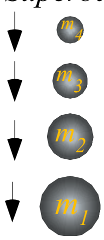
Physics 3922H Physics Colloquium Thur.2.18.2016   Exercise Set 5 Due Thur 2.25

The first two problems appeared on the 2016 PhD physics qualifying exam
(...and I did not suggest them but I did vote for them.)

Superball tower IBM phenomena (Independent Bang Model with initial V_k=-1)



The 100% energy transfer limit

1.7.1 Suppose each m_k has just the right mass ratio m_k/m_{k+1} with the m_{k+1} above it to pass on all its energy to m_{k+1} so the top ball-*N*, a *1gm* pellet, goes off with the total energy. Construct velocity-velocity diagrams, indicate velocity at each stage, and derive the required intermediate mass values for (a) $N=2$, (b) $N=3$, (c) $N=4$.

(d) Give algebraic formula for this *Maximum Amplified Velocity* factor in terms of *N* (*MAV(N)* =___________*?).*

(e) Give algebraic formula neighbor-mass ratios $R=M_{N-1}/M_N$ in terms of *N* (*R(N)*=_________*?).*

The towering limit

1.7.2 Suppose each m_k is very much larger than m_{k+1} above it so that final v_{k+1} approaches its upper limit. Then top m_N goes off with nearly the highest velocity v_N attainable. Construct the velocity-velocity diagrams. Indicate each intermediate velocity limit value at each stage and the limiting top value for (a) $N=2$, (b) $N=3$, (c) $N=4$.

(d) Give algebraic formula for *Absolute Maximum Amplified Velocity* factor in terms of *N* (*AMAV(N)* =____*?).*

The optimal idler (An algebra/calculus problem)

1.7.3 Assume the usual initial conditions for IBM. Find optimum mass m_2 in terms of masses m_1 and m_3 that will get the maximum final v_3 for mass m_3. Also, find that v_3 value.

The last problem was considered too difficult for the 2016 PhD qualifying exam.

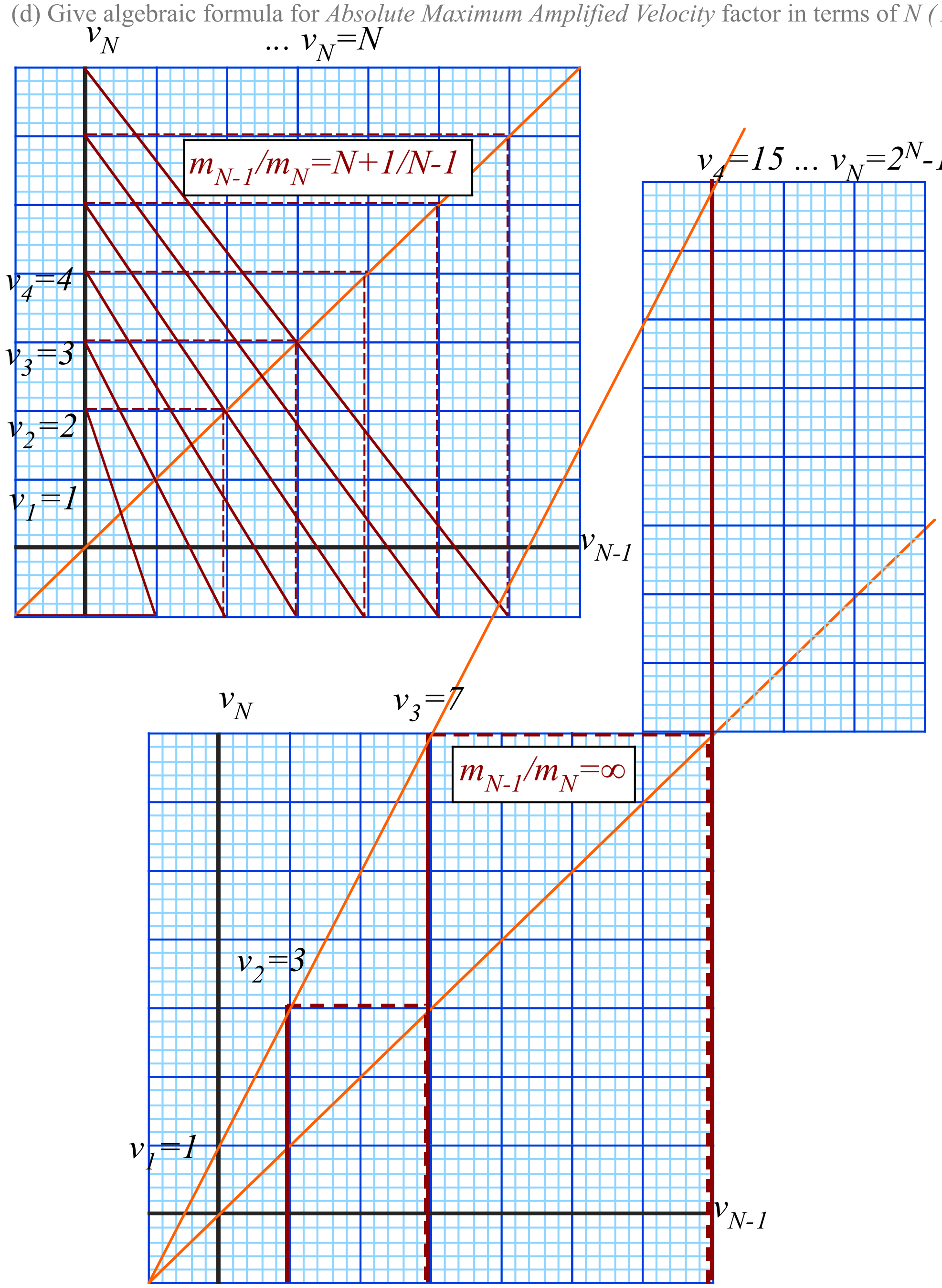
Superball tower IBM model constructions (Independent Bang Model with initial V_k=-1)

The 100% energy transfer limit

1.7.1 Suppose each m_k has just the right mass ratio m_k/m_{k+1} with the m_{k+1} above it to pass on all its energy to m_{k+1} so the top ball-*N*, a *1gm* pellet, goes off with the total energy. Construct velocity-velocity diagrams, indicate velocity at each stage, and derive the required intermediate mass values for (a) $N=2$, (b) $N=3$, (c) $N=4$.
(d) Give algebraic formula for this *Maximum Amplified Velocity* factor in terms of *N* (*MAV(N)=___________?*).
(e) Give algebraic formula neighbor-mass ratios $R=M_{N-1}/M_N$ in terms of *N* (*R(N)=_________?*).

The towering limit

1.7.2 Suppose each m_k is very much larger than m_{k+1} above it so that final v_{k+1} approaches its upper limit. Then top m_N goes off with nearly the highest velocity v_N attainable. Construct the velocity-velocity diagrams. Indicate each intermediate velocity limit value at each stage and the limiting top value for (a) $N=2$, (b) $N=3$, (c) $N=4$.
(d) Give algebraic formula for *Absolute Maximum Amplified Velocity* factor in terms of *N* (*AMAV(N)=____?*).



1st case shows *linear* series of final velocity. 2nd case shows *geometric* or *exponential* series of velocity.

(Solutions to Assignment 5 contd) The optimum idler:

1.7.3 To get highest final v_3 of mass m_3 find optimum mass m_2 in terms of masses m_1 and m_3 that will do that.

Let $m_1 = M$, $m_2 = x$ and $m_3 = m$. Then use (5.1b): $\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} = \frac{1}{m_1+m_2}\begin{pmatrix} m_1-m_2 & 2m_2 \\ 2m_1 & m_2-m_1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ in stages. 1st stage gives:

$$v_x^{FIN} = \frac{3M-x}{M+x}$$

$$\begin{pmatrix} v_M^{FIN} \\ v_x^{FIN} \end{pmatrix} = \frac{1}{M+x}\begin{pmatrix} M-x & 2x \\ 2M & x-M \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{M+x}\begin{pmatrix} M-3x \\ 3M-x \end{pmatrix}$$. The 2nd stage:

$$\begin{pmatrix} v_x^{FIN} \\ v_m^{FIN} \end{pmatrix} = \frac{1}{x+m}\begin{pmatrix} x-m & 2m \\ 2x & m-x \end{pmatrix}\begin{pmatrix} \frac{3M-x}{M+x} \\ -1 \end{pmatrix}$$

The velocity v_m is to be maximized.

$$v_m^{FIN} = \frac{2x\frac{3M-x}{M+x}-(m-x)}{x+m} = \frac{6Mx-2x^2+(x-m)(M+x)}{(M+x)(x+m)} = \frac{-x^2+(7M-m)x-mM}{x^2+(M+m)x+mM} = \frac{N(x)}{D(x)}$$

Derivative $\frac{1}{D(x)}\frac{dN}{dx} - N(x)\frac{d}{dx}\frac{1}{D(x)} = \frac{D(x)\frac{dN(x)}{dx} - N(x)\frac{dD(x)}{dx}}{D(x)^2}$ is set to zero.

$$\left(x^2+(M+m)x+mM\right)\left(-2x+(7M-m)\right) \qquad -\left(-x^2+(7M-m)x-mM\right)\left(2x+(M+m)\right)=0$$

	x^2	$+(M+m)x$	mM
$-2x$	$-2x^3$	$-2(M+m)x^2$	$-2mMx$
$(7M-m)$	$(7M-m)x^2$	$(7M-m)(M+m)x$	$(7M-m)mM$

	x^2	$-(7M-m)x$	mM
$2x$	$-2x^3$	$-2(7M-m)x^2$	$2mMx$
$(M+m)$	$(M+m)x^2$	$-(M+m)(7M-m)x$	$(M+m)mM$

Cancellations simplify it.

$$\left(x^2+(M+m)x+mM\right)\left(-2x+(7M-m)\right) \qquad -\left(-x^2+(7M-m)x-mM\right)\left(2x+(M+m)\right)=0$$

	x^2	$+(M+m)x$	mM
$-2x$		$-2(M)x^2$	
$(7M-m)$	$(7M)x^2$		$(7M)mM$

	x^2	$-(7M-m)x$	mM
$2x$		$-2(7M)x^2$	
$(M+m)$	$(M)x^2$		$(M)mM$

Result is quadratic and not cubic equation: $-8Mx^2+8M^2m=0$ or $-x^2+Mm=0$.

The result is geometric mean! $x=\sqrt{(Mm)}$ *or:* $m_2=\sqrt{(m_1 m_3)}$. The resulting final velocity is as follows:

$$v_m^{FIN} = \frac{-\sqrt{mM}^2+(7M-m)\sqrt{mM}-mM}{\sqrt{mM}^2+(M+m)\sqrt{mM}+mM} = \frac{-mM+(7M-m)\sqrt{mM}-mM}{mM+(M+m)\sqrt{mM}+mM} = \frac{(7M-m)\sqrt{mM}-2mM}{(M+m)\sqrt{mM}+2mM}$$

Xtra-Credit (Not assigned)

Now try more difficult problem for next stage where lowest mass is coming up with higher speed S but top one is still falling at speed -1.

Let $m_1 = M$, $m_2 = x$ and $m_3 = m$. Use (5.1b): $\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} = \frac{1}{m_1+m_2}\begin{pmatrix} m_1-m_2 & 2m_2 \\ 2m_1 & m_2-m_1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ in stages. 1st stage gives: $v_x^{FIN} = \frac{3M-x}{M+x}$

$$\begin{pmatrix} v_M^{FIN} \\ v_x^{FIN} \end{pmatrix} = \frac{1}{M+x}\begin{pmatrix} M-x & 2x \\ 2M & x-M \end{pmatrix}\begin{pmatrix} S \\ -1 \end{pmatrix} = \frac{1}{M+x}\begin{pmatrix} SM-(S+2)x \\ (2S+1)M-x \end{pmatrix}$$. 2nd stage: $\begin{pmatrix} v_x^{FIN} \\ v_m^{FIN} \end{pmatrix} = \frac{1}{x+m}\begin{pmatrix} x-m & 2m \\ 2x & m-x \end{pmatrix}\begin{pmatrix} \frac{3M-x}{M+x} \\ -1 \end{pmatrix}$

Again, velocity v_m is to be maximized.