Group Theory in Quantum Mechanics

Lecture 1 (1.15.13)

Introduction to quantum amplitudes and analyzers

(Quantum Theory for Computer Age - Ch. 1 of Unit 1)
(Principles of Symmetry, Dynamics, and Spectroscopy - Sec. 1-2 of Ch. 1)

Beam Sorters - Optical polarization sorting

Beam Sorters in Series and Transformation Matrices
  Introducing Dirac bra-ket notation

"Abstraction" of bra and ket vectors from a Transformation Matrix
  Introducing scalar and matrix products

Beam Sorters - Optical polarization sorting

2-State Sorters: spin-1/2 vs. optical polarization

Beam Sorters in Series and Transformation Matrices

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Beam Sorters

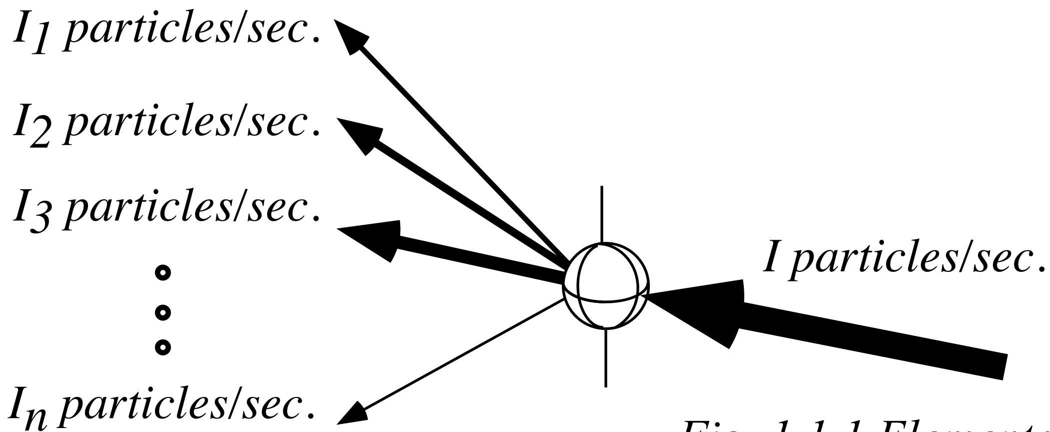


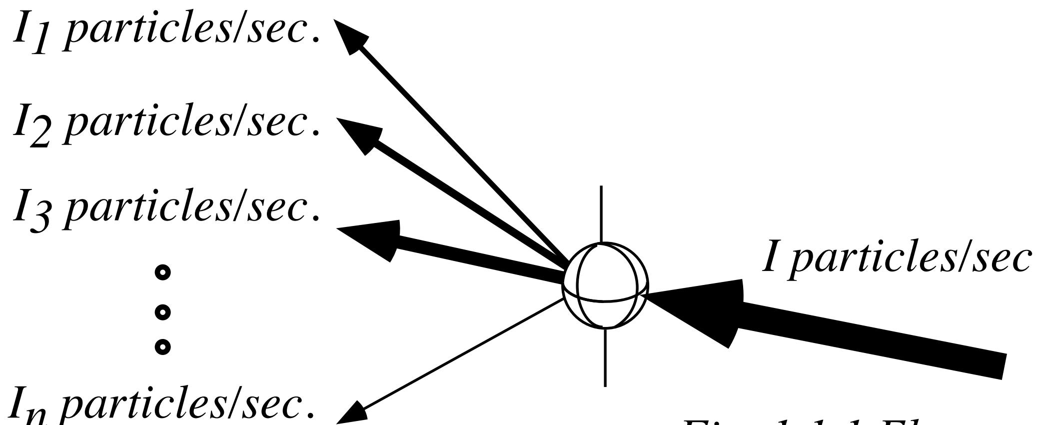
Fig. 1.1.1 Elementary beam sorter for n-state beam

$$I = I_1 + I_2 + I_3 + \ldots + I_n$$

One job of quantum mechanics is to compute *relative intensities* or *probabilities* P_k defined by

$$P_k = I_k / I \qquad \text{where:} \qquad 1 = P_1 + P_2 + P_3 + \ldots + P_n$$

Beam Sorters



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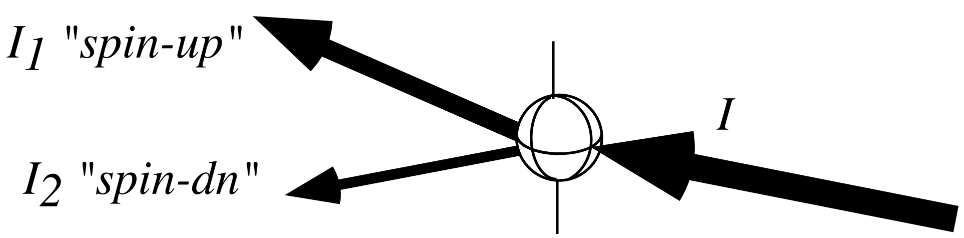
2-State Beam Sorters

Spin-1/2



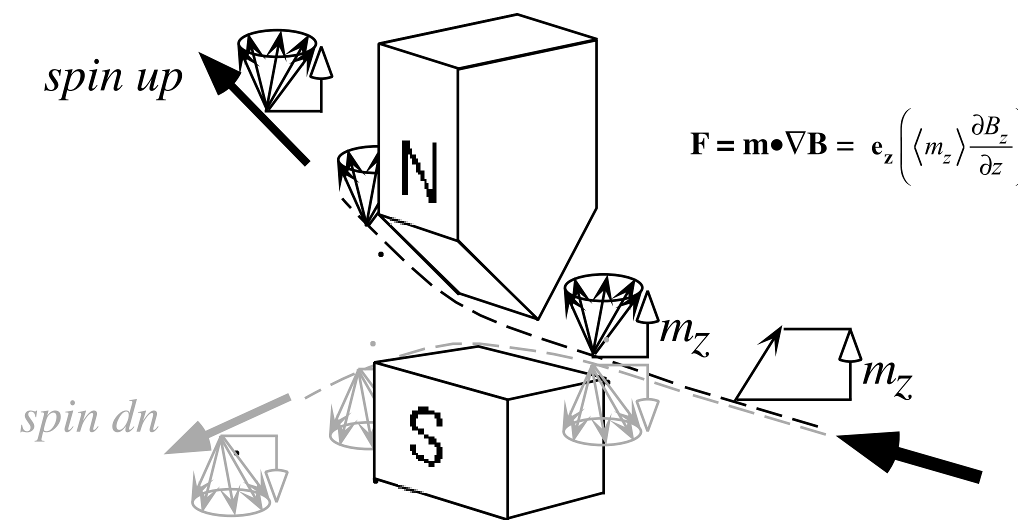
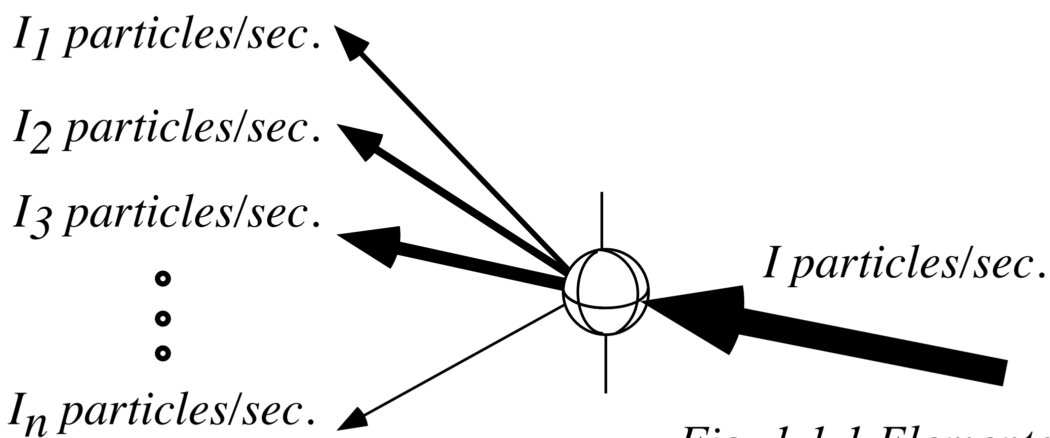


Fig. 1.1.2 Stern-Gerlach beam sorter for 2-state electron spin beam

Beam Sorters



$$I = I_1 + I_2 + I_3 + \ldots + I_n$$

Fig. 1.1.1 Elementary beam sorter for n-state beam

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2-State Beam Sorters

Spin-1/2

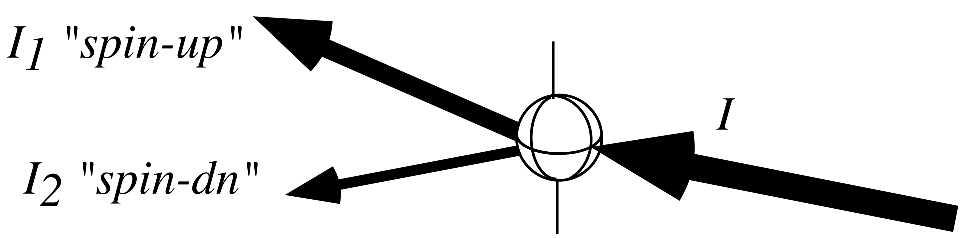


Fig. 1.1.2 Stern-Gerlach beam sorter for 2-state electron spin beam

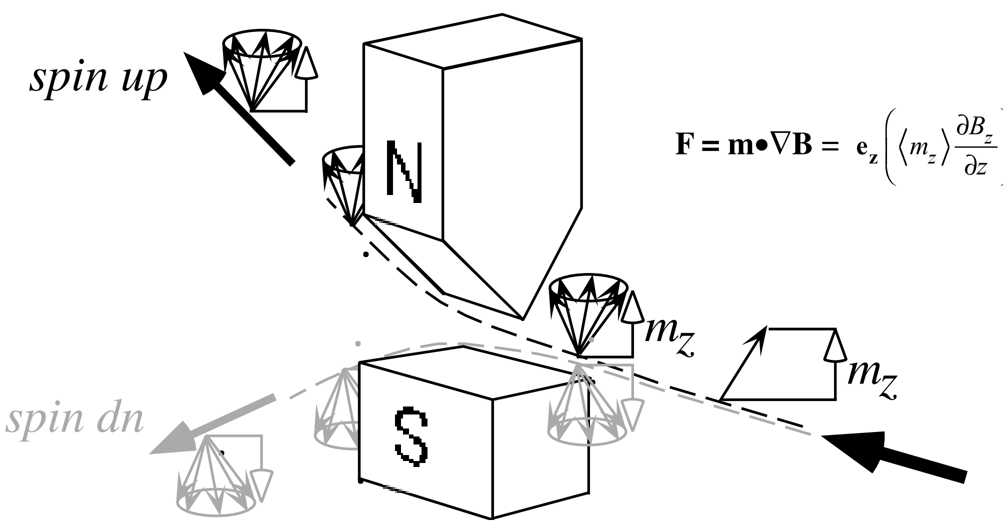


*Fig. 1.1.6 Sketch of electron beam sorting by non-uniform **B**-field: (Stern-Gerlach polarizer)*

Optical polarization

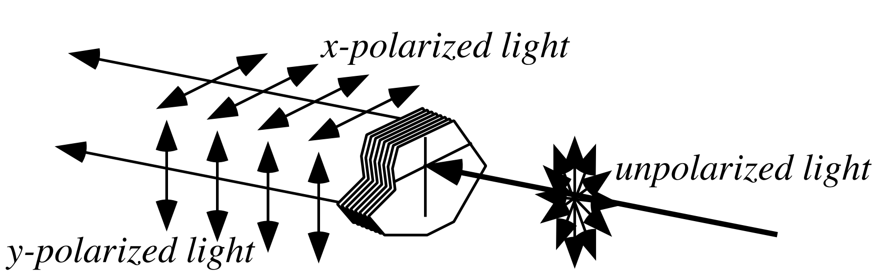


Fig. 1.1.3 Primitive photon beam sorter for 2-state polarization

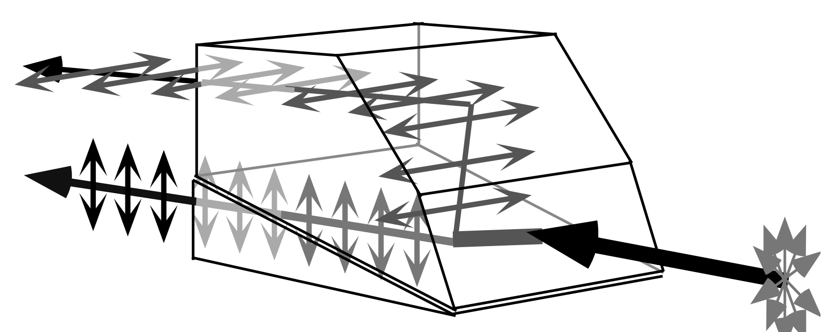
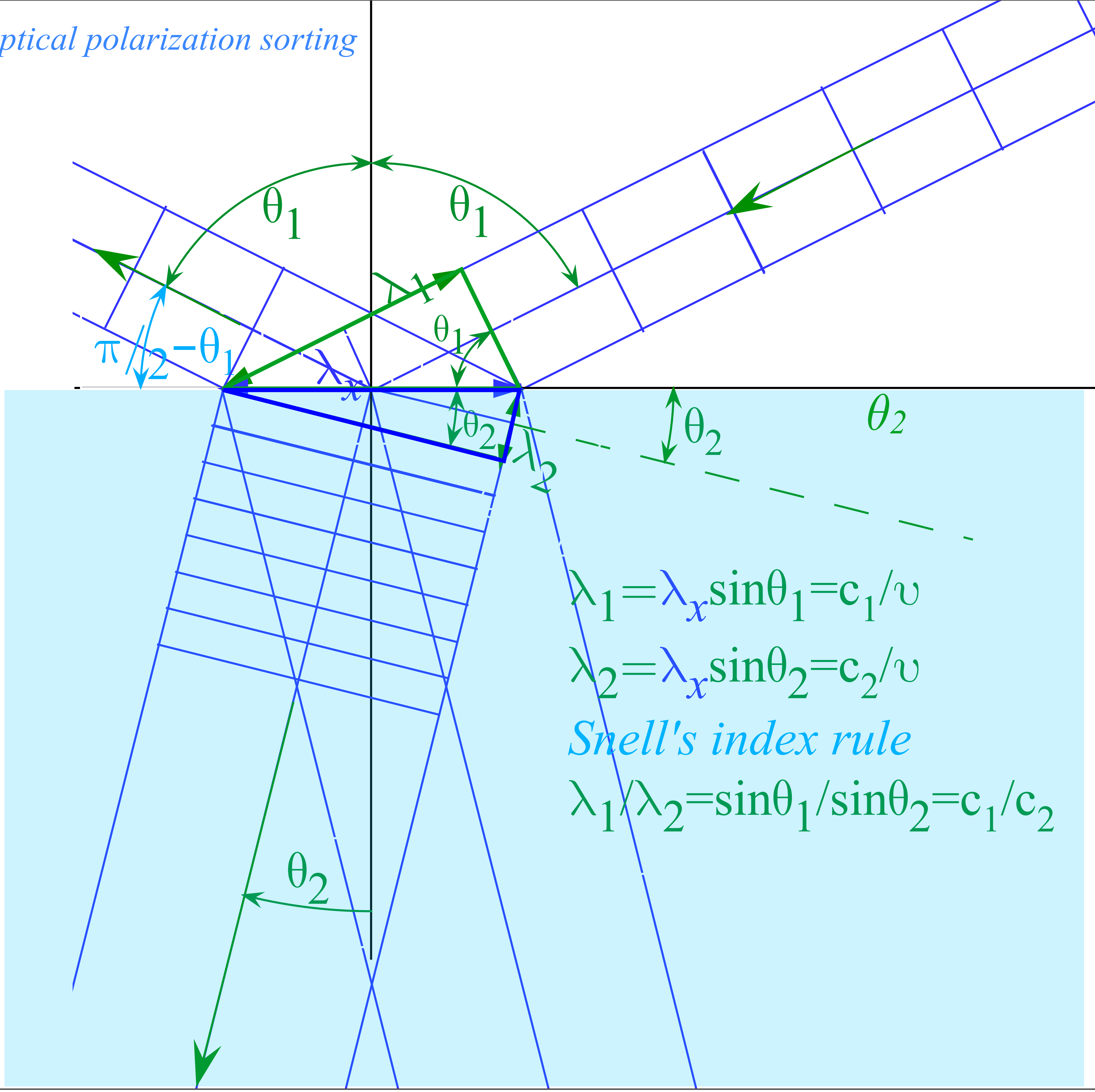
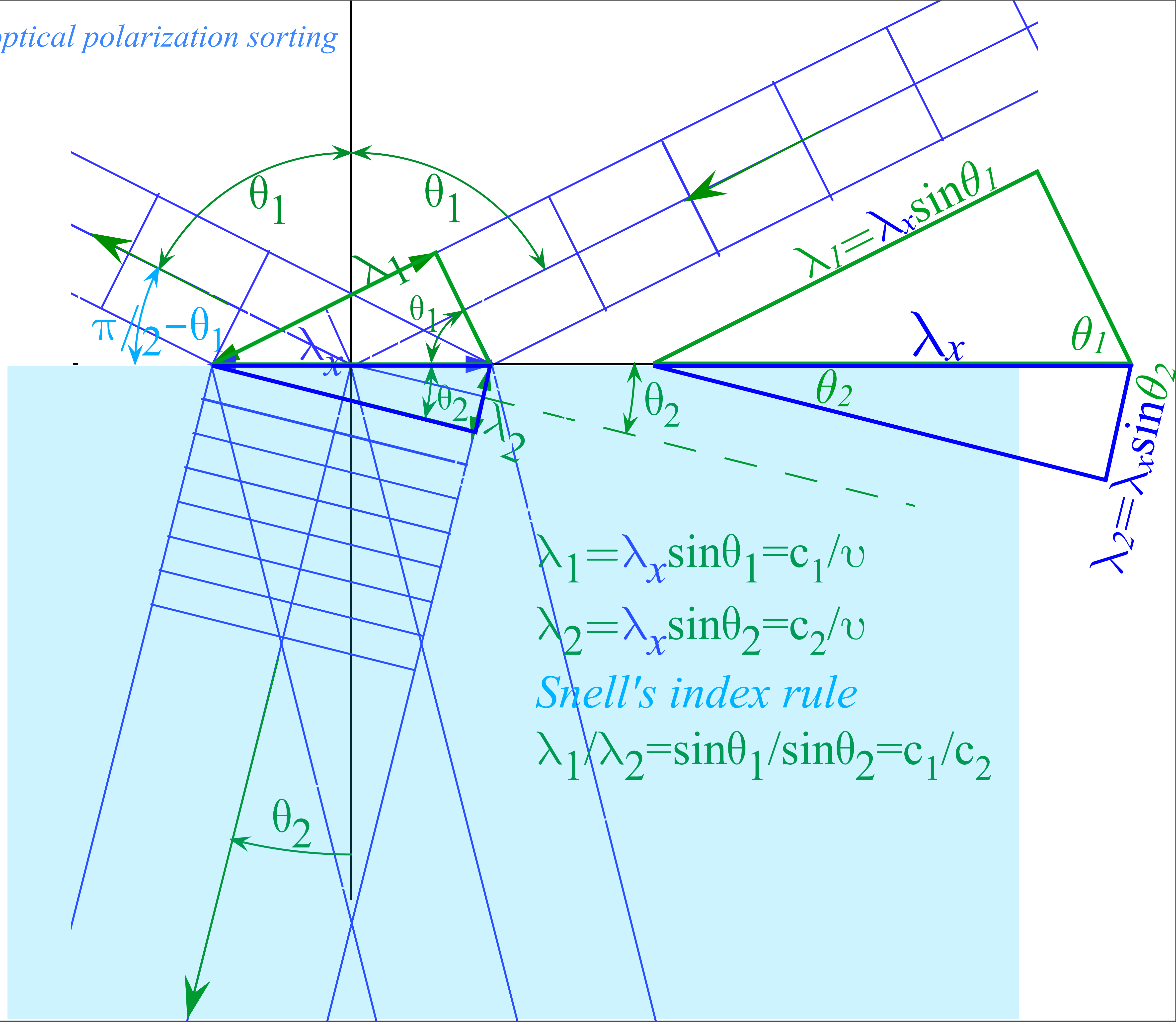


Fig. 1.1.5 Sketch of modern optical polarization sorter: (The Brewster prism)

Geometry of optical polarization sorting



Geometry of optical polarization sorting

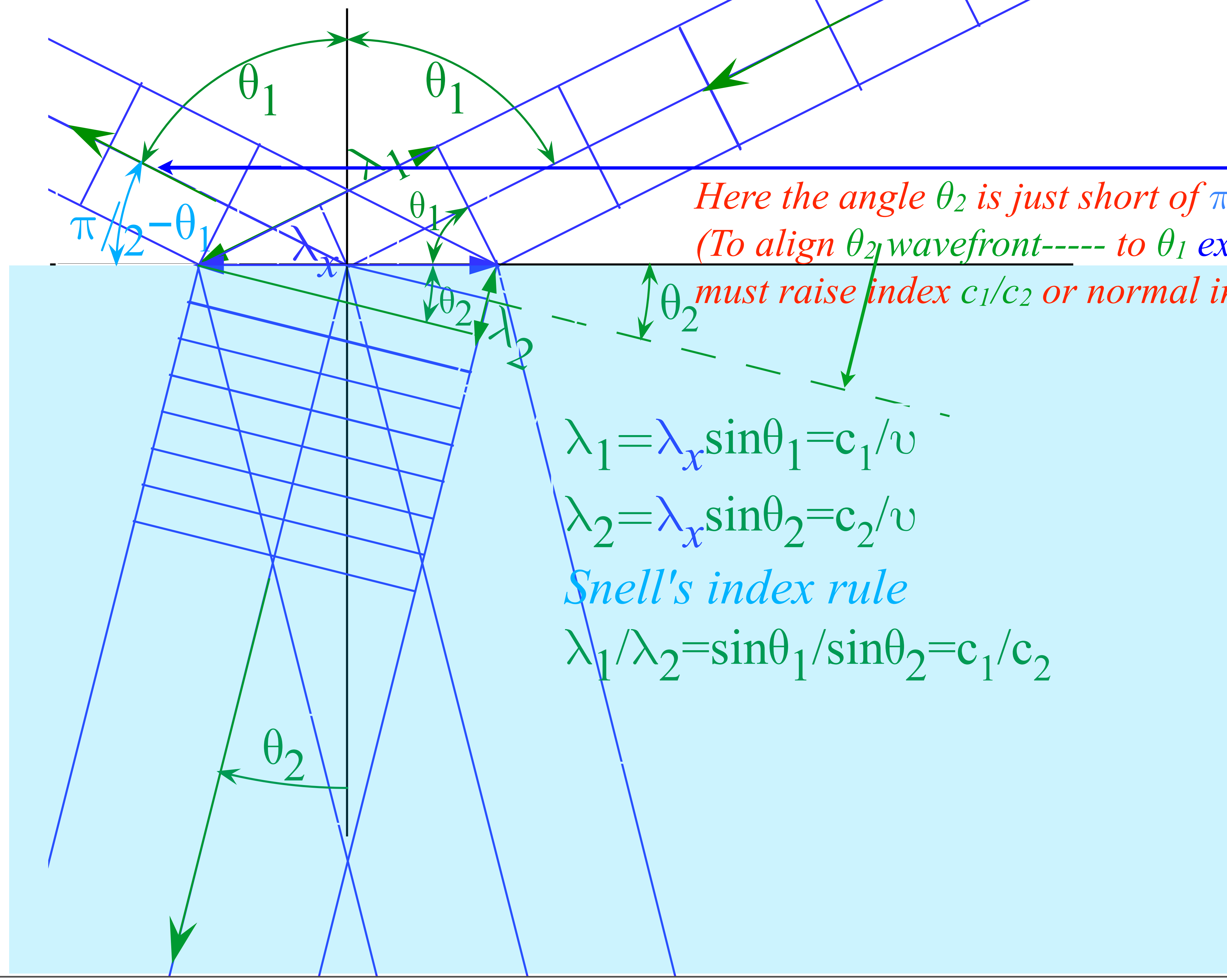


Brewsters's angle (Make $\theta_2=\pi/2-\theta_1$)

$\lambda_1/\lambda_2=\sin\theta_1/\sin\theta_2=c_1/c_2$

becomes:

$\lambda_1/\lambda_2=\sin\theta_1/\cos\theta_1=c_1/c_2=\tan\theta_1$



Here the angle θ_2 is just short of $\pi/2-\theta_1$
(To align θ_2 wavefront----- to θ_1 exit beam ←
must raise index c_1/c_2 or normal incidence θ_1)

$\lambda_1=\lambda_x\sin\theta_1=c_1/\upsilon$

$\lambda_2=\lambda_x\sin\theta_2=c_2/\upsilon$

Snell's index rule

$\lambda_1/\lambda_2=\sin\theta_1/\sin\theta_2=c_1/c_2$

Brewsters's angle (Make $\theta_2=\pi/2-\theta_1$)

$$\lambda_1/\lambda_2=\sin\theta_1/\sin\theta_2=c_1/c_2$$

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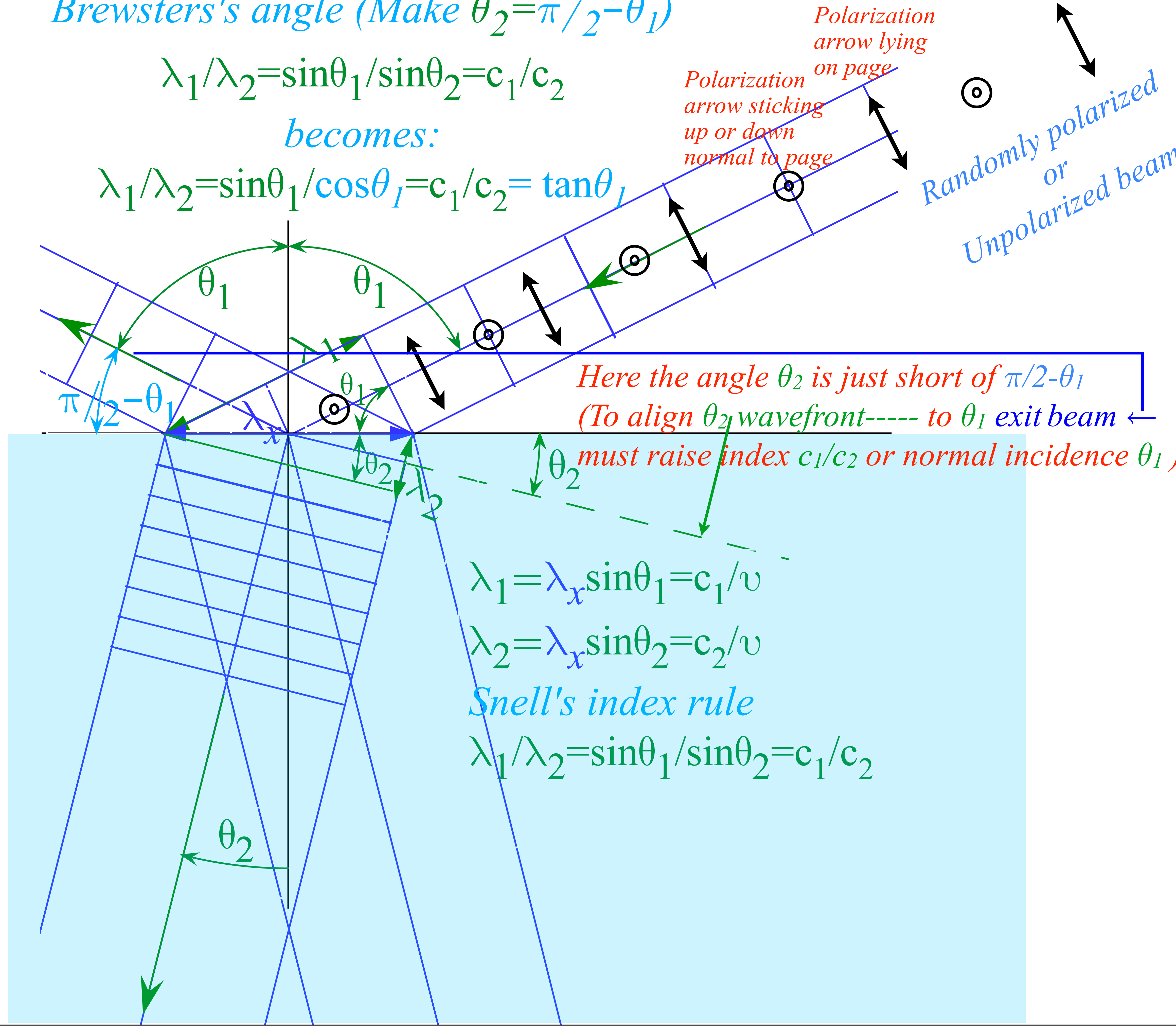
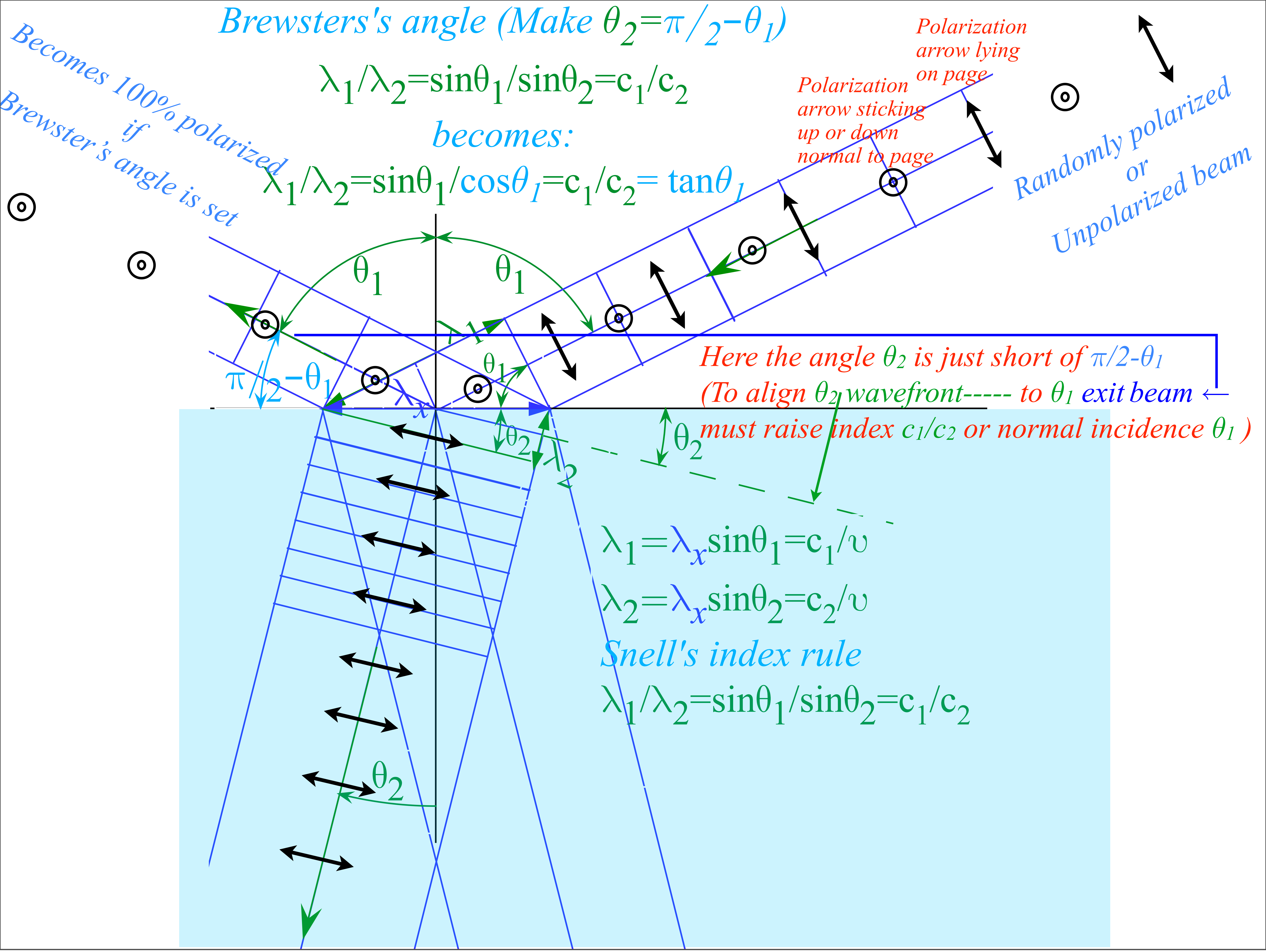


$$\lambda_1=\lambda_x\sin\theta_1=c_1/\upsilon$$

$$\lambda_2=\lambda_x\sin\theta_2=c_2/\upsilon$$

Snell's index rule

$$\lambda_1/\lambda_2=\sin\theta_1/\sin\theta_2=c_1/c_2$$


Brewsters's angle (Make θ2=π/2−θ1)
λ1/λ2=sinθ1/sinθ2=c1/c2
becomes:
λ1/λ2=sinθ1/cosθ1=c1/c2= tanθ1
Becomes 100% polarized if Brewster's angle is set
Polarization arrow lying on page
Polarization arrow sticking up or down normal to page
Randomly polarized or Unpolarized beam
θ1
θ1
λ1
θ1
π/2−θ1
λx
θ2
λ2
θ2
Here the angle θ2 is just short of π/2-θ1
(To align θ2 wavefront----- to θ1 exit beam ←
must raise index c1/c2 or normal incidence θ1)
λ1=λxsinθ1=c1/υ
λ2=λxsinθ2=c2/υ
Snell's index rule
λ1/λ2=sinθ1/sinθ2=c1/c2
θ2

Beam Sorters - Optical polarization sorting

2-State Sorters: spin-1/2 vs. optical polarization

→ *Beam Sorters in Series and Transformation Matrices*

Introducing Dirac bra-ket notation

"Abstraction" of bra and ket vectors from a Transformation Matrix

Introducing scalar and matrix products

Beam Sorters in Series and Transformation Matrices

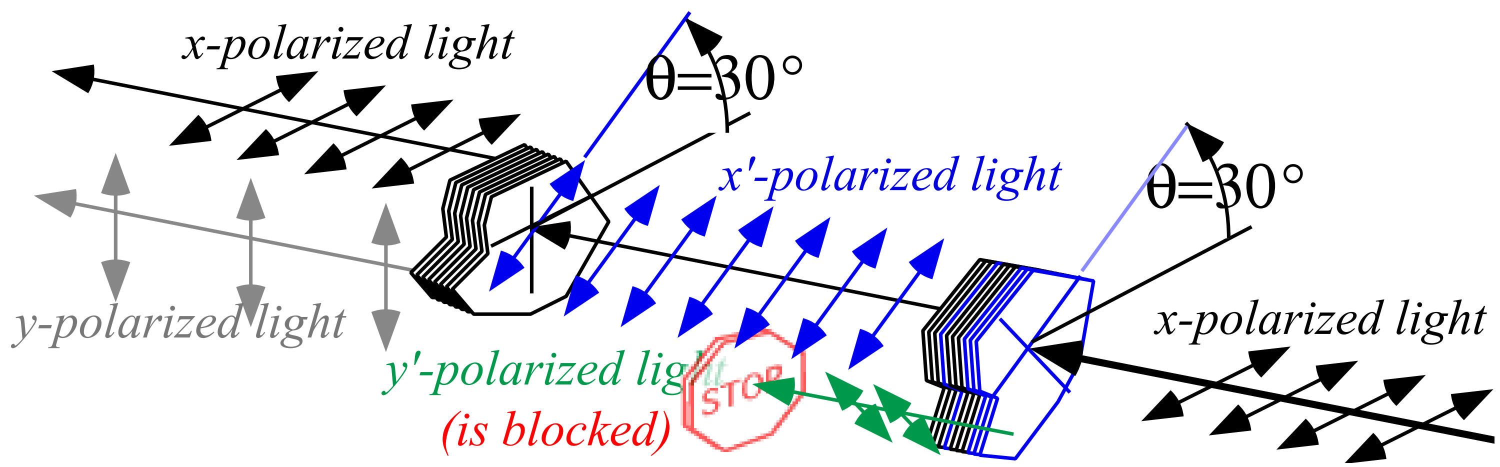


Fig. 1.2.1 Photon beam sorters in series with the first one y-blocked and tilted by angle θ=30°.

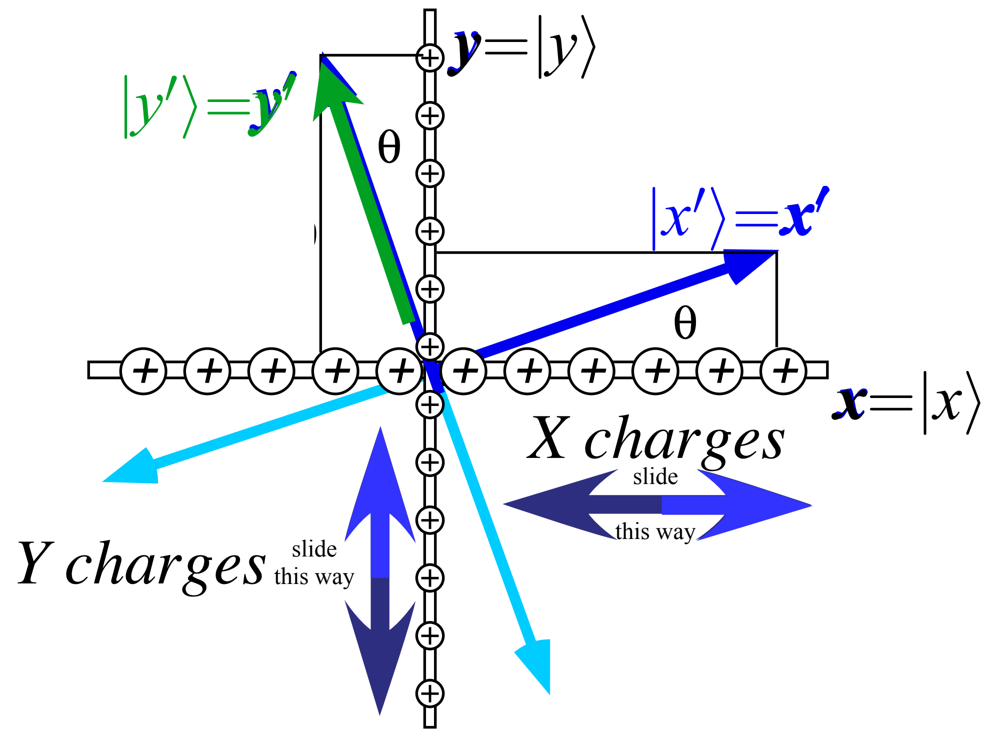


Fig. 1.2.2 Geometry of photon beam sorter for input polarizations (x',y') tilted by angle θ [relative to (*x*,*y*)].

Beam Sorters in Series and Transformation Matrices

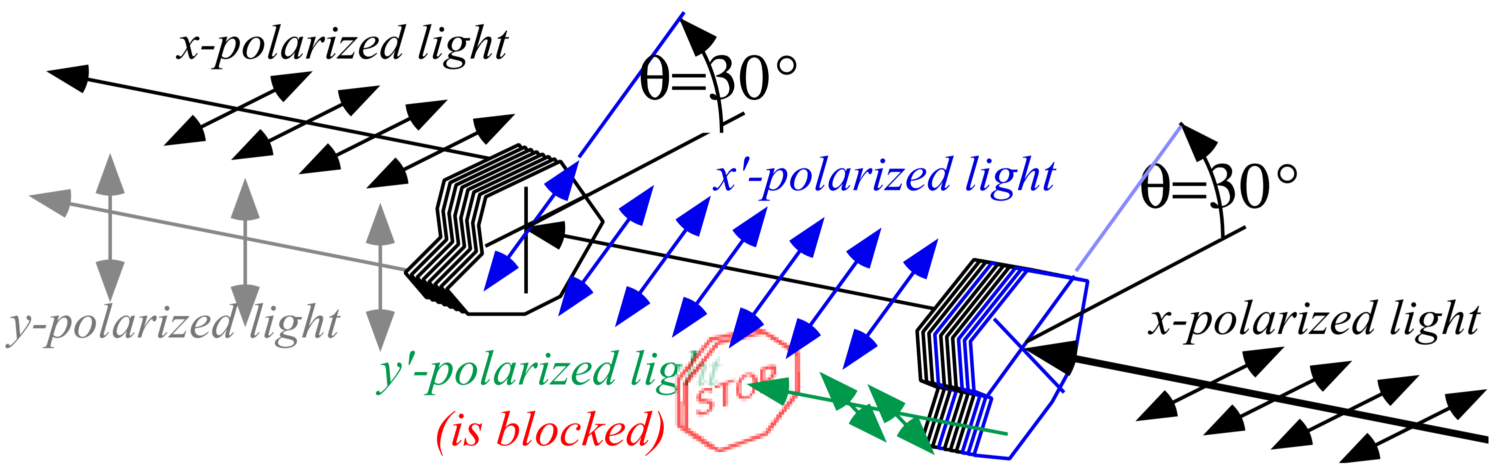


Fig. 1.2.1 Photon beam sorters in series with the first one y-blocked and tilted by angle θ=30°.

If we y′-blocked and let x′ through: *(to be (x,y)-analyzed*

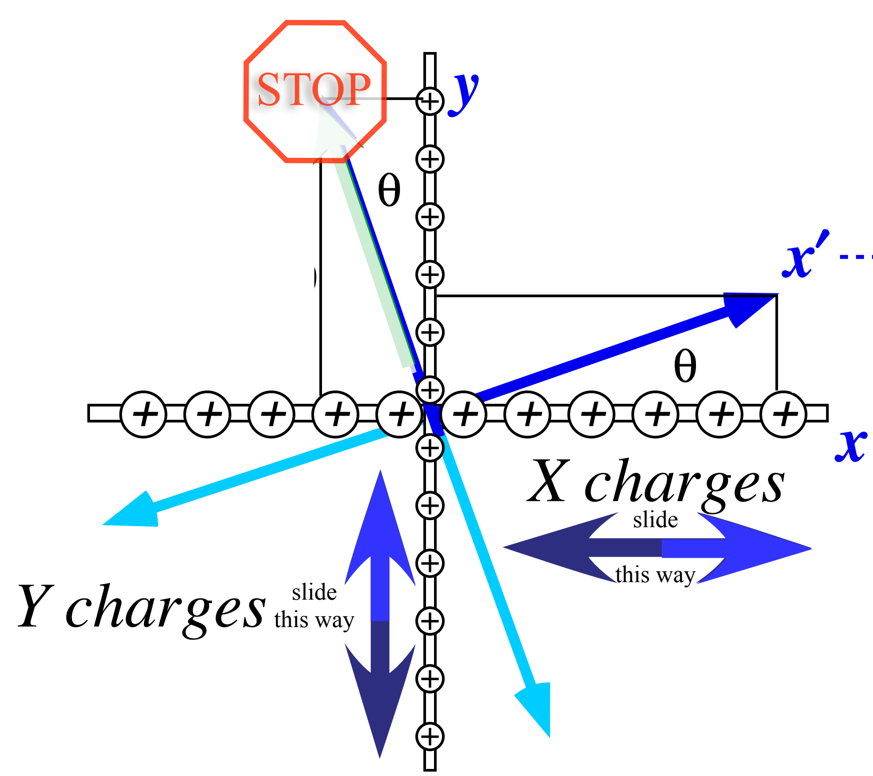


Fig. 1.2.2 Geometry of photon beam sorter for input polarizations (x′,y′) tilted by angle θ [relative to (*x*,*y*)].

Beam Sorters in Series and Transformation Matrices

Feynman-Dirac Interpretation of

$$\langle m | n' \rangle$$

=Amplitude of state-m after state-n′ is forced to choose from available m-type states

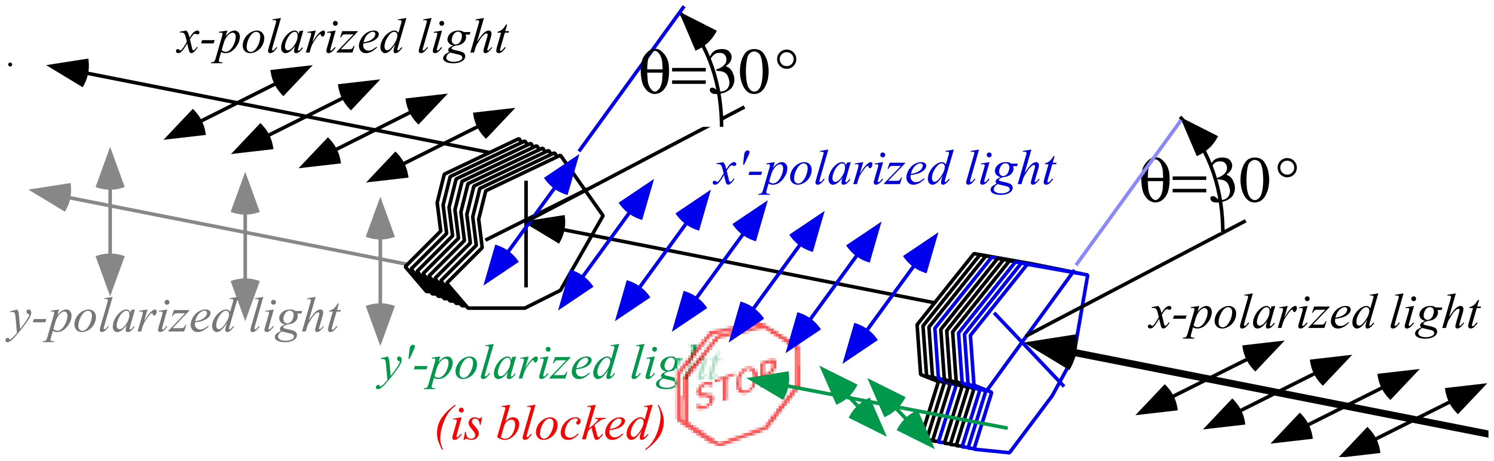
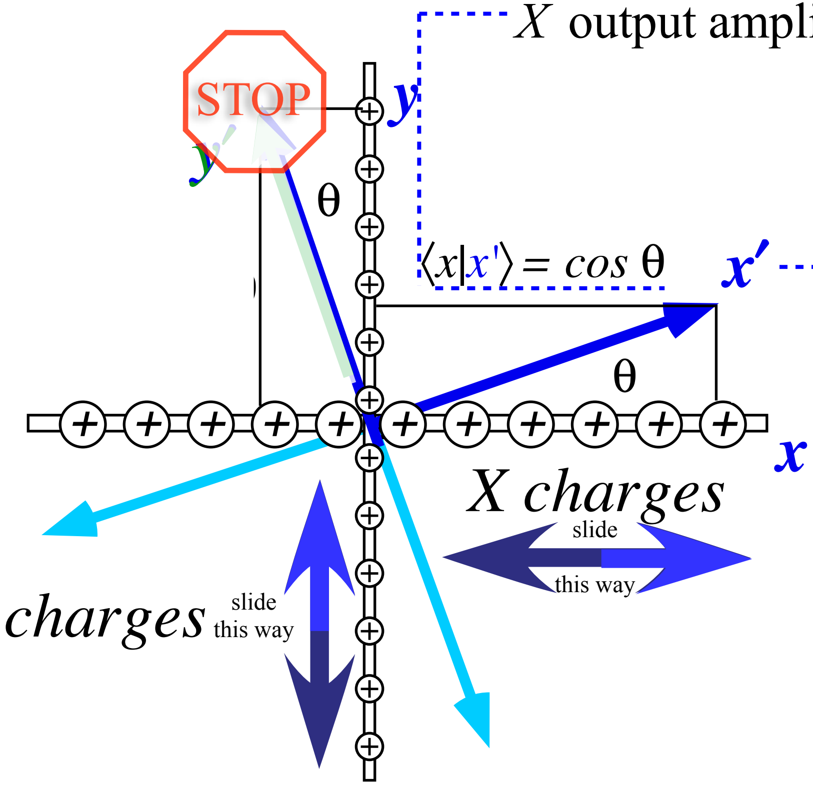


Fig. 1.2.1 Photon beam sorters in series with the first one y-blocked and tilted by angle θ=30°.

Introducing Dirac bra-ket notation.

If we y′-blocked and let x′ through: *(to be (x,y)-analyzed*

$$X \text{ output amplitude due to } x' \text{ input} = \langle x | x' \rangle = \cos\theta$$



$sin\theta = sin30° = 1/2 = 0.5$

$cos\theta = cos30° = \sqrt{3}/2 = 0.866$

Fig. 1.2.2 Geometry of photon beam sorter for input polarizations (x′,y′) tilted by angle θ [relative to (x,y)].

Beam Sorters in Series and Transformation Matrices

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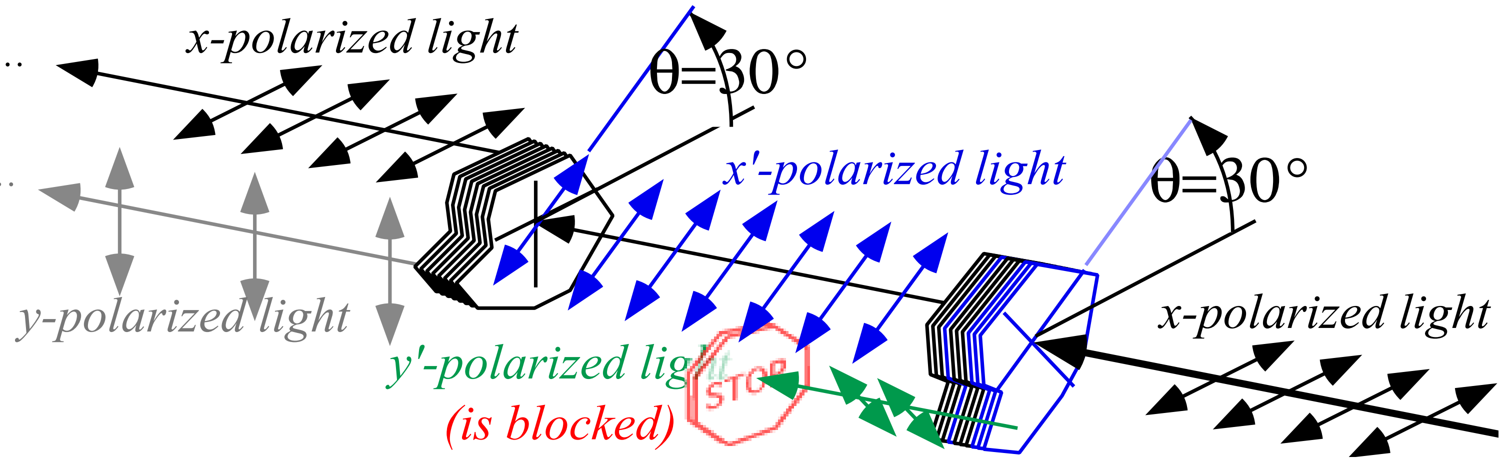

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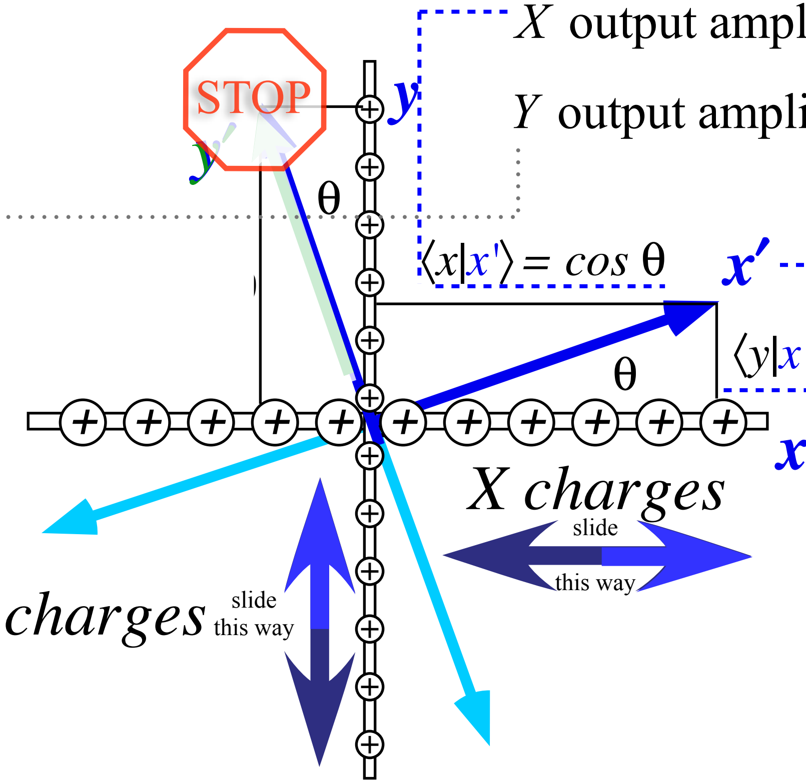
If we y′-blocked and let x′ through: *(to be (x,y)-analyzed*

$$X \text{ output amplitude due to } x' \text{ input} = \langle x | x' \rangle = \cos\theta,$$

$$Y \text{ output amplitude due to } x' \text{ input} = \langle y | x' \rangle = \sin\theta$$



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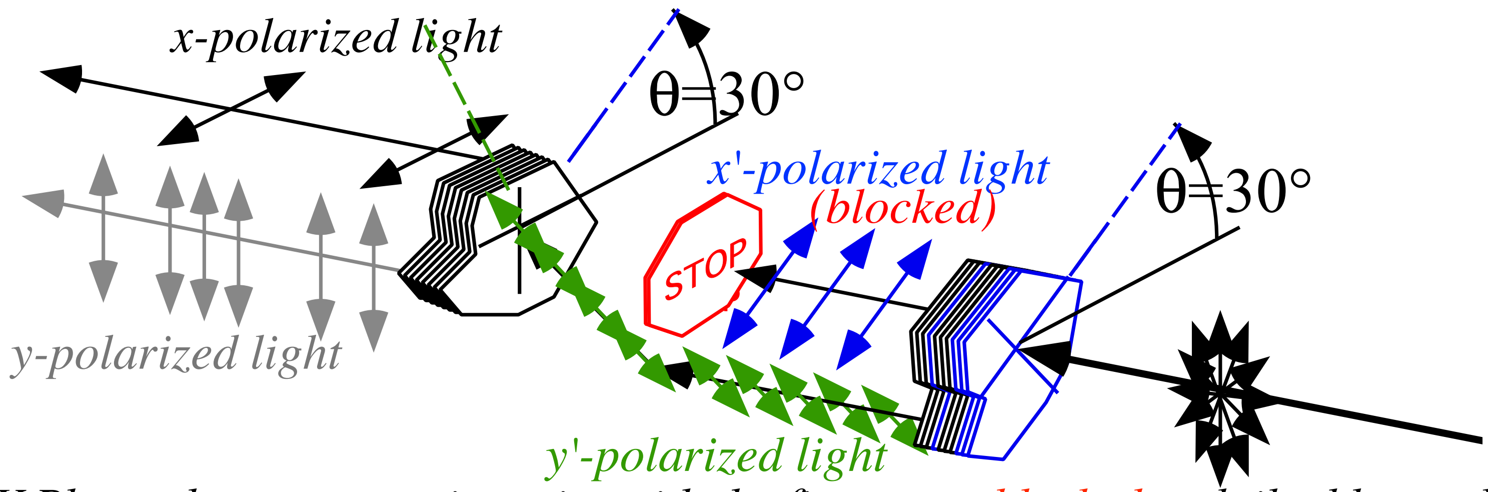

Fig. 1.2.X Photon beam sorters in series with the first one x-blocked and tilted by angle θ=30°.

Introducing Dirac bra-ket notation.

If we x′-blocked and let y′ through instead:

If we y′-blocked and let x′ through:

X output amplitude due to x' input $=\langle x|x'\rangle=\cos\theta,$

Y output amplitude due to x' input $=\langle y|x'\rangle=\sin\theta$

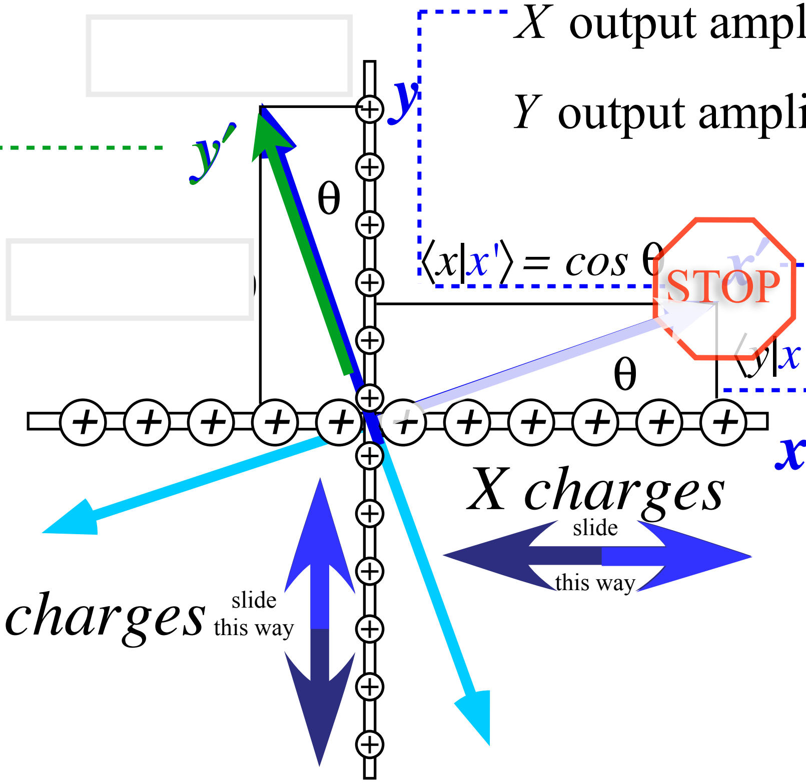

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Fig. 1.2.2 Geometry of photon beam sorter for input polarizations (x′,y′) tilted by angle θ [relative to (x,y)].

Beam Sorters in Series and Transformation Matrices

Feynman-Dirac Interpretation of

$$\langle m | n' \rangle$$

=Amplitude of state-m after state-n′ is forced to choose from available m-type states

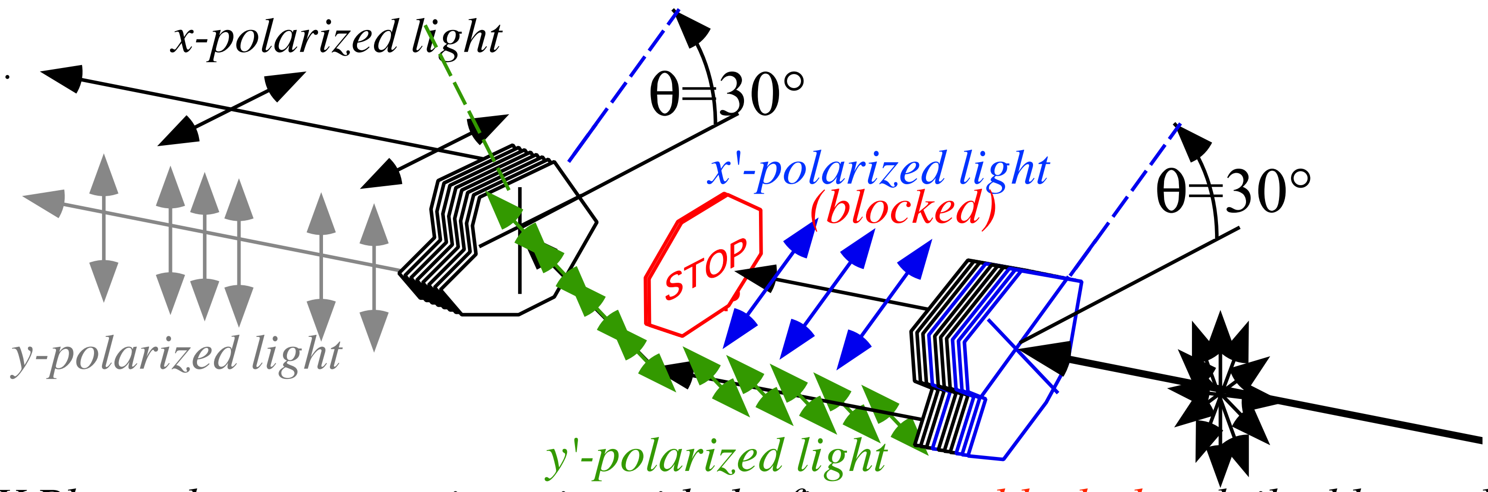

Fig. 1.2.X Photon beam sorters in series with the first one x-blocked and tilted by angle θ=30°.

Introducing Dirac bra-ket notation.

If we x′-blocked and let y′ through instead:

X output amplitude due to y' input $= \langle x | y' \rangle = -\sin\theta$

If we y′-blocked and let x′ through:

X output amplitude due to x' input $= \langle x | x' \rangle = \cos\theta$,

Y output amplitude due to x' input $= \langle y | x' \rangle = \sin\theta$

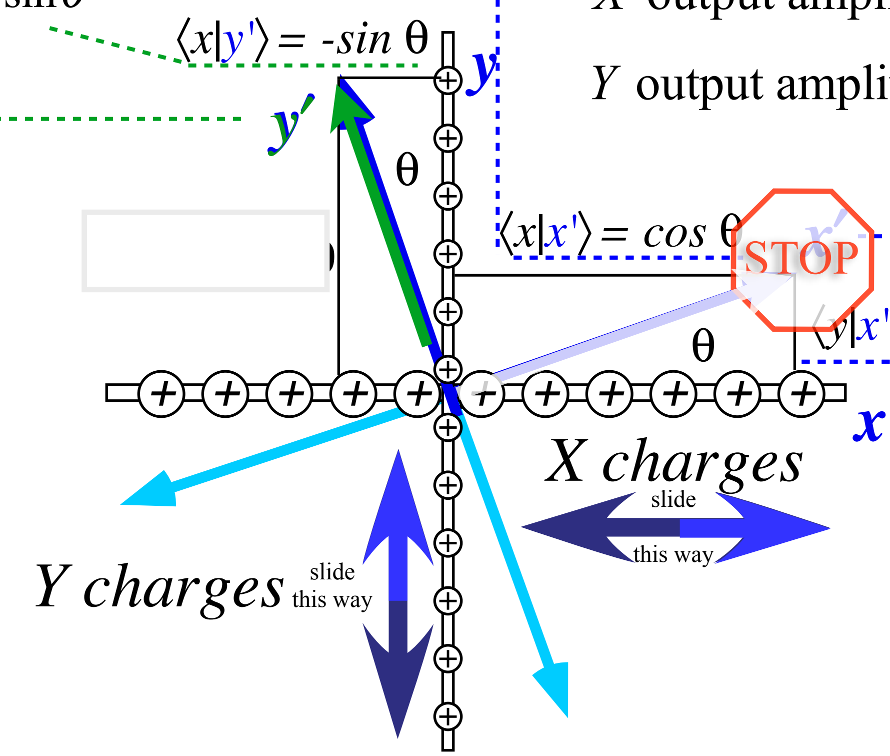

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Fig. 1.2.2 Geometry of photon beam sorter for input polarizations (x′,y′) tilted by angle θ [relative to (x,y)].

Beam Sorters in Series and Transformation Matrices

Feynman-Dirac Interpretation of

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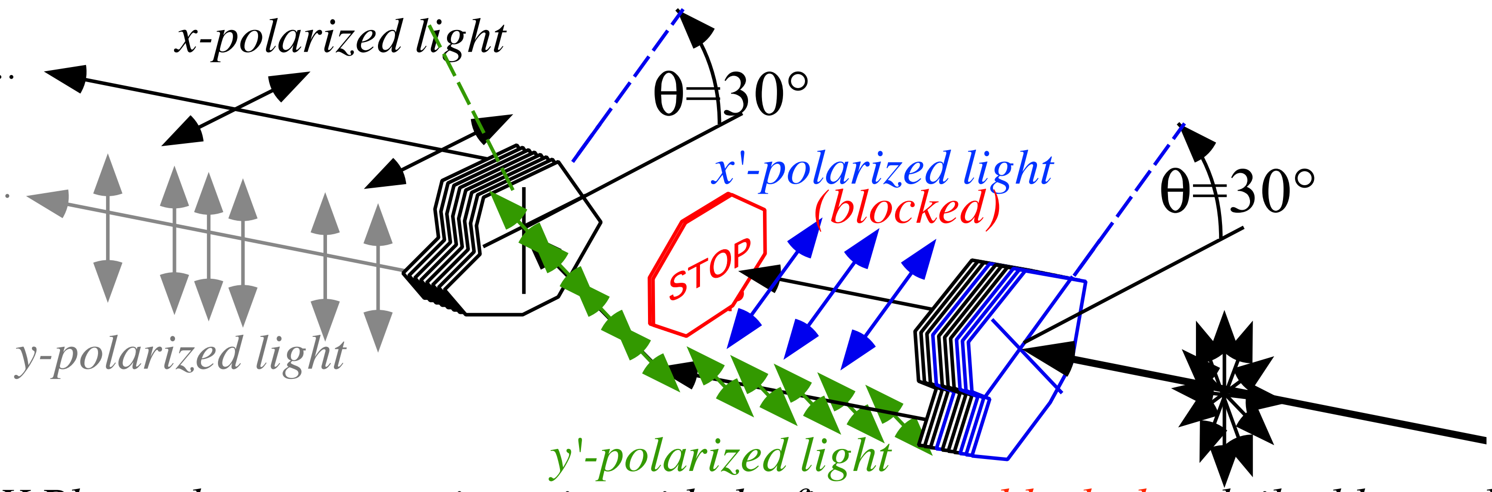


Fig. 1.2.X Photon beam sorters in series with the first one x-blocked and tilted by angle θ=30°.

Introducing Dirac bra-ket notation.

If we x′-blocked and let y′ through instead:

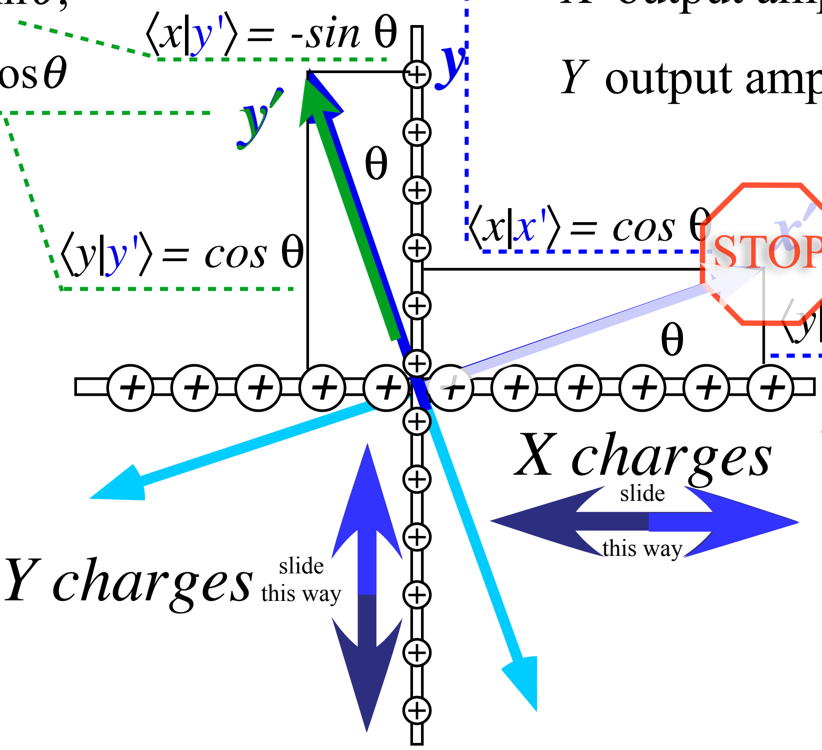
X output amplitude due to y' input $= \langle x | y' \rangle = -\sin\theta,$

Y output amplitude due to y' input $= \langle y | y' \rangle = \cos\theta$

If we y′-blocked and let x′ through:

X output amplitude due to x' input $= \langle x | x' \rangle = \cos\theta,$

Y output amplitude due to x' input $= \langle y | x' \rangle = \sin\theta$



$sin\theta = sin 30° = 1/2 = 0.5$

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Fig. 1.2.2 Geometry of photon beam sorter for input polarizations (x′,y′) tilted by angle θ [relative to (x,y)].

Beam Sorters in Series and Transformation Matrices

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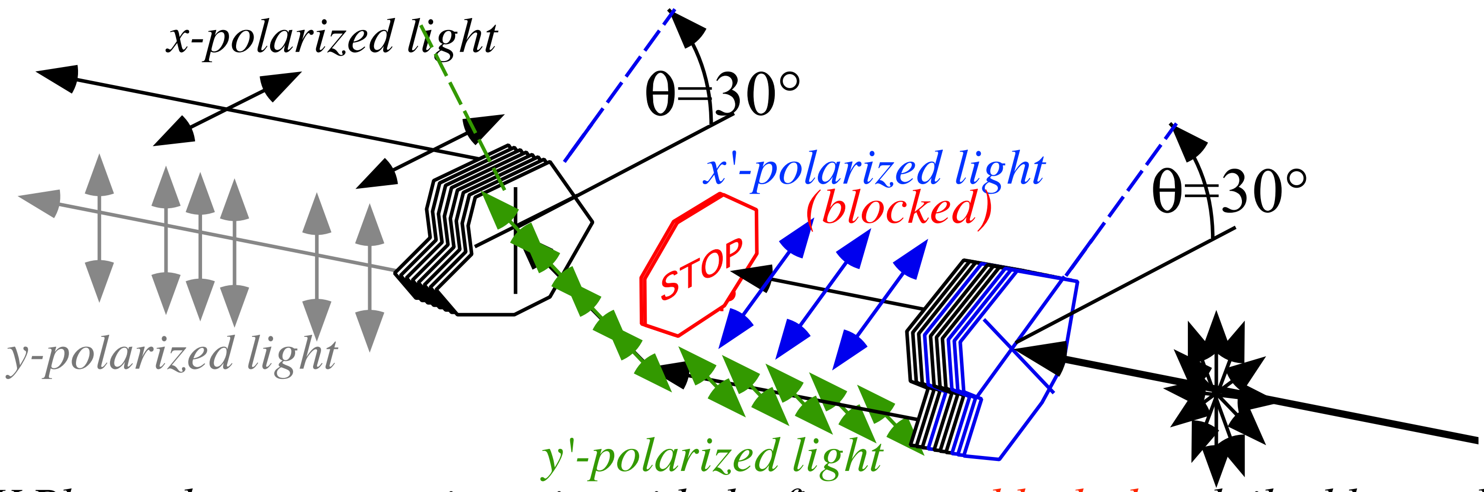

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Introducing Dirac bra-ket notation.

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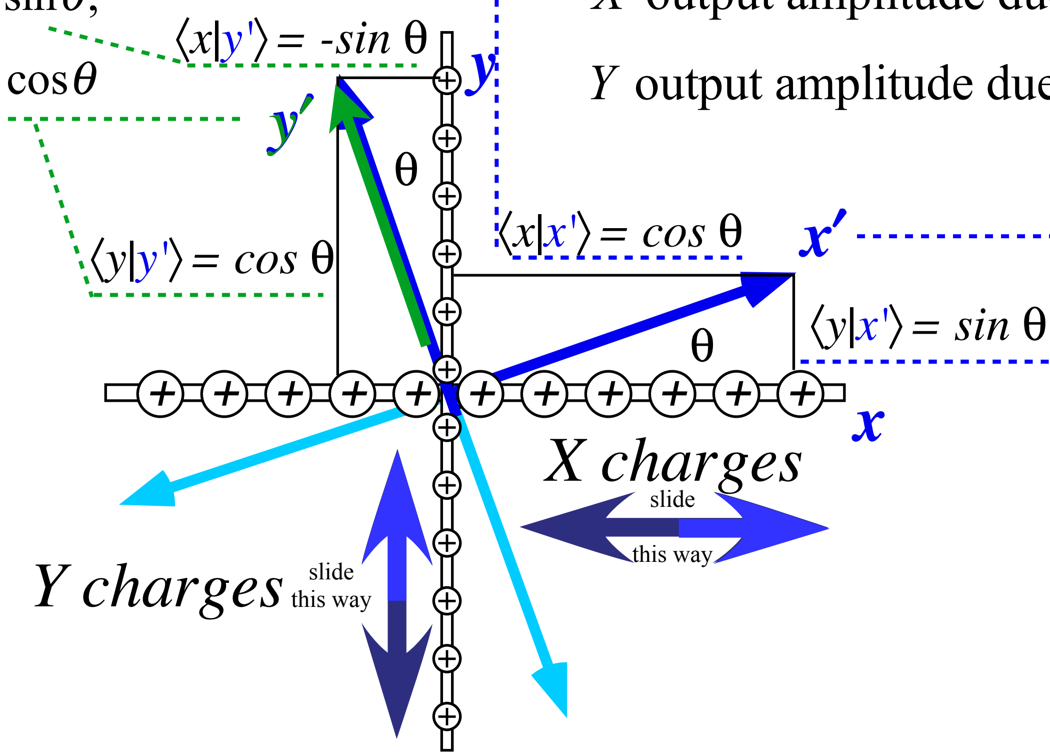

$\sin\theta = \sin 30° = 1/2 = 0.5$

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Fig. 1.2.2 Geometry of photon beam sorter for input polarizations (x′,y′) tilted by angle θ [relative to (x,y)].

$$\begin{pmatrix} \langle x | x' \rangle & \langle x | y' \rangle \\ \langle y | x' \rangle & \langle y | y' \rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Introducing bra-ket Transformation Matrix

$$T_{m,n'} = \langle m | n' \rangle$$

Beam Sorters - Optical polarization sorting

2-State Sorters: spin-1/2 vs. optical polarization

Beam Sorters in Series and Transformation Matrices

Introducing Dirac bra-ket notation

➡ *"Abstraction" of bra and ket vectors from a Transformation Matrix*

Introducing scalar and matrix products

"Abstraction" of bra and ket vectors from a Transformation Matrix

Ket or column vectors

Bra or row vectors

Given Transformation Matrix $T_{m,n'}$:

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

"Abstraction" of bra and ket vectors from a Transformation Matrix

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Abstracting ket $|\, n'\rangle$ state vectors
from
Transformation Matrix
$T_{m,n'} = \langle m|\, n'\rangle$

"Abstraction" of bra and ket vectors from a Transformation Matrix

Ket or column vectors

Bra or row vectors

Given Transformation Matrix $T_{m,n'}$:

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$\Downarrow$

$$|x'\rangle = \begin{pmatrix} \langle x|x'\rangle \\ \langle y|x'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

Abstracting ket $|n'\rangle$ *state vectors*

from

Transformation Matrix

$$T_{m,n'} = \langle m|n'\rangle$$

"Abstraction" of bra and ket vectors from a Transformation Matrix

Ket or column vectors

Bra or row vectors

Given Transformation Matrix $T_{m,n'}$:

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\Downarrow \qquad \Downarrow$$

$$|x'\rangle = \begin{pmatrix} \langle x|x'\rangle \\ \langle y|x'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad |y'\rangle = \begin{pmatrix} \langle x|y'\rangle \\ \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

Abstracting ket $|n'\rangle$ state vectors
from
Transformation Matrix
$T_{m,n'} = \langle m|n'\rangle$

"Abstraction" of bra and ket vectors from a Transformation Matrix

Ket or column vectors

Bra or row vectors

Given Transformation Matrix $T_{m,n'}$:

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

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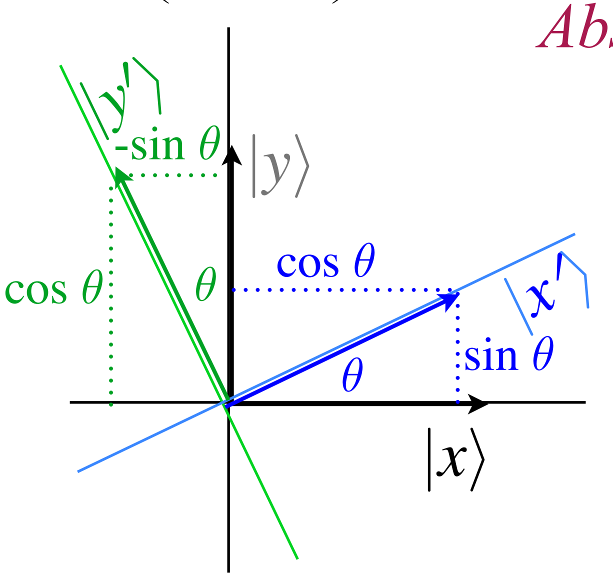


Abstracting ket $|n'\rangle$ state vectors
from
Transformation Matrix
$T_{m,n'} = \langle m|n'\rangle$

(θ=+30°)-Rotated kets $\{|\mathbf{x}'\rangle, |\mathbf{y}'\rangle\}$ or $\{\mathbf{x}', \mathbf{y}'\}$
represented in page-aligned $\{|\mathrm{x}\rangle, |\mathrm{y}\rangle\}$ basis.

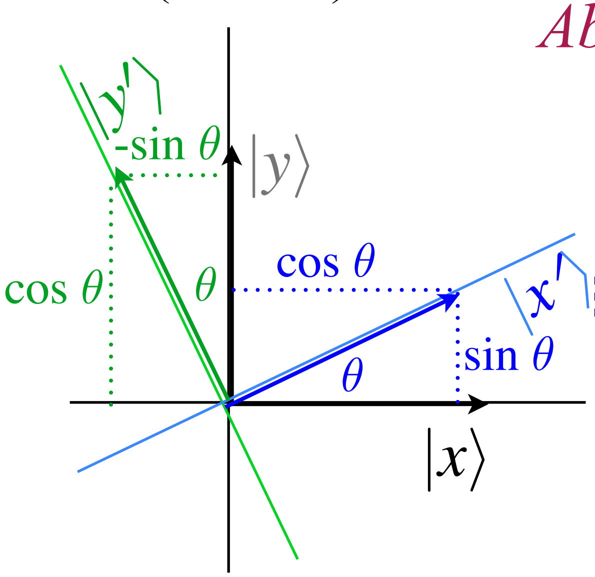
"Abstraction" of bra and ket vectors from a Transformation Matrix

Ket or column vectors *Bra or row vectors*

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow \qquad \Downarrow$$

$$|x'\rangle = \begin{pmatrix} \langle x|x'\rangle \\ \langle y|x'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad |y'\rangle = \begin{pmatrix} \langle x|y'\rangle \\ \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$



Abstracting ket $|n'\rangle$ *state vectors*
from
Transformation Matrix
$T_{m,n'} = \langle m|n'\rangle$

$$|x'\rangle = |x\rangle\langle x|x'\rangle + |y\rangle\langle y|x'\rangle$$
$$= |x\rangle(\cos\theta) + |y\rangle(\sin\theta)$$

($\theta=+30°$)-Rotated kets $\{|\mathbf{x}'\rangle, |\mathbf{y}'\rangle\}$ or $\{\mathbf{x}', \mathbf{y}'\}$
represented in page-aligned $\{|\mathrm{x}\rangle, |\mathrm{y}\rangle\}$ basis.

"Abstraction" of bra and ket vectors from a Transformation Matrix

Ket or column vectors | *Bra or row vectors*

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

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Abstracting ket $|n'\rangle$ state vectors
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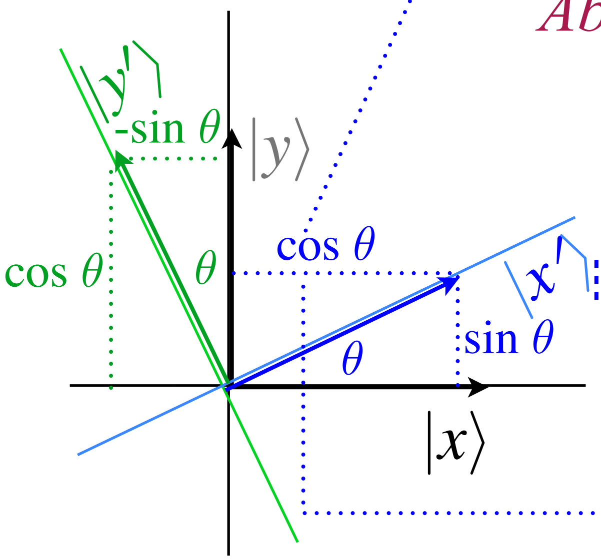
Transformation Matrix
$T_{m,n'} = \langle m|n'\rangle$



$$|x'\rangle = |x\rangle\langle x|x'\rangle + |y\rangle\langle y|x'\rangle$$
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($\theta=+30°$)-Rotated kets $\{|\mathbf{x}'\rangle, |\mathbf{y}'\rangle\}$ or $\{\mathbf{x}', \mathbf{y}'\}$
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"Abstraction" of bra and ket vectors from a Transformation Matrix

Ket or column vectors | *Bra or row vectors*

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow \qquad \Downarrow$$

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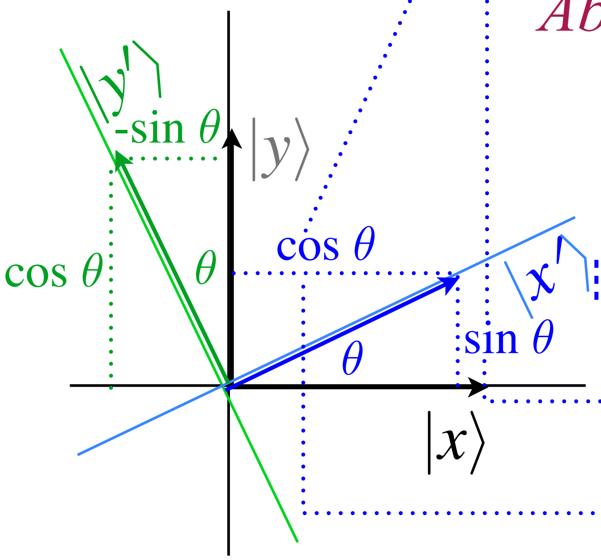

Abstracting ket $|n'\rangle$ *state vectors*
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Transformation Matrix
$T_{m,n'} = \langle m|n'\rangle$

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“Abstraction” of bra and ket vectors from a Transformation Matrix

Ket or column vectors

Bra or row vectors

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

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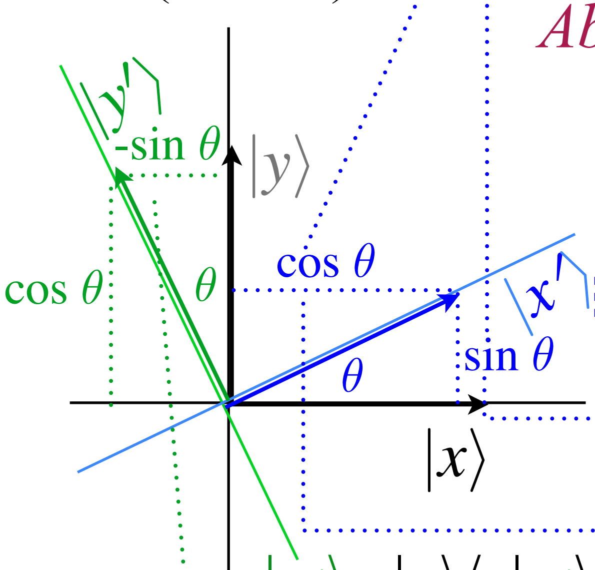
Abstracting ket $|n'\rangle$ state vectors
from
Transformation Matrix
$T_{m,n'} = \langle m|n'\rangle$



$$|x'\rangle = |x\rangle\langle x|x'\rangle + |y\rangle\langle y|x'\rangle$$
$$= |x\rangle(\cos\theta) + |y\rangle(\sin\theta)$$

$$|y'\rangle = |x\rangle\langle x|y'\rangle + |y\rangle\langle y|y'\rangle$$
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($\theta = +30°$)-Rotated kets $\{|\mathbf{x}'\rangle, |\mathbf{y}'\rangle\}$ or $\{\mathbf{x}', \mathbf{y}'\}$ represented in page-aligned $\{|\mathrm{x}\rangle, |\mathrm{y}\rangle\}$ basis.

"Abstraction" of bra and ket vectors from a Transformation Matrix

Ket or column vectors | *Bra or row vectors*

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\Downarrow \qquad \Downarrow$$

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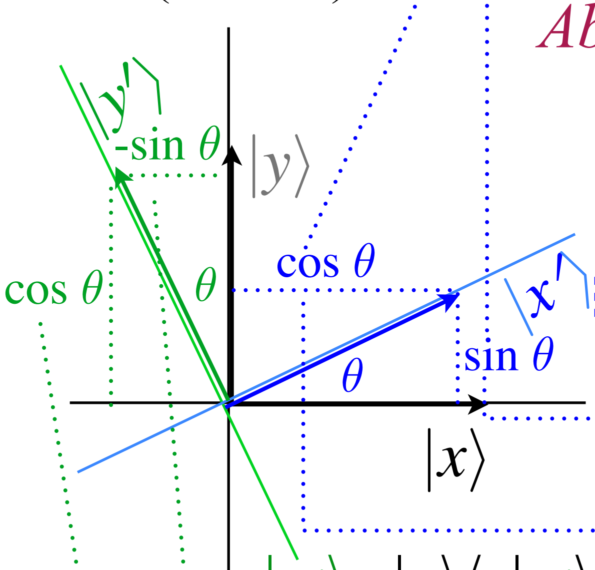
Abstracting ket $|n'\rangle$ state vectors
from
Transformation Matrix
$T_{m,n'} = \langle m|n'\rangle$



$$|x'\rangle = |x\rangle\langle x|x'\rangle + |y\rangle\langle y|x'\rangle$$
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($\theta=+30°$)-Rotated kets $\{|\mathbf{x}'\rangle, |\mathbf{y}'\rangle\}$ or $\{\mathbf{x}', \mathbf{y}'\}$
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"Abstraction" of bra and ket vectors from a Transformation Matrix

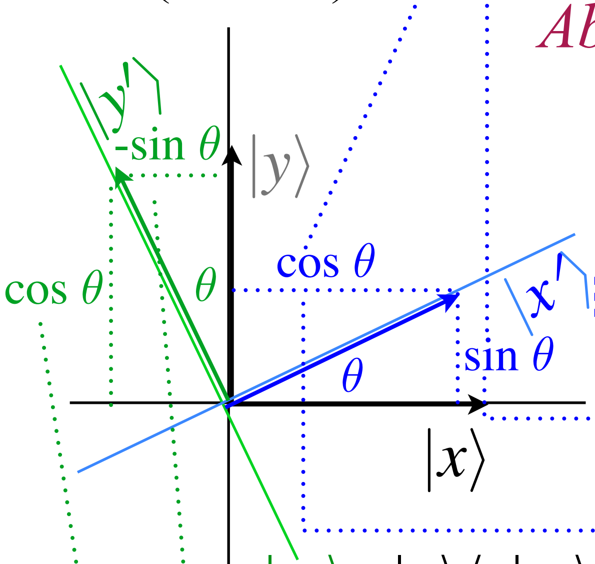
Ket or column vectors — *Bra or row vectors*

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$|x'\rangle = \begin{pmatrix} \langle x|x'\rangle \\ \langle y|x'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad |y'\rangle = \begin{pmatrix} \langle x|y'\rangle \\ \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

Abstracting ket $|n'\rangle$ state vectors from Transformation Matrix

$$T_{m,n'} = \langle m|n'\rangle$$



$$|x'\rangle = |x\rangle\langle x|x'\rangle + |y\rangle\langle y|x'\rangle = |x\rangle(\cos\theta) + |y\rangle(\sin\theta)$$

$$|y'\rangle = |x\rangle\langle x|y'\rangle + |y\rangle\langle y|y'\rangle = |x\rangle(-\sin\theta) + |y\rangle(\cos\theta)$$

($\theta=+30°$)-Rotated kets $\{|\mathbf{x}'\rangle, |\mathbf{y}'\rangle\}$ or $\{\mathbf{x}', \mathbf{y}'\}$ represented in page-aligned $\{|\mathbf{x}\rangle, |\mathbf{y}\rangle\}$ basis.

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$$= \mathbf{x}(\cos\theta) + \mathbf{y}(\sin\theta), \qquad = \mathbf{x}(-\sin\theta) + \mathbf{y}(\cos\theta).$$

"Abstraction" of bra and ket vectors from a Transformation Matrix

Ket or column vectors

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$|x'\rangle = \begin{pmatrix} \langle x|x'\rangle \\ \langle y|x'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad |y'\rangle = \begin{pmatrix} \langle x|y'\rangle \\ \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

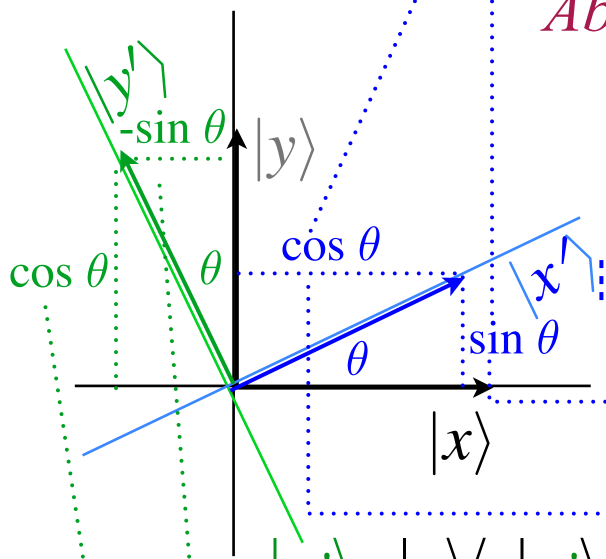
Abstracting ket $|n'\rangle$ state vectors from Transformation Matrix $T_{m,n'} = \langle m|n'\rangle$

Bra or row vectors

$$\langle x| = \begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \end{pmatrix}$$

$$\langle y| = \begin{pmatrix} \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \sin\theta & \cos\theta \end{pmatrix}$$

Abstracting bra $\langle m|$ state vectors from Transformation Matrix $T_{m,n'} = \langle m|n'\rangle$



$$|x'\rangle = |x\rangle\langle x|x'\rangle + |y\rangle\langle y|x'\rangle$$
$$= |x\rangle(\cos\theta) + |y\rangle(\sin\theta)$$

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"Abstraction" of bra and ket vectors from a Transformation Matrix

Ket or column vectors

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

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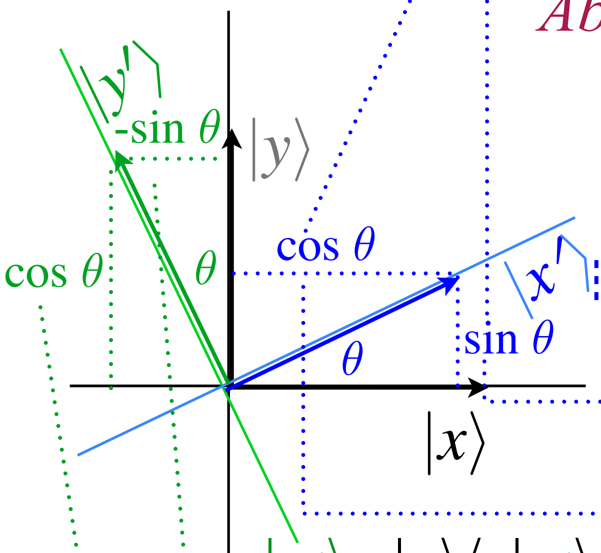
Abstracting ket $|n'\rangle$ state vectors from Transformation Matrix $T_{m,n'} = \langle m|n'\rangle$



$$|x'\rangle = |x\rangle\langle x|x'\rangle + |y\rangle\langle y|x'\rangle = |x\rangle(\cos\theta) + |y\rangle(\sin\theta)$$

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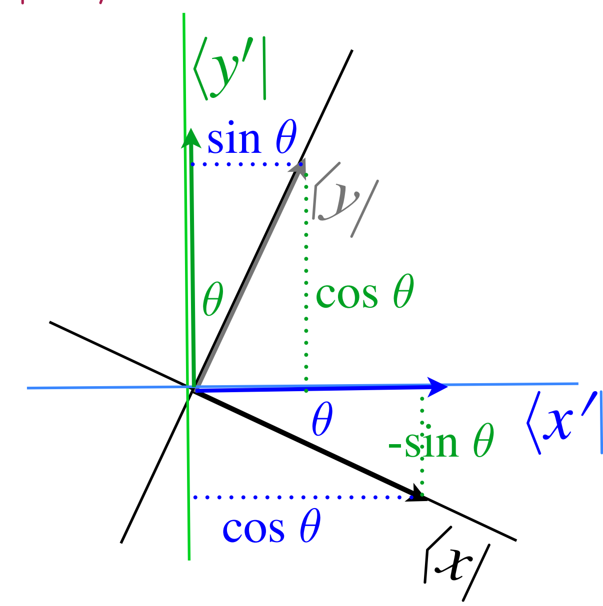
(θ=+30°)-Rotated kets $\{|\mathbf{x}'\rangle, |\mathbf{y}'\rangle\}$ or $\{\mathbf{x}', \mathbf{y}'\}$ represented in page-aligned $\{|\mathbf{x}\rangle, |\mathbf{y}\rangle\}$ basis.

Bra or row vectors

$$\langle x| = \begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \end{pmatrix}$$

$$\langle y| = \begin{pmatrix} \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \sin\theta & \cos\theta \end{pmatrix}$$

Abstracting bra $\langle m|$ state vectors from Transformation Matrix $T_{m,n'} = \langle m|n'\rangle$



(θ=-30°)-Rotated bras $\{\langle\mathbf{x}|, \langle\mathbf{y}|\}$ or $\{\mathbf{x}, \mathbf{y}\}$ represented in page-aligned $\{|\mathbf{x}'\rangle, |\mathbf{y}'\rangle\}$ basis.

The same thing in Gibbs vector notation:

$$\mathbf{x}' = \mathbf{x}(\mathbf{x}\bullet\mathbf{x}') + \mathbf{y}(\mathbf{y}\bullet\mathbf{x}'), \quad \mathbf{y}' = \mathbf{x}(\mathbf{x}\bullet\mathbf{y}') + \mathbf{y}(\mathbf{y}\bullet\mathbf{y}'),$$

$$= \mathbf{x}(\cos\theta) + \mathbf{y}(\sin\theta), \qquad = \mathbf{x}(-\sin\theta) + \mathbf{y}(\cos\theta).$$

"Abstraction" of bra and ket vectors from a Transformation Matrix

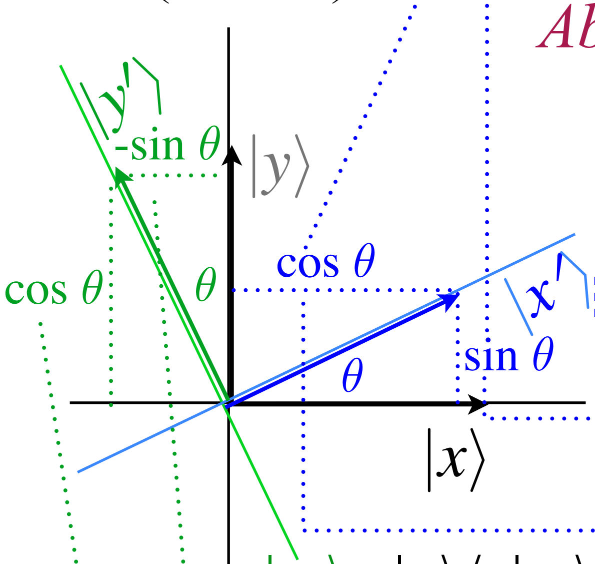
Ket or column vectors

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$|x'\rangle = \begin{pmatrix} \langle x|x'\rangle \\ \langle y|x'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad |y'\rangle = \begin{pmatrix} \langle x|y'\rangle \\ \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

Abstracting ket $|n'\rangle$ state vectors from Transformation Matrix

$$T_{m,n'} = \langle m|n'\rangle$$



$$|x'\rangle = |x\rangle\langle x|x'\rangle + |y\rangle\langle y|x'\rangle$$
$$= |x\rangle(\cos\theta) + |y\rangle(\sin\theta)$$

$$|y'\rangle = |x\rangle\langle x|y'\rangle + |y\rangle\langle y|y'\rangle$$
$$= |x\rangle(-\sin\theta) + |y\rangle(\cos\theta)$$

(θ=+30°)-Rotated kets $\{|\mathbf{x}'\rangle, |\mathbf{y}'\rangle\}$ or $\{\mathbf{x}', \mathbf{y}'\}$ represented in page-aligned $\{|\mathbf{x}\rangle, |\mathbf{y}\rangle\}$ basis.

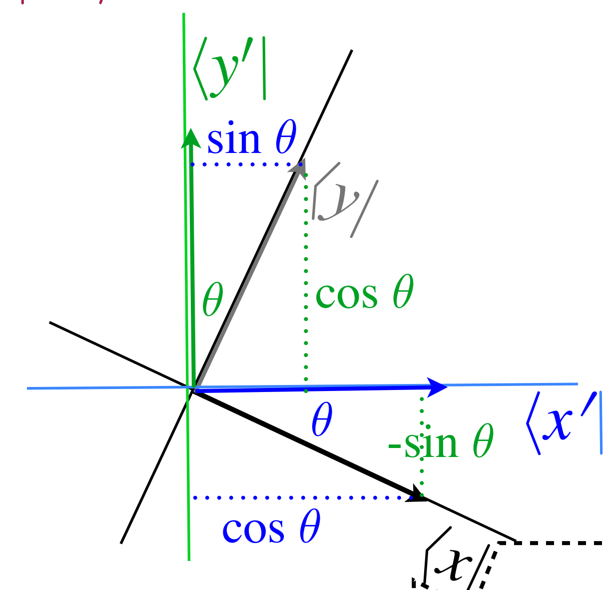
Bra or row vectors

$$\langle x| = \begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \end{pmatrix}$$

$$\langle y| = \begin{pmatrix} \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \sin\theta & \cos\theta \end{pmatrix}$$

Abstracting bra $\langle m|$ state vectors from Transformation Matrix

$$T_{m,n'} = \langle m|n'\rangle$$



$$\langle x| = \langle x|x'\rangle\langle x'| + \langle x|y'\rangle\langle y'|$$
$$= (\cos\theta)\langle x'| + (-\sin\theta)\langle y'|$$

(θ=-30°)-Rotated bras $\{\langle\mathbf{x}|, \langle\mathbf{y}|\}$ or $\{\mathbf{x}, \mathbf{y}\}$ represented in page-aligned $\{|\mathbf{x}'\rangle, |\mathbf{y}'\rangle\}$ basis.

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$$\mathbf{x}' = \mathbf{x}(\mathbf{x}\bullet\mathbf{x}') + \mathbf{y}(\mathbf{y}\bullet\mathbf{x}'), \quad \mathbf{y}' = \mathbf{x}(\mathbf{x}\bullet\mathbf{y}') + \mathbf{y}(\mathbf{y}\bullet\mathbf{y}'),$$
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"Abstraction" of bra and ket vectors from a Transformation Matrix

Ket or column vectors

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

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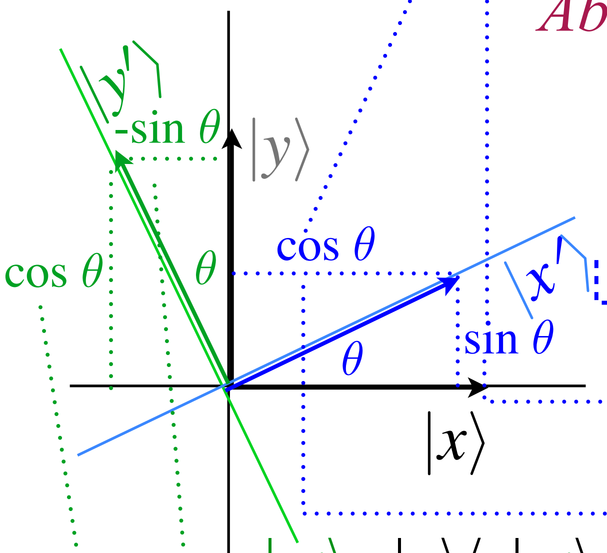
Abstracting ket $|n'\rangle$ state vectors from Transformation Matrix

$T_{m,n'} = \langle m|n'\rangle$



$$|x'\rangle = |x\rangle\langle x|x'\rangle + |y\rangle\langle y|x'\rangle = |x\rangle(\cos\theta) + |y\rangle(\sin\theta)$$

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Bra or row vectors

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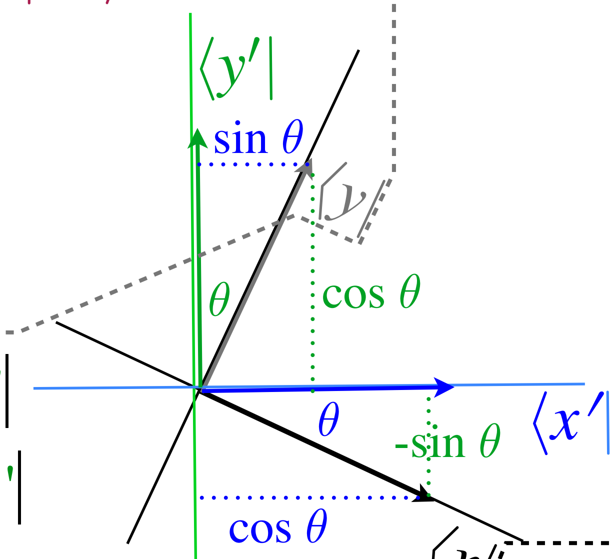
Abstracting bra $\langle m|$ state vectors from Transformation Matrix

$T_{m,n'} = \langle m|n'\rangle$



$$\langle y| = \langle y|x'\rangle\langle x'| + \langle y|y'\rangle\langle y'| = (\sin\theta)\langle x'| + (\cos\theta)\langle y'|$$

$$\langle x| = \langle x|x'\rangle\langle x'| + \langle x|y'\rangle\langle y'| = (\cos\theta)\langle x'| + (-\sin\theta)\langle y'|$$

(θ=+30°)-Rotated kets $\{|\mathbf{x}'\rangle, |\mathbf{y}'\rangle\}$ or $\{\mathbf{x}', \mathbf{y}'\}$ represented in page-aligned $\{|\mathbf{x}\rangle, |\mathbf{y}\rangle\}$ basis.

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"Abstraction" of bra and ket vectors from a Transformation Matrix

Ket or column vectors

Bra or row vectors

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

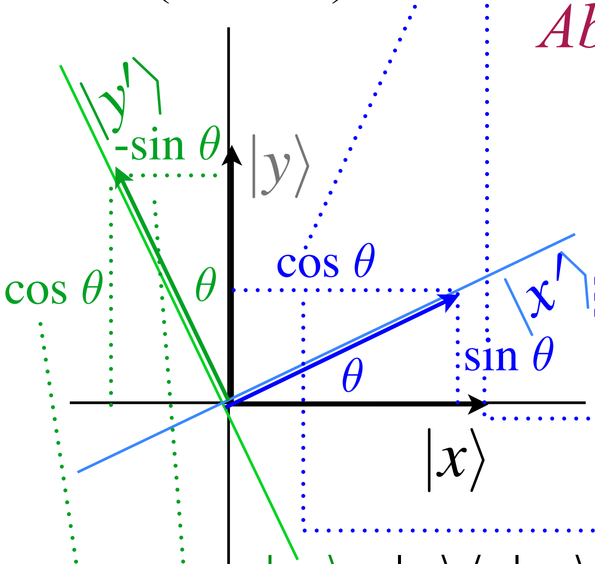
$$\Rightarrow \langle x| = \begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \end{pmatrix}$$

$$\Rightarrow \langle y| = \begin{pmatrix} \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \sin\theta & \cos\theta \end{pmatrix}$$

Abstracting bra $\langle m|$ state vectors from Transformation Matrix $T_{m,n'} = \langle m | n' \rangle$

$$|x'\rangle = \begin{pmatrix} \langle x|x'\rangle \\ \langle y|x'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad |y'\rangle = \begin{pmatrix} \langle x|y'\rangle \\ \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

Abstracting ket $|n'\rangle$ state vectors from Transformation Matrix $T_{m,n'} = \langle m | n' \rangle$



$$|x'\rangle = |x\rangle\langle x|x'\rangle + |y\rangle\langle y|x'\rangle = |x\rangle(\cos\theta) + |y\rangle(\sin\theta)$$

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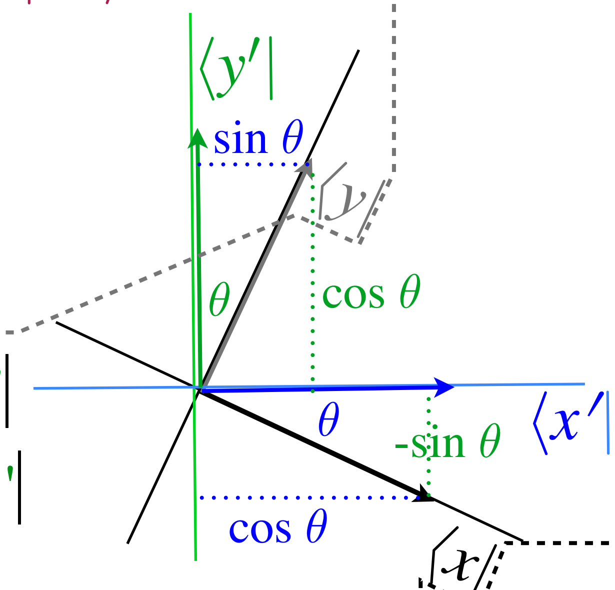


$$\langle y| = \langle y|x'\rangle\langle x'| + \langle y|y'\rangle\langle y'| = (\sin\theta)\langle x'| + (\cos\theta)\langle y'|$$

$$\langle x| = \langle x|x'\rangle\langle x'| + \langle x|y'\rangle\langle y'| = (\cos\theta)\langle x'| + (-\sin\theta)\langle y'|$$

(θ=+30°)-Rotated kets $\{|\mathbf{x}'\rangle, |\mathbf{y}'\rangle\}$ or $\{\mathbf{x}', \mathbf{y}'\}$ represented in page-aligned $\{|\mathbf{x}\rangle, |\mathbf{y}\rangle\}$ basis.

(θ=-30°)-Rotated bras $\{\langle\mathbf{x}|, \langle\mathbf{y}|\}$ or $\{\mathbf{x}, \mathbf{y}\}$ represented in page-aligned $\{|\mathbf{x}'\rangle, |\mathbf{y}'\rangle\}$ basis.

The same thing in Gibbs vector notation:

$$\mathbf{x}' = \mathbf{x}(\mathbf{x}\bullet\mathbf{x}') + \mathbf{y}(\mathbf{y}\bullet\mathbf{x}'), \quad \mathbf{y}' = \mathbf{x}(\mathbf{x}\bullet\mathbf{y}') + \mathbf{y}(\mathbf{y}\bullet\mathbf{y}'),$$
$$= \mathbf{x}(\cos\theta) + \mathbf{y}(\sin\theta), \quad = \mathbf{x}(-\sin\theta) + \mathbf{y}(\cos\theta).$$

The same thing in Gibbs vector notation:

$$\mathbf{x} = (\mathbf{x}\bullet\mathbf{x}')\mathbf{x}' + (\mathbf{x}\bullet\mathbf{y}')\mathbf{y}', \quad \mathbf{y} = (\mathbf{y}\bullet\mathbf{x}')\mathbf{x}' + (\mathbf{y}\bullet\mathbf{y}')\mathbf{y}',$$
$$\mathbf{x} = (\cos\theta)\mathbf{x}' + (-\sin\theta)\mathbf{y}', \quad \mathbf{y} = (\sin\theta)\mathbf{x}' + (\cos\theta)\mathbf{y}'.$$

"Abstraction" of bra and ket vectors from a Transformation Matrix

Ket or column vectors

Bra or row vectors

Given Transformation Matrix $T_{m,n'}$:
$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

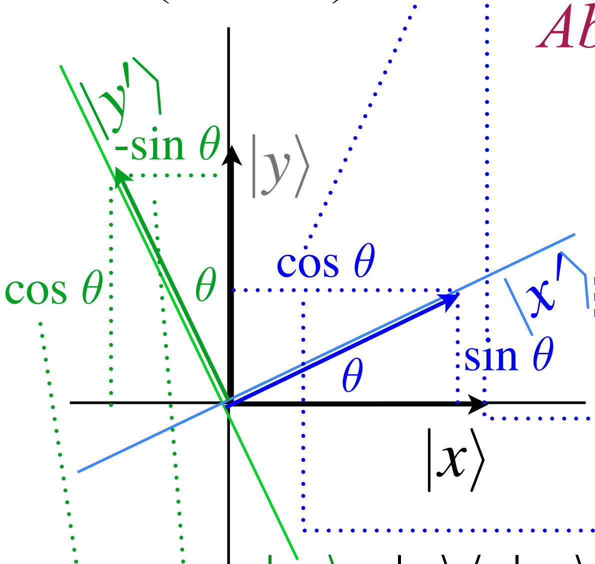
$$\Rightarrow \langle x| = \begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \end{pmatrix}$$

$$\Rightarrow \langle y| = \begin{pmatrix} \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \sin\theta & \cos\theta \end{pmatrix}$$

Abstracting bra $\langle m|$ state vectors from Transformation Matrix $T_{m,n'} = \langle m|n'\rangle$

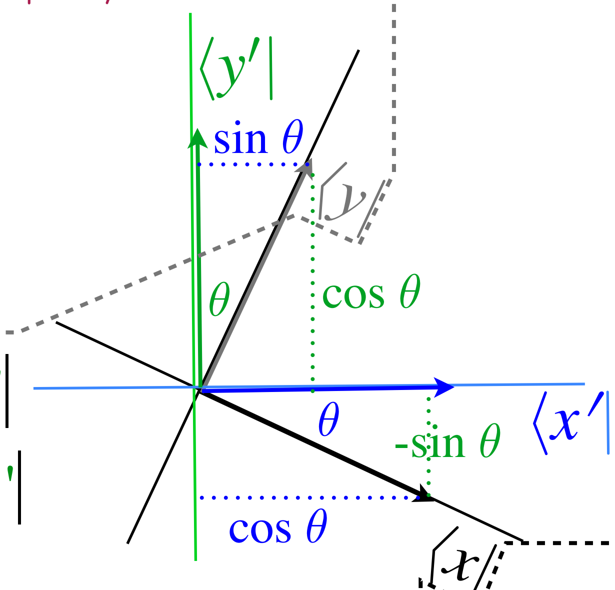
$$|x'\rangle = \begin{pmatrix} \langle x|x'\rangle \\ \langle y|x'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad |y'\rangle = \begin{pmatrix} \langle x|y'\rangle \\ \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

Abstracting ket $|n'\rangle$ state vectors from Transformation Matrix $T_{m,n'} = \langle m|n'\rangle$



$$|x'\rangle = |x\rangle\langle x|x'\rangle + |y\rangle\langle y|x'\rangle = |x\rangle(\cos\theta) + |y\rangle(\sin\theta)$$

$$|y'\rangle = |x\rangle\langle x|y'\rangle + |y\rangle\langle y|y'\rangle = |x\rangle(-\sin\theta) + |y\rangle(\cos\theta)$$



$$\langle y| = \langle y|x'\rangle\langle x'| + \langle y|y'\rangle\langle y'| = (\sin\theta)\langle x'| + (\cos\theta)\langle y'|$$

$$\langle x| = \langle x|x'\rangle\langle x'| + \langle x|y'\rangle\langle y'| = (\cos\theta)\langle x'| + (-\sin\theta)\langle y'|$$

(θ=+30°)-Rotated kets $\{|\mathbf{x}'\rangle, |\mathbf{y}'\rangle\}$ or $\{\mathbf{x}', \mathbf{y}'\}$ represented in page-aligned $\{|\mathbf{x}\rangle, |\mathbf{y}\rangle\}$ basis.

(θ=-30°)-Rotated bras $\{\langle\mathbf{x}|, \langle\mathbf{y}|\}$ or $\{\mathbf{x}, \mathbf{y}\}$ represented in page-aligned $\{|\mathbf{x}'\rangle, |\mathbf{y}'\rangle\}$ basis.

Ket vector algebra has the order of $T_{m,n'}$ transposed

$$|x'\rangle = |x\rangle\langle x|x'\rangle + |y\rangle\langle y|x'\rangle = |x\rangle(\cos\theta) + |y\rangle(\sin\theta)$$
$$|y'\rangle = |x\rangle\langle x|y'\rangle + |y\rangle\langle y|y'\rangle = |x\rangle(-\sin\theta) + |y\rangle(\cos\theta)$$

Bra vector algebra has the same order as $T_{m,n'}$

$$\langle x| = \langle x|x'\rangle\langle x'| + \langle x|y'\rangle\langle y'| = (\cos\theta)\langle x'| + (-\sin\theta)\langle y'|$$
$$\langle y| = \langle y|x'\rangle\langle x'| + \langle y|y'\rangle\langle y'| = (\sin\theta)\langle x'| + (\cos\theta)\langle y'|$$

Unit vector kets $|\mathbf{x}\rangle$ and $|\mathrm{y}\rangle$ or $\mathbf{x'}$ and $\mathbf{y'}$ are represented (in their own $|\mathrm{x}\rangle$and$|\mathrm{y}\rangle$ basis) as follows.

$$|x\rangle = \begin{pmatrix} \langle x|x\rangle \\ \langle y|x\rangle \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |y\rangle = \begin{pmatrix} \langle x|y\rangle \\ \langle y|y\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Beam Sorters - Optical polarization sorting

2-State Sorters: spin-1/2 vs. optical polarization

Beam Sorters in Series and Transformation Matrices

Introducing Dirac bra-ket notation

"Abstraction" of bra and ket vectors from a Transformation Matrix

→ *Introducing scalar and matrix products*

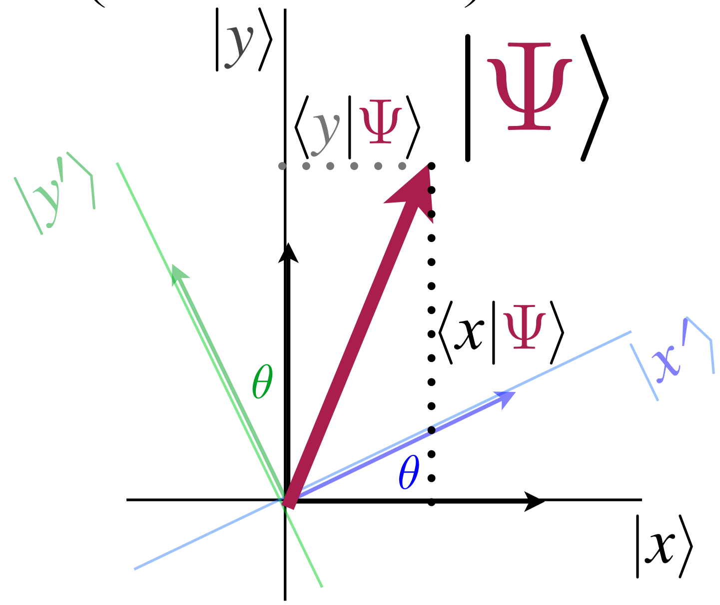
Transformation matrix $T_{m,n'} = \langle m | n' \rangle$ is array of dot or *scalar products* (dot products) of unit vectors or *direction cosines*.

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} (\mathbf{x}\bullet\mathbf{x'}) & (\mathbf{x}\bullet\mathbf{y'}) \\ (\mathbf{y}\bullet\mathbf{x'}) & (\mathbf{y}\bullet\mathbf{y'}) \end{pmatrix}$$

$\{\langle x|,\langle y|\}$ components of $|\Psi\rangle$:

$\langle x|\Psi\rangle = \Psi_x$

$\langle y|\Psi\rangle = \Psi_y$



Any state $|\Psi\rangle$ can be expanded in any basis $\{\langle x|,\langle y|\}$

$$|\Psi\rangle = |x\rangle\langle x|\Psi\rangle + |y\rangle\langle y|\Psi\rangle$$

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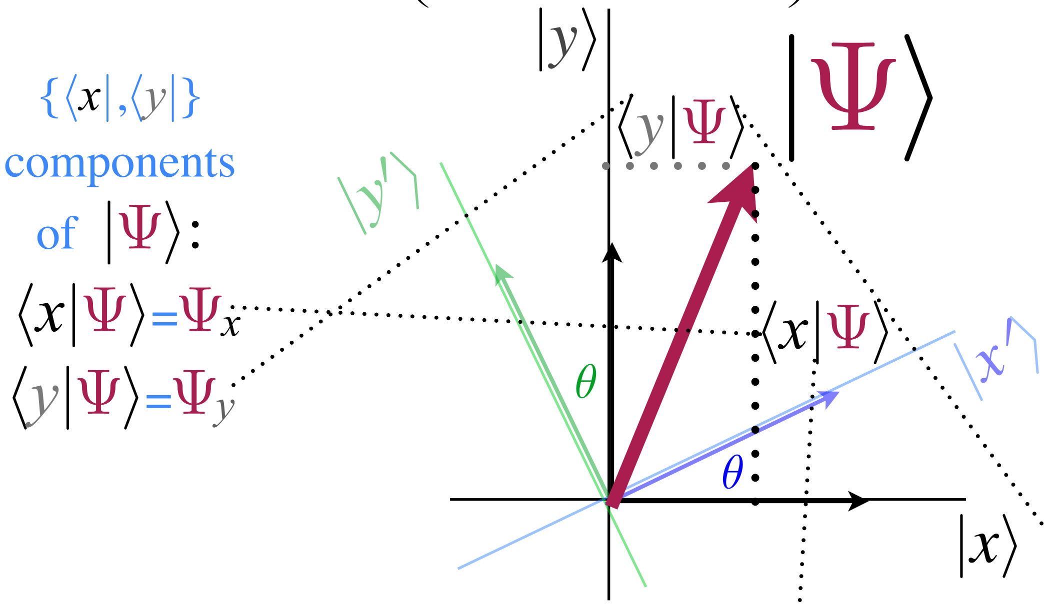

Any state $|\Psi\rangle$ can be expanded in any basis $\{\langle x|, \langle y|\}$

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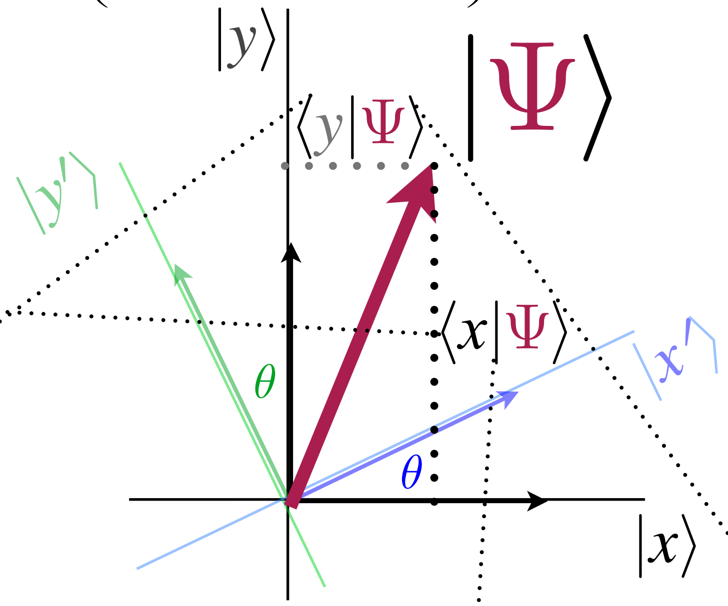

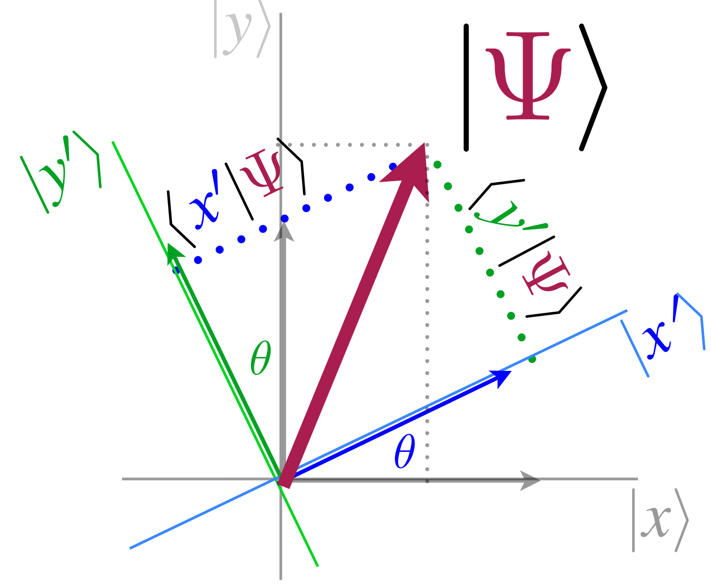

$\{\langle x'|, \langle y'|\}$ components of $|\Psi\rangle$:

$\langle x'|\Psi\rangle = \Psi_{x'}$

$\langle y'|\Psi\rangle = \Psi_{y'}$

Any state $|\Psi\rangle$ can be expanded in any basis $\{\langle x|, \langle y|\}$, or $\{\langle x'|, \langle y'|\}$, ...*etc*.

$$|\Psi\rangle = |x\rangle\langle x|\Psi\rangle + |y\rangle\langle y|\Psi\rangle = |x'\rangle\langle x'|\Psi\rangle + |y'\rangle\langle y'|\Psi\rangle$$

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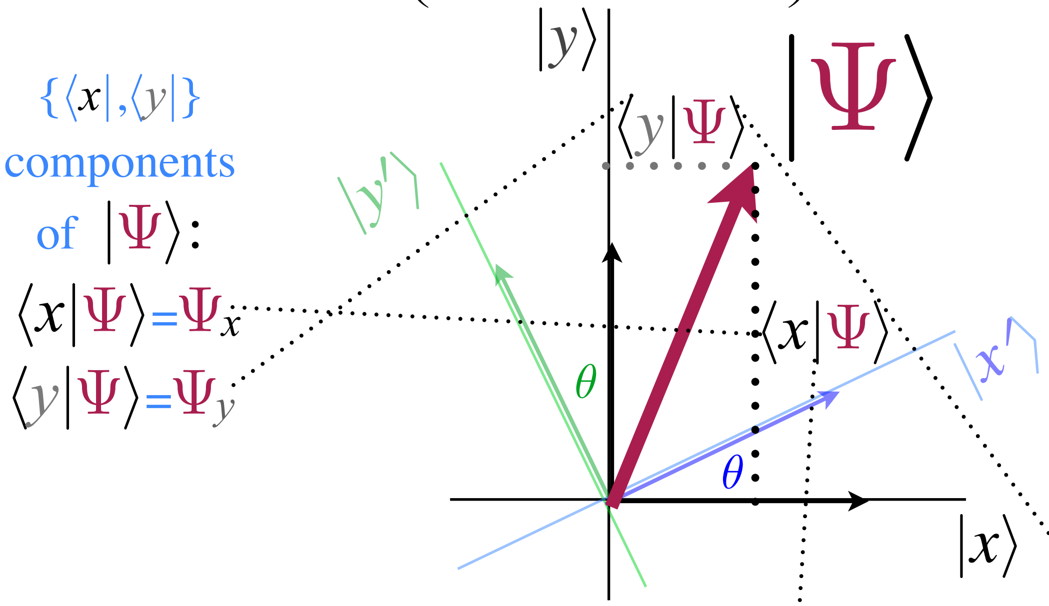

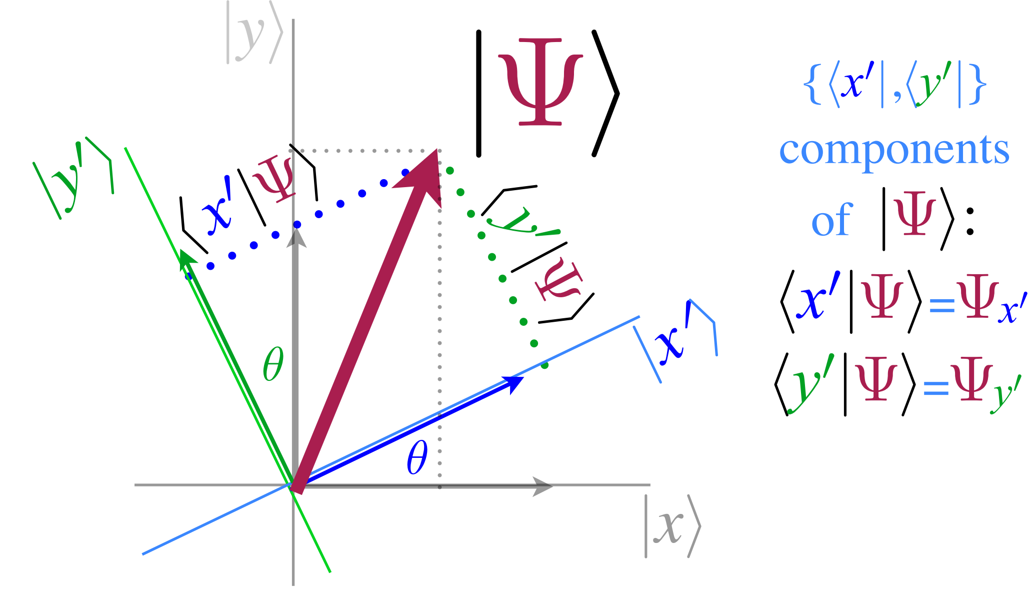

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Hybrid Gibbs-Dirac notation *(Ug-ly!)*

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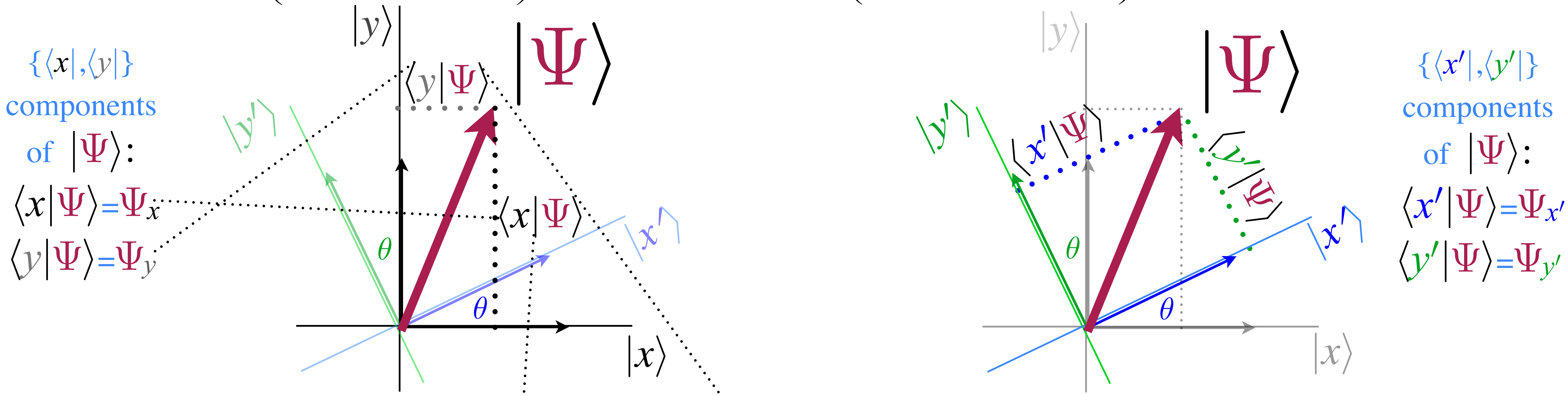

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Proof: $\langle x| = \langle x|x'\rangle\langle x'| + \langle x|y'\rangle\langle y'|$ implies: $\langle x|\Psi\rangle = \langle x|x'\rangle\langle x'|\Psi\rangle + \langle x|y'\rangle\langle y'|\Psi\rangle$

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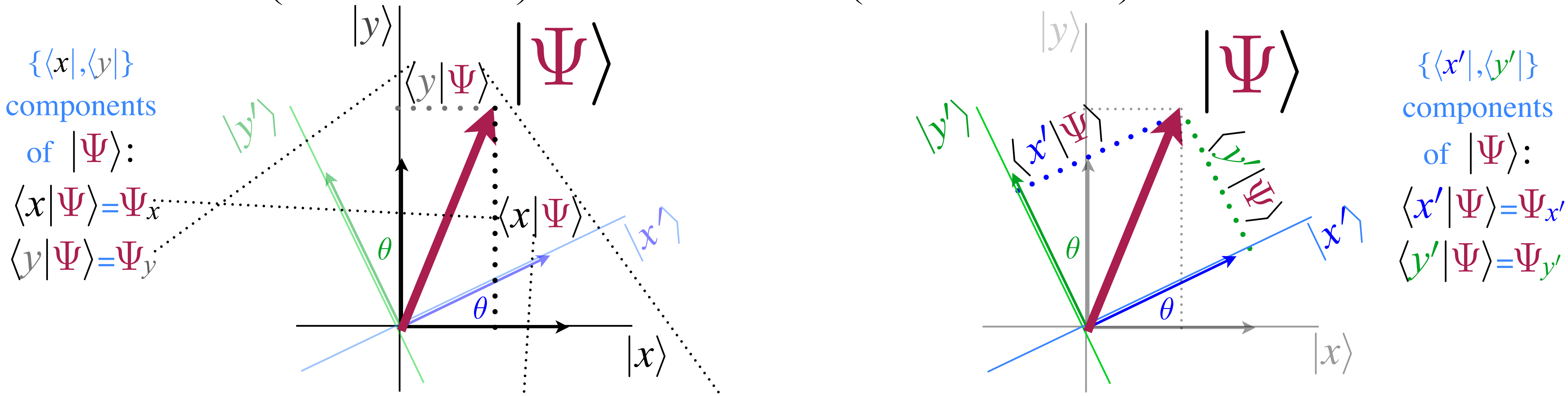

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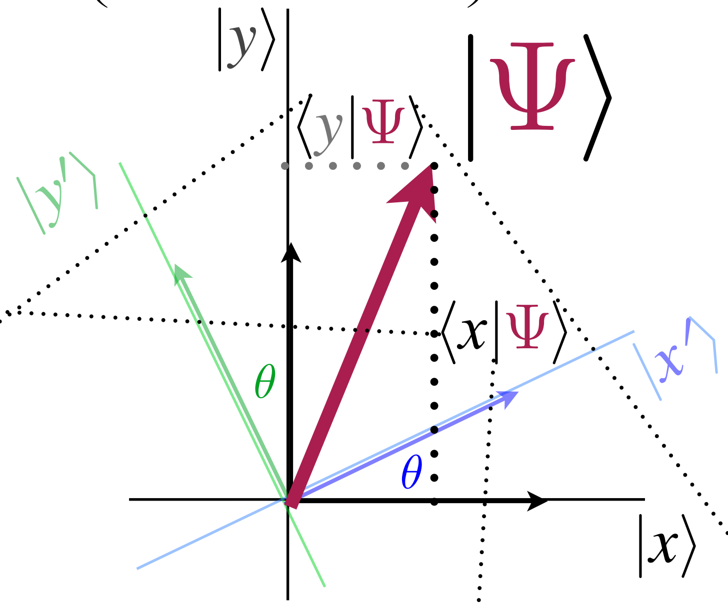

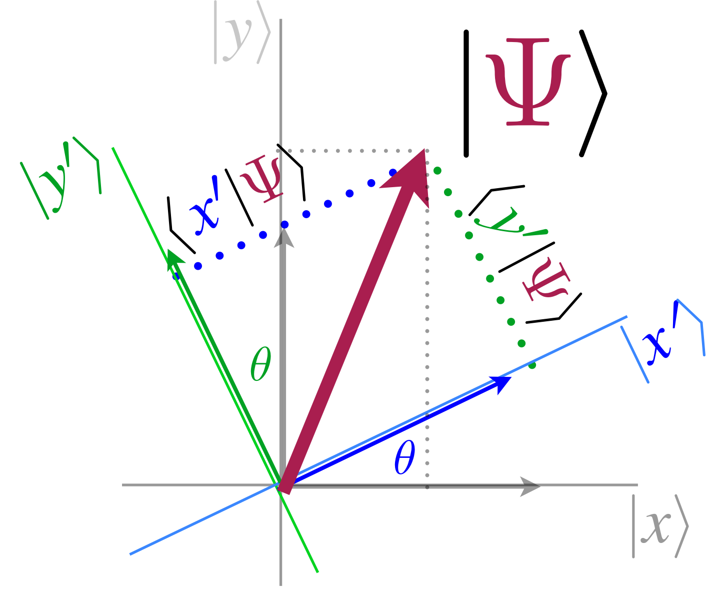

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Hybrid Gibbs-Dirac notation *(Ug-ly!)*

Inverse (†=T*=-1) matrix $T_{n',m}$ relates $\{\langle x'|\Psi\rangle,\langle y'|\Psi\rangle\}$ amplitudes to $\{\langle x|\Psi\rangle,\langle y|\Psi\rangle\}$.

$$\begin{pmatrix} \langle x'|\Psi\rangle \\ \langle y'|\Psi\rangle \end{pmatrix} = \begin{pmatrix} \langle x'|x\rangle & \langle x'|y\rangle \\ \langle y'|x\rangle & \langle y'|y\rangle \end{pmatrix} \begin{pmatrix} \langle x|\Psi\rangle \\ \langle y|\Psi\rangle \end{pmatrix} \quad \text{or:} \quad \begin{pmatrix} \Psi_{x'} \\ \Psi_{y'} \end{pmatrix} = \begin{pmatrix} \langle x'|x\rangle & \langle x'|y\rangle \\ \langle y'|x\rangle & \langle y'|y\rangle \end{pmatrix} \begin{pmatrix} \Psi_x \\ \Psi_y \end{pmatrix}$$

Hybrid Gibbs-Dirac notation *(Still Ug-ly!)*