Group Theory in Quantum Mechanics

Lecture 6 (1.31.13)

Spectral Decomposition of Bi-Cyclic ($C_2 \subset U(2)$) Operators

(Quantum Theory for Computer Age - Ch. 7-9 of Unit 3)

(Principles of Symmetry, Dynamics, and Spectroscopy - Sec. 1-3 of Ch. 2)

Review:How symmetry groups become eigen-solvers

C_2 Symmetric two-dimensional harmonic oscillators (2DHO)

C_2 (Bilateral σ_B reflection) symmetry conditions:

Minimal equation of σ_B and spectral decomposition of $C_2(\sigma_B)$

C_2 Symmetric 2DHO eigensolutions

C_2 Mode phase character table

C_2 Symmetric 2DHO uncoupling

2D-HO beats and mixed mode geometry

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Suppose you need to diagonalize a complicated operator **K** and knew that **K** commutes with some other operators **G** and **H** for which irreducible projectors are more easily found.

$$\mathbf{KG} = \mathbf{GK} \text{ or } \mathbf{G}^\dagger\mathbf{KG} = \mathbf{K} \text{ or } \mathbf{GKG}^\dagger = \mathbf{K}$$

$$\mathbf{KH} = \mathbf{HK} \text{ or } \mathbf{H}^\dagger\mathbf{KH} = \mathbf{K} \text{ or } \mathbf{HKH}^\dagger = \mathbf{K}$$

(Here assuming *unitary* $\mathbf{G}^\dagger=\mathbf{G}^{-1}$ and $\mathbf{H}^\dagger=\mathbf{H}^{-1}$.)

This means **K** is *invariant* to the transformation by **G** and **H**
and all their products **GH**, $\mathbf{GH}^2$, $\mathbf{G}^2\mathbf{H}$,.. *etc*. and all their inverses $\mathbf{G}^\dagger$,$\mathbf{H}^\dagger$,.. etc.

The group $\mathscr{G}_{\mathbf{K}} = \{\mathbf{1}, \mathbf{G}, \mathbf{H},..\}$ so formed by such operators is called a *symmetry group* for **K**.

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In certain ideal cases a **K**-matrix $\langle\mathbf{K}\rangle$ is a linear combination of matrices $\langle\mathbf{1}\rangle,\langle\mathbf{G}\rangle,\langle\mathbf{H}\rangle,...$ from $\mathscr{G}_\mathbf{K}$.

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Then spectral resolution of $\{\langle\mathbf{1}\rangle, \langle\mathbf{G}\rangle, \langle\mathbf{H}\rangle, .. \}$ also resolves $\langle\mathbf{K}\rangle$.

We will study ideal cases first. More general cases are built from this idea.

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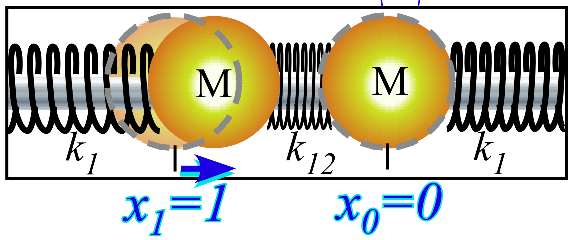
2D HO "binary" bases and coord. $\{x_0,x_1\}$

(a) unit base state

$|0\rangle = |x\rangle = |2\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$



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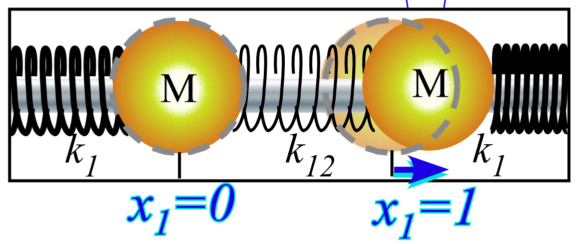
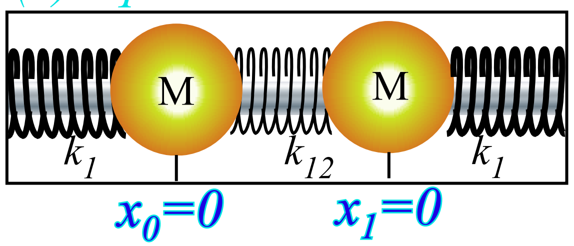


(c) equilibrium zero-state $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$



2D HO Matrix operator equations

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} \frac{k_1+k_{12}}{M} & \frac{-k_{12}}{M} \\ \frac{-k_{12}}{M} & \frac{k_1+k_{12}}{M} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

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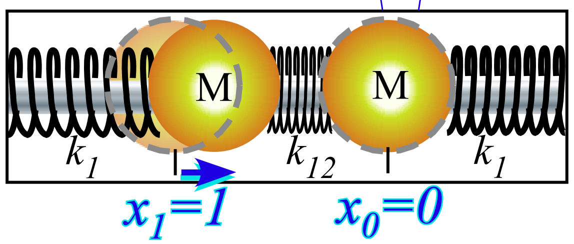

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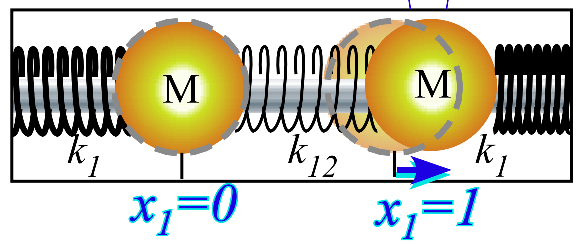

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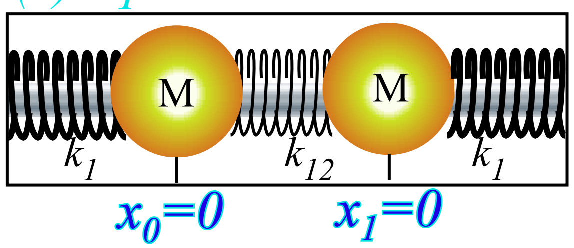

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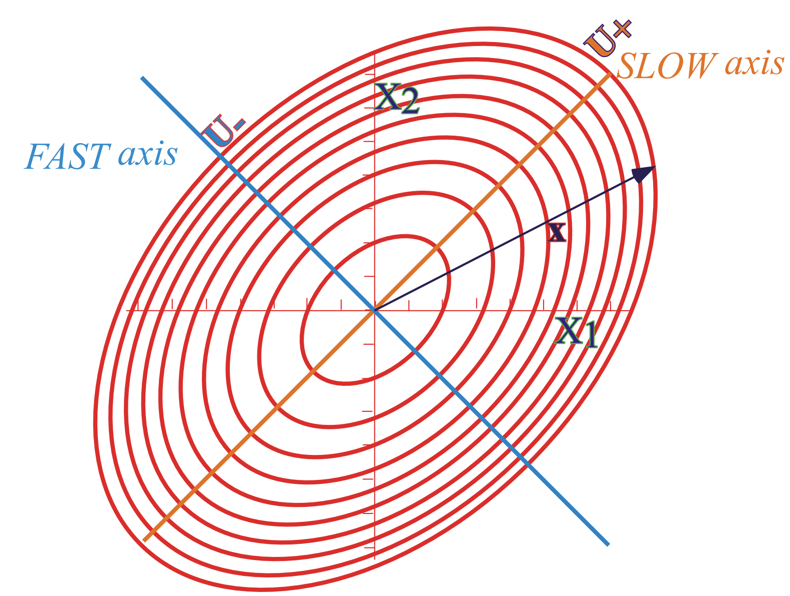
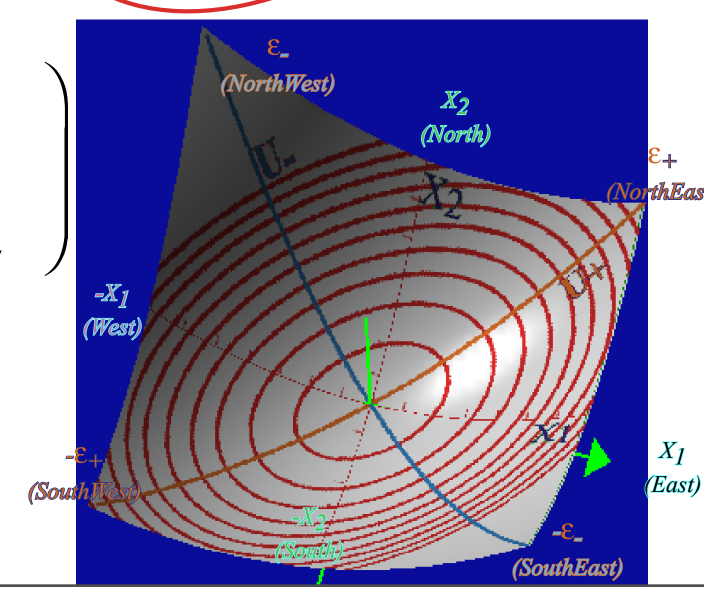
(a) PE Contours





2D HO kinetic energy $T(v_1, v_2)$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$= \frac{1}{2}\langle \dot{\mathbf{x}} | \mathbf{M} | \dot{\mathbf{x}} \rangle$$

2D HO potential energy $V(x_1, x_2)$

$$V = \frac{1}{2}(k_1 + k_{12})x_1^2 - k_{12}x_1x_2 + \frac{1}{2}(k_2 + k_{12})x_2^2$$

$$= \frac{1}{2}\langle \mathbf{x} | \mathbf{K} | \mathbf{x} \rangle \quad \text{where: } \mathbf{K} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = m_1 \ddot{x}_1 = F_1 = -\frac{\partial V}{\partial x_1} = -(k_1 + k_{12})x_1 + k_{12}x_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = m_2 \ddot{x}_2 = F_2 = -\frac{\partial V}{\partial x_2} = k_{12}x_1 - (k_2 + k_{12})x_2$$

2D HO Lagrange equations

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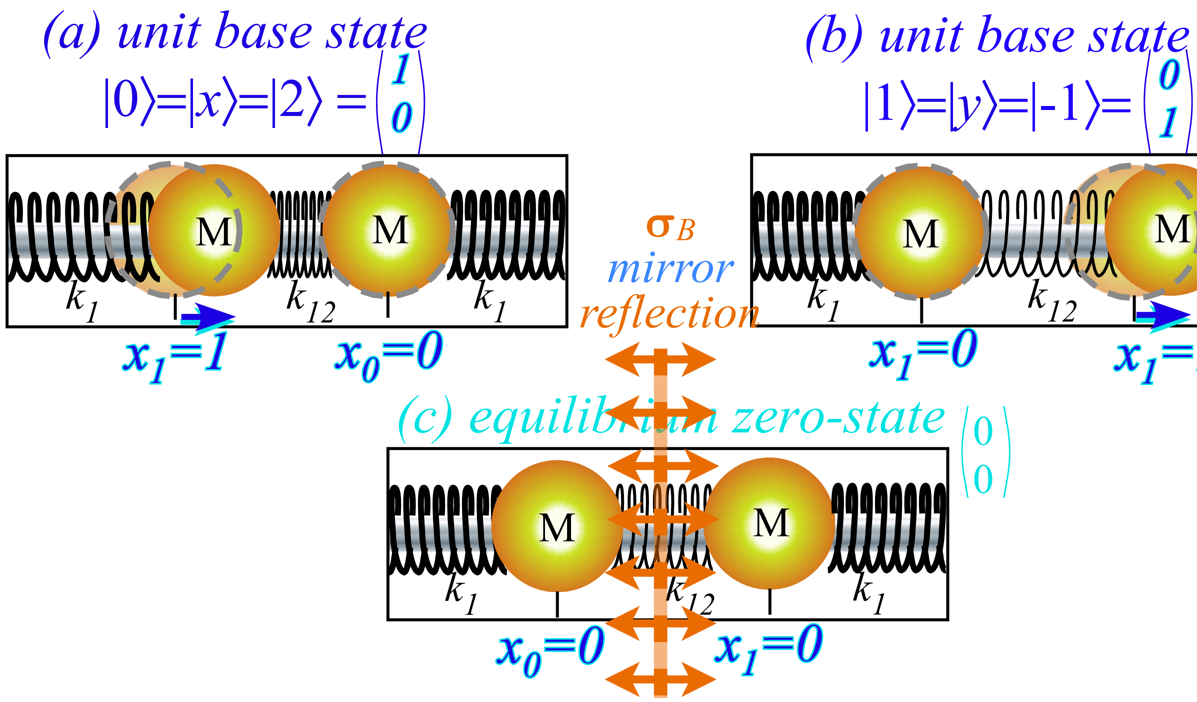

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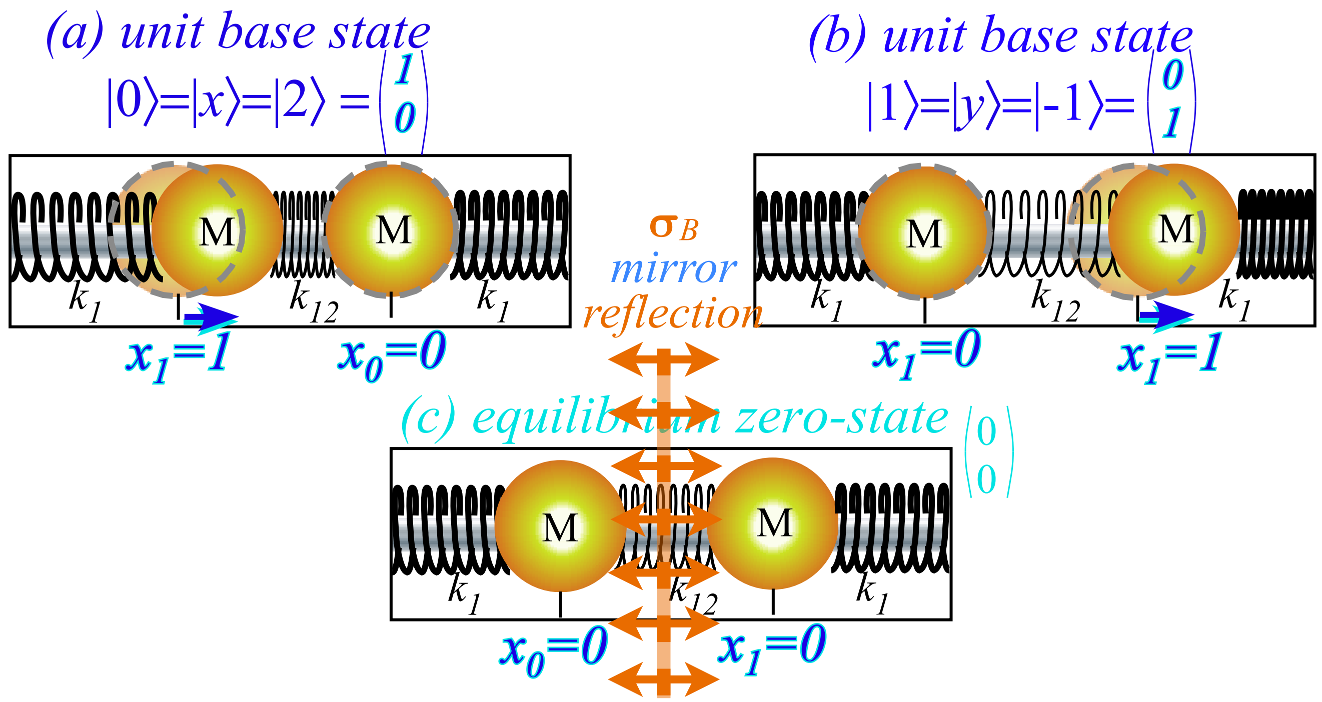

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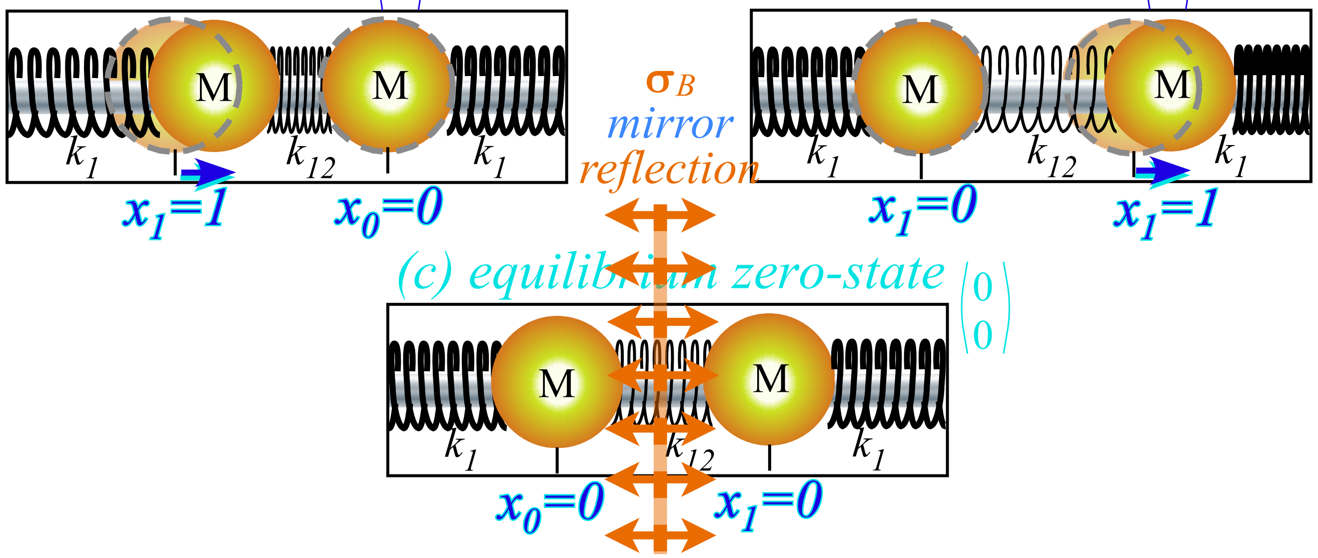

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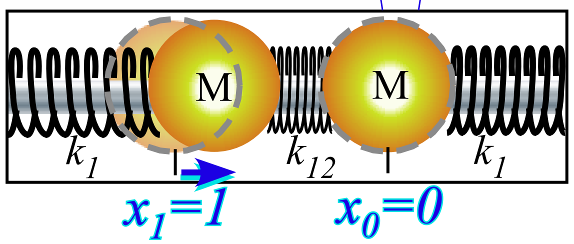

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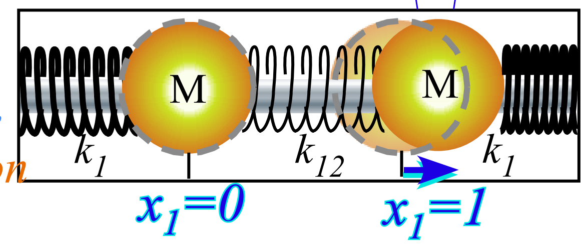

σ_B mirror reflection

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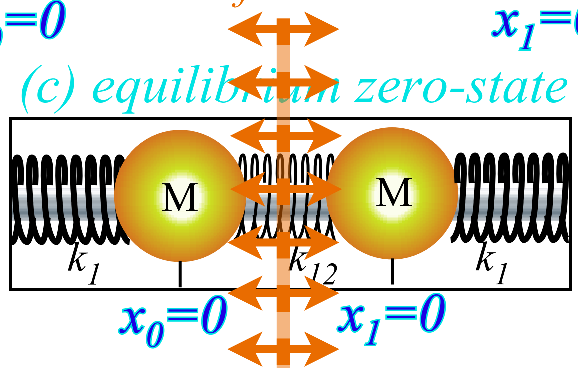

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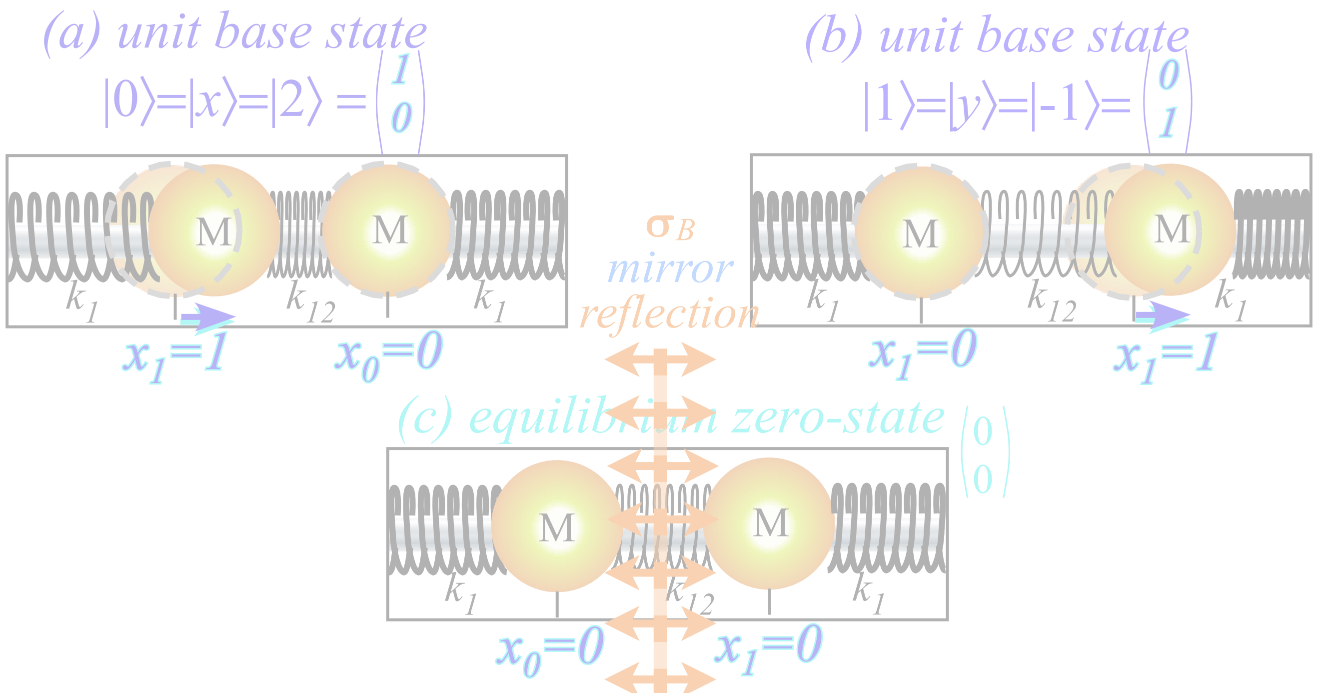

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Minimal equation of σ_B is: $\sigma_B{}^2 = 1$

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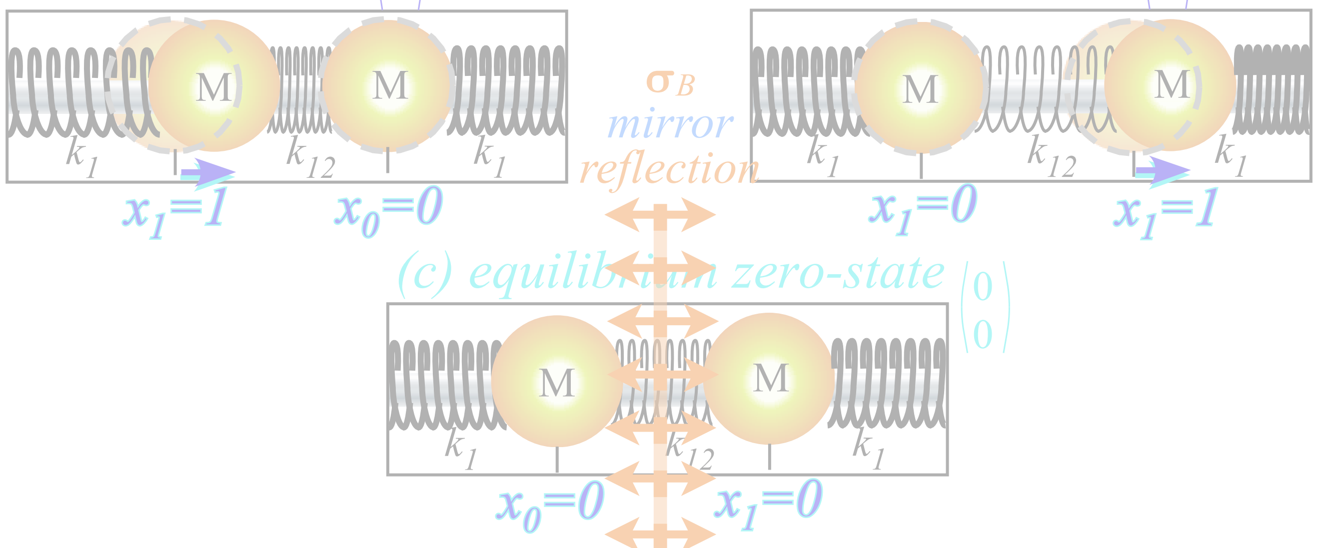

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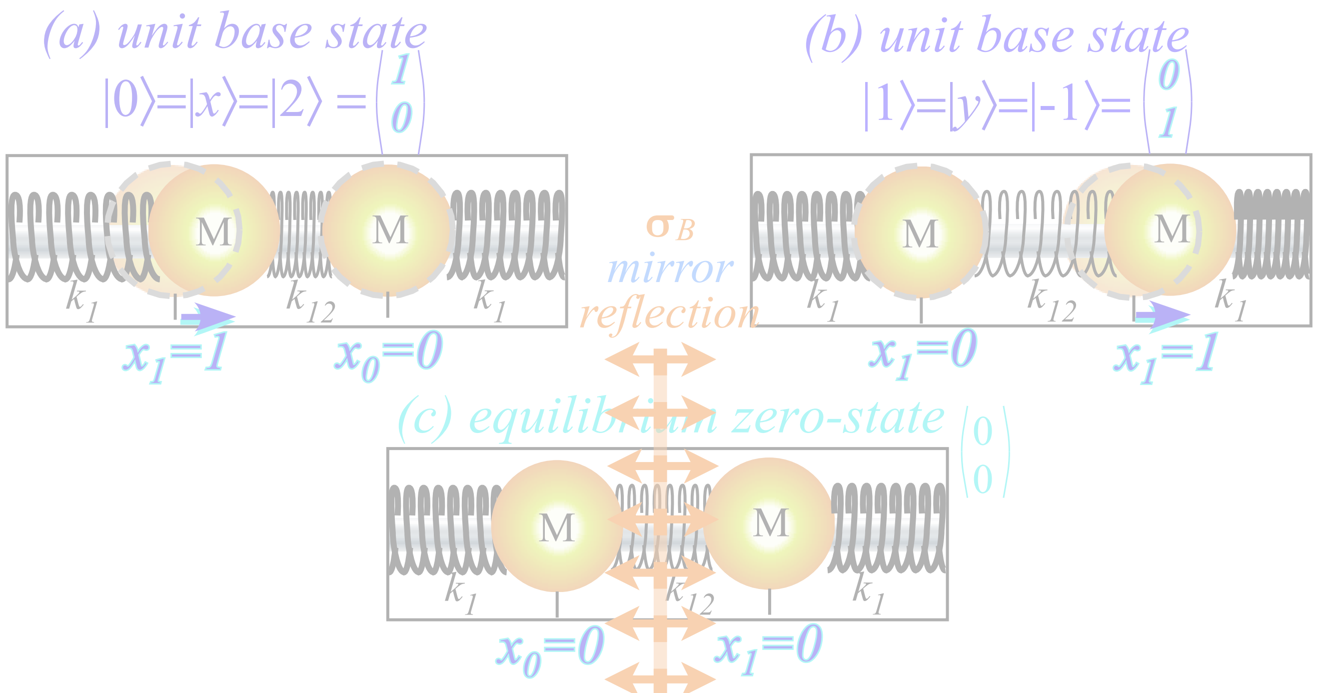
$$\begin{pmatrix}\langle\mathbf{1}|\mathbf{1}|\mathbf{1}\rangle & \langle\mathbf{1}|\mathbf{1}|\sigma_B\rangle\\ \langle\sigma_B|\mathbf{1}|\mathbf{1}\rangle & \langle\sigma_B|\mathbf{1}|\sigma_B\rangle\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\qquad \begin{pmatrix}\langle\mathbf{1}|\sigma_B|\mathbf{1}\rangle & \langle\mathbf{1}|\sigma_B|\sigma_B\rangle\\ \langle\sigma_B|\sigma_B|\mathbf{1}\rangle & \langle\sigma_B|\sigma_B|\sigma_B\rangle\end{pmatrix}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

Minimal equation of σ_B is: $\sigma_B{}^2=1$

or: $\sigma_B{}^2-1=0=(\sigma_B-1)(\sigma_B+1)$

C_2 Symmetric two-dimensional harmonic oscillators (2DHO)

2D HO "binary" bases and coord. $\{x_0, x_1\}$



2D HO Matrix operator equations

$$\begin{pmatrix}\ddot{x}_1\\ \ddot{x}_2\end{pmatrix} = -\begin{pmatrix}\frac{k_1+k_{12}}{M} & \frac{-k_{12}}{M}\\ \frac{-k_{12}}{M} & \frac{k_1+k_{12}}{M}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

$$= -\begin{pmatrix}K_{11} & K_{12}\\ K_{21} & K_{22}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

$$|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\,|\mathbf{x}\rangle$$

More conventional coordinate notation $\{x_0, x_1\} \rightarrow \{x_1, x_2\}$

C_2 (Bilateral σ_B reflection) symmetry conditions:

$K_{11} \equiv K \equiv K_{22}$ and: $K_{12} \equiv k \equiv K_{12} = -k_{12}$ (Let: $M=1$)

$$\begin{pmatrix}K_{11} & K_{12}\\ K_{21} & K_{22}\end{pmatrix} = \begin{pmatrix}K & k\\ k & K\end{pmatrix} = K\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + k\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$$\mathbf{K} = K\cdot\mathbf{1} + k\cdot\sigma_B$$

K-matrix is made of its symmetry operators in

group $C_2 = \{\mathbf{1}, \sigma_B\}$ with *product table*:

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

Symmetry *product table* gives C_2 group representations in *group basis* $\{|0\rangle = \mathbf{1}|0\rangle \equiv |\mathbf{1}\rangle,\ |1\rangle = \sigma_B|0\rangle \equiv |\sigma_B\rangle\}$

$$\begin{pmatrix}\langle\mathbf{1}|\mathbf{1}|\mathbf{1}\rangle & \langle\mathbf{1}|\mathbf{1}|\sigma_B\rangle\\ \langle\sigma_B|\mathbf{1}|\mathbf{1}\rangle & \langle\sigma_B|\mathbf{1}|\sigma_B\rangle\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \qquad \begin{pmatrix}\langle\mathbf{1}|\sigma_B|\mathbf{1}\rangle & \langle\mathbf{1}|\sigma_B|\sigma_B\rangle\\ \langle\sigma_B|\sigma_B|\mathbf{1}\rangle & \langle\sigma_B|\sigma_B|\sigma_B\rangle\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

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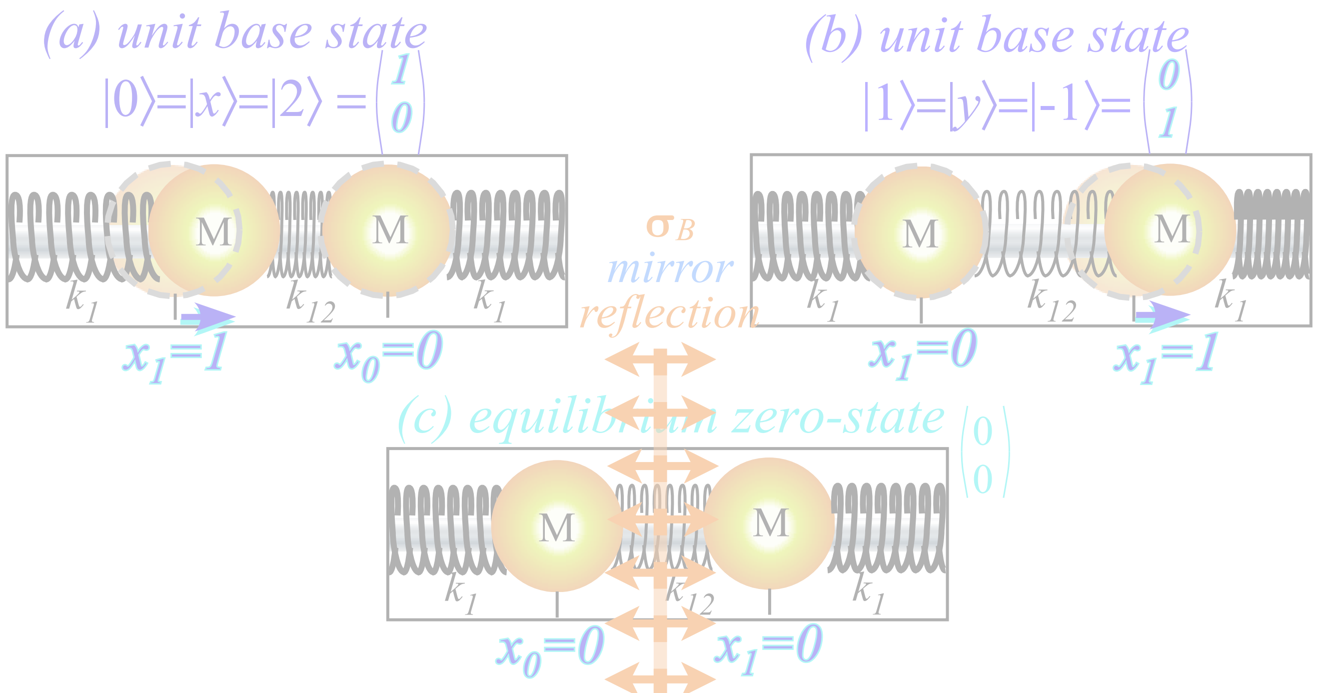
or: $\sigma_B{}^2 - 1 = 0 = (\sigma_B - 1)(\sigma_B + 1)$

with eigenvalues:
$\{\chi^+(\sigma_B) = +1,\ \chi^-(\sigma_B) = -1\}$

C_2 Symmetric two-dimensional harmonic oscillators (2DHO)

2D HO "binary" bases and coord. $\{x_0,x_1\}$



2D HO Matrix operator equations

$$\begin{pmatrix}\ddot{x}_1\\ \ddot{x}_2\end{pmatrix}=-\begin{pmatrix}\frac{k_1+k_{12}}{M} & \frac{-k_{12}}{M}\\ \frac{-k_{12}}{M} & \frac{k_1+k_{12}}{M}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

$$=-\begin{pmatrix}K_{11} & K_{12}\\ K_{21} & K_{22}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

$$|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\,|\mathbf{x}\rangle$$

More conventional coordinate notation $\{x_0,x_1\}\longrightarrow\{x_1,x_2\}$

C_2 (Bilateral σ_B reflection) symmetry conditions:

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$$\begin{pmatrix}K_{11} & K_{12}\\ K_{21} & K_{22}\end{pmatrix}=\begin{pmatrix}K & k\\ k & K\end{pmatrix}=K\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+k\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$$\mathbf{K}= K\cdot\mathbf{1} + k\cdot\sigma_B$$

K-matrix is made of its symmetry operators in

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C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
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$$\begin{pmatrix}\langle\mathbf{1}|\mathbf{1}|\mathbf{1}\rangle & \langle\mathbf{1}|\mathbf{1}|\sigma_B\rangle\\ \langle\sigma_B|\mathbf{1}|\mathbf{1}\rangle & \langle\sigma_B|\mathbf{1}|\sigma_B\rangle\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\qquad \begin{pmatrix}\langle\mathbf{1}|\sigma_B|\mathbf{1}\rangle & \langle\mathbf{1}|\sigma_B|\sigma_B\rangle\\ \langle\sigma_B|\sigma_B|\mathbf{1}\rangle & \langle\sigma_B|\sigma_B|\sigma_B\rangle\end{pmatrix}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

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with eigenvalues:
$\{\chi^+(\sigma_B)=+1,\ \chi^-(\sigma_B)=-1\}$

Spectral decomposition of $C_2(\sigma_B)$ into $\{\mathbf{P}^+,\mathbf{P}^-\}$

$$\mathbf{1}\ =\mathbf{P}^+ + \mathbf{P}^-$$

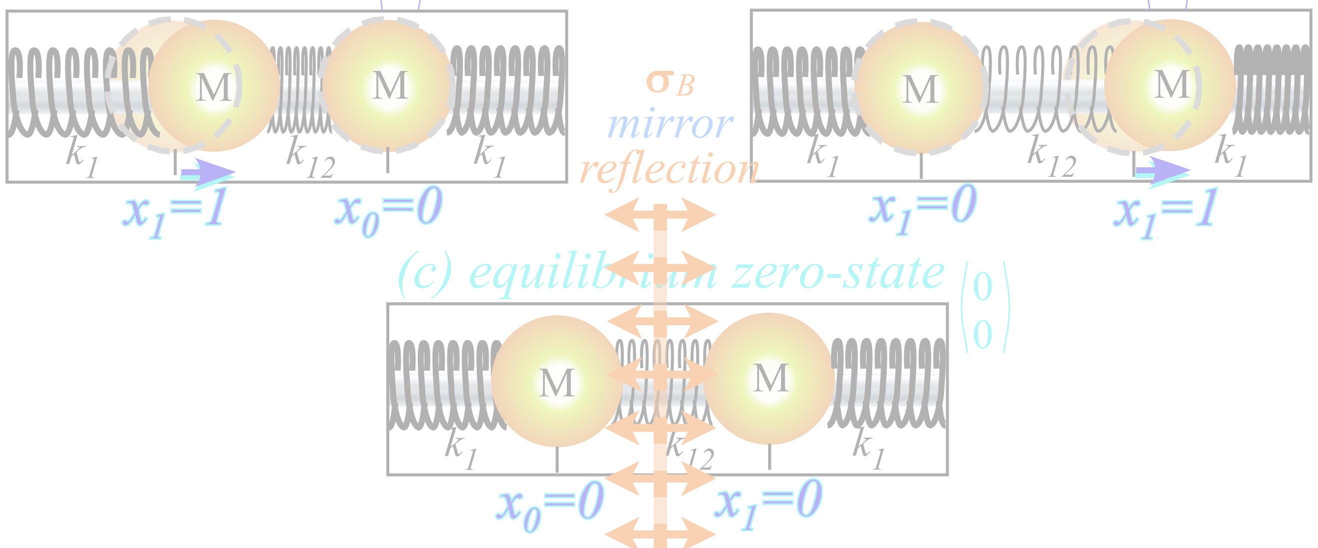
$$\sigma_B=\mathbf{P}^+ - \mathbf{P}^-$$

C_2 Symmetric two-dimensional harmonic oscillators (2DHO)

2D HO "binary" bases and coord. $\{x_0, x_1\}$

(a) unit base state $|0\rangle = |x\rangle = |2\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(b) unit base state $|1\rangle = |y\rangle = |-1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



(c) equilibrium zero-state $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

2D HO Matrix operator equations

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} \frac{k_1+k_{12}}{M} & \frac{-k_{12}}{M} \\ \frac{-k_{12}}{M} & \frac{k_1+k_{12}}{M} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= -\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\,|\mathbf{x}\rangle$$

More conventional coordinate notation $\{x_0, x_1\} \rightarrow \{x_1, x_2\}$

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$K_{11} \equiv K \equiv K_{22}$ and: $K_{12} \equiv k \equiv K_{12} = -k_{12}$ (Let: $M=1$)

$$\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} = \begin{pmatrix} K & k \\ k & K \end{pmatrix} = K \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + k \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\mathbf{K} = K \cdot \mathbf{1} + k \cdot \sigma_B$$

K-matrix is made of its symmetry operators in

group $C_2 = \{\mathbf{1}, \sigma_B\}$ with *product table*:

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
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$$\begin{pmatrix} \langle \mathbf{1}|\mathbf{1}|\mathbf{1}\rangle & \langle \mathbf{1}|\mathbf{1}|\sigma_B\rangle \\ \langle \sigma_B|\mathbf{1}|\mathbf{1}\rangle & \langle \sigma_B|\mathbf{1}|\sigma_B\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} \langle \mathbf{1}|\sigma_B|\mathbf{1}\rangle & \langle \mathbf{1}|\sigma_B|\sigma_B\rangle \\ \langle \sigma_B|\sigma_B|\mathbf{1}\rangle & \langle \sigma_B|\sigma_B|\sigma_B\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Minimal equation of σ_B is: $\sigma_B{}^2 = 1$

or: $\sigma_B{}^2 - 1 = 0 = (\sigma_B - 1)(\sigma_B + 1)$

with eigenvalues:

$\{\chi^+(\sigma_B) = +1,\ \chi^-(\sigma_B) = -1\}$

Spectral decomposition of $C_2(\sigma_B)$ into $\{\mathbf{P}^+, \mathbf{P}^-\}$

$$\mathbf{1} = \mathbf{P}^+ + \mathbf{P}^-$$

$$\sigma_B = \mathbf{P}^+ - \mathbf{P}^-$$

$\mathbf{P}^\pm$-projectors:

$$\mathbf{P}^+ = \frac{\mathbf{1} + \sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\mathbf{P}^+ = \frac{\mathbf{1} - \sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Review:How symmetry groups become eigen-solvers

C_2 Symmetric two-dimensional harmonic oscillators (2DHO)

C_2 (Bilateral σ_B reflection) symmetry conditions:

Minimal equation of σ_B and spectral decomposition of $C_2(\sigma_B)$

C_2 Symmetric 2DHO eigensolutions and D-Tran

C_2 Mode phase character table

C_2 Symmetric 2DHO uncoupling

2D-HO beats and mixed mode geometry

C_2 Symmetric 2DHO eigensolutions

$$\mathbf{K} \quad = \quad K\cdot\mathbf{1} \quad - \quad k_{12}\cdot\sigma_B$$

$$\begin{pmatrix} K & k \\ k & K \end{pmatrix} = K\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - k_{12}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_1+k_{12} \end{pmatrix}$$

$C_2(\sigma_B)$ spectrally decomposed into $\{\mathbf{P}^+,\mathbf{P}^-\}$ projectors:

$$\mathbf{P}^+ = \frac{\mathbf{1}+\sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\mathbf{1} = \mathbf{P}^+ + \mathbf{P}^-$$

$$\sigma_B = \mathbf{P}^+ - \mathbf{P}^-$$

$$\mathbf{P}^- = \frac{\mathbf{1}-\sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Eigenvalues of σ_B :

$\{\chi^+(\sigma_B) = +1,\ \chi^-(\sigma_B) = -1\}$

C_2 Symmetric 2DHO eigensolutions

$$\mathbf{K} = K\cdot\mathbf{1} - k_{12}\cdot\sigma_B$$

$$K\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - k_{12}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_1 + k_{12} \end{pmatrix}$$

K-matrix is made of its symmetry operators in *group* $C_2 = \{\mathbf{1}, \sigma_B\}$ with *product table*:

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

$C_2(\sigma_B)$ spectrally decomposed into $\{\mathbf{P}^+, \mathbf{P}^-\}$ projectors:

$$\mathbf{P}^+ = \frac{\mathbf{1}+\sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\mathbf{P}^- = \frac{\mathbf{1}-\sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\mathbf{1} = \mathbf{P}^+ + \mathbf{P}^-$$

$$\sigma_B = \mathbf{P}^+ - \mathbf{P}^-$$

Eigenvalues of σ_B :

$\{\chi^+(\sigma_B) = +1,\ \chi^-(\sigma_B) = -1\}$

C_2 Symmetric 2DHO eigensolutions

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

$$\mathbf{K} = K\cdot\mathbf{1} - k_{12}\cdot\sigma_B$$

$$K\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - k_{12}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_1+k_{12} \end{pmatrix}$$

K-matrix is made of its symmetry operators in *group* $C_2 = \{\mathbf{1}, \sigma_B\}$ with *product table*:

$C_2(\sigma_B)$ spectrally decomposed into $\{\mathbf{P}^+, \mathbf{P}^-\}$ projectors:

$$\mathbf{P}^+ = \frac{\mathbf{1}+\sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\mathbf{P}^- = \frac{\mathbf{1}-\sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\mathbf{1} = \mathbf{P}^+ + \mathbf{P}^-$$

$$\sigma_B = \mathbf{P}^+ - \mathbf{P}^-$$

Eigenvalues of σ_B :

$\{\chi^+(\sigma_B) = +1,\ \chi^-(\sigma_B) = -1\}$

Eigenvalues of $\mathbf{K} = K\cdot\mathbf{1} - k_{12}\cdot\sigma_B$:

$\varepsilon^+(\mathbf{K}) = K - k_{12}$, $\quad\varepsilon^-(\mathbf{K}) = K + k_{12}$

$= k_1$ $\quad\quad = k_1 + 2k_{12}$

C_2 Symmetric 2DHO eigensolutions

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

$$\mathbf{K} = K\cdot\mathbf{1} - k_{12}\cdot\sigma_B$$

$$K\begin{pmatrix}1&0\\0&1\end{pmatrix} - k_{12}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}k_1+k_{12}&-k_{12}\\-k_{12}&k_1+k_{12}\end{pmatrix}$$

K-matrix is made of its symmetry operators in *group* $C_2=\{\mathbf{1},\sigma_B\}$ with *product table*:

$C_2(\sigma_B)$ spectrally decomposed into $\{\mathbf{P}^+,\mathbf{P}^-\}$ projectors:

$$\mathbf{P}^+ = \frac{\mathbf{1}+\sigma_B}{2} = \begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{pmatrix} = |+\rangle\langle+|$$

factored projectors

$$\mathbf{P}^- = \frac{\mathbf{1}-\sigma_B}{2} = \begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\\-\frac{1}{2}&\frac{1}{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{-1}{\sqrt{2}}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{pmatrix} = |-\rangle\langle-|$$

$$\mathbf{1} = \mathbf{P}^+ + \mathbf{P}^-$$

$$\sigma_B = \mathbf{P}^+ - \mathbf{P}^-$$

Eigenvalues of σ_B :

$\{\chi^+(\sigma_B) = +1,\ \chi^-(\sigma_B) = -1\}$

Eigenvalues of $\mathbf{K}=K\cdot\mathbf{1} - k_{12}\cdot\sigma_B$:

$\varepsilon^+(\mathbf{K}) = K - k_{12}$, $\quad\varepsilon^-(\mathbf{K}) = K + k_{12}$

$= k_1 \qquad\qquad = k_1 + 2k_{12}$

C_2 Symmetric 2DHO eigensolutions

$$\mathbf{K} = K\cdot\mathbf{1} - k_{12}\cdot\sigma_B$$

$$K\begin{pmatrix}1&0\\0&1\end{pmatrix}-k_{12}\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}k_1+k_{12}&-k_{12}\\-k_{12}&k_1+k_{12}\end{pmatrix}$$

K-matrix is made of its symmetry operators in *group* $C_2=\{\mathbf{1},\sigma_B\}$ with *product table*:

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

$C_2(\sigma_B)$ spectrally decomposed into $\{\mathbf{P}^+,\mathbf{P}^-\}$ projectors:

$$\mathbf{P}^+=\frac{\mathbf{1}+\sigma_B}{2}=\begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{pmatrix}=|+\rangle\langle+|$$

factored projectors

$$\mathbf{P}^-=\frac{\mathbf{1}-\sigma_B}{2}=\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\\-\frac{1}{2}&\frac{1}{2}\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{-1}{\sqrt{2}}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{pmatrix}=|-\rangle\langle-|$$

$$\mathbf{1}=\mathbf{P}^++\mathbf{P}^-$$

$$\sigma_B=\mathbf{P}^+-\mathbf{P}^-$$

Eigenvalues of σ_B :

$\{\chi^+(\sigma_B)=+1,\ \chi^-(\sigma_B)=-1\}$

Eigenvalues of $\mathbf{K}=K\cdot\mathbf{1} - k_{12}\cdot\sigma_B$:

$\varepsilon^+(\mathbf{K})=K-k_{12},\quad \varepsilon^-(\mathbf{K})=K+k_{12}$

$=k_1 \quad\quad = k_1+2k_{12}$

Diagonalizing transformation (D-tran) of K-matrix:

$$\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}k_1+k_{12}&-k_{12}\\-k_{12}&k_1+k_{12}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{pmatrix}=\begin{pmatrix}k_1&0\\0&k_1+2k_{12}\end{pmatrix}$$

C_2 Symmetric 2DHO eigensolutions

$$\mathbf{K} = K\cdot\mathbf{1} - k_{12}\cdot\sigma_B$$

$$K\begin{pmatrix}1&0\\0&1\end{pmatrix} - k_{12}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}k_1+k_{12} & -k_{12}\\ -k_{12} & k_1+k_{12}\end{pmatrix}$$

K-matrix is made of its symmetry operators in *group* $C_2 = \{\mathbf{1}, \sigma_B\}$ with *product table*:

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

$C_2(\sigma_B)$ spectrally decomposed into $\{\mathbf{P}^+, \mathbf{P}^-\}$ projectors:

$$\mathbf{P}^+ = \frac{\mathbf{1}+\sigma_B}{2} = \begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\ \frac{1}{2}&\frac{1}{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix} = |+\rangle\langle+|$$

factored projectors

$$\mathbf{P}^- = \frac{\mathbf{1}-\sigma_B}{2} = \begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\\ -\frac{1}{2}&\frac{1}{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix} = |-\rangle\langle-|$$

$$\mathbf{1} = \mathbf{P}^+ + \mathbf{P}^-$$

$$\sigma_B = \mathbf{P}^+ - \mathbf{P}^-$$

Eigenvalues of σ_B :

$\{\chi^+(\sigma_B) = +1,\ \chi^-(\sigma_B) = -1\}$

Eigenvalues of $\mathbf{K} = K\cdot\mathbf{1}$ - $k_{12}\cdot\sigma_B$:

$\varepsilon^+(\mathbf{K}) = K - k_{12}$, $\quad \varepsilon^-(\mathbf{K}) = K + k_{12}$

$= k_1 \quad\quad = k_1 + 2k_{12}$

Diagonalizing transformation (D-tran) of K-matrix:

$$\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}k_1+k_{12} & -k_{12}\\ -k_{12} & k_1+k_{12}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{pmatrix} = \begin{pmatrix}k_1 & 0\\ 0 & k_1+2k_{12}\end{pmatrix}$$

(D-tran)

$$\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{pmatrix} = \begin{pmatrix}\langle x_1|+\rangle & \langle x_1|-\rangle\\ \langle x_2|+\rangle & \langle x_2|-\rangle\end{pmatrix}$$

C_2 Symmetric 2DHO eigensolutions

$$\mathbf{K} \quad = \quad K\cdot\mathbf{1} \quad - \quad k_{12}\cdot\sigma_B$$

$$K\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - k_{12}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_1+k_{12} \end{pmatrix}$$

K-matrix is made of its symmetry operators in *group* $C_2 = \{\mathbf{1}, \sigma_B\}$ with *product table*:

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

$C_2(\sigma_B)$ spectrally decomposed into $\{\mathbf{P}^+, \mathbf{P}^-\}$ projectors:

$$\mathbf{1} = \mathbf{P}^+ + \mathbf{P}^-$$

$$\sigma_B = \mathbf{P}^+ - \mathbf{P}^-$$

$$\mathbf{P}^+ = \frac{\mathbf{1}+\sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = |+\rangle\langle+|$$

factored projectors

$$\mathbf{P}^- = \frac{\mathbf{1}-\sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} = |-\rangle\langle-|$$

Eigenvalues of σ_B :

$$\{\chi^+(\sigma_B) = +1,\ \chi^-(\sigma_B) = -1\}$$

Eigenvalues of $\mathbf{K} = K\cdot\mathbf{1} - k_{12}\cdot\sigma_B$:

$$\varepsilon^+(\mathbf{K}) = K - k_{12}, \qquad \varepsilon^-(\mathbf{K}) = K + k_{12}$$

$$= k_1 \qquad\qquad\qquad = k_1 + 2k_{12}$$

Diagonalizing transformation (D-tran) of K-matrix:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_1+k_{12} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} k_1 & 0 \\ 0 & k_1+2k_{12} \end{pmatrix}$$

(D-tran)

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \langle x_1|+\rangle & \langle x_1|-\rangle \\ \langle x_2|+\rangle & \langle x_2|-\rangle \end{pmatrix}$$

(D-tran is its own inverse in this case!)

C_2 Symmetric 2DHO eigensolutions

$$\mathbf{K} = K\cdot\mathbf{1} - k_{12}\cdot\sigma_B$$

$$K\begin{pmatrix}1&0\\0&1\end{pmatrix}-k_{12}\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}k_1+k_{12}&-k_{12}\\-k_{12}&k_1+k_{12}\end{pmatrix}$$

K-matrix is made of its symmetry operators in *group* $C_2=\{\mathbf{1},\sigma_B\}$ with *product table*:

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

$C_2(\sigma_B)$ spectrally decomposed into $\{\mathbf{P}^+,\mathbf{P}^-\}$ projectors:

$$\mathbf{P}^+=\frac{\mathbf{1}+\sigma_B}{2}=\begin{pmatrix}\frac12&\frac12\\\frac12&\frac12\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\end{pmatrix}=|+\rangle\langle+|$$

$$\mathbf{1}=\mathbf{P}^++\mathbf{P}^-$$

factored projectors

$$\sigma_B=\mathbf{P}^+-\mathbf{P}^-$$

$$\mathbf{P}^-=\frac{\mathbf{1}-\sigma_B}{2}=\begin{pmatrix}\frac12&-\frac12\\-\frac12&\frac12\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt2}\\\frac{-1}{\sqrt2}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt2}&\frac{-1}{\sqrt2}\end{pmatrix}=|-\rangle\langle-|$$

Eigenvalues of σ_B :

$\{\chi^+(\sigma_B)=+1,\ \chi^-(\sigma_B)=-1\}$

Diagonalizing transformation (D-tran) of K-matrix:

$$\begin{pmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}&\frac{-1}{\sqrt2}\end{pmatrix}\begin{pmatrix}k_1+k_{12}&-k_{12}\\-k_{12}&k_1+k_{12}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}&\frac{-1}{\sqrt2}\end{pmatrix}=\begin{pmatrix}k_1&0\\0&k_1+2k_{12}\end{pmatrix}$$

Eigenvalues of $\mathbf{K}=K\cdot\mathbf{1}$ - $k_{12}\cdot\sigma_B$:

$\varepsilon^+(\mathbf{K})=K-k_{12}$, $\quad\varepsilon^-(\mathbf{K})=K+k_{12}$

$=k_1$ $\quad\quad = k_1+2k_{12}$

(D-tran)

$$\begin{pmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}&\frac{-1}{\sqrt2}\end{pmatrix}=\begin{pmatrix}\langle x_1|+\rangle&\langle x_1|-\rangle\\\langle x_2|+\rangle&\langle x_2|-\rangle\end{pmatrix}$$

(D-tran is its own inverse in this case!)

$$\begin{pmatrix}\langle+|x_1\rangle&\langle+|x_2\rangle\\\langle-|x_1\rangle&\langle-|x_2\rangle\end{pmatrix}\begin{pmatrix}k_1+k_{12}&k_{12}\\k_{12}&k_1+k_{12}\end{pmatrix}\begin{pmatrix}\langle x_1|+\rangle&\langle x_1|-\rangle\\\langle x_2|+\rangle&\langle x_2|-\rangle\end{pmatrix}=\begin{pmatrix}k_1&0\\0&k_1+2k_{12}\end{pmatrix}$$

$$\begin{pmatrix}\langle+|x_1\rangle&\langle+|x_2\rangle\\\langle-|x_1\rangle&\langle-|x_2\rangle\end{pmatrix}\begin{pmatrix}\langle x_1|\mathbf{K}|x_1\rangle&\langle x_1|\mathbf{K}|x_2\rangle\\\langle x_2|\mathbf{K}|x_1\rangle&\langle x_2|\mathbf{K}|x_2\rangle\end{pmatrix}\begin{pmatrix}\langle x_1|+\rangle&\langle x_1|-\rangle\\\langle x_2|+\rangle&\langle x_2|-\rangle\end{pmatrix}=\begin{pmatrix}\langle+|\mathbf{K}|+\rangle&\langle+|\mathbf{K}|-\rangle\\\langle-|\mathbf{K}|+\rangle&\langle-|\mathbf{K}|-\rangle\end{pmatrix}$$

Full Dirac notation

C_2 Symmetric 2DHO eigensolutions

$$\mathbf{K} = K\cdot\mathbf{1} - k_{12}\cdot\sigma_B$$

$$K\begin{pmatrix}1&0\\0&1\end{pmatrix} - k_{12}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}k_1+k_{12} & -k_{12}\\ -k_{12} & k_1+k_{12}\end{pmatrix}$$

K-matrix is made of its symmetry operators in *group* $C_2=\{\mathbf{1},\sigma_B\}$ with *product table*:

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

$C_2(\sigma_B)$ spectrally decomposed into $\{\mathbf{P}^+,\mathbf{P}^-\}$ projectors:

$$\mathbf{P}^+ = \frac{\mathbf{1}+\sigma_B}{2} = \begin{pmatrix}\frac12 & \frac12\\ \frac12 & \frac12\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt2}\\ \frac{1}{\sqrt2}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt2} & \frac{1}{\sqrt2}\end{pmatrix} = |+\rangle\langle+|$$

factored projectors

$$\mathbf{P}^- = \frac{\mathbf{1}-\sigma_B}{2} = \begin{pmatrix}\frac12 & -\frac12\\ -\frac12 & \frac12\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt2}\\ \frac{-1}{\sqrt2}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt2} & \frac{-1}{\sqrt2}\end{pmatrix} = |-\rangle\langle-|$$

$$\mathbf{1} = \mathbf{P}^+ + \mathbf{P}^-$$

$$\sigma_B = \mathbf{P}^+ - \mathbf{P}^-$$

Eigenvalues of σ_B :

$\{\chi^+(\sigma_B) = +1,\ \chi^-(\sigma_B) = -1\}$

Eigenvalues of $\mathbf{K}=K\cdot\mathbf{1}\ -\ k_{12}\cdot\sigma_B$:

$\varepsilon^+(\mathbf{K}) = K - k_{12},\qquad \varepsilon^-(\mathbf{K}) = K + k_{12}$

$\qquad = k_1 \qquad\qquad\qquad\qquad = k_1+2k_{12}$

Diagonalizing transformation (D-tran) of K-matrix:

$$\begin{pmatrix}\frac{1}{\sqrt2} & \frac{1}{\sqrt2}\\ \frac{1}{\sqrt2} & \frac{-1}{\sqrt2}\end{pmatrix}\begin{pmatrix}k_1+k_{12} & -k_{12}\\ -k_{12} & k_1+k_{12}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt2} & \frac{1}{\sqrt2}\\ \frac{1}{\sqrt2} & \frac{-1}{\sqrt2}\end{pmatrix} = \begin{pmatrix}k_1 & 0\\ 0 & k_1+2k_{12}\end{pmatrix}$$

(D-tran)

$$\begin{pmatrix}\frac{1}{\sqrt2} & \frac{1}{\sqrt2}\\ \frac{1}{\sqrt2} & \frac{-1}{\sqrt2}\end{pmatrix} = \begin{pmatrix}\langle x_1|+\rangle & \langle x_1|-\rangle\\ \langle x_2|+\rangle & \langle x_2|-\rangle\end{pmatrix}$$

(D-tran is its own inverse in this case!)

$$\begin{pmatrix}\langle +|x_1\rangle & \langle +|x_2\rangle\\ \langle -|x_1\rangle & \langle -|x_2\rangle\end{pmatrix}\begin{pmatrix}k_1+k_{12} & k_{12}\\ k_{12} & k_1+k_{12}\end{pmatrix}\begin{pmatrix}\langle x_1|+\rangle & \langle x_1|-\rangle\\ \langle x_2|+\rangle & \langle x_2|-\rangle\end{pmatrix} = \begin{pmatrix}k_1 & 0\\ 0 & k_1+2k_{12}\end{pmatrix}$$

$$\begin{pmatrix}\langle +|x_1\rangle & \langle +|x_2\rangle\\ \langle -|x_1\rangle & \langle -|x_2\rangle\end{pmatrix}\begin{pmatrix}\langle x_1|\mathbf{K}|x_1\rangle & \langle x_1|\mathbf{K}|x_2\rangle\\ \langle x_2|\mathbf{K}|x_1\rangle & \langle x_2|\mathbf{K}|x_2\rangle\end{pmatrix}\begin{pmatrix}\langle x_1|+\rangle & \langle x_1|-\rangle\\ \langle x_2|+\rangle & \langle x_2|-\rangle\end{pmatrix} = \begin{pmatrix}\langle +|\mathbf{K}|+\rangle & \langle +|\mathbf{K}|-\rangle\\ \langle -|\mathbf{K}|+\rangle & \langle -|\mathbf{K}|-\rangle\end{pmatrix}$$

Full Dirac notation

$$\mathbf{T}(\pm\leftarrow x_j)\cdot\mathbf{K}\cdot\mathbf{T}^\dagger(\pm\leftarrow x_j)$$

C_2 Symmetric 2DHO eigensolutions

$$\mathbf{K} = K\cdot\mathbf{1} - k_{12}\cdot\sigma_B$$

$$K\begin{pmatrix}1&0\\0&1\end{pmatrix} - k_{12}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}k_1+k_{12}&-k_{12}\\-k_{12}&k_1+k_{12}\end{pmatrix}$$

K-matrix is made of its symmetry operators in *group* $C_2=\{\mathbf{1},\sigma_B\}$ with *product table*:

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

$C_2(\sigma_B)$ spectrally decomposed into $\{\mathbf{P}^+,\mathbf{P}^-\}$ projectors:

$$\mathbf{P}^+ = \frac{\mathbf{1}+\sigma_B}{2} = \begin{pmatrix}\frac12&\frac12\\\frac12&\frac12\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\end{pmatrix} = |+\rangle\langle+|$$

factored projectors

$$\mathbf{P}^- = \frac{\mathbf{1}-\sigma_B}{2} = \begin{pmatrix}\frac12&-\frac12\\-\frac12&\frac12\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt2}\\\frac{-1}{\sqrt2}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt2}&\frac{-1}{\sqrt2}\end{pmatrix} = |-\rangle\langle-|$$

$$\mathbf{1} = \mathbf{P}^+ + \mathbf{P}^-$$

$$\sigma_B = \mathbf{P}^+ - \mathbf{P}^-$$

Eigenvalues of σ_B:

$$\{\chi^+(\sigma_B) = +1,\ \chi^-(\sigma_B) = -1\}$$

Eigenvalues of $\mathbf{K}=K\cdot\mathbf{1} - k_{12}\cdot\sigma_B$:

$$\varepsilon^+(\mathbf{K}) = K - k_{12} = k_1,\qquad \varepsilon^-(\mathbf{K}) = K + k_{12} = k_1+2k_{12}$$

Diagonalizing transformation (D-tran) of K-matrix:

$$\begin{pmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}&\frac{-1}{\sqrt2}\end{pmatrix}\begin{pmatrix}k_1+k_{12}&-k_{12}\\-k_{12}&k_1+k_{12}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}&\frac{-1}{\sqrt2}\end{pmatrix} = \begin{pmatrix}k_1&0\\0&k_1+2k_{12}\end{pmatrix}$$

$$\begin{pmatrix}\langle+|x_1\rangle&\langle+|x_2\rangle\\\langle-|x_1\rangle&\langle-|x_2\rangle\end{pmatrix}\begin{pmatrix}k_1+k_{12}&k_{12}\\k_{12}&k_1+k_{12}\end{pmatrix}\begin{pmatrix}\langle x_1|+\rangle&\langle x_1|-\rangle\\\langle x_2|+\rangle&\langle x_2|-\rangle\end{pmatrix} = \begin{pmatrix}k_1&0\\0&k_1+2k_{12}\end{pmatrix}$$

$$\begin{pmatrix}\langle+|x_1\rangle&\langle+|x_2\rangle\\\langle-|x_1\rangle&\langle-|x_2\rangle\end{pmatrix}\begin{pmatrix}\langle x_1|\mathbf{K}|x_1\rangle&\langle x_1|\mathbf{K}|x_2\rangle\\\langle x_2|\mathbf{K}|x_1\rangle&\langle x_2|\mathbf{K}|x_2\rangle\end{pmatrix}\begin{pmatrix}\langle x_1|+\rangle&\langle x_1|-\rangle\\\langle x_2|+\rangle&\langle x_2|-\rangle\end{pmatrix} = \begin{pmatrix}\langle+|\mathbf{K}|+\rangle&\langle+|\mathbf{K}|-\rangle\\\langle-|\mathbf{K}|+\rangle&\langle-|\mathbf{K}|-\rangle\end{pmatrix}$$

Full Dirac notation

(D-tran)

$$\begin{pmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}&\frac{-1}{\sqrt2}\end{pmatrix} = \begin{pmatrix}\langle x_1|+\rangle&\langle x_1|-\rangle\\\langle x_2|+\rangle&\langle x_2|-\rangle\end{pmatrix}$$

(D-tran is its own inverse in this case!)

$$\mathbf{T}(\pm\leftarrow x_j)|x_1\rangle = |+\rangle$$
$$\mathbf{T}(\pm\leftarrow x_j)|x_2\rangle = |-\rangle$$

$$\mathbf{T}(\pm\leftarrow x_j)\cdot\mathbf{K}\cdot\mathbf{T}^\dagger(\pm\leftarrow x_j)$$

$$|x_1\rangle = \mathbf{T}^\dagger(\pm\leftarrow x_j)|+\rangle$$
$$|x_2\rangle = \mathbf{T}^\dagger(\pm\leftarrow x_j)|-\rangle$$

$$\mathbf{T}^\dagger(\pm\leftarrow x_j) = \mathbf{T}(x_j\leftarrow\pm)$$

C_2 Symmetric 2DHO eigensolutions

$$\mathbf{K} = K\cdot\mathbf{1} - k_{12}\cdot\sigma_B$$

$$K\begin{pmatrix}1&0\\0&1\end{pmatrix} - k_{12}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}k_1+k_{12} & -k_{12}\\ -k_{12} & k_1+k_{12}\end{pmatrix}$$

K-matrix is made of its symmetry operators

in *group* $C_2=\{\mathbf{1},\sigma_B\}$ with *product table*:

$C_2(\sigma_B)$ spectrally decomposed into $\{\mathbf{P}^+,\mathbf{P}^-\}$projectors:

$$\mathbf{P}^+ = \frac{\mathbf{1}+\sigma_B}{2} = \begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\ \frac{1}{2}&\frac{1}{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix} = |+\rangle\langle+|$$

$$\mathbf{1} = \mathbf{P}^+ + \mathbf{P}^-$$

factored projectors

$$\sigma_B = \mathbf{P}^+ - \mathbf{P}^-$$

$$\mathbf{P}^- = \frac{\mathbf{1}-\sigma_B}{2} = \begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\\ -\frac{1}{2}&\frac{1}{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix} = |-\rangle\langle-|$$

Eigenvalues of σ_B :

$\{\chi^+(\sigma_B) = +1,\ \chi^-(\sigma_B) = -1\}$

Diagonalizing transformation (D-tran) of K-matrix:

$$\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}k_1+k_{12} & -k_{12}\\ -k_{12} & k_1+k_{12}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{pmatrix} = \begin{pmatrix}k_1 & 0\\ 0 & k_1+2k_{12}\end{pmatrix}$$

Eigenvalues of $\mathbf{K}=K\cdot\mathbf{1}\ -\ k_{12}\cdot\sigma_B$:

$$\varepsilon^+(\mathbf{K}) = K-k_{12}, \qquad \varepsilon^-(\mathbf{K}) = K+k_{12}$$
$$= k_1 \qquad\qquad\qquad = k_1+2k_{12}$$

(D-tran)

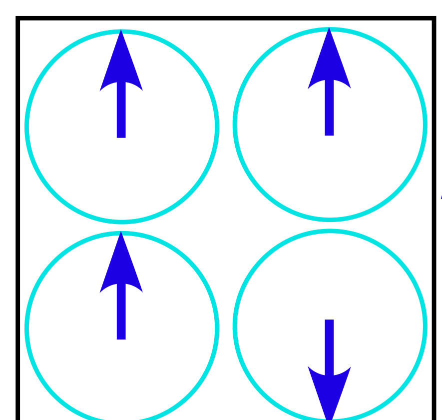
$$\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{pmatrix} = \begin{pmatrix}\langle x_1|+\rangle & \langle x_1|-\rangle\\ \langle x_2|+\rangle & \langle x_2|-\rangle\end{pmatrix}$$

(D-tran is its own inverse in this case!)

C_2 mode phase character tables

p is position

	$p=0$	$p=1$
$m=0_2$	1	1
$m=1_2$	1	-1



norm: $1/\sqrt{2}$

m is wave-number or "momentum"

C_2 Symmetric 2DHO eigensolutions

$$\mathbf{K} = K\cdot\mathbf{1} - k_{12}\cdot\sigma_B$$

$$K\begin{pmatrix}1&0\\0&1\end{pmatrix} - k_{12}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}k_1+k_{12} & -k_{12}\\ -k_{12} & k_1+k_{12}\end{pmatrix}$$

K-matrix is made of its symmetry operators

in *group* $C_2 = \{\mathbf{1}, \sigma_B\}$ with *product table*:

$C_2(\sigma_B)$ spectrally decomposed into $\{\mathbf{P}^+, \mathbf{P}^-\}$ projectors:

$$\mathbf{1} = \mathbf{P}^+ + \mathbf{P}^-$$

$$\sigma_B = \mathbf{P}^+ - \mathbf{P}^-$$

$$\mathbf{P}^+ = \frac{\mathbf{1}+\sigma_B}{2} = \begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\ \frac{1}{2}&\frac{1}{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix} = |+\rangle\langle+|$$

factored projectors

$$\mathbf{P}^- = \frac{\mathbf{1}-\sigma_B}{2} = \begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\\ -\frac{1}{2}&\frac{1}{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix} = |-\rangle\langle-|$$

Eigenvalues of σ_B :

$\{\chi^+(\sigma_B) = +1,\ \chi^-(\sigma_B) = -1\}$

Diagonalizing transformation (D-tran) of K-matrix:

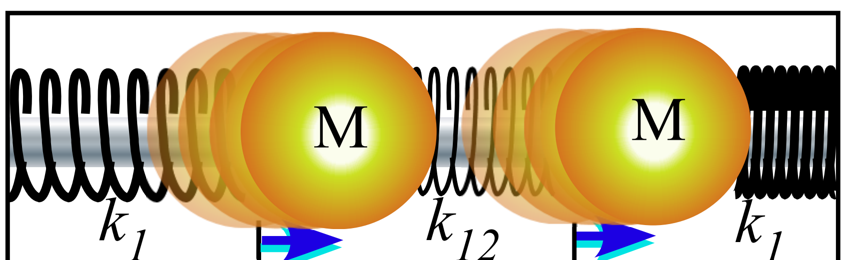
$$\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}k_1+k_{12} & -k_{12}\\ -k_{12} & k_1+k_{12}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{pmatrix} = \begin{pmatrix}k_1 & 0\\ 0 & k_1+2k_{12}\end{pmatrix}$$

Eigenvalues of $\mathbf{K}=K\cdot\mathbf{1}$ - $k_{12}\cdot\sigma_B$:

$\varepsilon^+(\mathbf{K}) = K - k_{12}$, $\quad\varepsilon^-(\mathbf{K}) = K + k_{12}$

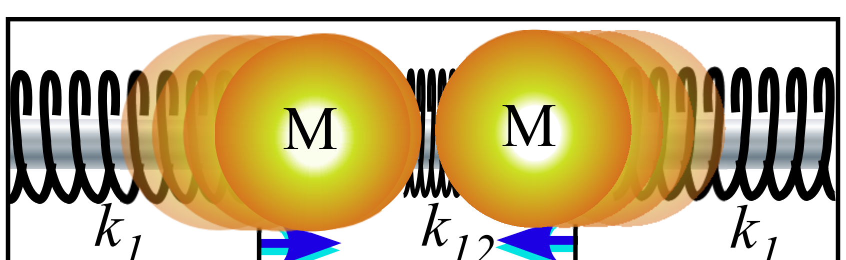
$= k_1$ $\quad\quad = k_1+2k_{12}$

Even mode $|+\rangle=|0_2\rangle = \begin{pmatrix}1\\1\end{pmatrix}/\sqrt{2}$



$x_0=1/\sqrt{2}$ $\quad x_1=1/\sqrt{2}$

Odd mode $|-\rangle=|1_2\rangle = \begin{pmatrix}1\\-1\end{pmatrix}/\sqrt{2}$

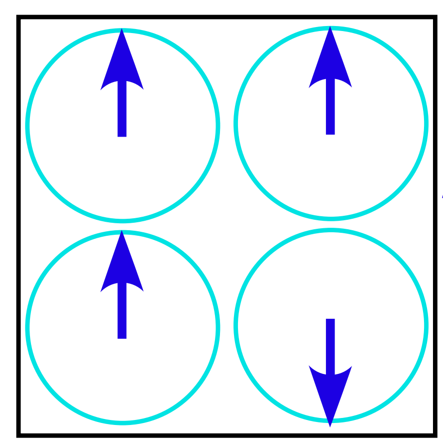


$x_0=1/\sqrt{2}$ $\quad x_1=-1/\sqrt{2}$

C_2 *mode phase character tables*

p is position

	$p=0$	$p=1$
$m=0_2$	1	1
$m=1_2$	1	-1



norm: $1/\sqrt{2}$

m is wave-number or "momentum"

(D-tran)

$$\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{pmatrix} = \begin{pmatrix}\langle x_1|+\rangle & \langle x_1|-\rangle\\ \langle x_2|+\rangle & \langle x_2|-\rangle\end{pmatrix}$$

(D-tran is its own inverse in this case!)

Review:How symmetry groups become eigen-solvers

C_2 Symmetric two-dimensional harmonic oscillators (2DHO)

C_2 (Bilateral σ_B reflection) symmetry conditions:

Minimal equation of σ_B and spectral decomposition of $C_2(\sigma_B)$

C_2 Symmetric 2DHO eigensolutions

C_2 Mode phase character table

→ *C_2 Symmetric 2DHO uncoupling and **mixed mode** projector algebra*

*2D-HO **beats and mixed mode geometry***

C_2 Symmetric 2DHO uncoupling

2D HO Matrix operator equations are coupled in $\{x_1,x_2\}$-basis

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_1+k_{12} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\left|\ddot{\mathbf{x}}\right\rangle \quad = \quad - \quad \mathbf{K} \quad \left|\mathbf{x}\right\rangle$$

$$\begin{pmatrix} \left\langle x_1\middle|\ddot{\mathbf{x}}\right\rangle \\ \left\langle x_2\middle|\ddot{\mathbf{x}}\right\rangle \end{pmatrix} = -\begin{pmatrix} \left\langle x_1\middle|\mathbf{K}\middle|x_1\right\rangle & \left\langle x_1\middle|\mathbf{K}\middle|x_2\right\rangle \\ \left\langle x_2\middle|\mathbf{K}\middle|x_1\right\rangle & \left\langle x_2\middle|\mathbf{K}\middle|x_2\right\rangle \end{pmatrix}\begin{pmatrix} \left\langle x_1\middle|\mathbf{x}\right\rangle \\ \left\langle x_2\middle|\mathbf{x}\right\rangle \end{pmatrix}$$

C_2 Symmetric 2DHO uncoupling

2D HO Matrix operator equations are coupled in $\{x_1,x_2\}$-basis *...but are **un**coupled in* $\{+,-\}$-basis

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_1+k_{12} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\left|\ddot{\mathbf{x}}\right\rangle = -\mathbf{K}\left|\mathbf{x}\right\rangle$$

$$\begin{pmatrix} \langle x_1|\ddot{\mathbf{x}}\rangle \\ \langle x_2|\ddot{\mathbf{x}}\rangle \end{pmatrix} = -\begin{pmatrix} \langle x_1|\mathbf{K}|x_1\rangle & \langle x_1|\mathbf{K}|x_2\rangle \\ \langle x_2|\mathbf{K}|x_1\rangle & \langle x_2|\mathbf{K}|x_2\rangle \end{pmatrix}\begin{pmatrix} \langle x_1|\mathbf{x}\rangle \\ \langle x_2|\mathbf{x}\rangle \end{pmatrix}$$

$$\begin{pmatrix} \ddot{x}_+ \\ \ddot{x}_- \end{pmatrix} = -\begin{pmatrix} k_1 & 0 \\ 0 & k_1+2k_{12} \end{pmatrix}\begin{pmatrix} x_+ \\ x_- \end{pmatrix}$$

$$\left|\ddot{\mathbf{x}}\right\rangle = -\mathbf{K}\left|\mathbf{x}\right\rangle$$

$$\begin{pmatrix} \langle +|\ddot{\mathbf{x}}\rangle \\ \langle -|\ddot{\mathbf{x}}\rangle \end{pmatrix} = -\begin{pmatrix} \langle +|\mathbf{K}|+\rangle & \cancel{\langle +|\mathbf{K}|-\rangle} \\ \cancel{\langle -|\mathbf{K}|+\rangle} & \langle -|\mathbf{K}|-\rangle \end{pmatrix}\begin{pmatrix} \langle +|\mathbf{x}\rangle \\ \langle -|\mathbf{x}\rangle \end{pmatrix}$$

C_2 Symmetric 2DHO uncoupling

2D HO Matrix operator equations are coupled in $\{x_1,x_2\}$-basis *...but are* ***un****coupled in* $\{+,-\}$-basis

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_1+k_{12} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$|\ddot{\mathbf{x}}\rangle \quad = \quad - \quad \mathbf{K} \quad |\mathbf{x}\rangle$$

$$\begin{pmatrix} \langle x_1|\ddot{\mathbf{x}}\rangle \\ \langle x_2|\ddot{\mathbf{x}}\rangle \end{pmatrix} = -\begin{pmatrix} \langle x_1|\mathbf{K}|x_1\rangle & \langle x_1|\mathbf{K}|x_2\rangle \\ \langle x_2|\mathbf{K}|x_1\rangle & \langle x_2|\mathbf{K}|x_2\rangle \end{pmatrix}\begin{pmatrix} \langle x_1|\mathbf{x}\rangle \\ \langle x_2|\mathbf{x}\rangle \end{pmatrix}$$

$$\begin{pmatrix} \ddot{x}_+ \\ \ddot{x}_- \end{pmatrix} = -\begin{pmatrix} k_1 & 0 \\ 0 & k_1+2k_{12} \end{pmatrix}\begin{pmatrix} x_+ \\ x_- \end{pmatrix}$$

$$|\ddot{\mathbf{x}}\rangle \quad = \quad - \quad \mathbf{K} \quad |\mathbf{x}\rangle$$

$$\begin{pmatrix} \langle +|\ddot{\mathbf{x}}\rangle \\ \langle -|\ddot{\mathbf{x}}\rangle \end{pmatrix} = -\begin{pmatrix} \langle +|\mathbf{K}|+\rangle & \cancel{\langle +|\mathbf{K}|-\rangle} \\ \cancel{\langle -|\mathbf{K}|+\rangle} & \langle -|\mathbf{K}|-\rangle \end{pmatrix}\begin{pmatrix} \langle +|\mathbf{x}\rangle \\ \langle -|\mathbf{x}\rangle \end{pmatrix}$$

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C_2 Symmetric 2DHO uncoupling

2D HO Matrix operator equations are coupled in $\{x_1,x_2\}$-basis *...but are* ***un****coupled in* $\{+,-\}$-basis

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_1+k_{12} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

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Each mode runs independently

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AM modulation results

$$\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}=\begin{pmatrix}\dfrac{e^{-i\omega_+t}+e^{-i\omega_-t}}{2}\\[2ex] \dfrac{e^{-i\omega_+t}-e^{-i\omega_-t}}{2}\end{pmatrix}$$

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AM modulation results

$$\frac{e^{ia}+e^{ib}}{2}=e^{i\frac{a+b}{2}}\frac{e^{i\frac{a-b}{2}}+e^{-i\frac{a-b}{2}}}{2}$$

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$$\mathbf{x}(0) = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}\left(\tfrac{1}{\sqrt{2}}\right) + \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{pmatrix}\left(\tfrac{1}{\sqrt{2}}\right) = \tfrac{1}{\sqrt{2}}|+\rangle + \tfrac{1}{\sqrt{2}}|-\rangle = \begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\end{pmatrix} + \begin{pmatrix}\frac{1}{2}\\ -\frac{1}{2}\end{pmatrix}$$

$$\text{so: } \mathbf{x}(t) = e^{-i\omega_+ t}\begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\end{pmatrix} + e^{-i\omega_- t}\begin{pmatrix}\frac{1}{2}\\ -\frac{1}{2}\end{pmatrix}$$

AM modulation results

$$\frac{e^{ia}+e^{ib}}{2} = e^{i\frac{a+b}{2}}\frac{e^{i\frac{a-b}{2}} + e^{-i\frac{a-b}{2}}}{2}$$

$$\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix} = \begin{pmatrix}\dfrac{e^{-i\omega_+ t}+e^{-i\omega_- t}}{2}\\[2ex] \dfrac{e^{-i\omega_+ t}-e^{-i\omega_- t}}{2}\end{pmatrix} = \frac{e^{-i\frac{(\omega_+ + \omega_-)}{2}t}}{2}\begin{pmatrix}e^{-i\frac{(\omega_+ - \omega_-)}{2}t} + e^{i\frac{(\omega_+ - \omega_-)}{2}t}\\ e^{-i\frac{(\omega_+ - \omega_-)}{2}t} - e^{i\frac{(\omega_+ - \omega_-)}{2}t}\end{pmatrix}$$

C_2 Symmetric 2DHO uncoupling

2D HO Matrix operator equations are coupled in $\{x_1,x_2\}$-basis *...but are* ***un****coupled in* $\{+,-\}$-basis

$$\begin{pmatrix}\ddot{x}_1\\ \ddot{x}_2\end{pmatrix} = -\begin{pmatrix}k_1+k_{12} & -k_{12}\\ -k_{12} & k_1+k_{12}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

$$|\ddot{\mathbf{x}}\rangle \quad = \quad -\quad \mathbf{K} \quad |\mathbf{x}\rangle$$

$$\begin{pmatrix}\langle x_1|\ddot{\mathbf{x}}\rangle\\ \langle x_2|\ddot{\mathbf{x}}\rangle\end{pmatrix} = -\begin{pmatrix}\langle x_1|\mathbf{K}|x_1\rangle & \langle x_1|\mathbf{K}|x_2\rangle\\ \langle x_2|\mathbf{K}|x_1\rangle & \langle x_2|\mathbf{K}|x_2\rangle\end{pmatrix}\begin{pmatrix}\langle x_1|\mathbf{x}\rangle\\ \langle x_2|\mathbf{x}\rangle\end{pmatrix}$$

$$\begin{pmatrix}\ddot{x}_+\\ \ddot{x}_-\end{pmatrix} = -\begin{pmatrix}k_1 & 0\\ 0 & k_1+2k_{12}\end{pmatrix}\begin{pmatrix}x_+\\ x_-\end{pmatrix}$$

$$|\ddot{\mathbf{x}}\rangle \quad = \quad -\quad \mathbf{K} \quad |\mathbf{x}\rangle$$

$$\begin{pmatrix}\langle +|\ddot{\mathbf{x}}\rangle\\ \langle -|\ddot{\mathbf{x}}\rangle\end{pmatrix} = -\begin{pmatrix}\langle +|\mathbf{K}|+\rangle & \cancel{\langle +|\mathbf{K}|-\rangle}\\ \cancel{\langle -|\mathbf{K}|+\rangle} & \langle -|\mathbf{K}|-\rangle\end{pmatrix}\begin{pmatrix}\langle +|\mathbf{x}\rangle\\ \langle -|\mathbf{x}\rangle\end{pmatrix}$$

C_2 Symmetric 2DHO ***un****coupled dynamics*

Eigenbra vectors: $\langle +| = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}$, $\langle -| = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix}$

$$\begin{pmatrix}M\ddot{x}_+ + (k_1)x_+\\ M\ddot{x}_- + (k_1+2k_{12})x_-\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

Each mode runs independently

$(+)$-*mode at frequency* $\omega_+ = \sqrt{(k_1/M)}$

$(-)$-*mode at frequency* $\omega_- = \sqrt{(k_1+2k_{12})/M}$

Eigenket vectors: $|+\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$, $|-\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{pmatrix}$

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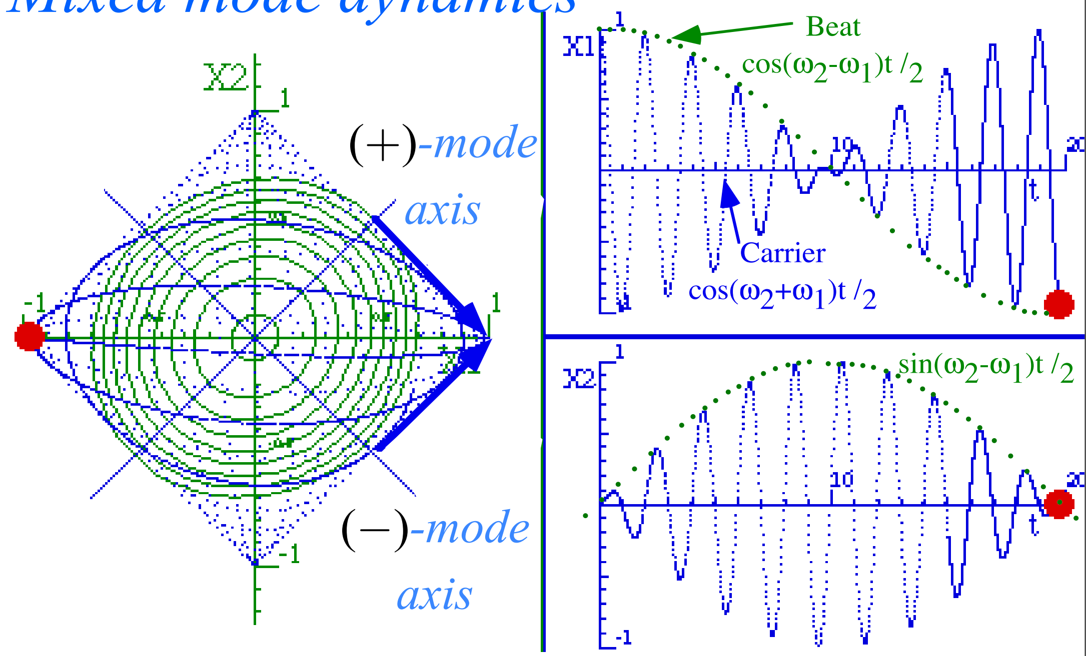
Mixed mode dynamics



100% AM modulation results

$$\frac{e^{ia}+e^{ib}}{2} = e^{i\frac{a+b}{2}}\frac{e^{i\frac{a-b}{2}}+e^{-i\frac{a-b}{2}}}{2} = e^{i\frac{a+b}{2}}\cos\left(\frac{a-b}{2}\right)$$

$$\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix} = \begin{pmatrix}\dfrac{e^{-i\omega_+ t}+e^{-i\omega_- t}}{2}\\ \dfrac{e^{-i\omega_+ t}-e^{-i\omega_- t}}{2}\end{pmatrix} = \frac{e^{-i\frac{(\omega_++\omega_-)}{2}t}}{2}\begin{pmatrix}e^{-i\frac{(\omega_+-\omega_-)}{2}t}+e^{i\frac{(\omega_+-\omega_-)}{2}t}\\ e^{-i\frac{(\omega_+-\omega_-)}{2}t}-e^{i\frac{(\omega_+-\omega_-)}{2}t}\end{pmatrix} = e^{-i\frac{(\omega_++\omega_-)}{2}t}\begin{pmatrix}\cos\frac{(\omega_- - \omega_+)t}{2}\\ i\sin\frac{(\omega_- - \omega_+)t}{2}\end{pmatrix}$$

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C_2 Symmetric 2DHO ***un****coupled dynamics*

Eigenbra vectors: $\langle +| = \left(\frac{1}{\sqrt2}\ \ \frac{1}{\sqrt2}\right),\quad \langle -| = \left(\frac{1}{\sqrt2}\ \ \frac{-1}{\sqrt2}\right)$

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(+)*-mode at frequency* $\omega_+=\sqrt{(k_1/M)}$

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$$\mathbf{x}(0) = \begin{pmatrix}\frac{1}{\sqrt2}\\ \frac{1}{\sqrt2}\end{pmatrix}\left(\tfrac{1}{\sqrt2}\right) + \begin{pmatrix}\frac{1}{\sqrt2}\\ \frac{-1}{\sqrt2}\end{pmatrix}\left(\tfrac{1}{\sqrt2}\right) = \tfrac{1}{\sqrt2}|+\rangle + \tfrac{1}{\sqrt2}|-\rangle = \begin{pmatrix}\frac12\\ \frac12\end{pmatrix} + \begin{pmatrix}\frac12\\ -\frac12\end{pmatrix}$$

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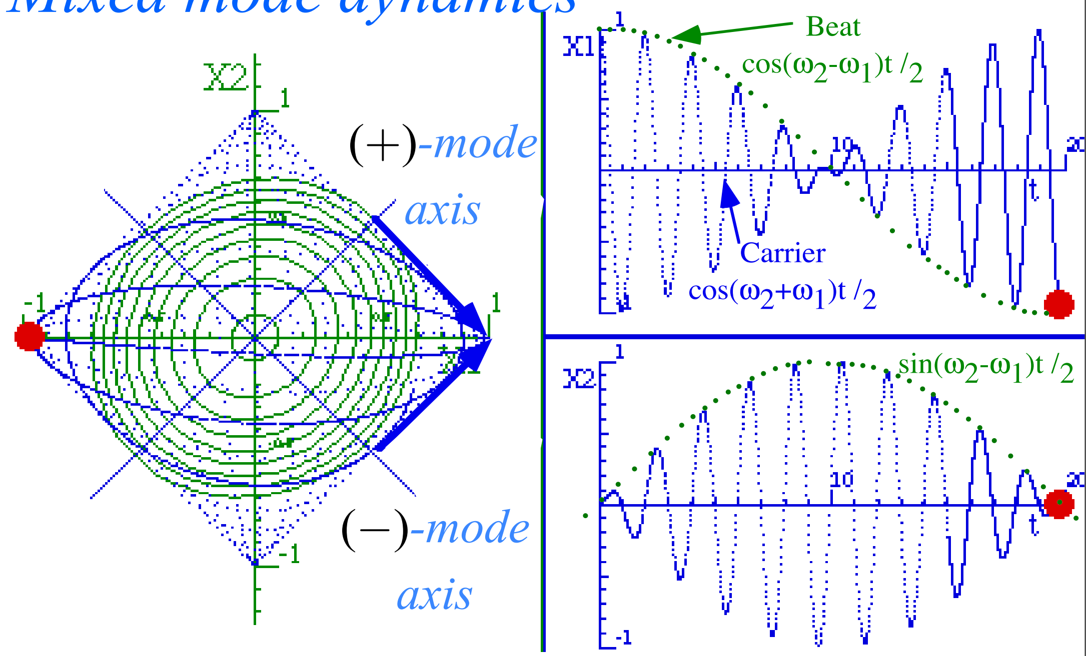
Mixed mode dynamics



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Note the i *phase*

Review:How symmetry groups become eigen-solvers

C_2 Symmetric two-dimensional harmonic oscillators (2DHO)

C_2 (Bilateral σ_B reflection) symmetry conditions:

Minimal equation of σ_B and spectral decomposition of $C_2(\sigma_B)$

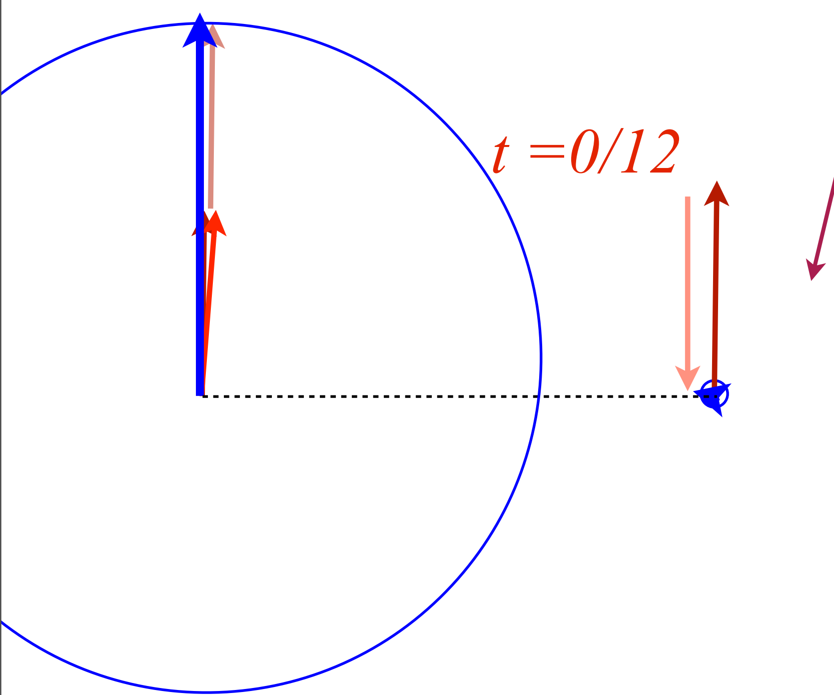
C_2 Symmetric 2DHO eigensolutions

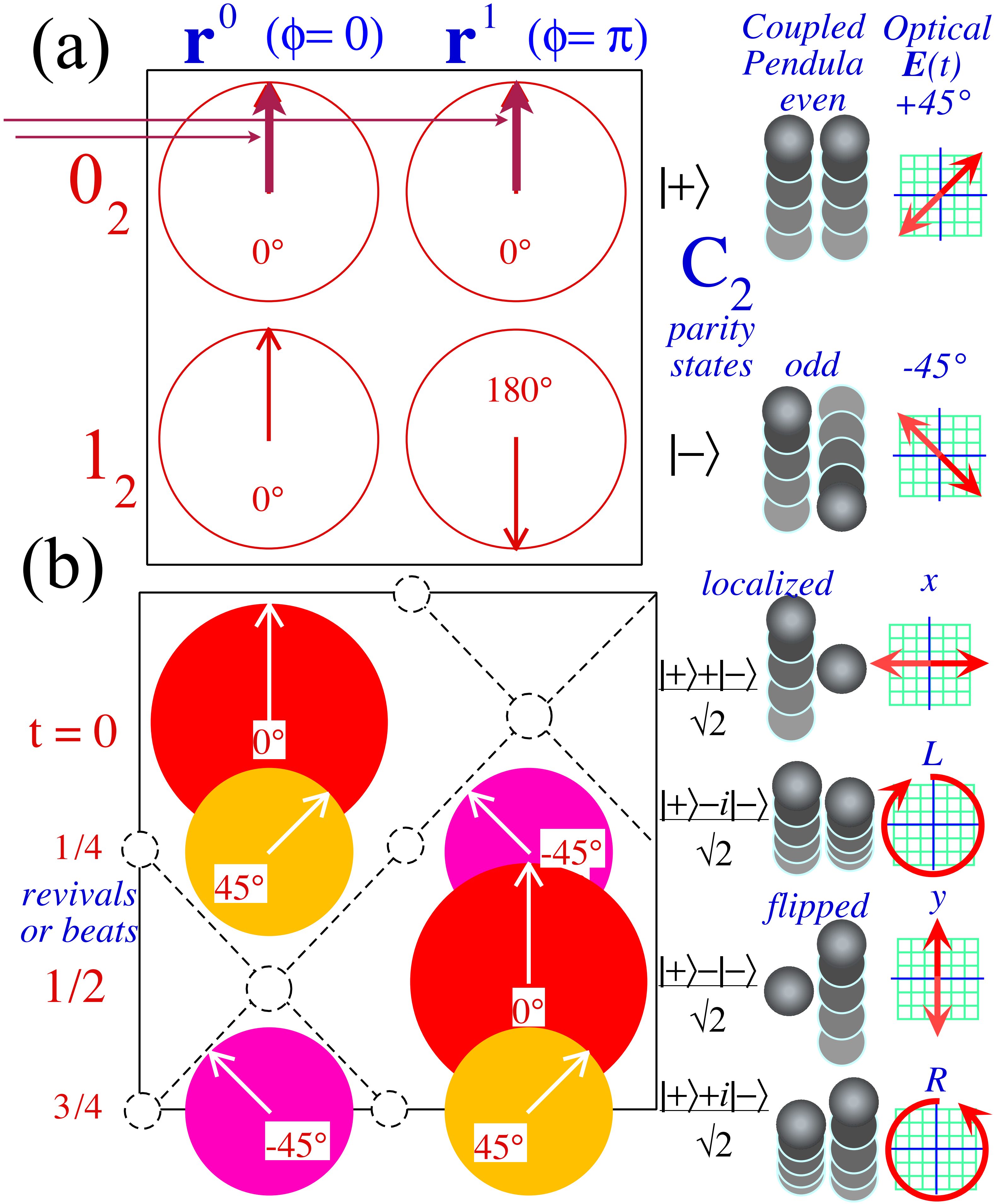
C_2 Mode phase character table

C_2 Symmetric 2DHO uncoupling

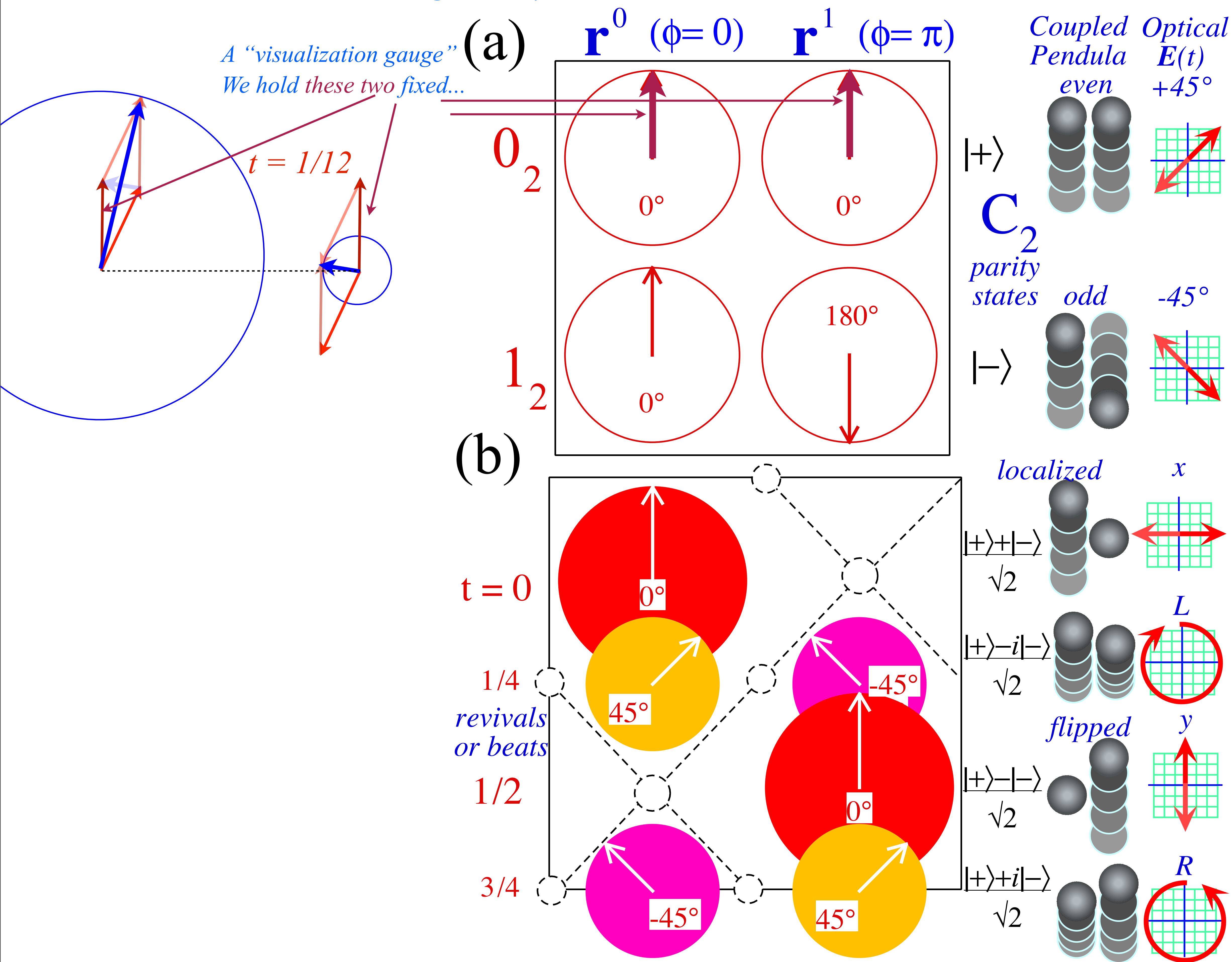
→ *2D-HO beats and mixed mode geometry*

2D-HO beats and mixed mode geometry

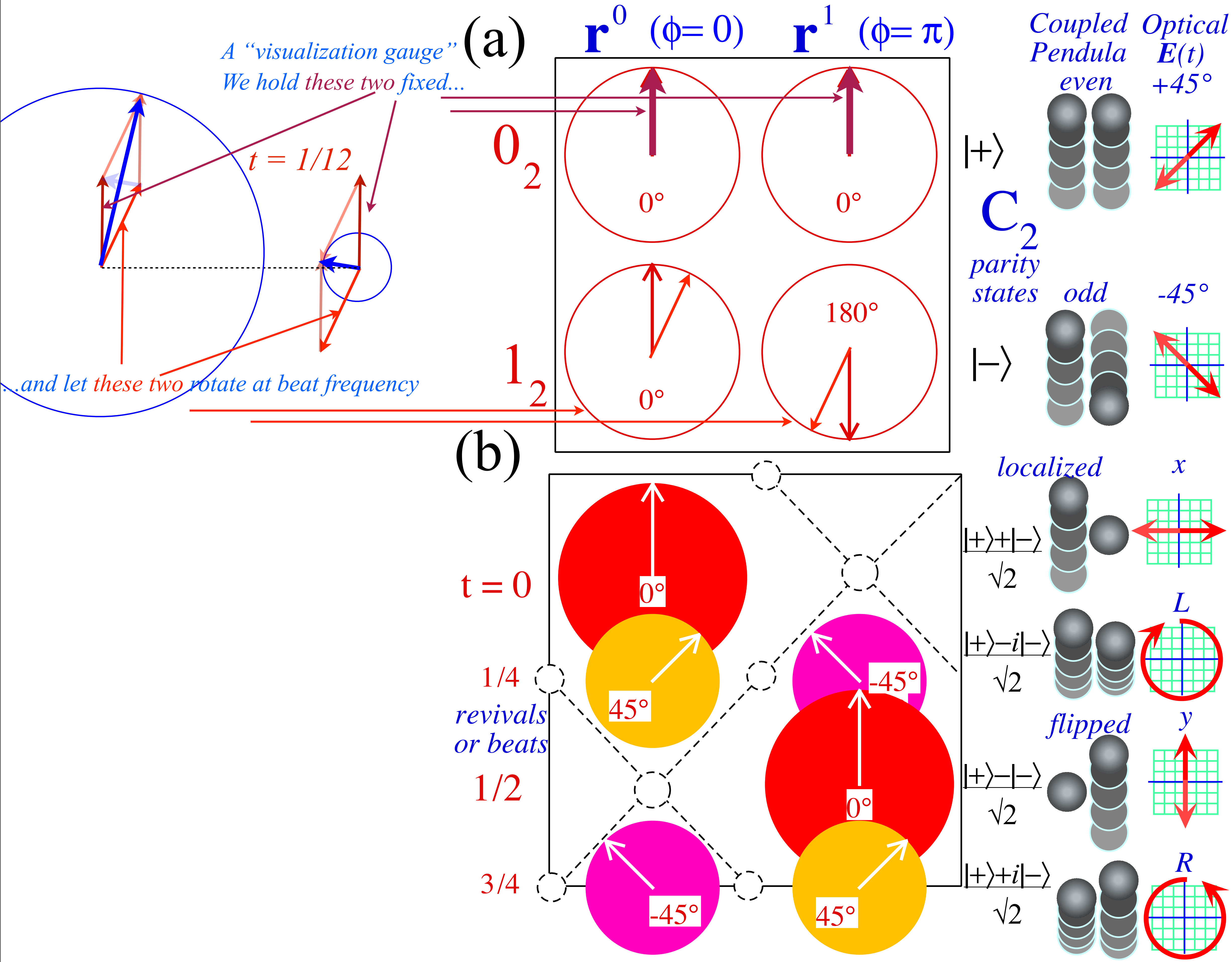




2D-HO beats and mixed mode geometry



2D-HO beats and mixed mode geometry



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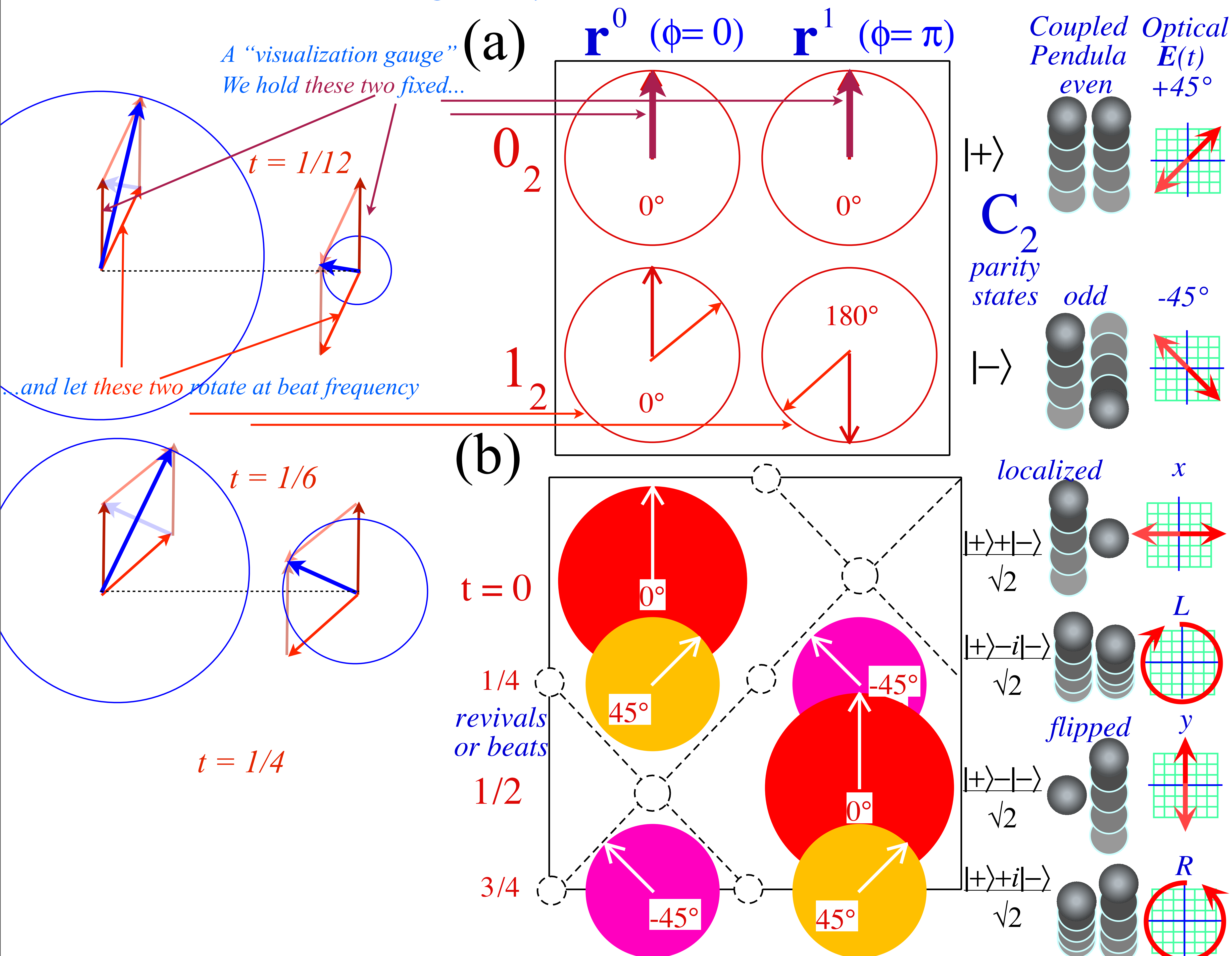

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