$-r_1^2$	r_4^2	R_1^2	R_3^2	$-R_2^2$	-1	r_3	$-r_2$	r_1	
r_1^2	-1	R_1^2	R_2^2	R_3^2	$-r_1$	r_3	r_4	r_2	r_4^2	r_2^2	r_3^2	
r_2^2	$-R_1^2$	-1	R_3^2	$-R_2^2$	r_4	$-r_2$	r_1	r_3	$-r_3^2$	$-r_1^2$	r_4^2	
r_3^2	$-R_2^2$	$-R_3^2$	-1	R_1^2	r_2	r_4	$-r_3$	r_1	r_2^2	$-r_4^2$	$-r_1^2$	
r_4^2	$-R_3^2$	R_2^2	$-R_1^2$	-1	r_3	r_1	r_2	$-r_4$	$-r_1^2$	r_3^2	$-r_2^2$	
R_1^2	$-r_4$	r_3	$-r_2$	r_1	r_2^2	$-r_1^2$	r_4^2	$-r_3^2$	-1	R_3^2	$-R_2^2$	
R_2^2	$-r_2$	r_1	r_4	$-r_3$	r_3^2	$-r_4^2$	$-r_1^2$	r_2^2	$-R_3^2$	-1	R_1^2	
R_3^2	$-r_3$	$-r_4$	r_1	r_2	r_4^2	r_3^2	$-r_2^2$	$-r_1^2$	R_2^2	$-R_1^2$	-1	

T class products

$\mathbf{1} = \mathbf{c}_1$	\mathbf{c}_r	\mathbf{c}_r^\dagger	\mathbf{c}_ρ
\mathbf{c}_r	$4\mathbf{c}_r^\dagger$	$4\mathbf{1} + 4\mathbf{c}_\rho$	$3\mathbf{c}_r$
\mathbf{c}_r^\dagger		$4\mathbf{c}_r$	$3\mathbf{c}_r^\dagger$
\mathbf{c}_ρ			$3\mathbf{1} + 2\mathbf{c}_\rho$

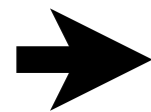
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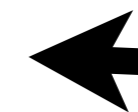
Octahedral O and spin- $O \subset U(2)$



Tetrahedral T class algebra

Tetrahedral T class minimal equations

Tetrahedral T class projectors and characters



Octahedral O class algebra

Octahedral O class minimal equations

Octahedral O class projectors and characters

Octahedral $O_h \supset O$ subgroup correlations

Octahedral $O_h \supset O$ subgroup correlations

$O_h \supset O \supset D_4$ subgroup correlations

$O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

Applications

Tetrahedral T class algebra

$$\mathbf{c}_1 = \mathbf{1}, \quad \mathbf{c}_r = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4, \quad \mathbf{c}_r^\dagger = \mathbf{r}_1^2 + \mathbf{r}_2^2 + \mathbf{r}_3^2 + \mathbf{r}_4^2, \quad \mathbf{c}_\rho = \mathbf{R}_1^2 + \mathbf{R}_2^2 + \mathbf{R}_3^2,$$

T group products

	$+120^\circ$				-120°				$\pm 180^\circ$ XYZ			
	$[111]$	$[\bar{1}\bar{1}1]$	$[1\bar{1}\bar{1}]$	$[\bar{1}1\bar{1}]$	$[\bar{1}\bar{1}\bar{1}]$	$[11\bar{1}]$	$[\bar{1}11]$	$[1\bar{1}1]$	$[\bar{1}\bar{1}1]$	$[100]$	$[010]$	$[001]$
1	r_1	r_2	r_3	r_4	\tilde{r}_1^2	\tilde{r}_2^2	\tilde{r}_3^2	\tilde{r}_4^2	ρ_x	ρ_y	ρ_z	
	r_1^2	r_2^2	r_3^2	r_4^2	R_1^2	R_2^2	R_3^2	R_4^2	R_1^2	R_2^2	R_3^2	
r_1	r_1^2	$-r_4^2$	$-r_2^2$	$-r_3^2$	-1	$-R_2^2$	$-R_3^2$	$-R_1^2$	$-r_2$	$-r_3$	$-r_4$	
r_2	$-r_3^2$	r_2^2	$-r_4^2$	$-r_1^2$	R_2^2	-1	R_1^2	$-R_3^2$	r_1	r_4	$-r_3$	
r_3	$-r_4^2$	$-r_1^2$	r_3^2	$-r_2^2$	R_3^2	$-R_1^2$	-1	R_2^2	$-r_4$	r_1	r_2	
r_4	$-r_2^2$	$-r_3^2$	$-r_1^2$	r_4^2	R_1^2	R_3^2	$-R_2^2$	-1	r_3	$-r_2$	r_1	
r_1^2	-1	R_1^2	R_2^2	R_3^2	$-r_1$	r_3	r_4	r_2	r_4^2	r_2^2	r_3^2	
r_2^2	$-R_1^2$	-1	R_3^2	$-R_2^2$	r_4	$-r_2$	r_1	r_3	$-r_3^2$	$-r_1^2$	r_4^2	
r_3^2	$-R_2^2$	$-R_3^2$	-1	R_1^2	r_2	r_4	$-r_3$	r_1	r_2^2	$-r_4^2$	$-r_1^2$	
r_4^2	$-R_3^2$	R_2^2	$-R_1^2$	-1	r_3	r_1	r_2	$-r_4$	$-r_1^2$	r_3^2	$-r_2^2$	
R_1^2	$-r_4$	r_3	$-r_2$	r_1	r_2^2	$-r_1^2$	r_4^2	$-r_3^2$	-1	R_3^2	$-R_2^2$	
R_2^2	$-r_2$	r_1	r_4	$-r_3$	r_3^2	$-r_4^2$	$-r_1^2$	r_2^2	$-R_3^2$	-1	R_1^2	
R_3^2	$-r_3$	$-r_4$	r_1	r_2	r_4^2	r_3^2	$-r_2^2$	$-r_1^2$	R_2^2	$-R_1^2$	-1	

T class products

$\mathbf{1} = \mathbf{c}_1$	\mathbf{c}_r	\mathbf{c}_r^\dagger	\mathbf{c}_ρ
\mathbf{c}_r	$4\mathbf{c}_r^\dagger$	$4\mathbf{1} + 4\mathbf{c}_\rho$	$3\mathbf{c}_r$
\mathbf{c}_r^\dagger		$4\mathbf{c}_r$	$3\mathbf{c}_r^\dagger$
\mathbf{c}_ρ			$3\mathbf{1} + 2\mathbf{c}_\rho$

Minimal equation for \mathbf{c}_ρ

$$\mathbf{c}_\rho^2 = 3\mathbf{1} + 2\mathbf{c}_\rho$$

$$\mathbf{c}_\rho^2 - 2\mathbf{c}_\rho - 3\mathbf{1} = \mathbf{0}$$

Tetrahedral T class algebra

$$\mathbf{c}_1 = \mathbf{1}, \quad \mathbf{c}_r = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4, \quad \mathbf{c}_r^\dagger = \mathbf{r}_1^2 + \mathbf{r}_2^2 + \mathbf{r}_3^2 + \mathbf{r}_4^2, \quad \mathbf{c}_\rho = \mathbf{R}_1^2 + \mathbf{R}_2^2 + \mathbf{R}_3^2,$$

T group products

	$+120^\circ$				-120°				$\pm 180^\circ$ XYZ			
	$[111]$	$[\bar{1}\bar{1}1]$	$[1\bar{1}\bar{1}]$	$[\bar{1}1\bar{1}]$	$[\bar{1}\bar{1}\bar{1}]$	$[11\bar{1}]$	$[\bar{1}11]$	$[1\bar{1}1]$	$[\bar{1}\bar{1}1]$	$[100]$	$[010]$	$[001]$
1	r_1	r_2	r_3	r_4	\tilde{r}_1^2	\tilde{r}_2^2	\tilde{r}_3^2	\tilde{r}_4^2	ρ_x	ρ_y	ρ_z	
	r_1^2	r_2^2	r_3^2	r_4^2	R_1^2	R_2^2	R_3^2	R_4^2	R_1^2	R_2^2	R_3^2	
r_1	r_1^2	$-r_4^2$	$-r_2^2$	$-r_3^2$	-1	$-R_2^2$	$-R_3^2$	$-R_1^2$	$-r_2$	$-r_3$	$-r_4$	
r_2	$-r_3^2$	r_2^2	$-r_4^2$	$-r_1^2$	R_2^2	-1	R_1^2	$-R_3^2$	r_1	r_4	$-r_3$	
r_3	$-r_4^2$	$-r_1^2$	r_3^2	$-r_2^2$	R_3^2	$-R_1^2$	-1	R_2^2	$-r_4$	r_1	r_2	
r_4	$-r_2^2$	$-r_3^2$	$-r_1^2$	r_4^2	R_1^2	R_3^2	$-R_2^2$	-1	r_3	$-r_2$	r_1	
r_1^2	-1	R_1^2	R_2^2	R_3^2	$-r_1$	r_3	r_4	r_2	r_4^2	r_2^2	r_3^2	
r_2^2	$-R_1^2$	-1	R_3^2	$-R_2^2$	r_4	$-r_2$	r_1	r_3	$-r_3^2$	$-r_1^2$	r_4^2	
r_3^2	$-R_2^2$	$-R_3^2$	-1	R_1^2	r_2	r_4	$-r_3$	r_1	r_2^2	$-r_4^2$	$-r_1^2$	
r_4^2	$-R_3^2$	R_2^2	$-R_1^2$	-1	r_3	r_1	r_2	$-r_4$	$-r_1^2$	r_3^2	$-r_2^2$	
R_1^2	$-r_4$	r_3	$-r_2$	r_1	r_2^2	$-r_1^2$	r_4^2	$-r_3^2$	-1	R_3^2	$-R_2^2$	
R_2^2	$-r_2$	r_1	r_4	$-r_3$	r_3^2	$-r_4^2$	$-r_1^2$	r_2^2	$-R_3^2$	-1	R_1^2	
R_3^2	$-r_3$	$-r_4$	r_1	r_2	r_4^2	r_3^2	$-r_2^2$	$-r_1^2$	R_2^2	$-R_1^2$	-1	

T class products

$\mathbf{1} = \mathbf{c}_1$	\mathbf{c}_r	\mathbf{c}_r^\dagger	\mathbf{c}_ρ
\mathbf{c}_r	$4\mathbf{c}_r^\dagger$	$4\mathbf{1} + 4\mathbf{c}_\rho$	$3\mathbf{c}_r$
\mathbf{c}_r^\dagger		$4\mathbf{c}_r$	$3\mathbf{c}_r^\dagger$
\mathbf{c}_ρ			$3\mathbf{1} + 2\mathbf{c}_\rho$

Minimal equation for \mathbf{c}_ρ

$$\mathbf{c}_\rho^2 = 3\mathbf{1} + 2\mathbf{c}_\rho$$

$$\mathbf{c}_\rho^2 - 2\mathbf{c}_\rho - 3\mathbf{1} = \mathbf{0}$$

$$(\mathbf{c}_\rho - 3\mathbf{1})(\mathbf{c}_\rho + \mathbf{1}) = \mathbf{0}$$

Tetrahedral T class algebra

$$\mathbf{c}_1 = \mathbf{1}, \quad \mathbf{c}_r = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4, \quad \mathbf{c}_r^\dagger = \mathbf{r}_1^2 + \mathbf{r}_2^2 + \mathbf{r}_3^2 + \mathbf{r}_4^2, \quad \mathbf{c}_\rho = \mathbf{R}_1^2 + \mathbf{R}_2^2 + \mathbf{R}_3^2,$$

T group products

	$+120^\circ$				-120°				$\pm 180^\circ$ XYZ			
	$[111]$	$[\bar{1}\bar{1}1]$	$[1\bar{1}\bar{1}]$	$[\bar{1}1\bar{1}]$	$[\bar{1}\bar{1}\bar{1}]$	$[11\bar{1}]$	$[\bar{1}11]$	$[1\bar{1}1]$	$[\bar{1}\bar{1}1]$	$[100]$	$[010]$	$[001]$
1	r_1	r_2	r_3	r_4	\tilde{r}_1^2	\tilde{r}_2^2	\tilde{r}_3^2	\tilde{r}_4^2	ρ_x	ρ_y	ρ_z	
	r_1^2	r_2^2	r_3^2	r_4^2	R_1^2	R_2^2	R_3^2	R_4^2	R_1^2	R_2^2	R_3^2	
r_1	r_1^2	$-r_4^2$	$-r_2^2$	$-r_3^2$	-1	$-R_2^2$	$-R_3^2$	$-R_1^2$	$-r_2$	$-r_3$	$-r_4$	
r_2	$-r_3^2$	r_2^2	$-r_4^2$	$-r_1^2$	R_2^2	-1	R_1^2	$-R_3^2$	r_1	r_4	$-r_3$	
r_3	$-r_4^2$	$-r_1^2$	r_3^2	$-r_2^2$	R_3^2	$-R_1^2$	-1	R_2^2	$-r_4$	r_1	r_2	
r_4	$-r_2^2$	$-r_3^2$	$-r_1^2$	r_4^2	R_1^2	R_3^2	$-R_2^2$	-1	r_3	$-r_2$	r_1	
r_1^2	-1	R_1^2	R_2^2	R_3^2	$-r_1$	r_3	r_4	r_2	r_4^2	r_2^2	r_3^2	
r_2^2	$-R_1^2$	-1	R_3^2	$-R_2^2$	r_4	$-r_2$	r_1	r_3	$-r_3^2$	$-r_1^2$	r_4^2	
r_3^2	$-R_2^2$	$-R_3^2$	-1	R_1^2	r_2	r_4	$-r_3$	r_1	r_2^2	$-r_4^2$	$-r_1^2$	
r_4^2	$-R_3^2$	R_2^2	$-R_1^2$	-1	r_3	r_1	r_2	$-r_4$	$-r_1^2$	r_3^2	$-r_2^2$	
R_1^2	$-r_4$	r_3	$-r_2$	r_1	r_2^2	$-r_1^2$	r_4^2	$-r_3^2$	-1	R_3^2	$-R_2^2$	
R_2^2	$-r_2$	r_1	r_4	$-r_3$	r_3^2	$-r_4^2$	$-r_1^2$	r_2^2	$-R_3^2$	-1	R_1^2	
R_3^2	$-r_3$	$-r_4$	r_1	r_2	r_4^2	r_3^2	$-r_2^2$	$-r_1^2$	R_2^2	$-R_1^2$	-1	

T class products

$\mathbf{1} = \mathbf{c}_1$	\mathbf{c}_r	\mathbf{c}_r^\dagger	\mathbf{c}_ρ
\mathbf{c}_r	$4\mathbf{c}_r^\dagger$	$4\mathbf{1} + 4\mathbf{c}_\rho$	$3\mathbf{c}_r$
\mathbf{c}_r^\dagger		$4\mathbf{c}_r$	$3\mathbf{c}_r^\dagger$
\mathbf{c}_ρ			$3\mathbf{1} + 2\mathbf{c}_\rho$

Minimal equation for \mathbf{c}_r

$$\mathbf{c}_r^2 = 4\tilde{\mathbf{c}}_r$$

Minimal equation for \mathbf{c}_ρ

$$\mathbf{c}_\rho^2 = 3\mathbf{1} + 2\mathbf{c}_\rho$$

$$\mathbf{c}_\rho^2 - 2\mathbf{c}_\rho - 3\mathbf{1} = \mathbf{0}$$

$$(\mathbf{c}_\rho - 3\mathbf{1})(\mathbf{c}_\rho + \mathbf{1}) = \mathbf{0}$$

Tetrahedral T class algebra

$$\mathbf{c}_1 = \mathbf{1}, \quad \mathbf{c}_r = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4, \quad \mathbf{c}_r^\dagger = \mathbf{r}_1^2 + \mathbf{r}_2^2 + \mathbf{r}_3^2 + \mathbf{r}_4^2, \quad \mathbf{c}_\rho = \mathbf{R}_1^2 + \mathbf{R}_2^2 + \mathbf{R}_3^2,$$

T group products

	$+120^\circ$				-120°				$\pm 180^\circ$ XYZ			
	$[111]$	$[\bar{1}\bar{1}1]$	$[1\bar{1}\bar{1}]$	$[\bar{1}1\bar{1}]$	$[\bar{1}\bar{1}\bar{1}]$	$[11\bar{1}]$	$[\bar{1}11]$	$[1\bar{1}1]$	$[\bar{1}\bar{1}1]$	$[100]$	$[010]$	$[001]$
1	r_1	r_2	r_3	r_4	\tilde{r}_1^2	\tilde{r}_2^2	\tilde{r}_3^2	\tilde{r}_4^2	ρ_x	ρ_y	ρ_z	
	r_1^2	r_2^2	r_3^2	r_4^2	R_1^2	R_2^2	R_3^2	R_4^2	R_1^2	R_2^2	R_3^2	
r_1	r_1^2	$-r_4^2$	$-r_2^2$	$-r_3^2$	-1	$-R_2^2$	$-R_3^2$	$-R_1^2$	$-r_2$	$-r_3$	$-r_4$	
r_2	$-r_3^2$	r_2^2	$-r_4^2$	$-r_1^2$	R_2^2	-1	R_1^2	$-R_3^2$	r_1	r_4	$-r_3$	
r_3	$-r_4^2$	$-r_1^2$	r_3^2	$-r_2^2$	R_3^2	$-R_1^2$	-1	R_2^2	$-r_4$	r_1	r_2	
r_4	$-r_2^2$	$-r_3^2$	$-r_1^2$	r_4^2	R_1^2	R_3^2	$-R_2^2$	-1	r_3	$-r_2$	r_1	
r_1^2	-1	R_1^2	R_2^2	R_3^2	$-r_1$	r_3	r_4	r_2	r_4^2	r_2^2	r_3^2	
r_2^2	$-R_1^2$	-1	R_3^2	$-R_2^2$	r_4	$-r_2$	r_1	r_3	$-r_3^2$	$-r_1^2$	r_4^2	
r_3^2	$-R_2^2$	$-R_3^2$	-1	R_1^2	r_2	r_4	$-r_3$	r_1	r_2^2	$-r_4^2$	$-r_1^2$	
r_4^2	$-R_3^2$	R_2^2	$-R_1^2$	-1	r_3	r_1	r_2	$-r_4$	$-r_1^2$	r_3^2	$-r_2^2$	
R_1^2	$-r_4$	r_3	$-r_2$	r_1	r_2^2	$-r_1^2$	r_4^2	$-r_3^2$	-1	R_3^2	$-R_2^2$	
R_2^2	$-r_2$	r_1	r_4	$-r_3$	r_3^2	$-r_4^2$	$-r_1^2$	r_2^2	$-R_3^2$	-1	R_1^2	
R_3^2	$-r_3$	$-r_4$	r_1	r_2	r_4^2	r_3^2	$-r_2^2$	$-r_1^2$	R_2^2	$-R_1^2$	-1	

T class products

$\mathbf{1} = \mathbf{c}_1$	\mathbf{c}_r	\mathbf{c}_r^\dagger	\mathbf{c}_ρ
\mathbf{c}_r	$4\mathbf{c}_r^\dagger$	$4\mathbf{1} + 4\mathbf{c}_\rho$	$3\mathbf{c}_r$
\mathbf{c}_r^\dagger		$4\mathbf{c}_r$	$3\mathbf{c}_r^\dagger$
\mathbf{c}_ρ			$3\mathbf{1} + 2\mathbf{c}_\rho$

Minimal equation for \mathbf{c}_r

$$\mathbf{c}_r^2 = 4\tilde{\mathbf{c}}_r$$

$$\mathbf{c}_r^3 = 4\tilde{\mathbf{c}}_r \mathbf{c}_r = 4(4\mathbf{1} + 4\mathbf{c}_\rho) = 16\mathbf{1} + 16\mathbf{c}_\rho$$

Minimal equation for \mathbf{c}_ρ

$$\mathbf{c}_\rho^2 = 3\mathbf{1} + 2\mathbf{c}_\rho$$

$$\mathbf{c}_\rho^2 - 2\mathbf{c}_\rho - 3\mathbf{1} = \mathbf{0}$$

$$(\mathbf{c}_\rho - 3\mathbf{1})(\mathbf{c}_\rho + \mathbf{1}) = \mathbf{0}$$

Tetrahedral T class algebra

$$\mathbf{c}_1 = \mathbf{1}, \quad \mathbf{c}_r = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4, \quad \mathbf{c}_r^\dagger = \mathbf{r}_1^2 + \mathbf{r}_2^2 + \mathbf{r}_3^2 + \mathbf{r}_4^2, \quad \mathbf{c}_\rho = \mathbf{R}_1^2 + \mathbf{R}_2^2 + \mathbf{R}_3^2,$$

T group products

	$+120^\circ$				-120°				$\pm 180^\circ$ XYZ		
	$[111]$	$[\bar{1}\bar{1}1]$	$[1\bar{1}\bar{1}]$	$[\bar{1}1\bar{1}]$	$[\bar{1}\bar{1}\bar{1}]$	$[11\bar{1}]$	$[\bar{1}11]$	$[1\bar{1}1]$	$[100]$	$[010]$	$[001]$
1	r_1	r_2	r_3	r_4	\tilde{r}_1^2	\tilde{r}_2^2	\tilde{r}_3^2	\tilde{r}_4^2	ρ_x	ρ_y	ρ_z
	r_1^2	r_2^2	r_3^2	r_4^2	R_1^2	R_2^2	R_3^2	R_4^2	R_1^2	R_2^2	R_3^2
r_1	r_1^2	$-r_4^2$	$-r_2^2$	$-r_3^2$	-1	$-R_2^2$	$-R_3^2$	$-R_1^2$	$-r_2$	$-r_3$	$-r_4$
r_2	$-r_3^2$	r_2^2	$-r_4^2$	$-r_1^2$	R_2^2	-1	R_1^2	$-R_3^2$	r_1	r_4	$-r_3$
r_3	$-r_4^2$	$-r_1^2$	r_3^2	$-r_2^2$	R_3^2	$-R_1^2$	-1	R_2^2	$-r_4$	r_1	r_2
r_4	$-r_2^2$	$-r_3^2$	$-r_1^2$	r_4^2	R_1^2	R_3^2	$-R_2^2$	-1	r_3	$-r_2$	r_1
r_1^2	-1	R_1^2	R_2^2	R_3^2	$-r_1$	r_3	r_4	r_2	r_4^2	r_2^2	r_3^2
r_2^2	$-R_1^2$	-1	R_3^2	$-R_2^2$	r_4	$-r_2$	r_1	r_3	$-r_3^2$	$-r_1^2$	r_4^2
r_3^2	$-R_2^2$	$-R_3^2$	-1	R_1^2	r_2	r_4	$-r_3$	r_1	r_2^2	$-r_4^2$	$-r_1^2$
r_4^2	$-R_3^2$	R_2^2	$-R_1^2$	-1	r_3	r_1	r_2	$-r_4$	$-r_1^2$	r_3^2	$-r_2^2$
R_1^2	$-r_4$	r_3	$-r_2$	r_1	r_2^2	$-r_1^2$	r_4^2	$-r_3^2$	-1	R_3^2	$-R_2^2$
R_2^2	$-r_2$	r_1	r_4	$-r_3$	r_3^2	$-r_4^2$	$-r_1^2$	r_2^2	$-R_3^2$	-1	R_1^2
R_3^2	$-r_3$	$-r_4$	r_1	r_2	r_4^2	r_3^2	$-r_2^2$	$-r_1^2$	R_2^2	$-R_1^2$	-1

T class products

$\mathbf{1} = \mathbf{c}_1$	\mathbf{c}_r	\mathbf{c}_r^\dagger	\mathbf{c}_ρ
\mathbf{c}_r	$4\mathbf{c}_r^\dagger$	$4\mathbf{1} + 4\mathbf{c}_\rho$	$3\mathbf{c}_r$
\mathbf{c}_r^\dagger		$4\mathbf{c}_r$	$3\mathbf{c}_r^\dagger$
\mathbf{c}_ρ			$3\mathbf{1} + 2\mathbf{c}_\rho$

Minimal equation for \mathbf{c}_r

$$\mathbf{c}_r^2 = 4\tilde{\mathbf{c}}_r$$

$$\mathbf{c}_r^3 = 4\tilde{\mathbf{c}}_r \mathbf{c}_r = 4(4\mathbf{1} + 4\mathbf{c}_\rho) = 16\mathbf{1} + 16\mathbf{c}_\rho$$

$$\mathbf{c}_r^4 = 16\mathbf{1}\mathbf{c}_r + 16\mathbf{c}_\rho \mathbf{c}_r = 16\mathbf{1}\mathbf{c}_r + 16(3\mathbf{c}_r)$$

Minimal equation for \mathbf{c}_ρ

$$\mathbf{c}_\rho^2 = 3\mathbf{1} + 2\mathbf{c}_\rho$$

$$\mathbf{c}_\rho^2 - 2\mathbf{c}_\rho - 3\mathbf{1} = \mathbf{0}$$

$$(\mathbf{c}_\rho - 3\mathbf{1})(\mathbf{c}_\rho + \mathbf{1}) = \mathbf{0}$$

Tetrahedral T class algebra

$$\mathbf{c}_1 = \mathbf{1}, \quad \mathbf{c}_r = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4, \quad \mathbf{c}_r^\dagger = \mathbf{r}_1^2 + \mathbf{r}_2^2 + \mathbf{r}_3^2 + \mathbf{r}_4^2, \quad \mathbf{c}_\rho = \mathbf{R}_1^2 + \mathbf{R}_2^2 + \mathbf{R}_3^2,$$

T group products

	$+120^\circ$				-120°				$\pm 180^\circ$ XYZ		
	$[111]$	$[\bar{1}\bar{1}1]$	$[1\bar{1}\bar{1}]$	$[\bar{1}1\bar{1}]$	$[\bar{1}\bar{1}\bar{1}]$	$[11\bar{1}]$	$[\bar{1}11]$	$[1\bar{1}1]$	$[100]$	$[010]$	$[001]$
1	r_1	r_2	r_3	r_4	\tilde{r}_1^2	\tilde{r}_2^2	\tilde{r}_3^2	\tilde{r}_4^2	ρ_x	ρ_y	ρ_z
	r_1^2	r_2^2	r_3^2	r_4^2	R_1^2	R_2^2	R_3^2	R_4^2	R_1^2	R_2^2	R_3^2
r_1	r_1^2	$-r_4^2$	$-r_2^2$	$-r_3^2$	-1	$-R_2^2$	$-R_3^2$	$-R_1^2$	$-r_2$	$-r_3$	$-r_4$
r_2	$-r_3^2$	r_2^2	$-r_4^2$	$-r_1^2$	R_2^2	-1	R_1^2	$-R_3^2$	r_1	r_4	$-r_3$
r_3	$-r_4^2$	$-r_1^2$	r_3^2	$-r_2^2$	R_3^2	$-R_1^2$	-1	R_2^2	$-r_4$	r_1	r_2
r_4	$-r_2^2$	$-r_3^2$	$-r_1^2$	r_4^2	R_1^2	R_3^2	$-R_2^2$	-1	r_3	$-r_2$	r_1
r_1^2	-1	R_1^2	R_2^2	R_3^2	$-r_1$	r_3	r_4	r_2	r_4^2	r_2^2	r_3^2
r_2^2	$-R_1^2$	-1	R_3^2	$-R_2^2$	r_4	$-r_2$	r_1	r_3	$-r_3^2$	$-r_1^2$	r_4^2
r_3^2	$-R_2^2$	$-R_3^2$	-1	R_1^2	r_2	r_4	$-r_3$	r_1	r_2^2	$-r_4^2$	$-r_1^2$
r_4^2	$-R_3^2$	R_2^2	$-R_1^2$	-1	r_3	r_1	r_2	$-r_4$	$-r_1^2$	r_3^2	$-r_2^2$
R_1^2	$-r_4$	r_3	$-r_2$	r_1	r_2^2	$-r_1^2$	r_4^2	$-r_3^2$	-1	R_3^2	$-R_2^2$
R_2^2	$-r_2$	r_1	r_4	$-r_3$	r_3^2	$-r_4^2$	$-r_1^2$	r_2^2	$-R_3^2$	-1	R_1^2
R_3^2	$-r_3$	$-r_4$	r_1	r_2	r_4^2	r_3^2	$-r_2^2$	$-r_1^2$	R_2^2	$-R_1^2$	-1

T class products

$\mathbf{1} = \mathbf{c}_1$	\mathbf{c}_r	\mathbf{c}_r^\dagger	\mathbf{c}_ρ
\mathbf{c}_r	$4\mathbf{c}_r^\dagger$	$4\mathbf{1} + 4\mathbf{c}_\rho$	$3\mathbf{c}_r$
\mathbf{c}_r^\dagger		$4\mathbf{c}_r$	$3\mathbf{c}_r^\dagger$
\mathbf{c}_ρ			$3\mathbf{1} + 2\mathbf{c}_\rho$

Minimal equation for \mathbf{c}_r

$$\mathbf{c}_r^2 = 4\tilde{\mathbf{c}}_r$$

$$\mathbf{c}_r^3 = 4\tilde{\mathbf{c}}_r \mathbf{c}_r = 4(4\mathbf{1} + 4\mathbf{c}_\rho) = 16\mathbf{1} + 16\mathbf{c}_\rho$$

$$\mathbf{c}_r^4 = 16\mathbf{1}\mathbf{c}_r + 16\mathbf{c}_\rho \mathbf{c}_r = 16\mathbf{1}\mathbf{c}_r + 16(3\mathbf{c}_r)$$

$$\mathbf{c}_r^4 - 64\mathbf{c}_r = (\mathbf{c}_r^3 - 64\mathbf{1})\mathbf{c}_r = \mathbf{0}$$

Minimal equation for \mathbf{c}_ρ

$$\mathbf{c}_\rho^2 = 3\mathbf{1} + 2\mathbf{c}_\rho$$

$$\mathbf{c}_\rho^2 - 2\mathbf{c}_\rho - 3\mathbf{1} = \mathbf{0}$$

$$(\mathbf{c}_\rho - 3\mathbf{1})(\mathbf{c}_\rho + \mathbf{1}) = \mathbf{0}$$

Tetrahedral T class algebra

$$\mathbf{c}_1 = \mathbf{1}, \quad \mathbf{c}_r = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4, \quad \mathbf{c}_r^\dagger = \mathbf{r}_1^2 + \mathbf{r}_2^2 + \mathbf{r}_3^2 + \mathbf{r}_4^2, \quad \mathbf{c}_\rho = \mathbf{R}_1^2 + \mathbf{R}_2^2 + \mathbf{R}_3^2,$$

T group products

	$+120^\circ$				-120°				$\pm 180^\circ$ XYZ		
	$[111]$	$[\bar{1}\bar{1}1]$	$[1\bar{1}\bar{1}]$	$[\bar{1}1\bar{1}]$	$[\bar{1}\bar{1}\bar{1}]$	$[11\bar{1}]$	$[\bar{1}11]$	$[1\bar{1}1]$	$[100]$	$[010]$	$[001]$
1	r_1	r_2	r_3	r_4	\tilde{r}_1^2	\tilde{r}_2^2	\tilde{r}_3^2	\tilde{r}_4^2	ρ_x	ρ_y	ρ_z
	r_1^2	r_2^2	r_3^2	r_4^2	R_1^2	R_2^2	R_3^2	R_4^2	R_1^2	R_2^2	R_3^2
r_1	r_1^2	$-r_4^2$	$-r_2^2$	$-r_3^2$	-1	$-R_2^2$	$-R_3^2$	$-R_1^2$	$-r_2$	$-r_3$	$-r_4$
r_2	$-r_3^2$	r_2^2	$-r_4^2$	$-r_1^2$	R_2^2	-1	R_1^2	$-R_3^2$	r_1	r_4	$-r_3$
r_3	$-r_4^2$	$-r_1^2$	r_3^2	$-r_2^2$	R_3^2	$-R_1^2$	-1	R_2^2	$-r_4$	r_1	r_2
r_4	$-r_2^2$	$-r_3^2$	$-r_1^2$	r_4^2	R_1^2	R_3^2	$-R_2^2$	-1	r_3	$-r_2$	r_1
r_1^2	-1	R_1^2	R_2^2	R_3^2	$-r_1$	r_3	r_4	r_2	r_4^2	r_2^2	r_3^2
r_2^2	$-R_1^2$	-1	R_3^2	$-R_2^2$	r_4	$-r_2$	r_1	r_3	$-r_3^2$	$-r_1^2$	r_4^2
r_3^2	$-R_2^2$	$-R_3^2$	-1	R_1^2	r_2	r_4	$-r_3$	r_1	r_2^2	$-r_4^2$	$-r_1^2$
r_4^2	$-R_3^2$	R_2^2	$-R_1^2$	-1	r_3	r_1	r_2	$-r_4$	$-r_1^2$	r_3^2	$-r_2^2$
R_1^2	$-r_4$	r_3	$-r_2$	r_1	r_2^2	$-r_1^2$	r_4^2	$-r_3^2$	-1	R_3^2	$-R_2^2$
R_2^2	$-r_2$	r_1	r_4	$-r_3$	r_3^2	$-r_4^2$	$-r_1^2$	r_2^2	$-R_3^2$	-1	R_1^2
R_3^2	$-r_3$	$-r_4$	r_1	r_2	r_4^2	r_3^2	$-r_2^2$	$-r_1^2$	R_2^2	$-R_1^2$	-1

T class products

$\mathbf{1} = \mathbf{c}_1$	\mathbf{c}_r	\mathbf{c}_r^\dagger	\mathbf{c}_ρ
\mathbf{c}_r	$4\mathbf{c}_r^\dagger$	$4\mathbf{1} + 4\mathbf{c}_\rho$	$3\mathbf{c}_r$
\mathbf{c}_r^\dagger		$4\mathbf{c}_r$	$3\mathbf{c}_r^\dagger$
\mathbf{c}_ρ			$3\mathbf{1} + 2\mathbf{c}_\rho$

Minimal equation for \mathbf{c}_r

$$\mathbf{c}_r^2 = 4\tilde{\mathbf{c}}_r$$

$$\mathbf{c}_r^3 = 4\tilde{\mathbf{c}}_r \mathbf{c}_r = 4(4\mathbf{1} + 4\mathbf{c}_\rho) = 16\mathbf{1} + 16\mathbf{c}_\rho$$

$$\mathbf{c}_r^4 = 16\mathbf{1}\mathbf{c}_r + 16\mathbf{c}_\rho \mathbf{c}_r = 16\mathbf{1}\mathbf{c}_r + 16(3\mathbf{c}_r)$$

$$\mathbf{c}_r^4 - 64\mathbf{c}_r = (\mathbf{c}_r^3 - 64\mathbf{1})\mathbf{c}_r = \mathbf{0}$$

$$(\mathbf{c}_r - 4e^{+2\pi i/3}\mathbf{1})(\mathbf{c}_r - 4e^{-2\pi i/3}\mathbf{1})(\mathbf{c}_r - 4\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}$$

Minimal equation for \mathbf{c}_ρ

$$\mathbf{c}_\rho^2 = 3\mathbf{1} + 2\mathbf{c}_\rho$$

$$\mathbf{c}_\rho^2 - 2\mathbf{c}_\rho - 3\mathbf{1} = \mathbf{0}$$

$$(\mathbf{c}_\rho - 3\mathbf{1})(\mathbf{c}_\rho + \mathbf{1}) = \mathbf{0}$$

Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$

Octahedral-cubic O symmetry and group operations

Tetrahedral symmetry becomes Icosahedral

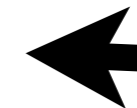
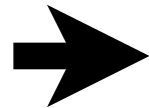
Octahedral groups $O_h \supset O \sim T_d \supset T$

Octahedral O and spin- $O \subset U(2)$

Tetrahedral T class algebra

Tetrahedral T class minimal equations

Tetrahedral T class projectors and characters



Octahedral O class algebra

Octahedral O class minimal equations

Octahedral O class projectors and characters

Octahedral $O_h \supset O$ subgroup correlations

Octahedral $O_h \supset O$ subgroup correlations

$O_h \supset O \supset D_4$ subgroup correlations

$O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

Applications

Tetrahedral T class projectors

$$\mathbf{c}_g = \sum_{\mu} \frac{\chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{\chi_g^{\mu}} \chi_g^{\mu*} \mathbf{c}_g$$

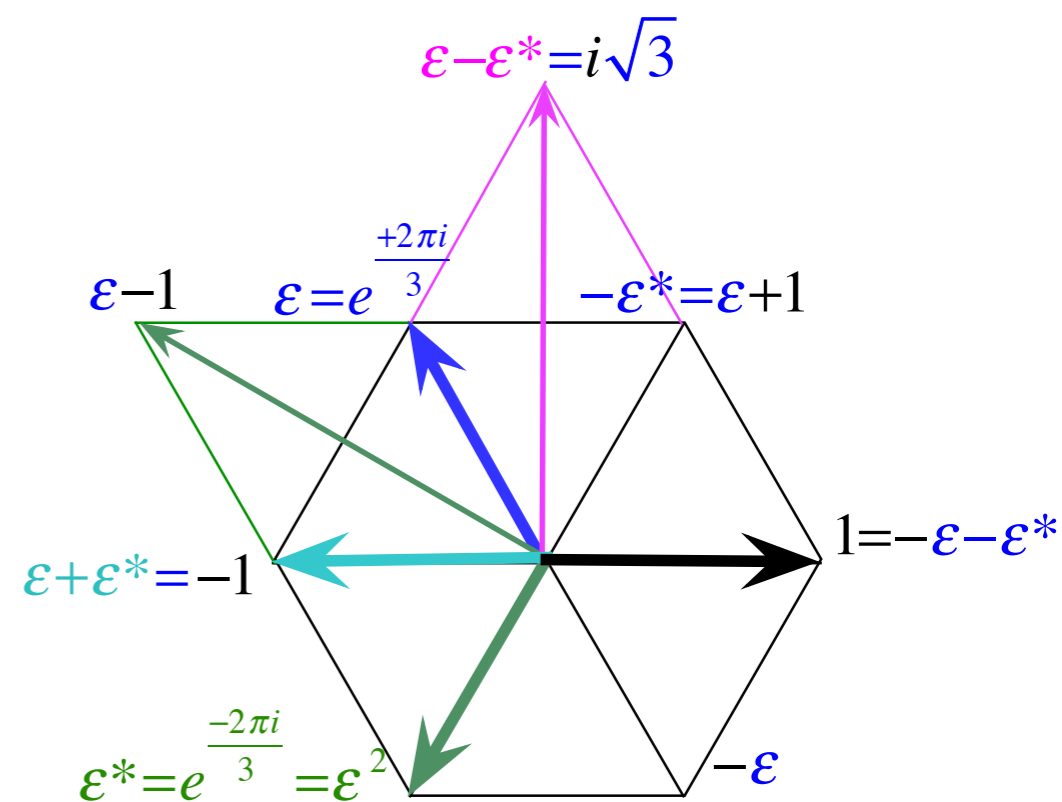
Minimal equation for \mathbf{c}_r : $(\mathbf{c}_r - 4\epsilon \mathbf{1})(\mathbf{c}_r - 4\epsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0) = 0$

T class products

$\mathbf{1} = \mathbf{c}_1$	\mathbf{c}_r	$\tilde{\mathbf{c}}_r$	\mathbf{c}_{ρ}
\mathbf{c}_r	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1} + 4\mathbf{c}_{\rho}$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
\mathbf{c}_{ρ}			$3\mathbf{1} + 2\mathbf{c}_{\rho}$

Minimal equation for \mathbf{c}_r

$$(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0) = 0$$



Tetrahedral T class projectors

$$\mathbf{c}_g = \sum_{\mu} \frac{\chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{\chi_g^{\mu}} \chi_g^{\mu*} \mathbf{c}_g$$

Minimal equation for \mathbf{c}_r : $(\mathbf{c}_r - 4\epsilon \mathbf{1})(\mathbf{c}_r - 4\epsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0) = 0$

$$\mathbf{P}^{(4\epsilon)} = \frac{(\mathbf{c}_r - 4\epsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0)}{(4\epsilon - 4\epsilon^*)(4\epsilon - 4)(4\epsilon - 0)}$$

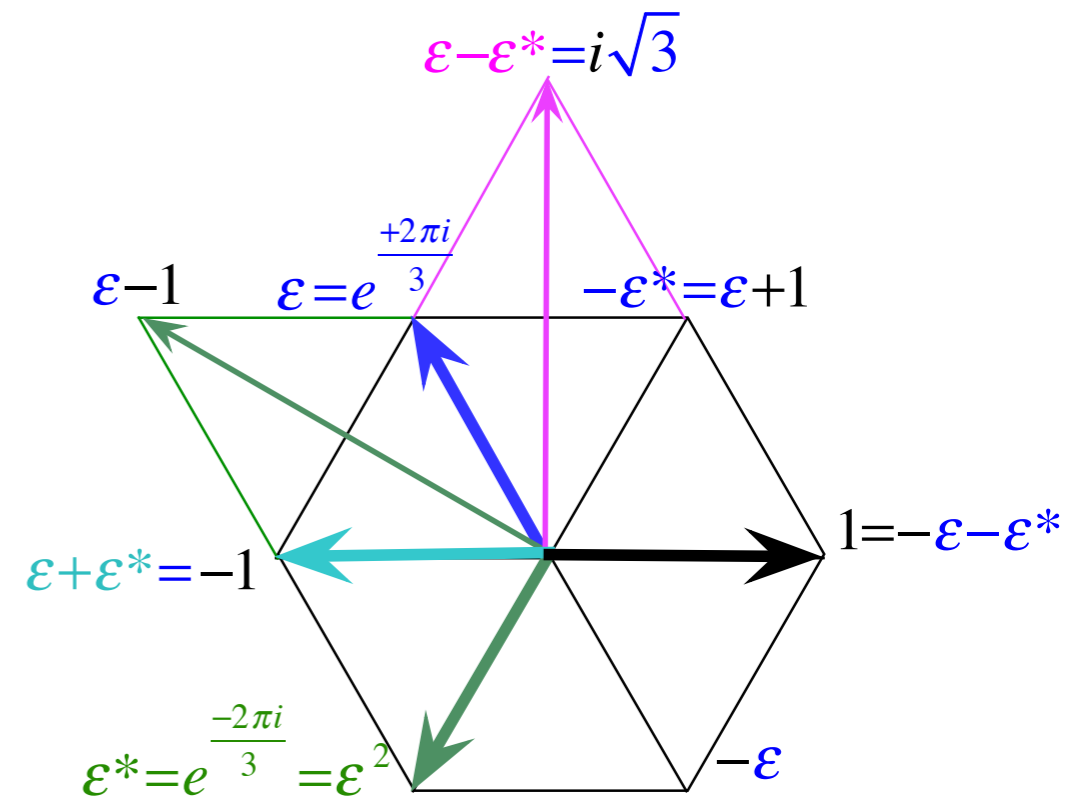
$$\mathbf{P}^{(4\epsilon^*)} = \frac{(\mathbf{c}_r - 4\epsilon \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0)}{(4\epsilon^* - 4\epsilon)(4\epsilon^* - 4)(4\epsilon^* - 0)}$$

T class products

$\mathbf{1} = \mathbf{c}_1$	\mathbf{c}_r	$\tilde{\mathbf{c}}_r$	\mathbf{c}_ρ
\mathbf{c}_r	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1} + 4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}_r$
\mathbf{c}_ρ			$3\mathbf{1} + 2\mathbf{c}_\rho$

Minimal equation for \mathbf{c}_r

$$(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0) = 0$$



Tetrahedral T class projectors

$$\mathbf{c}_g = \sum_{\mu} \frac{\circ c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{\circ G} \chi_g^{\mu*} c_g$$

Minimal equation for \mathbf{c}_r : $(\mathbf{c}_r - 4\epsilon \mathbf{1})(\mathbf{c}_r - 4\epsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0) = 0$

$$\mathbf{P}^{(4\epsilon)} = \frac{(\mathbf{c}_r - 4\epsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0)}{(4\epsilon - 4\epsilon^*)(4\epsilon - 4)(4\epsilon - 0)}$$

$$\mathbf{P}^{(4\epsilon^*)} = \frac{(\mathbf{c}_r - 4\epsilon \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0)}{(4\epsilon^* - 4\epsilon)(4\epsilon^* - 4)(4\epsilon^* - 0)}$$

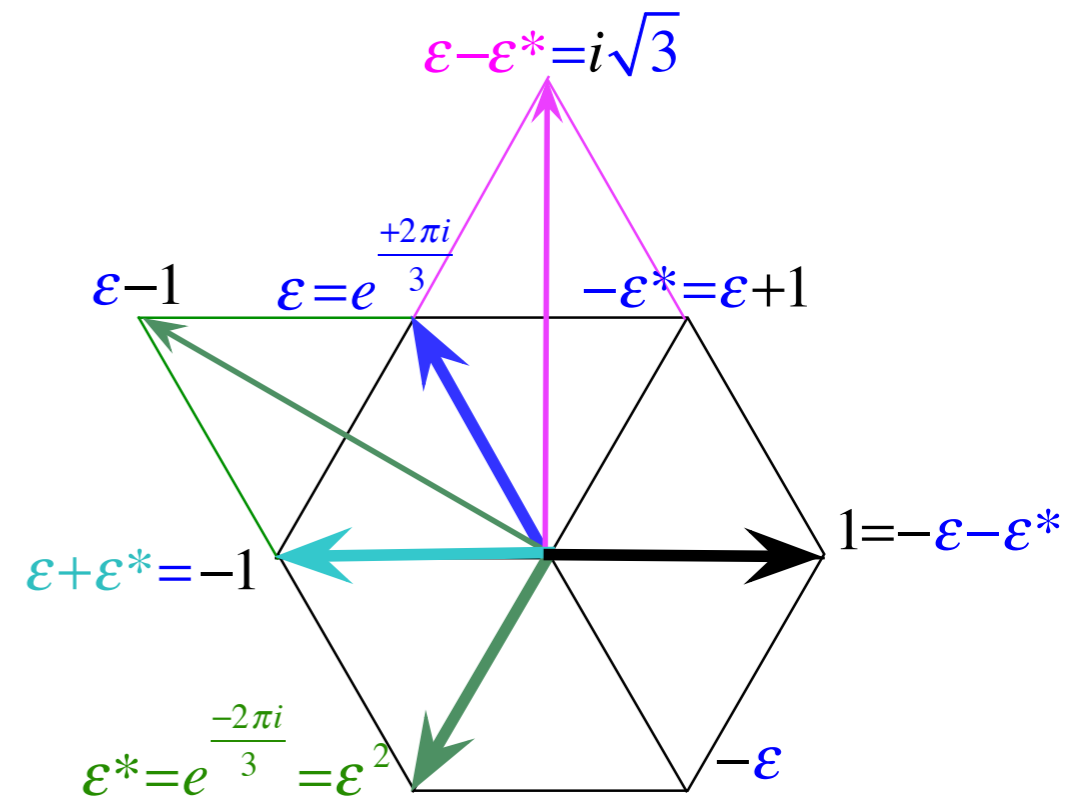
$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\epsilon \mathbf{1})(\mathbf{c}_r - 4\epsilon^* \mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\epsilon)(4 - 4\epsilon^*)(4 - 0)}$$

Minimal equation for \mathbf{c}_r

$$(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0) = 0$$

T class products

$\mathbf{1} = \mathbf{c}_1$	\mathbf{c}_r	$\tilde{\mathbf{c}}_r$	\mathbf{c}_ρ
\mathbf{c}_r	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1} + 4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
\mathbf{c}_ρ			$3\mathbf{1} + 2\mathbf{c}_\rho$



Tetrahedral T class projectors

$$\mathbf{c}_g = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{{}^{\circ}G \chi_g^{\mu*}} \mathbf{c}_g$$

Minimal equation for \mathbf{c}_r : $(\mathbf{c}_r - 4\epsilon \mathbf{1})(\mathbf{c}_r - 4\epsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0) = 0$

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$$\mathbf{P}^{(4\epsilon^*)} = \frac{(\mathbf{c}_r - 4\epsilon \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0)}{(4\epsilon^* - 4\epsilon)(4\epsilon^* - 4)(4\epsilon^* - 0)}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\epsilon \mathbf{1})(\mathbf{c}_r - 4\epsilon^* \mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\epsilon)(4 - 4\epsilon^*)(4 - 0)}$$

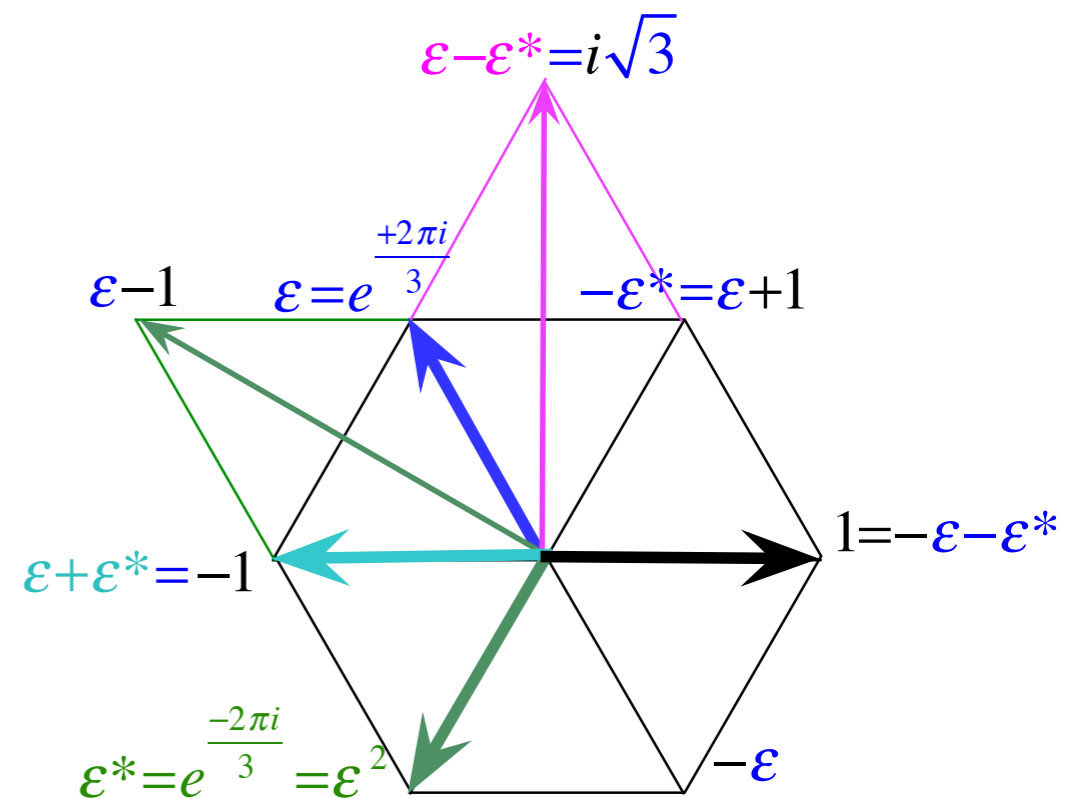
$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\epsilon \mathbf{1})(\mathbf{c}_r - 4\epsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})}{(0 - 4\epsilon)(0 - 4\epsilon^*)(0 - 4)}$$

Minimal equation for \mathbf{c}_r

$$(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0) = 0$$

T class products

$\mathbf{1} = \mathbf{c}_1$	\mathbf{c}_r	$\tilde{\mathbf{c}}_r$	\mathbf{c}_{ρ}
\mathbf{c}_r	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1} + 4\mathbf{c}_{\rho}$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
\mathbf{c}_{ρ}			$3\mathbf{1} + 2\mathbf{c}_{\rho}$



Tetrahedral T class projectors

$$\mathbf{c}_g = \sum_{\mu} \frac{\circ c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

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$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)}$$

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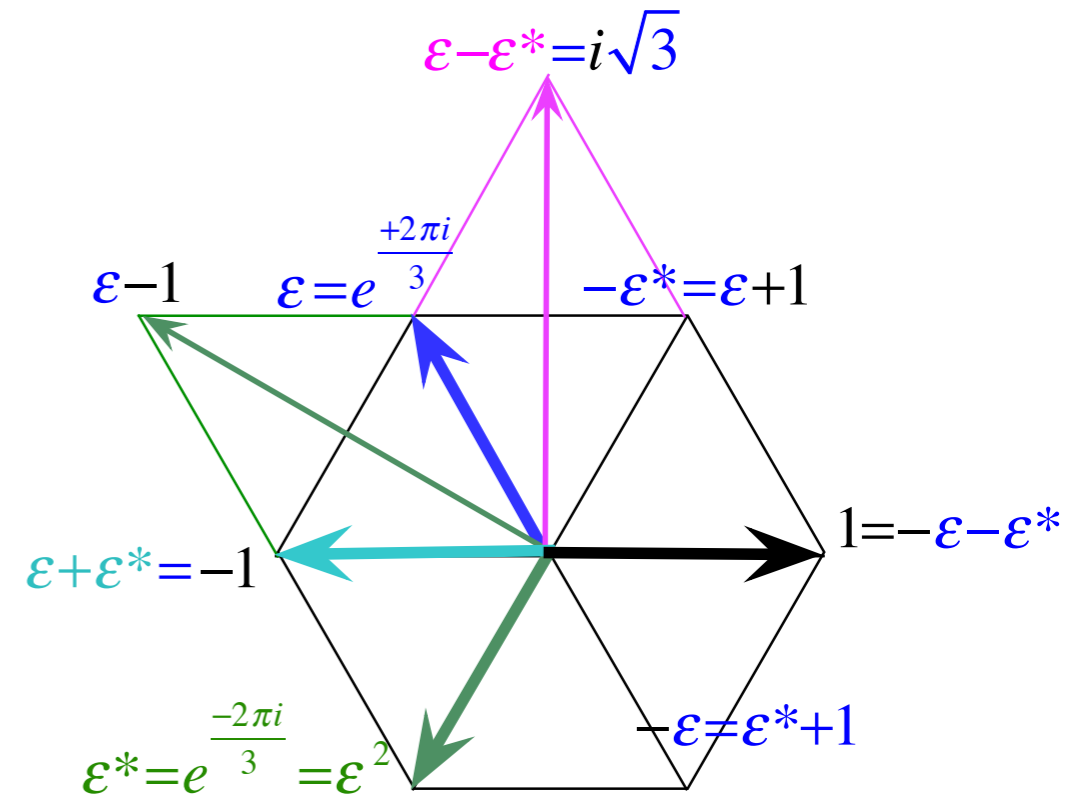
$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon \mathbf{1})(\mathbf{c}_r - 4\varepsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)}$$

Minimal equation for \mathbf{c}_r

$$(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0) = 0$$

T class products

$\mathbf{1} = \mathbf{c}_1$	\mathbf{c}_r	$\tilde{\mathbf{c}}_r$	\mathbf{c}_ρ
\mathbf{c}_r	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1} + 4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
\mathbf{c}_ρ			$3\mathbf{1} + 2\mathbf{c}_\rho$



Tetrahedral T class projectors

$$\mathbf{c}_g = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

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$$= \frac{4\tilde{\mathbf{c}}_r \mathbf{c}_r + 4(\varepsilon)4\tilde{\mathbf{c}}_r + 16 \cdot \varepsilon^* \mathbf{c}_r}{64i\sqrt{3}(\varepsilon^2 - \varepsilon)}$$

$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon \mathbf{1})(\mathbf{c}_r - 4\varepsilon^* \mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)}$$

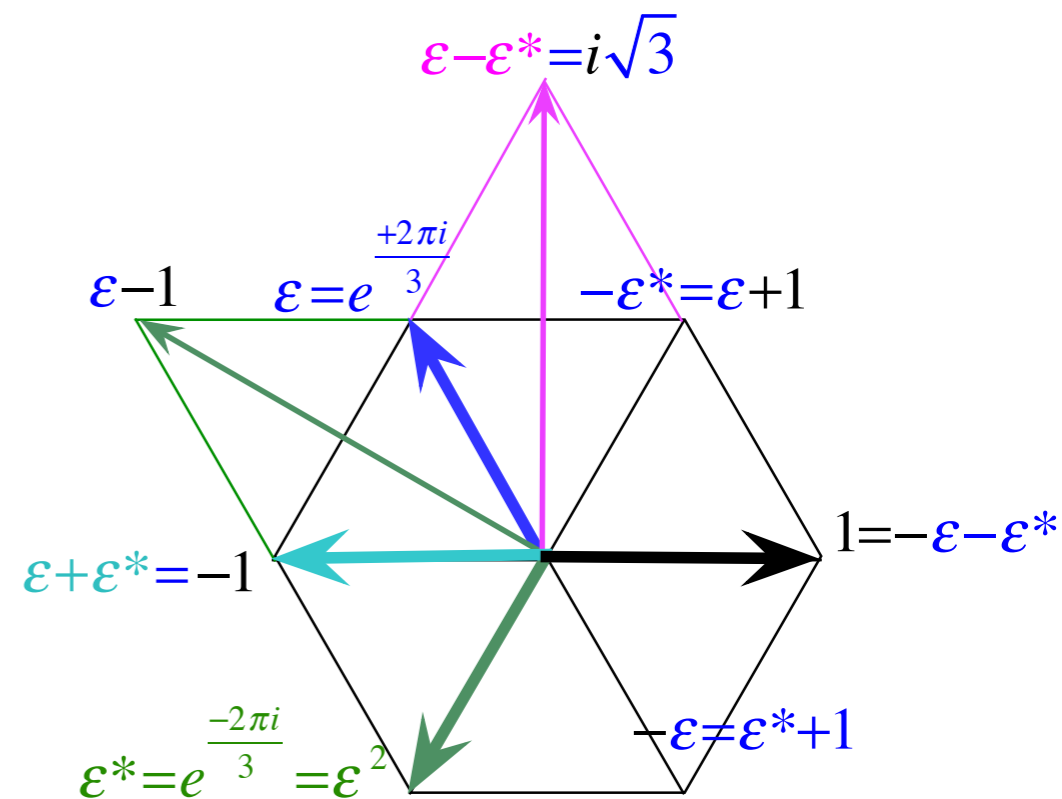
$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon \mathbf{1})(\mathbf{c}_r - 4\varepsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)}$$

Minimal equation for \mathbf{c}_r

$$(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0) = 0$$

T class products

$\mathbf{1} = \mathbf{c}_1$	\mathbf{c}_r	$\tilde{\mathbf{c}}_r$	\mathbf{c}_ρ
\mathbf{c}_r	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1} + 4\mathbf{c}_\rho$	$3\mathbf{c}_r$
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\mathbf{c}_ρ			$3\mathbf{1} + 2\mathbf{c}_\rho$



Tetrahedral T class projectors

T class products

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$$= \frac{4\tilde{\mathbf{c}}_r \mathbf{c}_r + 4(\varepsilon)4\tilde{\mathbf{c}}_r + 16 \cdot \varepsilon^* \mathbf{c}_r}{64i\sqrt{3}(\varepsilon^2 - \varepsilon)} = \frac{16(\mathbf{1} + \mathbf{c}_{\rho}) + 16\varepsilon\tilde{\mathbf{c}}_r + 16 \cdot \varepsilon^* \mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})}$$

$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)}$$

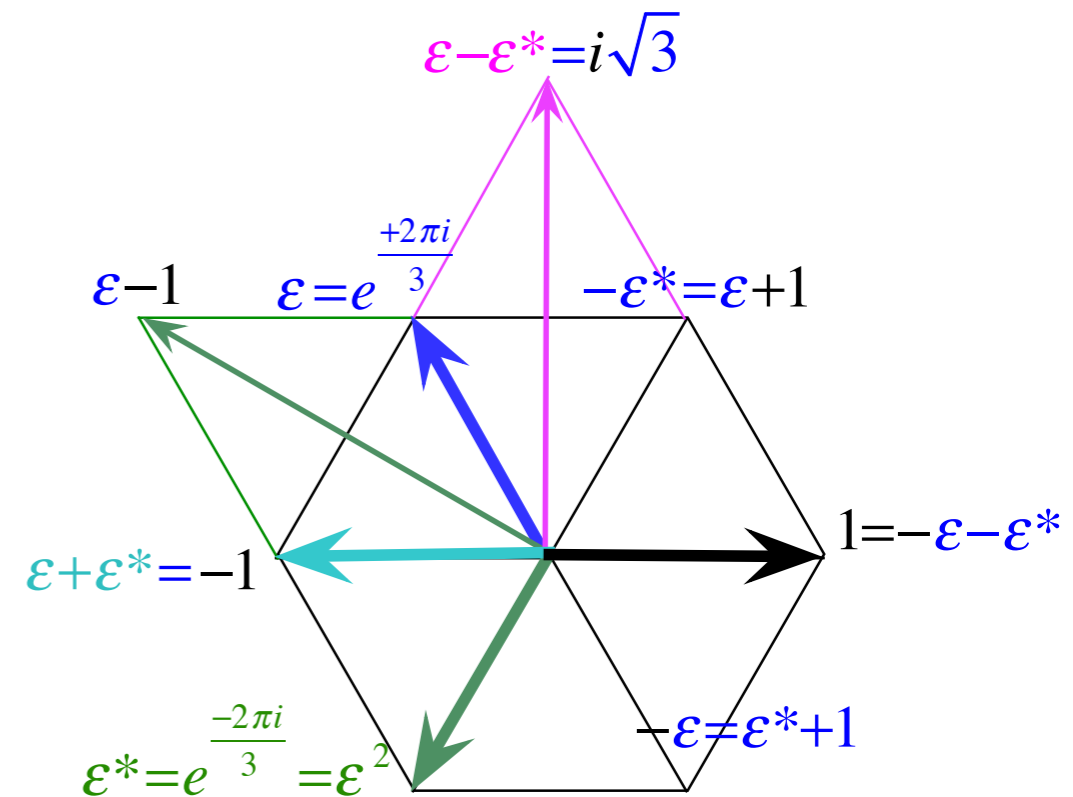
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$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon \mathbf{1})(\mathbf{c}_r - 4\varepsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)}$$

Minimal equation for \mathbf{c}_r

$$(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0) = 0$$

$\mathbf{1} = \mathbf{c}_1$	\mathbf{c}_r	$\tilde{\mathbf{c}}_r$	\mathbf{c}_{ρ}
\mathbf{c}_r	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1} + 4\mathbf{c}_{\rho}$	$3\mathbf{c}_r$
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Tetrahedral T class projectors

T class products

$$\mathbf{c}_g = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

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$$= \frac{4\tilde{\mathbf{c}}_r \mathbf{c}_r + 4(\epsilon)4\tilde{\mathbf{c}}_r + 16 \cdot \epsilon^* \mathbf{c}_r}{64i\sqrt{3}(\epsilon^2 - \epsilon)} = \frac{16(\mathbf{1} + \mathbf{c}_{\rho}) + 16\epsilon\tilde{\mathbf{c}}_r + 16 \cdot \epsilon^* \mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})} = \frac{\mathbf{1} + \mathbf{c}_{\rho} + \epsilon\tilde{\mathbf{c}}_r + \epsilon^* \mathbf{c}_r}{12}$$

$$\mathbf{P}^{(4\epsilon^*)} = \frac{(\mathbf{c}_r - 4\epsilon \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0)}{(4\epsilon^* - 4\epsilon)(4\epsilon^* - 4)(4\epsilon^* - 0)}$$

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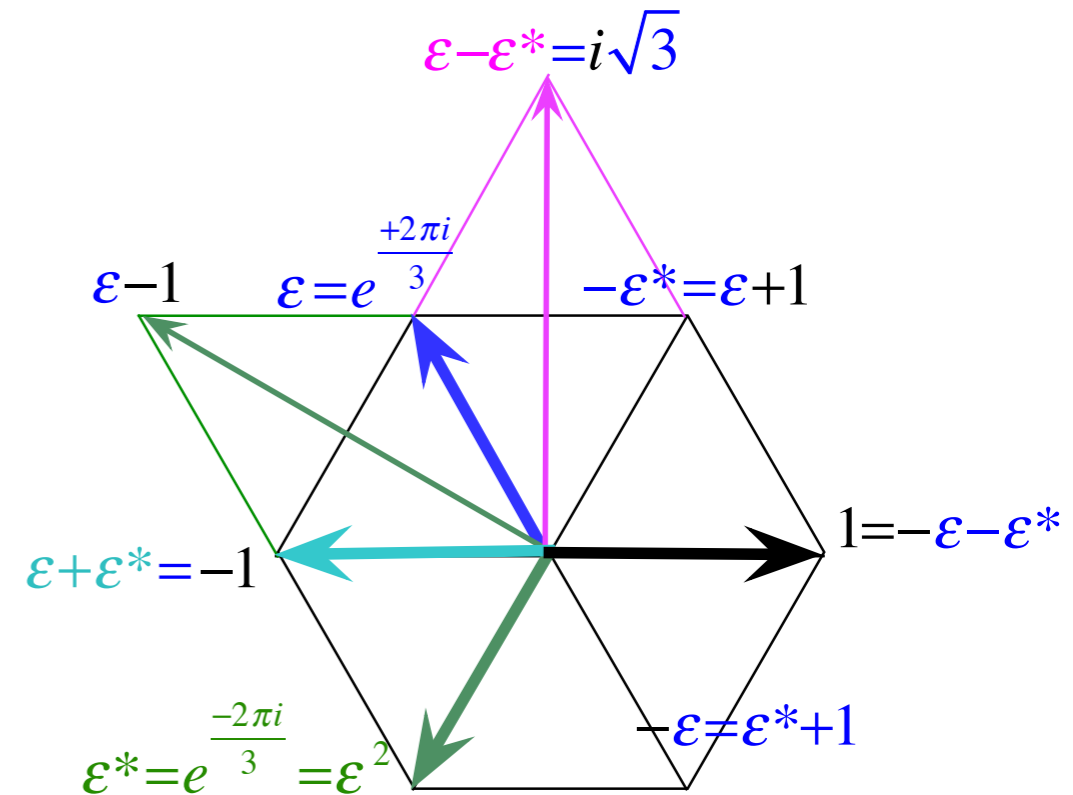
$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\epsilon \mathbf{1})(\mathbf{c}_r - 4\epsilon^* \mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\epsilon)(4 - 4\epsilon^*)(4 - 0)}$$

$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\epsilon \mathbf{1})(\mathbf{c}_r - 4\epsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})}{(0 - 4\epsilon)(0 - 4\epsilon^*)(0 - 4)}$$

Minimal equation for \mathbf{c}_r

$$(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0) = 0$$

$\mathbf{1} = \mathbf{c}_1$	\mathbf{c}_r	$\tilde{\mathbf{c}}_r$	\mathbf{c}_{ρ}
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$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
\mathbf{c}_{ρ}			$3\mathbf{1} + 2\mathbf{c}_{\rho}$



Tetrahedral T class projectors

T class products

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$$\mathbf{P}^{(4\epsilon)} = \frac{(\mathbf{c}_r - 4\epsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0)}{(4\epsilon - 4\epsilon^*)(4\epsilon - 4)(4\epsilon - 0)} = \frac{(\mathbf{c}_r^2 - 4(\epsilon^* + 1)\mathbf{c}_r + 16 \cdot \epsilon^*)\mathbf{c}_r}{64(\epsilon - \epsilon^*)(\epsilon - 1)\epsilon}$$

$$= \frac{4\tilde{\mathbf{c}}_r \mathbf{c}_r + 4(\epsilon)4\tilde{\mathbf{c}}_r + 16 \cdot \epsilon^* \mathbf{c}_r}{64i\sqrt{3}(\epsilon^2 - \epsilon)} = \frac{16(\mathbf{1} + \mathbf{c}_\rho) + 16\epsilon\tilde{\mathbf{c}}_r + 16 \cdot \epsilon^* \mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})} = \frac{\mathbf{1} + \mathbf{c}_\rho + \epsilon\tilde{\mathbf{c}}_r + \epsilon^* \mathbf{c}_r}{12}$$

$$\mathbf{P}^{(4\epsilon^*)} = \frac{(\mathbf{c}_r - 4\epsilon \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0)}{(4\epsilon^* - 4\epsilon)(4\epsilon^* - 4)(4\epsilon^* - 0)}$$

$$= \frac{\mathbf{1} + \mathbf{c}_\rho + \epsilon^* \tilde{\mathbf{c}}_r + \epsilon \mathbf{c}_r}{12}$$

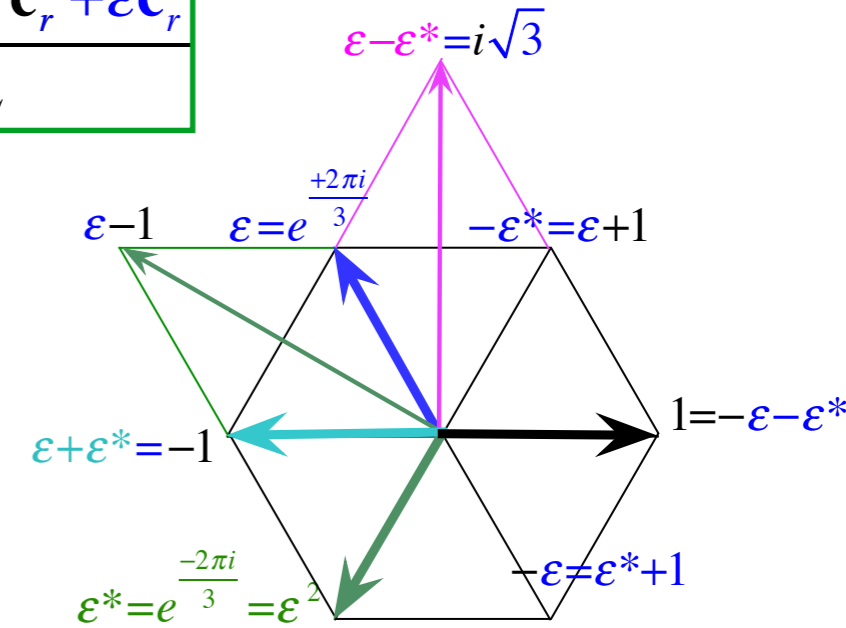
$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\epsilon \mathbf{1})(\mathbf{c}_r - 4\epsilon^* \mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\epsilon)(4 - 4\epsilon^*)(4 - 0)}$$

$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\epsilon \mathbf{1})(\mathbf{c}_r - 4\epsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})}{(0 - 4\epsilon)(0 - 4\epsilon^*)(0 - 4)}$$

Minimal equation for \mathbf{c}_r

$$(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0) = 0$$

$\mathbf{1} = \mathbf{c}_1$	\mathbf{c}_r	$\tilde{\mathbf{c}}_r$	\mathbf{c}_ρ
\mathbf{c}_r	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1} + 4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}_r$
\mathbf{c}_ρ			$3\mathbf{1} + 2\mathbf{c}_\rho$



$T : \mathbf{c}_g =$	\mathbf{c}_1	\mathbf{c}_r	$\tilde{\mathbf{c}}_r$	\mathbf{c}_ρ
\cdot				
$\chi_g^\epsilon =$	1	ϵ^*	ϵ	1
$\chi_g^{\epsilon^*} =$	1	ϵ	ϵ^*	1
\cdot				

Tetrahedral T class projectors

T class products

$$\mathbf{c}_g = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{{}^{\circ}G \chi_g^{\mu*}} \mathbf{c}_g$$

Minimal equation for \mathbf{c}_r : $(\mathbf{c}_r - 4\varepsilon \mathbf{1})(\mathbf{c}_r - 4\varepsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0) = 0$

$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon^* + 1)\mathbf{c}_r + 16 \cdot \varepsilon^*)\mathbf{c}_r}{64(\varepsilon - \varepsilon^*)(\varepsilon - 1)\varepsilon}$$

$$= \frac{4\tilde{\mathbf{c}}_r \mathbf{c}_r + 4(\varepsilon)4\tilde{\mathbf{c}}_r + 16 \cdot \varepsilon^* \mathbf{c}_r}{64i\sqrt{3}(\varepsilon^2 - \varepsilon)} = \frac{16(\mathbf{1} + \mathbf{c}_{\rho}) + 16\varepsilon\tilde{\mathbf{c}}_r + 16 \cdot \varepsilon^* \mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})} = \frac{\mathbf{1} + \mathbf{c}_{\rho} + \varepsilon\tilde{\mathbf{c}}_r + \varepsilon^* \mathbf{c}_r}{12}$$

$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)}$$

$$= \frac{\mathbf{1} + \mathbf{c}_{\rho} + \varepsilon^* \tilde{\mathbf{c}}_r + \varepsilon \mathbf{c}_r}{12}$$

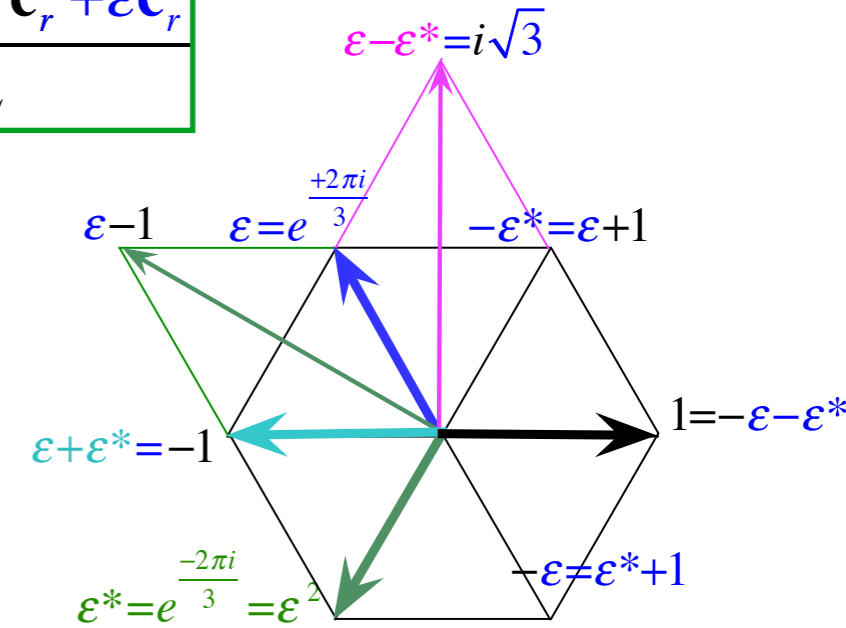
$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon \mathbf{1})(\mathbf{c}_r - 4\varepsilon^* \mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon + \varepsilon^*)\mathbf{c}_r + 16\mathbf{1})\mathbf{c}_r}{64(1 - (\varepsilon + \varepsilon^*) + 1)}$$

$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon \mathbf{1})(\mathbf{c}_r - 4\varepsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)}$$

Minimal equation for \mathbf{c}_r

$$(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0) = 0$$

$\mathbf{1} = \mathbf{c}_1$	\mathbf{c}_r	$\tilde{\mathbf{c}}_r$	\mathbf{c}_{ρ}
\mathbf{c}_r	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1} + 4\mathbf{c}_{\rho}$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}_r$
\mathbf{c}_{ρ}			$3\mathbf{1} + 2\mathbf{c}_{\rho}$



$T : \mathbf{c}_g =$	\mathbf{c}_1	\mathbf{c}_r	$\tilde{\mathbf{c}}_r$	\mathbf{c}_{ρ}
\cdot				
$\chi_g^{\varepsilon} =$	1	ε^*	ε	1
$\chi_g^{\varepsilon^*} =$	1	ε	ε^*	1
\cdot				

Tetrahedral T class projectors

T class products

$$\mathbf{c}_g = \sum_{\mu} \frac{{}^{\circ}\mathbf{c}_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } \mathbf{c}_g} \frac{\ell^{\mu}}{{}^{\circ}\mathbf{G} \chi_g^{\mu*}} \mathbf{c}_g$$

Minimal equation for \mathbf{c}_r : $(\mathbf{c}_r - 4\varepsilon \mathbf{1})(\mathbf{c}_r - 4\varepsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0) = 0$

$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon^* + 1)\mathbf{c}_r + 16 \cdot \varepsilon^*)\mathbf{c}_r}{64(\varepsilon - \varepsilon^*)(\varepsilon - 1)\varepsilon}$$

$$= \frac{4\tilde{\mathbf{c}}_r \mathbf{c}_r + 4(\varepsilon)4\tilde{\mathbf{c}}_r + 16 \cdot \varepsilon^* \mathbf{c}_r}{64i\sqrt{3}(\varepsilon^2 - \varepsilon)} = \frac{16(\mathbf{1} + \mathbf{c}_\rho) + 16\varepsilon\tilde{\mathbf{c}}_r + 16 \cdot \varepsilon^* \mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})} = \frac{\mathbf{1} + \mathbf{c}_\rho + \varepsilon\tilde{\mathbf{c}}_r + \varepsilon^* \mathbf{c}_r}{12}$$

$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)}$$

$$= \frac{\mathbf{1} + \mathbf{c}_\rho + \varepsilon^* \tilde{\mathbf{c}}_r + \varepsilon \mathbf{c}_r}{12}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon \mathbf{1})(\mathbf{c}_r - 4\varepsilon^* \mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon + \varepsilon^*)\mathbf{c}_r + 16\mathbf{1})\mathbf{c}_r}{64(1 - (\varepsilon + \varepsilon^*) + 1)}$$

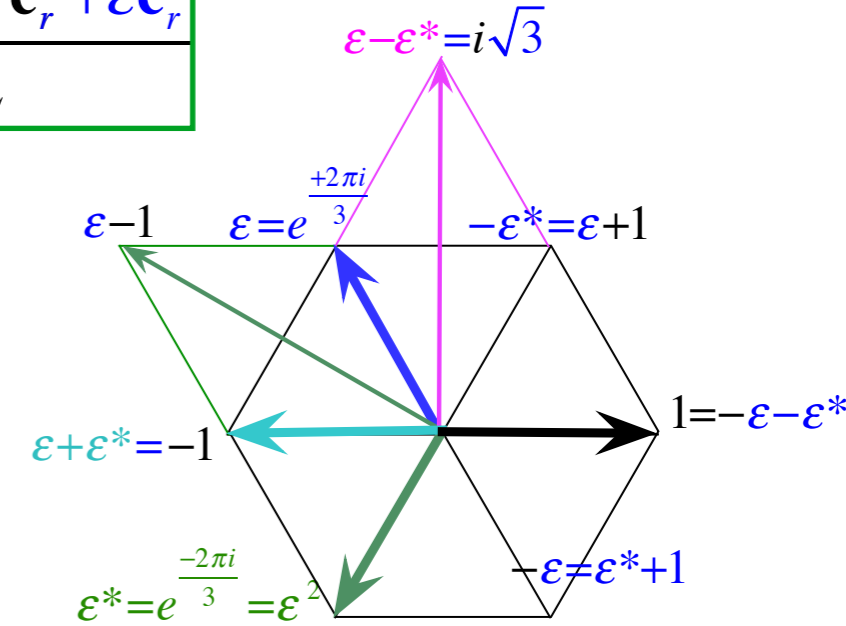
$$= \frac{4\tilde{\mathbf{c}}_r \mathbf{c}_r + 4 \cdot 4\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1 + 1 + 1)}$$

$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon \mathbf{1})(\mathbf{c}_r - 4\varepsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)}$$

Minimal equation for \mathbf{c}_r

$$(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0) = 0$$

$\mathbf{1} = \mathbf{c}_1$	\mathbf{c}_r	$\tilde{\mathbf{c}}_r$	\mathbf{c}_ρ
\mathbf{c}_r	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1} + 4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}_r$
\mathbf{c}_ρ			$3\mathbf{1} + 2\mathbf{c}_\rho$



$T : \mathbf{c}_g =$	\mathbf{c}_1	\mathbf{c}_r	$\tilde{\mathbf{c}}_r$	\mathbf{c}_ρ
\cdot				
$\chi_g^{\varepsilon} =$	1	ε^*	ε	1
$\chi_g^{\varepsilon^*} =$	1	ε	ε^*	1
\cdot				

Tetrahedral T class projectors

T class products

$$\mathbf{c}_g = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{{}^{\circ}G \chi_g^{\mu*}} \mathbf{c}_g$$

Minimal equation for \mathbf{c}_r : $(\mathbf{c}_r - 4\epsilon \mathbf{1})(\mathbf{c}_r - 4\epsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0) = 0$

$$\mathbf{P}^{(4\epsilon)} = \frac{(\mathbf{c}_r - 4\epsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0)}{(4\epsilon - 4\epsilon^*)(4\epsilon - 4)(4\epsilon - 0)} = \frac{(\mathbf{c}_r^2 - 4(\epsilon^* + 1)\mathbf{c}_r + 16 \cdot \epsilon^*)\mathbf{c}_r}{64(\epsilon - \epsilon^*)(\epsilon - 1)\epsilon}$$

$$= \frac{4\tilde{\mathbf{c}}_r \mathbf{c}_r + 4(\epsilon)4\tilde{\mathbf{c}}_r + 16 \cdot \epsilon^* \mathbf{c}_r}{64i\sqrt{3}(\epsilon^2 - \epsilon)} = \frac{16(\mathbf{1} + \mathbf{c}_{\rho}) + 16\epsilon\tilde{\mathbf{c}}_r + 16 \cdot \epsilon^* \mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})} = \frac{\mathbf{1} + \mathbf{c}_{\rho} + \epsilon\tilde{\mathbf{c}}_r + \epsilon^* \mathbf{c}_r}{12}$$

$$\mathbf{P}^{(4\epsilon^*)} = \frac{(\mathbf{c}_r - 4\epsilon \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0)}{(4\epsilon^* - 4\epsilon)(4\epsilon^* - 4)(4\epsilon^* - 0)}$$

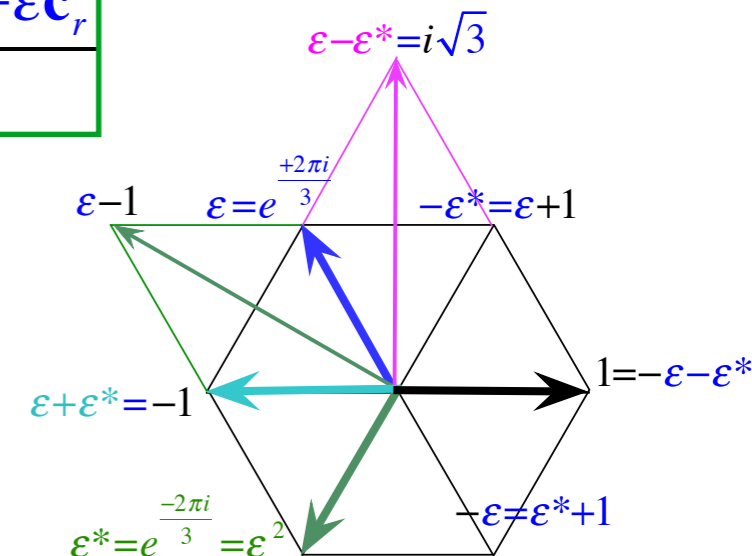
$$= \frac{\mathbf{1} + \mathbf{c}_{\rho} + \epsilon^* \tilde{\mathbf{c}}_r + \epsilon \mathbf{c}_r}{12}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\epsilon \mathbf{1})(\mathbf{c}_r - 4\epsilon^* \mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\epsilon)(4 - 4\epsilon^*)(4 - 0)} = \frac{(\mathbf{c}_r^2 - 4(\epsilon + \epsilon^*)\mathbf{c}_r + 16\mathbf{1})\mathbf{c}_r}{64(1 - (\epsilon + \epsilon^*) + 1)}$$

$$= \frac{4\tilde{\mathbf{c}}_r \mathbf{c}_r + 4 \cdot 4\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1+1+1)} = \frac{16(\mathbf{1} + \mathbf{c}_{\rho}) + 16\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1+1+1)} = \frac{\mathbf{1} + \mathbf{c}_{\rho} + \tilde{\mathbf{c}}_r + \mathbf{c}_r}{12}$$

$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\epsilon \mathbf{1})(\mathbf{c}_r - 4\epsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})}{(0 - 4\epsilon)(0 - 4\epsilon^*)(0 - 4)}$$

$\mathbf{1} = \mathbf{c}_1$	\mathbf{c}_r	$\tilde{\mathbf{c}}_r$	\mathbf{c}_{ρ}
\mathbf{c}_r	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1} + 4\mathbf{c}_{\rho}$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}_r$
\mathbf{c}_{ρ}			$3\mathbf{1} + 2\mathbf{c}_{\rho}$



Minimal equation for \mathbf{c}_r

$$(\mathbf{c}_r - 4e^{+2\pi i/3} \mathbf{1})(\mathbf{c}_r - 4e^{-2\pi i/3} \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0) = 0$$

$T : \mathbf{c}_g =$	\mathbf{c}_1	\mathbf{c}_r	$\tilde{\mathbf{c}}_r$	\mathbf{c}_{ρ}
$\chi_g^A =$	1	1	1	1
$\chi_g^{\epsilon} =$	1	ϵ^*	ϵ	1
$\chi_g^{\epsilon^*} =$	1	ϵ	ϵ^*	1
.				

Tetrahedral T class projectors

T class products

$$\mathbf{c}_g = \sum_{\mu} \frac{{}^{\circ}\mathbf{c}_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } \mathbf{c}_g} \frac{\ell^{\mu}}{{}^{\circ}\mathbf{G} \chi_g^{\mu*}} \mathbf{c}_g$$

Minimal equation for \mathbf{c}_r : $(\mathbf{c}_r - 4\epsilon \mathbf{1})(\mathbf{c}_r - 4\epsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0) = 0$

$$\mathbf{P}^{(4\epsilon)} = \frac{(\mathbf{c}_r - 4\epsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0)}{(4\epsilon - 4\epsilon^*)(4\epsilon - 4)(4\epsilon - 0)} = \frac{(\mathbf{c}_r^2 - 4(\epsilon^* + 1)\mathbf{c}_r + 16 \cdot \epsilon^*)\mathbf{c}_r}{64(\epsilon - \epsilon^*)(\epsilon - 1)\epsilon}$$

$$= \frac{4\tilde{\mathbf{c}}_r \mathbf{c}_r + 4(\epsilon)4\tilde{\mathbf{c}}_r + 16 \cdot \epsilon^* \mathbf{c}_r}{64i\sqrt{3}(\epsilon^2 - \epsilon)} = \frac{16(\mathbf{1} + \mathbf{c}_{\rho}) + 16\epsilon\tilde{\mathbf{c}}_r + 16 \cdot \epsilon^* \mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})} = \frac{\mathbf{1} + \mathbf{c}_{\rho} + \epsilon\tilde{\mathbf{c}}_r + \epsilon^* \mathbf{c}_r}{12}$$

$$\mathbf{P}^{(4\epsilon^*)} = \frac{(\mathbf{c}_r - 4\epsilon \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0)}{(4\epsilon^* - 4\epsilon)(4\epsilon^* - 4)(4\epsilon^* - 0)}$$

$$= \frac{\mathbf{1} + \mathbf{c}_{\rho} + \epsilon^* \tilde{\mathbf{c}}_r + \epsilon \mathbf{c}_r}{12}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\epsilon \mathbf{1})(\mathbf{c}_r - 4\epsilon^* \mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\epsilon)(4 - 4\epsilon^*)(4 - 0)} = \frac{(\mathbf{c}_r^2 - 4(\epsilon + \epsilon^*)\mathbf{c}_r + 16\mathbf{1})\mathbf{c}_r}{64(1 - (\epsilon + \epsilon^*) + 1)}$$

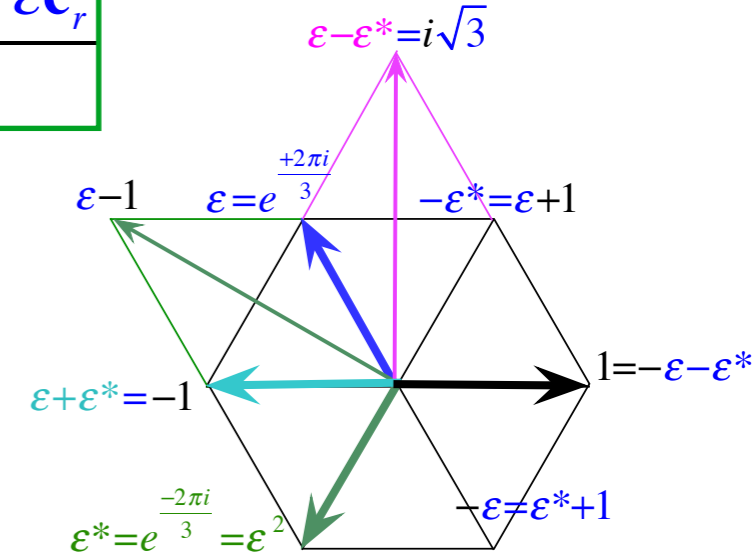
$$= \frac{4\tilde{\mathbf{c}}_r \mathbf{c}_r + 4 \cdot 4\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1+1+1)} = \frac{16(\mathbf{1} + \mathbf{c}_{\rho}) + 16\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1+1+1)} = \frac{\mathbf{1} + \mathbf{c}_{\rho} + \tilde{\mathbf{c}}_r + \mathbf{c}_r}{12}$$

$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\epsilon \mathbf{1})(\mathbf{c}_r - 4\epsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})}{(0 - 4\epsilon)(0 - 4\epsilon^*)(0 - 4)} = \frac{(\mathbf{c}_r^2 - 4(\epsilon + \epsilon^*)\mathbf{c}_r + 16 \cdot \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})}{64(-\epsilon)(-\epsilon^*)(-1)}$$

Minimal equation for \mathbf{c}_r

$$(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0) = 0$$

$\mathbf{1} = \mathbf{c}_1$	\mathbf{c}_r	$\tilde{\mathbf{c}}_r$	\mathbf{c}_{ρ}
\mathbf{c}_r	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1} + 4\mathbf{c}_{\rho}$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}_r$
\mathbf{c}_{ρ}			$3\mathbf{1} + 2\mathbf{c}_{\rho}$



$T : \mathbf{c}_g =$	\mathbf{c}_1	\mathbf{c}_r	$\tilde{\mathbf{c}}_r$	\mathbf{c}_{ρ}
$\chi_g^A =$	1	1	1	1
$\chi_g^{\epsilon} =$	1	ϵ^*	ϵ	1
$\chi_g^{\epsilon^*} =$	1	ϵ	ϵ^*	1
.				

Tetrahedral T class projectors

T class products

$$\mathbf{c}_g = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

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Minimal equation for \mathbf{c}_r : $(\mathbf{c}_r - 4\epsilon\mathbf{1})(\mathbf{c}_r - 4\epsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = 0$

$$\mathbf{P}^{(4\epsilon)} = \frac{(\mathbf{c}_r - 4\epsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\epsilon - 4\epsilon^*)(4\epsilon - 4)(4\epsilon - 0)} = \frac{(\mathbf{c}_r^2 - 4(\epsilon^* + 1)\mathbf{c}_r + 16\cdot\epsilon^*)\mathbf{c}_r}{64(\epsilon - \epsilon^*)(\epsilon - 1)\epsilon}$$

$$= \frac{4\tilde{\mathbf{c}}_r \mathbf{c}_r + 4(\epsilon)4\tilde{\mathbf{c}}_r + 16\cdot\epsilon^*\mathbf{c}_r}{64i\sqrt{3}(\epsilon^2 - \epsilon)} = \frac{16(\mathbf{1} + \mathbf{c}_{\rho}) + 16\epsilon\tilde{\mathbf{c}}_r + 16\cdot\epsilon^*\mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})} = \frac{\mathbf{1} + \mathbf{c}_{\rho} + \epsilon\tilde{\mathbf{c}}_r + \epsilon^*\mathbf{c}_r}{12}$$

$$\mathbf{P}^{(4\epsilon^*)} = \frac{(\mathbf{c}_r - 4\epsilon\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\epsilon^* - 4\epsilon)(4\epsilon^* - 4)(4\epsilon^* - 0)}$$

$$= \frac{\mathbf{1} + \mathbf{c}_{\rho} + \epsilon^*\tilde{\mathbf{c}}_r + \epsilon\mathbf{c}_r}{12}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\epsilon\mathbf{1})(\mathbf{c}_r - 4\epsilon^*\mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\epsilon)(4 - 4\epsilon^*)(4 - 0)} = \frac{(\mathbf{c}_r^2 - 4(\epsilon + \epsilon^*)\mathbf{c}_r + 16\mathbf{1})\mathbf{c}_r}{64(1 - (\epsilon + \epsilon^*) + 1)}$$

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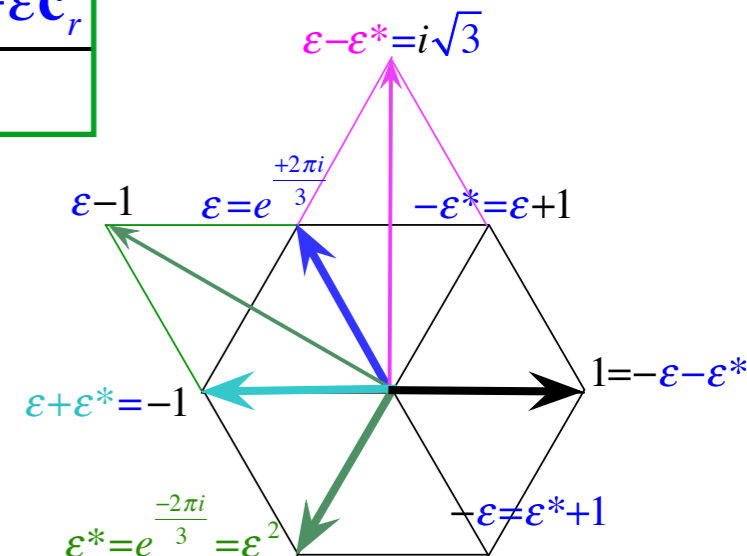
$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\epsilon\mathbf{1})(\mathbf{c}_r - 4\epsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{(0 - 4\epsilon)(0 - 4\epsilon^*)(0 - 4)} = \frac{(\mathbf{c}_r^2 - 4(\epsilon + \epsilon^*)\mathbf{c}_r + 16\cdot\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{64(-\epsilon)(-\epsilon^*)(-1)}$$

$$= \frac{4\tilde{\mathbf{c}}_r + 4\mathbf{c}_r + 16\cdot\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{-64}$$

Minimal equation for \mathbf{c}_r

$$(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = 0$$

$\mathbf{1} = \mathbf{c}_1$	\mathbf{c}_r	$\tilde{\mathbf{c}}_r$	\mathbf{c}_{ρ}
\mathbf{c}_r	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1} + 4\mathbf{c}_{\rho}$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}_r$
\mathbf{c}_{ρ}			$3\mathbf{1} + 2\mathbf{c}_{\rho}$



$T : \mathbf{c}_g =$	\mathbf{c}_1	\mathbf{c}_r	$\tilde{\mathbf{c}}_r$	\mathbf{c}_{ρ}
$\chi_g^A =$	1	1	1	1
$\chi_g^{\epsilon} =$	1	ϵ^*	ϵ	1
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.				

Tetrahedral T class projectors

T class products

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$$= \frac{4\tilde{\mathbf{c}}_r \mathbf{c}_r + 4(\epsilon)4\tilde{\mathbf{c}}_r + 16\cdot\epsilon^*\mathbf{c}_r}{64i\sqrt{3}(\epsilon^2 - \epsilon)} = \frac{16(1 + \mathbf{c}_\rho) + 16\epsilon\tilde{\mathbf{c}}_r + 16\cdot\epsilon^*\mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})} = \frac{1 + \mathbf{c}_\rho + \epsilon\tilde{\mathbf{c}}_r + \epsilon^*\mathbf{c}_r}{12}$$

$$\mathbf{P}^{(4\epsilon^*)} = \frac{(\mathbf{c}_r - 4\epsilon\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\epsilon^* - 4\epsilon)(4\epsilon^* - 4)(4\epsilon^* - 0)}$$

$$= \frac{1 + \mathbf{c}_\rho + \epsilon^*\tilde{\mathbf{c}}_r + \epsilon\mathbf{c}_r}{12}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\epsilon\mathbf{1})(\mathbf{c}_r - 4\epsilon^*\mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\epsilon)(4 - 4\epsilon^*)(4 - 0)} = \frac{(\mathbf{c}_r^2 - 4(\epsilon + \epsilon^*)\mathbf{c}_r + 16\mathbf{1})\mathbf{c}_r}{64(1 - (\epsilon + \epsilon^*) + 1)}$$

$$= \frac{4\tilde{\mathbf{c}}_r \mathbf{c}_r + 4\cdot 4\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1 + 1 + 1)} = \frac{16(1 + \mathbf{c}_\rho) + 16\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1 + 1 + 1)} = \frac{1 + \mathbf{c}_\rho + \tilde{\mathbf{c}}_r + \mathbf{c}_r}{12}$$

$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\epsilon\mathbf{1})(\mathbf{c}_r - 4\epsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{(0 - 4\epsilon)(0 - 4\epsilon^*)(0 - 4)} = \frac{(\mathbf{c}_r^2 - 4(\epsilon + \epsilon^*)\mathbf{c}_r + 16\cdot\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{64(-\epsilon)(-\epsilon^*)(-1)}$$

$$= \frac{4\tilde{\mathbf{c}}_r + 4\mathbf{c}_r + 16\cdot\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{-64}$$

$$= \frac{4\tilde{\mathbf{c}}_r \mathbf{c}_r + 4\mathbf{c}_r^2 + 16\mathbf{c}_r - 16\tilde{\mathbf{c}}_r - 16\mathbf{c}_r - 64\cdot\mathbf{1}}{-64} =$$

$\mathbf{1} = \mathbf{c}_1$	\mathbf{c}_r	$\tilde{\mathbf{c}}_r$	\mathbf{c}_ρ
\mathbf{c}_r	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1} + 4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}_r$
\mathbf{c}_ρ			$3\mathbf{1} + 2\mathbf{c}_\rho$

$T : \mathbf{c}_g =$	\mathbf{c}_1	\mathbf{c}_r	$\tilde{\mathbf{c}}_r$	\mathbf{c}_ρ
$\chi_g^A =$	1	1	1	1
$\chi_g^\epsilon =$	1	ϵ^*	ϵ	1
$\chi_g^{\epsilon^*} =$	1	ϵ	ϵ^*	1
.				

Tetrahedral T class projectors

T class products

$$\mathbf{c}_g = \sum_{\mu} \frac{{}^{\circ}\mathbf{c}_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } \mathbf{c}_g} \frac{\ell^{\mu}}{{}^{\circ}\mathbf{G} \chi_g^{\mu*}} \mathbf{c}_g$$

Minimal equation for \mathbf{c}_r : $(\mathbf{c}_r - 4\epsilon\mathbf{1})(\mathbf{c}_r - 4\epsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = 0$

$$\mathbf{P}^{(4\epsilon)} = \frac{(\mathbf{c}_r - 4\epsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\epsilon - 4\epsilon^*)(4\epsilon - 4)(4\epsilon - 0)} = \frac{(\mathbf{c}_r^2 - 4(\epsilon^* + 1)\mathbf{c}_r + 16\cdot\epsilon^*)\mathbf{c}_r}{64(\epsilon - \epsilon^*)(\epsilon - 1)\epsilon}$$

$$= \frac{4\tilde{\mathbf{c}}_r \mathbf{c}_r + 4(\epsilon)4\tilde{\mathbf{c}}_r + 16\cdot\epsilon^*\mathbf{c}_r}{64i\sqrt{3}(\epsilon^2 - \epsilon)} = \frac{16(1 + \mathbf{c}_{\rho}) + 16\epsilon\tilde{\mathbf{c}}_r + 16\cdot\epsilon^*\mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})} = \frac{1 + \mathbf{c}_{\rho} + \epsilon\tilde{\mathbf{c}}_r + \epsilon^*\mathbf{c}_r}{12}$$

$$\mathbf{P}^{(4\epsilon^*)} = \frac{(\mathbf{c}_r - 4\epsilon\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\epsilon^* - 4\epsilon)(4\epsilon^* - 4)(4\epsilon^* - 0)} = \frac{1 + \mathbf{c}_{\rho} + \epsilon^*\tilde{\mathbf{c}}_r + \epsilon\mathbf{c}_r}{12}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\epsilon\mathbf{1})(\mathbf{c}_r - 4\epsilon^*\mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\epsilon)(4 - 4\epsilon^*)(4 - 0)} = \frac{(\mathbf{c}_r^2 - 4(\epsilon + \epsilon^*)\mathbf{c}_r + 16\mathbf{1})\mathbf{c}_r}{64(1 - (\epsilon + \epsilon^*) + 1)}$$

$$= \frac{4\tilde{\mathbf{c}}_r \mathbf{c}_r + 4\cdot 4\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1 + 1 + 1)} = \frac{16(1 + \mathbf{c}_{\rho}) + 16\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1 + 1 + 1)} = \frac{1 + \mathbf{c}_{\rho} + \tilde{\mathbf{c}}_r + \mathbf{c}_r}{12}$$

$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\epsilon\mathbf{1})(\mathbf{c}_r - 4\epsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{(0 - 4\epsilon)(0 - 4\epsilon^*)(0 - 4)} = \frac{(\mathbf{c}_r^2 - 4(\epsilon + \epsilon^*)\mathbf{c}_r + 16\cdot\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{64(-\epsilon)(-\epsilon^*)(-1)}$$

$$= \frac{4\tilde{\mathbf{c}}_r + 4\mathbf{c}_r + 16\cdot\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{-64}$$

$$= \frac{4\tilde{\mathbf{c}}_r \mathbf{c}_r + 4\mathbf{c}_r^2 + 16\mathbf{c}_r - 16\tilde{\mathbf{c}}_r - 16\mathbf{c}_r - 64\cdot\mathbf{1}}{-64}$$

$$= \frac{4(4\mathbf{1} + 4\mathbf{c}_{\rho}) + 16\tilde{\mathbf{c}}_r + 16\mathbf{c}_r - 16\tilde{\mathbf{c}}_r - 16\mathbf{c}_r - 64\cdot\mathbf{1}}{-64}$$

$\mathbf{1} = \mathbf{c}_1$	\mathbf{c}_r	$\tilde{\mathbf{c}}_r$	\mathbf{c}_{ρ}
\mathbf{c}_r	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1} + 4\mathbf{c}_{\rho}$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}_r$
\mathbf{c}_{ρ}			$3\mathbf{1} + 2\mathbf{c}_{\rho}$

$T : \mathbf{c}_g =$	\mathbf{c}_1	\mathbf{c}_r	$\tilde{\mathbf{c}}_r$	\mathbf{c}_{ρ}
$\chi_g^A =$	1	1	1	1
$\chi_g^{\epsilon} =$	1	ϵ^*	ϵ	1
$\chi_g^{\epsilon^*} =$	1	ϵ	ϵ^*	1
.				

Tetrahedral T class characters

T class products

$$\mathbf{c}_g = \sum_{\mu} \frac{{}^{\circ}\mathbf{c}_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } \mathbf{c}_g} \frac{\ell^{\mu}}{{}^{\circ}\mathbf{G} \chi_g^{\mu*}} \mathbf{c}_g$$

Minimal equation for \mathbf{c}_r : $(\mathbf{c}_r - 4\epsilon\mathbf{1})(\mathbf{c}_r - 4\epsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = 0$

$$\mathbf{P}^{(4\epsilon)} = \frac{(\mathbf{c}_r - 4\epsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\epsilon - 4\epsilon^*)(4\epsilon - 4)(4\epsilon - 0)} = \frac{(\mathbf{c}_r^2 - 4(\epsilon^* + 1)\mathbf{c}_r + 16\cdot\epsilon^*)\mathbf{c}_r}{64(\epsilon - \epsilon^*)(\epsilon - 1)\epsilon}$$

$$= \frac{4\tilde{\mathbf{c}}_r \mathbf{c}_r + 4(\epsilon)4\tilde{\mathbf{c}}_r + 16\cdot\epsilon^*\mathbf{c}_r}{64i\sqrt{3}(\epsilon^2 - \epsilon)} = \frac{16(\mathbf{1} + \mathbf{c}_\rho) + 16\epsilon\tilde{\mathbf{c}}_r + 16\cdot\epsilon^*\mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})}$$

$$= \frac{\mathbf{1} + \mathbf{c}_\rho + \epsilon\tilde{\mathbf{c}}_r + \epsilon^*\mathbf{c}_r}{12}$$

$$\mathbf{P}^{(4\epsilon^*)} = \frac{(\mathbf{c}_r - 4\epsilon\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\epsilon^* - 4\epsilon)(4\epsilon^* - 4)(4\epsilon^* - 0)}$$

$$= \frac{\mathbf{1} + \mathbf{c}_\rho + \epsilon^*\tilde{\mathbf{c}}_r + \epsilon\mathbf{c}_r}{12}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\epsilon\mathbf{1})(\mathbf{c}_r - 4\epsilon^*\mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\epsilon)(4 - 4\epsilon^*)(4 - 0)} = \frac{(\mathbf{c}_r^2 - 4(\epsilon + \epsilon^*)\mathbf{c}_r + 16\mathbf{1})\mathbf{c}_r}{64(1 - (\epsilon + \epsilon^*) + 1)}$$

$$= \frac{4\tilde{\mathbf{c}}_r \mathbf{c}_r + 4\cdot 4\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1 + 1 + 1)} = \frac{16(\mathbf{1} + \mathbf{c}_\rho) + 16\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1 + 1 + 1)}$$

$$= \frac{\mathbf{1} + \mathbf{c}_\rho + \tilde{\mathbf{c}}_r + \mathbf{c}_r}{12}$$

$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\epsilon\mathbf{1})(\mathbf{c}_r - 4\epsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{(0 - 4\epsilon)(0 - 4\epsilon^*)(0 - 4)} = \frac{(\mathbf{c}_r^2 - 4(\epsilon + \epsilon^*)\mathbf{c}_r + 16\cdot\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{64(-\epsilon)(-\epsilon^*)(-1)}$$

$$= \frac{4\tilde{\mathbf{c}}_r + 4\mathbf{c}_r + 16\cdot\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{-64}$$

$$= \frac{4\tilde{\mathbf{c}}_r \mathbf{c}_r + 4\mathbf{c}_r^2 + 16\mathbf{c}_r - 16\tilde{\mathbf{c}}_r - 16\mathbf{c}_r - 64\cdot\mathbf{1}}{-64}$$

$$= \frac{4(4\mathbf{1} + 4\mathbf{c}_\rho) + 16\tilde{\mathbf{c}}_r + 16\mathbf{c}_r - 16\tilde{\mathbf{c}}_r - 16\mathbf{c}_r - 64\cdot\mathbf{1}}{-64} = \frac{-48\cdot\mathbf{1} + 16\mathbf{c}_\rho}{-64}$$

$$= \frac{3}{4}\mathbf{1} - \frac{1}{4}\mathbf{c}_\rho$$

$\mathbf{1} = \mathbf{c}_1$	\mathbf{c}_r	$\tilde{\mathbf{c}}_r$	\mathbf{c}_ρ
\mathbf{c}_r	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1} + 4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}_r$
\mathbf{c}_ρ			$3\mathbf{1} + 2\mathbf{c}_\rho$

$T : \mathbf{c}_g =$	\mathbf{c}_1	\mathbf{c}_r	$\tilde{\mathbf{c}}_r$	\mathbf{c}_ρ
$\chi_g^A =$	1	1	1	1
$\chi_g^\epsilon =$	1	ϵ^*	ϵ	1
$\chi_g^{\epsilon^*} =$	1	ϵ	ϵ^*	1
$\chi_g^T =$	3	0	0	-1

Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$

Octahedral-cubic O symmetry and group operations

Tetrahedral symmetry becomes Icosahedral

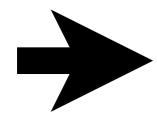
Octahedral groups $O_h \supset O \sim T_d \supset T$

Octahedral O and spin- $O \subset U(2)$

Tetrahedral T class algebra

Tetrahedral T class minimal equations

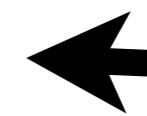
Tetrahedral T class projectors and characters



Octahedral O class algebra

Octahedral O class minimal equations

Octahedral O class projectors and characters



Octahedral $O_h \supset O$ subgroup correlations

Octahedral $O_h \supset O$ subgroup correlations

$O_h \supset O \supset D_4$ subgroup correlations

$O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

Applications

Octahedral O class algebra

$$\mathbf{c}_I = \mathbf{1}, \quad \mathbf{c}_r = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4 + \mathbf{r}_1^2 + \mathbf{r}_2^2 + \mathbf{r}_3^2 + \mathbf{r}_4^2, \quad \mathbf{c}_\rho = \mathbf{R}_1^2 + \mathbf{R}_2^2 + \mathbf{R}_3^2,$$

$$\mathbf{c}_R = \mathbf{R}_1 + \mathbf{R}_2 + \mathbf{R}_3 + \mathbf{R}_1^3 + \mathbf{R}_2^3 + \mathbf{R}_3^3, \quad \mathbf{c}_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3 + \mathbf{i}_4 + \mathbf{i}_5 + \mathbf{i}_6$$

O class products

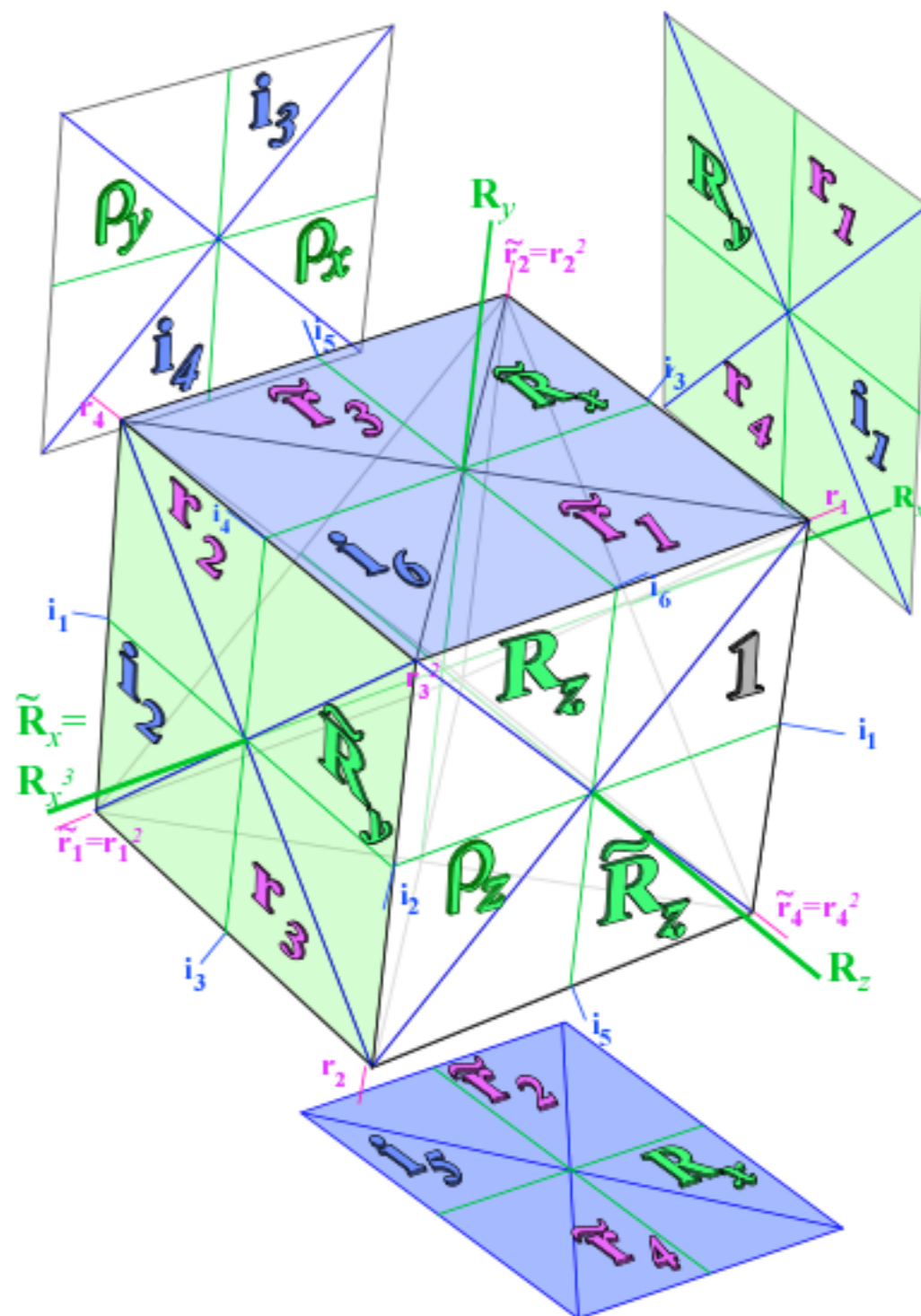
Unnecessary to do $24^2 = 576$ products since each row (or column) of $\mathbf{c}_A \mathbf{c}_B$ has same class proportion

For example:

$$\mathbf{c}_\rho \mathbf{c}_i = \mathbf{R}_1^2 \mathbf{i}_1 + \dots = \mathbf{R}_2 + \dots$$

$$+ \mathbf{R}_2^2 \mathbf{i}_1 + \dots + \mathbf{i}_2 + \dots$$

$$+ \mathbf{R}_3^2 \mathbf{i}_1 + \dots + \mathbf{R}_2^3 + \dots$$



Octahedral O class algebra

$$\mathbf{c}_I = \mathbf{1}, \quad \mathbf{c}_r = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4 + \mathbf{r}_1^2 + \mathbf{r}_2^2 + \mathbf{r}_3^2 + \mathbf{r}_4^2, \quad \mathbf{c}_\rho = \mathbf{R}_1^2 + \mathbf{R}_2^2 + \mathbf{R}_3^2,$$

$$\mathbf{c}_R = \mathbf{R}_1 + \mathbf{R}_2 + \mathbf{R}_3 + \mathbf{R}_1^3 + \mathbf{R}_2^3 + \mathbf{R}_3^3, \quad \mathbf{c}_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3 + \mathbf{i}_4 + \mathbf{i}_5 + \mathbf{i}_6$$

O class products

Unnecessary to do $24^2 = 576$ products since each row (or column) of $\mathbf{c}_A \mathbf{c}_B$ has same class proportion

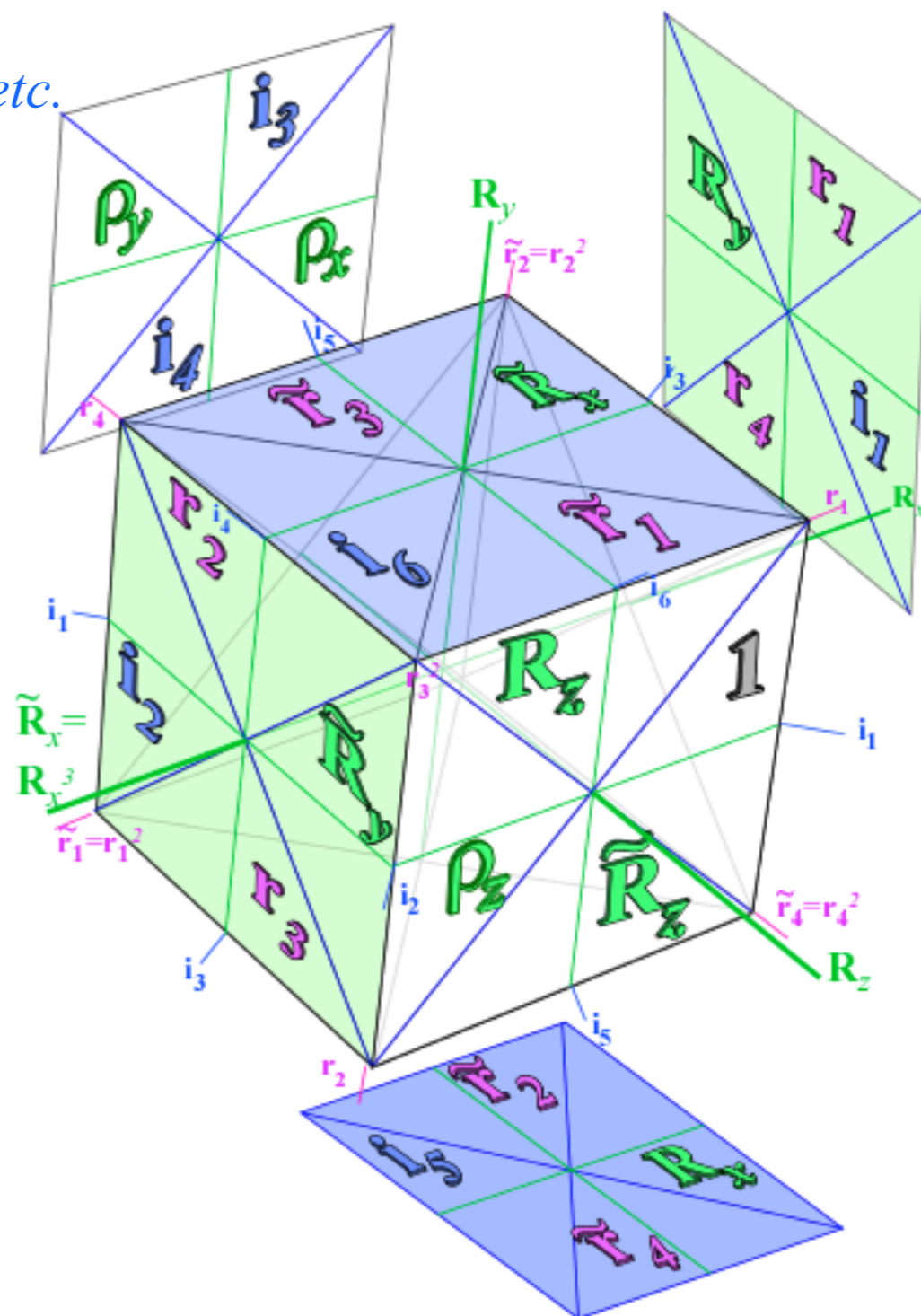
For example:

So there are $2\mathbf{c}_R$ for each \mathbf{c}_i :

$$\mathbf{c}_{R^2} \mathbf{c}_i = \mathbf{R}_1^2 \mathbf{i}_1 + \dots = \mathbf{R}_2 + \dots \quad \mathbf{c}_\rho \mathbf{c}_i = 2\mathbf{c}_R + \mathbf{c}_i \quad \text{or: } 4\mathbf{c}_R + 2\mathbf{c}_i \text{ etc.}$$

$$+ \mathbf{R}_2^2 \mathbf{i}_1 + \dots + \mathbf{i}_2 + \dots$$

$$+ \mathbf{R}_3^2 \mathbf{i}_1 + \dots + \mathbf{R}_2^3 + \dots$$



Octahedral O class algebra

$$\mathbf{c}_I = \mathbf{1}, \quad \mathbf{c}_r = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4 + \mathbf{r}_1^2 + \mathbf{r}_2^2 + \mathbf{r}_3^2 + \mathbf{r}_4^2, \quad \mathbf{c}_\rho = \mathbf{R}_1^2 + \mathbf{R}_2^2 + \mathbf{R}_3^2,$$

$$\mathbf{c}_R = \mathbf{R}_1 + \mathbf{R}_2 + \mathbf{R}_3 + \mathbf{R}_1^3 + \mathbf{R}_2^3 + \mathbf{R}_3^3, \quad \mathbf{c}_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3 + \mathbf{i}_4 + \mathbf{i}_5 + \mathbf{i}_6$$

O class products

Unnecessary to do $24^2 = 576$ products since each row (or column) of $\mathbf{c}_A \mathbf{c}_B$ has same class proportion

For example:

So there are $2\mathbf{c}_R$ for each \mathbf{c}_i in $({}^\circ\mathbf{c}_\rho) \cdot ({}^\circ\mathbf{c}_i) = (3) \cdot (6) = 18$ terms

$$\mathbf{c}_{R^2} \mathbf{c}_i = \mathbf{R}_1^2 \mathbf{i}_1 + \dots = \mathbf{R}_2 + \dots$$

$$+ \mathbf{R}_2^2 \mathbf{i}_1 + \dots + \mathbf{i}_2 + \dots$$

$$+ \mathbf{R}_3^2 \mathbf{i}_1 + \dots + \mathbf{R}_2^3 + \dots$$

$\mathbf{c}_\rho \mathbf{c}_i = 2\mathbf{c}_R + \mathbf{c}_i$ or: $4\mathbf{c}_R + 2\mathbf{c}_i$ etc.

$$\text{So: } 2({}^\circ\mathbf{c}_R) + ({}^\circ\mathbf{c}_i) = 2 \cdot 6 + 6 = 18$$

Proof that class proportion cannot vary:

$$\mathbf{c}_g \mathbf{c}_h = \mathbf{g}_1 \mathbf{h}_1 + \mathbf{g}_2 \mathbf{h}_1 + \dots = \mathbf{g}_1 \mathbf{h}_1 + \mathbf{t} \mathbf{g}_1 \mathbf{h}_1 \mathbf{t}^{-1} + \dots = \mathbf{g}_1 \mathbf{h}_1 + \mathbf{t} \mathbf{g}_1 \mathbf{t}^{-1} \mathbf{t} \mathbf{h}_1 \mathbf{t}^{-1} + \dots = \mathbf{g}_1 \mathbf{h}_1 + \mathbf{g}_2 \mathbf{t} \mathbf{h}_1 \mathbf{t}^{-1} + \dots$$

$$+ \mathbf{g}_1 \mathbf{h}_2 + \mathbf{g}_2 \mathbf{h}_2 + \dots + \mathbf{g}_1 \mathbf{h}_2 + \mathbf{t} \mathbf{g}_1 \mathbf{h}_2 \mathbf{t}^{-1} + \dots + \mathbf{g}_1 \mathbf{h}_2 + \mathbf{t} \mathbf{g}_1 \mathbf{t}^{-1} \mathbf{t} \mathbf{h}_2 \mathbf{t}^{-1} + \dots + \mathbf{g}_1 \mathbf{h}_2 + \mathbf{g}_2 \mathbf{t} \mathbf{h}_2 \mathbf{t}^{-1} + \dots$$

$$= \mathbf{g}_1 \mathbf{h}_3 + \mathbf{g}_2 \mathbf{h}_3 + \dots + \mathbf{g}_1 \mathbf{h}_2 + \mathbf{t} \mathbf{g}_1 \mathbf{h}_3 \mathbf{t}^{-1} + \dots + \mathbf{g}_1 \mathbf{h}_2 + \mathbf{t} \mathbf{g}_1 \mathbf{t}^{-1} \mathbf{t} \mathbf{h}_3 \mathbf{t}^{-1} + \dots + \mathbf{g}_1 \mathbf{h}_2 + \mathbf{g}_2 \mathbf{t} \mathbf{h}_3 \mathbf{t}^{-1} + \dots$$

O class product table

$\mathbf{1} = \mathbf{c}_1$	\mathbf{c}_r	\mathbf{c}_ρ	\mathbf{c}_R	\mathbf{c}_i
\mathbf{c}_r	$8\mathbf{1} + 4\mathbf{c}_r + 8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R + 4\mathbf{c}_i$	$4\mathbf{c}_R + 4\mathbf{c}_i$
\mathbf{c}_ρ		$3\mathbf{1} + 2\mathbf{c}_\rho$	$\mathbf{c}_R + 2\mathbf{c}_i$	$2\mathbf{c}_R + \mathbf{c}_i$
\mathbf{c}_R			$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$	$3\mathbf{c}_r + 4\mathbf{c}_\rho$
\mathbf{c}_i				$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$

Minimal equation for \mathbf{c}_ρ

$$\mathbf{c}_\rho^2 = 3\mathbf{1} + 2\mathbf{c}_\rho$$

$$\mathbf{c}_\rho^2 - 2\mathbf{c}_\rho - 3\mathbf{1} = \mathbf{0}$$

$$(\mathbf{c}_\rho - 3\mathbf{1})(\mathbf{c}_\rho + \mathbf{1}) = \mathbf{0}$$

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Tetrahedral T class algebra

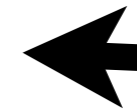
Tetrahedral T class minimal equations

Tetrahedral T class projectors and characters

→ *Octahedral O class algebra*

Octahedral O class minimal equations

Octahedral O class projectors and characters



Octahedral $O_h \supset O$ subgroup correlations

Octahedral $O_h \supset O$ subgroup correlations

$O_h \supset O \supset D_4$ subgroup correlations

$O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

Applications

Octahedral O class minimal equations

$$\mathbf{c}_g = \sum_{\mu} \frac{\chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{\chi_g^{\mu*}} \mathbf{c}_g$$

O class product table

$\mathbf{1} = c_1$	\mathbf{c}_r	\mathbf{c}_ρ	\mathbf{c}_R	\mathbf{c}_i
\mathbf{c}_r	$8\mathbf{1} + 4\mathbf{c}_r + 8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R + 4\mathbf{c}_i$	$4\mathbf{c}_R + 4\mathbf{c}_i$
\mathbf{c}_ρ		$3\mathbf{1} + 2\mathbf{c}_\rho$	$\mathbf{c}_R + 2\mathbf{c}_i$	$2\mathbf{c}_R + \mathbf{c}_i$
\mathbf{c}_R			$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$	$3\mathbf{c}_r + 4\mathbf{c}_\rho$
\mathbf{c}_i				$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$

Minimal equation for \mathbf{c}_i

$$\mathbf{c}_i^2 = 6 \cdot \mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$$

Octahedral O class minimal equations

$$\mathbf{c}_g = \sum_{\mu} \frac{\chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{\chi_g^{\mu*}} \mathbf{c}_g$$

O class product table

$\mathbf{1} = c_1$	\mathbf{c}_r	\mathbf{c}_ρ	\mathbf{c}_R	\mathbf{c}_i
\mathbf{c}_r	$8\mathbf{1} + 4\mathbf{c}_r + 8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R + 4\mathbf{c}_i$	$4\mathbf{c}_R + 4\mathbf{c}_i$
\mathbf{c}_ρ		$3\mathbf{1} + 2\mathbf{c}_\rho$	$\mathbf{c}_R + 2\mathbf{c}_i$	$2\mathbf{c}_R + \mathbf{c}_i$
\mathbf{c}_R			$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$	$3\mathbf{c}_r + 4\mathbf{c}_\rho$
\mathbf{c}_i				$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$

Minimal equation for \mathbf{c}_i

$$\mathbf{c}_i^2 = 6 \cdot \mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$$

$$\mathbf{c}_i^3 = 6 \cdot \mathbf{1} \mathbf{c}_i + 3\mathbf{c}_r \mathbf{c}_i + 2\mathbf{c}_\rho \mathbf{c}_i$$

Octahedral O class minimal equations

$$\mathbf{c}_g = \sum_{\mu} \frac{\chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{\chi_g^{\mu*}} \mathbf{c}_g$$

O class product table

$\mathbf{1} = c_1$	\mathbf{c}_r	\mathbf{c}_ρ	\mathbf{c}_R	\mathbf{c}_i
\mathbf{c}_r	$8\mathbf{1} + 4\mathbf{c}_r + 8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R + 4\mathbf{c}_i$	$4\mathbf{c}_R + 4\mathbf{c}_i$
\mathbf{c}_ρ		$3\mathbf{1} + 2\mathbf{c}_\rho$	$\mathbf{c}_R + 2\mathbf{c}_i$	$2\mathbf{c}_R + \mathbf{c}_i$
\mathbf{c}_R			$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$	$3\mathbf{c}_r + 4\mathbf{c}_\rho$
\mathbf{c}_i				$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$

Minimal equation for \mathbf{c}_i

$$\mathbf{c}_i^2 = 6 \cdot \mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$$

$$\mathbf{c}_i^3 = 6 \cdot \mathbf{1} \mathbf{c}_i + 3\mathbf{c}_r \mathbf{c}_i + 2\mathbf{c}_\rho \mathbf{c}_i$$

$$= 6\mathbf{c}_i + 3(4\mathbf{c}_R + 4\mathbf{c}_i) + 2(2\mathbf{c}_R + \mathbf{c}_i)$$

Octahedral O class minimal equations

$$\mathbf{c}_g = \sum_{\mu} \frac{\chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } \mathbf{c}_g} \frac{\ell^{\mu}}{\chi_g^{\mu*}} \mathbf{c}_g$$

O class product table

$\mathbf{1} = c_1$	\mathbf{c}_r	\mathbf{c}_ρ	\mathbf{c}_R	\mathbf{c}_i
\mathbf{c}_r	$8\mathbf{1} + 4\mathbf{c}_r + 8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R + 4\mathbf{c}_i$	$4\mathbf{c}_R + 4\mathbf{c}_i$
\mathbf{c}_ρ		$3\mathbf{1} + 2\mathbf{c}_\rho$	$\mathbf{c}_R + 2\mathbf{c}_i$	$2\mathbf{c}_R + \mathbf{c}_i$
\mathbf{c}_R			$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$	$3\mathbf{c}_r + 4\mathbf{c}_\rho$
\mathbf{c}_i				$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$

Minimal equation for \mathbf{c}_i

$$\mathbf{c}_i^2 = 6 \cdot \mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$$

$$\mathbf{c}_i^3 = 6 \cdot \mathbf{1} \mathbf{c}_i + 3\mathbf{c}_r \mathbf{c}_i + 2\mathbf{c}_\rho \mathbf{c}_i$$

$$= 6\mathbf{c}_i + 3(4\mathbf{c}_R + 4\mathbf{c}_i) + 2(2\mathbf{c}_R + \mathbf{c}_i)$$

$$\mathbf{c}_i^3 = 16\mathbf{c}_R + 20\mathbf{c}_i$$

Octahedral O class minimal equations

$$\mathbf{c}_g = \sum_{\mu} \frac{\chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } \mathbf{c}_g} \frac{\ell^{\mu}}{\chi_g^{\mu*}} \mathbf{c}_g$$

O class product table

$\mathbf{1} = c_1$	\mathbf{c}_r	\mathbf{c}_ρ	\mathbf{c}_R	\mathbf{c}_i
\mathbf{c}_r	$8\mathbf{1} + 4\mathbf{c}_r + 8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R + 4\mathbf{c}_i$	$4\mathbf{c}_R + 4\mathbf{c}_i$
\mathbf{c}_ρ		$3\mathbf{1} + 2\mathbf{c}_\rho$	$\mathbf{c}_R + 2\mathbf{c}_i$	$2\mathbf{c}_R + \mathbf{c}_i$
\mathbf{c}_R			$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$	$3\mathbf{c}_r + 4\mathbf{c}_\rho$
\mathbf{c}_i				$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$

Minimal equation for \mathbf{c}_i

$$\mathbf{c}_i^2 = 6 \cdot \mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$$

$$\mathbf{c}_i^3 = 6 \cdot \mathbf{1} \mathbf{c}_i + 3\mathbf{c}_r \mathbf{c}_i + 2\mathbf{c}_\rho \mathbf{c}_i$$

$$= 6\mathbf{c}_i + 3(4\mathbf{c}_R + 4\mathbf{c}_i) + 2(2\mathbf{c}_R + \mathbf{c}_i)$$

$$\mathbf{c}_i^3 = 16\mathbf{c}_R + 20\mathbf{c}_i$$

$$\mathbf{c}_i^4 = 16\mathbf{c}_R \mathbf{c}_i + 20\mathbf{c}_i \mathbf{c}_i$$

Octahedral O class minimal equations

$$\mathbf{c}_g = \sum_{\mu} \frac{\chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{\chi_g^{\mu*}} \mathbf{c}_g$$

O class product table

$\mathbf{1} = c_1$	\mathbf{c}_r	\mathbf{c}_ρ	\mathbf{c}_R	\mathbf{c}_i
\mathbf{c}_r	$8\mathbf{1} + 4\mathbf{c}_r + 8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R + 4\mathbf{c}_i$	$4\mathbf{c}_R + 4\mathbf{c}_i$
\mathbf{c}_ρ		$3\mathbf{1} + 2\mathbf{c}_\rho$	$\mathbf{c}_R + 2\mathbf{c}_i$	$2\mathbf{c}_R + \mathbf{c}_i$
\mathbf{c}_R			$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$	$3\mathbf{c}_r + 4\mathbf{c}_\rho$
\mathbf{c}_i				$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$

Minimal equation for \mathbf{c}_i

$$\mathbf{c}_i^2 = 6 \cdot \mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$$

$$\mathbf{c}_i^3 = 6 \cdot \mathbf{1} \mathbf{c}_i + 3\mathbf{c}_r \mathbf{c}_i + 2\mathbf{c}_\rho \mathbf{c}_i$$

$$= 6\mathbf{c}_i + 3(4\mathbf{c}_R + 4\mathbf{c}_i) + 2(2\mathbf{c}_R + \mathbf{c}_i)$$

$$\mathbf{c}_i^3 = 16\mathbf{c}_R + 20\mathbf{c}_i$$

$$\mathbf{c}_i^4 = 16\mathbf{c}_R \mathbf{c}_i + 20\mathbf{c}_i \mathbf{c}_i$$

$$= 16(3\mathbf{c}_r + 4\mathbf{c}_\rho) + 20(6 \cdot \mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho)$$

Octahedral O class minimal equations

$$\mathbf{c}_g = \sum_{\mu} \frac{\chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{\chi_g^{\mu*}} \mathbf{c}_g$$

O class product table

$\mathbf{1} = c_1$	\mathbf{c}_r	\mathbf{c}_ρ	\mathbf{c}_R	\mathbf{c}_i
\mathbf{c}_r	$8\mathbf{1} + 4\mathbf{c}_r + 8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R + 4\mathbf{c}_i$	$4\mathbf{c}_R + 4\mathbf{c}_i$
\mathbf{c}_ρ		$3\mathbf{1} + 2\mathbf{c}_\rho$	$\mathbf{c}_R + 2\mathbf{c}_i$	$2\mathbf{c}_R + \mathbf{c}_i$
\mathbf{c}_R			$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$	$3\mathbf{c}_r + 4\mathbf{c}_\rho$
\mathbf{c}_i				$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$

Minimal equation for \mathbf{c}_i

$$\mathbf{c}_i^2 = 6 \cdot \mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$$

$$\mathbf{c}_i^3 = 6 \cdot \mathbf{1} \mathbf{c}_i + 3\mathbf{c}_r \mathbf{c}_i + 2\mathbf{c}_\rho \mathbf{c}_i$$

$$= 6\mathbf{c}_i + 3(4\mathbf{c}_R + 4\mathbf{c}_i) + 2(2\mathbf{c}_R + \mathbf{c}_i)$$

$$\mathbf{c}_i^3 = 16\mathbf{c}_R + 20\mathbf{c}_i$$

$$\mathbf{c}_i^4 = 16\mathbf{c}_R \mathbf{c}_i + 20\mathbf{c}_i \mathbf{c}_i$$

$$= 16(3\mathbf{c}_r + 4\mathbf{c}_\rho) + 20(6 \cdot \mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho)$$

$$= 48\mathbf{c}_r + 64\mathbf{c}_\rho + 120 \cdot \mathbf{1} + 60\mathbf{c}_r + 40\mathbf{c}_\rho$$

Octahedral O class minimal equations

$$\mathbf{c}_g = \sum_{\mu} \frac{\chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{\chi_g^{\mu*}} \mathbf{c}_g$$

O class product table

$\mathbf{1} = c_1$	\mathbf{c}_r	\mathbf{c}_ρ	\mathbf{c}_R	\mathbf{c}_i
\mathbf{c}_r	$8\mathbf{1} + 4\mathbf{c}_r + 8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R + 4\mathbf{c}_i$	$4\mathbf{c}_R + 4\mathbf{c}_i$
\mathbf{c}_ρ		$3\mathbf{1} + 2\mathbf{c}_\rho$	$\mathbf{c}_R + 2\mathbf{c}_i$	$2\mathbf{c}_R + \mathbf{c}_i$
\mathbf{c}_R			$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$	$3\mathbf{c}_r + 4\mathbf{c}_\rho$
\mathbf{c}_i				$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$

Minimal equation for \mathbf{c}_i

$$\mathbf{c}_i^2 = 6 \cdot \mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$$

$$\mathbf{c}_i^3 = 6 \cdot \mathbf{1} \mathbf{c}_i + 3\mathbf{c}_r \mathbf{c}_i + 2\mathbf{c}_\rho \mathbf{c}_i$$

$$= 6\mathbf{c}_i + 3(4\mathbf{c}_R + 4\mathbf{c}_i) + 2(2\mathbf{c}_R + \mathbf{c}_i)$$

$$\mathbf{c}_i^3 = 16\mathbf{c}_R + 20\mathbf{c}_i$$

$$\mathbf{c}_i^4 = 16\mathbf{c}_R \mathbf{c}_i + 20\mathbf{c}_i \mathbf{c}_i$$

$$= 16(3\mathbf{c}_r + 4\mathbf{c}_\rho) + 20(6 \cdot \mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho)$$

$$= 48\mathbf{c}_r + 64\mathbf{c}_\rho + 120 \cdot \mathbf{1} + 60\mathbf{c}_r + 40\mathbf{c}_\rho$$

$$= 120 \cdot \mathbf{1} + 108\mathbf{c}_r + 104\mathbf{c}_\rho$$

Octahedral O class minimal equations

$$\mathbf{c}_g = \sum_{\mu} \frac{\chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } \mathbf{c}_g} \frac{\ell^{\mu}}{\chi_g^{\mu*}} \mathbf{c}_g$$

O class product table

$\mathbf{1} = c_1$	\mathbf{c}_r	\mathbf{c}_ρ	\mathbf{c}_R	\mathbf{c}_i
\mathbf{c}_r	$8\mathbf{1} + 4\mathbf{c}_r + 8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R + 4\mathbf{c}_i$	$4\mathbf{c}_R + 4\mathbf{c}_i$
\mathbf{c}_ρ		$3\mathbf{1} + 2\mathbf{c}_\rho$	$\mathbf{c}_R + 2\mathbf{c}_i$	$2\mathbf{c}_R + \mathbf{c}_i$
\mathbf{c}_R			$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$	$3\mathbf{c}_r + 4\mathbf{c}_\rho$
\mathbf{c}_i				$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$

Minimal equation for \mathbf{c}_i

$$\mathbf{c}_i^2 = 6 \cdot \mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$$

$$\mathbf{c}_i^3 = 6 \cdot \mathbf{1} \mathbf{c}_i + 3\mathbf{c}_r \mathbf{c}_i + 2\mathbf{c}_\rho \mathbf{c}_i$$

$$= 6\mathbf{c}_i + 3(4\mathbf{c}_R + 4\mathbf{c}_i) + 2(2\mathbf{c}_R + \mathbf{c}_i)$$

$$\mathbf{c}_i^3 = 16\mathbf{c}_R + 20\mathbf{c}_i$$

$$\mathbf{c}_i^4 = 16\mathbf{c}_R \mathbf{c}_i + 20\mathbf{c}_i \mathbf{c}_i$$

$$= 16(3\mathbf{c}_r + 4\mathbf{c}_\rho) + 20(6 \cdot \mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho)$$

$$= 48\mathbf{c}_r + 64\mathbf{c}_\rho + 120 \cdot \mathbf{1} + 60\mathbf{c}_r + 40\mathbf{c}_\rho$$

$$= 120 \cdot \mathbf{1} + 108\mathbf{c}_r + 104\mathbf{c}_\rho$$

$$\mathbf{c}_i^5 = 120\mathbf{c}_i + 108\mathbf{c}_r \mathbf{c}_i + 104\mathbf{c}_\rho \mathbf{c}_i$$

$$= 120\mathbf{c}_i + 108(4\mathbf{c}_R + 4\mathbf{c}_i) + 104(2\mathbf{c}_R + \mathbf{c}_i)$$

$$= 640\mathbf{c}_R + 656\mathbf{c}_i$$

Octahedral O class minimal equations

$$\mathbf{c}_g = \sum_{\mu} \frac{\chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } \mathbf{c}_g} \frac{\ell^{\mu}}{\chi_g^{\mu*}} \mathbf{c}_g$$

O class product table

$\mathbf{1} = c_1$	\mathbf{c}_r	\mathbf{c}_ρ	\mathbf{c}_R	\mathbf{c}_i
\mathbf{c}_r	$8\mathbf{1} + 4\mathbf{c}_r + 8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R + 4\mathbf{c}_i$	$4\mathbf{c}_R + 4\mathbf{c}_i$
\mathbf{c}_ρ		$3\mathbf{1} + 2\mathbf{c}_\rho$	$\mathbf{c}_R + 2\mathbf{c}_i$	$2\mathbf{c}_R + \mathbf{c}_i$
\mathbf{c}_R			$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$	$3\mathbf{c}_r + 4\mathbf{c}_\rho$
\mathbf{c}_i				$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$

Minimal equation for \mathbf{c}_i

$$\mathbf{c}_i^2 = 6 \cdot \mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$$

$$\mathbf{c}_i^3 = 6 \cdot \mathbf{1} \mathbf{c}_i + 3\mathbf{c}_r \mathbf{c}_i + 2\mathbf{c}_\rho \mathbf{c}_i$$

$$= 6\mathbf{c}_i + 3(4\mathbf{c}_R + 4\mathbf{c}_i) + 2(2\mathbf{c}_R + \mathbf{c}_i)$$

$$\mathbf{c}_i^3 = 16\mathbf{c}_R + 20\mathbf{c}_i$$

$$\mathbf{c}_i^4 = 16\mathbf{c}_R \mathbf{c}_i + 20\mathbf{c}_i \mathbf{c}_i$$

$$= 16(3\mathbf{c}_r + 4\mathbf{c}_\rho) + 20(6 \cdot \mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho)$$

$$= 48\mathbf{c}_r + 64\mathbf{c}_\rho + 120 \cdot \mathbf{1} + 60\mathbf{c}_r + 40\mathbf{c}_\rho$$

$$= 120 \cdot \mathbf{1} + 108\mathbf{c}_r + 104\mathbf{c}_\rho$$

$$\mathbf{c}_i^5 = 120\mathbf{c}_i + 108\mathbf{c}_r \mathbf{c}_i + 104\mathbf{c}_\rho \mathbf{c}_i$$

$$= 120\mathbf{c}_i + 108(4\mathbf{c}_R + 4\mathbf{c}_i) + 104(2\mathbf{c}_R + \mathbf{c}_i)$$

$$= 640\mathbf{c}_R + 656\mathbf{c}_i$$

$$40\mathbf{c}_i^3 = 640\mathbf{c}_R + 800\mathbf{c}_i$$

Octahedral O class minimal equations

$$\mathbf{c}_g = \sum_{\mu} \frac{\chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } \mathbf{c}_g} \frac{\ell^{\mu}}{\chi_g^{\mu*}} \mathbf{c}_g$$

O class product table

$\mathbf{1} = c_1$	\mathbf{c}_r	\mathbf{c}_ρ	\mathbf{c}_R	\mathbf{c}_i
\mathbf{c}_r	$8\mathbf{1} + 4\mathbf{c}_r + 8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R + 4\mathbf{c}_i$	$4\mathbf{c}_R + 4\mathbf{c}_i$
\mathbf{c}_ρ		$3\mathbf{1} + 2\mathbf{c}_\rho$	$\mathbf{c}_R + 2\mathbf{c}_i$	$2\mathbf{c}_R + \mathbf{c}_i$
\mathbf{c}_R			$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$	$3\mathbf{c}_r + 4\mathbf{c}_\rho$
\mathbf{c}_i				$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$

Minimal equation for \mathbf{c}_i

$$\mathbf{c}_i^2 = 6 \cdot \mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$$

$$\mathbf{c}_i^3 = 6 \cdot \mathbf{1} \mathbf{c}_i + 3\mathbf{c}_r \mathbf{c}_i + 2\mathbf{c}_\rho \mathbf{c}_i$$

$$= 6\mathbf{c}_i + 3(4\mathbf{c}_R + 4\mathbf{c}_i) + 2(2\mathbf{c}_R + \mathbf{c}_i)$$

$$\mathbf{c}_i^3 = 16\mathbf{c}_R + 20\mathbf{c}_i$$

$$\mathbf{c}_i^4 = 16\mathbf{c}_R \mathbf{c}_i + 20\mathbf{c}_i \mathbf{c}_i$$

$$= 16(3\mathbf{c}_r + 4\mathbf{c}_\rho) + 20(6 \cdot \mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho)$$

$$= 48\mathbf{c}_r + 64\mathbf{c}_\rho + 120 \cdot \mathbf{1} + 60\mathbf{c}_r + 40\mathbf{c}_\rho$$

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$$\mathbf{c}_i^5 = 120\mathbf{c}_i + 108\mathbf{c}_r \mathbf{c}_i + 104\mathbf{c}_\rho \mathbf{c}_i$$

$$= 120\mathbf{c}_i + 108(4\mathbf{c}_R + 4\mathbf{c}_i) + 104(2\mathbf{c}_R + \mathbf{c}_i)$$

$$= 640\mathbf{c}_R + 656\mathbf{c}_i$$

$$40\mathbf{c}_i^3 = 640\mathbf{c}_R + 800\mathbf{c}_i$$

$$\mathbf{c}_i^5 - 40\mathbf{c}_i^3 + 144\mathbf{c}_i = 0$$

Octahedral O class minimal equations

$$\mathbf{c}_g = \sum_{\mu} \frac{\chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } \mathbf{c}_g} \frac{\ell^{\mu}}{\chi_g^{\mu*}} \mathbf{c}_g$$

O class product table

$\mathbf{1} = c_1$	\mathbf{c}_r	\mathbf{c}_ρ	\mathbf{c}_R	\mathbf{c}_i
\mathbf{c}_r	$8\mathbf{1} + 4\mathbf{c}_r + 8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R + 4\mathbf{c}_i$	$4\mathbf{c}_R + 4\mathbf{c}_i$
\mathbf{c}_ρ		$3\mathbf{1} + 2\mathbf{c}_\rho$	$\mathbf{c}_R + 2\mathbf{c}_i$	$2\mathbf{c}_R + \mathbf{c}_i$
\mathbf{c}_R			$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$	$3\mathbf{c}_r + 4\mathbf{c}_\rho$
\mathbf{c}_i				$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$

Minimal equation for \mathbf{c}_i

$$\mathbf{c}_i^2 = 6 \cdot \mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$$

$$\mathbf{c}_i^3 = 6 \cdot \mathbf{1} \mathbf{c}_i + 3\mathbf{c}_r \mathbf{c}_i + 2\mathbf{c}_\rho \mathbf{c}_i$$

$$= 6\mathbf{c}_i + 3(4\mathbf{c}_R + 4\mathbf{c}_i) + 2(2\mathbf{c}_R + \mathbf{c}_i)$$

$$\mathbf{c}_i^3 = 16\mathbf{c}_R + 20\mathbf{c}_i$$

$$\mathbf{c}_i^4 = 16\mathbf{c}_R \mathbf{c}_i + 20\mathbf{c}_i \mathbf{c}_i$$

$$= 16(3\mathbf{c}_r + 4\mathbf{c}_\rho) + 20(6 \cdot \mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho)$$

$$= 48\mathbf{c}_r + 64\mathbf{c}_\rho + 120 \cdot \mathbf{1} + 60\mathbf{c}_r + 40\mathbf{c}_\rho$$

$$= 120 \cdot \mathbf{1} + 108\mathbf{c}_r + 104\mathbf{c}_\rho$$

$$\mathbf{c}_i^5 = 120\mathbf{c}_i + 108\mathbf{c}_r \mathbf{c}_i + 104\mathbf{c}_\rho \mathbf{c}_i$$

$$= 120\mathbf{c}_i + 108(4\mathbf{c}_R + 4\mathbf{c}_i) + 104(2\mathbf{c}_R + \mathbf{c}_i)$$

$$= 640\mathbf{c}_R + 656\mathbf{c}_i$$

$$40\mathbf{c}_i^3 = 640\mathbf{c}_R + 800\mathbf{c}_i$$

$$\mathbf{c}_i^5 - 40\mathbf{c}_i^3 + 144\mathbf{c}_i = 0 = (\mathbf{c}_i^2 - 36 \cdot \mathbf{1})(\mathbf{c}_i^2 - 4 \cdot \mathbf{1})(\mathbf{c}_i - 0 \cdot \mathbf{1})$$

Octahedral O class minimal equations

$$\mathbf{c}_g = \sum_{\mu} \frac{\chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } \mathbf{c}_g} \frac{\ell^{\mu}}{\chi_g^{\mu*}} \mathbf{c}_g$$

O class product table

$\mathbf{1} = c_1$	\mathbf{c}_r	\mathbf{c}_ρ	\mathbf{c}_R	\mathbf{c}_i
\mathbf{c}_r	$8\mathbf{1} + 4\mathbf{c}_r + 8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R + 4\mathbf{c}_i$	$4\mathbf{c}_R + 4\mathbf{c}_i$
\mathbf{c}_ρ		$3\mathbf{1} + 2\mathbf{c}_\rho$	$\mathbf{c}_R + 2\mathbf{c}_i$	$2\mathbf{c}_R + \mathbf{c}_i$
\mathbf{c}_R			$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$	$3\mathbf{c}_r + 4\mathbf{c}_\rho$
\mathbf{c}_i				$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$

Minimal equation for \mathbf{c}_i

$$\mathbf{c}_i^2 = 6 \cdot \mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$$

$$\mathbf{c}_i^3 = 6 \cdot \mathbf{1} \mathbf{c}_i + 3\mathbf{c}_r \mathbf{c}_i + 2\mathbf{c}_\rho \mathbf{c}_i$$

$$= 6\mathbf{c}_i + 3(4\mathbf{c}_R + 4\mathbf{c}_i) + 2(2\mathbf{c}_R + \mathbf{c}_i)$$

$$\mathbf{c}_i^3 = 16\mathbf{c}_R + 20\mathbf{c}_i$$

$$\mathbf{c}_i^4 = 16\mathbf{c}_R \mathbf{c}_i + 20\mathbf{c}_i \mathbf{c}_i$$

$$= 16(3\mathbf{c}_r + 4\mathbf{c}_\rho) + 20(6 \cdot \mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho)$$

$$= 48\mathbf{c}_r + 64\mathbf{c}_\rho + 120 \cdot \mathbf{1} + 60\mathbf{c}_r + 40\mathbf{c}_\rho$$

$$= 120 \cdot \mathbf{1} + 108\mathbf{c}_r + 104\mathbf{c}_\rho$$

$$\mathbf{c}_i^5 = 120\mathbf{c}_i + 108\mathbf{c}_r \mathbf{c}_i + 104\mathbf{c}_\rho \mathbf{c}_i$$

$$= 120\mathbf{c}_i + 108(4\mathbf{c}_R + 4\mathbf{c}_i) + 104(2\mathbf{c}_R + \mathbf{c}_i)$$

$$= 640\mathbf{c}_R + 656\mathbf{c}_i$$

$$40\mathbf{c}_i^3 = 640\mathbf{c}_R + 800\mathbf{c}_i$$

$$\mathbf{c}_i^5 - 40\mathbf{c}_i^3 + 144\mathbf{c}_i = 0 = (\mathbf{c}_i^2 - 36 \cdot \mathbf{1})(\mathbf{c}_i^2 - 4 \cdot \mathbf{1})(\mathbf{c}_i - 0 \cdot \mathbf{1})$$

$$0 = (\mathbf{c}_i + 6 \cdot \mathbf{1})(\mathbf{c}_i - 6 \cdot \mathbf{1})(\mathbf{c}_i + 2 \cdot \mathbf{1})(\mathbf{c}_i - 2 \cdot \mathbf{1})(\mathbf{c}_i - 0 \cdot \mathbf{1})$$

Minimal equation for \mathbf{c}_i

Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$

Octahedral-cubic O symmetry and group operations

Tetrahedral symmetry becomes Icosahedral

Octahedral groups $O_h \supset O \sim T_d \supset T$

Octahedral O and spin- $O \subset U(2)$

Tetrahedral T class algebra

Tetrahedral T class minimal equations

Tetrahedral T class projectors and characters

Octahedral O class algebra

Octahedral O class minimal equations

Octahedral O class projectors and characters

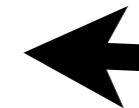
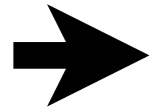
Octahedral $O_h \supset O$ subgroup correlations

Octahedral $O_h \supset O$ subgroup correlations

$O_h \supset O \supset D_4$ subgroup correlations

$O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

Applications



Octahedral O projector algebra

Begin with minimal equation:

$$0 = (\mathbf{c}_i + 2 \cdot \mathbf{1})(\mathbf{c}_i - 2 \cdot \mathbf{1})(\mathbf{c}_i + 6 \cdot \mathbf{1})(\mathbf{c}_i - 6 \cdot \mathbf{1})(\mathbf{c}_i - 0 \cdot \mathbf{1})$$

$$\mathbf{c}_g = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{{}^{\circ}G} \chi_g^{\mu*} \mathbf{c}_{g_g}$$

$$\mathbf{P}^{(2)} = \frac{(\mathbf{c}_i + 2 \cdot \mathbf{1})(\mathbf{c}_i - 6 \cdot \mathbf{1})(\mathbf{c}_i + 6 \cdot \mathbf{1})(\mathbf{c}_i - 0)}{(2 + 2)(2 - 6)(2 + 6)(2 - 0)}$$

O class product table

$\mathbf{1} = c_1$	\mathbf{c}_r	\mathbf{c}_{ρ}	\mathbf{c}_R	\mathbf{c}_i
\mathbf{c}_r	$8\mathbf{1} + 4\mathbf{c}_r + 8\mathbf{c}_{\rho}$	$3\mathbf{c}_r$	$4\mathbf{c}_R + 4\mathbf{c}_i$	$4\mathbf{c}_R + 4\mathbf{c}_i$
\mathbf{c}_{ρ}		$3\mathbf{1} + 2\mathbf{c}_{\rho}$	$\mathbf{c}_R + 2\mathbf{c}_i$	$2\mathbf{c}_R + \mathbf{c}_i$
\mathbf{c}_R			$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_{\rho}$	$3\mathbf{c}_r + 4\mathbf{c}_{\rho}$
\mathbf{c}_i				$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_{\rho}$

Octahedral O projector algebra

Begin with minimal equation:

$$0 = (c_i + 2 \cdot 1)(c_i - 2 \cdot 1)(c_i + 6 \cdot 1)(c_i - 6 \cdot 1)(c_i - 0 \cdot 1)$$

$$c_g = \sum_{\mu} \frac{c_g \chi_g^{\mu}}{\ell^{\mu}} P^{\mu}$$

$$P^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{c_g} \chi_g^{\mu*} c_g$$

$$P^{(2)} = \frac{(c_i + 2 \cdot 1)(c_i - 6 \cdot 1)(c_i + 6 \cdot 1)(c_i - 0)}{(2 + 2)(2 - 6)(2 + 6)(2 - 0)} = \frac{(c_i + 2 \cdot 1)(c_i^2 - 36 \cdot 1)c_i}{-256}$$

O class product table

$\mathbf{1} = c_1$	\mathbf{c}_r	\mathbf{c}_ρ	\mathbf{c}_R	\mathbf{c}_i
\mathbf{c}_r	$8\mathbf{1} + 4\mathbf{c}_r + 8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R + 4\mathbf{c}_i$	$4\mathbf{c}_R + 4\mathbf{c}_i$
\mathbf{c}_ρ		$3\mathbf{1} + 2\mathbf{c}_\rho$	$\mathbf{c}_R + 2\mathbf{c}_i$	$2\mathbf{c}_R + \mathbf{c}_i$
\mathbf{c}_R			$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$	$3\mathbf{c}_r + 4\mathbf{c}_\rho$
\mathbf{c}_i				$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$

Octahedral O projector algebra

Begin with minimal equation:

$$0 = (\mathbf{c}_i + 2 \cdot \mathbf{1})(\mathbf{c}_i - 2 \cdot \mathbf{1})(\mathbf{c}_i + 6 \cdot \mathbf{1})(\mathbf{c}_i - 6 \cdot \mathbf{1})(\mathbf{c}_i - 0 \cdot \mathbf{1})$$

$$\mathbf{c}_g = \sum_{\mu} \frac{\circ c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{\circ G} \chi_g^{\mu*} \mathbf{c}_g$$

$$\mathbf{P}^{(2)} = \frac{(\mathbf{c}_i + 2 \cdot \mathbf{1})(\mathbf{c}_i - 6 \cdot \mathbf{1})(\mathbf{c}_i + 6 \cdot \mathbf{1})(\mathbf{c}_i - 0)}{(2 + 2)(2 - 6)(2 + 6)(2 - 0)} = \frac{(\mathbf{c}_i + 2 \cdot \mathbf{1})(\mathbf{c}_i^2 - 36 \cdot \mathbf{1})\mathbf{c}_i}{-256} = \frac{\mathbf{c}_i^4 + 2 \cdot \mathbf{c}_i^3 - 36 \mathbf{c}_i^2 - 72 \mathbf{c}_i}{-256}$$

O class product table

$\mathbf{1} = c_1$	\mathbf{c}_r	\mathbf{c}_ρ	\mathbf{c}_R	\mathbf{c}_i
\mathbf{c}_r	$8\mathbf{1} + 4\mathbf{c}_r + 8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R + 4\mathbf{c}_i$	$4\mathbf{c}_R + 4\mathbf{c}_i$
\mathbf{c}_ρ		$3\mathbf{1} + 2\mathbf{c}_\rho$	$\mathbf{c}_R + 2\mathbf{c}_i$	$2\mathbf{c}_R + \mathbf{c}_i$
\mathbf{c}_R			$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$	$3\mathbf{c}_r + 4\mathbf{c}_\rho$
\mathbf{c}_i				$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$

Octahedral O projector algebra

Begin with minimal equation:

$$0 = (c_i + 2 \cdot 1)(c_i - 2 \cdot 1)(c_i + 6 \cdot 1)(c_i - 6 \cdot 1)(c_i - 0 \cdot 1)$$

$$c_g = \sum_{\mu} \frac{c_g \chi_g^{\mu}}{\ell^{\mu}} P^{\mu}$$

$$P^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{c_g} \chi_g^{\mu*} c_g$$

$$P^{(2)} = \frac{(c_i + 2 \cdot 1)(c_i - 6 \cdot 1)(c_i + 6 \cdot 1)(c_i - 0)}{(2 + 2)(2 - 6)(2 + 6)(2 - 0)} = \frac{(c_i + 2 \cdot 1)(c_i^2 - 36 \cdot 1)c_i}{-256} = \frac{c_i^4 + 2 \cdot c_i^3 - 36 c_i^2 - 72 c_i}{-256}$$

$$c_i = \quad + \quad c_i$$

O class product table

$\mathbf{1} = c_1$	c_r	c_{ρ}	c_R	c_i
c_r	$8\mathbf{1} + 4c_r + 8c_{\rho}$	$3c_r$	$4c_R + 4c_i$	$4c_R + 4c_i$
c_{ρ}		$3\mathbf{1} + 2c_{\rho}$	$c_R + 2c_i$	$2c_R + c_i$
c_R			$6\mathbf{1} + 3c_r + 2c_{\rho}$	$3c_r + 4c_{\rho}$
c_i				$6\mathbf{1} + 3c_r + 2c_{\rho}$

Octahedral O projector algebra

Begin with minimal equation:

$$0 = (c_i + 2 \cdot 1)(c_i - 2 \cdot 1)(c_i + 6 \cdot 1)(c_i - 6 \cdot 1)(c_i - 0 \cdot 1)$$

$$c_g = \sum_{\mu} \frac{c_g \chi_g^{\mu}}{\ell^{\mu}} P^{\mu}$$

$$P^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{c_g} \chi_g^{\mu*} c_g$$

$$P^{(2)} = \frac{(c_i + 2 \cdot 1)(c_i - 6 \cdot 1)(c_i + 6 \cdot 1)(c_i - 0)}{(2 + 2)(2 - 6)(2 + 6)(2 - 0)} = \frac{(c_i + 2 \cdot 1)(c_i^2 - 36 \cdot 1)c_i}{-256} = \frac{c_i^4 + 2 \cdot c_i^3 - 36 c_i^2 - 72 c_i}{-256}$$

$$c_i^2 = 6 \cdot 1 + 3c_r + 2c_{\rho}$$

$$c_i = \quad + \quad c_i$$

O class product table

$1 = c_1$	c_r	c_{ρ}	c_R	c_i
c_r	$81 + 4c_r + 8c_{\rho}$	$3c_r$	$4c_R + 4c_i$	$4c_R + 4c_i$
c_{ρ}		$31 + 2c_{\rho}$	$c_R + 2c_i$	$2c_R + c_i$
c_R			$61 + 3c_r + 2c_{\rho}$	$3c_r + 4c_{\rho}$
c_i				$61 + 3c_r + 2c_{\rho}$

Octahedral O projector algebra

Begin with minimal equation:

$$0 = (\mathbf{c}_i + 2 \cdot \mathbf{1})(\mathbf{c}_i - 2 \cdot \mathbf{1})(\mathbf{c}_i + 6 \cdot \mathbf{1})(\mathbf{c}_i - 6 \cdot \mathbf{1})(\mathbf{c}_i - 0 \cdot \mathbf{1})$$

$$\mathbf{c}_g = \sum_{\mu} \frac{\circ c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{\circ G} \chi_g^{\mu*} \mathbf{c}_g$$

$$\mathbf{P}^{(2)} = \frac{(\mathbf{c}_i + 2 \cdot \mathbf{1})(\mathbf{c}_i - 6 \cdot \mathbf{1})(\mathbf{c}_i + 6 \cdot \mathbf{1})(\mathbf{c}_i - 0)}{(2 + 2)(2 - 6)(2 + 6)(2 - 0)} = \frac{(\mathbf{c}_i + 2 \cdot \mathbf{1})(\mathbf{c}_i^2 - 36 \cdot \mathbf{1})\mathbf{c}_i}{-256} = \frac{\mathbf{c}_i^4 + 2 \cdot \mathbf{c}_i^3 - 36 \mathbf{c}_i^2 - 72 \mathbf{c}_i}{-256}$$

$$\begin{aligned} \mathbf{c}_i^3 &= && + 16\mathbf{c}_R + 20\mathbf{c}_i \\ \mathbf{c}_i^2 &= 6 \cdot \mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho \\ \mathbf{c}_i &= && + \mathbf{c}_i \end{aligned}$$

O class product table

$\mathbf{1} = c_1$	\mathbf{c}_r	\mathbf{c}_ρ	\mathbf{c}_R	\mathbf{c}_i
\mathbf{c}_r	$8\mathbf{1} + 4\mathbf{c}_r + 8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R + 4\mathbf{c}_i$	$4\mathbf{c}_R + 4\mathbf{c}_i$
\mathbf{c}_ρ		$3\mathbf{1} + 2\mathbf{c}_\rho$	$\mathbf{c}_R + 2\mathbf{c}_i$	$2\mathbf{c}_R + \mathbf{c}_i$
\mathbf{c}_R			$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$	$3\mathbf{c}_r + 4\mathbf{c}_\rho$
\mathbf{c}_i				$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$

Octahedral O projector algebra

Begin with minimal equation:

$$0 = (c_i + 2 \cdot 1)(c_i - 2 \cdot 1)(c_i + 6 \cdot 1)(c_i - 6 \cdot 1)(c_i - 0 \cdot 1)$$

$$c_g = \sum_{\mu} \frac{\circ c_g \chi_g^{\mu}}{\ell^{\mu}} P^{\mu}$$

$$P^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{\circ G} \chi_g^{\mu*} c_g$$

$$P^{(2)} = \frac{(c_i + 2 \cdot 1)(c_i - 6 \cdot 1)(c_i + 6 \cdot 1)(c_i - 0)}{(2 + 2)(2 - 6)(2 + 6)(2 - 0)} = \frac{(c_i + 2 \cdot 1)(c_i^2 - 36 \cdot 1)c_i}{-256} = \frac{c_i^4 + 2 \cdot c_i^3 - 36 c_i^2 - 72 c_i}{-256}$$

$$c_i^4 = 120 \cdot 1 + 108 c_r + 104 c_{\rho}$$

$$c_i^3 = \quad \quad \quad + 16 c_R + 20 c_i$$

$$c_i^2 = 6 \cdot 1 + 3 c_r + 2 c_{\rho}$$

$$c_i = \quad \quad \quad + \quad \quad c_i$$

O class product table

$1 = c_1$	c_r	c_{ρ}	c_R	c_i
c_r	$81 + 4c_r + 8c_{\rho}$	$3c_r$	$4c_R + 4c_i$	$4c_R + 4c_i$
c_{ρ}		$31 + 2c_{\rho}$	$c_R + 2c_i$	$2c_R + c_i$
c_R			$61 + 3c_r + 2c_{\rho}$	$3c_r + 4c_{\rho}$
c_i				$61 + 3c_r + 2c_{\rho}$

Octahedral O projector algebra

Begin with minimal equation:

$$0 = (c_i + 2 \cdot 1)(c_i - 2 \cdot 1)(c_i + 6 \cdot 1)(c_i - 6 \cdot 1)(c_i - 0 \cdot 1)$$

$$c_g = \sum_{\mu} \frac{\circ c_g \chi_g^{\mu}}{\ell^{\mu}} P^{\mu}$$

$$P^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{\circ G} \chi_g^{\mu*} c_g$$

$$P^{(2)} = \frac{(c_i + 2 \cdot 1)(c_i - 6 \cdot 1)(c_i + 6 \cdot 1)(c_i - 0)}{(2 + 2)(2 - 6)(2 + 6)(2 - 0)} = \frac{(c_i + 2 \cdot 1)(c_i^2 - 36 \cdot 1)c_i}{-256} = \frac{c_i^4 + 2 \cdot c_i^3 - 36 c_i^2 - 72 c_i}{-256}$$

$$c_i^4 = 120 \cdot 1 + 108 c_r + 104 c_{\rho}$$

$$c_i^4 = 120 \cdot 1 + 108 c_r + 104 c_{\rho}$$

$$c_i^3 = \quad \quad \quad + 16 c_R + 20 c_i$$

$$c_i^2 = 6 \cdot 1 + 3 c_r + 2 c_{\rho}$$

$$c_i = \quad \quad \quad + \quad \quad c_i$$

O class product table

$1 = c_1$	c_r	c_{ρ}	c_R	c_i
c_r	$81 + 4c_r + 8c_{\rho}$	$3c_r$	$4c_R + 4c_i$	$4c_R + 4c_i$
c_{ρ}		$31 + 2c_{\rho}$	$c_R + 2c_i$	$2c_R + c_i$
c_R			$61 + 3c_r + 2c_{\rho}$	$3c_r + 4c_{\rho}$
c_i				$61 + 3c_r + 2c_{\rho}$

Octahedral O projector algebra

Begin with minimal equation:

$$0 = (c_i + 2 \cdot 1)(c_i - 2 \cdot 1)(c_i + 6 \cdot 1)(c_i - 6 \cdot 1)(c_i - 0 \cdot 1)$$

$$c_g = \sum_{\mu} \frac{\circ c_g \chi_g^{\mu}}{\ell^{\mu}} P^{\mu}$$

$$P^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{\circ G} \chi_g^{\mu*} c_g$$

$$P^{(2)} = \frac{(c_i + 2 \cdot 1)(c_i - 6 \cdot 1)(c_i + 6 \cdot 1)(c_i - 0)}{(2 + 2)(2 - 6)(2 + 6)(2 - 0)} = \frac{(c_i + 2 \cdot 1)(c_i^2 - 36 \cdot 1)c_i}{-256} = \frac{c_i^4 + 2 \cdot c_i^3 - 36 c_i^2 - 72 c_i}{-256}$$

$$c_i^4 = 120 \cdot 1 + 108 c_r + 104 c_{\rho}$$

$$c_i^3 = \quad \quad \quad + 16 c_R + 20 c_i$$

$$c_i^2 = 6 \cdot 1 + 3 c_r + 2 c_{\rho}$$

$$c_i = \quad \quad \quad + \quad \quad c_i$$

$$c_i^4 = 120 \cdot 1 + 108 c_r + 104 c_{\rho}$$

$$+ 2 c_i^3 = \quad \quad \quad + 32 c_R + 40 c_i$$

O class product table

$1 = c_1$	c_r	c_{ρ}	c_R	c_i
c_r	$81 + 4c_r + 8c_{\rho}$	$3c_r$	$4c_R + 4c_i$	$4c_R + 4c_i$
c_{ρ}		$31 + 2c_{\rho}$	$c_R + 2c_i$	$2c_R + c_i$
c_R			$61 + 3c_r + 2c_{\rho}$	$3c_r + 4c_{\rho}$
c_i				$61 + 3c_r + 2c_{\rho}$

Octahedral O projector algebra

Begin with minimal equation: $0 = (c_i + 2 \cdot 1)(c_i - 2 \cdot 1)(c_i + 6 \cdot 1)(c_i - 6 \cdot 1)(c_i - 0 \cdot 1)$

$$c_g = \sum_{\mu} \frac{\circ c_g \chi_g^{\mu}}{\ell^{\mu}} P^{\mu}$$

$$P^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{\circ G} \chi_g^{\mu*} c_g$$

$$P^{(2)} = \frac{(c_i + 2 \cdot 1)(c_i - 6 \cdot 1)(c_i + 6 \cdot 1)(c_i - 0)}{(2 + 2)(2 - 6)(2 + 6)(2 - 0)} = \frac{(c_i + 2 \cdot 1)(c_i^2 - 36 \cdot 1)c_i}{-256} = \frac{c_i^4 + 2 \cdot c_i^3 - 36 c_i^2 - 72 c_i}{-256}$$

$$c_i^4 = 120 \cdot 1 + 108 c_r + 104 c_{\rho}$$

$$c_i^3 = \quad \quad \quad + 16 c_R + 20 c_i$$

$$c_i^2 = 6 \cdot 1 + 3 c_r + 2 c_{\rho}$$

$$c_i = \quad \quad \quad + \quad \quad c_i$$

$$c_i^4 = 120 \cdot 1 + 108 c_r + 104 c_{\rho}$$

$$+ 2 c_i^3 = \quad \quad \quad + 32 c_R + 40 c_i$$

$$- 36 c_i^2 = -216 \cdot 1 - 108 c_r - 72 c_{\rho}$$

O class product table

$1 = c_1$	c_r	c_{ρ}	c_R	c_i
c_r	$81 + 4c_r + 8c_{\rho}$	$3c_r$	$4c_R + 4c_i$	$4c_R + 4c_i$
c_{ρ}		$31 + 2c_{\rho}$	$c_R + 2c_i$	$2c_R + c_i$
c_R			$61 + 3c_r + 2c_{\rho}$	$3c_r + 4c_{\rho}$
c_i				$61 + 3c_r + 2c_{\rho}$

Octahedral O projector algebra

Begin with minimal equation:

$$0 = (c_i + 2 \cdot 1)(c_i - 2 \cdot 1)(c_i + 6 \cdot 1)(c_i - 6 \cdot 1)(c_i - 0 \cdot 1)$$

$$c_g = \sum_{\mu} \frac{\circ c_g \chi_g^{\mu}}{\ell^{\mu}} P^{\mu}$$

$$P^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{\circ G} \chi_g^{\mu*} c_g$$

$$P^{(2)} = \frac{(c_i + 2 \cdot 1)(c_i - 6 \cdot 1)(c_i + 6 \cdot 1)(c_i - 0)}{(2 + 2)(2 - 6)(2 + 6)(2 - 0)} = \frac{(c_i + 2 \cdot 1)(c_i^2 - 36 \cdot 1)c_i}{-256} = \frac{c_i^4 + 2 \cdot c_i^3 - 36 c_i^2 - 72 c_i}{-256}$$

$$c_i^4 = 120 \cdot 1 + 108 c_r + 104 c_{\rho}$$

$$c_i^3 = \quad \quad \quad + 16 c_R + 20 c_i$$

$$c_i^2 = 6 \cdot 1 + 3 c_r + 2 c_{\rho}$$

$$c_i = \quad \quad \quad + \quad \quad c_i$$

$$c_i^4 = 120 \cdot 1 + 108 c_r + 104 c_{\rho}$$

$$+ 2 c_i^3 = \quad \quad \quad + 32 c_R + 40 c_i$$

$$- 36 c_i^2 = -216 \cdot 1 - 108 c_r - 72 c_{\rho}$$

$$- 72 c_i = \quad \quad \quad - 72 c_i$$

O class product table

$1 = c_1$	c_r	c_{ρ}	c_R	c_i
c_r	$81 + 4c_r + 8c_{\rho}$	$3c_r$	$4c_R + 4c_i$	$4c_R + 4c_i$
c_{ρ}		$31 + 2c_{\rho}$	$c_R + 2c_i$	$2c_R + c_i$
c_R			$61 + 3c_r + 2c_{\rho}$	$3c_r + 4c_{\rho}$
c_i				$61 + 3c_r + 2c_{\rho}$

Octahedral O projector algebra

Begin with minimal equation:

$$0 = (c_i + 2 \cdot 1)(c_i - 2 \cdot 1)(c_i + 6 \cdot 1)(c_i - 6 \cdot 1)(c_i - 0 \cdot 1)$$

$$c_g = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} P^{\mu}$$

$$P^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{{}^{\circ}G} \chi_g^{\mu*} c_g$$

$$P^{(2)} = \frac{(c_i + 2 \cdot 1)(c_i - 6 \cdot 1)(c_i + 6 \cdot 1)(c_i - 0)}{(2 + 2)(2 - 6)(2 + 6)(2 - 0)} = \frac{(c_i + 2 \cdot 1)(c_i^2 - 36 \cdot 1)c_i}{-256} = \frac{c_i^4 + 2 \cdot c_i^3 - 36 c_i^2 - 72 c_i}{-256}$$

$$c_i^4 = 120 \cdot 1 + 108 c_r + 104 c_{\rho}$$

$$c_i^3 = \quad \quad \quad + 16 c_R + 20 c_i$$

$$c_i^2 = 6 \cdot 1 + 3 c_r + 2 c_{\rho}$$

$$c_i = \quad \quad \quad + \quad \quad c_i$$

$$c_i^4 = 120 \cdot 1 + 108 c_r + 104 c_{\rho}$$

$$+ 2 c_i^3 = \quad \quad \quad + 32 c_R + 40 c_i$$

$$- 36 c_i^2 = -216 \cdot 1 - 108 c_r - 72 c_{\rho}$$

$$- 72 c_i = \quad \quad \quad - 72 c_i$$

$$-256 P^{(2)} = -96 \cdot 1 + 0 c_r + 32 c_{\rho} + 32 c_R - 32 c_i$$

O class product table

$1 = c_1$	c_r	c_{ρ}	c_R	c_i
c_r	$81 + 4c_r + 8c_{\rho}$	$3c_r$	$4c_R + 4c_i$	$4c_R + 4c_i$
c_{ρ}		$31 + 2c_{\rho}$	$c_R + 2c_i$	$2c_R + c_i$
c_R			$61 + 3c_r + 2c_{\rho}$	$3c_r + 4c_{\rho}$
c_i				$61 + 3c_r + 2c_{\rho}$

Octahedral O projector algebra

Begin with minimal equation:

$$0 = (c_i + 2 \cdot 1)(c_i - 2 \cdot 1)(c_i + 6 \cdot 1)(c_i - 6 \cdot 1)(c_i - 0 \cdot 1)$$

$$c_g = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} P^{\mu}$$

$$P^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{{}^{\circ}G} \chi_g^{\mu*} c_g$$

$$P^{(2)} = \frac{(c_i + 2 \cdot 1)(c_i - 6 \cdot 1)(c_i + 6 \cdot 1)(c_i - 0)}{(2 + 2)(2 - 6)(2 + 6)(2 - 0)} = \frac{(c_i + 2 \cdot 1)(c_i^2 - 36 \cdot 1)c_i}{-256} = \frac{c_i^4 + 2 \cdot c_i^3 - 36c_i^2 - 72c_i}{-256}$$

$$c_i^4 = 120 \cdot 1 + 108c_r + 104c_{\rho}$$

$$c_i^3 = \quad \quad \quad + 16c_R + 20c_i$$

$$c_i^2 = 6 \cdot 1 + 3c_r + 2c_{\rho}$$

$$c_i = \quad \quad \quad + \quad \quad c_i$$

$$c_i^4 = 120 \cdot 1 + 108c_r + 104c_{\rho}$$

$$+ 2c_i^3 = \quad \quad \quad + 32c_R + 40c_i$$

$$- 36c_i^2 = -216 \cdot 1 - 108c_r - 72c_{\rho}$$

$$- 72c_i = \quad \quad \quad - 72c_i$$

$$-256P^{(2)} = -96 \cdot 1 + 0c_r + 32c_{\rho} + 32c_R - 32c_i$$

$$P^{(2)} = \frac{3}{8} \mathbf{1} - \frac{0}{8} c_r + \frac{1}{8} c_{\rho} - \frac{1}{8} c_R + \frac{1}{8} c_i$$

O class product table

$\mathbf{1} = c_1$	c_r	c_{ρ}	c_R	c_i
c_r	$8\mathbf{1} + 4c_r + 8c_{\rho}$	$3c_r$	$4c_R + 4c_i$	$4c_R + 4c_i$
c_{ρ}		$3\mathbf{1} + 2c_{\rho}$	$c_R + 2c_i$	$2c_R + c_i$
c_R			$6\mathbf{1} + 3c_r + 2c_{\rho}$	$3c_r + 4c_{\rho}$
c_i				$6\mathbf{1} + 3c_r + 2c_{\rho}$

Octahedral O projector algebra

Begin with minimal equation: $0 = (c_i + 2 \cdot 1)(c_i - 2 \cdot 1)(c_i + 6 \cdot 1)(c_i - 6 \cdot 1)(c_i - 0 \cdot 1)$

$$c_g = \sum_{\mu} \frac{c_g \chi_g^{\mu}}{\ell^{\mu}} P^{\mu}$$

$$P^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{c_g} \chi_g^{\mu*} c_g$$

$$P^{(2)} = \frac{(c_i + 2 \cdot 1)(c_i - 6 \cdot 1)(c_i + 6 \cdot 1)(c_i - 0)}{(2 + 2)(2 - 6)(2 + 6)(2 - 0)} = \frac{(c_i + 2 \cdot 1)(c_i^2 - 36 \cdot 1)c_i}{-256} = \frac{c_i^4 + 2 \cdot c_i^3 - 36 c_i^2 - 72 c_i}{-256}$$

$$\begin{aligned} c_i^4 &= 120 \cdot 1 + 108 c_r + 104 c_{\rho} \\ c_i^3 &= + 16 c_R + 20 c_i \\ c_i^2 &= 6 \cdot 1 + 3 c_r + 2 c_{\rho} \\ c_i &= + c_i \end{aligned}$$

$$\begin{aligned} c_i^4 &= 120 \cdot 1 + 108 c_r + 104 c_{\rho} \\ + 2 c_i^3 &= + 32 c_R + 40 c_i \\ - 36 c_i^2 &= -216 \cdot 1 - 108 c_r - 72 c_{\rho} \\ - 72 c_i &= - 72 c_i \\ \hline -256 P^{(2)} &= -96 \cdot 1 + 0 c_r + 32 c_{\rho} + 32 c_R - 32 c_i \\ P^{(2)} &= \frac{3}{8} \mathbf{1} + \frac{0}{8} c_r - \frac{1}{8} c_{\rho} - \frac{1}{8} c_R + \frac{1}{8} c_i \end{aligned}$$

O class product table

$\mathbf{1} = c_1$	c_r	c_{ρ}	c_R	c_i
c_r	$8\mathbf{1} + 4c_r + 8c_{\rho}$	$3c_r$	$4c_R + 4c_i$	$4c_R + 4c_i$
c_{ρ}		$3\mathbf{1} + 2c_{\rho}$	$c_R + 2c_i$	$2c_R + c_i$
c_R			$6\mathbf{1} + 3c_r + 2c_{\rho}$	$3c_r + 4c_{\rho}$
c_i				$6\mathbf{1} + 3c_r + 2c_{\rho}$

χ_g^{μ}	$\mathbf{g} = \mathbf{1}$	$\mathbf{r}_{1..4}$	ρ_{xyz}	\mathbf{R}_{xyz}	$\mathbf{i}_{1..6}$
\cdot					
\cdot					
\cdot					
χ^{T_2}	3	0	-1	-1	1

Octahedral O projector algebra

Begin with minimal equation: $0 = (\mathbf{c}_i + 2 \cdot \mathbf{1})(\mathbf{c}_i - 2 \cdot \mathbf{1})(\mathbf{c}_i + 6 \cdot \mathbf{1})(\mathbf{c}_i - 6 \cdot \mathbf{1})(\mathbf{c}_i - 0 \cdot \mathbf{1})$

$$\mathbf{c}_g = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{{}^{\circ}G} \chi_g^{\mu*} \mathbf{c}_g$$

$$\mathbf{P}^{(2)} = \frac{(\mathbf{c}_i + 2 \cdot \mathbf{1})(\mathbf{c}_i - 6 \cdot \mathbf{1})(\mathbf{c}_i + 6 \cdot \mathbf{1})(\mathbf{c}_i - 0)}{(2 + 2)(2 - 6)(2 + 6)(2 - 0)} = \frac{(\mathbf{c}_i + 2 \cdot \mathbf{1})(\mathbf{c}_i^2 - 36 \cdot \mathbf{1})\mathbf{c}_i}{-256} = \frac{\mathbf{c}_i^4 + 2 \cdot \mathbf{c}_i^3 - 36 \mathbf{c}_i^2 - 72 \mathbf{c}_i}{-256}$$

$$\mathbf{c}_i^4 = 120 \cdot \mathbf{1} + 108 \mathbf{c}_r + 104 \mathbf{c}_\rho$$

$$\mathbf{c}_i^3 = \quad \quad \quad + 16 \mathbf{c}_R + 20 \mathbf{c}_i$$

$$\mathbf{c}_i^2 = 6 \cdot \mathbf{1} + 3 \mathbf{c}_r + 2 \mathbf{c}_\rho$$

$$\mathbf{c}_i = \quad \quad \quad + \quad \quad \quad \mathbf{c}_i$$

$$\mathbf{c}_i^4 = 120 \cdot \mathbf{1} + 108 \mathbf{c}_r + 104 \mathbf{c}_\rho$$

$$+ 2 \mathbf{c}_i^3 = \quad \quad \quad + 32 \mathbf{c}_R + 40 \mathbf{c}_i$$

$$- 36 \mathbf{c}_i^2 = -216 \cdot \mathbf{1} - 108 \mathbf{c}_r - 72 \mathbf{c}_\rho$$

$$- 72 \mathbf{c}_i = \quad \quad \quad - 72 \mathbf{c}_i$$

$$-256 \mathbf{P}^{(2)} = -96 \cdot \mathbf{1} + 0 \mathbf{c}_r + 32 \mathbf{c}_\rho + 32 \mathbf{c}_R - 32 \mathbf{c}_i$$

$$\mathbf{P}^{(2)} = \frac{3}{8} \mathbf{1} + \frac{0}{8} \mathbf{c}_r - \frac{1}{8} \mathbf{c}_\rho - \frac{1}{8} \mathbf{c}_R + \frac{1}{8} \mathbf{c}_i$$

$$\mathbf{P}^{(-2)} = \frac{3}{8} \mathbf{1} + \frac{0}{8} \mathbf{c}_r - \frac{1}{8} \mathbf{c}_\rho + \frac{1}{8} \mathbf{c}_R - \frac{1}{8} \mathbf{c}_i$$

O class product table

Octahedral O characters

$\mathbf{1} = c_1$	\mathbf{c}_r	\mathbf{c}_ρ	\mathbf{c}_R	\mathbf{c}_i
\mathbf{c}_r	$8\mathbf{1} + 4\mathbf{c}_r + 8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R + 4\mathbf{c}_i$	$4\mathbf{c}_R + 4\mathbf{c}_i$
\mathbf{c}_ρ		$3\mathbf{1} + 2\mathbf{c}_\rho$	$\mathbf{c}_R + 2\mathbf{c}_i$	$2\mathbf{c}_R + \mathbf{c}_i$
\mathbf{c}_R			$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$	$3\mathbf{c}_r + 4\mathbf{c}_\rho$
\mathbf{c}_i				$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$

χ_g^{μ}	$\mathbf{g} = \mathbf{1}$	$\mathbf{r}_{1..4}$	$\mathbf{\rho}_{xyz}$	\mathbf{R}_{xyz}	$\mathbf{i}_{1..6}$
\cdot					
\cdot					
\cdot					
χ^{T_1}	3	0	-1	1	-1
χ^{T_2}	3	0	-1	-1	1

Octahedral O projector algebra

Begin with minimal equation: $0 = (\mathbf{c}_i + 2 \cdot \mathbf{1})(\mathbf{c}_i - 2 \cdot \mathbf{1})(\mathbf{c}_i + 6 \cdot \mathbf{1})(\mathbf{c}_i - 6 \cdot \mathbf{1})(\mathbf{c}_i - 0 \cdot \mathbf{1})$

$$\mathbf{c}_g = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{{}^{\circ}G} \chi_g^{\mu*} \mathbf{c}_g$$

$$\mathbf{P}^{(2)} = \frac{(\mathbf{c}_i + 2 \cdot \mathbf{1})(\mathbf{c}_i - 6 \cdot \mathbf{1})(\mathbf{c}_i + 6 \cdot \mathbf{1})(\mathbf{c}_i - 0)}{(2 + 2)(2 - 6)(2 + 6)(2 - 0)} = \frac{(\mathbf{c}_i + 2 \cdot \mathbf{1})(\mathbf{c}_i^2 - 36 \cdot \mathbf{1})\mathbf{c}_i}{-256} = \frac{\mathbf{c}_i^4 + 2 \cdot \mathbf{c}_i^3 - 36 \mathbf{c}_i^2 - 72 \mathbf{c}_i}{-256}$$

$$\mathbf{c}_i^4 = 120 \cdot \mathbf{1} + 108 \mathbf{c}_r + 104 \mathbf{c}_\rho$$

$$\mathbf{c}_i^3 = \phantom{120 \cdot \mathbf{1} + 108 \mathbf{c}_r + 104 \mathbf{c}_\rho} + 16 \mathbf{c}_R + 20 \mathbf{c}_i$$

$$\mathbf{c}_i^2 = 6 \cdot \mathbf{1} + 3 \mathbf{c}_r + 2 \mathbf{c}_\rho$$

$$\mathbf{c}_i = \phantom{6 \cdot \mathbf{1} + 3 \mathbf{c}_r + 2 \mathbf{c}_\rho} + \phantom{16 \mathbf{c}_R + 20 \mathbf{c}_i} \mathbf{c}_i$$

$$\mathbf{c}_i^4 = 120 \cdot \mathbf{1} + 108 \mathbf{c}_r + 104 \mathbf{c}_\rho$$

$$+ 2 \mathbf{c}_i^3 = \phantom{120 \cdot \mathbf{1} + 108 \mathbf{c}_r + 104 \mathbf{c}_\rho} + 32 \mathbf{c}_R + 40 \mathbf{c}_i$$

$$- 36 \mathbf{c}_i^2 = -216 \cdot \mathbf{1} - 108 \mathbf{c}_r - 72 \mathbf{c}_\rho$$

$$- 72 \mathbf{c}_i = \phantom{-216 \cdot \mathbf{1} - 108 \mathbf{c}_r - 72 \mathbf{c}_\rho} - 72 \mathbf{c}_i$$

$$-256 \mathbf{P}^{(2)} = -96 \cdot \mathbf{1} + 0 \mathbf{c}_r + 32 \mathbf{c}_\rho + 32 \mathbf{c}_R - 32 \mathbf{c}_i$$

$$\mathbf{P}^{(2)} = \frac{3}{8} \mathbf{1} + \frac{0}{8} \mathbf{c}_r - \frac{1}{8} \mathbf{c}_\rho - \frac{1}{8} \mathbf{c}_R + \frac{1}{8} \mathbf{c}_i$$

$$\mathbf{P}^{(-2)} = \frac{3}{8} \mathbf{1} + \frac{0}{8} \mathbf{c}_r - \frac{1}{8} \mathbf{c}_\rho + \frac{1}{8} \mathbf{c}_R - \frac{1}{8} \mathbf{c}_i$$

O class product table

Octahedral O characters

$\mathbf{1} = c_1$	\mathbf{c}_r	\mathbf{c}_ρ	\mathbf{c}_R	\mathbf{c}_i
\mathbf{c}_r	$8\mathbf{1} + 4\mathbf{c}_r + 8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R + 4\mathbf{c}_i$	$4\mathbf{c}_R + 4\mathbf{c}_i$
\mathbf{c}_ρ		$3\mathbf{1} + 2\mathbf{c}_\rho$	$\mathbf{c}_R + 2\mathbf{c}_i$	$2\mathbf{c}_R + \mathbf{c}_i$
\mathbf{c}_R			$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$	$3\mathbf{c}_r + 4\mathbf{c}_\rho$
\mathbf{c}_i				$6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$

(Remaining character derivations left as an exercise)

χ_g^{μ}	$\mathbf{g} = \mathbf{1}$	$\mathbf{r}_{1..4}$	$\mathbf{\rho}_{xyz}$	\mathbf{R}_{xyz}	$\mathbf{i}_{1..6}$
χ^{A_1}	1	1	1	1	1
χ^{A_2}	1	1	1	-1	-1
χ^E	2	-1	2	0	0
χ^{T_1}	3	0	-1	1	-1
χ^{T_2}	3	0	-1	-1	1

Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$

Octahedral-cubic O symmetry and group operations

Tetrahedral symmetry becomes Icosahedral

Octahedral groups $O_h \supset O \sim T_d \supset T$

Octahedral O and spin- $O \subset U(2)$

Tetrahedral T class algebra

Tetrahedral T class minimal equations

Tetrahedral T class projectors and characters

Octahedral O class algebra

Octahedral O class minimal equations

Octahedral O class projectors and characters

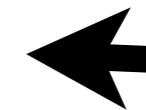
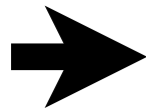
Octahedral $O_h \supset O$ subgroup correlations

Octahedral $O_h \supset O$ subgroup correlations

$O_h \supset O \supset D_4$ subgroup correlations

$O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

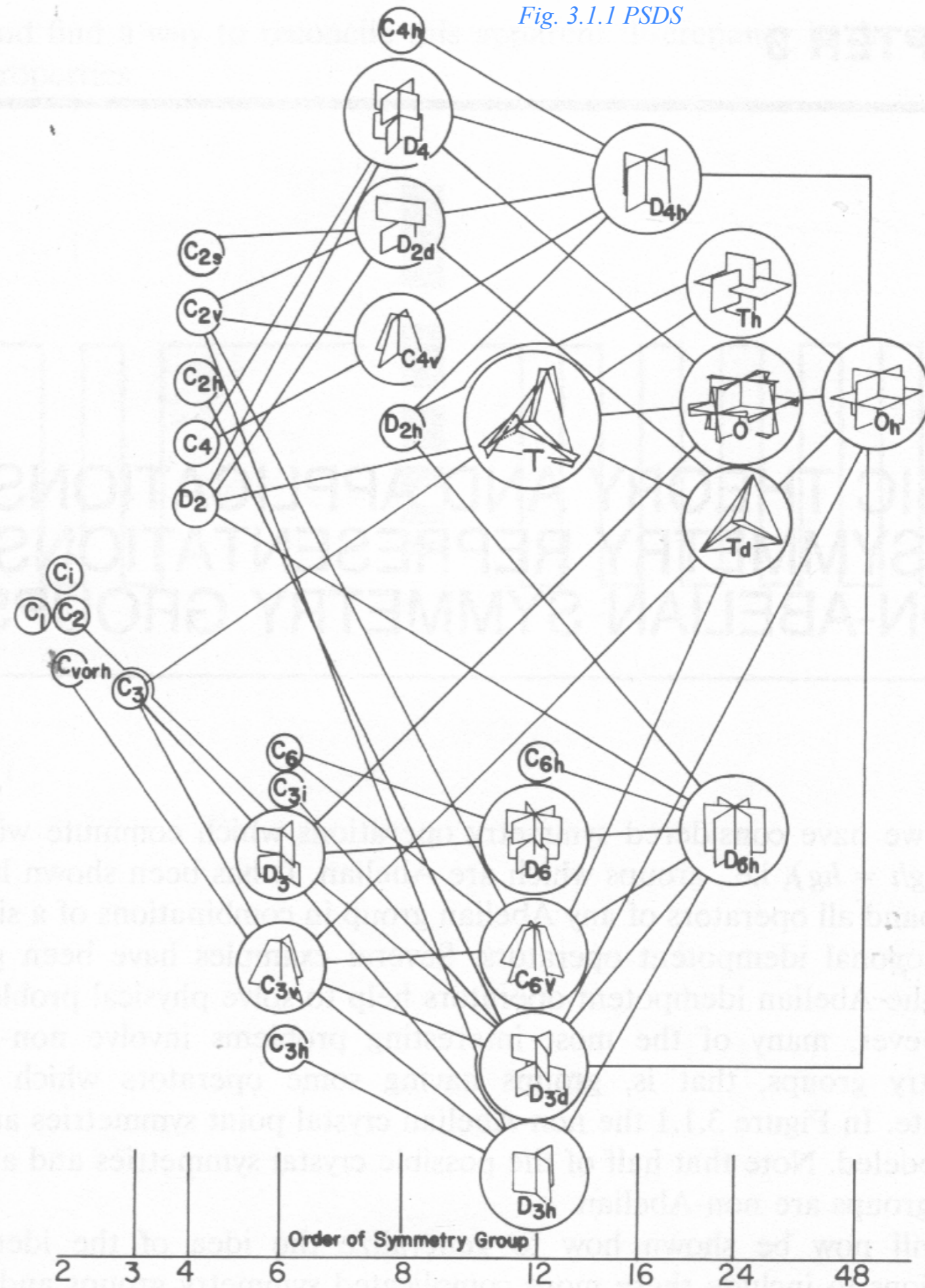
Applications



Octahedral $O_h \supset O$ subgroup correlations

χ_g^μ	$g=1$	$r_{1...4}$	ρ_{xyz}	R_{xyz}	$i_{1...6}$
χ^{A_1}	1	1	1	1	1
χ^{A_2}	1	1	1	-1	-1
χ^E	2	-1	2	0	0
χ^{T_1}	3	0	-1	1	-1
χ^{T_2}	3	0	-1	-1	1

Fig. 3.1.1 PSDS

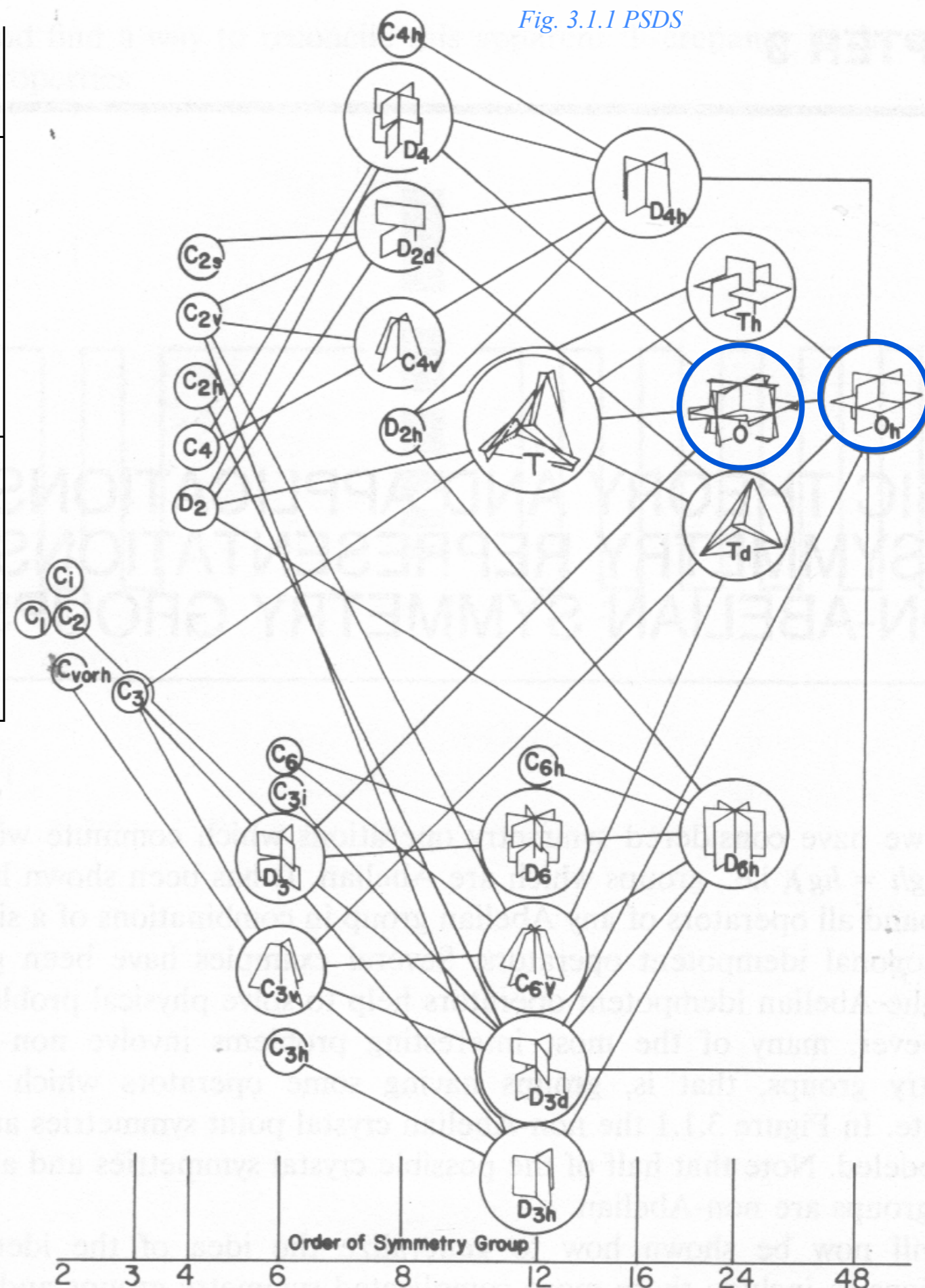


Octahedral $O_h \supset O$ subgroup correlations

Fig. 3.1.1 PSDS

$\chi_g^{\mu_p}$	1	r _{1...4}	ρ _{xyz}	R _{xyz}	i _{1..6}	I	Ir= s _{1...4}	Ip= σ _{xyz}	IR= S _{xyz}	Ii= σ _{1..6}
$\chi^{A_{1g}}$	1	1	1	1	1	1	1	1	1	1
$\chi^{A_{2g}}$	1	1	1	-1	-1	1	1	1	-1	-1
χ^{E_g}	2	-1	2	0	0	2	-1	2	0	0
$\chi^{T_{1g}}$	3	0	-1	1	-1	3	0	-1	1	-1
$\chi^{T_{2g}}$	3	0	-1	-1	1	3	0	-1	-1	1
$\chi^{A_{1u}}$	1	1	1	1	1	-1	-1	-1	-1	-1
$\chi^{A_{2u}}$	1	1	1	-1	-1	-1	-1	-1	1	1
χ^{E_u}	2	-1	2	0	0	-2	1	-2	0	0
$\chi^{T_{1u}}$	3	0	-1	1	-1	-3	0	1	-1	1
$\chi^{T_{2u}}$	3	0	-1	-1	1	-3	0	1	1	-1

$O_h \supset O$	A ₁	A ₂	E	T ₁	T ₂
A_{1g}	1
A_{2g}	.	1	.	.	.
E_g	.	.	1	.	.
T_{1g}	.	.	.	1	.
T_{2g}	1
A_{1u}	1
A_{2u}	.	1	.	.	.
E_u	.	.	1	.	.
T_{1u}	.	.	.	1	.
T_{2u}	1



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Octahedral O class algebra

Octahedral O class minimal equations

Octahedral O class projectors and characters

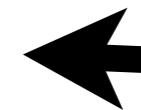
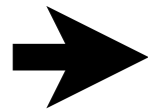
Octahedral $O_h \supset O$ subgroup correlations

Octahedral $O_h \supset O$ subgroup correlations

$O_h \supset O \supset D_4$ subgroup correlations

$O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

Applications



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

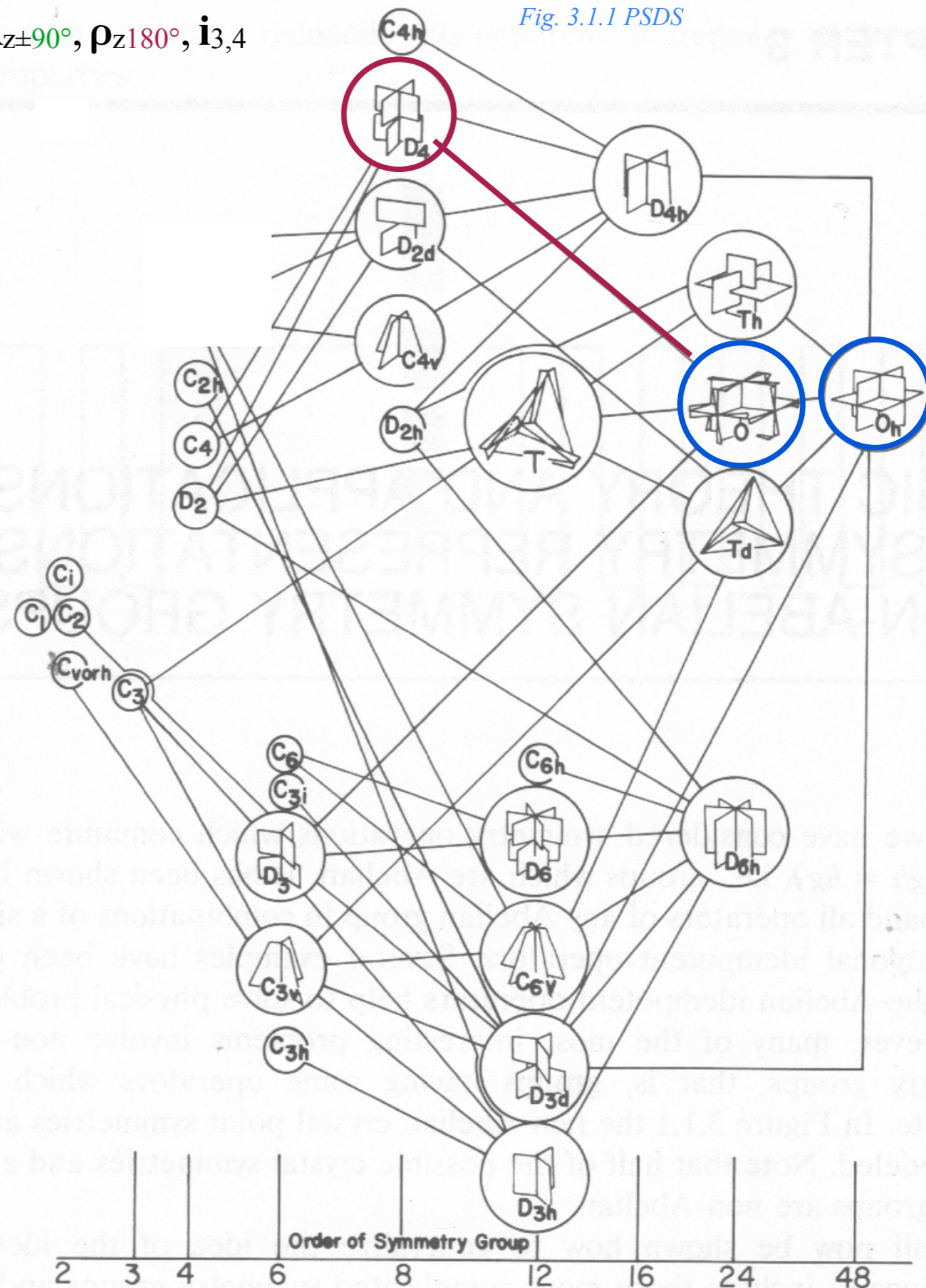
$O \downarrow D_4$ subduction

$D_4: \mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{z180^\circ}, \mathbf{i}_{3,4}$

Fig. 3.1.1 PSDS

$\chi_g^\mu(O)$	$g=1$	$r_{1..4}$	180° ρ_{xyz}	90° \mathbf{R}_{xyz}	180° $\mathbf{i}_{1..6}$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

$\chi_g^\mu(D_4)$	$g=1$	ρ_{z180°	$\mathbf{R}_{z\pm 90^\circ}$	$\rho_{x,y180^\circ}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	-1	1	-1
A_2	1	1	1	-1	-1
B_2	1	1	-1	-1	1
E	2	-2	0	0	0



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

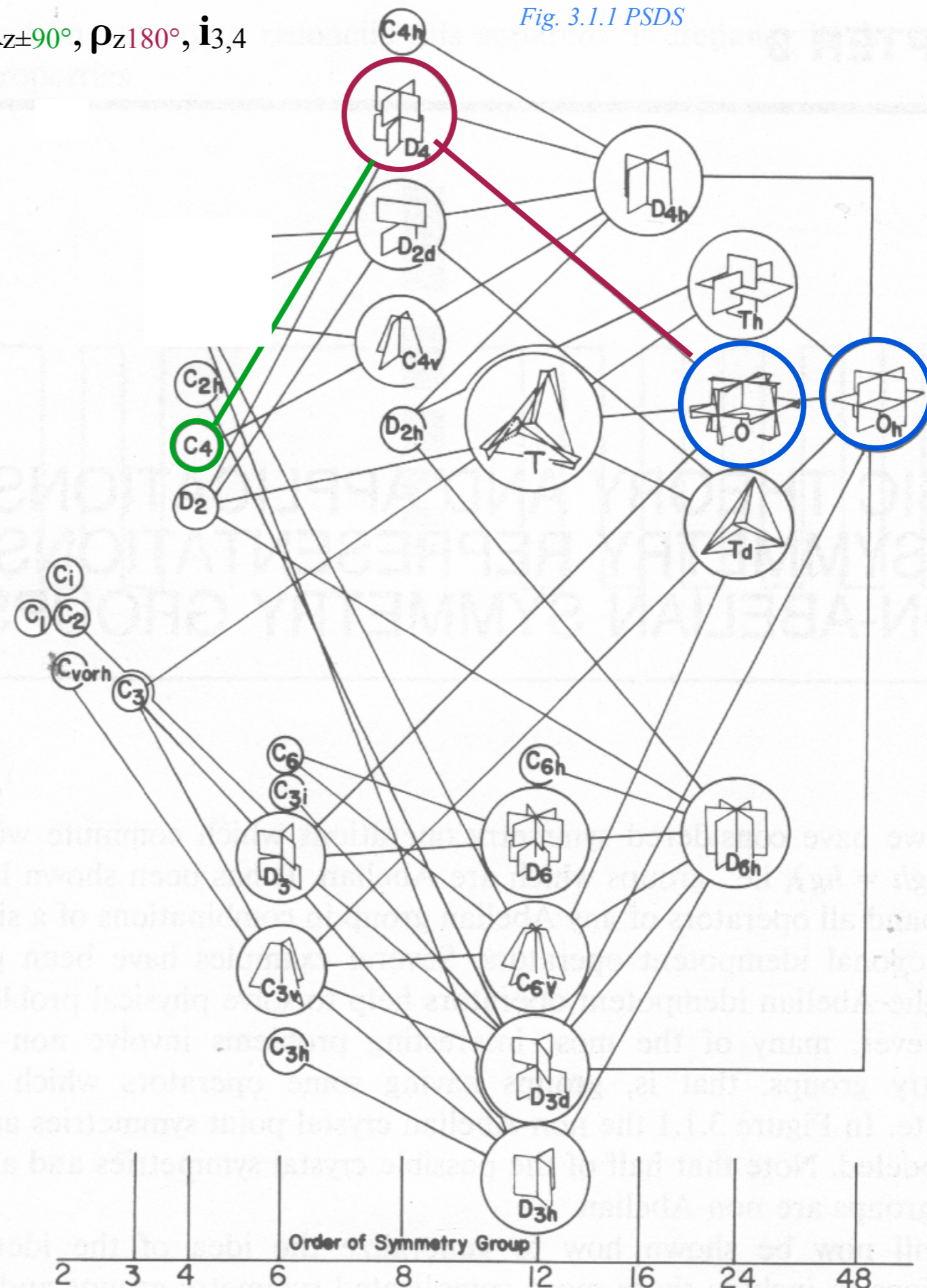
D_4 : $\mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{z180^\circ}, \mathbf{i}_{3,4}$

Fig. 3.1.1 PSDS

$\chi_g^\mu(O)$	$g=1$	$r_{1..4}$	180° ρ_{xyz}	90° \mathbf{R}_{xyz}	180° $\mathbf{i}_{1..6}$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

$\chi_g^\mu(D_4)$	$g=1$	ρ_{z180°	$\mathbf{R}_{z\pm 90^\circ}$	$\rho_{x,y180^\circ}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	-1	1	-1
A_2	1	1	1	-1	-1
B_2	1	1	-1	-1	1
E	2	-2	0	0	0

$\chi_g^\mu(C_4)$	$g=1$	\mathbf{R}_{z+90°	\mathbf{R}_{z+180°	\mathbf{R}_{z-90°
$(0)_4$	1	1	1	1
$(1)_4$	1	i	-1	$-i$
$(2)_4$	1	-1	1	-1
$(3)_4$	1	$-i$	-1	i



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

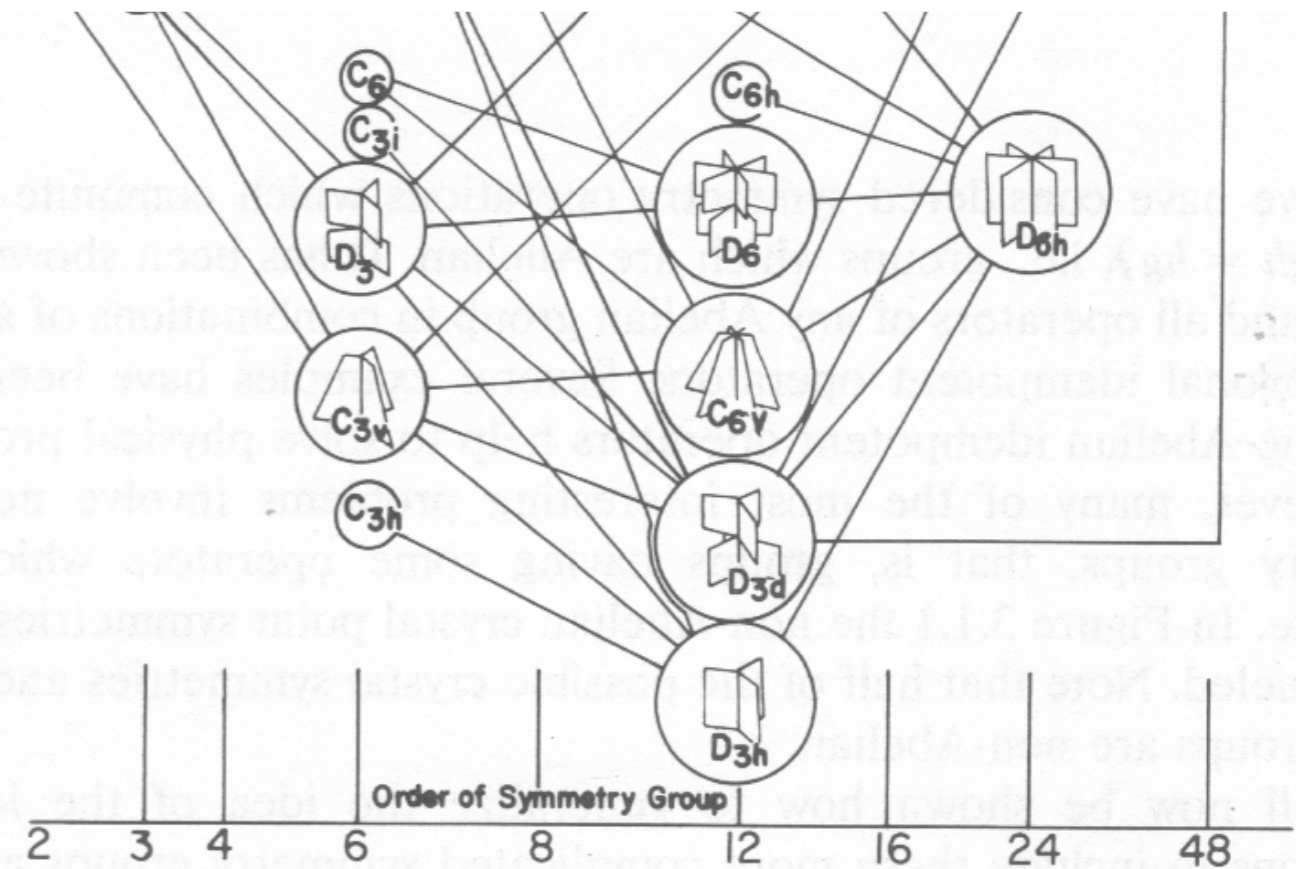
$O \downarrow D_4$ subduction

D_4 : $\mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{z180^\circ}, \mathbf{i}_{3,4}$

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1$.

$\chi_g^\mu(O)$	$g=1$	$r_{1..4}$	180° ρ_{xyz}	90° \mathbf{R}_{xyz}	180° $\mathbf{i}_{1..6}$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

$\chi_g^\mu(D_4)$	$g=1$	ρ_{z180°	$\mathbf{R}_{z\pm 90^\circ}$	$\rho_{x,y180^\circ}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	-1	1	-1
A_2	1	1	1	-1	-1
B_2	1	1	-1	-1	1
E	2	-2	0	0	0



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

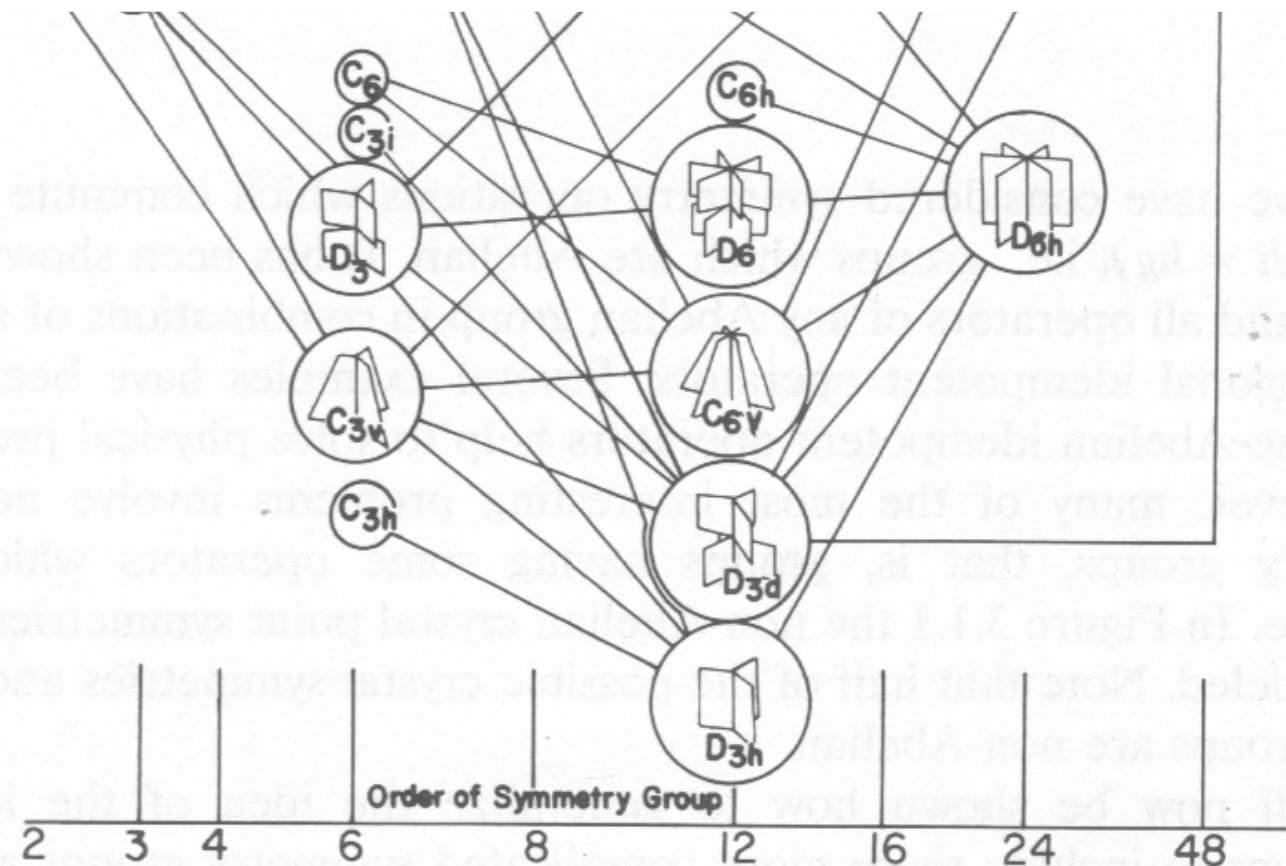
$O \downarrow D_4$ subduction

D_4 : $\mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{z180^\circ}, \mathbf{i}_{3,4}$

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1$. $A_1(O) \downarrow D_4 = A_1(D_4)$

$\chi_g^\mu(O)$	$g=1$	$r_{1..4}$	180° ρ_{xyz}	90° \mathbf{R}_{xyz}	180° $\mathbf{i}_{1..6}$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

$\chi_g^\mu(D_4)$	$g=1$	ρ_{z180°	$\mathbf{R}_{z\pm 90^\circ}$	$\rho_{x,y180^\circ}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	-1	1	-1
A_2	1	1	1	-1	-1
B_2	1	1	-1	-1	1
E	2	-2	0	0	0



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

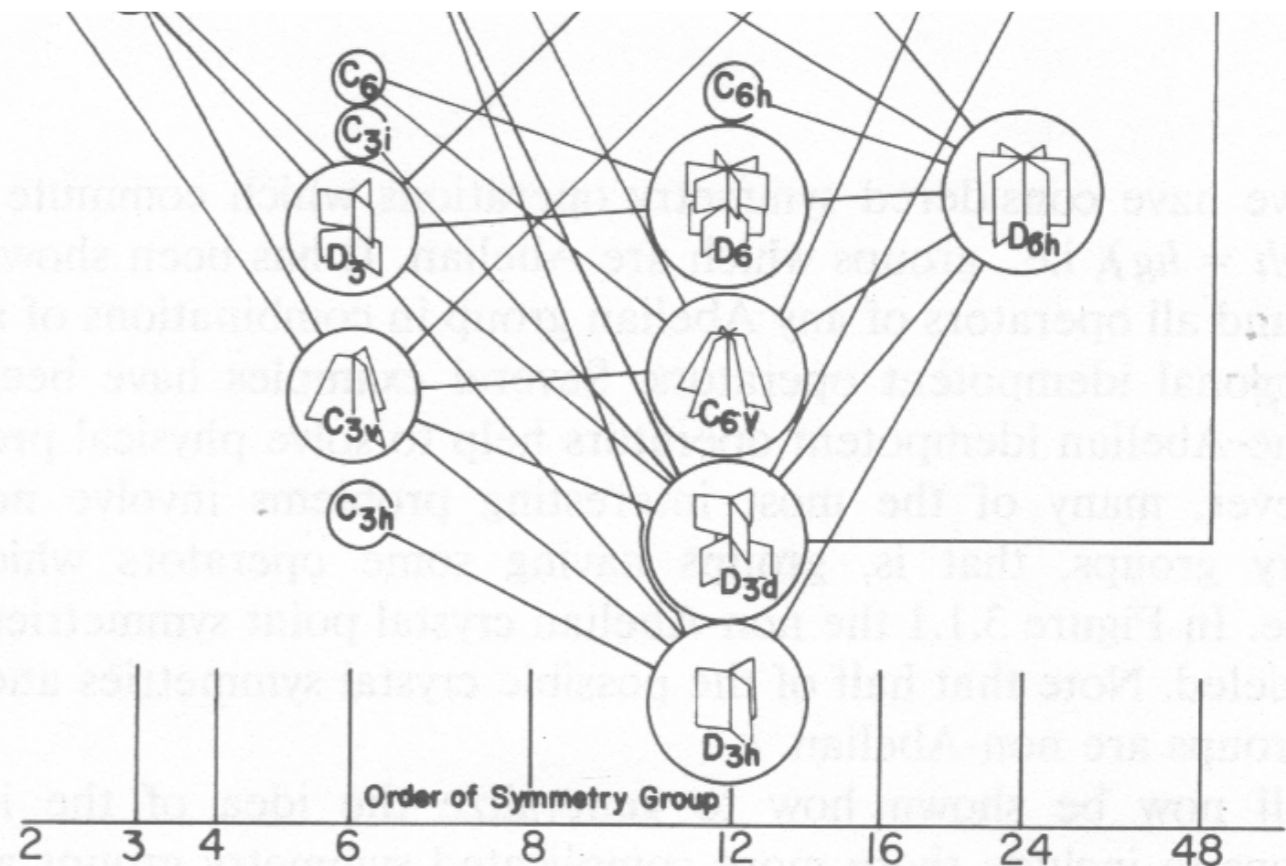
$O \downarrow D_4$ subduction

D_4 : $\mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{z180^\circ}, \mathbf{i}_{3,4}$

$\chi_g^\mu(O)$	$g=1$	$\mathbf{r}_{1..4}$	180° ρ_{xyz}	90° \mathbf{R}_{xyz}	180° $\mathbf{i}_{1..6}$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1$. $A_1(O) \downarrow D_4 = A_1(D_4)$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1$.

$\chi_g^\mu(D_4)$	$g=1$	ρ_{z180°	$\mathbf{R}_{z\pm 90^\circ}$	$\rho_{x,y180^\circ}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	-1	1	-1
A_2	1	1	1	-1	-1
B_2	1	1	-1	-1	1
E	2	-2	0	0	0



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

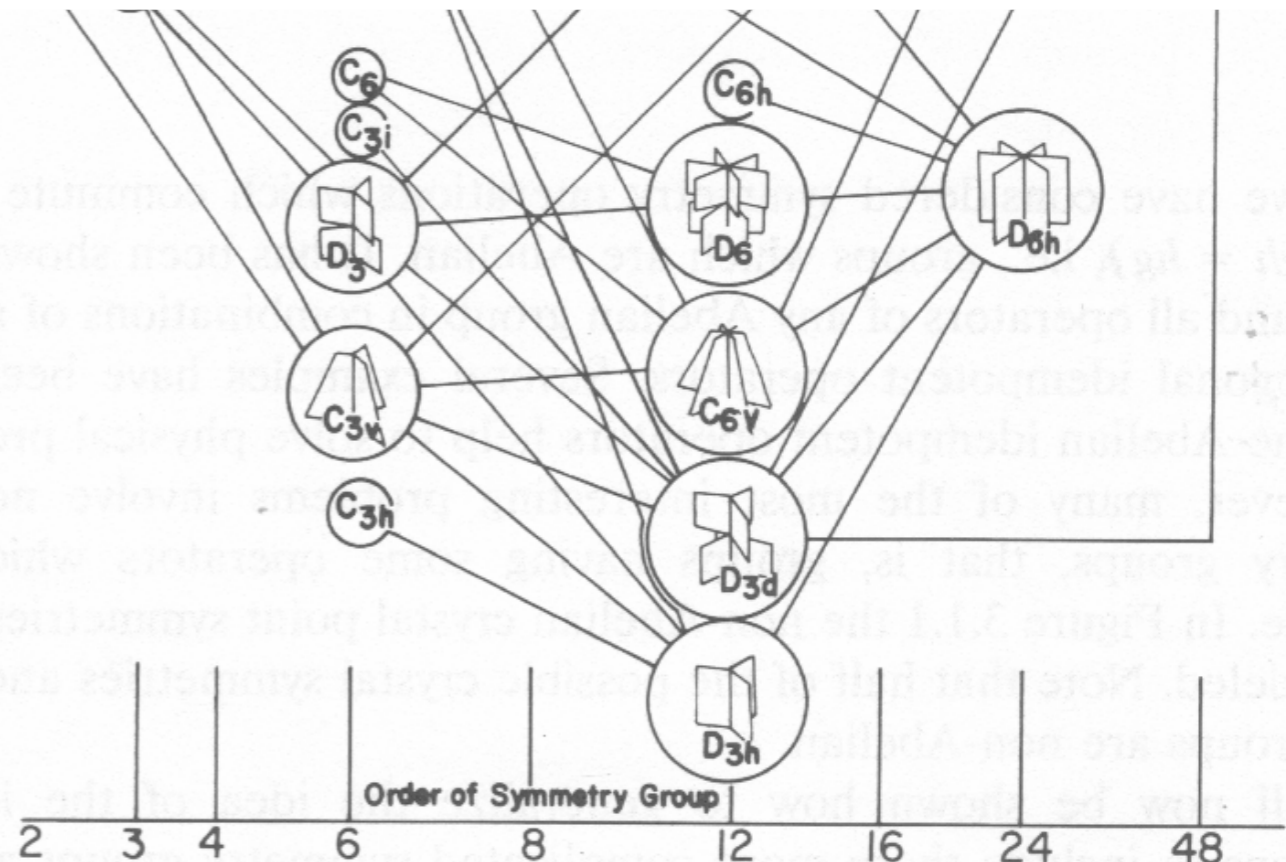
$O \downarrow D_4$ subduction

D_4 : $\mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{z180^\circ}, \mathbf{i}_{3,4}$

$\chi_g^\mu(O)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1..4}$	180° ρ_{xyz}	90° \mathbf{R}_{xyz}	180° $\mathbf{i}_{1..6}$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1$. $A_1(O) \downarrow D_4 = A_1(D_4)$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1$. $A_2(O) \downarrow D_4 = B_1(D_4)$

$\chi_g^\mu(D_4)$	$\mathbf{g}=\mathbf{1}$	ρ_{z180°	$\mathbf{R}_{z\pm 90^\circ}$	$\rho_{x,y180^\circ}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	-1	1	-1
A_2	1	1	1	-1	-1
B_2	1	1	-1	-1	1
E	2	-2	0	0	0



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

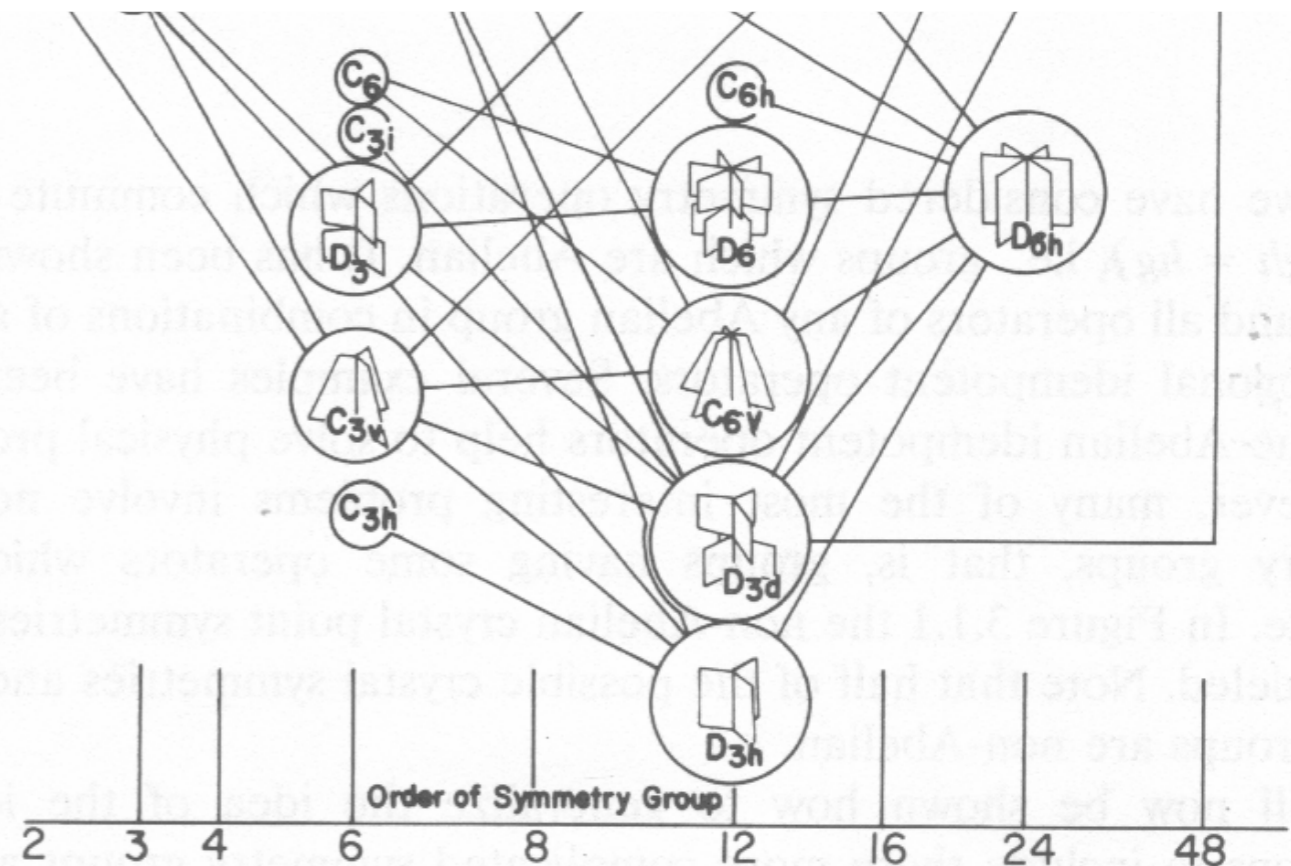
$O \downarrow D_4$ subduction

D_4 : $\mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{z180^\circ}, \mathbf{i}_{3,4}$

$\chi_g^\mu(O)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1..4}$	180° ρ_{xyz}	90° \mathbf{R}_{xyz}	180° $\mathbf{i}_{1..6}$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1$. $A_1(O) \downarrow D_4 = A_1(D_4)$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1$. $A_2(O) \downarrow D_4 = B_1(D_4)$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0$.

$\chi_g^\mu(D_4)$	$\mathbf{g}=\mathbf{1}$	ρ_{z180°	$\mathbf{R}_{z\pm 90^\circ}$	$\rho_{x,y180^\circ}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	-1	1	-1
A_2	1	1	1	-1	-1
B_2	1	1	-1	-1	1
E	2	-2	0	0	0



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

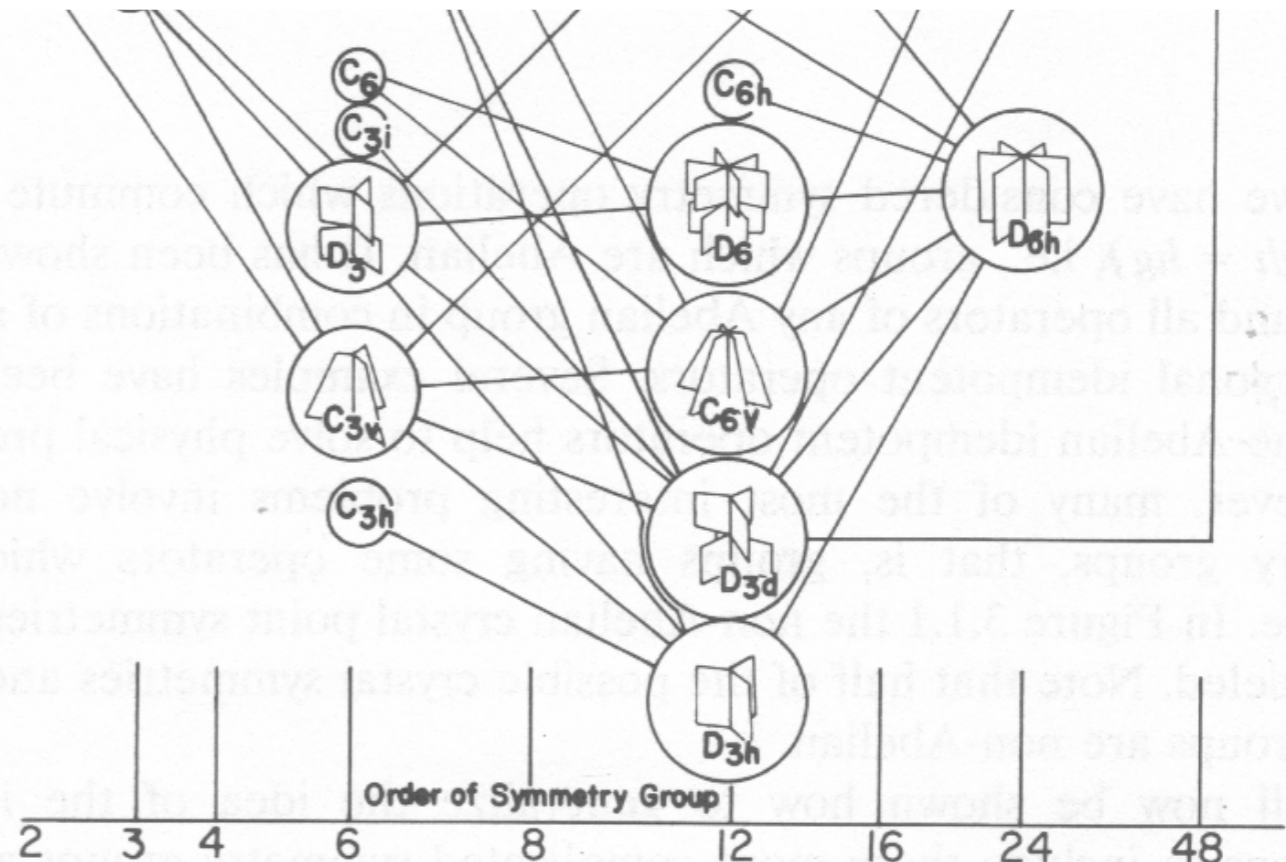
$O \downarrow D_4$ subduction

D_4 : $\mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{z180^\circ}, \mathbf{i}_{3,4}$

$\chi_g^\mu(O)$	$g=1$	$r_{1..4}$	180° ρ_{xyz}	90° \mathbf{R}_{xyz}	180° $\mathbf{i}_{1..6}$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1$. $A_1(O) \downarrow D_4 = A_1(D_4)$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1$. $A_2(O) \downarrow D_4 = B_1(D_4)$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0$. $E(O) \downarrow D_4 = A_1 \oplus B_1(D_4)$

$\chi_g^\mu(D_4)$	$g=1$	ρ_{z180°	$\mathbf{R}_{z\pm 90^\circ}$	$\rho_{x,y180^\circ}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	-1	1	-1
A_2	1	1	1	-1	-1
B_2	1	1	-1	-1	1
E	2	-2	0	0	0



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

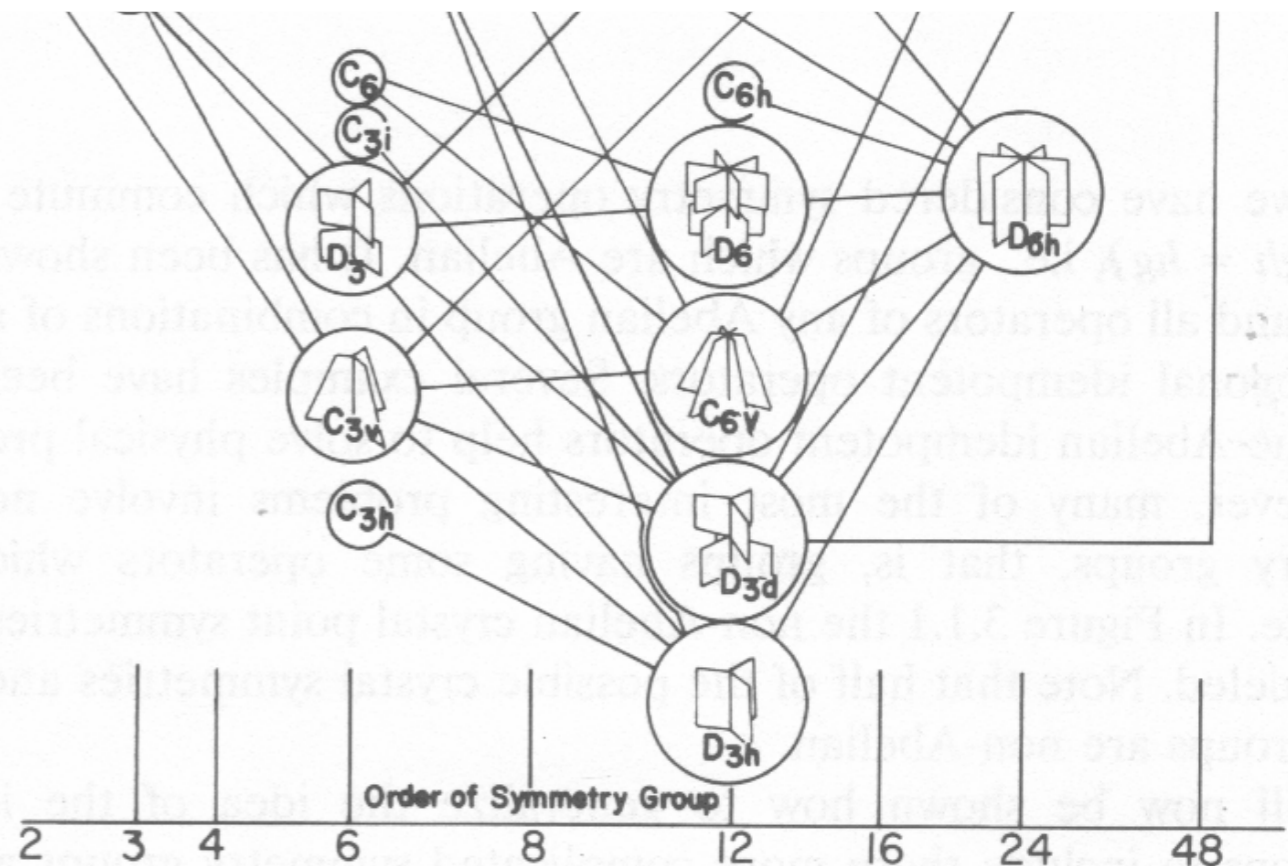
$O \downarrow D_4$ subduction

D_4 : $\mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{z180^\circ}, \mathbf{i}_{3,4}$

$\chi_g^\mu(O)$	$g=1$	$r_{1..4}$	180° ρ_{xyz}	90° \mathbf{R}_{xyz}	180° $\mathbf{i}_{1..6}$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1$. $A_1(O) \downarrow D_4 = A_1(D_4)$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1$. $A_2(O) \downarrow D_4 = B_1(D_4)$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0$. $E(O) \downarrow D_4 = A_1 \oplus B_1(D_4)$
 $T_1(O) \downarrow D_4 = 3, -1, 1, -1, -1$.

$\chi_g^\mu(D_4)$	$g=1$	ρ_{z180°	$\mathbf{R}_{z\pm 90^\circ}$	$\rho_{x,y180^\circ}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	-1	1	-1
A_2	1	1	1	-1	-1
B_2	1	1	-1	-1	1
E	2	-2	0	0	0



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

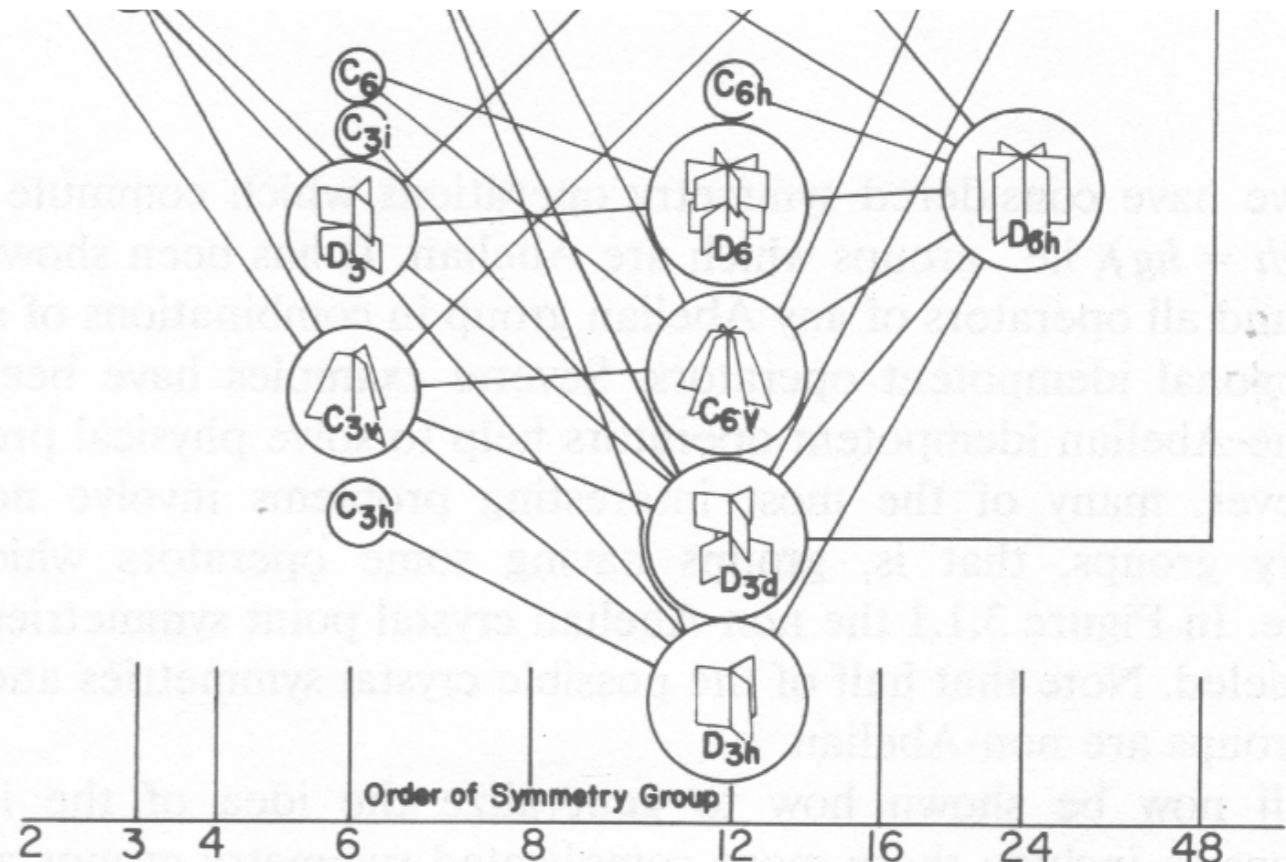
$O \downarrow D_4$ subduction

D_4 : $\mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{z180^\circ}, \mathbf{i}_{3,4}$

$\chi_g^\mu(O)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1..4}$	180° ρ_{xyz}	90° \mathbf{R}_{xyz}	180° $\mathbf{i}_{1..6}$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1$. $A_1(O) \downarrow D_4 = A_1(D_4)$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1$. $A_2(O) \downarrow D_4 = B_1(D_4)$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0$. $E(O) \downarrow D_4 = A_1 \oplus B_1(D_4)$
 $T_1(O) \downarrow D_4 = 3, -1, 1, -1, -1$. $T_1(O) \downarrow D_4 = E \oplus A_2(D_4)$

$\chi_g^\mu(D_4)$	$\mathbf{g}=\mathbf{1}$	ρ_{z180°	$\mathbf{R}_{z\pm 90^\circ}$	$\rho_{x,y180^\circ}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	-1	1	-1
A_2	1	1	1	-1	-1
B_2	1	1	-1	-1	1
E	2	-2	0	0	0



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

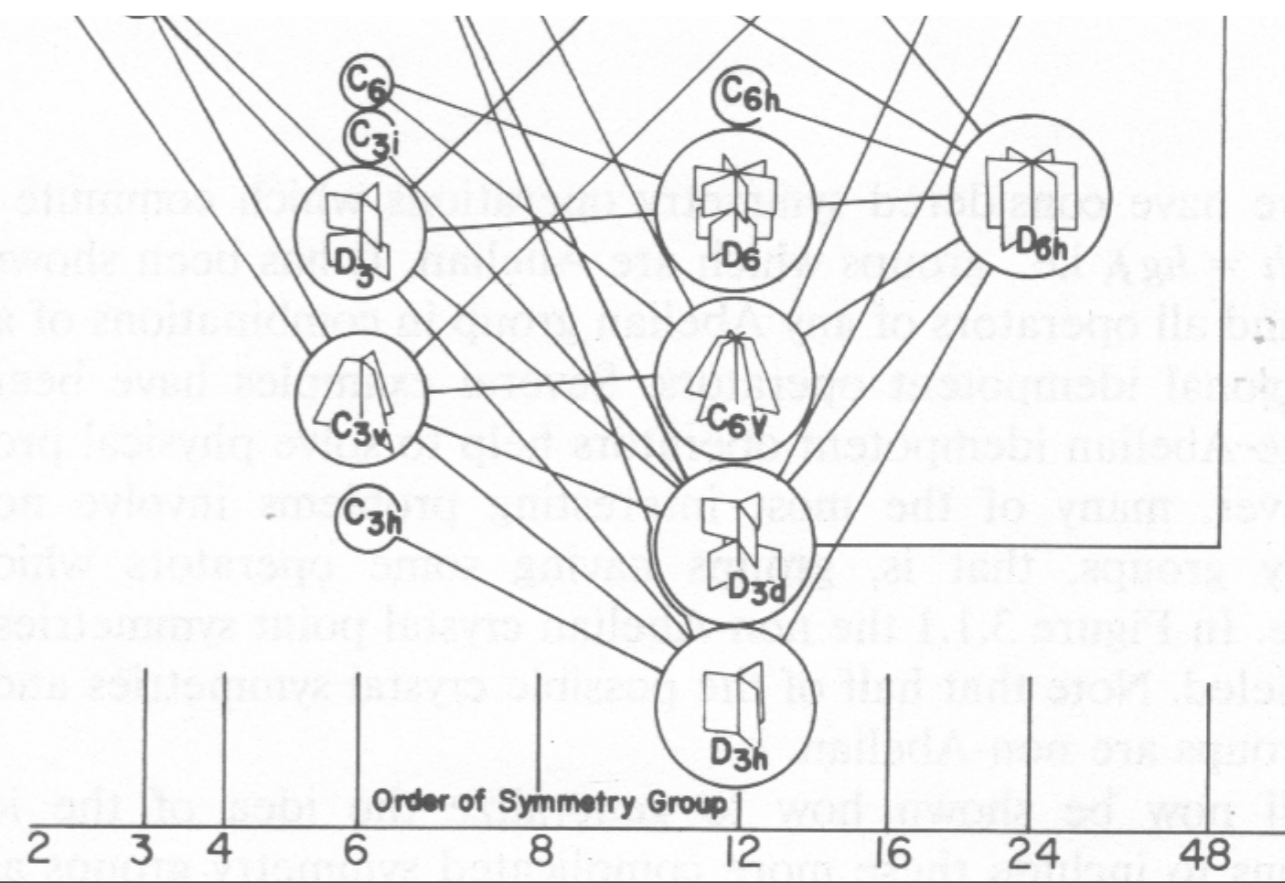
$O \downarrow D_4$ subduction

D_4 : $\mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{z180^\circ}, \mathbf{i}_{3,4}$

$\chi_g^\mu(O)$	$g=1$	$\mathbf{r}_{1..4}$	180° ρ_{xyz}	90° \mathbf{R}_{xyz}	180° $\mathbf{i}_{1..6}$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1$. $A_1(O) \downarrow D_4 = A_1(D_4)$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1$. $A_2(O) \downarrow D_4 = B_1(D_4)$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0$. $E(O) \downarrow D_4 = A_1 \oplus B_1(D_4)$
 $T_1(O) \downarrow D_4 = 3, -1, 1, -1, -1$. $T_1(O) \downarrow D_4 = E \oplus A_2(D_4)$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1$.

$\chi_g^\mu(D_4)$	$g=1$	ρ_{z180°	$\mathbf{R}_{z\pm 90^\circ}$	$\rho_{x,y180^\circ}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	-1	1	-1
A_2	1	1	1	-1	-1
B_2	1	1	-1	-1	1
E	2	-2	0	0	0



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

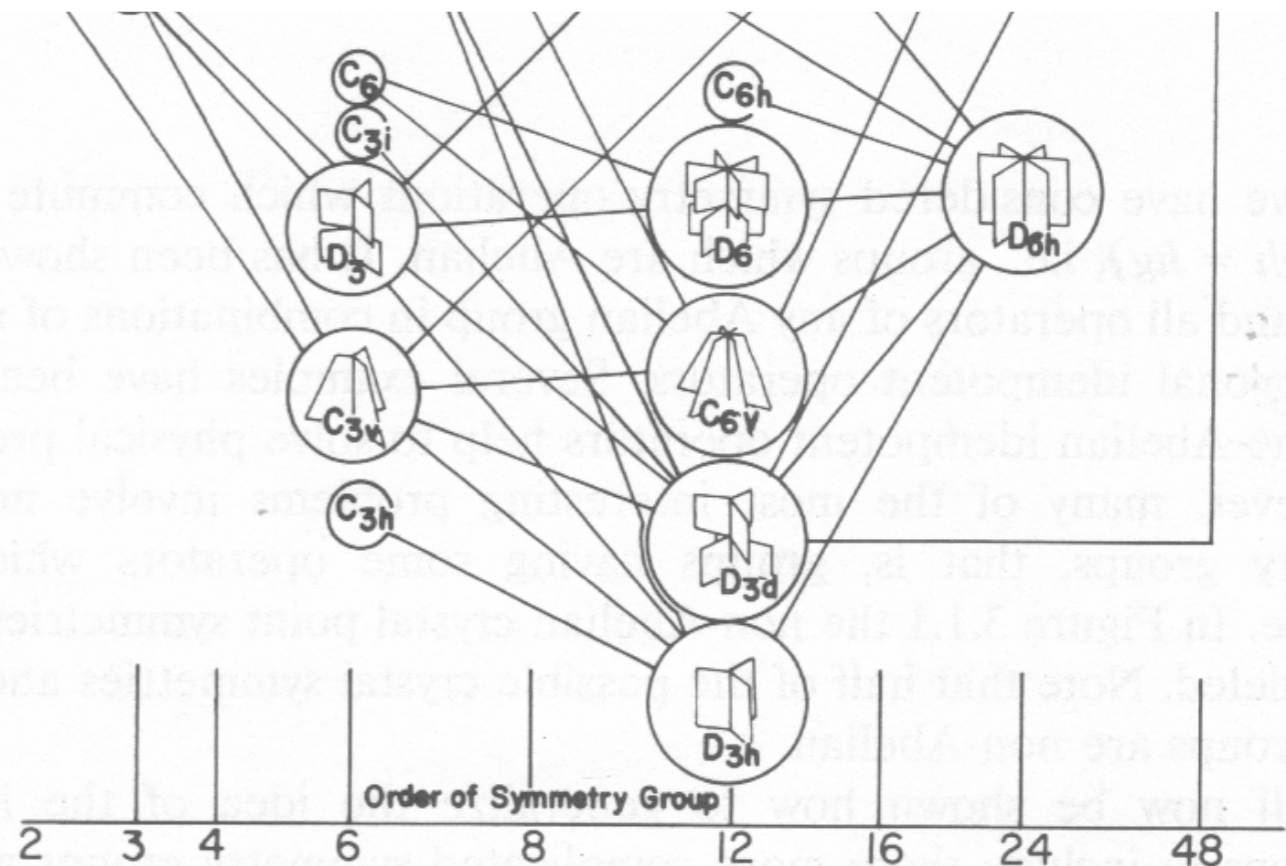
$O \downarrow D_4$ subduction

D_4 : $\mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{z180^\circ}, \mathbf{i}_{3,4}$

$\chi_g^\mu(O)$	$g=1$	$\mathbf{r}_{1..4}$	180° ρ_{xyz}	90° \mathbf{R}_{xyz}	180° $\mathbf{i}_{1..6}$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1$. $A_1(O) \downarrow D_4 = A_1(D_4)$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1$. $A_2(O) \downarrow D_4 = B_1(D_4)$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0$. $E(O) \downarrow D_4 = A_1 \oplus B_1(D_4)$
 $T_1(O) \downarrow D_4 = 3, -1, 1, -1, -1$. $T_1(O) \downarrow D_4 = E \oplus A_2(D_4)$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1$. $T_2(O) \downarrow D_4 = E \oplus B_2(D_4)$

$\chi_g^\mu(D_4)$	$g=1$	ρ_{z180°	$\mathbf{R}_{z\pm 90^\circ}$	$\rho_{x,y180^\circ}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	-1	1	-1
A_2	1	1	1	-1	-1
B_2	1	1	-1	-1	1
E	2	-2	0	0	0



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

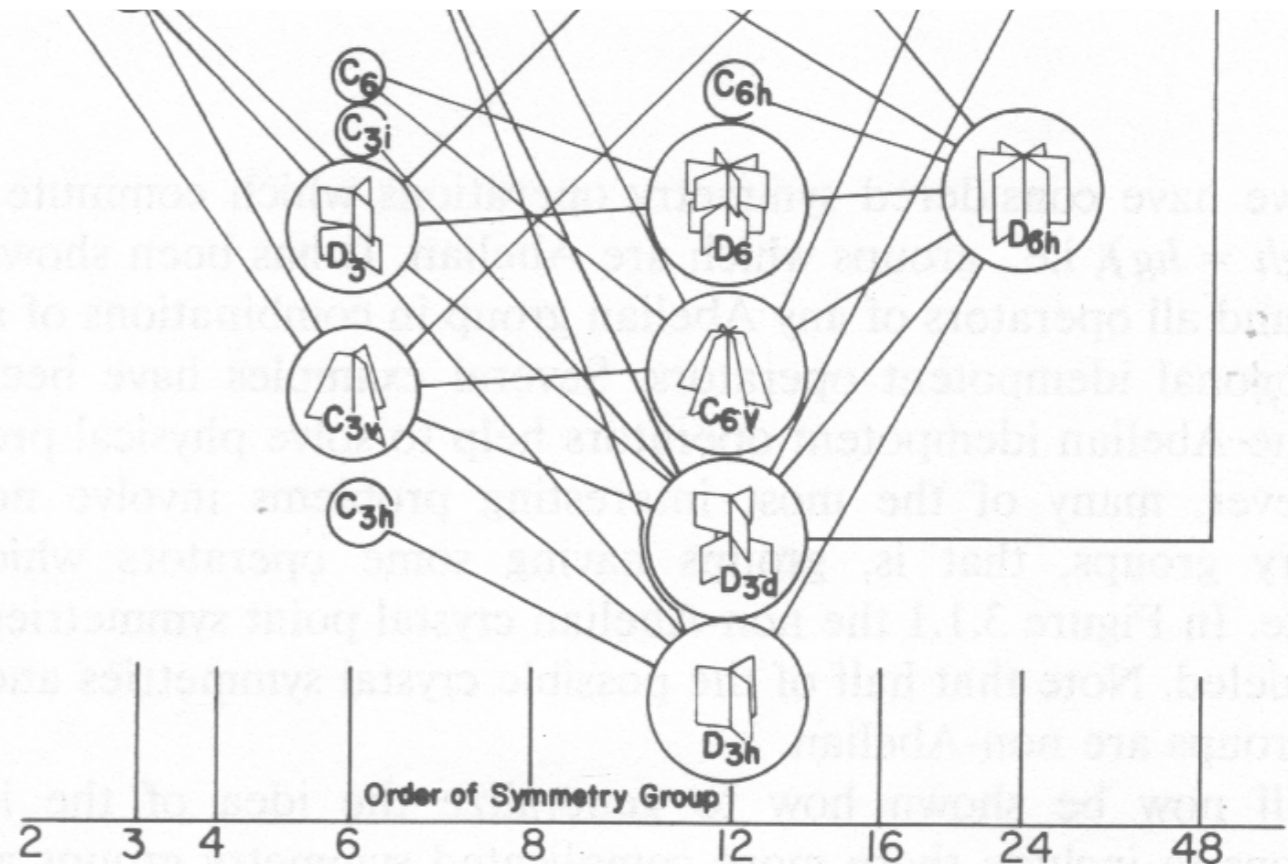
D_4 : $\mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{z180^\circ}, \mathbf{i}_{3,4}$

$\chi_g^\mu(O)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1..4}$	180° ρ_{xyz}	90° \mathbf{R}_{xyz}	180° $\mathbf{i}_{1..6}$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1$. $A_1(O) \downarrow D_4 = A_1(D_4)$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1$. $A_2(O) \downarrow D_4 = B_1(D_4)$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0$. $E(O) \downarrow D_4 = A_1 \oplus B_1(D_4)$
 $T_1(O) \downarrow D_4 = 3, -1, 1, -1, -1$. $T_1(O) \downarrow D_4 = E \oplus A_2(D_4)$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1$. $T_2(O) \downarrow D_4 = E \oplus B_2(D_4)$

$\chi_g^\mu(D_4)$	$\mathbf{g}=\mathbf{1}$	ρ_{z180°	$\mathbf{R}_{z\pm 90^\circ}$	$\rho_{x,y180^\circ}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	-1	1	-1
A_2	1	1	1	-1	-1
B_2	1	1	-1	-1	1
E	2	-2	0	0	0

$O \downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1
A_2	.	1	.	.	.
E	1	1	.	.	.
T_1	.	.	1	.	1
T_2	.	.	.	1	1



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

D_4 : $\mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{z180^\circ}, \mathbf{i}_{3,4}$

$\chi_g^\mu(O)$	$\mathbf{g} = \mathbf{1}$	$\mathbf{r}_{1..4}$	180° ρ_{xyz}	90° \mathbf{R}_{xyz}	180° $\mathbf{i}_{1..6}$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1.$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1.$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0.$
 $T_1(O) \downarrow D_4 = 3, -1, 1, -1, -1.$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1.$

$\chi_g^\mu(D_4)$	$\mathbf{g} = \mathbf{1}$	ρ_{z180°	$\mathbf{R}_{z\pm 90^\circ}$	$\rho_{x,y180^\circ}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	-1	1	-1
A_2	1	1	1	-1	-1
B_2	1	1	-1	-1	1
E	2	-2	0	0	0

$\chi_g^\mu(C_4)$	$\mathbf{g} = \mathbf{1}$	\mathbf{R}_{z+90°	\mathbf{R}_{z+180°	\mathbf{R}_{z-90°
$(0)_4$	1	1	1	1
$(1)_4$	1	i	-1	$-i$
$(2)_4$	1	-1	1	-1
$(3)_4$	1	$-i$	-1	i

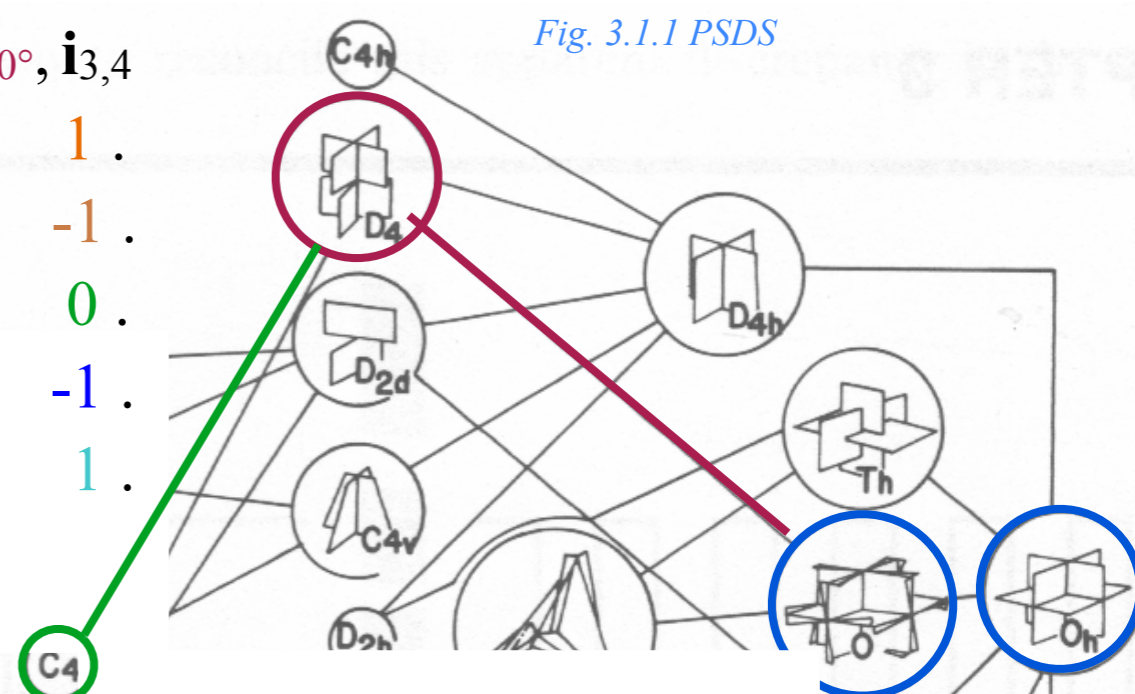
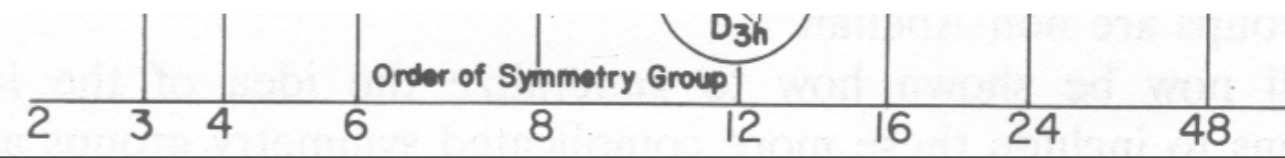


Fig. 3.1.1 PSDS

$O \downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1



Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$

Octahedral-cubic O symmetry and group operations

Tetrahedral symmetry becomes Icosahedral

Octahedral groups $O_h \supset O \sim T_d \supset T$

Octahedral O and spin- $O \subset U(2)$

Tetrahedral T class algebra

Tetrahedral T class minimal equations

Tetrahedral T class projectors and characters

Octahedral O class algebra

Octahedral O class minimal equations

Octahedral O class projectors and characters

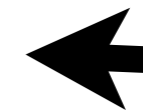
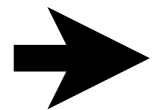
Octahedral $O_h \supset O$ subgroup correlations

Octahedral $O_h \supset O$ subgroup correlations

$O_h \supset O \supset D_4$ subgroup correlations

$O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

Applications



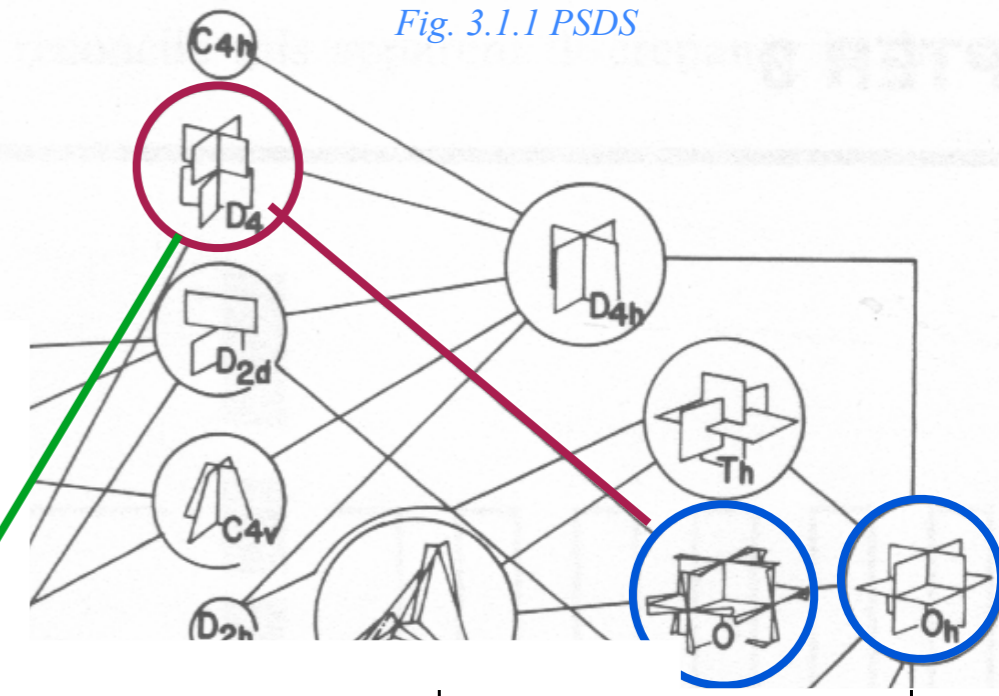
Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

D_4 : $\mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{z180^\circ}, \mathbf{i}_{3,4}$

$\chi_g^\mu(O)$	$\mathbf{g} = \mathbf{1}$	$\mathbf{r}_{1..4}$	180° ρ_{xyz}	90° \mathbf{R}_{xyz}	180° $\mathbf{i}_{1..6}$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

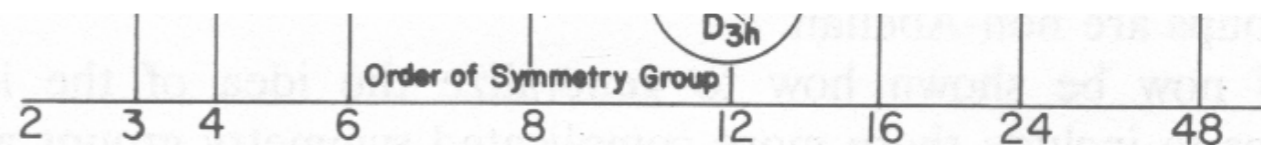
$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1.$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1.$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0.$
 $T_1(O) \downarrow D_4 = 3, -1, 1, -1, -1.$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1.$



$\chi_g^\mu(D_4)$	$\mathbf{g} = \mathbf{1}$	ρ_{z180°	$\mathbf{R}_{z\pm 90^\circ}$	$\rho_{x,y180^\circ}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	-1	1	-1
A_2	1	1	1	-1	-1
B_2	1	1	-1	-1	1
E	2	-2	0	0	0

$O \downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$\chi_g^\mu(C_4)$	$\mathbf{g} = \mathbf{1}$	\mathbf{R}_{z+90°	\mathbf{R}_{z+180°	\mathbf{R}_{z-90°
$(0)_4$	1	1	1	1
$(1)_4$	1	i	-1	$-i$
$(2)_4$	1	-1	1	-1
$(3)_4$	1	$-i$	-1	i



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

$\chi_g^\mu(O)$	$g=1$	$r_{1..4}$	180° ρ_{xyz}	90° R_{xyz}	180° $i_{1..6}$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

$D_4: \mathbf{1}, \rho_{z180^\circ}, R_{z\pm 90^\circ}, \rho_{z180^\circ}, i_{3,4}$

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1.$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1.$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0.$
 $T_2(O) \downarrow D_4 = 3, -1, 1, -1, -1.$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1.$

$\chi_g^\mu(D_4)$	$g=1$	ρ_{z180°	$R_{z\pm 90^\circ}$	$\rho_{x,y180^\circ}$	$i_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	-1	-1	-1
A_2	1	1	1	-1	-1
B_2	1	1	-1	1	1
E	2	-2	0	0	0

$D_4 \downarrow C_4$ subduction

$\mathbf{1}, R_{z+90^\circ}, \rho_{z180^\circ}, R_{z-90^\circ}$

$A_1(D_4) \downarrow C_4 = 1, 1, 1, 1.$

$\chi_g^\mu(C_4)$	$g=1$	R_{z+90°	R_{z+180°	R_{z-90°
$(0)_4$	1	1	1	1
$(1)_4$	1	i	-1	$-i$
$(2)_4$	1	-1	1	-1
$(3)_4$	1	$-i$	-1	i

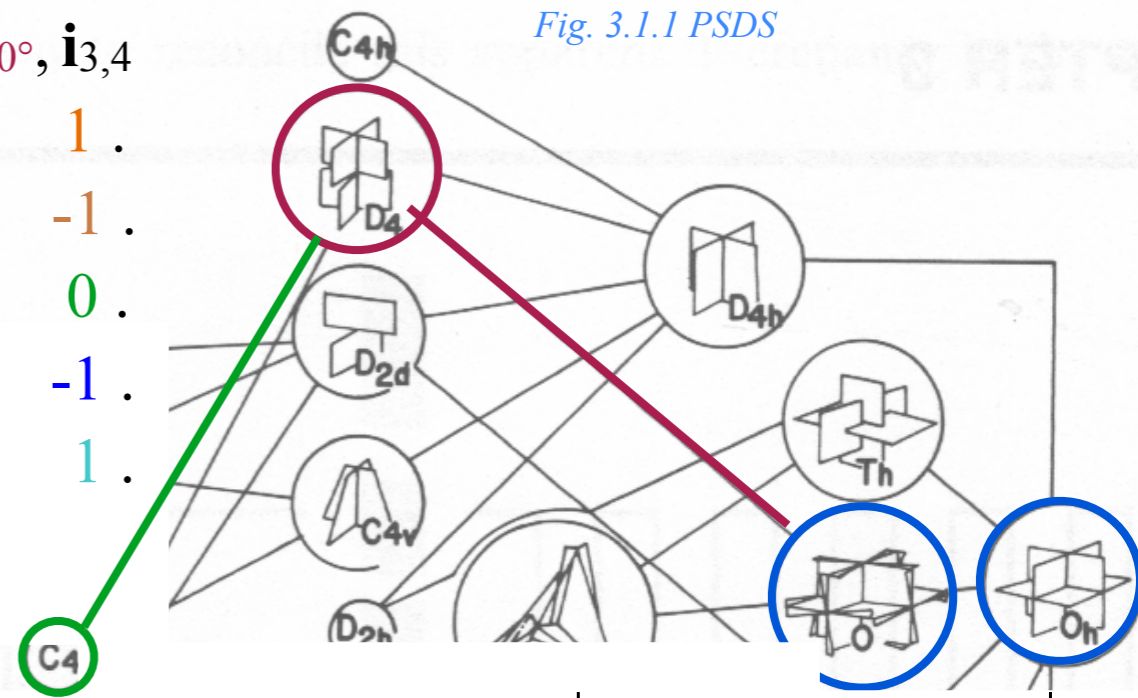
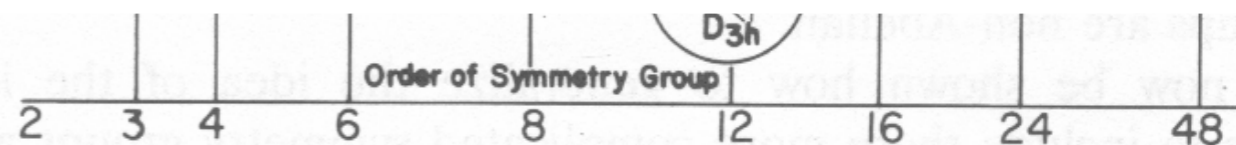


Fig. 3.1.1 PSDS

$O \downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

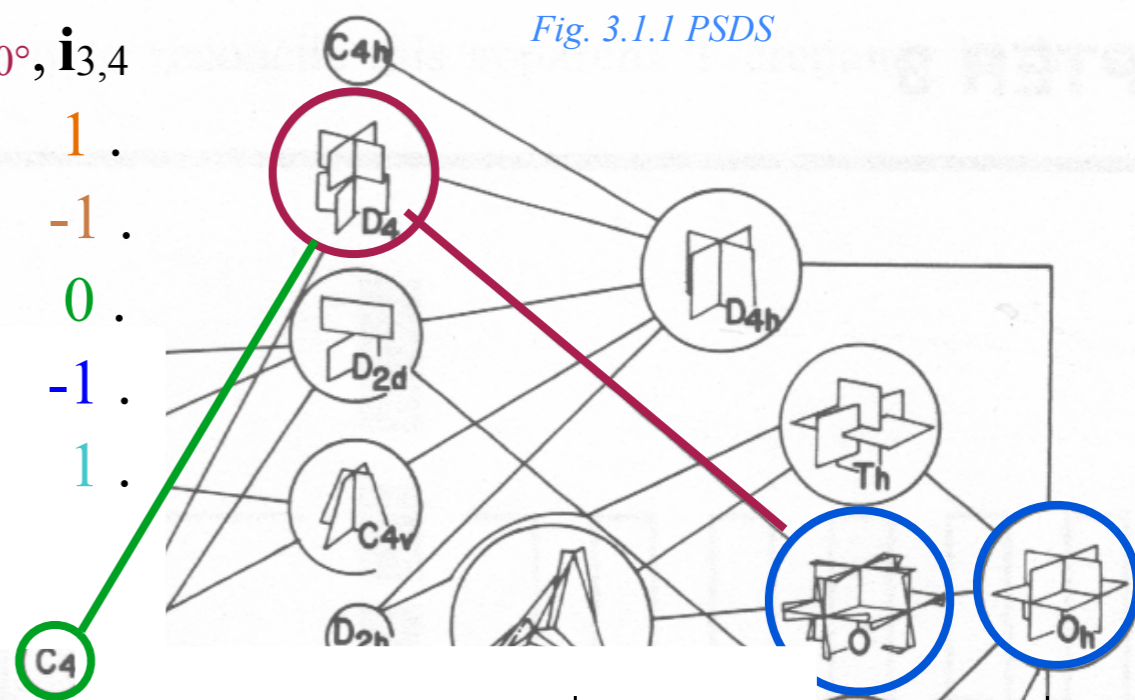
$O \downarrow D_4$ subduction

$\chi_g^\mu(O)$	$g=1$	$r_{1..4}$	180° ρ_{xyz}	90° R_{xyz}	180° $i_{1..6}$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

$D_4: 1, \rho_{z180^\circ}, R_{z\pm 90^\circ}, \rho_{z180^\circ}, i_{3,4}$

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1.$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1.$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0.$
 $T_2(O) \downarrow D_4 = 3, -1, 1, -1, -1.$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1.$

Fig. 3.1.1 PSDS



$\chi_g^\mu(D_4)$	$g=1$	ρ_{z180°	$R_{z\pm 90^\circ}$	$\rho_{x,y180^\circ}$	$i_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	-1	-1	-1
A_2	1	1	1	-1	-1
B_2	1	1	-1	1	1
E	2	-2	0	0	0

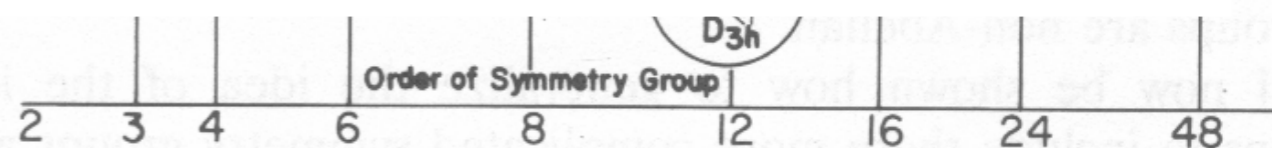
$D_4 \downarrow C_4$ subduction

$1, R_{z+90^\circ}, \rho_{z180^\circ}, R_{z-90^\circ}$

$A_1(D_4) \downarrow C_4 = 1, 1, 1, 1. = (0)_4$

$O \downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1
A_2	.	1	.	.	.
E	1	1	.	.	.
T_1	.	.	1	.	1
T_2	.	.	.	1	1

$\chi_g^\mu(C_4)$	$g=1$	R_{z+90°	R_{z+180°	R_{z-90°
$(0)_4$	1	1	1	1
$(1)_4$	1	i	-1	$-i$
$(2)_4$	1	-1	1	-1
$(3)_4$	1	$-i$	-1	i



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

$D_4: \mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{z180^\circ}, \mathbf{i}_{3,4}$

$\chi_g^\mu(O)$	$g=1$	$r_{1..4}$	180° ρ_{xyz}	90° \mathbf{R}_{xyz}	180° $\mathbf{i}_{1..6}$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1.$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1.$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0.$
 $T_1(O) \downarrow D_4 = 3, -1, 1, -1, -1.$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1.$

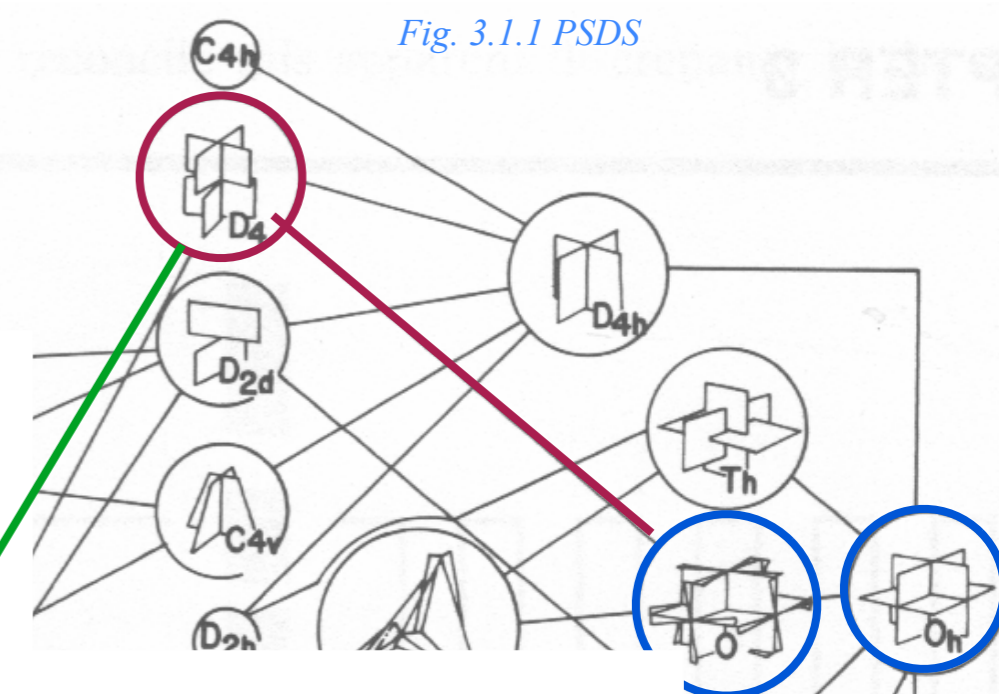


Fig. 3.1.1 PSDS

$\chi_g^\mu(D_4)$	$g=1$	ρ_{z180°	$\mathbf{R}_{z\pm 90^\circ}$	$\rho_{x,y180^\circ}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	-1	1	-1
A_2	1	1	1	-1	-1
B_2	1	1	-1	-1	1
E	2	-2	0	0	0

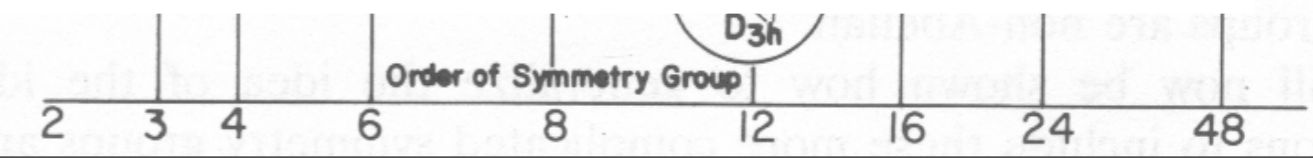
$D_4 \downarrow C_4$ subduction

$\mathbf{1}, \mathbf{R}_{z+90^\circ}, \rho_{z180^\circ}, \mathbf{R}_{z-90^\circ}$

$A_1(D_4) \downarrow C_4 = 1, 1, 1, 1. = (0)_4$
 $B_1(D_4) \downarrow C_4 = 1, -1, 1, -1.$

$O \downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1
A_2	.	1	.	.	.
E	1	1	.	.	.
T_1	.	.	1	.	1
T_2	.	.	.	1	1

$\chi_g^\mu(C_4)$	$g=1$	\mathbf{R}_{z+90°	\mathbf{R}_{z+180°	\mathbf{R}_{z-90°
$(0)_4$	1	1	1	1
$(1)_4$	1	i	-1	$-i$
$(2)_4$	1	-1	1	-1
$(3)_4$	1	$-i$	-1	i



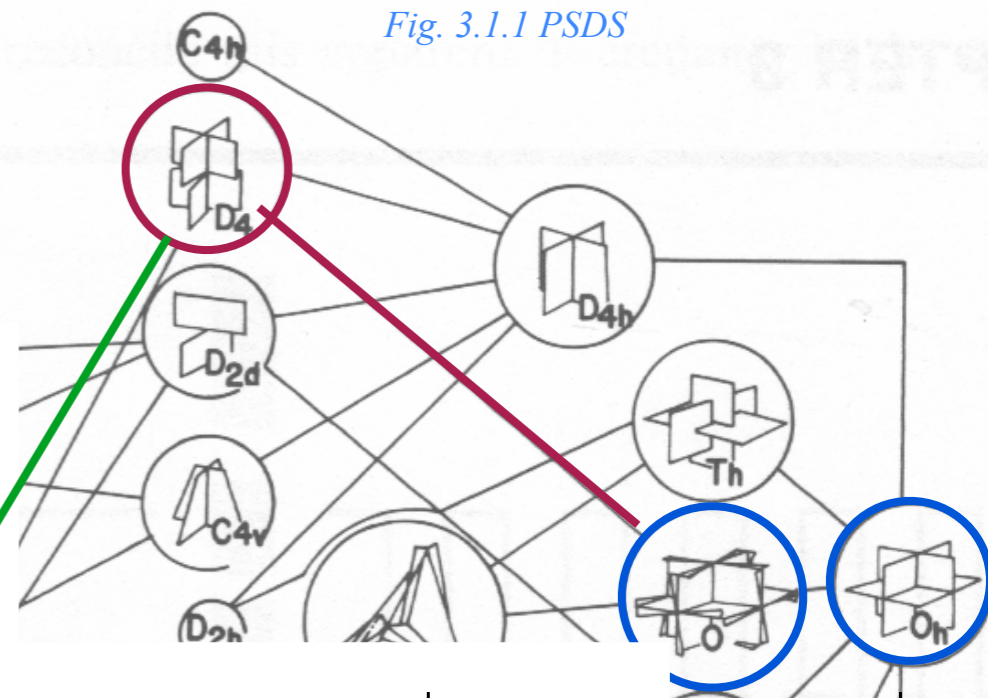
Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

D_4 : $1, \rho_{z180^\circ}, R_{z\pm 90^\circ}, \rho_{z180^\circ}, i_{3,4}$

$\chi_g^\mu(O)$	$g=1$	$r_{1..4}$	180° ρ_{xyz}	90° R_{xyz}	180° $i_{1..6}$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0$
 $T_1(O) \downarrow D_4 = 3, -1, 1, -1, -1$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1$



$\chi_g^\mu(D_4)$	$g=1$	ρ_{z180°	$R_{z\pm 90^\circ}$	$\rho_{x,y180^\circ}$	$i_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	-1	1	-1
A_2	1	1	1	-1	-1
B_2	1	1	-1	-1	1
E	2	-2	0	0	0

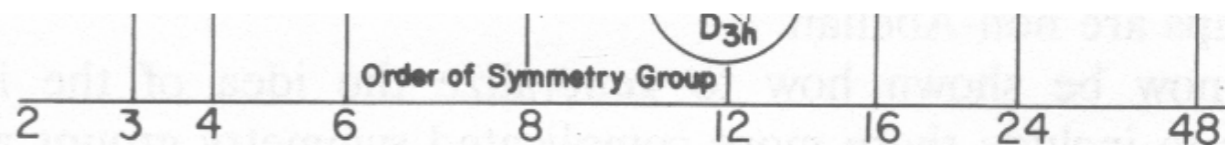
$D_4 \downarrow C_4$ subduction

$1, R_{z+90^\circ}, \rho_{z180^\circ}, R_{z-90^\circ}$

$A_1(D_4) \downarrow C_4 = 1, 1, 1, 1 = (0)_4$
 $B_1(D_4) \downarrow C_4 = 1, -1, 1, -1 = (2)_4$

$O \downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$\chi_g^\mu(C_4)$	$g=1$	R_{z+90°	R_{z+180°	R_{z-90°
$(0)_4$	1	1	1	1
$(1)_4$	1	i	-1	$-i$
$(2)_4$	1	-1	1	-1
$(3)_4$	1	$-i$	-1	i



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

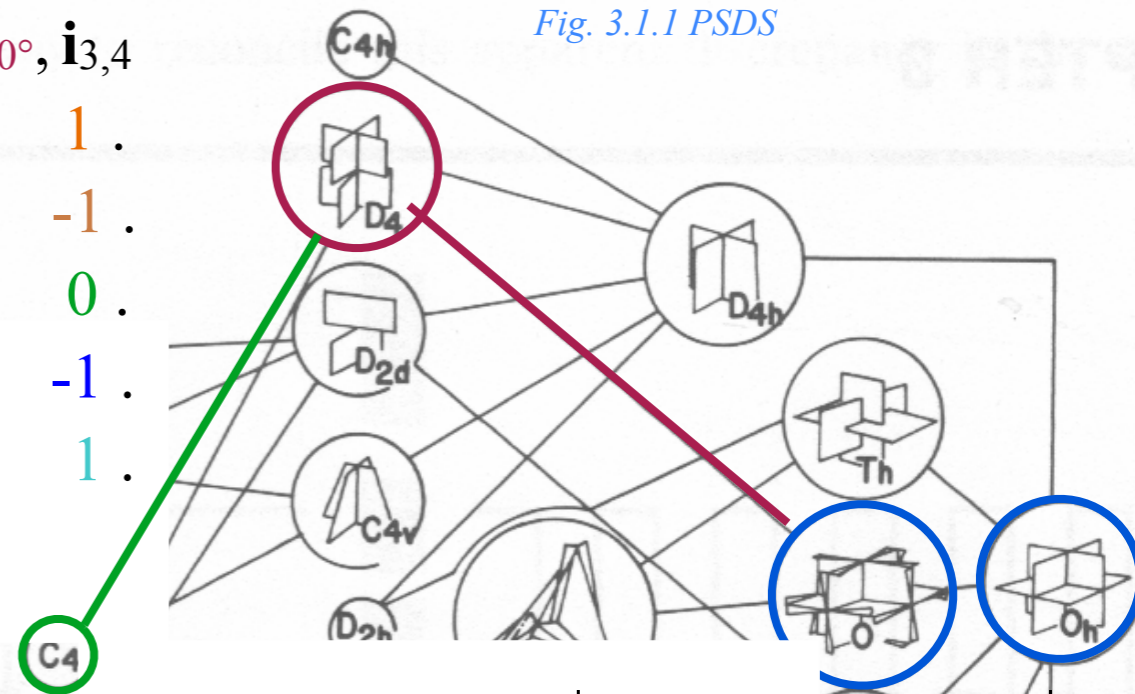
$O \downarrow D_4$ subduction

$\chi_g^\mu(O)$	$g=1$	$r_{1..4}$	180° ρ_{xyz}	90° R_{xyz}	180° $i_{1..6}$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

$D_4: 1, \rho_{z180^\circ}, R_{z\pm 90^\circ}, \rho_{z180^\circ}, i_{3,4}$

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1.$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1.$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0.$
 $T_2(O) \downarrow D_4 = 3, -1, 1, -1, -1.$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1.$

Fig. 3.1.1 PSDS



$\chi_g^\mu(D_4)$	$g=1$	ρ_{z180°	$R_{z\pm 90^\circ}$	$\rho_{x,y180^\circ}$	$i_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	-1	1	-1
A_2	1	1	1	-1	-1
B_2	1	1	-1	-1	1
E	2	-2	0	0	0

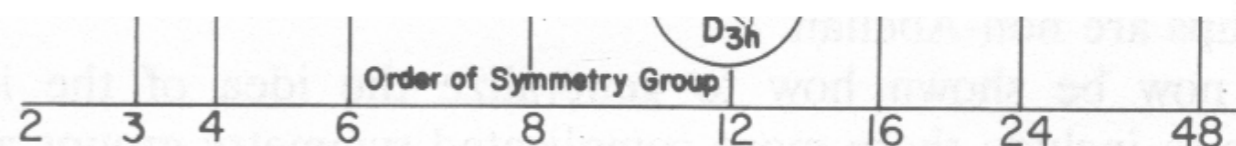
$D_4 \downarrow C_4$ subduction

$1, R_{z+90^\circ}, \rho_{z180^\circ}, R_{z-90^\circ}$

$A_1(D_4) \downarrow C_4 = 1, 1, 1, 1. = (0)_4$
 $B_1(D_4) \downarrow C_4 = 1, -1, 1, -1. = (2)_4$
 $A_2(D_4) \downarrow C_4 = 1, 1, 1, 1.$

$O \downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1
A_2	.	1	.	.	.
E	1	1	.	.	.
T_1	.	.	1	.	1
T_2	.	.	.	1	1

$\chi_g^\mu(C_4)$	$g=1$	R_{z+90°	R_{z+180°	R_{z-90°
$(0)_4$	1	1	1	1
$(1)_4$	1	i	-1	$-i$
$(2)_4$	1	-1	1	-1
$(3)_4$	1	$-i$	-1	i



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

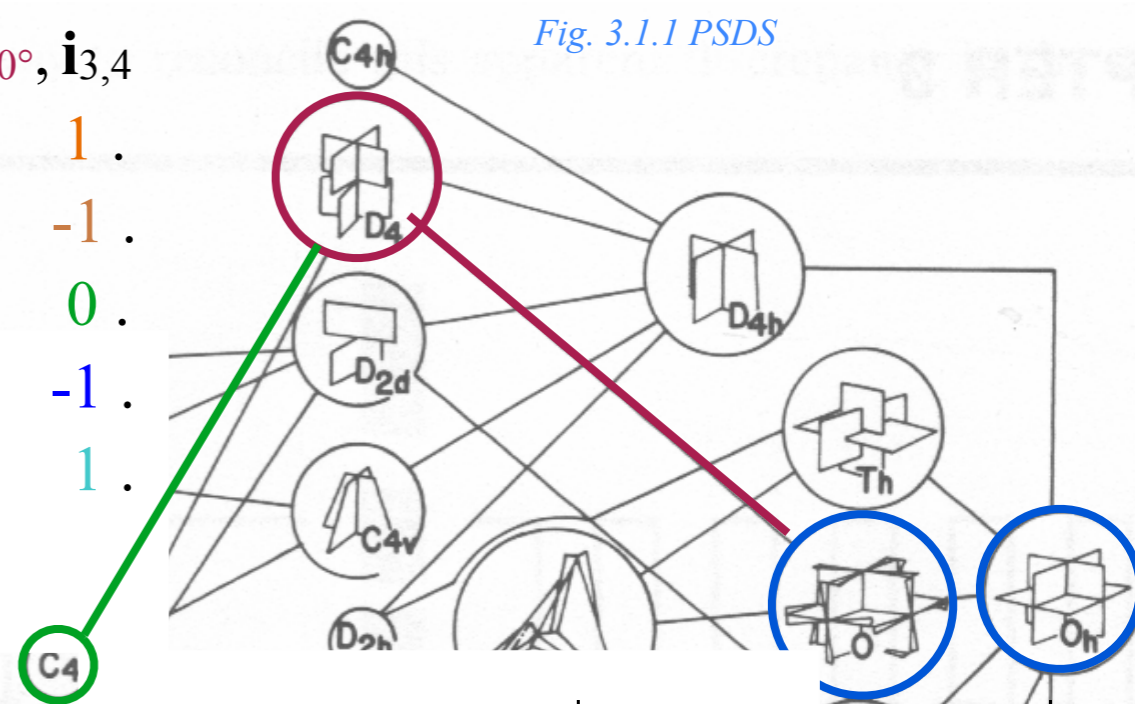
$O \downarrow D_4$ subduction

$\chi_g^\mu(O)$	$g=1$	$r_{1..4}$	180° ρ_{xyz}	90° R_{xyz}	180° $i_{1..6}$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

$D_4: 1, \rho_{z180^\circ}, R_{z\pm 90^\circ}, \rho_{z180^\circ}, i_{3,4}$

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1.$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1.$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0.$
 $T_2(O) \downarrow D_4 = 3, -1, 1, -1, -1.$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1.$

Fig. 3.1.1 PSDS



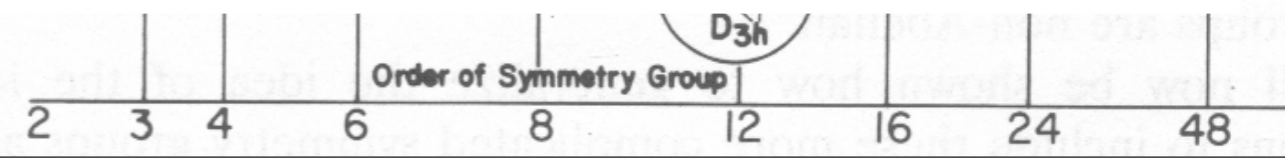
$\chi_g^\mu(D_4)$	$g=1$	ρ_{z180°	$R_{z\pm 90^\circ}$	$\rho_{x,y180^\circ}$	$i_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	-1	1	-1
A_2	1	1	1	-1	-1
B_2	1	1	-1	-1	1
E	2	-2	0	0	0

$D_4 \downarrow C_4$ subduction

$1, R_{z+90^\circ}, \rho_{z180^\circ}, R_{z-90^\circ}$
 $A_1(D_4) \downarrow C_4 = 1, 1, 1, 1. = (0)_4$
 $B_1(D_4) \downarrow C_4 = 1, -1, 1, -1. = (2)_4$
 $A_2(D_4) \downarrow C_4 = 1, 1, 1, 1. = (0)_4$

$O \downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1
A_2	.	1	.	.	.
E	1	1	.	.	.
T_1	.	.	1	.	1
T_2	.	.	.	1	1

$\chi_g^\mu(C_4)$	$g=1$	R_{z+90°	R_{z+180°	R_{z-90°
$(0)_4$	1	1	1	1
$(1)_4$	1	i	-1	$-i$
$(2)_4$	1	-1	1	-1
$(3)_4$	1	$-i$	-1	i



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

$\chi_g^\mu(O)$	$g=1$	$r_{1..4}$	180° ρ_{xyz}	90° R_{xyz}	180° $i_{1..6}$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

$D_4: \mathbf{1}, \rho_{z180^\circ}, R_{z\pm 90^\circ}, \rho_{z180^\circ}, i_{3,4}$

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1.$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1.$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0.$
 $T_2(O) \downarrow D_4 = 3, -1, 1, -1, -1.$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1.$

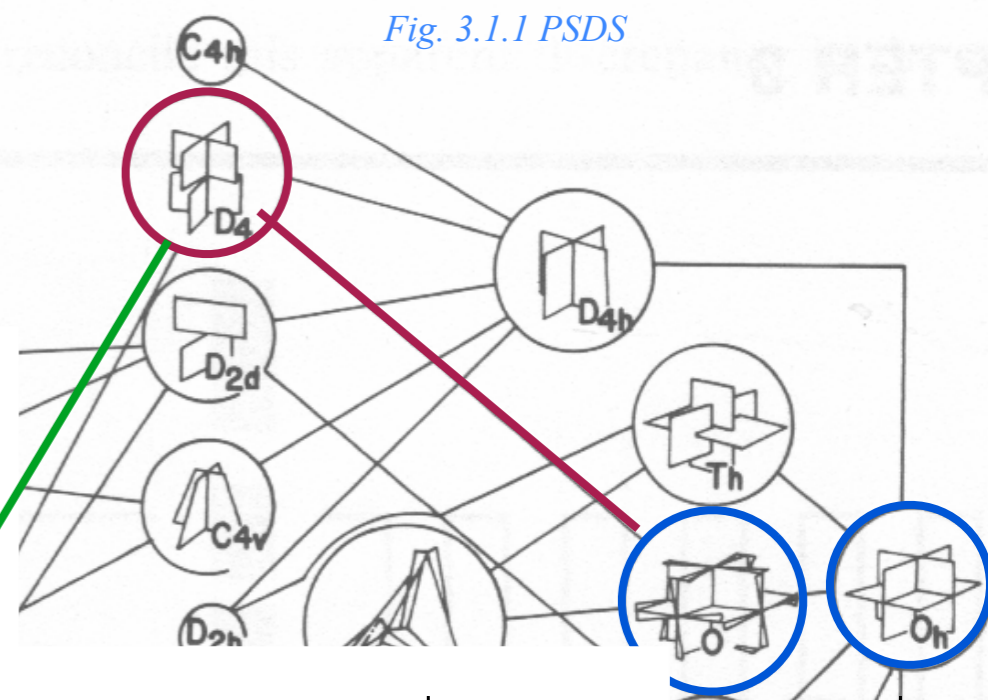


Fig. 3.1.1 PSDS

$\chi_g^\mu(D_4)$	$g=1$	ρ_{z180°	$R_{z\pm 90^\circ}$	$\rho_{x,y180^\circ}$	$i_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	-1	1	-1
A_2	1	1	1	-1	-1
B_2	1	1	-1	-1	1
E	2	-2	0	0	0

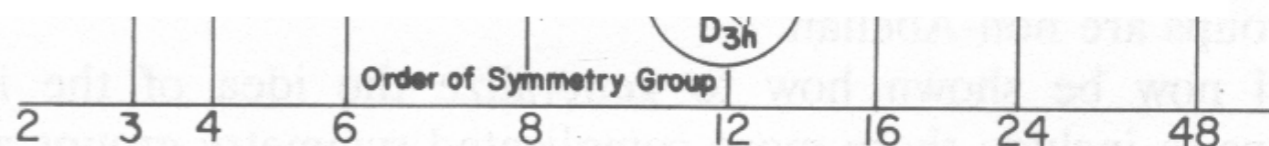
$D_4 \downarrow C_4$ subduction

$\mathbf{1}, R_{z+90^\circ}, \rho_{z180^\circ}, R_{z-90^\circ}$

$A_1(D_4) \downarrow C_4 = 1, 1, 1, 1. = (0)_4$
 $B_1(D_4) \downarrow C_4 = 1, -1, 1, -1. = (2)_4$
 $A_2(D_4) \downarrow C_4 = 1, 1, 1, 1. = (0)_4$
 $B_2(D_4) \downarrow C_4 = 1, -1, 1, -1.$

$O \downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1
A_2	.	1	.	.	.
E	1	1	.	.	.
T_1	.	.	1	.	1
T_2	.	.	.	1	1

$\chi_g^\mu(C_4)$	$g=1$	R_{z+90°	R_{z+180°	R_{z-90°
$(0)_4$	1	1	1	1
$(1)_4$	1	i	-1	$-i$
$(2)_4$	1	-1	1	-1
$(3)_4$	1	$-i$	-1	i



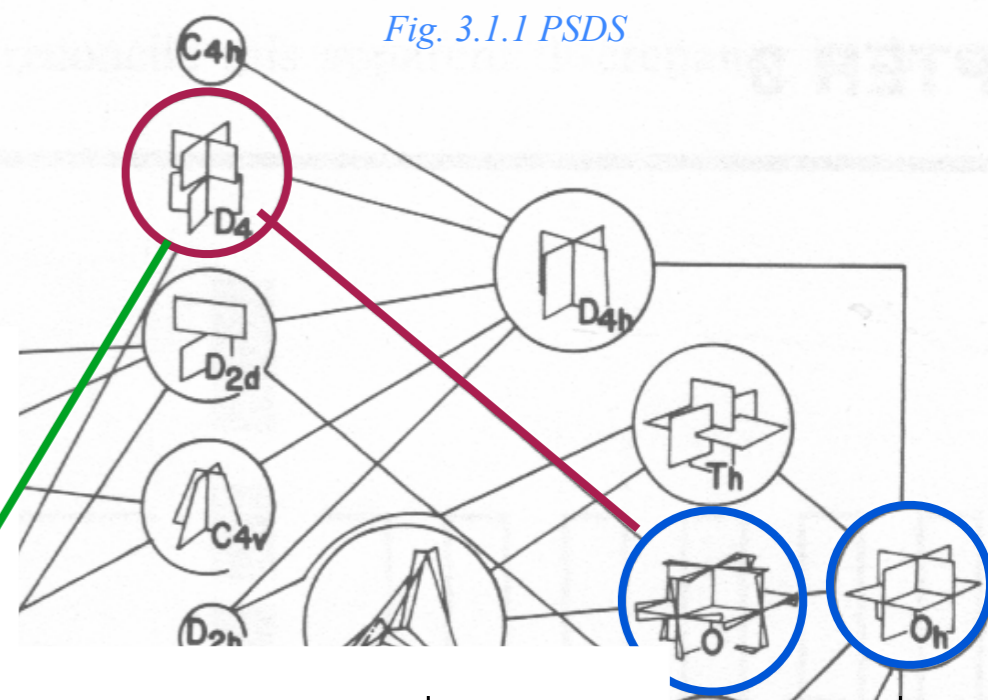
Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

$\chi_g^\mu(O)$	$g=1$	$r_{1..4}$	180° ρ_{xyz}	90° R_{xyz}	180° $i_{1..6}$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

$D_4: 1, \rho_{z180^\circ}, R_{z\pm 90^\circ}, \rho_{z180^\circ}, i_{3,4}$

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1.$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1.$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0.$
 $T_2(O) \downarrow D_4 = 3, -1, 1, -1, -1.$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1.$



$\chi_g^\mu(D_4)$	$g=1$	ρ_{z180°	$R_{z\pm 90^\circ}$	$\rho_{x,y180^\circ}$	$i_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	-1	1	-1
A_2	1	1	1	-1	-1
B_2	1	1	-1	-1	1
E	2	-2	0	0	0

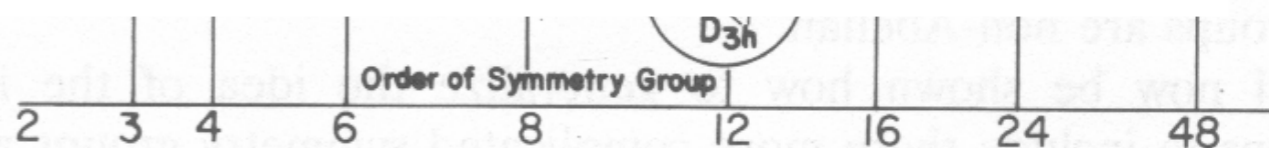
$D_4 \downarrow C_4$ subduction

$1, R_{z+90^\circ}, \rho_{z180^\circ}, R_{z-90^\circ}$

$A_1(D_4) \downarrow C_4 = 1, 1, 1, 1. = (0)_4$
 $B_1(D_4) \downarrow C_4 = 1, -1, 1, -1. = (2)_4$
 $A_2(D_4) \downarrow C_4 = 1, 1, 1, 1. = (0)_4$
 $B_2(D_4) \downarrow C_4 = 1, -1, 1, -1. = (2)_4$

$O \downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1
A_2	.	1	.	.	.
E	1	1	.	.	.
T_1	.	.	1	.	1
T_2	.	.	.	1	1

$\chi_g^\mu(C_4)$	$g=1$	R_{z+90°	R_{z+180°	R_{z-90°
$(0)_4$	1	1	1	1
$(1)_4$	1	i	-1	$-i$
$(2)_4$	1	-1	1	-1
$(3)_4$	1	$-i$	-1	i



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

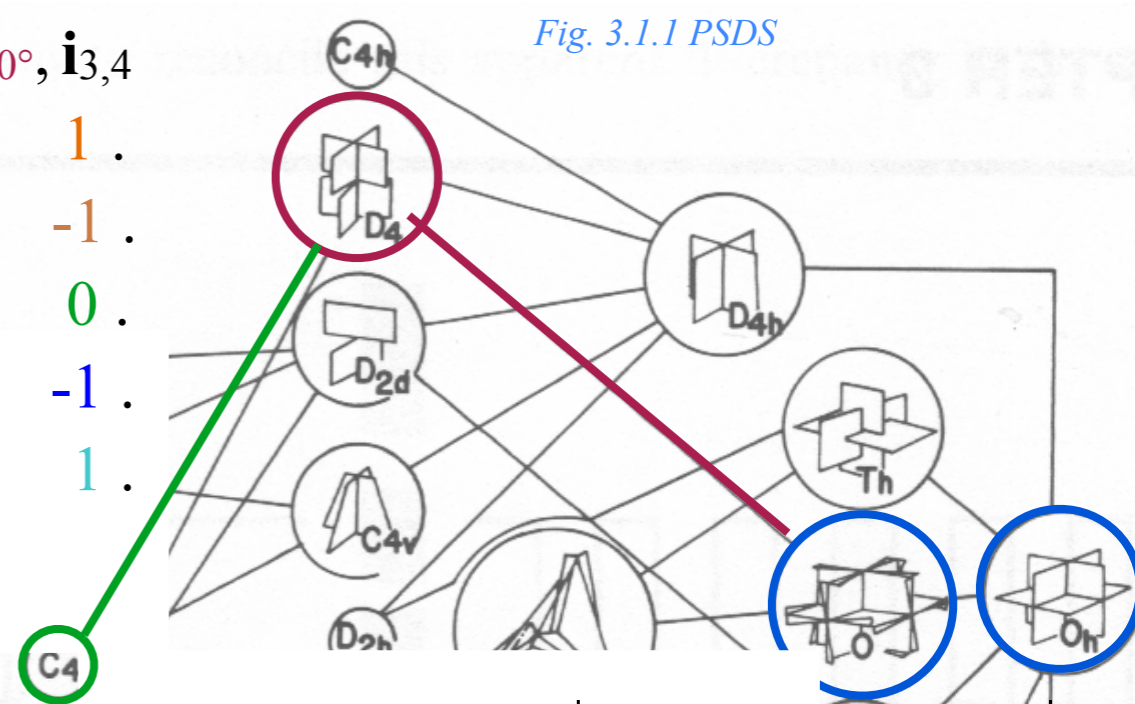
$O \downarrow D_4$ subduction

$\chi_g^\mu(O)$	$g=1$	$r_{1..4}$	180° ρ_{xyz}	90° R_{xyz}	180° $i_{1..6}$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

$D_4: 1, \rho_{z180^\circ}, R_{z\pm 90^\circ}, \rho_{z180^\circ}, i_{3,4}$

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1.$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1.$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0.$
 $T_2(O) \downarrow D_4 = 3, -1, 1, -1, -1.$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1.$

Fig. 3.1.1 PSDS



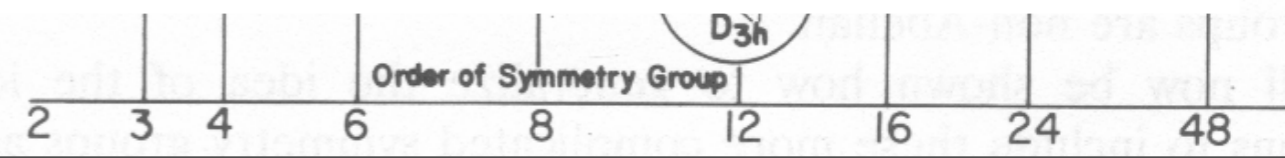
$\chi_g^\mu(D_4)$	$g=1$	ρ_{z180°	$R_{z\pm 90^\circ}$	$\rho_{x,y180^\circ}$	$i_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	-1	1	-1
A_2	1	1	1	-1	-1
B_2	1	1	-1	-1	1
E	2	-2	0	0	0

$D_4 \downarrow C_4$ subduction

$1, R_{z+90^\circ}, \rho_{z180^\circ}, R_{z-90^\circ}$
 $A_1(D_4) \downarrow C_4 = 1, 1, 1, 1. = (0)_4$
 $B_1(D_4) \downarrow C_4 = 1, -1, 1, -1. = (2)_4$
 $A_2(D_4) \downarrow C_4 = 1, 1, 1, 1. = (0)_4$
 $B_2(D_4) \downarrow C_4 = 1, -1, 1, -1. = (2)_4$
 $E(D_4) \downarrow C_4 = 2, 0, -2, 0.$

$O \downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1
A_2	.	1	.	.	.
E	1	1	.	.	.
T_1	.	.	1	.	1
T_2	.	.	.	1	1

$\chi_g^\mu(C_4)$	$g=1$	R_{z+90°	R_{z+180°	R_{z-90°
$(0)_4$	1	1	1	1
$(1)_4$	1	i	-1	$-i$
$(2)_4$	1	-1	1	-1
$(3)_4$	1	$-i$	-1	i



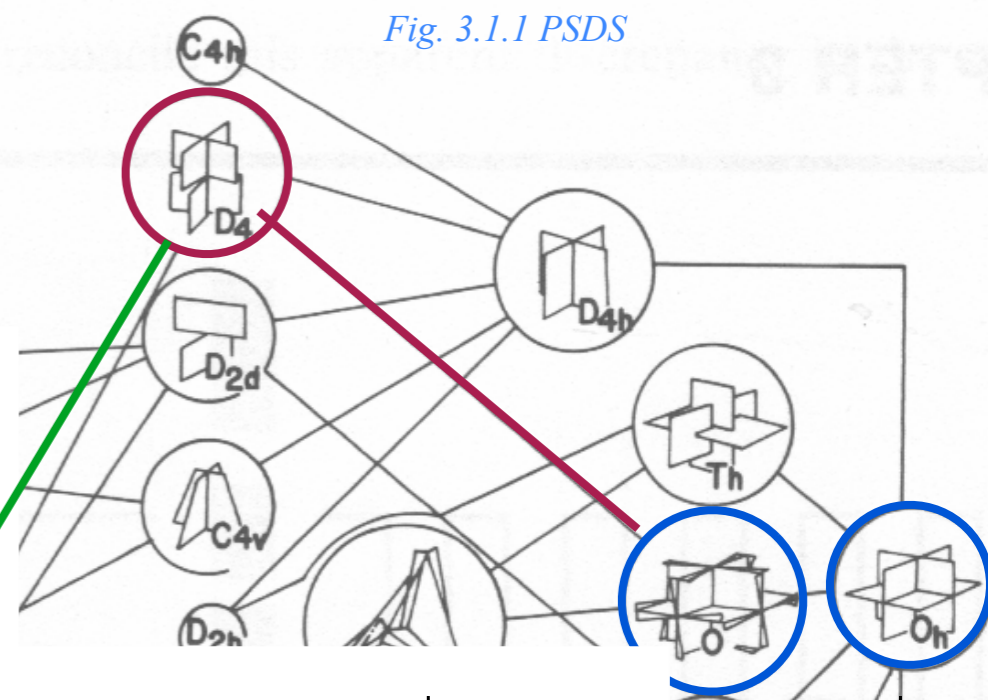
Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

$\chi_g^\mu(O)$	$g=1$	$r_{1..4}$	180° ρ_{xyz}	90° R_{xyz}	180° $i_{1..6}$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

$D_4: 1, \rho_{z180^\circ}, R_{z\pm 90^\circ}, \rho_{z180^\circ}, i_{3,4}$

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0$
 $T_2(O) \downarrow D_4 = 3, -1, 1, -1, -1$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1$



$\chi_g^\mu(D_4)$	$g=1$	ρ_{z180°	$R_{z\pm 90^\circ}$	$\rho_{x,y180^\circ}$	$i_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	-1	1	-1
A_2	1	1	1	-1	-1
B_2	1	1	-1	-1	1
E	2	-2	0	0	0

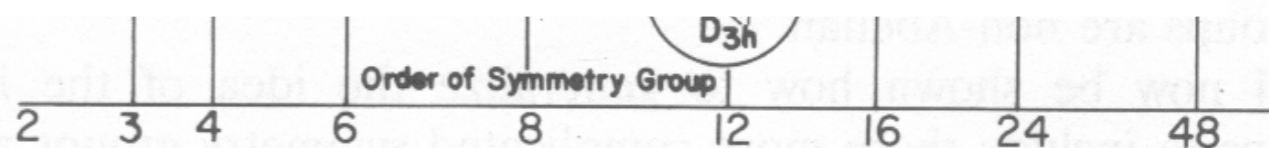
$D_4 \downarrow C_4$ subduction

$1, R_{z+90^\circ}, \rho_{z180^\circ}, R_{z-90^\circ}$

$A_1(D_4) \downarrow C_4 = 1, 1, 1, 1 = (0)_4$
 $B_1(D_4) \downarrow C_4 = 1, -1, 1, -1 = (2)_4$
 $A_2(D_4) \downarrow C_4 = 1, 1, 1, 1 = (0)_4$
 $B_2(D_4) \downarrow C_4 = 1, -1, 1, -1 = (2)_4$
 $E(D_4) \downarrow C_4 = 2, 0, -2, 0 = (1)_4 \oplus (3)_4$

$O \downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1
A_2	.	1	.	.	.
E	1	1	.	.	.
T_1	.	.	1	.	1
T_2	.	.	.	1	1

$\chi_g^\mu(C_4)$	$g=1$	R_{z+90°	R_{z+180°	R_{z-90°
$(0)_4$	1	1	1	1
$(1)_4$	1	i	-1	$-i$
$(2)_4$	1	-1	1	-1
$(3)_4$	1	$-i$	-1	i



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

$\chi_g^\mu(O)$	$g=1$	$r_{1..4}$	180° ρ_{xyz}	90° R_{xyz}	180° $i_{1..6}$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

$D_4: 1, \rho_{z180^\circ}, R_{z\pm 90^\circ}, \rho_{z180^\circ}, i_{3,4}$

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0$
 $T_2(O) \downarrow D_4 = 3, -1, 1, -1, -1$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1$

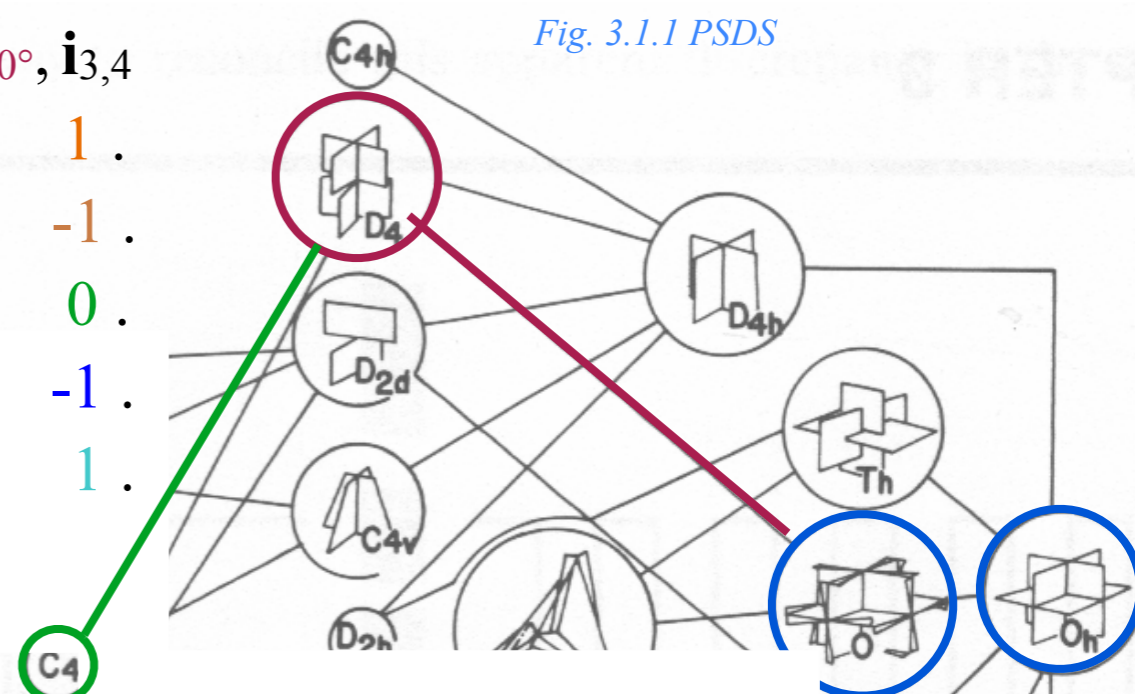


Fig. 3.1.1 PSDS

$\chi_g^\mu(D_4)$	$g=1$	ρ_{z180°	$R_{z\pm 90^\circ}$	$\rho_{x,y180^\circ}$	$i_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	-1	1	-1
A_2	1	1	1	-1	-1
B_2	1	1	-1	-1	1
E	2	-2	0	0	0

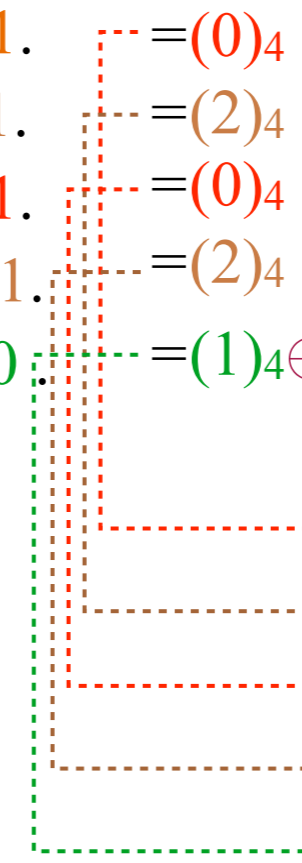
$D_4 \downarrow C_4$ subduction

$1, R_{z+90^\circ}, \rho_{z180^\circ}, R_{z-90^\circ}$

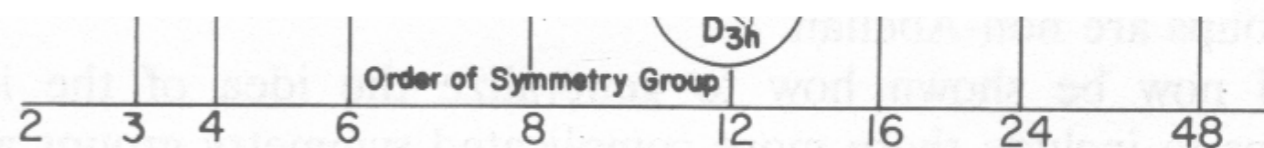
$A_1(D_4) \downarrow C_4 = 1, 1, 1, 1$
 $B_1(D_4) \downarrow C_4 = 1, -1, 1, -1$
 $A_2(D_4) \downarrow C_4 = 1, 1, 1, 1$
 $B_2(D_4) \downarrow C_4 = 1, -1, 1, -1$
 $E(D_4) \downarrow C_4 = 2, 0, -2, 0$

$O \downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$\chi_g^\mu(C_4)$	$g=1$	R_{z+90°	R_{z+180°	R_{z-90°
$(0)_4$	1	1	1	1
$(1)_4$	1	i	-1	$-i$
$(2)_4$	1	-1	1	-1
$(3)_4$	1	$-i$	-1	i



$D_4 \downarrow C_4$	0_4	1_4	2_4	$3_4 = \bar{1}_4$
A_1	1	·	·	·
B_1	·	·	1	·
A_2	1	·	·	·
B_2	·	·	1	·
E	·	1	·	1



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

$\chi_g^\mu(O)$	$g=1$	$r_{1..4}$	180° ρ_{xyz}	90° R_{xyz}	180° $i_{1..6}$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

$D_4: 1, \rho_{z180^\circ}, R_{z\pm 90^\circ}, \rho_{z180^\circ}, i_{3,4}$

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0$
 $T_2(O) \downarrow D_4 = 3, -1, 1, -1, -1$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1$

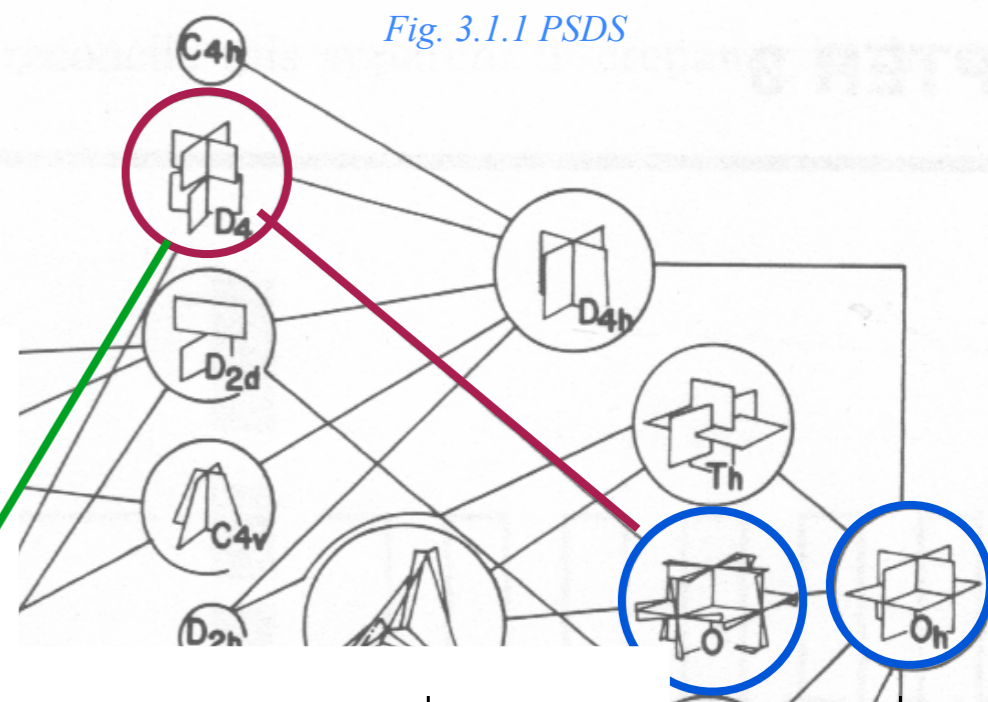


Fig. 3.1.1 PSDS

$\chi_g^\mu(D_4)$	$g=1$	ρ_{z180°	$R_{z\pm 90^\circ}$	$\rho_{x,y180^\circ}$	$i_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	-1	1	-1
A_2	1	1	1	-1	-1
B_2	1	1	-1	-1	1
E	2	-2	0	0	0

$D_4 \downarrow C_4$ subduction

$1, R_{z+90^\circ}, \rho_{z180^\circ}, R_{z-90^\circ}$

$A_1(D_4) \downarrow C_4 = 1, 1, 1, 1 = (0)_4$
 $B_1(D_4) \downarrow C_4 = 1, -1, 1, -1 = (2)_4$
 $A_2(D_4) \downarrow C_4 = 1, 1, 1, 1 = (0)_4$
 $B_2(D_4) \downarrow C_4 = 1, -1, 1, -1 = (2)_4$
 $E(D_4) \downarrow C_4 = 2, 0, -2, 0 = (1)_4 \oplus (3)_4$

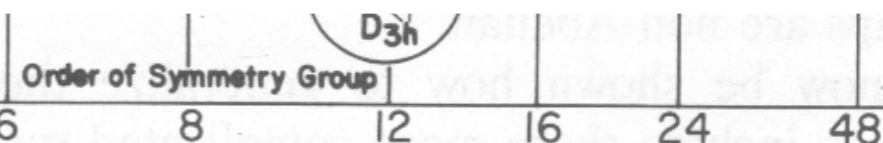
$O \downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1
A_2	.	1	.	.	.
E	1	1	.	.	.
T_1	.	.	1	.	1
T_2	.	.	.	1	1

$\chi_g^\mu(C_4)$	$g=1$	R_{z+90°	R_{z+180°	R_{z-90°
$(0)_4$	1	1	1	1
$(1)_4$	1	i	-1	$-i$
$(2)_4$	1	-1	1	-1
$(3)_4$	1	$-i$	-1	i

$O \downarrow C_4$ subduction

$O \downarrow C_4$	0_4	1_4	2_4	$3_4 = \bar{1}_4$
A_1	1	.	.	.
A_2	.	.	1	.
E	1	.	1	.
T_1	1	1	.	1
T_2	.	1	1	1

$D_4 \downarrow C_4$	0_4	1_4	2_4	$3_4 = \bar{1}_4$
A_1	1	.	.	.
B_1	.	.	1	.
A_2	1	.	.	.
B_2	.	.	1	.
E	.	1	.	1



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

D_4 : $1, \rho_{z180^\circ}, R_{z\pm 90^\circ}, \rho_{z180^\circ}, i_{3,4}$

$\chi_g^\mu(O)$	$g=1$	$r_{1..4}$	180° ρ_{xyz}	90° R_{xyz}	180° $i_{1..6}$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0$
 $T_2(O) \downarrow D_4 = 3, -1, 1, -1, -1$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1$

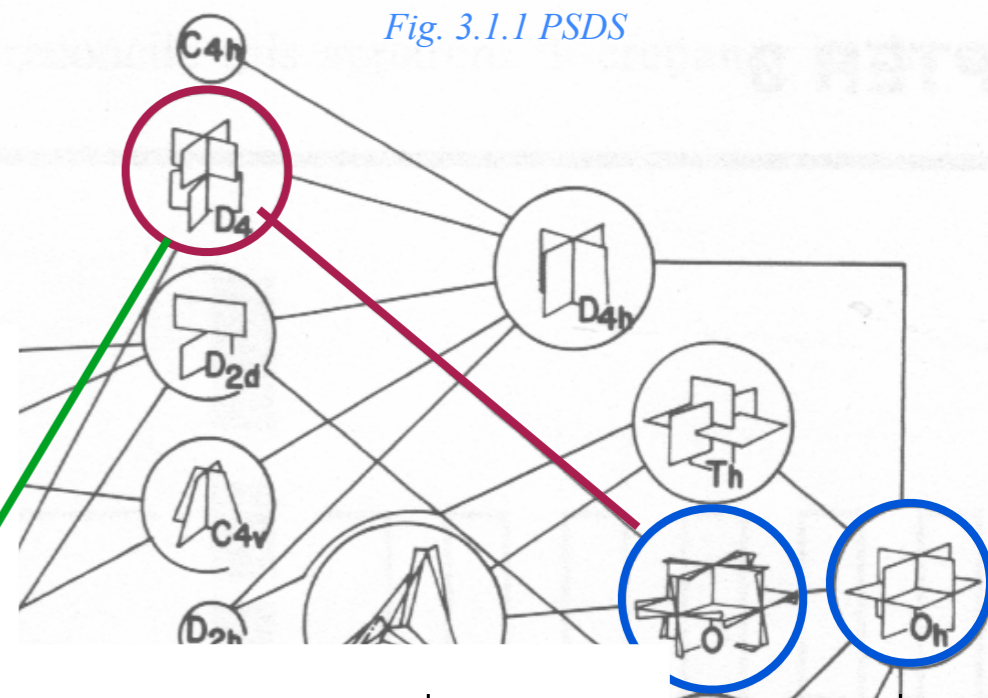


Fig. 3.1.1 PSDS

$\chi_g^\mu(D_4)$	$g=1$	ρ_{z180°	$R_{z\pm 90^\circ}$	$\rho_{x,y180^\circ}$	$i_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	-1	-1	-1
A_2	1	1	1	-1	-1
B_2	1	1	-1	1	1
E	2	-2	0	0	0

$D_4 \downarrow C_4$ subduction

$1, R_{z+90^\circ}, \rho_{z180^\circ}, R_{z-90^\circ}$

$A_1(D_4) \downarrow C_4 = 1, 1, 1, 1 = (0)_4$
 $B_1(D_4) \downarrow C_4 = 1, -1, 1, -1 = (2)_4$
 $A_2(D_4) \downarrow C_4 = 1, 1, 1, 1 = (0)_4$
 $B_2(D_4) \downarrow C_4 = 1, -1, 1, -1 = (2)_4$
 $E(D_4) \downarrow C_4 = 2, 0, -2, 0 = (1)_4 \oplus (3)_4$

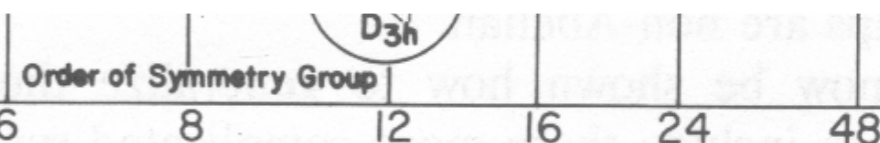
$O \downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1
A_2	.	1	.	.	.
E	1	1	.	.	.
T_1	.	.	1	.	1
T_2	.	.	.	1	1

$\chi_g^\mu(C_4)$	$g=1$	R_{z+90°	R_{z+180°	R_{z-90°
$(0)_4$	1	1	1	1
$(1)_4$	1	i	-1	$-i$
$(2)_4$	1	-1	1	-1
$(3)_4$	1	$-i$	-1	i

$O \downarrow C_4$ subduction

$O \downarrow C_4$	0_4	1_4	2_4	$3_4 = \bar{1}_4$
A_1	→ 1	.	.	.
A_2	.	.	1	.
E	1	.	1	.
T_1	1	1	.	1
T_2	.	1	1	1

$D_4 \downarrow C_4$	0_4	1_4	2_4	$3_4 = \bar{1}_4$
A_1	→ 1	.	.	.
B_1	.	.	1	.
A_2	1	.	.	.
B_2	.	.	1	.
E	.	1	.	1



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

D_4 : $1, \rho_{z180^\circ}, R_{z\pm 90^\circ}, \rho_{z180^\circ}, i_{3,4}$

$\chi_g^\mu(O)$	$g=1$	$r_{1..4}$	180° ρ_{xyz}	90° R_{xyz}	180° $i_{1..6}$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0$
 $T_2(O) \downarrow D_4 = 3, -1, 1, -1, -1$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1$

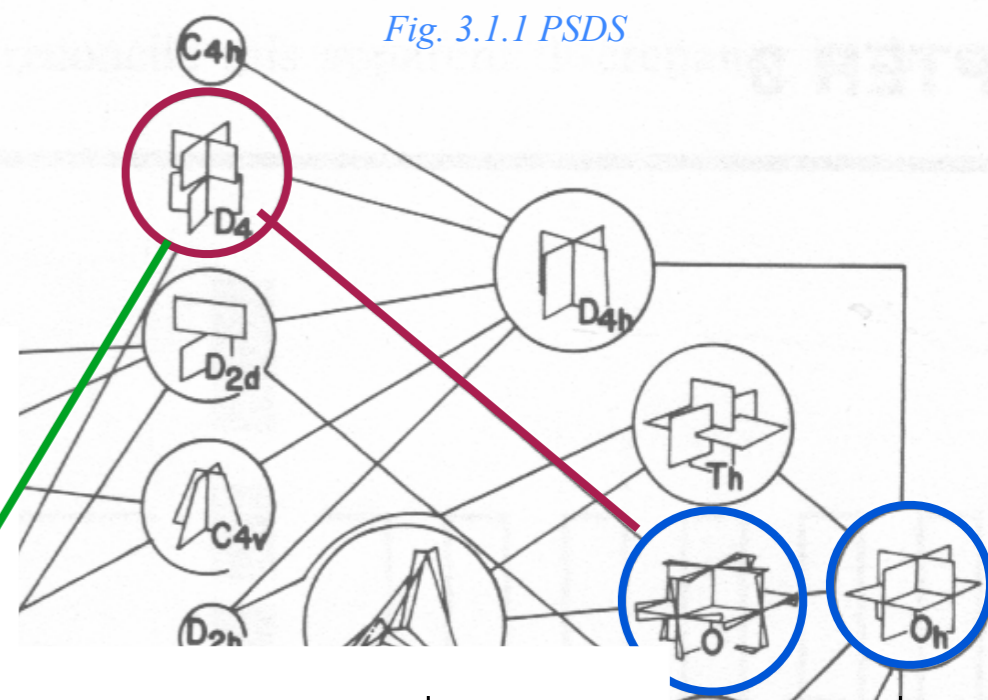


Fig. 3.1.1 PSDS

$\chi_g^\mu(D_4)$	$g=1$	ρ_{z180°	$R_{z\pm 90^\circ}$	$\rho_{x,y180^\circ}$	$i_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	-1	1	-1
A_2	1	1	1	-1	-1
B_2	1	1	-1	-1	1
E	2	-2	0	0	0

$D_4 \downarrow C_4$ subduction

$1, R_{z+90^\circ}, \rho_{z180^\circ}, R_{z-90^\circ}$

$A_1(D_4) \downarrow C_4 = 1, 1, 1, 1 = (0)_4$
 $B_1(D_4) \downarrow C_4 = 1, -1, 1, -1 = (2)_4$
 $A_2(D_4) \downarrow C_4 = 1, 1, 1, 1 = (0)_4$
 $B_2(D_4) \downarrow C_4 = 1, -1, 1, -1 = (2)_4$
 $E(D_4) \downarrow C_4 = 2, 0, -2, 0 = (1)_4 \oplus (3)_4$

$O \downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1
A_2	.	1	.	.	.
E	1	1	.	.	.
T_1	.	.	1	.	1
T_2	.	.	.	1	1

$\chi_g^\mu(C_4)$	$g=1$	R_{z+90°	R_{z+180°	R_{z-90°
$(0)_4$	1	1	1	1
$(1)_4$	1	i	-1	$-i$
$(2)_4$	1	-1	1	-1
$(3)_4$	1	$-i$	-1	i

$O \downarrow C_4$ subduction

$O \downarrow C_4$	0_4	1_4	2_4	$3_4 = \bar{1}_4$
A_1	1	.	.	.
A_2	.	.	→ 1	.
E	1	.	1	.
T_1	1	1	.	1
T_2	.	1	1	1

$D_4 \downarrow C_4$	0_4	1_4	2_4	$3_4 = \bar{1}_4$
A_1	1	.	.	.
B_1	.	→ 1	.	.
A_2	1	.	.	.
B_2	.	.	1	.
E	.	1	.	1

Order of Symmetry Group: 2, 3, 4, 6, 8, 12, 16, 24, 48

Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

D_4 : $1, \rho_{z180^\circ}, R_{z\pm 90^\circ}, \rho_{z180^\circ}, i_{3,4}$

$\chi_g^\mu(O)$	$g=1$	$r_{1..4}$	180° ρ_{xyz}	90° R_{xyz}	180° $i_{1..6}$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0$
 $T_2(O) \downarrow D_4 = 3, -1, 1, -1, -1$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1$

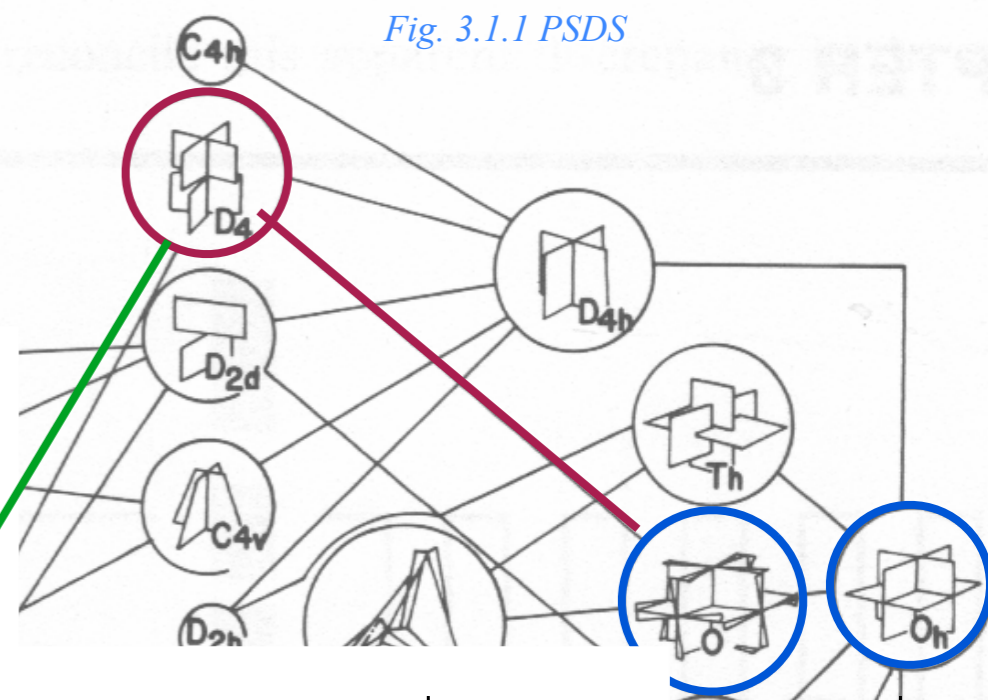


Fig. 3.1.1 PSDS

$\chi_g^\mu(D_4)$	$g=1$	ρ_{z180°	$R_{z\pm 90^\circ}$	$\rho_{x,y180^\circ}$	$i_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	-1	1	-1
A_2	1	1	1	-1	-1
B_2	1	1	-1	-1	1
E	2	-2	0	0	0

$D_4 \downarrow C_4$ subduction

$1, R_{z+90^\circ}, \rho_{z180^\circ}, R_{z-90^\circ}$

$A_1(D_4) \downarrow C_4 = 1, 1, 1, 1 = (0)_4$
 $B_1(D_4) \downarrow C_4 = 1, -1, 1, -1 = (2)_4$
 $A_2(D_4) \downarrow C_4 = 1, 1, 1, 1 = (0)_4$
 $B_2(D_4) \downarrow C_4 = 1, -1, 1, -1 = (2)_4$
 $E(D_4) \downarrow C_4 = 2, 0, -2, 0 = (1)_4 \oplus (3)_4$

$O \downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1
A_2	.	1	.	.	.
E	1	1	.	.	.
T_1	.	.	1	.	1
T_2	.	.	.	1	1

$\chi_g^\mu(C_4)$	$g=1$	R_{z+90°	R_{z+180°	R_{z-90°
$(0)_4$	1	1	1	1
$(1)_4$	1	i	-1	$-i$
$(2)_4$	1	-1	1	-1
$(3)_4$	1	$-i$	-1	i

$O \downarrow C_4$ subduction

$O \downarrow C_4$	0_4	1_4	2_4	$3_4 = \bar{1}_4$
A_1	1	.	.	.
A_2	.	.	1	.
E	→ 1	.	→ 1	.
T_1	1	1	.	1
T_2	.	1	1	1

$D_4 \downarrow C_4$	0_4	1_4	2_4	$3_4 = \bar{1}_4$
A_1	→ 1	.	.	.
B_1	.	→ 1	.	.
A_2	1	.	.	.
B_2	.	.	1	.
E	.	1	.	1

Order of Symmetry Group: 2, 3, 4, 6, 8, 12, 16, 24, 48

Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

$\chi_g^\mu(O)$	$g=1$	$r_{1..4}$	180° ρ_{xyz}	90° R_{xyz}	180° $i_{1..6}$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

$D_4: 1, \rho_{z180^\circ}, R_{z\pm 90^\circ}, \rho_{z180^\circ}, i_{3,4}$

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0$
 $T_2(O) \downarrow D_4 = 3, -1, 1, -1, -1$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1$

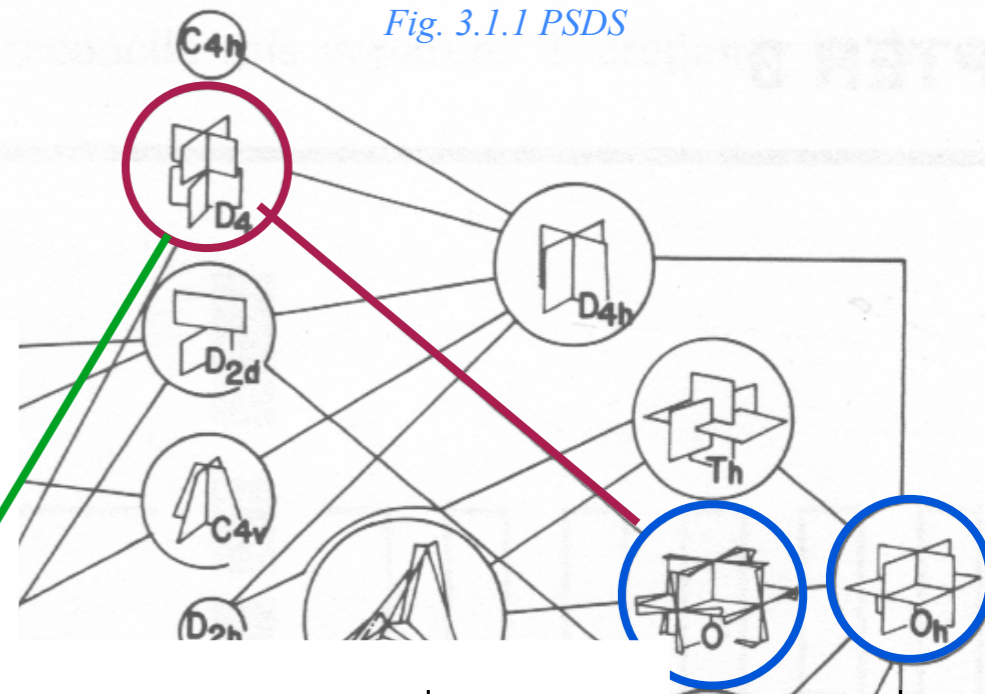


Fig. 3.1.1 PSDS

$\chi_g^\mu(D_4)$	$g=1$	ρ_{z180°	$R_{z\pm 90^\circ}$	$\rho_{x,y180^\circ}$	$i_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	-1	1	-1
A_2	1	1	1	-1	-1
B_2	1	1	-1	-1	1
E	2	-2	0	0	0

$D_4 \downarrow C_4$ subduction

$1, R_{z+90^\circ}, \rho_{z180^\circ}, R_{z-90^\circ}$

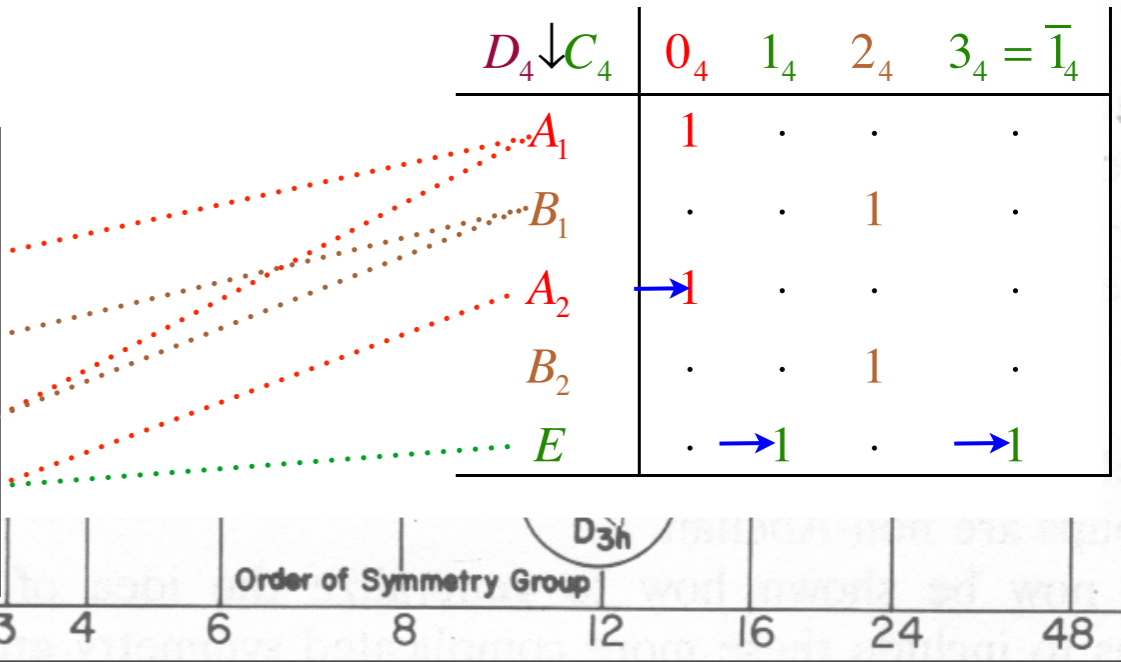
$A_1(D_4) \downarrow C_4 = 1, 1, 1, 1 = (0)_4$
 $B_1(D_4) \downarrow C_4 = 1, -1, 1, -1 = (2)_4$
 $A_2(D_4) \downarrow C_4 = 1, 1, 1, 1 = (0)_4$
 $B_2(D_4) \downarrow C_4 = 1, -1, 1, -1 = (2)_4$
 $E(D_4) \downarrow C_4 = 2, 0, -2, 0 = (1)_4 \oplus (3)_4$

$O \downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1
A_2	.	1	.	.	.
E	1	1	.	.	.
T_1	.	.	1	.	1
T_2	.	.	.	1	1

$\chi_g^\mu(C_4)$	$g=1$	R_{z+90°	R_{z+180°	R_{z-90°
$(0)_4$	1	1	1	1
$(1)_4$	1	i	-1	$-i$
$(2)_4$	1	-1	1	-1
$(3)_4$	1	$-i$	-1	i

$O \downarrow C_4$ subduction

$O \downarrow C_4$	0_4	1_4	2_4	$3_4 = \bar{1}_4$
A_1	1	.	.	.
A_2	.	.	1	.
E	1	.	1	.
T_1	$\rightarrow 1$	$\rightarrow 1$.	$\rightarrow 1$
T_2	.	1	1	1



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

D_4 : $1, \rho_{z180^\circ}, R_{z\pm 90^\circ}, \rho_{z180^\circ}, i_{3,4}$

$\chi_g^\mu(O)$	$g=1$	$r_{1..4}$	180° ρ_{xyz}	90° R_{xyz}	180° $i_{1..6}$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0$
 $T_2(O) \downarrow D_4 = 3, -1, 1, -1, -1$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1$

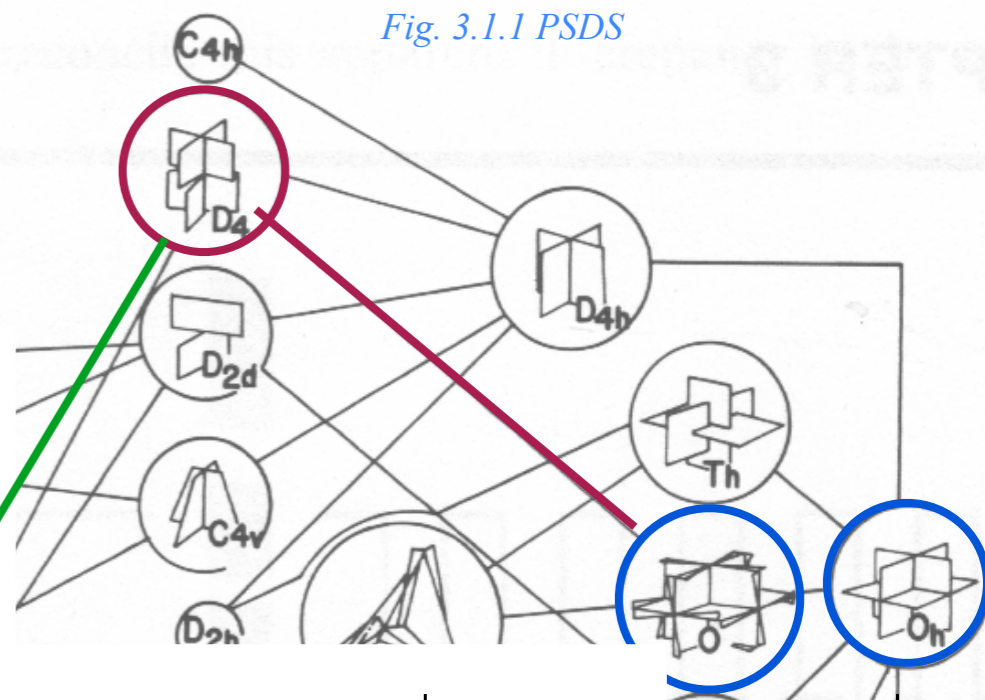


Fig. 3.1.1 PSDS

$\chi_g^\mu(D_4)$	$g=1$	ρ_{z180°	$R_{z\pm 90^\circ}$	$\rho_{x,y180^\circ}$	$i_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	-1	1	-1
A_2	1	1	1	-1	-1
B_2	1	1	-1	-1	1
E	2	-2	0	0	0

$D_4 \downarrow C_4$ subduction

$1, R_{z+90^\circ}, \rho_{z180^\circ}, R_{z-90^\circ}$

$A_1(D_4) \downarrow C_4 = 1, 1, 1, 1 = (0)_4$
 $B_1(D_4) \downarrow C_4 = 1, -1, 1, -1 = (2)_4$
 $A_2(D_4) \downarrow C_4 = 1, 1, 1, 1 = (0)_4$
 $B_2(D_4) \downarrow C_4 = 1, -1, 1, -1 = (2)_4$
 $E(D_4) \downarrow C_4 = 2, 0, -2, 0 = (1)_4 \oplus (3)_4$

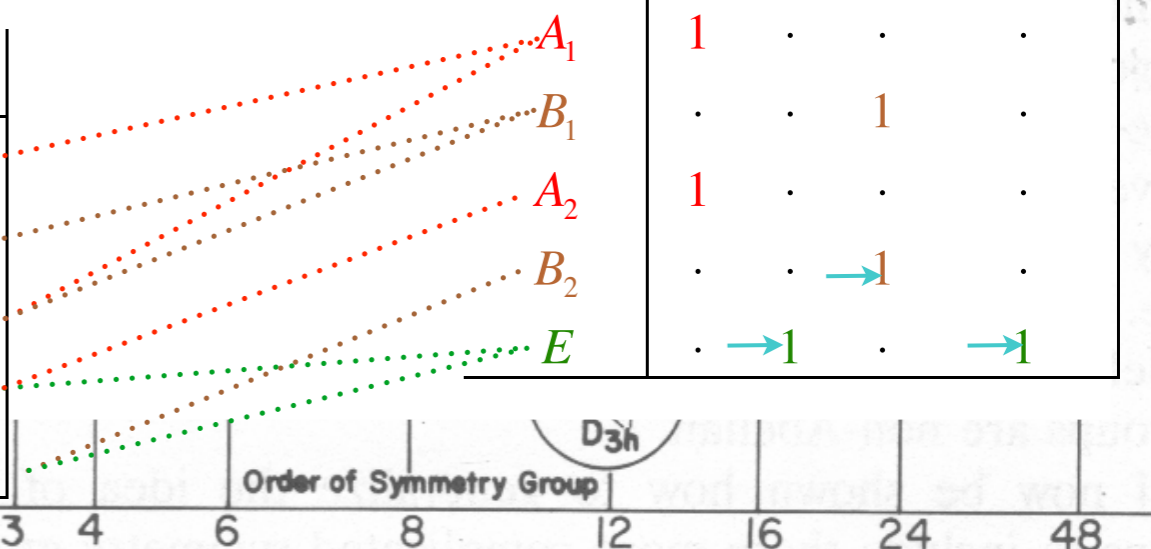
$O \downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1
A_2	.	1	.	.	.
E	1	1	.	.	.
T_1	.	.	1	.	1
T_2	.	.	.	1	1

$\chi_g^\mu(C_4)$	$g=1$	R_{z+90°	R_{z+180°	R_{z-90°
$(0)_4$	1	1	1	1
$(1)_4$	1	i	-1	$-i$
$(2)_4$	1	-1	1	-1
$(3)_4$	1	$-i$	-1	i

$O \downarrow C_4$ subduction

$O \downarrow C_4$	0_4	1_4	2_4	$3_4 = \bar{1}_4$
A_1	1	.	.	.
A_2	.	.	1	.
E	1	.	1	.
T_1	1	1	.	1
T_2	.	$\rightarrow 1$	$\rightarrow 1$	$\rightarrow 1$

$D_4 \downarrow C_4$	0_4	1_4	2_4	$3_4 = \bar{1}_4$
A_1	1	.	.	.
B_1	.	.	1	.
A_2	1	.	.	.
B_2	.	.	$\rightarrow 1$.
E	.	$\rightarrow 1$.	$\rightarrow 1$



Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$

Octahedral-cubic O symmetry and group operations

Tetrahedral symmetry becomes Icosahedral

Octahedral groups $O_h \supset O \sim T_d \supset T$

Octahedral O and spin- $O \subset U(2)$

Tetrahedral T class algebra

Tetrahedral T class minimal equations

Tetrahedral T class projectors and characters

Octahedral O class algebra

Octahedral O class minimal equations

Octahedral O class projectors and characters

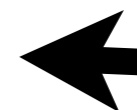
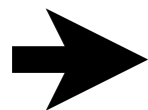
Octahedral $O_h \supset O$ subgroup correlations

Octahedral $O_h \supset O$ subgroup correlations

$O_h \supset O \supset D_4$ subgroup correlations

$O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

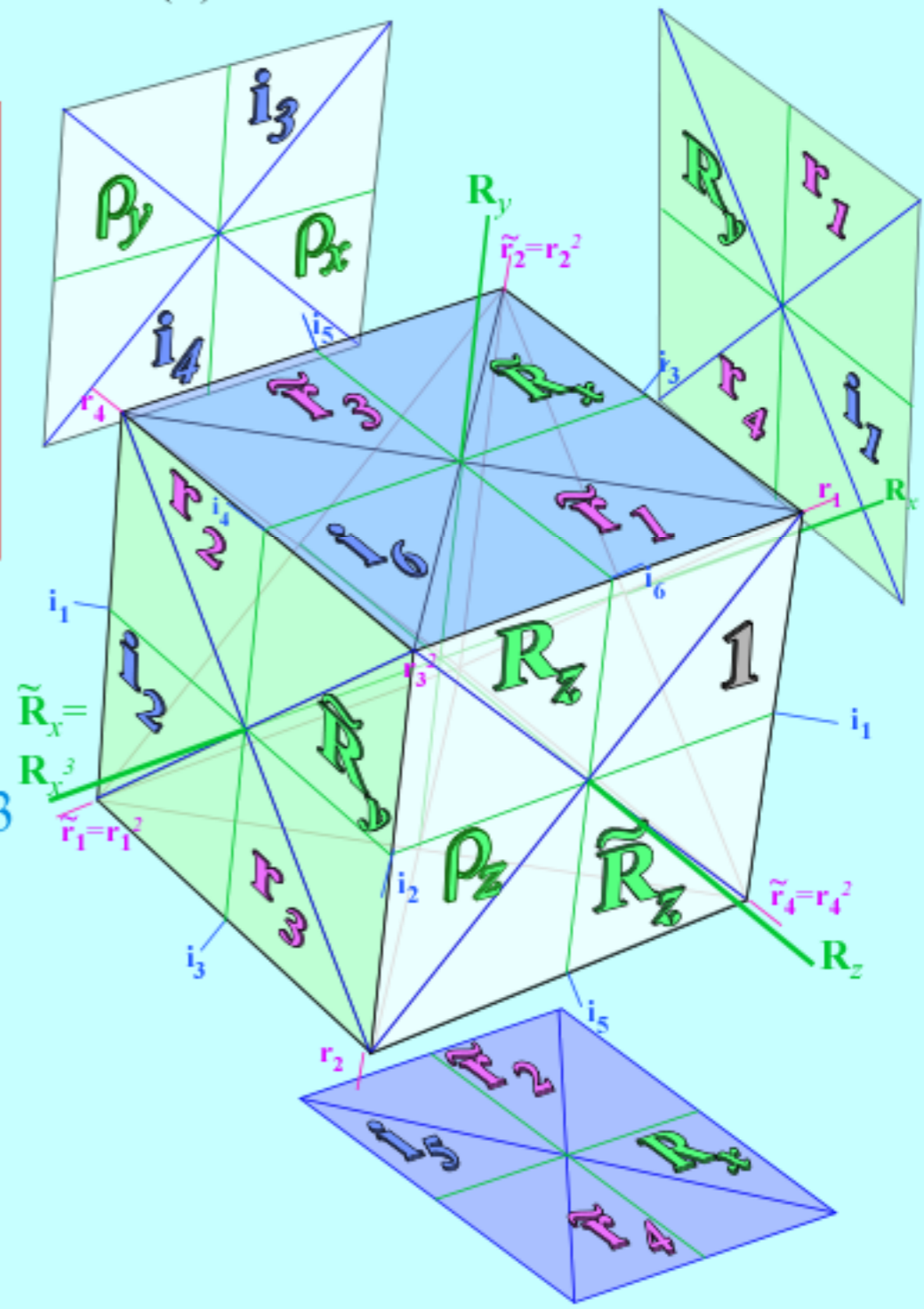
Applications



$$\begin{aligned} \ell^{A_1} &= 1 \\ \ell^{A_2} &= 1 \\ \ell^E &= 2 \\ \ell^{T_1} &= 3 \\ \ell^{T_2} &= 3 \end{aligned}$$

Example: $G=O$ Centrum: $\kappa(O) = \sum_{(\alpha)} (\ell^\alpha)^0 = 1^0 + 1^0 + 2^0 + 3^0 + 3^0 = 5$
 Cubic-Octahedral Rank: $\rho(O) = \sum_{(\alpha)} (\ell^\alpha)^1 = 1^1 + 1^1 + 2^1 + 3^1 + 3^1 = 10$
 Group O Order: $o(O) = \sum_{(\alpha)} (\ell^\alpha)^2 = 1^2 + 1^2 + 2^2 + 3^2 + 3^2 = 24$

O group	$g = 1$	r_{1-4}	ρ_{xyz}	R_{xyz}	i_{1-6}
$\chi_{\kappa_g}^\alpha$		\tilde{r}_{1-4}		\tilde{R}_{xyz}	
$\alpha = A_1$ s-orbital r^2	1	1	1	1	1
A_2 d-orbitals	1	1	1	-1	-1
E $\{x^2+y^2-2z^2, x^2-y^2\}$	2	-1	2	0	0
T_1 p-orbitals $\{x, y, z\}$	3	0	-1	1	-1
T_2 $\{xz, yz, xy\}$ d-orbitals	3	0	-1	-1	1

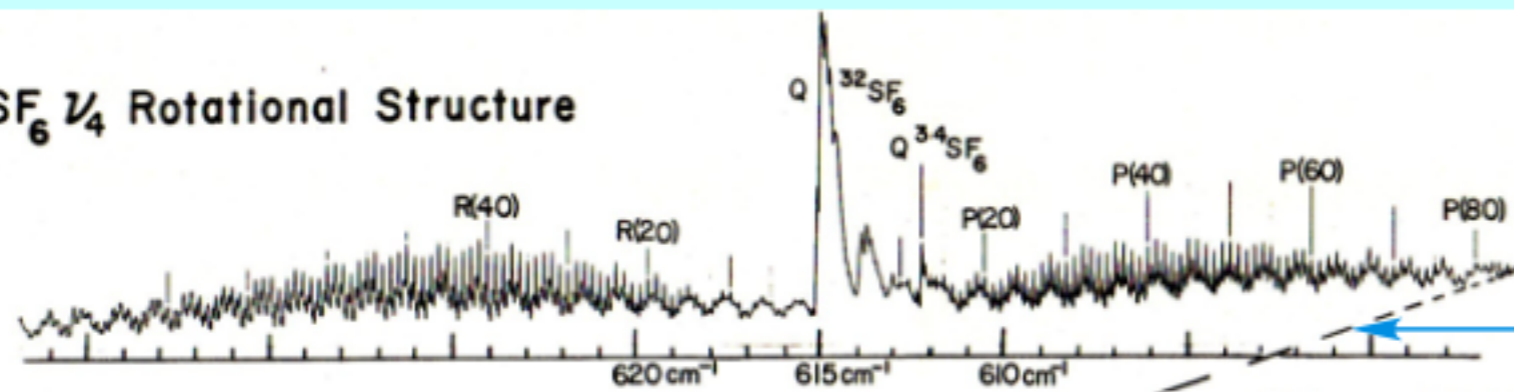


$O \supset C_4$ $(0)_4 (1)_4 (2)_4 (3)_4 = (-1)_4$ $O \supset C_3$ $(0)_3 (1)_3 (2)_3 = (-1)_3$

A_1	1	•	•	•
A_2	•	•	1	•
E	1	•	1	•
T_1	1	1	•	1
T_2	•	1	1	1

A_1	1	•	•
A_2	1	•	•
E	•	1	1
T_1	1	1	1
T_2	1	1	1

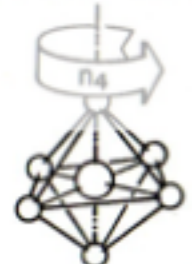
(a) SF₆ ν₄ Rotational Structure



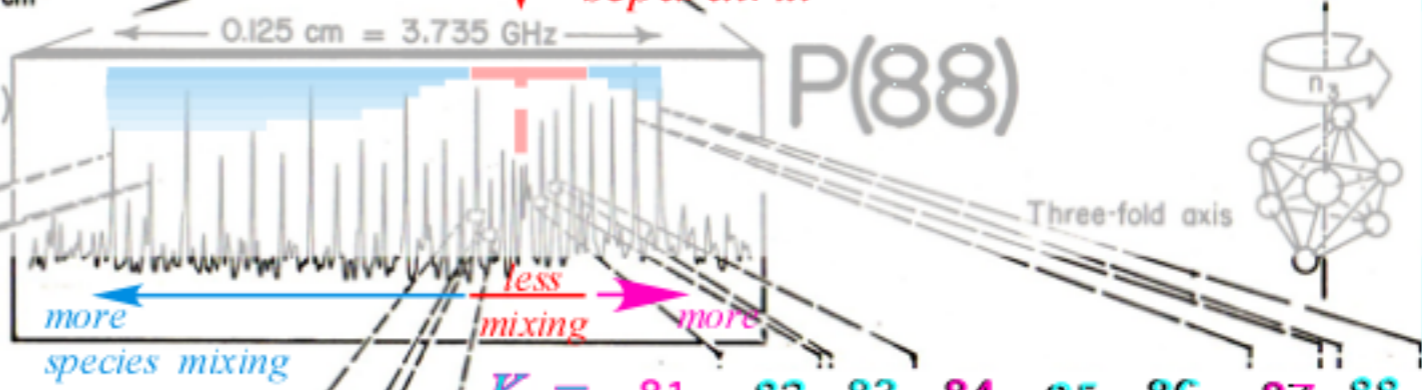
FT IR and Laser Diode Spectra
K.C. Kim, W.B. Person, D. Seitz, and B.J. Krohn
J. Mol. Spectrosc. 76, 322 (1979).

Primary AET species mixing increases with distance from "separatrix"

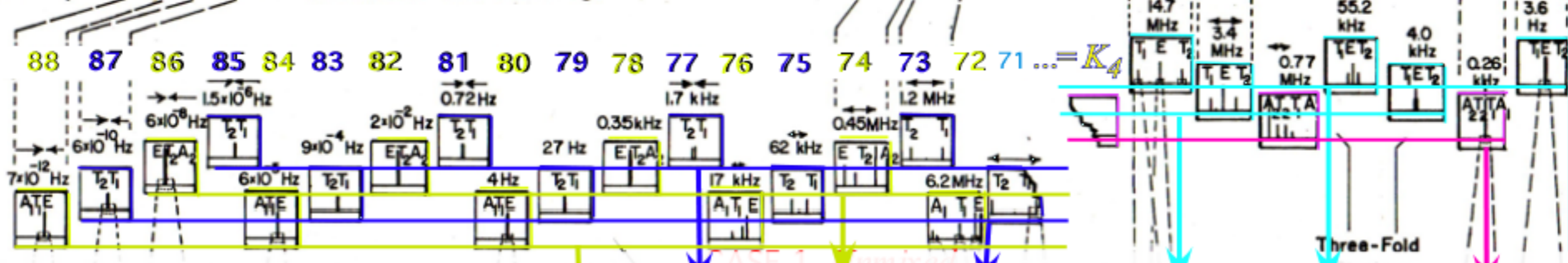
(b) P(88) Fine Structure (Rotational anisotropy effects)



SF₆ ν₃ P(88) ~ 16m



(c) Superfine Structure (Rotational axis tunneling)



Observed repeating sequence(s) .. A₁ T₁ E T₂ T₁ E T₂ A₂ T₂ T₁ A₁ T₁ E T₂ T₁ E T₂ A₂ T₂ T₁ A₁ ..

$O=C_4$ (0)₄ (1)₄ (2)₄ (3)₄ = (-1)₄

A ₁	1	•	•	•
A ₂	•	•	1	•
E	1	•	1	•
T ₁	1	1	•	1
T ₂	•	1	1	1

$O=C_3$ (0)₃ (1)₃ (2)₃ = (-1)₃

A ₁	1	•	•
A ₂	1	•	•
E	•	1	1
T ₁	1	1	1
T ₂	1	1	1

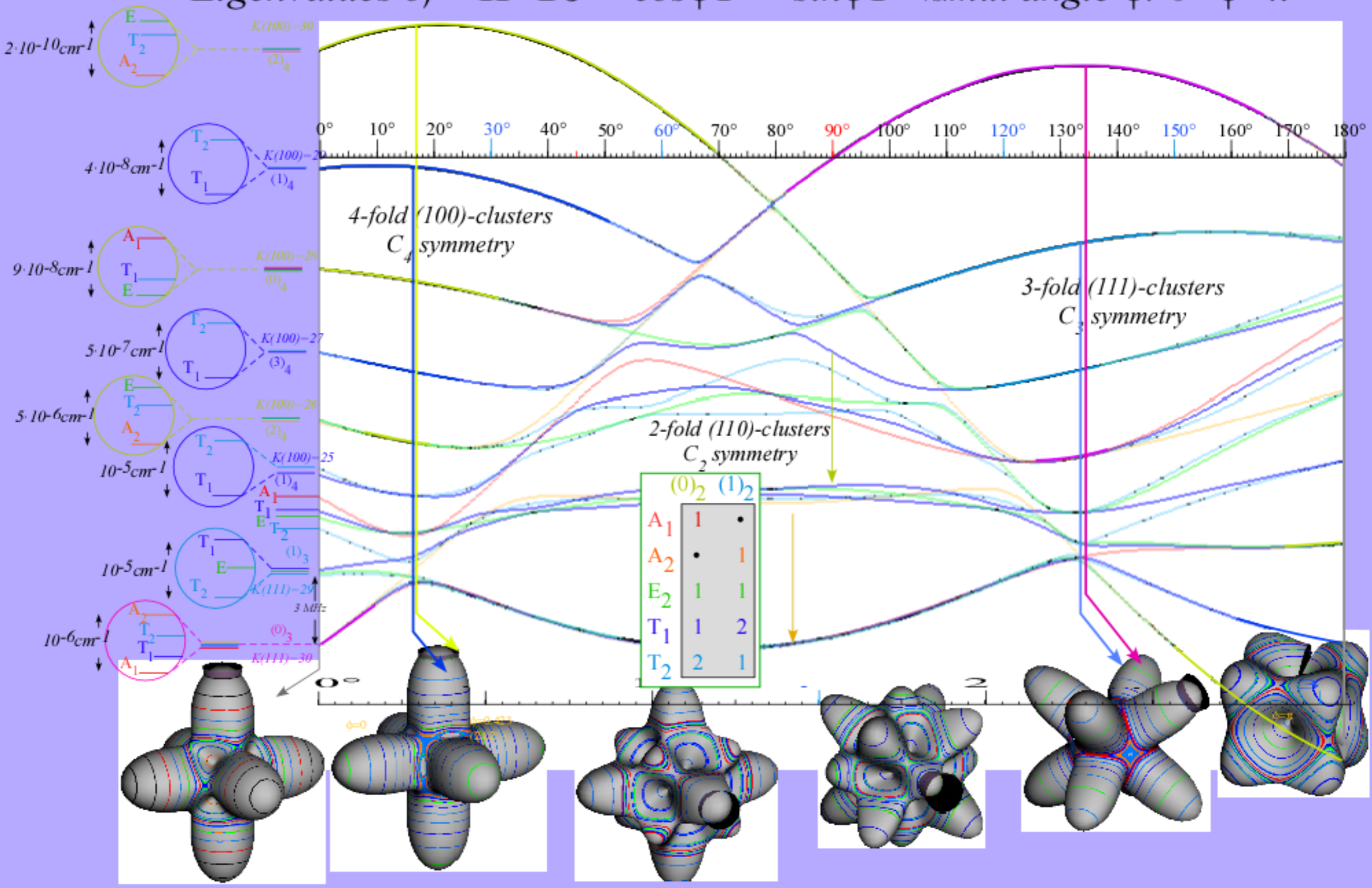
Local correlations explain clustering...
... but what about spacing and ordering?...

...and physical consequences?

Major mixing lowest two LUSTERS

(e) Superfine Structure on Correlation Frame

Eigenvalues of $\mathbf{H} = B\mathbf{J}^2 + \cos\phi\mathbf{T}^{[4]} + \sin\phi\mathbf{T}^{[6]}$ vs. mix angle $\phi: 0 < \phi < \pi$



	$(0)_2$	$(1)_2$
A_1	1	•
A_2	•	1
E_2	1	1
T_1	1	2
T_2	2	1