Lecture 12 advanced
Thur. 2.25.2016

Kepler Geometry of IHO (Isotropic Harmonic Oscillator) Elliptical Orbits

(Ch. 8 and Ch. 9 of Unit 1)

Kepler "laws" (Some that apply to all central (isotropic) F(r) force fields)

Angular momentum invariance of IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$ *(Derived here)*

Angular momentum invariance of Coulomb: $F(r)=-GMm/r^2$ with $U(r)=-GMm\cdot/r$ *(Derived later)*

Total energy E=KE+PE invariance of IHO: $F(r)=-k\cdot r$ *(Derived here)*

Total energy E=KE+PE invariance of Coulomb: $F(r)=-GMm/r^2$ *(Derived later)*

A confusing introduction to Coriolis-centrifugal force geometry *(Derived better later)*

Introduction to dual matrix operator contact geometry (based on IHO orbits)

Quadratic form ellipse $\mathbf{r}\bullet Q\bullet\mathbf{r}=1$ vs.inverse form ellipse $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$

Duality norm relations ($\mathbf{r}\bullet\mathbf{p}=1$)

Q-Ellipse tangents $\mathbf{r}'$ normal to dual Q^{-1}-ellipse position $\mathbf{p}$ ($\mathbf{r}'\bullet\mathbf{p}=0=\mathbf{r}\bullet\mathbf{p}'$)

Operator geometric sequences and eigenvectors

Alternative scaling of matrix operator geometry

Vector calculus of tensor operation

Q:Where is this headed? A: Lagrangian-Hamiltonian duality

Link ⇒ BoxIt simulation of IHO orbits
Link → IHO orbital time rates of change
Link → IHO Exegesis Plot

Kepler"laws" (Some that apply to all central (isotropic) F(r) force fields)

→ *Angular momentum invariance of* IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$ *(Derived here)*

Angular momentum invariance of Coulomb: $F(r)=-GMm/r^2$ with $U(r)=-GMm\cdot/r$ *(Derived in Unit 5)*

Total energy E=KE+PE invariance of IHO: $F(r)=-k\cdot r$ *(Derived here)*

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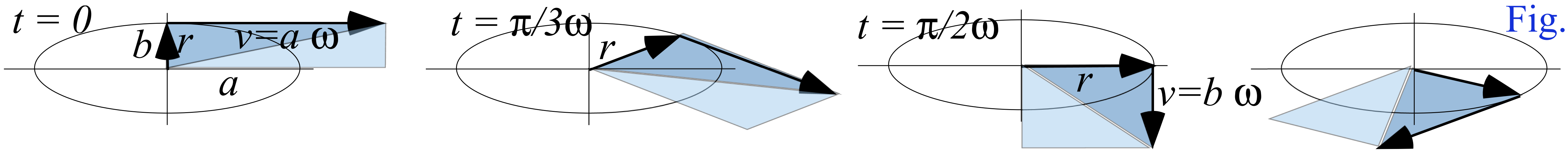
Some Kepler's "laws" for central (isotropic) force F(r)

...and certainly apply to the IHO: F(r)=-k·r with U(r)=k·r²/2 (Recall from Lect.12 p.19: $k=G\frac{4\pi}{3}m\rho_{\oplus}$)

Unit 1
Fig. 9.8



1. Area of triangle $\measuredangle_{\mathbf{r}}^{\mathbf{v}} = \mathbf{r}\times\mathbf{v}/2$ is constant

$$\mathbf{r}\times\mathbf{v} = r_x v_y - r_y v_x = a\cos\omega t\cdot\left(b\omega\cos\omega t\right) - b\sin\omega t\cdot\left(-a\omega\sin\omega t\right) = ab\cdot\omega\left(\cos^2\omega t+\sin^2\omega t\right)$$

✓ *for IHO*

$$\begin{pmatrix} r_x \\ r_y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\cos\omega t \\ b\sin\omega t \end{pmatrix} \qquad \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} -a\omega\sin\omega t \\ b\omega\cos\omega t \end{pmatrix}$$

Some Kepler's "laws" that apply to any central (isotropic) force F(r)

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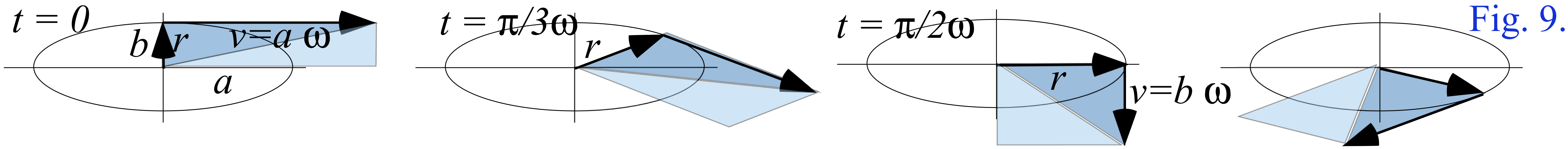

Unit 1
Fig. 9.8

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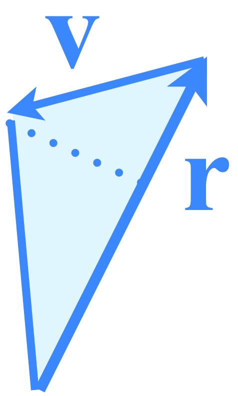
✓ *for IHO*

2. *Angular momentum* $\mathbf{L} = m\,\mathbf{r}\times\mathbf{v}$ is conserved

$$L = m\,|\mathbf{r}\times\mathbf{v}| = m\left(r_x v_y - r_y v_x\right) = m\cdot ab\cdot\omega$$

✓ *for IHO*

$$|\mathbf{r}\times\mathbf{v}| = r\cdot v\cdot\sin\measuredangle_r^y$$



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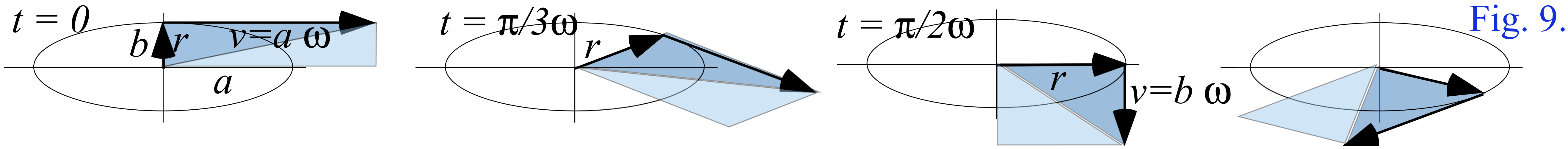

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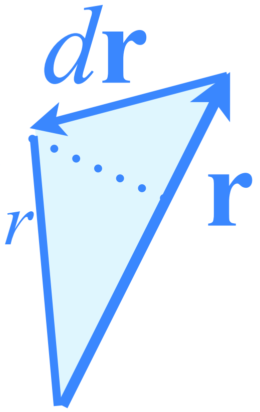
3. Equal area is swept by radius vector in each equal time interval T

$$A_T = \int_0^T \frac{\mathbf{r}\times d\mathbf{r}}{2} = \int_0^T \frac{\mathbf{r}\times\frac{d\mathbf{r}}{dt}}{2}dt = \int_0^T \frac{\mathbf{r}\times\mathbf{v}}{2}dt = \frac{L}{\underbrace{2m}_{\text{by 2.}}}\int_0^T dt = \frac{L}{2m}T$$

✓ *for IHO*

$$|\mathbf{r}\times d\mathbf{r}| = r\cdot dr\cdot\sin\measuredangle_{r}^{dr}$$



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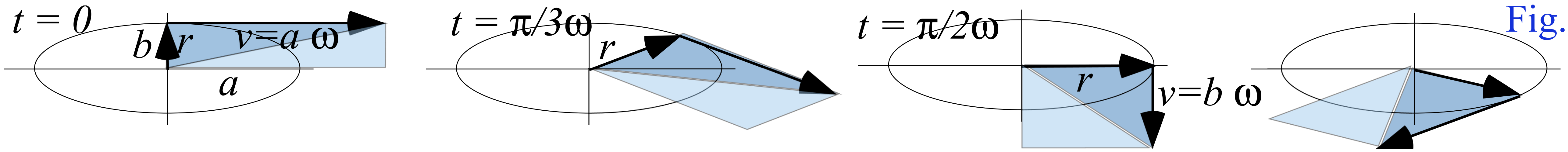

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In one period: $\tau = \frac{1}{\upsilon} = \frac{2\pi}{\omega} = \frac{2mA_\tau}{L}$ the area is: $A_\tau = \frac{L\tau}{2m}$ ($= ab\cdot\pi$ for ellipse orbit)

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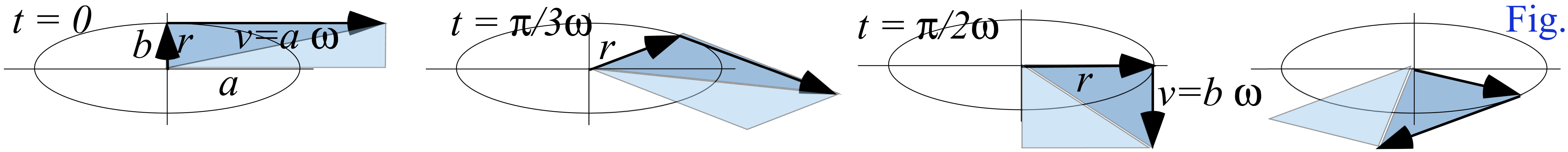

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(Recall from Lecture 7: $\omega = \sqrt{k/m} = \sqrt{G\rho_{\oplus}4\pi/3}$)

Some Kepler's "laws" for all central (isotropic) force F(r) fields

- *Angular momentum invariance of* IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$ *(Derived here)*
- → *Angular momentum invariance of* Coulomb: $F(r)=-GMm/r^2$ with $U(r)=-GMm\cdot/r$ *(Derived in Unit 5)*
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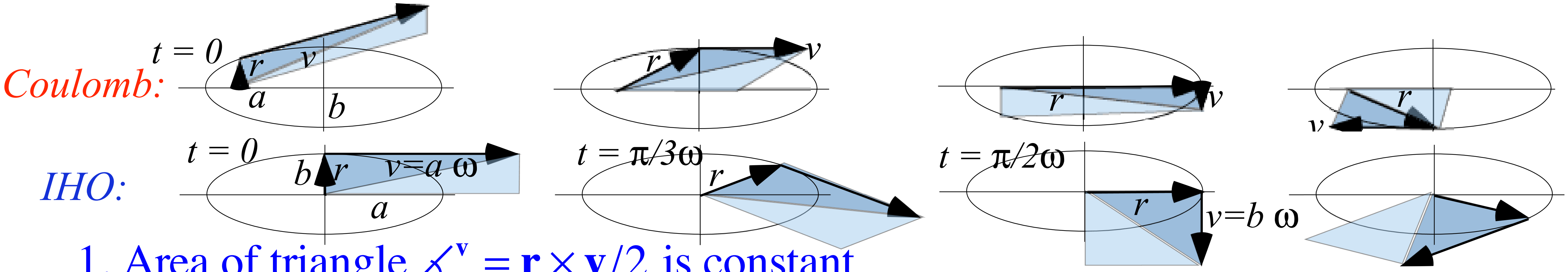
Apply to IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$ and Coulomb: $F(r)=-GMm/r^2$ with $U(r)=-GMm\cdot/r$



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✓ *for IHO*

(Derived in Unit 5) ✓ *for Coul.*

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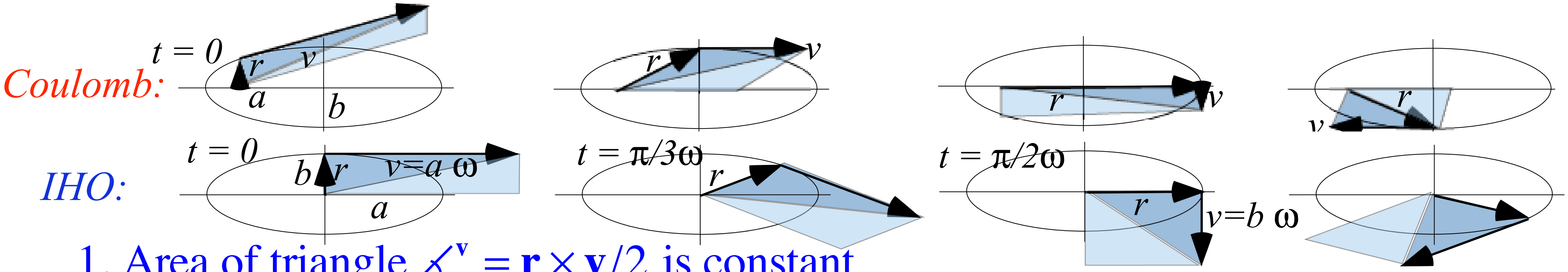

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(... in Unit 5)

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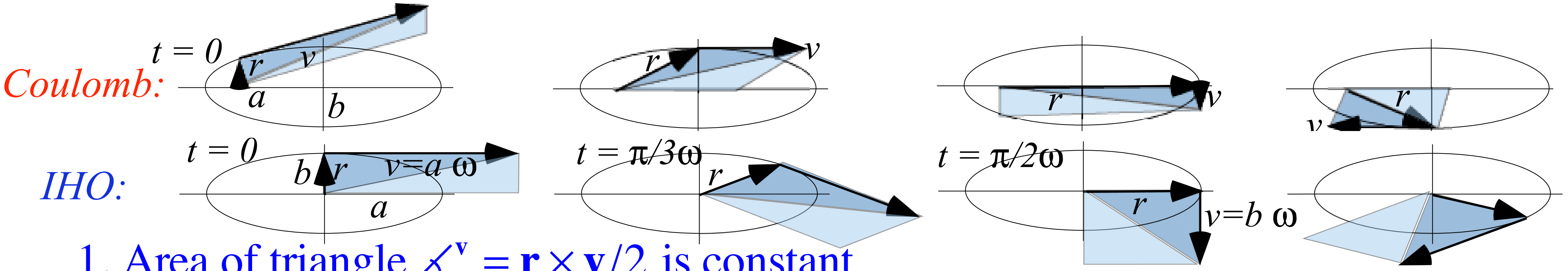

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(... in Unit 5)

✓ *for IHO*

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3. Equal area is swept by radius vector in each equal time interval T

In one period:

$$\tau = \frac{1}{\upsilon} = \frac{2\pi}{\omega} = \frac{2mA_{\tau}}{L} = \frac{2m\cdot ab\cdot\pi}{L} = \begin{cases} \dfrac{2m\cdot ab\cdot\pi}{m\cdot ab\cdot\sqrt{G\rho_{\oplus}4\pi/3}} \\ \dfrac{2m\cdot ab\cdot\pi}{m\cdot a^{-1/2}b\sqrt{GM_{\oplus}}} \end{cases}$$

Applies to any central F(r)

Applies to IHO and Coulomb

✓ *for IHO*

✓ *for Coul.*

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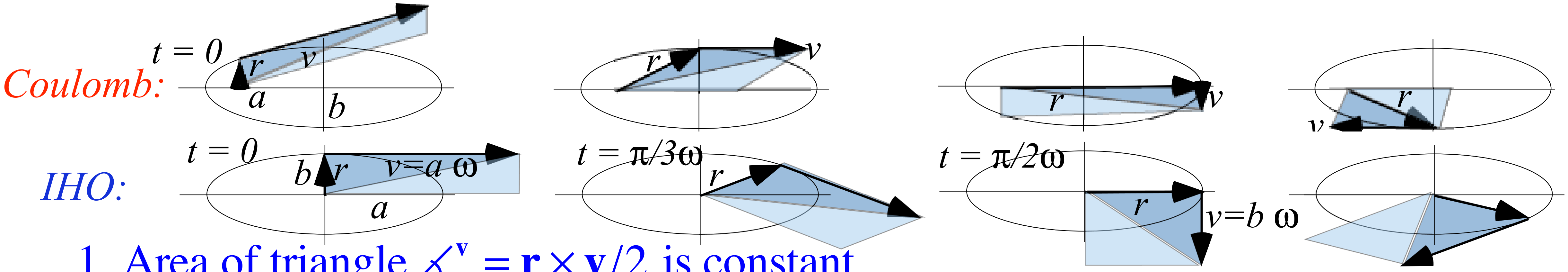

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(Derived in Unit 5)

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✓ for IHO

✓ for Coul.

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Applies to any central F(r)

Applies to IHO and Coulomb

(not a function of *a* or *b*)

that is ω_{IHO}

that is ω_{Coul}

(not a function of *b*)

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Kepler laws involve ∡-momentum conservation in isotropic force F(r)

Now consider orbital energy conservation of the IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$

Total energy=$KE+PE$ is constant

$$KE+PE=\frac{1}{2}\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}+\frac{1}{2}\mathbf{r}\bullet\mathbf{K}\bullet\mathbf{r}$$

$$=\frac{1}{2}\begin{pmatrix} v_x & v_y \end{pmatrix}\bullet\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\bullet\begin{pmatrix} v_x \\ v_y \end{pmatrix}+\begin{pmatrix} r_x & r_y \end{pmatrix}\bullet\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\bullet\begin{pmatrix} r_x \\ r_y \end{pmatrix}$$

$$=\frac{1}{2}mv_x^2+\frac{1}{2}mv_y^2+\frac{1}{2}kr_x^2+\frac{1}{2}kr_y^2$$

$$=\frac{1}{2}m(-a\omega\sin\omega t)^2+\frac{1}{2}m(b\omega\cos\omega t)^2+\frac{1}{2}k(a\cos\omega t)^2+\frac{1}{2}k(b\sin\omega t)^2$$

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix}=\begin{pmatrix} -a\omega\sin\omega t \\ b\omega\cos\omega t \end{pmatrix}$$

$$\begin{pmatrix} r_x \\ r_y \end{pmatrix}=\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} a\cos\omega t \\ b\sin\omega t \end{pmatrix}$$

Kepler laws involve ∡-momentum conservation in isotropic force F(r)

Now consider orbital energy conservation of the IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$

Total *IHO* energy=$KE+PE$ is constant

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$$=\frac{1}{2}mv_x^2+\frac{1}{2}mv_y^2+\frac{1}{2}kr_x^2+\frac{1}{2}kr_y^2$$

$$=\frac{1}{2}m(-a\omega\sin\omega t)^2+\frac{1}{2}m(b\omega\cos\omega t)^2+\frac{1}{2}k(a\cos\omega t)^2+\frac{1}{2}k(b\sin\omega t)^2$$

$$=\frac{1}{2}ma^2\omega^2(\sin^2\omega t)+\frac{1}{2}mb^2\omega^2(\cos^2\omega t)^2+\frac{1}{2}ka^2(\cos^2\omega t)+\frac{1}{2}kb^2(\sin^2\omega t)$$

$$=\frac{1}{2}m\omega^2(a^2+b^2)$$

$$Given: k=m\omega^2$$

Kepler laws involve ∡-momentum conservation in isotropic force F(r)

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$$=\frac{1}{2}mv_x^2+\frac{1}{2}mv_y^2+\frac{1}{2}kr_x^2+\frac{1}{2}kr_y^2$$

$$=\frac{1}{2}m(-a\omega\sin\omega t)^2+\frac{1}{2}m(b\omega\cos\omega t)^2+\frac{1}{2}k(a\cos\omega t)^2+\frac{1}{2}k(b\sin\omega t)^2$$

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$$=\frac{1}{2}m\omega^2(a^2+b^2)\qquad Given: k=m\omega^2$$

$$E=KE+PE=\frac{1}{2}m\omega^2(a^2+b^2)=\frac{1}{2}k(a^2+b^2)\quad \text{since: } \omega=\sqrt{\frac{k}{m}}=\sqrt{G\rho_{\oplus}4\pi/3}\quad \text{or: } m\omega^2=k$$

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$$=\frac{1}{2}\begin{pmatrix} v_x & v_y \end{pmatrix}\bullet\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\bullet\begin{pmatrix} v_x \\ v_y \end{pmatrix}+\begin{pmatrix} r_x & r_y \end{pmatrix}\bullet\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\bullet\begin{pmatrix} r_x \\ r_y \end{pmatrix}$$

$$=\frac{1}{2}mv_x^2+\frac{1}{2}mv_y^2+\frac{1}{2}kr_x^2+\frac{1}{2}kr_y^2$$

$$=\frac{1}{2}m(-a\omega\sin\omega t)^2+\frac{1}{2}m(b\omega\cos\omega t)^2+\frac{1}{2}k(a\cos\omega t)^2+\frac{1}{2}k(b\sin\omega t)^2$$

$$=\frac{1}{2}ma^2\omega^2(\sin^2\omega t)+\frac{1}{2}mb^2\omega^2(\cos^2\omega t)^2+\frac{1}{2}ka^2(\cos^2\omega t)+\frac{1}{2}kb^2(\sin^2\omega t)$$

$$=\frac{1}{2}m\omega^2(a^2+b^2)$$

$Given: k=m\omega^2$

$E=KE+PE=\frac{1}{2}m\omega^2(a^2+b^2)=\frac{1}{2}k(a^2+b^2)$ since: $\omega=\sqrt{\frac{k}{m}}=\sqrt{G\rho_\oplus 4\pi/3}$ or: $m\omega^2=k$

We'll see that the Coul. orbits are simpler: (*like the period*...not a function of *b*)

Kepler laws involve ∡-momentum conservation in isotropic force F(r)

Now consider orbital energy conservation of the IHO: F(r)=-k·r with U(r)=k·r²/2

Total *IHO* energy=$KE + PE$ is constant

$$KE + PE = \frac{1}{2}\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v} + \frac{1}{2}\mathbf{r}\bullet\mathbf{K}\bullet\mathbf{r}$$

$$= \frac{1}{2}\begin{pmatrix} v_x & v_y \end{pmatrix}\bullet\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\bullet\begin{pmatrix} v_x \\ v_y \end{pmatrix} + \begin{pmatrix} r_x & r_y \end{pmatrix}\bullet\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\bullet\begin{pmatrix} r_x \\ r_y \end{pmatrix}$$

$$= \frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2 + \frac{1}{2}kr_x^2 + \frac{1}{2}kr_y^2$$

$$= \frac{1}{2}m(-a\omega\sin\omega t)^2 + \frac{1}{2}m(b\omega\cos\omega t)^2 + \frac{1}{2}k(a\cos\omega t)^2 + \frac{1}{2}k(b\sin\omega t)^2$$

$$= \frac{1}{2}ma^2\omega^2(\sin^2\omega t) + \frac{1}{2}mb^2\omega^2(\cos^2\omega t)^2 + \frac{1}{2}ka^2(\cos^2\omega t) + \frac{1}{2}kb^2(\sin^2\omega t)$$

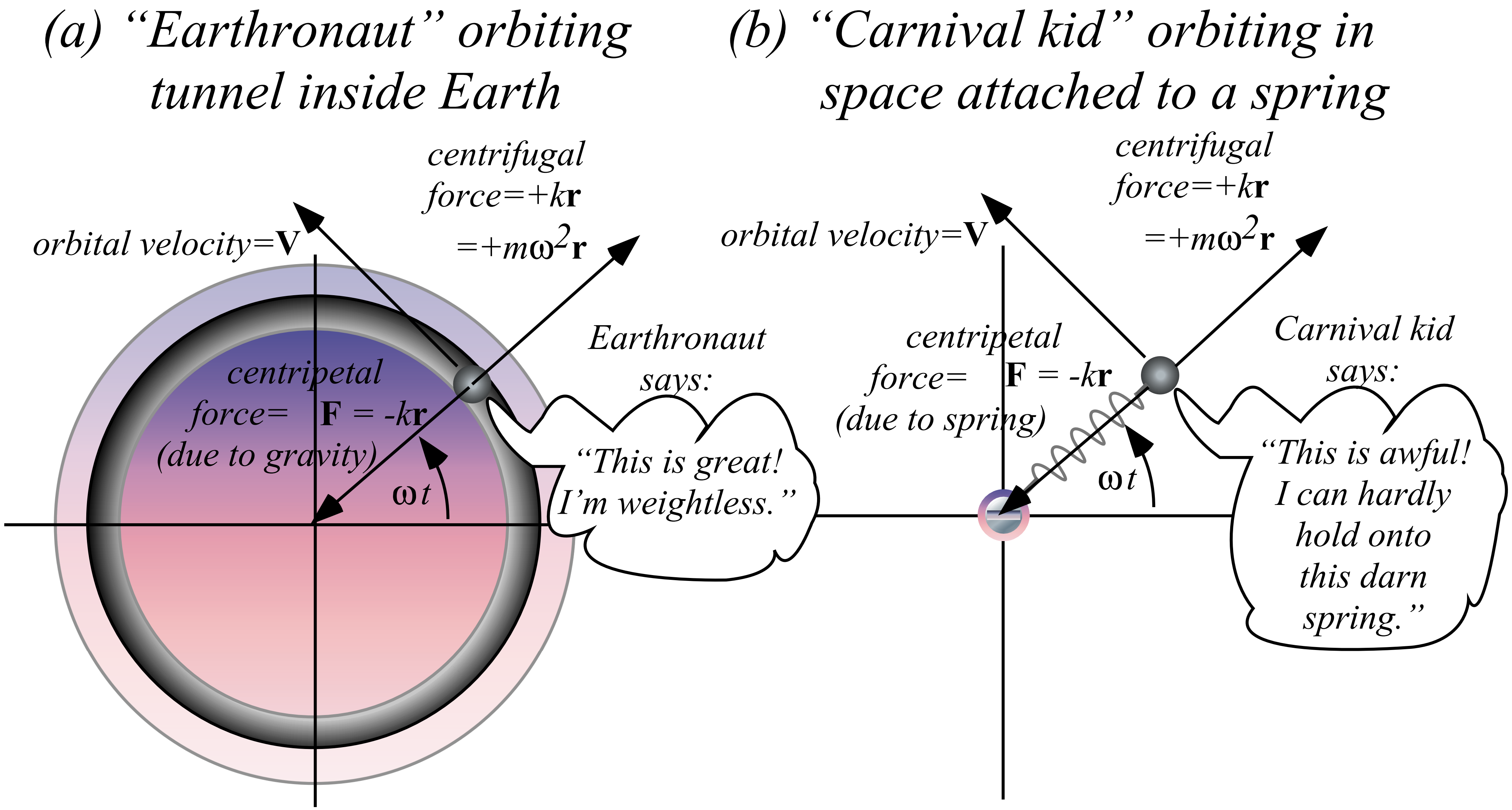
$$= \frac{1}{2}m\omega^2(a^2 + b^2) \qquad Given: k = m\omega^2$$

$$E = KE + PE = \frac{1}{2}m\omega^2(a^2 + b^2) = \frac{1}{2}k(a^2 + b^2)$$ since: $\omega = \sqrt{\frac{k}{m}} = \sqrt{G\rho_\oplus 4\pi/3}$ or: $m\omega^2 = k$

We'll see that the Coul. orbits are simpler: (*like the period*...not a function of *b*)

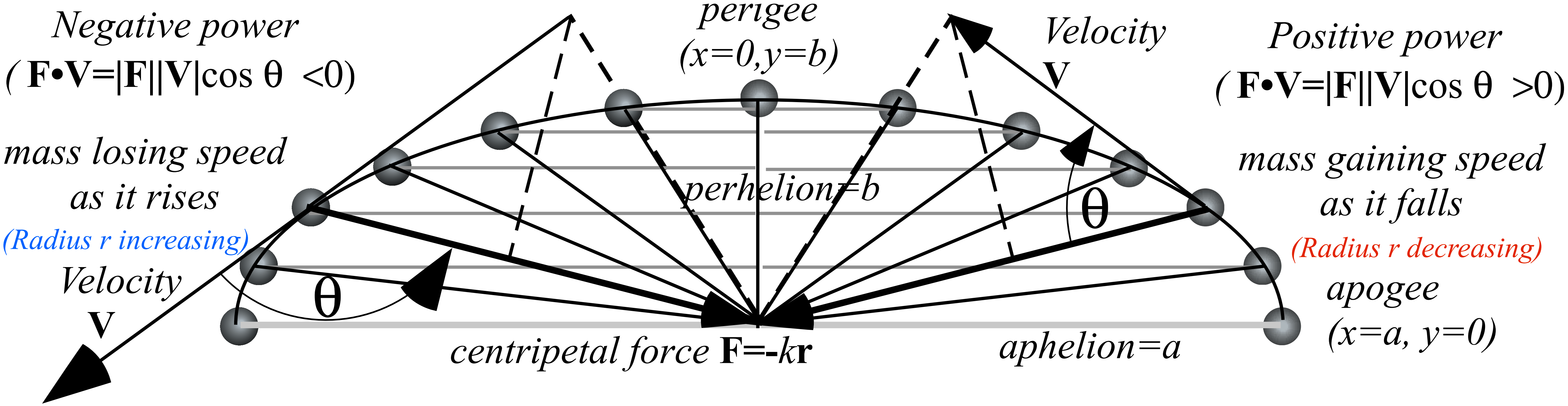
$$E = KE + PE = \frac{1}{2}mv^2{}_x + \frac{1}{2}mv^2{}_y - \frac{k}{r} = \frac{1}{2}mv^2{}_x + \frac{1}{2}mv^2{}_y - \frac{GM_\oplus m}{r} = -\frac{GM_\oplus m}{a}$$

→ *A confusing introduction to Coriolis-centrifugal force geometry* *(Derived better later)*

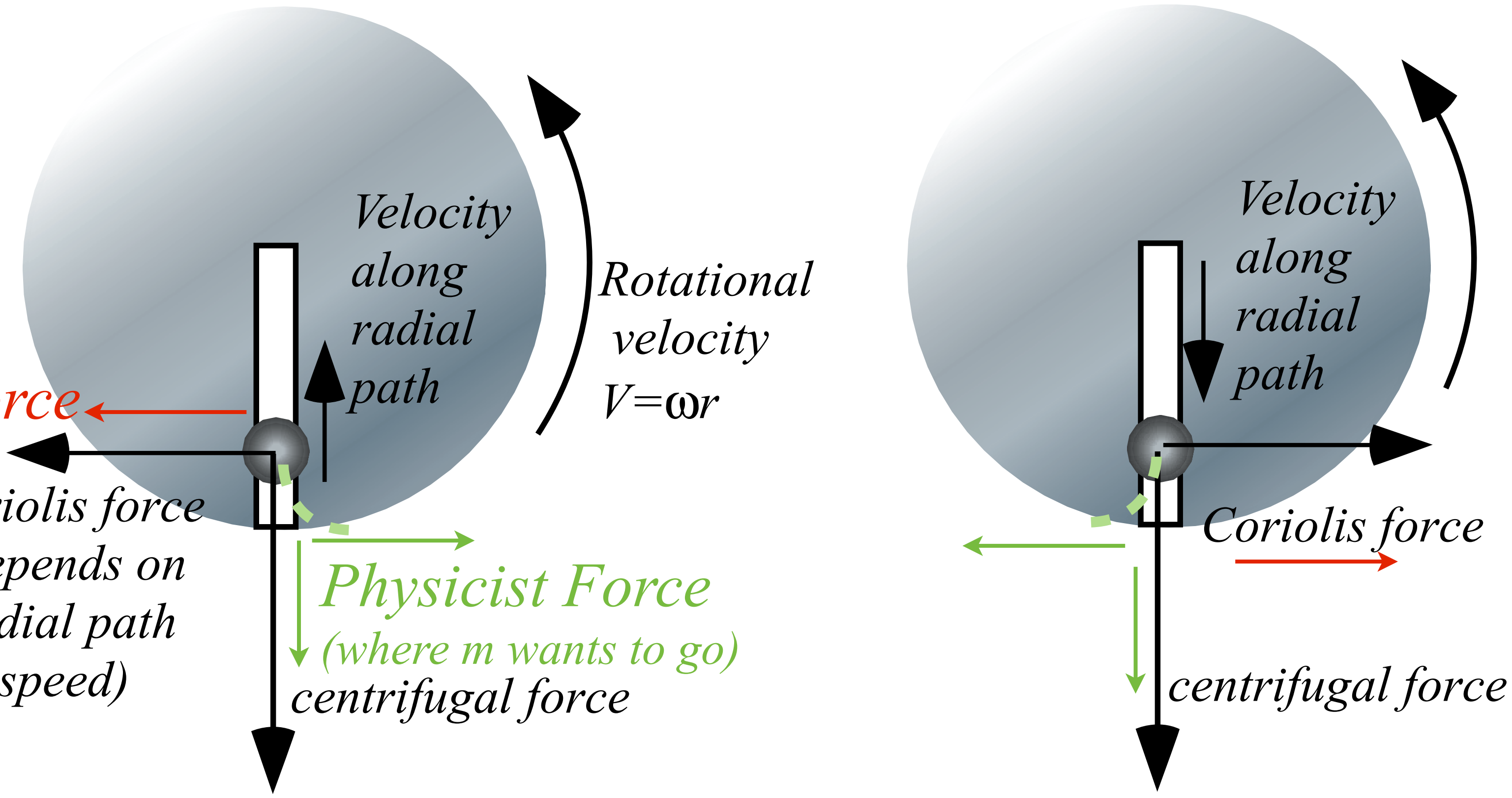


Unit 1
Fig. 9.2

Unit 1
Fig. 9.3



(a) Centrifugal and Coriolis Forces on Merry-Go-Round



Velocity along radial path

Rotational velocity $V=\omega r$

Mathematician Force

(to hold m back)

Constraint force keeps m in radial slot

Coriolis force (depends on radial path speed)

Physicist Force

(where m wants to go)

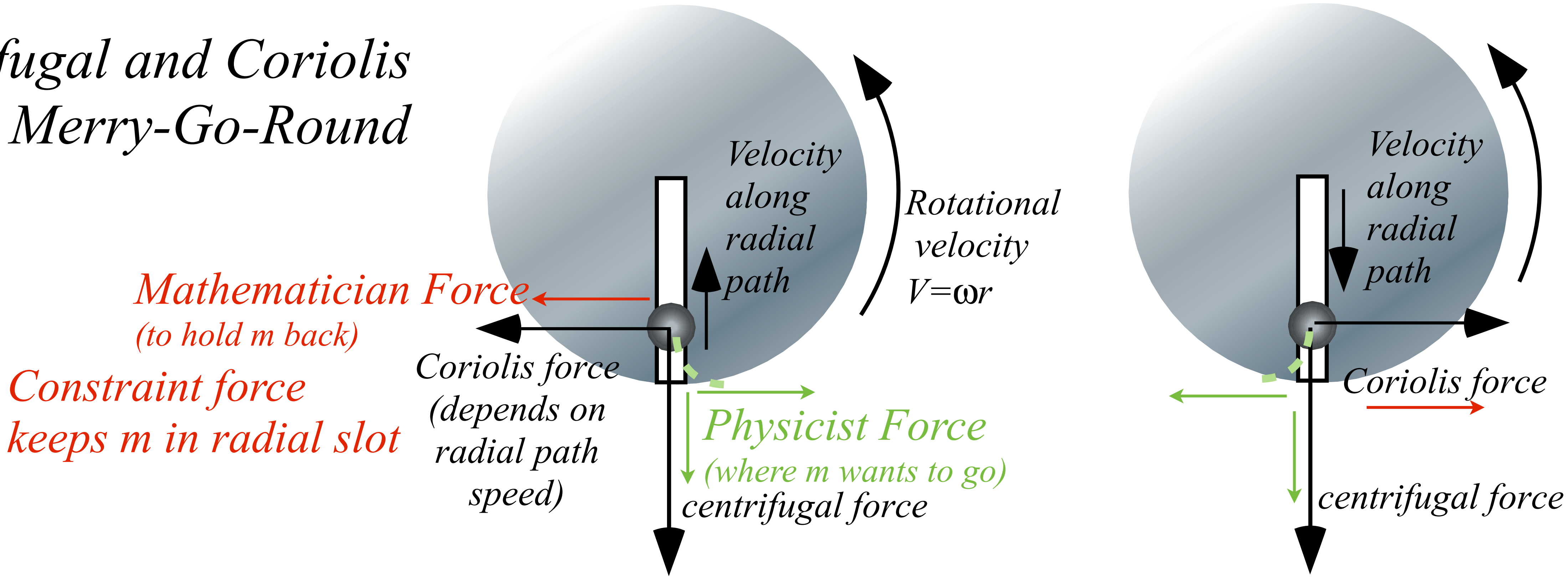
centrifugal force

Velocity along radial path

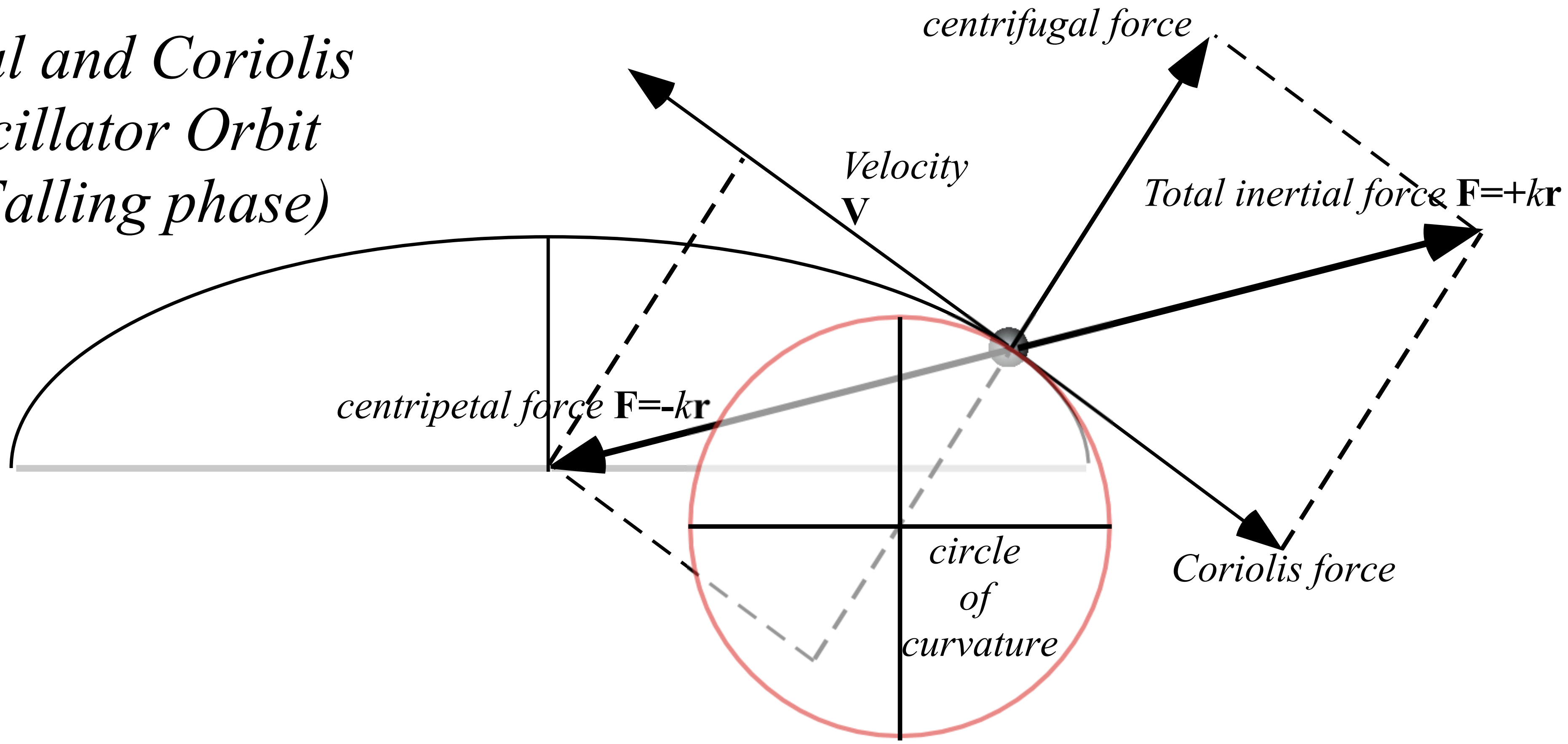
Coriolis force

centrifugal force

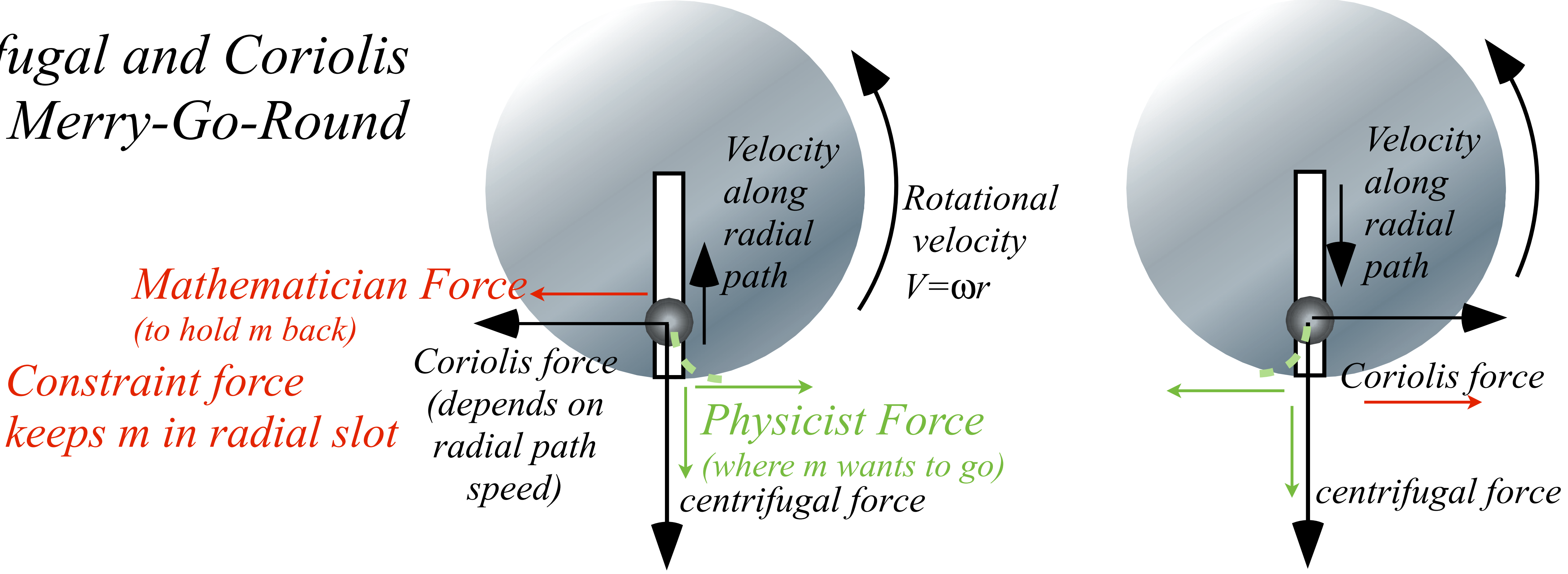
(a) Centrifugal and Coriolis Forces on Merry-Go-Round



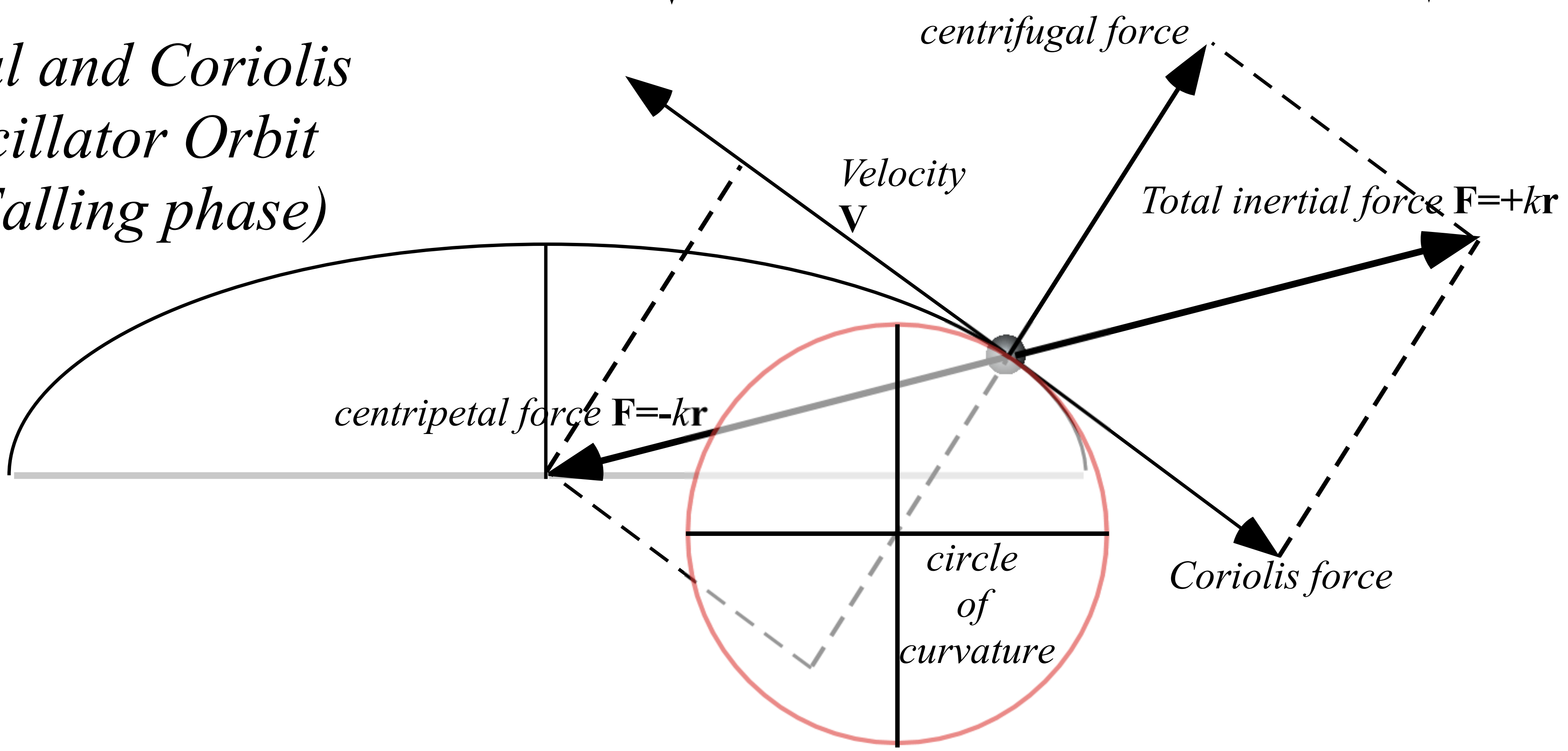
(b) Centrifugal and Coriolis Forces on Oscillator Orbit (Falling phase)



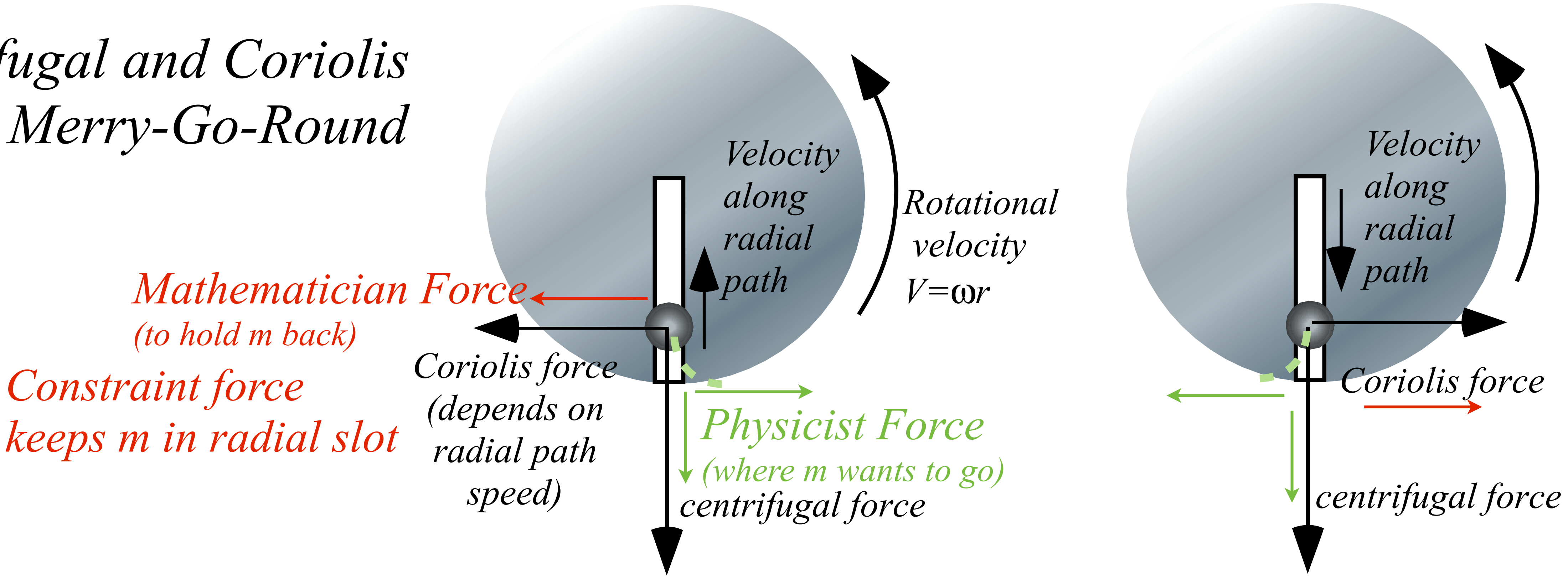
(a) Centrifugal and Coriolis Forces on Merry-Go-Round



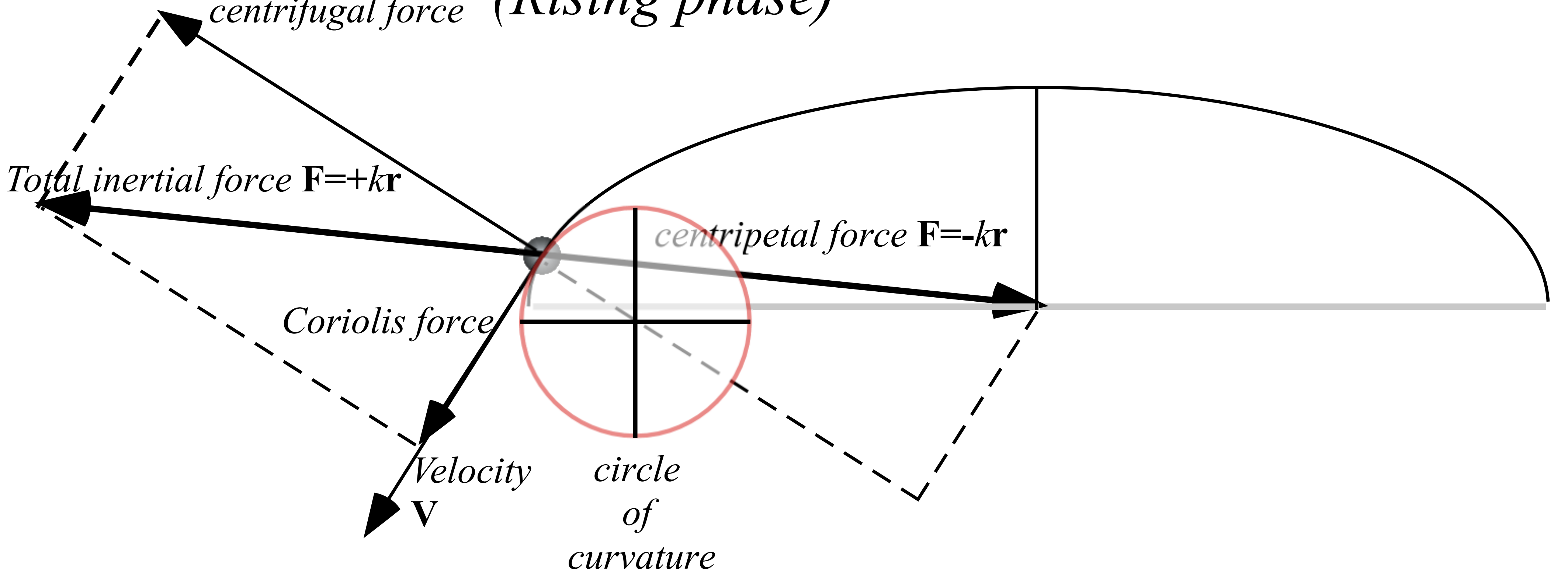
(b) Centrifugal and Coriolis Forces on Oscillator Orbit (Falling phase)

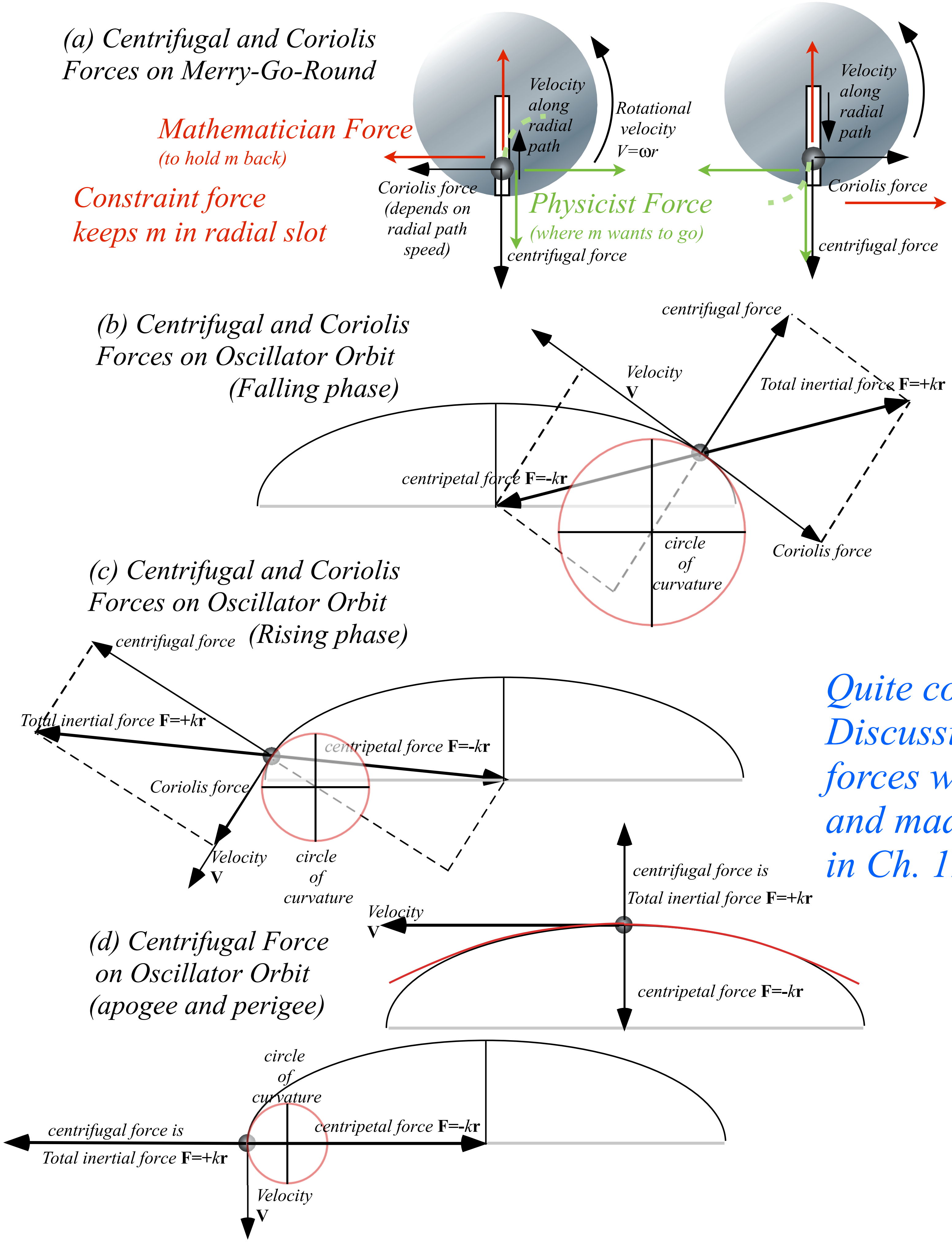


(a) Centrifugal and Coriolis Forces on Merry-Go-Round



(c) Centrifugal and Coriolis Forces on Oscillator Orbit (Rising phase)





(a) Centrifugal and Coriolis Forces on Merry-Go-Round

(b) Centrifugal and Coriolis Forces on Oscillator Orbit (Falling phase)

(c) Centrifugal and Coriolis Forces on Oscillator Orbit (Rising phase)

(d) Centrifugal Force on Oscillator Orbit (apogee and perigee)

Unit 1
Fig. 11.4
a-d

Quite confusing?
Discussion of Coriolis forces will be done more elegantly and made more physically intuitive in Ch. 12 of Unit1 and in Unit 6.

→ *Introduction to dual matrix operator contact geometry (based on IHO orbits)*

- *Quadratic form ellipse* $\mathbf{r}\bullet Q\bullet\mathbf{r}=1$ *vs.inverse form ellipse* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$
  - *Duality norm relations* $(\mathbf{r}\bullet\mathbf{p}=1)$
  - *Q-Ellipse tangents* $\mathbf{r}'$ *normal to dual* Q^{-1}*-ellipse position* $\mathbf{p}$ $(\mathbf{r}'\bullet\mathbf{p}=0=\mathbf{r}\bullet\mathbf{p}')$
- *Operator geometric sequences and eigenvectors*
- *Alternative scaling of matrix operator geometry*
  - *Vector calculus of tensor operation*

Quadratic forms and tangent contact geometry of their ellipses

A matrix Q that generates an ellipse by $\mathbf{r}\bullet Q\bullet\mathbf{r}=1$ *is called* positive-definite (*if* $\mathbf{r}\bullet Q\bullet\mathbf{r}$ *always* >0)

$$\begin{pmatrix} x & y \end{pmatrix}\bullet\overbrace{\begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix}}^{\mathbf{r}\bullet\mathbf{Q}\bullet\mathbf{r}}\bullet\begin{pmatrix} x \\ y \end{pmatrix}\overset{=1}{=1=}\overbrace{\begin{pmatrix} x & y \end{pmatrix}}^{\mathbf{r}}\bullet\overbrace{\begin{pmatrix} \frac{x}{a^2} \\ \frac{y}{b^2} \end{pmatrix}}^{\mathbf{Q}\bullet\mathbf{r}}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

A inverse matrix Q^{-1} *generates an ellipse by* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$ *called* inverse *or* dual *ellipse:*

$$\begin{pmatrix} p_x & p_y \end{pmatrix}\bullet\overbrace{\begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}}^{\mathbf{p}\bullet\mathbf{Q}^{-1}\bullet\mathbf{p}}\bullet\begin{pmatrix} p_x \\ p_y \end{pmatrix}\overset{=1}{=1=}\overbrace{\begin{pmatrix} p_x & p_y \end{pmatrix}}^{\mathbf{p}}\bullet\overbrace{\begin{pmatrix} a^2 p_x \\ b^2 p_y \end{pmatrix}}^{\mathbf{Q}^{-1}\bullet\mathbf{p}}=a^2 p_x^2+b^2 p_y^2$$

Quadratic forms and tangent contact geometry of their ellipses

A matrix Q that generates an ellipse by **r**•*Q*•**r**=*1 is called* positive-definite (*if* **r**•*Q*•**r** *always >0*)

$$(x\quad y)\bullet\overbrace{\begin{pmatrix}\frac{1}{a^2} & 0\\ 0 & \frac{1}{b^2}\end{pmatrix}}^{\mathbf{r}\bullet\mathbf{Q}\bullet\mathbf{r}\qquad\qquad =1}\bullet\begin{pmatrix}x\\ y\end{pmatrix}=1=(\overbrace{x\quad y}^{\mathbf{r}})\bullet\overbrace{\begin{pmatrix}\frac{x}{a^2}\\ \frac{y}{b^2}\end{pmatrix}}^{\mathbf{Q}\bullet\mathbf{r}=\mathbf{p}}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

Defined mapping between ellipses

A inverse matrix Q^{-1} *generates an ellipse by* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$ *called* inverse *or* dual *ellipse:*

$$(p_x\quad p_y)\bullet\overbrace{\begin{pmatrix}a^2 & 0\\ 0 & b^2\end{pmatrix}}^{\mathbf{p}\bullet\mathbf{Q}^{-1}\bullet\mathbf{p}\qquad\qquad =1}\bullet\begin{pmatrix}p_x\\ p_y\end{pmatrix}=1=(\overbrace{p_x\quad p_y}^{\mathbf{p}})\bullet\overbrace{\begin{pmatrix}a^2p_x\\ b^2p_y\end{pmatrix}}^{\mathbf{Q}^{-1}\bullet\mathbf{p}=\mathbf{r}}=a^2p_x^2+b^2p_y^2$$

Introduction to dual matrix operator contact geometry (based on IHO orbits)

→ *Quadratic form ellipse* $\mathbf{r}\bullet Q\bullet\mathbf{r}=1$ *vs.inverse form ellipse* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$

- *Duality norm relations* $(\mathbf{r}\bullet\mathbf{p}=1)$
- *Q-Ellipse tangents* $\mathbf{r}'$ *normal to dual* Q^{-1}*-ellipse position* $\mathbf{p}$ $(\mathbf{r}'\bullet\mathbf{p}=0=\mathbf{r}\bullet\mathbf{p}')$

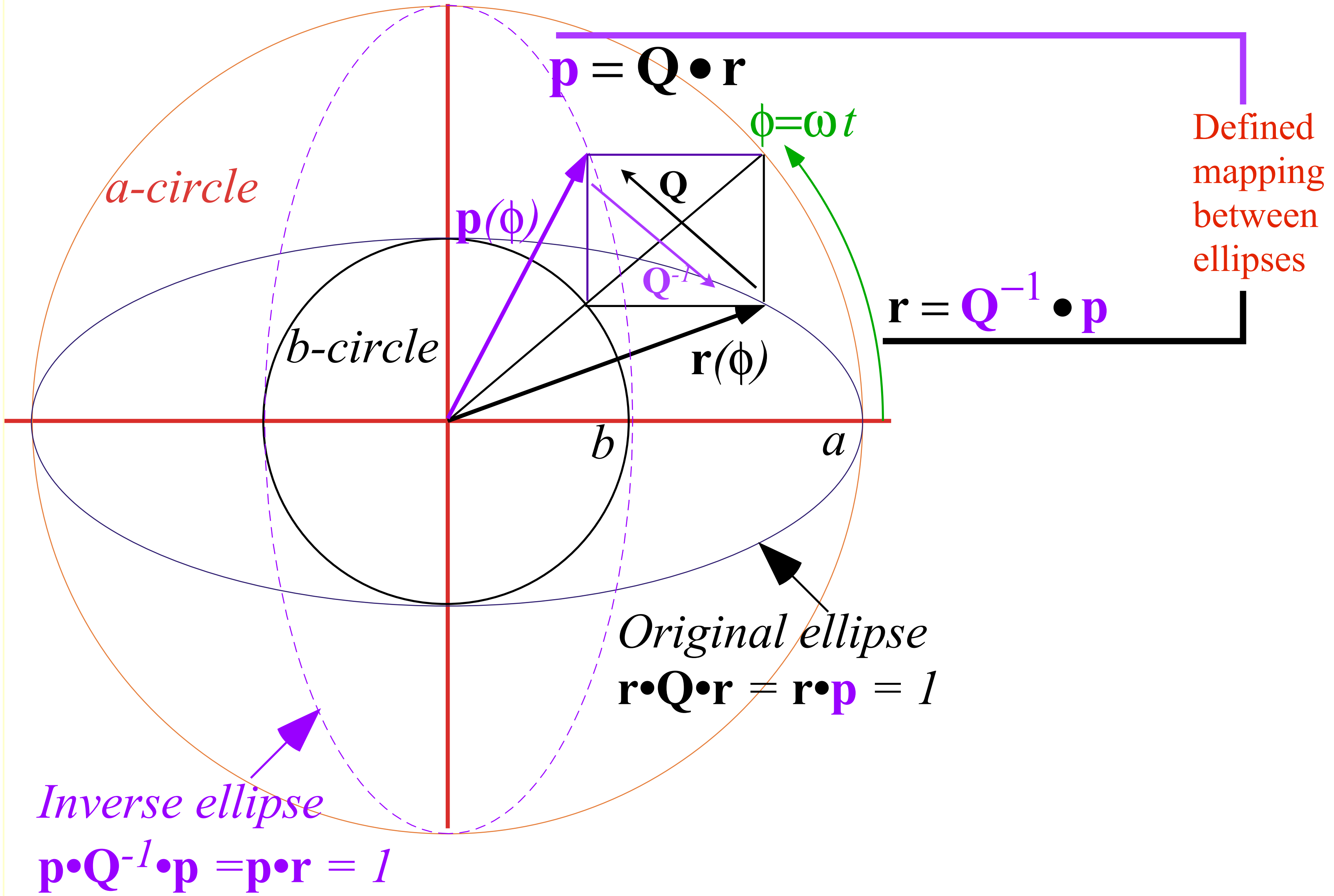
Operator geometric sequences and eigenvectors

Alternative scaling of matrix operator geometry

- *Vector calculus of tensor operation*

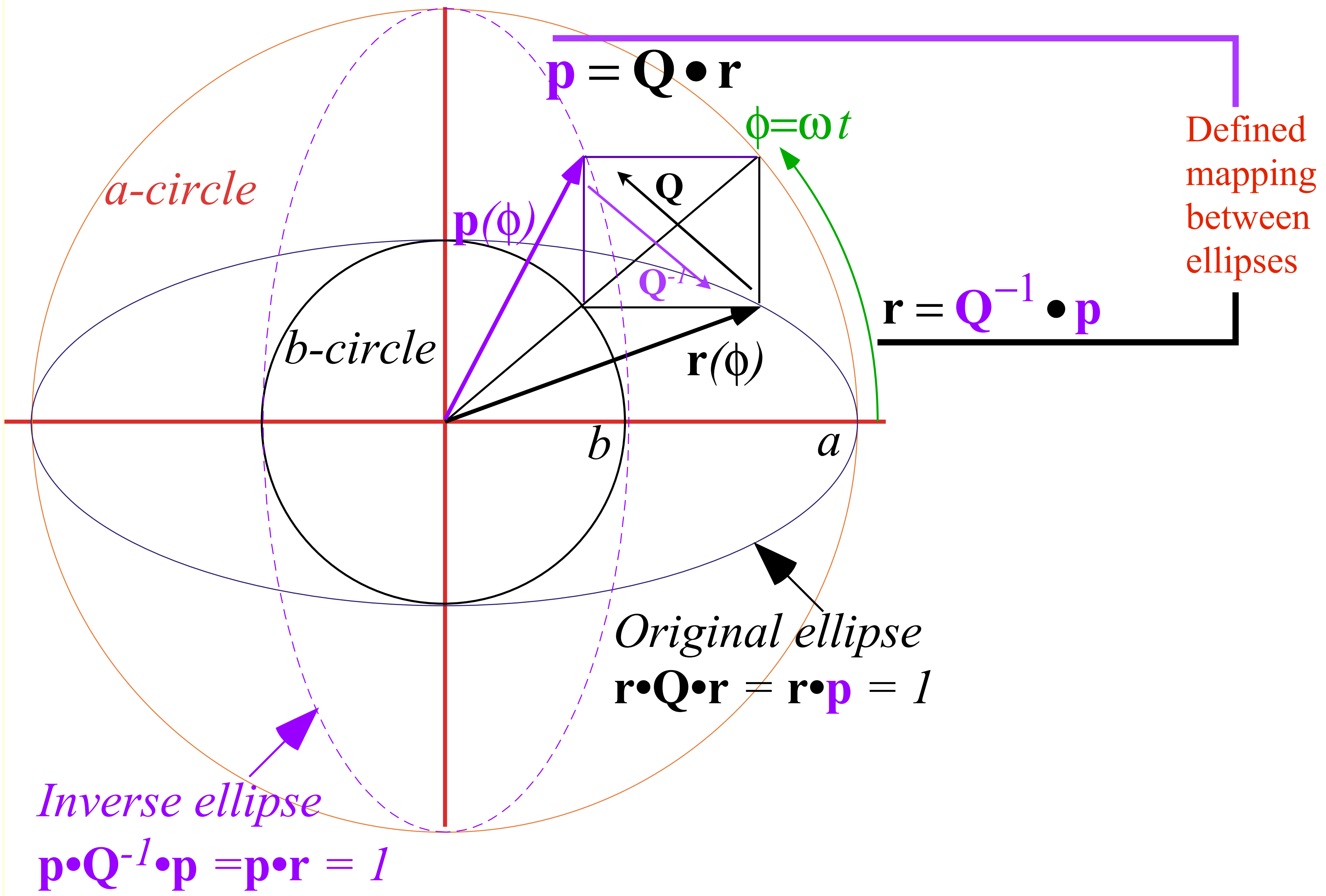
(a) Quadratic form ellipse and *Inverse quadratic form ellipse*

based on
Unit 1
Fig. 11.6



(a) Quadratic form ellipse and
Inverse quadratic form ellipse

based on
Unit 1
Fig. 11.6



Here plot of **p**-ellipse is re-scaled by scalefactor $S=a \cdot b$

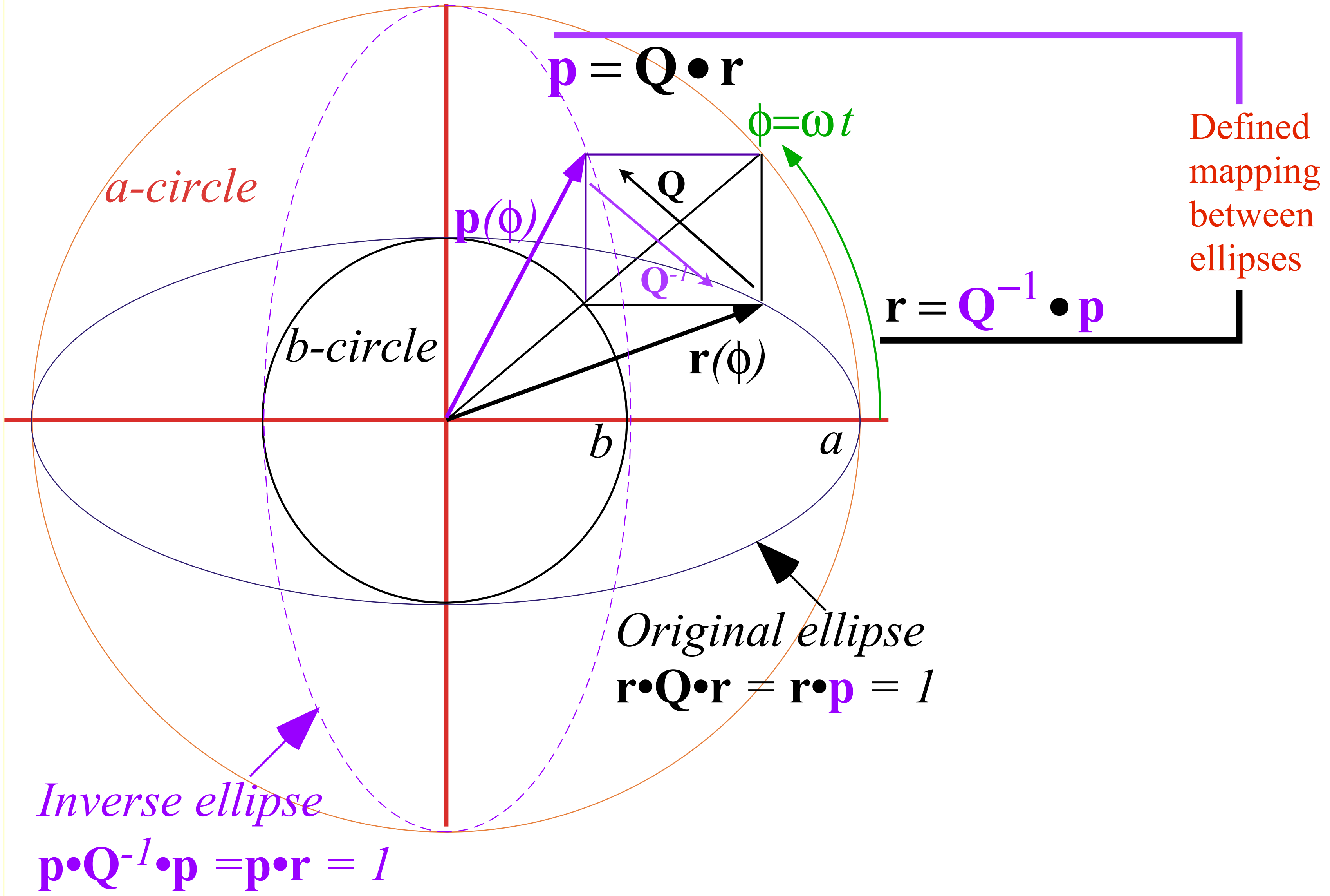
p-ellipse x-radius=$1/a$ plotted at: $S(1/a)=b$ (=1 for a=2, b=1)

p-ellipse y-radius=$1/b$ plotted at: $S(1/b)=a$ (=2 for a=2, b=1)

- *Introduction to dual matrix operator geometry (based on IHO orbits)*
  - *Quadratic form ellipse* $\mathbf{r}\bullet Q\bullet\mathbf{r}=1$ *vs.inverse form ellipse* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$
    - *Duality norm relations* ($\mathbf{r}\bullet\mathbf{p}=1$)
    - *Q-Ellipse tangents* $\mathbf{r}'$ *normal to dual* Q^{-1}*-ellipse position* $\mathbf{p}$ ($\mathbf{r}'\bullet\mathbf{p}=0=\mathbf{r}\bullet\mathbf{p}'$)
  - *Operator geometric sequences and eigenvectors*
  - *Alternative scaling of matrix operator geometry*
    - *Vector calculus of tensor operation*

(a) Quadratic form ellipse and Inverse quadratic form ellipse

based on
Unit 1
Fig. 11.6



Quadratic form $\mathbf{r} \bullet \mathbf{Q} \bullet \mathbf{r} = 1$ *has mutual duality relations with inverse form* $\mathbf{p} \bullet \mathbf{Q}^{-1} \bullet \mathbf{p} = 1 = \mathbf{p} \bullet \mathbf{r}$

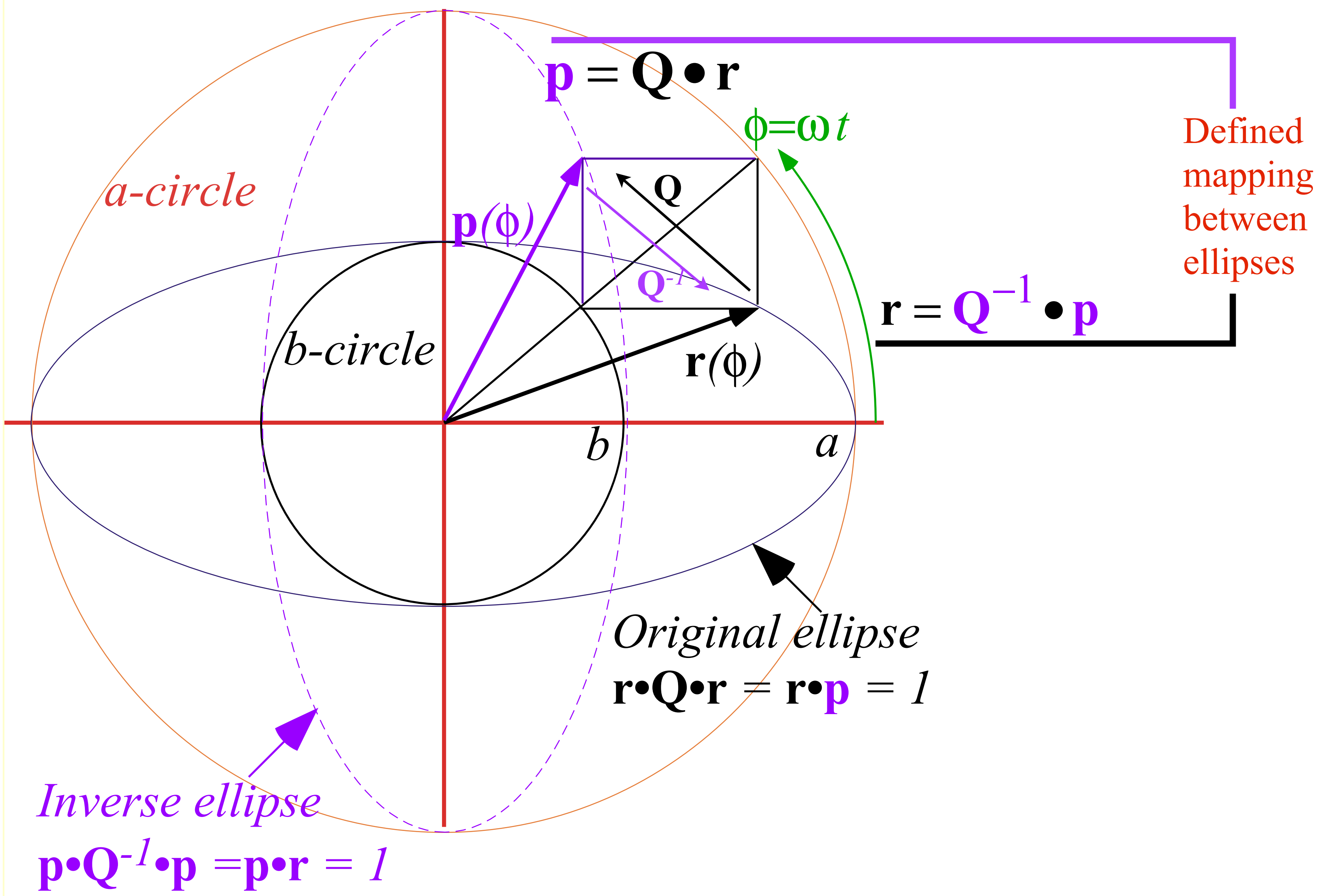
Here plot of **p**-ellipse is re-scaled by scalefactor $S = a \cdot b$

p-ellipse x-radius=1/a plotted at: $S(1/a) = b$ (=1 for a=2, b=1)

p-ellipse y-radius=1/b plotted at: $S(1/b) = a$ (=2 for a=2, b=1)

(a) Quadratic form ellipse and Inverse quadratic form ellipse

based on
Unit 1
Fig. 11.6



Quadratic form **r•Q•r** =*1* *has mutual duality relations with inverse form* **p•Q**$^{-1}$**•p** =*1*= **p•r**

$$\mathbf{p} = \mathbf{Q} \cdot \mathbf{r} = \overbrace{\begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix}}^{\mathbf{Q}} \bullet \overbrace{\begin{pmatrix} x \\ y \end{pmatrix}}^{\mathbf{r}} = \overbrace{\begin{pmatrix} x/a^2 \\ y/b^2 \end{pmatrix}}^{\mathbf{p}} = \begin{pmatrix} (1/a)\cos\phi \\ (1/b)\sin\phi \end{pmatrix} \text{ where: } \begin{array}{l} x = r_x = a\cos\phi = a\cos\omega t \\ y = r_y = b\sin\phi \;\; = b\sin\omega t \end{array} \text{ so: } \boxed{\mathbf{p} \cdot \mathbf{r} = 1}$$

Here plot of **p**-ellipse is re-scaled by scalefactor $S=a \cdot b$

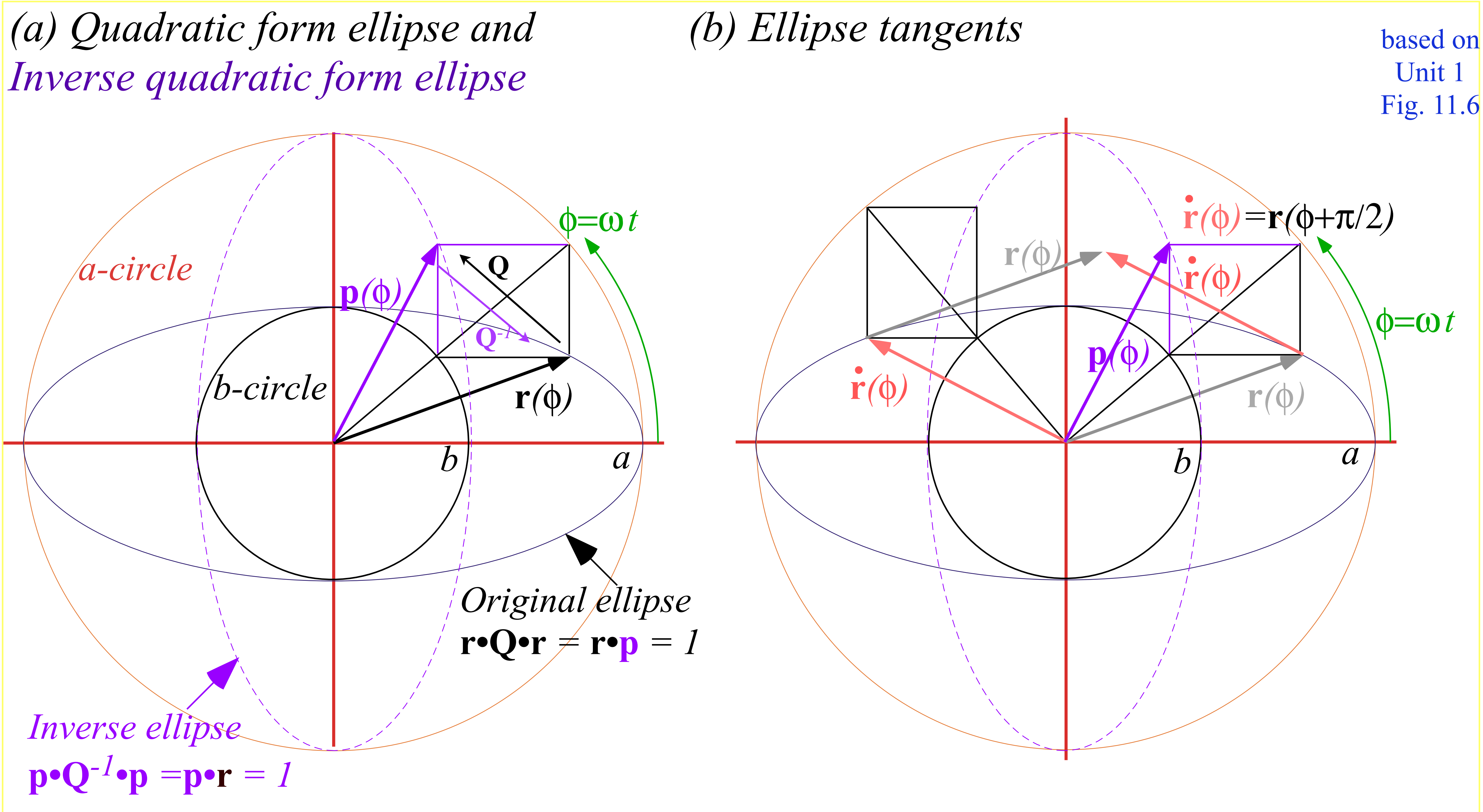
p-ellipse x-radius=$1/a$ plotted at: $S(1/a)=b$ (=1 for a=2, b=1)

p-ellipse y-radius=$1/b$ plotted at: $S(1/b)=a$ (=2 for a=2, b=1)

Link ⇒ BoxIt simulation of IHO orbits
Link → IHO orbital time rates of change
Link → IHO Exegesis Plot

Introduction to dual matrix operator geometry (based on IHO orbits)

- *Quadratic form ellipse* $\mathbf{r}\bullet Q\bullet\mathbf{r}=1$ *vs.inverse form ellipse* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$
  - *Duality norm relations* ($\mathbf{r}\bullet\mathbf{p}=1$)
  - → *Q-Ellipse tangents* $\mathbf{r}'$ *normal to dual* Q^{-1}*-ellipse position* $\mathbf{p}$ ($\mathbf{r}'\bullet\mathbf{p}=0=\mathbf{r}\bullet\mathbf{p}'$)
- *Operator geometric sequences and eigenvectors*
- *Alternative scaling of matrix operator geometry*
  - *Vector calculus of tensor operation*



based on Unit 1 Fig. 11.6

Quadratic form **r•Q•r** *=1 has mutual duality relations with inverse form* **p•Q⁻¹•p** *=1=* **p•r**

$$\mathbf{p}=\mathbf{Q}\bullet\mathbf{r}=\begin{pmatrix}1/a^2 & 0\\ 0 & 1/b^2\end{pmatrix}\bullet\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x/a^2\\ y/b^2\end{pmatrix}=\begin{pmatrix}(1/a)\cos\phi\\ (1/b)\sin\phi\end{pmatrix}\quad\text{where: }\begin{matrix}x=r_x=a\cos\phi=a\cos\omega t\\ y=r_y=b\sin\phi\ =b\sin\omega t\end{matrix}\quad\text{so: }\boxed{\mathbf{p}\bullet\mathbf{r}=1}$$

Here plot of **p**-ellipse is re-scaled by scalefactor $S=a\cdot b$

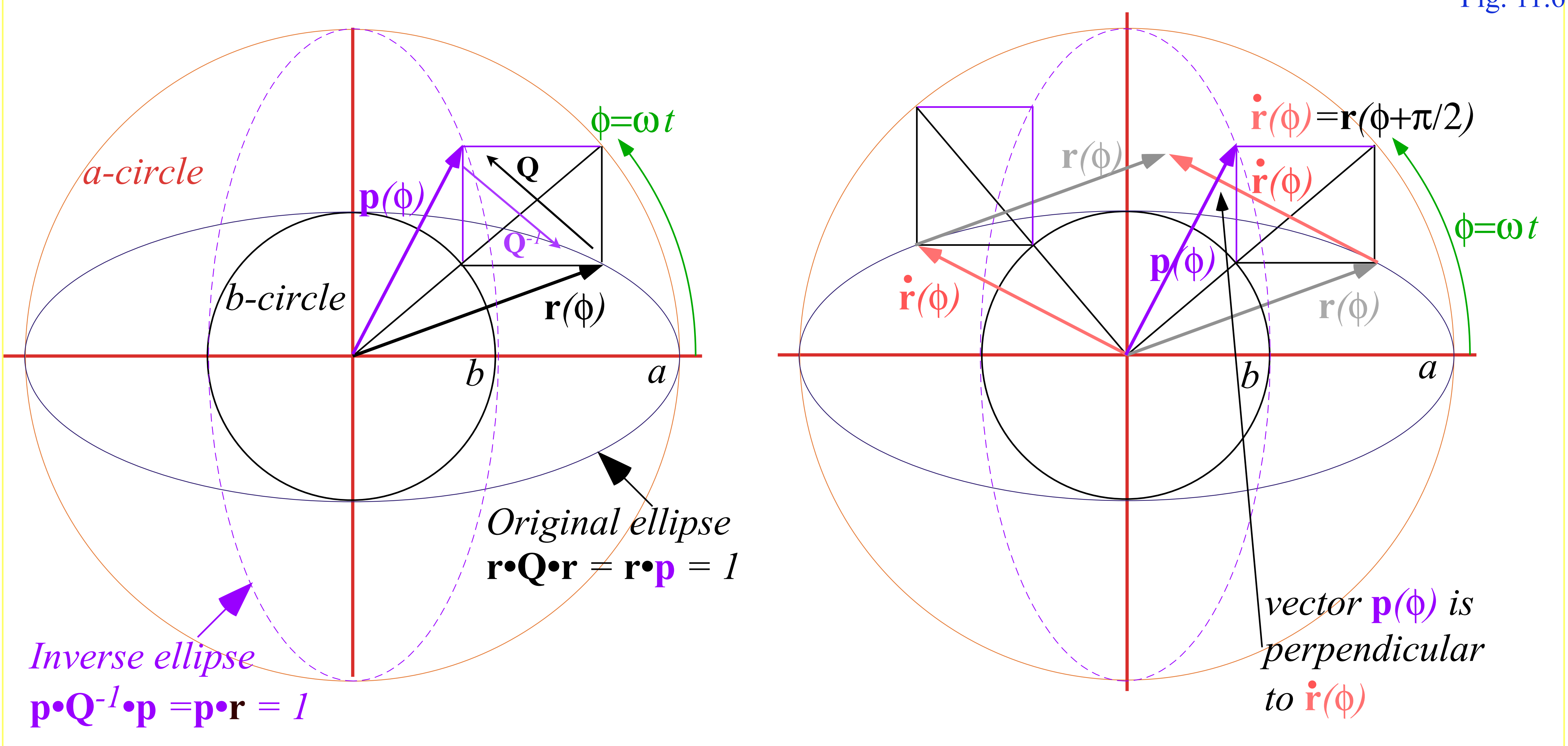
p-ellipse x-radius=1/a plotted at: $S(1/a)=b$ (=1 for a=2, b=1)

p-ellipse y-radius=1/b plotted at: $S(1/b)=a$ (=2 for a=2, b=1)

(a) Quadratic form ellipse and
Inverse quadratic form ellipse

(b) Ellipse tangents

based on
Unit 1
Fig. 11.6



Quadratic form $\mathbf{r}\bullet\mathbf{Q}\bullet\mathbf{r} = 1$ *has mutual duality relations with inverse form* $\mathbf{p}\bullet\mathbf{Q}^{-1}\bullet\mathbf{p} = 1 = \mathbf{p}\bullet\mathbf{r}$

$$\mathbf{p}=\mathbf{Q}\bullet\mathbf{r}=\begin{pmatrix}1/a^2 & 0\\ 0 & 1/b^2\end{pmatrix}\bullet\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x/a^2\\ y/b^2\end{pmatrix}=\begin{pmatrix}(1/a)\cos\phi\\ (1/b)\sin\phi\end{pmatrix}$$

where: $x = r_x = a\cos\phi = a\cos\omega t$
$y = r_y = b\sin\phi \ \ = b\sin\omega t$

so: $\boxed{\mathbf{p}\bullet\mathbf{r} = 1}$

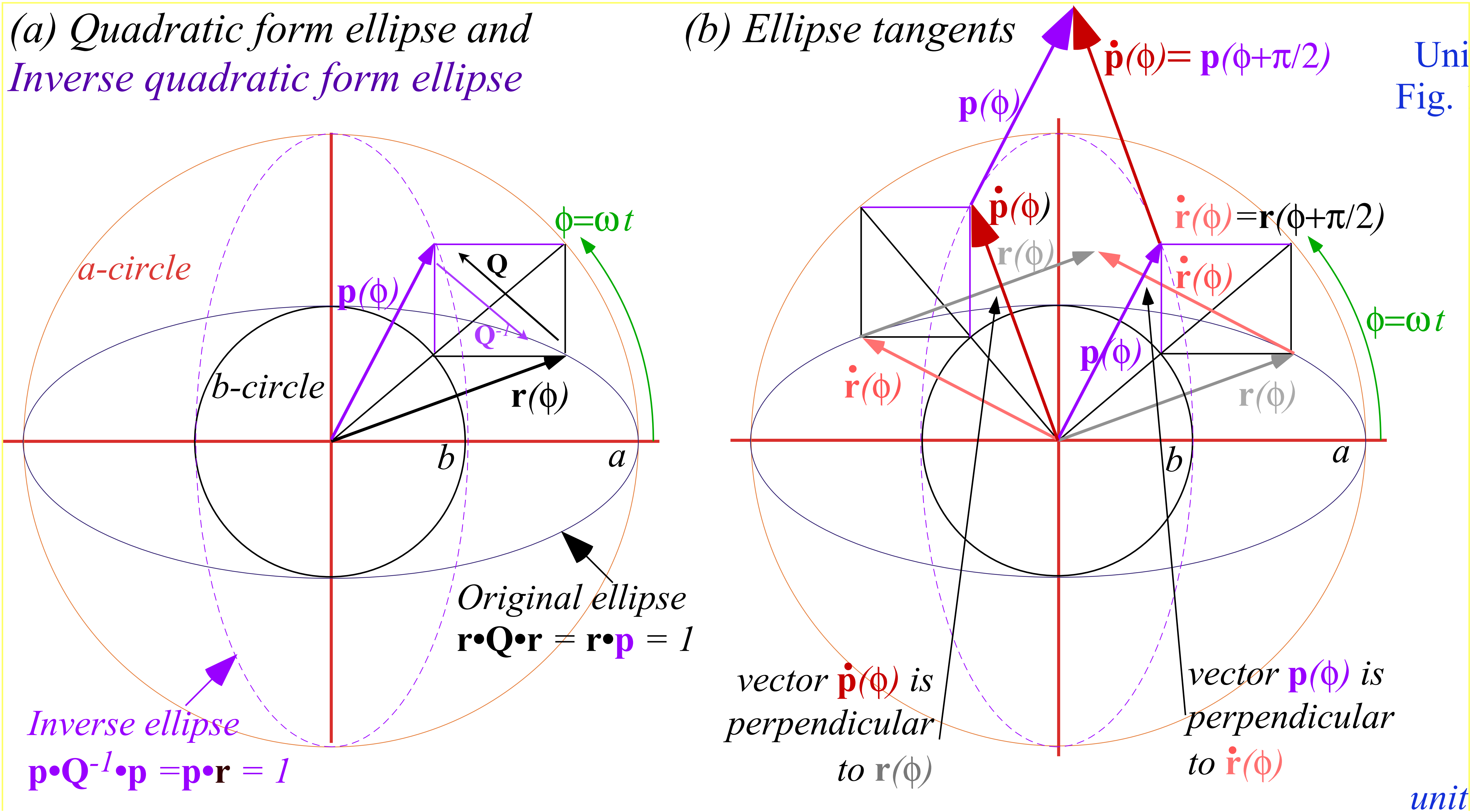
$\mathbf{p}$ is perpendicular to velocity $\mathbf{v}=\dot{\mathbf{r}}$, a mutual orthogonality

$$\boxed{\dot{\mathbf{r}}\bullet\mathbf{p}=0}=\begin{pmatrix}\dot{r}_x & \dot{r}_y\end{pmatrix}\bullet\begin{pmatrix}p_x\\ p_y\end{pmatrix}=\begin{pmatrix}-a\sin\phi & b\cos\phi\end{pmatrix}\bullet\begin{pmatrix}(1/a)\cos\phi\\ (1/b)\sin\phi\end{pmatrix}$$

where: $\dot{r}_x = -a\sin\phi$
$\dot{r}_y = b\cos\phi$

and: $p_x = (1/a)\cos\phi$
$p_y = (1/b)\sin\phi$

Unit 1
Fig. 11.6



(a) Quadratic form ellipse and Inverse quadratic form ellipse

(b) Ellipse tangents

unit mutual projection

Quadratic form $\mathbf{r}\bullet\mathbf{Q}\bullet\mathbf{r}=1$ *has mutual duality relations with inverse form* $\mathbf{p}\bullet\mathbf{Q}^{-1}\bullet\mathbf{p}=1$

$$\mathbf{p}=\mathbf{Q}\bullet\mathbf{r}=\begin{pmatrix}1/a^2 & 0\\ 0 & 1/b^2\end{pmatrix}\bullet\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x/a^2\\ y/b^2\end{pmatrix}=\begin{pmatrix}(1/a)\cos\phi\\ (1/b)\sin\phi\end{pmatrix}$$

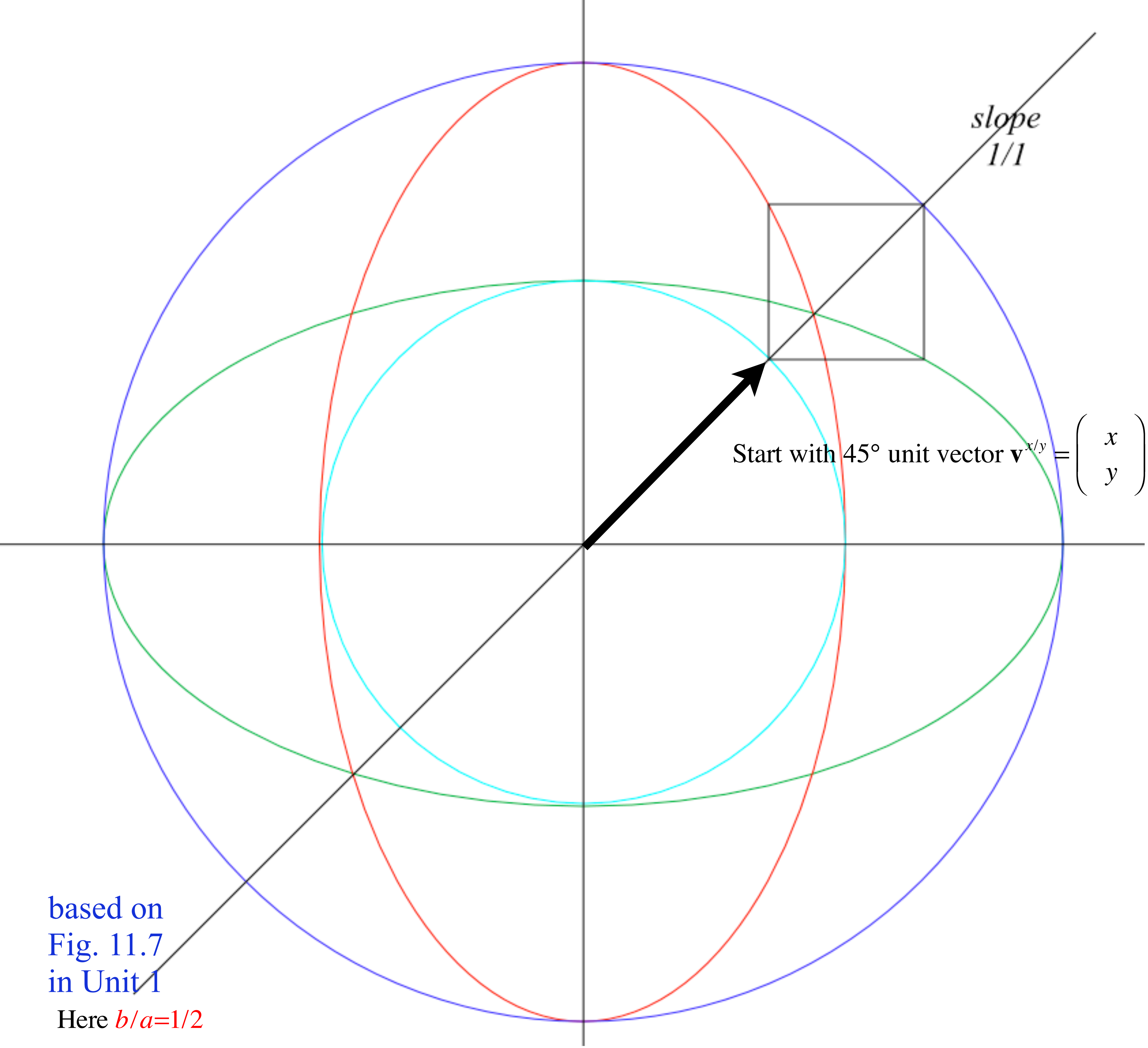
where: $x=r_x=a\cos\phi=a\cos\omega t$, $y=r_y=b\sin\phi\ =b\sin\omega t$ so: $\boxed{\mathbf{p}\bullet\mathbf{r}=1}$

$\mathbf{p}$ *is perpendicular to velocity* $\mathbf{v}=\dot{\mathbf{r}}$ *, a mutual orthogonality. So is* $\mathbf{r}$ *perpendicular to* $\dot{\mathbf{p}}$*:* $\boxed{\dot{\mathbf{p}}\bullet\mathbf{r}=0}$

$$\boxed{\dot{\mathbf{r}}\bullet\mathbf{p}=0}=\begin{pmatrix}\dot r_x & \dot r_y\end{pmatrix}\bullet\begin{pmatrix}p_x\\ p_y\end{pmatrix}=\begin{pmatrix}-a\sin\phi & b\cos\phi\end{pmatrix}\bullet\begin{pmatrix}(1/a)\cos\phi\\ (1/b)\sin\phi\end{pmatrix}$$

where: $\dot r_x=-a\sin\phi$, $\dot r_y=b\cos\phi$ and: $p_x=(1/a)\cos\phi$, $p_y=(1/b)\sin\phi$

Introduction to dual matrix operator geometry (based on IHO orbits)
- *Quadratic form ellipse* $\mathbf{r}\bullet Q\bullet\mathbf{r}=1$ *vs.inverse form ellipse* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$
    - *Duality norm relations (* $\mathbf{r}\bullet\mathbf{p}=1$*)*
    - *Q-Ellipse tangents* $\mathbf{r}'$ *normal to dual* Q^{-1}*-ellipse position* $\mathbf{p}$ *(* $\mathbf{r}'\bullet\mathbf{p}=0=\mathbf{r}\bullet\mathbf{p}'$*)*
- → *Operator geometric sequences and eigenvectors*
- *Alternative scaling of matrix operator geometry*
    - *Vector calculus of tensor operation*



Start with 45° unit vector $\mathbf{v}^{x/y} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$.

based on
Fig. 11.7
in Unit 1

Here $b/a=1/2$

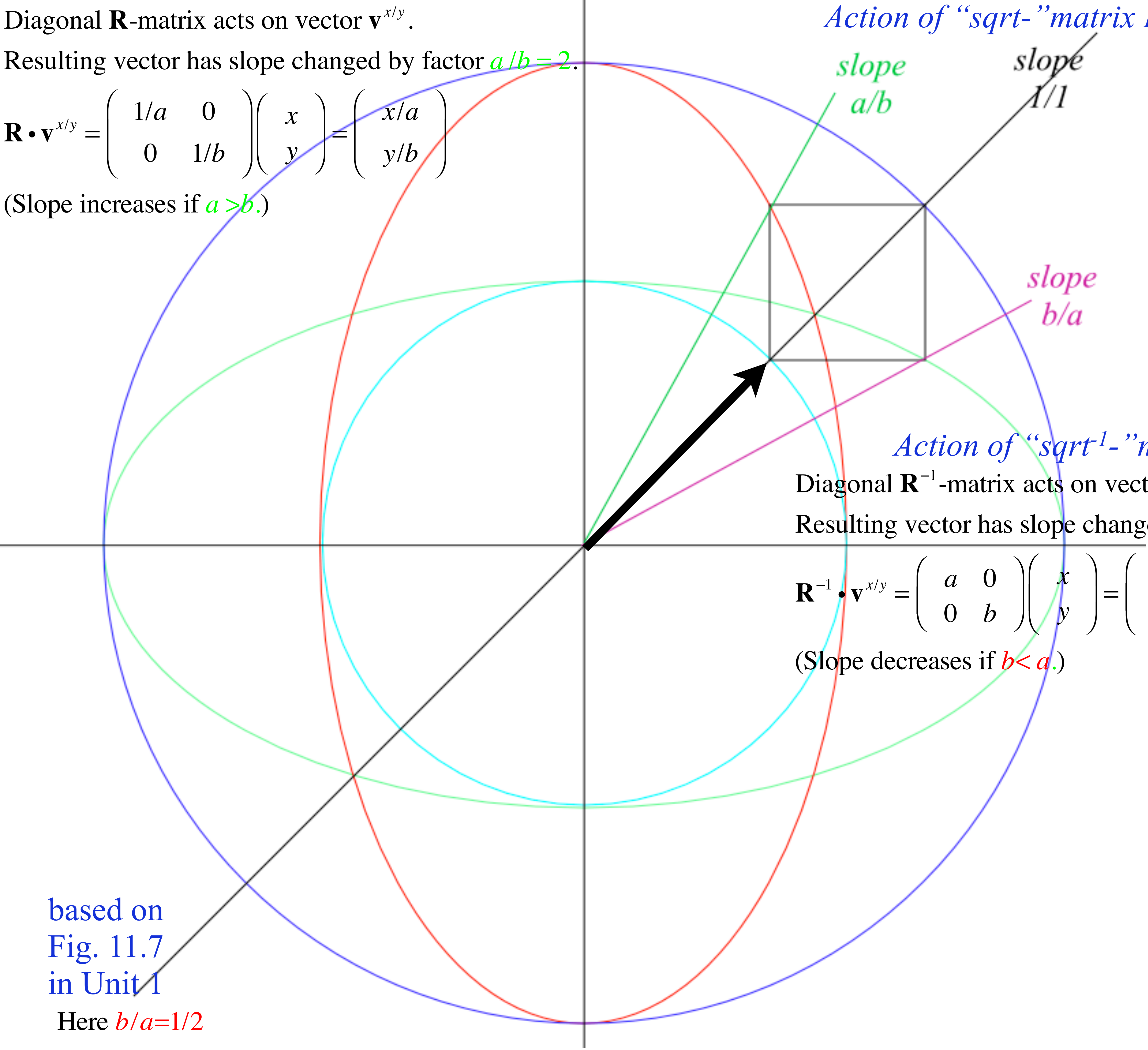
Action of “sqrt-”matrix $R=\sqrt{Q}$

Diagonal **R**-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a/b=2$.

$$\mathbf{R}\bullet\mathbf{v}^{x/y}=\begin{pmatrix}1/a & 0\\ 0 & 1/b\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x/a\\ y/b\end{pmatrix}$$

(Slope increases if $a>b$.)



Action of “sqrt^{-1}-”matrix $R^{-1}=\sqrt{Q}^{-1}$

Diagonal $\mathbf{R}^{-1}$-matrix acts on vector $\mathbf{v}^{x/y}$.

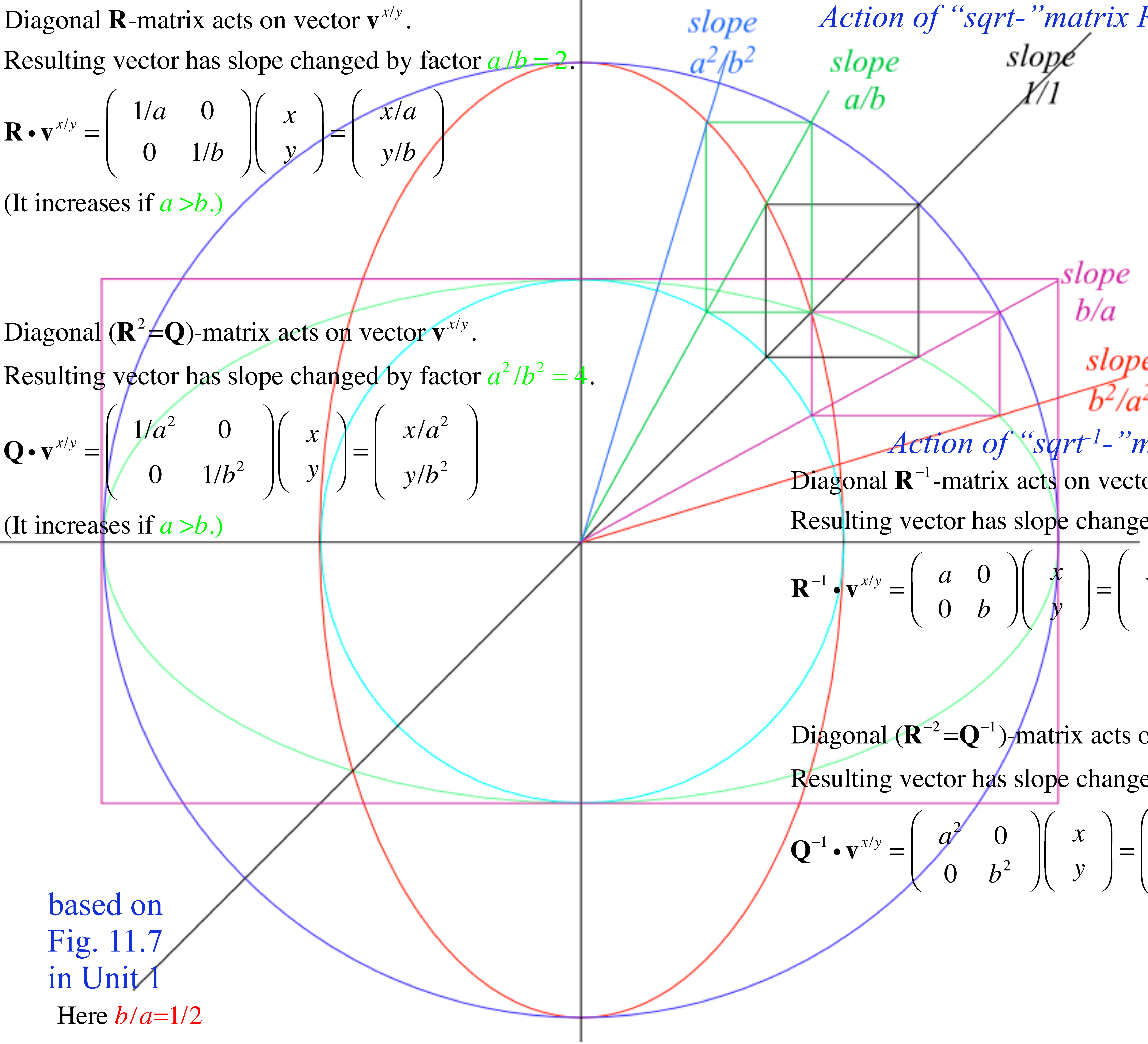
Resulting vector has slope changed by factor b/a.

$$\mathbf{R}^{-1}\bullet\mathbf{v}^{x/y}=\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x\cdot a\\ y\cdot b\end{pmatrix}$$

(Slope decreases if $b<a$.)

based on
Fig. 11.7
in Unit 1

Here $b/a=1/2$



Action of "sqrt-"matrix $R=\sqrt{Q}$

Diagonal **R**-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a/b=2$.

$$\mathbf{R}\bullet\mathbf{v}^{x/y}=\begin{pmatrix}1/a & 0\\ 0 & 1/b\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x/a\\ y/b\end{pmatrix}$$

(It increases if $a>b$.)

Diagonal ($\mathbf{R}^2=\mathbf{Q}$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a^2/b^2=4$.

$$\mathbf{Q}\bullet\mathbf{v}^{x/y}=\begin{pmatrix}1/a^2 & 0\\ 0 & 1/b^2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x/a^2\\ y/b^2\end{pmatrix}$$

(It increases if $a>b$.)

Action of "sqrt^{-1}-"matrix $R^{-1}=\sqrt{Q^{-1}}$

Diagonal $\mathbf{R}^{-1}$-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $b/a=1/2$.

$$\mathbf{R}^{-1}\bullet\mathbf{v}^{x/y}=\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x\cdot a\\ y\cdot b\end{pmatrix}$$

Diagonal ($\mathbf{R}^{-2}=\mathbf{Q}^{-1}$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $b^2/a^2=1/4$.

$$\mathbf{Q}^{-1}\bullet\mathbf{v}^{x/y}=\begin{pmatrix}a^2 & 0\\ 0 & b^2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x\cdot a^2\\ y\cdot b^2\end{pmatrix}$$

based on
Fig. 11.7
in Unit 1

Here $b/a=1/2$

Diagonal **R**-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a/b = 2$.

$$\mathbf{R}\bullet\mathbf{v}^{x/y} = \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/a \\ y/b \end{pmatrix}$$

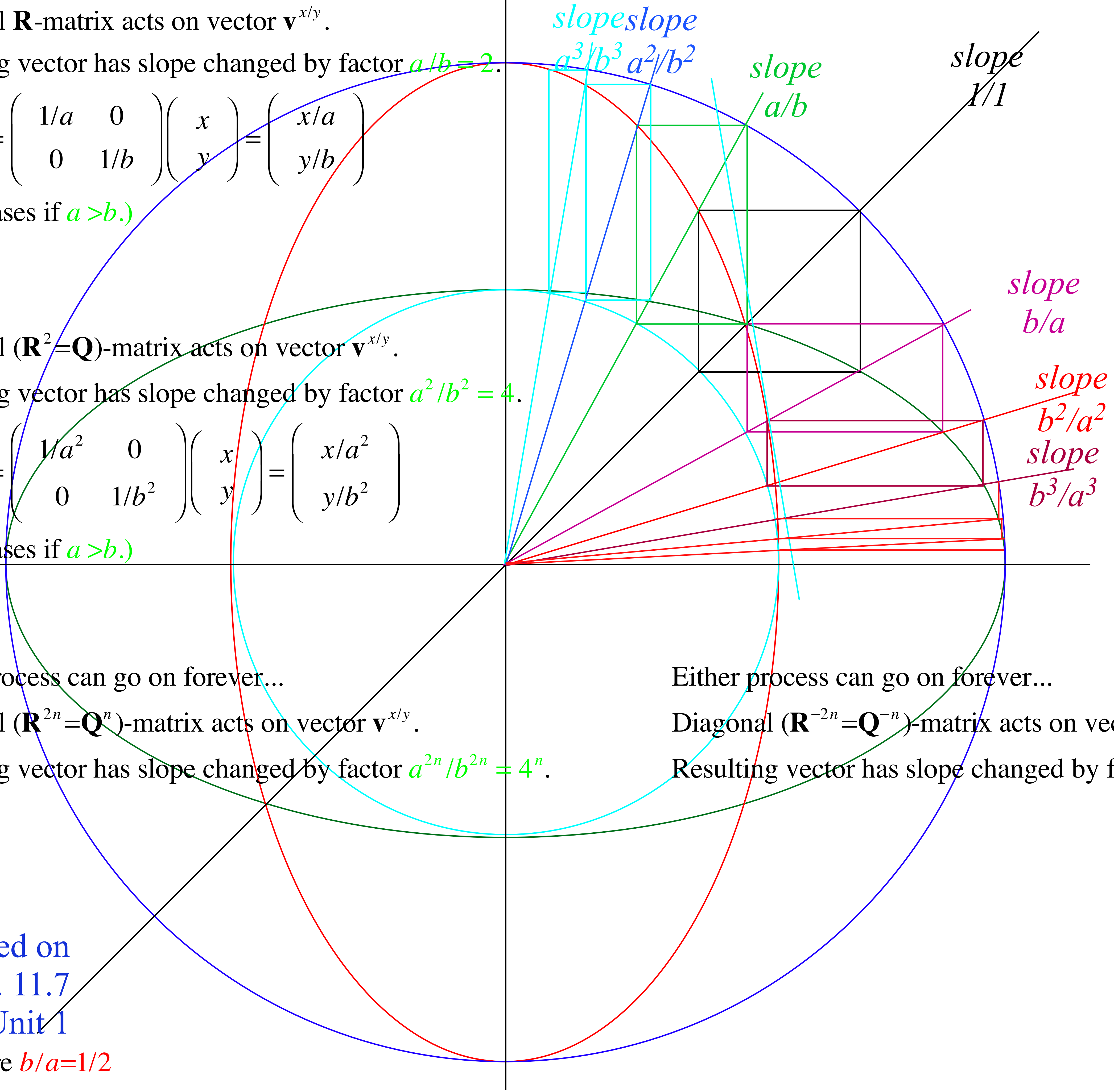
(It increases if $a > b$.)

Diagonal ($\mathbf{R}^2 = \mathbf{Q}$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a^2/b^2 = 4$.

$$\mathbf{Q}\bullet\mathbf{v}^{x/y} = \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/a^2 \\ y/b^2 \end{pmatrix}$$

(It increases if $a > b$.)



Either process can go on forever...

Diagonal ($\mathbf{R}^{2n} = \mathbf{Q}^n$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a^{2n}/b^{2n} = 4^n$.

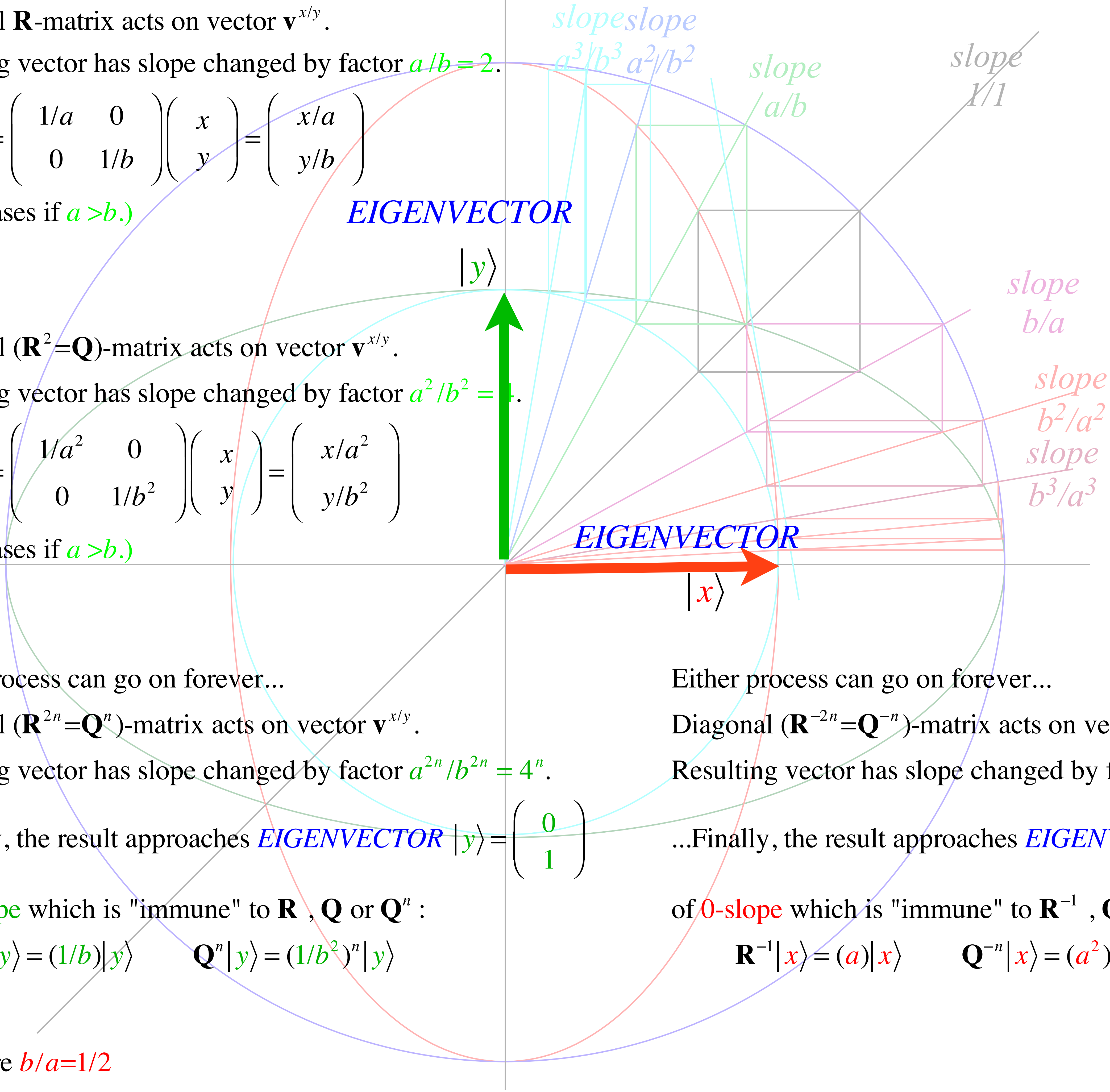
Either process can go on forever...

Diagonal ($\mathbf{R}^{-2n} = \mathbf{Q}^{-n}$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $b^{2n}/a^{2n} = 4^{-n}$.

based on
Fig. 11.7
in Unit 1

Here $b/a = 1/2$



Diagonal **R**-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a/b=2$.

$$\mathbf{R}\bullet\mathbf{v}^{x/y}=\begin{pmatrix}1/a & 0\\ 0 & 1/b\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x/a\\ y/b\end{pmatrix}$$

(It increases if $a>b$.)

Diagonal ($\mathbf{R}^2$=**Q**)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a^2/b^2=4$.

$$\mathbf{Q}\bullet\mathbf{v}^{x/y}=\begin{pmatrix}1/a^2 & 0\\ 0 & 1/b^2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x/a^2\\ y/b^2\end{pmatrix}$$

(It increases if $a>b$.)

Either process can go on forever...

Diagonal ($\mathbf{R}^{2n}$=$\mathbf{Q}^n$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a^{2n}/b^{2n}=4^n$.

...Finally, the result approaches *EIGENVECTOR* $|y\rangle=\begin{pmatrix}0\\ 1\end{pmatrix}$

of ∞-slope which is "immune" to **R** , **Q** or $\mathbf{Q}^n$:

$$\mathbf{R}|y\rangle=(1/b)|y\rangle \qquad \mathbf{Q}^n|y\rangle=(1/b^2)^n|y\rangle$$

Either process can go on forever...

Diagonal ($\mathbf{R}^{-2n}$=$\mathbf{Q}^{-n}$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $b^{2n}/a^{2n}=4^{-n}$.

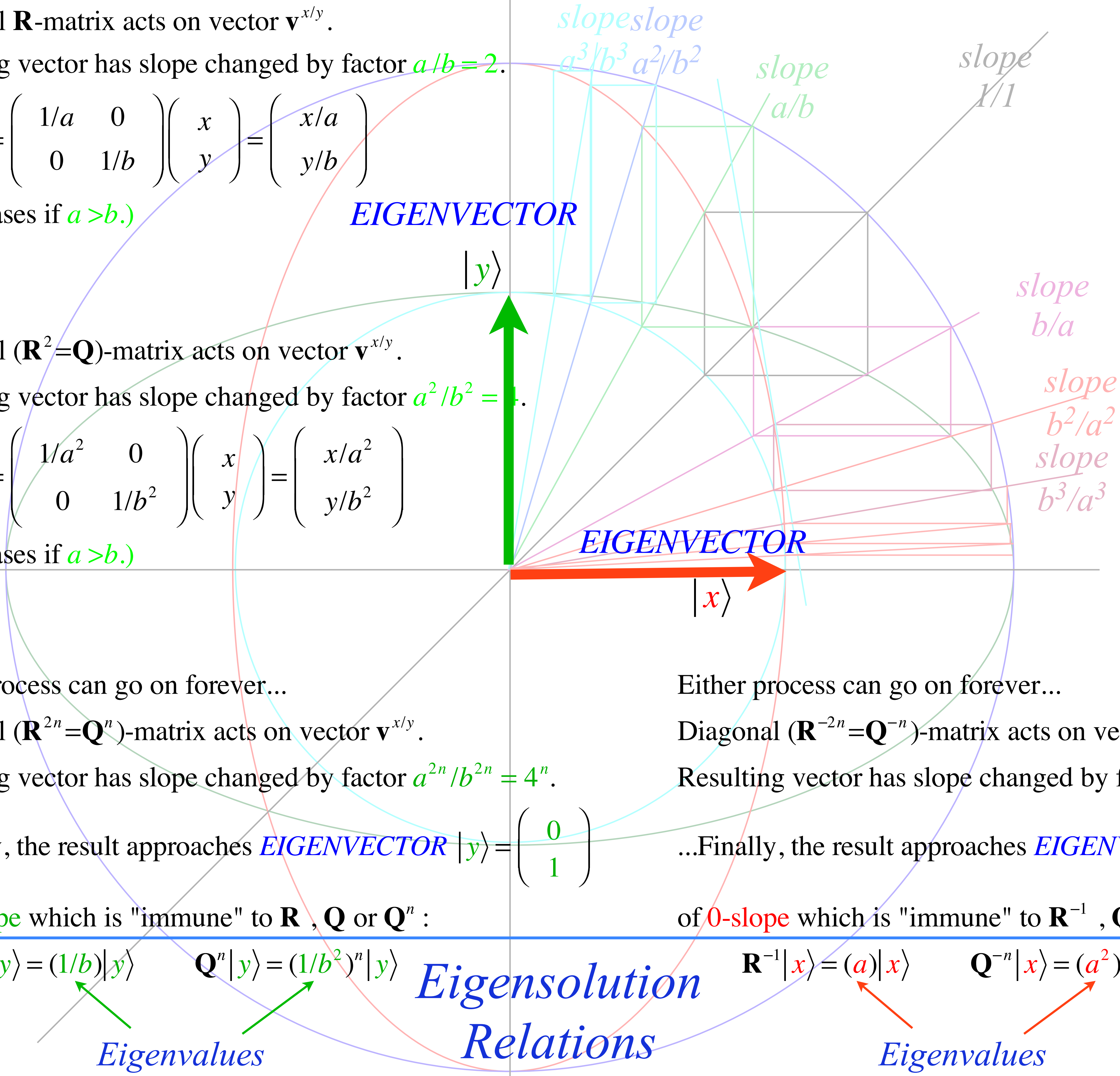
...Finally, the result approaches *EIGENVECTOR* $|x\rangle=\begin{pmatrix}1\\ 0\end{pmatrix}$

of 0-slope which is "immune" to $\mathbf{R}^{-1}$, $\mathbf{Q}^{-1}$ or $\mathbf{Q}^{-n}$:

$$\mathbf{R}^{-1}|x\rangle=(a)|x\rangle \qquad \mathbf{Q}^{-n}|x\rangle=(a^2)^n|x\rangle$$

Here b/a=1/2

Here $b/a=1/2$



Diagonal **R**-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a/b=2$.

$$\mathbf{R}\bullet\mathbf{v}^{x/y}=\begin{pmatrix}1/a & 0\\ 0 & 1/b\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x/a\\ y/b\end{pmatrix}$$

(It increases if $a>b$.)

Diagonal ($\mathbf{R}^2=\mathbf{Q}$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a^2/b^2=4$.

$$\mathbf{Q}\bullet\mathbf{v}^{x/y}=\begin{pmatrix}1/a^2 & 0\\ 0 & 1/b^2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x/a^2\\ y/b^2\end{pmatrix}$$

(It increases if $a>b$.)

Either process can go on forever...

Diagonal ($\mathbf{R}^{2n}=\mathbf{Q}^n$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a^{2n}/b^{2n}=4^n$.

...Finally, the result approaches *EIGENVECTOR* $|y\rangle=\begin{pmatrix}0\\ 1\end{pmatrix}$

of ∞-slope which is "immune" to **R** , **Q** or $\mathbf{Q}^n$:

Either process can go on forever...

Diagonal ($\mathbf{R}^{-2n}=\mathbf{Q}^{-n}$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $b^{2n}/a^{2n}=4^{-n}$.

...Finally, the result approaches *EIGENVECTOR* $|x\rangle=\begin{pmatrix}1\\ 0\end{pmatrix}$

of 0-slope which is "immune" to $\mathbf{R}^{-1}$, $\mathbf{Q}^{-1}$ or $\mathbf{Q}^{-n}$:

$\mathbf{R}|y\rangle=(1/b)|y\rangle$ $\quad$ $\mathbf{Q}^n|y\rangle=(1/b^2)^n|y\rangle$

Eigenvalues

Eigensolution Relations

$\mathbf{R}^{-1}|x\rangle=(a)|x\rangle$ $\quad$ $\mathbf{Q}^{-n}|x\rangle=(a^2)^n|x\rangle$

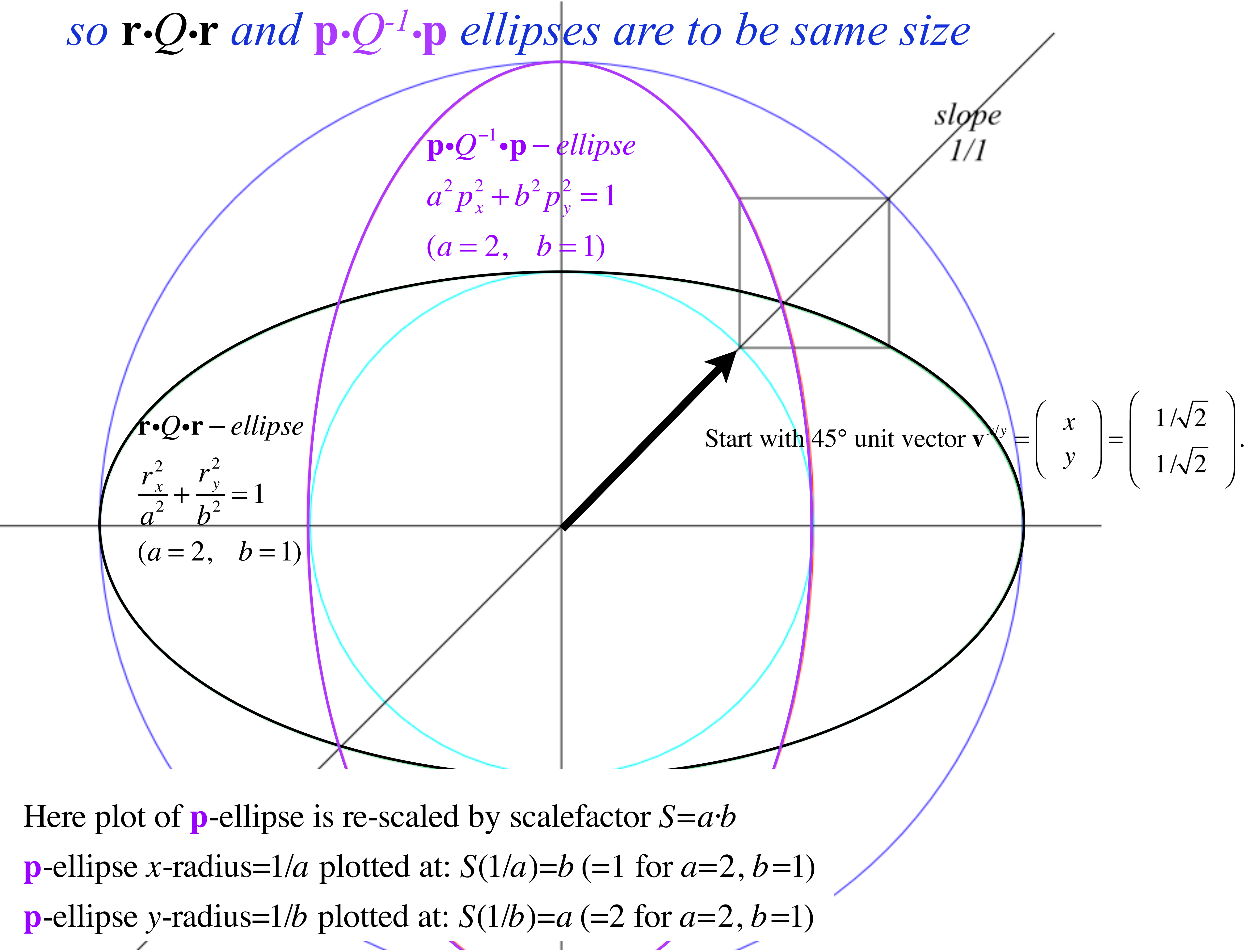
Eigenvalues

Introduction to dual matrix operator geometry (based on IHO orbits)

- *Quadratic form ellipse* $\mathbf{r}\bullet Q\bullet\mathbf{r}=1$ *vs.inverse form ellipse* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$
  - *Duality norm relations (* $\mathbf{r}\bullet\mathbf{p}=1$*)*
  - *Q-Ellipse tangents* $\mathbf{r}'$ *normal to dual* Q^{-1}*-ellipse position* $\mathbf{p}$ *(* $\mathbf{r}'\bullet\mathbf{p}=0=\mathbf{r}\bullet\mathbf{p}'$*)*
- *Operator geometric sequences and eigenvectors*
- → *Alternative scaling of matrix operator geometry*
  - *Vector calculus of tensor operation*

You may rescale **p***-plot by scale factor* $S=(a\cdot b)$ *so* **r**·Q·**r** *and* **p**·Q^{-1}·**p** *ellipses are to be same size*

Here $b/a=1/2$



$\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}-ellipse$

$$a^2p_x^2+b^2p_y^2=1$$

$$(a=2,\quad b=1)$$

$\mathbf{r}\bullet Q\bullet\mathbf{r}-ellipse$

$$\frac{r_x^2}{a^2}+\frac{r_y^2}{b^2}=1$$

$$(a=2,\quad b=1)$$

Start with 45° unit vector $\mathbf{v}^{x/y}=\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$.

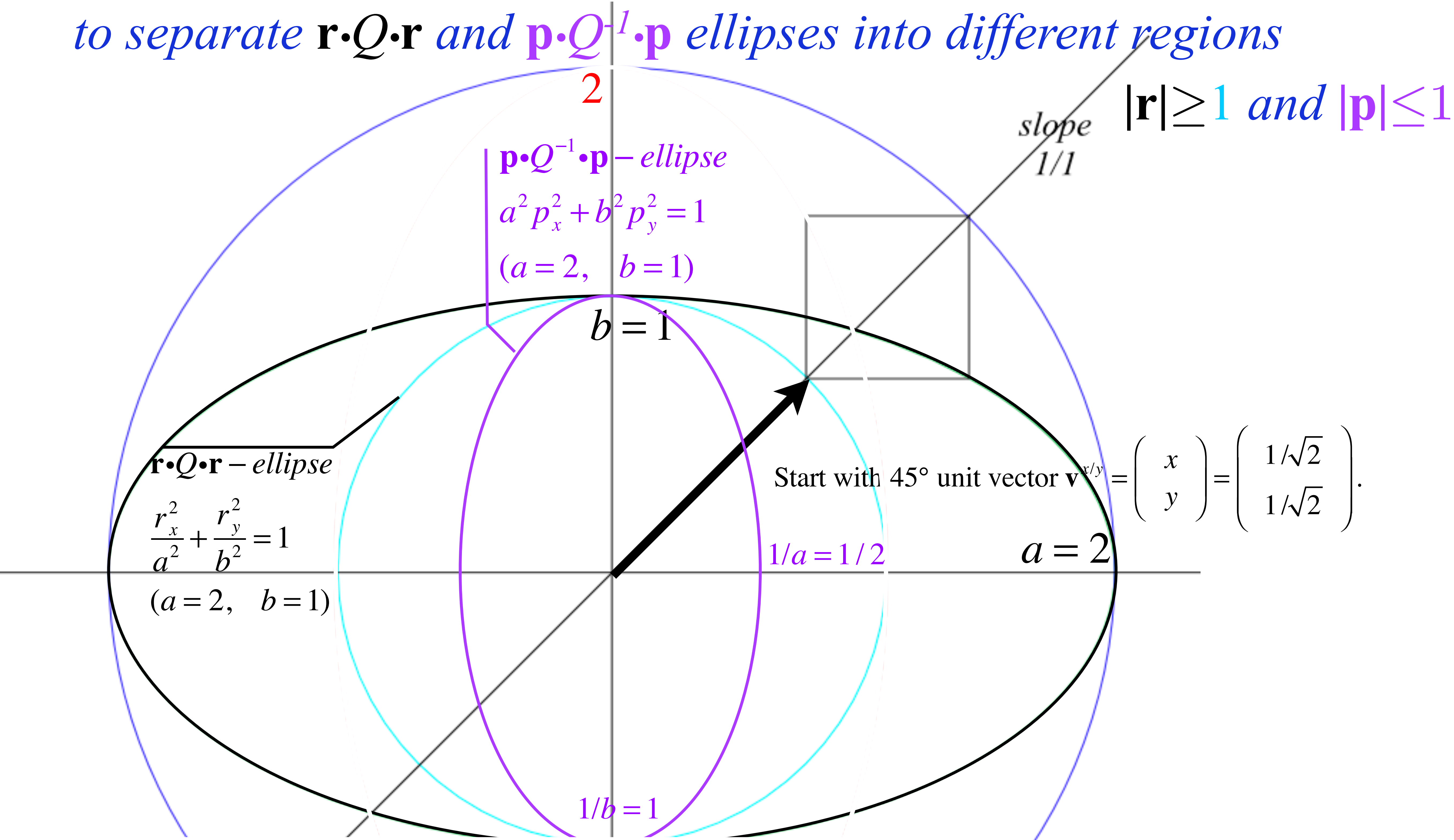
Here plot of **p**-ellipse is re-scaled by scalefactor $S=a\cdot b$

p-ellipse x-radius=$1/a$ plotted at: $S(1/a)=b$ (=1 for a=2, b=1)

p-ellipse y-radius=$1/b$ plotted at: $S(1/b)=a$ (=2 for a=2, b=1)

Here b/a=1/2

*..or else rescale **p**-plot by scale factor $S=b$ to separate **r**·Q·**r** and **p**·Q^{-1}·**p** ellipses into different regions*



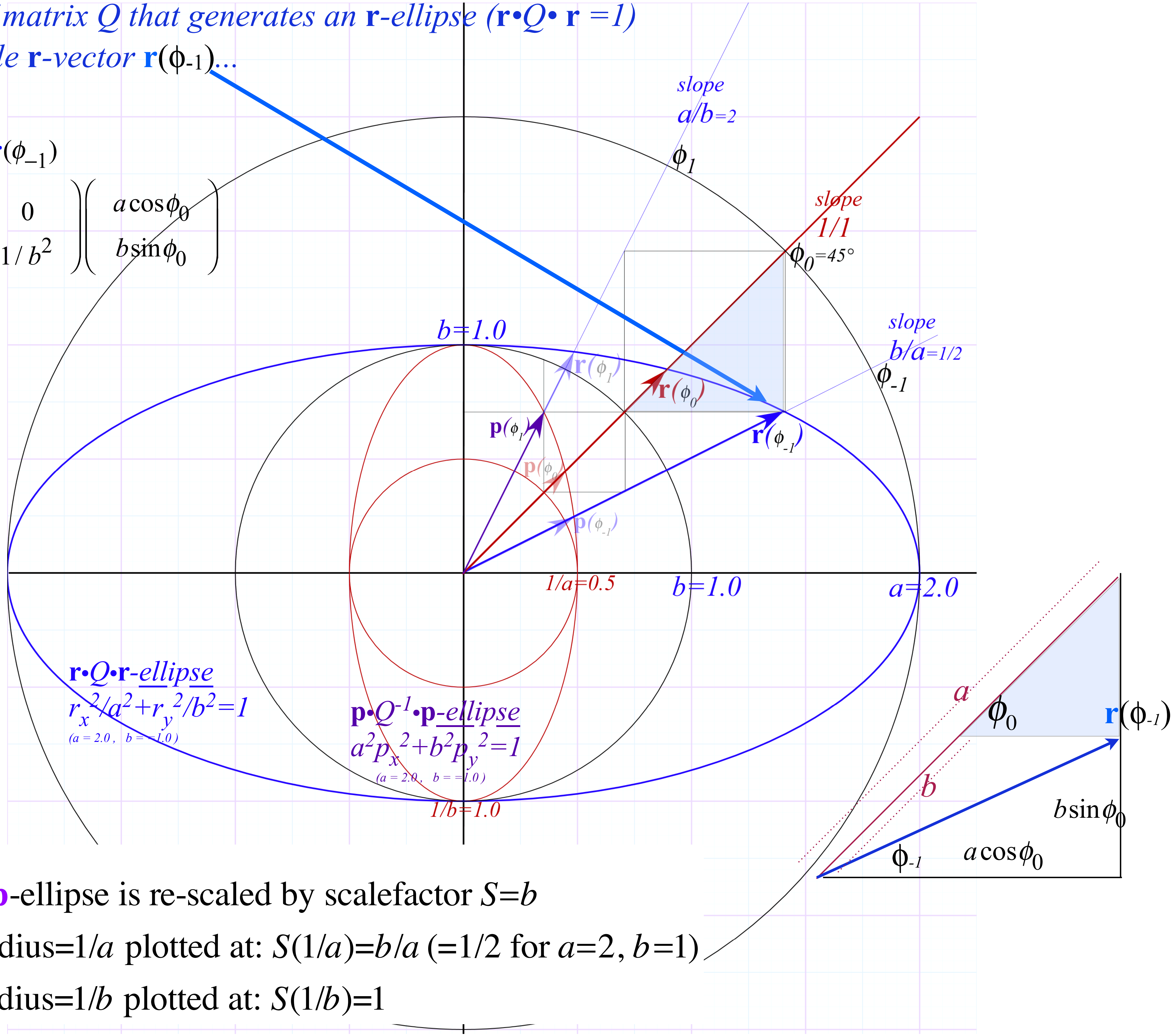
Here plot of **p**-ellipse is re-scaled by scalefactor $S=b$

p-ellipse x-radius=1/a plotted at: $S(1/a)=b/a$ (=1/2 for a=2, b=1)

p-ellipse y-radius=1/b plotted at: $S(1/b)$=1

Action of matrix Q that generates an **r***-ellipse (***r***•Q•* **r** *=1)*
on a single **r***-vector* **r**(ϕ-1)...

$$\mathbf{p}(\phi_1) = \mathbf{Q} \bullet \mathbf{r}(\phi_{-1})$$
$$= \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix} \begin{pmatrix} a\cos\phi_0 \\ b\sin\phi_0 \end{pmatrix}$$



Variation of
Fig. 11.7
in Unit 1

Here plot of **p**-ellipse is re-scaled by scalefactor $S=b$

p-ellipse x-radius=$1/a$ plotted at: $S(1/a)=b/a$ (=1/2 for a=2, b=1)

p-ellipse y-radius=$1/b$ plotted at: $S(1/b)=1$

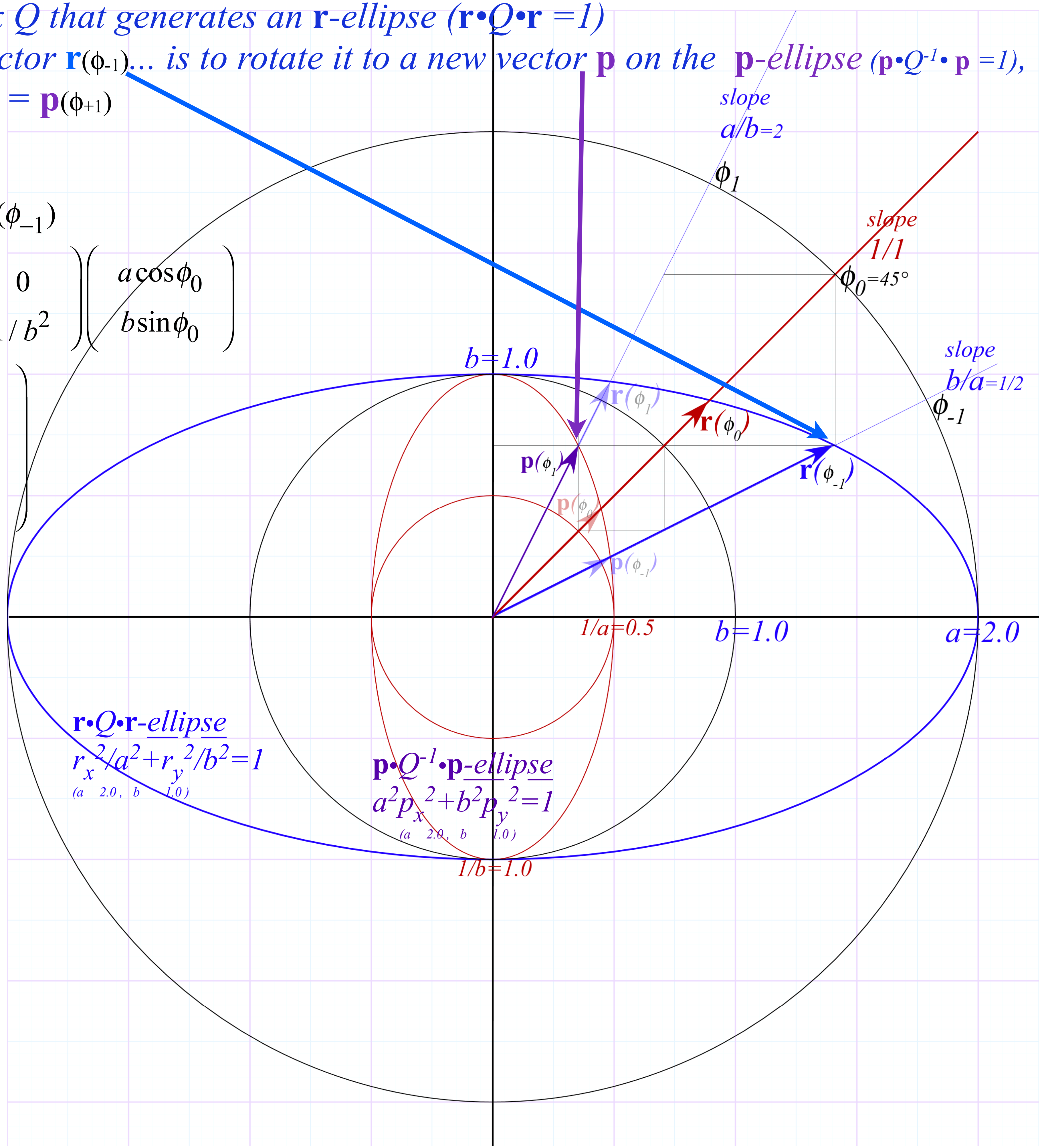
Action of matrix Q that generates an **r**-ellipse (**r**•Q•**r** =1) on a single **r**-vector **r**(ϕ_{-1})... is to rotate it to a new vector **p** on the **p**-ellipse (**p**•Q^{-1}• **p** =1), that is, Q•**r**(ϕ_{-1}) = **p**(ϕ_{+1})

$$\mathbf{p}(\phi_1) = \mathbf{Q} \bullet \mathbf{r}(\phi_{-1})$$

$$= \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix} \begin{pmatrix} a\cos\phi_0 \\ b\sin\phi_0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{a}\cos\phi_0 \\ \frac{1}{b}\sin\phi_0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}\frac{1}{\sqrt{2}} \\ \frac{1}{1}\frac{1}{\sqrt{2}} \end{pmatrix}$$



Variation of
Fig. 11.7
in Unit 1

*Action of matrix Q that generates an **r**-ellipse (**r**•Q•**r** =1) on a single **r**-vector **r**(ϕ-1)... is to rotate it to a new vector **p** on the **p**-ellipse (**p**•Q-1• **p** =1), that is, Q•**r**(ϕ-1) = **p**(ϕ+1)*

$$\mathbf{p}(\phi_1) = \mathbf{Q}\bullet\mathbf{r}(\phi_{-1})$$

$$= \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix}\begin{pmatrix} a\cos\phi_0 \\ b\sin\phi_0 \end{pmatrix}$$

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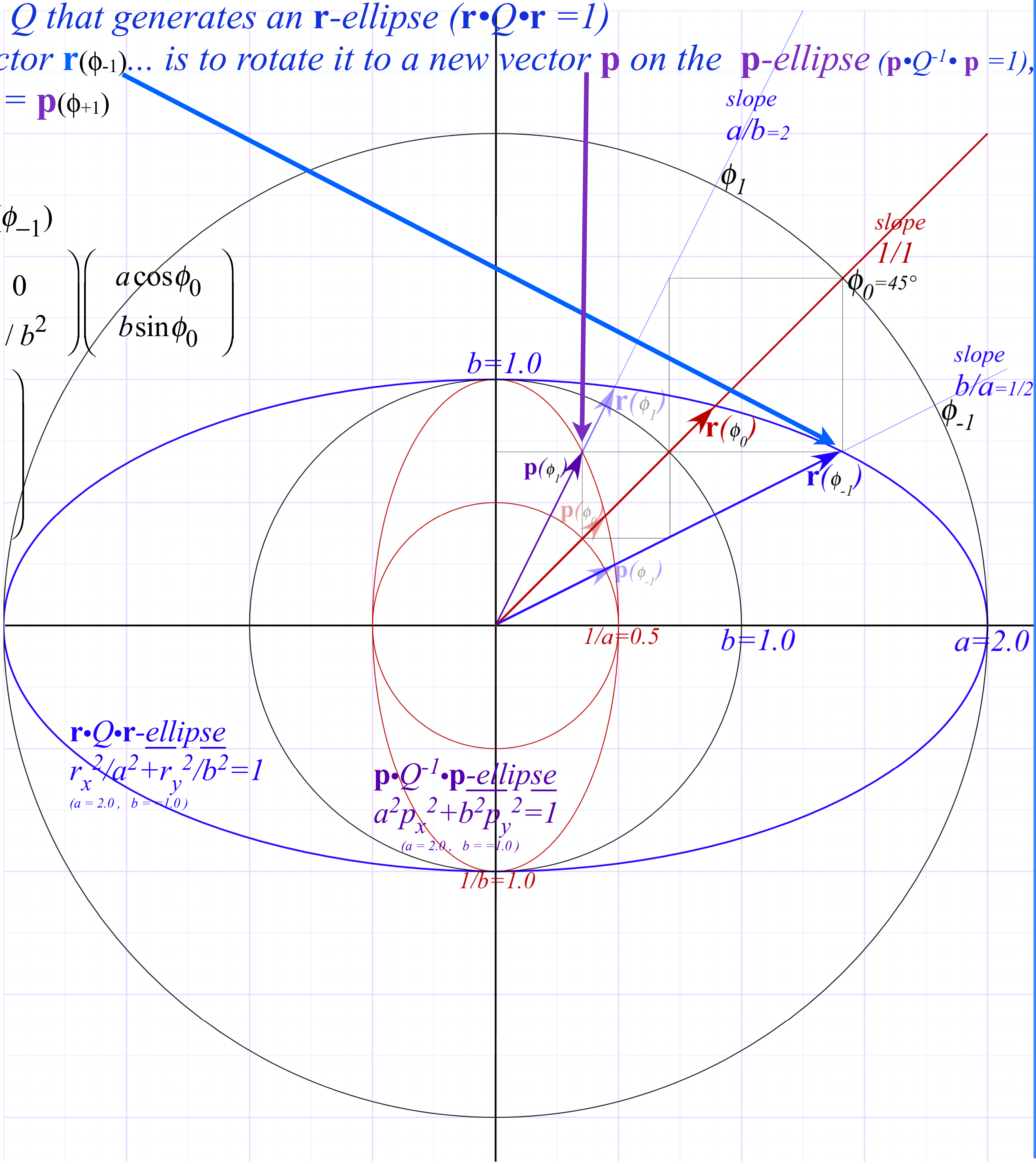
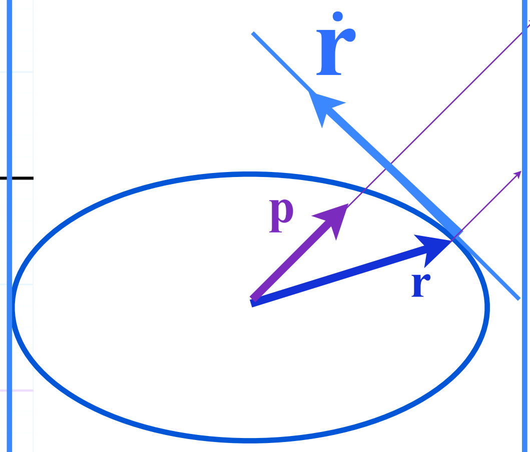


Key points of matrix geometry:

*Matrix Q maps any vector **r** to a new vector **p** normal to the tangent **ṙ** to its **r**•Q•**r**-ellipse.*



Variation of Fig. 11.7 in Unit 1

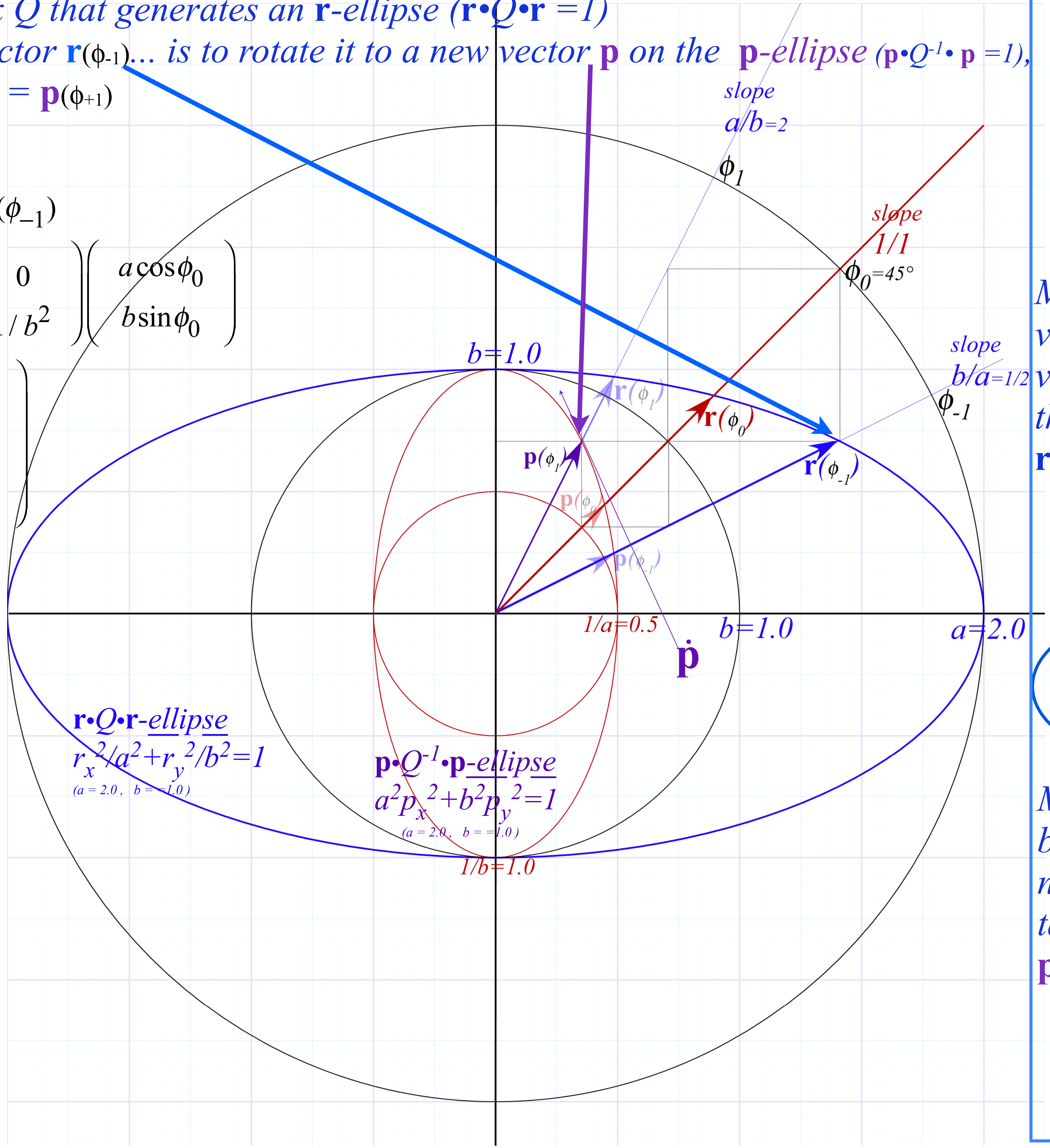
Action of matrix Q that generates an **r***-ellipse (***r***•Q•***r** *=1) on a single* **r***-vector* $\mathbf{r}(\phi_{-1})$*... is to rotate it to a new vector* **p** *on the* **p***-ellipse (***p***•*Q^{-1}*•* **p** *=1), that is,* $Q\bullet\mathbf{r}(\phi_{-1}) = \mathbf{p}(\phi_{+1})$

$$\mathbf{p}(\phi_1)=\mathbf{Q}\bullet\mathbf{r}(\phi_{-1})$$

$$=\begin{pmatrix}1/a^2 & 0\\ 0 & 1/b^2\end{pmatrix}\begin{pmatrix}a\cos\phi_0\\ b\sin\phi_0\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{a}\cos\phi_0\\ \frac{1}{b}\sin\phi_0\end{pmatrix}$$

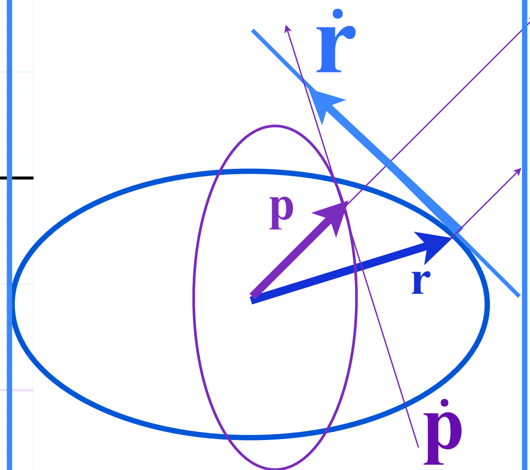
$$=\begin{pmatrix}\frac{1}{2}\frac{1}{\sqrt{2}}\\ \frac{1}{1}\frac{1}{\sqrt{2}}\end{pmatrix}$$



Variation of
Fig. 11.7
in Unit 1

Key points of matrix geometry:

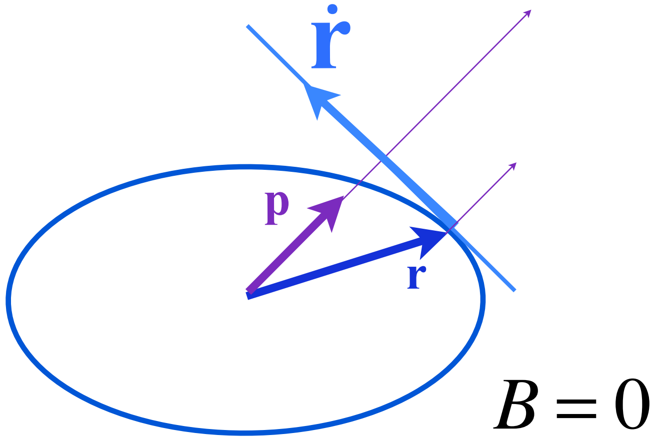
Matrix Q maps any vector **r** *to a new vector* **p** *normal to the tangent* $\dot{\mathbf{r}}$ *to its* **r***•Q•***r***-ellipse.*

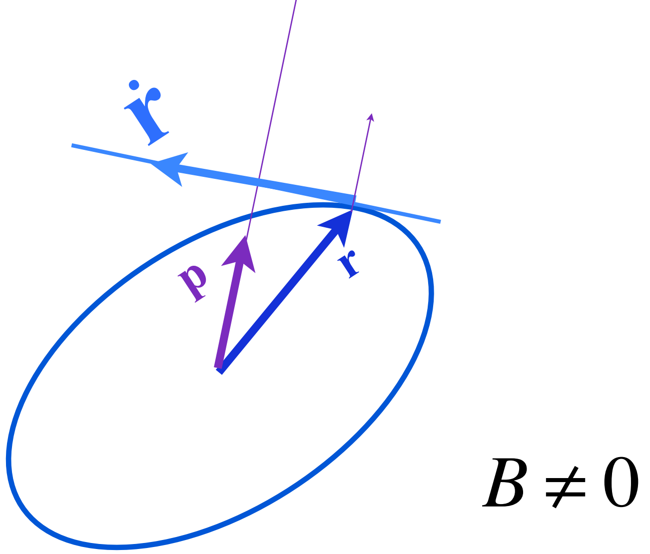


Matrix Q^{-1} *maps* **p** *back to* **r** *that is normal to the tangent* $\dot{\mathbf{p}}$ *to its* **p***•* Q^{-1}*•* **p***-ellipse.*

Introduction to dual matrix operator geometry (based on IHO orbits)

- *Quadratic form ellipse* $\mathbf{r}\bullet Q\bullet\mathbf{r}=1$ *vs.inverse form ellipse* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$
  - *Duality norm relations* $(\mathbf{r}\bullet\mathbf{p}=1)$
  - *Q-Ellipse tangents* $\mathbf{r}'$ *normal to dual* Q^{-1}*-ellipse position* $\mathbf{p}$ $(\mathbf{r}'\bullet\mathbf{p}=0=\mathbf{r}\bullet\mathbf{p}')$
- *Operator geometric sequences and eigenvectors*
- *Alternative scaling of matrix operator geometry*
  - → *Vector calculus of tensor operation*

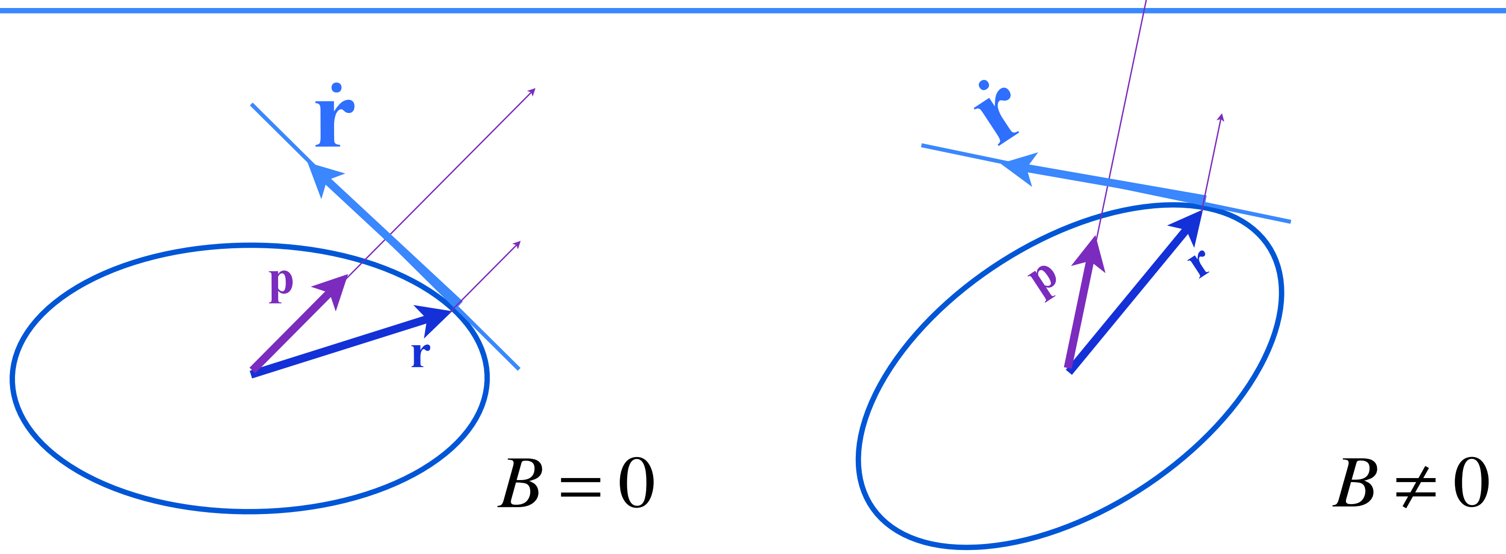




Derive matrix "normal-to-ellipse"geometry by vector calculus:

Let matrix $Q = \begin{pmatrix} A & B \\ B & D \end{pmatrix}$

define the ellipse $1 = \mathbf{r} \bullet Q \bullet \mathbf{r} = \begin{pmatrix} x & y \end{pmatrix} \bullet \begin{pmatrix} A & B \\ B & D \end{pmatrix} \bullet \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \bullet \begin{pmatrix} A \cdot x + B \cdot y \\ B \cdot x + D \cdot y \end{pmatrix} = A \cdot x^2 + 2B \cdot xy + D \cdot y^2 = 1$



Derive matrix “normal-to-ellipse”geometry by vector calculus:

Let matrix $Q = \begin{pmatrix} A & B \\ B & D \end{pmatrix}$

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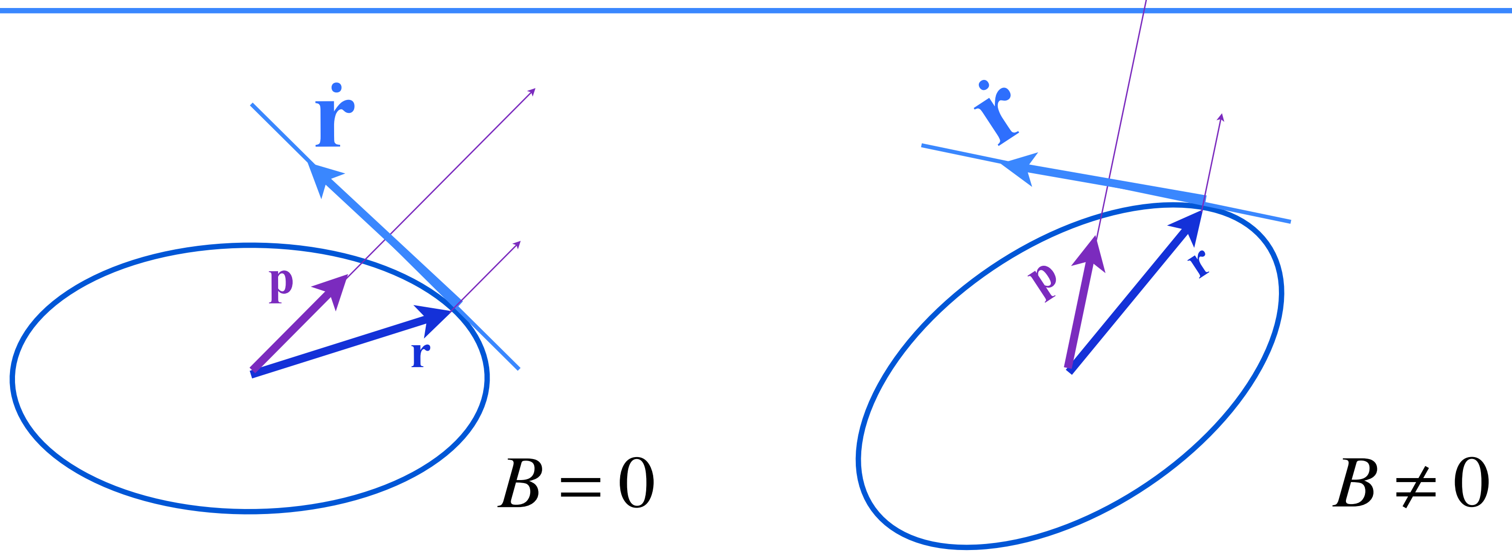
Compare operation by Q *on vector* $\mathbf{r}$ *with* *vector derivative or gradient of* $\mathbf{r}\bullet Q\bullet\mathbf{r}$

$$\frac{\partial}{\partial\mathbf{r}}(\mathbf{r}\bullet Q\bullet\mathbf{r})=\nabla(\mathbf{r}\bullet Q\bullet\mathbf{r})$$

$$\begin{pmatrix} A & B \\ B & D \end{pmatrix}\bullet\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} A\cdot x+B\cdot y \\ B\cdot x+D\cdot y \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}\left(A\cdot x^2+2B\cdot xy+D\cdot y^2\right)=\begin{pmatrix} 2A\cdot x+2B\cdot y \\ 2B\cdot x+2D\cdot y \end{pmatrix}$$



Derive matrix “normal-to-ellipse”geometry by vector calculus:

Let matrix $Q = \begin{pmatrix} A & B \\ B & D \end{pmatrix}$

define the ellipse 1=$\mathbf{r}\bullet Q\bullet\mathbf{r} = \begin{pmatrix} x & y \end{pmatrix}\bullet\begin{pmatrix} A & B \\ B & D \end{pmatrix}\bullet\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix}\bullet\begin{pmatrix} A\cdot x+B\cdot y \\ B\cdot x+D\cdot y \end{pmatrix} = A\cdot x^2+2B\cdot xy+D\cdot y^2=1$

Compare operation by Q on vector **r** *with* *vector derivative or gradient of* $\mathbf{r}\bullet Q\bullet\mathbf{r}$

$$\begin{pmatrix} A & B \\ B & D \end{pmatrix}\bullet\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A\cdot x+B\cdot y \\ B\cdot x+D\cdot y \end{pmatrix}$$

$$\frac{\partial}{\partial\mathbf{r}}(\mathbf{r}\bullet Q\bullet\mathbf{r}) = \nabla(\mathbf{r}\bullet Q\bullet\mathbf{r})$$

$$\begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{pmatrix}\left(A\cdot x^2+2B\cdot xy+D\cdot y^2\right) = \begin{pmatrix} 2A\cdot x+2B\cdot y \\ 2B\cdot x+2D\cdot y \end{pmatrix}$$

Very simple result:

$$\frac{\partial}{\partial\mathbf{r}}\left(\frac{\mathbf{r}\bullet Q\bullet\mathbf{r}}{2}\right) = \nabla\left(\frac{\mathbf{r}\bullet Q\bullet\mathbf{r}}{2}\right) = Q\bullet\mathbf{r}$$

- *Introduction to dual matrix operator geometry (based on IHO orbits)*
  - *Quadratic form ellipse* $\mathbf{r}\bullet Q\bullet\mathbf{r}=1$ *vs.inverse form ellipse* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$
    - *Duality norm relations* ($\mathbf{r}\bullet\mathbf{p}=1$)
    - *Q-Ellipse tangents* $\mathbf{r}'$ *normal to dual* Q^{-1}*-ellipse position* $\mathbf{p}$ ($\mathbf{r}'\bullet\mathbf{p}=0=\mathbf{r}\bullet\mathbf{p}'$)
  - *(Still more) Operator geometric sequences and eigenvectors*
  - *Alternative scaling of matrix operator geometry*
    - *Vector calculus of tensor operation*

Action of "sqrt-"matrix $R=\sqrt{Q}$ *(R generates another ellipse* $\mathbf{r}\bullet R\bullet\mathbf{r}=1$ *not shown)*

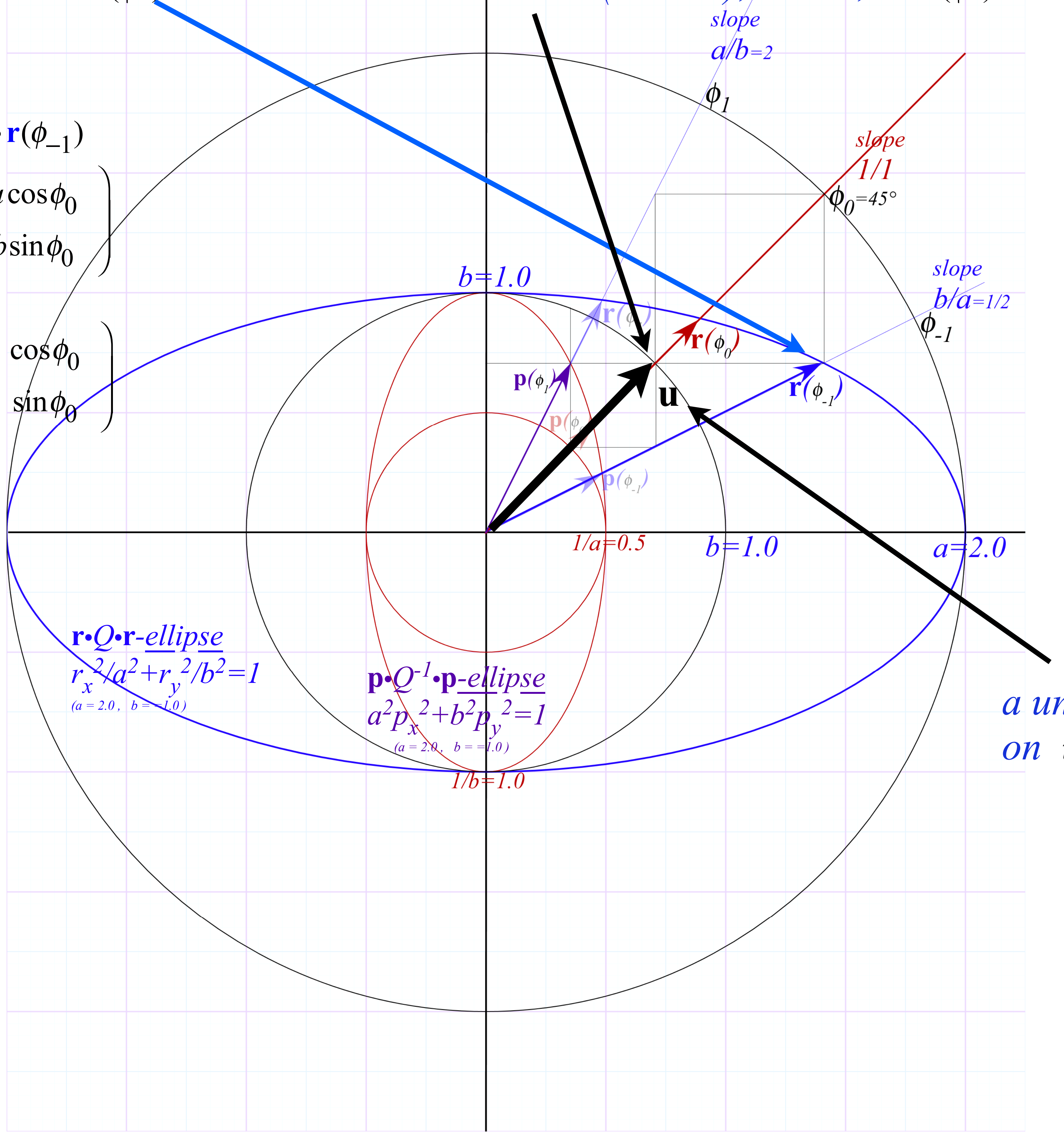
on a single **r***-vector* $\mathbf{r}(\phi_{-1})$*... is to rotate it to* **u***-circle (*$\mathbf{u}\bullet\mathbf{u}=1$*), that is,* $R\bullet\mathbf{r}(\phi_{-1})=\mathbf{u}=$ *(const.)*$\mathbf{r}(\phi_0)$

$$\mathbf{u}=\sqrt{\mathbf{Q}}\bullet\mathbf{r}(\phi_{-1})=\mathbf{R}\bullet\mathbf{r}(\phi_{-1})$$

$$=\begin{pmatrix}1/a & 0\\ 0 & 1/b\end{pmatrix}\begin{pmatrix}a\cos\phi_0\\ b\sin\phi_0\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{a}a\cos\phi_0\\ \frac{1}{b}b\sin\phi_0\end{pmatrix}=\begin{pmatrix}\cos\phi_0\\ \sin\phi_0\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$$

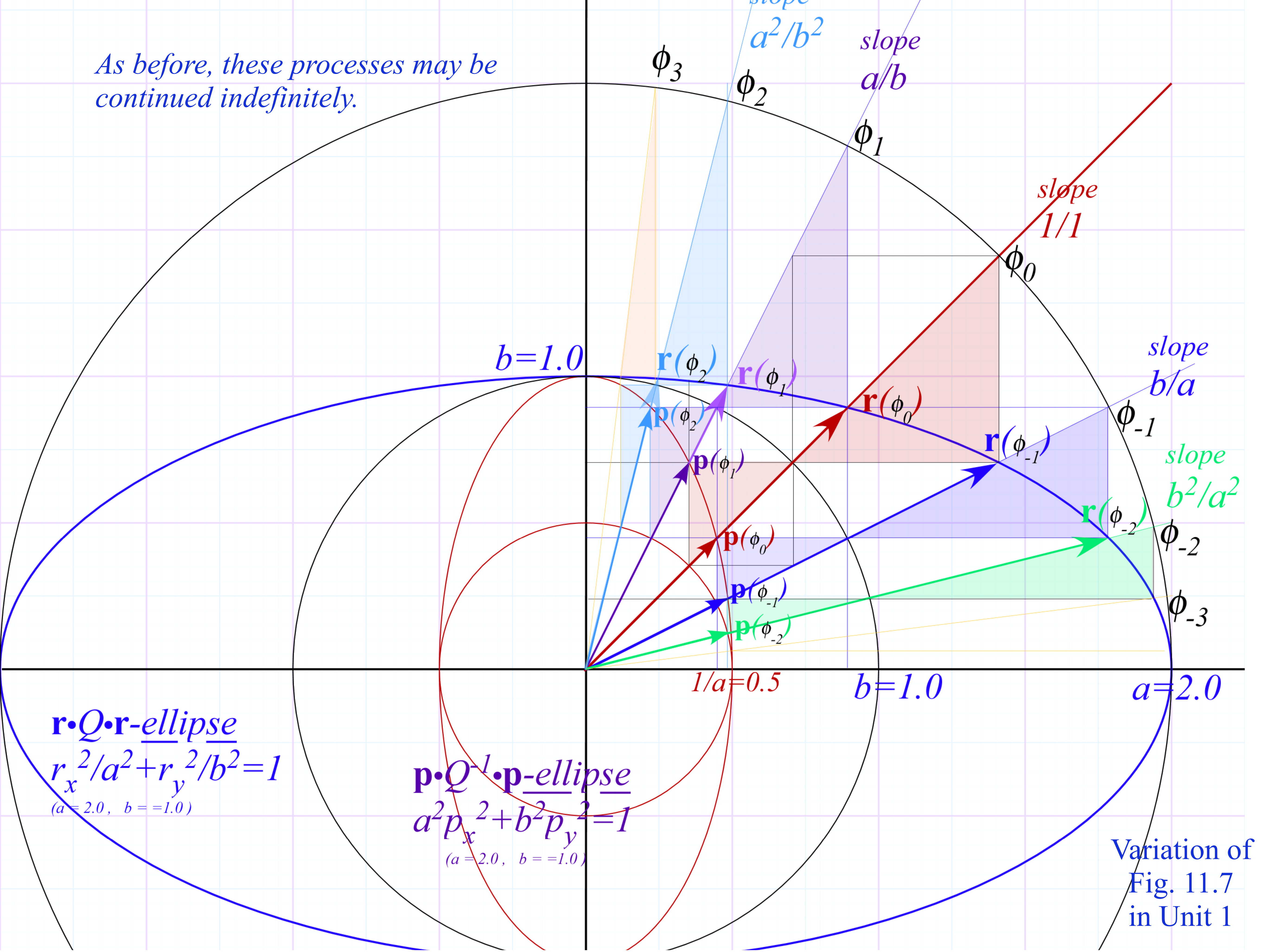


a unit vector

on unit-circle

Variation of
Fig. 11.7
in Unit 1

As before, these processes may be continued indefinitely.



$\mathbf{r}\bullet Q\bullet\mathbf{r}$-ellipse

$r_x{}^2/a^2+r_y{}^2/b^2=1$

$(a = 2.0 , \quad b = =1.0)$

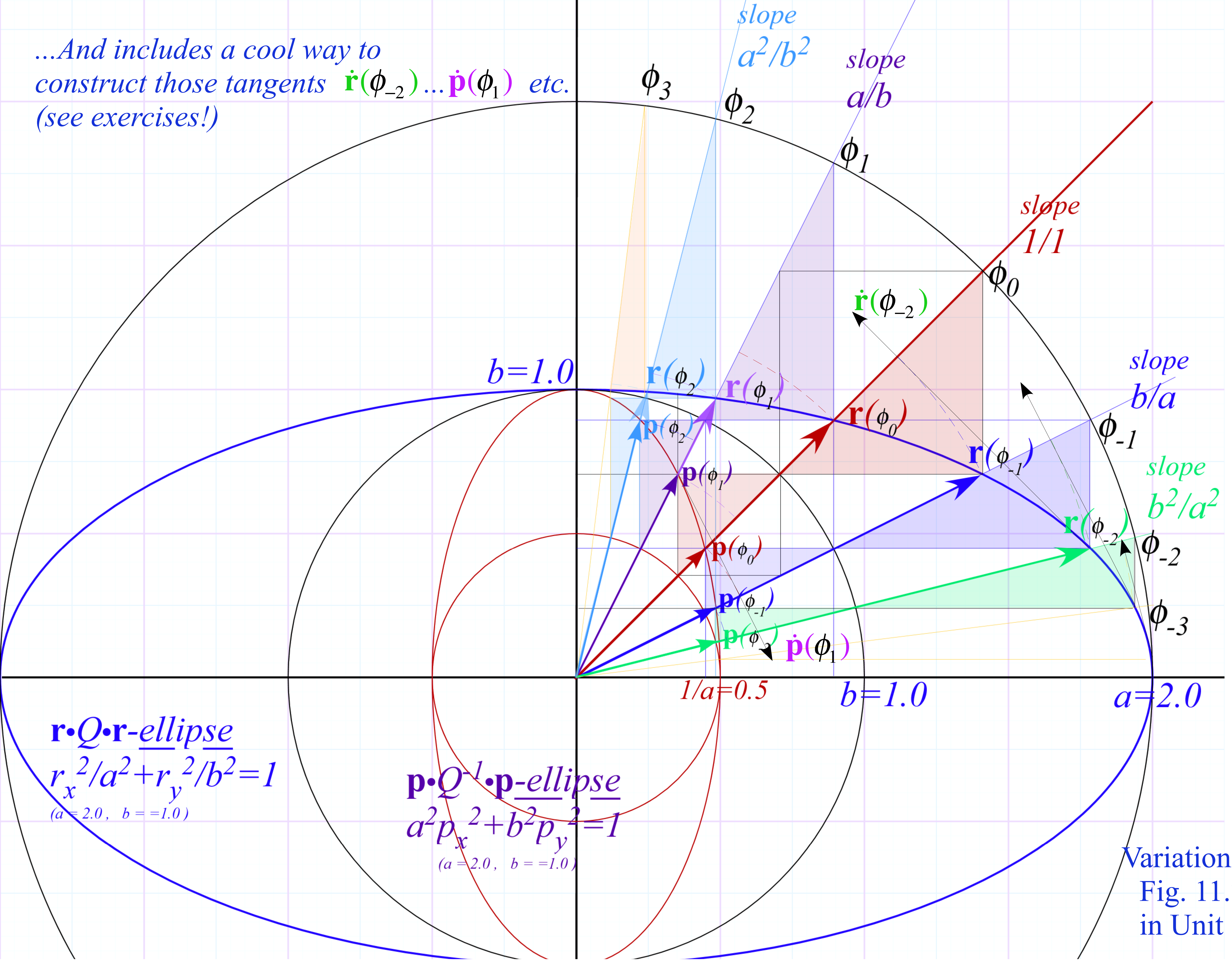
$\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}$-ellipse

$a^2p_x{}^2+b^2p_y{}^2=1$

$(a = 2.0 , \quad b = =1.0)$

Variation of Fig. 11.7 in Unit 1

...And includes a cool way to construct those tangents $\dot{\mathbf{r}}(\phi_{-2})\ldots\dot{\mathbf{p}}(\phi_1)$ *etc. (see exercises!)*



Variation of Fig. 11.7 in Unit 1

Q:Where is this headed? *A: Lagrangian-Hamiltonian duality*

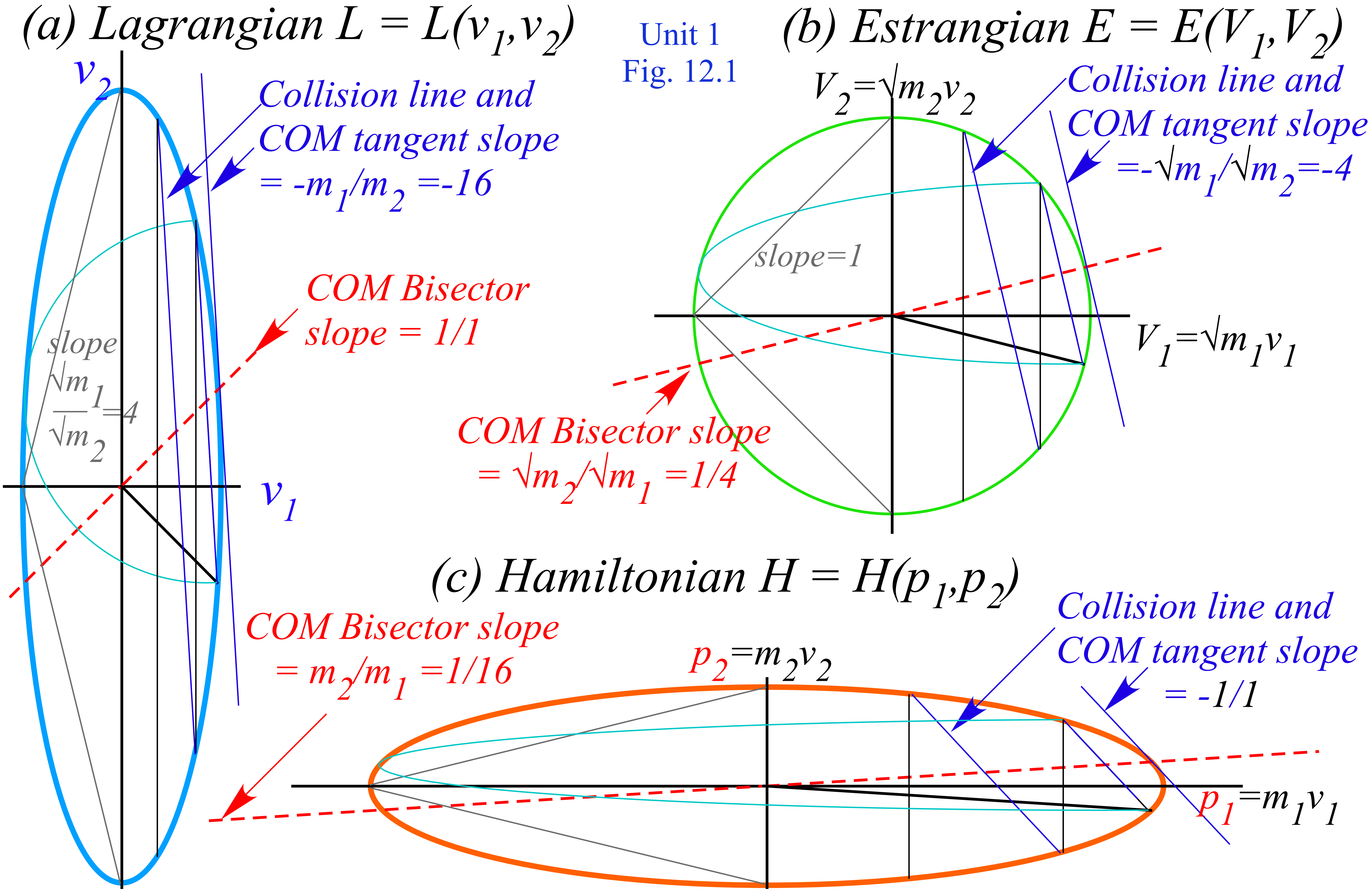
Preview of Lecture 9

The R and Q matrix transformations are like the mechanics rescaling matrices $\sqrt{\mathbf{M}}$ *and* $\mathbf{M}$:

Like $Q=R^2$: $\mathbf{M}=\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}=\mathbf{R}^2$

Like $\sqrt{Q}=R$: $\sqrt{\mathbf{M}}=\begin{pmatrix} \sqrt{m_1} & 0 \\ 0 & \sqrt{m_2} \end{pmatrix}=\mathbf{R}$

Like $Q^{-1}=R^{-2}$: $\mathbf{M}^{-1}=\begin{pmatrix} 1/m_1 & 0 \\ 0 & 1/m_2 \end{pmatrix}=\mathbf{R}^{-2}$



(a) Lagrangian $L = L(v_1,v_2)$

v_2

v_1

Collision line and COM tangent slope $= -m_1/m_2 = -16$

COM Bisector slope = 1/1

slope $\frac{\sqrt{m_1}}{\sqrt{m_2}}=4$

Unit 1 Fig. 12.1

(b) Estrangian $E = E(V_1,V_2)$

$V_2=\sqrt{m_2}v_2$

Collision line and COM tangent slope $=-\sqrt{m_1}/\sqrt{m_2}=-4$

slope=1

$V_1=\sqrt{m_1}v_1$

COM Bisector slope $=\sqrt{m_2}/\sqrt{m_1}=1/4$

(c) Hamiltonian $H = H(p_1,p_2)$

COM Bisector slope $= m_2/m_1 = 1/16$

$p_2=m_2v_2$

Collision line and COM tangent slope = -1/1

$p_1=m_1v_1$

Unit 1
Fig. 12.2