Group Theory in Quantum Mechanics

Lecture 1 *(1.13.15)*

Introduction to quantum amplitudes and analyzers

(Quantum Theory for Computer Age - Ch. 1 of Unit 1)

(Principles of Symmetry, Dynamics, and Spectroscopy - Sec. 1-2 of Ch. 1)

Beam Sorters

2-State Sorters: spin-1/2 vs. optical polarization

Geometry of optical polarization selection and Brewster's angle

Feynman's lever

Beam Sorters in Series and Transformation Matrices

Introducing Dirac bra-ket notation

"Abstraction" of bra and ket vectors from a Transformation Matrix

Introducing scalar and matrix products

Principles of Symmetry, Dynamics, and Spectroscopy *{Text}* - URL is "http://www.uark.edu/ua/modphys/markup/PSDSWeb.html"

Quantum Theory for the Computer Age - URL is "http://www.uark.edu/ua/modphys/markup/QTCAWeb.html"

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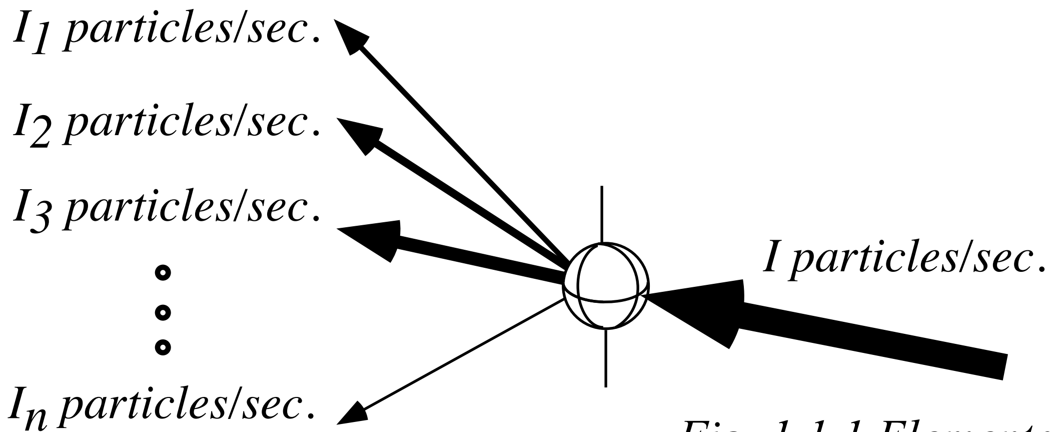
Beam Sorters



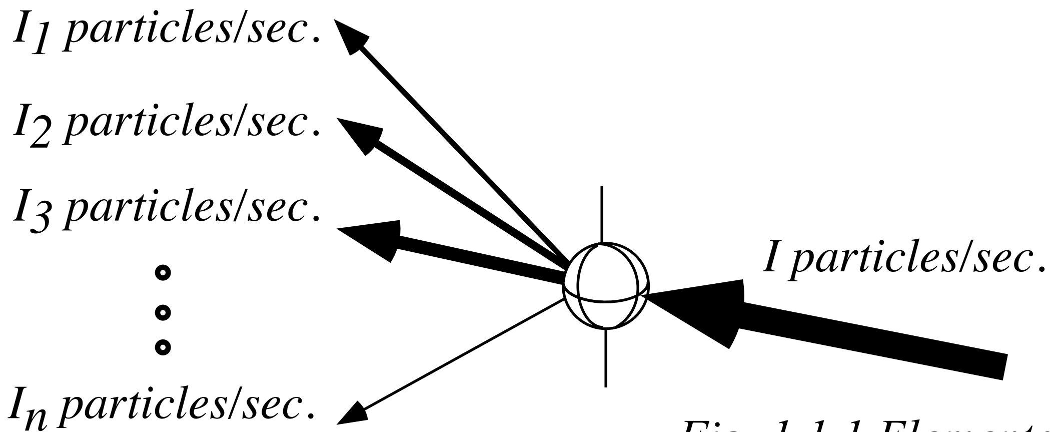
$$I = I_1 + I_2 + I_3 + \ldots + I_n$$

Fig. 1.1.1 Elementary beam sorter for n-state beam

One job of quantum mechanics is to compute *relative intensities* or *probabilities* P_k defined by

$$P_k = I_k / I \qquad \text{where:} \qquad 1 = P_1 + P_2 + P_3 + \ldots + P_n$$

Beam Sorters



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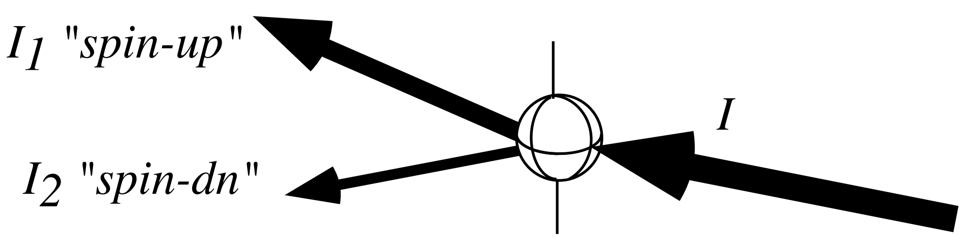
2-State Beam Sorters

Spin-1/2



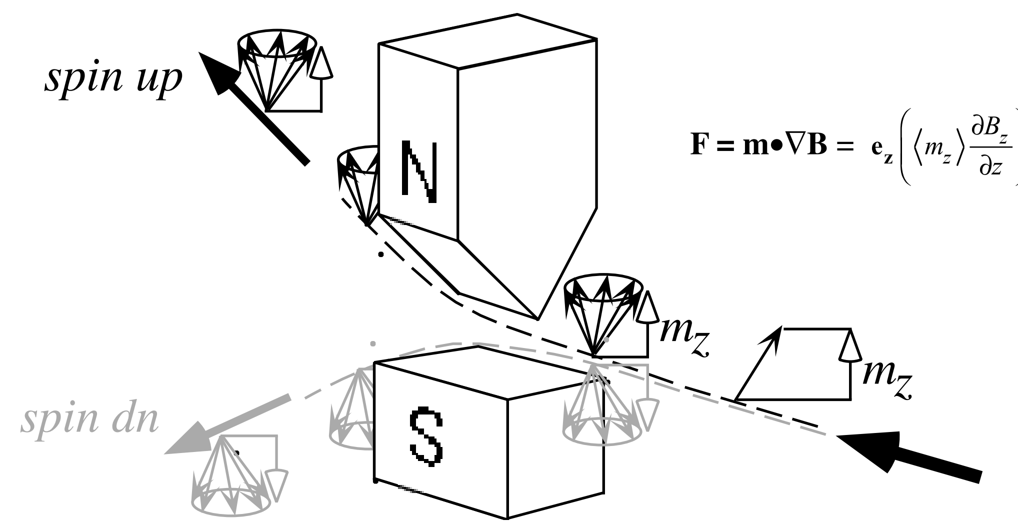


Fig. 1.1.2 Stern-Gerlach beam sorter for 2-state electron spin beam

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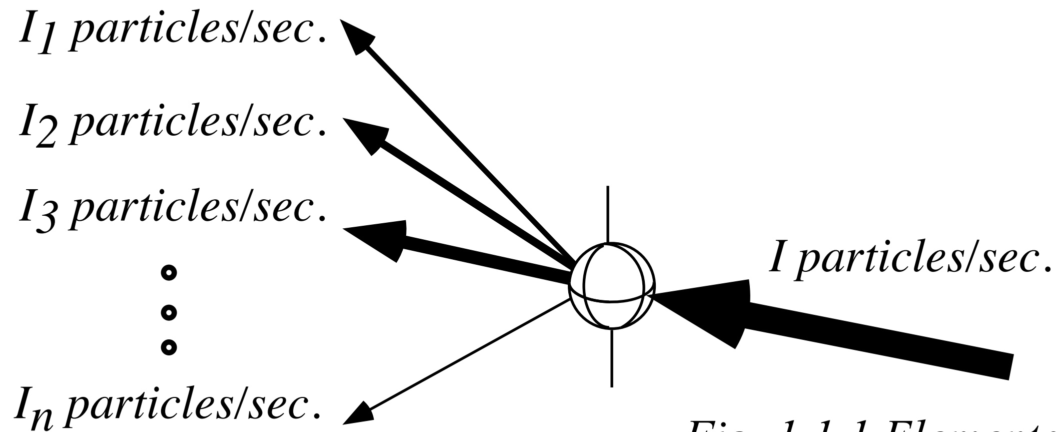
Beam Sorters



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2-State Beam Sorters

Spin-1/2

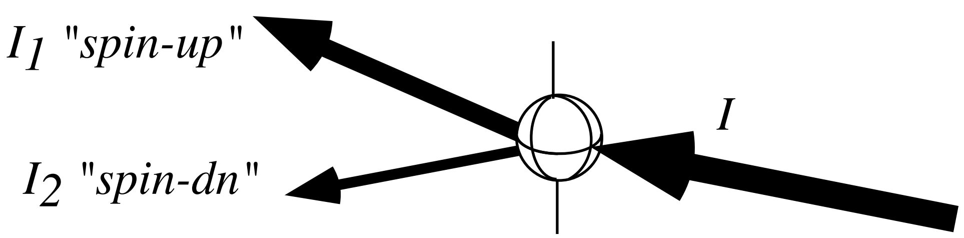

Fig. 1.1.2 Stern-Gerlach beam sorter for 2-state electron spin beam

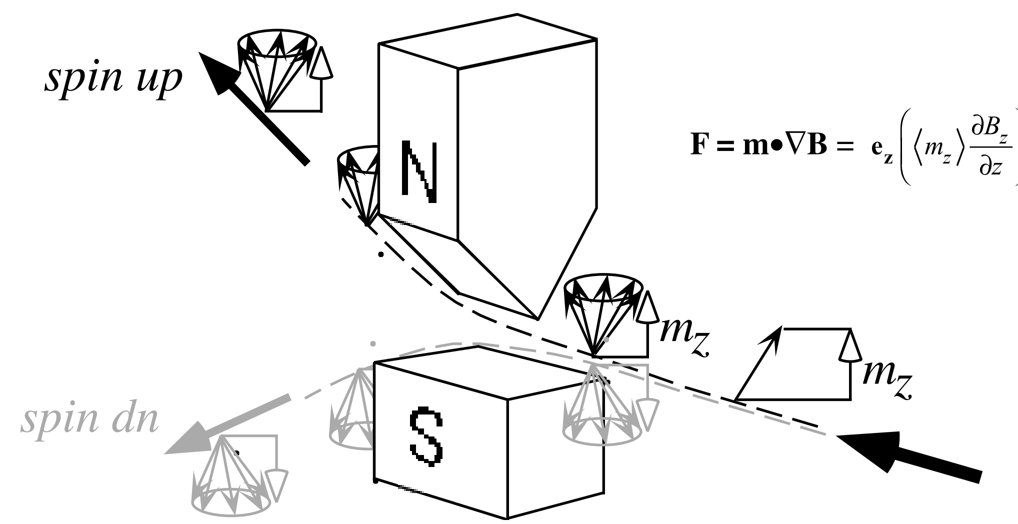

*Fig. 1.1.6 Sketch of electron beam sorting by non-uniform **B**-field: (Stern-Gerlach polarizer)*

Optical polarization

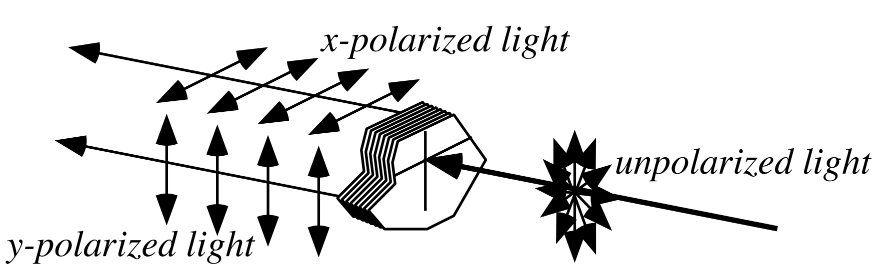

Fig. 1.1.3 Primitive photon beam sorter for 2-state polarization

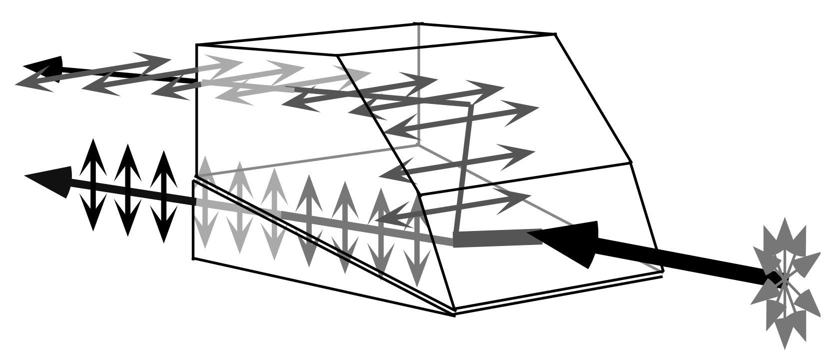

Fig. 1.1.5 Sketch of modern optical polarization sorter: (The Brewster prism)

Beam Sorters

2-State Sorters: spin-1/2 vs. optical polarization

→ *Geometry of optical polarization selection and Brewster's angle*

Feynman's lever

Beam Sorters in Series and Transformation Matrices

Introducing Dirac bra-ket notation

"Abstraction" of bra and ket vectors from a Transformation Matrix

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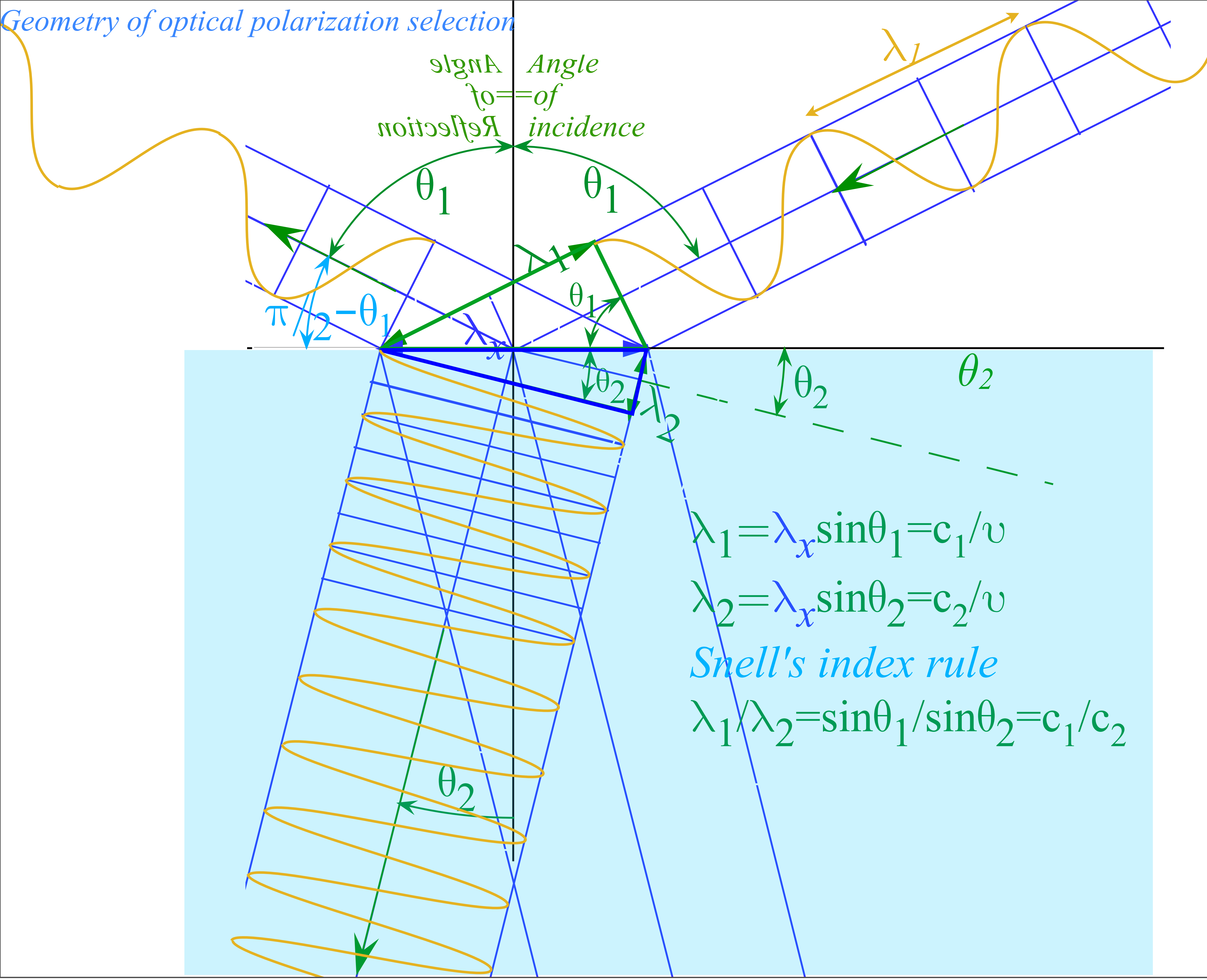
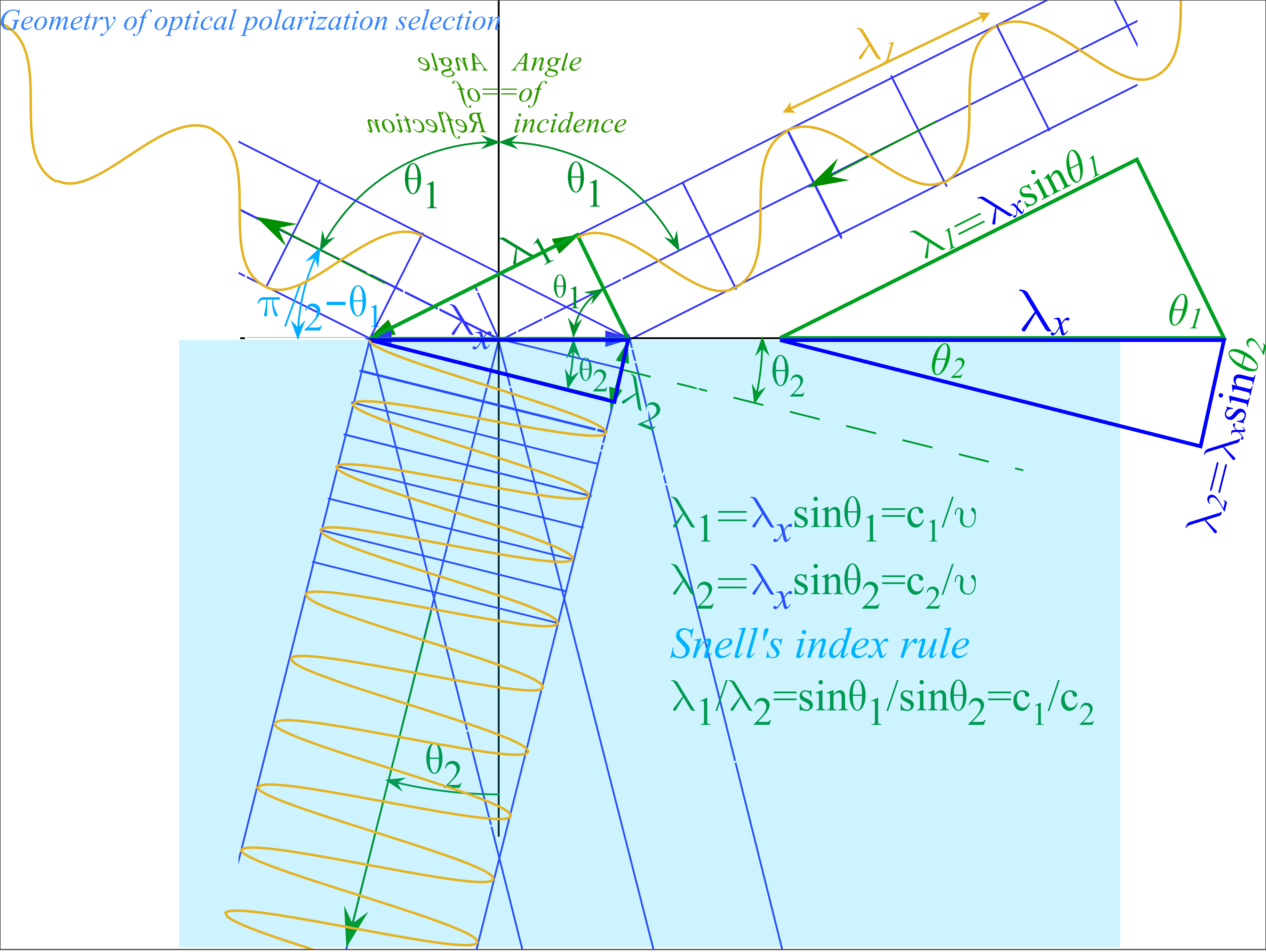
Geometry of optical polarization selection



Geometry of optical polarization selection

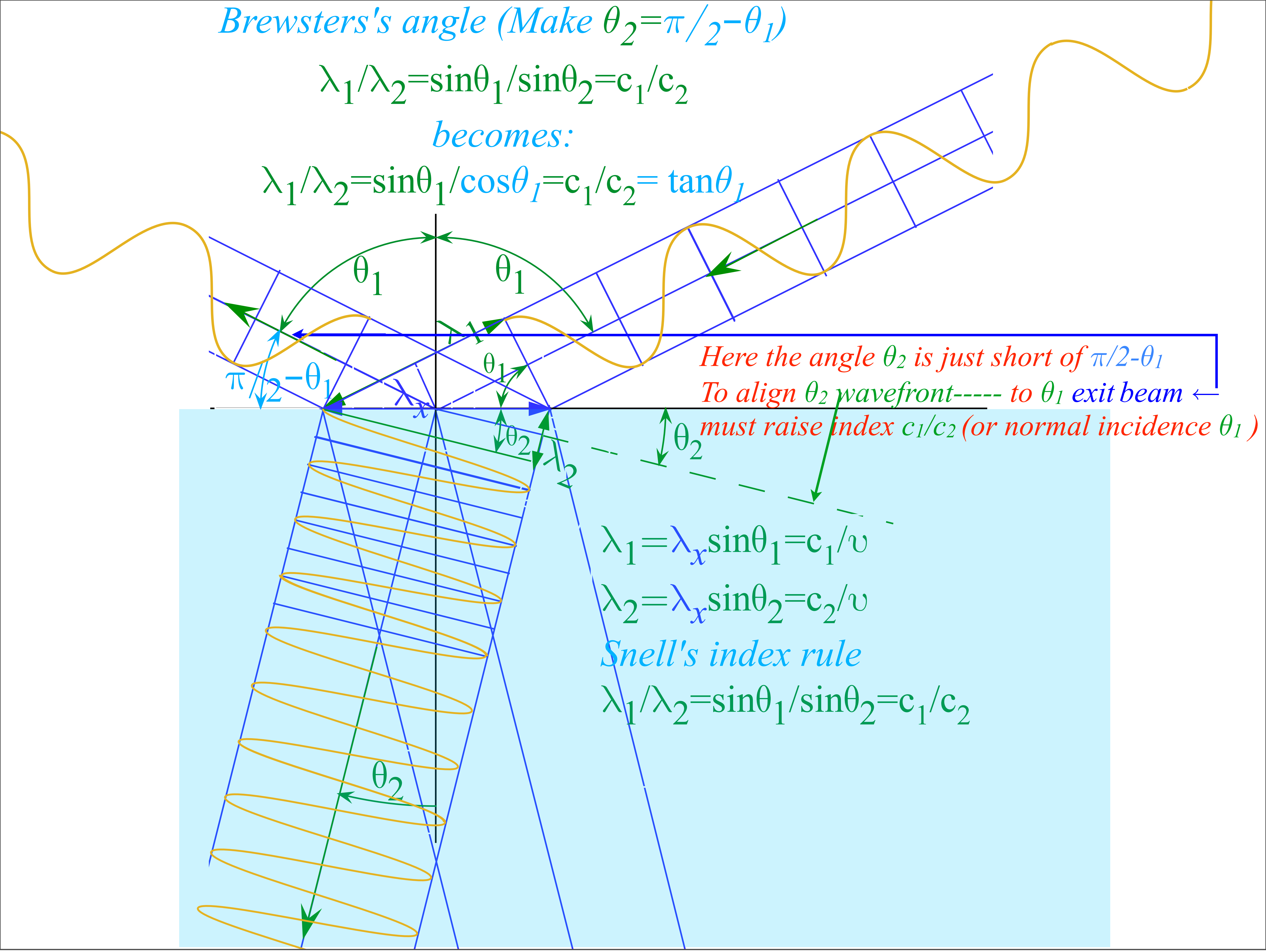


Brewsters's angle (Make $\theta_2 = \pi/2 - \theta_1$)

$$\lambda_1/\lambda_2 = \sin\theta_1/\sin\theta_2 = c_1/c_2$$

becomes:

$$\lambda_1/\lambda_2 = \sin\theta_1/\cos\theta_1 = c_1/c_2 = \tan\theta_1$$



Here the angle θ_2 is just short of $\pi/2-\theta_1$

To align θ_2 wavefront----- to θ_1 exit beam ←

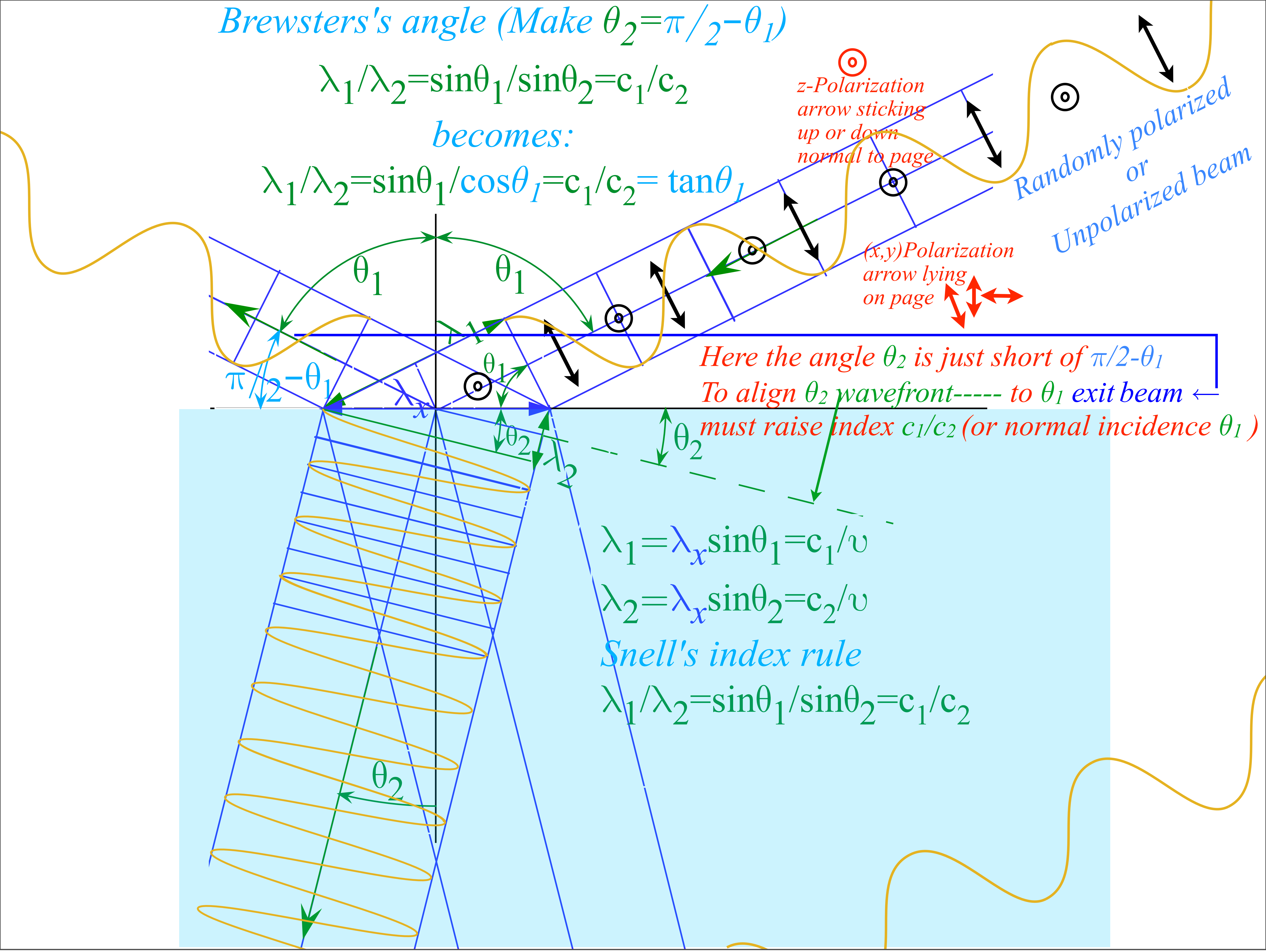
must raise index c_1/c_2 (or normal incidence θ_1)

$$\lambda_1 = \lambda_x \sin\theta_1 = c_1/\upsilon$$

$$\lambda_2 = \lambda_x \sin\theta_2 = c_2/\upsilon$$

Snell's index rule

$$\lambda_1/\lambda_2 = \sin\theta_1/\sin\theta_2 = c_1/c_2$$



Brewsters's angle (Make $\theta_2=\pi/2-\theta_1$)
$\lambda_1/\lambda_2=\sin\theta_1/\sin\theta_2=c_1/c_2$
becomes:
$\lambda_1/\lambda_2=\sin\theta_1/\cos\theta_1=c_1/c_2=\tan\theta_1$
z-Polarization arrow sticking up or down normal to page
Randomly polarized or Unpolarized beam
(x,y)Polarization arrow lying on page
θ_1
θ_1
λ_1
θ_1
$\pi/2-\theta_1$
λ_x
θ_2
λ_2
θ_2
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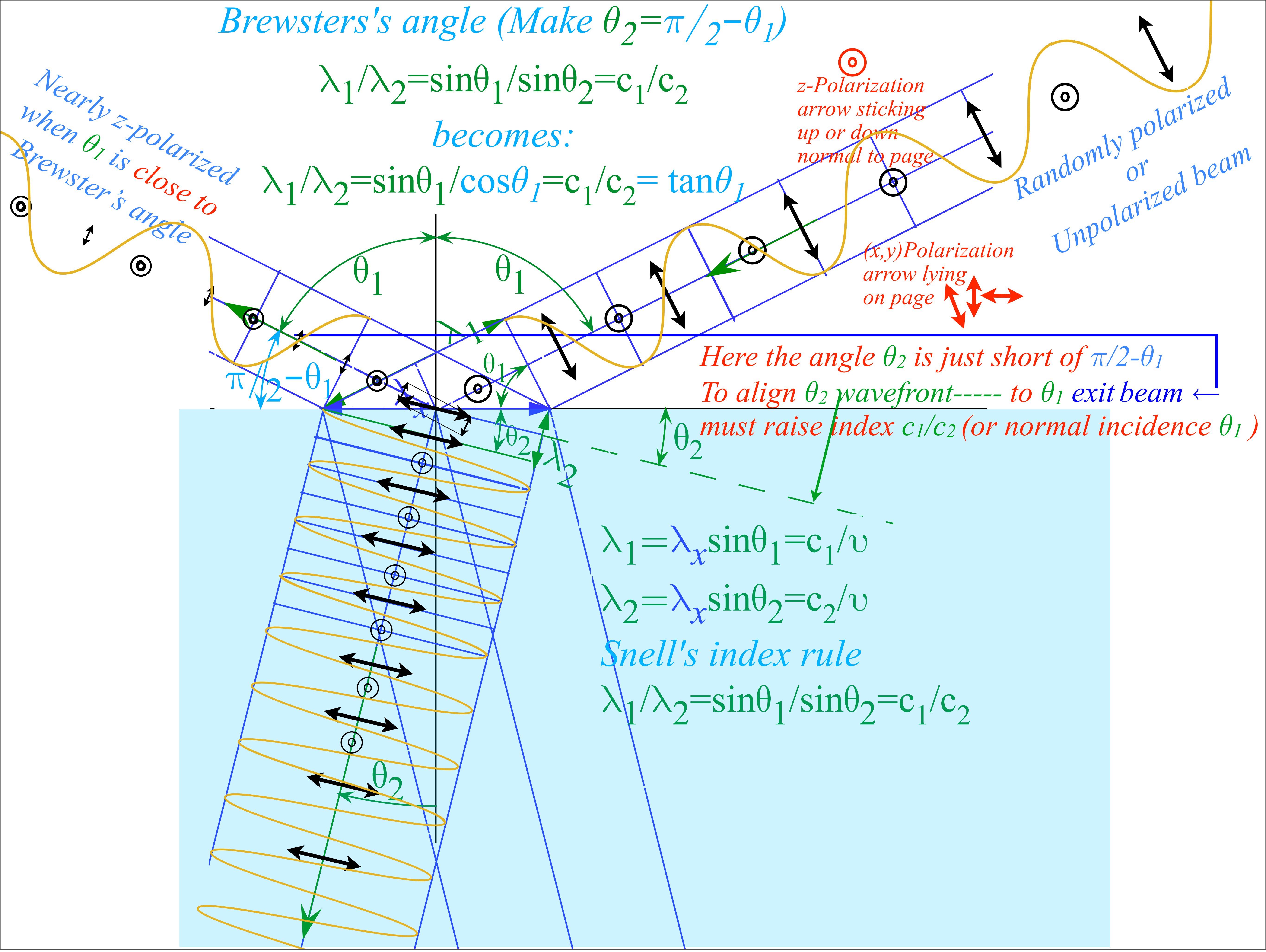

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Nearly z-polarized when θ_1 is close to Brewster's angle
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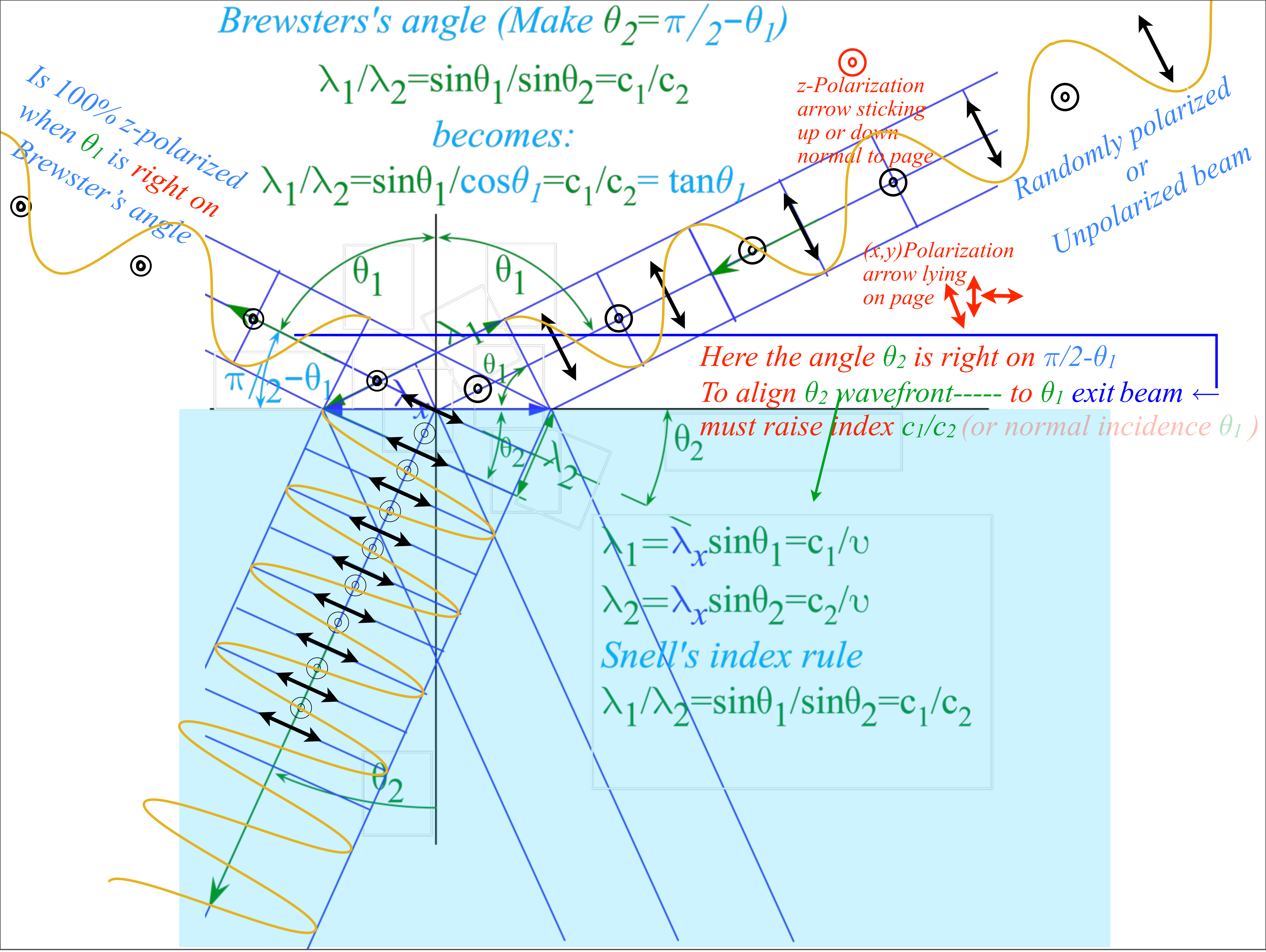

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$\lambda_1/\lambda_2=\sin\theta_1/\sin\theta_2=c_1/c_2$
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Is 100% z-polarized when θ_1 is right on Brewster's angle
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Beam Sorters

2-State Sorters: spin-1/2 vs. optical polarization

Geometry of optical polarization selection and Brewster's angle

➡ *Feynman's lever*

Beam Sorters in Series and Transformation Matrices

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Introducing scalar and matrix products

(c) Visualizing Radiation Coupling Using Feynman's Lever The detailed solutions of Newton's and Maxwell's equations for coupled particles and em fields are complicated. However, for small numbers of particles there is a graphical construction given in the Feynman Lectures (Section II-21) which is very instructive. It provides a way to tell exactly what the fields will be around an arbitrarily moving charge.

Imagine that you are holding a charge and moving it back and forth. Let the charge be attached to a ring which can slide on a long lever arm as shown in Figure 6.5.6(a). Let the lever have a unit vector $(-\hat{\mathbf{e}}_r)$ or pointer pointing in the opposite direction of the lever $\mathbf{r}$ on the other side of its swivel point (0) at origin. Feynman has shown that the $\mathbf{E}$ field at origin at time t depends on the position of the pointer $\hat{\mathbf{e}}'_r$ and lever $\mathbf{r}'$ at a slightly earlier time $(t' = t - r/c)$. The time delay is just the time it would take a signal traveling at c to propagate from r at t' to origin at t. The $\mathbf{E}$ field is given by

$$\mathbf{E}(0,t) = \frac{q}{4\pi\varepsilon_0}\left\{\frac{-\hat{\mathbf{e}}'_r}{(r')^2} + \frac{r'}{c}\frac{d}{dt}\left[\frac{-\hat{\mathbf{e}}'_r}{(r')^2}\right] + \frac{1}{c^2}\frac{d^2}{dt^2}[-\mathbf{e}'_r]\right\}$$

$$= \underset{\text{term}}{\text{Coulomb}} + \underset{\text{term}}{\text{induction}} + \underset{\text{term}}{\text{radiation}}. \qquad (6.5.25a)$$

The first term is just the usual Coulomb field. The second term gives rise to a magnetic induction field,

$$\mathbf{B}(0,t) = (\hat{\mathbf{e}}'_r \times \mathbf{E})/c, \qquad (6.5.25b)$$

at origin if the charge has velocity transverse to $\mathbf{r}$. Finally, the third radiation term contributes to $\mathbf{E}(0,t)$ and $\mathbf{B}(0,t)$ in (6.5.25) if the charge has acceleration transverse to $\mathbf{r}$. It is interesting to note that in some ways this term is the reverse of Newton's law. For Newton's law one is given a field $\mathbf{E}$ or

Feynman's Lectures now free online

http://www.feynmanlectures.caltech.edu/

See Volume II Chapter 21 for the lever

Feynman's lever as described in PSDS:

http://www.uark.edu/ua/modphys/pdfs/PSDS_Pdfs/PSDS_Ch.6_%284.20.10%29.pdf

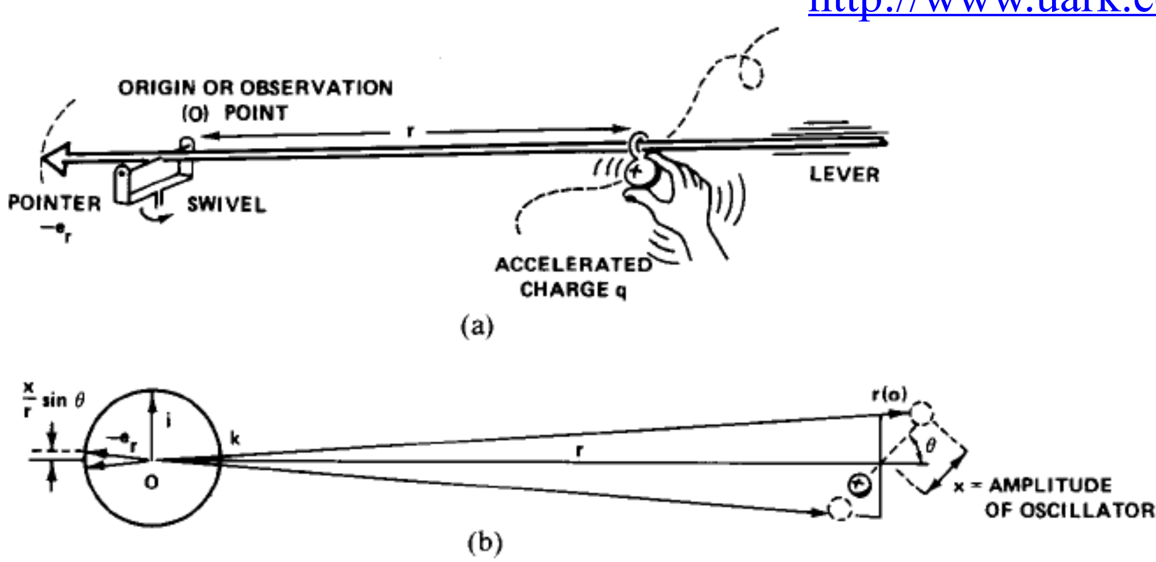

Figure 6.5.6 Feynman's lever. This construction provides a convenient way to visualize the field due to an accelerated or moving charge.

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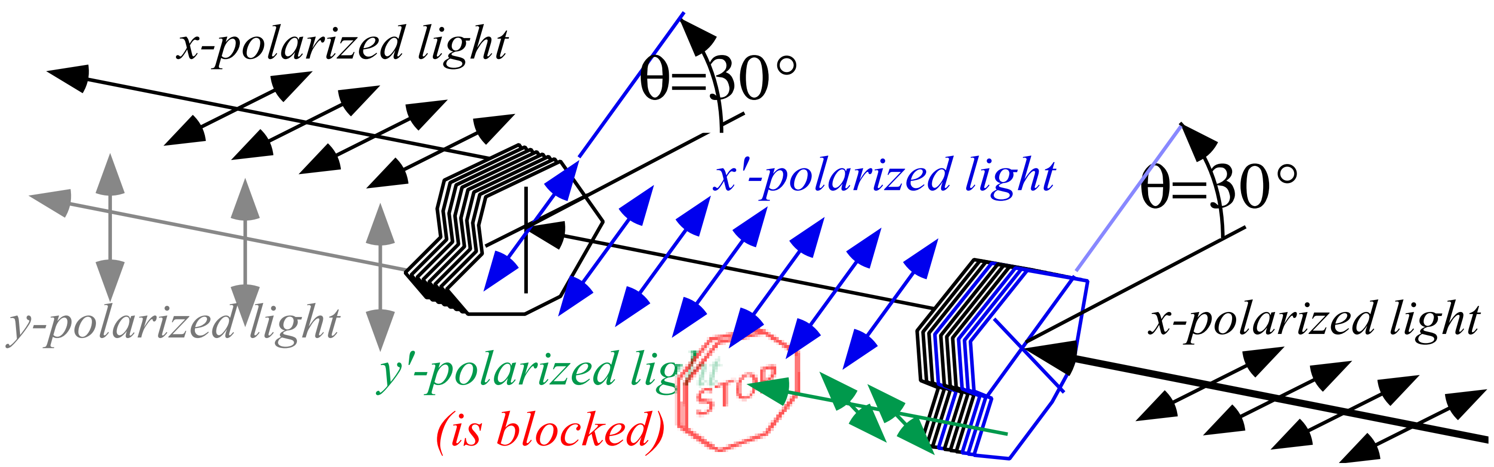


Fig. 1.2.1 Photon beam sorters in series with the first one y-blocked and tilted by angle θ=30°.

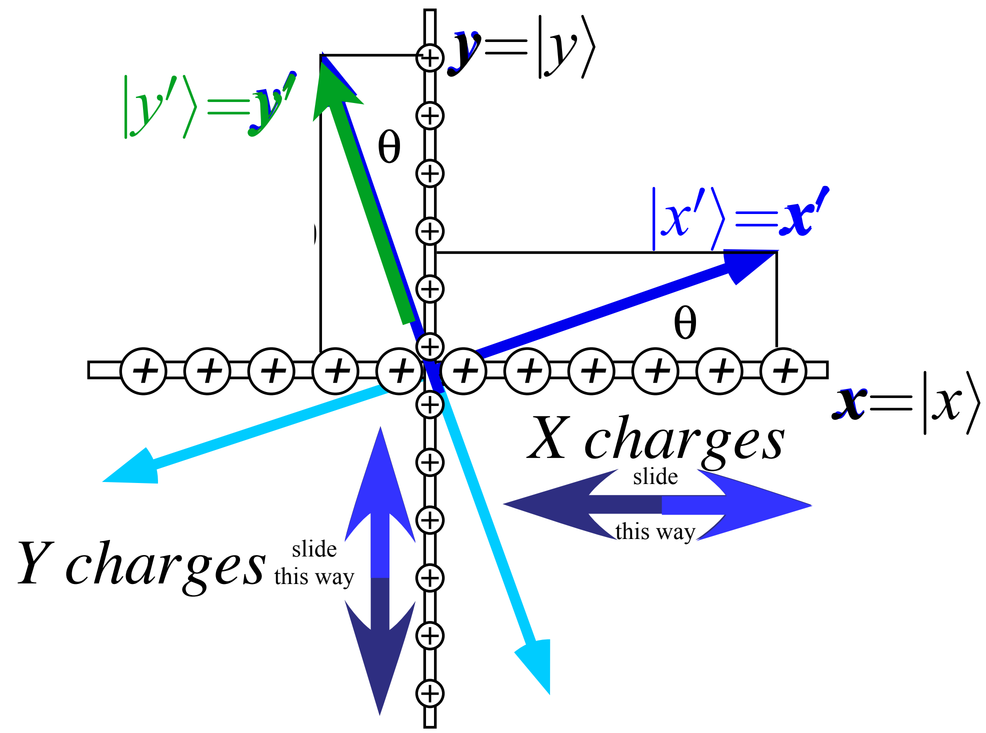
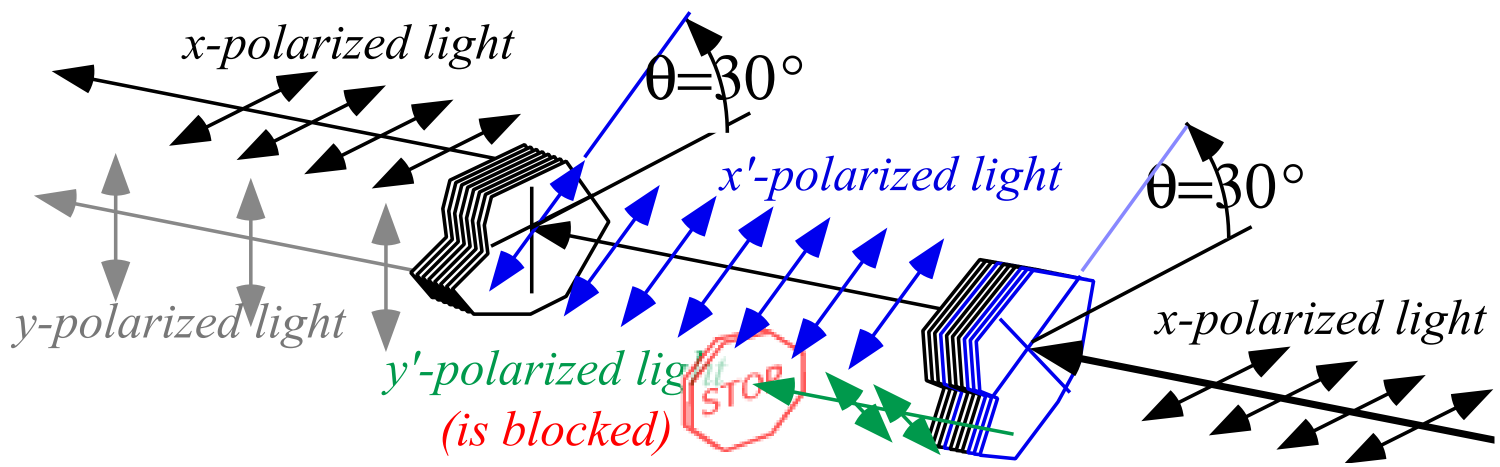


Fig. 1.2.2 Geometry of photon beam sorter for input polarizations (x',y') *tilted by angle* θ [relative to (x,y)].

http://www.uark.edu/ua/modphys/pdfs/QTCA_Pdfs/QTCA_Text_2013/QTCA_Unit_1_Ch._1-3_2013.pdf

Beam Sorters in Series and Transformation Matrices



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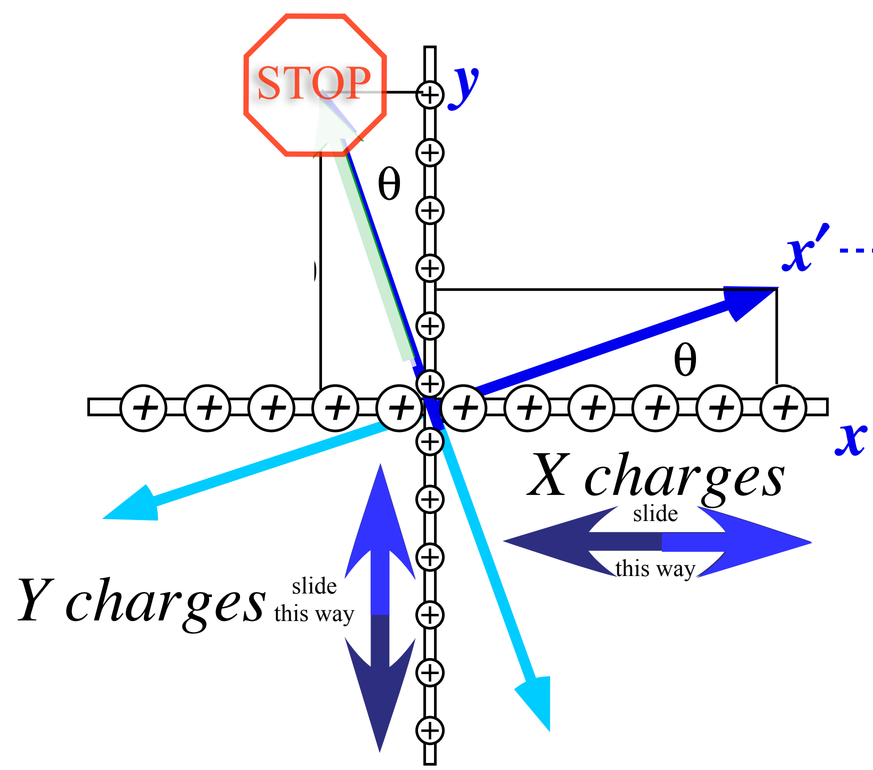
If we y'-blocked and let x' through: *(to be (x,y)-analyzed*



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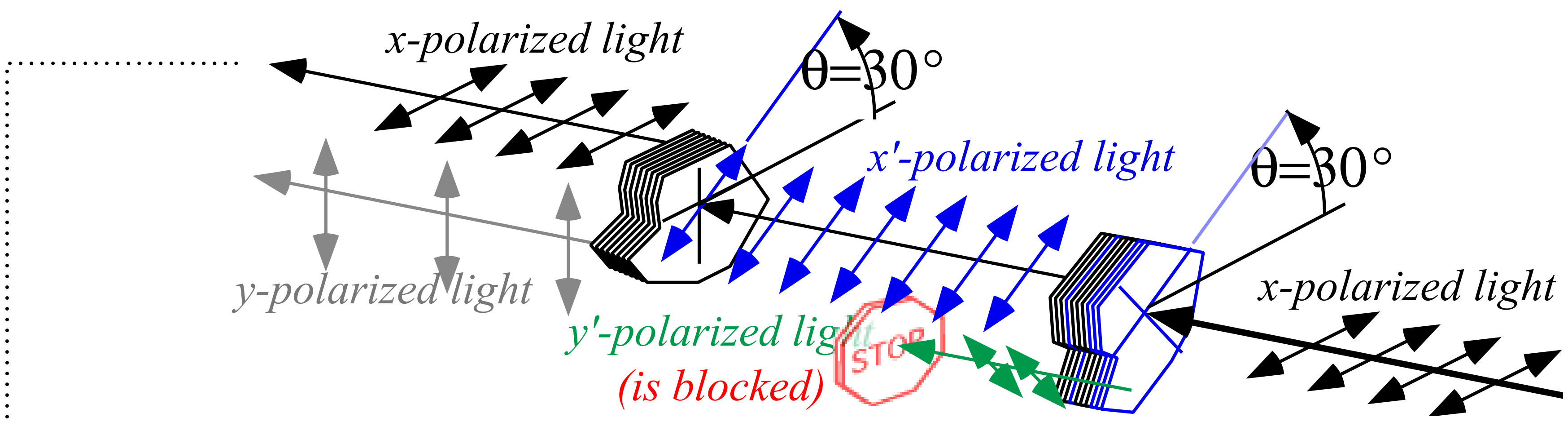


Fig. 1.2.1 Photon beam sorters in series with the first one y-blocked and tilted by angle θ=30°.

Feynman-Dirac Interpretation of

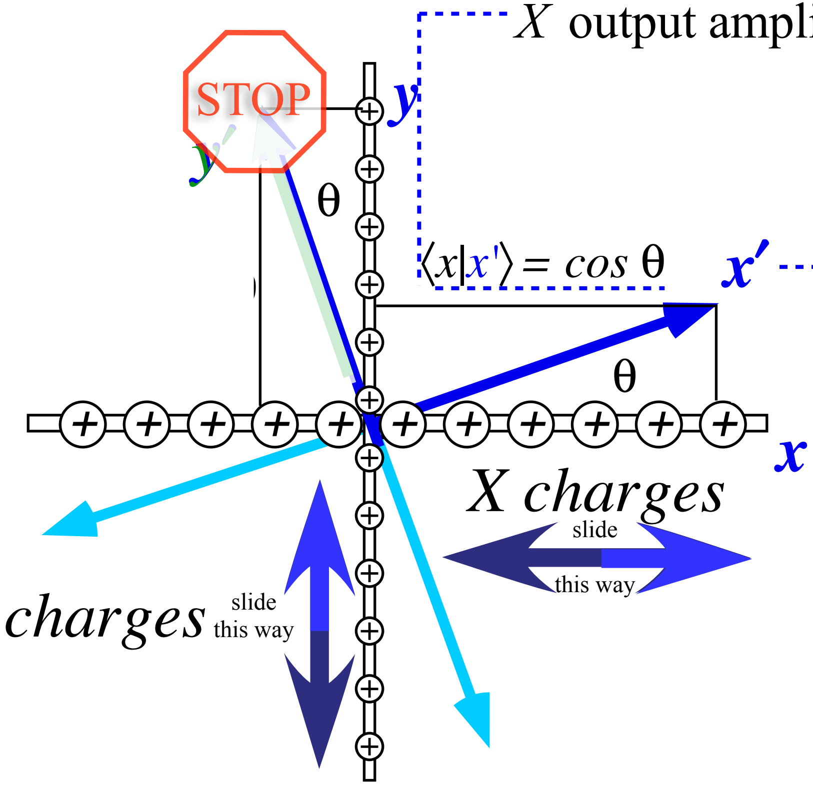
$$\langle m | n' \rangle$$

=Amplitude of state-m after state-n′ is forced to choose from available m-type states

Introducing Dirac bra-ket notation.

If we y′-blocked and let x′ through: *(to be (x,y)-analyzed*

X output amplitude due to x' input $= \langle x | x' \rangle = \cos\theta$

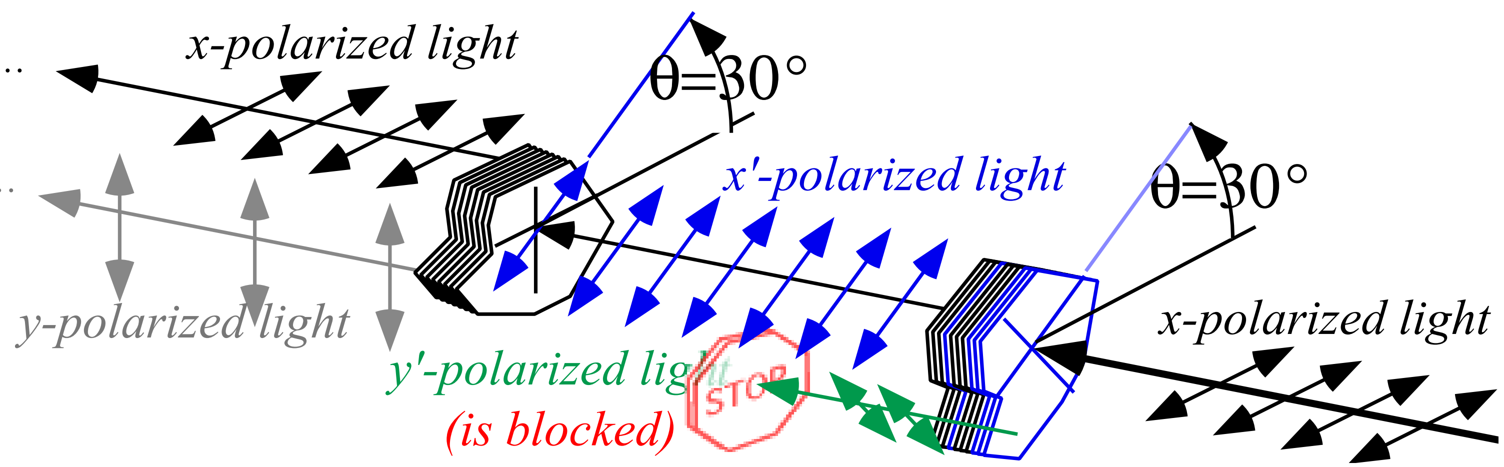


$sin\theta = sin30° = 1/2 = 0.5$

$cos\theta = cos30° = \sqrt{3}/2 = 0.866$

Fig. 1.2.2 Geometry of photon beam sorter for input polarizations (x′,y′) tilted by angle θ [relative to (*x*,*y*)].

Beam Sorters in Series and Transformation Matrices



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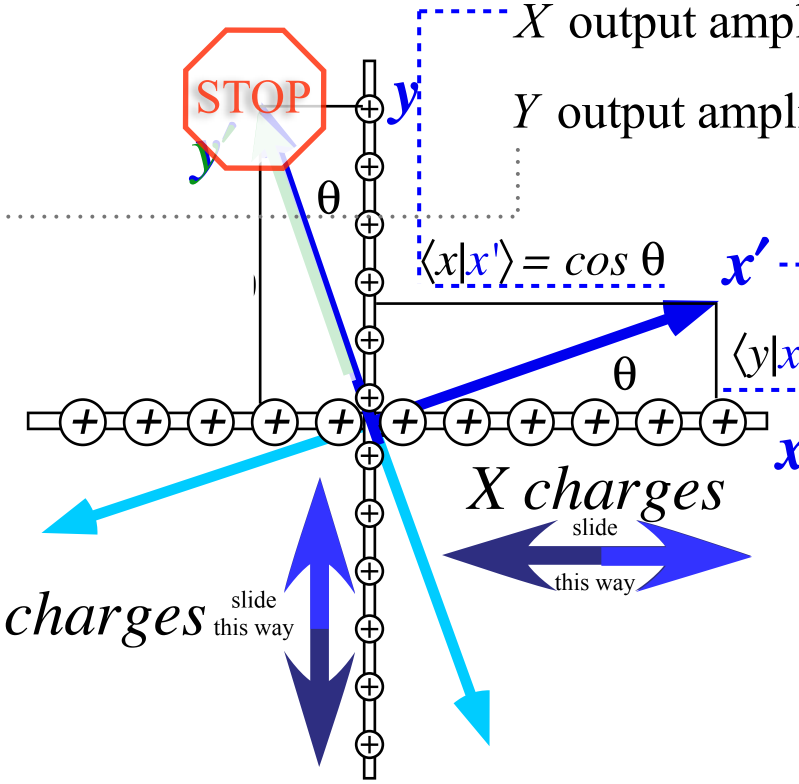
Introducing Dirac bra-ket notation.

If we y′-blocked and let x′ through: *(to be (x,y)-analyzed*

$$X \text{ output amplitude due to } x' \text{ input} = \langle x | x' \rangle = \cos\theta,$$

$$Y \text{ output amplitude due to } x' \text{ input} = \langle y | x' \rangle = \sin\theta$$



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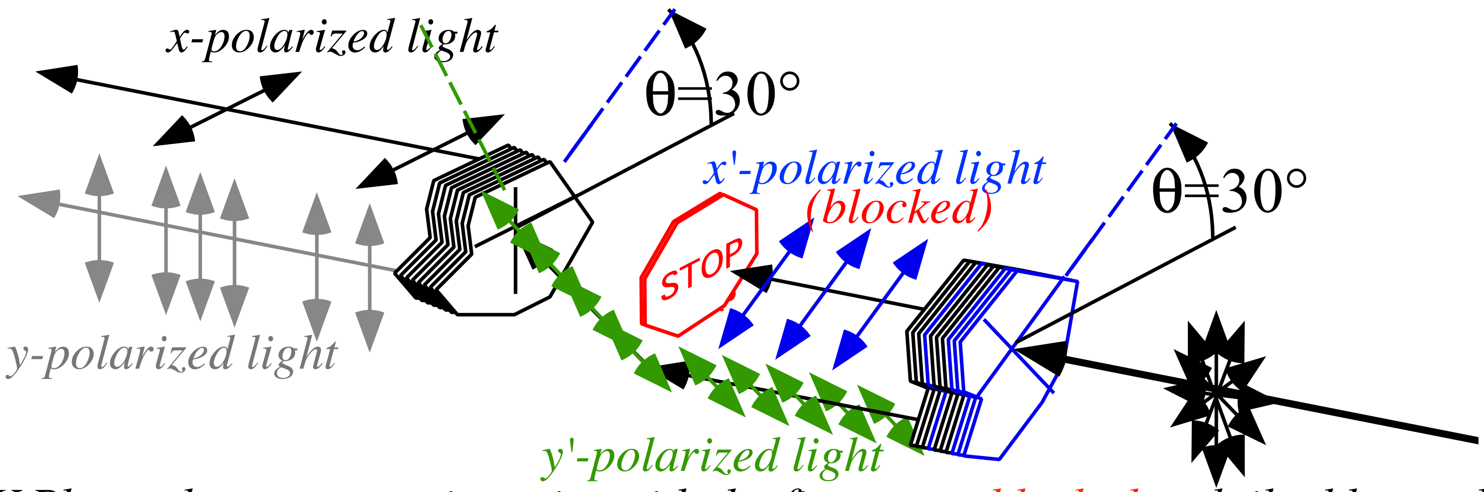


Fig. 1.2.X Photon beam sorters in series with the first one x-blocked and tilted by angle θ=30°.

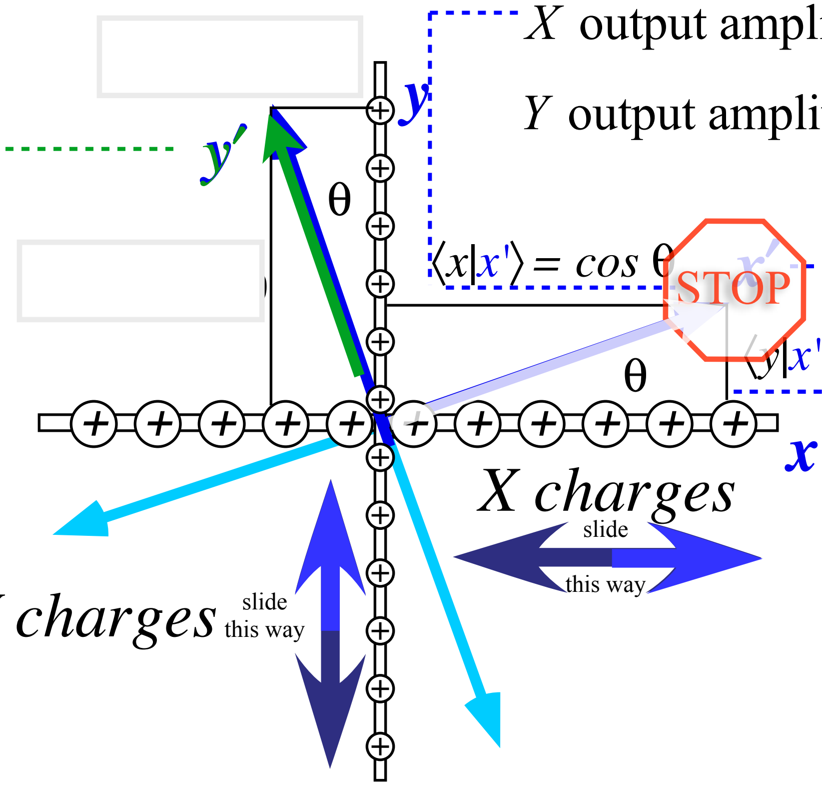
Introducing Dirac bra-ket notation.

If we x′-blocked and let y′ through instead:

If we y′-blocked and let x′ through:

X output amplitude due to x' input $= \langle x | x' \rangle = \cos\theta$,

Y output amplitude due to x' input $= \langle y | x' \rangle = \sin\theta$



$sin\theta = sin30° = 1/2 = 0.5$

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Fig. 1.2.2 Geometry of photon beam sorter for input polarizations (x′,y′) tilted by angle θ [relative to (x,y)].

Beam Sorters in Series and Transformation Matrices

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$$\langle m | n' \rangle$$

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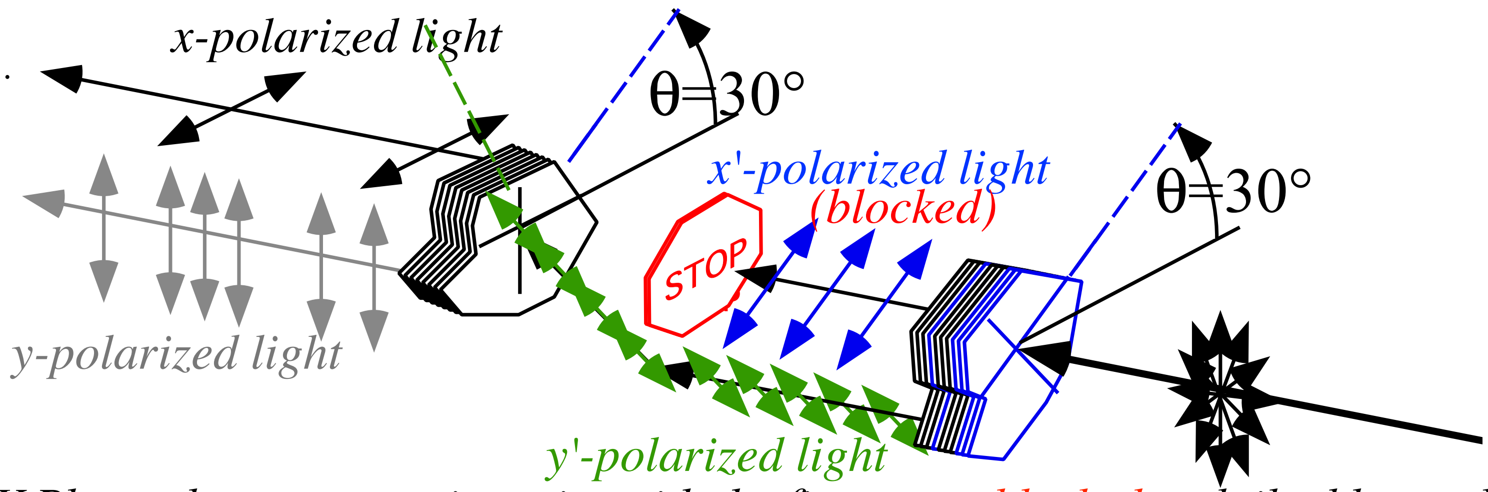

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Introducing Dirac bra-ket notation.

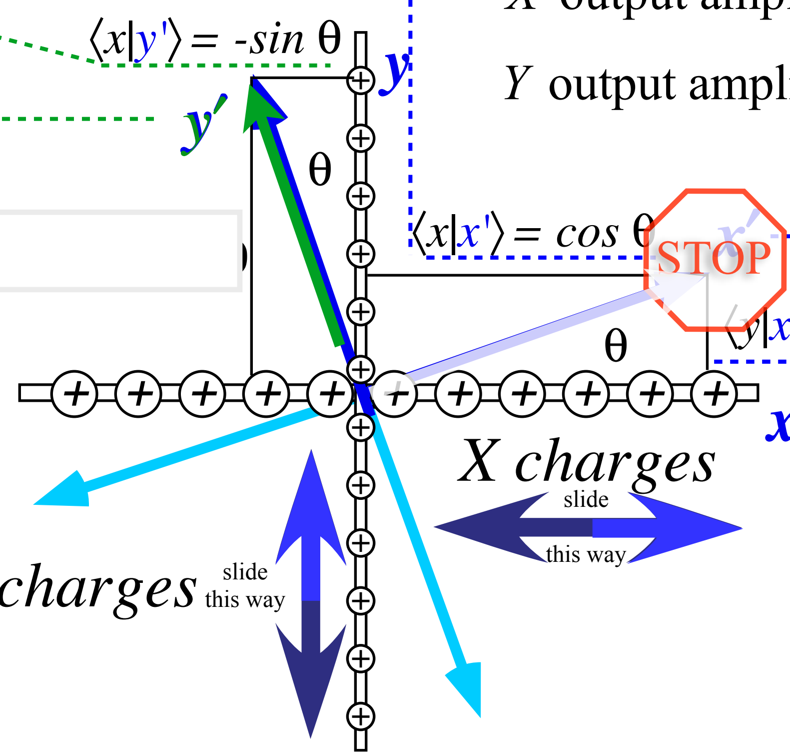
If we x′-blocked and let y′ through instead:

X output amplitude due to y' input $= \langle x | y' \rangle = -\sin\theta$

If we y′-blocked and let x′ through:

X output amplitude due to x' input $= \langle x | x' \rangle = \cos\theta$,

Y output amplitude due to x' input $= \langle y | x' \rangle = \sin\theta$



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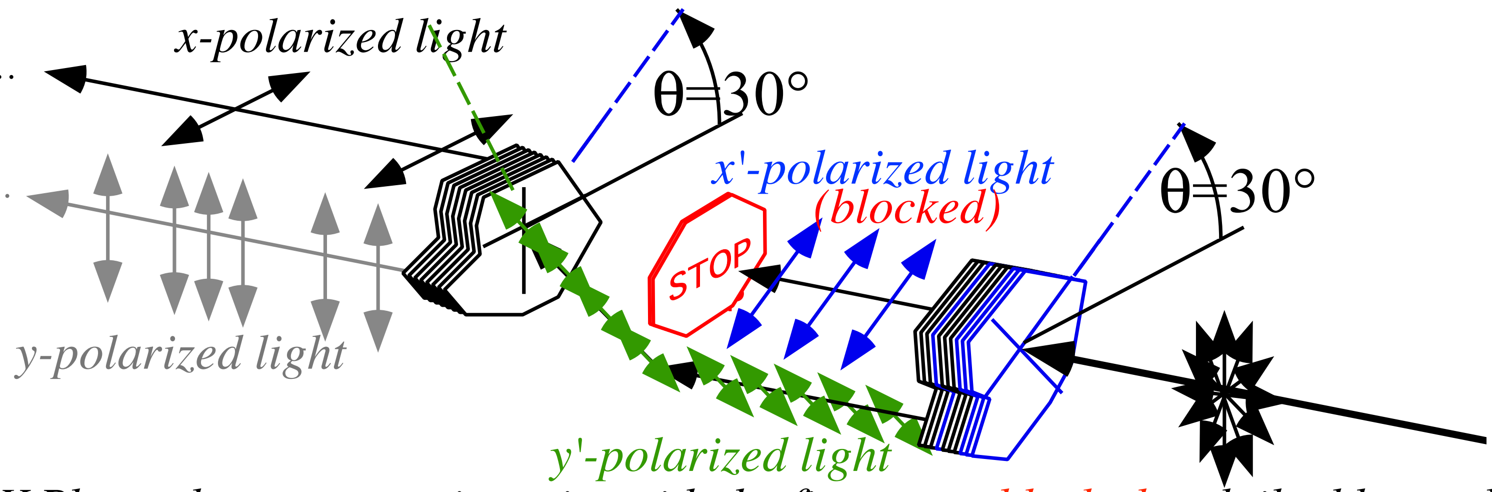

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Introducing Dirac bra-ket notation.

If we x′-blocked and let y′ through instead:

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Y output amplitude due to y' input $= \langle y | y' \rangle = \cos\theta$

If we y′-blocked and let x′ through:

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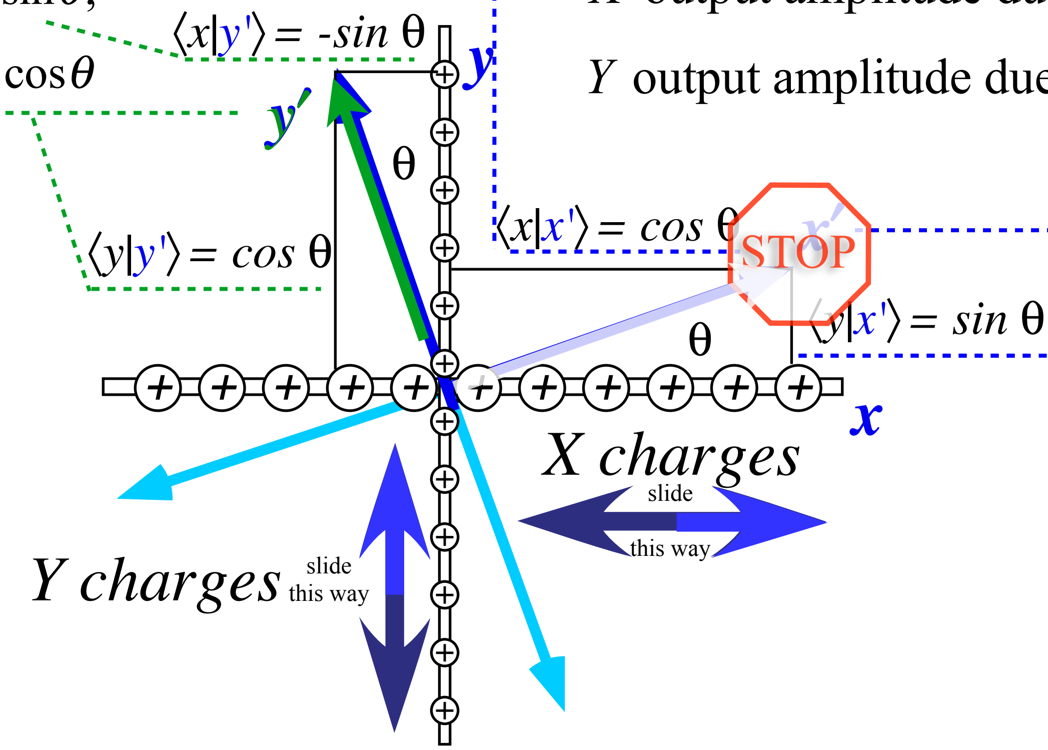

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$$\langle m | n' \rangle$$

=Amplitude of state-m after state-n′ is forced to choose from available m-type states

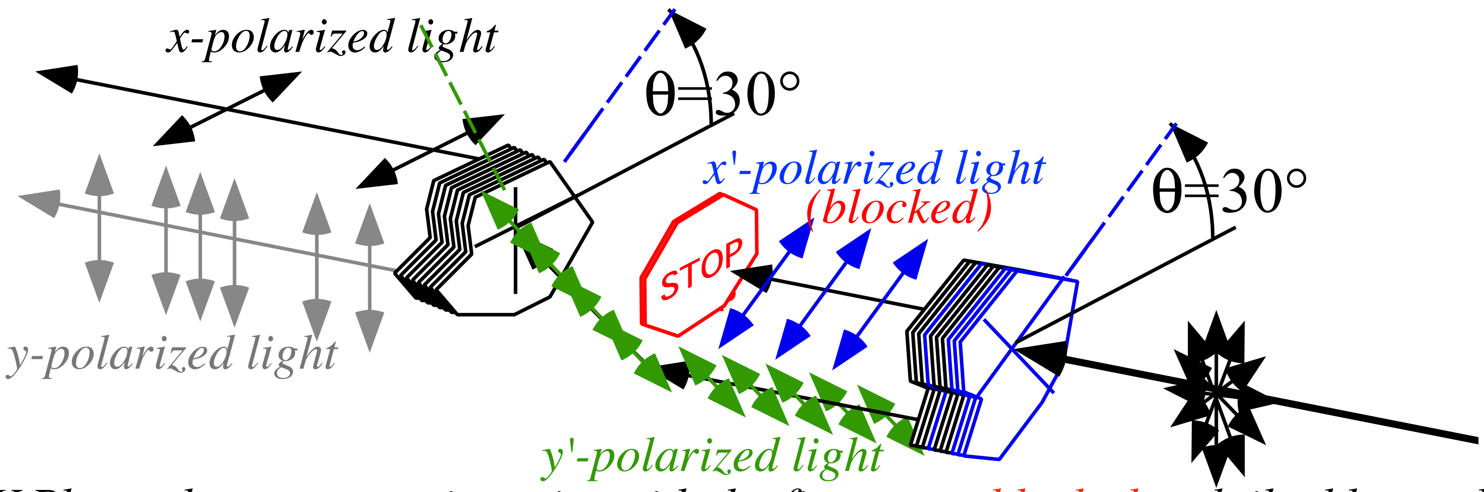


Fig. 1.2.X Photon beam sorters in series with the first one x-blocked and tilted by angle θ=30°.

Introducing Dirac bra-ket notation.

If we x′-blocked and let y′ through instead:

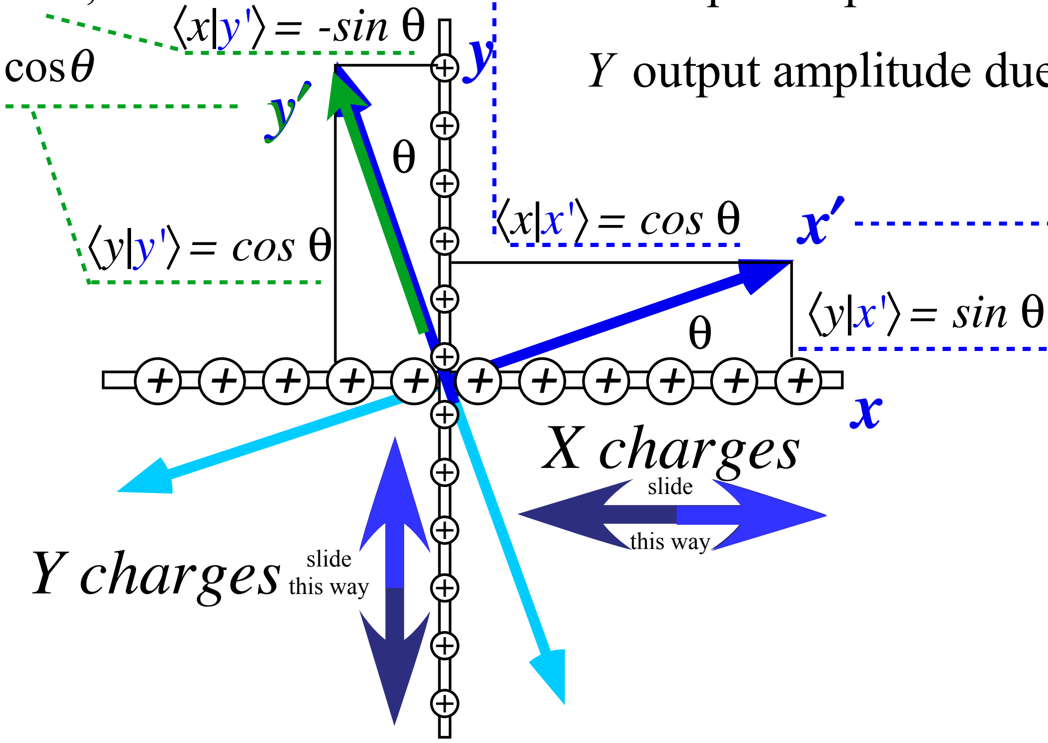
X output amplitude due to y' input $= \langle x | y' \rangle = -\sin\theta$,

Y output amplitude due to y' input $= \langle y | y' \rangle = \cos\theta$

If we y′-blocked and let x′ through:

X output amplitude due to x' input $= \langle x | x' \rangle = \cos\theta$,

Y output amplitude due to x' input $= \langle y | x' \rangle = \sin\theta$



$\sin\theta = \sin 30° = 1/2 = 0.5$

$\cos\theta = \cos 30° = \sqrt{3}/2 = 0.866$

Fig. 1.2.2 Geometry of photon beam sorter for input polarizations (x′,y′) tilted by angle θ [relative to (x,y)].

$$\begin{pmatrix} \langle x | x' \rangle & \langle x | y' \rangle \\ \langle y | x' \rangle & \langle y | y' \rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Introducing bra-ket Transformation Matrix

$T_{m,n'} = \langle m | n' \rangle$

Beam Sorters

2-State Sorters: spin-1/2 vs. optical polarization

Geometry of optical polarization selection and Brewster's angle

Feynman's lever

Beam Sorters in Series and Transformation Matrices

Introducing Dirac bra-ket notation

→ *"Abstraction" of bra and ket vectors from a Transformation Matrix*

Introducing scalar and matrix products

"Abstraction" of bra and ket vectors from a Transformation Matrix

Ket or column vectors

Bra or row vectors

Given Transformation Matrix $T_{m,n'}$:

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

"Abstraction" of bra and ket vectors from a Transformation Matrix

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Bra or row vectors

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Abstracting ket $|\,n'\rangle$ state vectors
from
Transformation Matrix

$$T_{m,n'} = \langle m|\,n'\rangle$$

"Abstraction" of bra and ket vectors from a Transformation Matrix

Ket or column vectors | *Bra or row vectors*

Given Transformation Matrix $T_{m,n'}$:
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$\Downarrow$

$$|x'\rangle = \begin{pmatrix} \langle x|x'\rangle \\ \langle y|x'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

Abstracting ket $|\,n'\rangle$ state vectors
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$\Downarrow$ $\Downarrow$

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Abstracting ket $|n'\rangle$ state vectors
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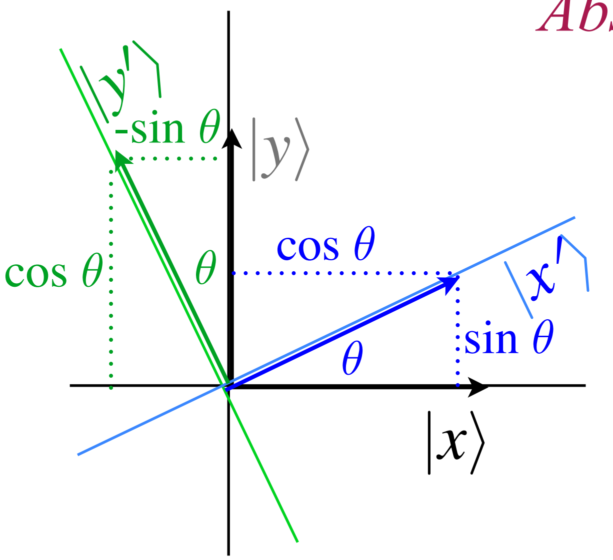
Transformation Matrix
$T_{m,n'} = \langle m|\, n'\rangle$



($\theta = +30°$)-Rotated kets $\{|\mathbf{x}'\rangle, |\mathbf{y}'\rangle\}$ or $\{\mathbf{x}', \mathbf{y}'\}$
represented in page-aligned $\{|\mathrm{x}\rangle, |\mathrm{y}\rangle\}$ basis.

"Abstraction" of bra and ket vectors from a Transformation Matrix

Ket or column vectors | *Bra or row vectors*

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

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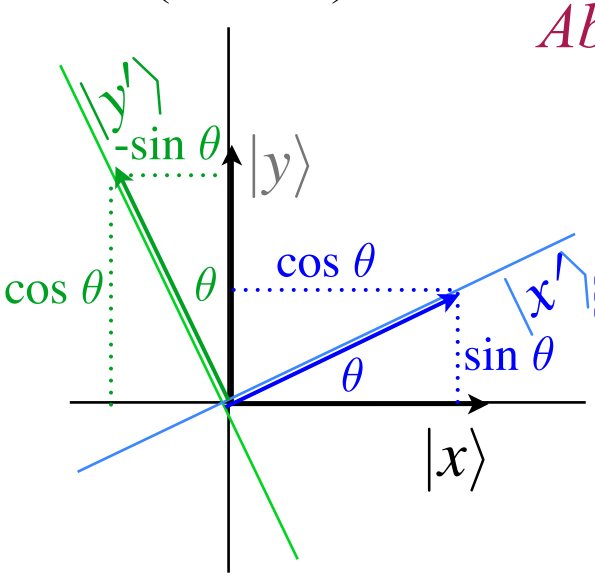

Abstracting ket $|\,n'\rangle$ *state vectors*
from
Transformation Matrix
$T_{m,n'} = \langle m|\,n'\rangle$

$$|x'\rangle = |x\rangle\langle x|x'\rangle + |y\rangle\langle y|x'\rangle$$
$$= |x\rangle(\cos\theta) + |y\rangle(\sin\theta)$$

($\theta = +30°$)-Rotated kets $\{|\mathbf{x}'\rangle, |\mathbf{y}'\rangle\}$ or $\{\mathbf{x}', \mathbf{y}'\}$
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"Abstraction" of bra and ket vectors from a Transformation Matrix

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$$\Downarrow \qquad \Downarrow$$

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Abstracting ket $|n'\rangle$ state vectors
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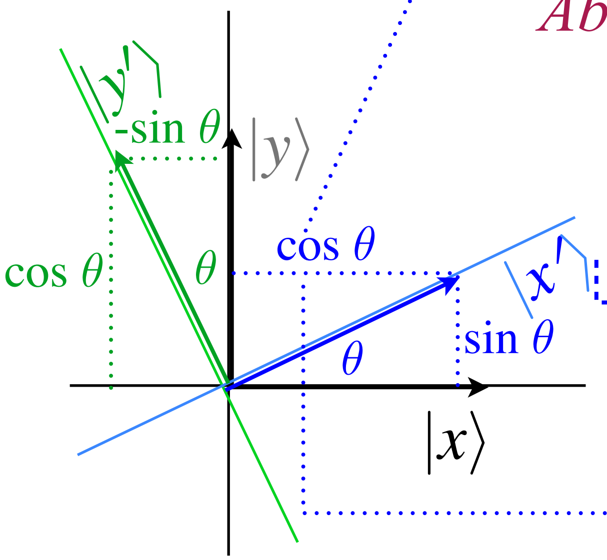
Transformation Matrix
$T_{m,n'} = \langle m|n'\rangle$



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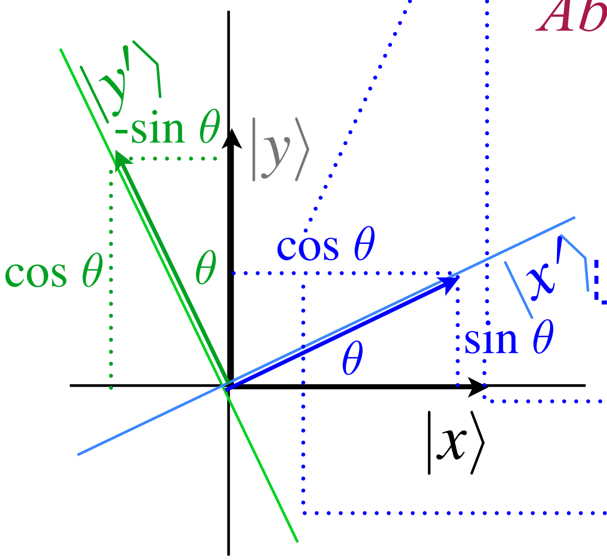

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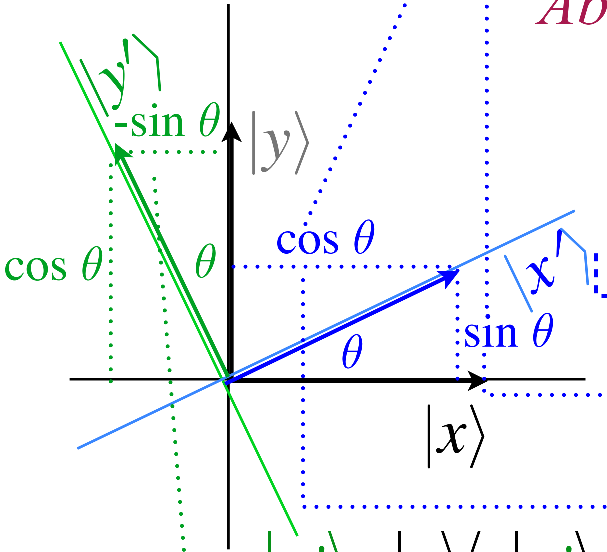
from
Transformation Matrix
$T_{m,n'} = \langle m|n'\rangle$



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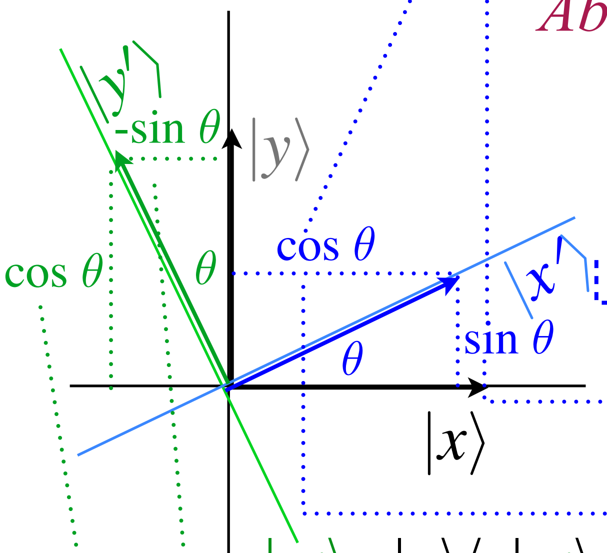

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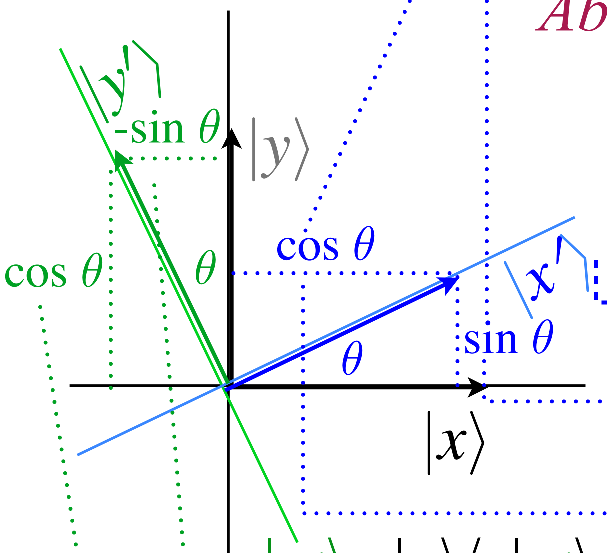

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Beam Sorters

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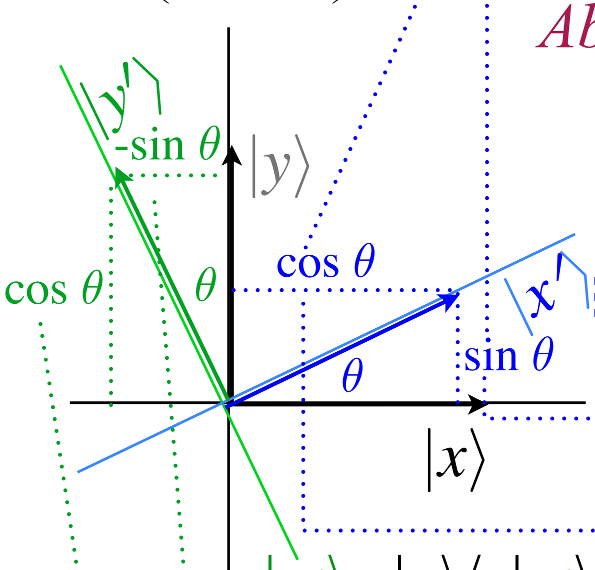
Abstracting ket $|n'\rangle$ state vectors from Transformation Matrix $T_{m,n'} = \langle m|n'\rangle$



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Bra or row vectors

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Abstracting bra $\langle m|$ state vectors from Transformation Matrix $T_{m,n'} = \langle m|n'\rangle$

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"Abstraction" of bra and ket vectors from a Transformation Matrix

Ket or column vectors | **Bra or row vectors**

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Abstracting bra $\langle m|$ state vectors from Transformation Matrix
$T_{m,n'} = \langle m|n'\rangle$

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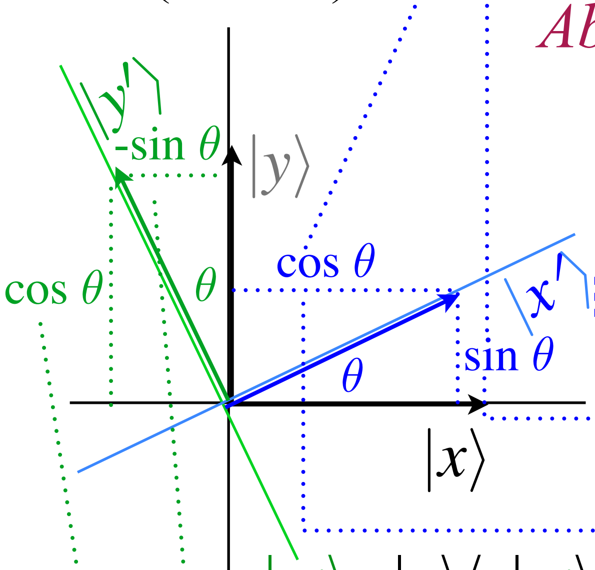
Abstracting ket $|n'\rangle$ state vectors from Transformation Matrix
$T_{m,n'} = \langle m|n'\rangle$



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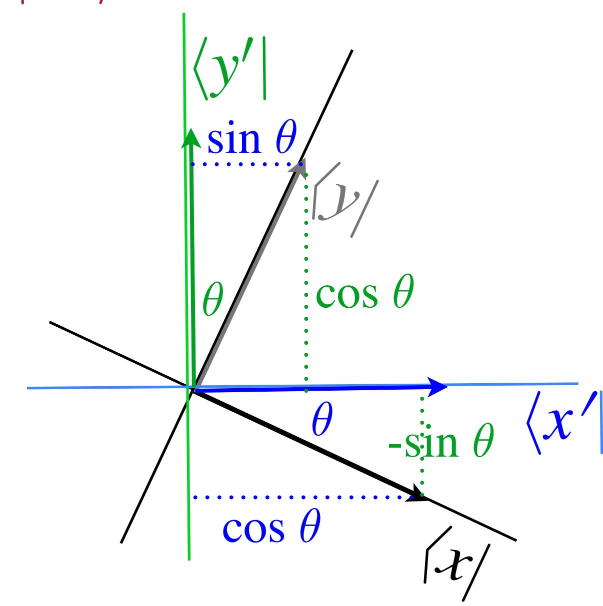

(θ=+30°)-Rotated kets $\{|\mathbf{x}'\rangle, |\mathbf{y}'\rangle\}$ or $\{\mathbf{x}', \mathbf{y}'\}$ represented in page-aligned $\{|\mathbf{x}\rangle, |\mathbf{y}\rangle\}$ basis.

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"Abstraction" of bra and ket vectors from a Transformation Matrix

Ket or column vectors | *Bra or row vectors*

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Abstracting bra $\langle m|$ state vectors
from
Transformation Matrix
$T_{m,n'} = \langle m|\ n'\rangle$

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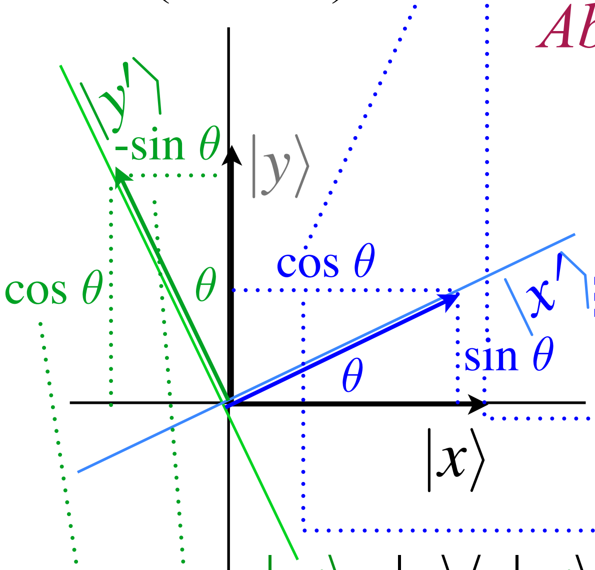
Abstracting ket $|\ n'\rangle$ state vectors
from
Transformation Matrix
$T_{m,n'} = \langle m|\ n'\rangle$



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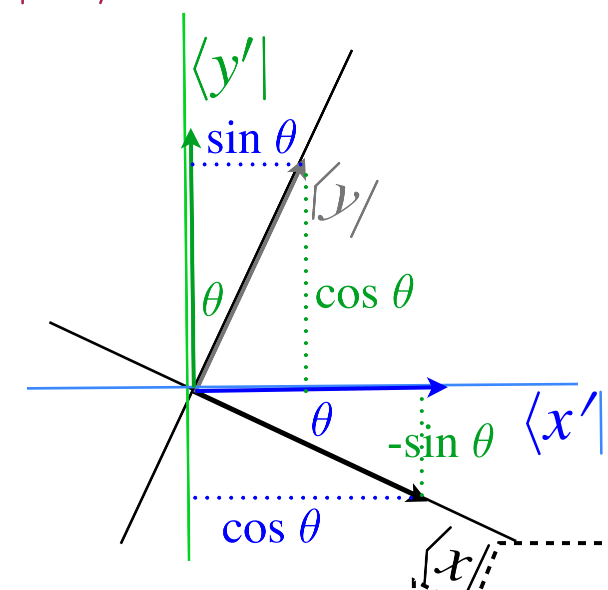

$$\langle x| = \langle x|x'\rangle\langle x'| + \langle x|y'\rangle\langle y'| = (\cos\theta)\langle x'| + (-\sin\theta)\langle y'|$$

(θ=+30°)-Rotated kets $\{|\mathbf{x}'\rangle, |\mathbf{y}'\rangle\}$ or $\{\mathbf{x}', \mathbf{y}'\}$ represented in page-aligned $\{|\mathbf{x}\rangle, |\mathbf{y}\rangle\}$ basis.

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"Abstraction" of bra and ket vectors from a Transformation Matrix

Ket or column vectors

Bra or row vectors

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Abstracting bra $\langle m|$ *state vectors*
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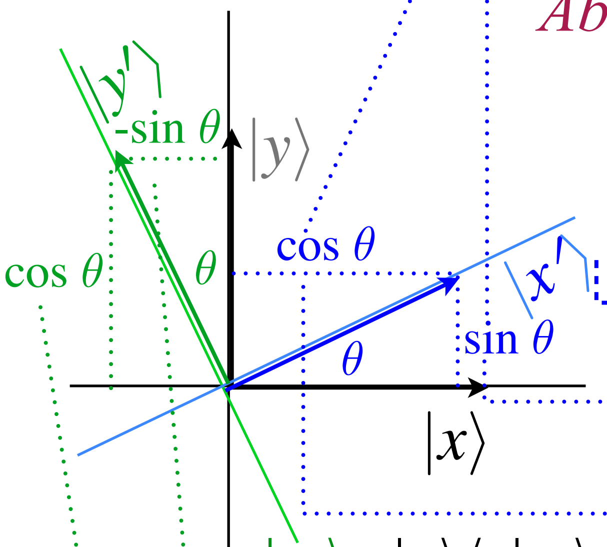
Transformation Matrix
$T_{m,n'}=\langle m|n'\rangle$

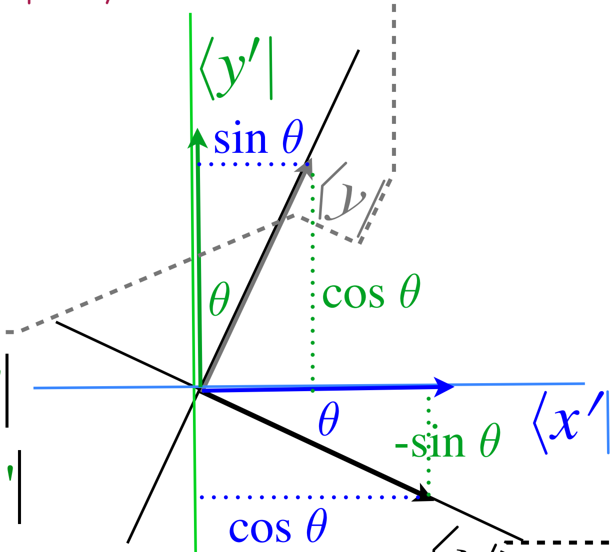
$$|x'\rangle = \begin{pmatrix}\langle x|x'\rangle\\ \langle y|x'\rangle\end{pmatrix} = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}, \quad |y'\rangle = \begin{pmatrix}\langle x|y'\rangle\\ \langle y|y'\rangle\end{pmatrix} = \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$$

Abstracting ket $|n'\rangle$ *state vectors*
from
Transformation Matrix
$T_{m,n'}=\langle m|n'\rangle$



$$|x'\rangle = |x\rangle\langle x|x'\rangle + |y\rangle\langle y|x'\rangle = |x\rangle(\cos\theta) + |y\rangle(\sin\theta)$$

$$|y'\rangle = |x\rangle\langle x|y'\rangle + |y\rangle\langle y|y'\rangle = |x\rangle(-\sin\theta) + |y\rangle(\cos\theta)$$



$$\langle y| = \langle y|x'\rangle\langle x'| + \langle y|y'\rangle\langle y'| = (\sin\theta)\langle x'| + (\cos\theta)\langle y'|$$

$$\langle x| = \langle x|x'\rangle\langle x'| + \langle x|y'\rangle\langle y'| = (\cos\theta)\langle x'| + (-\sin\theta)\langle y'|$$

(θ=+30°)-Rotated kets $\{|\mathbf{x}'\rangle, |\mathbf{y}'\rangle\}$ or $\{\mathbf{x}', \mathbf{y}'\}$ represented in page-aligned $\{|\mathbf{x}\rangle, |\mathbf{y}\rangle\}$ basis.

(θ=-30°)-Rotated bras $\{\langle\mathbf{x}|, \langle\mathbf{y}|\}$ or $\{\mathbf{x}, \mathbf{y}\}$ represented in page-aligned $\{|\mathbf{x}'\rangle, |\mathbf{y}'\rangle\}$ basis.

The same thing in Gibbs vector notation:

$$\mathbf{x}' = \mathbf{x}(\mathbf{x}\bullet\mathbf{x}') + \mathbf{y}(\mathbf{y}\bullet\mathbf{x}'), \quad \mathbf{y}' = \mathbf{x}(\mathbf{x}\bullet\mathbf{y}') + \mathbf{y}(\mathbf{y}\bullet\mathbf{y}'),$$
$$= \mathbf{x}(\cos\theta) + \mathbf{y}(\sin\theta), \qquad = \mathbf{x}(-\sin\theta) + \mathbf{y}(\cos\theta).$$

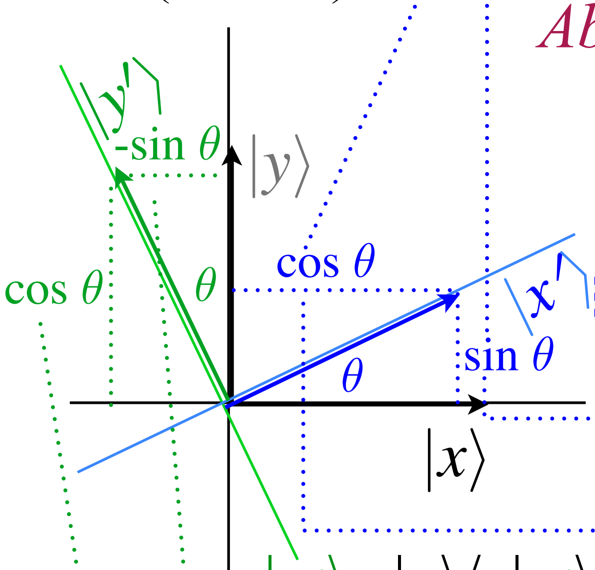
"Abstraction" of bra and ket vectors from a Transformation Matrix

Ket or column vectors

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$|x'\rangle = \begin{pmatrix} \langle x|x'\rangle \\ \langle y|x'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad |y'\rangle = \begin{pmatrix} \langle x|y'\rangle \\ \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

Abstracting ket $|n'\rangle$ state vectors from Transformation Matrix $T_{m,n'} = \langle m|n'\rangle$



$$|x'\rangle = |x\rangle\langle x|x'\rangle + |y\rangle\langle y|x'\rangle = |x\rangle(\cos\theta) + |y\rangle(\sin\theta)$$

$$|y'\rangle = |x\rangle\langle x|y'\rangle + |y\rangle\langle y|y'\rangle = |x\rangle(-\sin\theta) + |y\rangle(\cos\theta)$$

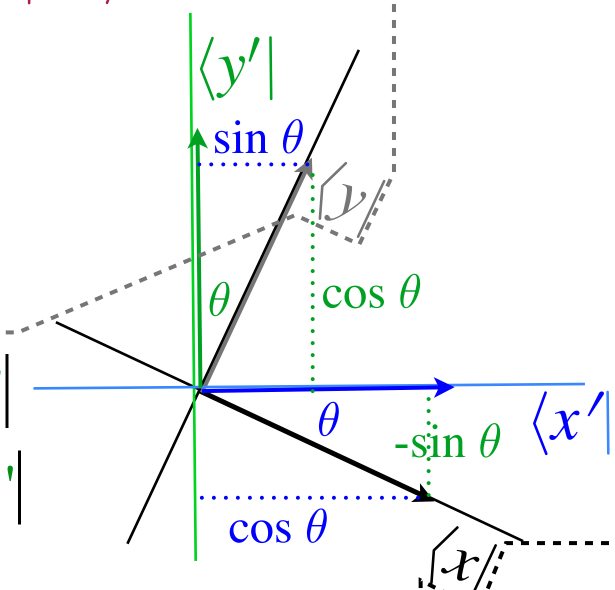
(θ=+30°)-Rotated kets $\{|\mathbf{x}'\rangle, |\mathbf{y}'\rangle\}$ or $\{\mathbf{x}', \mathbf{y}'\}$ represented in page-aligned $\{|\mathbf{x}\rangle, |\mathbf{y}\rangle\}$ basis.

Bra or row vectors

$$\Rightarrow \langle x| = \begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \end{pmatrix}$$

$$\Rightarrow \langle y| = \begin{pmatrix} \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \sin\theta & \cos\theta \end{pmatrix}$$

Abstracting bra $\langle m|$ state vectors from Transformation Matrix $T_{m,n'} = \langle m|n'\rangle$



$$\langle y| = \langle y|x'\rangle\langle x'| + \langle y|y'\rangle\langle y'| = (\sin\theta)\langle x'| + (\cos\theta)\langle y'|$$

$$\langle x| = \langle x|x'\rangle\langle x'| + \langle x|y'\rangle\langle y'| = (\cos\theta)\langle x'| + (-\sin\theta)\langle y'|$$

(θ=-30°)-Rotated bras $\{\langle\mathbf{x}|, \langle\mathbf{y}|\}$ or $\{\mathbf{x}, \mathbf{y}\}$ represented in page-aligned $\{|\mathbf{x}'\rangle, |\mathbf{y}'\rangle\}$ basis.

The same thing in Gibbs vector notation:

$$\mathbf{x}' = \mathbf{x}(\mathbf{x}\bullet\mathbf{x}') + \mathbf{y}(\mathbf{y}\bullet\mathbf{x}'), \quad \mathbf{y}' = \mathbf{x}(\mathbf{x}\bullet\mathbf{y}') + \mathbf{y}(\mathbf{y}\bullet\mathbf{y}'),$$

$$= \mathbf{x}(\cos\theta) + \mathbf{y}(\sin\theta), \quad = \mathbf{x}(-\sin\theta) + \mathbf{y}(\cos\theta).$$

The same thing in Gibbs vector notation:

$$\mathbf{x} = (\mathbf{x}\bullet\mathbf{x}')\mathbf{x}' + (\mathbf{x}\bullet\mathbf{y}')\mathbf{y}', \quad \mathbf{y} = (\mathbf{y}\bullet\mathbf{x}')\mathbf{x}' + (\mathbf{y}\bullet\mathbf{y}')\mathbf{y}',$$

$$\mathbf{x} = (\cos\theta)\mathbf{x}' + (-\sin\theta)\mathbf{y}', \quad \mathbf{y} = (\sin\theta)\mathbf{x}' + (\cos\theta)\mathbf{y}'.$$

"Abstraction" of bra and ket vectors from a Transformation Matrix

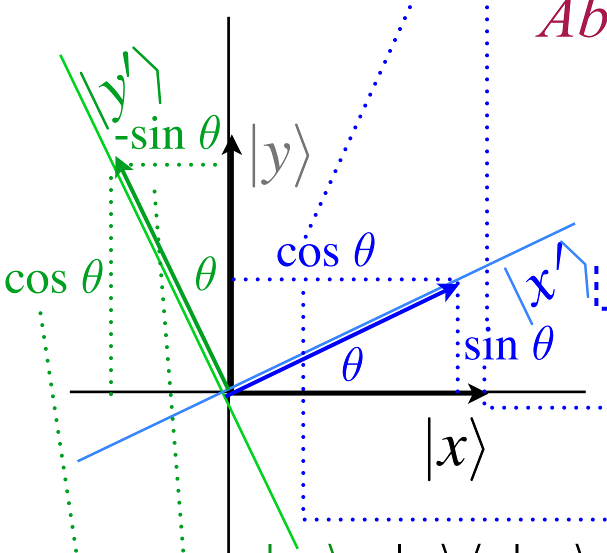
Ket or column vectors

Given Transformation Matrix $T_{m,n'}$:

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$|x'\rangle = \begin{pmatrix} \langle x|x'\rangle \\ \langle y|x'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad |y'\rangle = \begin{pmatrix} \langle x|y'\rangle \\ \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

Abstracting ket $|n'\rangle$ state vectors from Transformation Matrix $T_{m,n'} = \langle m|n'\rangle$



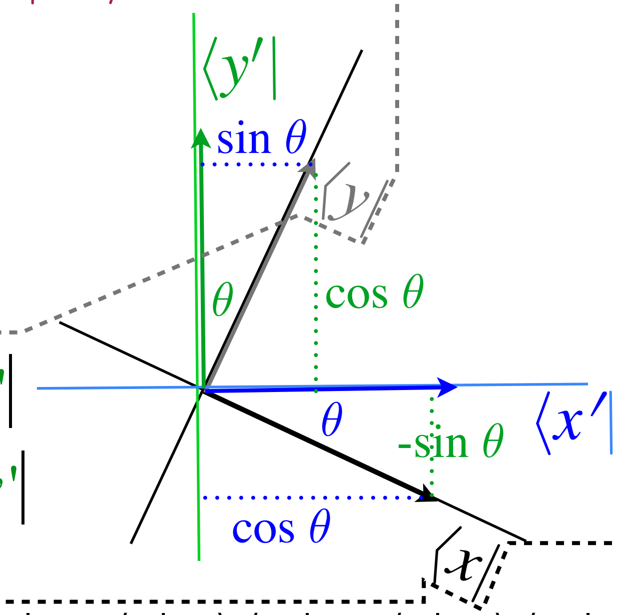
$$|x'\rangle = |x\rangle\langle x|x'\rangle + |y\rangle\langle y|x'\rangle$$
$$= |x\rangle(\cos\theta) + |y\rangle(\sin\theta)$$

$$|y'\rangle = |x\rangle\langle x|y'\rangle + |y\rangle\langle y|y'\rangle$$
$$= |x\rangle(-\sin\theta) + |y\rangle(\cos\theta)$$

Bra or row vectors

$$\langle x| = \begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \end{pmatrix}$$
$$\langle y| = \begin{pmatrix} \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \sin\theta & \cos\theta \end{pmatrix}$$

Abstracting bra $\langle m|$ state vectors from Transformation Matrix $T_{m,n'} = \langle m|n'\rangle$



$$\langle y| = \langle y|x'\rangle\langle x'| + \langle y|y'\rangle\langle y'|$$
$$= (\sin\theta)\langle x'| + (\cos\theta)\langle y'|$$

$$\langle x| = \langle x|x'\rangle\langle x'| + \langle x|y'\rangle\langle y'|$$
$$= (\cos\theta)\langle x'| + (-\sin\theta)\langle y'|$$

(θ=+30°)-Rotated kets $\{|\mathbf{x}'\rangle, |\mathbf{y}'\rangle\}$ or $\{\mathbf{x}', \mathbf{y}'\}$ represented in page-aligned $\{|\mathbf{x}\rangle, |\mathbf{y}\rangle\}$ basis.

(θ=-30°)-Rotated bras $\{\langle\mathbf{x}|, \langle\mathbf{y}|\}$ or $\{\mathbf{x}, \mathbf{y}\}$ represented in page-aligned $\{|\mathbf{x}'\rangle, |\mathbf{y}'\rangle\}$ basis.

Ket vector algebra has the order of $T_{m,n'}$ transposed

$$|x'\rangle = |x\rangle\langle x|x'\rangle + |y\rangle\langle y|x'\rangle = |x\rangle(\cos\theta) + |y\rangle(\sin\theta)$$
$$|y'\rangle = |x\rangle\langle x|y'\rangle + |y\rangle\langle y|y'\rangle = |x\rangle(-\sin\theta) + |y\rangle(\cos\theta)$$

Bra vector algebra has the same order as $T_{m,n'}$

$$\langle x| = \langle x|x'\rangle\langle x'| + \langle x|y'\rangle\langle y'| = (\cos\theta)\langle x'| + (-\sin\theta)\langle y'|$$
$$\langle y| = \langle y|x'\rangle\langle x'| + \langle y|y'\rangle\langle y'| = (\sin\theta)\langle x'| + (\cos\theta)\langle y'|$$

Unit vector kets $|\mathbf{x}\rangle$ and $|\mathrm{y}\rangle$ or $\mathbf{x}'$ and $\mathbf{y}'$ are represented (in their own $|\mathrm{x}\rangle$and$|\mathrm{y}\rangle$ basis) as follows.

$$|x\rangle = \begin{pmatrix} \langle x|x\rangle \\ \langle y|x\rangle \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |y\rangle = \begin{pmatrix} \langle x|y\rangle \\ \langle y|y\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Beam Sorters

2-State Sorters: spin-1/2 vs. optical polarization

Geometry of optical polarization selection and Brewster's angle

Feynman's lever

Beam Sorters in Series and Transformation Matrices

Introducing Dirac bra-ket notation

"Abstraction" of bra and ket vectors from a Transformation Matrix

→ *Introducing scalar and matrix products*

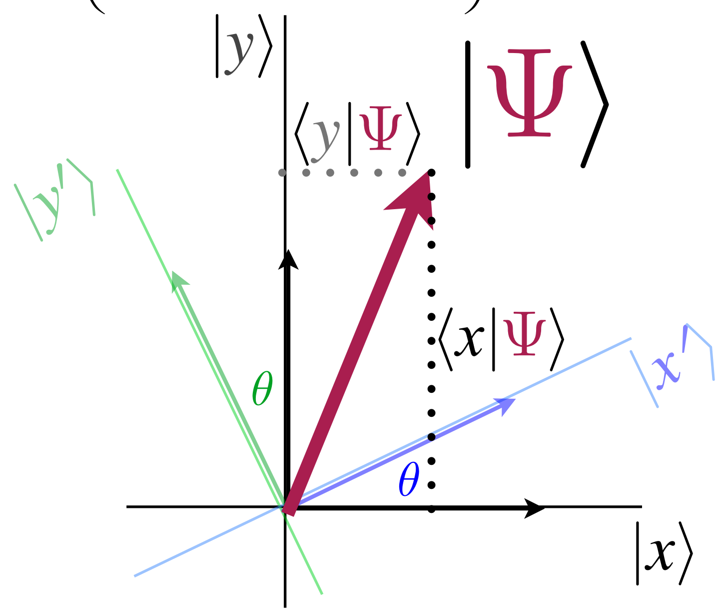
Transformation matrix $T_{m,n'} = \langle m | n' \rangle$ is array of dot or *scalar products* (dot products) of unit vectors or *direction cosines*.

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} (\mathbf{x}\bullet\mathbf{x'}) & (\mathbf{x}\bullet\mathbf{y'}) \\ (\mathbf{y}\bullet\mathbf{x'}) & (\mathbf{y}\bullet\mathbf{y'}) \end{pmatrix}$$

$\{\langle x|, \langle y|\}$ components of $|\Psi\rangle$:

$\langle x|\Psi\rangle = \Psi_x$

$\langle y|\Psi\rangle = \Psi_y$

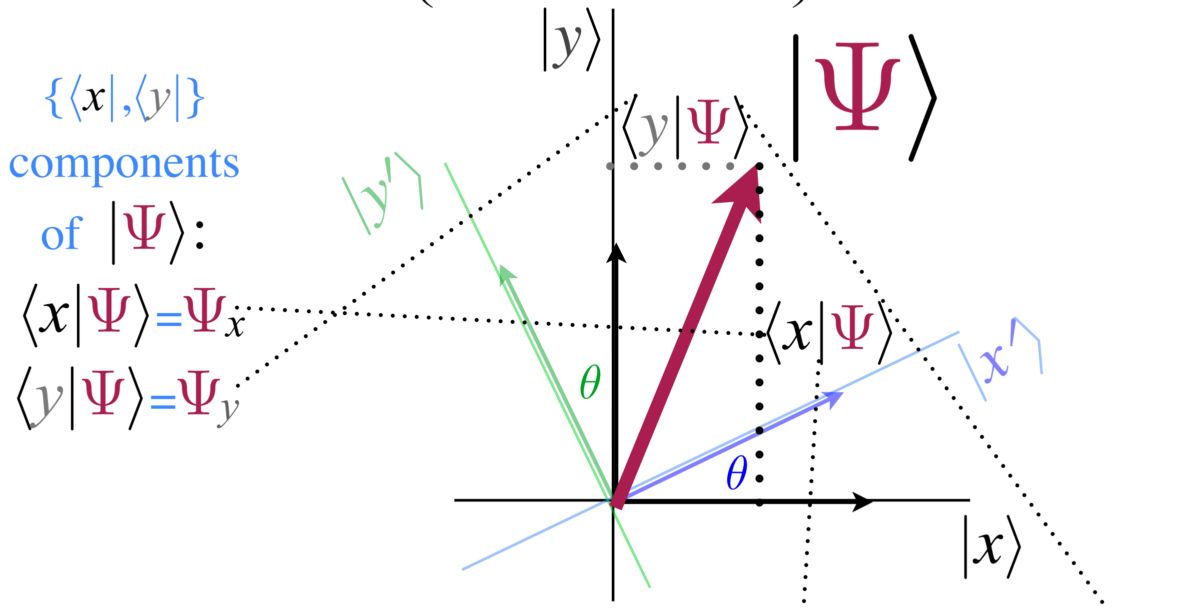


Any state $|\Psi\rangle$ can be expanded in any basis $\{\langle x|, \langle y|\}$

$$|\Psi\rangle = |x\rangle\langle x|\Psi\rangle + |y\rangle\langle y|\Psi\rangle$$

Transformation matrix $T_{m,n'} = \langle m | n' \rangle$ is array of dot or *scalar products* (dot products) of unit vectors or *direction cosines*.

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} (\mathbf{x}\bullet\mathbf{x}') & (\mathbf{x}\bullet\mathbf{y}') \\ (\mathbf{y}\bullet\mathbf{x}') & (\mathbf{y}\bullet\mathbf{y}') \end{pmatrix}$$



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$$|\Psi\rangle = |x\rangle\langle x|\Psi\rangle + |y\rangle\langle y|\Psi\rangle$$

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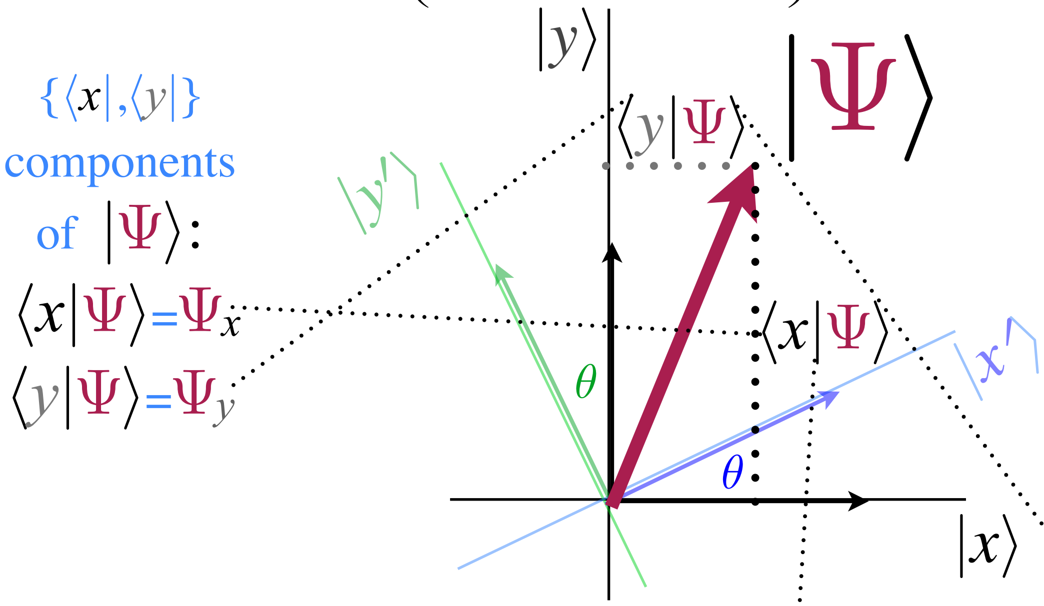
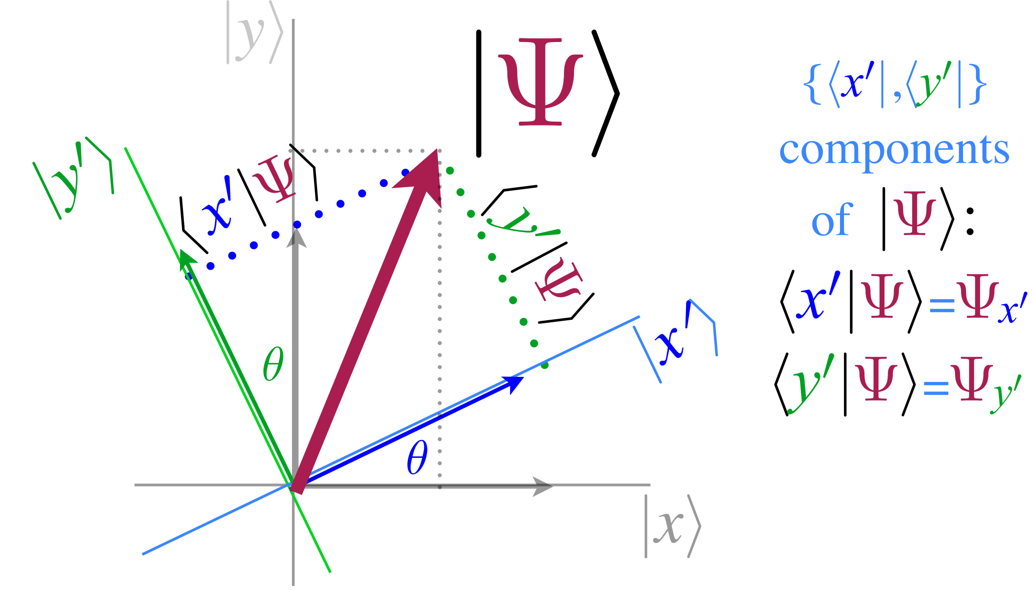


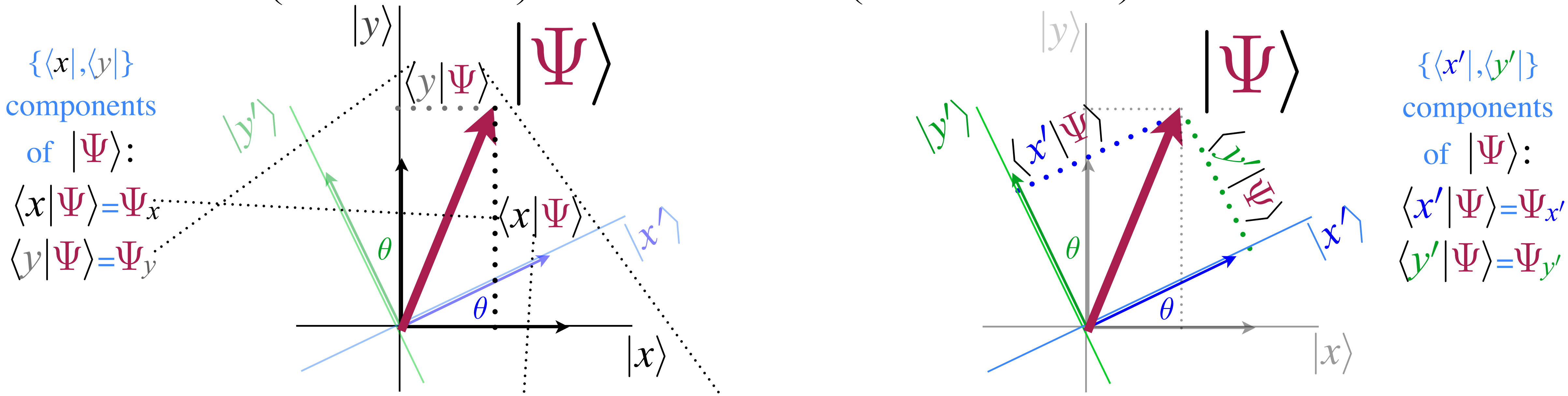


Any state $|\Psi\rangle$ can be expanded in any basis $\{\langle x|,\langle y|\}$, or $\{\langle x'|,\langle y'|\}$, ...*etc*.

$$|\Psi\rangle = |x\rangle\langle x|\Psi\rangle + |y\rangle\langle y|\Psi\rangle = |x'\rangle\langle x'|\Psi\rangle + |y'\rangle\langle y'|\Psi\rangle$$

Transformation matrix $T_{m,n'} = \langle m | n' \rangle$ is array of dot or *scalar products* (dot products) of unit vectors or *direction cosines*.

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} (\mathbf{x}\bullet\mathbf{x}') & (\mathbf{x}\bullet\mathbf{y}') \\ (\mathbf{y}\bullet\mathbf{x}') & (\mathbf{y}\bullet\mathbf{y}') \end{pmatrix}$$



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$$|\Psi\rangle = |x\rangle\langle x|\Psi\rangle + |y\rangle\langle y|\Psi\rangle = |x'\rangle\langle x'|\Psi\rangle + |y'\rangle\langle y'|\Psi\rangle$$

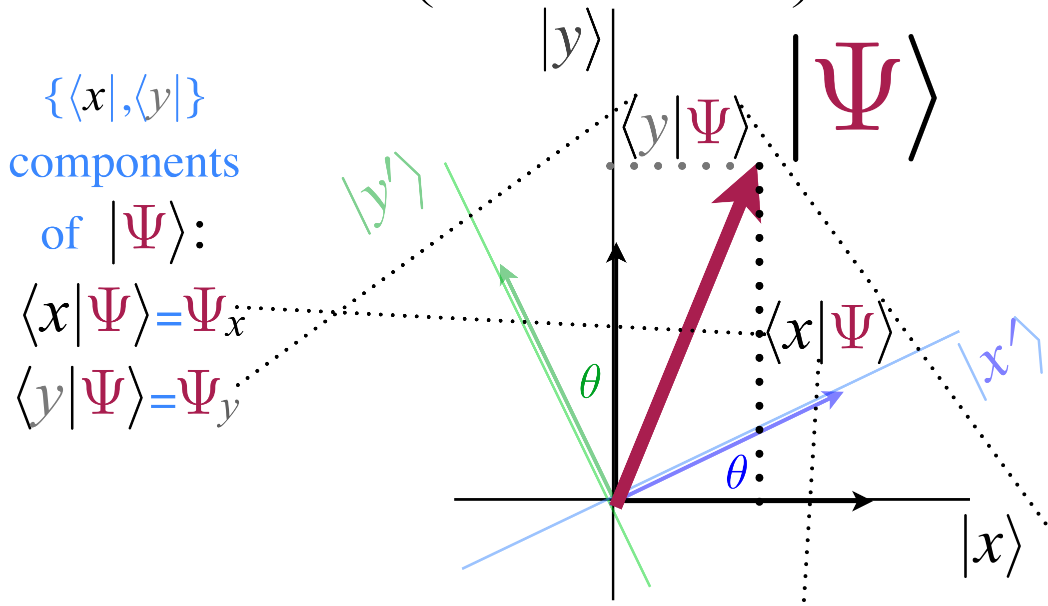
Transformation matrix $T_{m,n'}$ relates $\{\langle x|\Psi\rangle, \langle y|\Psi\rangle\}$ amplitudes to $\{\langle x'|\Psi\rangle, \langle y'|\Psi\rangle\}$.

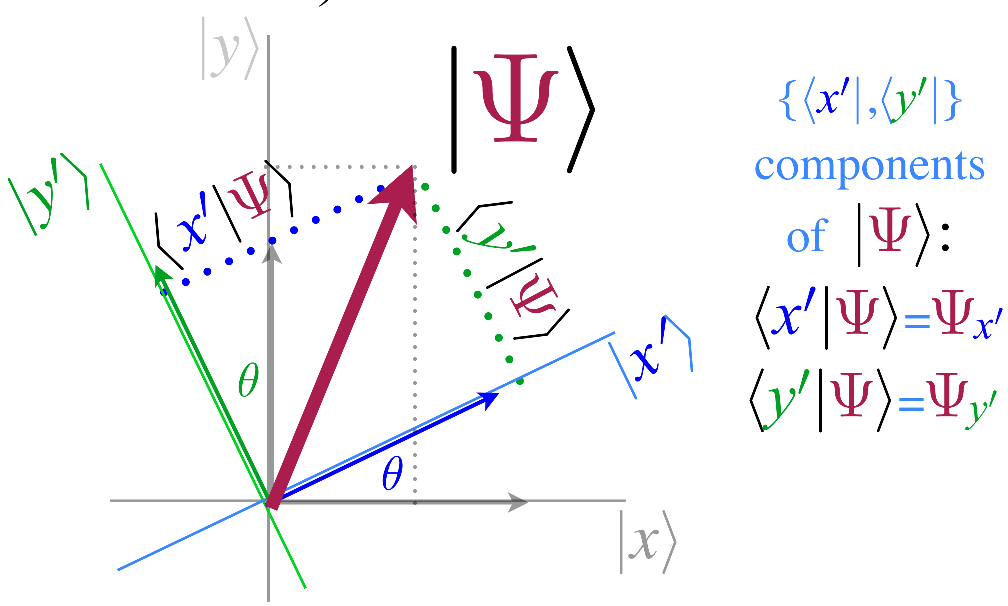
$$\begin{pmatrix} \langle x|\Psi\rangle \\ \langle y|\Psi\rangle \end{pmatrix} = \begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} \begin{pmatrix} \langle x'|\Psi\rangle \\ \langle y'|\Psi\rangle \end{pmatrix} \quad \text{or:} \quad \begin{pmatrix} \Psi_x \\ \Psi_y \end{pmatrix} = \begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} \begin{pmatrix} \Psi_{x'} \\ \Psi_{y'} \end{pmatrix}$$

Hybrid Gibbs-Dirac notation *(Ug-ly!)*

Transformation matrix $T_{m,n'}=\langle m|n'\rangle$ is array of dot or *scalar products* (dot products) of unit vectors or *direction cosines*.

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} (\mathbf{x}\bullet\mathbf{x'}) & (\mathbf{x}\bullet\mathbf{y'}) \\ (\mathbf{y}\bullet\mathbf{x'}) & (\mathbf{y}\bullet\mathbf{y'}) \end{pmatrix}$$





Any state $|\Psi\rangle$ can be expanded in any basis $\{\langle x|,\langle y|\}$, or $\{\langle x'|,\langle y'|\}$, ...*etc*.

$$|\Psi\rangle=|x\rangle\langle x|\Psi\rangle + |y\rangle\langle y|\Psi\rangle = |x'\rangle\langle x'|\Psi\rangle + |y'\rangle\langle y'|\Psi\rangle$$

Transformation matrix $T_{m,n'}$ relates $\{\langle x|\Psi\rangle,\langle y|\Psi\rangle\}$ amplitudes to $\{\langle x'|\Psi\rangle,\langle y'|\Psi\rangle\}$.

$$\begin{pmatrix} \langle x|\Psi\rangle \\ \langle y|\Psi\rangle \end{pmatrix} = \begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix}\begin{pmatrix} \langle x'|\Psi\rangle \\ \langle y'|\Psi\rangle \end{pmatrix} \quad \text{or:} \quad \begin{pmatrix} \Psi_x \\ \Psi_y \end{pmatrix} = \begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix}\begin{pmatrix} \Psi_{x'} \\ \Psi_{y'} \end{pmatrix}$$

Hybrid Gibbs-Dirac notation *(Ug-ly!)*

Proof: $\langle x|=\langle x|x'\rangle\langle x'|+\langle x|y'\rangle\langle y'|$ implies: $\langle x|\Psi\rangle=\langle x|x'\rangle\langle x'|\Psi\rangle+\langle x|y'\rangle\langle y'|\Psi\rangle$

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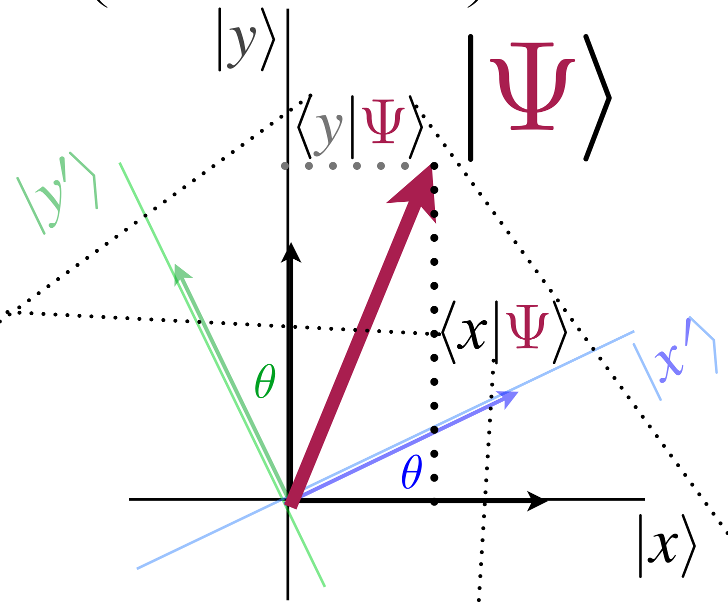
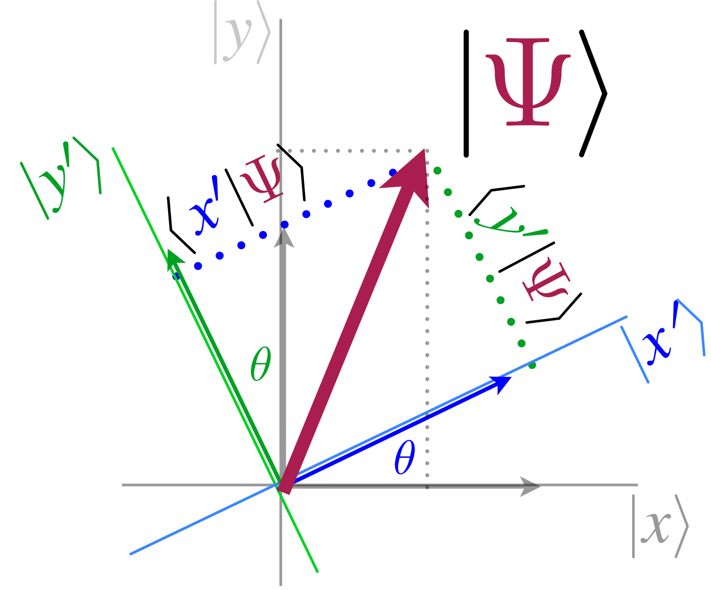
$\{\langle x|,\langle y|\}$ components of $|\Psi\rangle$:

$\langle x|\Psi\rangle = \Psi_x$

$\langle y|\Psi\rangle = \Psi_y$





$\{\langle x'|,\langle y'|\}$ components of $|\Psi\rangle$:

$\langle x'|\Psi\rangle = \Psi_{x'}$

$\langle y'|\Psi\rangle = \Psi_{y'}$

Any state $|\Psi\rangle$ can be expanded in any basis $\{\langle x|,\langle y|\}$, or $\{\langle x'|,\langle y'|\}$, ...*etc*.

$$|\Psi\rangle = |x\rangle\langle x|\Psi\rangle + |y\rangle\langle y|\Psi\rangle = |x'\rangle\langle x'|\Psi\rangle + |y'\rangle\langle y'|\Psi\rangle$$

Transformation matrix $T_{m,n'}$ relates $\{\langle x|\Psi\rangle,\langle y|\Psi\rangle\}$ amplitudes to $\{\langle x'|\Psi\rangle,\langle y'|\Psi\rangle\}$.

$$\begin{pmatrix} \langle x|\Psi\rangle \\ \langle y|\Psi\rangle \end{pmatrix} = \begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} \begin{pmatrix} \langle x'|\Psi\rangle \\ \langle y'|\Psi\rangle \end{pmatrix} \quad \text{or:} \quad \begin{pmatrix} \Psi_x \\ \Psi_y \end{pmatrix} = \begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} \begin{pmatrix} \Psi_{x'} \\ \Psi_{y'} \end{pmatrix}$$

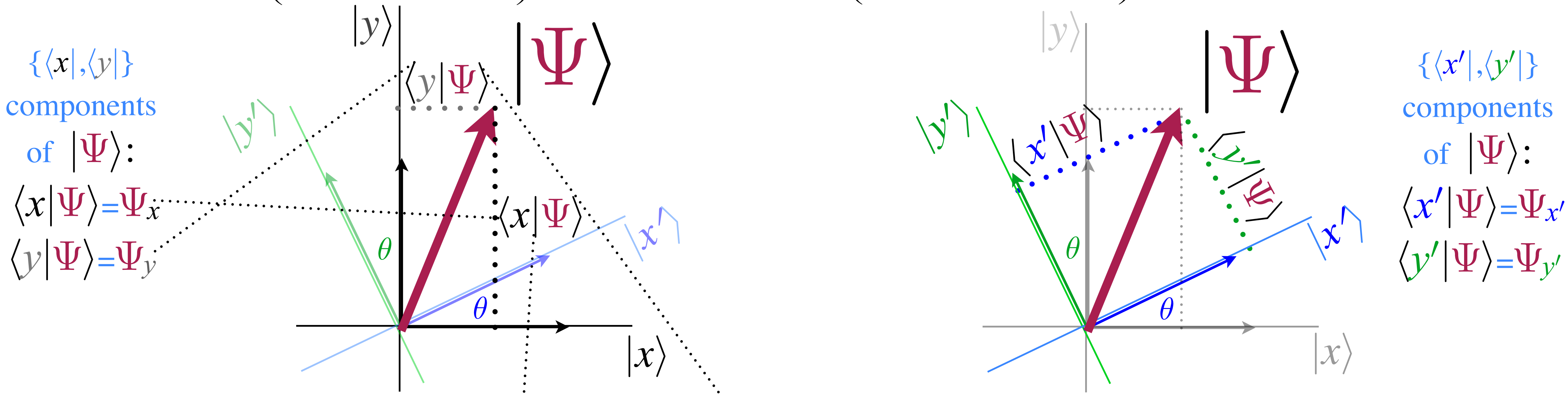
Hybrid Gibbs-Dirac notation *(Ug-ly!)*

Proof: $\langle x| = \langle x|x'\rangle\langle x'| + \langle x|y'\rangle\langle y'|$ implies: $\langle x|\Psi\rangle = \langle x|x'\rangle\langle x'|\Psi\rangle + \langle x|y'\rangle\langle y'|\Psi\rangle$

$\langle y| = \langle y|x'\rangle\langle x'| + \langle y|y'\rangle\langle y'|$ implies: $\langle y|\Psi\rangle = \langle y|x'\rangle\langle x'|\Psi\rangle + \langle y|y'\rangle\langle y'|\Psi\rangle$

Transformation matrix $T_{m,n'} = \langle m | n' \rangle$ is array of dot or *scalar products* (dot products) of unit vectors or *direction cosines*.

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} (\mathbf{x}\bullet\mathbf{x}') & (\mathbf{x}\bullet\mathbf{y}') \\ (\mathbf{y}\bullet\mathbf{x}') & (\mathbf{y}\bullet\mathbf{y}') \end{pmatrix}$$



Any state $|\Psi\rangle$ can be expanded in any basis $\{\langle x|,\langle y|\}$, or $\{\langle x'|,\langle y'|\}$, ...*etc*.

$$|\Psi\rangle = |x\rangle\langle x|\Psi\rangle + |y\rangle\langle y|\Psi\rangle = |x'\rangle\langle x'|\Psi\rangle + |y'\rangle\langle y'|\Psi\rangle$$

Transformation matrix $T_{m,n'}$ relates $\{\langle x|\Psi\rangle,\langle y|\Psi\rangle\}$ amplitudes to $\{\langle x'|\Psi\rangle,\langle y'|\Psi\rangle\}$.

$$\begin{pmatrix} \langle x|\Psi\rangle \\ \langle y|\Psi\rangle \end{pmatrix} = \begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} \begin{pmatrix} \langle x'|\Psi\rangle \\ \langle y'|\Psi\rangle \end{pmatrix} \quad \text{or:} \quad \begin{pmatrix} \Psi_x \\ \Psi_y \end{pmatrix} = \begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} \begin{pmatrix} \Psi_{x'} \\ \Psi_{y'} \end{pmatrix}$$

Hybrid Gibbs-Dirac notation *(Ug-ly!)*

Inverse (†=T*=-1) matrix $T_{n',m}$ relates $\{\langle x'|\Psi\rangle,\langle y'|\Psi\rangle\}$ amplitudes to $\{\langle x|\Psi\rangle,\langle y|\Psi\rangle\}$.

$$\begin{pmatrix} \langle x'|\Psi\rangle \\ \langle y'|\Psi\rangle \end{pmatrix} = \begin{pmatrix} \langle x'|x\rangle & \langle x'|y\rangle \\ \langle y'|x\rangle & \langle y'|y\rangle \end{pmatrix} \begin{pmatrix} \langle x|\Psi\rangle \\ \langle y|\Psi\rangle \end{pmatrix} \quad \text{or:} \quad \begin{pmatrix} \Psi_{x'} \\ \Psi_{y'} \end{pmatrix} = \begin{pmatrix} \langle x'|x\rangle & \langle x'|y\rangle \\ \langle y'|x\rangle & \langle y'|y\rangle \end{pmatrix} \begin{pmatrix} \Psi_x \\ \Psi_y \end{pmatrix}$$

Hybrid Gibbs-Dirac notation *(Still Ug-ly!)*