Group Theory in Quantum Mechanics

Lecture 8 *(2.7.13)*

Spinor and vector representations of U(2) and R(3) Operators

(Quantum Theory for Computer Age - Ch. 10A-B of Unit 3)
(Principles of Symmetry, Dynamics, and Spectroscopy - Sec. 1-3 of Ch. 5)

Review:How "Crazy-Thing"-Theorem makes spinor and vector representation matrices
- *Half-angle $\Theta/2 = \varphi$ replacement and Darboux crank axis operators*

Operator-on-Operator transformations

Product algebra for Pauli's σ_μ and Hamilton's $\boldsymbol{q}_\mu = -i\sigma_\mu$
- *Group product algebra*
  - *Jordan-Pauli identity and U(2) product* $\mathbf{R}[\boldsymbol{\Theta}]\mathbf{R}[\boldsymbol{\Theta}'] = \mathbf{R}[\boldsymbol{\Theta}''']$*- formula*
  - *Transformation* $\mathbf{R}[\boldsymbol{\Theta}]\sigma_\mu\mathbf{R}[\boldsymbol{\Theta}]^\dagger$ *of spinor σ_μ-operators*
  - *Transformation* $\mathbf{R}[\boldsymbol{\Theta}]\mathbf{R}[\boldsymbol{\Theta}']\mathbf{R}[\boldsymbol{\Theta}]^\dagger$ *of group-operators*
  - *Operator-on-Operator transformations*

Geometry of groups: Hamilton's turns and It's all done with mirrors!
- *Group product geometry*
  - *U(2) product* $\mathbf{R}[\boldsymbol{\Theta}]\mathbf{R}[\boldsymbol{\Theta}'] = \mathbf{R}[\boldsymbol{\Theta}''']$*- geometry*
  - *Transformation* $\mathbf{R}[\boldsymbol{\Theta}]\mathbf{R}[\boldsymbol{\Theta}']\mathbf{R}[\boldsymbol{\Theta}]^\dagger$ *geometry*

Euler $\mathbf{R}(\alpha\beta\gamma)$ versus *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$
- *Euler* $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$
- *Euler* $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\boldsymbol{\Theta}=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed*
- *R(3)-U(2) slide rule for converting* $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$
- *Euler* $\mathbf{R}(\alpha\beta\gamma)$ *Sundial*

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Operator-on-Operator transformations

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Group product algebra

Jordan-Pauli identity and U(2) product $\mathbf{R}[\Theta]\mathbf{R}[\Theta']=\mathbf{R}[\Theta''']$*- formula*

Transformation $\mathbf{R}[\Theta]\sigma_\mu\mathbf{R}[\Theta]^\dagger$ *of spinor σ_μ-operators*

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ *of group-operators*

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Geometry of groups: Hamilton's turns and It's all done with mirrors!

Group product geometry

U(2) product $\mathbf{R}[\Theta]\mathbf{R}[\Theta']=\mathbf{R}[\Theta''']$*- geometry*

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ *geometry*

Euler $\mathbf{R}(\alpha\beta\gamma)$ versus *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

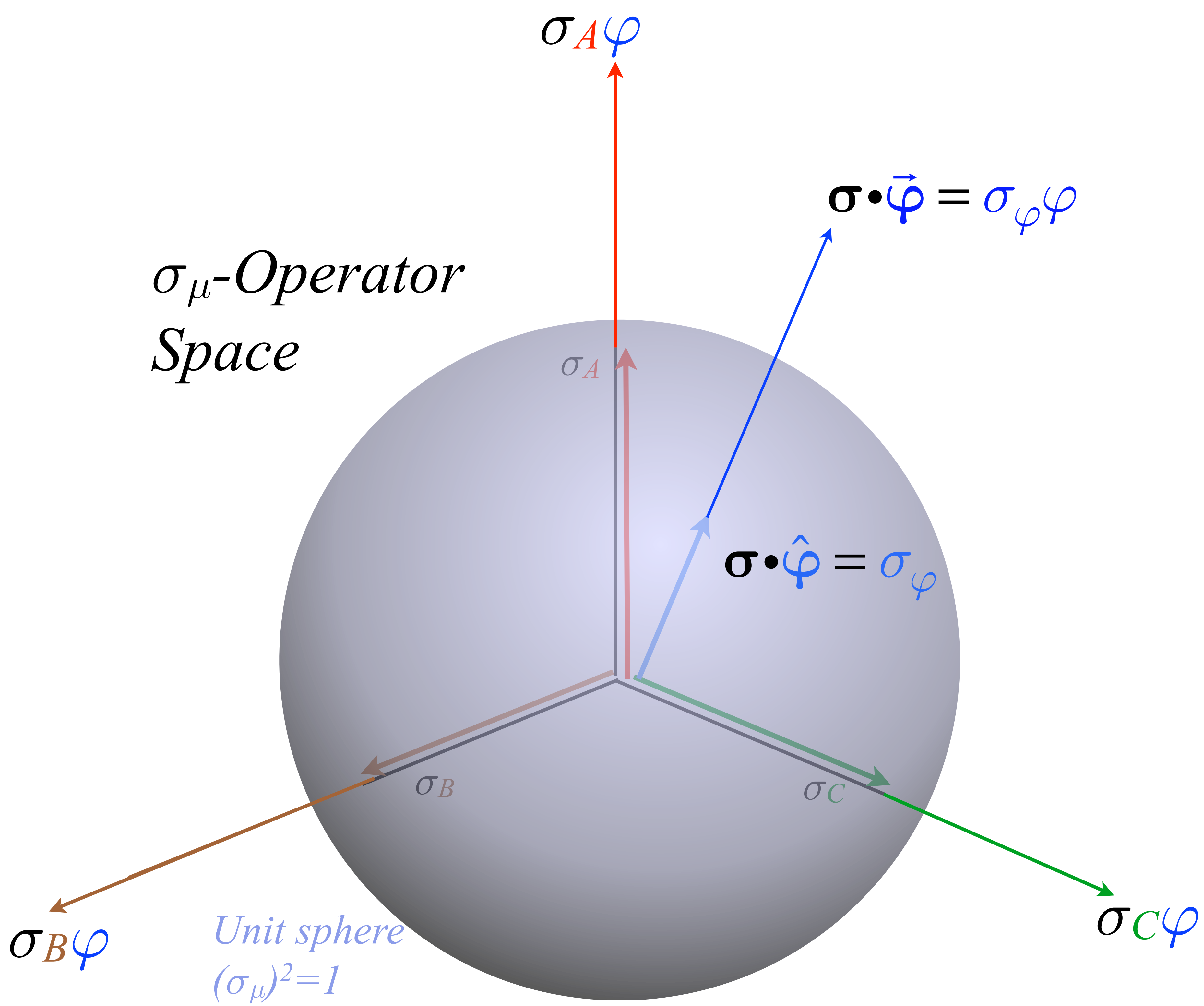
Euler $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed*

R(3)-U(2) slide rule for converting $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$

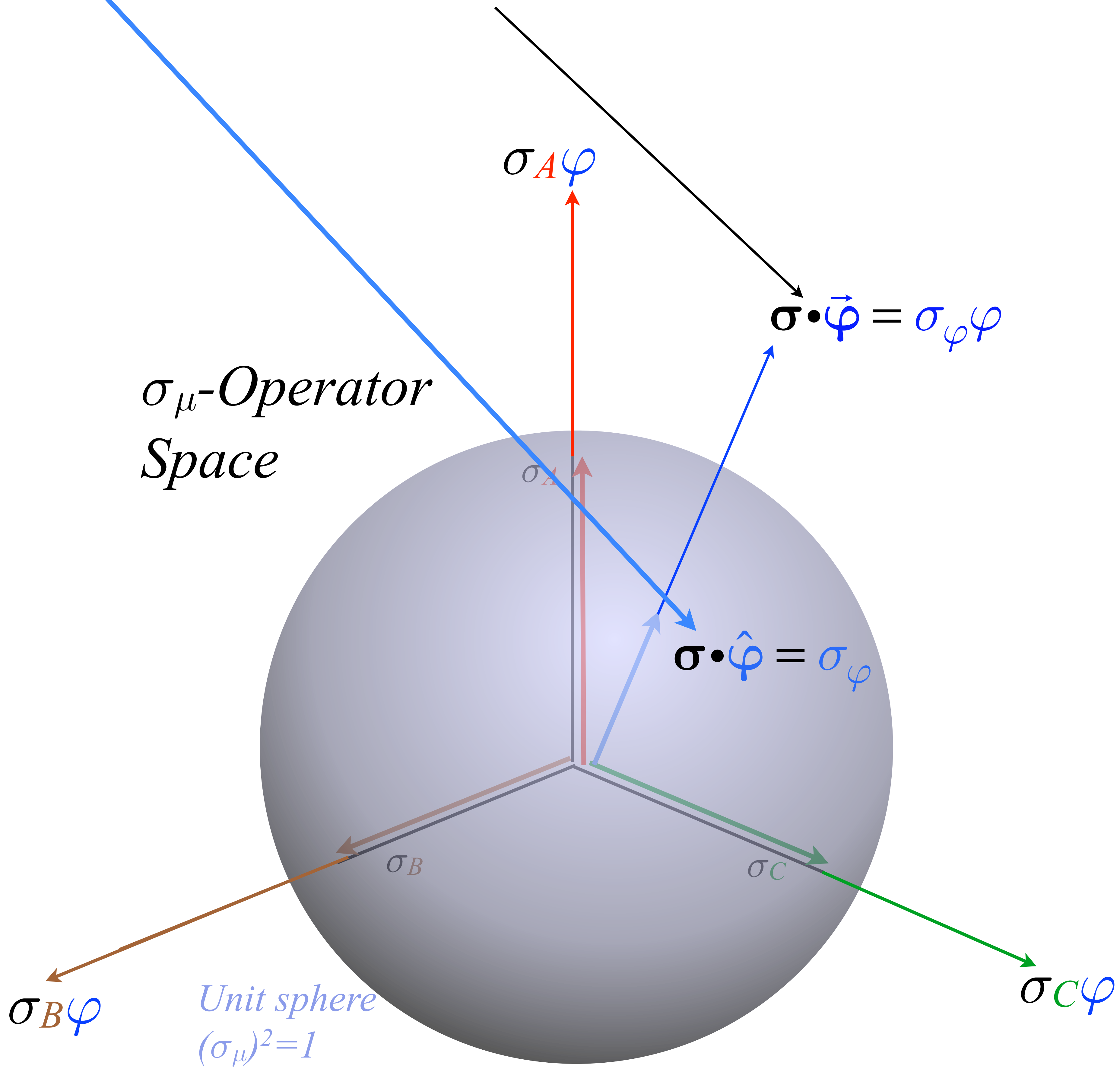
Euler $\mathbf{R}(\alpha\beta\gamma)$ *Sundial*

Not-so-Crazy Thing: $\sigma_\varphi = \boldsymbol{\sigma}_A \hat{\varphi}_A + \boldsymbol{\sigma}_B \hat{\varphi}_B + \boldsymbol{\sigma}_C \hat{\varphi}_C = \boldsymbol{\sigma} \bullet \hat{\boldsymbol{\varphi}} = \boldsymbol{\sigma} \bullet \vec{\boldsymbol{\varphi}} / \varphi$ where: $(\sigma_\varphi)^2 = \mathbf{1}$

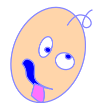


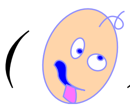
Not-so-Crazy Thing: $\sigma_\varphi = \boldsymbol{\sigma}_A\hat{\varphi}_A + \boldsymbol{\sigma}_B\hat{\varphi}_B + \boldsymbol{\sigma}_C\hat{\varphi}_C = \boldsymbol{\sigma}\bullet\hat{\boldsymbol{\varphi}} = \boldsymbol{\sigma}\bullet\vec{\boldsymbol{\varphi}}/\varphi$ where: $(\sigma_\varphi)^2 = \mathbf{1}$

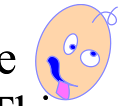
$\sigma_\varphi\varphi = \boldsymbol{\sigma}_A\varphi_A + \boldsymbol{\sigma}_B\varphi_B + \boldsymbol{\sigma}_C\varphi_C = \boldsymbol{\sigma}\bullet\boldsymbol{\varphi} = \boldsymbol{\sigma}\bullet\vec{\boldsymbol{\varphi}}$ where: $(\sigma_\varphi)^2 = \mathbf{1}\varphi^2$

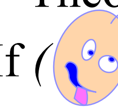


Not-so-Crazy Thing: $\sigma_\varphi = \boldsymbol{\sigma}_A\hat{\varphi}_A + \boldsymbol{\sigma}_B\hat{\varphi}_B + \boldsymbol{\sigma}_C\hat{\varphi}_C = \boldsymbol{\sigma}\bullet\hat{\boldsymbol{\varphi}} = \boldsymbol{\sigma}\bullet\vec{\boldsymbol{\varphi}}/\varphi$ where: $(\sigma_\varphi)^2=\mathbf{1}$

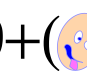
Crazy Thing:  $= -i\sigma_\varphi = -i\sigma_A\hat{\varphi}_A - i\sigma_B\hat{\varphi}_B - i\sigma_C\hat{\varphi}_C = -i\boldsymbol{\sigma}\bullet\hat{\boldsymbol{\varphi}} = -i\boldsymbol{\sigma}\bullet\vec{\boldsymbol{\varphi}}/\varphi$

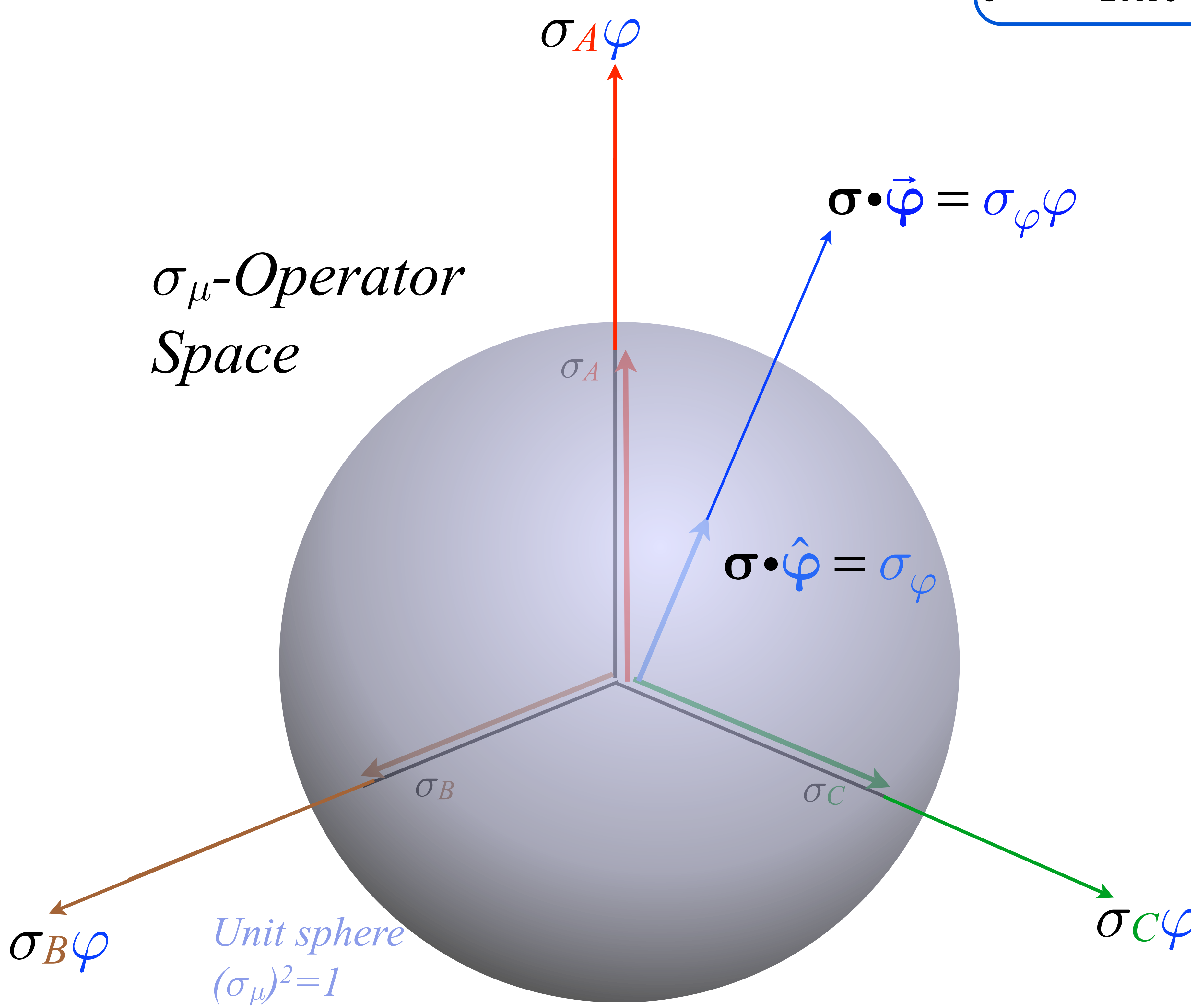
satisfies crazy requirement: ()$^2 = (-i\sigma_\varphi)^2 = -\mathbf{1}$

The Crazy Thing Theorem: 

If ()$^2 = -\mathbf{1}$

Then:

e()θ $= \mathbf{1}\cos\theta +$ ()$\sin\theta$



Not-so-Crazy Thing: $\sigma_\varphi = \boldsymbol{\sigma}_A\hat{\varphi}_A + \boldsymbol{\sigma}_B\hat{\varphi}_B + \boldsymbol{\sigma}_C\hat{\varphi}_C = \boldsymbol{\sigma}\bullet\hat{\boldsymbol{\varphi}} = \boldsymbol{\sigma}\bullet\vec{\boldsymbol{\varphi}}/\varphi$ where: $(\sigma_\varphi)^2 = \mathbf{1}$

Crazy Thing: $= -i\sigma_\varphi = -i\boldsymbol{\sigma}_A\hat{\varphi}_A - i\boldsymbol{\sigma}_B\hat{\varphi}_B - i\boldsymbol{\sigma}_C\hat{\varphi}_C = -i\boldsymbol{\sigma}\bullet\hat{\boldsymbol{\varphi}} = -i\boldsymbol{\sigma}\bullet\vec{\boldsymbol{\varphi}}/\varphi$

satisfies crazy requirement: $(\quad)^2 = (-i\sigma_\varphi)^2 = \mathbf{-1}$

So:

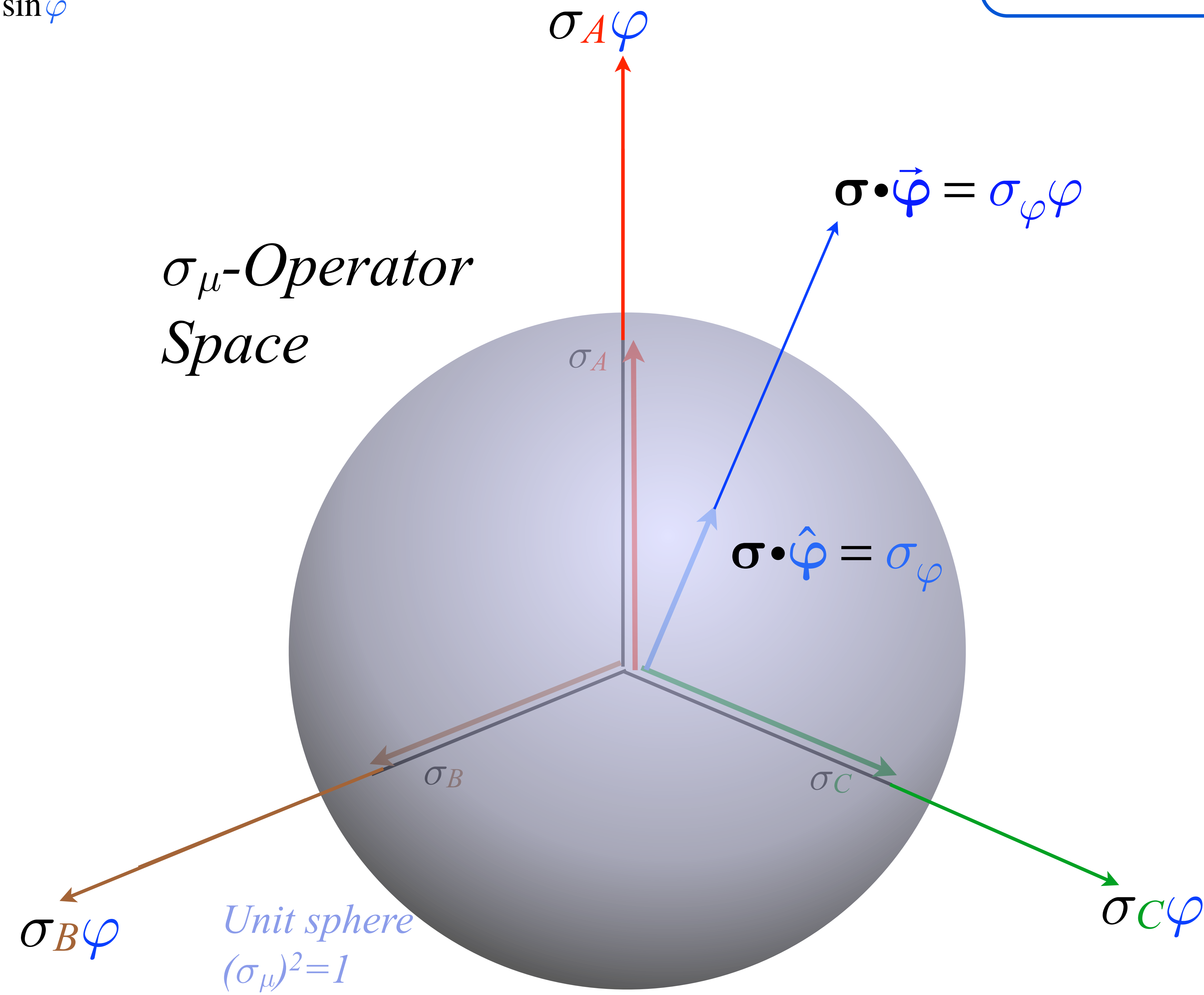
$$e^{-i\sigma_\varphi\varphi} = \mathbf{1}\cos\varphi \quad - i\sigma_\varphi \sin\varphi$$

The Crazy Thing Theorem:

If $(\quad)^2 = \mathbf{-1}$

Then:

$e^{(\quad)\theta} = \mathbf{1}\cos\theta + (\quad)\sin\theta$



Not-so-Crazy Thing: $\sigma_\varphi = \sigma_A\hat{\varphi}_A + \sigma_B\hat{\varphi}_B + \sigma_C\hat{\varphi}_C = \boldsymbol{\sigma}\bullet\hat{\boldsymbol{\varphi}} = \boldsymbol{\sigma}\bullet\vec{\boldsymbol{\varphi}}/\varphi$ where: $(\sigma_\varphi)^2 = \mathbf{1}$

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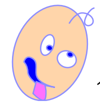
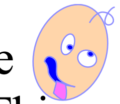
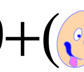
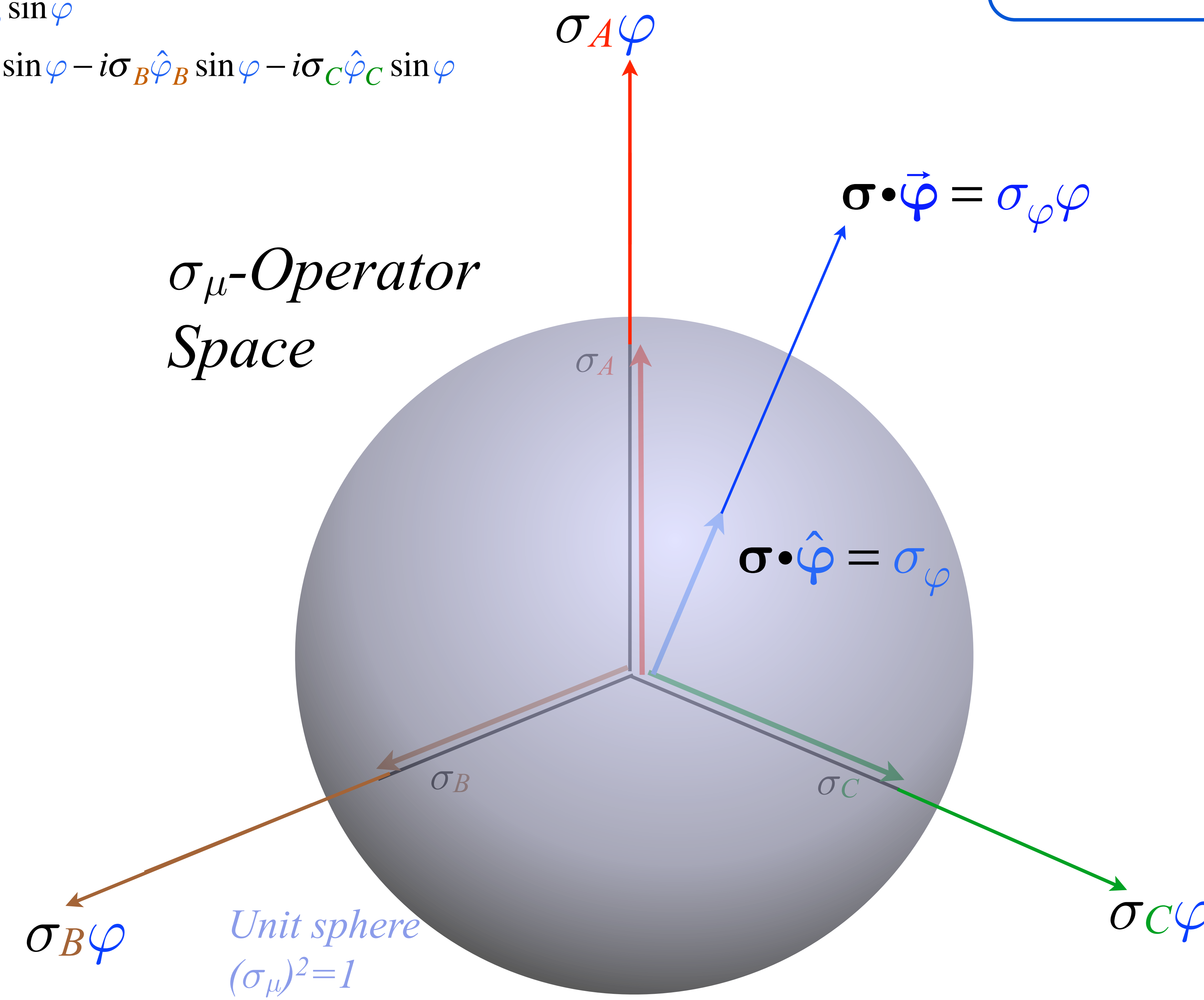
satisfies crazy requirement: ( $)^2 = (-i\sigma_\varphi)^2 = \mathbf{-1}$

So:

$$e^{-i\sigma_\varphi\varphi} = \mathbf{1}\cos\varphi \quad - i\sigma_\varphi \sin\varphi$$

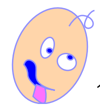
$$= \mathbf{1}\cos\varphi - i\sigma_A\hat{\varphi}_A\sin\varphi - i\sigma_B\hat{\varphi}_B\sin\varphi - i\sigma_C\hat{\varphi}_C\sin\varphi$$

The 
Crazy Thing
Theorem:
If ( $)^2 = -\mathbf{1}$
Then:
e()θ $= \mathbf{1}\cos\theta +$ ()sinθ



Not-so-Crazy Thing: $\sigma_\varphi = \sigma_A\hat{\varphi}_A + \sigma_B\hat{\varphi}_B + \sigma_C\hat{\varphi}_C = \boldsymbol{\sigma}\bullet\hat{\boldsymbol{\varphi}} = \boldsymbol{\sigma}\bullet\vec{\boldsymbol{\varphi}}/\varphi$ where: $(\sigma_\varphi)^2=\mathbf{1}$

Crazy Thing:  $=-i\sigma_\varphi = -i\sigma_A\hat{\varphi}_A - i\sigma_B\hat{\varphi}_B - i\sigma_C\hat{\varphi}_C = -i\boldsymbol{\sigma}\bullet\hat{\boldsymbol{\varphi}} = -i\boldsymbol{\sigma}\bullet\vec{\boldsymbol{\varphi}}/\varphi$

satisfies crazy requirement: ( $)^2 = (-i\sigma_\varphi)^2 = \mathbf{-1}$

So:

$$e^{-i\sigma_\varphi\varphi} = \mathbf{1}\cos\varphi \quad -i\sigma_\varphi\sin\varphi$$

$$= \mathbf{1}\cos\varphi \quad -i\sigma_A\hat{\varphi}_A\sin\varphi \quad -i\sigma_B\hat{\varphi}_B\sin\varphi \quad -i\sigma_C\hat{\varphi}_C\sin\varphi$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\varphi - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\hat{\varphi}_A\sin\varphi - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\hat{\varphi}_B\sin\varphi - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\hat{\varphi}_C\sin\varphi$$

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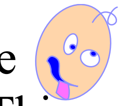
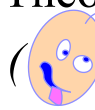
The 
Crazy Thing
Theorem:
If ( $)^2= \mathbf{-1}$
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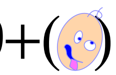
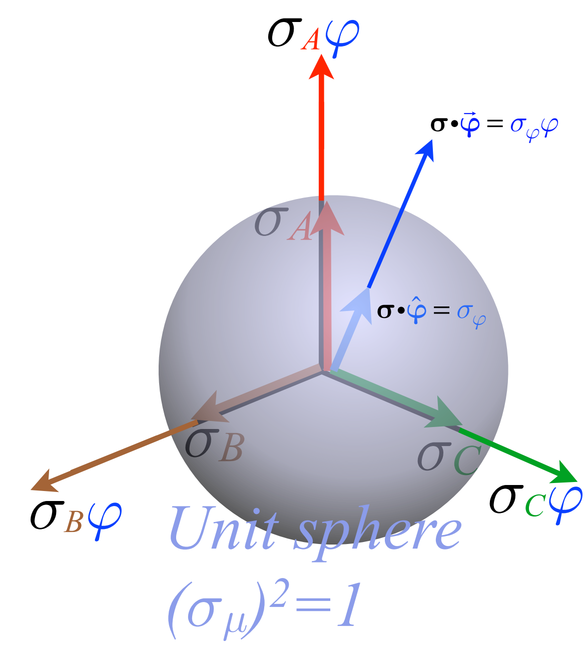
e()θ $=\mathbf{1}\cos\theta + ($  $)\sin\theta$

σ_μ-Operator Space



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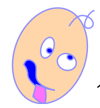
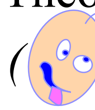
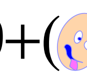
satisfies crazy requirement: ()$^2 = (-i\sigma_\varphi)^2 = \mathbf{-1}$

The Crazy Thing  Theorem:

If ()^2= **-1**

Then:

e()θ = **1** cosθ+()sinθ

So:

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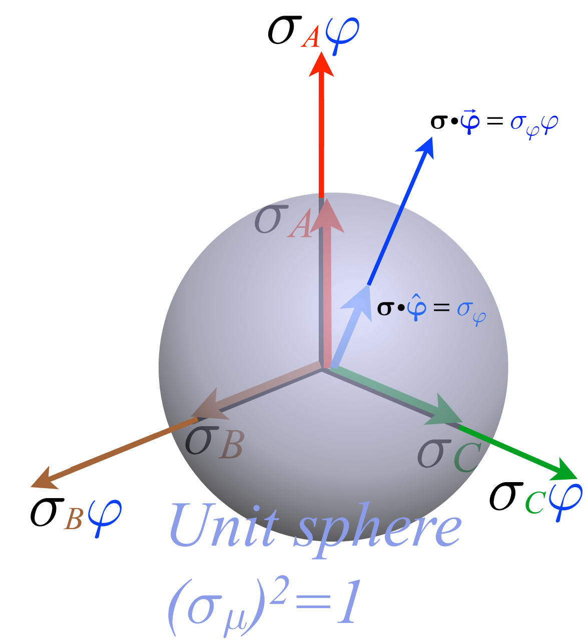
$$= \begin{pmatrix} \cos\varphi - i\hat{\varphi}_A \sin\varphi & (-i\hat{\varphi}_B - \hat{\varphi}_C)\sin\varphi \\ (-i\hat{\varphi}_B + \hat{\varphi}_C)\sin\varphi & \cos\varphi + i\hat{\varphi}_A \sin\varphi \end{pmatrix}$$

$$= \begin{pmatrix} \cos\varphi - i\sin\varphi & 0 \\ 0 & \cos\varphi + i\sin\varphi \end{pmatrix} = \begin{pmatrix} e^{-i\varphi} & 0 \\ 0 & e^{+i\varphi} \end{pmatrix} \text{ for: } \hat{\varphi} = \begin{pmatrix} \hat{\varphi}_A \\ \hat{\varphi}_B \\ \hat{\varphi}_C \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ or: } \boldsymbol{\sigma}\bullet\hat{\boldsymbol{\varphi}} = \sigma_A$$

$$= \mathbf{R}_A = e^{-i\sigma_A\varphi}$$

σ_μ-Operator Space



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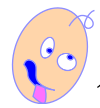
Crazy Thing:  $= -i\sigma_\varphi = -i\boldsymbol{\sigma}_A\hat{\varphi}_A - i\boldsymbol{\sigma}_B\hat{\varphi}_B - i\boldsymbol{\sigma}_C\hat{\varphi}_C = -i\boldsymbol{\sigma}\bullet\hat{\boldsymbol{\varphi}} = -i\boldsymbol{\sigma}\bullet\vec{\boldsymbol{\varphi}}/\varphi$

satisfies crazy requirement: ( $)^2 = (-i\sigma_\varphi)^2 = \mathbf{-1}$

So:

$$e^{-i\sigma_\varphi\varphi} = \mathbf{1}\cos\varphi \quad - i\sigma_\varphi\sin\varphi$$

$$= \mathbf{1}\cos\varphi \quad - i\boldsymbol{\sigma}_A\hat{\varphi}_A\sin\varphi \quad - i\boldsymbol{\sigma}_B\hat{\varphi}_B\sin\varphi \quad - i\boldsymbol{\sigma}_C\hat{\varphi}_C\sin\varphi$$

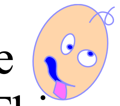
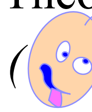
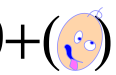
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\varphi - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\hat{\varphi}_A\sin\varphi - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\hat{\varphi}_B\sin\varphi - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\hat{\varphi}_C\sin\varphi$$

$$= \begin{pmatrix} \cos\varphi - i\hat{\varphi}_A\sin\varphi & (-i\hat{\varphi}_B - \hat{\varphi}_C)\sin\varphi \\ (-i\hat{\varphi}_B + \hat{\varphi}_C)\sin\varphi & \cos\varphi + i\hat{\varphi}_A\sin\varphi \end{pmatrix}$$

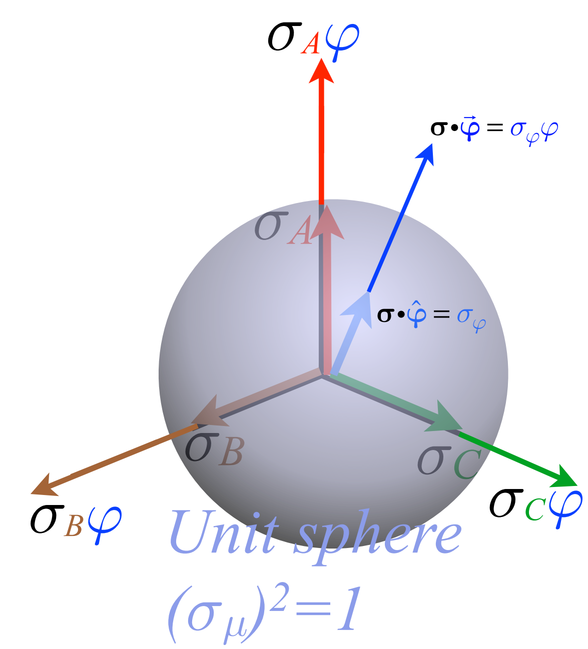
$$= \begin{pmatrix} \cos\varphi - i\sin\varphi & 0 \\ 0 & \cos\varphi + i\sin\varphi \end{pmatrix} = \begin{pmatrix} e^{-i\varphi} & 0 \\ 0 & e^{+i\varphi} \end{pmatrix} \text{ for: } \hat{\varphi} = \begin{pmatrix} \hat{\varphi}_A \\ \hat{\varphi}_B \\ \hat{\varphi}_C \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ or: } \boldsymbol{\sigma}\bullet\hat{\boldsymbol{\varphi}} = \boldsymbol{\sigma}_A$$

$$= \mathbf{R}_A = e^{-i\sigma_A\varphi}$$

$$= \mathbf{R}_B = e^{-i\sigma_B\varphi} \qquad = \begin{pmatrix} \cos\varphi & -i\sin\varphi \\ -i\sin\varphi & \cos\varphi \end{pmatrix} \text{ for: } \hat{\varphi} = \begin{pmatrix} \hat{\varphi}_A \\ \hat{\varphi}_B \\ \hat{\varphi}_C \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ or: } \boldsymbol{\sigma}\bullet\hat{\boldsymbol{\varphi}} = \boldsymbol{\sigma}_B$$

The 
Crazy Thing
Theorem:
If ( $)^2 = \mathbf{-1}$
Then:
e()θ $= \mathbf{1}\cos\theta +$ ()$\sin\theta$

σ_μ-Operator Space



Not-so-Crazy Thing: $\sigma_\varphi = \sigma_A\hat{\varphi}_A + \sigma_B\hat{\varphi}_B + \sigma_C\hat{\varphi}_C = \boldsymbol{\sigma}\bullet\hat{\boldsymbol{\varphi}} = \boldsymbol{\sigma}\bullet\vec{\boldsymbol{\varphi}}/\varphi$ where: $(\sigma_\varphi)^2 = \mathbf{1}$

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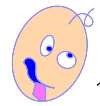
satisfies crazy requirement: ( $)^2 = (-i\sigma_\varphi)^2 = \mathbf{-1}$

So:

$$e^{-i\sigma_\varphi\varphi} = \mathbf{1}\cos\varphi - i\sigma_\varphi\sin\varphi$$

$$= \mathbf{1}\cos\varphi - i\sigma_A\hat{\varphi}_A\sin\varphi - i\sigma_B\hat{\varphi}_B\sin\varphi - i\sigma_C\hat{\varphi}_C\sin\varphi$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\varphi - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\hat{\varphi}_A\sin\varphi - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\hat{\varphi}_B\sin\varphi - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\hat{\varphi}_C\sin\varphi$$

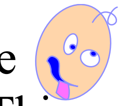
$$= \begin{pmatrix} \cos\varphi - i\hat{\varphi}_A\sin\varphi & (-i\hat{\varphi}_B - \hat{\varphi}_C)\sin\varphi \\ (-i\hat{\varphi}_B + \hat{\varphi}_C)\sin\varphi & \cos\varphi + i\hat{\varphi}_A\sin\varphi \end{pmatrix}$$

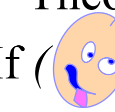
$$= \begin{pmatrix} \cos\varphi - i\sin\varphi & 0 \\ 0 & \cos\varphi + i\sin\varphi \end{pmatrix} = \begin{pmatrix} e^{-i\varphi} & 0 \\ 0 & e^{+i\varphi} \end{pmatrix} \text{ for: } \hat{\varphi} = \begin{pmatrix} \hat{\varphi}_A \\ \hat{\varphi}_B \\ \hat{\varphi}_C \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ or: } \boldsymbol{\sigma}\bullet\hat{\boldsymbol{\varphi}} = \sigma_A$$

$$= \mathbf{R}_A = e^{-i\sigma_A\varphi}$$

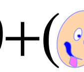
$$= \mathbf{R}_B = e^{-i\sigma_B\varphi} = \begin{pmatrix} \cos\varphi & -i\sin\varphi \\ -i\sin\varphi & \cos\varphi \end{pmatrix} \text{ for: } \hat{\varphi} = \begin{pmatrix} \hat{\varphi}_A \\ \hat{\varphi}_B \\ \hat{\varphi}_C \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ or: } \boldsymbol{\sigma}\bullet\hat{\boldsymbol{\varphi}} = \sigma_B$$

$$= \mathbf{R}_C = e^{-i\sigma_C\varphi} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \text{ for: } \hat{\varphi} = \begin{pmatrix} \hat{\varphi}_A \\ \hat{\varphi}_B \\ \hat{\varphi}_C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ or: } \boldsymbol{\sigma}\bullet\hat{\boldsymbol{\varphi}} = \sigma_C$$

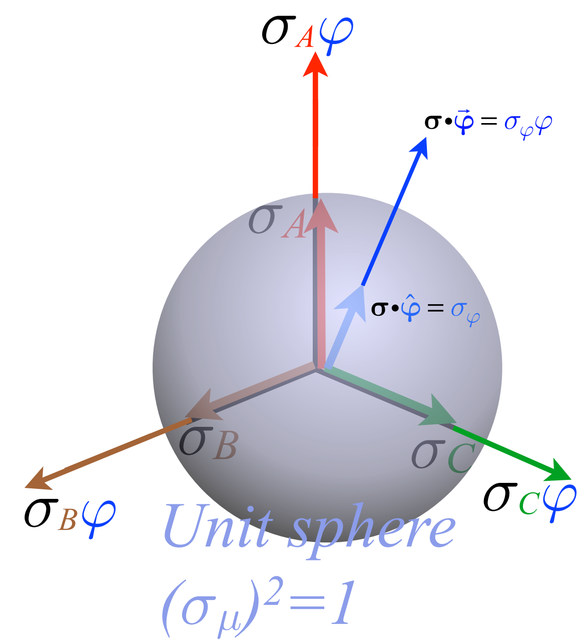
The Crazy Thing Theorem: 

If ( $)^2 = \mathbf{-1}$

Then:

e()θ $= \mathbf{1}\cos\theta +$ ()$\sin\theta$

σ_μ-Operator Space



Review:How "Crazy-Thing"-Theorem makes spinor and vector representation matrices

Half-angle $\Theta/2 = \varphi$ *replacement and Darboux crank axis operators*

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Product algebra for Pauli's σ_μ *and Hamilton's* $q_\mu = -i\sigma_\mu$

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Jordan-Pauli identity and U(2) product $\mathbf{R}[\Theta]\mathbf{R}[\Theta']=\mathbf{R}[\Theta''']$*- formula*

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Euler $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed*

R(3)-U(2) slide rule for converting $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$

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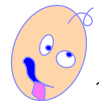
Not-so-Crazy Thing: $\sigma_\varphi = \sigma_A \hat{\varphi}_A + \sigma_B \hat{\varphi}_B + \sigma_C \hat{\varphi}_C = \boldsymbol{\sigma} \bullet \hat{\varphi} = \boldsymbol{\sigma} \bullet \vec{\varphi} / \varphi$ where: $(\sigma_\varphi)^2 = \mathbf{1}$

Crazy Thing:  $= -i\sigma_\varphi = -i\sigma_A \hat{\varphi}_A - i\sigma_B \hat{\varphi}_B - i\sigma_C \hat{\varphi}_C = -i\boldsymbol{\sigma} \bullet \hat{\varphi} = -i\boldsymbol{\sigma} \bullet \vec{\varphi} / \varphi$

satisfies crazy requirement: ( $)^2 = (-i\sigma_\varphi)^2 = \mathbf{-1}$

So:

$$e^{-i\sigma_\varphi \varphi} = \mathbf{1}\cos\varphi \quad - i\sigma_\varphi \sin\varphi$$

$$= \mathbf{1}\cos\varphi \quad - i\sigma_A \hat{\varphi}_A \sin\varphi \quad - i\sigma_B \hat{\varphi}_B \sin\varphi \quad - i\sigma_C \hat{\varphi}_C \sin\varphi$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos\varphi - i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \hat{\varphi}_A \sin\varphi - i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \hat{\varphi}_B \sin\varphi - i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \hat{\varphi}_C \sin\varphi$$

$$= \begin{pmatrix} \cos\varphi - i\hat{\varphi}_A \sin\varphi & (-i\hat{\varphi}_B - \hat{\varphi}_C)\sin\varphi \\ (-i\hat{\varphi}_B + \hat{\varphi}_C)\sin\varphi & \cos\varphi + i\hat{\varphi}_A \sin\varphi \end{pmatrix}$$

$$= \begin{pmatrix} \cos\varphi - i\sin\varphi & 0 \\ 0 & \cos\varphi + i\sin\varphi \end{pmatrix} = \begin{pmatrix} e^{-i\varphi} & 0 \\ 0 & e^{+i\varphi} \end{pmatrix} \text{ for: } \hat{\varphi} = \begin{pmatrix} \hat{\varphi}_A \\ \hat{\varphi}_B \\ \hat{\varphi}_C \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ or: } \boldsymbol{\sigma} \bullet \hat{\varphi} = \sigma_A$$

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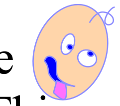
The Crazy Thing Theorem: 

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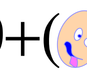
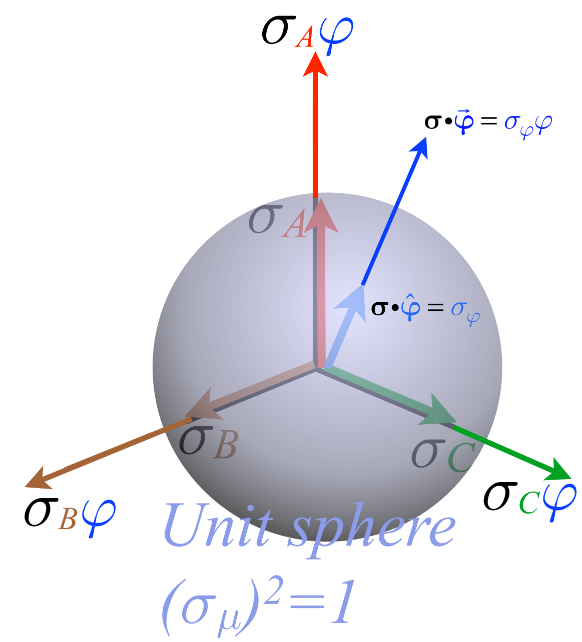
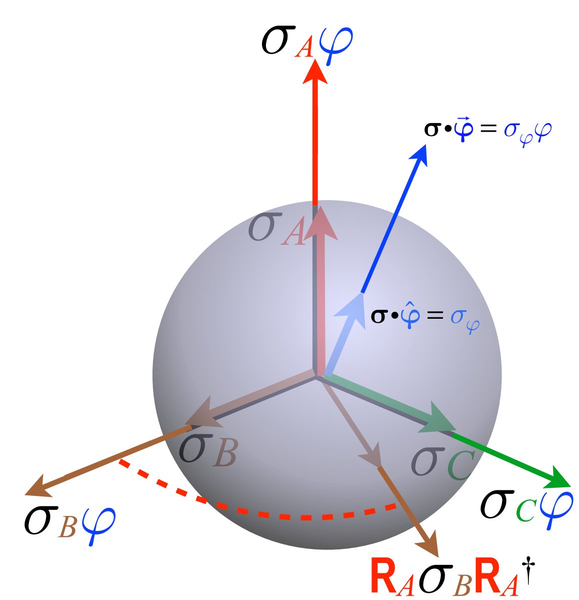
e()θ $= \mathbf{1}\cos\theta +$ ()$\sin\theta$

σ_μ-Operator Space





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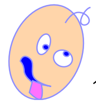
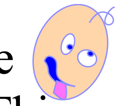
satisfies crazy requirement: ( $)^2 = (-i\sigma_\varphi)^2 = \mathbf{-1}$

The Crazy Thing  Theorem:

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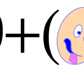
e()θ $= \mathbf{1}\cos\theta +$ ()$\sin\theta$

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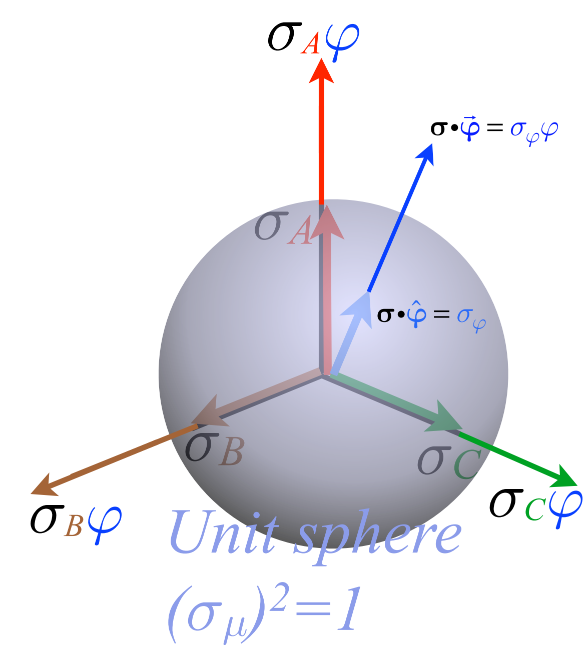
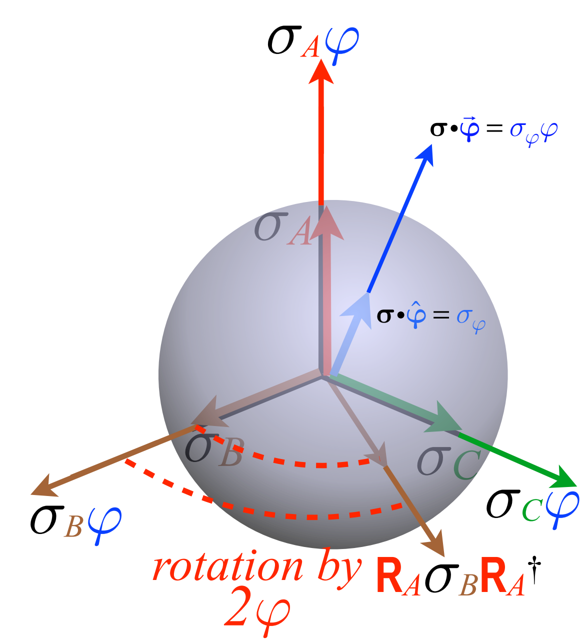
σ_μ-Operator Space





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$$e^{-i\sigma_A \varphi}\sigma_B\, e^{+i\sigma_A \varphi} = \begin{pmatrix} e^{-i\varphi} & 0 \\ 0 & e^{+i\varphi} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^{+i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} = \begin{pmatrix} 0 & e^{-i\varphi} \\ e^{+i\varphi} & 0 \end{pmatrix}\begin{pmatrix} e^{+i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} = \begin{pmatrix} 0 & e^{-i2\varphi} \\ e^{+i2\varphi} & 0 \end{pmatrix} = \sigma_B \cos 2\varphi + \sigma_C \sin 2\varphi$$

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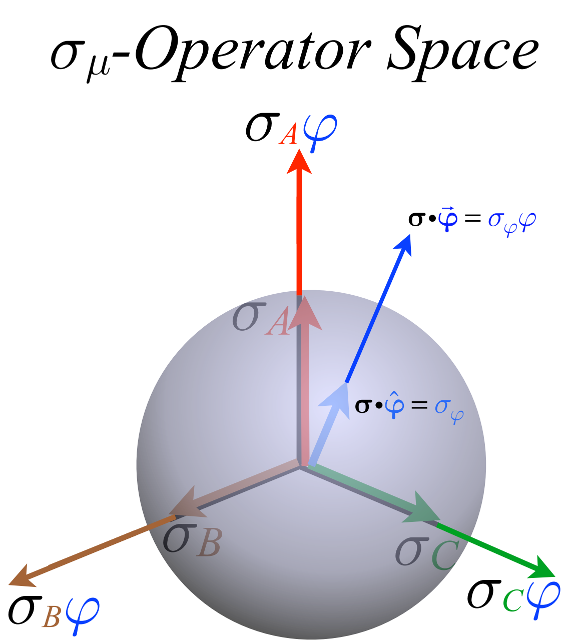
Replace spinor angle φ in: $e^{-i\boldsymbol{\sigma}\bullet\vec{\boldsymbol{\varphi}}} = \mathbf{R}_\varphi = \mathbf{1}\cos\varphi - i\boldsymbol{\sigma}\bullet\hat{\boldsymbol{\varphi}}\sin\varphi$

with 3D $\frac{1}{2}$-angle $\dfrac{\Theta}{2} = \varphi$



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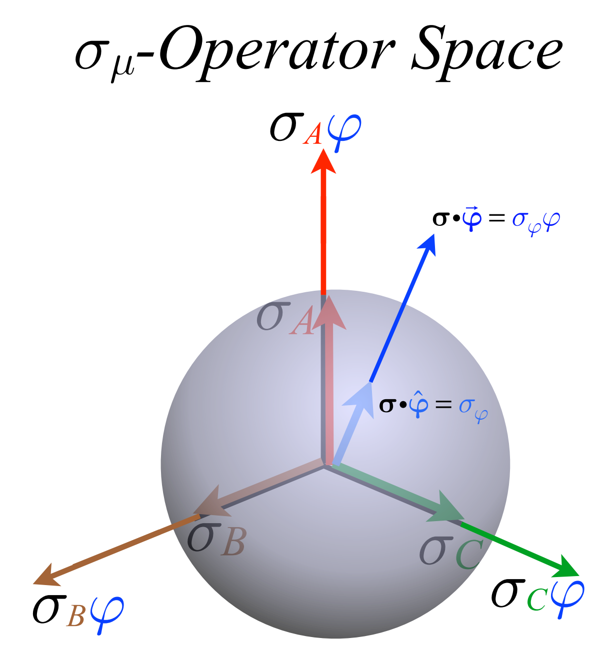

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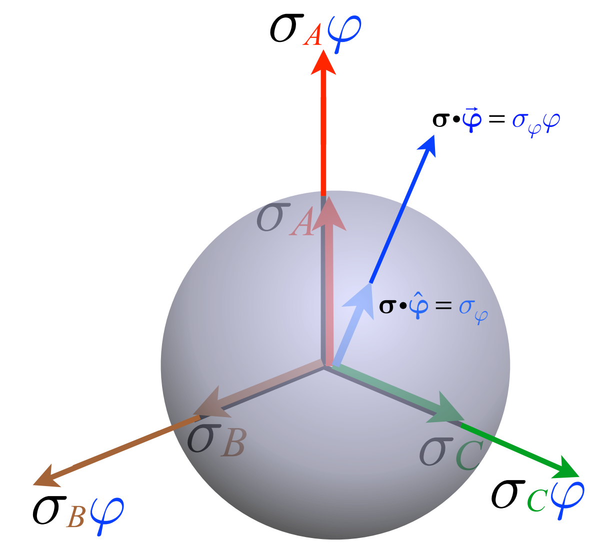
Unit rotation axis vector $\hat{\boldsymbol{\Theta}} = \begin{pmatrix} \hat{\Theta}_X, & \hat{\Theta}_Y, & \hat{\Theta}_Z \end{pmatrix} = \begin{pmatrix} \cos\varphi\sin\vartheta & \sin\varphi\sin\vartheta & \cos\vartheta \end{pmatrix}$ is the same as the unit $\hat{\boldsymbol{\varphi}}$.

σ_μ-Operator Space



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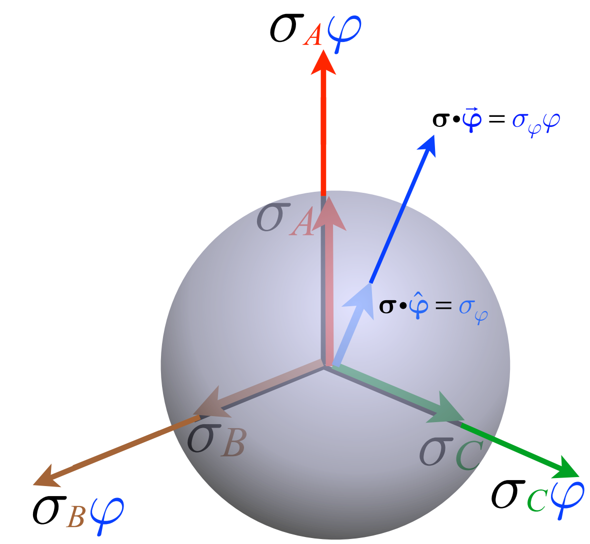
σ_μ-Operator Space



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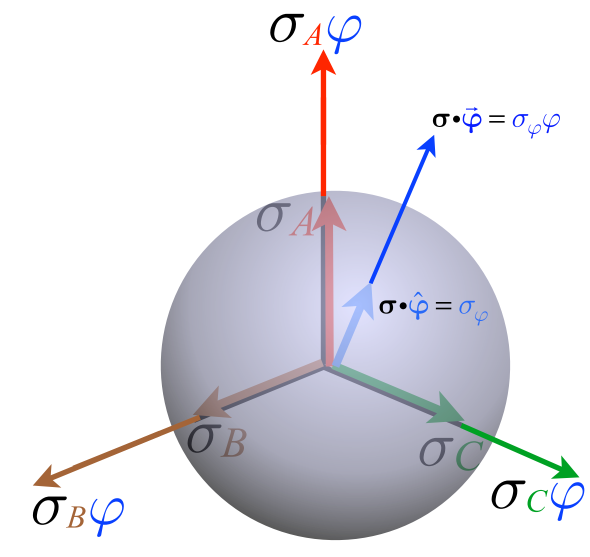
with 3D $\frac{1}{2}$-angle $\frac{\Theta}{2} = \varphi$ in: $e^{-i\boldsymbol{\sigma}\bullet\frac{\vec{\Theta}}{2}} = \mathbf{R}[\vec{\Theta}] = \mathbf{1}\cos\frac{\Theta}{2} - i\boldsymbol{\sigma}\bullet\hat{\Theta}\sin\frac{\Theta}{2} = e^{-i\frac{\boldsymbol{\sigma}}{2}\bullet\vec{\Theta}} = e^{-i\mathbf{S}\bullet\vec{\Theta}} = e^{-i\sigma_\varphi\varphi} = e^{-i\boldsymbol{\sigma}\bullet\vec{\varphi}}$

Unit rotation axis vector $\hat{\Theta} = \begin{pmatrix} \hat{\Theta}_X, & \hat{\Theta}_Y, & \hat{\Theta}_Z \end{pmatrix} = \begin{pmatrix} \cos\varphi\sin\vartheta & \sin\varphi\sin\vartheta & \cos\vartheta \end{pmatrix}$ is the same as the unit $\hat{\varphi}$.

σ_μ-Operator Space



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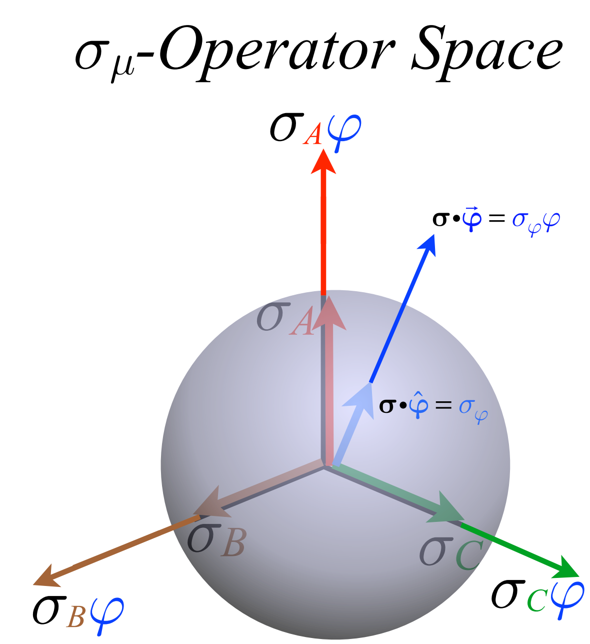
$$\mathbf{R}[\vec{\Theta}] = \cos\frac{\Theta}{2}\,\mathbf{1} - i\,\sigma_X\,\hat{\Theta}_X\sin\frac{\Theta}{2} - i\,\sigma_Y\,\hat{\Theta}_Y\sin\frac{\Theta}{2} - i\,\sigma_Z\,\hat{\Theta}_Z\sin\frac{\Theta}{2}$$

$$= \cos\frac{\Theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\hat{\Theta}_X\sin\frac{\Theta}{2} - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\hat{\Theta}_Y\sin\frac{\Theta}{2} - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\hat{\Theta}_Z\sin\frac{\Theta}{2}$$



Half-angle $\Theta/2=\varphi$ replacement in rotation $\mathbf{R}_\varphi = e^{-i\sigma_\varphi \varphi} = e^{-i\boldsymbol{\sigma}\bullet\boldsymbol{\varphi}}$ *where:* $\sigma_\varphi = \boldsymbol{\sigma}\bullet\vec{\boldsymbol{\varphi}} = \sigma_A\varphi_A + \sigma_B\varphi_B + \sigma_C\varphi_C = \boldsymbol{\sigma}\bullet\hat{\boldsymbol{\varphi}}\varphi$

Replace spinor angle φ in: $e^{-i\boldsymbol{\sigma}\bullet\vec{\boldsymbol{\varphi}}} = \mathbf{R}_\varphi = \mathbf{1}\cos\varphi - i\boldsymbol{\sigma}\bullet\hat{\boldsymbol{\varphi}}\sin\varphi$

with 3D $\frac{1}{2}$-angle $\frac{\Theta}{2}=\varphi$ in: $e^{-i\boldsymbol{\sigma}\bullet\frac{\vec{\boldsymbol{\Theta}}}{2}} = \mathbf{R}[\vec{\boldsymbol{\Theta}}] = \mathbf{1}\cos\frac{\Theta}{2} - i\boldsymbol{\sigma}\bullet\hat{\boldsymbol{\Theta}}\sin\frac{\Theta}{2} = e^{-i\frac{\boldsymbol{\sigma}}{2}\bullet\vec{\boldsymbol{\Theta}}} = e^{-i\mathbf{S}\bullet\vec{\boldsymbol{\Theta}}} = e^{-i\sigma_\varphi\varphi} = e^{-i\boldsymbol{\sigma}\bullet\vec{\boldsymbol{\varphi}}}$

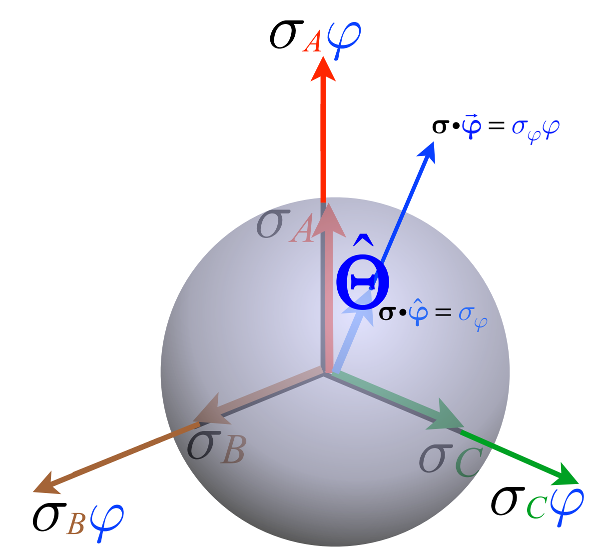
Unit rotation axis vector $\hat{\boldsymbol{\Theta}} = \left(\hat{\Theta}_X, \hat{\Theta}_Y, \hat{\Theta}_Z\right) = \left(\cos\varphi\sin\vartheta \quad \sin\varphi\sin\vartheta \quad \cos\vartheta\right)$ is the same as the unit $\hat{\boldsymbol{\varphi}}$.

$$\mathbf{R}[\vec{\boldsymbol{\Theta}}] = \cos\frac{\Theta}{2}\,\mathbf{1} - i\,\sigma_X\,\hat{\Theta}_X\sin\frac{\Theta}{2} - i\,\sigma_Y\,\hat{\Theta}_Y\sin\frac{\Theta}{2} - i\,\sigma_Z\,\hat{\Theta}_Z\sin\frac{\Theta}{2}$$

$$= \cos\frac{\Theta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix} - i\begin{pmatrix}0&1\\1&0\end{pmatrix}\hat{\Theta}_X\sin\frac{\Theta}{2} - i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\hat{\Theta}_Y\sin\frac{\Theta}{2} - i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\hat{\Theta}_Z\sin\frac{\Theta}{2}$$

$$\begin{pmatrix}\langle 1|\mathbf{R}[\vec{\boldsymbol{\Theta}}]|1\rangle & \langle 1|\mathbf{R}[\vec{\boldsymbol{\Theta}}]|2\rangle\\ \langle 2|\mathbf{R}[\vec{\boldsymbol{\Theta}}]|1\rangle & \langle 2|\mathbf{R}[\vec{\boldsymbol{\Theta}}]|2\rangle\end{pmatrix} = \begin{pmatrix}\cos\frac{\Theta}{2} - i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X - i\hat{\Theta}_Y\right)\\ -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X + i\hat{\Theta}_Y\right) & \cos\frac{\Theta}{2} + i\hat{\Theta}_Z\sin\frac{\Theta}{2}\end{pmatrix}$$

σ_μ-Operator Space



$\hat{\Theta}_X = \cos\varphi\ \sin\vartheta$

$\hat{\Theta}_Y = \sin\varphi\ \sin\vartheta$

$\hat{\Theta}_Z = \cos\vartheta$

Polar coordinates for unit axis vector $\hat{\boldsymbol{\Theta}}$

Half-angle $\Theta/2=\varphi$ replacement in rotation $\mathbf{R}_\varphi = e^{-i\sigma_\varphi \varphi} = e^{-i\boldsymbol{\sigma}\bullet\vec{\varphi}}$ *where:* $\sigma_\varphi = \boldsymbol{\sigma}\bullet\vec{\varphi} = \sigma_A\varphi_A + \sigma_B\varphi_B + \sigma_C\varphi_C = \boldsymbol{\sigma}\bullet\hat{\varphi}\varphi$

Replace spinor angle φ in: $e^{-i\boldsymbol{\sigma}\bullet\vec{\varphi}} = \mathbf{R}_\varphi = \mathbf{1}\cos\varphi - i\boldsymbol{\sigma}\bullet\hat{\varphi}\sin\varphi$

with 3D $\frac{1}{2}$-angle $\frac{\Theta}{2} = \varphi$ in: $e^{-i\boldsymbol{\sigma}\bullet\frac{\vec{\Theta}}{2}} = \mathbf{R}[\vec{\Theta}] = \mathbf{1}\cos\frac{\Theta}{2} - i\boldsymbol{\sigma}\bullet\hat{\Theta}\sin\frac{\Theta}{2} = e^{-i\frac{\boldsymbol{\sigma}}{2}\bullet\vec{\Theta}} = e^{-i\mathbf{S}\bullet\vec{\Theta}} = e^{-i\sigma_\varphi\varphi} = e^{-i\boldsymbol{\sigma}\bullet\vec{\varphi}}$

Unit rotation axis vector $\hat{\Theta} = \left(\hat{\Theta}_X, \hat{\Theta}_Y, \hat{\Theta}_Z\right) = \left(\cos\varphi\sin\vartheta \quad \sin\varphi\sin\vartheta \quad \cos\vartheta\right)$ is the same as the unit $\hat{\varphi}$.

$$\mathbf{R}[\vec{\Theta}] = \cos\frac{\Theta}{2}\,\mathbf{1} - i\,\sigma_X\,\hat{\Theta}_X\sin\frac{\Theta}{2} - i\,\sigma_Y\,\hat{\Theta}_Y\sin\frac{\Theta}{2} - i\,\sigma_Z\,\hat{\Theta}_Z\sin\frac{\Theta}{2}$$

$$= \cos\frac{\Theta}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\hat{\Theta}_X\sin\frac{\Theta}{2} - i\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\hat{\Theta}_Y\sin\frac{\Theta}{2} - i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\hat{\Theta}_Z\sin\frac{\Theta}{2}$$

$$\begin{pmatrix}\langle 1|\mathbf{R}[\vec{\Theta}]|1\rangle & \langle 1|\mathbf{R}[\vec{\Theta}]|2\rangle\\ \langle 2|\mathbf{R}[\vec{\Theta}]|1\rangle & \langle 2|\mathbf{R}[\vec{\Theta}]|2\rangle\end{pmatrix} = \begin{pmatrix}\cos\frac{\Theta}{2} - i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X - i\hat{\Theta}_Y\right)\\ -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X + i\hat{\Theta}_Y\right) & \cos\frac{\Theta}{2} + i\hat{\Theta}_Z\sin\frac{\Theta}{2}\end{pmatrix}$$

$$= \begin{pmatrix}\cos\frac{\Theta}{2} - i\cos\vartheta\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}(\cos\varphi\sin\vartheta - i\sin\varphi\sin\vartheta)\\ -i\sin\frac{\Theta}{2}(\cos\varphi\sin\vartheta + i\sin\varphi\sin\vartheta) & \cos\frac{\Theta}{2} + i\cos\vartheta\sin\frac{\Theta}{2}\end{pmatrix}$$

$$= \mathbf{R}[\varphi\vartheta\Theta] = e^{-i\vec{\Theta}\bullet\mathbf{S}} = e^{-i\mathbf{H}t}$$

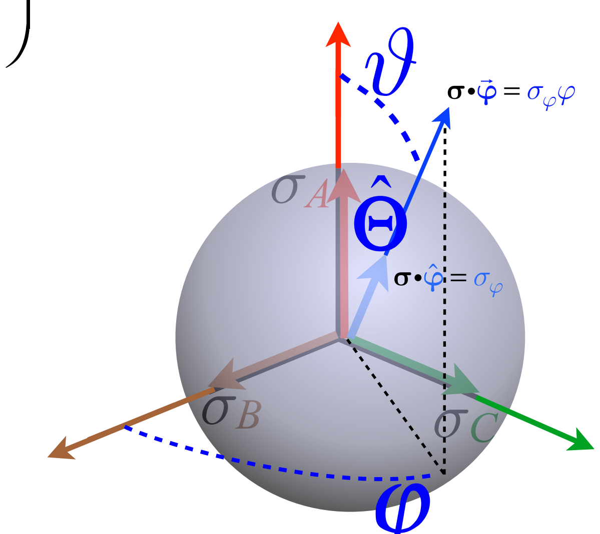
Polar coordinates for unit axis vector $\hat{\Theta}$

$\hat{\Theta}_X = \cos\varphi\ \sin\vartheta$

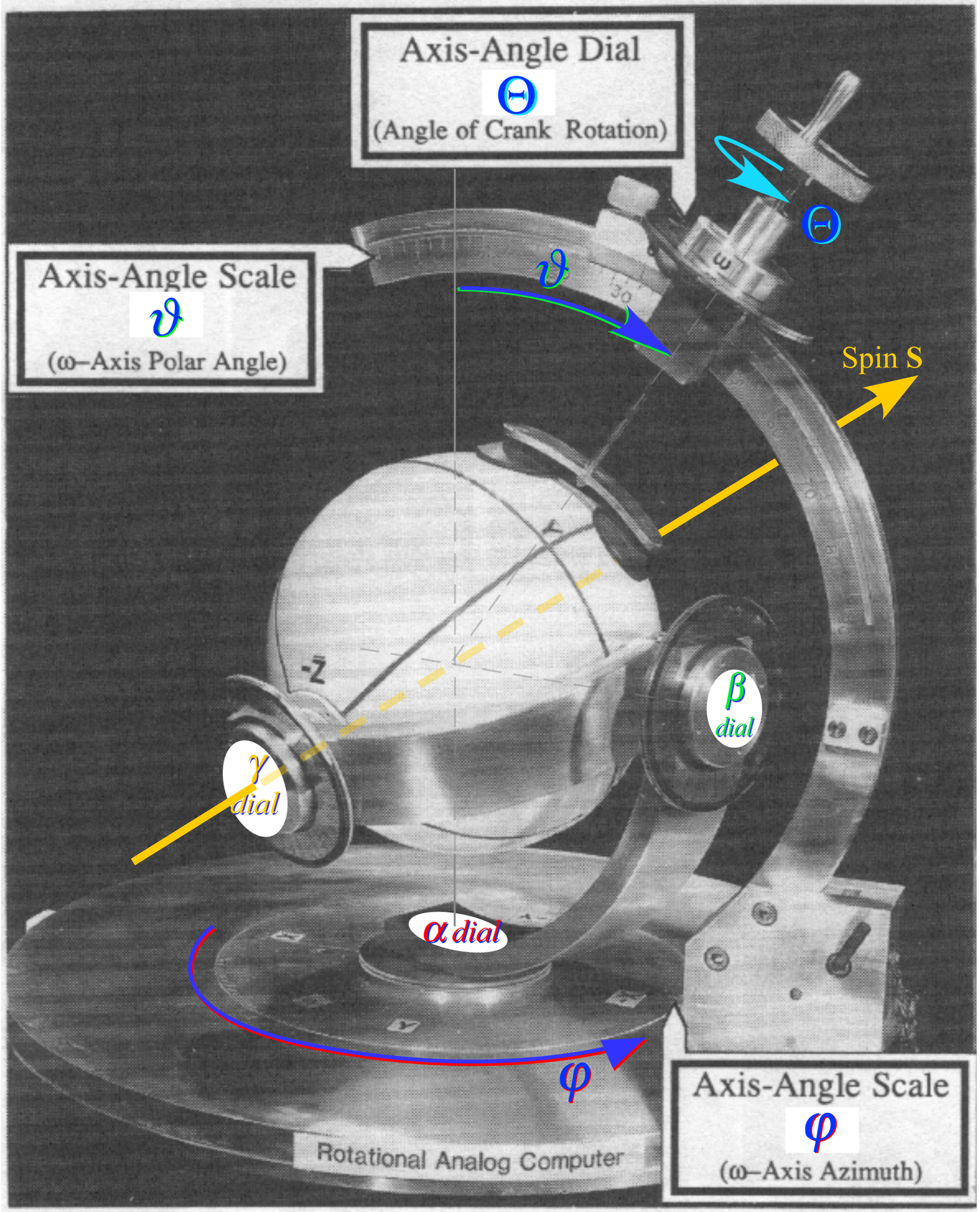
$\hat{\Theta}_Y = \sin\varphi\ \sin\vartheta$

$\hat{\Theta}_Z = \cos\vartheta$

Θ-crank polar angle



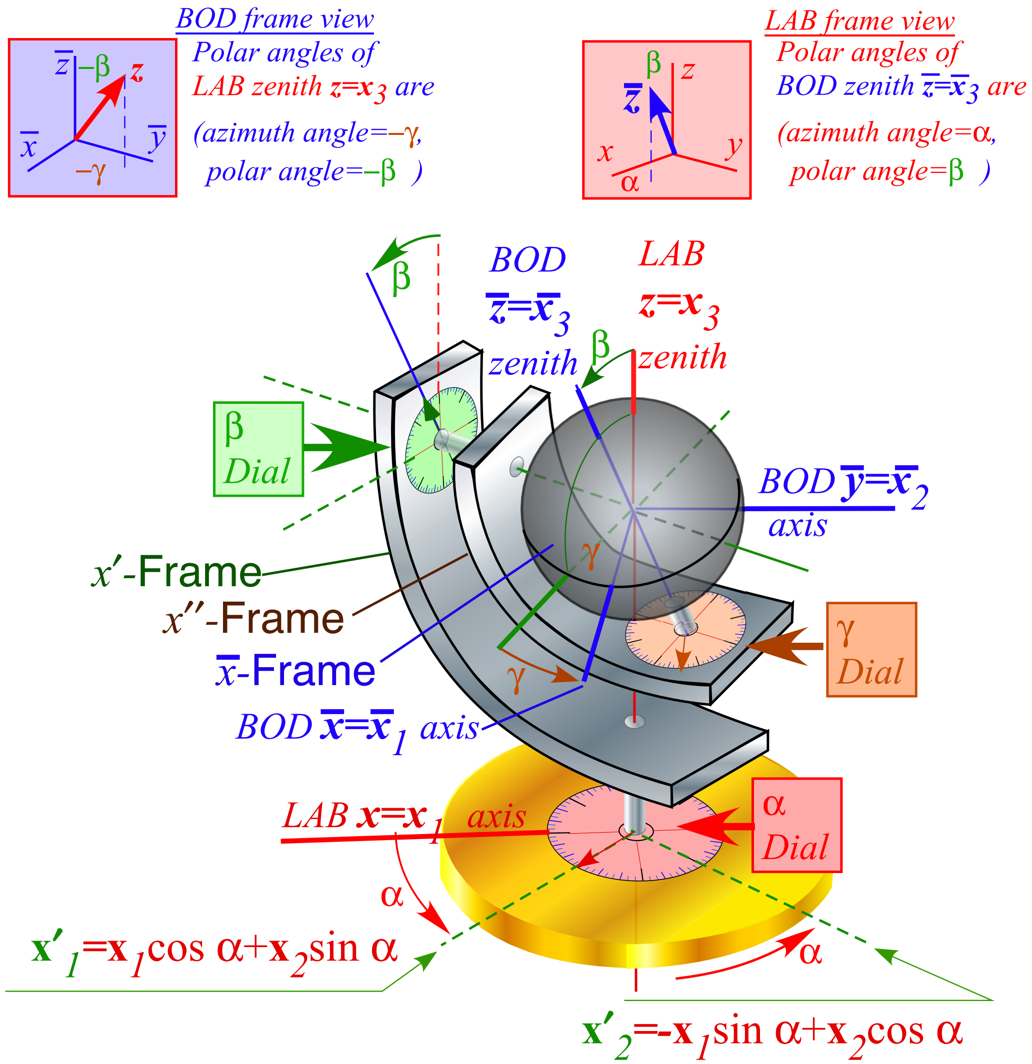
Θ-crank azimuth angle

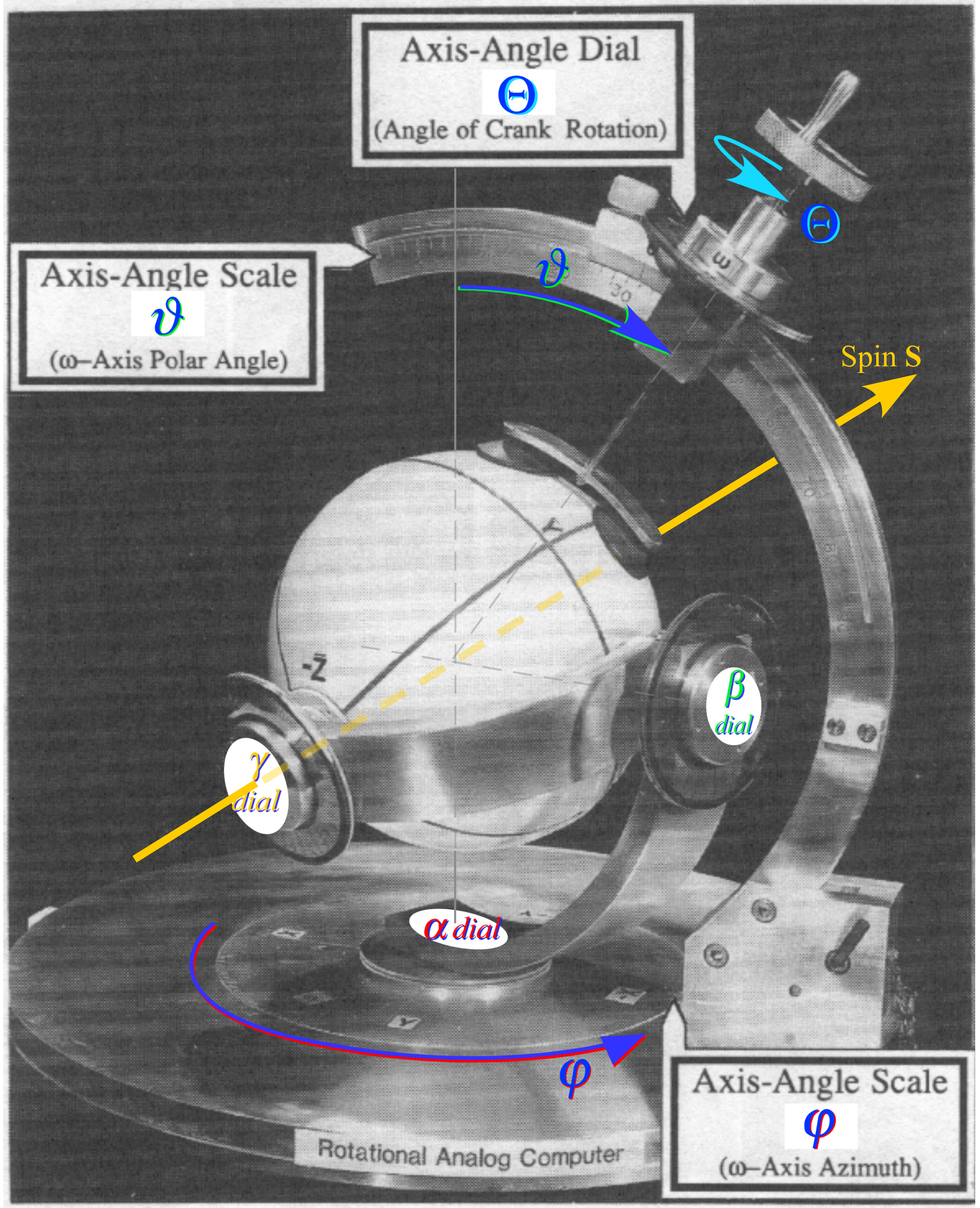


Polar coordinates for unit axis vector $\hat{\Theta}$

$$\hat{\Theta}_X = \cos\varphi \ \sin\vartheta$$
$$\hat{\Theta}_Y = \sin\varphi \ \sin\vartheta$$
$$\hat{\Theta}_Z = \cos\vartheta$$

Here spin-rotor S-polar coordinates are Euler angles





Polar coordinates for unit axis vector $\hat{\Theta}$

$$\hat{\Theta}_X = \cos\varphi \; \sin\vartheta$$
$$\hat{\Theta}_Y = \sin\varphi \; \sin\vartheta$$
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Review:How "Crazy-Thing"-Theorem makes spinor and vector representation matrices

Half-angle $\Theta/2 = \varphi$ replacement and Darboux crank axis operators

Operator-on-Operator transformations

Product algebra for Pauli's σ_μ and Hamilton's $\boldsymbol{q}_\mu = -i\sigma_\mu$

Group product algebra

Jordan-Pauli identity and U(2) product $\mathbf{R}[\Theta]\mathbf{R}[\Theta']=\mathbf{R}[\Theta''']$*- formula*

Transformation $\mathbf{R}[\Theta]\sigma_\mu\mathbf{R}[\Theta]^\dagger$ *of spinor σ_μ-operators*

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ *of group-operators*

Operator-on-Operator transformations

Geometry of groups: Hamilton's turns and It's all done with mirrors!

Group product geometry

U(2) product $\mathbf{R}[\Theta]\mathbf{R}[\Theta']=\mathbf{R}[\Theta''']$*- geometry*

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ *geometry*

Euler $\mathbf{R}(\alpha\beta\gamma)$ versus *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *rotation $\Theta=0-4\pi$-sequence $[\varphi\vartheta]$ fixed*

R(3)-U(2) slide rule for converting $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *Sundial*

Operator-on-Operator transformations

Product algebra Multiplication rules for Pauli's "σ_μ-quaternions" and Hamilton's $\mathbf{q}_\mu = -i\sigma_\mu$.

•	**1**	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
1	**1**	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
$\mathbf{q}_X$	$\mathbf{q}_X$	$-\mathbf{1}$	$\mathbf{q}_Z$	$-\mathbf{q}_Y$
$\mathbf{q}_Y$	$\mathbf{q}_Y$	$-\mathbf{q}_Z$	$-\mathbf{1}$	$\mathbf{q}_X$
$\mathbf{q}_Z$	$\mathbf{q}_Z$	$\mathbf{q}_Y$	$-\mathbf{q}_X$	$-\mathbf{1}$

,

•	**1**	σ_X	σ_Y	σ_Z
1	**1**	σ_X	σ_Y	σ_Z
σ_X	σ_X	**1**	$i\sigma_Z$	$-i\sigma_Y$
σ_Y	σ_Y	$-i\sigma_Z$	**1**	$i\sigma_X$
σ_Z	σ_Z	$i\sigma_Y$	$-i\sigma_X$	**1**

Operator-on-Operator transformations

Product algebra Multiplication rules for Pauli's "σ_μ-quaternions" and Hamilton's $\mathbf{q}_\mu = -i\sigma_\mu$.

$\bullet$	$\mathbf{1}$	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
$\mathbf{q}_X$	$\mathbf{q}_X$	$-\mathbf{1}$	$\mathbf{q}_Z$	$-\mathbf{q}_Y$
$\mathbf{q}_Y$	$\mathbf{q}_Y$	$-\mathbf{q}_Z$	$-\mathbf{1}$	$\mathbf{q}_X$
$\mathbf{q}_Z$	$\mathbf{q}_Z$	$\mathbf{q}_Y$	$-\mathbf{q}_X$	$-\mathbf{1}$

,

$\bullet$	$\mathbf{1}$	σ_X	σ_Y	σ_Z
$\mathbf{1}$	$\mathbf{1}$	σ_X	σ_Y	σ_Z
σ_X	σ_X	$\mathbf{1}$	$i\sigma_Z$	$-i\sigma_Y$
σ_Y	σ_Y	$-i\sigma_Z$	$\mathbf{1}$	$i\sigma_X$
σ_Z	σ_Z	$i\sigma_Y$	$-i\sigma_X$	$\mathbf{1}$

$$\mathbf{q}_\mu\, \mathbf{q}_\nu = -\delta_{\mu\nu}\, \mathbf{1} + \varepsilon_{\mu\nu\lambda}\, \mathbf{q}_\lambda$$
$$\sigma_\mu\, \sigma_\nu = \delta_{\mu\nu}\, \mathbf{1} + i\, \varepsilon_{\mu\nu\lambda}\, \sigma_\lambda$$

Operator-on-Operator transformations

Product algebra Multiplication rules for Pauli's "σ_μ-quaternions" and Hamilton's $\mathbf{q}_\mu = -i\sigma_\mu$.

•	1	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
1	1	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
$\mathbf{q}_X$	$\mathbf{q}_X$	-1	$\mathbf{q}_Z$	$-\mathbf{q}_Y$
$\mathbf{q}_Y$	$\mathbf{q}_Y$	$-\mathbf{q}_Z$	-1	$\mathbf{q}_X$
$\mathbf{q}_Z$	$\mathbf{q}_Z$	$\mathbf{q}_Y$	$-\mathbf{q}_X$	-1

,

•	1	σ_X	σ_Y	σ_Z
1	1	σ_X	σ_Y	σ_Z
σ_X	σ_X	1	$i\sigma_Z$	$-i\sigma_Y$
σ_Y	σ_Y	$-i\sigma_Z$	1	$i\sigma_X$
σ_Z	σ_Z	$i\sigma_Y$	$-i\sigma_X$	1

$$\mathbf{q}_\mu \, \mathbf{q}_\nu = -\delta_{\mu\nu} \, \mathbf{1} + \varepsilon_{\mu\nu\lambda} \, \mathbf{q}_\lambda$$

$$\sigma_\mu \, \sigma_\nu = \delta_{\mu\nu} \, \mathbf{1} + i \, \varepsilon_{\mu\nu\lambda} \, \sigma_\lambda$$

Commutation rules for Pauli ops: σ_μ

$$\sigma_\mu\sigma_\nu - \sigma_\nu\sigma_\mu = [\sigma_\mu, \sigma_\nu] = 2i \, \varepsilon_{\mu\nu\lambda} \, \sigma_\lambda$$

Operator-on-Operator transformations

Product algebra Multiplication rules for Pauli's "σ_μ-quaternions" and Hamilton's $\mathbf{q}_\mu = -i\sigma_\mu$.

•	1	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
1	1	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
$\mathbf{q}_X$	$\mathbf{q}_X$	-1	$\mathbf{q}_Z$	$-\mathbf{q}_Y$
$\mathbf{q}_Y$	$\mathbf{q}_Y$	$-\mathbf{q}_Z$	-1	$\mathbf{q}_X$
$\mathbf{q}_Z$	$\mathbf{q}_Z$	$\mathbf{q}_Y$	$-\mathbf{q}_X$	-1

,

•	1	σ_X	σ_Y	σ_Z
1	1	σ_X	σ_Y	σ_Z
σ_X	σ_X	1	$i\sigma_Z$	$-i\sigma_Y$
σ_Y	σ_Y	$-i\sigma_Z$	1	$i\sigma_X$
σ_Z	σ_Z	$i\sigma_Y$	$-i\sigma_X$	1

$$\mathbf{q}_\mu\,\mathbf{q}_\nu = -\delta_{\mu\nu}\,\mathbf{1} + \varepsilon_{\mu\nu\lambda}\,\mathbf{q}_\lambda$$

$$\sigma_\mu\,\sigma_\nu = \delta_{\mu\nu}\,\mathbf{1} + i\,\varepsilon_{\mu\nu\lambda}\,\sigma_\lambda$$

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$$\mathbf{S}_\mu\mathbf{S}_\nu - \mathbf{S}_\nu\mathbf{S}_\mu = [\mathbf{S}_\mu, \mathbf{S}_\nu] = i\,\varepsilon_{\mu\nu\lambda}\,\mathbf{S}_\lambda$$

Review:How "Crazy-Thing"-Theorem makes spinor and vector representation matrices

Half-angle $\Theta/2 = \varphi$ replacement and Darboux crank axis operators

Operator-on-Operator transformations

Product algebra for Pauli's σ_μ and Hamilton's $\boldsymbol{q}_\mu = -i\sigma_\mu$

Group product algebra

Jordan-Pauli identity and U(2) product $\mathbf{R}[\boldsymbol{\Theta}]\mathbf{R}[\boldsymbol{\Theta}']=\mathbf{R}[\boldsymbol{\Theta}''']$*- formula*

Transformation $\mathbf{R}[\boldsymbol{\Theta}]\sigma_\mu\mathbf{R}[\boldsymbol{\Theta}]^\dagger$ *of spinor σ_μ-operators*

Transformation $\mathbf{R}[\boldsymbol{\Theta}]\mathbf{R}[\boldsymbol{\Theta}']\mathbf{R}[\boldsymbol{\Theta}]^\dagger$ *of group-operators*

Operator-on-Operator transformations

Geometry of groups: Hamilton's turns and It's all done with mirrors!

Group product geometry

U(2) product $\mathbf{R}[\boldsymbol{\Theta}]\mathbf{R}[\boldsymbol{\Theta}']=\mathbf{R}[\boldsymbol{\Theta}''']$*- geometry*

Transformation $\mathbf{R}[\boldsymbol{\Theta}]\mathbf{R}[\boldsymbol{\Theta}']\mathbf{R}[\boldsymbol{\Theta}]^\dagger$ *geometry*

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Euler $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

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R(3)-U(2) slide rule for converting $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *Sundial*

Operator-on-Operator transformations

Product algebra Multiplication rules for Pauli's "σ_μ-quaternions" and Hamilton's $\mathbf{q}_\mu = -i\sigma_\mu$.

•	1	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
1	1	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
$\mathbf{q}_X$	$\mathbf{q}_X$	-1	$\mathbf{q}_Z$	$-\mathbf{q}_Y$
$\mathbf{q}_Y$	$\mathbf{q}_Y$	$-\mathbf{q}_Z$	-1	$\mathbf{q}_X$
$\mathbf{q}_Z$	$\mathbf{q}_Z$	$\mathbf{q}_Y$	$-\mathbf{q}_X$	-1

,

•	1	σ_X	σ_Y	σ_Z
1	1	σ_X	σ_Y	σ_Z
σ_X	σ_X	1	$i\sigma_Z$	$-i\sigma_Y$
σ_Y	σ_Y	$-i\sigma_Z$	1	$i\sigma_X$
σ_Z	σ_Z	$i\sigma_Y$	$-i\sigma_X$	1

$$\mathbf{q}_\mu \mathbf{q}_\nu = -\delta_{\mu\nu}\mathbf{1} + \varepsilon_{\mu\nu\lambda}\mathbf{q}_\lambda$$

$$\sigma_\mu \sigma_\nu = \delta_{\mu\nu}\mathbf{1} + i\,\varepsilon_{\mu\nu\lambda}\sigma_\lambda$$

Commutation rules for Pauli ops: σ_μ

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$$\mathbf{S}_\mu\mathbf{S}_\nu - \mathbf{S}_\nu\mathbf{S}_\mu = [\mathbf{S}_\mu, \mathbf{S}_\nu] = i\,\varepsilon_{\mu\nu\lambda}\mathbf{S}_\lambda$$

Group products

$$\mathbf{R}[\vec{\Theta}']\mathbf{R}[\vec{\Theta}] = \left(\cos\frac{\Theta'}{2}\mathbf{1} - i\sin\frac{\Theta'}{2}\hat{\Theta}'\bullet\boldsymbol{\sigma}\right)\left(\cos\frac{\Theta}{2}\mathbf{1} - i\sin\frac{\Theta}{2}\hat{\Theta}\bullet\boldsymbol{\sigma}\right)$$

Operator-on-Operator transformations

Product algebra Multiplication rules for Pauli's "σ_μ-quaternions" and Hamilton's $\mathbf{q}_\mu = -i\sigma_\mu$.

•	1	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
1	1	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
$\mathbf{q}_X$	$\mathbf{q}_X$	-1	$\mathbf{q}_Z$	$-\mathbf{q}_Y$
$\mathbf{q}_Y$	$\mathbf{q}_Y$	$-\mathbf{q}_Z$	-1	$\mathbf{q}_X$
$\mathbf{q}_Z$	$\mathbf{q}_Z$	$\mathbf{q}_Y$	$-\mathbf{q}_X$	-1

,

•	1	σ_X	σ_Y	σ_Z
1	1	σ_X	σ_Y	σ_Z
σ_X	σ_X	1	$i\sigma_Z$	$-i\sigma_Y$
σ_Y	σ_Y	$-i\sigma_Z$	1	$i\sigma_X$
σ_Z	σ_Z	$i\sigma_Y$	$-i\sigma_X$	1

$$\mathbf{q}_\mu \, \mathbf{q}_\nu = -\delta_{\mu\nu}\,\mathbf{1} + \varepsilon_{\mu\nu\lambda}\,\mathbf{q}_\lambda$$

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Group products

$$\mathbf{R}[\vec{\Theta}']\mathbf{R}[\vec{\Theta}] = \left(\cos\frac{\Theta'}{2}\,\mathbf{1} - i\sin\frac{\Theta'}{2}\,\hat{\Theta}'\bullet\boldsymbol{\sigma}\right)\left(\cos\frac{\Theta}{2}\,\mathbf{1} - i\sin\frac{\Theta}{2}\,\hat{\Theta}\bullet\boldsymbol{\sigma}\right)$$

$$= \cos\frac{\Theta'}{2}\cos\frac{\Theta}{2}\,\mathbf{1} - i\left[\cos\frac{\Theta'}{2}\sin\frac{\Theta}{2}\,\hat{\Theta} + \cos\frac{\Theta}{2}\sin\frac{\Theta'}{2}\,\hat{\Theta}'\right]\bullet\boldsymbol{\sigma} - \sin\frac{\Theta'}{2}\sin\frac{\Theta}{2}\left(\hat{\Theta}'\bullet\boldsymbol{\sigma}\right)\left(\hat{\Theta}\bullet\boldsymbol{\sigma}\right)$$

Operator-on-Operator transformations

Product algebra Multiplication rules for Pauli's "σ_μ-quaternions" and Hamilton's $\mathbf{q}_\mu = -i\sigma_\mu$.

•	1	q_X	q_Y	q_Z
1	1	q_X	q_Y	q_Z
q_X	q_X	-1	q_Z	$-q_Y$
q_Y	q_Y	$-q_Z$	-1	q_X
q_Z	q_Z	q_Y	$-q_X$	-1

,

•	1	σ_X	σ_Y	σ_Z
1	1	σ_X	σ_Y	σ_Z
σ_X	σ_X	1	$i\sigma_Z$	$-i\sigma_Y$
σ_Y	σ_Y	$-i\sigma_Z$	1	$i\sigma_X$
σ_Z	σ_Z	$i\sigma_Y$	$-i\sigma_X$	1

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Group products

$$\mathbf{R}[\vec{\Theta}']\mathbf{R}[\vec{\Theta}] = \left(\cos\frac{\Theta'}{2}\mathbf{1} - i\sin\frac{\Theta'}{2}\hat{\Theta}'\bullet\sigma\right)\left(\cos\frac{\Theta}{2}\mathbf{1} - i\sin\frac{\Theta}{2}\hat{\Theta}\bullet\sigma\right)$$

$$= \cos\frac{\Theta'}{2}\cos\frac{\Theta}{2}\mathbf{1} - i\left[\cos\frac{\Theta'}{2}\sin\frac{\Theta}{2}\hat{\Theta} + \cos\frac{\Theta}{2}\sin\frac{\Theta'}{2}\hat{\Theta}'\right]\bullet\sigma - \sin\frac{\Theta'}{2}\sin\frac{\Theta}{2}\left(\hat{\Theta}'\bullet\sigma\right)\left(\hat{\Theta}\bullet\sigma\right)$$

Jordan-Pauli identity is used to reduce $(\Theta'\bullet\sigma)(\Theta\bullet\sigma)$ to $(\Theta'\bullet\Theta)\mathbf{1} + (\Theta'\times\Theta)\sigma$

$$\mathbf{R}[\vec{\Theta}']\mathbf{R}[\vec{\Theta}] = \left(\cos\frac{\Theta''}{2}\right)\mathbf{1} - i\left[\sin\frac{\Theta''}{2}\hat{\Theta}''\right]\bullet\sigma = \mathbf{R}[\Theta'']$$

$$= \left(\cos\frac{\Theta'}{2}\cos\frac{\Theta}{2} - \sin\frac{\Theta'}{2}\sin\frac{\Theta}{2}\right)\mathbf{1} - i\left[\left[\cos\frac{\Theta'}{2}\sin\frac{\Theta}{2}\hat{\Theta} + \cos\frac{\Theta}{2}\sin\frac{\Theta'}{2}\hat{\Theta}'\right] + \sin\frac{\Theta'}{2}\sin\frac{\Theta}{2}\hat{\Theta}'\times\hat{\Theta}\right]\bullet\sigma$$

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Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ *of group-operators*

Operator-on-Operator transformations

Geometry of groups: Hamilton's turns and It's all done with mirrors!

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U(2) product $\mathbf{R}[\Theta]\mathbf{R}[\Theta']=\mathbf{R}[\Theta''']$- *geometry*

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ *geometry*

Euler $\mathbf{R}(\alpha\beta\gamma)$ versus *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed*

R(3)-U(2) slide rule for converting $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *Sundial*

Operator-on-Operator transformations

Product algebra Multiplication rules for Pauli's "σ_μ-quaternions" and Hamilton's $\mathbf{q}_\mu = -i\sigma_\mu$.

•	1	q_X	q_Y	q_Z
1	1	q_X	q_Y	q_Z
q_X	q_X	−1	q_Z	$-q_Y$
q_Y	q_Y	$-q_Z$	−1	q_X
q_Z	q_Z	q_Y	$-q_X$	−1

,

•	1	σ_X	σ_Y	σ_Z
1	1	σ_X	σ_Y	σ_Z
σ_X	σ_X	1	$i\sigma_Z$	$-i\sigma_Y$
σ_Y	σ_Y	$-i\sigma_Z$	1	$i\sigma_X$
σ_Z	σ_Z	$i\sigma_Y$	$-i\sigma_X$	1

$$\mathbf{q}_\mu \mathbf{q}_\nu = -\delta_{\mu\nu}\mathbf{1} + \varepsilon_{\mu\nu\lambda}\mathbf{q}_\lambda$$
$$\sigma_\mu \sigma_\nu = \delta_{\mu\nu}\mathbf{1} + i\,\varepsilon_{\mu\nu\lambda}\sigma_\lambda$$

Commutation rules for Pauli ops: σ_μ

$$\sigma_\mu\sigma_\nu - \sigma_\nu\sigma_\mu = [\sigma_\mu, \sigma_\nu] = 2i\,\varepsilon_{\mu\nu\lambda}\sigma_\lambda$$

Jordan's spin-ops: $\mathbf{J}_\mu = \mathbf{S}_\mu = \sigma_\mu/2$.

$$\mathbf{S}_\mu\mathbf{S}_\nu - \mathbf{S}_\nu\mathbf{S}_\mu = [\mathbf{S}_\mu, \mathbf{S}_\nu] = i\,\varepsilon_{\mu\nu\lambda}\mathbf{S}_\lambda$$

Group products

$$\mathbf{R}[\vec{\Theta}']\mathbf{R}[\vec{\Theta}] = \left(\cos\frac{\Theta'}{2}\mathbf{1} - i\sin\frac{\Theta'}{2}\hat{\Theta}'\bullet\sigma\right)\left(\cos\frac{\Theta}{2}\mathbf{1} - i\sin\frac{\Theta}{2}\hat{\Theta}\bullet\sigma\right)$$

$$= \cos\frac{\Theta'}{2}\cos\frac{\Theta}{2}\mathbf{1} - i\left[\cos\frac{\Theta'}{2}\sin\frac{\Theta}{2}\hat{\Theta} + \cos\frac{\Theta}{2}\sin\frac{\Theta'}{2}\hat{\Theta}'\right]\bullet\sigma - \sin\frac{\Theta'}{2}\sin\frac{\Theta}{2}(\hat{\Theta}'\bullet\sigma)(\hat{\Theta}\bullet\sigma)$$

Jordan-Pauli identity is used to reduce $(\Theta'\bullet\sigma)(\Theta\bullet\sigma)$ to $(\Theta'\bullet\Theta)\mathbf{1} + (\Theta'\times\Theta)\sigma$

$$\mathbf{R}[\vec{\Theta}']\mathbf{R}[\vec{\Theta}] = \left(\cos\frac{\Theta''}{2}\right)\mathbf{1} - i\left[\sin\frac{\Theta''}{2}\hat{\Theta}''\right]\bullet\sigma = \mathbf{R}[\Theta'']$$

$$= \left(\cos\frac{\Theta'}{2}\cos\frac{\Theta}{2} - \sin\frac{\Theta'}{2}\sin\frac{\Theta}{2}\hat{\Theta}'\bullet\hat{\Theta}\right)\mathbf{1} - i\left[\left[\cos\frac{\Theta'}{2}\sin\frac{\Theta}{2}\hat{\Theta} + \cos\frac{\Theta}{2}\sin\frac{\Theta'}{2}\hat{\Theta}'\right] + \sin\frac{\Theta'}{2}\sin\frac{\Theta}{2}\hat{\Theta}'\times\hat{\Theta}\right]\bullet\sigma$$

Now easy to solve for the new *product angle* Θ'' .

$$\left(\cos\frac{\Theta''}{2}\right) = \left(\cos\frac{\Theta'}{2}\cos\frac{\Theta}{2} - \sin\frac{\Theta'}{2}\sin\frac{\Theta}{2}\hat{\Theta}'\bullet\hat{\Theta}\right)$$

U(2) and R(3) Group Product Formulae

Operator-on-Operator transformations

Product algebra Multiplication rules for Pauli's "σ_μ-quaternions" and Hamilton's $\mathbf{q}_\mu = -i\sigma_\mu$.

•	1	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
1	1	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
$\mathbf{q}_X$	$\mathbf{q}_X$	-1	$\mathbf{q}_Z$	$-\mathbf{q}_Y$
$\mathbf{q}_Y$	$\mathbf{q}_Y$	$-\mathbf{q}_Z$	-1	$\mathbf{q}_X$
$\mathbf{q}_Z$	$\mathbf{q}_Z$	$\mathbf{q}_Y$	$-\mathbf{q}_X$	-1

,

•	1	σ_X	σ_Y	σ_Z
1	1	σ_X	σ_Y	σ_Z
σ_X	σ_X	1	$i\sigma_Z$	$-i\sigma_Y$
σ_Y	σ_Y	$-i\sigma_Z$	1	$i\sigma_X$
σ_Z	σ_Z	$i\sigma_Y$	$-i\sigma_X$	1

$$\mathbf{q}_\mu \mathbf{q}_\nu = -\delta_{\mu\nu}\mathbf{1} + \varepsilon_{\mu\nu\lambda}\mathbf{q}_\lambda$$
$$\sigma_\mu \sigma_\nu = \delta_{\mu\nu}\mathbf{1} + i\,\varepsilon_{\mu\nu\lambda}\sigma_\lambda$$

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$$\mathbf{S}_\mu\mathbf{S}_\nu - \mathbf{S}_\nu\mathbf{S}_\mu = [\mathbf{S}_\mu, \mathbf{S}_\nu] = i\,\varepsilon_{\mu\nu\lambda}\mathbf{S}_\lambda$$

Group products

$$\mathbf{R}[\vec{\Theta}']\mathbf{R}[\vec{\Theta}] = \left(\cos\frac{\Theta'}{2}\mathbf{1} - i\sin\frac{\Theta'}{2}\hat{\Theta}'\bullet\sigma\right)\left(\cos\frac{\Theta}{2}\mathbf{1} - i\sin\frac{\Theta}{2}\hat{\Theta}\bullet\sigma\right)$$

$$= \cos\frac{\Theta'}{2}\cos\frac{\Theta}{2}\mathbf{1} - i\left[\cos\frac{\Theta'}{2}\sin\frac{\Theta}{2}\hat{\Theta} + \cos\frac{\Theta}{2}\sin\frac{\Theta'}{2}\hat{\Theta}'\right]\bullet\sigma - \sin\frac{\Theta'}{2}\sin\frac{\Theta}{2}\left(\hat{\Theta}'\bullet\sigma\right)\left(\hat{\Theta}\bullet\sigma\right)$$

Jordan-Pauli identity is used to reduce $(\Theta'\bullet\sigma)(\Theta\bullet\sigma)$ to $(\Theta'\bullet\Theta)\mathbf{1} + (\Theta'\times\Theta)\sigma$

$$\mathbf{R}[\vec{\Theta}']\mathbf{R}[\vec{\Theta}] = \left(\cos\frac{\Theta''}{2}\right)\mathbf{1} - i\left[\sin\frac{\Theta''}{2}\hat{\Theta}''\right]\bullet\sigma = \mathbf{R}[\Theta'']$$

$$= \left(\cos\frac{\Theta'}{2}\cos\frac{\Theta}{2} - \sin\frac{\Theta'}{2}\sin\frac{\Theta}{2}\hat{\Theta}'\bullet\hat{\Theta}\right)\mathbf{1} - i\left[\left[\cos\frac{\Theta'}{2}\sin\frac{\Theta}{2}\hat{\Theta} + \cos\frac{\Theta}{2}\sin\frac{\Theta'}{2}\hat{\Theta}'\right] + \sin\frac{\Theta'}{2}\sin\frac{\Theta}{2}\hat{\Theta}'\times\hat{\Theta}\right]\bullet\sigma$$

Now easy to solve for the new *product angle* Θ'' and new *crank unit vector* $\hat{\Theta}''$.

U(2) and R(3) Group Product Formulae

$$\left(\cos\frac{\Theta''}{2}\right) = \left(\cos\frac{\Theta'}{2}\cos\frac{\Theta}{2} - \sin\frac{\Theta'}{2}\sin\frac{\Theta}{2}\hat{\Theta}'\bullet\hat{\Theta}\right)$$

$$\left[\sin\frac{\Theta''}{2}\hat{\Theta}''\right] = \left[\cos\frac{\Theta'}{2}\sin\frac{\Theta}{2}\hat{\Theta} + \cos\frac{\Theta}{2}\sin\frac{\Theta'}{2}\hat{\Theta}' + \sin\frac{\Theta'}{2}\sin\frac{\Theta}{2}\hat{\Theta}'\times\hat{\Theta}\right]$$

Review:How "Crazy-Thing"-Theorem makes spinor and vector representation matrices

Half-angle $\Theta/2 = \varphi$ replacement and Darboux crank axis operators

Operator-on-Operator transformations

Product algebra for Pauli's σ_μ and Hamilton's $q_\mu = -i\sigma_\mu$

Group product algebra

Jordan-Pauli identity and U(2) product $\mathbf{R}[\Theta]\mathbf{R}[\Theta']=\mathbf{R}[\Theta''']$- *formula*

→ *Transformation* $\mathbf{R}[\Theta]\sigma_\mu\mathbf{R}[\Theta]^\dagger$ *of spinor* σ_μ*-operators*

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ *of group-operators*

Operator-on-Operator transformations

Geometry of groups: Hamilton's turns and It's all done with mirrors!

Group product geometry

U(2) product $\mathbf{R}[\Theta]\mathbf{R}[\Theta']=\mathbf{R}[\Theta''']$- *geometry*

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ *geometry*

Euler $\mathbf{R}(\alpha\beta\gamma)$ versus *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed*

R(3)-U(2) slide rule for converting $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *Sundial*

Transformation of spinor σ_μ-operators

$$\mathbf{R}[\vec{\boldsymbol{\Theta}}]\sigma_L\mathbf{R}[\vec{\boldsymbol{\Theta}}]^\dagger = \left(\cos\frac{\boldsymbol{\Theta}}{2}\,\mathbf{1} - i\sin\frac{\boldsymbol{\Theta}}{2}\,\hat{\boldsymbol{\Theta}}\bullet\boldsymbol{\sigma}\right)\sigma_L\left(\cos\frac{\boldsymbol{\Theta}}{2}\,\mathbf{1} - i\sin\frac{\boldsymbol{\Theta}}{2}\,\hat{\boldsymbol{\Theta}}\bullet\boldsymbol{\sigma}\right)^\dagger$$

Transformation of spinor σ_μ-operators

$$\mathbf{R}[\vec{\Theta}]\sigma_L\mathbf{R}[\vec{\Theta}]^\dagger = \left(\cos\frac{\Theta}{2}\,\mathbf{1} - i\sin\frac{\Theta}{2}\,\hat{\Theta}\bullet\boldsymbol{\sigma}\right)\boldsymbol{\sigma}_L\left(\cos\frac{\Theta}{2}\,\mathbf{1} - i\sin\frac{\Theta}{2}\,\hat{\Theta}\bullet\boldsymbol{\sigma}\right)^\dagger$$

$$= \left(\cos\frac{\Theta}{2}\,\mathbf{1} - i\sin\frac{\Theta}{2}\hat{\Theta}_K\boldsymbol{\sigma}_K\right)\boldsymbol{\sigma}_L\left(\cos\frac{\Theta}{2}\,\mathbf{1} + i\sin\frac{\Theta}{2}\hat{\Theta}_M\boldsymbol{\sigma}_M\right)$$

Transformation of spinor σ_μ-operators

$$\mathbf{R}[\vec{\Theta}]\sigma_L\mathbf{R}[\vec{\Theta}]^\dagger = \left(\cos\frac{\Theta}{2}\,\mathbf{1} - i\sin\frac{\Theta}{2}\,\hat{\boldsymbol{\Theta}}\bullet\boldsymbol{\sigma}\right)\sigma_L\left(\cos\frac{\Theta}{2}\,\mathbf{1} - i\sin\frac{\Theta}{2}\,\hat{\boldsymbol{\Theta}}\bullet\boldsymbol{\sigma}\right)^\dagger$$

$$= \left(\cos\frac{\Theta}{2}\,\mathbf{1} - i\sin\frac{\Theta}{2}\hat{\Theta}_K\sigma_K\right)\sigma_L\left(\cos\frac{\Theta}{2}\,\mathbf{1} + i\sin\frac{\Theta}{2}\hat{\Theta}_M\sigma_M\right)$$

$$= \sigma'_L \qquad = \sigma_L\cos\Theta - \varepsilon_{LKN}\hat{\Theta}_K\sigma_N\sin\Theta + (1-\cos\Theta)\hat{\Theta}_L\left(\hat{\Theta}_M\sigma_M\right)$$

(Left as an exercise)

Review:How "Crazy-Thing"-Theorem makes spinor and vector representation matrices

Half-angle $\Theta/2=\varphi$ replacement and Darboux crank axis operators

Operator-on-Operator transformations

Product algebra for Pauli's σ_μ and Hamilton's $q_\mu = -i\sigma_\mu$

Group product algebra

Jordan-Pauli identity and U(2) product $\mathbf{R}[\Theta]\mathbf{R}[\Theta']=\mathbf{R}[\Theta''']$*- formula*

Transformation $\mathbf{R}[\Theta]\sigma_\mu\mathbf{R}[\Theta]^\dagger$ *of spinor σ_μ-operators*

→ *Transformation* $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ *of group-operators*

Operator-on-Operator transformations

Geometry of groups: Hamilton's turns and It's all done with mirrors!

Group product geometry

U(2) product $\mathbf{R}[\Theta]\mathbf{R}[\Theta']=\mathbf{R}[\Theta''']$*- geometry*

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ *geometry*

Euler $\mathbf{R}(\alpha\beta\gamma)$ versus *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

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Euler $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed*

R(3)-U(2) slide rule for converting $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *Sundial*

Transformation of spinor σ_μ-operators

$$\mathbf{R}[\vec{\Theta}]\sigma_L\mathbf{R}[\vec{\Theta}]^\dagger = \left(\cos\frac{\Theta}{2}\,\mathbf{1} - i\sin\frac{\Theta}{2}\,\hat{\Theta}\bullet\boldsymbol{\sigma}\right)\sigma_L\left(\cos\frac{\Theta}{2}\,\mathbf{1} - i\sin\frac{\Theta}{2}\,\hat{\Theta}\bullet\boldsymbol{\sigma}\right)^\dagger$$

$$= \left(\cos\frac{\Theta}{2}\,\mathbf{1} - i\sin\frac{\Theta}{2}\hat{\Theta}_K\sigma_K\right)\sigma_L\left(\cos\frac{\Theta}{2}\,\mathbf{1} + i\sin\frac{\Theta}{2}\hat{\Theta}_M\sigma_M\right)$$

$$= \sigma'_L \qquad = \sigma_L\cos\Theta - \varepsilon_{LKN}\hat{\Theta}_K\sigma_N\sin\Theta + (1-\cos\Theta)\hat{\Theta}_L\left(\hat{\Theta}_M\sigma_M\right)$$

General transformation of rotational **R[Θ′]**-*operators*

$$\mathbf{R}[\vec{\Theta}]\mathbf{R}[\vec{\Theta}']\mathbf{R}[\vec{\Theta}]^\dagger = \left(\cos\frac{\Theta}{2}\,\mathbf{1} - i\sin\frac{\Theta}{2}\,\hat{\Theta}\bullet\boldsymbol{\sigma}\right)\mathbf{R}[\vec{\Theta}']\left(\cos\frac{\Theta}{2}\,\mathbf{1} - i\sin\frac{\Theta}{2}\,\hat{\Theta}\bullet\boldsymbol{\sigma}\right)^\dagger$$

$$\mathbf{R}\left[\mathbf{R}[\Theta]\cdot\vec{\Theta}'\right] = \left(\cos\frac{\Theta}{2}\,\mathbf{1} - i\sin\frac{\Theta}{2}\hat{\Theta}_K\sigma_K\right)\mathbf{R}[\vec{\Theta}']\left(\cos\frac{\Theta}{2}\,\mathbf{1} + i\sin\frac{\Theta}{2}\hat{\Theta}_M\sigma_M\right)$$

Transformation of spinor σ_μ-operators

$$\mathbf{R}[\vec{\Theta}]\sigma_L\mathbf{R}[\vec{\Theta}]^\dagger = \left(\cos\frac{\Theta}{2}\mathbf{1} - i\sin\frac{\Theta}{2}\hat{\boldsymbol{\Theta}}\bullet\boldsymbol{\sigma}\right)\sigma_L\left(\cos\frac{\Theta}{2}\mathbf{1} - i\sin\frac{\Theta}{2}\hat{\boldsymbol{\Theta}}\bullet\boldsymbol{\sigma}\right)^\dagger$$

$$= \left(\cos\frac{\Theta}{2}\mathbf{1} - i\sin\frac{\Theta}{2}\hat{\Theta}_K\sigma_K\right)\sigma_L\left(\cos\frac{\Theta}{2}\mathbf{1} + i\sin\frac{\Theta}{2}\hat{\Theta}_M\sigma_M\right)$$

$$= \sigma'_L \qquad = \sigma_L\cos\Theta - \varepsilon_{LKN}\hat{\Theta}_K\sigma_N\sin\Theta + (1-\cos\Theta)\hat{\Theta}_L\left(\hat{\Theta}_M\sigma_M\right)$$

General transformation of rotational **R[Θ′]***-operators*

$$\mathbf{R}[\vec{\Theta}]\mathbf{R}[\vec{\Theta}']\mathbf{R}[\vec{\Theta}]^\dagger = \left(\cos\frac{\Theta}{2}\mathbf{1} - i\sin\frac{\Theta}{2}\hat{\boldsymbol{\Theta}}\bullet\boldsymbol{\sigma}\right)\mathbf{R}[\vec{\Theta}']\left(\cos\frac{\Theta}{2}\mathbf{1} - i\sin\frac{\Theta}{2}\hat{\boldsymbol{\Theta}}\bullet\boldsymbol{\sigma}\right)^\dagger$$

$$\mathbf{R}\left[\mathbf{R}[\Theta]\cdot\vec{\Theta}'\right] = \left(\cos\frac{\Theta}{2}\mathbf{1} - i\sin\frac{\Theta}{2}\hat{\Theta}_K\sigma_K\right)\mathbf{R}[\vec{\Theta}']\left(\cos\frac{\Theta}{2}\mathbf{1} + i\sin\frac{\Theta}{2}\hat{\Theta}_M\sigma_M\right)$$

This one is better seen geometrically. Algebra not so quick.

Review:How "Crazy-Thing"-Theorem makes spinor and vector representation matrices

Half-angle $\Theta/2=\varphi$ replacement and Darboux crank axis operators

Operator-on-Operator transformations

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Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ *of group-operators*

Operator-on-Operator transformations

Geometry of groups: Hamilton's turns and It's all done with mirrors!

→ *Group product geometry*

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Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ *geometry*

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Euler $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

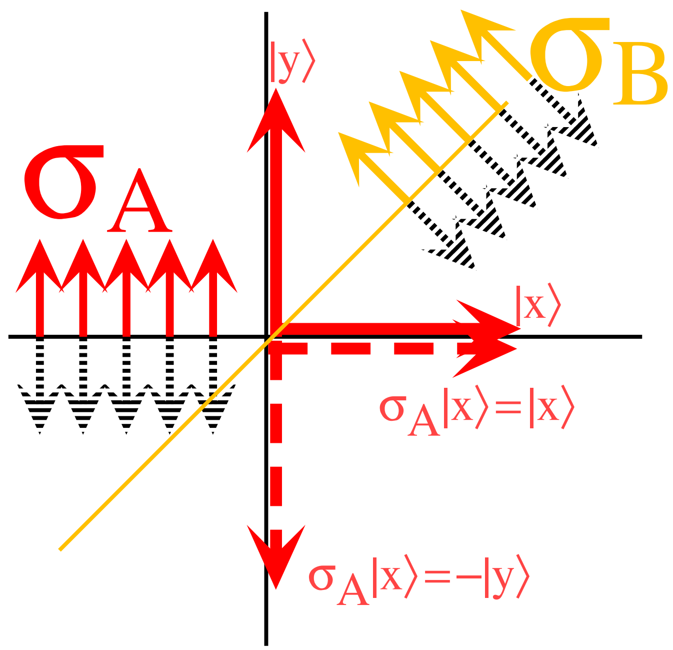
Euler $\mathbf{R}(\alpha\beta\gamma)$ *rotation $\Theta=0-4\pi$-sequence $[\varphi\vartheta]$ fixed*

R(3)-U(2) slide rule for converting $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *Sundial*

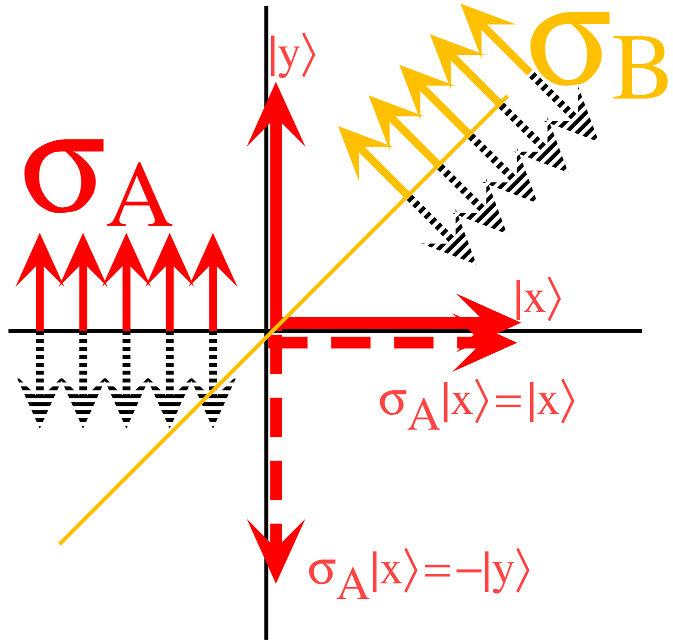
Geometry of U(2) transformations. It's all done with mirrors!

$$\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



Geometry of U(2) transformations. It's all done with mirrors!

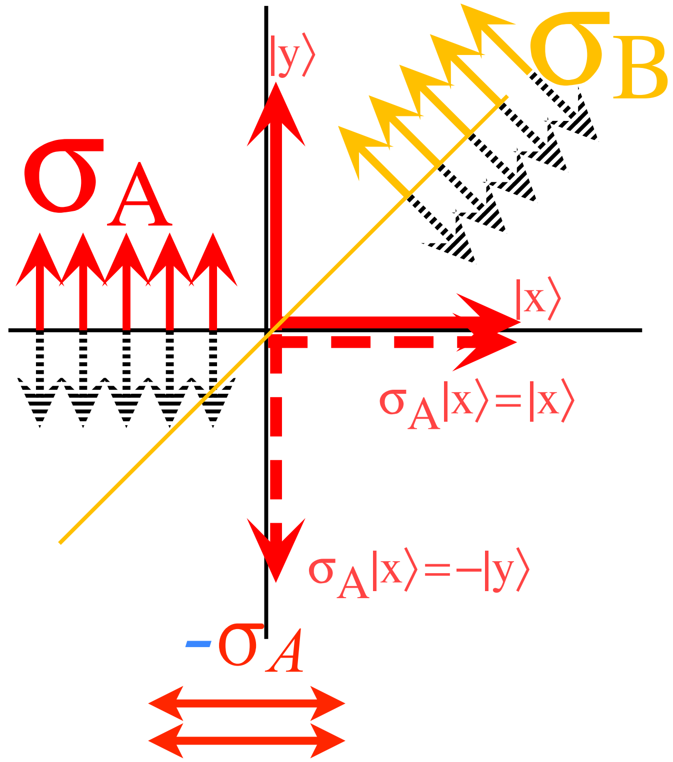
$$\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



Note that -σ_A is a y-plane mirror

Geometry of U(2) transformations. It's all done with mirrors!

$$\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

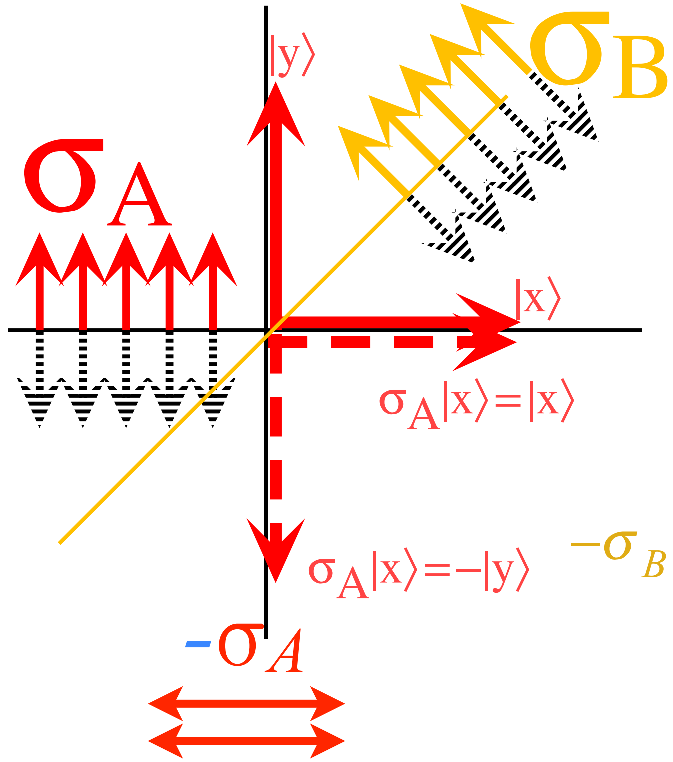


Note that is a y-plane mirror

$$-\sigma_A = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$$

Geometry of U(2) transformations. It's all done with mirrors!

$$\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



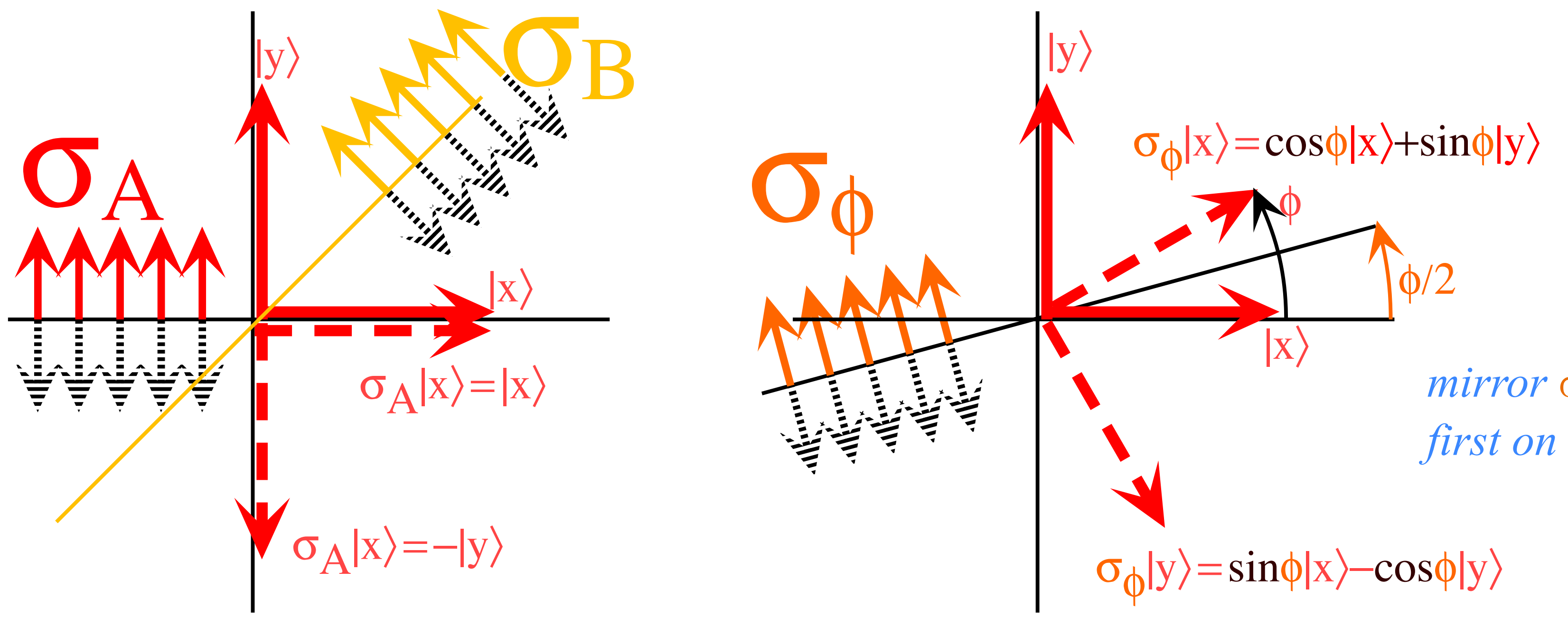
$$-\sigma_B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = ?$$

Note that is a y-plane mirror

$$-\sigma_A = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$$

Geometry of U(2) transformations. It's all done with mirrors!

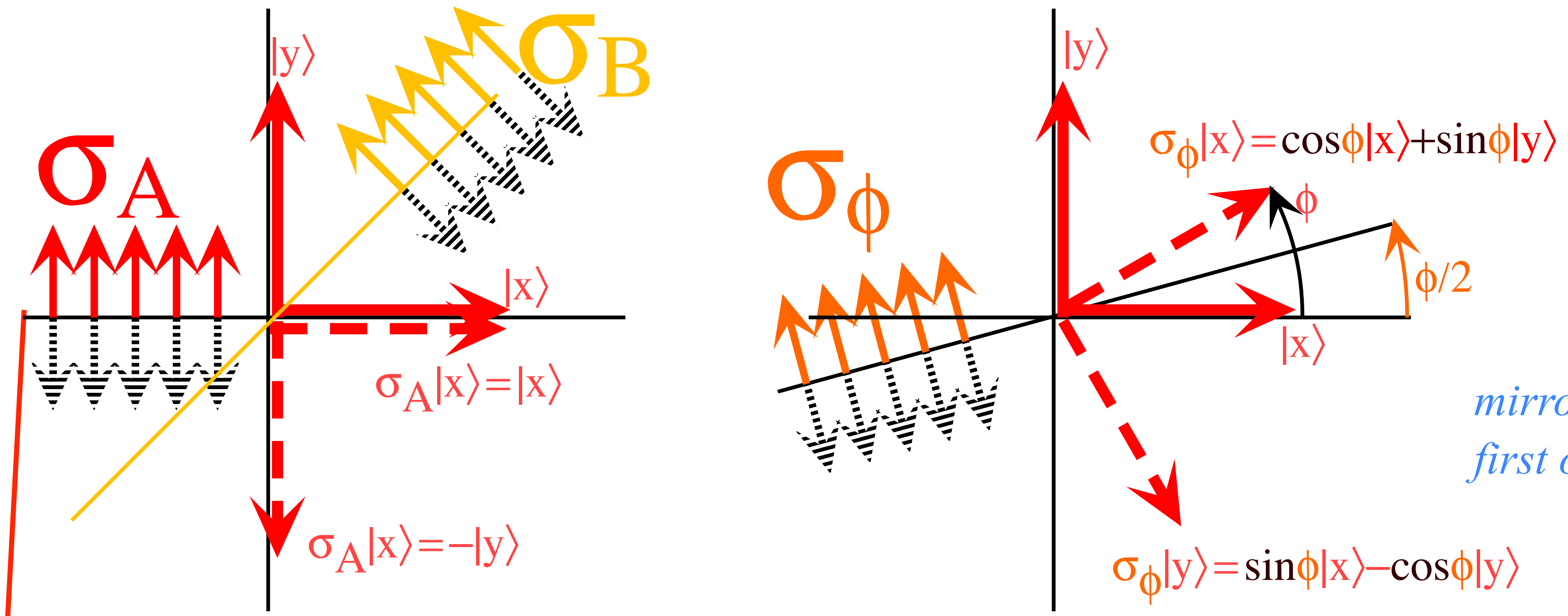
$$\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} = \sigma_A \cos\phi + \sigma_B \sin\phi$$



mirror σ_ϕ acts first on kets

Geometry of U(2) transformations. It's all done with mirrors!

$$\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} = \sigma_A \cos\phi + \sigma_B \sin\phi$$



mirror σ_A acts first on kets

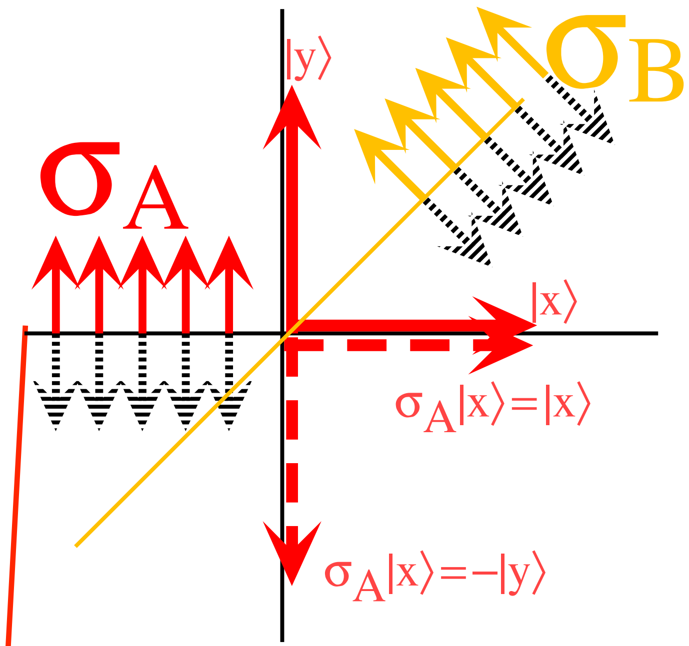
mirror σ_ϕ acts first on kets

$$\sigma_\phi \sigma_A = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

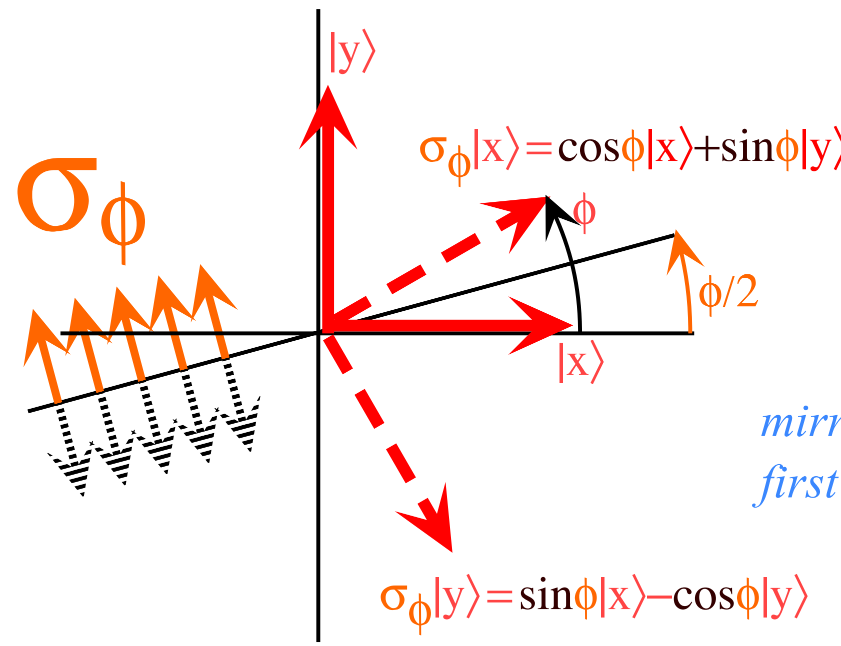
mirror σ_ϕ goes 2nd

Geometry of U(2) transformations. It's all done with mirrors!

$$\sigma_A=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \sigma_B=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_\phi=\begin{pmatrix}\cos\phi & \sin\phi\\ \sin\phi & -\cos\phi\end{pmatrix}=\sigma_A\cos\phi+\sigma_B\sin\phi$$



mirror σ_A acts first on kets



mirror σ_ϕ acts first on kets

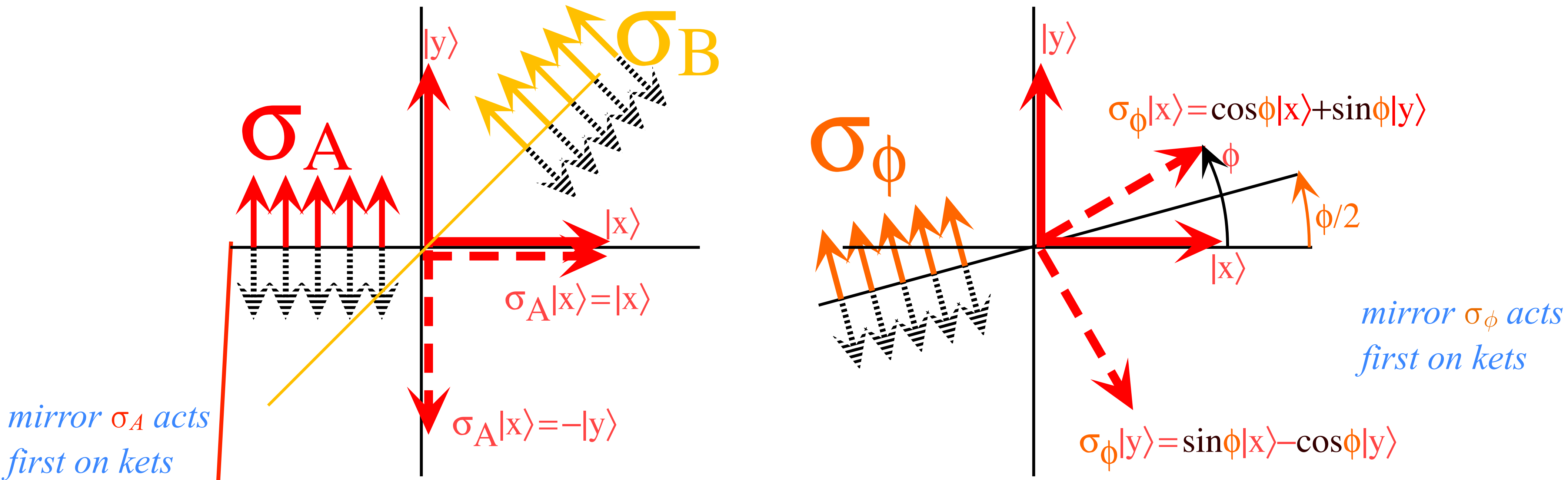
mirror σ_ϕ goes 2nd

$$\sigma_\phi\sigma_A=\begin{pmatrix}\cos\phi & \sin\phi\\ \sin\phi & -\cos\phi\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$=\begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}=R[\phi]\ ,$$

Geometry of U(2) transformations. It's all done with mirrors!

$$\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} = \sigma_A \cos\phi + \sigma_B \sin\phi$$



mirror σ_ϕ goes 2nd

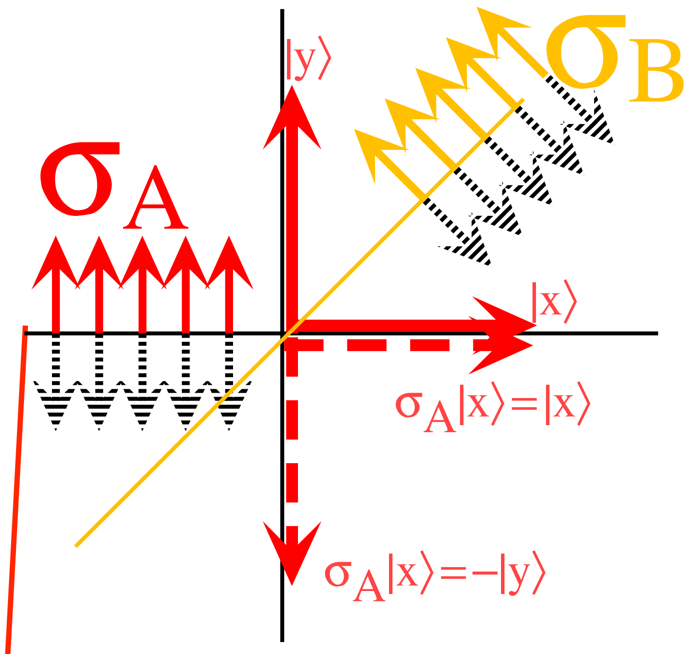
$$\sigma_\phi \sigma_A = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

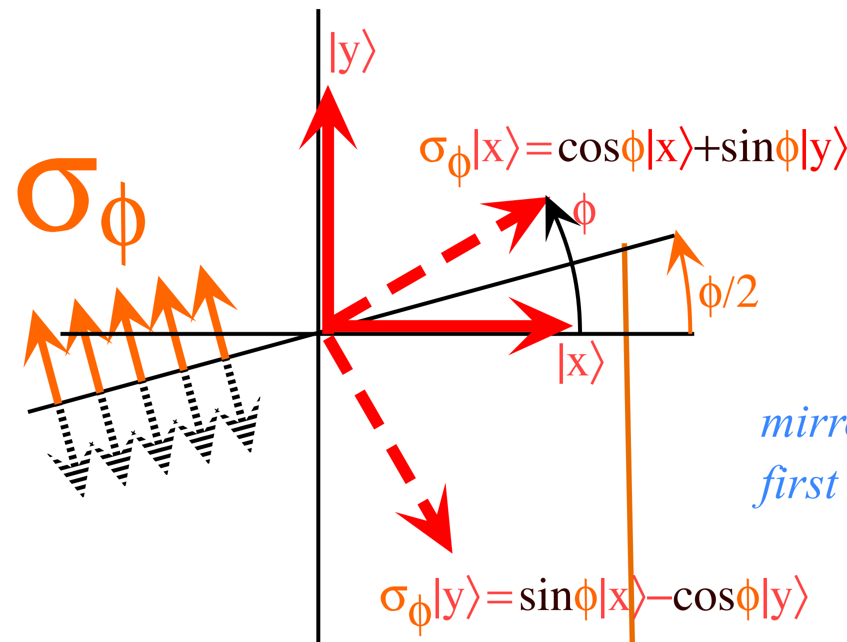
$$= \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = R[\phi] \ ,$$

Rotation angle ϕ is TWICE the angle $\phi/2$ between mirror σ_A and mirror σ_ϕ

Geometry of U(2) transformations. It's all done with mirrors!

$$\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} = \sigma_A \cos\phi + \sigma_B \sin\phi$$





mirror σ_A acts first on kets

mirror σ_ϕ acts first on kets

mirror σ_ϕ goes 2nd

$$\sigma_\phi \sigma_A = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = R[\phi],$$

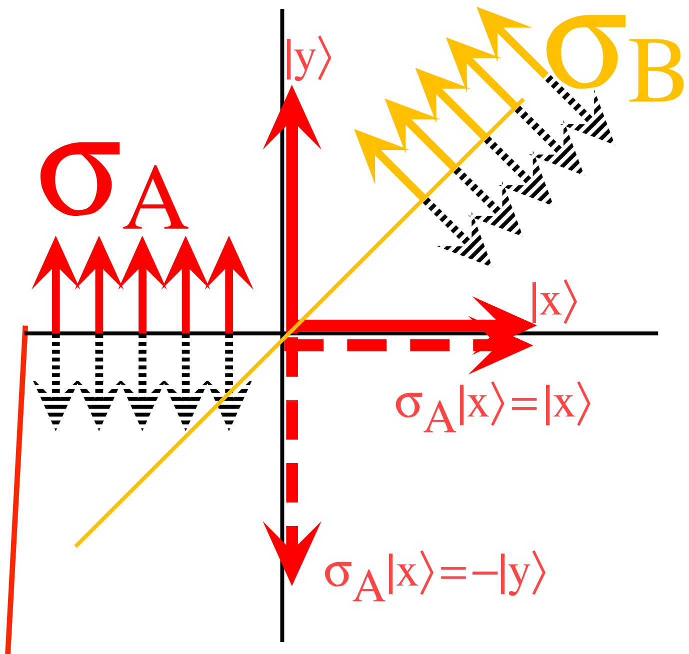
mirror σ_A goes 2nd

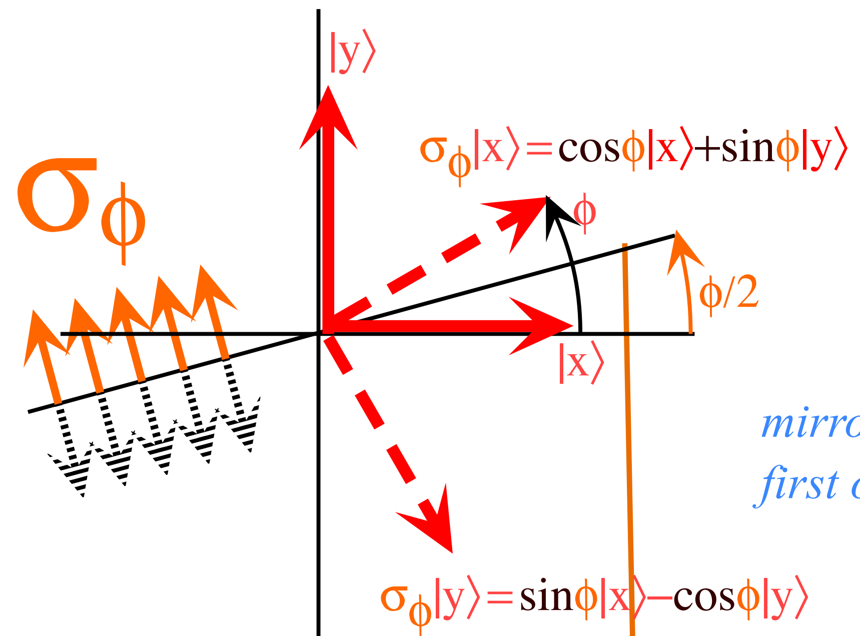
$$\sigma_A \sigma_\phi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} = R[-\phi]$$

Rotation angle ϕ is TWICE the angle $\phi/2$ between mirror σ_A and mirror σ_ϕ

Geometry of U(2) transformations. It's all done with mirrors!

$$\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} = \sigma_A \cos\phi + \sigma_B \sin\phi$$





mirror σ_A acts first on kets

mirror σ_ϕ acts first on kets

mirror σ_ϕ goes 2nd

$$\sigma_\phi \sigma_A = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = R[\phi] ,$$

mirror σ_A goes 2nd

$$\sigma_A \sigma_\phi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} = R[-\phi]$$

Rotation angle ϕ is TWICE the angle $\phi/2$ between mirror σ_A and mirror σ_ϕ

$$\sigma_A \sigma_B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} = i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = ?$$

Review:How "Crazy-Thing"-Theorem makes spinor and vector representation matrices

Half-angle $\Theta/2 = \varphi$ replacement and Darboux crank axis operators

Operator-on-Operator transformations

Product algebra for Pauli's σ_μ and Hamilton's $q_\mu = -i\sigma_\mu$

Group product algebra

Jordan-Pauli identity and U(2) product $\mathbf{R}[\Theta]\mathbf{R}[\Theta']=\mathbf{R}[\Theta''']$*- formula*

Transformation $\mathbf{R}[\Theta]\sigma_\mu\mathbf{R}[\Theta]^\dagger$ *of spinor σ_μ-operators*

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ *of group-operators*

Operator-on-Operator transformations

Geometry of groups: Hamilton's turns and It's all done with mirrors!

→ *Group product geometry*

U(2) product $\mathbf{R}[\Theta]\mathbf{R}[\Theta']=\mathbf{R}[\Theta''']$*- geometry*

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ *geometry*

Euler $\mathbf{R}(\alpha\beta\gamma)$ versus *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *rotation $\Theta=0-4\pi$-sequence $[\varphi\vartheta]$ fixed*

R(3)-U(2) slide rule for converting $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *Sundial*

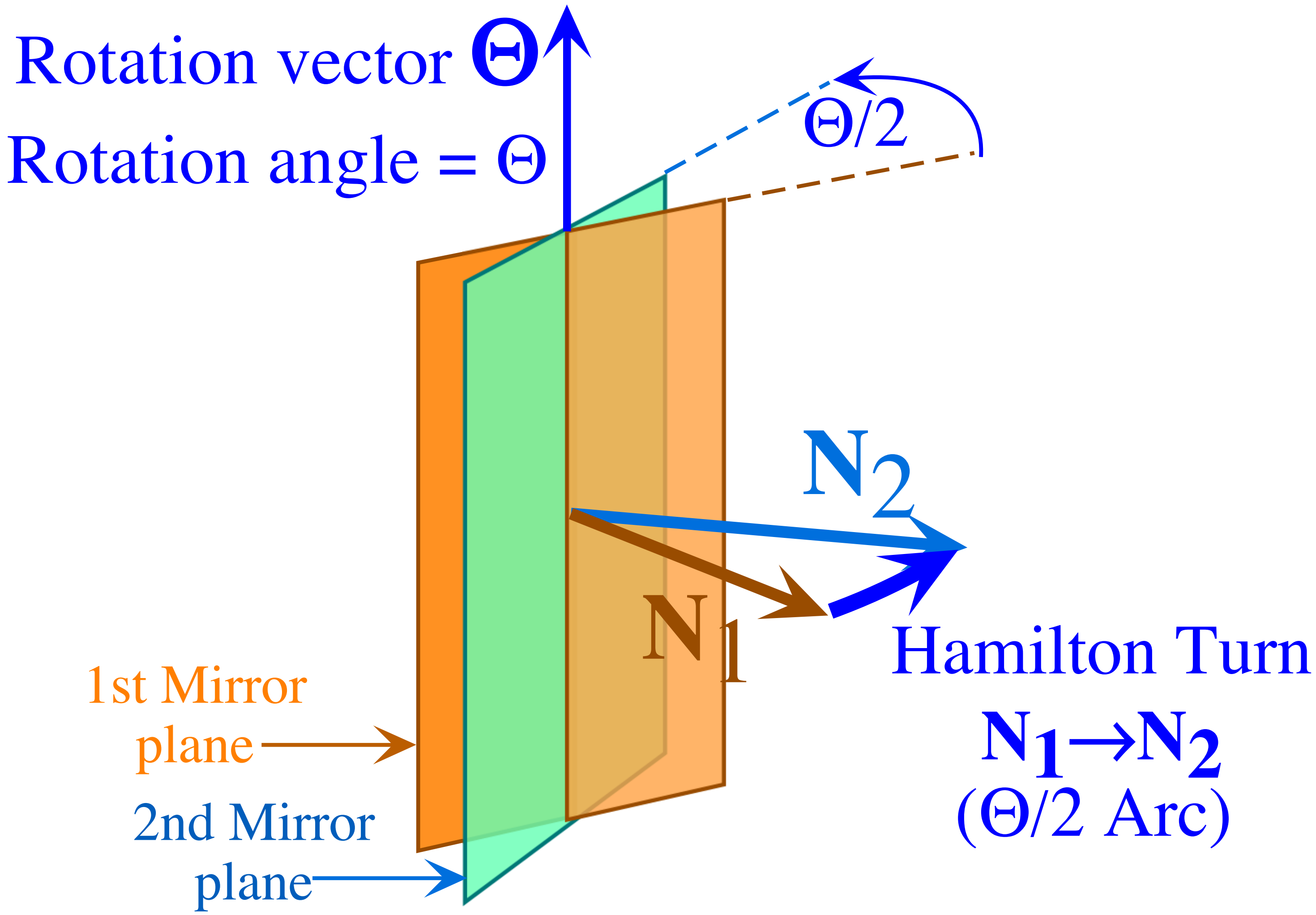


Fig. 10.A.7 Mirror reflection planes, normals, and Hamilton-turn arc vector.

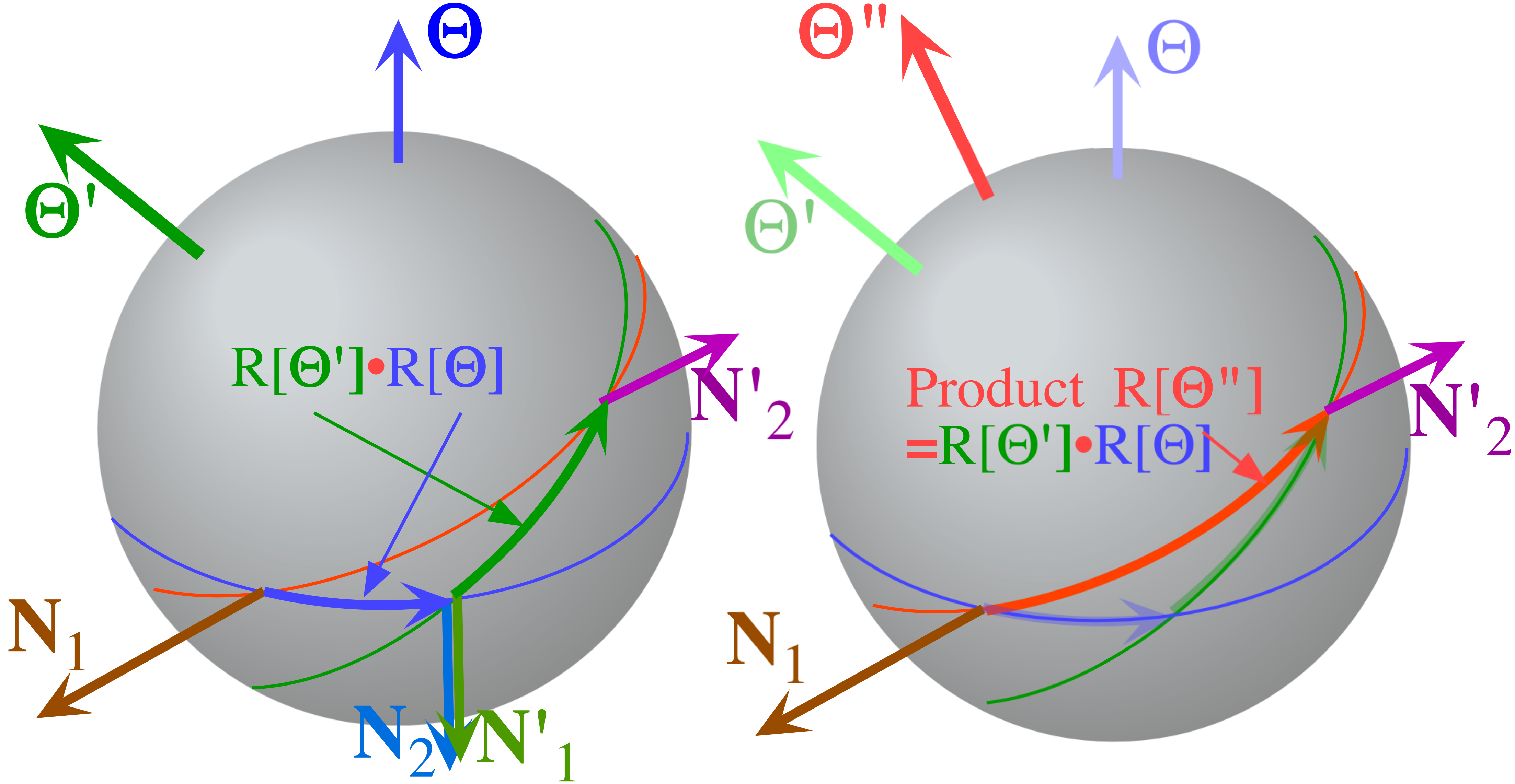
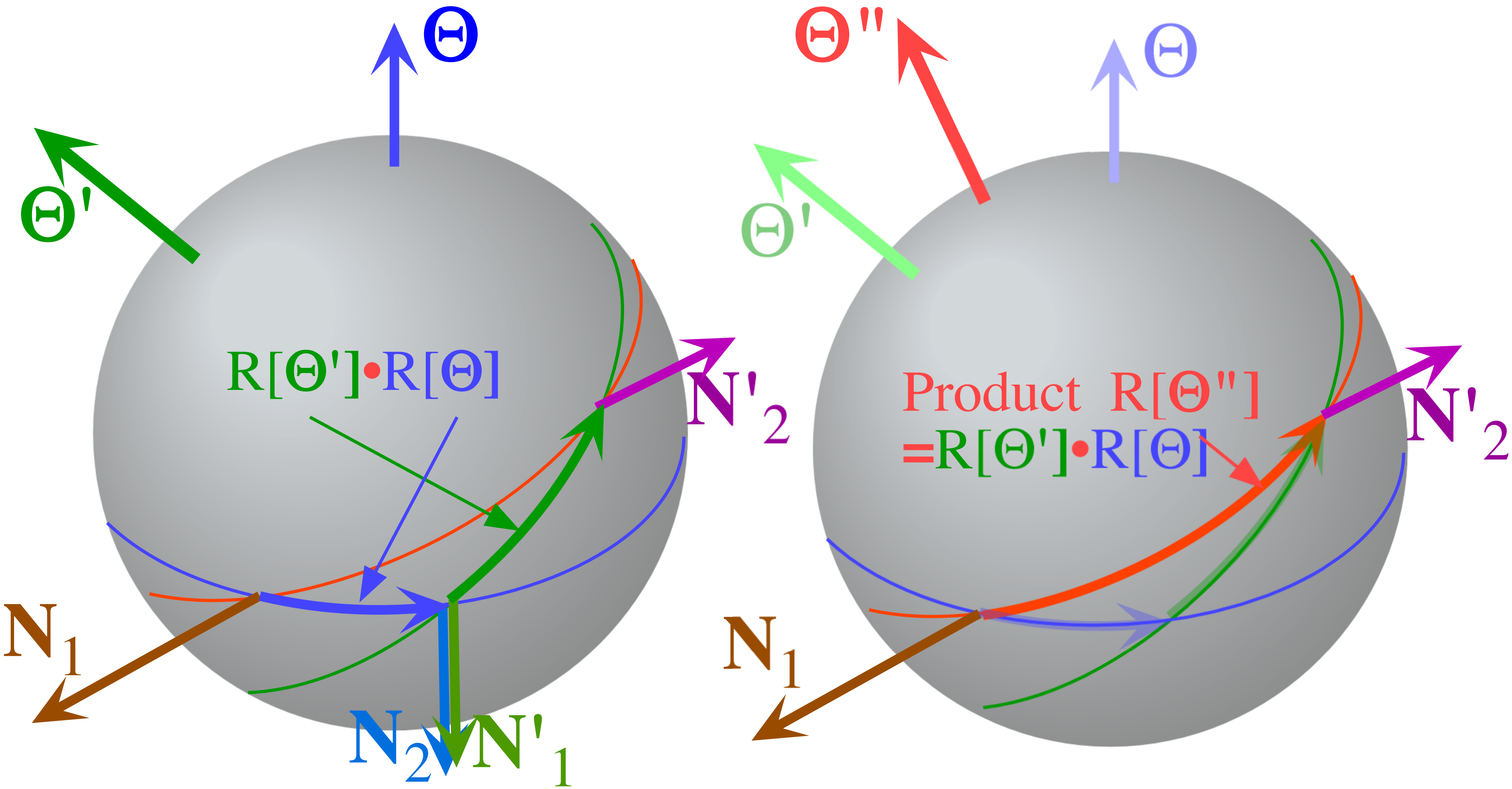


Fig. 10.A.8 Adding Hamilton-turn arcs to compute a U(2) product **R**[Θ"]=**R**[Θ']**R**[Θ].

Each arc Θ/2, Θ'/2, or Θ"/2 is 1/2 actual angle Θ, Θ', or Θ" of rotation **R**[Θ], **R**[Θ'], or **R**[Θ"].



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Each arc Θ/2, Θ'/2, or Θ"/2 is 1/2 actual angle Θ, Θ', or Θ" of rotation **R**[Θ], **R**[Θ'], or **R**[Θ"].
Arc Θ/2 between **N1** and **N2** and its supplement (Θ±2π)/2=Θ/2±π between **N1** and **-N2** represent the same classical rotation by Θ.

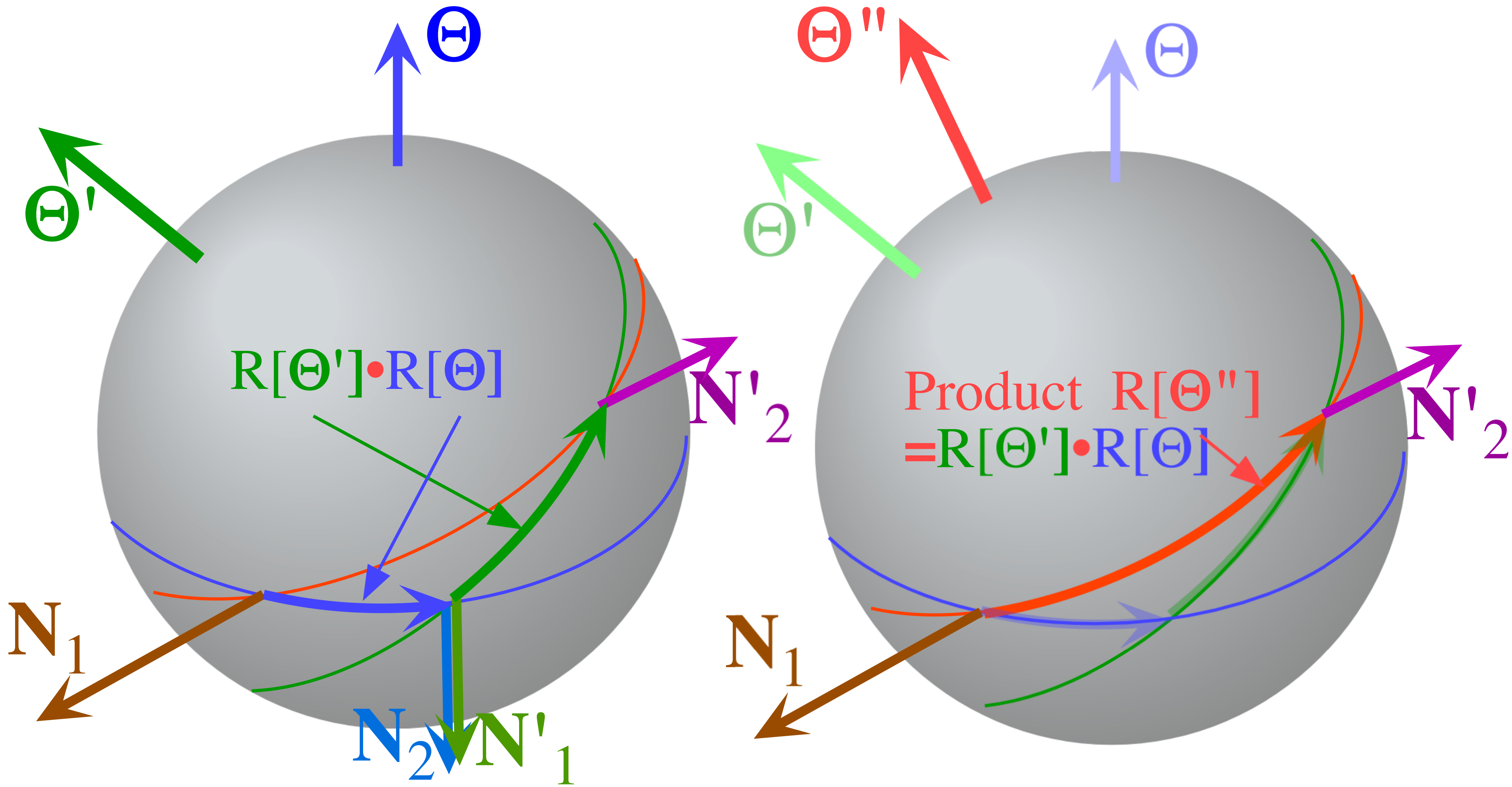

Fig. 10.A.8 Adding Hamilton-turn arcs to compute a U(2) product **R**[Θ"]=**R**[Θ']**R**[Θ].

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For quantum spin-1/2 object, the arc pointing from **N1** to the antipodal normal -**N2** represents a Θ-rotation with an extra π-phase factor $e^{\pm i\pi} = -1$, that is, -**R**[Θ] .

Review:How "Crazy-Thing"-Theorem makes spinor and vector representation matrices

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Transformation $\mathbf{R}[\Theta]\sigma_\mu\mathbf{R}[\Theta]^\dagger$ *of spinor* σ_μ*-operators*

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ *of group-operators*

Operator-on-Operator transformations

Geometry of groups: Hamilton's turns and It's all done with mirrors!

Group product geometry

U(2) product $\mathbf{R}[\Theta]\mathbf{R}[\Theta']=\mathbf{R}[\Theta''']$*- geometry*

→ *Transformation* $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ *geometry*

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Euler $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed*

R(3)-U(2) slide rule for converting $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *Sundial*

Geometry of transformation of rotational **R[Θ]**-*operators*

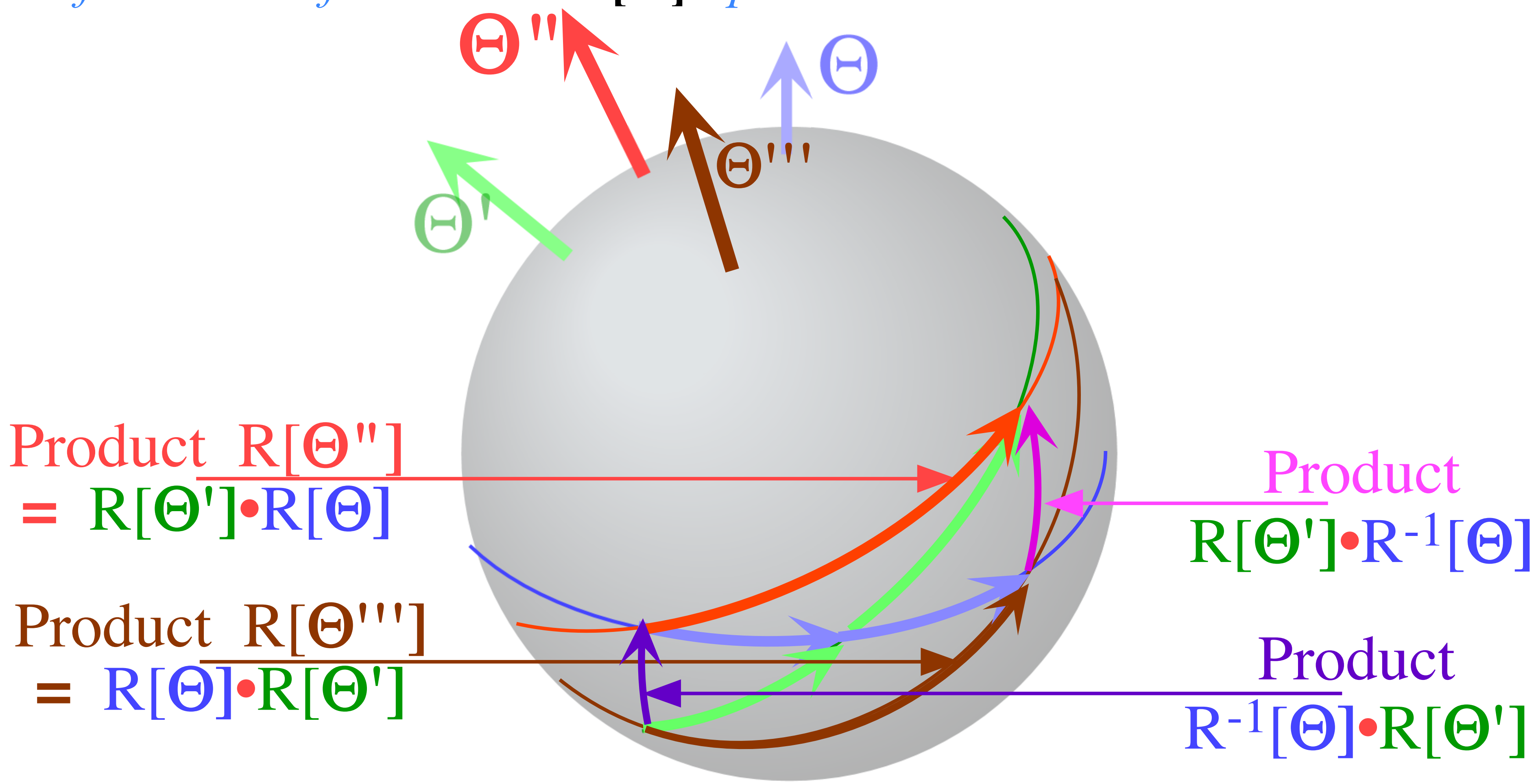

Fig. 10.A.9 Hamilton-turn arc parallelogram with **R**[Θ'']=**R**[Θ']**R**[Θ] *and* **R**[Θ''']=**R**[Θ]**R**[Θ']

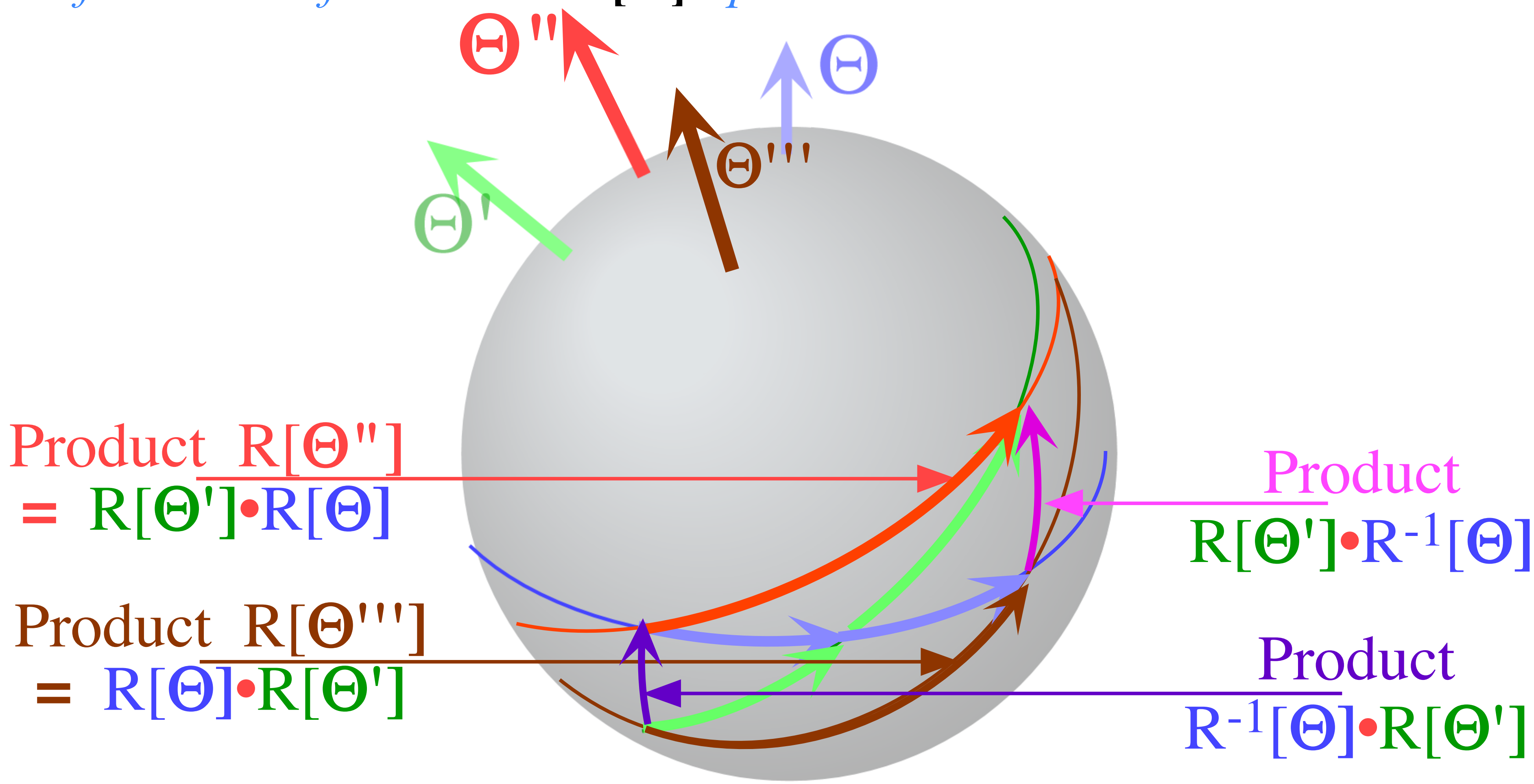

Fig. 10.A.9 Hamilton-turn arc parallelogram with **R**[Θ'']=**R**[Θ']**R**[Θ] *and* **R**[Θ''']=**R**[Θ]**R**[Θ']

Vectors added in the reverse order give **R**[Θ''']=**R**[Θ]**R**[Θ'] instead of **R**[Θ'']=**R**[Θ']**R**[Θ].

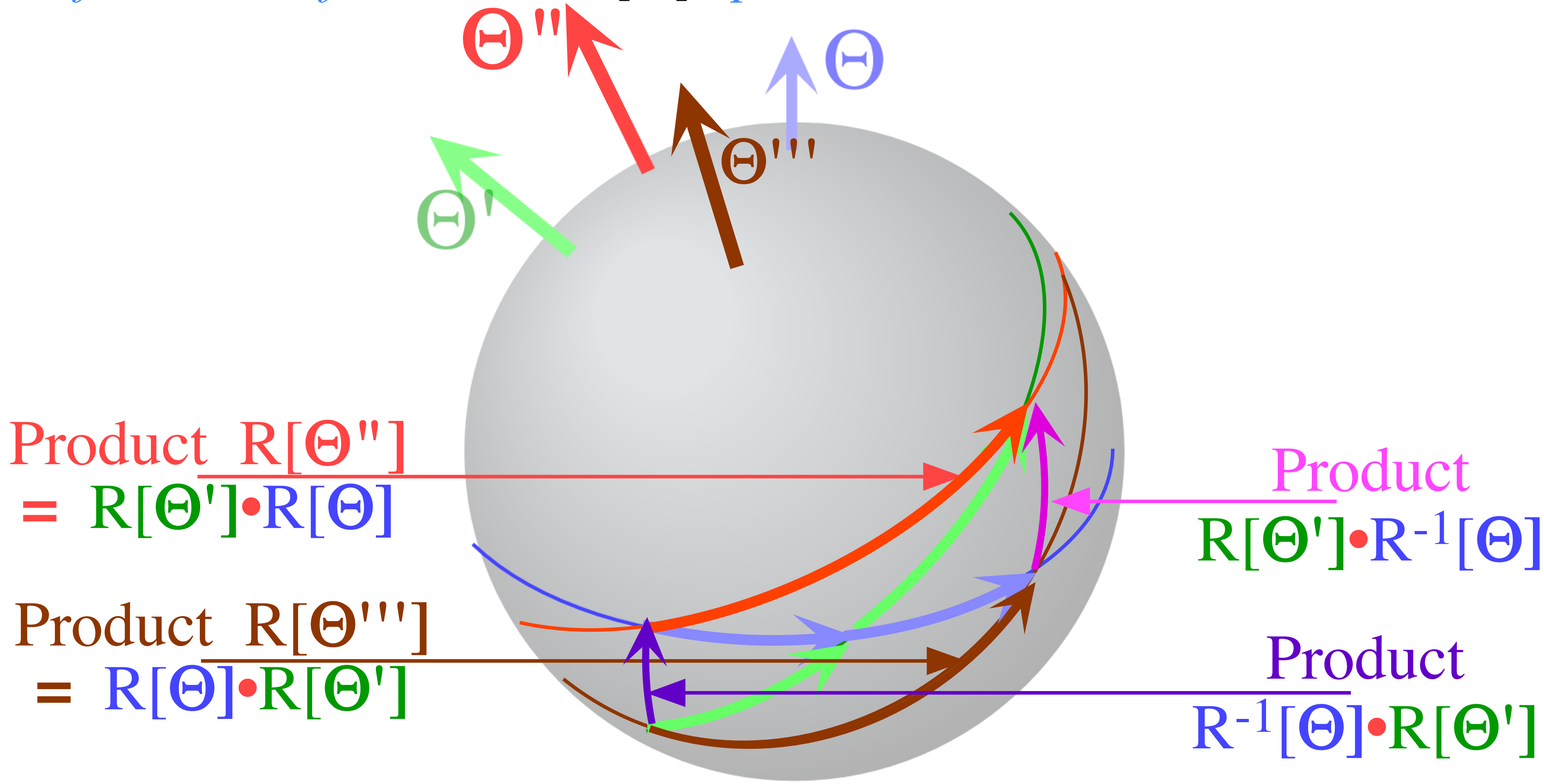

Fig. 10.A.9 Hamilton-turn arc parallelogram with **R**[Θ'']=**R**[Θ']**R**[Θ] *and* **R**[Θ''']=**R**[Θ]**R**[Θ']

Vectors added in the reverse order give **R**[Θ''']=**R**[Θ]**R**[Θ'] instead of **R**[Θ'']=**R**[Θ']**R**[Θ].

A similarity transformation of rotation **R**[Θ''] by rotation **R**[Θ] gives rotation **R**[Θ''']

R[Θ] **R**[Θ''] **R**[-Θ] = **R**[Θ''']

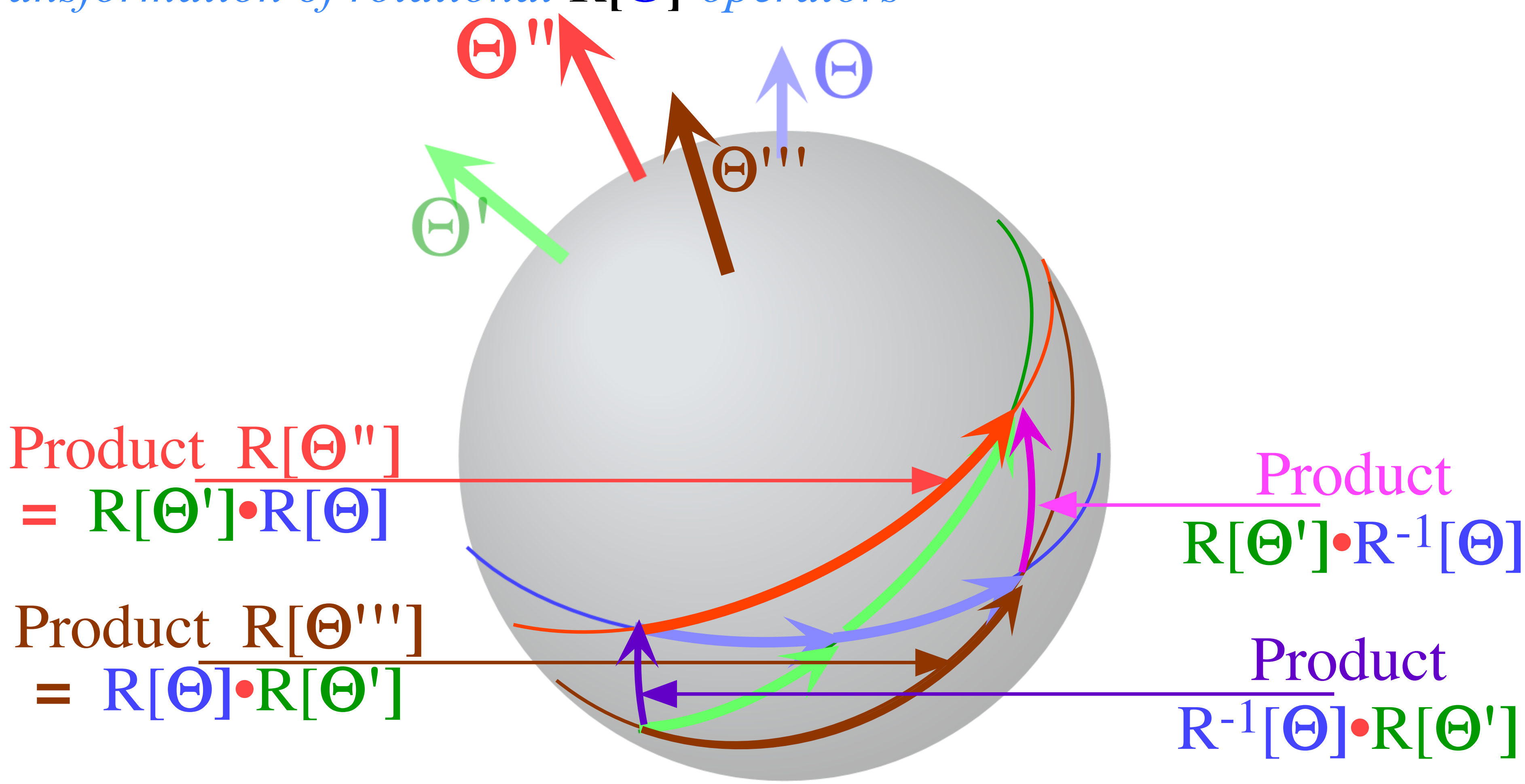

Fig. 10.A.9 Hamilton-turn arc parallelogram with **R**[Θ'']=**R**[Θ']**R**[Θ] *and* **R**[Θ''']=**R**[Θ]**R**[Θ']

Vectors added in the reverse order give **R**[Θ''']=**R**[Θ]**R**[Θ'] instead of **R**[Θ'']=**R**[Θ']**R**[Θ].

A similarity transformation of rotation **R**[Θ''] by rotation **R**[Θ] gives rotation **R**[Θ'''] and *vice-versa*:

R[Θ] **R**[Θ''] **R**[-Θ] = **R**[Θ''']   **R**[-Θ] **R**[Θ'''] **R**[Θ] = **R**[Θ'']

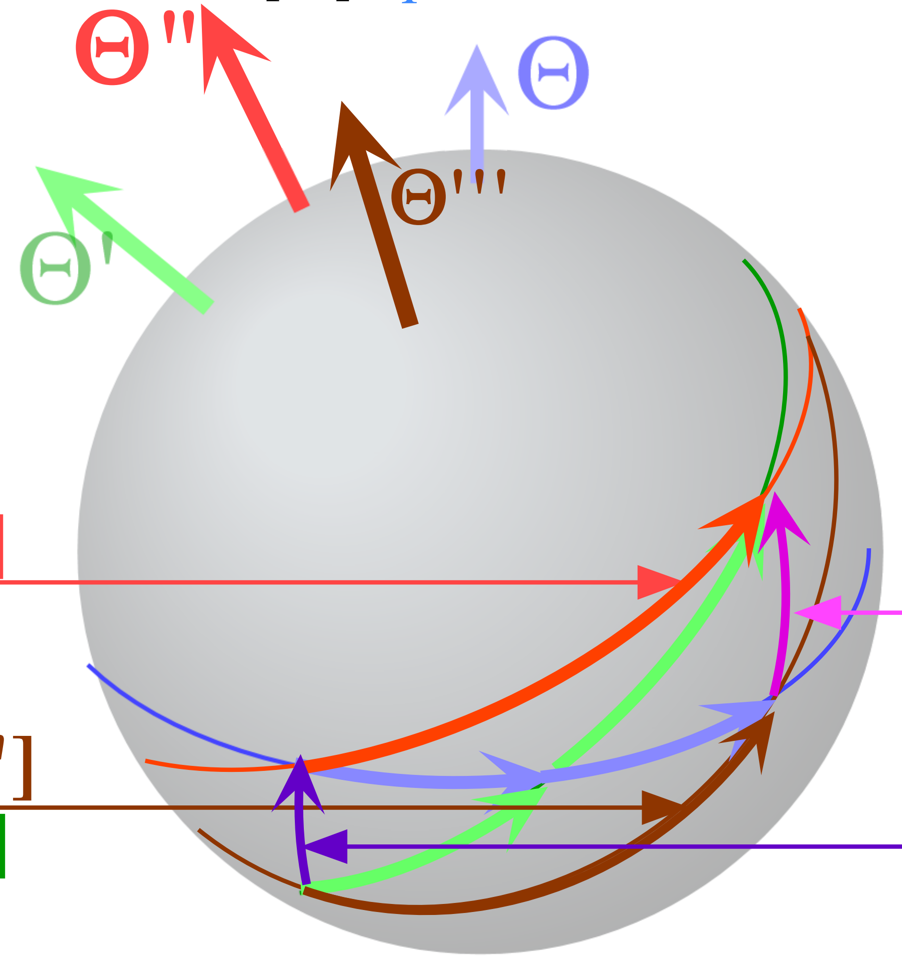

Fig. 10.A.9 Hamilton-turn arc parallelogram with **R**[Θ'']=**R**[Θ']**R**[Θ] *and* **R**[Θ''']=**R**[Θ]**R**[Θ']

Vectors added in the reverse order give **R**[Θ''']=**R**[Θ]**R**[Θ'] instead of **R**[Θ'']=**R**[Θ']**R**[Θ].

A similarity transformation of rotation **R**[Θ''] by rotation **R**[Θ] gives rotation **R**[Θ'''] and *vice-versa*:

R[Θ] **R**[Θ''] **R**[-Θ] = **R**[Θ''']   **R**[-Θ] **R**[Θ'''] **R**[Θ] = **R**[Θ'']

Everything associated with rotation **R**[Θ''] is rotated by full angle Θ around axis Θ.

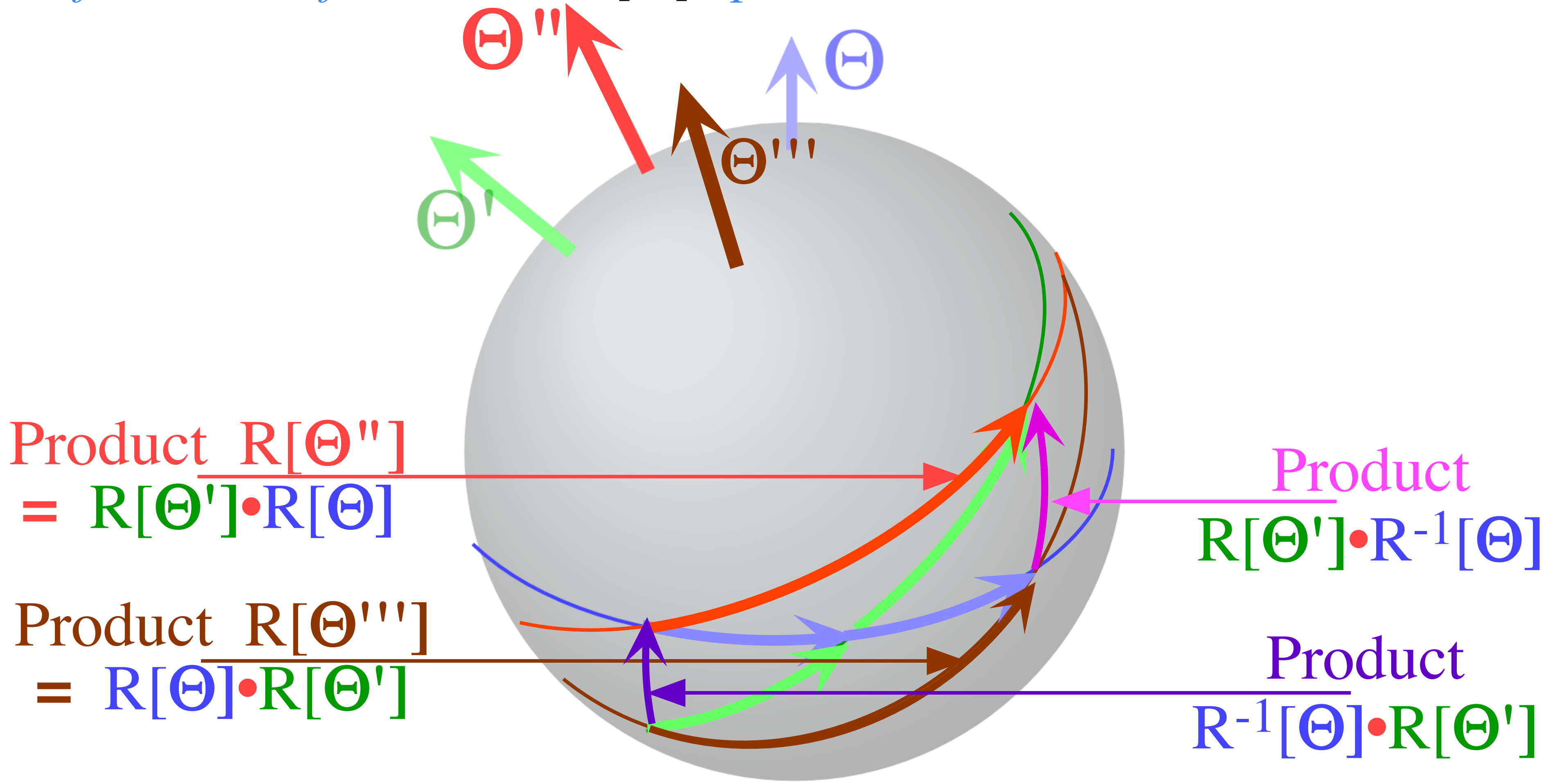

Fig. 10.A.9 Hamilton-turn arc parallelogram with **R**[Θ"]=**R**[Θ']**R**[Θ] *and* **R**[Θ''']=**R**[Θ]**R**[Θ']

Vectors added in the reverse order give **R**[Θ''']=**R**[Θ]**R**[Θ'] instead of **R**[Θ"]=**R**[Θ']**R**[Θ].

A similarity transformation of rotation **R**[Θ"] by rotation **R**[Θ] gives rotation **R**[Θ'''] and *vice-versa*:

R[Θ] **R**[Θ"] **R**[-Θ] = **R**[Θ''']   **R**[-Θ] **R**[Θ'''] **R**[Θ] = **R**[Θ"]

Everything associated with rotation **R**[Θ"] is rotated by full angle Θ around axis Θ.

Crank vector Θ and its turn arc moved by <u>two</u> **R**[Θ] turn arcs into turn arc of **R**[Θ'''] below it.

Geometry of transformation of rotational **R[Θ]**-*operators*

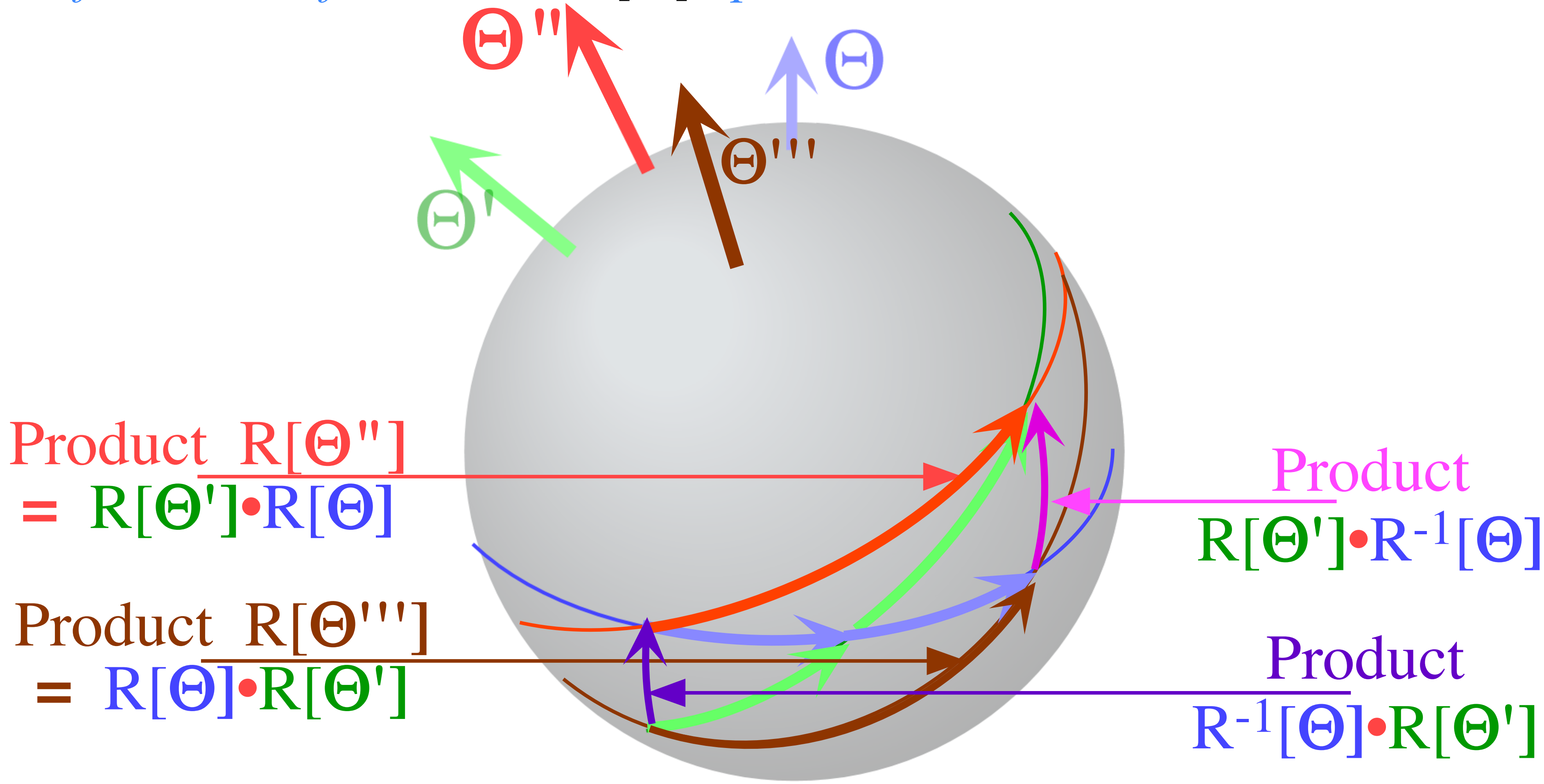

Fig. 10.A.9 Hamilton-turn arc parallelogram with **R**[Θ'']=**R**[Θ']**R**[Θ] *and* **R**[Θ''']=**R**[Θ]**R**[Θ']

Vectors added in the reverse order give **R**[Θ''']=**R**[Θ]**R**[Θ'] instead of **R**[Θ'']=**R**[Θ']**R**[Θ].

A similarity transformation of rotation **R**[Θ''] by rotation **R**[Θ] gives rotation **R**[Θ'''] and *vice-versa*:

R[Θ] **R**[Θ''] **R**[-Θ] = **R**[Θ''']   **R**[-Θ] **R**[Θ'''] **R**[Θ] = **R**[Θ'']

Everything associated with rotation **R**[Θ''] is rotated by full angle Θ around axis Θ.

Crank vector Θ and its turn arc moved by <u>two</u> **R**[Θ] turn arcs into turn arc of **R**[Θ'''] below it.

Another similarity transformation of rotation **R**[Θ'''] by rotation **R**[Θ'] to **R**[Θ'']

R[Θ'] **R**[Θ'''] **R**[-Θ'] = **R**[Θ'']

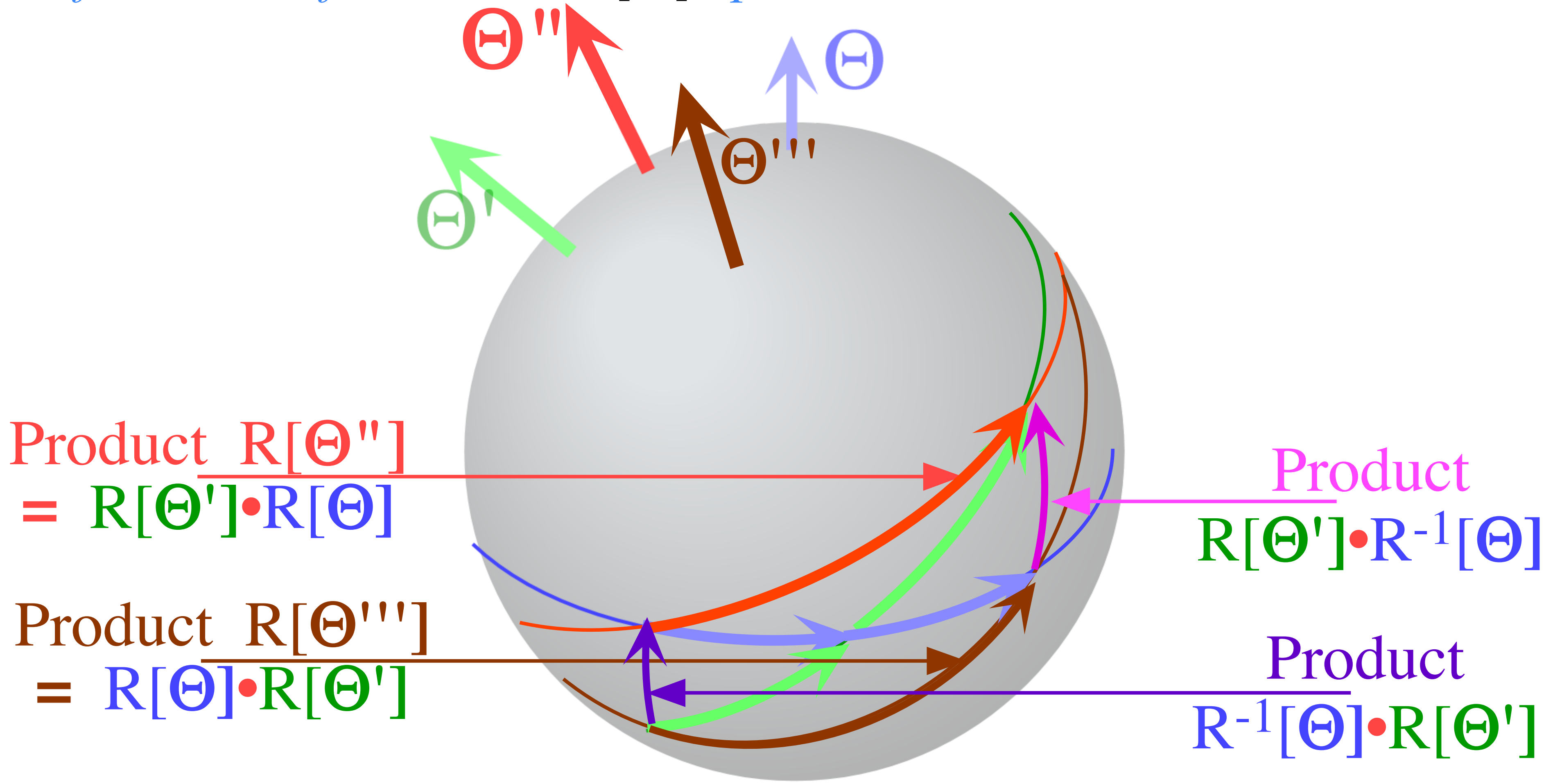

Fig. 10.A.9 Hamilton-turn arc parallelogram with **R**[Θ'']=**R**[Θ']**R**[Θ] *and* **R**[Θ''']=**R**[Θ]**R**[Θ']

Vectors added in the reverse order give **R**[Θ''']=**R**[Θ]**R**[Θ'] instead of **R**[Θ'']=**R**[Θ']**R**[Θ].

A similarity transformation of rotation **R**[Θ''] by rotation **R**[Θ] gives rotation **R**[Θ'''] and *vice-versa*:

R[Θ] **R**[Θ''] **R**[-Θ] = **R**[Θ'''] **R**[-Θ] **R**[Θ'''] **R**[Θ] = **R**[Θ'']

Everything associated with rotation **R**[Θ''] is rotated by full angle Θ around axis Θ.

Crank vector Θ and its turn arc moved by <u>two</u> **R**[Θ] turn arcs into turn arc of **R**[Θ'''] below it.

Another similarity transformation of rotation **R**[Θ'''] by rotation **R**[Θ'] to **R**[Θ''] and *vice-versa*:

R[Θ'] **R**[Θ'''] **R**[-Θ'] = **R**[Θ''] **R**[-Θ'] **R**[Θ''] **R**[Θ'] = **R**[Θ''']

Geometry of transformation of rotational **R[Θ]**-*operators*

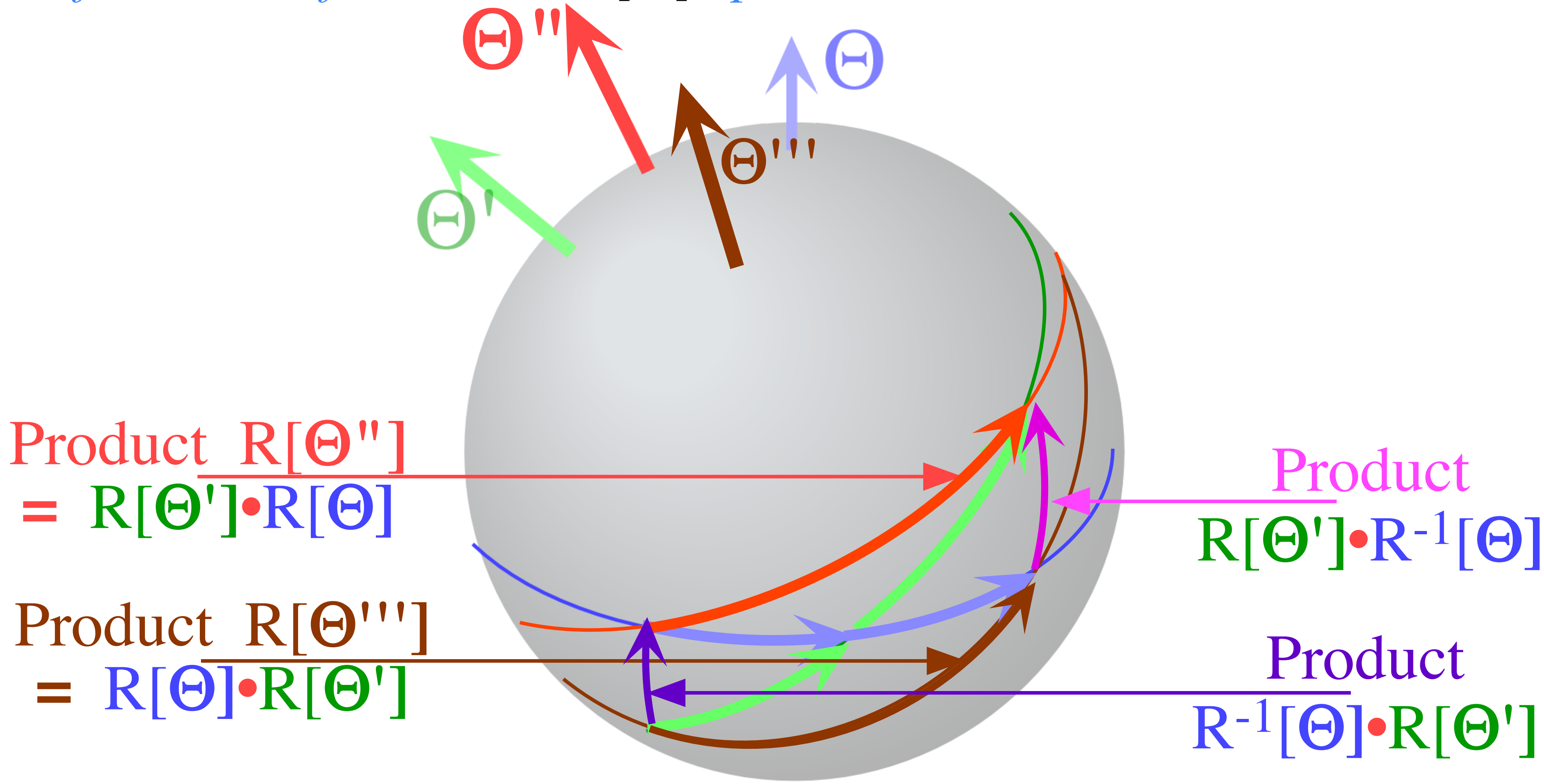

Fig. 10.A.9 Hamilton-turn arc parallelogram with **R**[Θ'']=**R**[Θ']**R**[Θ] *and* **R**[Θ''']=**R**[Θ]**R**[Θ']

Vectors added in the reverse order give **R**[Θ''']=**R**[Θ]**R**[Θ'] instead of **R**[Θ'']=**R**[Θ']**R**[Θ].

A similarity transformation of rotation **R**[Θ''] by rotation **R**[Θ] gives rotation **R**[Θ'''] and *vice-versa*:

R[Θ] **R**[Θ''] **R**[-Θ] = **R**[Θ''']   **R**[-Θ] **R**[Θ'''] **R**[Θ] = **R**[Θ'']

Everything associated with rotation **R**[Θ''] is rotated by full angle Θ around axis Θ.

Crank vector Θ and its turn arc moved by <u>two</u> **R**[Θ] turn arcs into turn arc of **R**[Θ'''] below it.

Another similarity transformation of rotation **R**[Θ'''] by rotation **R**[Θ'] to **R**[Θ''] and *vice-versa*:

R[Θ'] **R**[Θ'''] **R**[-Θ'] = **R**[Θ'']   **R**[-Θ'] **R**[Θ''] **R**[Θ'] = **R**[Θ''']

Many (∞) rotations transform **R**[Θ''] into **R**[Θ'''] . Of these, there is one with the least angle Θ_{min}.

Review:How "Crazy-Thing"-Theorem makes spinor and vector representation matrices

Half-angle $\Theta/2 = \varphi$ *replacement and Darboux crank axis operators*

Operator-on-Operator transformations

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Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ *of group-operators*

Operator-on-Operator transformations

Geometry of groups: Hamilton's turns and It's all done with mirrors!

Group product geometry

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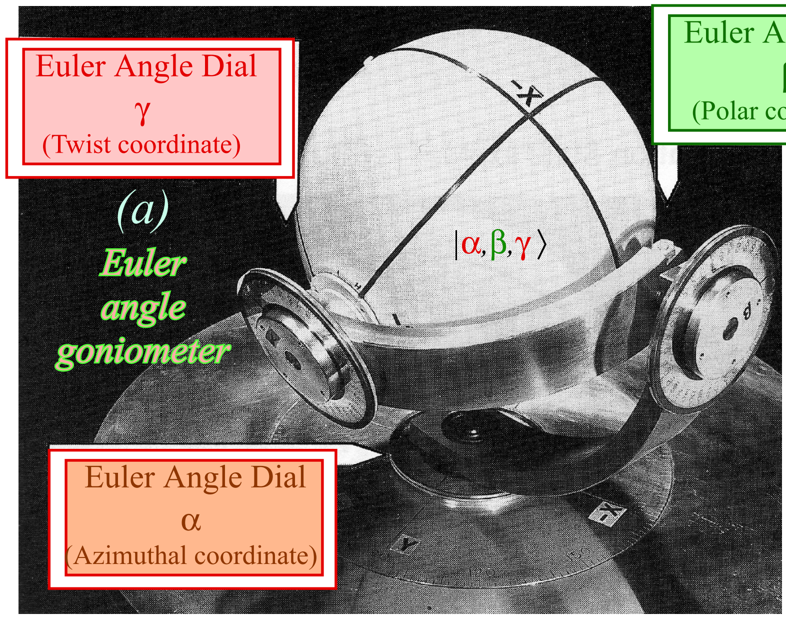
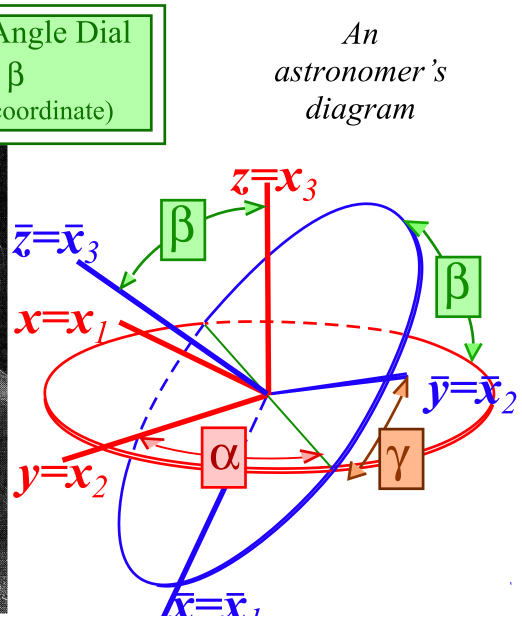
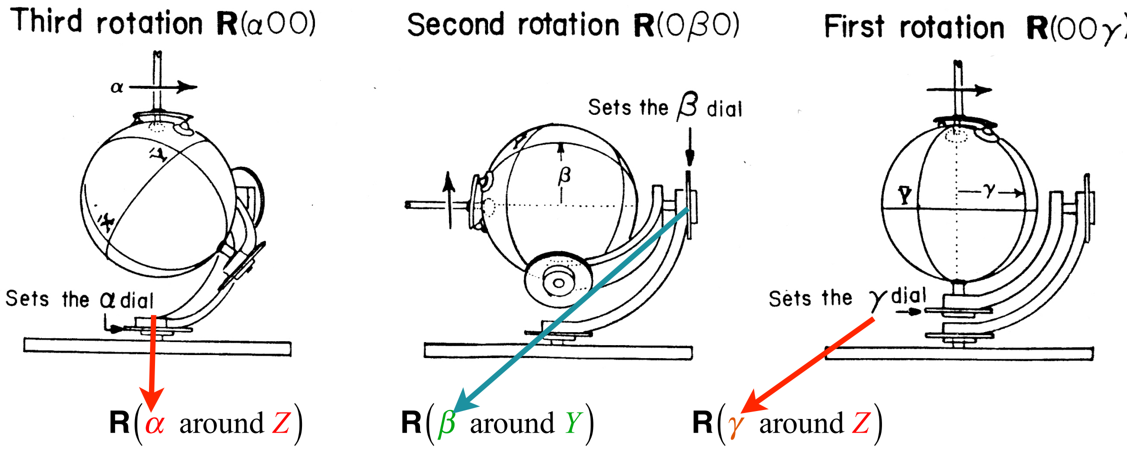
R(3)-U(2) slide rule for converting $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *Sundial*

Euler R(αβγ) versus *Darboux* R[φϑΘ]



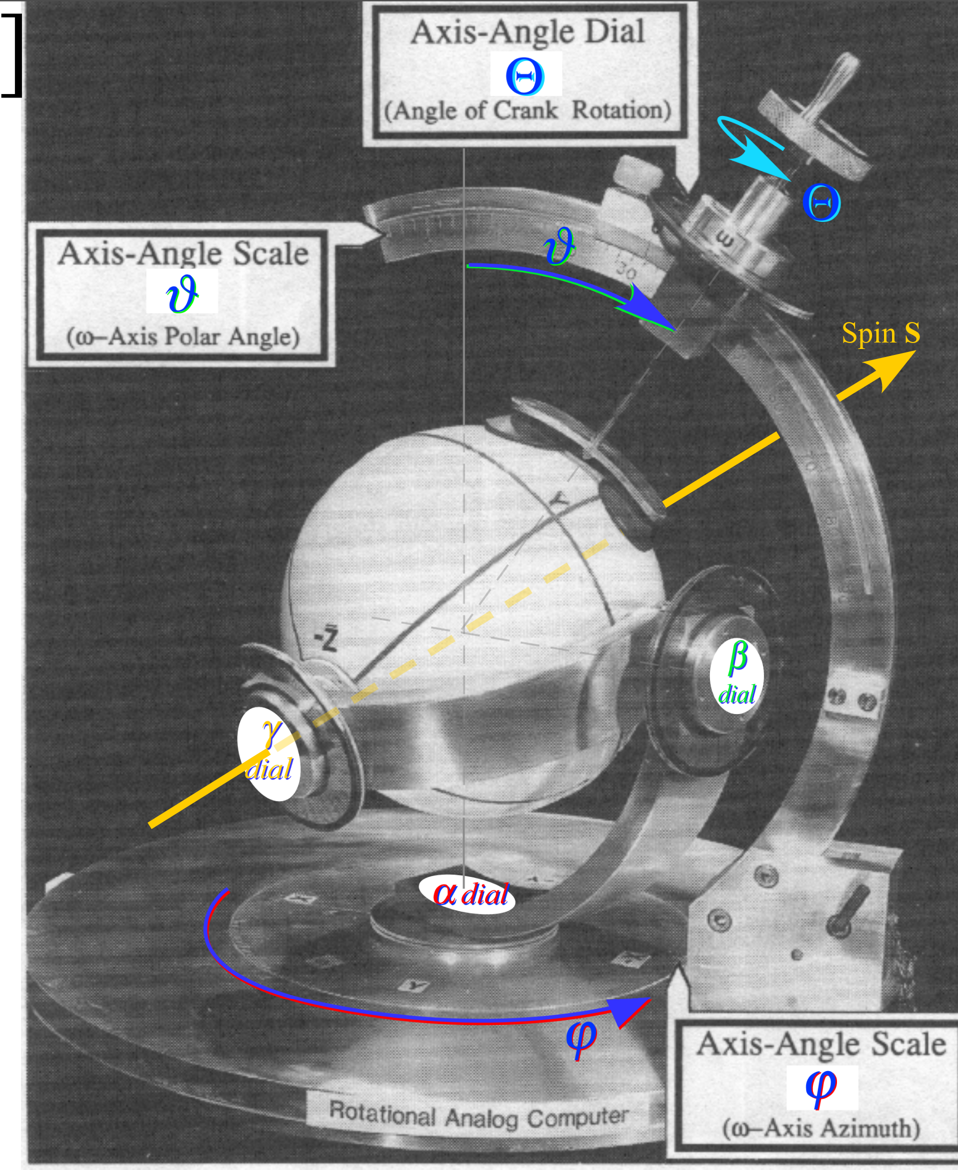




$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}}\end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}}\end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}=$$

$$=\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

Euler R(αβγ) is simpler to form than Θ-axis *Darboux* R[φϑΘ].

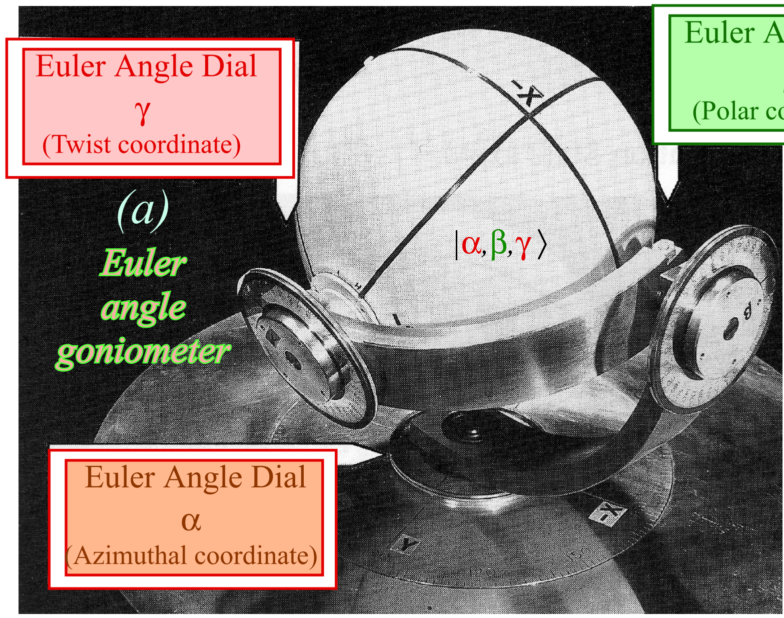


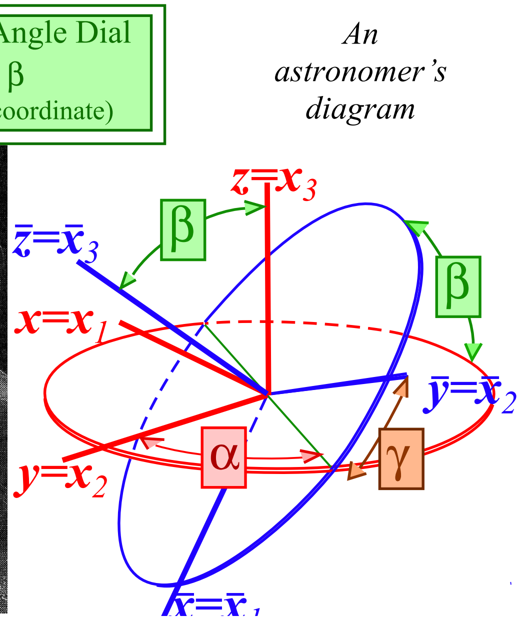
$$\mathbf{R}[\vec{\Theta}]=\begin{pmatrix}\cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}(\hat{\Theta}_X-i\hat{\Theta}_Y) \\ -i\sin\frac{\Theta}{2}(\hat{\Theta}_X+i\hat{\Theta}_Y) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2}\end{pmatrix}=\mathbf{R}[\varphi\vartheta\Theta]=e^{-i\mathbf{H}t}$$

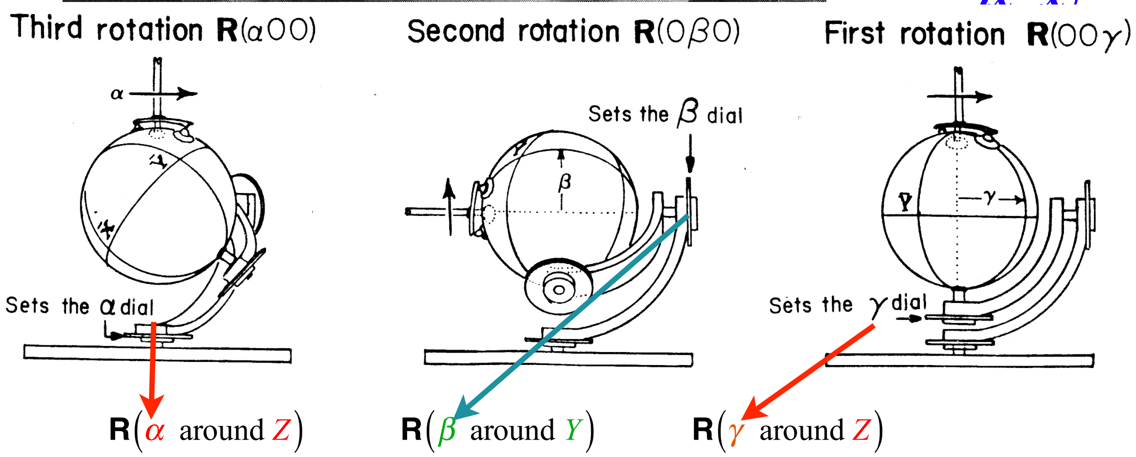
$$=\cos\frac{\Theta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

$$=\begin{pmatrix}\cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta) \\ \sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2}\end{pmatrix}$$

Euler R(αβγ) versus *Darboux* R[φϑΘ]







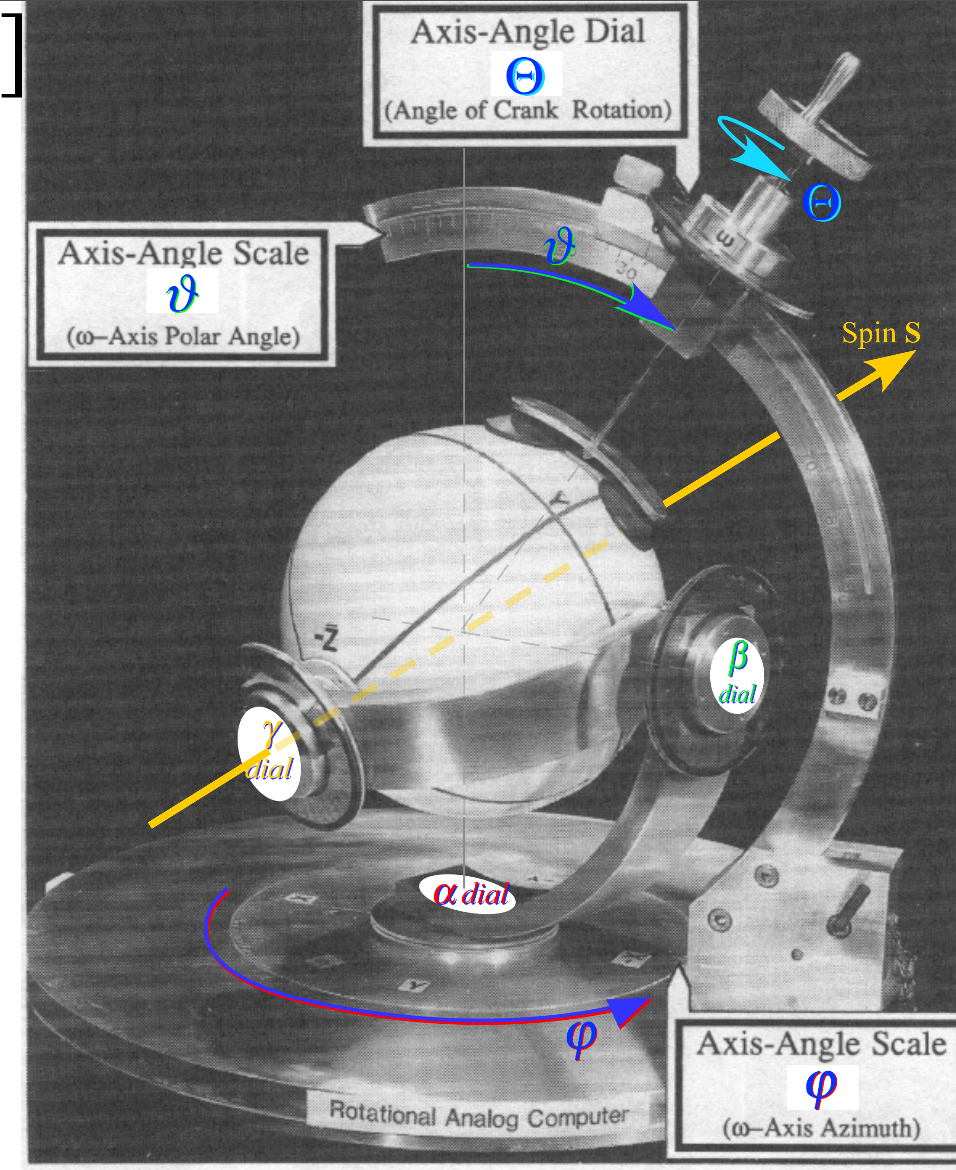
$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}}\end{pmatrix}\begin{pmatrix}\cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}}\end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}=$$

$$=\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

Euler R(αβγ) is simpler to form than Θ-axis *Darboux* R[φϑΘ].

Euler *state definition*:

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ (αβγ *make better coordinates)*

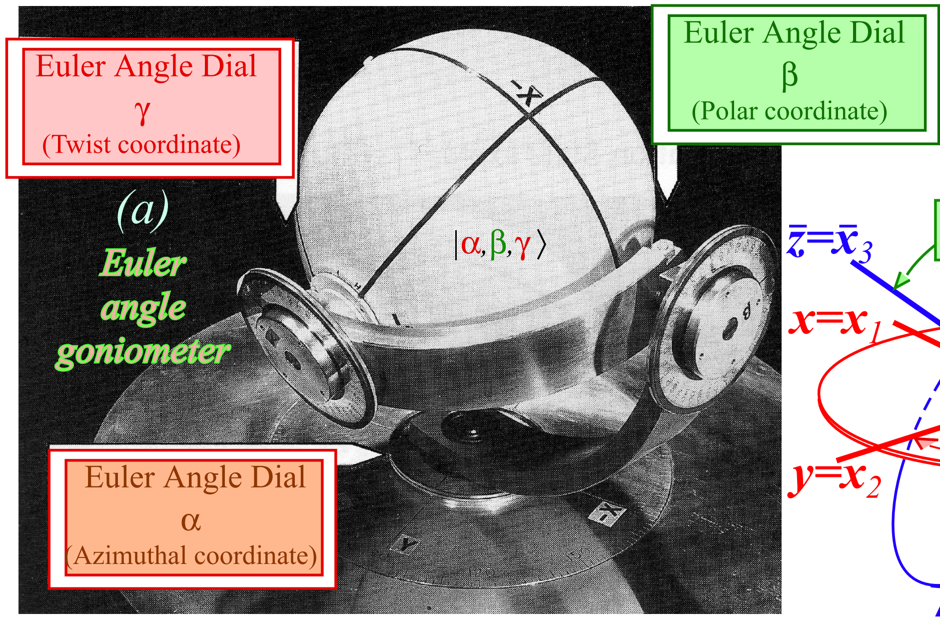


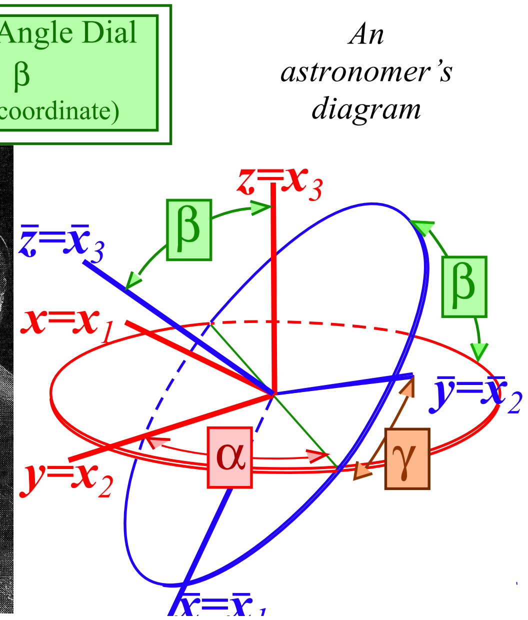
$$\mathbf{R}[\vec{\Theta}]=\begin{pmatrix}\cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}(\hat{\Theta}_X-i\hat{\Theta}_Y) \\ -i\sin\frac{\Theta}{2}(\hat{\Theta}_X+i\hat{\Theta}_Y) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2}\end{pmatrix}=\mathbf{R}[\varphi\vartheta\Theta]=e^{-i\mathbf{H}t}$$

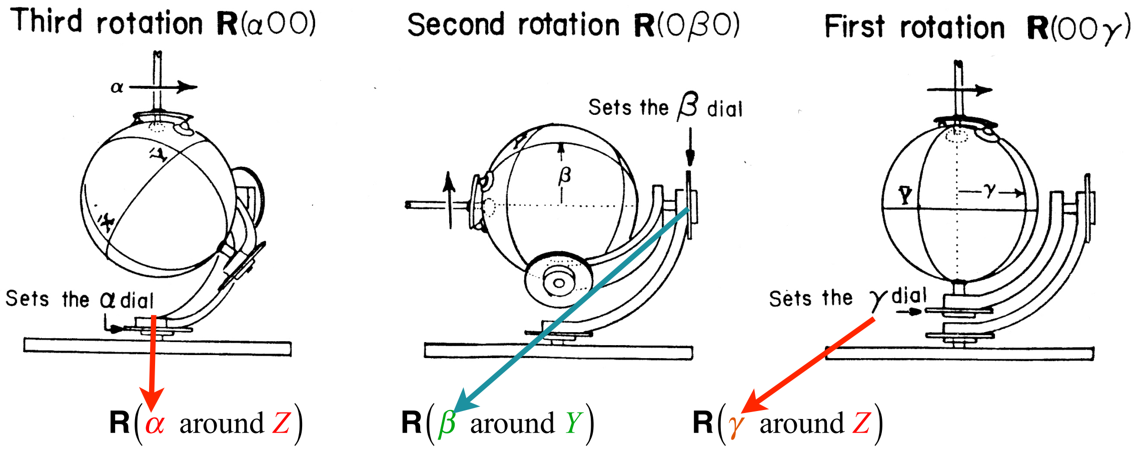
$$=\cos\frac{\Theta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

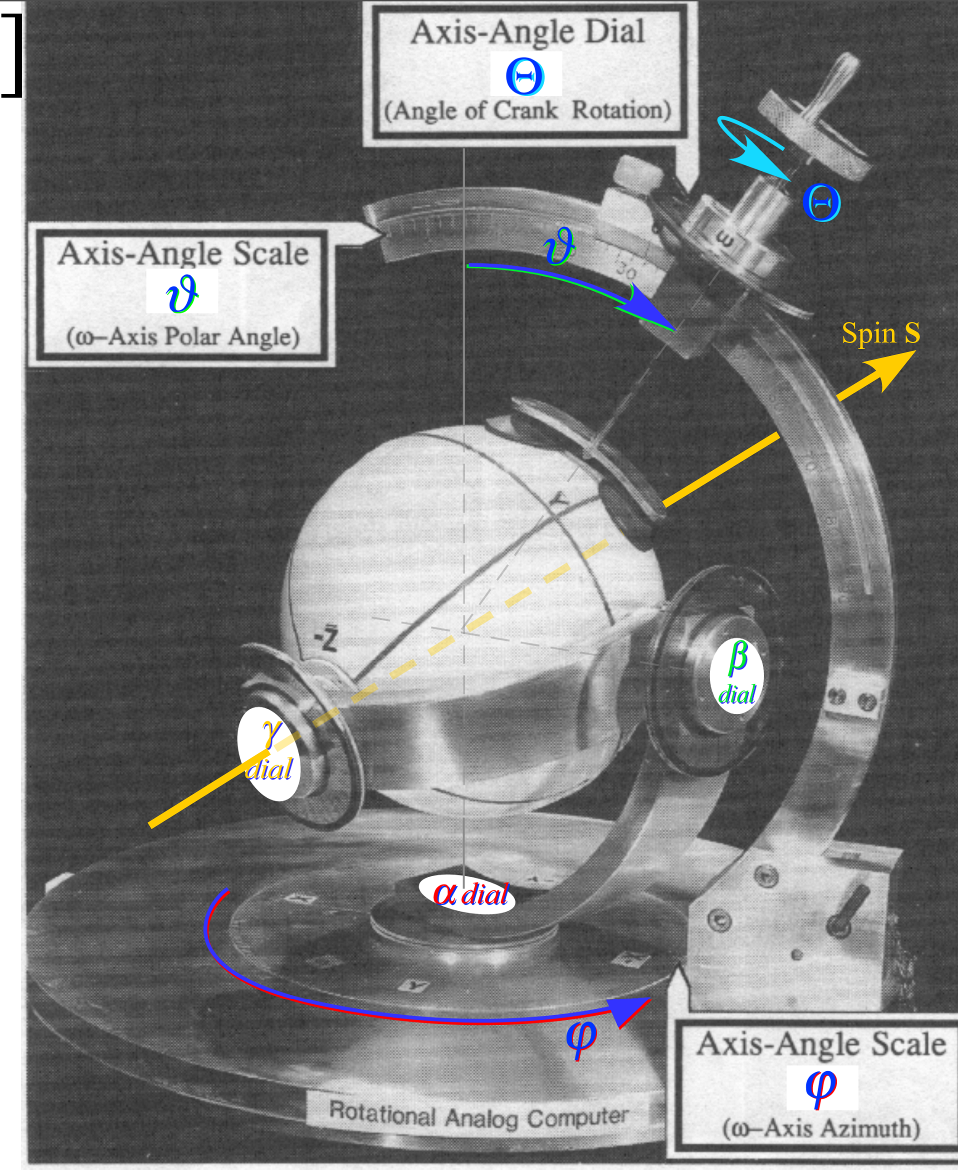
$$=\begin{pmatrix}\cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta) \\ \sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2}\end{pmatrix}$$

Euler R(αβγ) versus *Darboux* R[φϑΘ]









$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}=$$

$$=\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}-i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

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$$=\cos\frac{\Theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}-i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

$$=\begin{pmatrix} \cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta\right) \\ \sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta\right) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2} \end{pmatrix}$$

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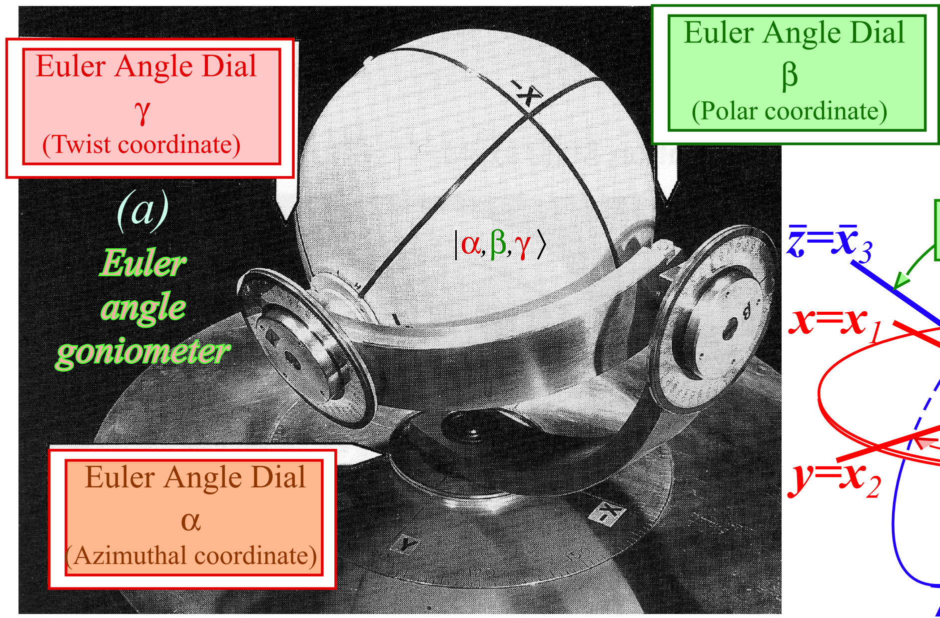
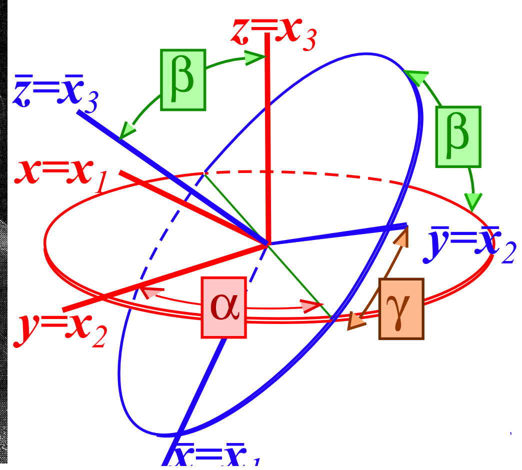
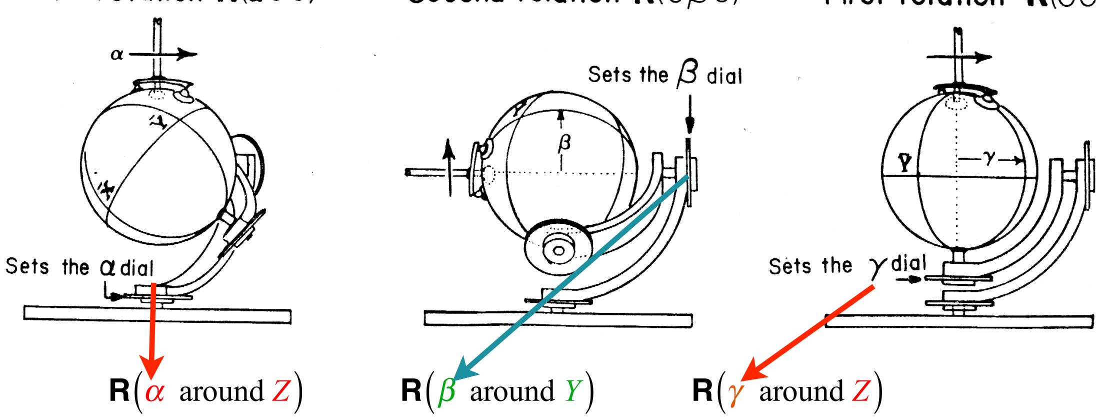
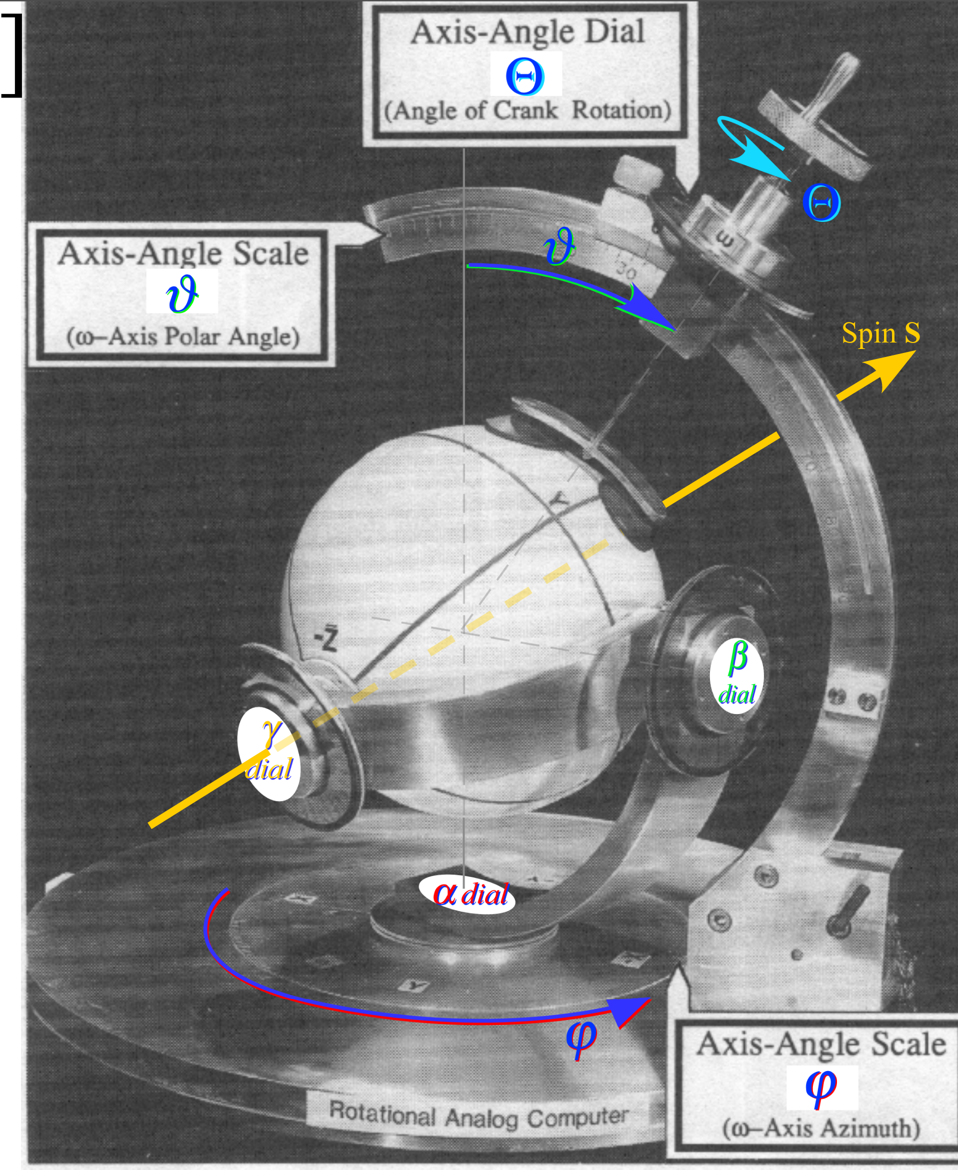
Euler R(αβγ) versus *Darboux* R[φϑΘ]



An astronomer's diagram







$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}=$$

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$$\mathbf{R}[\vec{\Theta}]=\begin{pmatrix} \cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}(\hat{\Theta}_X-i\hat{\Theta}_Y) \\ -i\sin\frac{\Theta}{2}(\hat{\Theta}_X+i\hat{\Theta}_Y) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2} \end{pmatrix}=\mathbf{R}[\varphi\vartheta\Theta]=e^{-i\mathbf{H}t}$$

$$=\cos\frac{\Theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}-i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\,\sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\,\sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

$$=\begin{pmatrix} \cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta) \\ \sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2} \end{pmatrix}$$

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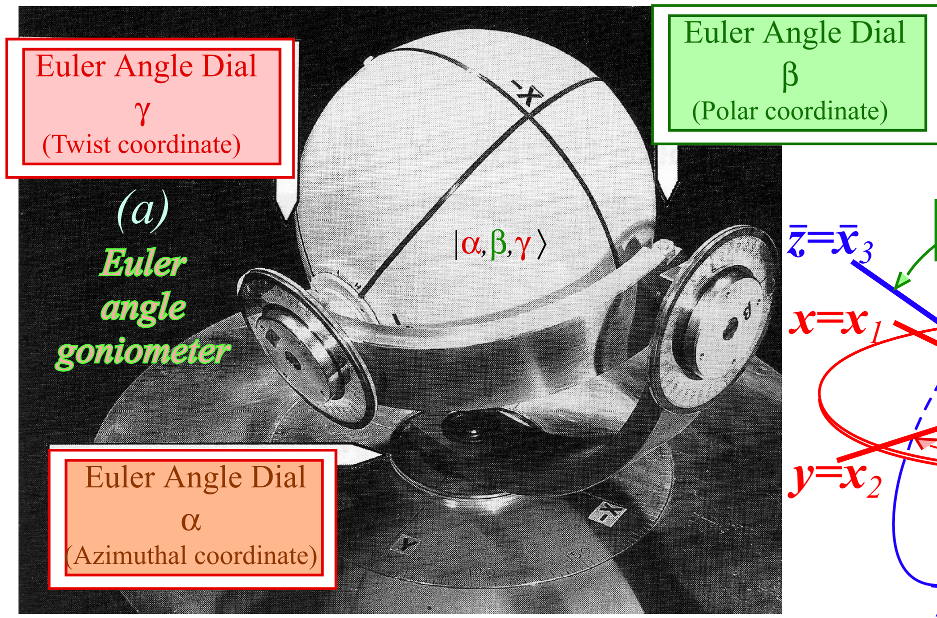
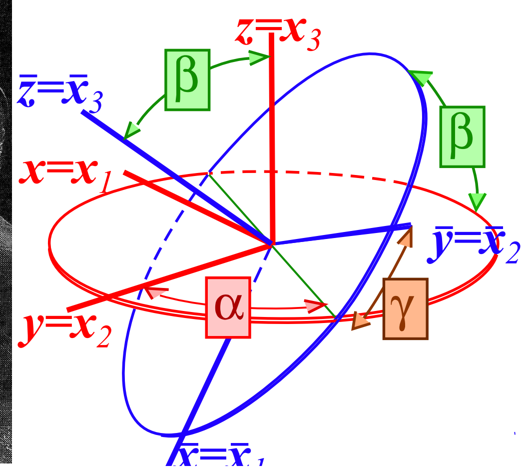
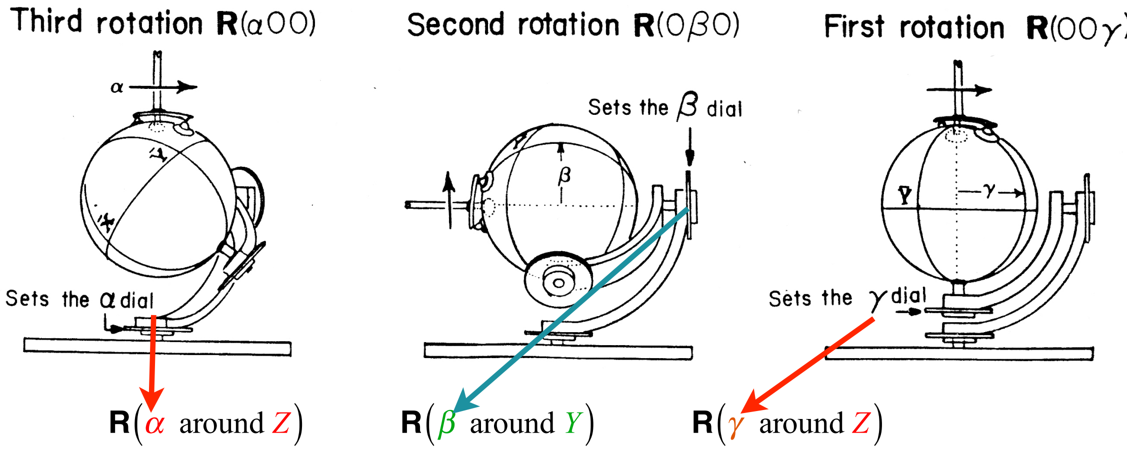
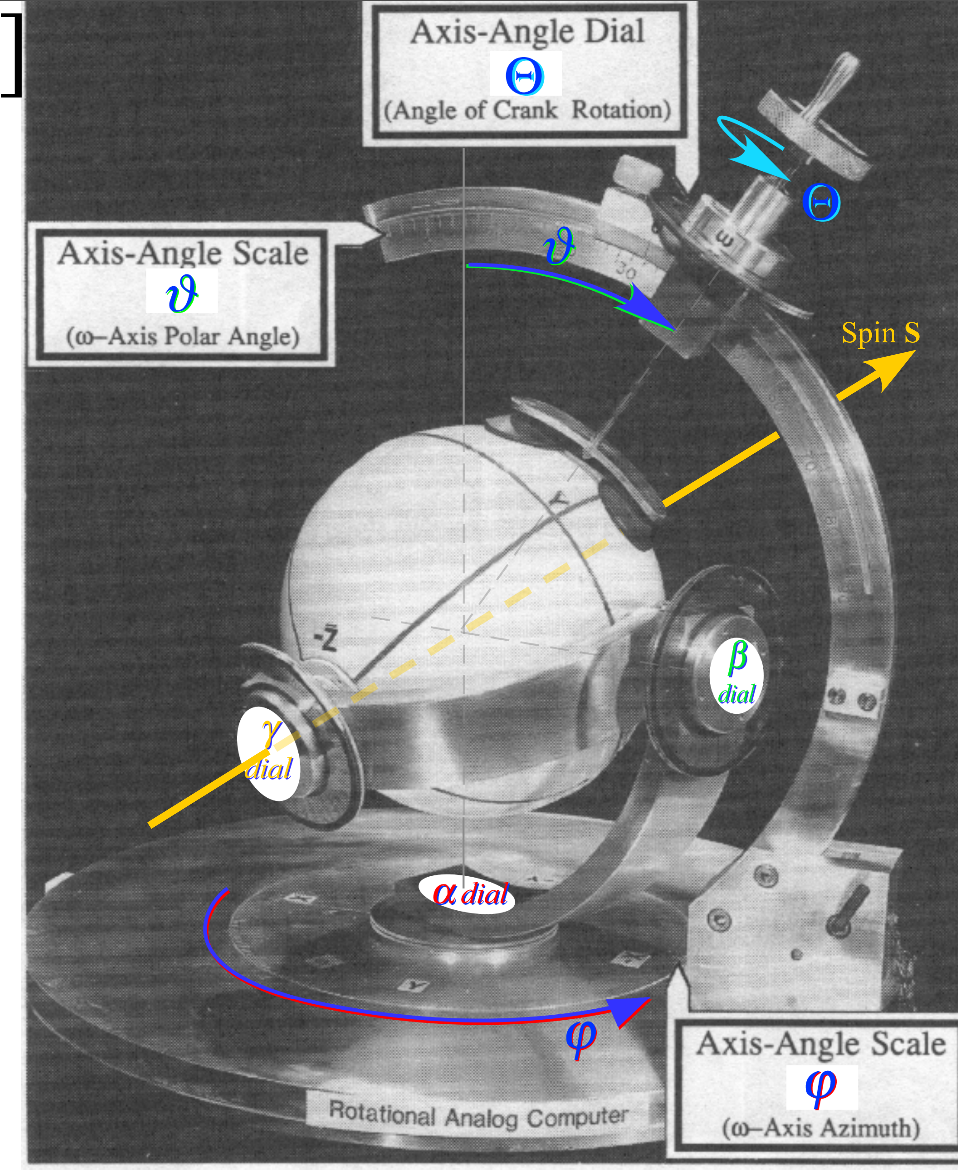
Euler R(αβγ) versus *Darboux* R[φϑΘ]



An astronomer's diagram







$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}}\end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}}\end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}=$$

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$$\mathbf{R}[\vec{\Theta}]=\begin{pmatrix}\cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}(\hat{\Theta}_X-i\hat{\Theta}_Y) \\ -i\sin\frac{\Theta}{2}(\hat{\Theta}_X+i\hat{\Theta}_Y) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2}\end{pmatrix}=\mathbf{R}[\varphi\vartheta\Theta]=e^{-i\mathbf{H}t}$$

$$=\cos\frac{\Theta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

$$=\begin{pmatrix}\cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta) \\ \sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2}\end{pmatrix}$$

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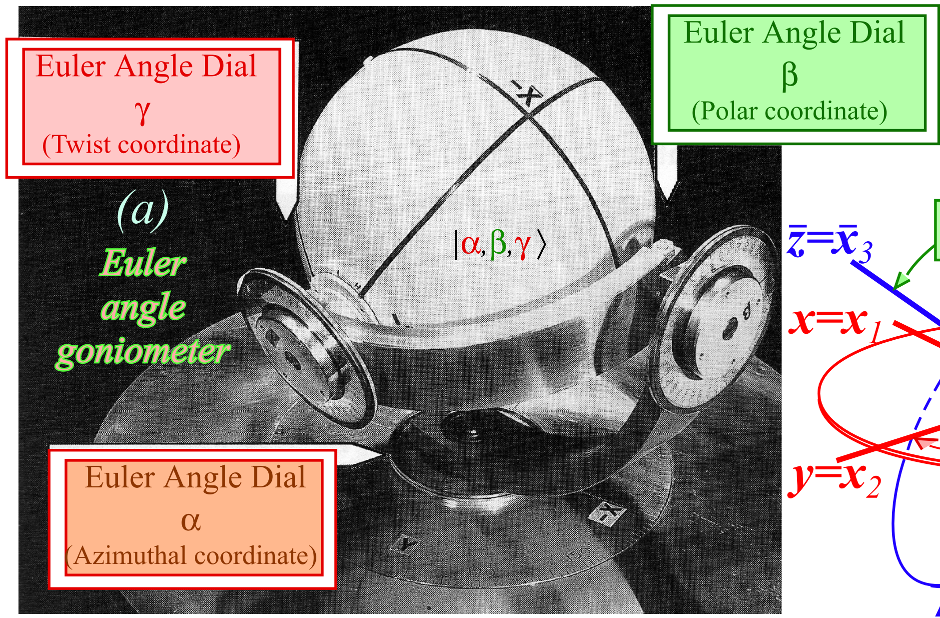
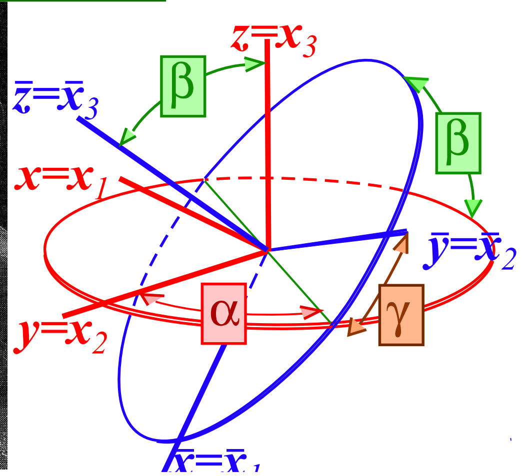
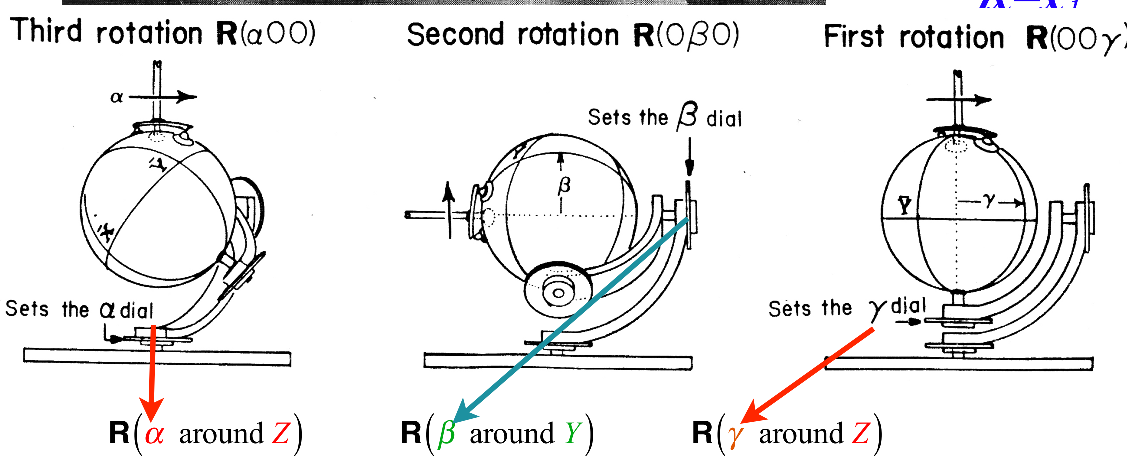
x_1 = cos[(γ+α)/2] cosβ/2 = cosΘ/2

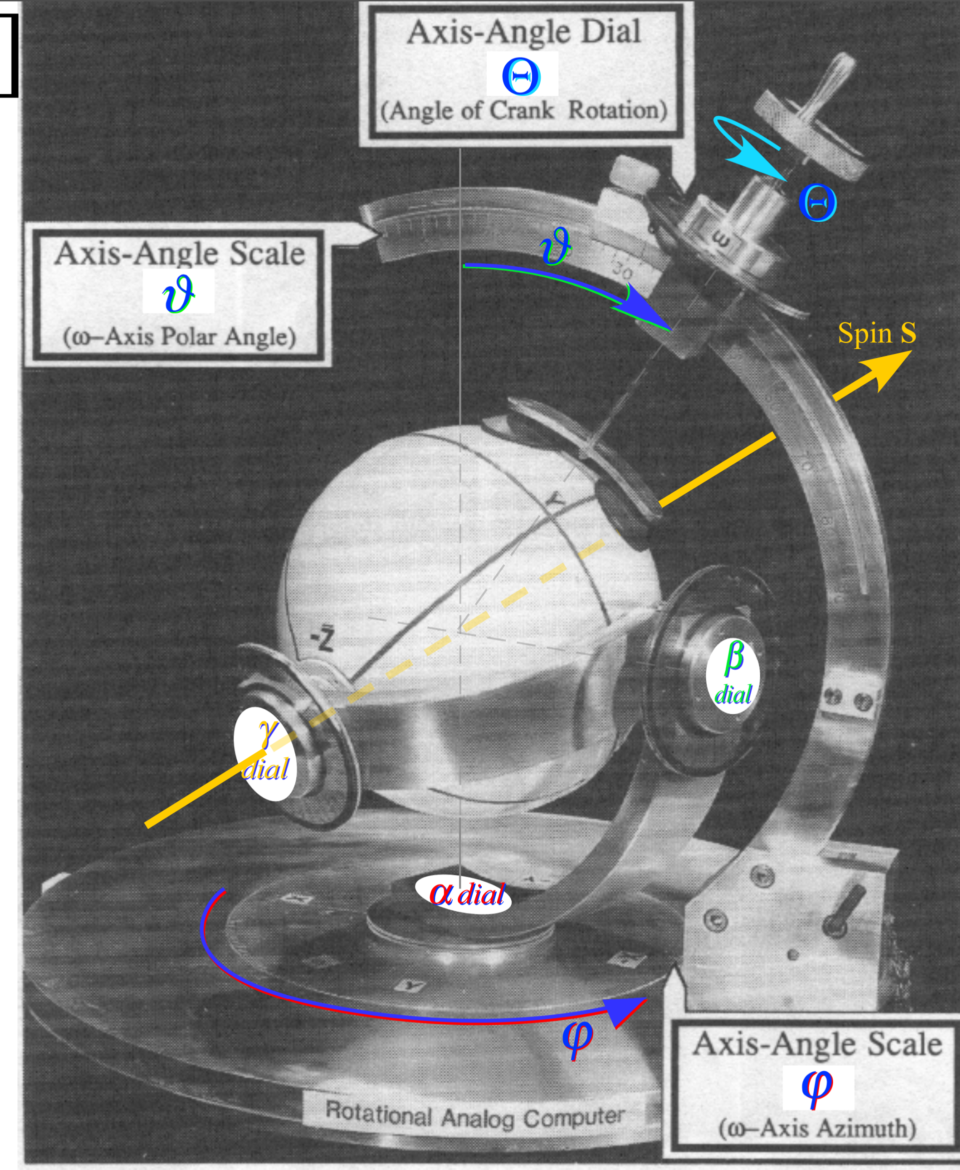
Euler R(αβγ) versus *Darboux* R[φϑΘ]



An astronomer's diagram







$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}=$$

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$$\mathbf{R}[\vec{\Theta}]=\begin{pmatrix} \cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}(\hat{\Theta}_X-i\hat{\Theta}_Y) \\ -i\sin\frac{\Theta}{2}(\hat{\Theta}_X+i\hat{\Theta}_Y) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2} \end{pmatrix}=\mathbf{R}[\varphi\vartheta\Theta]=e^{-i\mathbf{H}t}$$

$$=\cos\frac{\Theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}-i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\hat{\Theta}_X\sin\frac{\Theta}{2}-i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\hat{\Theta}_Y\sin\frac{\Theta}{2}-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\hat{\Theta}_Z\sin\frac{\Theta}{2}$$

$\hat{\Theta}_X = \cos\varphi\ \sin\vartheta$, $\hat{\Theta}_Y = \sin\varphi\ \sin\vartheta$, $\hat{\Theta}_Z = \cos\vartheta$

$$=\begin{pmatrix} \cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta) \\ \sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2} \end{pmatrix}$$

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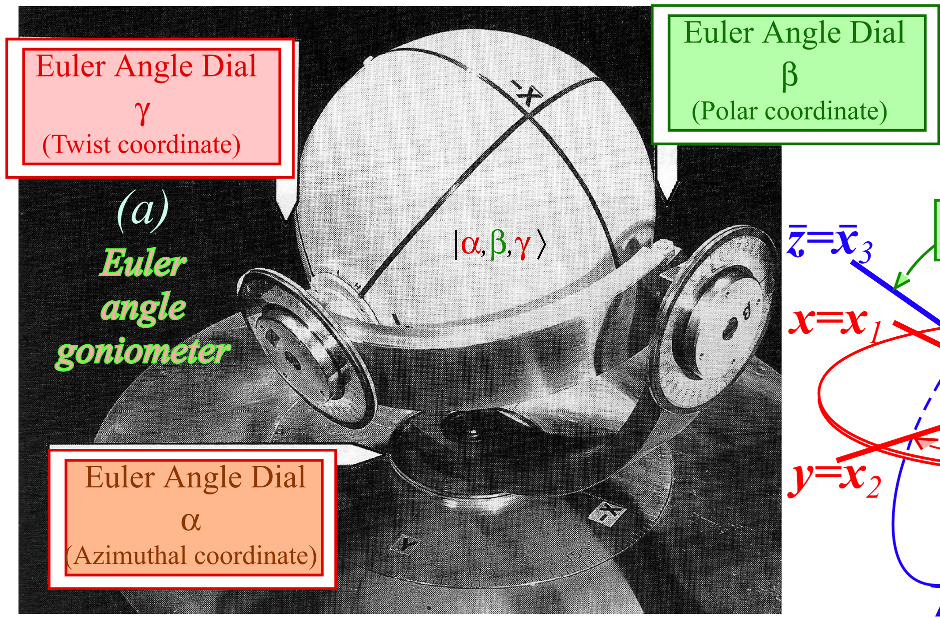
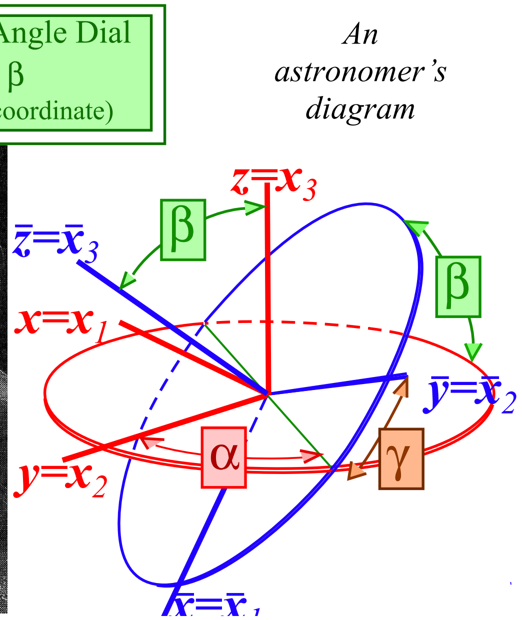
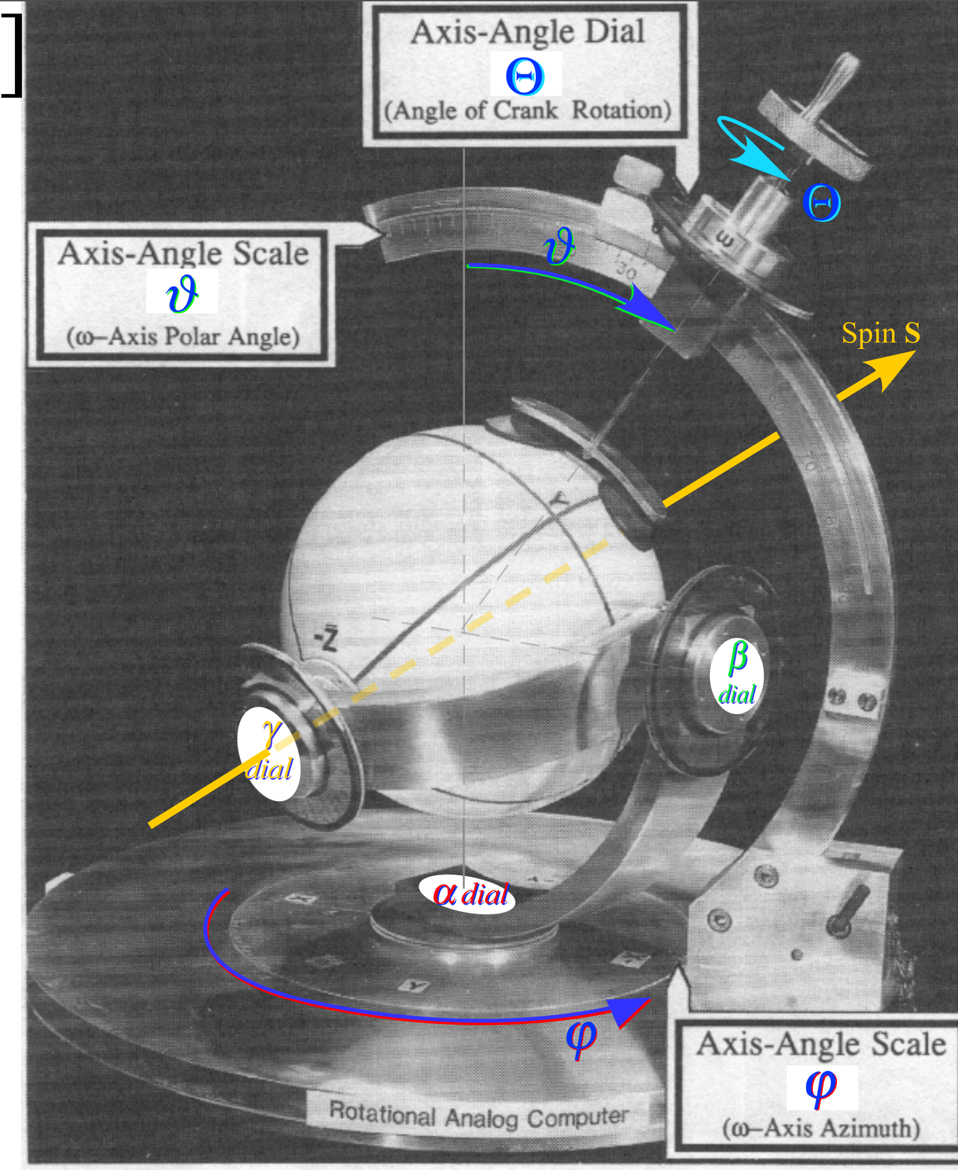
$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$

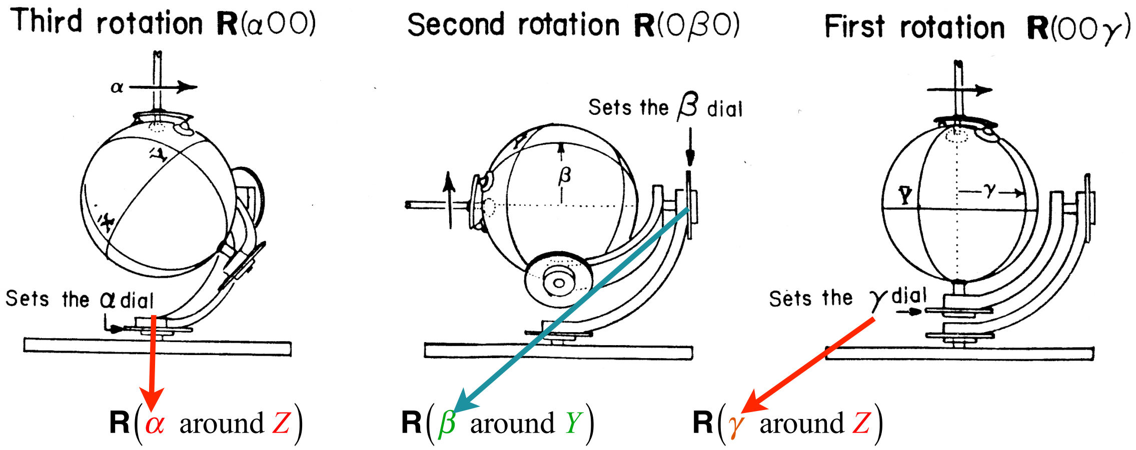
$-p_2 = \sin[(\gamma-\alpha)/2]\ \sin\beta/2 = \hat{\Theta}_X \sin\Theta/2 = \cos\varphi\ \sin\vartheta\ \sin\Theta/2$

Euler R(αβγ) versus *Darboux* R[φϑΘ]









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$$\mathbf{R}[\vec{\Theta}]=\begin{pmatrix}\cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}(\hat{\Theta}_X-i\hat{\Theta}_Y)\\ -i\sin\frac{\Theta}{2}(\hat{\Theta}_X+i\hat{\Theta}_Y) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2}\end{pmatrix}=\mathbf{R}[\varphi\vartheta\Theta]=e^{-i\mathbf{H}t}$$

$$=\cos\frac{\Theta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

$$=\begin{pmatrix}\cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta)\\ \sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2}\end{pmatrix}$$

Euler R(αβγ) is simpler to form than Θ-axis *Darboux* R[φϑΘ].

Euler *state definition* lets us relate R(αβγ) to R[φϑΘ] ...

$|\alpha\beta\gamma\rangle=\mathbf{R}(\alpha\beta\gamma)|000\rangle$ (αβγ *make better coordinates)*

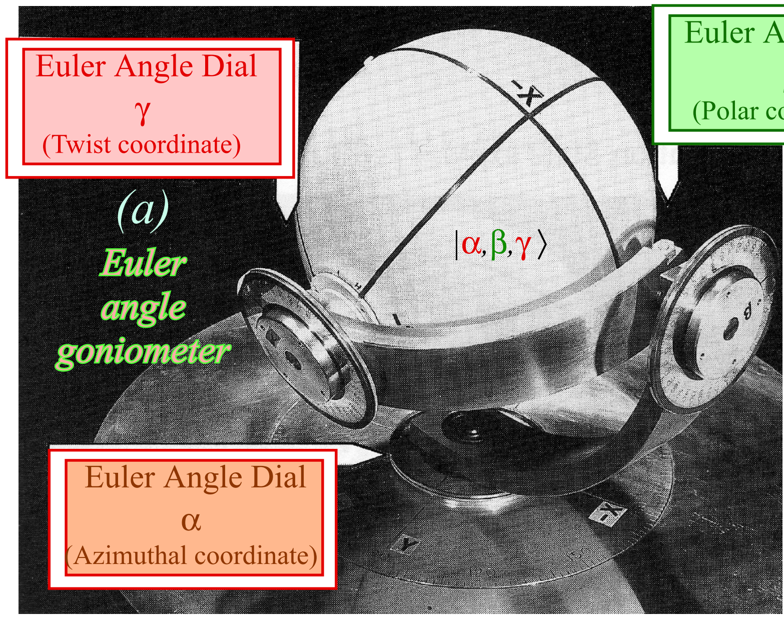
$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}x_1+ip_1\\x_2+ip_2\end{pmatrix}$$

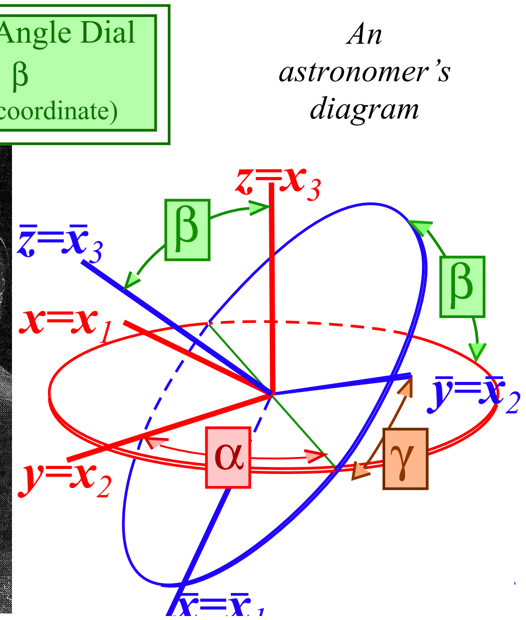
$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$

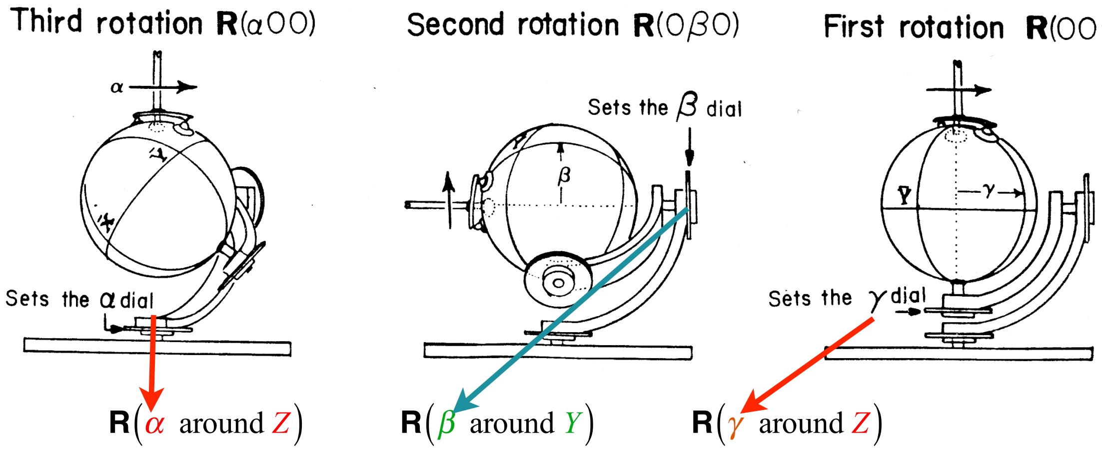
$-p_2 = \sin[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_X\sin\Theta/2 = \cos\varphi\ \sin\vartheta\ \sin\Theta/2$

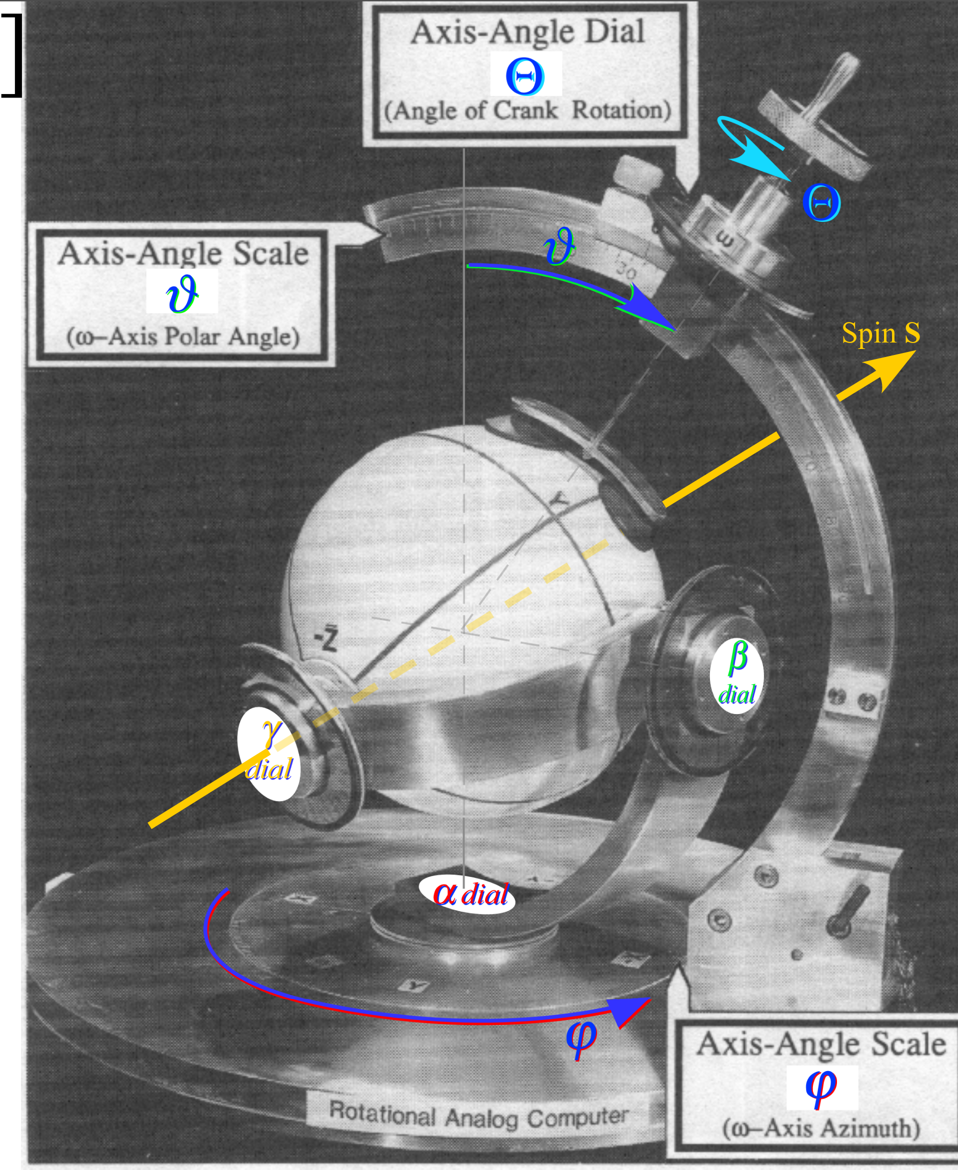
$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y\sin\Theta/2 = \sin\varphi\ \sin\vartheta\ \sin\Theta/2$

Euler R(αβγ) versus *Darboux* R[φϑΘ]









$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}=$$

$$=\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}-i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

$$\mathbf{R}[\vec{\Theta}]=\begin{pmatrix} \cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}(\hat{\Theta}_X-i\hat{\Theta}_Y) \\ -i\sin\frac{\Theta}{2}(\hat{\Theta}_X+i\hat{\Theta}_Y) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2} \end{pmatrix}=\mathbf{R}[\varphi\vartheta\Theta]=e^{-i\mathbf{H}t}$$

$$=\cos\frac{\Theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}-i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

$$=\begin{pmatrix} \cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta) \\ \sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2} \end{pmatrix}$$

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$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}=\begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$

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$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y\sin\Theta/2 = \sin\varphi\ \sin\vartheta\ \sin\Theta/2$

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Review:How "Crazy-Thing"-Theorem makes spinor and vector representation matrices

Half-angle $\Theta/2=\varphi$ replacement and Darboux crank axis operators

Operator-on-Operator transformations

Product algebra for Pauli's σ_μ and Hamilton's $\boldsymbol{q}_\mu = -i\sigma_\mu$

Group product algebra

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Transformation $\mathbf{R}[\Theta]\sigma_\mu\mathbf{R}[\Theta]^\dagger$ *of spinor σ_μ-operators*

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ *of group-operators*

Operator-on-Operator transformations

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U(2) product $\mathbf{R}[\Theta]\mathbf{R}[\Theta']=\mathbf{R}[\Theta''']$*- geometry*

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ *geometry*

Euler $\mathbf{R}(\alpha\beta\gamma)$ versus *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

→ *Euler* $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed*

R(3)-U(2) slide rule for converting $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *Sundial*

Euler **R**$(\alpha\beta\gamma)$ *related to Darboux* **R**$[\varphi\vartheta\Theta]$

Euler *state definition* lets us relate **R**$(\alpha\beta\gamma)$ to **R**$[\varphi\vartheta\Theta]$ ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ $\alpha\beta\gamma$ *make better coordinates but:* $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\mathbf{1}\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\mathbf{1}\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$$
$$-p_2 = \sin[(\gamma-\alpha)/2]\ \sin\beta/2 = \hat{\Theta}_X \sin\Theta/2 = \cos\varphi\ \sin\vartheta\ \sin\Theta/2$$
$$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y \sin\Theta/2 = \sin\varphi\ \sin\vartheta\ \sin\Theta/2$$
$$-p_1 = \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z \sin\Theta/2 = \cos\vartheta\ \sin\Theta/2$$

Solving these relations yields *Euler angles* $(\alpha\beta\gamma)$ in terms of *Darboux axis angles* $[\varphi\vartheta\Theta]$

$$\alpha = \varphi - \pi/2 + T, \quad \text{where: } T = \tan^{-1}(\tan(\Theta/2)\cos\vartheta)$$

$$\beta = 2\sin^{-1}(\sin\Theta/2\ \sin\vartheta)$$

$$\gamma = \pi/2 - \varphi + T$$

Euler **R**(αβγ) *related to Darboux* **R**[φϑΘ]

Euler *state definition* lets us relate **R**(αβγ) to **R**[φϑΘ] ...

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$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$\begin{aligned} x_1 &= \cos[(\gamma+\alpha)/2]\cos\beta/2 = && \cos\Theta/2 \\ -p_2 &= \sin[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_X\sin\Theta/2 &&= \cos\varphi\ \sin\vartheta\ \sin\Theta/2 \\ x_2 &= \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y\sin\Theta/2 &&= \sin\varphi\ \sin\vartheta\ \sin\Theta/2 \\ -p_1 &= \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z\sin\Theta/2 &&= \cos\vartheta\ \sin\Theta/2 \end{aligned}$$

Solving these relations yields *Euler angles* (αβγ) in terms of *Darboux axis angles* [φϑΘ]

$$\alpha = \varphi - \pi/2 + T, \quad \text{where: } T = \tan^{-1}(\tan(\Theta/2)\cos\vartheta)$$

$$\beta = 2\sin^{-1}(\sin\Theta/2\,\sin\vartheta)$$

$$\gamma = \pi/2 - \varphi + T$$

Inverse relations have *Darboux axis angles* [φϑΘ] in terms of *Euler angles* (αβγ)

$$\varphi = (\alpha - \gamma + \pi)/2$$

$$\vartheta = \tan^{-1}[\tan\beta/2/\sin(\alpha+\gamma)/2]$$

$$\Theta = 2\cos^{-1}[\cos\beta/2\,\cos(\alpha+\gamma)/2]$$

Euler R(αβγ) related to Darboux R[φϑΘ]

Euler *state definition* lets us relate **R**(αβγ) to **R**[φϑΘ] ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ αβγ *make better coordinates but:* $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\mathbf{1}\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\mathbf{1}\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$

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$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y \sin\Theta/2 = \sin\varphi\ \sin\vartheta\ \sin\Theta/2$

$-p_1 = \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z \sin\Theta/2 = \cos\vartheta\ \sin\Theta/2$

Solving these relations yields *Euler angles* (αβγ) in terms of *Darboux axis angles* [φϑΘ]

$\alpha = \varphi - \pi/2 + T$, where: $T = \tan^{-1}(\tan(\Theta/2)\cos\vartheta)$

$\beta = 2\sin^{-1}(\sin\Theta/2\ \sin\vartheta)$

$\gamma = \pi/2 - \varphi + T$

Inverse relations have *Darboux axis angles* [φϑΘ] in terms of *Euler angles* (αβγ)

$\varphi = (\alpha - \gamma + \pi)/2$

$\vartheta = \tan^{-1}[\tan\beta/2/\sin(\alpha+\gamma)/2]$

$\Theta = 2\cos^{-1}[\cos\beta/2\cos(\alpha+\gamma)/2]$

Example: *Euler angles* (α=50° β=60° γ=70°)

$\varphi = (50° - 70° + 180°)/2 = 80°$

$\vartheta = \tan^{-1}[\tan 60°/2/\sin(50°+\gamma)/2] = 33.7°$

$\Theta = 2\cos^{-1}[\cos 60°/2\cos(50°+\gamma)/2] = 128.7°$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma)$ to $\mathbf{R}[\varphi\vartheta\Theta]$ ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ $\alpha\beta\gamma$ *make better coordinates but:* $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\mathbf{1}\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\mathbf{1}\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

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Solving these relations yields *Euler angles* $(\alpha\beta\gamma)$ in terms of *Darboux axis angles* $[\varphi\vartheta\Theta]$

$\alpha = \varphi - \pi/2 + T$, where: $T = \tan^{-1}(\tan(\Theta/2)\cos\vartheta)$

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$\gamma = \pi/2 - \varphi + T$

Inverse relations have *Darboux axis angles* $[\varphi\vartheta\Theta]$ in terms of *Euler angles* $(\alpha\beta\gamma)$

$\varphi = (\alpha - \gamma + \pi)/2$

$\vartheta = \tan^{-1}[\tan\beta/2/\ \sin(\alpha+\gamma)/2]$

$\Theta = 2\cos^{-1}[\cos\beta/2\ \cos(\alpha+\gamma)/2]$

Example: *Euler angles* $(\alpha=50^\circ\ \beta=60^\circ\ \gamma=70^\circ)$

$\varphi = (50^\circ - 70^\circ + 180^\circ)/2 \qquad = 80^\circ$

$\vartheta = \tan^{-1}[\tan 60^\circ/2/\ \sin(50^\circ+\gamma)/2] \qquad = 33.7^\circ$

$\Theta = 2\cos^{-1}[\cos 60^\circ/2\ \cos(50^\circ+\gamma)/2] \qquad = 128.7^\circ$

Reverse check: $(\alpha\beta\gamma)$ in terms of $[\varphi\vartheta\Theta]$

$\alpha = 80^\circ - 90^\circ + \tan^{-1}(\tan(128.7^\circ/2)\cos 33.7^\circ) = 50.007^\circ$

$\beta = 2\sin^{-1}(\sin 128.7^\circ/2\ \sin 33.7^\circ) = 60.022^\circ$

$\gamma = \pi/2 - 128.7^\circ + \tan^{-1}(\tan(128.7^\circ/2) = 70.007^\circ$

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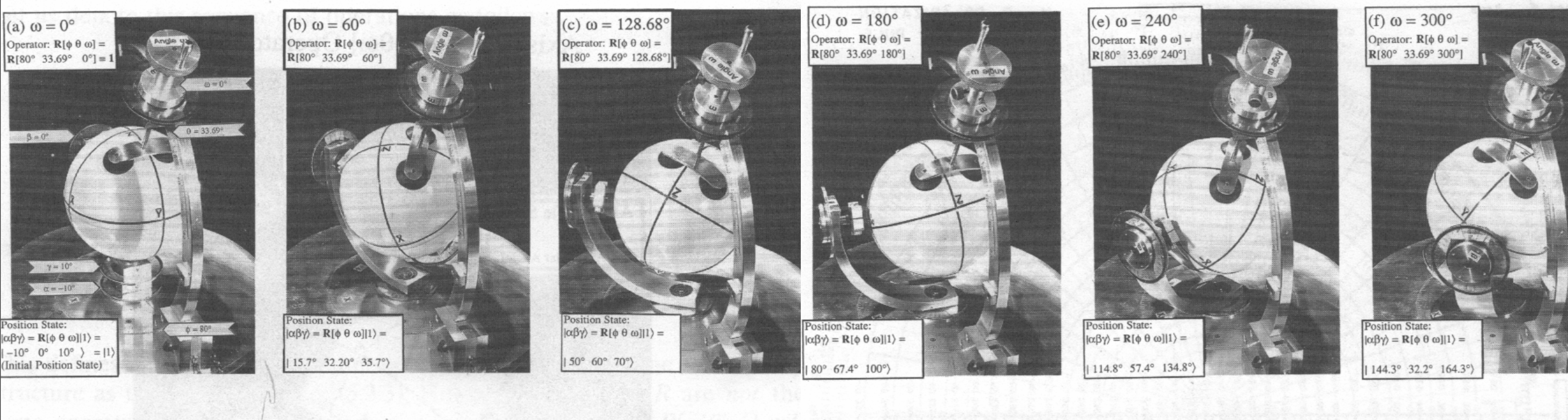
→ *Euler* $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed*

R(3)-U(2) slide rule for converting $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *Sundial*

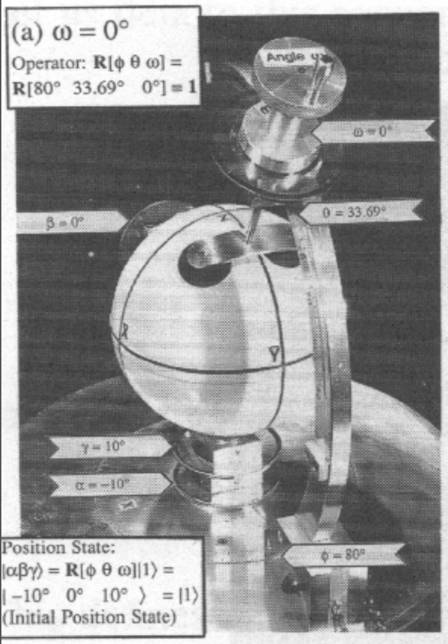
Euler **R**$(\alpha\beta\gamma)$ *rotation* Θ=0−4π*-sequence* [φϑ] *fixed*

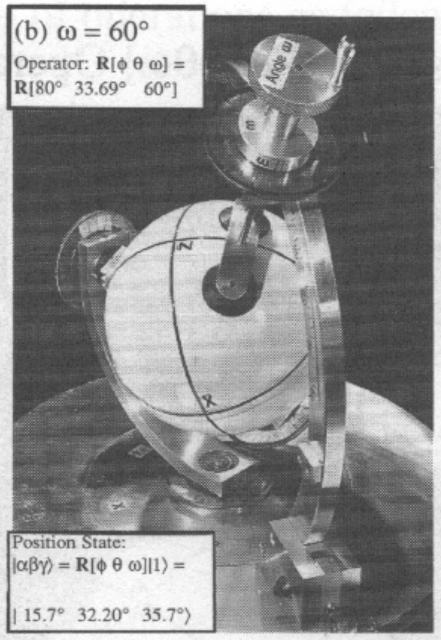
Θ=0° Θ=60° Θ=128.7° Θ=180° Θ=240° Θ=300°

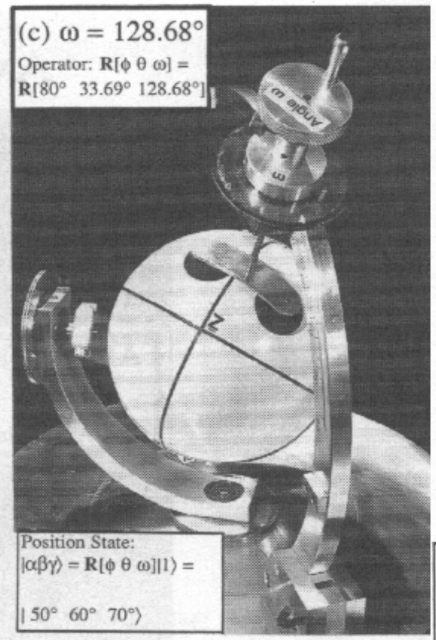


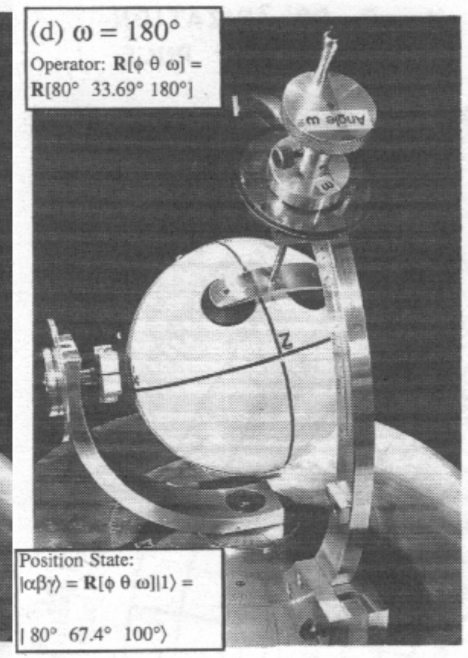
Euler **R**($\alpha\beta\gamma$) *rotation* Θ=0–4π-*sequence* [$\varphi\vartheta$] *fixed*

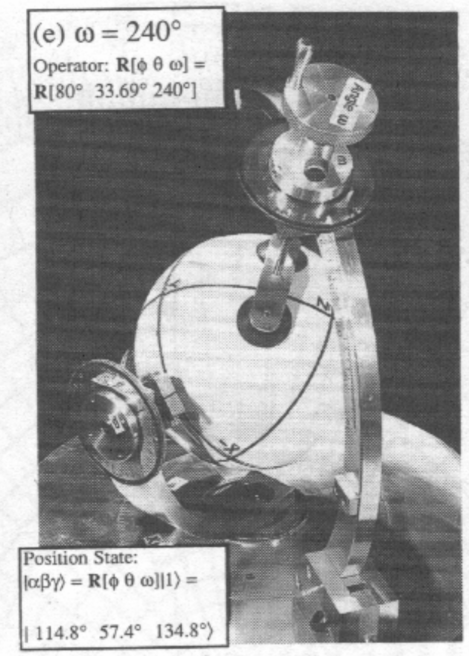
Θ=0° Θ=60° Θ=128.7° Θ=180° Θ=240° Θ=300°

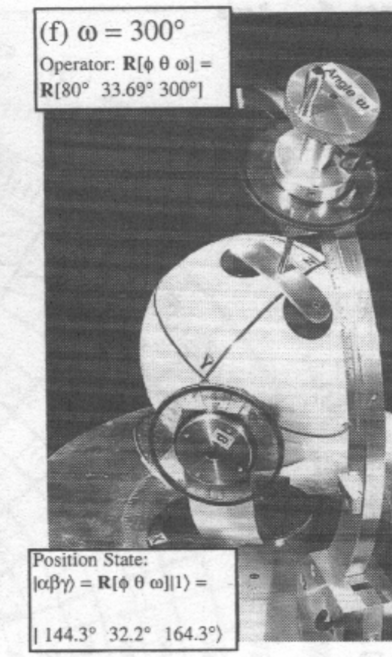


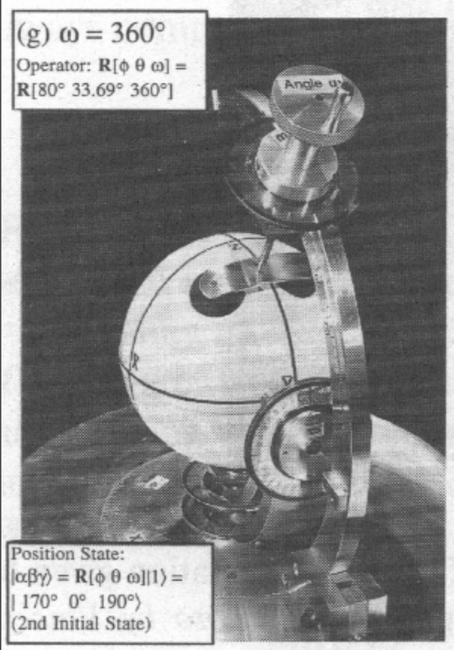


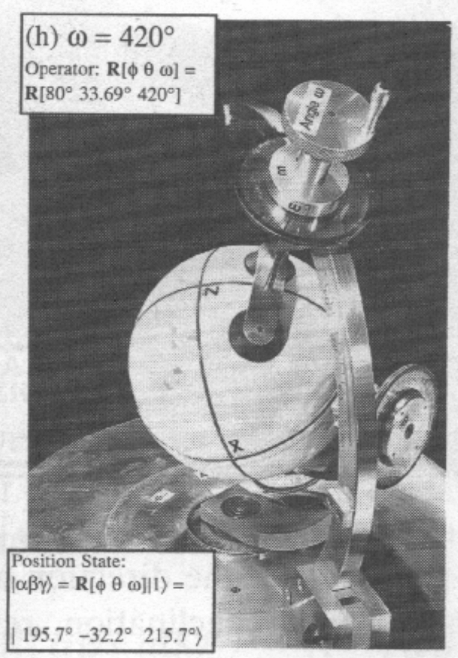


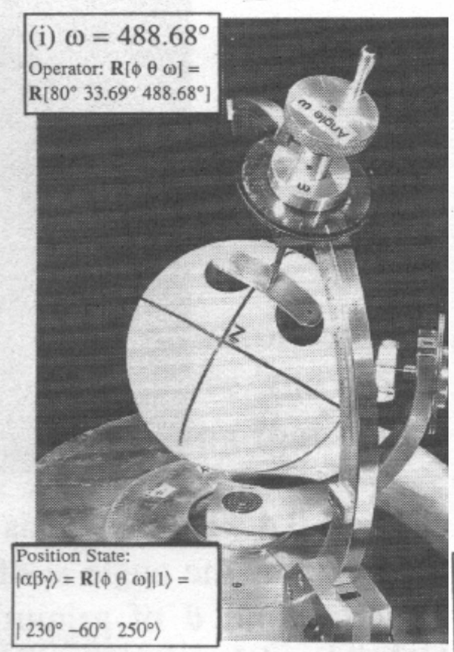


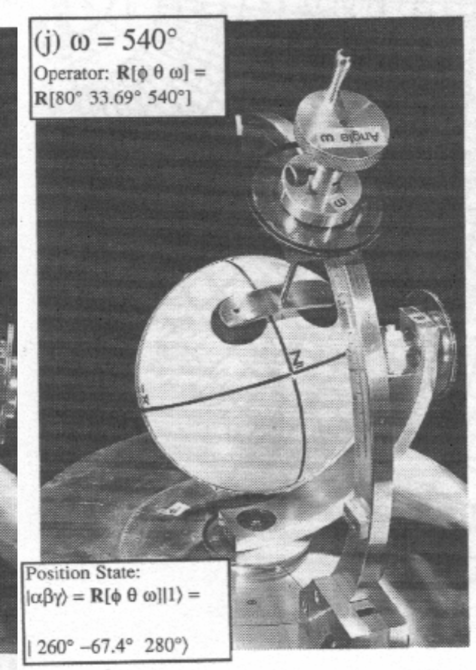


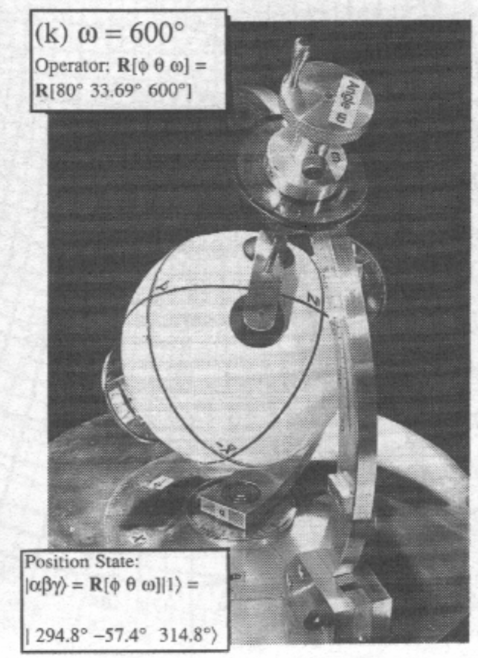


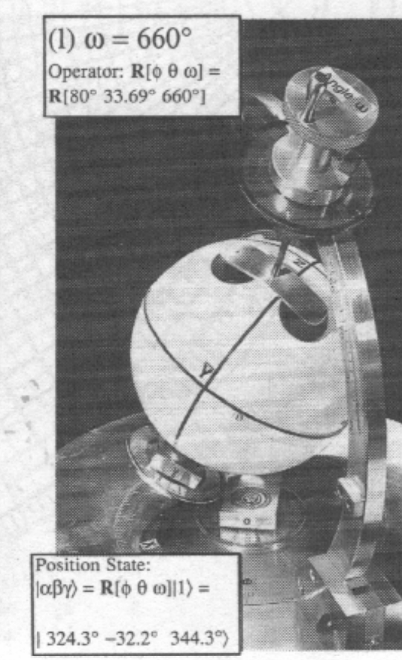












Θ=360° Θ=420° Θ=488.7° Θ=540° Θ=600° Θ=660°

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➡ *R(3)-U(2) slide rule for converting* $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *Sundial*

R(3)-U(2) slide rule for converting R(αβγ) ↔ R[φϑΘ]

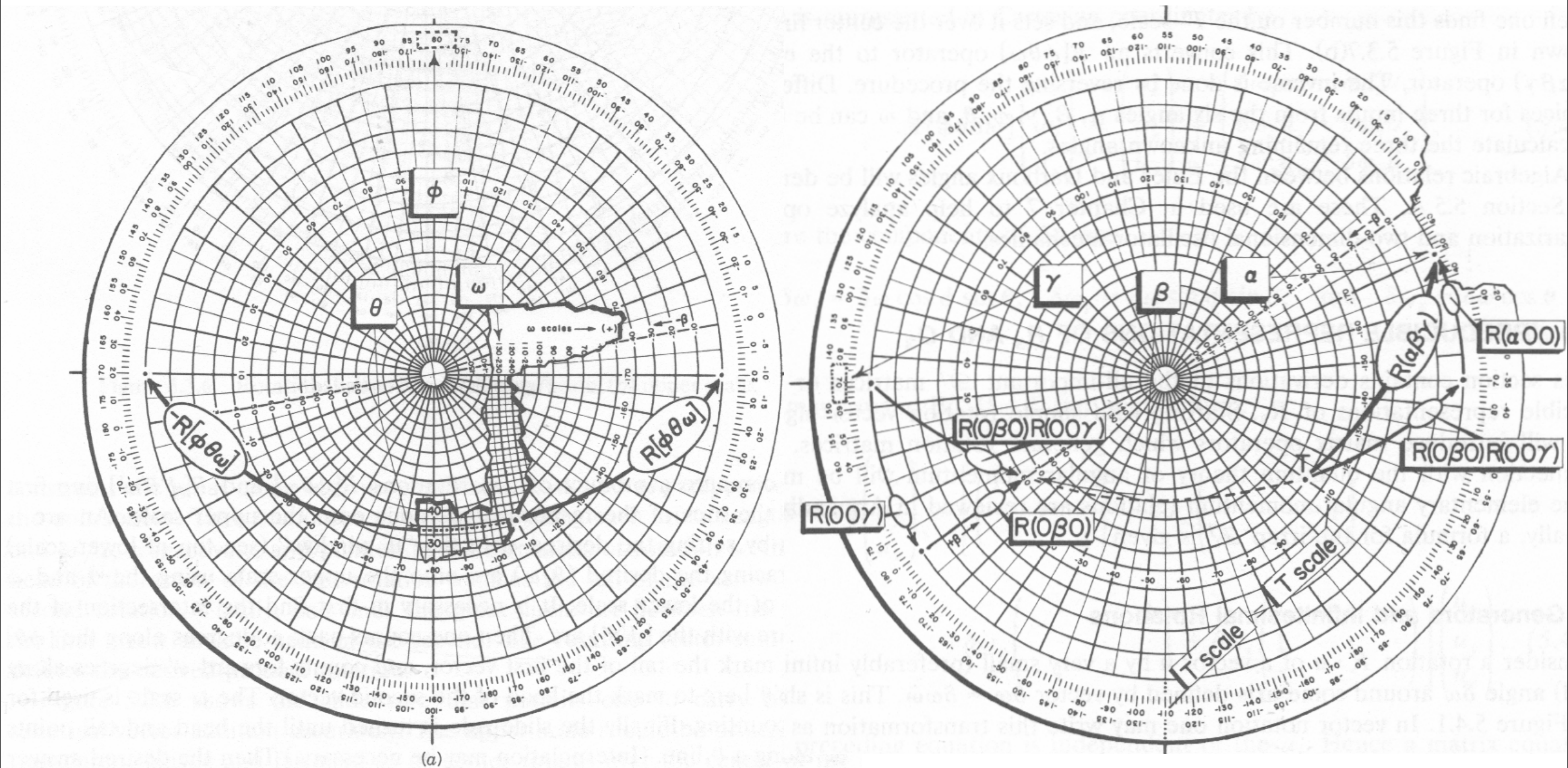


Figure 5.3.7 Setting the rotational slide rule. (a) Darboux or axis angles. (b) Euler angles.

Review:How "Crazy-Thing"-Theorem makes spinor and vector representation matrices

Half-angle $\Theta/2=\varphi$ replacement and Darboux crank axis operators

Operator-on-Operator transformations

Product algebra for Pauli's σ_μ and Hamilton's $\boldsymbol{q}_\mu = -i\sigma_\mu$

Group product algebra

Jordan-Pauli identity and U(2) product $\mathbf{R}[\Theta]\mathbf{R}[\Theta']=\mathbf{R}[\Theta''']$*- formula*

Transformation $\mathbf{R}[\Theta]\sigma_\mu\mathbf{R}[\Theta]^\dagger$ *of spinor σ_μ-operators*

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ *of group-operators*

Operator-on-Operator transformations

Geometry of groups: Hamilton's turns and It's all done with mirrors!

Group product geometry

U(2) product $\mathbf{R}[\Theta]\mathbf{R}[\Theta']=\mathbf{R}[\Theta''']$*- geometry*

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ *geometry*

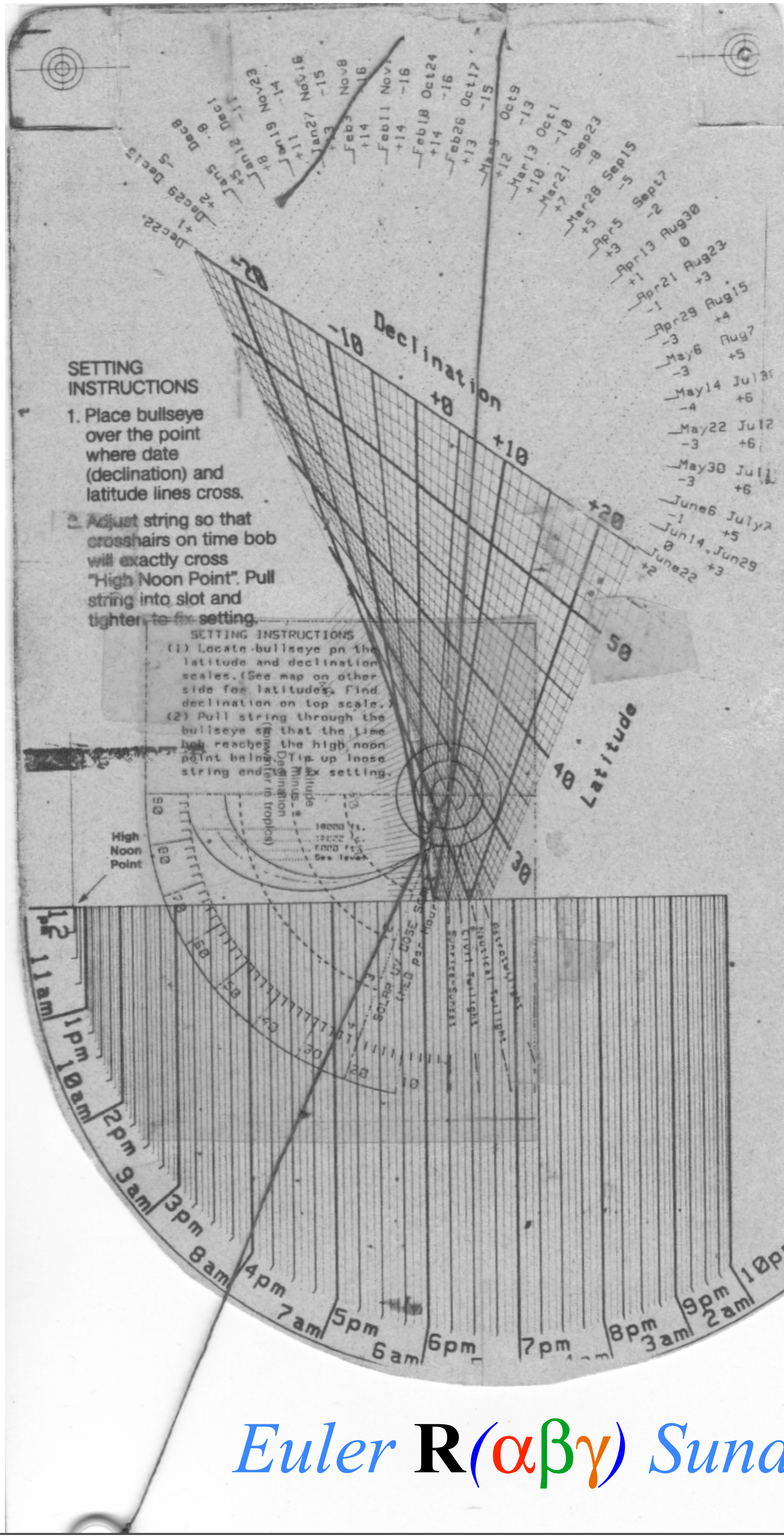
Euler $\mathbf{R}(\alpha\beta\gamma)$ versus *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

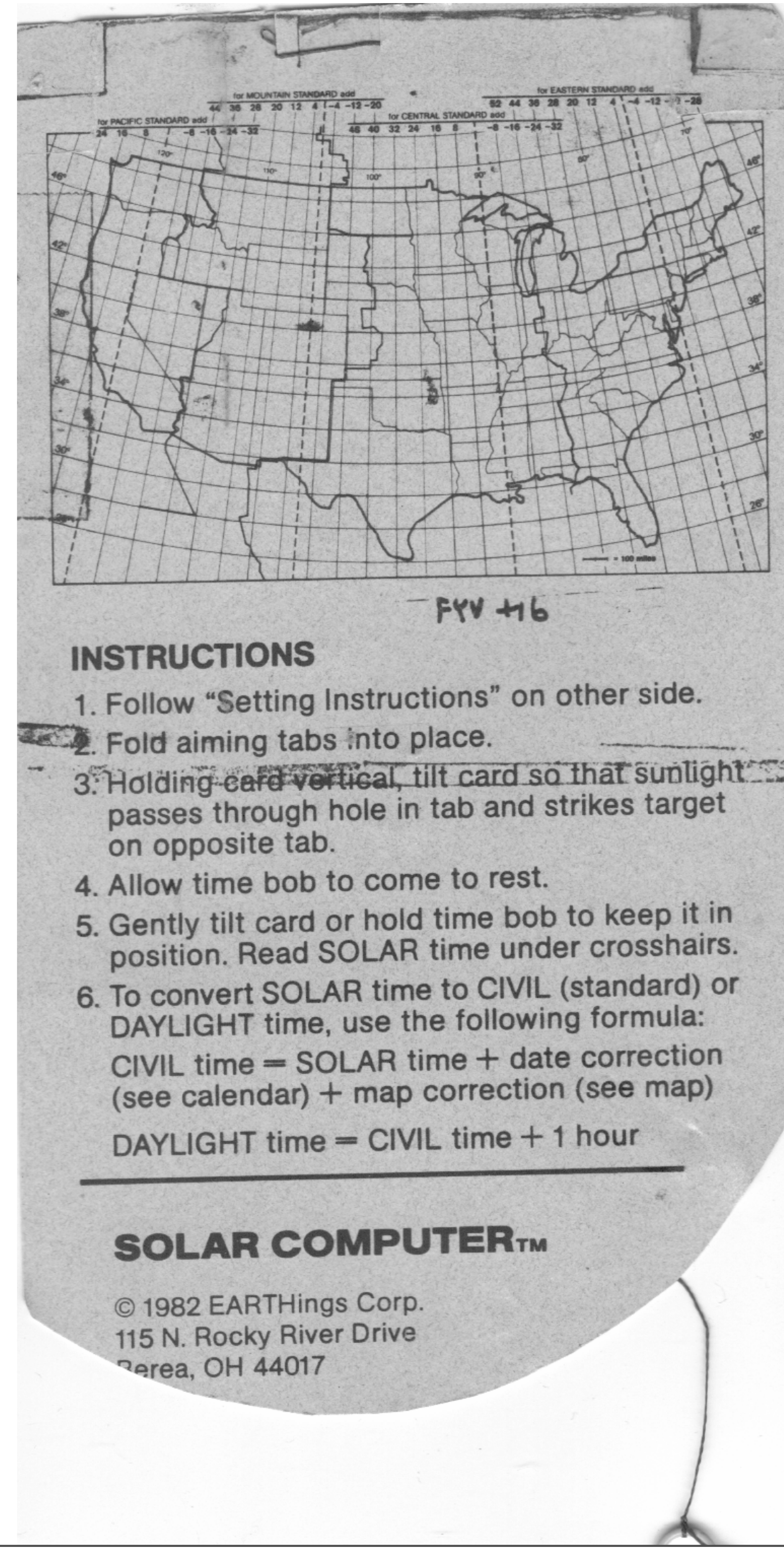
Euler $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed*

→ *R(3)-U(2) slide rule for converting* $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *Sundial*





Euler **R**(αβγ) *Sundial*