Group Theory in Quantum Mechanics

Lecture 23 *(4.16.13)*

Harmonic oscillator symmetry U(1)⊂U(2)⊂U(3)...

(Int.J.Mol.Sci, 14, 714(2013) p.755-774 , QTCA Unit 7 Ch. 21-22)
(PSDS - Ch. 8)

Review : *1-D* $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(1) representations*
Review : *Translate* **T***(a) and/or Boost* **B***(b) to construct coherent state*
Review : *Time evolution of coherent state (and "squeezed" states)*

2-D $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(2) representations and R(3) angular momentum operators*

2D-Oscillator basic states and operations
Commutation relations
Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry
Anti-commutation relations
Two-dimensional (or 2-particle) base states: ket-kets and bra-bras
Outer product arrays
Entangled 2-particle states

Mostly Notation and Bookkeeping :

Two-particle (or 2-dimensional) matrix operators
U(2) Hamiltonian and irreducible representations
2D-Oscillator states and related 3D angular momentum multiplets
ND multiplets

R(3) Angular momentum generators by U(2) analysis
Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$
SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

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Review : *1-D $\mathbf{a}^\dagger\mathbf{a}$ algebra of U(1) representations*

Define

$$\mathbf{a} = \frac{(\mathbf{X}+i\mathbf{P})}{\sqrt{\hbar\omega}} = \frac{\left(\sqrt{M\omega}\,\mathbf{x}+i\mathbf{p}/\sqrt{M\omega}\right)}{\sqrt{2\hbar}}$$

Destruction operator

and

$$\mathbf{a}^\dagger = \frac{(\mathbf{X}-i\mathbf{P})}{\sqrt{\hbar\omega}} = \frac{\left(\sqrt{M\omega}\,\mathbf{x}-i\mathbf{p}/\sqrt{M\omega}\right)}{\sqrt{2\hbar}}$$

Creation Operator

Commutation relations between $\mathbf{a} = (\mathbf{X}+i\mathbf{P})/2$ and $\mathbf{a}^\dagger = (\mathbf{X}-i\mathbf{P})/2$ with $\mathbf{X}\equiv\sqrt{M\omega}\mathbf{x}/\sqrt{2}$ and $\mathbf{P}\equiv\mathbf{p}/\sqrt{2M}$:

$$\left[\mathbf{a},\mathbf{a}^\dagger\right] \equiv \mathbf{a}\mathbf{a}^\dagger - \mathbf{a}^\dagger\mathbf{a} = \frac{1}{2\hbar}\left(\sqrt{M\omega}\,\mathbf{x}+i\mathbf{p}/\sqrt{M\omega}\right)\left(\sqrt{M\omega}\,\mathbf{x}-i\mathbf{p}/\sqrt{M\omega}\right) - \frac{1}{2\hbar}\left(\sqrt{M\omega}\,\mathbf{x}-i\mathbf{p}/\sqrt{M\omega}\right)\left(\sqrt{M\omega}\,\mathbf{x}+i\mathbf{p}/\sqrt{M\omega}\right)$$

$$\left[\mathbf{a},\mathbf{a}^\dagger\right] = \frac{2i}{2\hbar}(\mathbf{p}\mathbf{x}-\mathbf{x}\mathbf{p}) = \frac{-i}{\hbar}\left[\mathbf{x},\mathbf{p}\right] = \mathbf{1}$$

$$\boxed{\left[\mathbf{a},\mathbf{a}^\dagger\right] = \mathbf{1}}$$ or $$\boxed{\mathbf{a}\mathbf{a}^\dagger = \mathbf{a}^\dagger\mathbf{a} + \mathbf{1}}$$

$$\left[\mathbf{x},\mathbf{p}\right] \equiv \mathbf{x}\mathbf{p} - \mathbf{p}\mathbf{x} = \hbar i\,\mathbf{1}$$

Review : *1-D* **a†a** *algebra of U(1) representations*

1st excited state wavefunction $\psi_1(x) = \langle x | 1 \rangle$

$$\langle x | \mathbf{a}^\dagger | 0 \rangle = \langle x | 1 \rangle = \psi_1(x)$$

Expanding the creation operator

$$\langle x | \mathbf{a}^\dagger | 0 \rangle = \frac{1}{\sqrt{2\hbar}} \left(\sqrt{M\omega}\, \langle x | \mathbf{x} | 0 \rangle - i \langle x | \mathbf{p} | 0 \rangle / \sqrt{M\omega} \right) = \langle x | 1 \rangle = \psi_1(x)$$

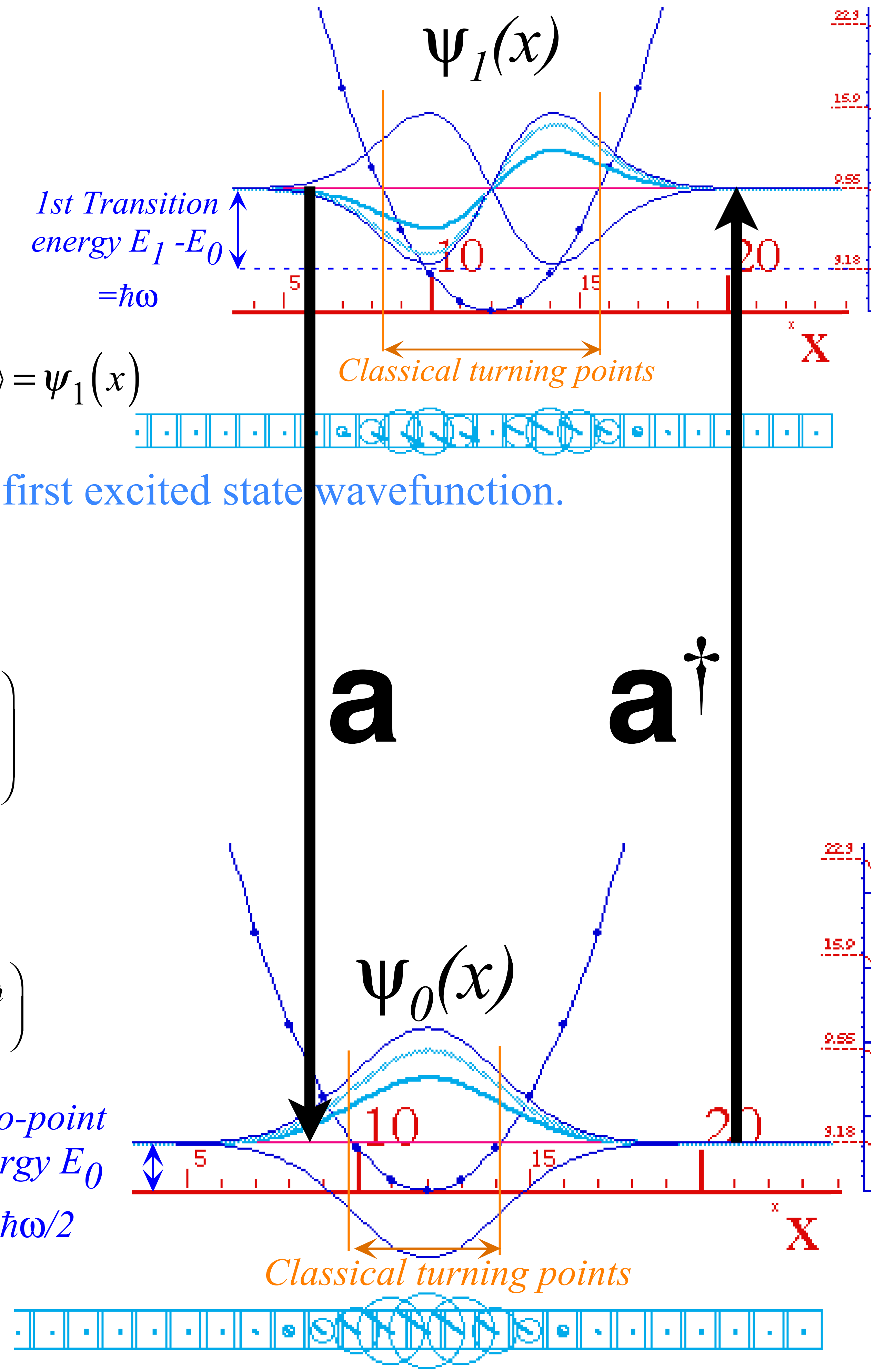
The operator coordinate representations generate the first excited state wavefunction.

$$\langle x | 1 \rangle = \psi_1(x) = \frac{1}{\sqrt{2\hbar}} \left(\sqrt{M\omega}\; x\psi_0(x) - i \frac{\hbar}{i} \frac{\partial \psi_0(x)}{\partial x} / \sqrt{M\omega} \right)$$

$$= \frac{1}{\sqrt{2\hbar}} \left(\sqrt{M\omega}\; x \frac{e^{-M\omega x^2/2\hbar}}{const.} - i \frac{\hbar}{i} \frac{\partial}{\partial x} \frac{e^{-M\omega x^2/2\hbar}}{const.} / \sqrt{M\omega} \right)$$

$$= \frac{1}{\sqrt{2\hbar}} \frac{e^{-M\omega x^2/2\hbar}}{const.} \left(\sqrt{M\omega}\; x + i \frac{\hbar}{i} \frac{M\omega x}{\hbar} / \sqrt{M\omega} \right)$$

$$= \frac{\sqrt{M\omega}}{\sqrt{2\hbar}} \frac{e^{-M\omega x^2/2\hbar}}{const.} (2x) = \left(\frac{M\omega}{\pi\hbar} \right)^{3/4} \sqrt{2\pi} \left(x\, e^{-M\omega x^2/2\hbar} \right)$$



Review : *1-D $\mathbf{a}^\dagger\mathbf{a}$ algebra of U(1) representations*

Derive normalization for n^{th} state obtained by $(\mathbf{a}^\dagger)^n$ operator:

Use: $\mathbf{a}^n\mathbf{a}^{\dagger n} = n!\left(\mathbf{1} + n\mathbf{a}^\dagger\mathbf{a} + \frac{n(n-1)}{2!\cdot 2!}\mathbf{a}^{\dagger 2}\mathbf{a}^2 + \ldots\right)$

$$|n\rangle = \frac{\mathbf{a}^{\dagger n}|0\rangle}{const.},\quad \text{where:}\quad 1 = \langle n|n\rangle = \frac{\langle 0|\mathbf{a}^n\mathbf{a}^{\dagger n}|0\rangle}{(const.)^2} = n!\frac{\langle 0|\mathbf{1}+n\mathbf{a}^\dagger\mathbf{a}+..|0\rangle}{(const.)^2} = \frac{n!}{(const.)^2}$$

$$|n\rangle = \frac{\mathbf{a}^{\dagger n}|0\rangle}{\sqrt{n!}}$$ *Root-factorial normalization*

Use: $\mathbf{a}\mathbf{a}^{\dagger n} = n\mathbf{a}^{\dagger n-1} + \mathbf{a}^{\dagger n}\mathbf{a}$

Apply creation $\mathbf{a}^\dagger$:

$$\mathbf{a}^\dagger|n\rangle = \frac{\mathbf{a}^{\dagger n+1}|0\rangle}{\sqrt{n!}} = \sqrt{n+1}\,\frac{\mathbf{a}^{\dagger n+1}|0\rangle}{\sqrt{(n+1)!}}$$

Apply destruction $\mathbf{a}$:

$$\mathbf{a}|n\rangle = \frac{\mathbf{a}\mathbf{a}^{\dagger n}|0\rangle}{\sqrt{n!}} = \frac{(n\mathbf{a}^{\dagger n-1} + \mathbf{a}^{\dagger n}\mathbf{a})|0\rangle}{\sqrt{n!}} = \sqrt{n}\,\frac{\mathbf{a}^{\dagger n-1}|0\rangle}{\sqrt{(n-1)!}}$$

$$\mathbf{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle \qquad \mathbf{a}|n\rangle = \sqrt{n}\,|n-1\rangle$$

Feynman's mnemonic rule: Larger of two quanta goes in radical factor

$$\langle \mathbf{a}^\dagger \rangle = \begin{pmatrix} \cdot & & & & & \\ 1 & \cdot & & & & \\ & \sqrt{2} & \cdot & & & \\ & & \sqrt{3} & \cdot & & \\ & & & \sqrt{4} & \cdot & \\ & & & & \ddots & \cdot \end{pmatrix} \qquad \langle \mathbf{a} \rangle = \begin{pmatrix} \cdot & 1 & & & & \\ & \cdot & \sqrt{2} & & & \\ & & \cdot & \sqrt{3} & & \\ & & & \cdot & \sqrt{4} & \\ & & & & \cdot & \ddots \\ & & & & & \cdot \end{pmatrix}$$

Use: $\mathbf{a}\mathbf{a}^{\dagger n} = n\mathbf{a}^{\dagger n-1} + \mathbf{a}^{\dagger n}\mathbf{a}$

Number operator and Hamiltonian operator

Number operator $\mathbf{N}=\mathbf{a}^\dagger\mathbf{a}$ counts quanta.

$$\mathbf{a}^\dagger\mathbf{a}|n\rangle = \frac{\mathbf{a}^\dagger\mathbf{a}\mathbf{a}^{\dagger n}|0\rangle}{\sqrt{n!}} = n\frac{\mathbf{a}^\dagger\mathbf{a}^{\dagger n-1}|0\rangle}{\sqrt{n!}} = n\frac{\mathbf{a}^{\dagger n}|0\rangle}{\sqrt{n!}} = n|n\rangle$$

Hamiltonian operator

$$\mathbf{H}\,|n\rangle = \hbar\omega\,\mathbf{a}^\dagger\mathbf{a}\,|n\rangle + \hbar\omega/2\,\mathbf{1}\,|n\rangle = \hbar\omega(n+1/2)|n\rangle$$

$$\langle \mathbf{H} \rangle = \hbar\omega\langle \mathbf{a}^\dagger\mathbf{a} + \tfrac{1}{2}\mathbf{1}\rangle = \hbar\omega\begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & 2 & & \\ & & & 3 & \\ & & & & \ddots \end{pmatrix} + \hbar\omega\begin{pmatrix} 1/2 & & & & \\ & 1/2 & & & \\ & & 1/2 & & \\ & & & 1/2 & \\ & & & & \ddots \end{pmatrix}$$

Hamiltonian operator is $\hbar\omega$ $\mathbf{N}$ plus zero-point energy $\mathbf{1}\hbar\omega/2$.

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T(*a*) and **B**(*b*) operations do not commute. $\mathbf{T}(a)=e^{-ia\mathbf{p}/\hbar}$ *or* $\mathbf{B}(b)=e^{ib\mathbf{x}/\hbar}$

Define a *combined boost-translation operation:* $\mathbf{C}(a,b)=e^{i(b\mathbf{x}-a\mathbf{p})/\hbar}$

Use *Baker-Campbell-Hausdorf identity* since $[\mathbf{x},\mathbf{p}]=i\hbar\mathbf{1}$ and $[[\mathbf{x},\mathbf{p}],\mathbf{x}]=[[\mathbf{x},\mathbf{p}],\mathbf{p}]=\mathbf{0}$.

$$e^{\mathbf{A}+\mathbf{B}}=e^{\mathbf{A}}e^{\mathbf{B}}e^{-[\mathbf{A},\mathbf{B}]/2}=e^{\mathbf{B}}e^{\mathbf{A}}e^{[\mathbf{A},\mathbf{B}]/2}\ ,\ \text{where: }\left[\mathbf{A},\left[\mathbf{A},\mathbf{B}\right]\right]=\mathbf{0}=\left[\mathbf{B},\left[\mathbf{A},\mathbf{B}\right]\right]$$

$$\mathbf{C}(a,b)=e^{i(b\mathbf{x}-a\mathbf{p})/\hbar}=e^{ib\mathbf{x}/\hbar}e^{-ia\mathbf{p}/\hbar}e^{-ab[\mathbf{x},\mathbf{p}]/2\hbar^2}=e^{ib\mathbf{x}/\hbar}e^{-ia\mathbf{p}/\hbar}e^{-iab/2\hbar}$$

$$\mathbf{C}(a,b)=\mathbf{B}(b)\mathbf{T}(a)e^{-iab/2\hbar}=\mathbf{T}(a)\mathbf{B}(b)e^{iab/2\hbar}$$

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$$e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{A}}e^{\mathbf{B}}e^{-[\mathbf{A},\mathbf{B}]/2} = e^{\mathbf{B}}e^{\mathbf{A}}e^{[\mathbf{A},\mathbf{B}]/2}, \text{ where: } [\mathbf{A},[\mathbf{A},\mathbf{B}]] = \mathbf{0} = [\mathbf{B},[\mathbf{A},\mathbf{B}]]$$

$$\mathbf{C}(a,b) = e^{i(b\mathbf{x}-a\mathbf{p})/\hbar} = e^{ib\mathbf{x}/\hbar}e^{-ia\mathbf{p}/\hbar}e^{-ab[\mathbf{x},\mathbf{p}]/2\hbar^2} = e^{ib\mathbf{x}/\hbar}e^{-ia\mathbf{p}/\hbar}e^{-iab/2\hbar}$$

$$\mathbf{C}(a,b) = \mathbf{B}(b)\mathbf{T}(a)e^{-iab/2\hbar} = \mathbf{T}(a)\mathbf{B}(b)e^{iab/2\hbar}$$

Complex *phasor coordinate* $\alpha(a,b)$ defined by: $\alpha(a,b)$

$$= a\sqrt{M\omega/2\hbar} + ib/\sqrt{2\hbar M\omega}$$

$$= \left[a + i\frac{b}{M\omega}\right]\sqrt{M\omega/2\hbar}$$

Reordering only affects the overall phase.

$$\mathbf{C}(a,b) = e^{i(b\mathbf{x}-a\mathbf{p})/\hbar} = e^{ib(\mathbf{a}^\dagger+\mathbf{a})/\sqrt{2\hbar M\omega} + a(\mathbf{a}^\dagger-\mathbf{a})\sqrt{M\omega/2\hbar}}$$

$$= e^{\alpha\mathbf{a}^\dagger - \alpha^*\mathbf{a}} = e^{-|\alpha|^2/2}e^{\alpha\mathbf{a}^\dagger}e^{-\alpha^*\mathbf{a}} = e^{|\alpha|^2/2}e^{-\alpha^*\mathbf{a}}e^{\alpha\mathbf{a}^\dagger}$$

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$$= e^{\alpha\mathbf{a}^\dagger-\alpha^*\mathbf{a}} = e^{-|\alpha|^2/2}e^{\alpha\mathbf{a}^\dagger}e^{-\alpha^*\mathbf{a}} = e^{|\alpha|^2/2}e^{-\alpha^*\mathbf{a}}e^{\alpha\mathbf{a}^\dagger}$$

Coherent wavepacket state $|\alpha(x_0,p_0)\rangle$: $\left|\alpha_0(x_0,p_0)\right\rangle = \mathbf{C}(x_0,p_0)|0\rangle = e^{i(x_0\mathbf{x}-p_0\mathbf{p})/\hbar}|0\rangle$

$$= e^{-|\alpha_0|^2/2}e^{\alpha_0\mathbf{a}^\dagger}e^{-\alpha_0^*\mathbf{a}}|0\rangle$$

$$= e^{-|\alpha_0|^2/2}e^{\alpha_0\mathbf{a}^\dagger}|0\rangle$$

$$= e^{-|\alpha_0|^2/2}\sum_{n=0}^{\infty}\left(\alpha_0\mathbf{a}^\dagger\right)^n|0\rangle/n!$$

$$= e^{-|\alpha_0|^2/2}\sum_{n=0}^{\infty}\frac{(\alpha_0)^n}{\sqrt{n!}}|n\rangle\ , \quad \text{where: } |n\rangle = \frac{\mathbf{a}^{\dagger n}|0\rangle}{\sqrt{n!}}$$

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Time evolution operator for constant **H** has general form : $\mathbf{U}(t,0) = e^{-i\mathbf{H}t/\hbar}$

Oscillator eigenstate time evolution is simply determined by harmonic phases.

$$\mathbf{U}(t,0)\left|n\right\rangle = e^{-i\mathbf{H}t/\hbar}\left|n\right\rangle = e^{-i(n+1/2)\omega t}\left|n\right\rangle$$

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$$= e^{-i\omega t/2}e^{-|\alpha_0|^2/2}\sum_{n=0}^{\infty}\frac{\left(\alpha_0 e^{-i\omega t}\right)^n}{\sqrt{n!}}\left|n\right\rangle$$

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Evolution simplifies to a variable-α_0 coherent state with a *time dependent phasor coordinate* α_t:

$$\mathbf{U}(t,0)\left|\alpha_0(x_0,p_0)\right\rangle = e^{-i\omega t/2}\left|\alpha_t(x_t,p_t)\right\rangle$$ where:

$$\alpha_t(x_t,p_t) = e^{-i\omega t}\,\alpha_0(x_0,p_0)$$

$$\left[x_t + i\frac{p_t}{M\omega}\right] = e^{-i\omega t}\left[x_0 + i\frac{p_0}{M\omega}\right]$$

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$$\alpha_t(x_t,p_t) = e^{-i\omega t}\ \alpha_0(x_0,p_0)$$

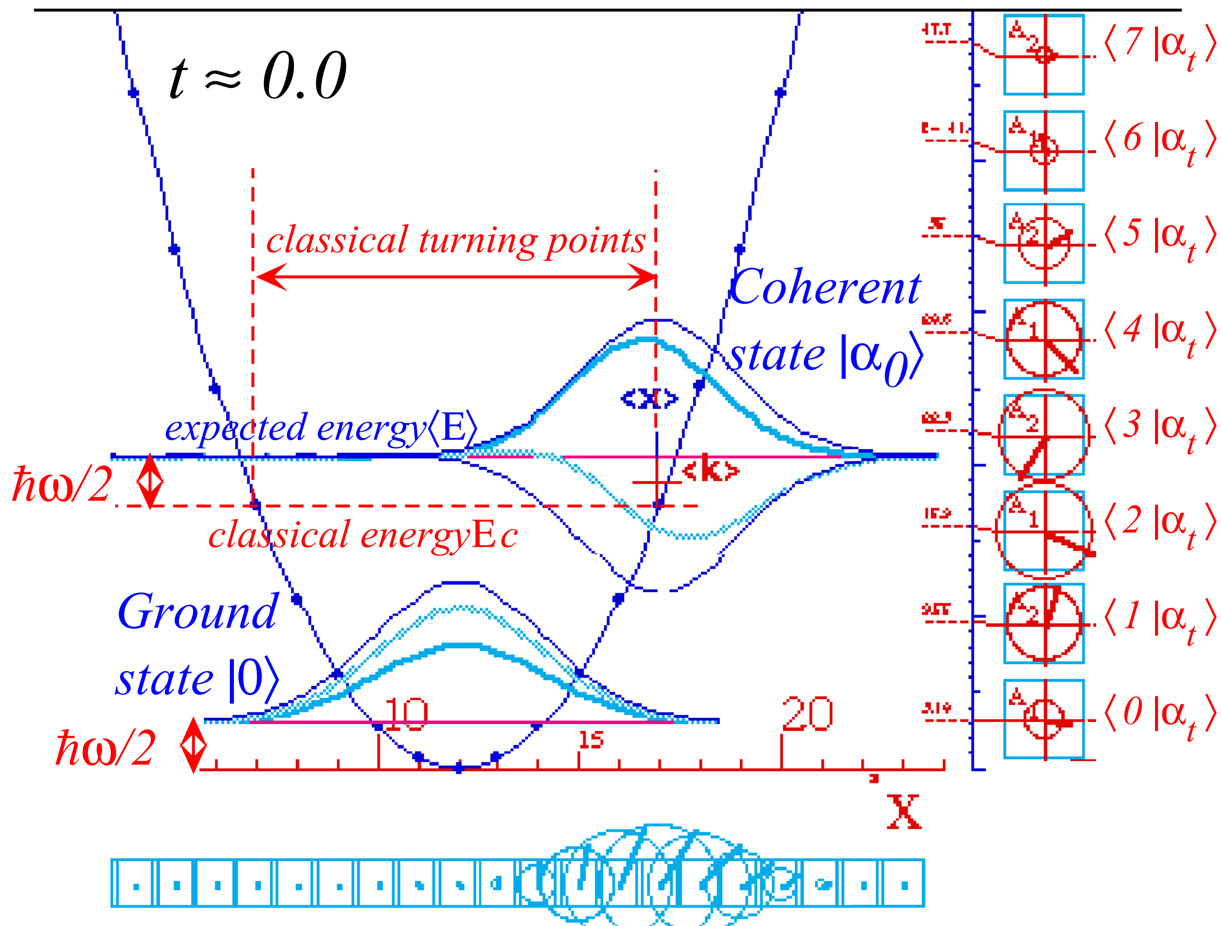
$$\left[x_t + i\frac{p_t}{M\omega}\right] = e^{-i\omega t}\left[x_0 + i\frac{p_0}{M\omega}\right]$$

(x_t,p_t) mimics classical oscillator

$$x_t = x_0\cos\omega t + \frac{p_0}{M\omega}\sin\omega t$$

$$\frac{p_t}{M\omega} = -x_0\sin\omega t + \frac{p_0}{M\omega}\cos\omega t$$

(Real and imaginary parts (x_t and $p_t/M\omega$) of α_t go clockwise on phasor circle.)



Coherent ket $|\alpha(x0,p0)\rangle$ is eigenvector of destruct-op. **a**.

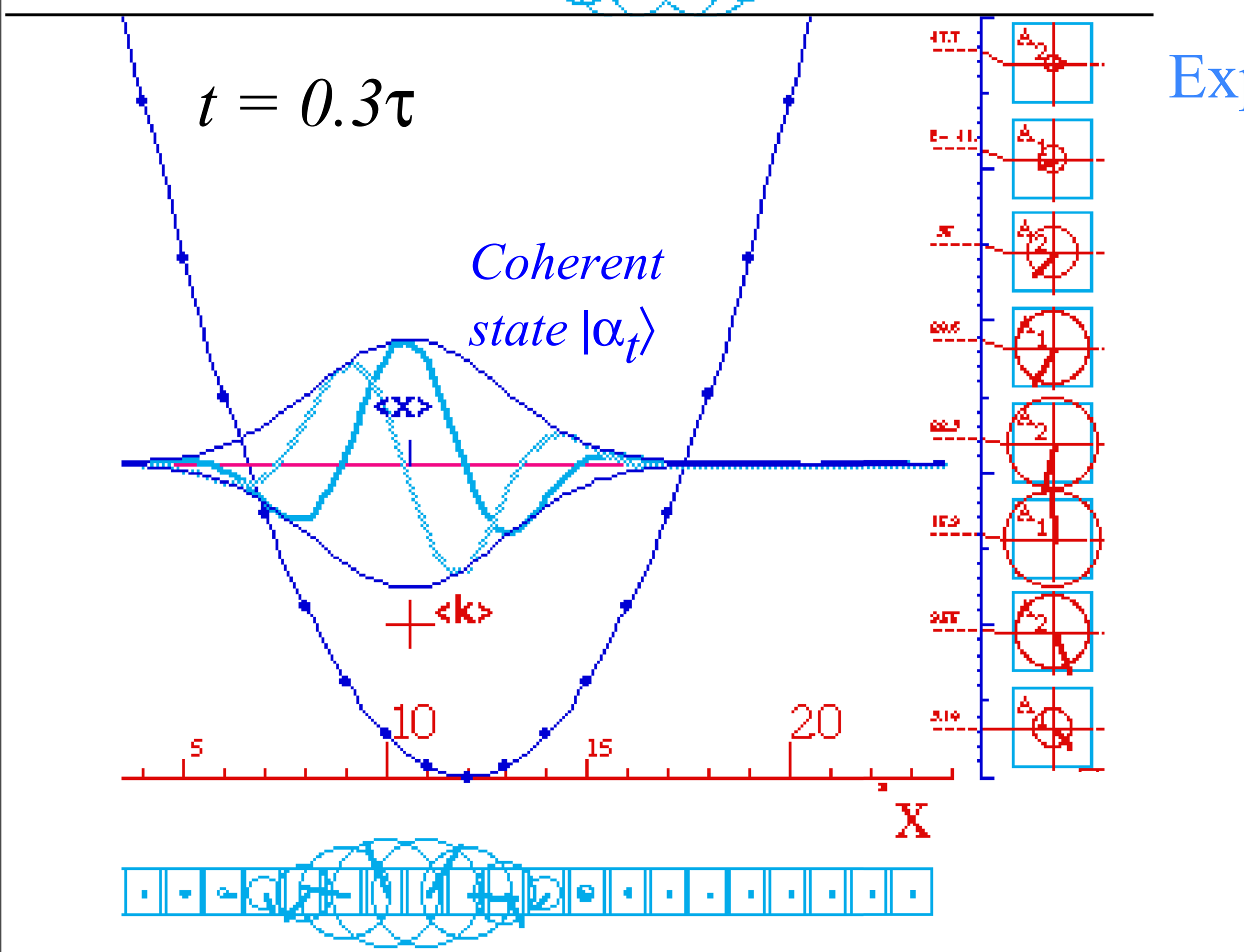
$$\mathbf{a}\left|\alpha_0(x_0,p_0)\right\rangle = e^{-|\alpha_0|^2/2}\sum_{n=0}^{\infty}\frac{(\alpha_0)^n}{\sqrt{n!}}\mathbf{a}|n\rangle$$

$$= e^{-|\alpha_0|^2/2}\sum_{n=0}^{\infty}\frac{(\alpha_0)^n}{\sqrt{n!}}\sqrt{n}|n-1\rangle$$

$$= \alpha_0\left|\alpha_0(x_0,p_0)\right\rangle \quad \text{with eigenvalue } \alpha_0$$

Coherent bra $\langle\alpha(x0,p0)|$ is eigenvector of create-op. $\mathbf{a}^\dagger$.

$$\left\langle\alpha_0(x_0,p_0)\right|\mathbf{a}^\dagger = \left\langle\alpha_0(x_0,p_0)\right|\alpha_0^*$$



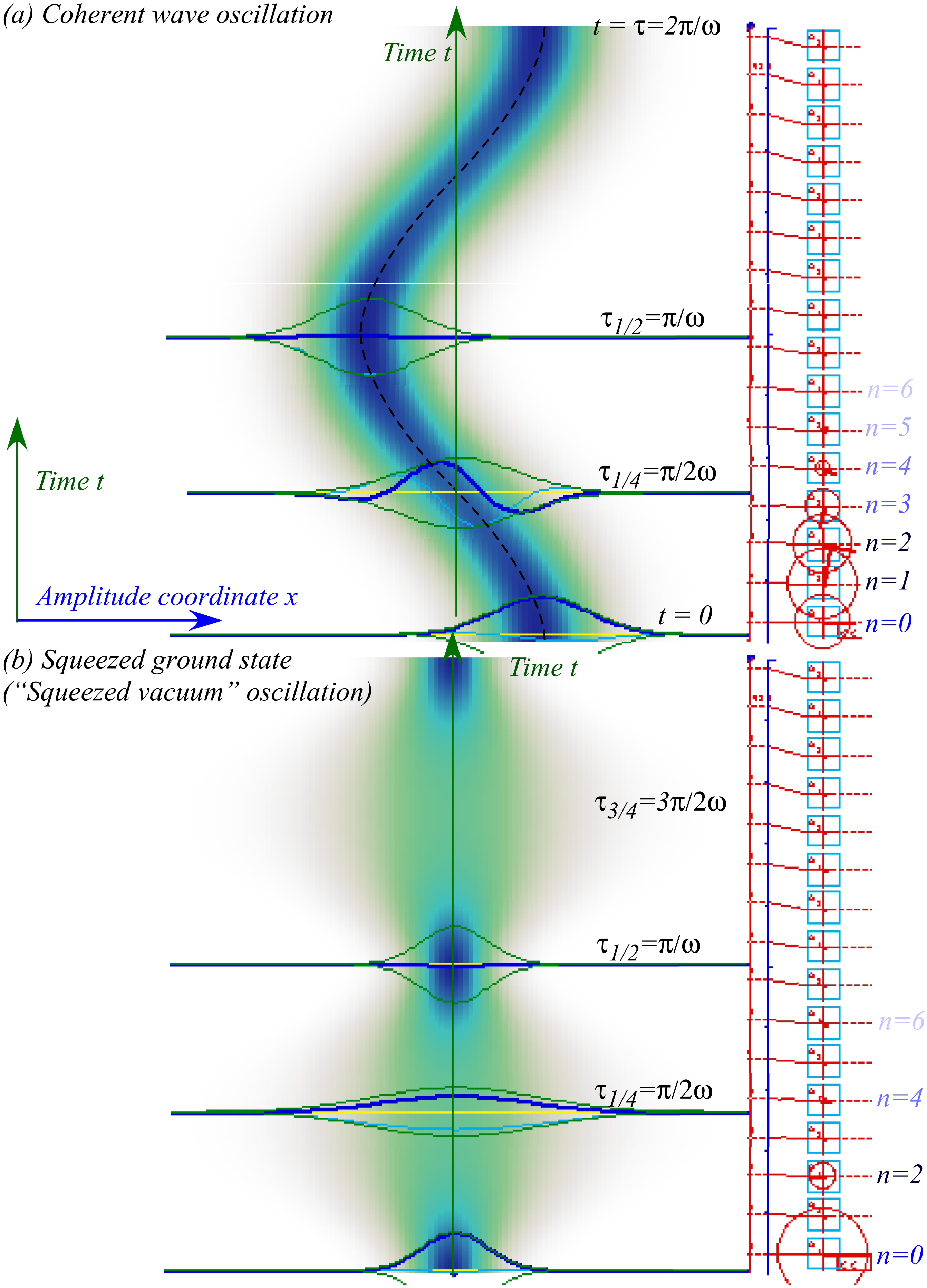
Expected quantum energy has simple time independent form.

$$\langle E\rangle\big|_{\alpha_0} = \left\langle\alpha_0(x_0,p_0)\right|\mathbf{H}\left|\alpha_0(x_0,p_0)\right\rangle$$

$$= \left\langle\alpha_0(x_0,p_0)\right|\left(\hbar\omega\mathbf{a}^\dagger\mathbf{a} + \frac{\hbar\omega}{2}\mathbf{1}\right)\left|\alpha_0(x_0,p_0)\right\rangle$$

$$= \hbar\omega\alpha_0^*\alpha_0 + \frac{\hbar\omega}{2}$$

Properties of "squeezed" coherent states



Yay! Classical Cosine trajectory!

what happens if you apply operators with non-linear "tensor" exponents $exp(s\mathbf{x}^2)$, $exp(f\,\mathbf{p}^2)$, *etc.*

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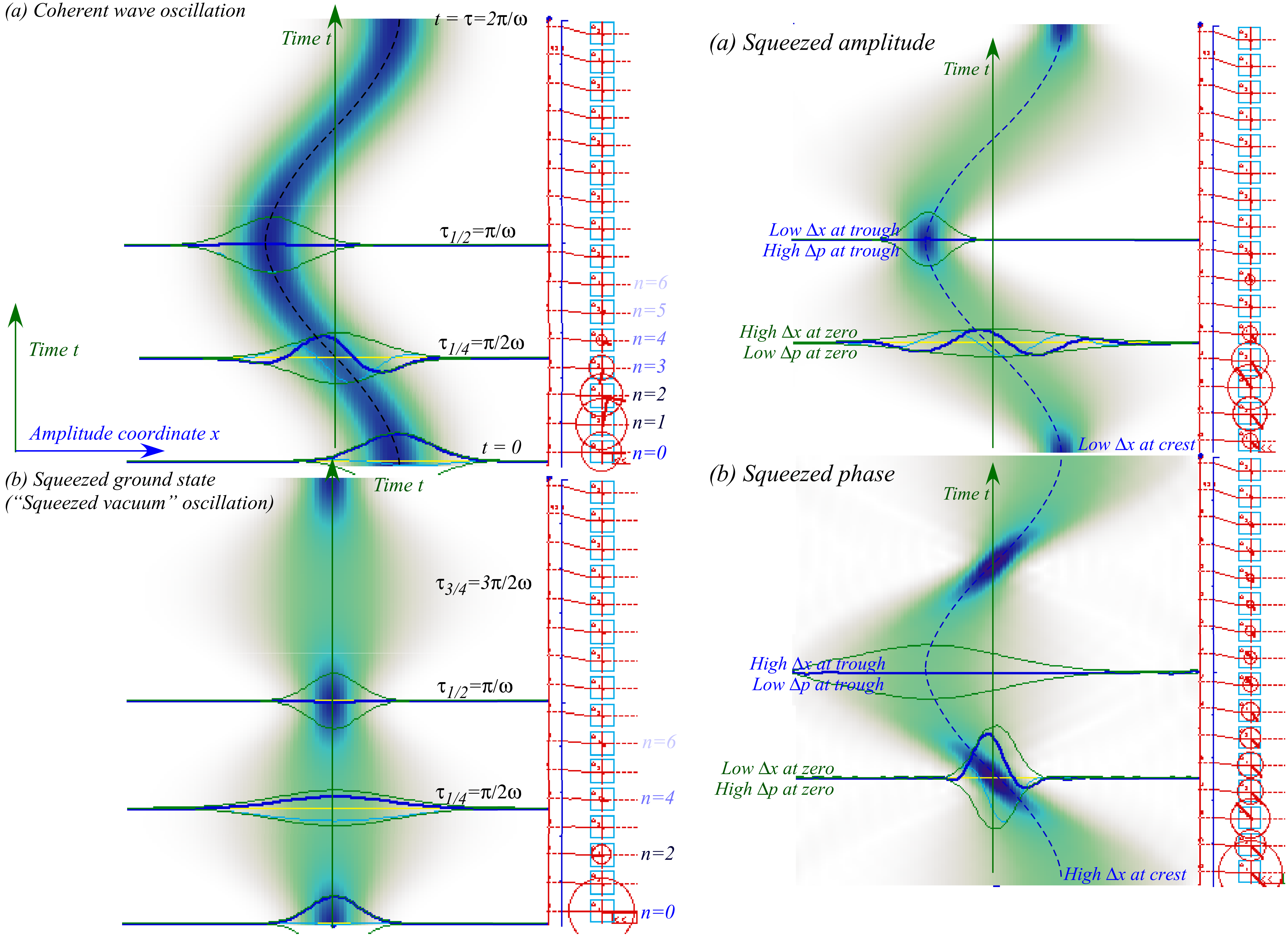

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$$= A\left(\mathbf{a}_1^\dagger\mathbf{a}_1+\mathbf{1}/2\right)+(B-iC)\mathbf{a}_1^\dagger\mathbf{a}_2 + (B+iC)\mathbf{a}_2^\dagger\mathbf{a}_1+D\left(\mathbf{a}_2^\dagger\mathbf{a}_2+\mathbf{1}/2\right)$$

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2D-Oscillator basic states and operations - Commutattion

First rewrite a classical 2-D Hamiltonian (Lecture. 6-9) with a thick-tip pen! (They're **operators** now!)

$$\mathbf{H} = \frac{A}{2}\left(\mathbf{p}_1^2 + \mathbf{x}_1^2\right) + B\left(\mathbf{x}_1\mathbf{x}_2 + \mathbf{p}_1\mathbf{p}_2\right) + C\left(\mathbf{x}_1\mathbf{p}_2 - \mathbf{x}_2\mathbf{p}_1\right) + \frac{D}{2}\left(\mathbf{p}_2^2 + \mathbf{x}_2^2\right)$$

(Mass factors $\sqrt{M}$, spring constants K_{ij}, and Planck $\hbar$ constants are absorbed into A, B, C, and D constants used in Lectures 6-9.)

Define **a** and **a**† **operators**

$$\mathbf{a}_1 = (\mathbf{x}_1 + i\,\mathbf{p}_1)/\sqrt{2} \qquad \mathbf{a}^\dagger_1 = (\mathbf{x}_1 - i\,\mathbf{p}_1)/\sqrt{2} \qquad \mathbf{a}_2 = (\mathbf{x}_2 + i\,\mathbf{p}_2)/\sqrt{2} \qquad \mathbf{a}^\dagger_2 = (\mathbf{x}_2 - i\,\mathbf{p}_2)/\sqrt{2}$$

$$\mathbf{x}_1 = (\mathbf{a}^\dagger_1 + \mathbf{a}_1)/\sqrt{2} \qquad \mathbf{p}_1 = i\,(\mathbf{a}^\dagger_1 - \mathbf{a}_1)/\sqrt{2} \qquad \mathbf{x}_2 = (\mathbf{a}^\dagger_2 + \mathbf{a}_2)/\sqrt{2} \qquad \mathbf{p}_2 = i\,(\mathbf{a}^\dagger_2 - \mathbf{a}_2)/\sqrt{2}$$

Each system dimension $\mathbf{x}_1$ and $\mathbf{x}_2$ is assumed orthogonal, neither being constrained by the other. This includes an axiom of *inter-dimensional commutivity*.

$$[\,\mathbf{x}_1, \mathbf{p}_2] = \mathbf{0} = [\,\mathbf{x}_2, \mathbf{p}_1], \quad [\,\mathbf{a}_1, \mathbf{a}^\dagger_2] = \mathbf{0} = [\,\mathbf{a}_2, \mathbf{a}^\dagger_1]$$

Commutation relations within space-1 or space-2 space are those of a 1D-oscillator.

$$[\,\mathbf{a}_1, \mathbf{a}^\dagger_1] = \mathbf{1}, \quad [\,\mathbf{a}_2, \mathbf{a}^\dagger_2] = \mathbf{1}$$

This applies in general to N-dimensional oscillator problems.

$$[\,\mathbf{a}_m, \mathbf{a}_n] = \mathbf{a}_m\mathbf{a}_n - \mathbf{a}_n\mathbf{a}_m = \mathbf{0} \qquad [\,\mathbf{a}_m, \mathbf{a}^\dagger_n] = \mathbf{a}_m\mathbf{a}^\dagger_n - \mathbf{a}^\dagger_n\mathbf{a}_m = \delta_{mn}\mathbf{1} \qquad [\,\mathbf{a}^\dagger_m, \mathbf{a}^\dagger_n] = \mathbf{a}^\dagger_m\mathbf{a}^\dagger_n - \mathbf{a}^\dagger_n\mathbf{a}^\dagger_m = \mathbf{0}$$

New symmetrized $\mathbf{a}^\dagger_m\mathbf{a}_n$ operators replace the old ket-bras $|m\rangle\langle n|$ that define semi-classical **H** matrix.

$$\mathbf{H} = H_{11}\left(\mathbf{a}_1^\dagger\mathbf{a}_1 + \mathbf{1}/2\right) + H_{12}\mathbf{a}_1^\dagger\mathbf{a}_2 + H_{21}\mathbf{a}_2^\dagger\mathbf{a}_1 + H_{22}\left(\mathbf{a}_2^\dagger\mathbf{a}_2 + \mathbf{1}/2\right)$$

$$= A\left(\mathbf{a}_1^\dagger\mathbf{a}_1 + \mathbf{1}/2\right) + (B - iC)\mathbf{a}_1^\dagger\mathbf{a}_2 + (B + iC)\mathbf{a}_2^\dagger\mathbf{a}_1 + D\left(\mathbf{a}_2^\dagger\mathbf{a}_2 + \mathbf{1}/2\right)$$

$$\mathbf{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix}$$

Both are elementary "place-holders" for parameters H_{mn} or A, $B \pm iC$, and D.

$$|m\rangle\langle n| \rightarrow \left(\mathbf{a}_m^\dagger\mathbf{a}_n + \mathbf{a}_n\mathbf{a}_m^\dagger\right)/2 = \mathbf{a}_m^\dagger\mathbf{a}_n + \delta_{m,n}\mathbf{1}/2$$

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  - *Commutation relations*
  - *Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry*
    - *Anti-commutation relations*
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Mostly Notation and Bookkeeping :

Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry

Commutivity is known as *Bose symmetry.* Bose and Einstein discovered this symmetry of light quanta.
($\mathbf{a}_m$, $\mathbf{a}^\dagger{}_n$) operators called *Boson operators* create or destroy *quanta* or "particles" known as *Bosons*.

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Fermi operators $(\mathbf{c}_m,\mathbf{c}_n)$ are defined to create *Fermions* and use *anti-commutators* $\{\mathbf{A},\mathbf{B}\} = \mathbf{AB}+\mathbf{BA}$.

$$\{\mathbf{c}_m,\mathbf{c}_n\}=\mathbf{c}_m\mathbf{c}_n+\mathbf{c}_n\mathbf{c}_m=\mathbf{0} \qquad \{\mathbf{c}_m,\mathbf{c}^\dagger_n\}=\mathbf{c}_m\mathbf{c}^\dagger_n+\mathbf{c}^\dagger_n\mathbf{c}_m=\delta_{mn}\mathbf{1} \qquad \{\mathbf{c}^\dagger_m,\mathbf{c}^\dagger_n\}=\mathbf{c}^\dagger_m\mathbf{c}^\dagger_n+\mathbf{c}^\dagger_n\mathbf{c}^\dagger_m=\mathbf{0}$$

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Fermi $\mathbf{c}^\dagger_n$ has a rigid birth-control policy; they are allowed just one Fermion or else, none at all.

Creating two Fermions of the same type is punished by death. This is because $x=-x$ implies $x=0$.

$$\mathbf{c}^\dagger_m\mathbf{c}^\dagger_m\,|0\rangle = -\,\mathbf{c}^\dagger_m\mathbf{c}^\dagger_m\,|0\rangle = \mathbf{0}$$

That no two indistinguishable Fermions can be in the same state, is called the *Pauli exclusion principle*.

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Quantum numbers of $n=0$ and $n=1$ are the only allowed eigenvalues of the number operator $\mathbf{c}^\dagger_m\mathbf{c}_m$.

$$\mathbf{c}^\dagger_m\mathbf{c}_m\,|0\rangle = \mathbf{0}\ ,\ \ \mathbf{c}^\dagger_m\mathbf{c}_m\,|1\rangle = |1\rangle\ ,\ \ \mathbf{c}^\dagger_m\mathbf{c}_m\,|n\rangle = \mathbf{0}\ \ \text{for: } n>1$$

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Two-dimensional (or 2-particle) base states: ket-kets and bra-bras

A state for a particle in two-dimensions (or two one-dimensional particles) is a *"ket-ket"* $|n_1\rangle|n_2\rangle$

It is outer product of the kets for each single dimension or particle.

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Must ask a perennial modern question: *"How are these structures stored in a computer program?"*
The usual answer is in *outer product* or *tensor arrays*. Next pages show sketches of these objects.

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Outer product arrays

Start with an elementary ket basis for each dimension or particle type-1 and type-2.

$$Type-1 \qquad\qquad\qquad\qquad Type-2 \qquad\qquad \cdots$$

$$|0_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, |1_1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, |2_1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{pmatrix}, \cdots \qquad |0_2\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, |1_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, |2_2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{pmatrix}, \cdots$$

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$$|0_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, |1_1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, |2_1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{pmatrix}, \cdots \qquad |0_2\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, |1_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, |2_2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{pmatrix}, \cdots$$

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$$|0_1\rangle|0_2\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ \hline 0 \\ 0 \\ 0 \\ \vdots \\ \hline 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, |0_1\rangle|1_2\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ \hline 0 \\ 0 \\ 0 \\ \vdots \\ \hline 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \cdots |1_1\rangle|0_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \hline 1 \\ 0 \\ 0 \\ \vdots \\ \hline 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \cdots |1_1\rangle|2_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \hline 0 \\ 0 \\ 1 \\ \vdots \\ \hline 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix},$$

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Least significant digit at (right) END

or anti-lexicographic *(00, 10, 20, ...01, 11, 21,..., 02, 12, 22, ..)* array indexing

"Big-Endian" indexing *(...00,10,20..01,11,21,31 ... 02,12,22,32...)*

Most significant digit at (right) END

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shorthand *big-bra-big-ket* notation

$$\left|\Psi\right\rangle=\begin{pmatrix}\langle 0_1 0_2|\Psi\rangle\\\langle 0_1 1_2|\Psi\rangle\\\langle 0_1 2_2|\Psi\rangle\\\vdots\\\hline\langle 1_1 0_2|\Psi\rangle\\\langle 1_1 1_2|\Psi\rangle\\\langle 1_1 2_2|\Psi\rangle\\\vdots\\\hline\langle 2_1 0_2|\Psi\rangle\\\langle 2_1 1_2|\Psi\rangle\\\langle 2_1 2_2|\Psi\rangle\\\vdots\end{pmatrix}=\begin{pmatrix}\Psi_{00}\\\Psi_{01}\\\Psi_{02}\\\vdots\\\hline\Psi_{10}\\\Psi_{11}\\\Psi_{12}\\\vdots\\\hline\Psi_{20}\\\Psi_{21}\\\Psi_{22}\\\vdots\end{pmatrix}$$

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Review : *1-D* **a†a** *algebra of U(1) representations*
Review : *Translate* **T**(a) *and/or Boost* **B**(b) *to construct coherent state*
Review : *Time evolution of coherent state (and "squeezed" states)*

2-D **a†a** *algebra of U(2) representations and R(3) angular momentum operators*
2D-Oscillator basic states and operations
Commutation relations
Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry
Anti-commutation relations
Two-dimensional (or 2-particle) base states: ket-kets and bra-bras
Outer product arrays
Entangled 2-particle states
Two-particle (or 2-dimensional) matrix operators
U(2) Hamiltonian and irreducible representations
2D-Oscillator states and related 3D angular momentum multiplets
ND multiplets

R(3) Angular momentum generators by U(2) analysis
Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$
SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Entangled 2-particle states

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ANALOGY:

A general *n-by-n* matrix **M** operator is a combination of n^2 terms: $\mathbf{M} = \sum_{j=1}^{n}\sum_{k=1}^{n} M_{j,k}\left|j\right\rangle\left\langle k\right|$

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So a general two-particle state $|\Psi\rangle$ is a combination of *entangled* products: $\left|\Psi\right\rangle = \sum_j \sum_k \psi_{j,k}|\Psi_j\rangle|\Psi_k\rangle$

Entangled 2-particle states

A matrix operator **M** is rarely a single nilpotent operator $|1\rangle\langle 2|$ or idempotent $|1\rangle\langle 1|$.

A two-particle state $|\Psi\rangle$ is rarely a single outer product $|\Psi_1\rangle|\Psi_2\rangle$ of 1-particle states $|\Psi_1\rangle$ and $|\Psi_2\rangle$.
(Even rarer is $|\Psi_1\rangle|\Psi_1\rangle$.)

ANALOGY:

A general *n-by-n* matrix **M** operator is a combination of n^2 terms: $\mathbf{M} = \sum_{j=1}^{n}\sum_{k=1}^{n} M_{j,k}|j\rangle\langle k|$

...that *might* be diagonalized to a combination of n projectors: $\mathbf{M} = \sum_{e=1}^{n} \mu_e|e\rangle\langle e|$

So a general two-particle state $|\Psi\rangle$ is a combination of *entangled* products: $|\Psi\rangle = \sum_j\sum_k \psi_{j,k}|\Psi_j\rangle|\Psi_k\rangle$

...that *might* be *de-entangled* to a combination of n terms: $|\Psi\rangle = \sum_e \phi_e|\varphi_e\rangle|\varphi_e\rangle$

Review : *1-D* $a^\dagger a$ *algebra of U(1) representations*
Review : *Translate* **T**(*a*) *and/or Boost* **B**(*b*) *to construct coherent state*
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Two-particle (or 2-dimensional) matrix operators

When 2-particle operator $\mathbf{a}_k$ acts on a 2-particle state, $\mathbf{a}_k$ "finds" its type-k state but ignores the others.

$$\mathbf{a}_1^\dagger|n_1 n_2\rangle = \mathbf{a}_1^\dagger|n_1\rangle|n_2\rangle = \sqrt{n_1+1}|n_1+1\ n_2\rangle \qquad \mathbf{a}_2^\dagger|n_1 n_2\rangle = |n_1\rangle\mathbf{a}_2^\dagger|n_2\rangle = \sqrt{n_2+1}|n_1\ n_2+1\rangle$$

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$$|n_1 n_2\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{n_1}\left(\mathbf{a}_2^\dagger\right)^{n_2}}{\sqrt{n_1!\,n_2!}}|0\;0\rangle$$

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The $\mathbf{a}_m{}^\dagger\mathbf{a}_n$ combinations in the *ABCD* Hamiltonian **H** have fairly simple matrix elements.

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	$\vert 00\rangle$	$\vert 01\rangle$	$\vert 02\rangle$	$\cdots$	$\vert 10\rangle$	$\vert 11\rangle$	$\vert 12\rangle$	$\cdots$	$\vert 20\rangle$	$\vert 21\rangle$	$\vert 22\rangle$	$\cdots$
$\langle 00\vert$	0			$\cdots$	$\cdot$			$\cdots$				
$\langle 01\vert$		D		$\cdots$	$B+iC$	$\cdot$		$\cdots$				
$\langle 02\vert$			$2D$	$\cdots$		$\sqrt{2}(B+iC)$	$\cdot$	$\cdots$				$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$				$\cdots$
$\langle 10\vert$	$\cdot$	$B-iC$		$\cdots$	A			$\cdots$	$\cdot$			$\cdots$
$\langle 11\vert$		$\cdot$	$\sqrt{2}(B-iC)$	$\cdots$		$A+D$		$\cdots$	$\sqrt{2}(B+iC)$	$\cdot$		$\cdots$
$\langle 12\vert$			$\cdot$	$\cdots$			$A+2D$	$\cdots$		$\sqrt{4}(B+iC)$	$\cdot$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$\langle 20\vert$					$\cdot$	$\sqrt{2}(B-iC)$		$\cdots$	$2A$			$\cdots$
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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$				$\cdots$
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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
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$\langle 12\|$			$\cdot$	$\cdots$			$A+2D$	$\cdots$		$\sqrt{4}(B+iC)$	$\cdot$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
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$\langle 11\vert$		$\cdot$	$\sqrt{2}(B-iC)$	$\cdots$		$A+D$		$\cdots$	$\sqrt{2}(B+iC)$	$\cdot$		$\cdots$
$\langle 12\vert$		*Example:*	$\cdot$	$\cdots$			$A+2D$	$\cdots$		$\sqrt{4}(B+iC)$	$\cdot$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$\langle 20\vert$	$\mathbf{a}_1^\dagger\mathbf{a}_2\vert 02\rangle=\sqrt{0+1}\sqrt{2}\vert 0+1\ 2-1\rangle=\sqrt{2}\vert 1\,1\rangle$				$\cdot$	$\sqrt{2}(B-iC)$		$\cdots$	$2A$			$\cdots$
$\langle 21\vert$	$\mathbf{a}_1^\dagger\mathbf{a}_2\vert n_1n_2\rangle=\sqrt{n_1+1}\sqrt{n_2}\vert n_1+1\ n_2-1\rangle$					$\cdot$	$\sqrt{4}(B-iC)$	$\cdots$		$2A+D$		$\cdots$
$\langle 22\vert$							$\cdot$	$\cdots$			$2A+2D$	$\cdots$
$\vdots$					$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

$\mathbf{a}_1^\dagger\mathbf{a}_1\left|n_1n_2\right\rangle=n_1\left|n_1n_2\right\rangle$

$\mathbf{a}_2^\dagger\mathbf{a}_1\left|n_1n_2\right\rangle=\sqrt{n_1}\sqrt{n_2+1}\left|n_1-1\,n_2+1\right\rangle$

$\mathbf{a}_1^\dagger\mathbf{a}_2\left|n_1n_2\right\rangle=\sqrt{n_1+1}\sqrt{n_2}\left|n_1+1\,n_2-1\right\rangle$

$\mathbf{a}_2^\dagger\mathbf{a}_2\left|n_1n_2\right\rangle=n_2\left|n_1n_2\right\rangle$

Rearrangement of rows and columns brings the matrix to a block-diagonal form.

U(2)-2D-HO Hamiltonian and irreducible representations

"Little-Endian" indexing
(...01,02,03..10,11,12,13 ...
20,21,22,23,...)

$$\mathbf{H} = A\left(\mathbf{a}_1^\dagger \mathbf{a}_1 + \mathbf{1}/2\right) + (B - iC)\,\mathbf{a}_1^\dagger \mathbf{a}_2 + (B + iC)\,\mathbf{a}_2^\dagger \mathbf{a}_1 + D\left(\mathbf{a}_2^\dagger \mathbf{a}_2 + \mathbf{1}/2\right)$$

$$\langle \mathbf{H} \rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$$

	$\vert 00\rangle$	$\vert 01\rangle$	$\vert 02\rangle$	$\cdots$	$\vert 10\rangle$	$\vert 11\rangle$	$\vert 12\rangle$	$\cdots$	$\vert 20\rangle$	$\vert 21\rangle$	$\vert 22\rangle$	$\cdots$
$\langle 00\vert$	0			$\cdots$	$\cdot$			$\cdots$				$\cdots$
$\langle 01\vert$		D		$\cdots$	$B+iC$	$\cdot$		$\cdots$				$\cdots$
$\langle 02\vert$			$2D$	$\cdots$		$\sqrt{2}(B+iC)$	$\cdot$	$\cdots$				$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$				$\cdots$
$\langle 10\vert$	$\cdot$	$B-iC$		$\cdots$	A			$\cdots$	$\cdot$			$\cdots$
$\langle 11\vert$		$\cdot$	$\sqrt{2}(B-iC)$	$\cdots$		$A+D$		$\cdots$	$\sqrt{2}(B+iC)$	$\cdot$		$\cdots$
$\langle 12\vert$		*Example:*	$\cdot$	$\cdots$			$A+2D$	$\cdots$		$\sqrt{4}(B+iC)$	$\cdot$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$\langle 20\vert$					$\cdot$	$\sqrt{2}(B-iC)$		$\cdots$	$2A$			$\cdots$
$\langle 21\vert$						$\cdot$	$\sqrt{4}(B-iC)$	$\cdots$		$2A+D$		$\cdots$
$\langle 22\vert$							$\cdot$	$\cdots$			$2A+2D$	$\cdots$
$\vdots$					$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Example: $\mathbf{a}_1^\dagger \mathbf{a}_2 \vert 02\rangle = \sqrt{0+1}\sqrt{2}\,\vert 0{+}1\ 2{-}1\rangle = \sqrt{2}\,\vert 1\,1\rangle$

$\mathbf{a}_1^\dagger \mathbf{a}_2 \vert n_1 n_2\rangle = \sqrt{n_1+1}\sqrt{n_2}\,\vert n_1{+}1\ n_2{-}1\rangle$

$$\mathbf{a}_1^\dagger \mathbf{a}_1 \vert n_1 n_2\rangle = n_1 \vert n_1 n_2\rangle$$
$$\mathbf{a}_2^\dagger \mathbf{a}_1 \vert n_1 n_2\rangle = \sqrt{n_1}\sqrt{n_2+1}\,\vert n_1{-}1\ n_2{+}1\rangle$$
$$\mathbf{a}_1^\dagger \mathbf{a}_2 \vert n_1 n_2\rangle = \sqrt{n_1+1}\sqrt{n_2}\,\vert n_1{+}1\ n_2{-}1\rangle$$
$$\mathbf{a}_2^\dagger \mathbf{a}_2 \vert n_1 n_2\rangle = n_2 \vert n_1 n_2\rangle$$

Rearrangement of rows and columns brings the matrix to a block-diagonal form.

Base states $\vert n_1\rangle \vert n_2\rangle$ with the same *total quantum number* $\nu = n_1 + n_2$ define each block.

U(2)-2D-HO Hamiltonian and irreducible representations

"Little-Endian" indexing (...01,02,03..10,11,12,13 ... 20,21,22,23,...)

$\mathbf{H} =$

$$A\left(\mathbf{a}_1^\dagger\mathbf{a}_1+\mathbf{1}/2\right)+\left(B-iC\right)\mathbf{a}_1^\dagger\mathbf{a}_2 +\left(B+iC\right)\mathbf{a}_2^\dagger\mathbf{a}_1+D\left(\mathbf{a}_2^\dagger\mathbf{a}_2+\mathbf{1}/2\right)$$

$\langle\mathbf{H}\rangle = A(\mathbf{1}/2)+D(\mathbf{1}/2)+$

	$\|00\rangle$	$\|01\rangle$	$\|02\rangle$	$\cdots$	$\|10\rangle$	$\|11\rangle$	$\|12\rangle$	$\cdots$	$\|20\rangle$	$\|21\rangle$	$\|22\rangle$	$\cdots$
$\langle 00\|$	0			$\cdots$	$\cdot$			$\cdots$				$\cdots$
$\langle 01\|$		D		$\cdots$	$B+iC$	$\cdot$		$\cdots$				$\cdots$
$\langle 02\|$			$2D$	$\cdots$		$\sqrt{2}(B+iC)$	$\cdot$	$\cdots$				$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$				$\cdots$
$\langle 10\|$	$\cdot$	$B-iC$		$\cdots$	A			$\cdots$	$\cdot$			$\cdots$
$\langle 11\|$		$\cdot$	$\sqrt{2}(B-iC)$	$\cdots$		$A+D$		$\cdots$	$\sqrt{2}(B+iC)$	$\cdot$		$\cdots$
$\langle 12\|$		*Example:*		$\cdots$			$A+2D$	$\cdots$		$\sqrt{4}(B+iC)$	$\cdot$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$\langle 20\|$	$\mathbf{a}_1^\dagger\mathbf{a}_2\|02\rangle=\sqrt{0+1}\sqrt{2}\|0+1\ 2-1\rangle=\sqrt{2}\|1\ 1\rangle$				$\cdot$	$\sqrt{2}(B-iC)$		$\cdots$	$2A$			$\cdots$
$\langle 21\|$	$\mathbf{a}_1^\dagger\mathbf{a}_2\|n_1n_2\rangle=\sqrt{n_1+1}\sqrt{n_2}\|n_1+1\ n_2-1\rangle$					$\cdot$	$\sqrt{4}(B-iC)$	$\cdots$		$2A+D$		$\cdots$
$\langle 22\|$							$\cdot$	$\cdots$			$2A+2D$	$\cdots$
$\vdots$					$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

$$\mathbf{a}_1^\dagger\mathbf{a}_1|n_1n_2\rangle=n_1|n_1n_2\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_1|n_1n_2\rangle=\sqrt{n_1}\sqrt{n_2+1}\,|n_1-1\,n_2+1\rangle$$

$$\mathbf{a}_1^\dagger\mathbf{a}_2|n_1n_2\rangle=\sqrt{n_1+1}\sqrt{n_2}\,|n_1+1\ n_2-1\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_2|n_1n_2\rangle=n_2|n_1n_2\rangle$$

Rearrangement of rows and columns brings the matrix to a block-diagonal form.

Base states $|n_1\rangle|n_2\rangle$ with the same *total quantum number* $\upsilon = n_1 + n_2$ define each block.

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$\langle\mathbf{H}\rangle = A(\mathbf{1}/2)+D(\mathbf{1}/2)+$

	$\|00\rangle$	$\|01\rangle$	$\|10\rangle$	$\|02\rangle$	$\|11\rangle$	$\|20\rangle$	$\|03\rangle$	$\|12\rangle$	$\|21\rangle$	$\|30\rangle$	$\cdots$
$\langle 00\|$	0	*Vacuum* ($\nu=0$)									
$\langle 01\|$		D	$B+iC$	*Fundamental* ($\nu=1$) vibrational sub-space							
$\langle 10\|$		$B-iC$	A								
$\langle 02\|$				$2D$	$\sqrt{2}(B+iC)$						
$\langle 11\|$				$\sqrt{2}(B-iC)$	$A+D$	$\sqrt{2}(B+iC)$	*Overtone* ($\nu=2$) vibrational sub-space				
$\langle 20\|$					$\sqrt{2}(B-iC)$	$2A$					
$\langle 03\|$							$3D$	$\sqrt{3}(B+iC)$			
$\langle 12\|$							$\sqrt{3}(B-iC)$	$A+2D$	$\sqrt{4}(B+iC)$		*Overtone* ($\nu=3$) vibrational sub-space
$\langle 21\|$								$\sqrt{4}(B-iC)$	$2A+D$	$\sqrt{3}(B+iC)$	
$\langle 30\|$									$\sqrt{3}(B-iC)$	$3A$	
$\vdots$											

$$\mathbf{H}^A = A\left(\mathbf{a}_1^\dagger\mathbf{a}_1+\mathbf{1}/2\right)+D\left(\mathbf{a}_2^\dagger\mathbf{a}_2+\mathbf{1}/2\right)$$

$$\varepsilon^A_{n_1n_2} = A\left(n_1+\frac{1}{2}\right)+D\left(n_2+\frac{1}{2}\right)=\frac{A+D}{2}(n_1+n_2+1)+\frac{A-D}{2}(n_1-n_2)$$

Review : *1-D* **a†a** *algebra of U(1) representations*
Review : *Translate* **T***(a) and/or Boost* **B***(b) to construct coherent state*
Review : *Time evolution of coherent state (and "squeezed" states)*

2-D **a†a** *algebra of U(2) representations and R(3) angular momentum operators*
- *2D-Oscillator basic states and operations*
  - *Commutation relations*
  - *Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry*
    - *Anti-commutation relations*
  - *Two-dimensional (or 2-particle) base states: ket-kets and bra-bras*
    - *Outer product arrays*
    - *Entangled 2-particle states*
- *Two-particle (or 2-dimensional) matrix operators*
  - *U(2) Hamiltonian and irreducible representations*
  - ➡ *2D-Oscillator states and related 3D angular momentum multiplets* ⬅
    - *ND multiplets*

R(3) Angular momentum generators by U(2) analysis
Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$
SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

2D-Oscillator states and related 3D angular momentum multiplets

Fundamental eigenstates

The first step is to diagonalize the fundamental 2-by-2 matrix .

$$\langle \mathbf{H} \rangle_{\upsilon=1}^{Fundamental} = \begin{array}{c|cc} n_1,n_2 & |1,0\rangle & |0,1\rangle \\ \hline \langle 1,0| & A & B-iC \\ \langle 0,1| & B+iC & D \end{array} + \frac{A+D}{2}\mathbf{1}$$

Group reorganized "Big-Endian" indexing

(...*00*,*10*,*20*..*01*,*11*,*21*,*31* ...*02*,*12*,*22*,*32*...)

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The first step is to diagonalize the fundamental 2-by-2 matrix .

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$\langle 0,1\vert$	$B+iC$	D

$+\frac{A+D}{2}\mathbf{1}$

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(...00,10,20..01,11,21,31 ...02,12,22,32...)

Recall decomposition of **H** (Lectures 6-10)

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} + \frac{A+D}{2}\mathbf{1} = (A+D)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{2} + 2C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{1}{2} + (A-D)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{2}$$

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$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}+\frac{A+D}{2}\mathbf{1}=(A+D)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+2B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{2}+2C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{1}{2}+(A-D)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{2}$$

in terms of Jordan-Pauli spin operators.

$$\mathbf{H}=\Omega_0\mathbf{1}+\boldsymbol{\Omega}\bullet\vec{\mathbf{S}}=\Omega_0\mathbf{1}+\Omega_B\mathbf{S}_B+\Omega_C\mathbf{S}_C+\Omega_A\mathbf{S}_A \quad (ABC \text{ Optical vector notation})$$

$$=\Omega_0\mathbf{1}+\Omega_X\mathbf{S}_X+\Omega_Y\mathbf{S}_Y+\Omega_Z\mathbf{S}_Z \quad (XYZ \text{ Electron spin notation})$$

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Frequency eigenvalues $\omega_\pm$ of $\mathbf{H}\text{-}\Omega_0\mathbf{1}/2$ and *fundamental transition frequency* $\Omega = \omega_+ - \omega_-$:

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Polar angles (φ,ϑ) of $+\mathbf{\Omega}$-vector (or polar angles $(\varphi,\vartheta\pm\pi)$ of $-\mathbf{\Omega}$-vector) gives **H** eigenvectors.

$$|\omega_+\rangle = \begin{pmatrix} e^{-i\varphi/2}\cos\frac{\vartheta}{2} \\ e^{i\varphi/2}\sin\frac{\vartheta}{2} \end{pmatrix}, \quad |\omega_-\rangle = \begin{pmatrix} -e^{-i\varphi/2}\sin\frac{\vartheta}{2} \\ e^{i\varphi/2}\cos\frac{\vartheta}{2} \end{pmatrix} \quad \text{where: } \begin{cases} \cos\vartheta = \frac{A\text{-}D}{\Omega} \\ \tan\varphi = \frac{C}{B} \end{cases}$$

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More important for the general solution, are the *eigen-creation operators* $\mathbf{a}^\dagger_+$ *and* $\mathbf{a}^\dagger_-$ defined by

$$\mathbf{a}_+^\dagger = e^{-i\varphi/2}\left(\cos\frac{\vartheta}{2}\mathbf{a}_1^\dagger + e^{i\varphi}\sin\frac{\vartheta}{2}\mathbf{a}_2^\dagger\right), \quad \mathbf{a}_-^\dagger = e^{-i\varphi/2}\left(-\sin\frac{\vartheta}{2}\mathbf{a}_1^\dagger + e^{i\varphi}\cos\frac{\vartheta}{2}\mathbf{a}_2^\dagger\right)$$

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More important for the general solution, are the *eigen-creation operators* $\mathbf{a}^\dagger_+$ *and* $\mathbf{a}^\dagger_-$ defined by

$$\mathbf{a}_+^\dagger = e^{-i\varphi/2}\left(\cos\frac{\vartheta}{2}\mathbf{a}_1^\dagger + e^{i\varphi}\sin\frac{\vartheta}{2}\mathbf{a}_2^\dagger\right), \quad \mathbf{a}_-^\dagger = e^{-i\varphi/2}\left(-\sin\frac{\vartheta}{2}\mathbf{a}_1^\dagger + e^{i\varphi}\cos\frac{\vartheta}{2}\mathbf{a}_2^\dagger\right)$$

$\mathbf{a}_\pm^\dagger$ create **H** eigenstates directly from the ground state.

$$\mathbf{a}_+^\dagger|0\rangle = |\omega_+\rangle, \quad \mathbf{a}_-^\dagger|0\rangle = |\omega_-\rangle$$

2D-Oscillator states and related 3D angular momentum multiplets

Group reorganized "Little-Endian" indexing (...01,02,03..10,11,12,13 ... 20,21,22,23,...)

Setting (B=0=C) and (A=ω_+) and (D=ω_-) gives diagonal block matrices.

$\langle \mathbf{H} \rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$

	$\vert 00\rangle$	$\vert 01\rangle$	$\vert 10\rangle$	$\vert 02\rangle$	$\vert 11\rangle$	$\vert 20\rangle$	$\vert 03\rangle$	$\vert 12\rangle$	$\vert 21\rangle$	$\vert 30\rangle$	$\cdots$
$\langle 00\vert$	0										
$\langle 01\vert$		ω_-									
$\langle 10\vert$			ω_+								
$\langle 02\vert$				$2\omega_-$							
$\langle 11\vert$					$\omega_+ + \omega_-$						
$\langle 20\vert$						$2\omega_+$					
$\langle 03\vert$							$3\omega_-$				
$\langle 12\vert$								$\omega_+ + 2\omega_-$			
$\langle 21\vert$									$2\omega_+ + \omega_-$		
$\langle 30\vert$										$3\omega_+$	
$\vdots$											

$$\omega_+ - \omega_- = \Omega$$
$$= \sqrt{(2B)^2 + (2C)^2 + (A-D)^2}$$
$$= A - D$$

$$\mathbf{H}^A = A\left(\mathbf{a}_1^\dagger \mathbf{a}_1 + \mathbf{1}/2\right) + D\left(\mathbf{a}_2^\dagger \mathbf{a}_2 + \mathbf{1}/2\right)$$

2D-Oscillator states *and related 3D angular momentum multiplets*

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$$\varepsilon^A_{n_1 n_2} = A\left(n_1 + \frac{1}{2}\right) + D\left(n_2 + \frac{1}{2}\right) = \frac{A+D}{2}(n_1 + n_2 + 1) + \frac{A-D}{2}(n_1 - n_2)$$

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Define *total quantum number* $\upsilon=2j$ and half-difference or *asymmetry quantum number m*

$$\upsilon = n_1 + n_2 = 2j \qquad j = \frac{n_1 + n_2}{2} = \frac{\upsilon}{2} \qquad m = \frac{n_1 - n_2}{2}$$

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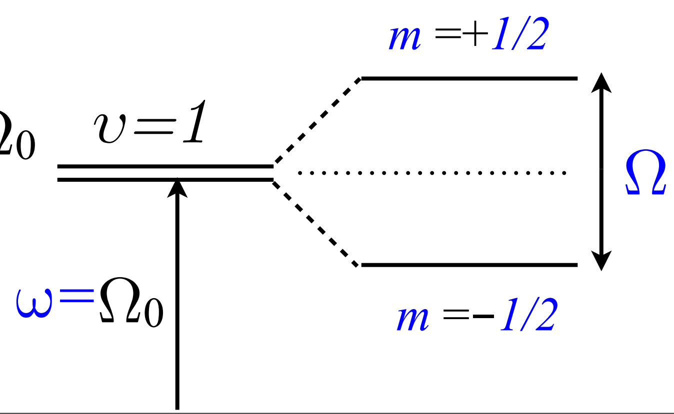
$\upsilon+1=2j+1$ multiplies *base frequency* $\omega=\Omega_0$

m multiplies *beat frequency* Ω



$$\omega_+ = \Omega_0 + \Omega\left(+\tfrac{1}{2}\right)$$
$$\omega_- = \Omega_0 + \Omega\left(-\tfrac{1}{2}\right)$$

Review : *1-D* **a**†**a** *algebra of U(1) representations*

Review : *Translate* **T**(*a*) *and/or Boost* **B**(*b*) *to construct coherent state*

Review : *Time evolution of coherent state (and "squeezed" states)*

2-D **a**†**a** *algebra of U(2) representations and R(3) angular momentum operators*

- *2D-Oscillator basic states and operations*
  - *Commutation relations*
  - *Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry*
    - *Anti-commutation relations*
  - *Two-dimensional (or 2-particle) base states: ket-kets and bra-bras*
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  - ➡ *2D-Oscillator states and related 3D angular momentum multiplets* ⬅
    - *ND multiplets*

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Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$

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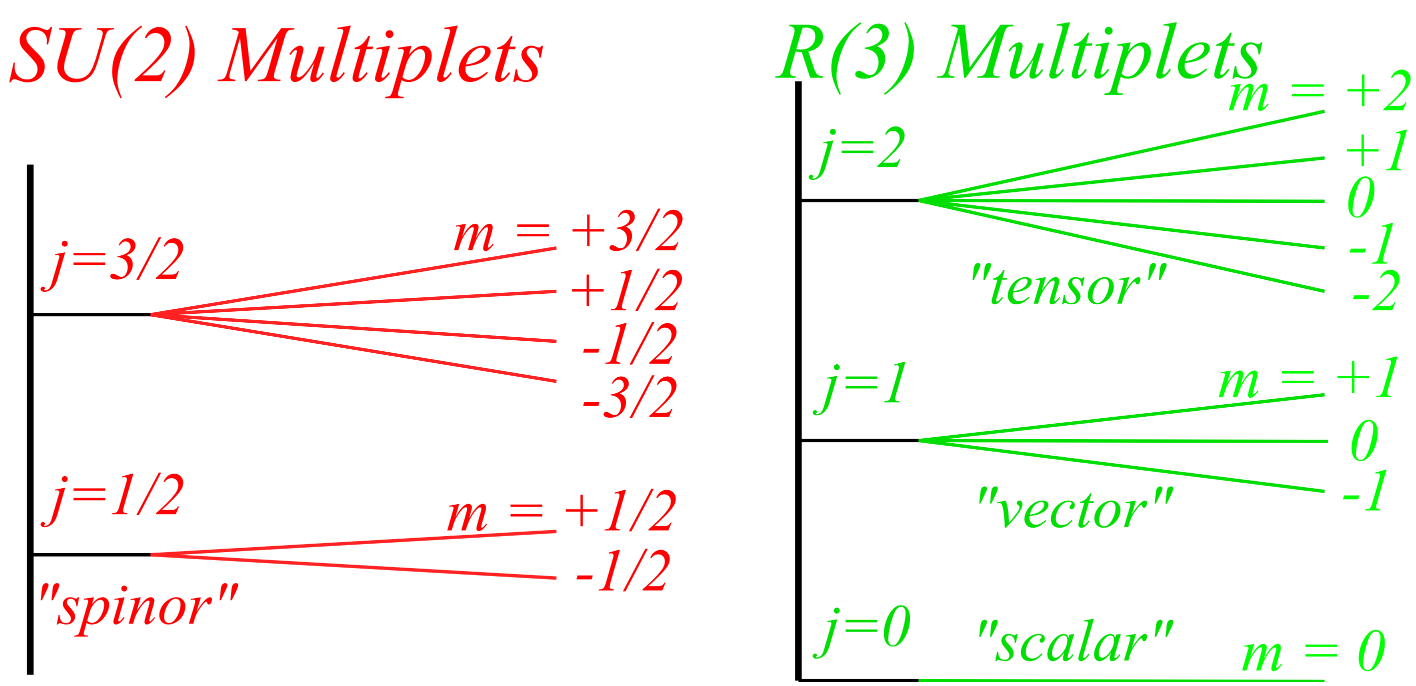

2D-Oscillator states and related 3D angular momentum multiplets

Setting (B=0=C) and (A=ω_+) and (D=ω_-) gives diagonal block matrices.

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$\vdots$											

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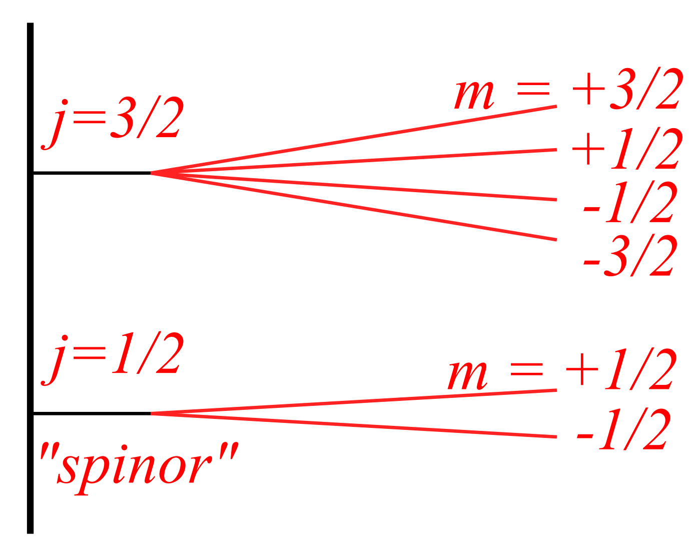
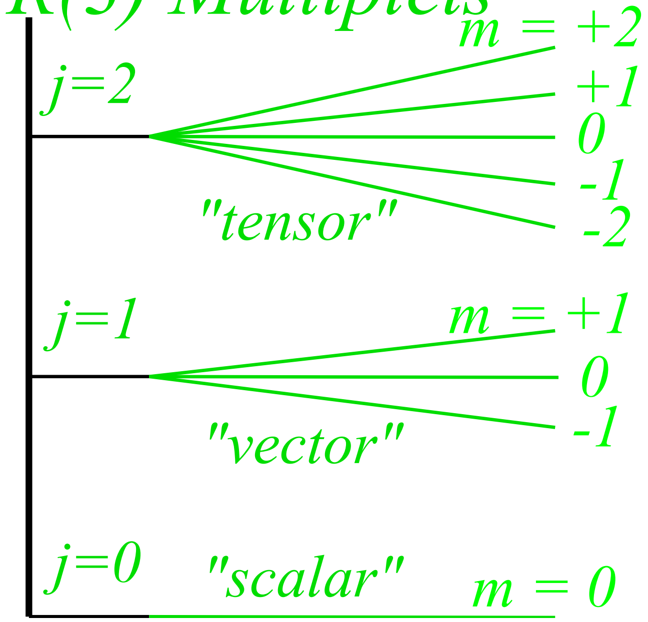
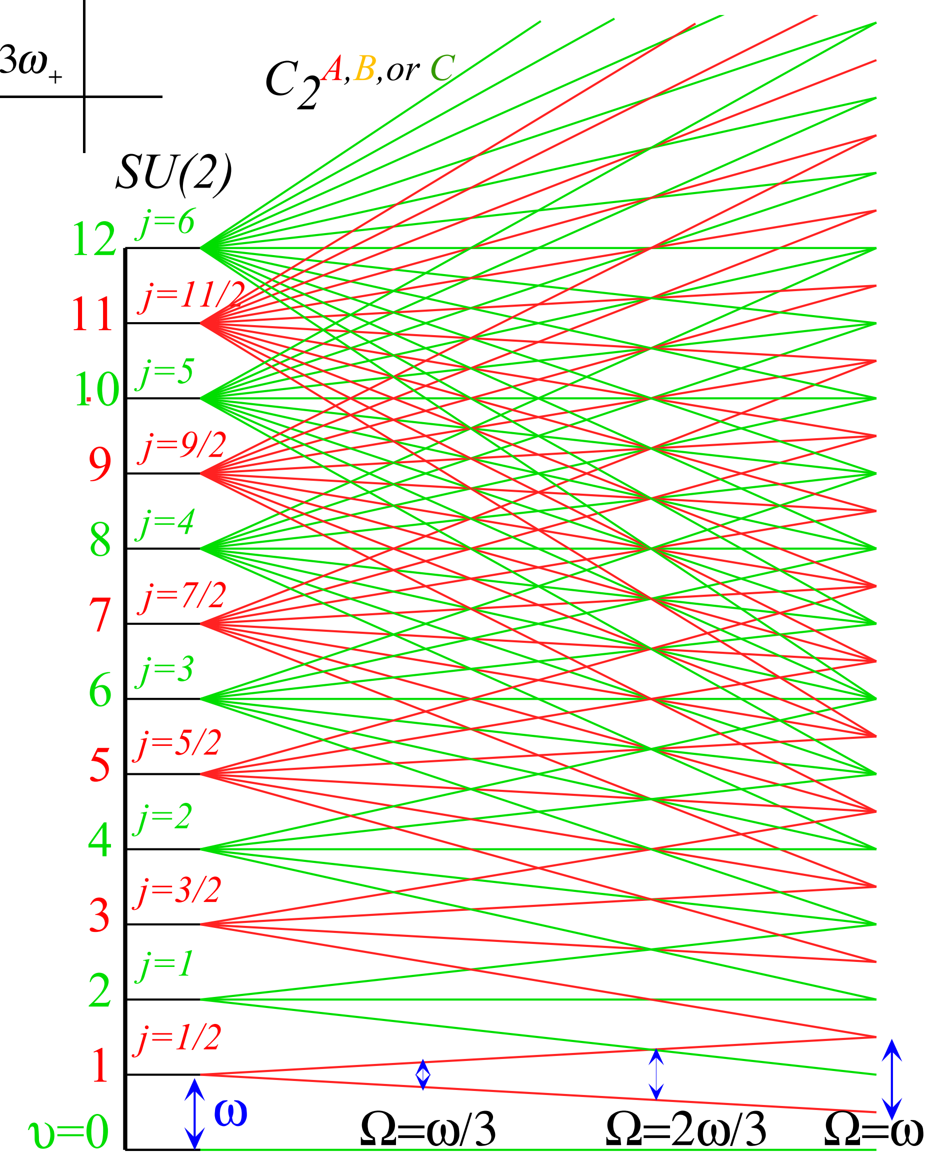
SU(2) Multiplets



R(3) Multiplets





2D-Oscillator states and related 3D angular momentum multiplets

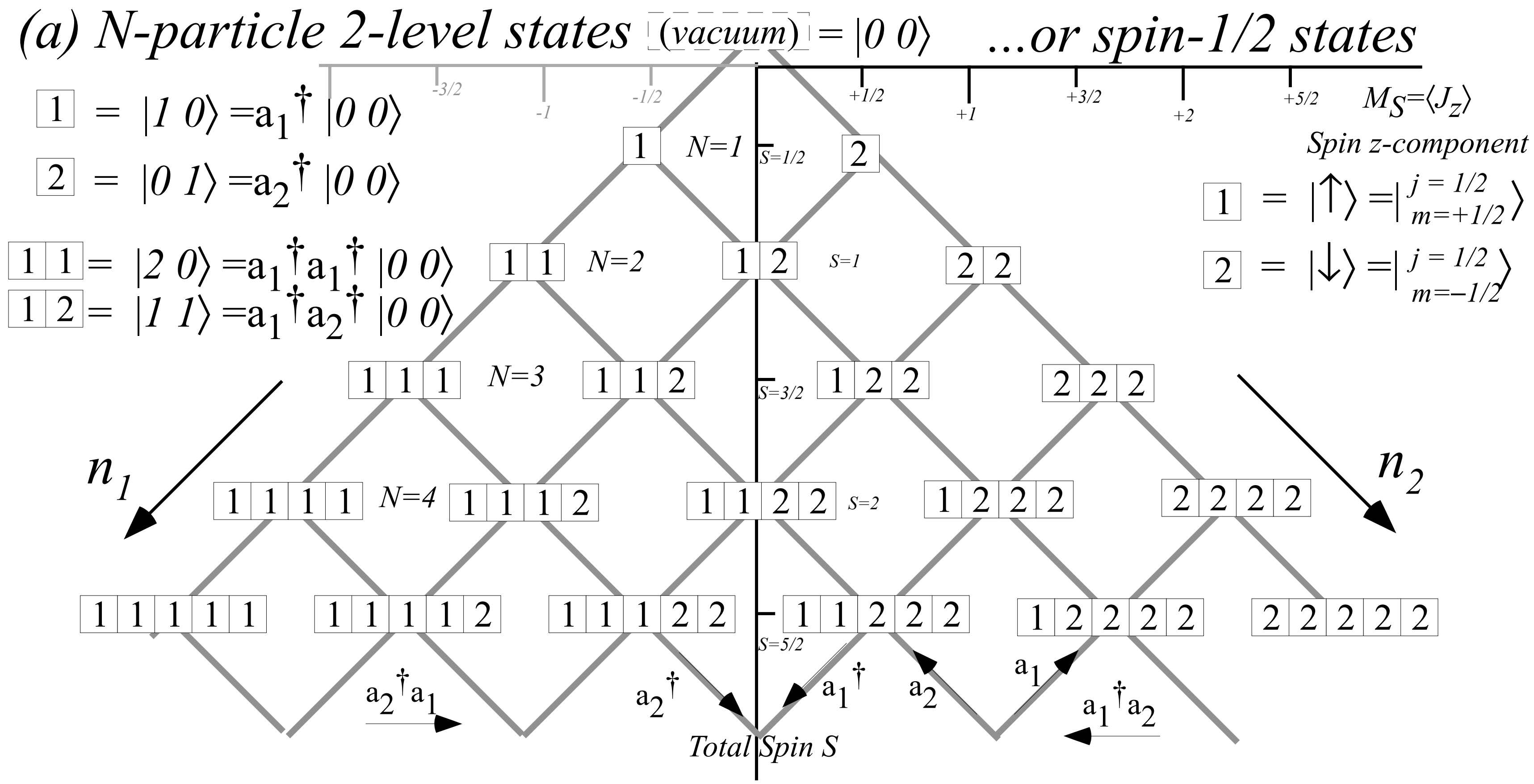
Structure of U(2)

$$\left|{j \atop m}\right\rangle = |n_1 n_2\rangle$$

$j=0$	$\left\|{0 \atop 0}\right\rangle = \|00\rangle$	"scalar"
$j=\frac{1}{2}$	$\left\|{1/2 \atop 1/2}\right\rangle = \|10\rangle = \|\uparrow\rangle$ $\left\|{1/2 \atop -1/2}\right\rangle = \|01\rangle = \|\downarrow\rangle$	"spinor"
$j=1$	$\left\|{1 \atop 1}\right\rangle = \|20\rangle$ $\left\|{1 \atop 0}\right\rangle = \|11\rangle$ $\left\|{1 \atop -1}\right\rangle = \|02\rangle$	"3-vector"
$j=\frac{3}{2}$	$\left\|{3/2 \atop 3/2}\right\rangle = \|30\rangle$ $\left\|{3/2 \atop 1/2}\right\rangle = \|21\rangle$ $\left\|{3/2 \atop -1/2}\right\rangle = \|12\rangle$ $\left\|{3/2 \atop -3/2}\right\rangle = \|03\rangle$	"4-spinor"
$j=2$	$\left\|{2 \atop 2}\right\rangle = \|40\rangle$ $\left\|{2 \atop 1}\right\rangle = \|31\rangle$ $\left\|{2 \atop 0}\right\rangle = \|22\rangle$ $\left\|{2 \atop -1}\right\rangle = \|13\rangle$ $\left\|{2 \atop -2}\right\rangle = \|04\rangle$	"tensor"
	$\vdots$	

$$j = \frac{\upsilon}{2} = \frac{n_1 + n_2}{2} \qquad n_1 = j + m = 2\upsilon + m$$

$$m = \frac{n_1 - n_2}{2} \qquad n_2 = j - m = 2\upsilon - m$$



Review : *1-D* **a†a** *algebra of U(1) representations*
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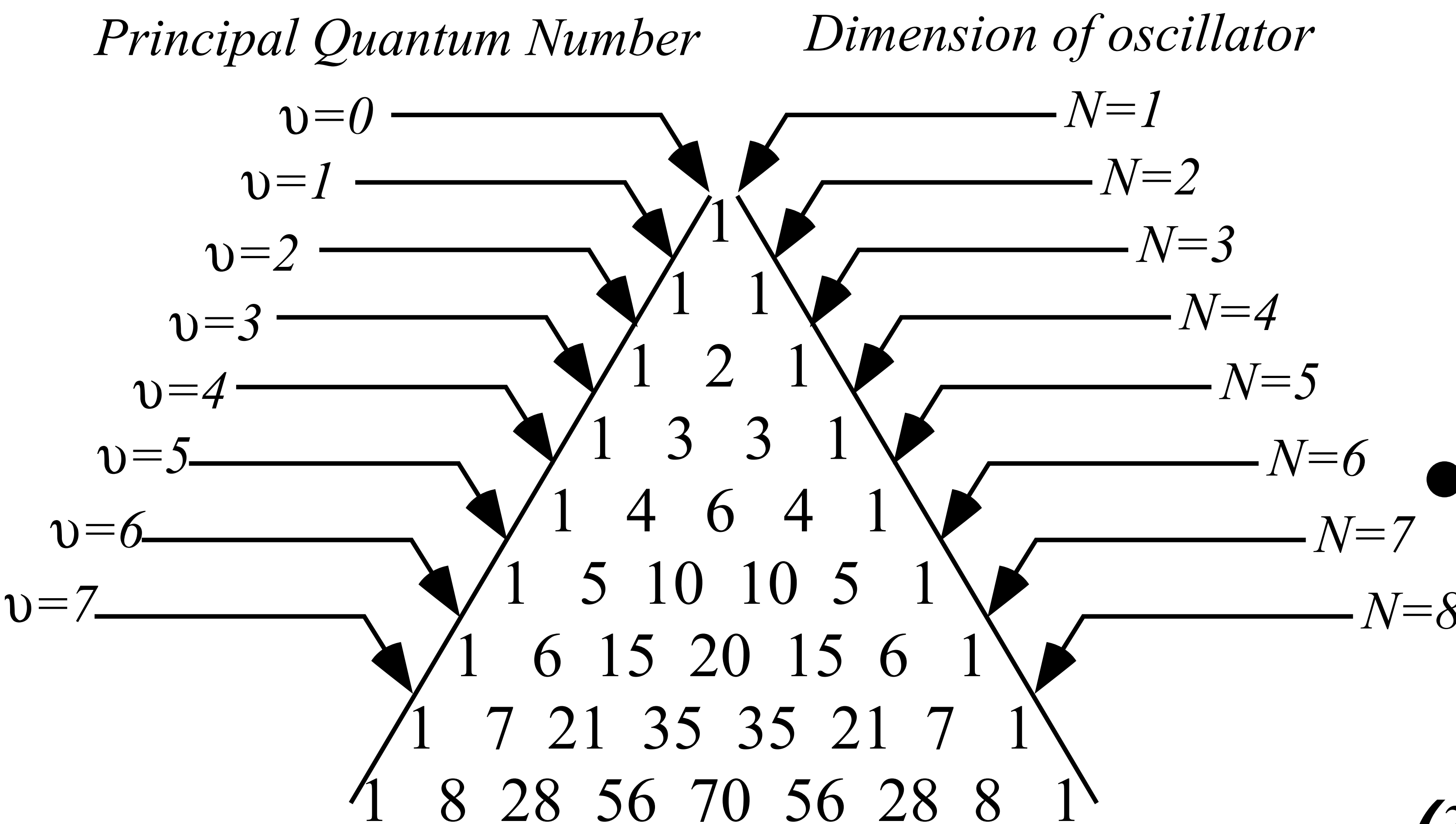
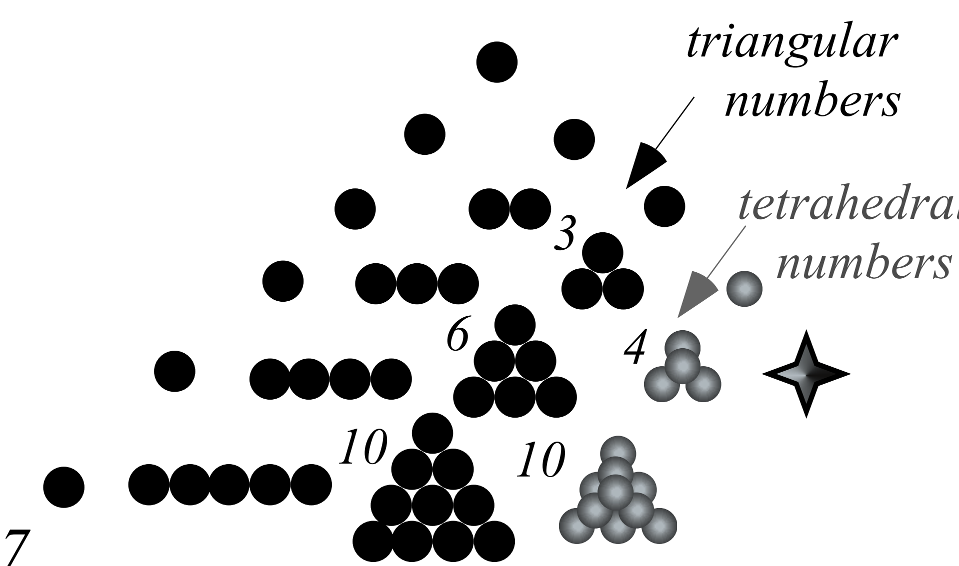
Introducing U(N)

(a) N-D Oscillator Degeneracy ℓ of quamtum level υ

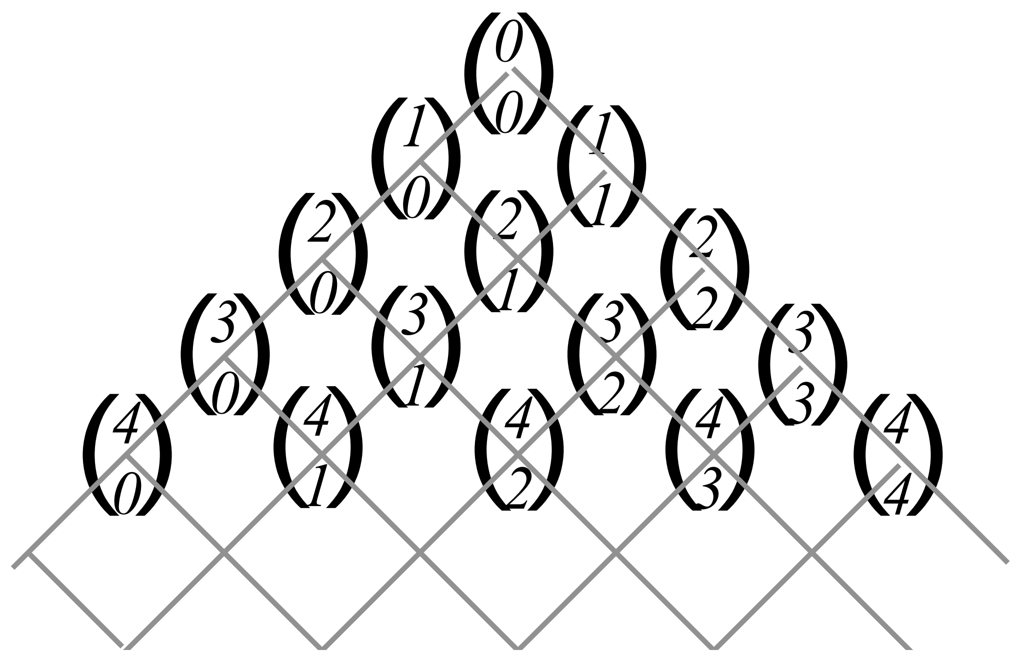


(b) Stacking numbers



(c) Binomial coefficients

$$\frac{(N-1+\upsilon)!}{(N-1)!\,\upsilon!} = \binom{N-1+\upsilon}{\upsilon} = \binom{N-1+\upsilon}{N-1}$$



Introducing U(3)

(b) N-particle 3-level states ...or spin-1 states

[1] $= |1\ 0\ 0\rangle = a_1^\dagger |0\ 0\ 0\rangle$

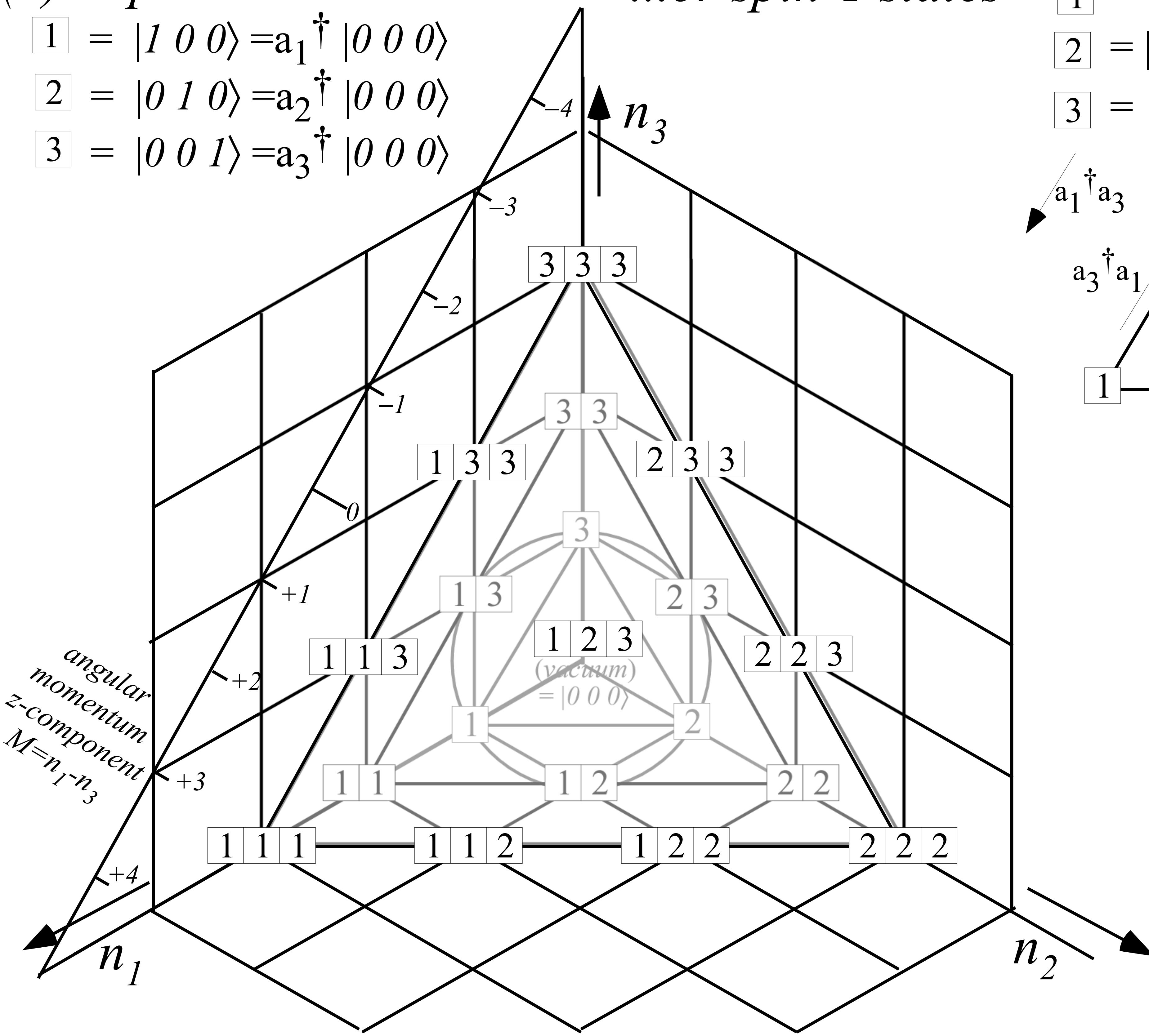
[2] $= |0\ 1\ 0\rangle = a_2^\dagger |0\ 0\ 0\rangle$

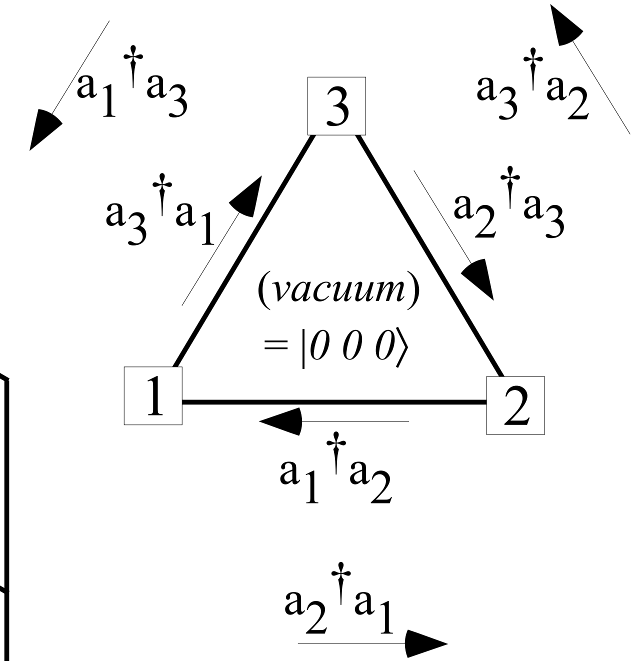
[3] $= |0\ 0\ 1\rangle = a_3^\dagger |0\ 0\ 0\rangle$

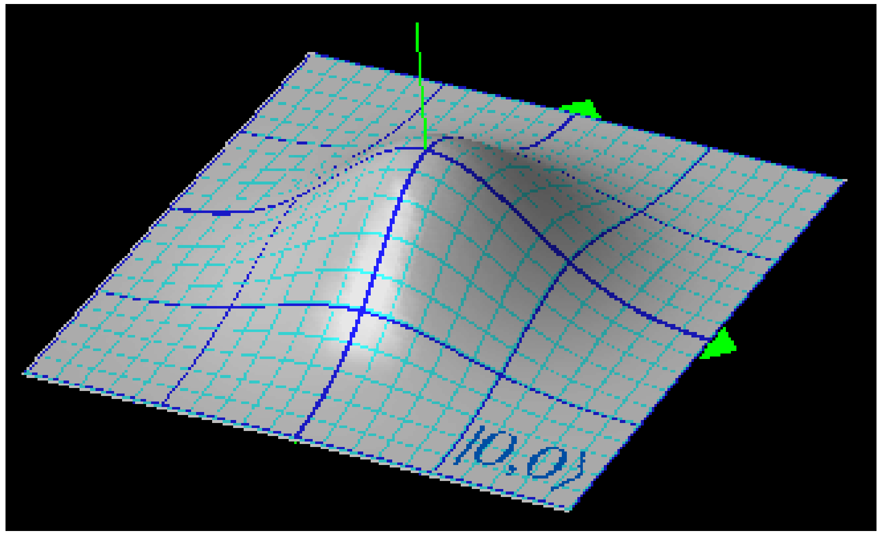
[1] $= |\uparrow\rangle = \left|{j=1 \atop m=+1}\right\rangle$

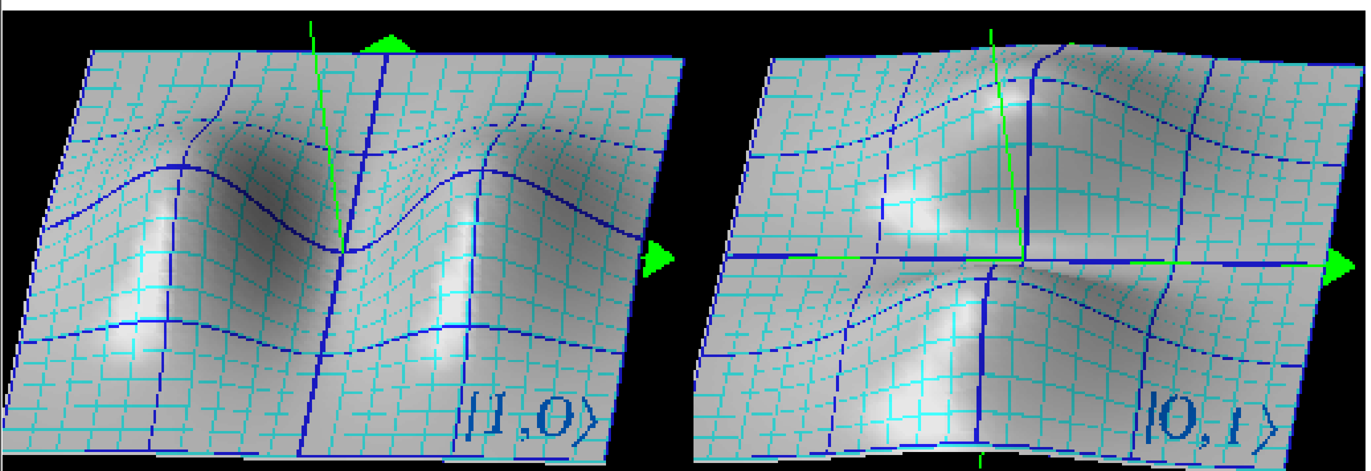
[2] $= |\leftrightarrow\rangle = \left|{j=1 \atop m=0}\right\rangle$

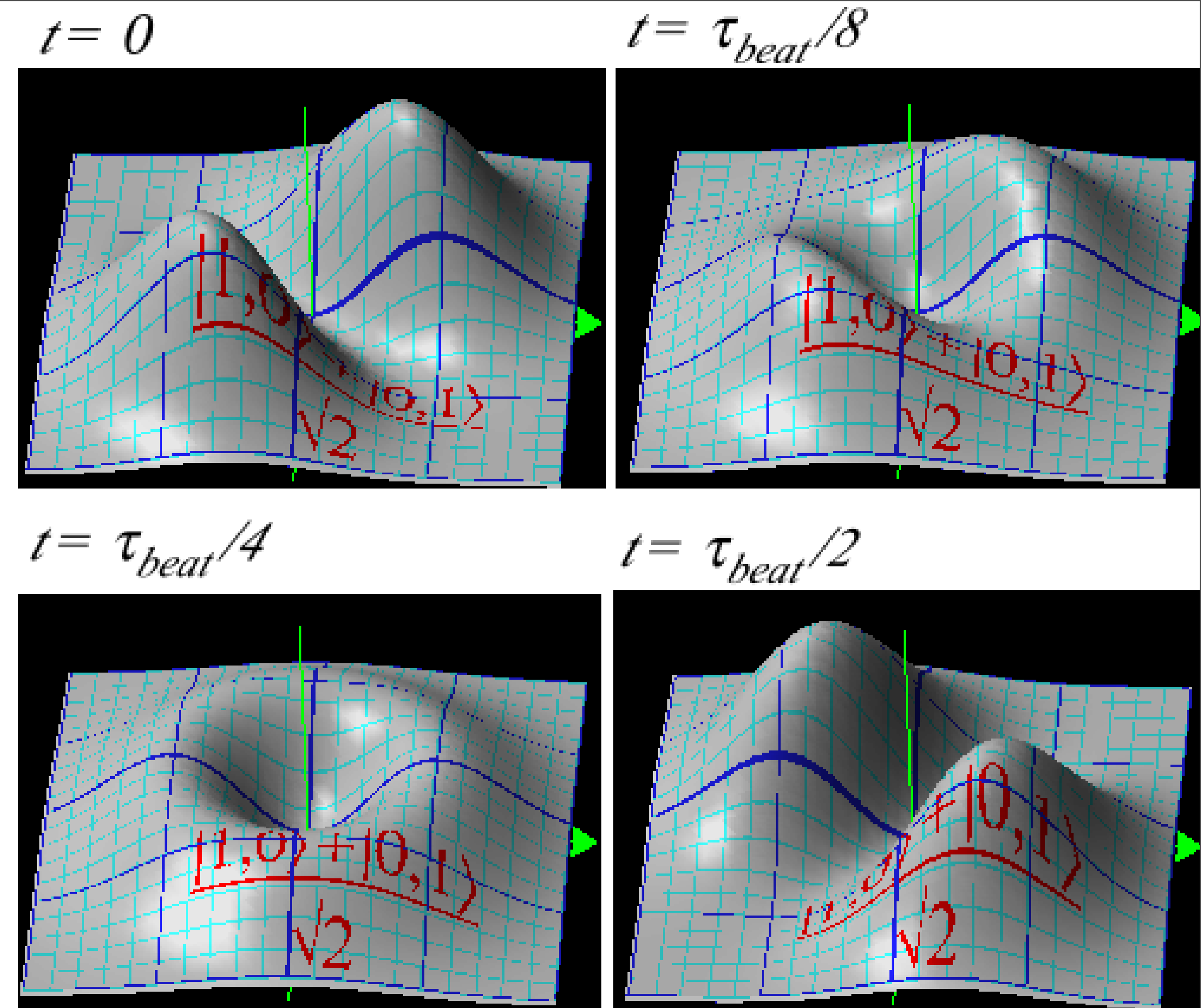
[3] $= |\downarrow\rangle = \left|{j=1 \atop m=-1}\right\rangle$











$$\Psi(x_1,x_2,t)=\frac{1}{2}\left|\psi_{10}(x_1,x_2)e^{-i\omega_{10}t}+\psi_{01}(x_1,x_2)e^{-i\omega_{01}t}\right|^2 e^{-\left(x_1^2+x_2^2\right)}=\frac{e^{-\left(x_1^2+x_2^2\right)}}{2\pi}\left|\sqrt{2}x_1e^{-i\omega_{10}t}+\sqrt{2}x_1e^{-i\omega_{01}t}\right|^2$$

$$=\frac{e^{-\left(x_1^2+x_2^2\right)}}{\pi}\left(x_1^2+x_2^2+2x_1x_2\cos\left(\omega_{10}-\omega_{01}\right)t\right)=\frac{e^{-\left(x_1^2+x_2^2\right)}}{\pi}\begin{cases}\left|x_1+x_2\right|^2 & for\text{: } t=0\\ x_1^2+x_2^2 & for\text{: } t=\tau_{\text{beat}}/4\\ \left|x_1-x_2\right|^2 & for\text{: } t=\tau_{\text{beat}}/2\end{cases}\qquad(21.1.30)$$

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(*...00,10,20..01,11,21,31 ...02,12,22,32...*)
(*...00,10, 01, 20,11, 02, 30, 21, 12, 03, 40, 31,22,...*)

(υ=*1*) or (*j*=*1/2*) block **H** matrices of U(2) oscillator

Use irreps of unit operator $\mathbf{S}_0 = \mathbf{1}$ and spin operators $\{ \mathbf{S}_X, \mathbf{S}_Y, \mathbf{S}_Z \}$. (also known as: $\{ \mathbf{S}_B, \mathbf{S}_C, \mathbf{S}_A \}$)

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + 2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} + (A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

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(*...00,10, 01, 20,11, 02, 30, 21, 12, 03, 40, 31,22,...*)

(υ=*1*) or (*j*=*1/2*) block **H** matrices of U(2) oscillator

Use irreps of unit operator $\mathbf{S}_0 = \mathbf{1}$ and spin operators $\{\mathbf{S}_X, \mathbf{S}_Y, \mathbf{S}_Z\}$. (also known as: $\{\mathbf{S}_B, \mathbf{S}_C, \mathbf{S}_A\}$)

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + 2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} + (A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

(υ=*2*) or (*j*=*1*) 3-by-3 block uses their vector irreps.

$$\begin{pmatrix} 2A & \sqrt{2}(B-iC) & \cdot \\ \sqrt{2}(B+iC) & A+D & \sqrt{2}(B-iC) \\ \cdot & \sqrt{2}(B+iC) & 2D \end{pmatrix} = (A+D)\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} + 2B\begin{pmatrix} \cdot & \frac{\sqrt{2}}{2} & \cdot \\ \frac{\sqrt{2}}{2} & \cdot & \frac{\sqrt{2}}{2} \\ \cdot & \frac{\sqrt{2}}{2} & \cdot \end{pmatrix} + 2C\begin{pmatrix} \cdot & -i\frac{\sqrt{2}}{2} & \cdot \\ i\frac{\sqrt{2}}{2} & \cdot & -i\frac{\sqrt{2}}{2} \\ \cdot & i\frac{\sqrt{2}}{2} & \cdot \end{pmatrix} + (A-D)\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

R(3) Angular momentum generators by U(2) analysis

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$$\begin{pmatrix} 3A & \sqrt{3}(B-iC) & & \\ \sqrt{3}(B+iC) & 2A+D & \sqrt{4}(B-iC) & \\ & \sqrt{4}(B+iC) & A+2D & \sqrt{3}(B-iC) \\ & & \sqrt{3}(B+iC) & 3D \end{pmatrix} = \frac{3(A+D)}{2}\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} + 2B\begin{pmatrix} \cdot & \frac{\sqrt{3}}{2} & \cdot & \cdot \\ \frac{\sqrt{3}}{2} & \cdot & \frac{\sqrt{4}}{2} & \cdot \\ \cdot & \frac{\sqrt{4}}{2} & \cdot & \frac{\sqrt{3}}{2} \\ \cdot & \cdot & \frac{\sqrt{3}}{2} & \cdot \end{pmatrix} + 2C\begin{pmatrix} \cdot & -i\frac{\sqrt{3}}{2} & \cdot & \cdot \\ i\frac{\sqrt{3}}{2} & \cdot & -i\frac{\sqrt{4}}{2} & \cdot \\ \cdot & i\frac{\sqrt{4}}{2} & \cdot & -i\frac{\sqrt{3}}{2} \\ \cdot & \cdot & i\frac{\sqrt{3}}{2} & \cdot \end{pmatrix} + (A-D)\begin{pmatrix} \frac{3}{2} & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{2} & \cdot & \cdot \\ \cdot & \cdot & -\frac{1}{2} & \cdot \\ \cdot & \cdot & \cdot & -\frac{3}{2} \end{pmatrix}$$

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(υ=*2j*) or (2j+*1*)-by-(2j+*1*) block uses $D^{(j)}(\mathbf{s}_\mu)$ irreps of U(2) or R(3).

$$\langle \mathbf{H} \rangle^{j-block} = 2j\Omega_0 \langle \mathbf{1} \rangle^j + \Omega_X \langle \mathbf{s}_X \rangle^j + \Omega_Y \langle \mathbf{s}_Y \rangle^j + \Omega_Z \langle \mathbf{s}_Z \rangle^j$$

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All j-block matrix operators factor into *raise-n-lower* operators $\mathbf{s}_\pm = \mathbf{s}_X \pm i\mathbf{s}_Y$ plus the diagonal $\mathbf{s}_Z$

$$\langle \mathbf{H} \rangle^{j-block} = 2j\Omega_0 \langle \mathbf{1} \rangle^j + \left[(\Omega_X - i\Omega_Y)\langle \mathbf{s}_X + i\mathbf{s}_Y \rangle^j + (\Omega_X + i\Omega_Y)\langle \mathbf{s}_X - i\mathbf{s}_Y \rangle^j \right]/2 + \Omega_Z \langle \mathbf{s}_Z \rangle^j$$

Review : *1-D* $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(1) representations*
Review : *Translate* $\mathbf{T}(a)$ *and/or Boost* $\mathbf{B}(b)$ *to construct coherent state*
Review : *Time evolution of coherent state (and "squeezed" states)*

2-D $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(2) representations and R(3) angular momentum operators*
2D-Oscillator basic states and operations
Commutation relations
Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry
Anti-commutation relations
Two-dimensional (or 2-particle) base states: ket-kets and bra-bras
Outer product arrays
Entangled 2-particle states
Two-particle (or 2-dimensional) matrix operators
U(2) Hamiltonian and irreducible representations
2D-Oscillator states and related 3D angular momentum multiplets
ND multiplets
R(3) Angular momentum generators by U(2) analysis
→ *Angular momentum raise-n-lower operators* $\mathbf{s}_+$ and $\mathbf{s}_-$ ←
SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Angular momentum raise-n-lower operators $\mathbf{S}_+$ and $\mathbf{S}_-$

$$\mathbf{s}_+ = \mathbf{s}_X + i\mathbf{s}_Y \quad \text{and} \quad \mathbf{s}_- = \mathbf{s}_X - i\mathbf{s}_Y = \mathbf{s}_+^\dagger$$

Starting with *j=1/2* we see that $\mathbf{S}+$ is an *elementary projection operator* $\mathbf{e}_{12} = |1\rangle\langle 2| = \mathbf{P}_{12}$

$$\langle \mathbf{s}_+ \rangle^{\frac{1}{2}} = D^{\frac{1}{2}}(\mathbf{s}_+) = D^{\frac{1}{2}}(\mathbf{s}_X + i\mathbf{s}_Y) = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + i\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathbf{P}_{12}$$

Such operators can be upgraded to *creation-destruction operator* combinations $\mathbf{a}^\dagger\mathbf{a}$

$$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2 = \mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow \quad , \qquad \mathbf{s}_- = (\mathbf{a}_1^\dagger \mathbf{a}_2)^\dagger = \mathbf{a}_2^\dagger \mathbf{a}_1 = \mathbf{a}_\downarrow^\dagger \mathbf{a}_\uparrow$$

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Hamilton-Pauli-Jordan representation of $\mathbf{s}_Z$ is: $\langle \mathbf{s}_Z \rangle^{\left(\frac{1}{2}\right)} = D^{\left(\frac{1}{2}\right)}(\mathbf{s}_Z) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

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Let $\mathbf{a}_1^\dagger = \mathbf{a}_\uparrow^\dagger$ create up-spin $\uparrow$

$$|1\rangle = |\uparrow\rangle = \left| \begin{matrix} 1/2 \\ +1/2 \end{matrix} \right\rangle = \mathbf{a}_1^\dagger|0\rangle = \mathbf{a}_\uparrow^\dagger|0\rangle$$

Angular momentum raise-n-lower operators $\mathbf{S}_+$ and $\mathbf{S}_-$

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Let $\mathbf{a}_2^\dagger = \mathbf{a}_\downarrow^\dagger$ create dn-spin $\downarrow$

$$|2\rangle = |\downarrow\rangle = \left| \begin{matrix} 1/2 \\ -1/2 \end{matrix} \right\rangle = \mathbf{a}_2^\dagger |0\rangle = \mathbf{a}_\downarrow^\dagger |0\rangle$$

Angular momentum raise-n-lower operators $\mathbf{S}_+$ and $\mathbf{S}_-$

$$\mathbf{s}_+ = \mathbf{s}_X + i\mathbf{s}_Y \quad \text{and} \quad \mathbf{s}_- = \mathbf{s}_X - i\mathbf{s}_Y = \mathbf{s}_+^\dagger$$

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$$\langle \mathbf{s}_+ \rangle^{\frac{1}{2}} = D^{\frac{1}{2}}(\mathbf{s}_+) = D^{\frac{1}{2}}(\mathbf{s}_X + i\mathbf{s}_Y) = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + i \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathbf{P}_{12}$$

Such operators can be upgraded to *creation-destruction operator* combinations $\mathbf{a}^\dagger\mathbf{a}$

$$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2 = \mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow \quad , \qquad \mathbf{s}_- = (\mathbf{a}_1^\dagger \mathbf{a}_2)^\dagger = \mathbf{a}_2^\dagger \mathbf{a}_1 = \mathbf{a}_\downarrow^\dagger \mathbf{a}_\uparrow$$

Hamilton-Pauli-Jordan representation of $\mathbf{s}_Z$ is: $\langle \mathbf{s}_Z \rangle^{(\frac{1}{2})} = D^{(\frac{1}{2})}(\mathbf{s}_Z) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

This suggests an $\mathbf{a}^\dagger\mathbf{a}$ form for $\mathbf{s}_Z$.

$$\mathbf{s}_Z = \tfrac{1}{2}(\mathbf{a}_1^\dagger \mathbf{a}_1 - \mathbf{a}_2^\dagger \mathbf{a}_2) = \tfrac{1}{2}(\mathbf{a}_\uparrow^\dagger \mathbf{a}_\uparrow - \mathbf{a}_\downarrow^\dagger \mathbf{a}_\downarrow)$$

Let $\mathbf{a}_1^\dagger = \mathbf{a}_\uparrow^\dagger$ create up-spin $\uparrow$

$$|1\rangle = |\uparrow\rangle = \left| \begin{matrix} 1/2 \\ +1/2 \end{matrix} \right\rangle = \mathbf{a}_1^\dagger |0\rangle = \mathbf{a}_\uparrow^\dagger |0\rangle$$

Let $\mathbf{a}_2^\dagger = \mathbf{a}_\downarrow^\dagger$ create dn-spin $\downarrow$

$$|2\rangle = |\downarrow\rangle = \left| \begin{matrix} 1/2 \\ -1/2 \end{matrix} \right\rangle = \mathbf{a}_2^\dagger |0\rangle = \mathbf{a}_\downarrow^\dagger |0\rangle$$

$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2 = \mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow$ destroys dn-spin $\downarrow$

creates up-spin $\uparrow$

to raise angular momentum by one $\hbar$ unit

$$\mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow |\downarrow\rangle = |\uparrow\rangle \quad \text{or:} \quad \mathbf{a}_1^\dagger \mathbf{a}_2 |2\rangle = |1\rangle$$

Angular momentum raise-n-lower operators $\mathbf{S}_+$ and $\mathbf{S}_-$

$$\boxed{\mathbf{s}_+ = \mathbf{s}_X + i\mathbf{s}_Y} \quad \text{and} \quad \boxed{\mathbf{s}_- = \mathbf{s}_X - i\mathbf{s}_Y = \mathbf{s}_+^\dagger}$$

Starting with $j=1/2$ we see that $\mathbf{S}+$ is an *elementary projection operator* $\mathbf{e}_{12} = |1\rangle\langle 2| = \mathbf{P}_{12}$

$$\langle \mathbf{s}_+ \rangle^{\frac{1}{2}} = D^{\frac{1}{2}}(\mathbf{s}_+) = D^{\frac{1}{2}}(\mathbf{s}_X + i\mathbf{s}_Y) = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + i \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathbf{P}_{12}$$

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Let $\mathbf{a}_1^\dagger = \mathbf{a}_\uparrow^\dagger$ create up-spin $\uparrow$

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$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2 = \mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow$ destroys dn-spin $\downarrow$

creates up-spin $\uparrow$

to <u>raise</u> angular momentum by one $\hbar$ unit

$$\mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow |\downarrow\rangle = |\uparrow\rangle \quad \text{or:} \quad \mathbf{a}_1^\dagger \mathbf{a}_2 |2\rangle = |1\rangle$$

Let $\mathbf{a}_2^\dagger = \mathbf{a}_\downarrow^\dagger$ create dn-spin $\downarrow$

$$|2\rangle = |\downarrow\rangle = \left| \begin{matrix} 1/2 \\ -1/2 \end{matrix} \right\rangle = \mathbf{a}_2^\dagger |0\rangle = \mathbf{a}_\downarrow^\dagger |0\rangle$$

$\mathbf{s}_- = \mathbf{a}_2^\dagger \mathbf{a}_1 = \mathbf{a}_\downarrow^\dagger \mathbf{a}_\uparrow$ destroys up-spin $\uparrow$

creates dn-spin $\downarrow$

to <u>lower</u> angular momentum by one $\hbar$ unit

$$\mathbf{a}_\downarrow^\dagger \mathbf{a}_\uparrow |\uparrow\rangle = |\downarrow\rangle \quad \text{or:} \quad \mathbf{a}_2^\dagger \mathbf{a}_1 |1\rangle = |2\rangle$$

Review : *1-D* $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(1) representations*
Review : *Translate* $\mathbf{T}(a)$ *and/or Boost* $\mathbf{B}(b)$ *to construct coherent state*
Review : *Time evolution of coherent state (and "squeezed" states)*

2-D $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(2) representations and R(3) angular momentum operators*
2D-Oscillator basic states and operations
Commutation relations
Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry
Anti-commutation relations
Two-dimensional (or 2-particle) base states: ket-kets and bra-bras
Outer product arrays
Entangled 2-particle states
Two-particle (or 2-dimensional) matrix operators
U(2) Hamiltonian and irreducible representations
2D-Oscillator states and related 3D angular momentum multiplets
ND multiplets
R(3) Angular momentum generators by U(2) analysis
Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$
➡ *SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors* ⬅

SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

U(2) boson oscillator states $|n_1,n_2\rangle$

Oscillator total quanta: $\upsilon=(n_1+n_2)$

$$|n_1 n_2\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{n_1}\left(\mathbf{a}_2^\dagger\right)^{n_2}}{\sqrt{n_1!\,n_2!}}|0\ 0\rangle$$

SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

U(2) boson oscillator states $|n_1,n_2\rangle$= R(3) spin or rotor states $\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$

Oscillator total quanta: $\upsilon=(n_1+n_2)$ Rotor total momenta: $j=\upsilon/2$

$$\left|n_1 n_2\right\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{n_1}\left(\mathbf{a}_2^\dagger\right)^{n_2}}{\sqrt{n_1!\,n_2!}}\left|0\,0\right\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{j+m}\left(\mathbf{a}_2^\dagger\right)^{j-m}}{\sqrt{(j+m)!(j-m)!}}\left|0\,0\right\rangle = \left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$$

SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

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Oscillator total quanta: $\upsilon=(n_1+n_2)$ Rotor total momenta: $j=\upsilon/2$ and z-momenta: $m=(n_1-n_2)/2$

$$\left|n_1 n_2\right\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{n_1}\left(\mathbf{a}_2^\dagger\right)^{n_2}}{\sqrt{n_1!\,n_2!}}\left|0\,0\right\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{j+m}\left(\mathbf{a}_2^\dagger\right)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}\left|0\,0\right\rangle = \left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$$

$j=\upsilon/2=(n_1-n_2)/2$

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SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

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$j=\upsilon/2=(n_1+n_2)/2$

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$n_1=j+m$

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$U(2)$ boson oscillator states = $U(2)$ spinor states

$$\left|n_\uparrow n_\downarrow\right\rangle = \frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{n_1}\left(\mathbf{a}_\downarrow^\dagger\right)^{n_2}}{\sqrt{n_\uparrow!\,n_2!}}\left|0\,0\right\rangle = \frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{j+m}\left(\mathbf{a}_\downarrow^\dagger\right)^{j-m}}{\sqrt{(j+m)!(j-m)!}}\left|0\,0\right\rangle = \left|{j \atop m}\right\rangle$$

SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

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U(2) boson oscillator states = *U(2)* spinor states

$$\left|n_\uparrow n_\downarrow\right\rangle = \frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{n_1}\left(\mathbf{a}_\downarrow^\dagger\right)^{n_2}}{\sqrt{n_\uparrow!\,n_2!}}\left|0\,0\right\rangle = \frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{j+m}\left(\mathbf{a}_\downarrow^\dagger\right)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}\left|0\,0\right\rangle = \left|{j \atop m}\right\rangle$$

Oscillator $\mathbf{a}^\dagger\mathbf{a}$...

$$\mathbf{a}_1^\dagger\mathbf{a}_2\left|n_1 n_2\right\rangle=\sqrt{n_1+1}\sqrt{n_2}\left|n_1+1\; n_2-1\right\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_1\left|n_1 n_2\right\rangle=\sqrt{n_1}\sqrt{n_2+1}\left|n_1-1\; n_2+1\right\rangle$$

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$U(2)$ boson oscillator states = $U(2)$ spinor states

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Oscillator $\mathbf{a}^\dagger\mathbf{a}$ give $\mathbf{s}_+$ matrices.

$$\mathbf{a}_1^\dagger\mathbf{a}_2\left|n_1 n_2\right\rangle=\sqrt{n_1+1}\sqrt{n_2}\left|n_1+1\; n_2-1\right\rangle \Rightarrow \mathbf{s}_+\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle=\sqrt{j+m+1}\sqrt{j-m}\left|\begin{smallmatrix} j \\ m+1 \end{smallmatrix}\right\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_1\left|n_1 n_2\right\rangle=\sqrt{n_1}\sqrt{n_2+1}\left|n_1-1\; n_2+1\right\rangle$$

SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

U(2) boson oscillator states $|n_1,n_2\rangle$ = R(3) spin or rotor states $\left|{j \atop m}\right\rangle$

Oscillator total quanta: $\upsilon=(n_1+n_2)$ Rotor total momenta: $j=\upsilon/2$ and z-momenta: $m=(n_1-n_2)/2$

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U(2) boson oscillator states = *U(2)* spinor states

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SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

U(2) boson oscillator states $|n_1,n_2\rangle$= R(3) spin or rotor states $\left|{}^{j}_{m}\right\rangle$

Oscillator total quanta: $\upsilon=(n_1+n_2)$ Rotor total momenta: $j=\upsilon/2$ and z-momenta: $m=(n_1-n_2)/2$

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U(2) boson oscillator states = *U(2)* spinor states

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1/2-difference of number-ops is $\mathbf{s}_Z$ eigenvalue.

$$\left.\begin{array}{l}\mathbf{a}_1^\dagger\mathbf{a}_1\left|n_1 n_2\right\rangle=n_1\left|n_1 n_2\right\rangle\\ \mathbf{a}_2^\dagger\mathbf{a}_2\left|n_1 n_2\right\rangle=n_2\left|n_1 n_2\right\rangle\end{array}\right\}$$

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SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

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U(2) boson oscillator states = *U(2)* spinor states

$$|n_\uparrow n_\downarrow\rangle = \frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{n_1}\left(\mathbf{a}_\downarrow^\dagger\right)^{n_2}}{\sqrt{n_\uparrow!\,n_2!}}|0\,0\rangle = \frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{j+m}\left(\mathbf{a}_\downarrow^\dagger\right)^{j-m}}{\sqrt{(j+m)!(j-m)!}}|0\,0\rangle = \left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$$

Oscillator $\mathbf{a}^\dagger\mathbf{a}$ give $\mathbf{s}_+$ and $\mathbf{s}_-$ matrices.

$$\mathbf{a}_1^\dagger\mathbf{a}_2|n_1 n_2\rangle=\sqrt{n_1+1}\sqrt{n_2}|n_1+1\; n_2-1\rangle \Rightarrow \mathbf{s}_+\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle=\sqrt{j+m+1}\sqrt{j-m}\left|\begin{smallmatrix} j \\ m+1 \end{smallmatrix}\right\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_1|n_1 n_2\rangle=\sqrt{n_1}\sqrt{n_2+1}|n_1-1\; n_2+1\rangle \Rightarrow \mathbf{s}_-\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle=\sqrt{j+m}\sqrt{j-m+1}\left|\begin{smallmatrix} j \\ m-1 \end{smallmatrix}\right\rangle$$

1/2-difference of number-ops is $\mathbf{s}_Z$ eigenvalue.

$$\left.\begin{matrix}\mathbf{a}_1^\dagger\mathbf{a}_1|n_1 n_2\rangle=n_1|n_1 n_2\rangle \\ \mathbf{a}_2^\dagger\mathbf{a}_2|n_1 n_2\rangle=n_2|n_1 n_2\rangle\end{matrix}\right\} \mathbf{s}_Z\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle=\tfrac{1}{2}\left(\mathbf{a}_1^\dagger\mathbf{a}_1-\mathbf{a}_2^\dagger\mathbf{a}_2\right)\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle=\frac{n_1-n_2}{2}\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle=m\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$$

j=1 vector $\mathbf{s}_+$

$$D^1(\mathbf{s}_+)=D^1(\mathbf{s}_X+i\mathbf{s}_Y)=\begin{pmatrix} \cdot & \frac{\sqrt{2}}{2} & \cdot \\ \frac{\sqrt{2}}{2} & \cdot & \frac{\sqrt{2}}{2} \\ \cdot & \frac{\sqrt{2}}{2} & \cdot \end{pmatrix}+i\begin{pmatrix} \cdot & -i\frac{\sqrt{2}}{2} & \cdot \\ i\frac{\sqrt{2}}{2} & \cdot & -i\frac{\sqrt{2}}{2} \\ \cdot & i\frac{\sqrt{2}}{2} & \cdot \end{pmatrix}=\begin{pmatrix} \cdot & \sqrt{2} & \cdot \\ 0 & \cdot & \sqrt{2} \\ \cdot & 0 & \cdot \end{pmatrix},$$

...and $\mathbf{s}_Z$

$$D^1(\mathbf{s}_Z)=\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

U(2) boson oscillator states $|n_1,n_2\rangle$= R(3) spin or rotor states $\left|{j \atop m}\right\rangle$

Oscillator total quanta: $\upsilon=(n_1+n_2)$ Rotor total momenta: $j=\upsilon/2$ and z-momenta: $m=(n_1-n_2)/2$

$$\left|n_1 n_2\right\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{n_1}\left(\mathbf{a}_2^\dagger\right)^{n_2}}{\sqrt{n_1!\,n_2!}}\left|0\,0\right\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{j+m}\left(\mathbf{a}_2^\dagger\right)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}\left|0\,0\right\rangle = \left|{j \atop m}\right\rangle$$

$$j=\upsilon/2=(n_1+n_2)/2 \qquad m=(n_1-n_2)/2$$

$$n_1=j+m \qquad n_2=j-m$$

U(2) boson oscillator states = *U(2)* spinor states

$$\left|n_\uparrow n_\downarrow\right\rangle = \frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{n_1}\left(\mathbf{a}_\downarrow^\dagger\right)^{n_2}}{\sqrt{n_\uparrow!\,n_2!}}\left|0\,0\right\rangle = \frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{j+m}\left(\mathbf{a}_\downarrow^\dagger\right)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}\left|0\,0\right\rangle = \left|{j \atop m}\right\rangle$$

Oscillator $\mathbf{a}^\dagger\mathbf{a}$ give $\mathbf{s}_+$ and $\mathbf{s}_-$ matrices.

$$\mathbf{a}_1^\dagger\mathbf{a}_2\left|n_1 n_2\right\rangle=\sqrt{n_1+1}\sqrt{n_2}\left|n_1+1\ n_2-1\right\rangle \Rightarrow \mathbf{s}_+\left|{j \atop m}\right\rangle=\sqrt{j+m+1}\sqrt{j-m}\left|{j \atop m+1}\right\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_1\left|n_1 n_2\right\rangle=\sqrt{n_1}\sqrt{n_2+1}\left|n_1-1\ n_2+1\right\rangle \Rightarrow \mathbf{s}_-\left|{j \atop m}\right\rangle=\sqrt{j+m}\sqrt{j-m+1}\left|{j \atop m-1}\right\rangle$$

1/2-difference of number-ops is $\mathbf{s}_Z$ eigenvalue.

$$\mathbf{a}_1^\dagger\mathbf{a}_1\left|n_1 n_2\right\rangle=n_1\left|n_1 n_2\right\rangle \qquad \mathbf{a}_2^\dagger\mathbf{a}_2\left|n_1 n_2\right\rangle=n_2\left|n_1 n_2\right\rangle$$

$$\mathbf{s}_Z\left|{j \atop m}\right\rangle=\tfrac{1}{2}\left(\mathbf{a}_1^\dagger\mathbf{a}_1-\mathbf{a}_2^\dagger\mathbf{a}_2\right)\left|{j \atop m}\right\rangle=\frac{n_1-n_2}{2}\left|{j \atop m}\right\rangle=m\left|{j \atop m}\right\rangle$$

j=1 vector $\mathbf{s}_+$ ...*and* $\mathbf{s}_Z$

$$D^1(\mathbf{s}_+)=D^1(\mathbf{s}_X+i\mathbf{s}_Y)=\begin{pmatrix} \cdot & \frac{\sqrt2}{2} & \cdot \\ \frac{\sqrt2}{2} & \cdot & \frac{\sqrt2}{2} \\ \cdot & \frac{\sqrt2}{2} & \cdot \end{pmatrix}+i\begin{pmatrix} \cdot & -i\frac{\sqrt2}{2} & \cdot \\ i\frac{\sqrt2}{2} & \cdot & -i\frac{\sqrt2}{2} \\ \cdot & i\frac{\sqrt2}{2} & \cdot \end{pmatrix}=\begin{pmatrix} \cdot & \sqrt2 & \cdot \\ 0 & \cdot & \sqrt2 \\ \cdot & 0 & \cdot \end{pmatrix}, \qquad D^1(\mathbf{s}_Z)=\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

j=3/2 spinor $\mathbf{s}_+$ ...*and* $\mathbf{s}_Z$

$$D^{\frac32}(\mathbf{s}_+)=\begin{pmatrix} \cdot & \sqrt3 & \cdot & \cdot \\ 0 & \cdot & \sqrt4 & \cdot \\ \cdot & 0 & \cdot & \sqrt3 \\ \cdot & \cdot & 0 & \cdot \end{pmatrix}=\left(D^{\frac32}(\mathbf{s}_-)\right)^\dagger, \qquad D^{\frac32}(\mathbf{s}_Z)=\begin{pmatrix} \frac32 & \cdot & \cdot & \cdot \\ \cdot & \frac12 & \cdot & \cdot \\ \cdot & \cdot & -\frac12 & \cdot \\ \cdot & \cdot & \cdot & -\frac32 \end{pmatrix}$$

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$j=\upsilon/2=(n_1+n_2)/2$

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$n_1=j+m$

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U(2) boson oscillator states = *U(2)* spinor states

$$\left|n_\uparrow n_\downarrow\right\rangle=\frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{n_1}\left(\mathbf{a}_\downarrow^\dagger\right)^{n_2}}{\sqrt{n_\uparrow!\,n_2!}}\left|0\,0\right\rangle=\frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{j+m}\left(\mathbf{a}_\downarrow^\dagger\right)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}\left|0\,0\right\rangle=\left|{j \atop m}\right\rangle$$

Oscillator $\mathbf{a}^\dagger\mathbf{a}$ give $\mathbf{s}_+$ and $\mathbf{s}_-$ matrices.

$$\mathbf{a}_1^\dagger\mathbf{a}_2\left|n_1 n_2\right\rangle=\sqrt{n_1+1}\sqrt{n_2}\left|n_1+1\; n_2-1\right\rangle\Rightarrow\mathbf{s}_+\left|{j \atop m}\right\rangle=\sqrt{j+m+1}\sqrt{j-m}\left|{j \atop m+1}\right\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_1\left|n_1 n_2\right\rangle=\sqrt{n_1}\sqrt{n_2+1}\left|n_1-1\; n_2+1\right\rangle\Rightarrow\mathbf{s}_-\left|{j \atop m}\right\rangle=\sqrt{j+m}\sqrt{j-m+1}\left|{j \atop m-1}\right\rangle$$

1/2-difference of number-ops is $\mathbf{s}_Z$ eigenvalue.

$$\mathbf{a}_1^\dagger\mathbf{a}_1\left|n_1 n_2\right\rangle=n_1\left|n_1 n_2\right\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_2\left|n_1 n_2\right\rangle=n_2\left|n_1 n_2\right\rangle$$

$$\mathbf{s}_Z\left|{j \atop m}\right\rangle=\tfrac{1}{2}\left(\mathbf{a}_1^\dagger\mathbf{a}_1-\mathbf{a}_2^\dagger\mathbf{a}_2\right)\left|{j \atop m}\right\rangle=\frac{n_1-n_2}{2}\left|{j \atop m}\right\rangle=m\left|{j \atop m}\right\rangle$$

j=1 vector $\mathbf{s}_+$ ...and $\mathbf{s}_Z$

$$D^1(\mathbf{s}_+)=D^1(\mathbf{s}_X+i\mathbf{s}_Y)=\begin{pmatrix} \cdot & \frac{\sqrt{2}}{2} & \cdot \\ \frac{\sqrt{2}}{2} & \cdot & \frac{\sqrt{2}}{2} \\ \cdot & \frac{\sqrt{2}}{2} & \cdot \end{pmatrix}+i\begin{pmatrix} \cdot & -i\frac{\sqrt{2}}{2} & \cdot \\ i\frac{\sqrt{2}}{2} & \cdot & -i\frac{\sqrt{2}}{2} \\ \cdot & i\frac{\sqrt{2}}{2} & \cdot \end{pmatrix}=\begin{pmatrix} \cdot & \sqrt{2} & \cdot \\ 0 & \cdot & \sqrt{2} \\ \cdot & 0 & \cdot \end{pmatrix},\qquad D^1(\mathbf{s}_Z)=\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

j=3/2 spinor $\mathbf{s}_+$ ...and $\mathbf{s}_Z$

$$D^{\frac{3}{2}}(\mathbf{s}_+)=\begin{pmatrix} \cdot & \sqrt{3} & \cdot & \cdot \\ 0 & \cdot & \sqrt{4} & \cdot \\ \cdot & 0 & \cdot & \sqrt{3} \\ \cdot & \cdot & 0 & \cdot \end{pmatrix}=\left(D^{\frac{3}{2}}(\mathbf{s}_-)\right)^\dagger,\qquad D^{\frac{3}{2}}(\mathbf{s}_Z)=\begin{pmatrix} \frac{3}{2} & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{2} & \cdot & \cdot \\ \cdot & \cdot & -\frac{1}{2} & \cdot \\ \cdot & \cdot & \cdot & -\frac{3}{2} \end{pmatrix}$$

j=2 tensor $\mathbf{s}_+$ ...and $\mathbf{s}_Z$

$$D^2(\mathbf{s}_+)=\begin{pmatrix} \cdot & \sqrt{4} & \cdot & \cdot & \cdot \\ 0 & \cdot & \sqrt{3} & \cdot & \cdot \\ \cdot & 0 & \cdot & \sqrt{3} & \cdot \\ \cdot & \cdot & 0 & \cdot & \sqrt{4} \\ \cdot & \cdot & \cdot & 0 & \cdot \end{pmatrix}=\left(D^2(\mathbf{s}_-)\right)^\dagger,\qquad D^2(\mathbf{s}_Z)=\begin{pmatrix} 2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & -2 \end{pmatrix}$$