The following is to acquaint you with some lesser known properties of exponentials and logarithms

1.8.1 Backsides of exponentials

(a) Follow zig-zag scheme shown at the beginning of Lect. 11 to make plots of exponential $y=e^x$ at as many integer points $x=-2, -1, 0, 1, 2,..$ as is practical on full page graph paper provided online or in lab. Then add to the plot precise half-way points $x=-2.5, -1.5, -0.5, etc...$ as is practical. Show how a plot of $y=\log_e x$ function is obtained from the graph

(b) By algebra or geometry find tangent lines and their slope at integer points $x=-2, -1, 0, 1, 2,..$ (This is equivalent to solving the part (c) of this exercise.)

(c) As a roller-coaster car moves down a track $y=e^x$ it shines one laser headlight beam along the track and another droplight beam vertically downward so both make spots on baseline $y=0$. Find the distance between spots as function of $x$.

1.8.2 Sophomore-Physics-Earth

(a) Follow the zig-zag scheme in Lect. 11 (or in Fig. 8.5 and 8.7 of text) to construct the potential and force curves of the Ideal Uniform Density Earth inside ($PE(x)=kx^2/2+PE(0)$) and outside ($PE(x)=-x^{-1}$).

(b) On graph show focal point, latus-radius , and directrix of the inside PE parabola. Draw as accurately as possible the parabola’s circle of curvature contacting it at $x=0$.

(c) Draw a “kite” (see Fig. 8.4 in text) tangent to parabola at $x=1$ and another tangent at $x=1/2$.

1.8.3 Tunnels to UK (5600 miles as an earthworm crawls) are shown below. One high-road is a direct route. A low-road turns at the Earth center. (Travel and turn-around are assumed frictionless and survivable.)

(a) What is the time for each trip? Discuss using geometry or algebra arguments.

   (a) Hi-road & low-road
   
   (b) Lots of roads

(b) Assume cars in subway tunnels depart Ark. at time $t=0$ as indicated above. Describe curve (thru dots shown) locating car positions at a later mid-trip time $t$ before arrival and at arrival. (Thales geometry of circular chords may help. Recall superball figure 6.1 in text.)

(c) What if the half-way turn-around point is above the Earth-center. Is trip quicker or slower?
(a) Follow zig-zag scheme shown at the beginning of Lect. 11 to make plots of exponential $y = e^x$ at as many integer points $x = -2, -1, 0, 1, 2, \ldots$ as is practical on full page graph paper provided online or in lab. Then add to the plot precise half-way points $x = -2.5, -1.5, -0.5, \ldots$ as is practical. Show how a plot of $y = \log_\alpha x$ function is obtained from the graph.
Each slope line intersects $y=0$ exactly -1 unit distance from their x-coordinate point.

If the graph is expanded it clearly shows that there is unit distance ($\Delta x = 1$) between x-axis intersections of any tangent to point $(x, y=e^x)$ and the vertical line $x=x_1$ going thru that point. 

Quite remarkable! All tangents to $y=e^x$ have unit footprints.
Exercise 1.8.3. Tunnels to UK (5600 miles away as an earthworm crawls) are shown below. One high-road is a direct route. The other low-road turns around at the Earth center. Travel and turn-around are assumed frictionless and survivable. (a) How long is each trip? Discuss. Both the same.

(a) Hi-road & low-road

(b) Lots of roads

(b) A network of subways leaving Ark. at time t=0. What curve (like the dots) describe each moment? Each is on a circle at distance \( r_A = D \cos \theta \) from A with \( D = R_{\text{earth}} (1 - \cos \omega_{\text{earth}} t) \). \( \theta \) is subway polar angle and \( \pi / \omega_{\text{earth}} = 42 \) minutes is the one-way surface-to-surface trip on each \( \theta \) path having length \( L = R_{\text{earth}} \cos \theta \).

(c) What if the half-way turn-around point is above the Earth-center. Is trip quicker or slower? There is a point nearly midway between the bend at Earth-center and the center of the straight Ark. to U.K. track where the bend should be to achieve a minimum travel time and shorter than the others’.
The more difficult problem of deep-V-tunnel global travel is solved similarly, but a geometric solution sketched below is quick (once you see the trick!). The trick is to imagine a pencil of competing tunnels going out from both point A and point B so that the trial runs form two expanding circles that finally touch on a tangent that bisects the A-to-B longitude angle \( \phi_{AOB} = \Delta \phi \). We find the angle \( \alpha = \pi/4 - \Delta \phi/4 \) between shortest path and quickest path. It approaches \( \alpha = 45^\circ \) in the local limit \( \Delta \phi \to 0 \). The \( \text{AMB} \) vertex angle is \( \phi_{\text{AMB}} = \pi/2 + \phi_{\text{AOB}}/2 \) and approaches a local 90° limit. Half the \( \text{AMB} \) vertex angle is \( \phi_{\text{AMB}}/2 = \pi/2 - \alpha = \alpha_c \) (compliment of \( \alpha \)) that is also horizon dip angle between the horizon and the quickest path. For short trips: \( \alpha = \alpha_c = 45^\circ \). For longer trips: \( \alpha < 45^\circ \) and \( \alpha_c > 45^\circ \).

Each circle diameter \( D = 2r \) (in units of Earth radius \( R_\oplus \)) expands as \( D = 1 - \cos \theta \) where \( \theta = \omega t \) is the circular orbit angle subtended by projecting the diameter point to the Earth circle. Travel time \( T \) is proportional to angle \( \theta \) with \( \theta = \pi \) corresponding to 42 minutes of a half-circle orbit and \( \theta = \pi/2 \) to 21 min. (Going half-way between A and B by the straight tunnel takes 21 minutes.)
The following is a general solution with the $\phi = \Phi/2 = 45^\circ$ case given numerically.

Here:

- $\Phi = 90^\circ$
- $\phi = \Phi/2 = 45^\circ$
- $\phi_{AMB} = \Phi/2 + \pi/2 = 135^\circ$

Circle radius $r$ relates to Earth radius $R$ and angle $\alpha$ (or $\phi$):

- $R\sin \phi = r + r \cos 2\alpha = r(1 + \cos[\pi/2 - \phi])$
- $= r(1 + \sin \phi)$

This gives time phase angle $\Omega_T$ vs $\phi$:

$$\Omega_T = \omega \cdot t = \cos^{-1} \left( \frac{R - 2r}{R} \right)$$
$$= \cos^{-1} \left( \frac{1 - \sin \phi}{1 + \sin \phi} \right)$$

Here:

- $\Omega_T = 80.12^\circ$

and: $t_T = 80.12/90 \cdot 21 = 18.7$ min.

(21 min. is 1/4 of 84 min. orbit period.)