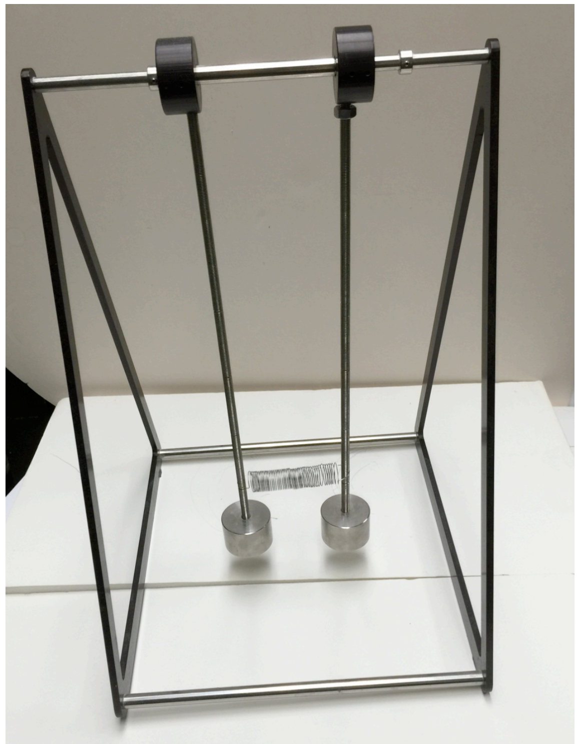
Lecture 16
Thur. 3.10.2016

Introduction to coupled oscillation and eigenmodes

(Ch. 3-4 of Unit 2)

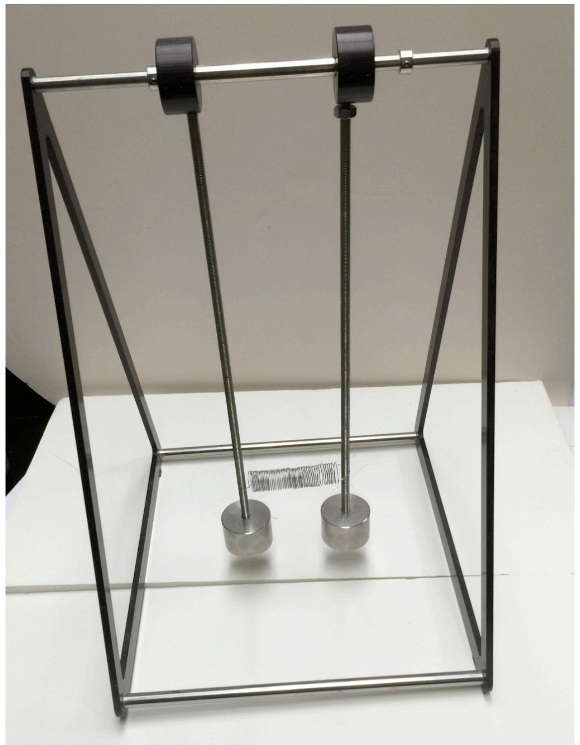


Review of 1D FDHO (Forced-Damped-Harmonic Oscillator) response

2D harmonic oscillator (2D-HO) equations of motion
Lagrangian and matrix forms

2D harmonic oscillator equation eigensolutions (normal modes)
Eigensolutions by geometry for 2D-HO with bilateral (B-Type) symmetry
Symmetric (low frequency) mode versus antisymmetric (high frequency) mode
Mixed mode beat dynamics (with constant $\pi/2$ phase-lag)

Eigensolutions by matrix-algebra with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$
Secular equation
Hamilton-Cayley equation and projectors
Idempotent projectors (how eigenvalues⇒eigenvectors)
Operator orthonormality and Completeness (Idempotent means: $\mathbf{P}\cdot\mathbf{P}=\mathbf{P}$*)*
Spectral Decompositions
Functional spectral decomposition
Orthonormality vs. Completeness vis-a`-vis Operator vs. State
Lagrange functional interpolation formula
Diagonalizing Transformations (D-Ttran) from projectors



➔ *Review of 1D FDHO (Forced-Damped-Harmonic Oscillator) response*

2D harmonic oscillator (2D-HO) equations of motion
Lagrangian and matrix forms

2D harmonic oscillator equation eigensolutions (normal modes)
Eigensolutions by geometry for 2D-HO with bilateral (B-Type) symmetry
Symmetric (low frequency) mode versus antisymmetric (high frequency) mode
Mixed mode beat dynamics (with constant π/2 phase-lag)

Eigensolutions by matrix-algebra with example M=
Secular equation
Hamilton-Cayley equation and projectors
Idempotent projectors (how eigenvalues⇒eigenvectors)
Operator orthonormality and Completeness (Idempotent means: **P·P=P***)*
Spectral Decompositions
Functional spectral decomposition
Orthonormality vs. Completeness vis-a`-vis Operator vs. State
Lagrange functional interpolation formula
Diagonalizing Transformations (D-Ttran) from projectors

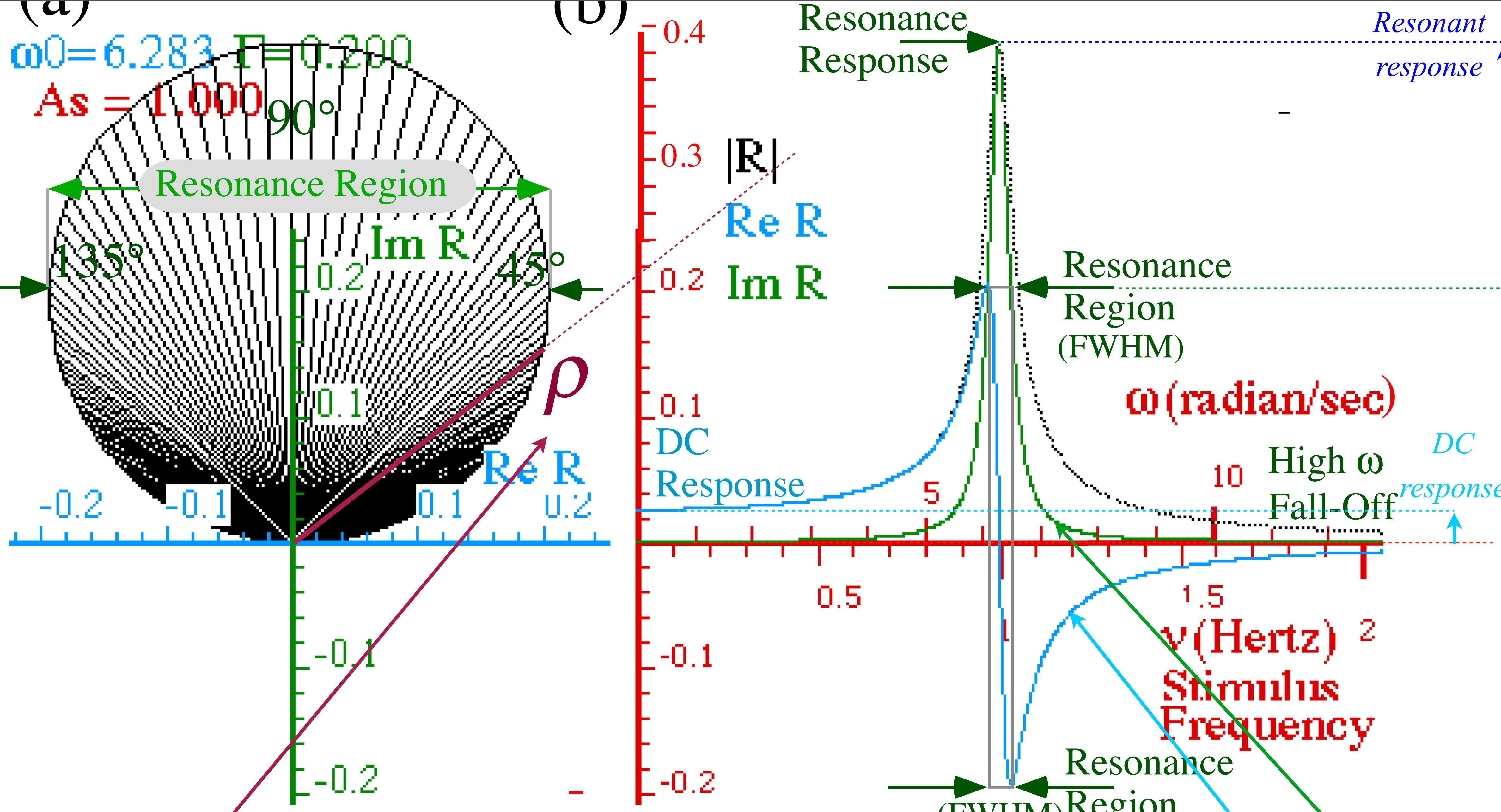


Fig. 2.2.6 Anatomy of oscillator Green-Lorentz response function plots

Phase lag angle

$$\rho = \tan^{-1}\left(\frac{2\Gamma\omega_s}{\omega_0^2 - \omega_s^2}\right)$$

$$\mathrm{Re}\, G_{\omega_0}(\omega_s) = \frac{\omega_0^2 - \omega_s^2}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

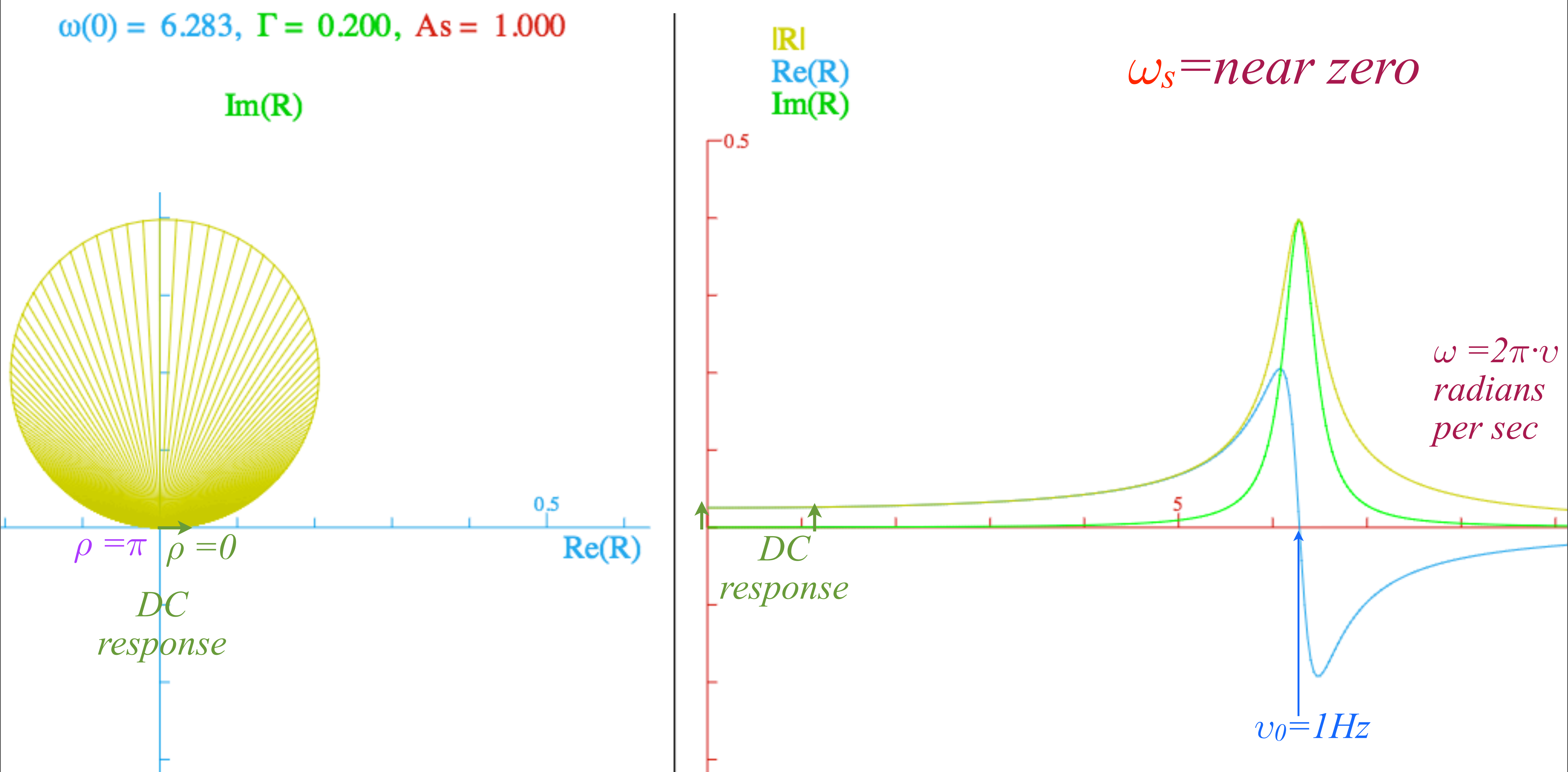
Real part

$$\mathrm{Im}\, G_{\omega_0}(\omega_s) = \frac{2\Gamma\omega_s}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

Imaginary part

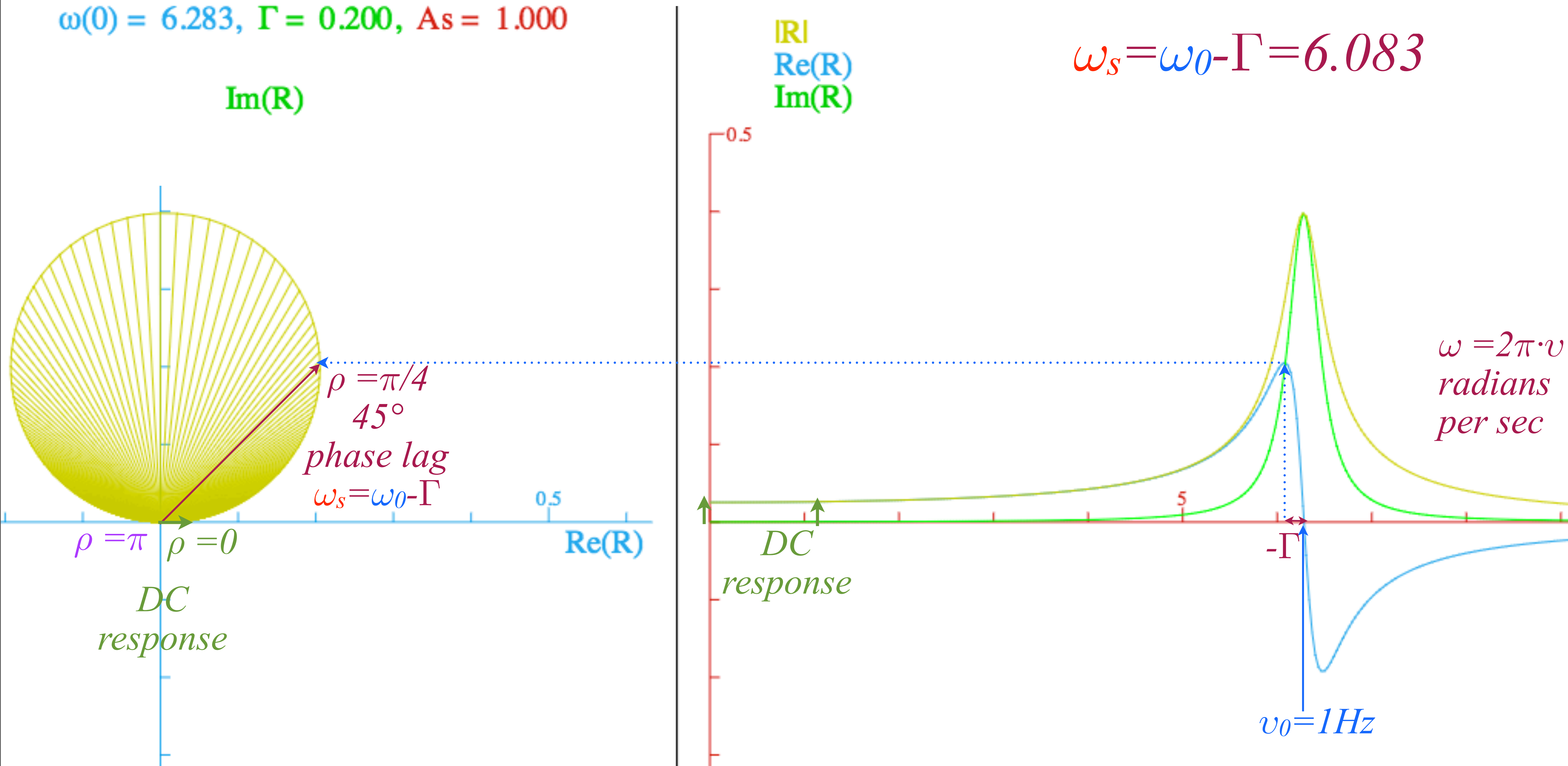
$$AAF = \frac{\text{Resonant response}}{\text{DC response}} = \frac{\left|G_{\omega_0}(\omega_s = \omega_0)\right|}{\left|G_{\omega_0}(0)\right|} = \frac{1/(2\Gamma\omega_0)}{1/\omega_0^2} = \frac{\omega_0}{2\Gamma} \equiv q \quad (\textit{angular quality factor})$$

OscillIt Web Simulation - *Lorentz Response (Γ=0.2)*



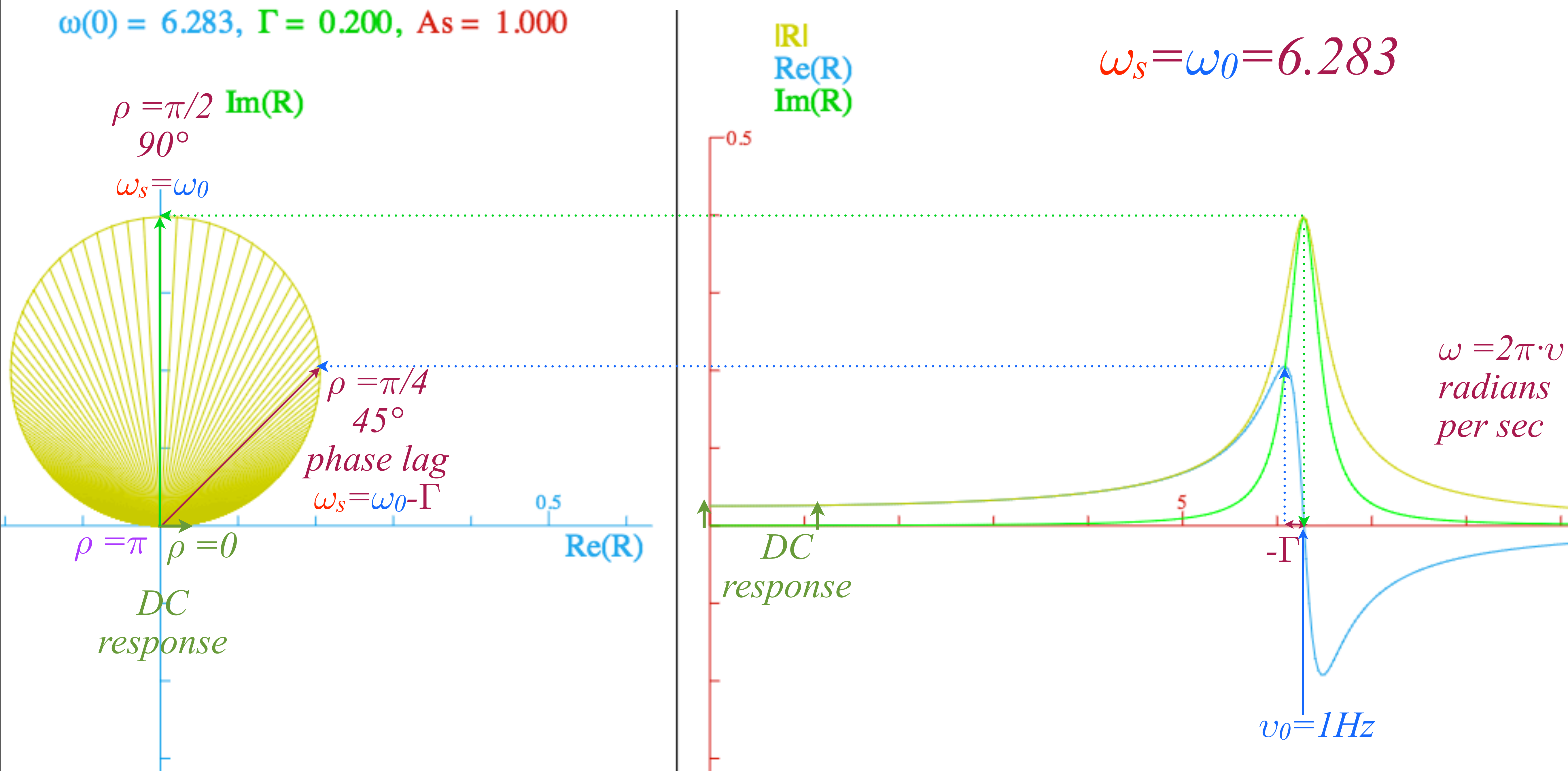
[OscillIt Web Simulation](#)

[*Lorentz Response (Γ=0.2)*](#)

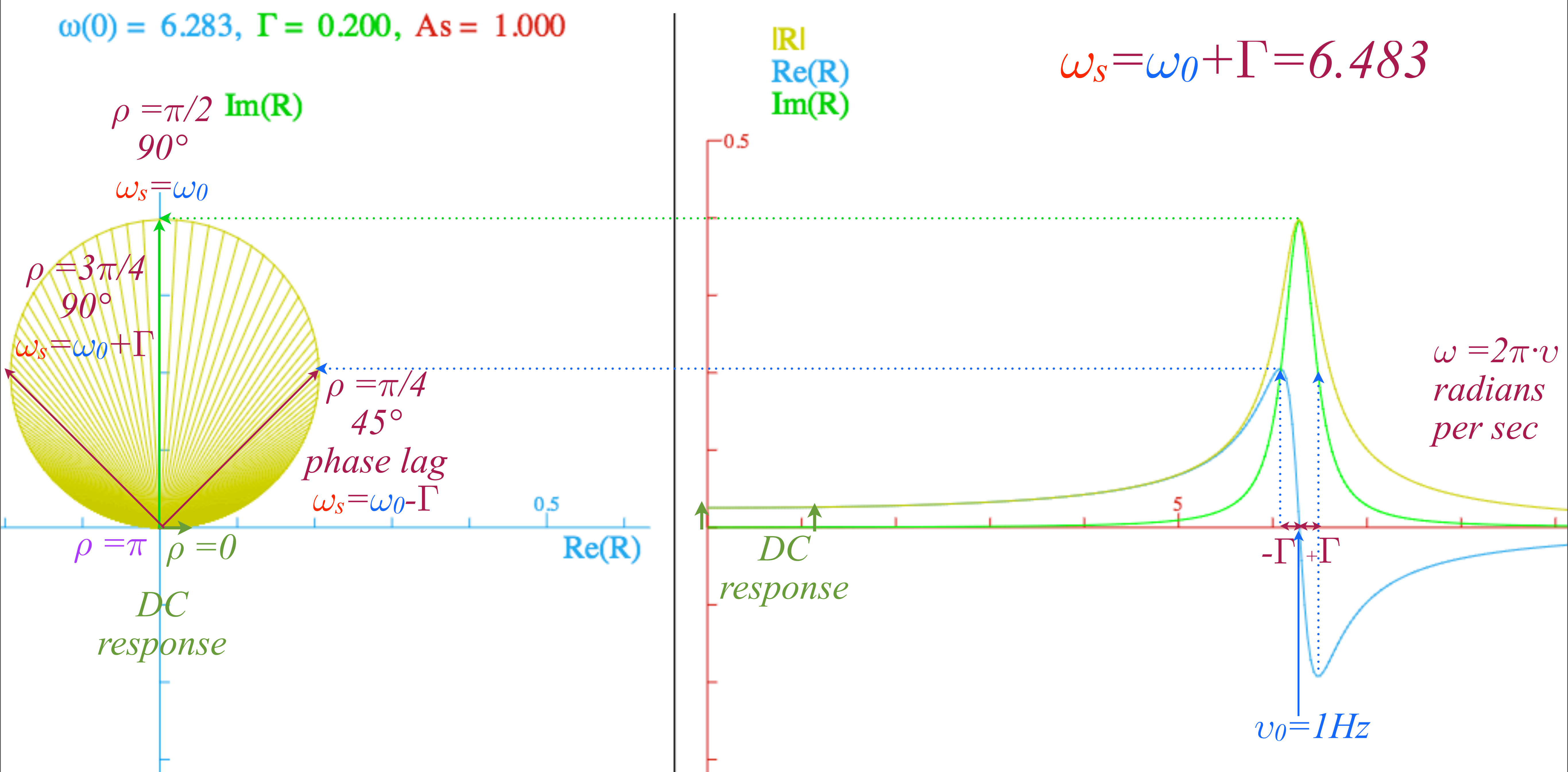


OscillIt Web Simulation

Lorentz Response (Γ=0.2)

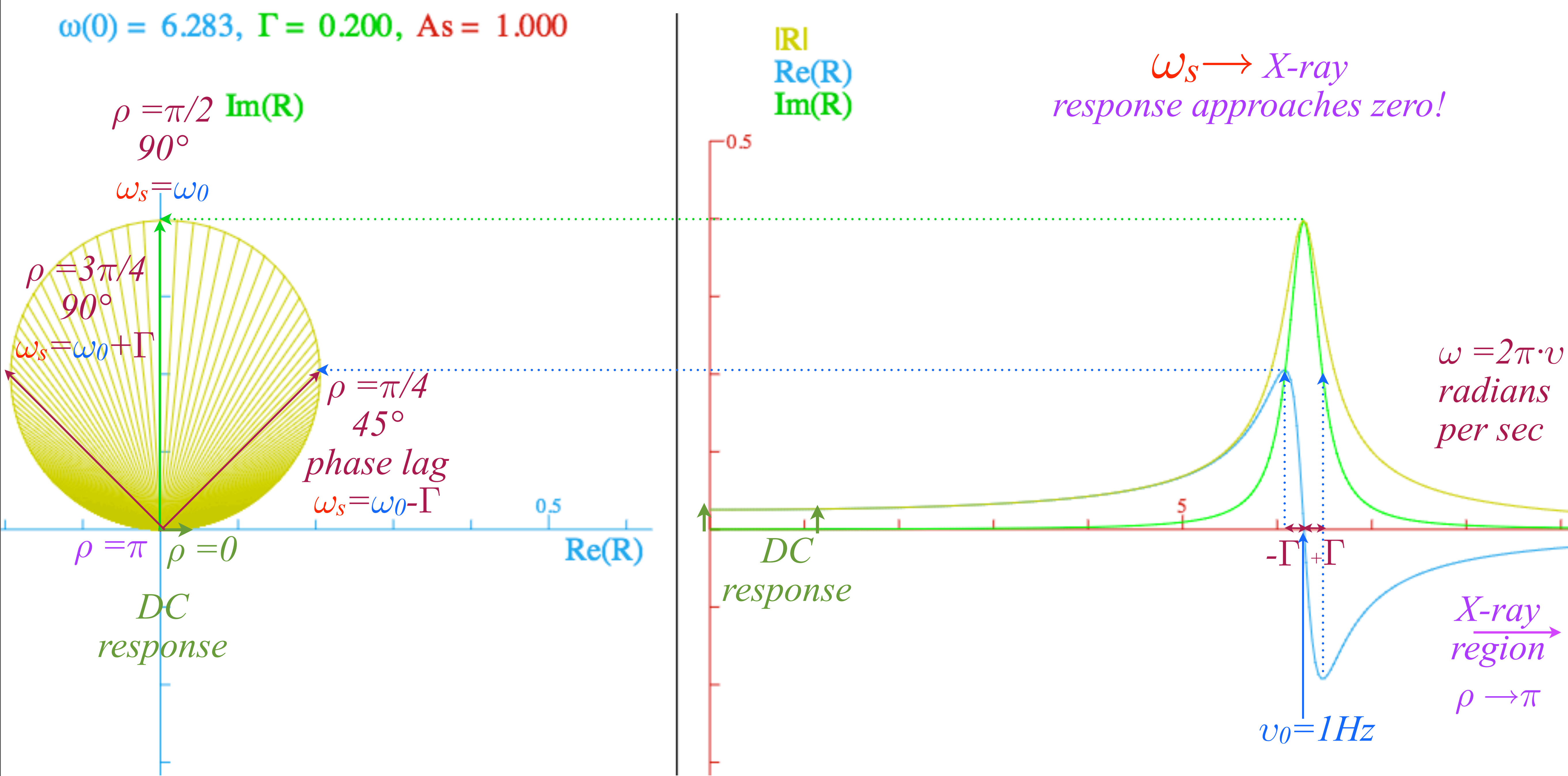


[OscillIt Web Simulation](#)
[*Lorentz Response (Γ=0.2)*](#)



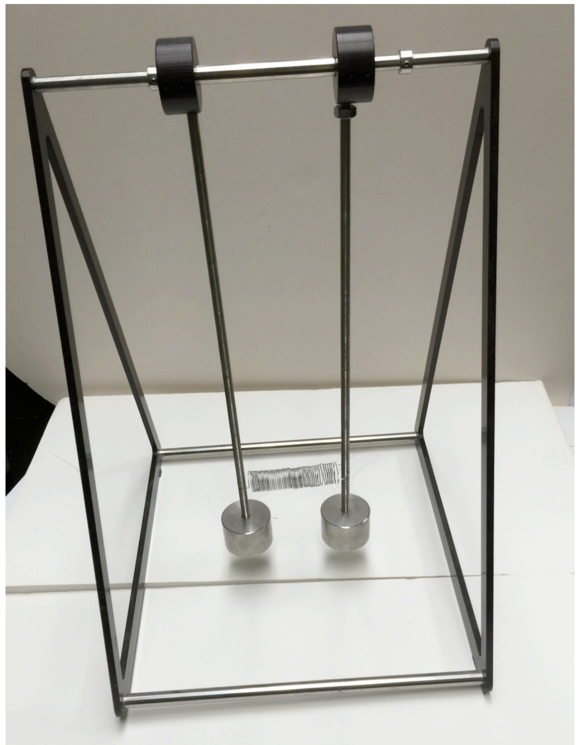
[OscillIt Web Simulation](#)

[*Lorentz Response (Γ=0.2)*](#)



OscillIt Web Simulation

Lorentz Response (Γ=0.2)



Review of 1D FDHO (Forced-Damped-Harmonic Oscillator) response

➔ 2D harmonic oscillator (2D-HO) equations of motion
Lagrangian and matrix forms

2D harmonic oscillator equation eigensolutions (normal modes)
Eigensolutions by geometry for 2D-HO with bilateral (B-Type) symmetry
Symmetric (low frequency) mode versus antisymmetric (high frequency) mode
Mixed mode beat dynamics (with constant $\pi/2$ phase-lag)

Eigensolutions by matrix-algebra with example M=
Secular equation
Hamilton-Cayley equation and projectors
Idempotent projectors (how eigenvalues⇒eigenvectors)
Operator orthonormality and Completeness (Idempotent means: **P·P=P**)
Spectral Decompositions
Functional spectral decomposition
Orthonormality vs. Completeness vis-a`-vis Operator vs. State
Lagrange functional interpolation formula
Diagonalizing Transformations (D-Ttran) from projectors

2D harmonic oscillators

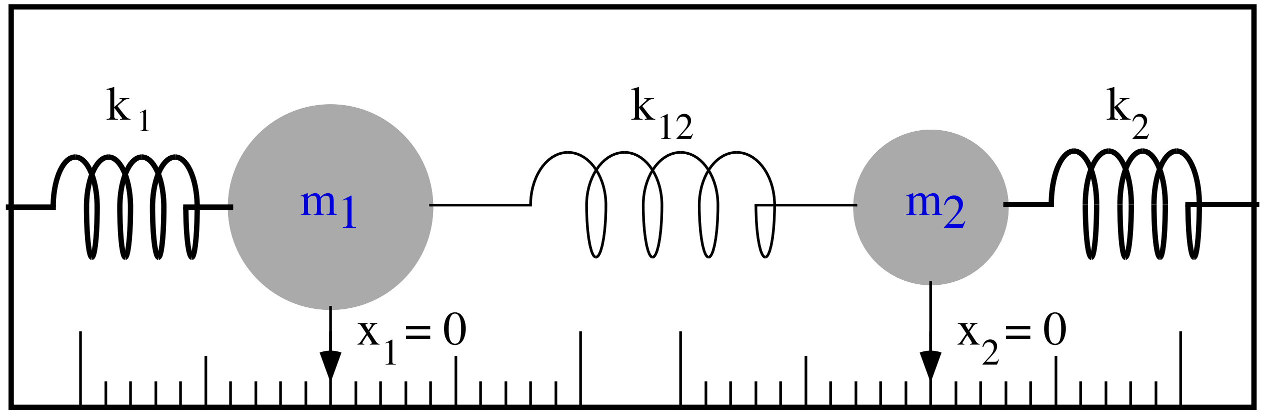
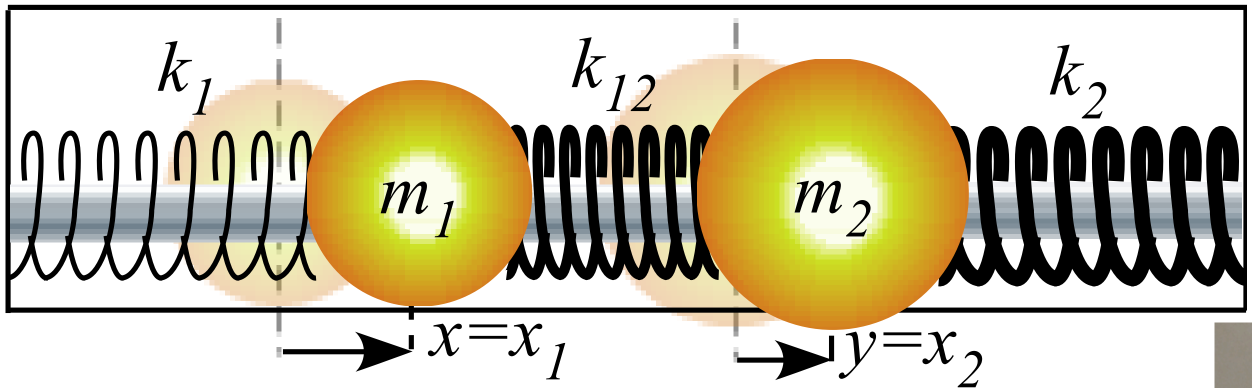


Fig. 3.3.1 Two 1-dimensional coupled oscillators



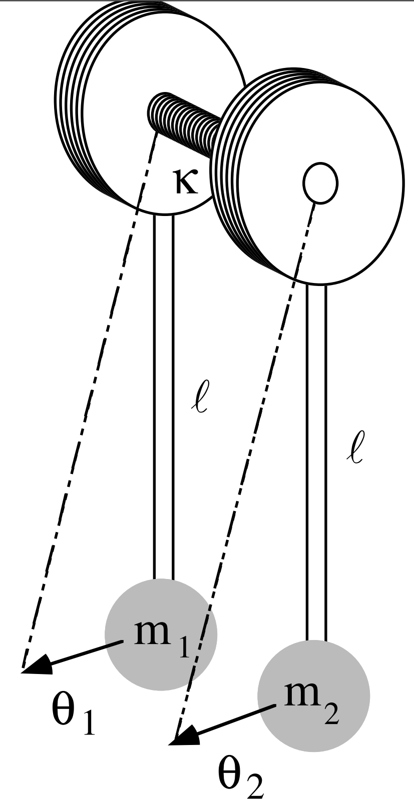


Fig. 3.3.2 Coupled pendulums

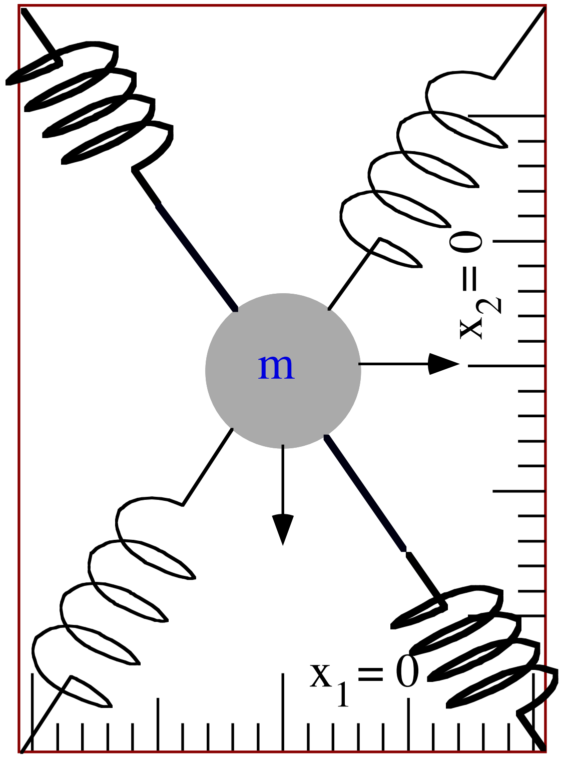
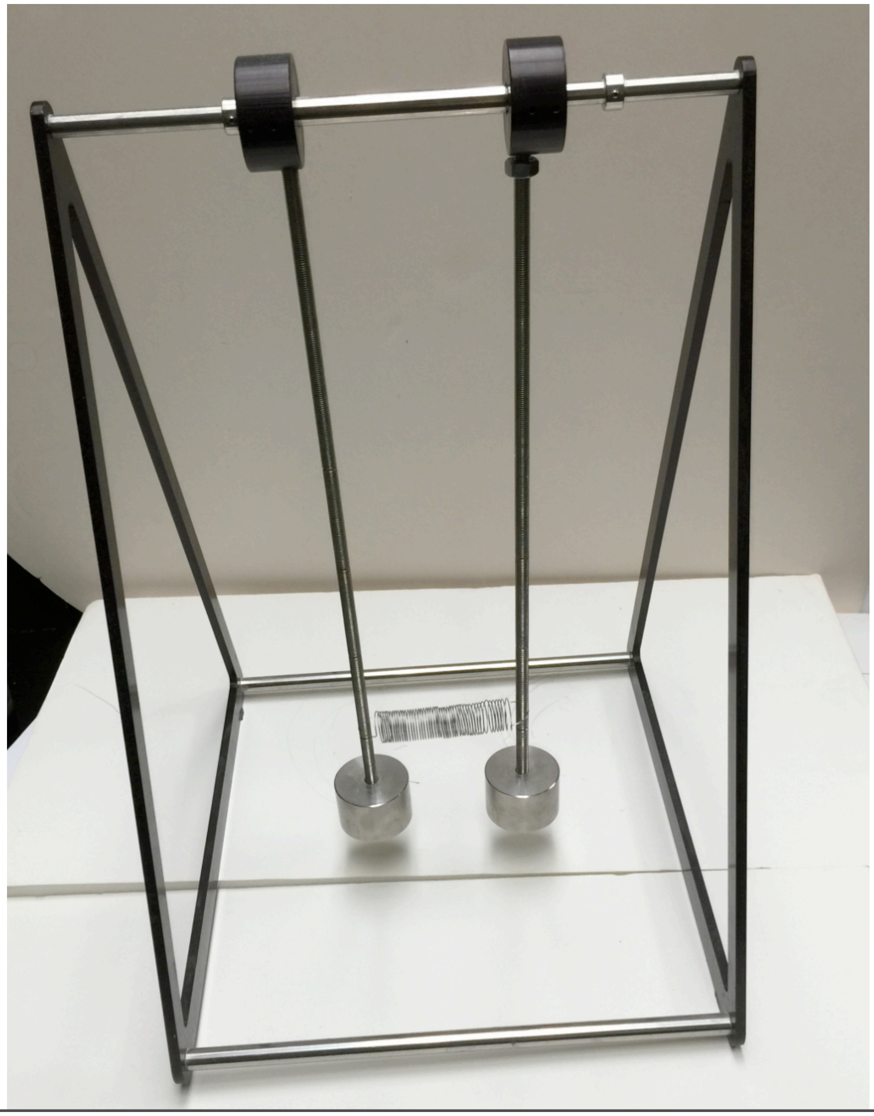


Fig. 3.3.3 One 2-dimensional coupled oscillator



2D harmonic oscillator energy

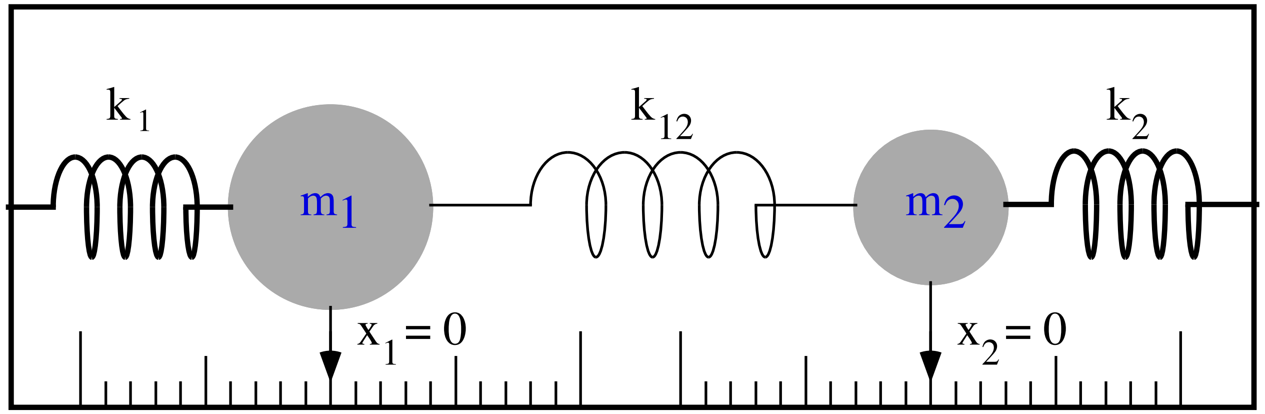
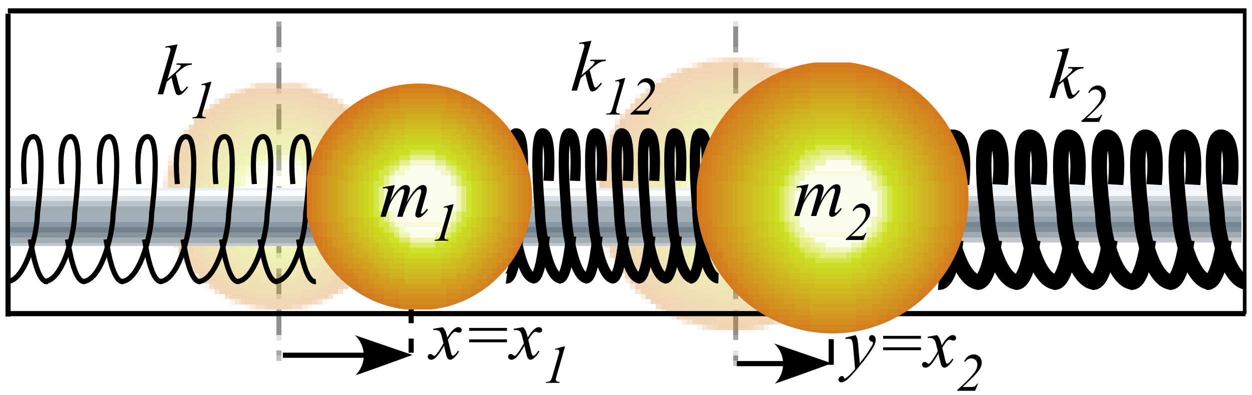


Fig. 3.3.1 Two 1-dimensional coupled oscillators



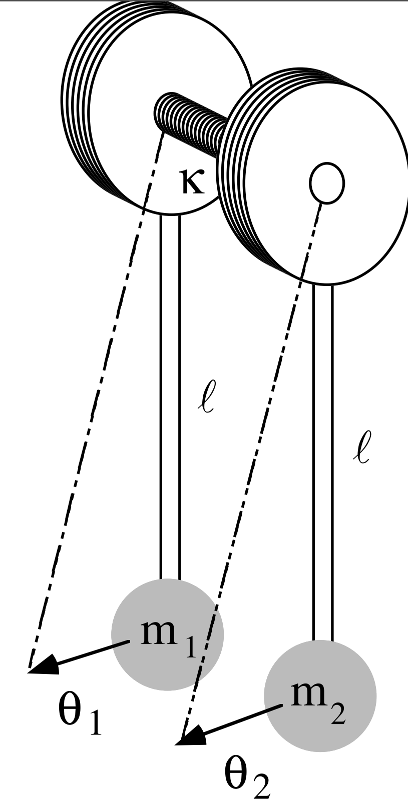


Fig. 3.3.2 Coupled pendulums

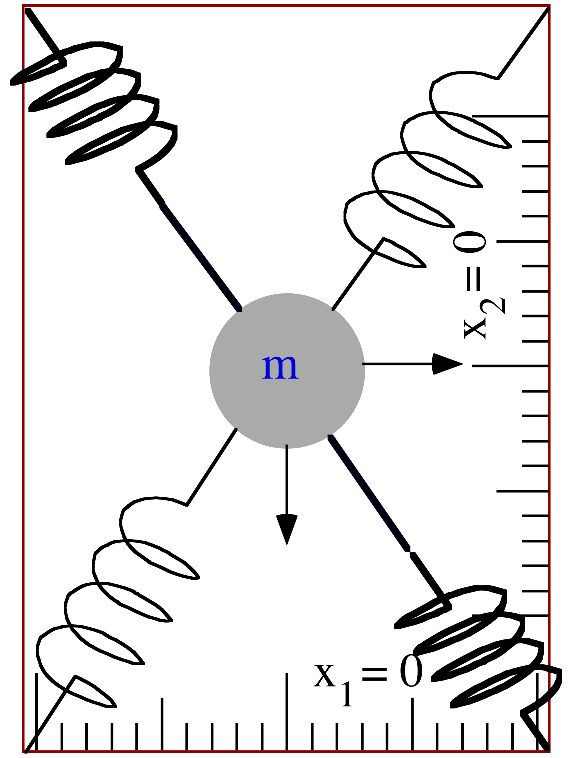


Fig. 3.3.3 One 2-dimensional coupled oscillator

2D HO kinetic energy $T(v_1, v_2)$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

2D harmonic oscillator energy

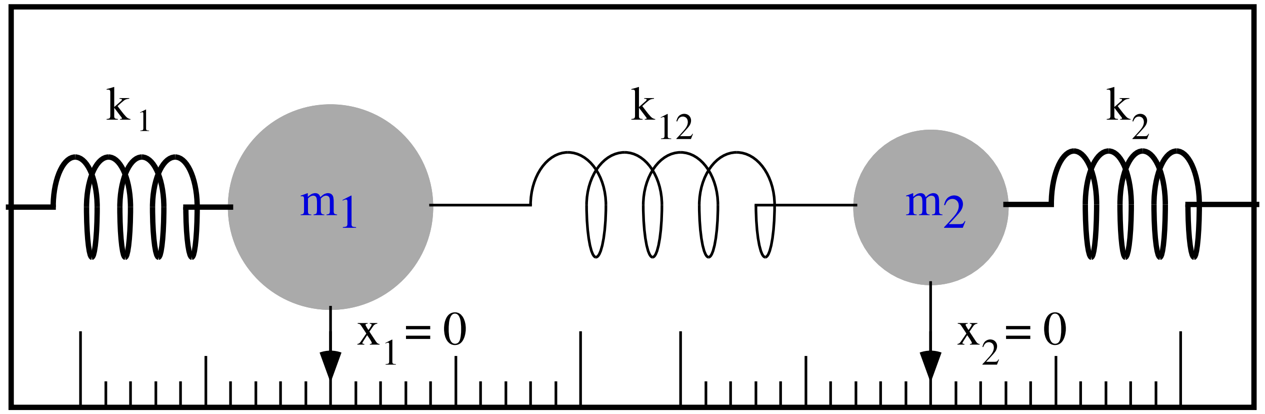
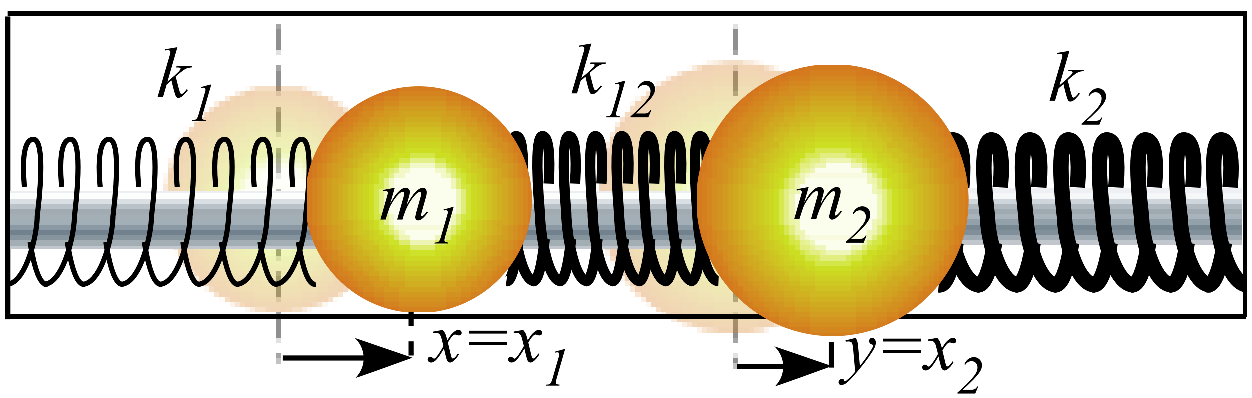


Fig. 3.3.1 Two 1-dimensional coupled oscillators



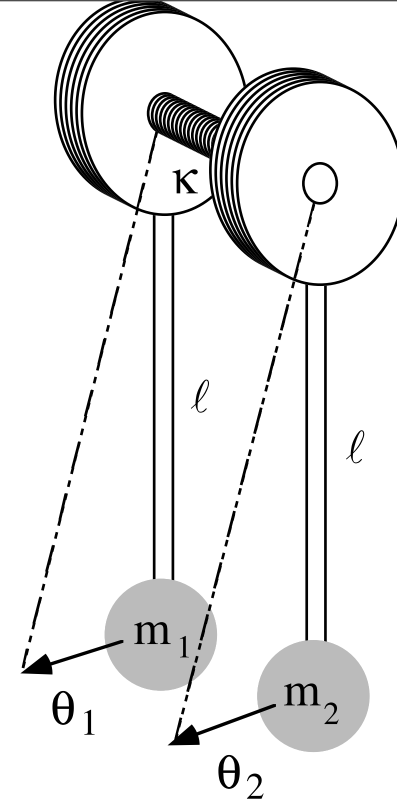


Fig. 3.3.2 Coupled pendulums

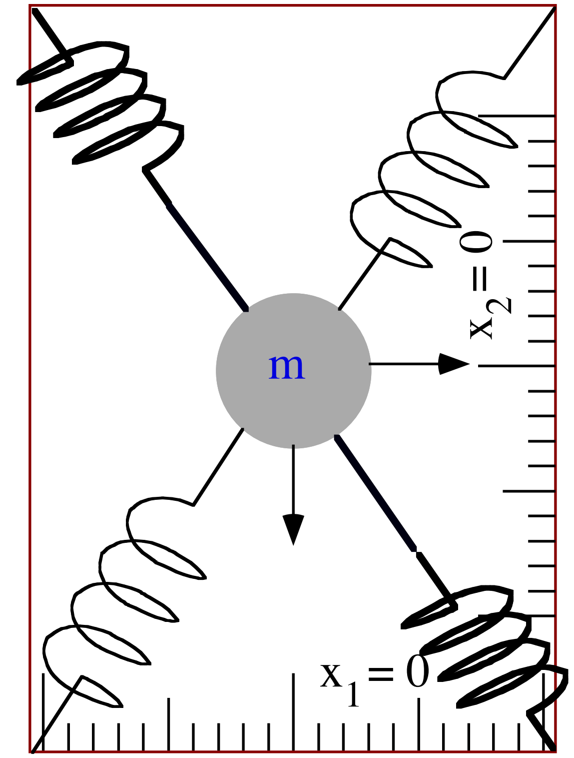


Fig. 3.3.3 One 2-dimensional coupled oscillator

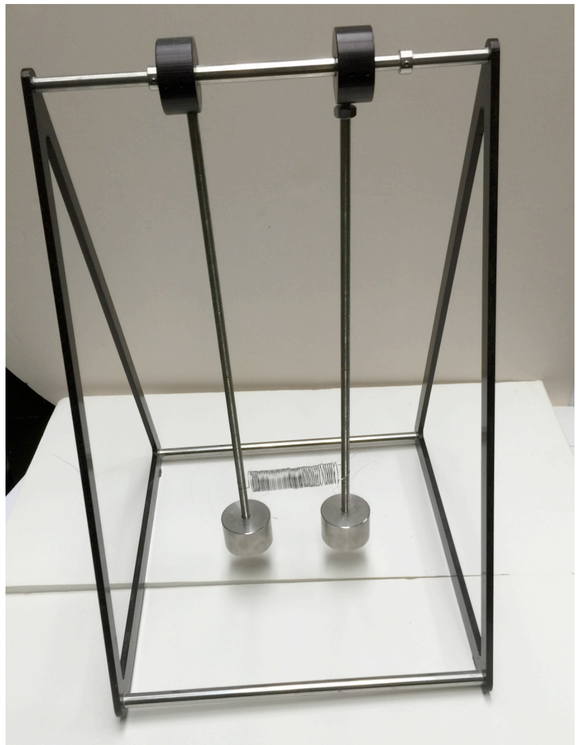
2D HO kinetic energy $T(v_1, v_2)$

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2$$

2D HO potential energy $V(x_1, x_2)$

$$V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2x_2^2 + \frac{1}{2}k_{12}\left(x_1 - x_2\right)^2$$

$$= \frac{1}{2}\left(k_1 + k_{12}\right)x_1^2 - k_{12}x_1x_2 + \frac{1}{2}\left(k_2 + k_{12}\right)x_2^2$$



Review of 1D FDHO (Forced-Damped-Harmonic Oscillator) response

2D harmonic oscillator (2D-HO) equations of motion
➔ *Lagrangian and matrix forms*

2D harmonic oscillator equation eigensolutions (normal modes)
Eigensolutions by geometry for 2D-HO with bilateral (B-Type) symmetry
Symmetric (low frequency) mode versus antisymmetric (high frequency) mode
Mixed mode beat dynamics (with constant π/2 phase-lag)

Eigensolutions by matrix-algebra with example M=
Secular equation
Hamilton-Cayley equation and projectors
Idempotent projectors (how eigenvalues⇒eigenvectors)
Operator orthonormality and Completeness (Idempotent means: **P·P=P***)*
Spectral Decompositions
Functional spectral decomposition
Orthonormality vs. Completeness vis-a`-vis Operator vs. State
Lagrange functional interpolation formula
Diagonalizing Transformations (D-Ttran) from projectors

2D harmonic oscillator equations

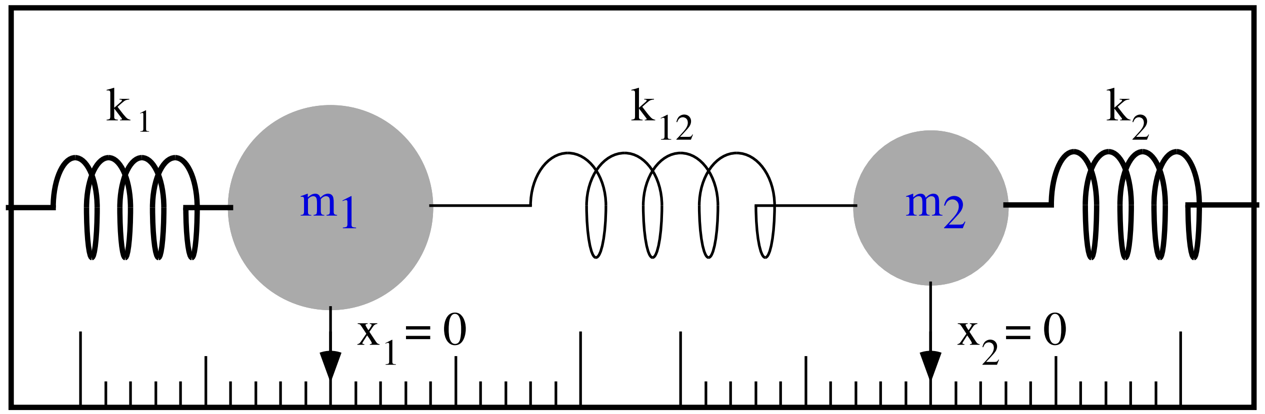
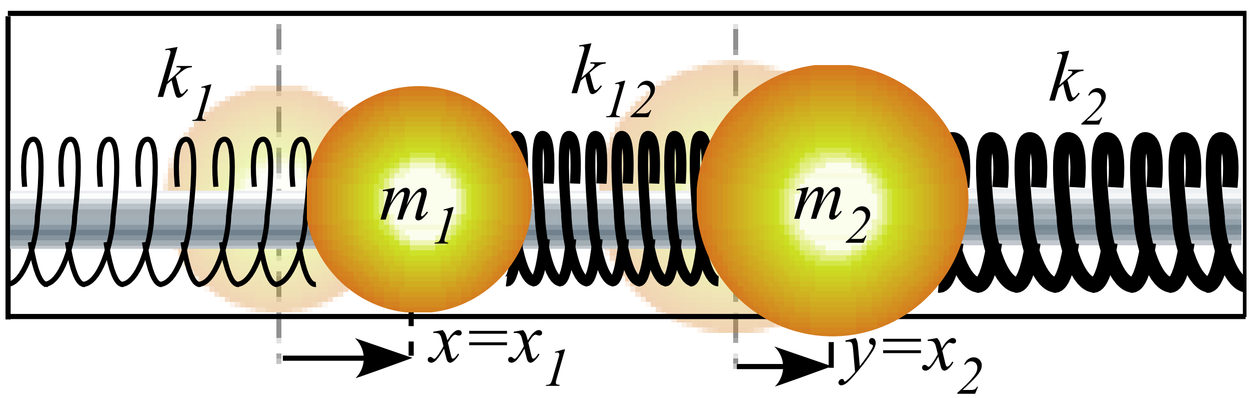


Fig. 3.3.1 Two 1-dimensional coupled oscillators



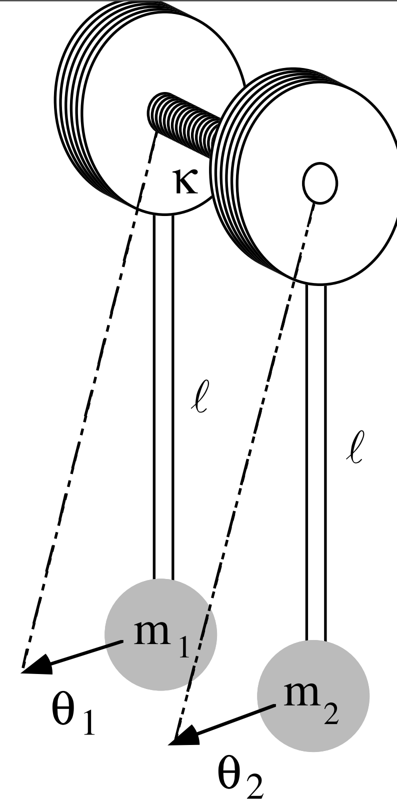


Fig. 3.3.2 Coupled pendulums

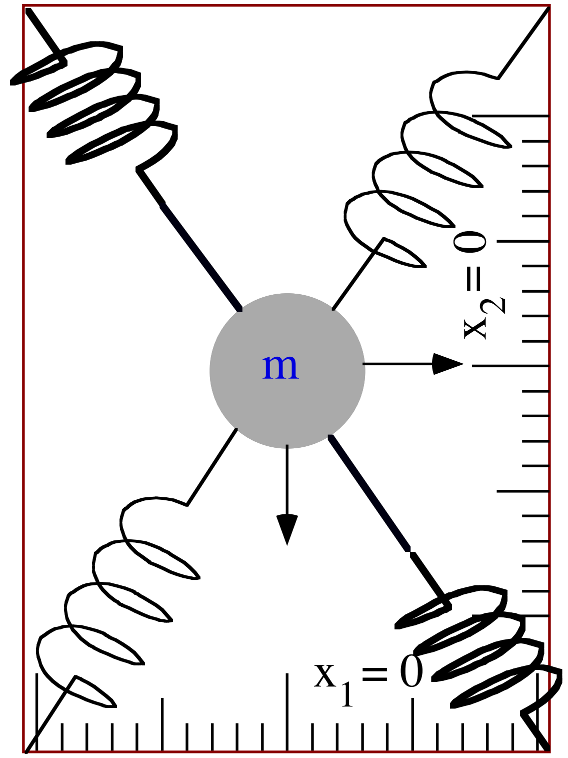


Fig. 3.3.3 One 2-dimensional coupled oscillator

2D HO kinetic energy $T(v_1, v_2)$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

2D HO potential energy $V(x_1, x_2)$

$$V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2 + \frac{1}{2} k_{12} \left(x_1 - x_2 \right)^2$$

$$= \frac{1}{2} \left(k_1 + k_{12} \right) x_1^2 - k_{12} x_1 x_2 + \frac{1}{2} \left(k_2 + k_{12} \right) x_2^2$$

Lagrange-Newton equations for 2D HO

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_1} \right) = m_1 \ddot{x}_1 = F_1 = -\frac{\partial V}{\partial x_1} = -\left(k_1 + k_{12} \right) x_1 + k_{12} x_2$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_2} \right) = m_2 \ddot{x}_2 = F_2 = -\frac{\partial V}{\partial x_2} = k_{12} x_1 - \left(k_2 + k_{12} \right) x_2$$

2D harmonic oscillator equations

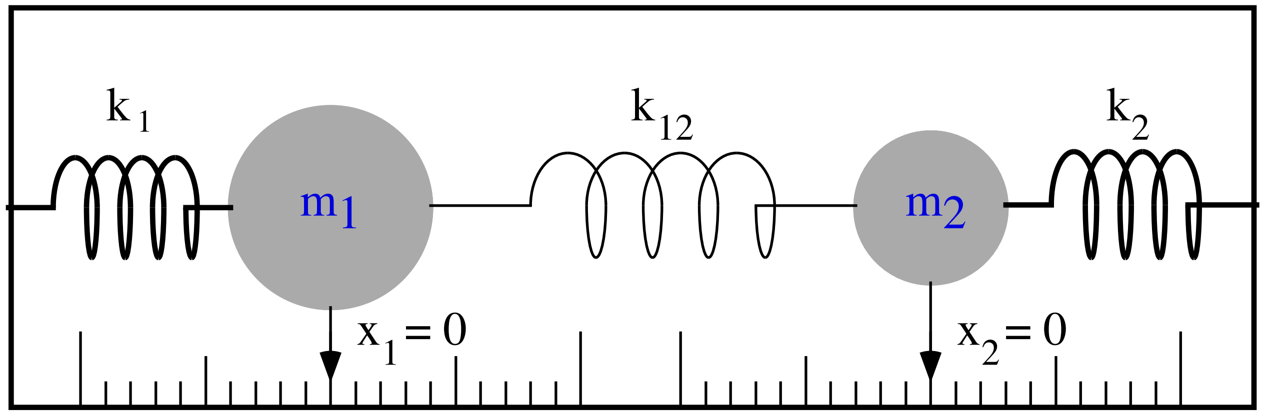
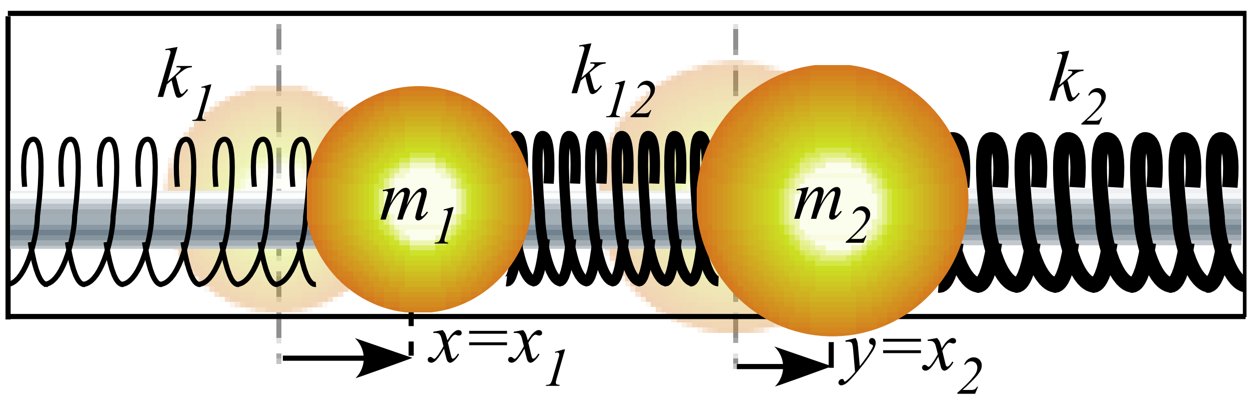


Fig. 3.3.1 Two 1-dimensional coupled oscillators



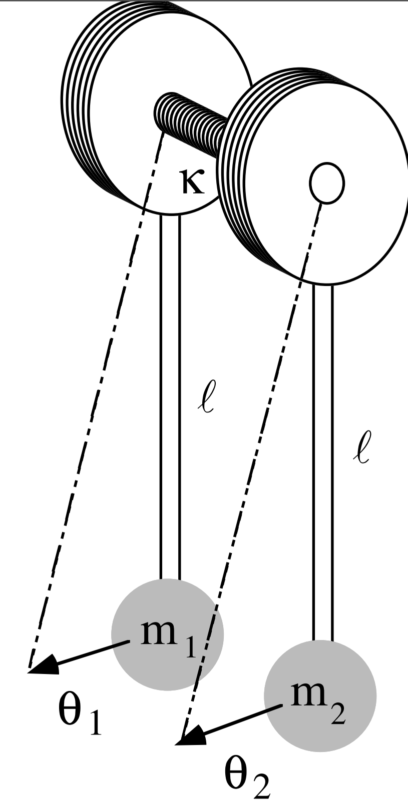


Fig. 3.3.2 Coupled pendulums

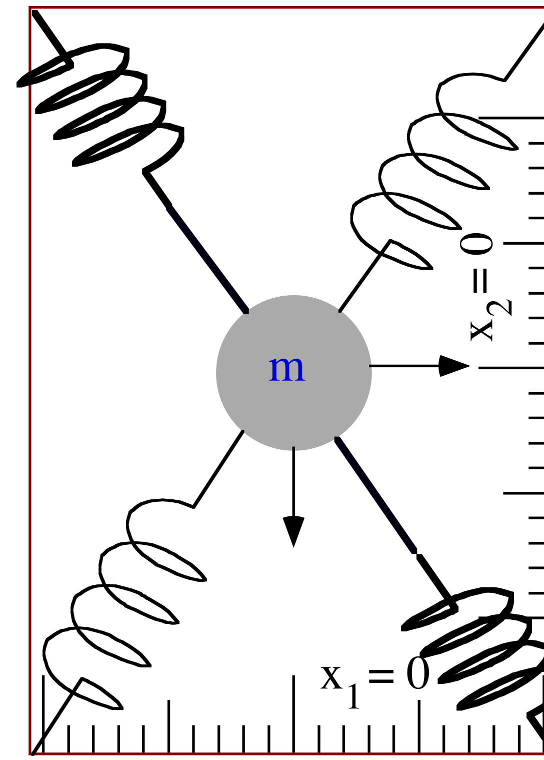


Fig. 3.3.3 One 2-dimensional coupled oscillator

2D HO kinetic energy $T(v_1, v_2)$

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2$$

2D HO potential energy $V(x_1, x_2)$

$$V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2x_2^2 + \frac{1}{2}k_{12}\left(x_1 - x_2\right)^2$$

$$= \frac{1}{2}\left(k_1 + k_{12}\right)x_1^2 - k_{12}x_1x_2 + \frac{1}{2}\left(k_2 + k_{12}\right)x_2^2$$

Lagrange-Newton equations for 2D HO

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = m_1\ddot{x}_1 = F_1 = -\frac{\partial V}{\partial x_1} = -\left(k_1 + k_{12}\right)x_1 + k_{12}x_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = m_2\ddot{x}_2 = F_2 = -\frac{\partial V}{\partial x_2} = k_{12}x_1 - \left(k_2 + k_{12}\right)x_2$$

2D HO Matrix operator equations

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

2D harmonic oscillator equations

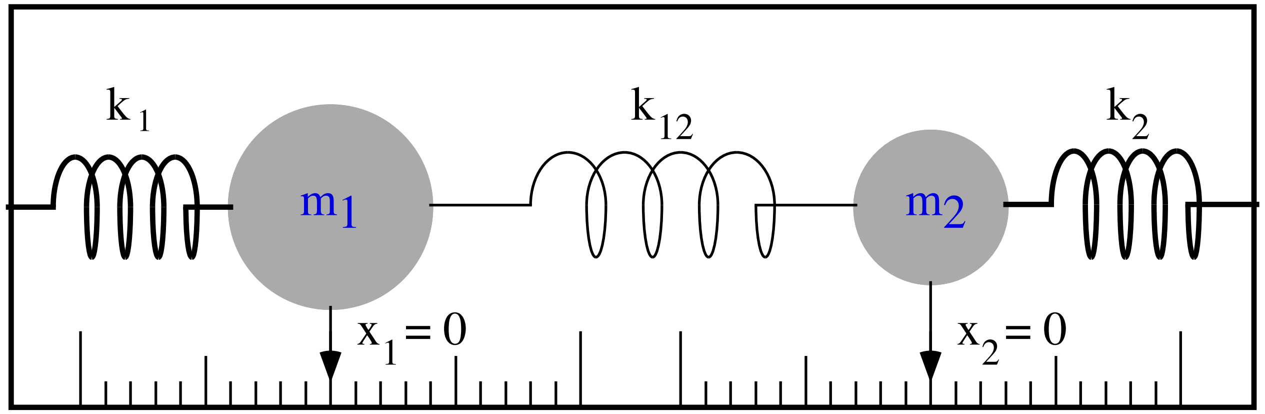
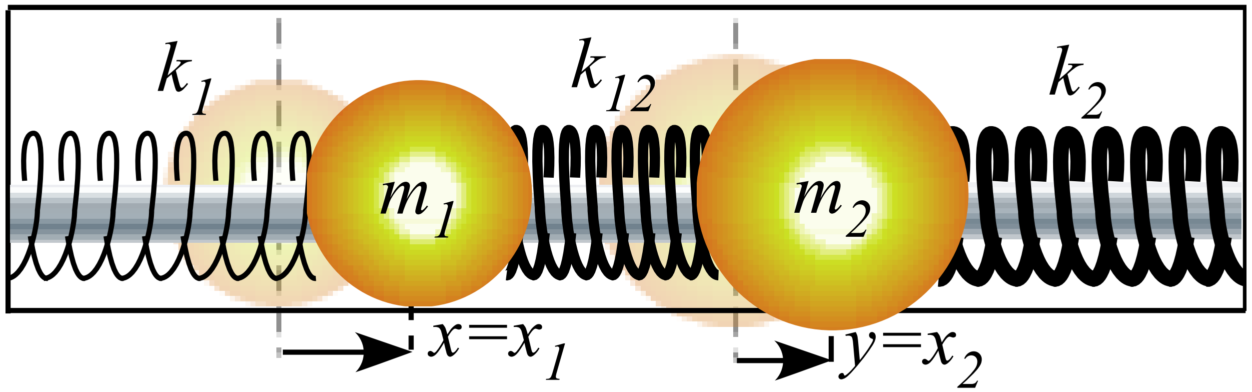


Fig. 3.3.1 Two 1-dimensional coupled oscillators



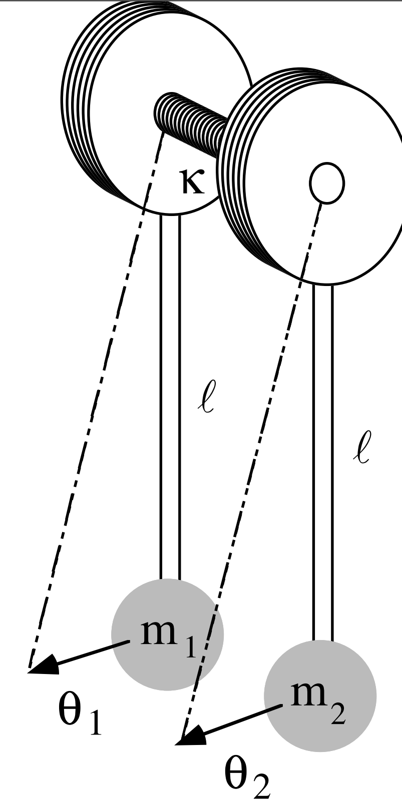


Fig. 3.3.2 Coupled pendulums

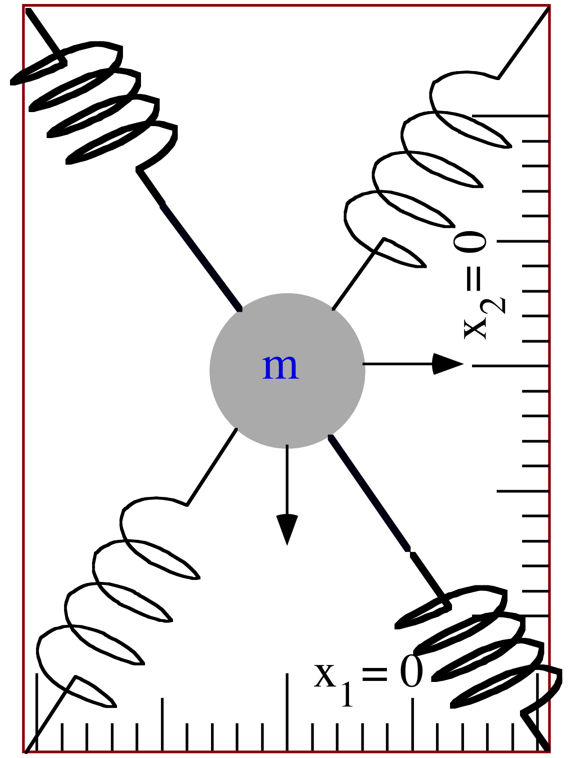


Fig. 3.3.3 One 2-dimensional coupled oscillator

2D HO kinetic energy $T(v_1, v_2)$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

2D HO potential energy $V(x_1, x_2)$

$$V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2 + \frac{1}{2} k_{12} \left(x_1 - x_2\right)^2$$

$$= \frac{1}{2}\left(k_1 + k_{12}\right) x_1^2 - k_{12} x_1 x_2 + \frac{1}{2}\left(k_2 + k_{12}\right) x_2^2$$

Lagrange-Newton equations for 2D HO

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = m_1 \ddot{x}_1 = F_1 = -\frac{\partial V}{\partial x_1} = -\left(k_1 + k_{12}\right) x_1 + k_{12} x_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = m_2 \ddot{x}_2 = F_2 = -\frac{\partial V}{\partial x_2} = k_{12} x_1 - \left(k_2 + k_{12}\right) x_2$$

2D HO Matrix operator equations

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Matrix operator notation:

$$\mathbf{M} \bullet \left|\ddot{\mathbf{x}}\right\rangle = -\mathbf{K} \bullet \left|\mathbf{x}\right\rangle$$

2D harmonic oscillator equations

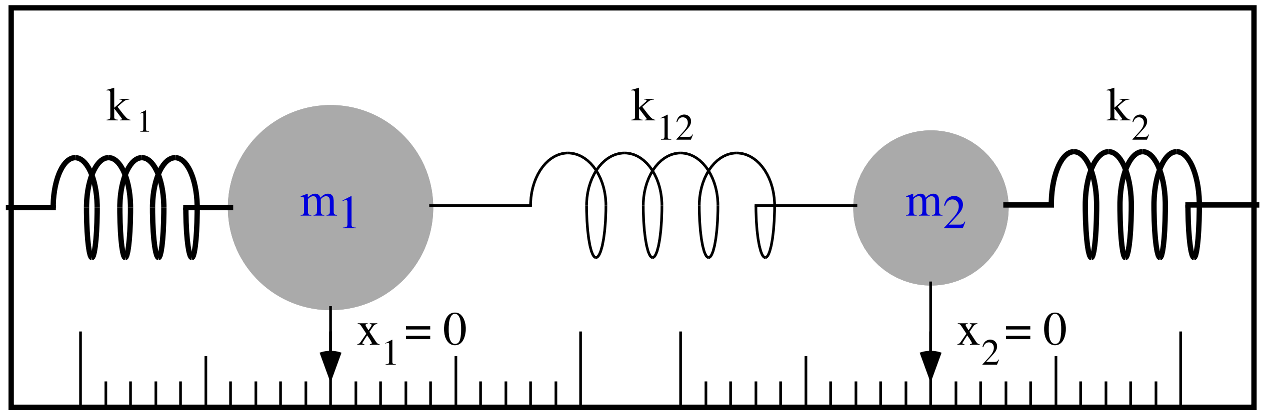
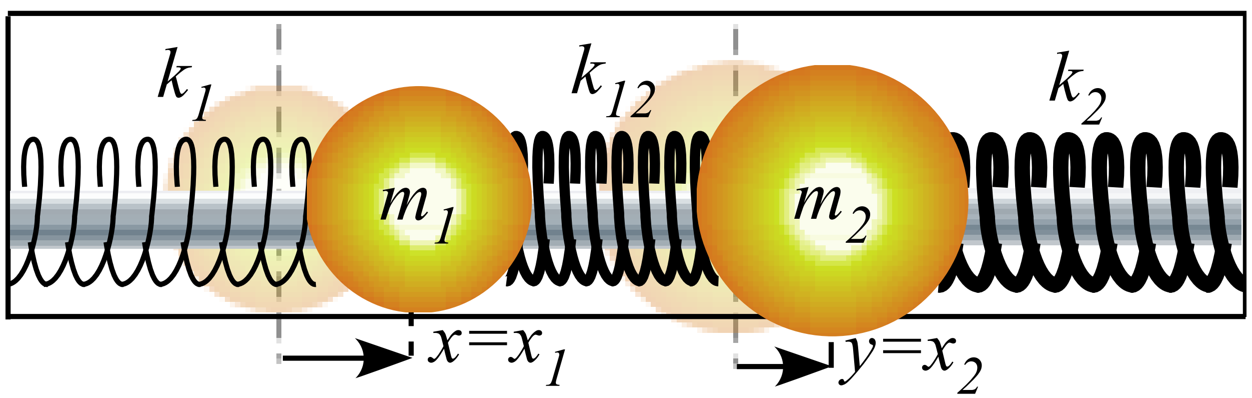


Fig. 3.3.1 Two 1-dimensional coupled oscillators



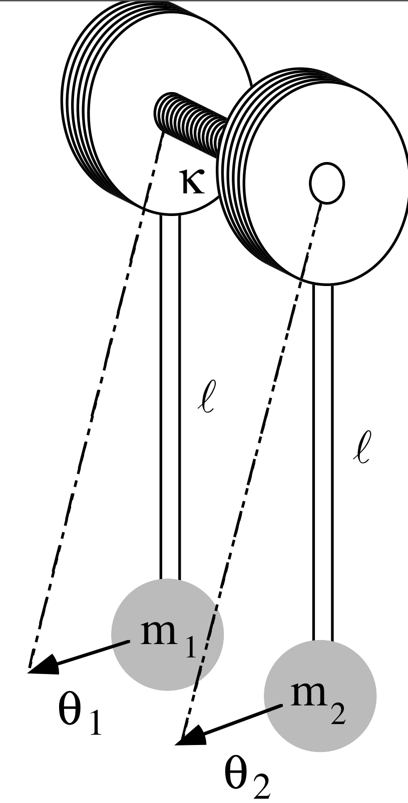


Fig. 3.3.2 Coupled pendulums

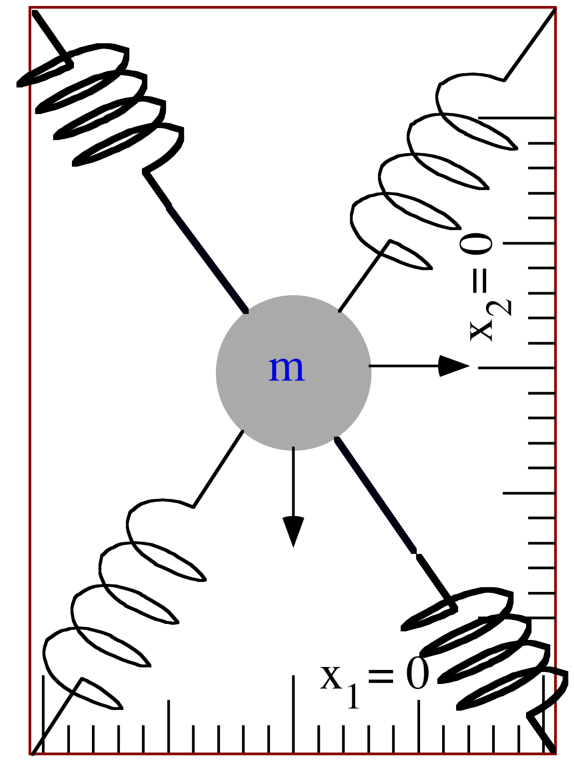


Fig. 3.3.3 One 2-dimensional coupled oscillator

2D HO kinetic energy $T(v_1, v_2)$

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2$$

$$= \frac{1}{2}\langle\dot{\mathbf{x}}|\mathbf{M}|\dot{\mathbf{x}}\rangle$$

2D HO potential energy $V(x_1, x_2)$

$$V = \frac{1}{2}(k_1 + k_{12})x_1^2 - k_{12}x_1x_2 + \frac{1}{2}(k_2 + k_{12})x_2^2$$

$$= \frac{1}{2}\langle\mathbf{x}|\mathbf{K}|\mathbf{x}\rangle \quad \text{where: } \mathbf{K} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix}$$

Lagrange-Newton equations for 2D HO

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = m_1\ddot{x}_1 = F_1 = -\frac{\partial V}{\partial x_1} = -(k_1 + k_{12})x_1 + k_{12}x_2$$

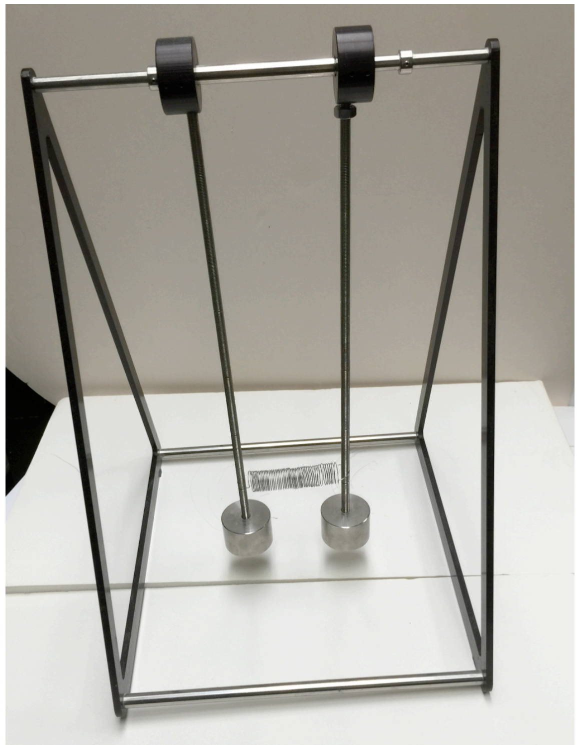
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = m_2\ddot{x}_2 = F_2 = -\frac{\partial V}{\partial x_2} = k_{12}x_1 - (k_2 + k_{12})x_2$$

2D HO Matrix operator equations

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Matrix operator notation:

$$\mathbf{M}\bullet|\ddot{\mathbf{x}}\rangle \quad = \quad - \quad \mathbf{K}\bullet|\mathbf{x}\rangle$$



Review of 1D FDHO (Forced-Damped-Harmonic Oscillator) response

2D harmonic oscillator (2D-HO) equations of motion
Lagrangian and matrix forms

➤ *2D harmonic oscillator equation eigensolutions (normal modes)*
Eigensolutions by geometry for 2D-HO with bilateral (B-Type) symmetry
Symmetric (low frequency) mode versus antisymmetric (high frequency) mode
Mixed mode beat dynamics (with constant π/2 phase-lag)

Eigensolutions by matrix-algebra with example M=
Secular equation
Hamilton-Cayley equation and projectors
Idempotent projectors (how eigenvalues⇒eigenvectors)
Operator orthonormality and Completeness (Idempotent means: **P·P=P***)*
Spectral Decompositions
Functional spectral decomposition
Orthonormality vs. Completeness vis-a`-vis Operator vs. State
Lagrange functional interpolation formula
Diagonalizing Transformations (D-Ttran) from projectors

2D harmonic oscillator equation solutions

1. May rewrite equation $\mathbf{M}\bullet|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$ in *acceleration* matrix form: $|\ddot{\mathbf{x}}\rangle = -\mathbf{A}|\mathbf{x}\rangle$ where: $\mathbf{A} = \mathbf{M}^{-1}\bullet\mathbf{K}$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}^{-1} \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\begin{pmatrix} \dfrac{k_1 + k_{12}}{m_1} & \dfrac{-k_{12}}{m_1} \\ \dfrac{-k_{12}}{m_2} & \dfrac{k_2 + k_{12}}{m_2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

2. Need to find *eigenvectors* $|\mathbf{e}_1\rangle, |\mathbf{e}_2\rangle, ...$ of acceleration matrix such that: $\mathbf{A}|\mathbf{e}_n\rangle = \varepsilon_n|\mathbf{e}_n\rangle = \omega_n^2|\mathbf{e}_n\rangle$

Then equations decouple to: $|\ddot{\mathbf{e}}_n\rangle = -\mathbf{A}|\mathbf{e}_n\rangle = -\varepsilon_n|\mathbf{e}_n\rangle = -\omega_n^2|\mathbf{e}_n\rangle$ where ε_n is an *eigenvalue*

and ω_n is an *eigenfrequency*

2D harmonic oscillator equation solutions

1. May rewrite equation $\mathbf{M}\bullet|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$ in *acceleration* matrix form: $|\ddot{\mathbf{x}}\rangle = -\mathbf{A}|\mathbf{x}\rangle$ where: $\mathbf{A} = \mathbf{M}^{-1}\bullet\mathbf{K}$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}^{-1} \begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\begin{pmatrix} \dfrac{k_1+k_{12}}{m_1} & \dfrac{-k_{12}}{m_1} \\ \dfrac{-k_{12}}{m_2} & \dfrac{k_2+k_{12}}{m_2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

2. Need to find *eigenvectors* $|\mathbf{e}_1\rangle, |\mathbf{e}_2\rangle, \ldots$ of acceleration matrix such that: $\mathbf{A}|\mathbf{e}_n\rangle = \varepsilon_n|\mathbf{e}_n\rangle = \omega_n^2|\mathbf{e}_n\rangle$

Then equations decouple to: $\dfrac{d^2}{dt^2}|\mathbf{e}_1\rangle \equiv |\ddot{\mathbf{e}}_1\rangle = -\mathbf{A}|\mathbf{e}_1\rangle = -\varepsilon_1|\mathbf{e}_1\rangle = -\omega_1^2|\mathbf{e}_1\rangle$ so: $|\mathbf{e}_1(t)\rangle = e^{-i\omega_1 t}|\mathbf{e}_1(0)\rangle$

where ε_1 is 1^{st} *eigenvalue* and ω_1 is 1^{st} *eigenfrequency*

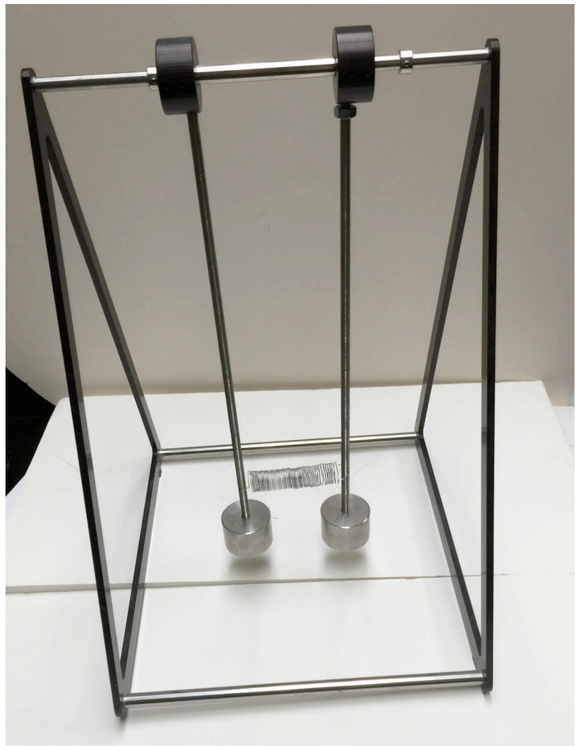
and: $\dfrac{d^2}{dt^2}|\mathbf{e}_2\rangle \equiv |\ddot{\mathbf{e}}_2\rangle = -\mathbf{A}|\mathbf{e}_2\rangle = -\varepsilon_2|\mathbf{e}_2\rangle = -\omega_2^2|\mathbf{e}_2\rangle$ so: $|\mathbf{e}_2(t)\rangle = e^{-i\omega_2 t}|\mathbf{e}_2(0)\rangle$

where ε_2 is 2^{nd} *eigenvalue* and ω_2 is 2^{nd} *eigenfrequency*

To introduce eigensolutions we take a simple case of unit masses *(m_1=1=m_2)*

So equation of motion is simply: $|\ddot{\mathbf{x}}\rangle = -\mathbf{K}|\mathbf{x}\rangle$

Eigenvectors $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$ are in special directions where $|\ddot{\mathbf{x}}\rangle = -\mathbf{K}|\mathbf{x}\rangle$ is in same direction as $|\mathbf{x}\rangle$



Review of 1D FDHO (Forced-Damped-Harmonic Oscillator) response

2D harmonic oscillator (2D-HO) equations of motion
Lagrangian and matrix forms

2D harmonic oscillator equation eigensolutions (normal modes)
Eigensolutions by geometry for 2D-HO with bilateral (B-Type) symmetry
Symmetric (low frequency) mode versus antisymmetric (high frequency) mode
Mixed mode beat dynamics (with constant π/2 phase-lag)

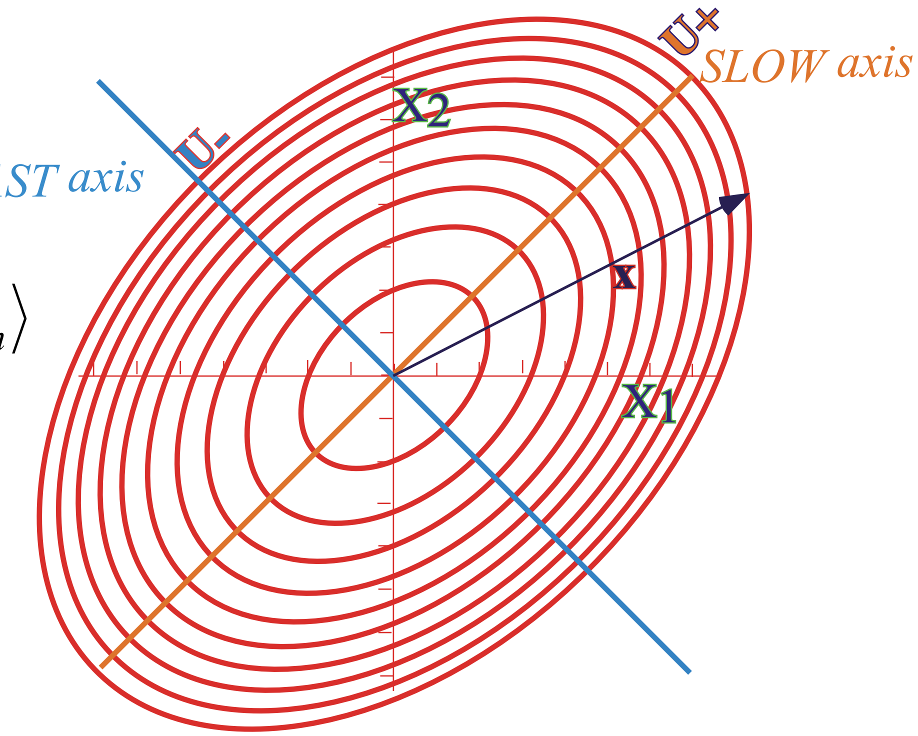
Eigensolutions by matrix-algebra with example M=
Secular equation
Hamilton-Cayley equation and projectors
Idempotent projectors (how eigenvalues⇒eigenvectors)
Operator orthonormality and Completeness (Idempotent means: **P·P=P***)*
Spectral Decompositions
Functional spectral decomposition
Orthonormality vs. Completeness vis-a`-vis Operator vs. State
Lagrange functional interpolation formula
Diagonalizing Transformations (D-Ttran) from projectors

2D HO potential energy $V(x_1, x_2)$ quadratic form defines layers of elliptical V-contours

$$V = \frac{1}{2}(k_1 + k_{12})x_1^2 - k_{12}x_1x_2 + \frac{1}{2}(k_2 + k_{12})x_2^2 = \frac{1}{2}\langle \mathbf{x}|\mathbf{K}|\mathbf{x}\rangle = \frac{1}{2}\mathbf{x}\bullet\mathbf{K}\bullet\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(a) PE Contours

What direction $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$ is the *same* as $\mathbf{K}|\mathbf{x}\rangle$??



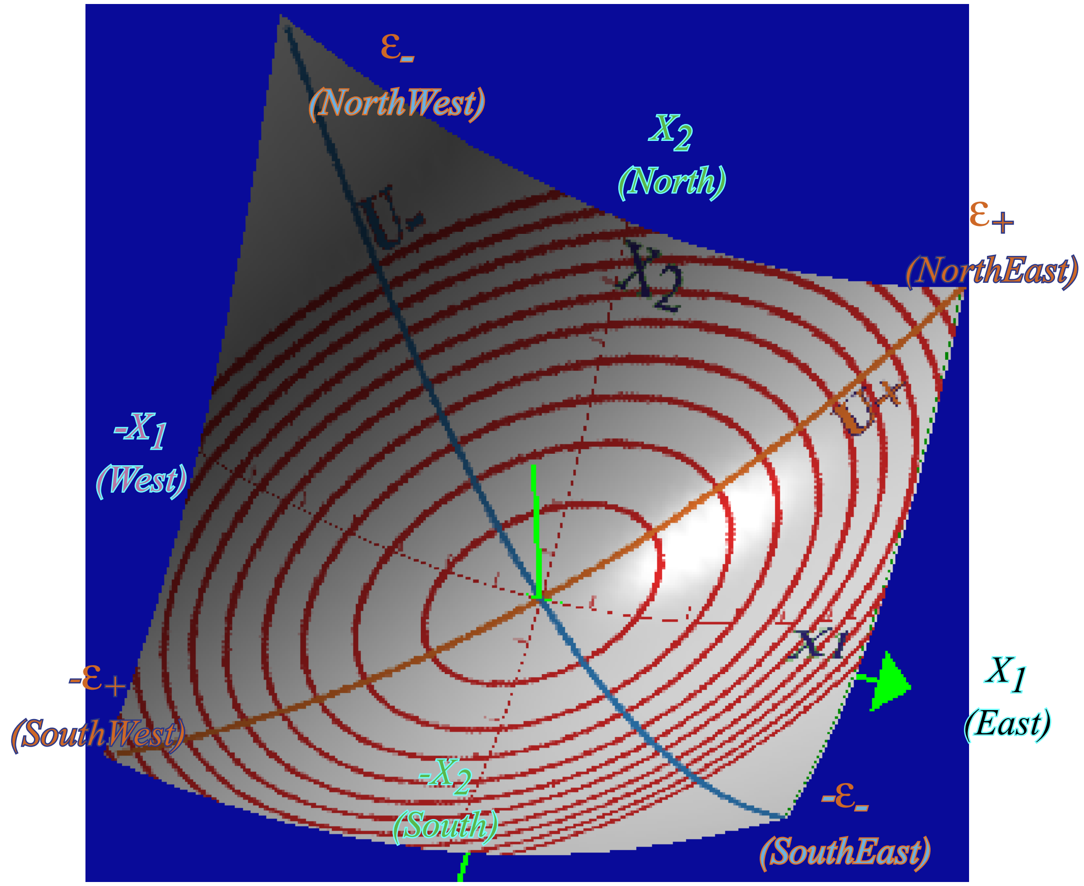


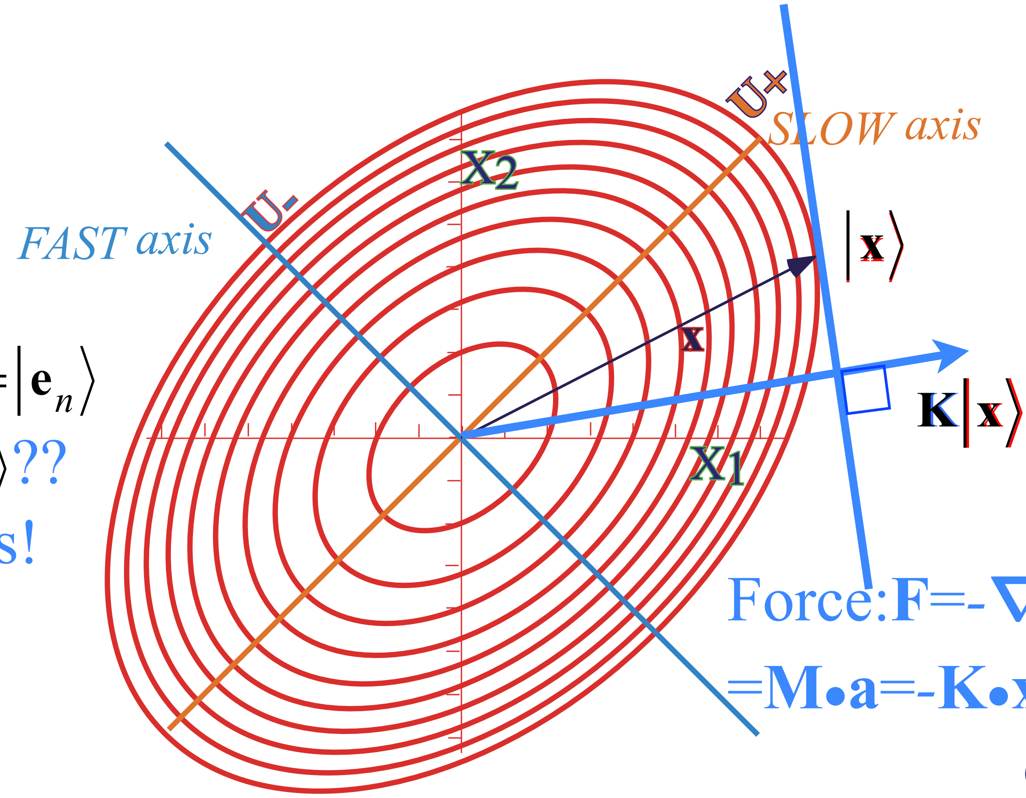
Fig. 3.3.4 Plot of potential function $V(x_1,x_2)$ showing elliptical $V(x_1,x_2)$=const. level curves.

2D HO potential energy $V(x_1, x_2)$ quadratic form defines layers of elliptical V-contours

$$V=\frac{1}{2}(k_1+k_{12})x_1^2-k_{12}x_1x_2+\frac{1}{2}(k_2+k_{12})x_2^2=\frac{1}{2}\langle\mathbf{x}|\mathbf{K}|\mathbf{x}\rangle=\frac{1}{2}\mathbf{x}\bullet\mathbf{K}\bullet\mathbf{x}=\begin{pmatrix} x_1 & x_2\end{pmatrix}\begin{pmatrix} k_1+k_{12} & -k_{12}\\ -k_{12} & k_2+k_{12}\end{pmatrix}\begin{pmatrix} x_1\\ x_2\end{pmatrix}$$

(a) PE Contours

What direction $|\mathbf{x}\rangle=|\mathbf{e}_n\rangle$
is the *same* as $\mathbf{K}|\mathbf{x}\rangle$??
Not <u>most</u> directions!



Force: $\mathbf{F}=-\nabla V$
$=\mathbf{M}\bullet\mathbf{a}=-\mathbf{K}\bullet\mathbf{x}$

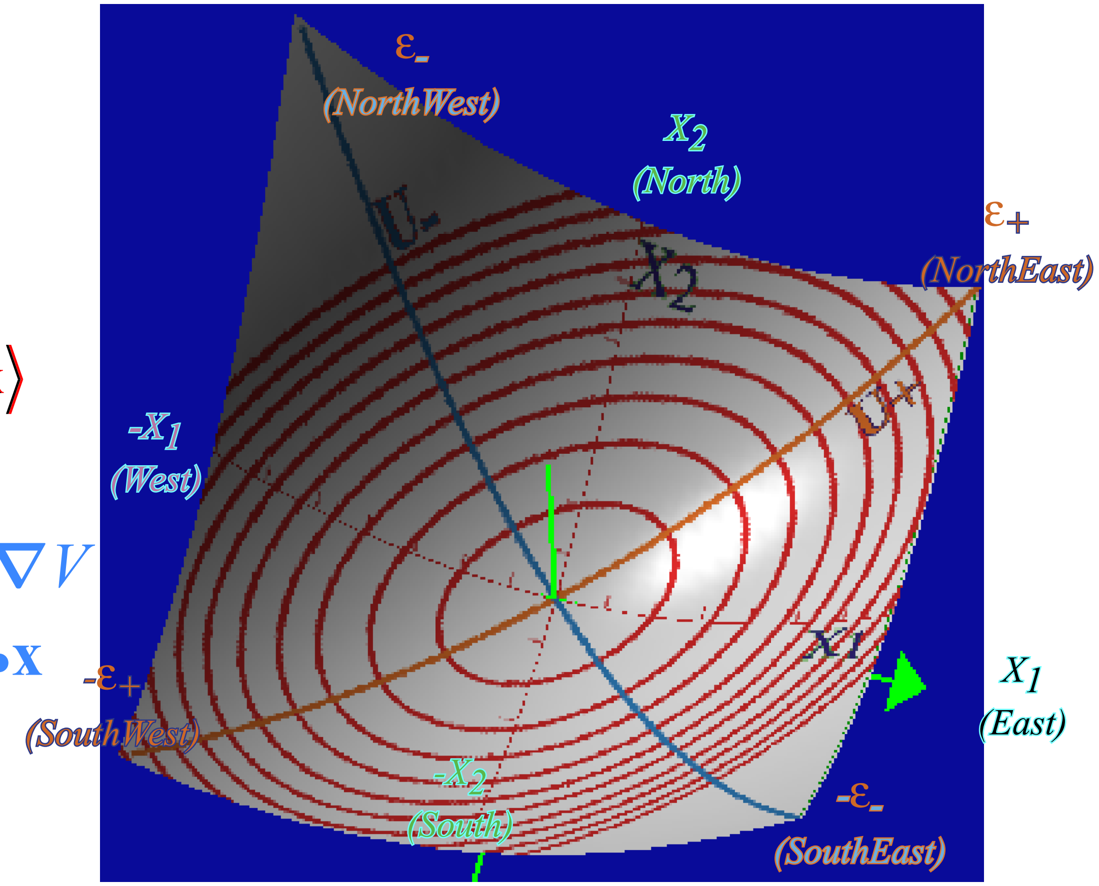
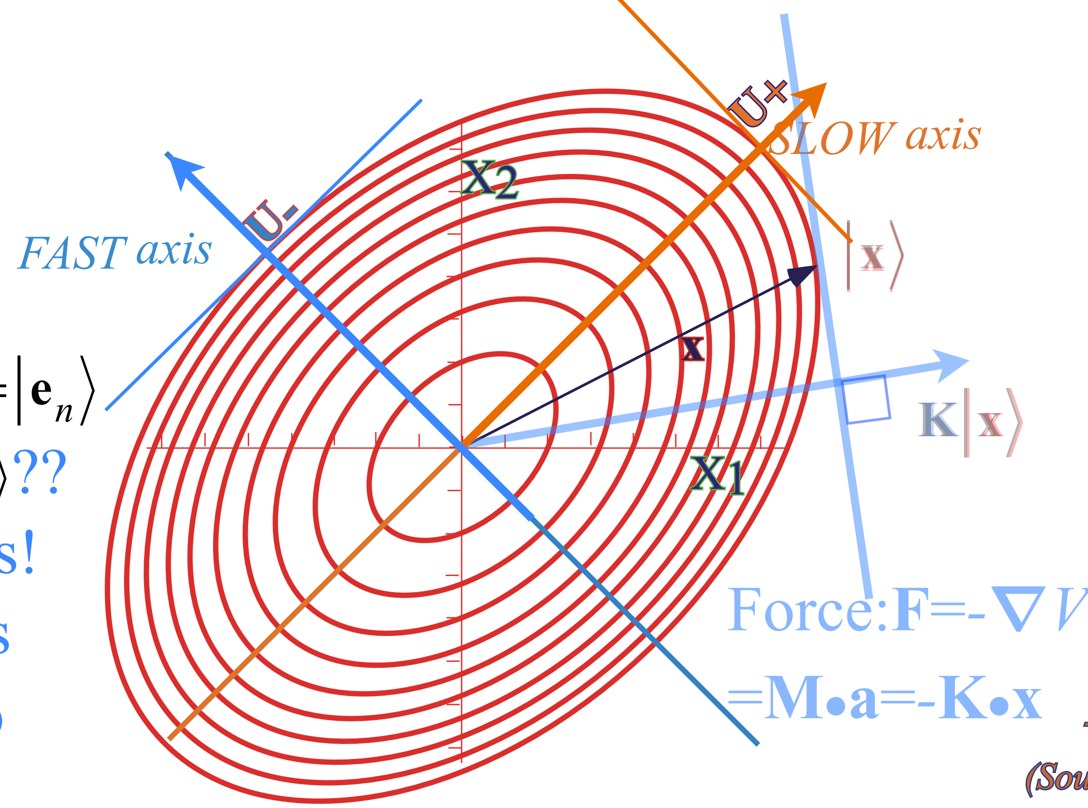


Fig. 3.3.4 Plot of potential function $V(x_1,x_2)$ showing elliptical $V(x_1,x_2)$=const. level curves.

2D HO potential energy $V(x_1, x_2)$ quadratic form defines layers of elliptical V-contours

$$V=\frac{1}{2}\left(k_1+k_{12}\right)x_1^2-k_{12}x_1x_2+\frac{1}{2}\left(k_2+k_{12}\right)x_2^2=\frac{1}{2}\langle\mathbf{x}|\mathbf{K}|\mathbf{x}\rangle=\frac{1}{2}\mathbf{x}\bullet\mathbf{K}\bullet\mathbf{x}=\begin{pmatrix}x_1 & x_2\end{pmatrix}\begin{pmatrix}k_1+k_{12} & -k_{12}\\ -k_{12} & k_2+k_{12}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

(a) PE Contours



What direction $|\mathbf{x}\rangle=|\mathbf{e}_n\rangle$ is the *same* as $\mathbf{K}|\mathbf{x}\rangle$??

Not <u>most</u> directions!

Only extremal axes work. (major or minor axes)

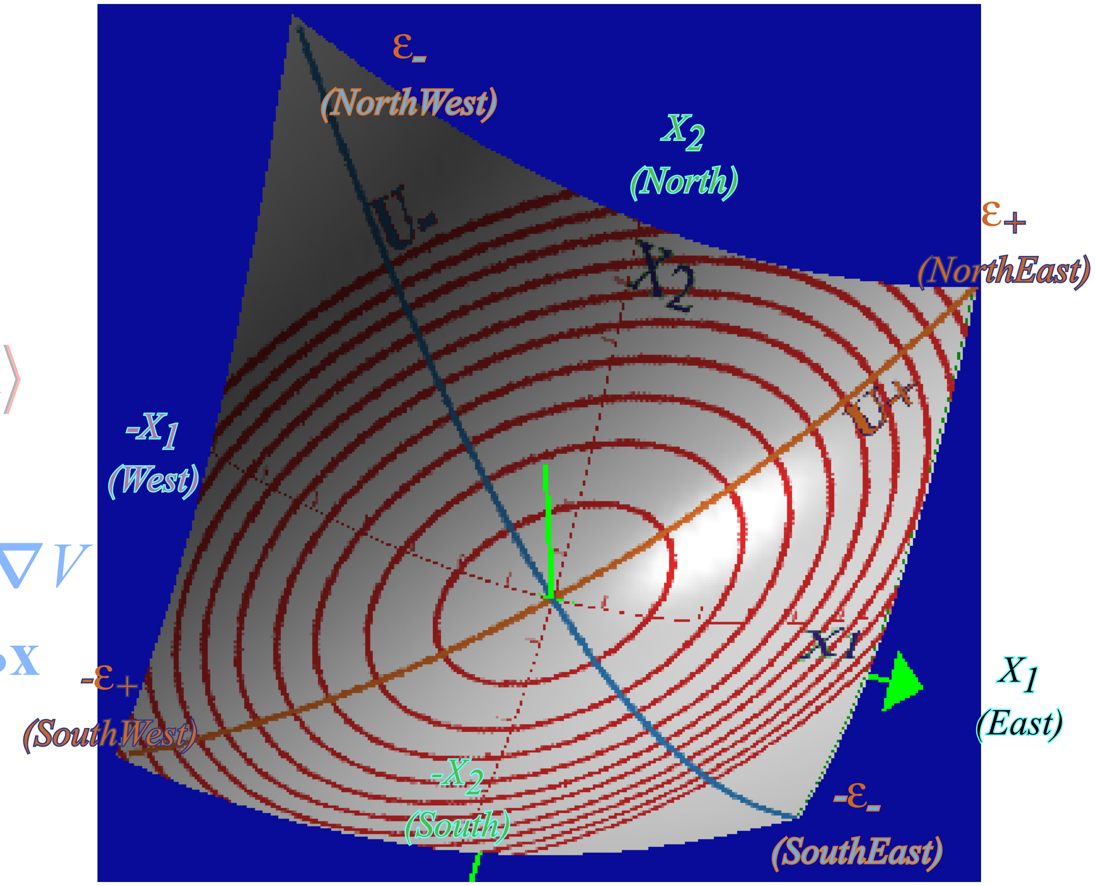
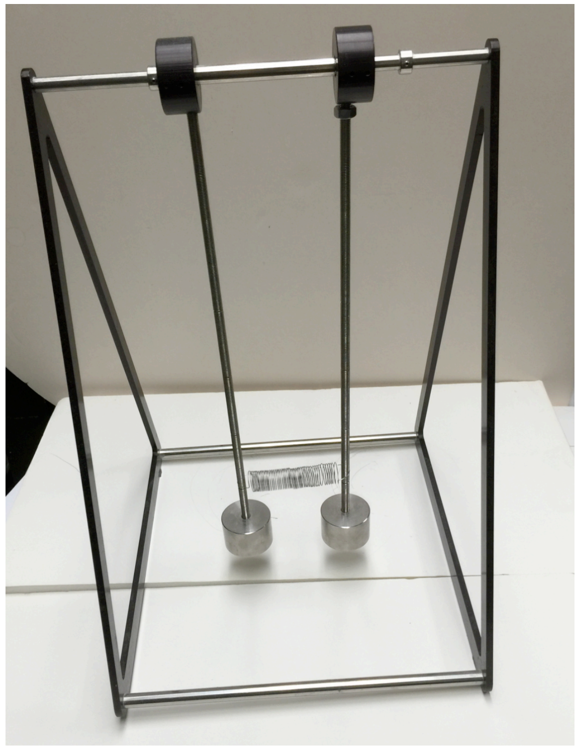


Fig. 3.3.4 Plot of potential function $V(x_1,x_2)$ showing elliptical $V(x_1,x_2)$=const. level curves.



Review of 1D FDHO (Forced-Damped-Harmonic Oscillator) response

2D harmonic oscillator (2D-HO) equations of motion
Lagrangian and matrix forms

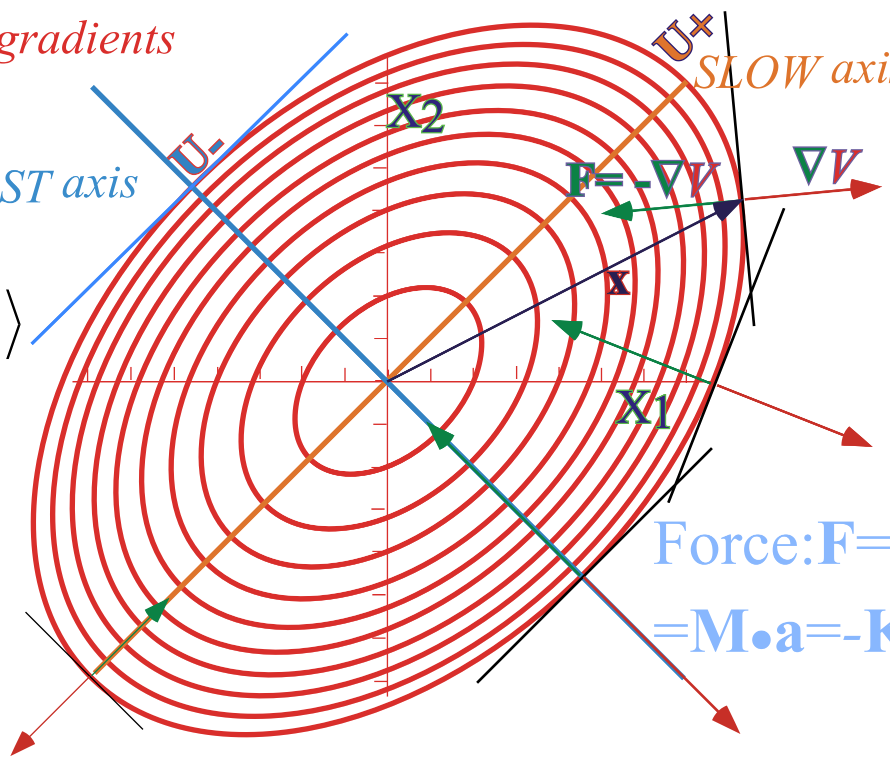
2D harmonic oscillator equation eigensolutions (normal modes)
Eigensolutions by geometry for 2D-HO with bilateral (B-Type) symmetry
Symmetric (low frequency) mode versus antisymmetric (high frequency) mode
Mixed mode beat dynamics (with constant $\pi/2$ phase-lag)

Eigensolutions by matrix-algebra with example M=
Secular equation
Hamilton-Cayley equation and projectors
Idempotent projectors (how eigenvalues⇒eigenvectors)
Operator orthonormality and Completeness (Idempotent means: **P·P=P**)
Spectral Decompositions
Functional spectral decomposition
Orthonormality vs. Completeness vis-a`-vis Operator vs. State
Lagrange functional interpolation formula
Diagonalizing Transformations (D-Ttran) from projectors

2D HO potential energy $V(x_1, x_2)$ quadratic form defines layers of elliptical V-contours

$$V = \frac{1}{2}\left(k + k_{12}\right)x_1^2 - k_{12}x_1x_2 + \frac{1}{2}\left(k + k_{12}\right)x_2^2 = \frac{1}{2}\langle\mathbf{x}|\mathbf{K}|\mathbf{x}\rangle = \frac{1}{2}\mathbf{x}\bullet\mathbf{K}\bullet\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} k+k_{12} & -k_{12} \\ -k_{12} & k+k_{12} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(a) PE Contours and gradients



What direction $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$ is the *same* as $\mathbf{K}|\mathbf{x}\rangle$??
Not <u>most</u> directions!
Only extremal axes work. (major or minor axes)

Force: $\mathbf{F} = -\nabla V$
$= \mathbf{M}\bullet\mathbf{a} = -\mathbf{K}\bullet\mathbf{x}$

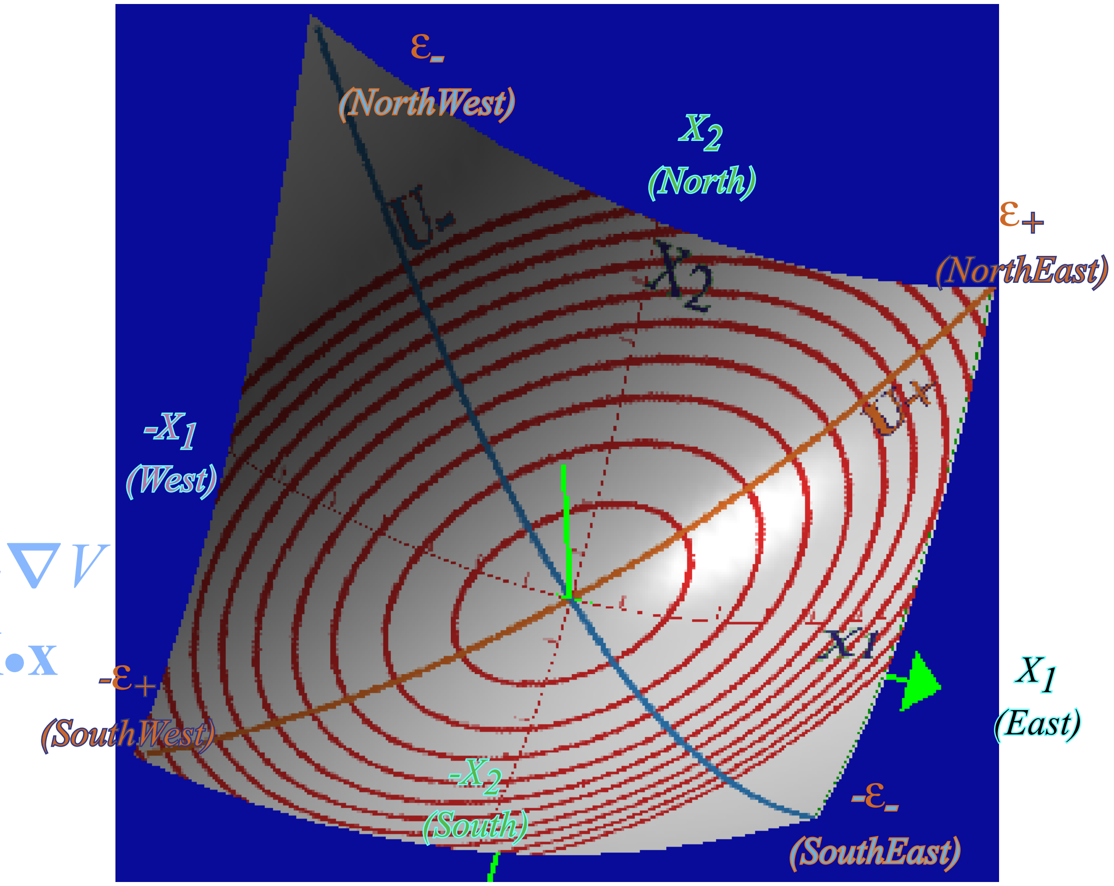
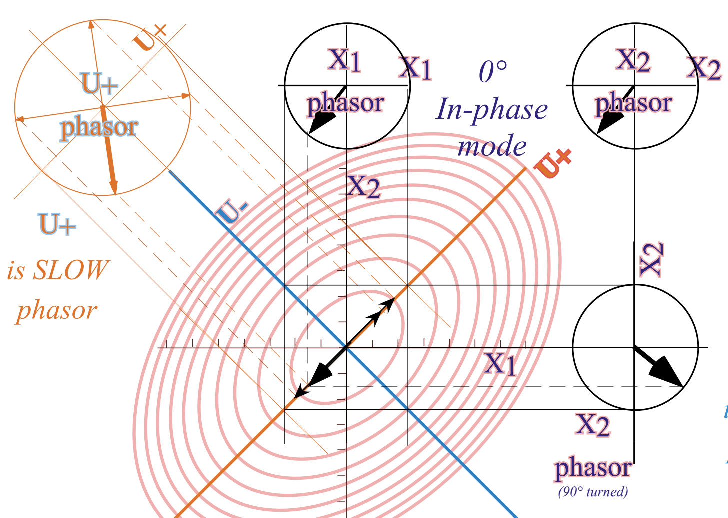
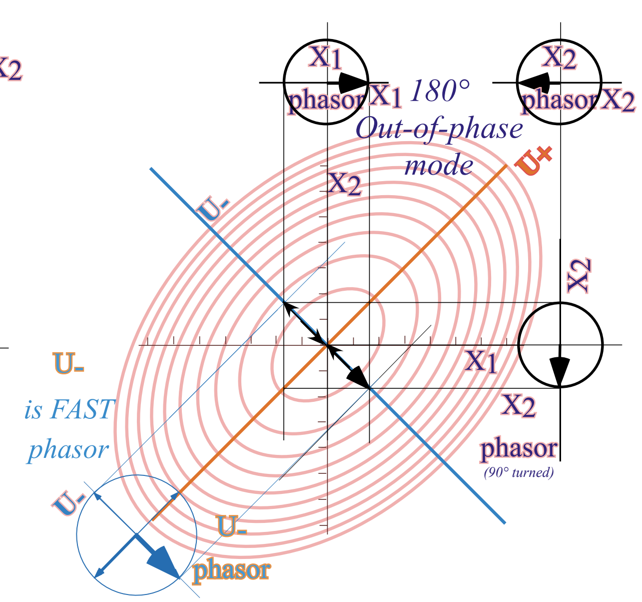


Fig. 3.3.4 Plot of potential function $V(x_1,x_2)$ showing elliptical $V(x_1,x_2)$=const. level curves.

(b) Symmetric U+ Coordinate SLOW Mode



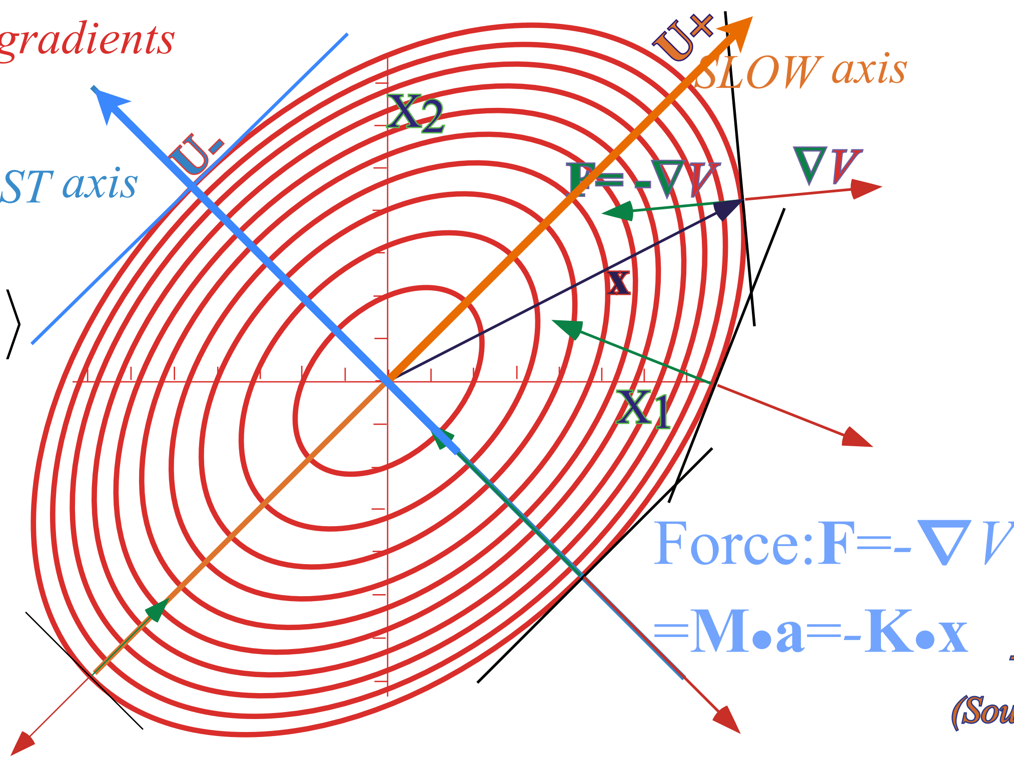
(c) Anti-symmetric U- Coordinate FAST Mode



2D HO potential energy $V(x_1, x_2)$ quadratic form defines layers of elliptical V-contours (Here: $k_1 = k = k_2$)

$$V = \frac{1}{2}(k+k_{12})x_1^2 - k_{12}x_1x_2 + \frac{1}{2}(k+k_{12})x_2^2 = \frac{1}{2}\langle\mathbf{x}|\mathbf{K}|\mathbf{x}\rangle = \frac{1}{2}\mathbf{x}\bullet\mathbf{K}\bullet\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} k+k_{12} & -k_{12} \\ -k_{12} & k+k_{12} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(a) PE Contours and gradients



What direction $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$ is the *same* as $\mathbf{K}|\mathbf{x}\rangle$??
Not <u>most</u> directions!
Only extremal axes work. (major or minor axes)

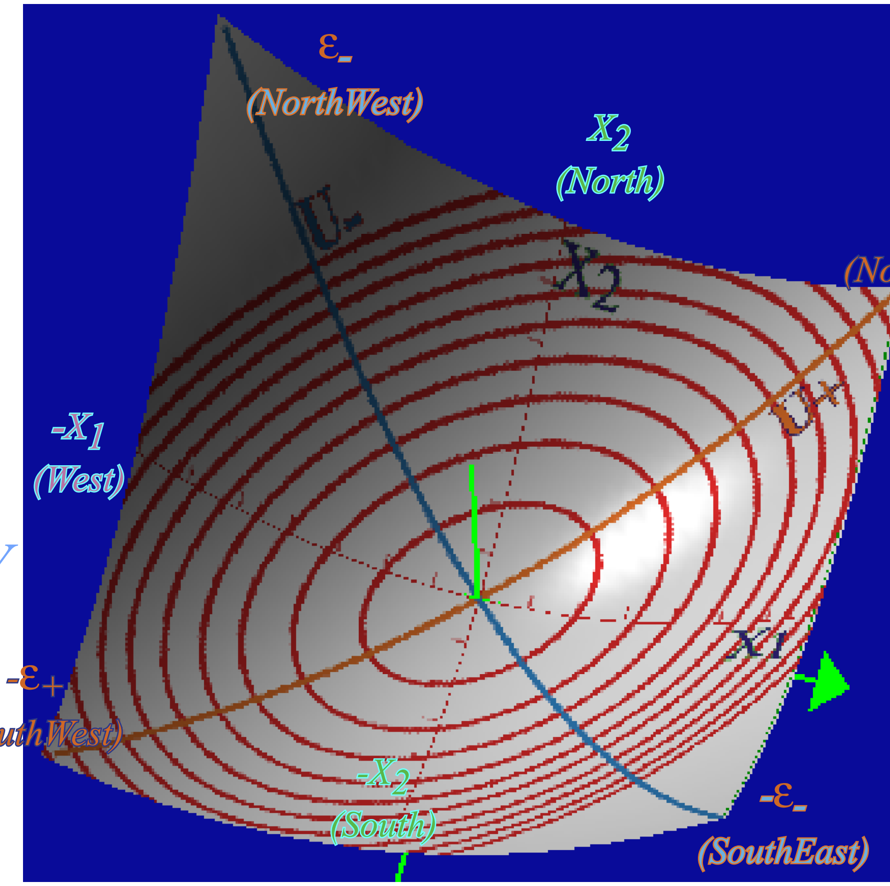
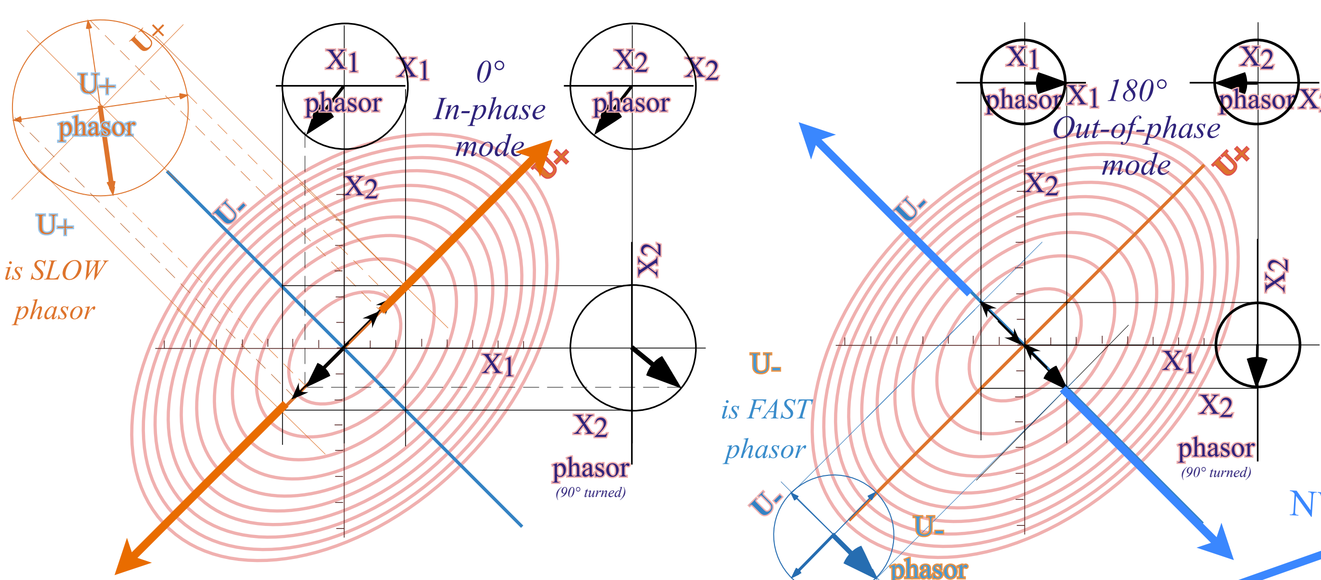
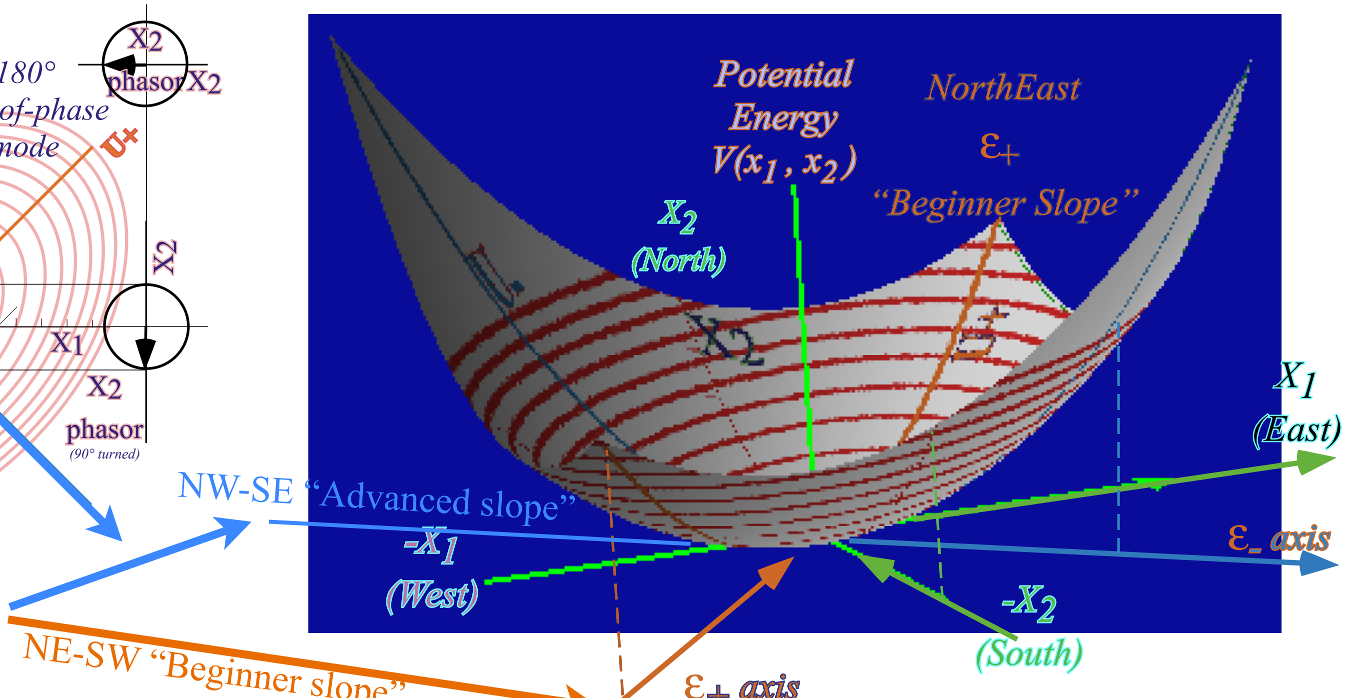


Fig. 3.3.4 Plot of potential function $V(x_1,x_2)$ showing elliptical $V(x_1,x_2)$=const. level curves.

(b) Symmetric U+ Coordinate SLOW Mode

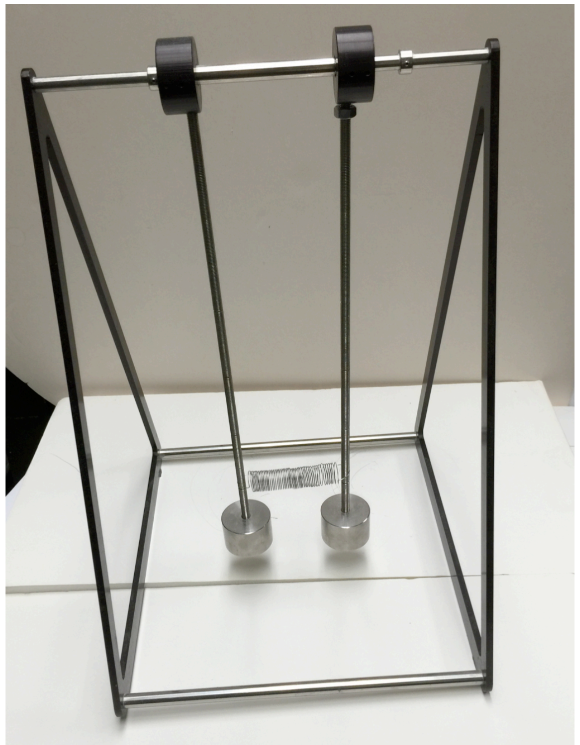
(c) Anti-symmetric U- Coordinate FAST Mode





With Bilateral symmetry ($k_1 = k = k_2$) the extremal axes lie at ±45°

Fig. 3.3.5 Topography lines of potential function $V(x_1,x_2)$ and orthogonal ε_+ and ε_- normal mode slopes

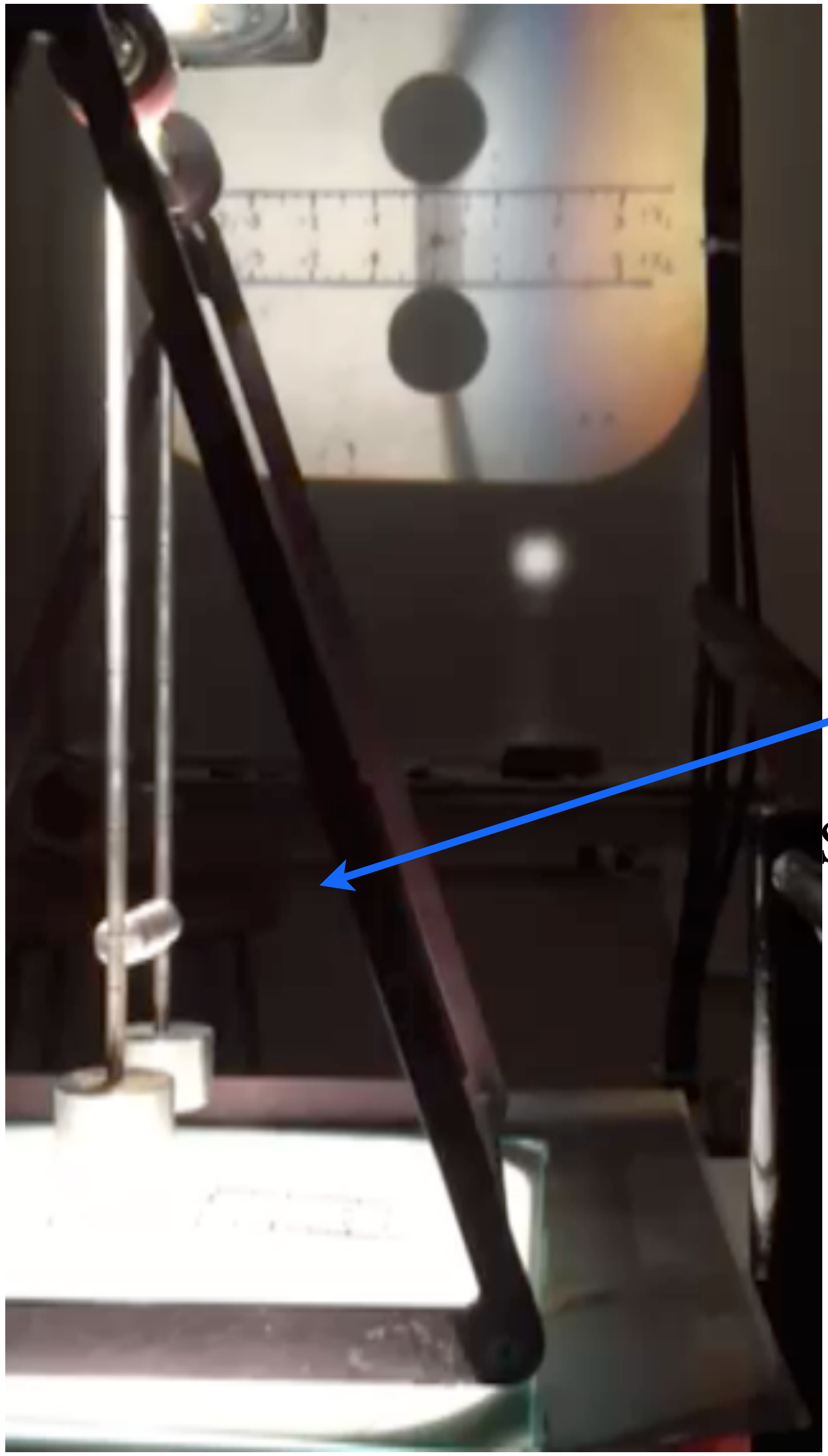


Review of 1D FDHO (Forced-Damped-Harmonic Oscillator) response

2D harmonic oscillator (2D-HO) equations of motion
Lagrangian and matrix forms

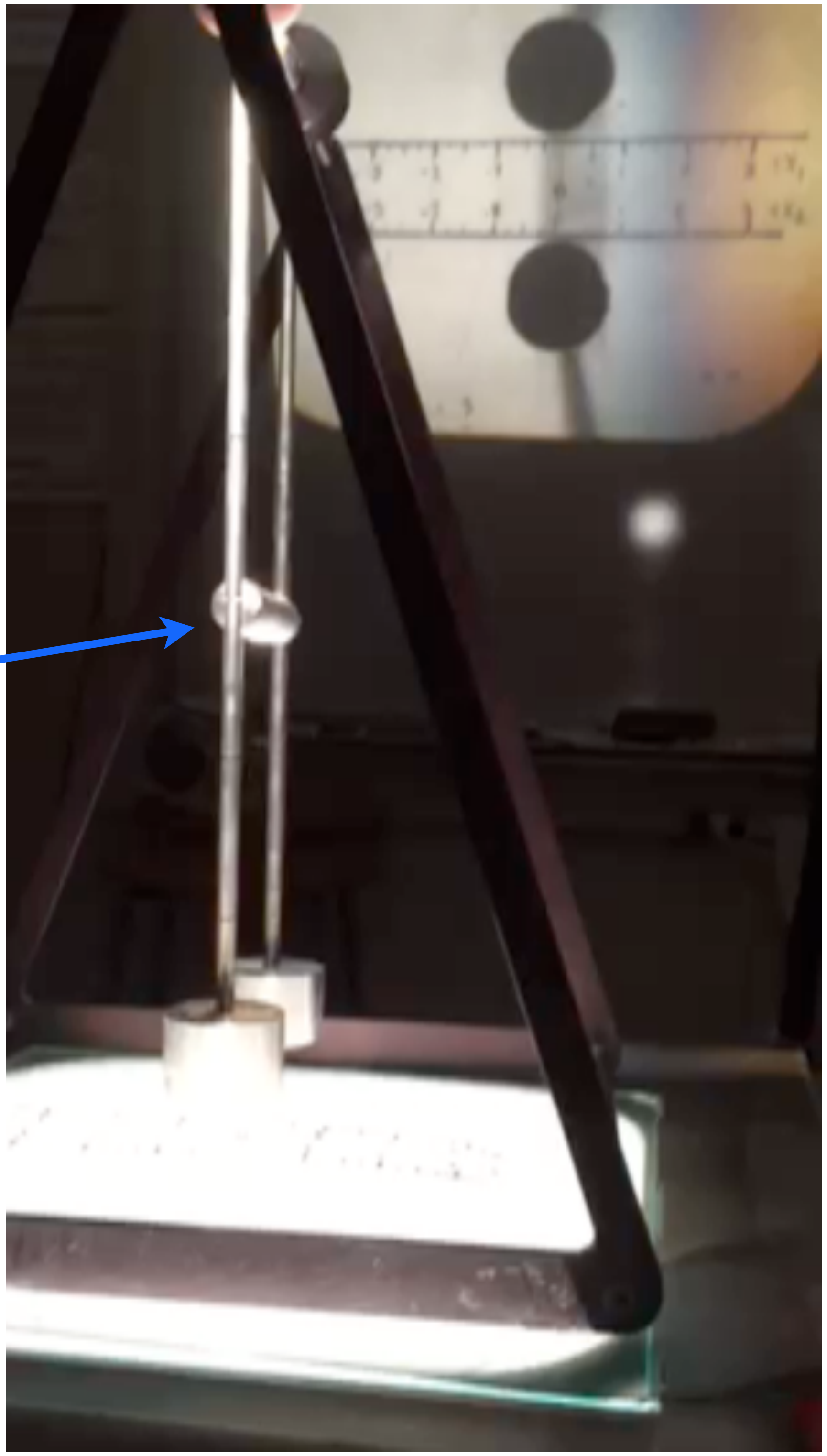
2D harmonic oscillator equation eigensolutions (normal modes)
Eigensolutions by geometry for 2D-HO with bilateral (B-Type) symmetry
Symmetric (low frequency) mode versus antisymmetric (high frequency) mode
➜ *Mixed mode beat dynamics (with constant $\pi/2$ phase-lag)*
Geometry of phase and polarization

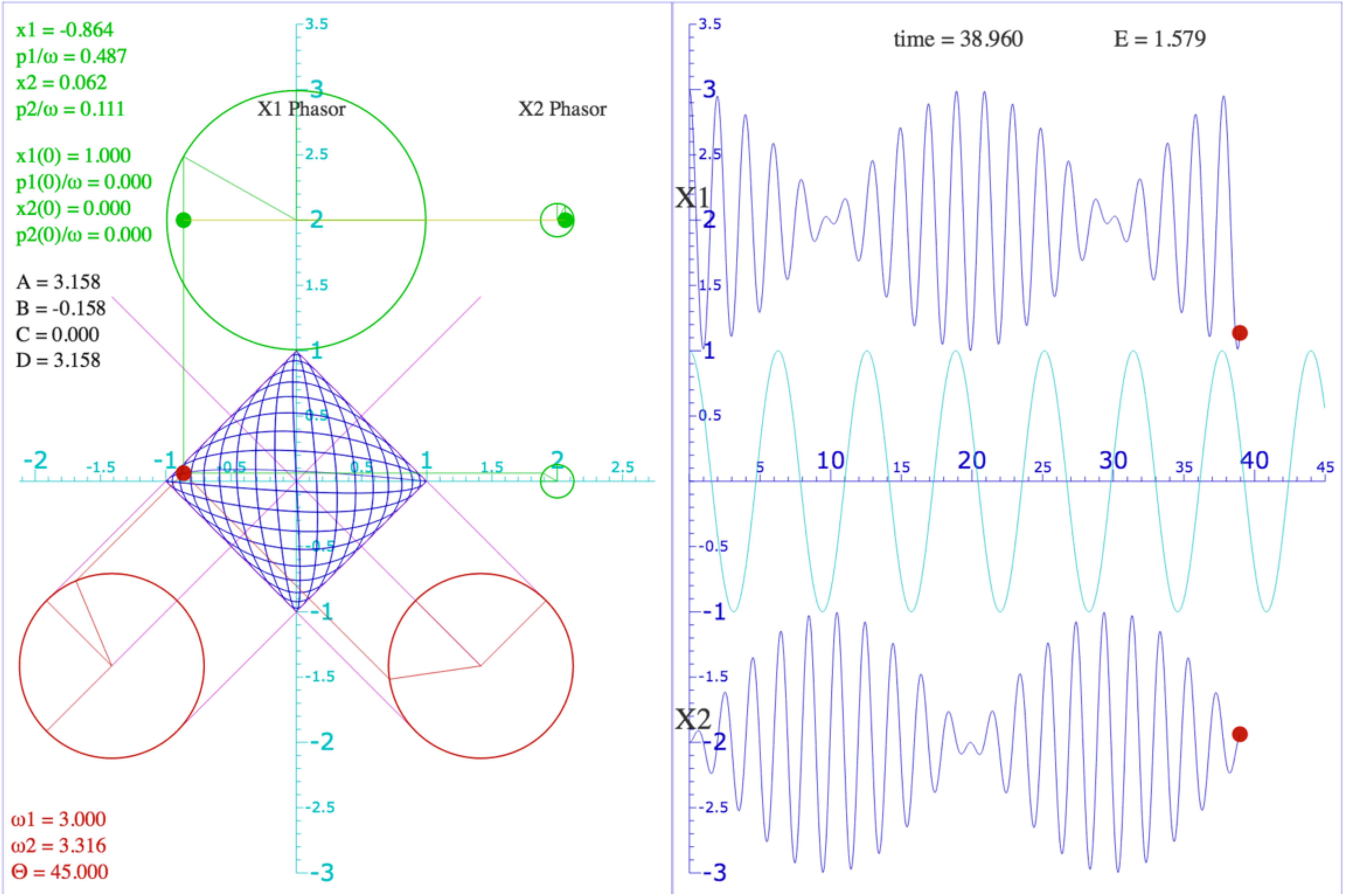
Eigensolutions by matrix-algebra with example M=
Secular equation
Hamilton-Cayley equation and projectors
Idempotent projectors (how eigenvalues⇒eigenvectors)
Operator orthonormality and Completeness (Idempotent means: **P·P=P***)*
Spectral Decompositions
Functional spectral decomposition
Orthonormality vs. Completeness vis-a`-vis Operator vs. State
Lagrange functional interpolation formula
Diagonalizing Transformations (D-Ttran) from projectors



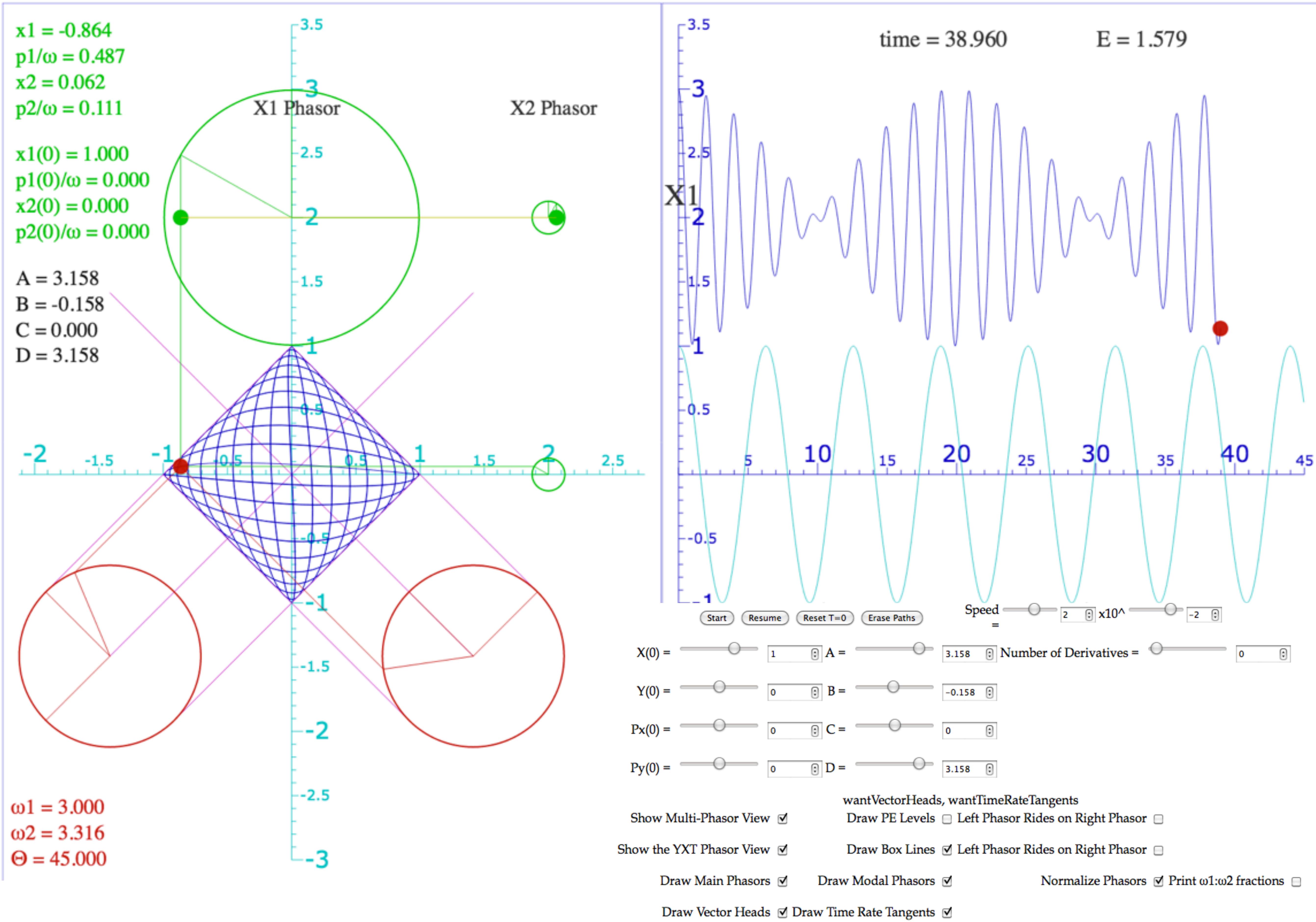
Spring set
down
for
STRONGER
coupling

Spring set
up
for
weaker
coupling

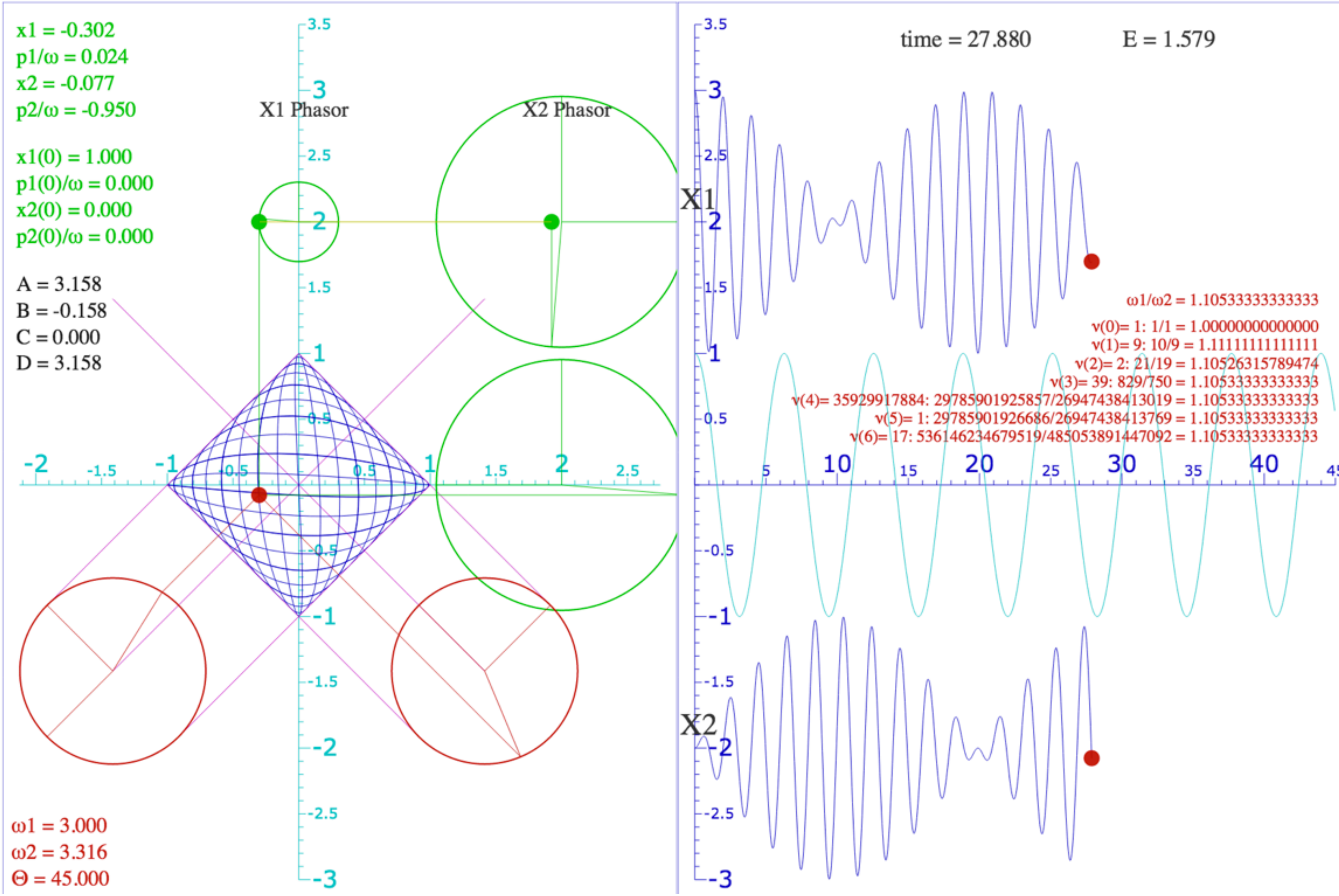




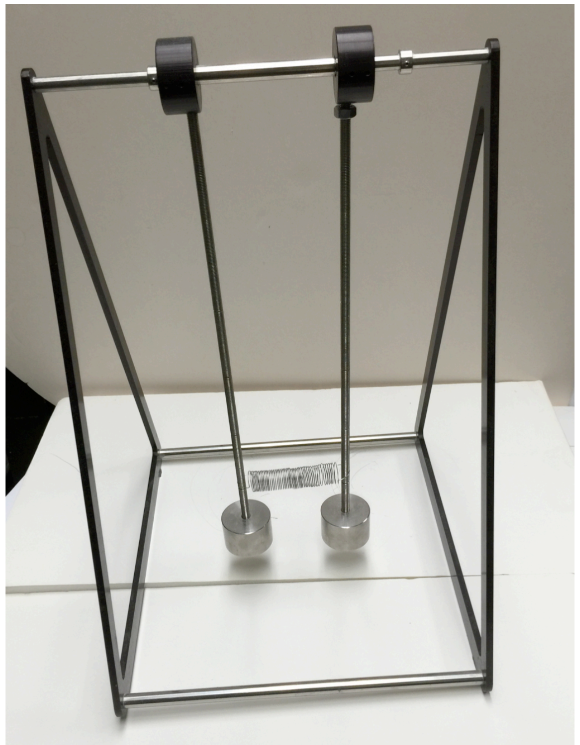
[BoxIt Web Simulation - *Coupled Oscillators Beating*]



BoxIt Web Simulation - *Coupled Oscillators Beating* *or* Generic URL: http://www.uark.edu/ua/modphys/markup/BoxItWeb.html



[BoxIt Web Simulation - *Coupled Oscillators w/Rationals*]



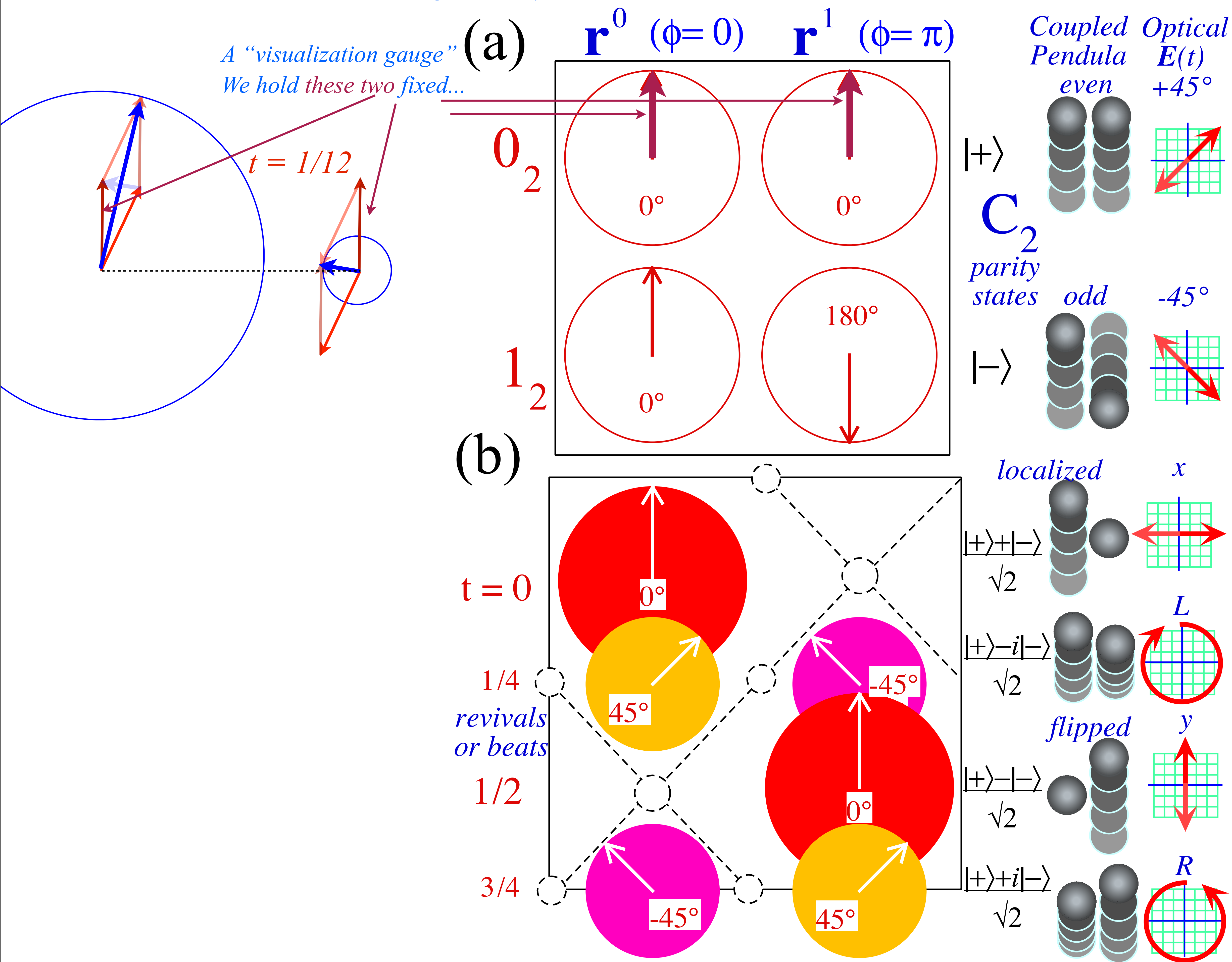
Review of 1D FDHO (Forced-Damped-Harmonic Oscillator) response

2D harmonic oscillator (2D-HO) equations of motion
Lagrangian and matrix forms

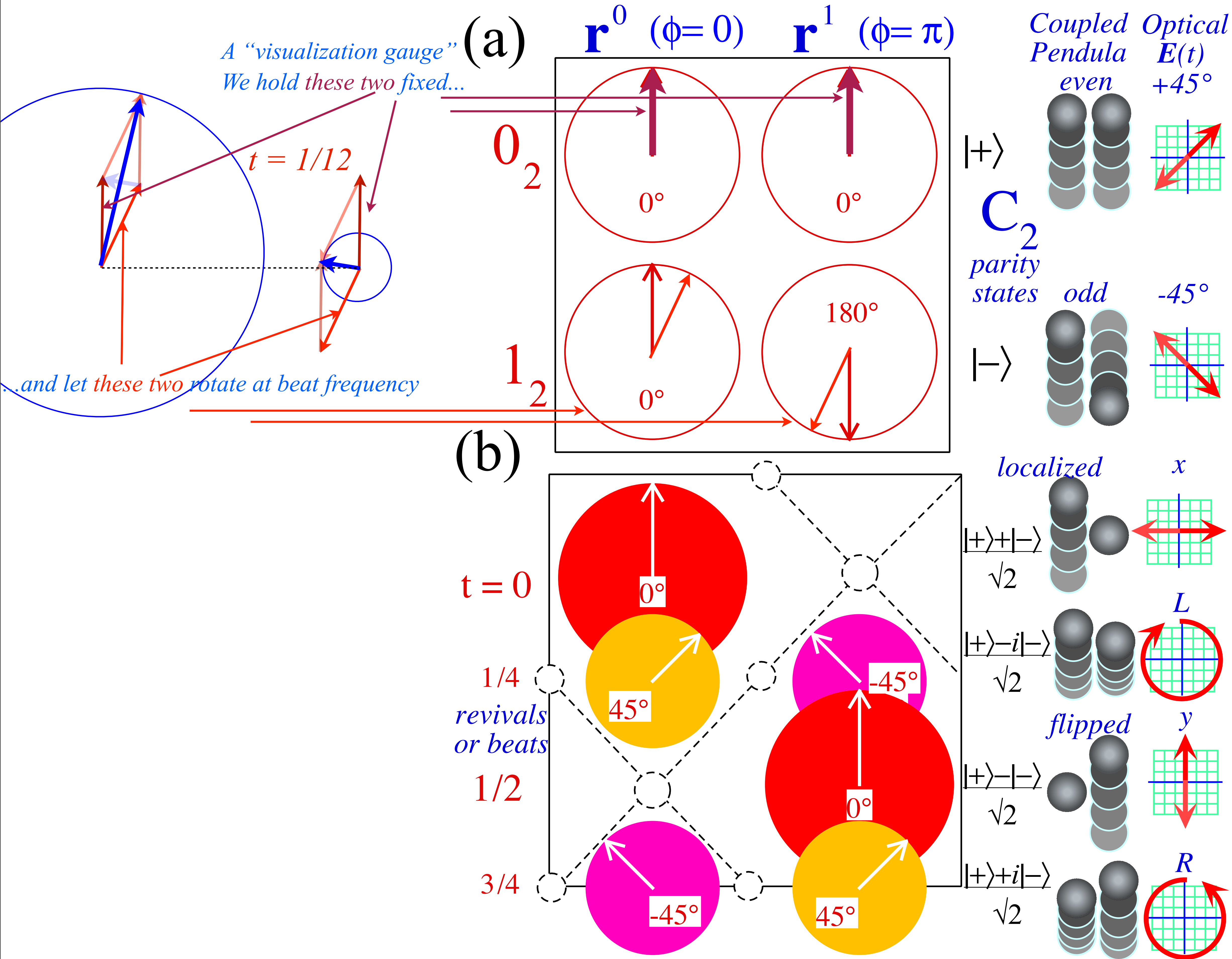
2D harmonic oscillator equation eigensolutions (normal modes)
Eigensolutions by geometry for 2D-HO with bilateral (B-Type) symmetry
Symmetric (low frequency) mode versus antisymmetric (high frequency) mode
Mixed mode beat dynamics (with constant π/2 phase-lag)
➔ *Geometry of phase and polarization*

Eigensolutions by matrix-algebra with example M=
Secular equation
Hamilton-Cayley equation and projectors
Idempotent projectors (how eigenvalues⇒eigenvectors)
Operator orthonormality and Completeness (Idempotent means: **P·P=P***)*
Spectral Decompositions
Functional spectral decomposition
Orthonormality vs. Completeness vis-a`-vis Operator vs. State
Lagrange functional interpolation formula
Diagonalizing Transformations (D-Ttran) from projectors

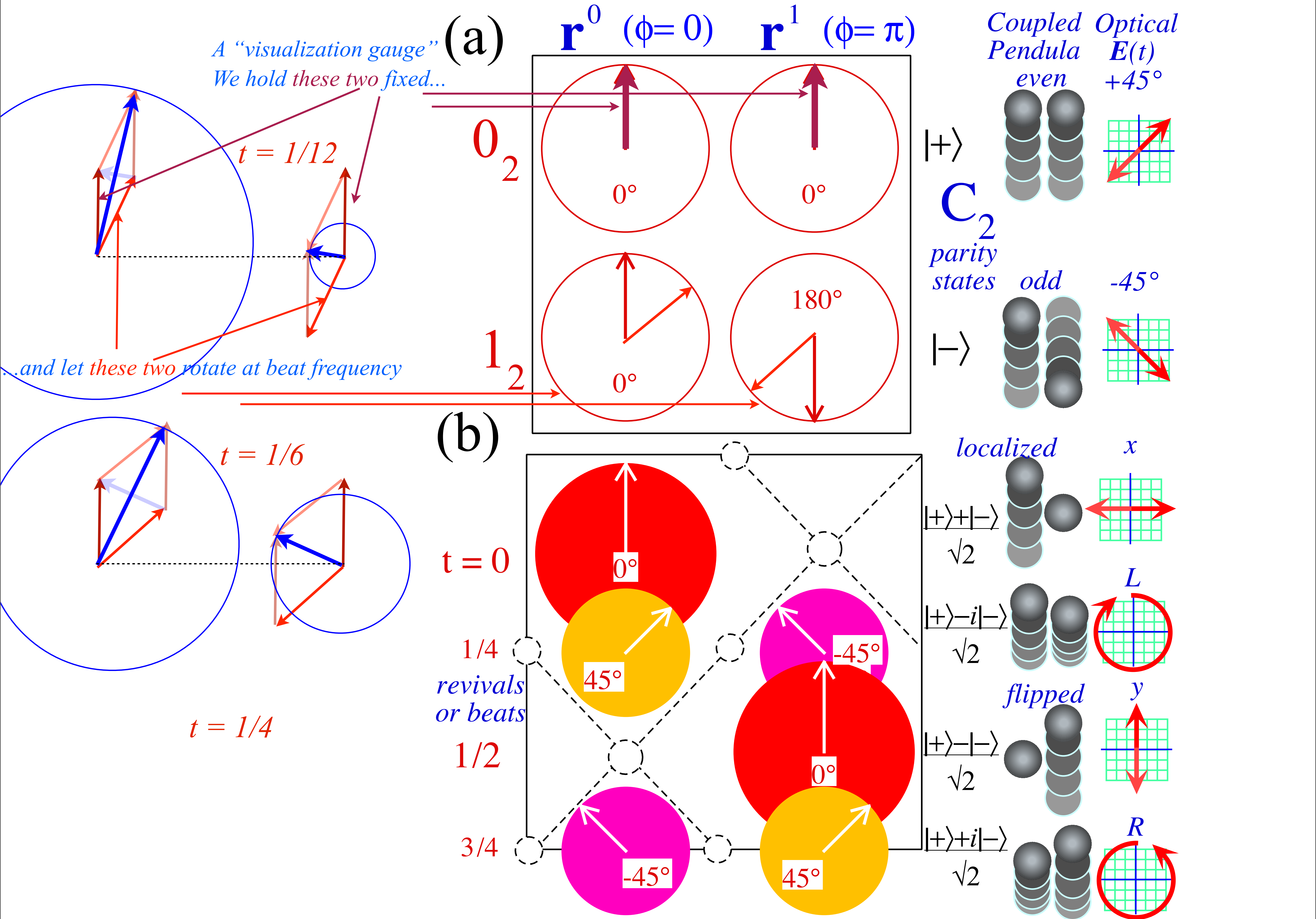
2D-HO beats and mixed mode geometry



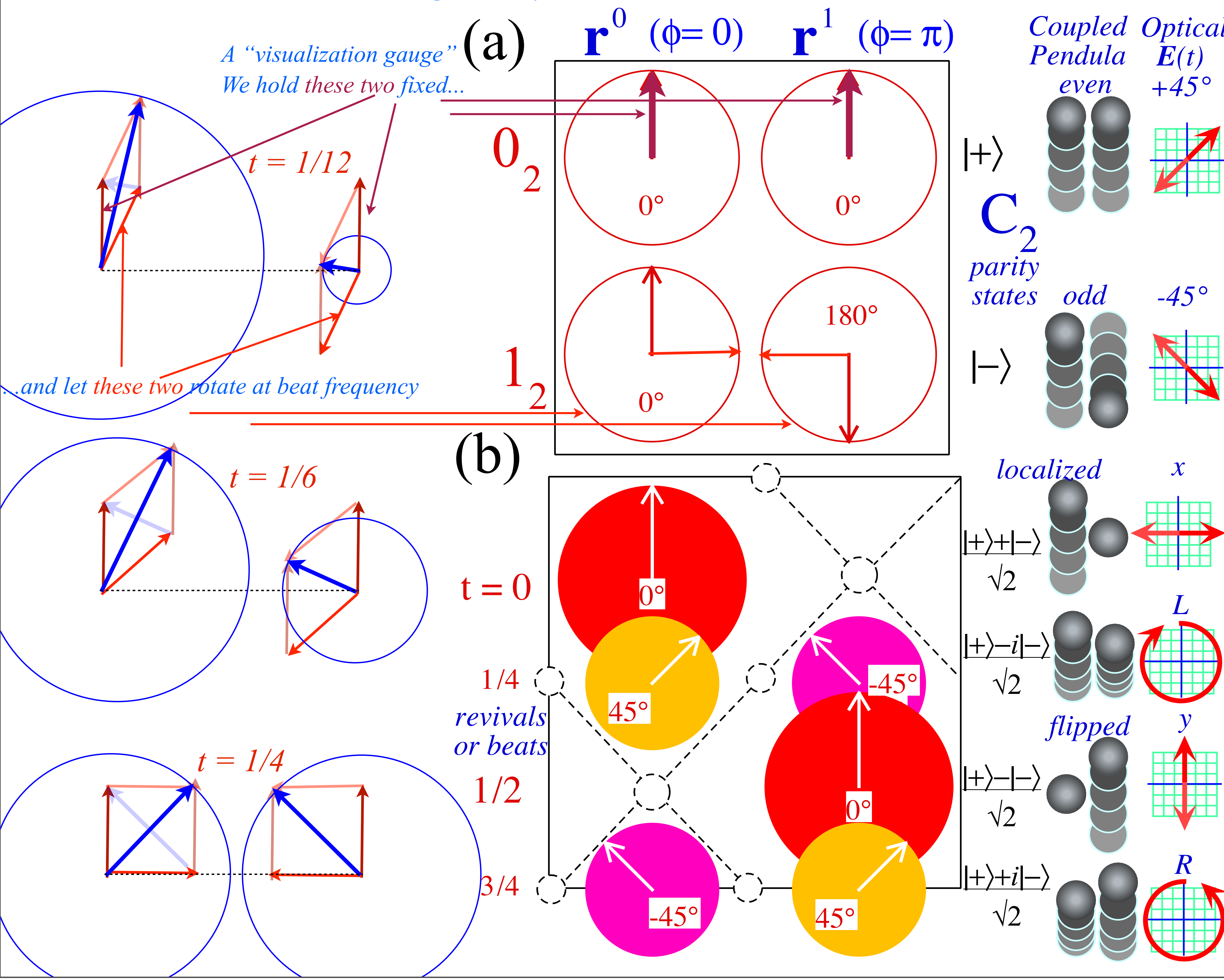
2D-HO beats and mixed mode geometry

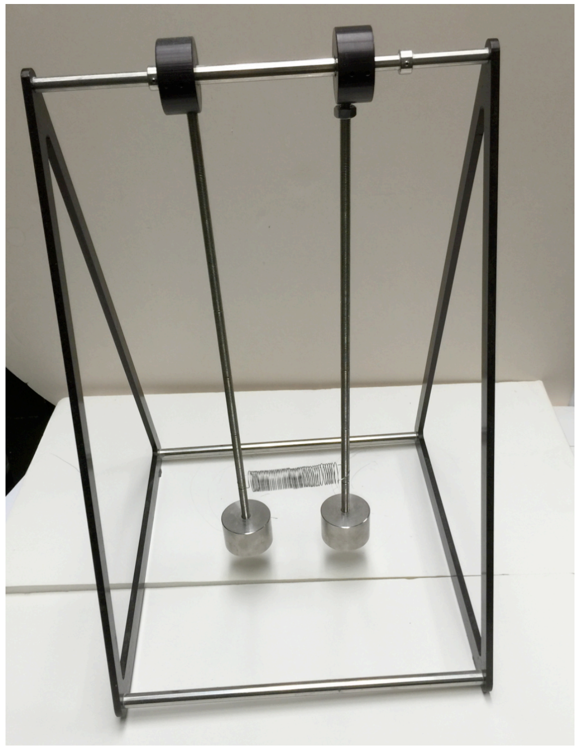


2D-HO beats and mixed mode geometry



2D-HO beats and mixed mode geometry





Review of 1D FDHO (Forced-Damped-Harmonic Oscillator) response

2D harmonic oscillator (2D-HO) equations of motion
Lagrangian and matrix forms

2D harmonic oscillator equation eigensolutions (normal modes)
Eigensolutions by geometry for 2D-HO with bilateral (B-Type) symmetry
Symmetric (low frequency) mode versus antisymmetric (high frequency) mode
Mixed mode beat dynamics (with constant π/2 phase-lag)
Geometry of phase and polarization

➡ Eigensolutions by matrix-algebra with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$
Secular equation
Hamilton-Cayley equation and projectors
Idempotent projectors (how eigenvalues⇒eigenvectors)
Operator orthonormality and Completeness (Idempotent means: $\mathbf{P}\cdot\mathbf{P}=\mathbf{P}$)
Spectral Decompositions
Functional spectral decomposition
Orthonormality vs. Completeness vis-a`-vis Operator vs. State
Lagrange functional interpolation formula
Diagonalizing Transformations (D-Ttran) from projectors

Matrix-algebraic method for finding eigenvector and eigenvalues *With example matrix* $\mathbf{M}=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

An *eigenvector* $|\varepsilon_k\rangle$ of **M** is in a direction that is left unchanged by **M**.

$$\mathbf{M}|\varepsilon_k\rangle=\varepsilon_k|\varepsilon_k\rangle,\ \text{or: } (\mathbf{M}-\varepsilon_k\mathbf{1})|\varepsilon_k\rangle=\mathbf{0}$$

$$\mathbf{M}|\varepsilon\rangle=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\varepsilon\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{or: } \begin{pmatrix} 4-\varepsilon & 1 \\ 3 & 2-\varepsilon \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

ε_k is *eigenvalue* associated with eigenvector $|\varepsilon_k\rangle$ direction.

A change of basis to $\{|\varepsilon_1\rangle,|\varepsilon_2\rangle,\cdots|\varepsilon_n\rangle\}$ called *diagonalization* gives

$$\begin{pmatrix} \langle\varepsilon_1|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_1|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_1|\mathbf{M}|\varepsilon_n\rangle \\ \langle\varepsilon_2|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_2|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_2|\mathbf{M}|\varepsilon_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\varepsilon_n|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_n|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_n|\mathbf{M}|\varepsilon_n\rangle \end{pmatrix}=\begin{pmatrix} \varepsilon_1 & 0 & \cdots & 0 \\ 0 & \varepsilon_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varepsilon_n \end{pmatrix}$$

Matrix-algebraic method for finding eigenvector and eigenvalues

With example matrix $\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}$

An *eigenvector* $|\varepsilon_k\rangle$ of **M** is in a direction that is left unchanged by **M**.

$$\mathbf{M}|\varepsilon_k\rangle=\varepsilon_k|\varepsilon_k\rangle,\ \text{ or: }\ (\mathbf{M}-\varepsilon_k\mathbf{1})|\varepsilon_k\rangle=\mathbf{0}$$

ε_k is *eigenvalue* associated with eigenvector $|\varepsilon_k\rangle$ direction.

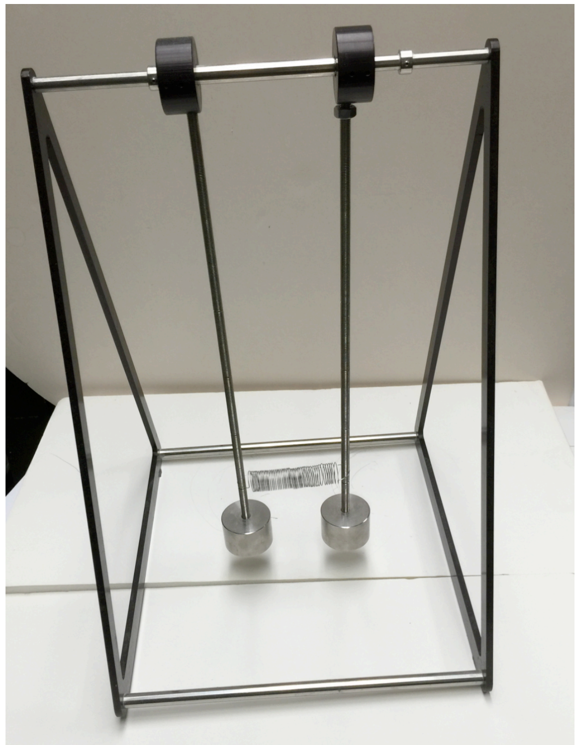
A change of basis to $\{|\varepsilon_1\rangle,|\varepsilon_2\rangle,\cdots|\varepsilon_n\rangle\}$ called *diagonalization* gives

$$\begin{pmatrix}\langle\varepsilon_1|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_1|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_1|\mathbf{M}|\varepsilon_n\rangle\\ \langle\varepsilon_2|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_2|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_2|\mathbf{M}|\varepsilon_n\rangle\\ \vdots & \vdots & \ddots & \vdots\\ \langle\varepsilon_n|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_n|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_n|\mathbf{M}|\varepsilon_n\rangle\end{pmatrix}=\begin{pmatrix}\varepsilon_1 & 0 & \cdots & 0\\ 0 & \varepsilon_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \varepsilon_n\end{pmatrix}$$

$$\mathbf{M}|\varepsilon\rangle=\begin{pmatrix}4&1\\3&2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\varepsilon\begin{pmatrix}x\\y\end{pmatrix}\ \text{ or: }\ \begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

Trying to solve by Kramer's inversion:

$$x=\frac{\det\left|\begin{pmatrix}0&1\\0&2-\varepsilon\end{pmatrix}\right|}{\det\left|\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\right|}\quad\text{and}\quad y=\frac{\det\left|\begin{pmatrix}4-\varepsilon&0\\3&0\end{pmatrix}\right|}{\det\left|\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\right|}$$



Review of 1D FDHO (Forced-Damped-Harmonic Oscillator) response

2D harmonic oscillator (2D-HO) equations of motion
Lagrangian and matrix forms

2D harmonic oscillator equation eigensolutions (normal modes)
Eigensolutions by geometry for 2D-HO with bilateral (B-Type) symmetry
Symmetric (low frequency) mode versus antisymmetric (high frequency) mode
Mixed mode beat dynamics (with constant π/2 phase-lag)
Geometry of phase and polarization

Eigensolutions by matrix-algebra with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$
➔ *Secular equation*
Hamilton-Cayley equation and projectors
Idempotent projectors (how eigenvalues⇒eigenvectors)
Operator orthonormality and Completeness (Idempotent means: **P·P=P***)*
Spectral Decompositions
Functional spectral decomposition
Orthonormality vs. Completeness vis-a`-vis Operator vs. State
Lagrange functional interpolation formula
Diagonalizing Transformations (D-Ttran) from projectors

Matrix-algebraic method for finding eigenvector and eigenvalues *With example matrix* $\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}$

An *eigenvector* $|\varepsilon_k\rangle$ of **M** is in a direction that is left unchanged by **M**.

$$\mathbf{M}|\varepsilon_k\rangle=\varepsilon_k|\varepsilon_k\rangle,\ \text{or: } (\mathbf{M}-\varepsilon_k\mathbf{1})|\varepsilon_k\rangle=\mathbf{0}$$

ε_k is *eigenvalue* associated with eigenvector $|\varepsilon_k\rangle$ direction.

A change of basis to $\{|\varepsilon_1\rangle,|\varepsilon_2\rangle,\cdots|\varepsilon_n\rangle\}$ called *diagonalization* gives

$$\begin{pmatrix}\langle\varepsilon_1|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_1|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_1|\mathbf{M}|\varepsilon_n\rangle\\ \langle\varepsilon_2|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_2|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_2|\mathbf{M}|\varepsilon_n\rangle\\ \vdots & \vdots & \ddots & \vdots\\ \langle\varepsilon_n|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_n|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_n|\mathbf{M}|\varepsilon_n\rangle\end{pmatrix}=\begin{pmatrix}\varepsilon_1 & 0 & \cdots & 0\\ 0 & \varepsilon_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \varepsilon_n\end{pmatrix}$$

First step in finding eigenvalues: Solve *secular equation*

$$\det|\mathbf{M}-\varepsilon\mathbf{1}|=0=(-1)^n\left(\varepsilon^n+a_1\varepsilon^{n-1}+a_2\varepsilon^{n-2}+\ldots+a_{n-1}\varepsilon\ +a_n\right)$$

where:

$$\boxed{a_1=-Trace\mathbf{M},\cdots,\ a_k=(-1)^k\sum\text{diagonal k-by-k minors of }\mathbf{M},\cdots,\ \ a_n=(-1)^n\det|\mathbf{M}|}$$

$$\mathbf{M}|\varepsilon\rangle=\begin{pmatrix}4&1\\3&2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\varepsilon\begin{pmatrix}x\\y\end{pmatrix}\quad\text{or: }\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

Trying to solve by Kramer's inversion:

$$x=\frac{\det\left|\begin{pmatrix}0&1\\0&2-\varepsilon\end{pmatrix}\right|}{\det\left|\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\right|}\quad\text{and}\quad y=\frac{\det\left|\begin{pmatrix}4-\varepsilon&0\\3&0\end{pmatrix}\right|}{\det\left|\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\right|}$$

Only possible non-zero $\{x,y\}$ if denominator is zero, too!

$$0=\det|\mathbf{M}-\varepsilon\cdot\mathbf{1}|=\det\left|\begin{pmatrix}4&1\\3&2\end{pmatrix}-\varepsilon\begin{pmatrix}1&0\\0&1\end{pmatrix}\right|=\det\left|\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\right|$$

$$0=(4-\varepsilon)(2-\varepsilon)-1\cdot3=8-6\varepsilon+\varepsilon^2-1\cdot3=\varepsilon^2-6\varepsilon+5$$

$$0=\varepsilon^2-Trace(\mathbf{M})\varepsilon+\det(\mathbf{M})$$

Matrix-algebraic method for finding eigenvector and eigenvalues *With example matrix* $\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}$

An *eigenvector* $|\varepsilon_k\rangle$ of **M** is in a direction that is left unchanged by **M**.

$$\mathbf{M}|\varepsilon_k\rangle=\varepsilon_k|\varepsilon_k\rangle,\ \text{ or: }\ (\mathbf{M}-\varepsilon_k\mathbf{1})|\varepsilon_k\rangle=\mathbf{0}$$

ε_k is *eigenvalue* associated with eigenvector $|\varepsilon_k\rangle$ direction.

A change of basis to $\{|\varepsilon_1\rangle,|\varepsilon_2\rangle,\cdots|\varepsilon_n\rangle\}$ called *diagonalization* gives

$$\begin{pmatrix}\langle\varepsilon_1|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_1|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_1|\mathbf{M}|\varepsilon_n\rangle\\ \langle\varepsilon_2|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_2|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_2|\mathbf{M}|\varepsilon_n\rangle\\ \vdots & \vdots & \ddots & \vdots\\ \langle\varepsilon_n|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_n|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_n|\mathbf{M}|\varepsilon_n\rangle\end{pmatrix}=\begin{pmatrix}\varepsilon_1 & 0 & \cdots & 0\\ 0 & \varepsilon_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \varepsilon_n\end{pmatrix}$$

First step in finding eigenvalues: Solve *secular equation*

$$\det|\mathbf{M}-\varepsilon\mathbf{1}|=0=(-1)^n\left(\varepsilon^n+a_1\varepsilon^{n-1}+a_2\varepsilon^{n-2}+\ldots+a_{n-1}\varepsilon\ +a_n\right)$$

where:

$$\boxed{a_1=-Trace\mathbf{M},\cdots,\ a_k=(-1)^k\sum\text{diagonal k-by-k minors of }\mathbf{M},\cdots,\ \ a_n=(-1)^n\det|\mathbf{M}|}$$

Secular equation has n-factors, one for each eigenvalue.

$$\det|\mathbf{M}-\varepsilon\mathbf{1}|=0=(-1)^n(\varepsilon-\varepsilon_1)(\varepsilon-\varepsilon_2)\cdots(\varepsilon-\varepsilon_n)$$

$$\mathbf{M}|\varepsilon\rangle=\begin{pmatrix}4&1\\3&2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\varepsilon\begin{pmatrix}x\\y\end{pmatrix}\ \text{ or: }\ \begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

Trying to solve by Kramer's inversion:

$$x=\frac{\det\left|\begin{pmatrix}0&1\\0&2-\varepsilon\end{pmatrix}\right|}{\det\left|\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\right|}\quad\text{and}\quad y=\frac{\det\left|\begin{pmatrix}4-\varepsilon&0\\3&0\end{pmatrix}\right|}{\det\left|\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\right|}$$

Only possible non-zero $\{x,y\}$ if denominator is zero, too!

$$0=\det|\mathbf{M}-\varepsilon\cdot\mathbf{1}|=\det\left|\begin{pmatrix}4&1\\3&2\end{pmatrix}-\varepsilon\begin{pmatrix}1&0\\0&1\end{pmatrix}\right|=\det\left|\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\right|$$

$$0=(4-\varepsilon)(2-\varepsilon)-1\cdot3=8-6\varepsilon+\varepsilon^2-1\cdot3=\varepsilon^2-6\varepsilon+5$$

$$0=\varepsilon^2-Trace(\mathbf{M})\varepsilon+\det(\mathbf{M})=\varepsilon^2-6\varepsilon+5$$

$$0=(\varepsilon-1)(\varepsilon-5)\ \text{ so let: }\ \varepsilon_1=1\ \text{ and: }\ \varepsilon_2=5$$

Review of 1D FDHO (Forced-Damped-Harmonic Oscillator) response

2D harmonic oscillator (2D-HO) equations of motion
Lagrangian and matrix forms

2D harmonic oscillator equation eigensolutions (normal modes)
Eigensolutions by geometry for 2D-HO with bilateral (B-Type) symmetry
Symmetric (low frequency) mode versus antisymmetric (high frequency) mode
Mixed mode beat dynamics (with constant π/2 phase-lag)
Geometry of phase and polarization

Eigensolutions by matrix-algebra with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$
Secular equation
Hamilton-Cayley equation ~~and projectors~~
Idempotent projectors (how eigenvalues⇒eigenvectors)
Operator orthonormality and Completeness (Idempotent means: **P·P=P***)*
Spectral Decompositions
Functional spectral decomposition
Orthonormality vs. Completeness vis-a`-vis Operator vs. State
Lagrange functional interpolation formula
Diagonalizing Transformations (D-Ttran) from projectors

Matrix-algebraic method for finding eigenvector and eigenvalues *With example matrix* $\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}$

An *eigenvector* $|\varepsilon_k\rangle$ of **M** is in a direction that is left unchanged by **M**.

$$\mathbf{M}|\varepsilon_k\rangle=\varepsilon_k|\varepsilon_k\rangle,\ \text{ or: }\ (\mathbf{M}-\varepsilon_k\mathbf{1})|\varepsilon_k\rangle=\mathbf{0}$$

ε_k is *eigenvalue* associated with eigenvector $|\varepsilon_k\rangle$ direction.

A change of basis to $\{|\varepsilon_1\rangle,|\varepsilon_2\rangle,\cdots|\varepsilon_n\rangle\}$ called *diagonalization* gives

$$\begin{pmatrix}\langle\varepsilon_1|\mathbf{M}|\varepsilon_1\rangle&\langle\varepsilon_1|\mathbf{M}|\varepsilon_2\rangle&\cdots&\langle\varepsilon_1|\mathbf{M}|\varepsilon_n\rangle\\\langle\varepsilon_2|\mathbf{M}|\varepsilon_1\rangle&\langle\varepsilon_2|\mathbf{M}|\varepsilon_2\rangle&\cdots&\langle\varepsilon_2|\mathbf{M}|\varepsilon_n\rangle\\\vdots&\vdots&\ddots&\vdots\\\langle\varepsilon_n|\mathbf{M}|\varepsilon_1\rangle&\langle\varepsilon_n|\mathbf{M}|\varepsilon_2\rangle&\cdots&\langle\varepsilon_n|\mathbf{M}|\varepsilon_n\rangle\end{pmatrix}=\begin{pmatrix}\varepsilon_1&0&\cdots&0\\0&\varepsilon_2&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&\varepsilon_n\end{pmatrix}$$

First step in finding eigenvalues: Solve *secular equation*

$$\det|\mathbf{M}-\varepsilon\mathbf{1}|=0=(-1)^n\left(\varepsilon^n+a_1\varepsilon^{n-1}+a_2\varepsilon^{n-2}+\ldots+a_{n-1}\varepsilon\ +a_n\right)$$

where:

$$\boxed{a_1=-Trace\,\mathbf{M},\cdots,\ a_k=(-1)^k\sum\text{diagonal k-by-k minors of }\mathbf{M},\cdots,\ \ a_n=(-1)^n\det|\mathbf{M}|}$$

Secular equation has n-factors, one for each eigenvalue.

$$\det|\mathbf{M}-\varepsilon\mathbf{1}|=0=(-1)^n(\varepsilon-\varepsilon_1)(\varepsilon-\varepsilon_2)\cdots(\varepsilon-\varepsilon_n)$$

Each ε replaced by **M** and each ε_k by ε_k **1** gives *Hamilton-Cayley* matrix equation.

$$\mathbf{0}=(\mathbf{M}-\varepsilon_1\mathbf{1})(\mathbf{M}-\varepsilon_2\mathbf{1})\cdots(\mathbf{M}-\varepsilon_n\mathbf{1})$$

Obviously true if **M** has diagonal form. (But, that's circular logic. Faith needed!)

$$\mathbf{M}|\varepsilon\rangle=\begin{pmatrix}4&1\\3&2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\varepsilon\begin{pmatrix}x\\y\end{pmatrix}\ \text{ or: }\ \begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

Trying to solve by Kramer's inversion:

$$x=\frac{\det\begin{vmatrix}\begin{pmatrix}0&1\\0&2-\varepsilon\end{pmatrix}\end{vmatrix}}{\det\begin{vmatrix}\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\end{vmatrix}}\quad\text{and}\quad y=\frac{\det\begin{vmatrix}\begin{pmatrix}4-\varepsilon&0\\3&0\end{pmatrix}\end{vmatrix}}{\det\begin{vmatrix}\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\end{vmatrix}}$$

Only possible non-zero $\{x,y\}$ if denominator is zero, too!

$$0=\det|\mathbf{M}-\varepsilon\cdot\mathbf{1}|=\det\left|\begin{pmatrix}4&1\\3&2\end{pmatrix}-\varepsilon\begin{pmatrix}1&0\\0&1\end{pmatrix}\right|=\det\left|\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\right|$$

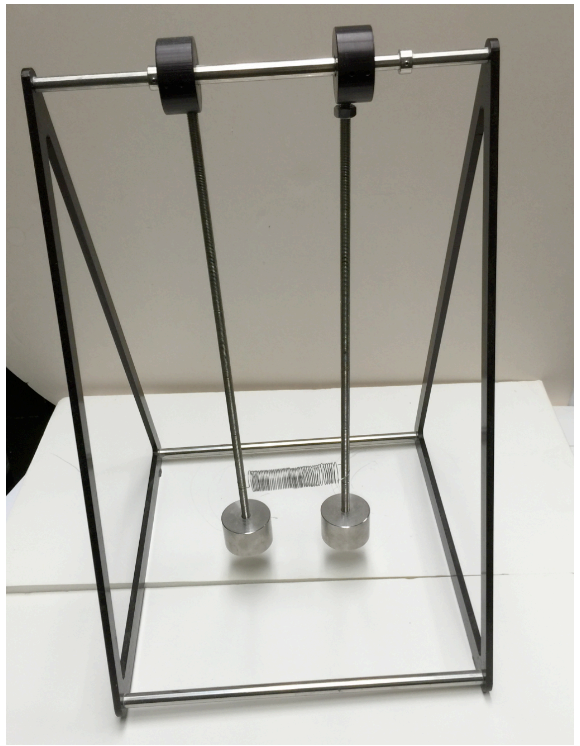
$$0=(4-\varepsilon)(2-\varepsilon)-1\cdot3=8-6\varepsilon+\varepsilon^2-1\cdot3=\varepsilon^2-6\varepsilon+5$$

$$0=\varepsilon^2-Trace(\mathbf{M})\varepsilon+\det(\mathbf{M})=\varepsilon^2-6\varepsilon+5$$

$$0=(\varepsilon-1)(\varepsilon-5)\ \text{ so let: }\ \varepsilon_1=1\ \text{ and: }\ \varepsilon_2=5$$

$$0=\mathbf{M}^2-6\mathbf{M}+5\mathbf{1}=(\mathbf{M}-1\cdot\mathbf{1})(\mathbf{M}-5\cdot\mathbf{1})$$

$$\begin{pmatrix}0&0\\0&0\end{pmatrix}=\begin{pmatrix}4&1\\3&2\end{pmatrix}^2-6\begin{pmatrix}4&1\\3&2\end{pmatrix}+5\begin{pmatrix}1&0\\0&1\end{pmatrix}$$



Review of 1D FDHO (Forced-Damped-Harmonic Oscillator) response

2D harmonic oscillator (2D-HO) equations of motion
Lagrangian and matrix forms

2D harmonic oscillator equation eigensolutions (normal modes)
Eigensolutions by geometry for 2D-HO with bilateral (B-Type) symmetry
Symmetric (low frequency) mode versus antisymmetric (high frequency) mode
Mixed mode beat dynamics (with constant π/2 phase-lag)
Geometry of phase and polarization

Eigensolutions by matrix-algebra with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$
Secular equation
Hamilton-Cayley equation and projectors
Idempotent projectors (how eigenvalues⇒eigenvectors)
Operator orthonormality and Completeness (Idempotent means: $\mathbf{P\cdot P=P}$*)*
Spectral Decompositions
Functional spectral decomposition
Orthonormality vs. Completeness vis-a`-vis Operator vs. State
Lagrange functional interpolation formula
Diagonalizing Transformations (D-Ttran) from projectors

Matrix-algebraic method for finding eigenvector and eigenvalues *With example matrix* $\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}$

An *eigenvector* $|\varepsilon_k\rangle$ of **M** is in a direction that is left unchanged by **M**.

$$\mathbf{M}|\varepsilon_k\rangle=\varepsilon_k|\varepsilon_k\rangle,\ \text{or:}\ (\mathbf{M}-\varepsilon_k\mathbf{1})|\varepsilon_k\rangle=\mathbf{0}$$

ε_k is *eigenvalue* associated with eigenvector $|\varepsilon_k\rangle$ direction.

A change of basis to $\{|\varepsilon_1\rangle,|\varepsilon_2\rangle,\cdots|\varepsilon_n\rangle\}$ called *diagonalization* gives

$$\begin{pmatrix}\langle\varepsilon_1|\mathbf{M}|\varepsilon_1\rangle&\langle\varepsilon_1|\mathbf{M}|\varepsilon_2\rangle&\cdots&\langle\varepsilon_1|\mathbf{M}|\varepsilon_n\rangle\\\langle\varepsilon_2|\mathbf{M}|\varepsilon_1\rangle&\langle\varepsilon_2|\mathbf{M}|\varepsilon_2\rangle&\cdots&\langle\varepsilon_2|\mathbf{M}|\varepsilon_n\rangle\\\vdots&\vdots&\ddots&\vdots\\\langle\varepsilon_n|\mathbf{M}|\varepsilon_1\rangle&\langle\varepsilon_n|\mathbf{M}|\varepsilon_2\rangle&\cdots&\langle\varepsilon_n|\mathbf{M}|\varepsilon_n\rangle\end{pmatrix}=\begin{pmatrix}\varepsilon_1&0&\cdots&0\\0&\varepsilon_2&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&\varepsilon_n\end{pmatrix}$$

First step in finding eigenvalues: Solve *secular equation*

$$\det|\mathbf{M}-\varepsilon\mathbf{1}|=0=(-1)^n\left(\varepsilon^n+a_1\varepsilon^{n-1}+a_2\varepsilon^{n-2}+\ldots+a_{n-1}\varepsilon\ +a_n\right)$$

where:

$$\boxed{a_1=-Trace\mathbf{M},\cdots,\ a_k=(-1)^k\sum\text{diagonal k-by-k minors of }\mathbf{M},\cdots,\ \ a_n=(-1)^n\det|\mathbf{M}|}$$

Secular equation has n-factors, one for each eigenvalue.

$$\det|\mathbf{M}-\varepsilon\mathbf{1}|=0=(-1)^n(\varepsilon-\varepsilon_1)(\varepsilon-\varepsilon_2)\cdots(\varepsilon-\varepsilon_n)$$

Each ε replaced by **M** and each ε_k by ε_k **1** gives *Hamilton-Cayley* matrix equation.

$$\mathbf{0}=(\mathbf{M}-\varepsilon_1\mathbf{1})(\mathbf{M}-\varepsilon_2\mathbf{1})\cdots(\mathbf{M}-\varepsilon_n\mathbf{1})$$

Obviously true if **M** has diagonal form. (But, that's circular logic. Faith needed!)

Replace j^{th} HC-factor by (**1**) to make *projection operators* .

$$\mathbf{p}_1=(\quad\mathbf{1}\quad)(\mathbf{M}-\varepsilon_2\mathbf{1})\cdots(\mathbf{M}-\varepsilon_n\mathbf{1})$$
$$\mathbf{p}_2=(\mathbf{M}-\varepsilon_1\mathbf{1})(\quad\mathbf{1}\quad)\cdots(\mathbf{M}-\varepsilon_n\mathbf{1})$$
$$\vdots$$
$$\mathbf{p}_n=(\mathbf{M}-\varepsilon_1\mathbf{1})(\mathbf{M}-\varepsilon_2\mathbf{1})\cdots(\quad\mathbf{1}\quad)$$

$$\mathbf{M}|\varepsilon\rangle=\begin{pmatrix}4&1\\3&2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\varepsilon\begin{pmatrix}x\\y\end{pmatrix}\quad\text{or:}\quad\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

Trying to solve by Kramer's inversion:

$$x=\frac{\det\left|\begin{pmatrix}0&1\\0&2-\varepsilon\end{pmatrix}\right|}{\det\left|\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\right|}\quad\text{and}\quad y=\frac{\det\left|\begin{pmatrix}4-\varepsilon&0\\3&0\end{pmatrix}\right|}{\det\left|\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\right|}$$

Only possible non-zero $\{x,y\}$ if denominator is zero, too!

$$0=\det|\mathbf{M}-\varepsilon\cdot\mathbf{1}|=\det\left|\begin{pmatrix}4&1\\3&2\end{pmatrix}-\varepsilon\begin{pmatrix}1&0\\0&1\end{pmatrix}\right|=\det\left|\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\right|$$

$$0=(4-\varepsilon)(2-\varepsilon)-1\cdot3=8-6\varepsilon+\varepsilon^2-1\cdot3=\varepsilon^2-6\varepsilon+5$$
$$0=\varepsilon^2-Trace(\mathbf{M})\varepsilon+\det(\mathbf{M})=\varepsilon^2-6\varepsilon+5$$

$$0=(\varepsilon-1)(\varepsilon-5)\ \text{ so let: }\ \varepsilon_1=1\ \text{ and: }\ \varepsilon_2=5$$

$$0=\mathbf{M}^2-6\mathbf{M}+5\mathbf{1}=(\mathbf{M}-1\cdot\mathbf{1})(\mathbf{M}-5\cdot\mathbf{1})$$

$$\begin{pmatrix}0&0\\0&0\end{pmatrix}=\begin{pmatrix}4&1\\3&2\end{pmatrix}^2-6\begin{pmatrix}4&1\\3&2\end{pmatrix}+5\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$\mathbf{p}_1=(\mathbf{1})(\mathbf{M}-5\cdot\mathbf{1})=\begin{pmatrix}4-5&1\\3&2-5\end{pmatrix}=\begin{pmatrix}-1&1\\3&-3\end{pmatrix}$$

$$\mathbf{p}_2=(\mathbf{M}-1\cdot\mathbf{1})(\mathbf{1})=\begin{pmatrix}4-1&1\\3&2-1\end{pmatrix}=\begin{pmatrix}3&1\\3&1\end{pmatrix}$$

Matrix-algebraic method for finding eigenvector and eigenvalues

With example matrix $\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}$

An *eigenvector* $|\varepsilon_k\rangle$ of $\mathbf{M}$ is in a direction that is left unchanged by $\mathbf{M}$.

$$\mathbf{M}|\varepsilon_k\rangle=\varepsilon_k|\varepsilon_k\rangle,\ \text{ or: }\ (\mathbf{M}-\varepsilon_k\mathbf{1})|\varepsilon_k\rangle=\mathbf{0}$$

ε_k is *eigenvalue* associated with eigenvector $|\varepsilon_k\rangle$ direction.

A change of basis to $\{|\varepsilon_1\rangle,|\varepsilon_2\rangle,\cdots|\varepsilon_n\rangle\}$ called *diagonalization* gives

$$\begin{pmatrix}\langle\varepsilon_1|\mathbf{M}|\varepsilon_1\rangle&\langle\varepsilon_1|\mathbf{M}|\varepsilon_2\rangle&\cdots&\langle\varepsilon_1|\mathbf{M}|\varepsilon_n\rangle\\\langle\varepsilon_2|\mathbf{M}|\varepsilon_1\rangle&\langle\varepsilon_2|\mathbf{M}|\varepsilon_2\rangle&\cdots&\langle\varepsilon_2|\mathbf{M}|\varepsilon_n\rangle\\\vdots&\vdots&\ddots&\vdots\\\langle\varepsilon_n|\mathbf{M}|\varepsilon_1\rangle&\langle\varepsilon_n|\mathbf{M}|\varepsilon_2\rangle&\cdots&\langle\varepsilon_n|\mathbf{M}|\varepsilon_n\rangle\end{pmatrix}=\begin{pmatrix}\varepsilon_1&0&\cdots&0\\0&\varepsilon_2&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&\varepsilon_n\end{pmatrix}$$

First step in finding eigenvalues: Solve *secular equation*

$$\det|\mathbf{M}-\varepsilon\mathbf{1}|=0=(-1)^n\left(\varepsilon^n+a_1\varepsilon^{n-1}+a_2\varepsilon^{n-2}+\ldots+a_{n-1}\varepsilon\ +a_n\right)$$

where:

$$a_1=-Trace\mathbf{M},\cdots,\ a_k=(-1)^k\sum\text{diagonal k-by-k minors of }\mathbf{M},\cdots,\ \ a_n=(-1)^n\det|\mathbf{M}|$$

Secular equation has n-factors, one for each eigenvalue.

$$\det|\mathbf{M}-\varepsilon\mathbf{1}|=0=(-1)^n(\varepsilon-\varepsilon_1)(\varepsilon-\varepsilon_2)\cdots(\varepsilon-\varepsilon_n)$$

Each ε replaced by $\mathbf{M}$ and each ε_k by $\varepsilon_k\mathbf{1}$ gives *Hamilton-Cayley* matrix equation.

$$\mathbf{0}=(\mathbf{M}-\varepsilon_1\mathbf{1})(\mathbf{M}-\varepsilon_2\mathbf{1})\cdots(\mathbf{M}-\varepsilon_n\mathbf{1})$$

Obviously true if $\mathbf{M}$ has diagonal form. (But, that's circular logic. Faith needed!)

Replace j^{th} HC-factor by $(\mathbf{1})$ to make *projection operators* $\mathbf{p}_k=\prod_{j\neq k}\left(\mathbf{M}-\varepsilon_j\mathbf{1}\right)$

$$\mathbf{p}_1=(\quad\mathbf{1}\quad)(\mathbf{M}-\varepsilon_2\mathbf{1})\cdots(\mathbf{M}-\varepsilon_n\mathbf{1})$$
$$\mathbf{p}_2=(\mathbf{M}-\varepsilon_1\mathbf{1})(\quad\mathbf{1}\quad)\cdots(\mathbf{M}-\varepsilon_n\mathbf{1})$$
$$\vdots$$
$$\mathbf{p}_n=(\mathbf{M}-\varepsilon_1\mathbf{1})(\mathbf{M}-\varepsilon_2\mathbf{1})\cdots(\quad\mathbf{1}\quad)$$

(Assume distinct e-values here: *Non-degeneracy clause*)
$\varepsilon_j\neq\varepsilon_k\neq\ldots$

Each $\mathbf{p}_k$ contains *eigen-bra-kets* since: $(\mathbf{M}-\varepsilon_k\mathbf{1})\mathbf{p}_k=0$ or: $\mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k=\mathbf{p}_k\mathbf{M}$.

$$\mathbf{M}|\varepsilon\rangle=\begin{pmatrix}4&1\\3&2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\varepsilon\begin{pmatrix}x\\y\end{pmatrix}\ \text{ or: }\ \begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

Trying to solve by Kramer's inversion:

$$x=\frac{\det\left|\begin{pmatrix}0&1\\0&2-\varepsilon\end{pmatrix}\right|}{\det\left|\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\right|}\quad\text{and}\quad y=\frac{\det\left|\begin{pmatrix}4-\varepsilon&0\\3&0\end{pmatrix}\right|}{\det\left|\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\right|}$$

Only possible non-zero $\{x,y\}$ if denominator is zero, too!

$$0=\det|\mathbf{M}-\varepsilon\cdot\mathbf{1}|=\det\left|\begin{pmatrix}4&1\\3&2\end{pmatrix}-\varepsilon\begin{pmatrix}1&0\\0&1\end{pmatrix}\right|=\det\left|\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\right|$$

$$0=(4-\varepsilon)(2-\varepsilon)-1\cdot3=8-6\varepsilon+\varepsilon^2-1\cdot3=\varepsilon^2-6\varepsilon+5$$
$$0=\varepsilon^2-Trace(\mathbf{M})\varepsilon+\det(\mathbf{M})=\varepsilon^2-6\varepsilon+5$$

$$0=(\varepsilon-1)(\varepsilon-5)\ \text{ so let: }\ \varepsilon_1=1\ \text{ and: }\ \varepsilon_2=5$$

$$0=\mathbf{M}^2-6\mathbf{M}+5\mathbf{1}=(\mathbf{M}-1\cdot\mathbf{1})(\mathbf{M}-5\cdot\mathbf{1})$$

$$\begin{pmatrix}0&0\\0&0\end{pmatrix}=\begin{pmatrix}4&1\\3&2\end{pmatrix}^2-6\begin{pmatrix}4&1\\3&2\end{pmatrix}+5\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$\mathbf{p}_1=(\mathbf{1})(\mathbf{M}-5\cdot\mathbf{1})=\begin{pmatrix}4-5&1\\3&2-5\end{pmatrix}=\begin{pmatrix}-1&1\\3&-3\end{pmatrix}$$

$$\mathbf{p}_2=(\mathbf{M}-1\cdot\mathbf{1})(\mathbf{1})=\begin{pmatrix}4-1&1\\3&2-1\end{pmatrix}=\begin{pmatrix}3&1\\3&1\end{pmatrix}$$

$$\mathbf{M}\mathbf{p}_1=\begin{pmatrix}4&1\\3&2\end{pmatrix}\cdot\begin{pmatrix}-1&1\\3&-3\end{pmatrix}=1\cdot\begin{pmatrix}-1&1\\3&-3\end{pmatrix}=1\cdot\mathbf{p}_1$$

$$\mathbf{M}\mathbf{p}_2=\begin{pmatrix}4&1\\3&2\end{pmatrix}\cdot\begin{pmatrix}3&1\\3&1\end{pmatrix}=5\cdot\begin{pmatrix}3&1\\3&1\end{pmatrix}=5\cdot\mathbf{p}_2$$

Matrix-algebraic method for finding eigenvector and eigenvalues *With example matrix* $\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}$

An *eigenvector* $|\varepsilon_k\rangle$ of $\mathbf{M}$ is in a direction that is left unchanged by $\mathbf{M}$.

$$\mathbf{M}|\varepsilon_k\rangle=\varepsilon_k|\varepsilon_k\rangle,\ \text{or: } (\mathbf{M}-\varepsilon_k\mathbf{1})|\varepsilon_k\rangle=\mathbf{0}$$

ε_k is *eigenvalue* associated with eigenvector $|\varepsilon_k\rangle$ direction.

A change of basis to $\{|\varepsilon_1\rangle,|\varepsilon_2\rangle,\cdots|\varepsilon_n\rangle\}$ called *diagonalization* gives

$$\begin{pmatrix}\langle\varepsilon_1|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_1|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_1|\mathbf{M}|\varepsilon_n\rangle\\ \langle\varepsilon_2|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_2|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_2|\mathbf{M}|\varepsilon_n\rangle\\ \vdots & \vdots & \ddots & \vdots\\ \langle\varepsilon_n|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_n|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_n|\mathbf{M}|\varepsilon_n\rangle\end{pmatrix}=\begin{pmatrix}\varepsilon_1&0&\cdots&0\\0&\varepsilon_2&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&\varepsilon_n\end{pmatrix}$$

First step in finding eigenvalues: Solve *secular equation*

$$\det|\mathbf{M}-\varepsilon\mathbf{1}|=0=(-1)^n\left(\varepsilon^n+a_1\varepsilon^{n-1}+a_2\varepsilon^{n-2}+\ldots+a_{n-1}\varepsilon\ +a_n\right)$$

where:

$$a_1=-Trace\mathbf{M},\cdots,\ a_k=(-1)^k\sum\text{diagonal k-by-k minors of }\mathbf{M},\cdots,\ \ a_n=(-1)^n\det|\mathbf{M}|$$

Secular equation has n-factors, one for each eigenvalue.

$$\det|\mathbf{M}-\varepsilon\mathbf{1}|=0=(-1)^n(\varepsilon-\varepsilon_1)(\varepsilon-\varepsilon_2)\cdots(\varepsilon-\varepsilon_n)$$

Each ε replaced by $\mathbf{M}$ and each ε_k by $\varepsilon_k\mathbf{1}$ gives *Hamilton-Cayley* matrix equation.

$$\mathbf{0}=(\mathbf{M}-\varepsilon_1\mathbf{1})(\mathbf{M}-\varepsilon_2\mathbf{1})\cdots(\mathbf{M}-\varepsilon_n\mathbf{1})$$

Obviously true if $\mathbf{M}$ has diagonal form. (But, that's circular logic. Faith needed!)

Replace j^{th} HC-factor by $(\mathbf{1})$ to make *projection operators* $\mathbf{p}_k=\prod_{j\neq k}(\mathbf{M}-\varepsilon_j\mathbf{1})$

$$\mathbf{p}_1=(\quad\mathbf{1}\quad)(\mathbf{M}-\varepsilon_2\mathbf{1})\cdots(\mathbf{M}-\varepsilon_n\mathbf{1})$$
$$\mathbf{p}_2=(\mathbf{M}-\varepsilon_1\mathbf{1})(\quad\mathbf{1}\quad)\cdots(\mathbf{M}-\varepsilon_n\mathbf{1})$$
$$\vdots$$
$$\mathbf{p}_n=(\mathbf{M}-\varepsilon_1\mathbf{1})(\mathbf{M}-\varepsilon_2\mathbf{1})\cdots(\quad\mathbf{1}\quad)$$

(Assume distinct e-values here: *Non-degeneracy clause*)
$\varepsilon_j\neq\varepsilon_k\neq\ldots$

Notice $\mathbf{p}_k$ *commutes with* $\mathbf{M}$,..

Each $\mathbf{p}_k$ contains *eigen-bra-kets* since: $(\mathbf{M}-\varepsilon_k\mathbf{1})\mathbf{p}_k=0$ or: $\mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k=\mathbf{p}_k\mathbf{M}$.

since $\mathbf{M}^1$, $\mathbf{M}^2$,..*commute with* $\mathbf{M}$.

$$\mathbf{M}|\varepsilon\rangle=\begin{pmatrix}4&1\\3&2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\varepsilon\begin{pmatrix}x\\y\end{pmatrix}\quad\text{or: }\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

Trying to solve by Kramer's inversion:

$$x=\frac{\det\left|\begin{pmatrix}0&1\\0&2-\varepsilon\end{pmatrix}\right|}{\det\left|\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\right|}\quad\text{and}\quad y=\frac{\det\left|\begin{pmatrix}4-\varepsilon&0\\3&0\end{pmatrix}\right|}{\det\left|\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\right|}$$

Only possible non-zero $\{x,y\}$ if denominator is zero, too!

$$0=\det|\mathbf{M}-\varepsilon\cdot\mathbf{1}|=\det\left|\begin{pmatrix}4&1\\3&2\end{pmatrix}-\varepsilon\begin{pmatrix}1&0\\0&1\end{pmatrix}\right|=\det\left|\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\right|$$

$$0=(4-\varepsilon)(2-\varepsilon)-1\cdot3=8-6\varepsilon+\varepsilon^2-1\cdot3=\varepsilon^2-6\varepsilon+5$$
$$0=\varepsilon^2-Trace(\mathbf{M})\varepsilon+\det(\mathbf{M})=\varepsilon^2-6\varepsilon+5$$

$$0=(\varepsilon-1)(\varepsilon-5)\ \text{ so let: }\ \varepsilon_1=1\ \text{ and: }\ \varepsilon_2=5$$

$$0=\mathbf{M}^2-6\mathbf{M}+5\mathbf{1}=(\mathbf{M}-1\cdot\mathbf{1})(\mathbf{M}-5\cdot\mathbf{1})$$

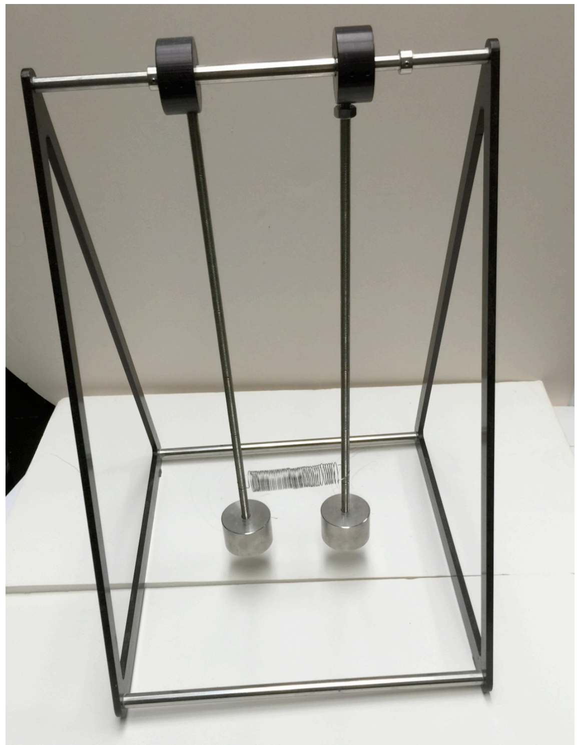
$$\begin{pmatrix}0&0\\0&0\end{pmatrix}=\begin{pmatrix}4&1\\3&2\end{pmatrix}^2-6\begin{pmatrix}4&1\\3&2\end{pmatrix}+5\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$\mathbf{p}_1=(\mathbf{1})(\mathbf{M}-5\cdot\mathbf{1})=\begin{pmatrix}4-5&1\\3&2-5\end{pmatrix}=\begin{pmatrix}-1&1\\3&-3\end{pmatrix}$$

$$\mathbf{p}_2=(\mathbf{M}-1\cdot\mathbf{1})(\mathbf{1})=\begin{pmatrix}4-1&1\\3&2-1\end{pmatrix}=\begin{pmatrix}3&1\\3&1\end{pmatrix}$$

$$\mathbf{M}\mathbf{p}_1=\begin{pmatrix}4&1\\3&2\end{pmatrix}\cdot\begin{pmatrix}-1&1\\3&-3\end{pmatrix}=1\cdot\begin{pmatrix}-1&1\\3&-3\end{pmatrix}=1\cdot\mathbf{p}_1$$

$$\mathbf{M}\mathbf{p}_2=\begin{pmatrix}4&1\\3&2\end{pmatrix}\cdot\begin{pmatrix}3&1\\3&1\end{pmatrix}=5\cdot\begin{pmatrix}3&1\\3&1\end{pmatrix}=5\cdot\mathbf{p}_2$$



Review of 1D FDHO (Forced-Damped-Harmonic Oscillator) response

2D harmonic oscillator (2D-HO) equations of motion
Lagrangian and matrix forms

2D harmonic oscillator equation eigensolutions (normal modes)
Eigensolutions by geometry for 2D-HO with bilateral (B-Type) symmetry
Symmetric (low frequency) mode versus antisymmetric (high frequency) mode
Mixed mode beat dynamics (with constant π/2 phase-lag)
Geometry of phase and polarization

Eigensolutions by matrix-algebra with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$
Secular equation
Hamilton-Cayley equation and projectors
Idempotent projectors ~~*(low eigenvalues→eigenvectors)*~~
Operator orthonormality and Completeness (Idempotent means: **P·P=P**)
Spectral Decompositions
Functional spectral decomposition
Orthonormality vs. Completeness vis-a`-vis Operator vs. State
Lagrange functional interpolation formula
Diagonalizing Transformations (D-Ttran) from projectors

Matrix-algebraic method for finding eigenvector and eigenvalues

$$\mathbf{p}_j\mathbf{p}_k = \mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1}) = \prod_{m\neq k}\left(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1}\right) \qquad \mathbf{M}\mathbf{p}_k = \varepsilon_k\mathbf{p}_k = \mathbf{p}_k\mathbf{M}$$

Multiplication properties of $\mathbf{p}_j$:

$$\mathbf{p}_j\mathbf{p}_k = \prod_{m\neq k}\left(\varepsilon_j\mathbf{p}_j - \varepsilon_m\mathbf{p}_j\right) = \mathbf{p}_j\prod_{m\neq k}\left(\varepsilon_j - \varepsilon_m\right) = \begin{cases} \mathbf{0} & \text{if} : j\neq k \\ \mathbf{p}_k\prod_{m\neq k}\left(\varepsilon_k-\varepsilon_m\right) & \text{if} : j=k \end{cases}$$

With example matrix $\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

$$\mathbf{p}_1 = (\mathbf{M}-5\cdot\mathbf{1}) = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}$$

$$\mathbf{p}_2 = (\mathbf{M}-1\cdot\mathbf{1}) = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$

$$\mathbf{p}_1\mathbf{p}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Matrix-algebraic method for finding eigenvector and eigenvalues

$$\mathbf{p}_j\mathbf{p}_k = \mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1}) = \prod_{m\neq k}\left(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1}\right) \qquad \mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k = \mathbf{p}_k\mathbf{M}$$

Multiplication properties of $\mathbf{p}_j$:

$$\mathbf{p}_j\mathbf{p}_k = \prod_{m\neq k}\left(\varepsilon_j\mathbf{p}_j - \varepsilon_m\mathbf{p}_j\right) = \mathbf{p}_j\prod_{m\neq k}\left(\varepsilon_j-\varepsilon_m\right) = \begin{cases} \mathbf{0} & if : j\neq k \\ \mathbf{p}_k\prod_{m\neq k}\left(\varepsilon_k-\varepsilon_m\right) & if : j=k \end{cases}$$

Last step:

make *Idempotent Projectors:* $\mathbf{P}_k = \dfrac{\mathbf{p}_k}{\prod_{m\neq k}\left(\varepsilon_k-\varepsilon_m\right)} = \dfrac{\prod_{m\neq k}\left(\mathbf{M}-\varepsilon_m\mathbf{1}\right)}{\prod_{m\neq k}\left(\varepsilon_k-\varepsilon_m\right)}$

(*Idempotent means:* **P·P=P**)

With example matrix $\mathbf{M}=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

$$\mathbf{p}_1 = (\mathbf{M}-5\cdot\mathbf{1}) = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}$$

$$\mathbf{p}_2 = (\mathbf{M}-1\cdot\mathbf{1}) = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$

$$\mathbf{p}_1\mathbf{p}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathbf{P}_1 = \frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix}$$

$$\mathbf{P}_2 = \frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)} = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$

Matrix-algebraic method for finding eigenvector and eigenvalues

$$\mathbf{p}_j\mathbf{p}_k = \mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1}) = \prod_{m\neq k}\left(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1}\right) \qquad \mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k=\mathbf{p}_k\mathbf{M}$$

Multiplication properties of $\mathbf{p}_j$:

$$\mathbf{p}_j\mathbf{p}_k = \prod_{m\neq k}\left(\varepsilon_j\mathbf{p}_j-\varepsilon_m\mathbf{p}_j\right) = \mathbf{p}_j\prod_{m\neq k}\left(\varepsilon_j-\varepsilon_m\right) = \begin{cases} \mathbf{0} & \mathit{if}: j\neq k \\ \mathbf{p}_k\prod_{m\neq k}\left(\varepsilon_k-\varepsilon_m\right) & \mathit{if}: j=k \end{cases}$$

Last step:

make *Idempotent Projectors:* $\mathbf{P}_k = \dfrac{\mathbf{p}_k}{\prod_{m\neq k}\left(\varepsilon_k-\varepsilon_m\right)} = \dfrac{\prod_{m\neq k}\left(\mathbf{M}-\varepsilon_m\mathbf{1}\right)}{\prod_{m\neq k}\left(\varepsilon_k-\varepsilon_m\right)}$

(*Idempotent means:* **P·P=P**)

$$\mathbf{P}_j\mathbf{P}_k = \begin{cases} \mathbf{0} & \mathit{if}: j\neq k \\ \mathbf{P}_k & \mathit{if}: j=k \end{cases}$$

$\mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k=\mathbf{p}_k\mathbf{M}$

implies:

$\mathbf{M}\mathbf{P}_k=\varepsilon_k\mathbf{P}_k=\mathbf{P}_k\mathbf{M}$

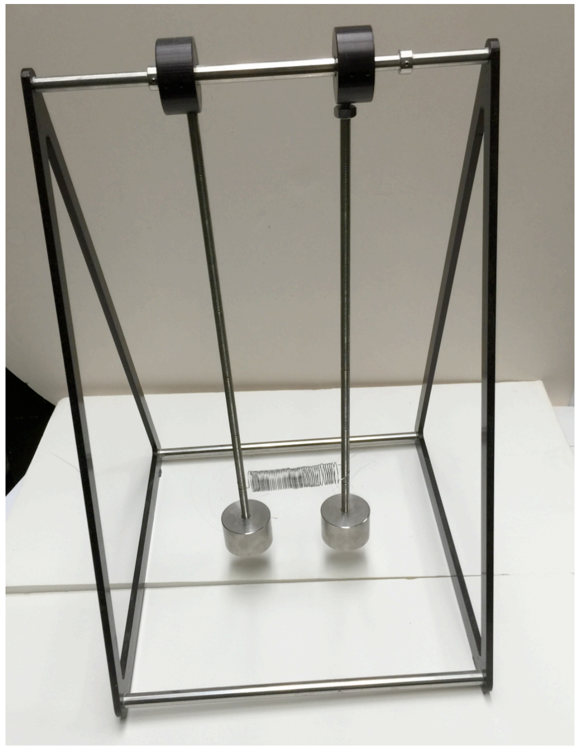
With example matrix $\mathbf{M}=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

$$\mathbf{p}_1 = (\mathbf{M}-5\cdot\mathbf{1}) = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} \qquad \mathbf{p}_1\mathbf{p}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathbf{p}_2 = (\mathbf{M}-1\cdot\mathbf{1}) = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$

$$\mathbf{P}_1 = \frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix}$$

$$\mathbf{P}_2 = \frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)} = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$



Review of 1D FDHO (Forced-Damped-Harmonic Oscillator) response

2D harmonic oscillator (2D-HO) equations of motion
Lagrangian and matrix forms

2D harmonic oscillator equation eigensolutions (normal modes)
Eigensolutions by geometry for 2D-HO with bilateral (B-Type) symmetry
Symmetric (low frequency) mode versus antisymmetric (high frequency) mode
Mixed mode beat dynamics (with constant $\pi/2$ phase-lag)
Geometry of phase and polarization

Eigensolutions by matrix-algebra with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$
Secular equation
Hamilton-Cayley equation and projectors
Idempotent projectors (how eigenvalues$\Rightarrow$eigenvectors)
Operator orthonormality and Completeness (Idempotent means: $\mathbf{P}\cdot\mathbf{P}=\mathbf{P}$)
Spectral Decompositions
Functional spectral decomposition
Orthonormality vs. Completeness vis-a`-vis Operator vs. State
Lagrange functional interpolation formula
Diagonalizing Transformations (D-Ttran) from projectors

Matrix-algebraic method for finding eigenvector and eigenvalues

$$\mathbf{p}_j\mathbf{p}_k = \mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1}) = \prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1}) \qquad \mathbf{M}\mathbf{p}_k = \varepsilon_k\mathbf{p}_k = \mathbf{p}_k\mathbf{M}$$

Multiplication properties of $\mathbf{p}_j$:

$$\mathbf{p}_j\mathbf{p}_k = \prod_{m\neq k}(\varepsilon_j\mathbf{p}_j - \varepsilon_m\mathbf{p}_j) = \mathbf{p}_j\prod_{m\neq k}(\varepsilon_j-\varepsilon_m) = \begin{cases} \mathbf{0} & if: j\neq k \\ \mathbf{p}_k\prod_{m\neq k}(\varepsilon_k-\varepsilon_m) & if: j=k \end{cases}$$

Last step:

make *Idempotent Projectors:* $\mathbf{P}_k = \dfrac{\mathbf{p}_k}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)} = \dfrac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$

(Idempotent means: **P·P=P)**

$$\mathbf{P}_j\mathbf{P}_k = \begin{cases} \mathbf{0} & if: j\neq k \\ \mathbf{P}_k & if: j=k \end{cases}$$

$\mathbf{M}\mathbf{p}_k = \varepsilon_k\mathbf{p}_k = \mathbf{p}_k\mathbf{M}$

implies:

$\mathbf{M}\mathbf{P}_k = \varepsilon_k\mathbf{P}_k = \mathbf{P}_k\mathbf{M}$

With example matrix $\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

$$\mathbf{p}_1 = (\mathbf{M}-5\cdot\mathbf{1}) = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} \qquad \mathbf{p}_1\mathbf{p}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathbf{p}_2 = (\mathbf{M}-1\cdot\mathbf{1}) = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$

Factoring bra-kets into "Ket-Bras:

$$\mathbf{P}_1 = \frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1\begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}{k_1} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

"Gauge" scale factors that only affect plots

$$\mathbf{P}_2 = \frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)} = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = k_2\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}}{k_2} = |\varepsilon_2\rangle\langle\varepsilon_2|$$

Matrix-algebraic method for finding eigenvector and eigenvalues

$$\mathbf{p}_j\mathbf{p}_k = \mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1}) = \prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1}) \qquad \mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k=\mathbf{p}_k\mathbf{M}$$

Multiplication properties of $\mathbf{p}_j$:

$$\mathbf{p}_j\mathbf{p}_k = \prod_{m\neq k}(\varepsilon_j\mathbf{p}_j-\varepsilon_m\mathbf{p}_j) = \mathbf{p}_j\prod_{m\neq k}(\varepsilon_j-\varepsilon_m) = \begin{cases}\mathbf{0} & if: j\neq k\\ \mathbf{p}_k\prod_{m\neq k}(\varepsilon_k-\varepsilon_m) & if: j=k\end{cases}$$

Last step:

make *Idempotent Projectors:* $\mathbf{P}_k = \dfrac{\mathbf{p}_k}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)} = \dfrac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$

(*Idempotent means:* $\mathbf{P}\cdot\mathbf{P}=\mathbf{P}$)

$$\mathbf{P}_j\mathbf{P}_k = \begin{cases}\mathbf{0} & if: j\neq k\\ \mathbf{P}_k & if: j=k\end{cases}$$

$\mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k=\mathbf{p}_k\mathbf{M}$

implies:

$\mathbf{M}\mathbf{P}_k=\varepsilon_k\mathbf{P}_k=\mathbf{P}_k\mathbf{M}$

With example matrix $\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}$

$$\mathbf{p}_1=(\mathbf{M}-5\cdot\mathbf{1})=\begin{pmatrix}-1&1\\3&-3\end{pmatrix}$$

$$\mathbf{p}_2=(\mathbf{M}-1\cdot\mathbf{1})=\begin{pmatrix}3&1\\3&1\end{pmatrix} \qquad \mathbf{p}_1\mathbf{p}_2=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

Factoring bra-kets into "Ket-Bras:

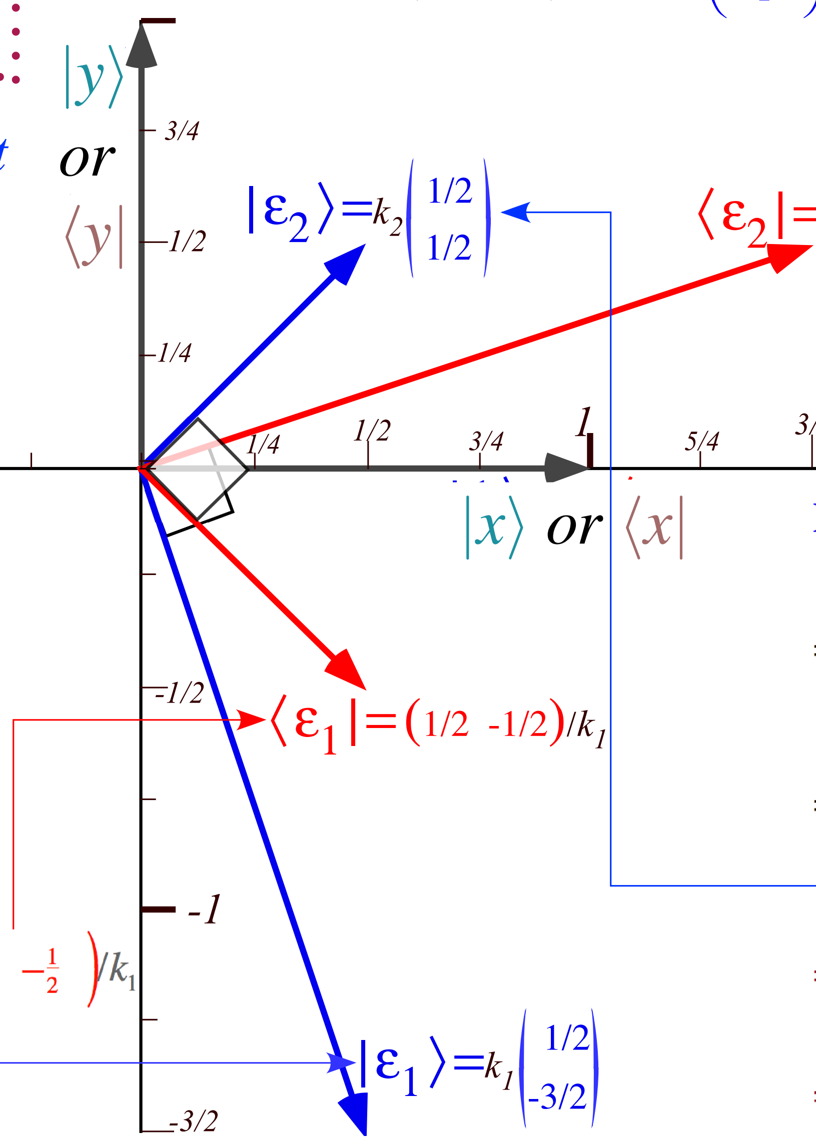
$$\mathbf{P}_1=\frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)}=\frac{1}{4}\begin{pmatrix}1&-1\\-3&3\end{pmatrix}=k_1\begin{pmatrix}\frac{1}{2}\\-\frac{3}{2}\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\end{pmatrix}}{k_1}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

"Gauge" scale factors that only affect plots

$$\mathbf{P}_2=\frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)}=\frac{1}{4}\begin{pmatrix}3&1\\3&1\end{pmatrix}=k_2\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac{3}{2}&\frac{1}{2}\end{pmatrix}}{k_2}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

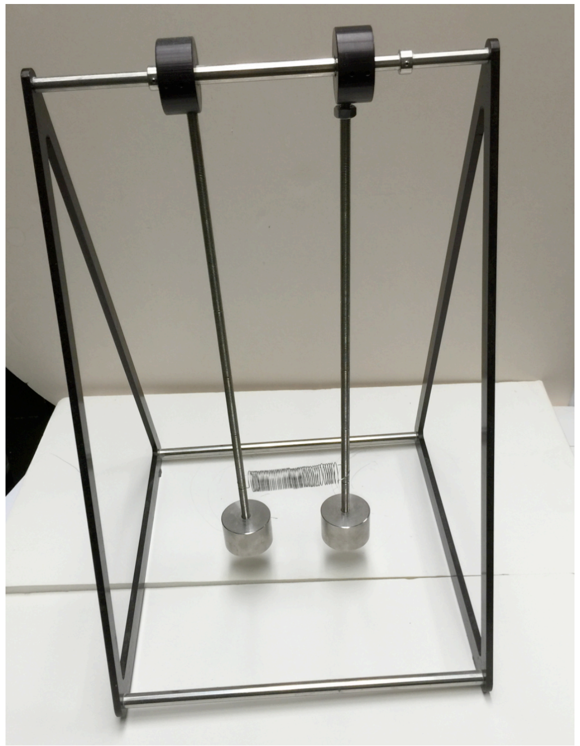
Eigen-bra-ket projectors of matrix:

$$\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}$$



$$\mathbf{P}_1=\frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)}=\frac{1}{4}\begin{pmatrix}1&-1\\-3&3\end{pmatrix}=\begin{pmatrix}\frac{1}{4}&-\frac{1}{4}\\-\frac{3}{4}&\frac{3}{4}\end{pmatrix}=k_1\begin{pmatrix}\frac{1}{2}\\-\frac{3}{2}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\end{pmatrix}/k_1=|\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2=\frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)}=\frac{1}{4}\begin{pmatrix}3&1\\3&1\end{pmatrix}=\begin{pmatrix}\frac{3}{4}&\frac{1}{4}\\\frac{3}{4}&\frac{1}{4}\end{pmatrix}=k_2\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\end{pmatrix}\otimes\begin{pmatrix}\frac{3}{2}&\frac{1}{2}\end{pmatrix}/k_2=|\varepsilon_2\rangle\langle\varepsilon_2|$$



Review of 1D FDHO (Forced-Damped-Harmonic Oscillator) response

2D harmonic oscillator (2D-HO) equations of motion
Lagrangian and matrix forms

2D harmonic oscillator equation eigensolutions (normal modes)
Eigensolutions by geometry for 2D-HO with bilateral (B-Type) symmetry
Symmetric (low frequency) mode versus antisymmetric (high frequency) mode
Mixed mode beat dynamics (with constant π/2 phase-lag)
Geometry of phase and polarization

Eigensolutions by matrix-algebra with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$
Secular equation
Hamilton-Cayley equation and projectors
Idempotent projectors (how eigenvalues⇒eigenvectors)
➤ *Operator orthonormality and Completeness (Idempotent means:* **P·P=P***)*
Spectral Decompositions
Functional spectral decomposition
Orthonormality vs. Completeness vis-a`-vis Operator vs. State
Lagrange functional interpolation formula
Diagonalizing Transformations (D-Ttran) from projectors

2D harmonic oscillator equations

Lagrangian and matrix forms and Reciprocity symmetry

2D harmonic oscillator equation eigensolutions

Geometric method

Matrix-algebraic eigensolutions with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

Secular equation

Hamilton-Cayley equation and projectors

Idempotent projectors (how eigenvalues⇒eigenvectors)

Operator orthonormality and Completeness (Idempotent means: $\mathbf{P}\cdot\mathbf{P}=\mathbf{P}$*)*

Spectral Decompositions

Functional spectral decomposition

Orthonormality vs. Completeness vis-a`-vis Operator vs. State

Lagrange functional interpolation formula

2D-HO eigensolution example with bilateral (B-Type) symmetry

Mixed mode beat dynamics and fixed π/2 phase

2D-HO eigensolution example with asymmetric (A-Type) symmetry

Initial state projection, mixed mode beat dynamics with variable phase

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry (ABCD-Types)

Matrix-algebraic method for finding eigenvector and eigenvalues

$$\mathbf{p}_j\mathbf{p}_k = \mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1}) = \prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1}) \qquad \mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k=\mathbf{p}_k\mathbf{M}$$

Multiplication properties of $\mathbf{p}_j$:

$$\mathbf{p}_j\mathbf{p}_k = \prod_{m\neq k}(\varepsilon_j\mathbf{p}_j-\varepsilon_m\mathbf{p}_j) = \mathbf{p}_j\prod_{m\neq k}(\varepsilon_j-\varepsilon_m) = \begin{cases}\mathbf{0} & \text{if}: j\neq k\\ \mathbf{p}_k\prod_{m\neq k}(\varepsilon_k-\varepsilon_m) & \text{if}: j=k\end{cases}$$

Last step:

make *Idempotent Projectors:* $\mathbf{P}_k = \dfrac{\mathbf{p}_k}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)} = \dfrac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$

(*Idempotent means:* **P·P=P**)

$$\mathbf{P}_j\mathbf{P}_k = \begin{cases}\mathbf{0} & \text{if}: j\neq k\\ \mathbf{P}_k & \text{if}: j=k\end{cases}$$

$\mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k=\mathbf{p}_k\mathbf{M}$

implies :

$\mathbf{M}\mathbf{P}_k=\varepsilon_k\mathbf{P}_k=\mathbf{P}_k\mathbf{M}$

With example matrix $\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}$

$$\mathbf{p}_1=(\mathbf{M}-5\cdot\mathbf{1})=\begin{pmatrix}-1&1\\3&-3\end{pmatrix}$$

$$\mathbf{p}_2=(\mathbf{M}-1\cdot\mathbf{1})=\begin{pmatrix}3&1\\3&1\end{pmatrix}$$

$$\mathbf{p}_1\mathbf{p}_2=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

Factoring bra-kets into "Ket-Bras:

$$\mathbf{P}_1=\frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)}=\frac{1}{4}\begin{pmatrix}1&-1\\-3&3\end{pmatrix}=k_1\begin{pmatrix}\frac12\\-\frac32\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac12&-\frac12\end{pmatrix}}{k_1}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

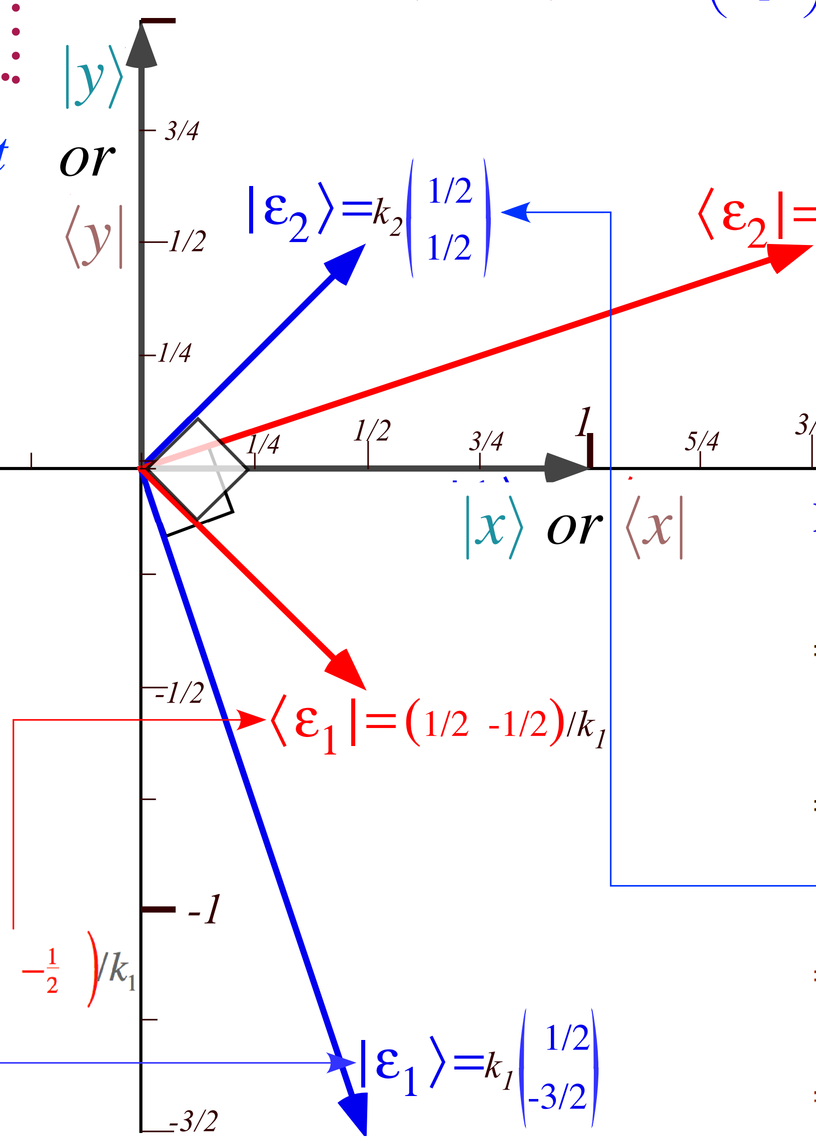
"Gauge" scale factors that only affect plots

$$\mathbf{P}_2=\frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)}=\frac{1}{4}\begin{pmatrix}3&1\\3&1\end{pmatrix}=k_2\begin{pmatrix}\frac12\\\frac12\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac32&\frac12\end{pmatrix}}{k_2}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

The $\mathbf{P}_j$ are *Mutually Ortho-Normal* as are bra-ket $\langle\varepsilon_j|$ and $|\varepsilon_j\rangle$ inside $\mathbf{P}_j$'s

$$\begin{pmatrix}\langle\varepsilon_1|\varepsilon_1\rangle & \langle\varepsilon_1|\varepsilon_2\rangle\\ \langle\varepsilon_2|\varepsilon_1\rangle & \langle\varepsilon_2|\varepsilon_2\rangle\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

Eigen-bra-ket projectors of matrix: $\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}$



$$\mathbf{P}_1=\frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)}=\frac14\begin{pmatrix}1&-1\\-3&3\end{pmatrix}=\begin{pmatrix}\frac14&-\frac14\\-\frac34&\frac34\end{pmatrix}=k_1\begin{pmatrix}\frac12\\-\frac32\end{pmatrix}\otimes\begin{pmatrix}\frac12&-\frac12\end{pmatrix}/k_1=|\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2=\frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)}=\frac14\begin{pmatrix}3&1\\3&1\end{pmatrix}=\begin{pmatrix}\frac34&\frac14\\\frac34&\frac14\end{pmatrix}=k_2\begin{pmatrix}\frac12\\\frac12\end{pmatrix}\otimes\begin{pmatrix}\frac32&\frac12\end{pmatrix}/k_2=|\varepsilon_2\rangle\langle\varepsilon_2|$$

Matrix-algebraic method for finding eigenvector and eigenvalues

$$\mathbf{p}_j\mathbf{p}_k = \mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1}) = \prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1}) \qquad \mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k=\mathbf{p}_k\mathbf{M}$$

Multiplication properties of $\mathbf{p}_j$:

$$\mathbf{p}_j\mathbf{p}_k = \prod_{m\neq k}(\varepsilon_j\mathbf{p}_j-\varepsilon_m\mathbf{p}_j) = \mathbf{p}_j\prod_{m\neq k}(\varepsilon_j-\varepsilon_m) = \begin{cases}\mathbf{0} & if: j\neq k\\ \mathbf{p}_k\prod_{m\neq k}(\varepsilon_k-\varepsilon_m) & if: j=k\end{cases}$$

Last step:

make *Idempotent Projectors:* $\mathbf{P}_k = \dfrac{\mathbf{p}_k}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)} = \dfrac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$

(Idempotent means: **P·P=P***)*

$$\mathbf{P}_j\mathbf{P}_k = \begin{cases}\mathbf{0} & if: j\neq k\\ \mathbf{P}_k & if: j=k\end{cases}$$

$\mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k=\mathbf{p}_k\mathbf{M}$

implies:

$\mathbf{M}\mathbf{P}_k=\varepsilon_k\mathbf{P}_k=\mathbf{P}_k\mathbf{M}$

With example matrix $\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}$

$\mathbf{p}_1=(\mathbf{M}-5\cdot\mathbf{1})=\begin{pmatrix}-1&1\\3&-3\end{pmatrix}$

$\mathbf{p}_2=(\mathbf{M}-1\cdot\mathbf{1})=\begin{pmatrix}3&1\\3&1\end{pmatrix}$

$\mathbf{p}_1\mathbf{p}_2=\begin{pmatrix}0&0\\0&0\end{pmatrix}$

Factoring bra-kets into "Ket-Bras:

$$\mathbf{P}_1=\frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)}=\frac{1}{4}\begin{pmatrix}1&-1\\-3&3\end{pmatrix}=k_1\begin{pmatrix}\frac{1}{2}\\-\frac{3}{2}\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\end{pmatrix}}{k_1}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

"Gauge" scale factors that only affect plots

$$\mathbf{P}_2=\frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)}=\frac{1}{4}\begin{pmatrix}3&1\\3&1\end{pmatrix}=k_2\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac{3}{2}&\frac{1}{2}\end{pmatrix}}{k_2}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

The $\mathbf{P}_j$ are *Mutually Ortho-Normal* as are bra-ket $\langle\varepsilon_j|$ and $|\varepsilon_j\rangle$ inside $\mathbf{P}_j$'s

$$\begin{pmatrix}\langle\varepsilon_1|\varepsilon_1\rangle&\langle\varepsilon_1|\varepsilon_2\rangle\\\langle\varepsilon_2|\varepsilon_1\rangle&\langle\varepsilon_2|\varepsilon_2\rangle\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

...and the $\mathbf{P}_j$ satisfy a *Completeness Relation:*

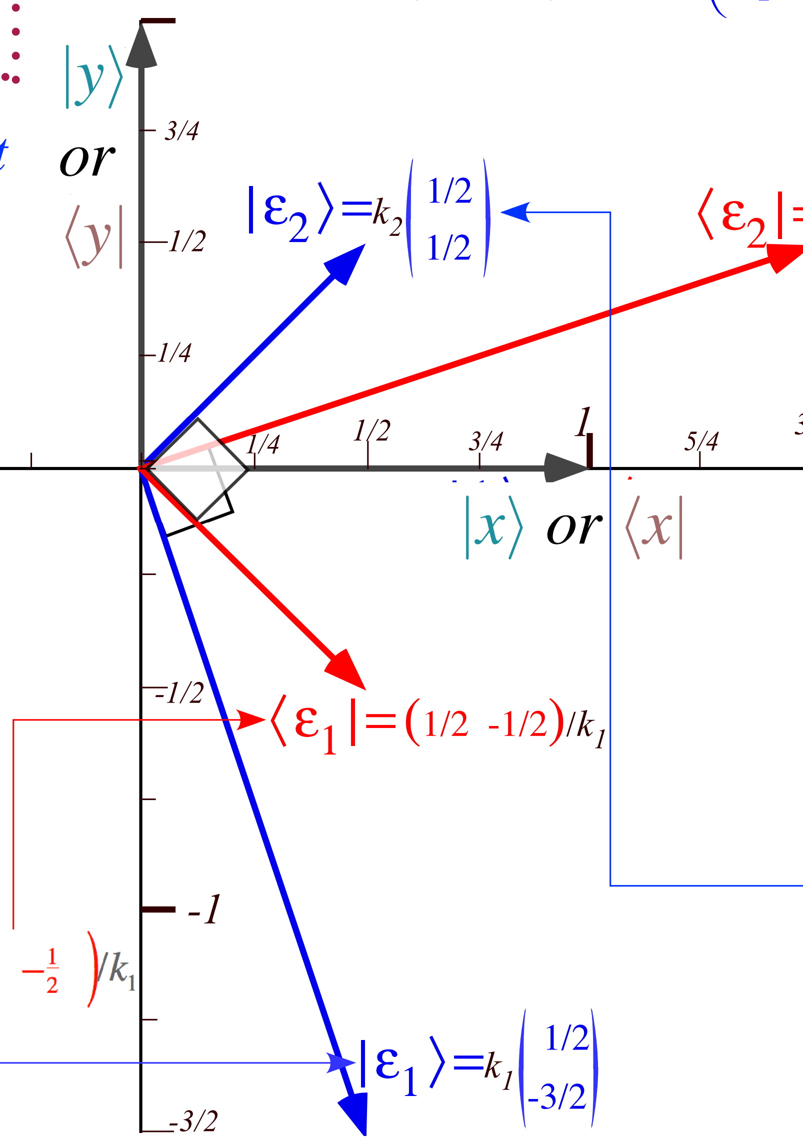
$\mathbf{1}= \mathbf{P}_1 + \mathbf{P}_2 + ...+ \mathbf{P}_n$

$=|\varepsilon_1\rangle\langle\varepsilon_1|+|\varepsilon_2\rangle\langle\varepsilon_2|+...+|\varepsilon_n\rangle\langle\varepsilon_n|$

$$\mathbf{P}_1+\mathbf{P}_2=\begin{pmatrix}1&0\\0&1\end{pmatrix}=|\varepsilon_1\rangle\langle\varepsilon_1|+|\varepsilon_2\rangle\langle\varepsilon_2|$$

Eigen-bra-ket projectors of matrix: $\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}$

$$\mathbf{P}_1=\frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)}=\frac{1}{4}\begin{pmatrix}1&-1\\-3&3\end{pmatrix}=\begin{pmatrix}\frac{1}{4}&-\frac{1}{4}\\-\frac{3}{4}&\frac{3}{4}\end{pmatrix}=k_1\begin{pmatrix}\frac{1}{2}\\-\frac{3}{2}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\end{pmatrix}/k_1=|\varepsilon_1\rangle\langle\varepsilon_1|$$



$$\mathbf{P}_2=\frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)}=\frac{1}{4}\begin{pmatrix}3&1\\3&1\end{pmatrix}=\begin{pmatrix}\frac{3}{4}&\frac{1}{4}\\\frac{3}{4}&\frac{1}{4}\end{pmatrix}=k_2\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\end{pmatrix}\otimes\begin{pmatrix}\frac{3}{2}&\frac{1}{2}\end{pmatrix}/k_2=|\varepsilon_2\rangle\langle\varepsilon_2|$$

2D harmonic oscillator equations
Lagrangian and matrix forms and Reciprocity symmetry

2D harmonic oscillator equation eigensolutions
Geometric method
Matrix-algebraic eigensolutions with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$
Secular equation
Hamilton-Cayley equation and projectors
Idempotent projectors (how eigenvalues⇒eigenvectors)
Operator orthonormality and Completeness (Idempotent means: $\mathbf{P\cdot P=P}$*)*
➨ *Spectral Decompositions* ⬅
Functional spectral decomposition
Orthonormality vs. Completeness vis-a`-vis Operator vs. State
Lagrange functional interpolation formula
Diagonalizing Transformations (D-Ttran) from projectors

2D-HO eigensolution example with bilateral (B-Type) symmetry
Mixed mode beat dynamics and fixed π/2 phase
2D-HO eigensolution example with asymmetric (A-Type) symmetry
Initial state projection, mixed mode beat dynamics with variable phase

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$
Hamilton-Pauli spinor symmetry (ABCD-Types)

Matrix-algebraic method for finding eigenvector and eigenvalues

$$\mathbf{p}_j\mathbf{p}_k = \mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1}) = \prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1}) \qquad \mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k=\mathbf{p}_k\mathbf{M}$$

Multiplication properties of $\mathbf{p}_j$:

$$\mathbf{p}_j\mathbf{p}_k = \prod_{m\neq k}(\varepsilon_j\mathbf{p}_j-\varepsilon_m\mathbf{p}_j) = \mathbf{p}_j\prod_{m\neq k}(\varepsilon_j-\varepsilon_m) = \begin{cases} \mathbf{0} & if: j\neq k \\ \mathbf{p}_k\prod_{m\neq k}(\varepsilon_k-\varepsilon_m) & if: j=k \end{cases}$$

Last step:

make *Idempotent Projectors:* $\mathbf{P}_k = \dfrac{\mathbf{p}_k}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)} = \dfrac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$

(Idempotent means: **P·P=P**)

$$\mathbf{P}_j\mathbf{P}_k = \begin{cases} \mathbf{0} & if: j\neq k \\ \mathbf{P}_k & if: j=k \end{cases}$$

$\mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k=\mathbf{p}_k\mathbf{M}$

implies:

$\mathbf{M}\mathbf{P}_k=\varepsilon_k\mathbf{P}_k=\mathbf{P}_k\mathbf{M}$

The $\mathbf{P}_j$ are *Mutually Ortho-Normal* as are bra-ket $\langle\varepsilon_j|$ and $|\varepsilon_j\rangle$ inside $\mathbf{P}_j$'s

$$\begin{pmatrix} \langle\varepsilon_1|\varepsilon_1\rangle & \langle\varepsilon_1|\varepsilon_2\rangle \\ \langle\varepsilon_2|\varepsilon_1\rangle & \langle\varepsilon_2|\varepsilon_2\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

...and the $\mathbf{P}_j$ satisfy a *Completeness Relation:*

$\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + ... + \mathbf{P}_n$

$= |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2| + ... + |\varepsilon_n\rangle\langle\varepsilon_n|$

$$\mathbf{P}_1+\mathbf{P}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2|$$

Eigen-operators $\mathbf{M}\mathbf{P}_k=\varepsilon_k\mathbf{P}_k$ then give *Spectral Decomposition* of operator **M**

$$\mathbf{M} = \mathbf{M}\mathbf{P}_1 + \mathbf{M}\mathbf{P}_2 + ... + \mathbf{M}\mathbf{P}_n = \varepsilon_1\mathbf{P}_1 + \varepsilon_2\mathbf{P}_2 + ... + \varepsilon_n\mathbf{P}_n$$

With example matrix $\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

$$\mathbf{p}_1 = (\mathbf{M}-5\cdot\mathbf{1}) = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}$$

$$\mathbf{p}_2 = (\mathbf{M}-1\cdot\mathbf{1}) = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$

$$\mathbf{p}_1\mathbf{p}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Factoring bra-kets into "Ket-Bras:

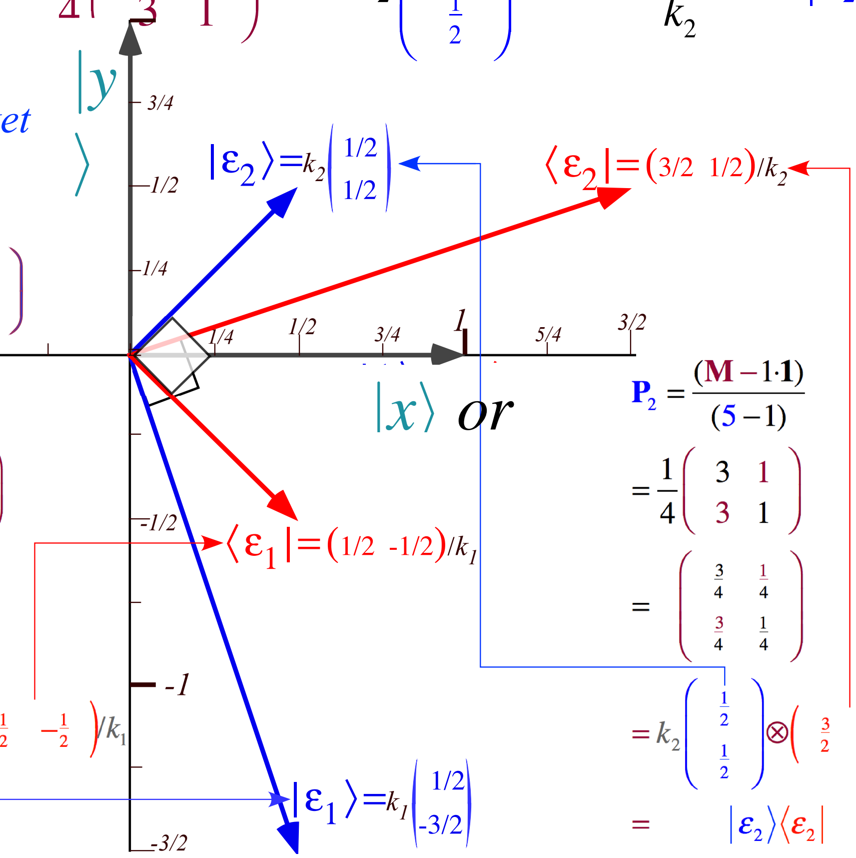
$$\mathbf{P}_1 = \frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1\begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}{k_1} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

"Gauge" scale factors that only affect plots

$$\mathbf{P}_2 = \frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)} = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = k_2\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}}{k_2} = |\varepsilon_2\rangle\langle\varepsilon_2|$$

Eigen-bra-ket projectors of matrix:

$\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$



$$\mathbf{P}_1 = \frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{3}{4} \end{pmatrix} = k_1\begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}/k_1 = |\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2 = \frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)} = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix} = k_2\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}/k_2 = |\varepsilon_2\rangle\langle\varepsilon_2|$$

Matrix-algebraic method for finding eigenvector and eigenvalues

$$\mathbf{p}_j\mathbf{p}_k = \mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1}) = \prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1}) \qquad \mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k=\mathbf{p}_k\mathbf{M}$$

Multiplication properties of $\mathbf{p}_j$:

$$\mathbf{p}_j\mathbf{p}_k = \prod_{m\neq k}(\varepsilon_j\mathbf{p}_j-\varepsilon_m\mathbf{p}_j) = \mathbf{p}_j\prod_{m\neq k}(\varepsilon_j-\varepsilon_m) = \begin{cases} \mathbf{0} & if: j\neq k \\ \mathbf{p}_k\prod_{m\neq k}(\varepsilon_k-\varepsilon_m) & if: j=k \end{cases}$$

Last step:

make *Idempotent Projectors:* $\mathbf{P}_k = \dfrac{\mathbf{p}_k}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)} = \dfrac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$

(Idempotent means: **P·P=P**)

$$\mathbf{P}_j\mathbf{P}_k = \begin{cases} \mathbf{0} & if: j\neq k \\ \mathbf{P}_k & if: j=k \end{cases}$$

$\mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k=\mathbf{p}_k\mathbf{M}$

implies:

$\mathbf{M}\mathbf{P}_k=\varepsilon_k\mathbf{P}_k=\mathbf{P}_k\mathbf{M}$

With example matrix $\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}$

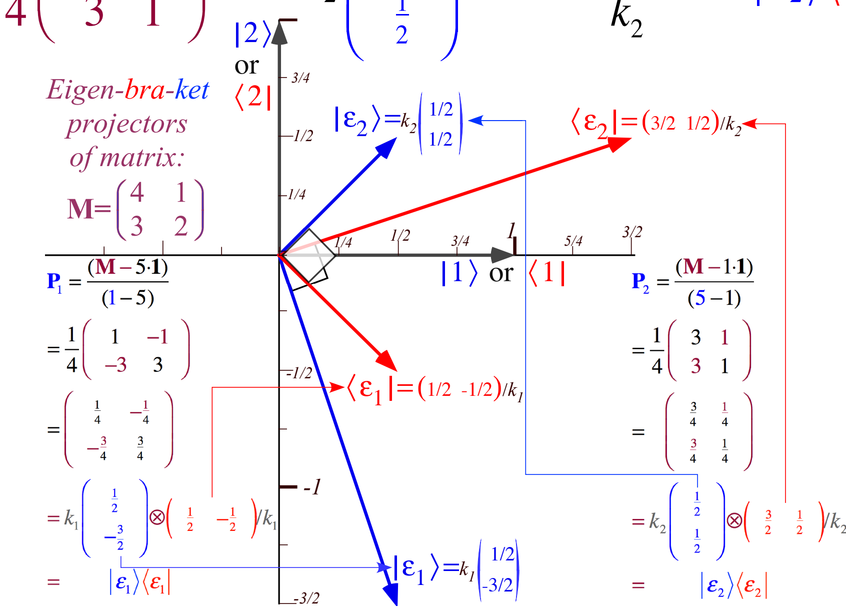
$$\mathbf{p}_1=(\mathbf{M}-5\cdot\mathbf{1})=\begin{pmatrix}-1&1\\3&-3\end{pmatrix} \qquad \mathbf{p}_1\mathbf{p}_2=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

$$\mathbf{p}_2=(\mathbf{M}-1\cdot\mathbf{1})=\begin{pmatrix}3&1\\3&1\end{pmatrix}$$

Factoring bra-kets into "Ket-Bras:

$$\mathbf{P}_1=\frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)}=\frac{1}{4}\begin{pmatrix}1&-1\\-3&3\end{pmatrix}=k_1\begin{pmatrix}\frac{1}{2}\\-\frac{3}{2}\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\end{pmatrix}}{k_1}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2=\frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)}=\frac{1}{4}\begin{pmatrix}3&1\\3&1\end{pmatrix}=k_2\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac{3}{2}&\frac{1}{2}\end{pmatrix}}{k_2}=|\varepsilon_2\rangle\langle\varepsilon_2|$$



The $\mathbf{P}_j$ are *Mutually Ortho-Normal* as are bra-ket $\langle\varepsilon_j|$and$|\varepsilon_j\rangle$ inside $\mathbf{P}_j$'s

$$\begin{pmatrix}\langle\varepsilon_1|\varepsilon_1\rangle & \langle\varepsilon_1|\varepsilon_2\rangle\\ \langle\varepsilon_2|\varepsilon_1\rangle & \langle\varepsilon_2|\varepsilon_2\rangle\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

...and the $\mathbf{P}_j$ satisfy a *Completeness Relation:*

$\mathbf{1}= \ \mathbf{P}_1 + \ \mathbf{P}_2 \ +...+ \ \mathbf{P}_n$

$=|\varepsilon_1\rangle\langle\varepsilon_1|+|\varepsilon_2\rangle\langle\varepsilon_2|+...+|\varepsilon_n\rangle\langle\varepsilon_n|$

$$\mathbf{P}_1+\mathbf{P}_2=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$=|\varepsilon_1\rangle\langle\varepsilon_1|+|\varepsilon_2\rangle\langle\varepsilon_2|$$

$$\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}=1\mathbf{P}_1+5\mathbf{P}_2=1|1\rangle\langle 1|+5|2\rangle\langle 2|=1\begin{pmatrix}\frac{1}{4}&-\frac{1}{4}\\-\frac{3}{4}&\frac{3}{4}\end{pmatrix}+5\begin{pmatrix}\frac{3}{4}&\frac{1}{4}\\\frac{3}{4}&\frac{1}{4}\end{pmatrix}$$

Eigen-operators $\mathbf{M}\mathbf{P}_k=\varepsilon_k\mathbf{P}_k$ then give *Spectral Decomposition* of operator **M**

$$\mathbf{M}=\mathbf{M}\mathbf{P}_1+\mathbf{M}\mathbf{P}_2+...+\mathbf{M}\mathbf{P}_n=\varepsilon_1\mathbf{P}_1+\varepsilon_2\mathbf{P}_2+...+\varepsilon_n\mathbf{P}_n$$

2D harmonic oscillator equations

Lagrangian and matrix forms and Reciprocity symmetry

2D harmonic oscillator equation eigensolutions

Geometric method

Matrix-algebraic eigensolutions with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

Secular equation

Hamilton-Cayley equation and projectors

Idempotent projectors (how eigenvalues⇒eigenvectors)

Operator orthonormality and Completeness (Idempotent means: $\mathbf{P\cdot P=P}$*)*

Spectral Decompositions

Functional spectral decomposition

Orthonormality vs. Completeness vis-a`-vis Operator vs. State

Lagrange functional interpolation formula

Diagonalizing Transformations (D-Ttran) from projectors

2D-HO eigensolution example with bilateral (B-Type) symmetry

Mixed mode beat dynamics and fixed π/2 phase

2D-HO eigensolution example with asymmetric (A-Type) symmetry

Initial state projection, mixed mode beat dynamics with variable phase

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry (ABCD-Types)

Matrix and operator Spectral Decompositons

$$\mathbf{p}_j\mathbf{p}_k = \mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1}) = \prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1}) \qquad \mathbf{M}\mathbf{p}_k = \varepsilon_k\mathbf{p}_k = \mathbf{p}_k\mathbf{M}$$

Multiplication properties of $\mathbf{p}_j$:

$$\mathbf{p}_j\mathbf{p}_k = \prod_{m\neq k}(\varepsilon_j\mathbf{p}_j-\varepsilon_m\mathbf{p}_j) = \mathbf{p}_j\prod_{m\neq k}(\varepsilon_j-\varepsilon_m) = \begin{cases}\mathbf{0} & if : j\neq k\\ \mathbf{p}_k\prod_{m\neq k}(\varepsilon_k-\varepsilon_m) & if : j=k\end{cases}$$

$$\mathbf{M} = \begin{pmatrix}4 & 1\\ 3 & 2\end{pmatrix}$$

$$\mathbf{p}_1 = (\mathbf{M}-5\cdot\mathbf{1}) = \begin{pmatrix}-1 & 1\\ 3 & -3\end{pmatrix} \qquad \mathbf{p}_1\mathbf{p}_2 = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

$$\mathbf{p}_2 = (\mathbf{M}-1\cdot\mathbf{1}) = \begin{pmatrix}3 & 1\\ 3 & 1\end{pmatrix}$$

Last step:

make *Idempotent Projectors:* $\mathbf{P}_k = \dfrac{\mathbf{p}_k}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)} = \dfrac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$

(*Idempotent means:* **P·P=P**)

$$\mathbf{P}_j\mathbf{P}_k = \begin{cases}\mathbf{0} & if : j\neq k\\ \mathbf{P}_k & if : j=k\end{cases}$$

$\mathbf{M}\mathbf{p}_k = \varepsilon_k\mathbf{p}_k = \mathbf{p}_k\mathbf{M}$
implies :
$\mathbf{M}\mathbf{P}_k = \varepsilon_k\mathbf{P}_k = \mathbf{P}_k\mathbf{M}$

Factoring bra-kets into "Ket-Bras:

$$\mathbf{P}_1 = \frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix}1 & -1\\ -3 & 3\end{pmatrix} = k_1\begin{pmatrix}\frac{1}{2}\\ -\frac{3}{2}\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\end{pmatrix}}{k_1} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2 = \frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)} = \frac{1}{4}\begin{pmatrix}3 & 1\\ 3 & 1\end{pmatrix} = k_2\begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac{3}{2} & \frac{1}{2}\end{pmatrix}}{k_2} = |\varepsilon_2\rangle\langle\varepsilon_2|$$

The $\mathbf{P}_j$ are *Mutually Ortho-Normal*
as are bra-ket $\langle\varepsilon_j|$and$|\varepsilon_j\rangle$ inside $\mathbf{P}_j$'s

$$\begin{pmatrix}\langle\varepsilon_1|\varepsilon_1\rangle & \langle\varepsilon_1|\varepsilon_2\rangle\\ \langle\varepsilon_2|\varepsilon_1\rangle & \langle\varepsilon_2|\varepsilon_2\rangle\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

...and the $\mathbf{P}_j$ satisfy a
Completeness Relation:

$$\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_n$$
$$= |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2| + \ldots + |\varepsilon_n\rangle\langle\varepsilon_n|$$

$$\mathbf{P}_1 + \mathbf{P}_2 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$
$$= |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2|$$

$$\mathbf{M} = \begin{pmatrix}4 & 1\\ 3 & 2\end{pmatrix} = 1\mathbf{P}_1 + 5\mathbf{P}_2 = 1|1\rangle\langle 1| + 5|2\rangle\langle 2| = 1\begin{pmatrix}\frac{1}{4} & -\frac{1}{4}\\ -\frac{3}{4} & \frac{3}{4}\end{pmatrix} + 5\begin{pmatrix}\frac{3}{4} & \frac{1}{4}\\ \frac{3}{4} & \frac{1}{4}\end{pmatrix}$$

Eigen-operators $\mathbf{M}\mathbf{P}_k = \varepsilon_k\mathbf{P}_k$ then give *Spectral Decomposition* of operator **M**

$$\mathbf{M} = \mathbf{M}\mathbf{P}_1 + \mathbf{M}\mathbf{P}_2 + \ldots + \mathbf{M}\mathbf{P}_n = \varepsilon_1\mathbf{P}_1 + \varepsilon_2\mathbf{P}_2 + \ldots + \varepsilon_n\mathbf{P}_n$$

...and *Functional Spectral Decomposition* of any function *f*(**M**) of **M**

$$f(\mathbf{M}) == f(\varepsilon_1)\mathbf{P}_1 + f(\varepsilon_2)\mathbf{P}_2 + \ldots + f(\varepsilon_n)\mathbf{P}_n$$

Matrix and operator Spectral Decompositons

$$\mathbf{p}_j\mathbf{p}_k = \mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1}) = \prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1}) \qquad \mathbf{M}\mathbf{p}_k = \varepsilon_k\mathbf{p}_k = \mathbf{p}_k\mathbf{M}$$

Multiplication properties of $\mathbf{p}_j$:

$$\mathbf{p}_j\mathbf{p}_k = \prod_{m\neq k}(\varepsilon_j\mathbf{p}_j - \varepsilon_m\mathbf{p}_j) = \mathbf{p}_j\prod_{m\neq k}(\varepsilon_j-\varepsilon_m) = \begin{cases} \mathbf{0} & if: j\neq k \\ \mathbf{p}_k\prod_{m\neq k}(\varepsilon_k-\varepsilon_m) & if: j=k \end{cases}$$

Last step:

make *Idempotent Projectors:* $\mathbf{P}_k = \dfrac{\mathbf{p}_k}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)} = \dfrac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$

(Idempotent means: **P·P=P***)*

$$\mathbf{P}_j\mathbf{P}_k = \begin{cases} \mathbf{0} & if: j\neq k \\ \mathbf{P}_k & if: j=k \end{cases}$$

$\mathbf{M}\mathbf{p}_k = \varepsilon_k\mathbf{p}_k = \mathbf{p}_k\mathbf{M}$ *implies:* $\mathbf{M}\mathbf{P}_k = \varepsilon_k\mathbf{P}_k = \mathbf{P}_k\mathbf{M}$

$$\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$$

$$\mathbf{p}_1 = (\mathbf{M}-5\cdot\mathbf{1}) = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} \qquad \mathbf{p}_1\mathbf{p}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathbf{p}_2 = (\mathbf{M}-1\cdot\mathbf{1}) = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$

Factoring bra-kets into "Ket-Bras:

$$\mathbf{P}_1 = \frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1\begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}{k_1} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2 = \frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)} = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = k_2\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}}{k_2} = |\varepsilon_2\rangle\langle\varepsilon_2|$$

The $\mathbf{P}_j$ are *Mutually Ortho-Normal* as are bra-ket $\langle\varepsilon_j|$and$|\varepsilon_j\rangle$ inside $\mathbf{P}_j$'s

$$\begin{pmatrix} \langle\varepsilon_1|\varepsilon_1\rangle & \langle\varepsilon_1|\varepsilon_2\rangle \\ \langle\varepsilon_2|\varepsilon_1\rangle & \langle\varepsilon_2|\varepsilon_2\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

...and the $\mathbf{P}_j$ satisfy a *Completeness Relation:*

$\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + ... + \mathbf{P}_n$

$= |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2| + ... + |\varepsilon_n\rangle\langle\varepsilon_n|$

$$\mathbf{P}_1 + \mathbf{P}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2|$$

$$\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = 1\mathbf{P}_1 + 5\mathbf{P}_2 = 1|1\rangle\langle 1| + 5|2\rangle\langle 2| = 1\begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{3}{4} \end{pmatrix} + 5\begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

Eigen-operators $\mathbf{M}\mathbf{P}_k = \varepsilon_k\mathbf{P}_k$ then give *Spectral Decomposition* of operator $\mathbf{M}$

$$\mathbf{M} = \mathbf{M}\mathbf{P}_1 + \mathbf{M}\mathbf{P}_2 + ... + \mathbf{M}\mathbf{P}_n = \varepsilon_1\mathbf{P}_1 + \varepsilon_2\mathbf{P}_2 + ... + \varepsilon_n\mathbf{P}_n$$

...and *Functional Spectral Decomposition* of any function *f*(**M**) of **M**

$$f(\mathbf{M}) == f(\varepsilon_1)\mathbf{P}_1 + f(\varepsilon_2)\mathbf{P}_2 + ... + f(\varepsilon_n)\mathbf{P}_n$$

Example:

$$\mathbf{M}^{50} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = 1^{50}\begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{3}{4} \end{pmatrix} + 5^{50}\begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1+3\cdot 5^{50} & 5^{50}-1 \\ 3\cdot 5^{50}-3 & 5^{50}+3 \end{pmatrix}$$

Matrix and operator Spectral Decompositons

$$\mathbf{p}_j\mathbf{p}_k = \mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1}) = \prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1}) \qquad \mathbf{M}\mathbf{p}_k = \varepsilon_k\mathbf{p}_k = \mathbf{p}_k\mathbf{M}$$

Multiplication properties of $\mathbf{p}_j$:

$$\mathbf{p}_j\mathbf{p}_k = \prod_{m\neq k}(\varepsilon_j\mathbf{p}_j - \varepsilon_m\mathbf{p}_j) = \mathbf{p}_j\prod_{m\neq k}(\varepsilon_j-\varepsilon_m) = \begin{cases} \mathbf{0} & if: j\neq k \\ \mathbf{p}_k\prod_{m\neq k}(\varepsilon_k-\varepsilon_m) & if: j=k \end{cases}$$

Last step:
make *Idempotent Projectors:* $\mathbf{P}_k = \dfrac{\mathbf{p}_k}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)} = \dfrac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$
(Idempotent means: **P·P=P**)

$$\mathbf{P}_j\mathbf{P}_k = \begin{cases} \mathbf{0} & if: j\neq k \\ \mathbf{P}_k & if: j=k \end{cases}$$

$\mathbf{M}\mathbf{p}_k = \varepsilon_k\mathbf{p}_k = \mathbf{p}_k\mathbf{M}$
implies:
$\mathbf{M}\mathbf{P}_k = \varepsilon_k\mathbf{P}_k = \mathbf{P}_k\mathbf{M}$

$$\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$$

$$\mathbf{p}_1 = (\mathbf{M}-5\cdot\mathbf{1}) = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} \qquad \mathbf{p}_1\mathbf{p}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathbf{p}_2 = (\mathbf{M}-1\cdot\mathbf{1}) = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$

Factoring bra-kets into "Ket-Bras:

$$\mathbf{P}_1 = \frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1\begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}{k_1} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2 = \frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)} = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = k_2\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}}{k_2} = |\varepsilon_2\rangle\langle\varepsilon_2|$$

The $\mathbf{P}_j$ are *Mutually Ortho-Normal*
as are bra-ket $\langle\varepsilon_j|$and$|\varepsilon_j\rangle$ inside $\mathbf{P}_j$'s

$$\begin{pmatrix} \langle\varepsilon_1|\varepsilon_1\rangle & \langle\varepsilon_1|\varepsilon_2\rangle \\ \langle\varepsilon_2|\varepsilon_1\rangle & \langle\varepsilon_2|\varepsilon_2\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

...and the $\mathbf{P}_j$ satisfy a
Completeness Relation:
$\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + ... + \mathbf{P}_n$
$= |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2| + ... + |\varepsilon_n\rangle\langle\varepsilon_n|$

$$\mathbf{P}_1 + \mathbf{P}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2|$$

$$\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = 1\mathbf{P}_1 + 5\mathbf{P}_2 = 1|1\rangle\langle 1| + 5|2\rangle\langle 2| = 1\begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{3}{4} \end{pmatrix} + 5\begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

Eigen-operators $\mathbf{M}\mathbf{P}_k = \varepsilon_k\mathbf{P}_k$ then give *Spectral Decomposition* of operator **M**

$$\mathbf{M} = \mathbf{M}\mathbf{P}_1 + \mathbf{M}\mathbf{P}_2 + ... + \mathbf{M}\mathbf{P}_n = \varepsilon_1\mathbf{P}_1 + \varepsilon_2\mathbf{P}_2 + ... + \varepsilon_n\mathbf{P}_n$$

...and *Functional Spectral Decomposition* of any function *f*(**M**) of **M**

$$f(\mathbf{M}) == f(\varepsilon_1)\mathbf{P}_1 + f(\varepsilon_2)\mathbf{P}_2 + ... + f(\varepsilon_n)\mathbf{P}_n$$

Examples:

$$\mathbf{M}^{50} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = 1^{50}\begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{3}{4} \end{pmatrix} + 5^{50}\begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1+3\cdot 5^{50} & 5^{50}-1 \\ 3\cdot 5^{50}-3 & 5^{50}+3 \end{pmatrix}$$

$$\sqrt{\mathbf{M}} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = \pm\sqrt{1}\begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{3}{4} \end{pmatrix} \pm\sqrt{5}\begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

2D harmonic oscillator equations
Lagrangian and matrix forms and Reciprocity symmetry

2D harmonic oscillator equation eigensolutions
Geometric method
Matrix-algebraic eigensolutions with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$
Secular equation
Hamilton-Cayley equation and projectors
Idempotent projectors (how eigenvalues⇒eigenvectors)
Operator orthonormality and Completeness (Idempotent means: $\mathbf{P\cdot P=P}$*)*
Spectral Decompositions
Functional spectral decomposition
Orthonormality vs. Completeness vis-a`-vis Operator vs. State
Lagrange functional interpolation formula
Diagonalizing Transformations (D-Ttran) from projectors

2D-HO eigensolution example with bilateral (B-Type) symmetry
Mixed mode beat dynamics and fixed π/2 phase
2D-HO eigensolution example with asymmetric (A-Type) symmetry
Initial state projection, mixed mode beat dynamics with variable phase

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$
Hamilton-Pauli spinor symmetry (ABCD-Types)

Orthonormality vs. Completeness

$$\mathbf{p}_j\mathbf{p}_k = \mathbf{p}_j \prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1}) = \prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1}) \qquad \mathbf{M}\mathbf{p}_k = \varepsilon_k\mathbf{p}_k = \mathbf{p}_k\mathbf{M}$$

Multiplication properties of $\mathbf{p}_j$:

$$\mathbf{p}_j\mathbf{p}_k = \prod_{m\neq k}(\varepsilon_j\mathbf{p}_j - \varepsilon_m\mathbf{p}_j) = \mathbf{p}_j\prod_{m\neq k}(\varepsilon_j-\varepsilon_m) = \begin{cases} \mathbf{0} & if : j\neq k \\ \mathbf{p}_k \prod_{m\neq k}(\varepsilon_k-\varepsilon_m) & if : j=k \end{cases}$$

Last step:

make *Idempotent Projectors:* $\mathbf{P}_k = \dfrac{\mathbf{p}_k}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)} = \dfrac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$

(*Idempotent means:* **P·P=P**)

$$\mathbf{P}_j\mathbf{P}_k = \begin{cases} \mathbf{0} & if : j\neq k \\ \mathbf{P}_k & if : j=k \end{cases}$$

$\mathbf{M}\mathbf{p}_k = \varepsilon_k\mathbf{p}_k = \mathbf{p}_k\mathbf{M}$

implies :

$\mathbf{M}\mathbf{P}_k = \varepsilon_k\mathbf{P}_k = \mathbf{P}_k\mathbf{M}$

$$\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$$

$$\mathbf{p}_1 = (\mathbf{M}-5\cdot\mathbf{1}) = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} \qquad \mathbf{p}_1\mathbf{p}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathbf{p}_2 = (\mathbf{M}-1\cdot\mathbf{1}) = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$

Factoring bra-kets into "Ket-Bras:

$$\mathbf{P}_1 = \frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1\begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}{k_1} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

"Gauge" scale factors that only affect plots

$$\mathbf{P}_2 = \frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)} = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = k_2\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}}{k_2} = |\varepsilon_2\rangle\langle\varepsilon_2|$$

The $\mathbf{P}_j$ are *Mutually Ortho-Normal* as are bra-ket $\langle\varepsilon_j|$and$|\varepsilon_j\rangle$ inside $\mathbf{P}_j$'s

$$\begin{pmatrix} \langle\varepsilon_1|\varepsilon_1\rangle & \langle\varepsilon_1|\varepsilon_2\rangle \\ \langle\varepsilon_2|\varepsilon_1\rangle & \langle\varepsilon_2|\varepsilon_2\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

...and the $\mathbf{P}_j$ satisfy a *Completeness Relation:*

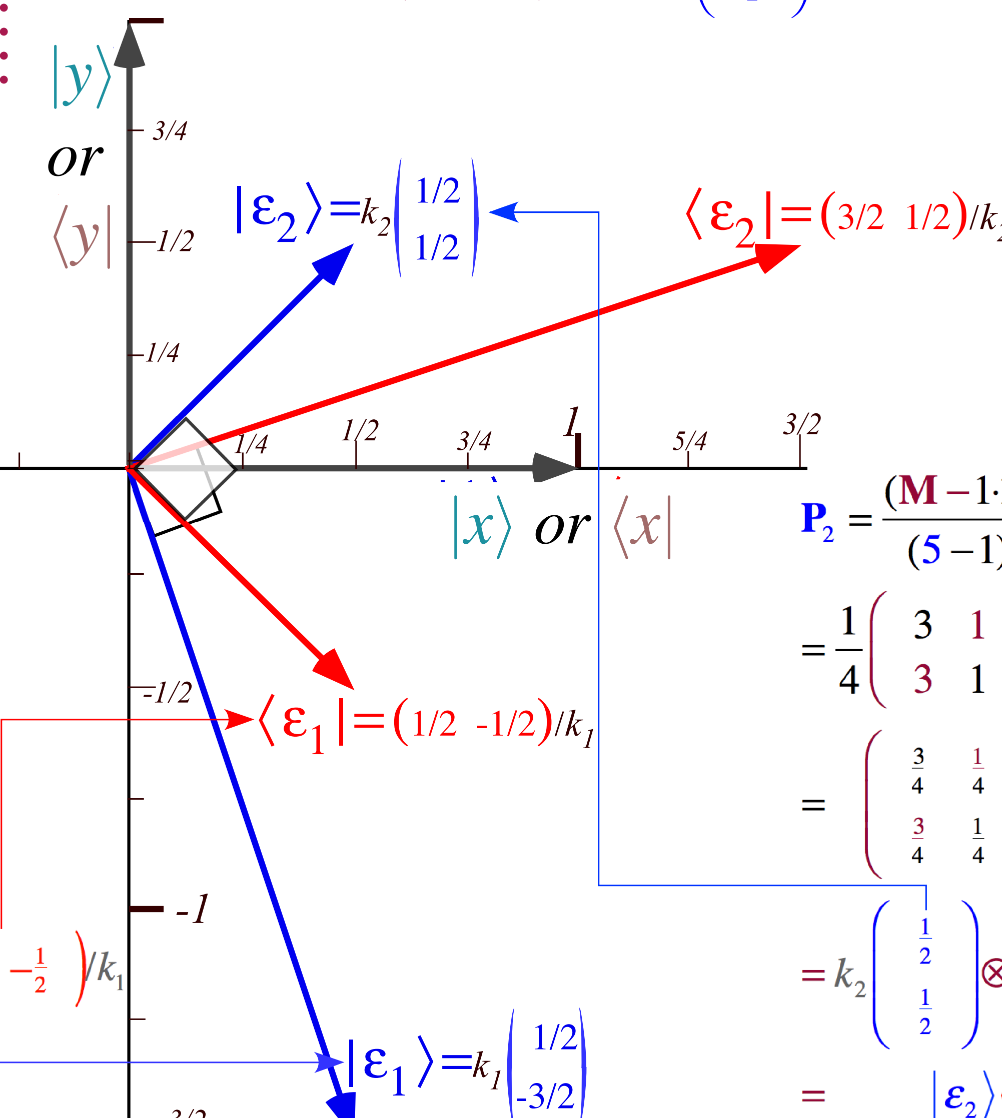
$\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + ... + \mathbf{P}_n$

$= |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2| + ... + |\varepsilon_n\rangle\langle\varepsilon_n|$

$$\mathbf{P}_1 + \mathbf{P}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2|$$

Eigen-bra-ket projectors of matrix:

$$\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$$



$$\mathbf{P}_1 = \frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{3}{4} \end{pmatrix} = k_1\begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}/k_1 = |\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2 = \frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)} = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix} = k_2\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}/k_2 = |\varepsilon_2\rangle\langle\varepsilon_2|$$

$\{|x\rangle,|y\rangle\}$-orthonormality with $\{|\varepsilon_1\rangle,|\varepsilon_2\rangle\}$-completeness

$$\langle x|y\rangle = \delta_{x,y} = \langle x|\mathbf{1}|y\rangle = \langle x|\varepsilon_1\rangle\langle\varepsilon_1|y\rangle + \langle x|\varepsilon_2\rangle\langle\varepsilon_2|y\rangle.$$

$\{|\varepsilon_1\rangle,|\varepsilon_2\rangle\}$-orthonormality with $\{|x\rangle,|y\rangle\}$-completeness

$$\langle\varepsilon_i|\varepsilon_j\rangle = \delta_{i,j} = \langle\varepsilon_i|\mathbf{1}|\varepsilon_j\rangle = \langle\varepsilon_i|x\rangle\langle x|\varepsilon_j\rangle + \langle\varepsilon_i|y\rangle\langle y|\varepsilon_j\rangle$$

Orthonormality vs. Completeness vis-a`-vis Operator vs. State

Operator expressions for orthonormality appear quite different from expressions for completeness.

$$\mathbf{P}_j\mathbf{P}_k = \delta_{jk}\mathbf{P}_k = \begin{cases} \mathbf{0} & if : j \neq k \\ \mathbf{P}_k & if : j = k \end{cases}$$

$$\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_n$$

Orthonormality vs. Completeness vis-a`-vis Operator vs. State

Operator expressions for orthonormality appear quite different from expressions for completeness.

$$\mathbf{P}_j\mathbf{P}_k = \delta_{jk}\mathbf{P}_k = \begin{cases} \mathbf{0} & if : j \neq k \\ \mathbf{P}_k & if : j = k \end{cases}$$

$$\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_n$$

$$|\varepsilon_j\rangle\langle\varepsilon_j|\varepsilon_k\rangle\langle\varepsilon_k| = \delta_{jk}|\varepsilon_k\rangle\langle\varepsilon_k| \quad \text{or:} \quad \langle\varepsilon_j|\varepsilon_k\rangle = \delta_{jk}$$

$$\mathbf{1} = |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2| + \ldots + |\varepsilon_n\rangle\langle\varepsilon_n|$$

Orthonormality vs. Completeness vis-a`-vis Operator vs. State

*Operator expressions for orthonormality appear quite **different** from expressions for completeness.*

$$\mathbf{P}_j\mathbf{P}_k = \delta_{jk}\mathbf{P}_k = \begin{cases} \mathbf{0} & if: j \neq k \\ \mathbf{P}_k & if: j = k \end{cases}$$

$$\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_n$$

$$|\varepsilon_j\rangle\langle\varepsilon_j|\varepsilon_k\rangle\langle\varepsilon_k| = \delta_{jk}|\varepsilon_k\rangle\langle\varepsilon_k| \quad \text{or:} \quad \langle\varepsilon_j|\varepsilon_k\rangle = \delta_{jk}$$

$$\mathbf{1} = |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2| + \ldots + |\varepsilon_n\rangle\langle\varepsilon_n|$$

*State vector representations of orthonormality are quite **similar** to representations of completeness.*
Like 2-sides of the same coin.

$\{|x\rangle, |y\rangle\}$-orthonormality with $\{|\varepsilon_1\rangle, |\varepsilon_2\rangle\}$-completeness

$$\langle x|y\rangle = \delta_{x,y} = \langle x|\mathbf{1}|y\rangle = \langle x|\varepsilon_1\rangle\langle\varepsilon_1|y\rangle + \langle x|\varepsilon_2\rangle\langle\varepsilon_2|y\rangle.$$

$\{|\varepsilon_1\rangle, |\varepsilon_2\rangle\}$-orthonormality with $\{|x\rangle, |y\rangle\}$-completeness

$$\langle\varepsilon_i|\varepsilon_j\rangle = \delta_{i,j} = \langle\varepsilon_i|\mathbf{1}|\varepsilon_j\rangle = \langle\varepsilon_i|x\rangle\langle x|\varepsilon_j\rangle + \langle\varepsilon_i|y\rangle\langle y|\varepsilon_j\rangle$$

Orthonormality vs. Completeness vis-a`-vis Operator vs. State

*Operator expressions for orthonormality appear quite **different** from expressions for completeness.*

$$\mathbf{P}_j\mathbf{P}_k = \delta_{jk}\mathbf{P}_k = \begin{cases} \mathbf{0} & if: j \neq k \\ \mathbf{P}_k & if: j = k \end{cases}$$

$$\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_n$$

$$|\varepsilon_j\rangle\langle\varepsilon_j|\varepsilon_k\rangle\langle\varepsilon_k| = \delta_{jk}|\varepsilon_k\rangle\langle\varepsilon_k| \quad \text{or:} \quad \langle\varepsilon_j|\varepsilon_k\rangle = \delta_{jk}$$

$$\mathbf{1} = |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2| + \ldots + |\varepsilon_n\rangle\langle\varepsilon_n|$$

*State vector representations of orthonormality are quite **similar** to representations of completeness.*
Like 2-sides of the same coin.

$\{|x\rangle, |y\rangle\}$-orthonormality with $\{|\varepsilon_1\rangle, |\varepsilon_2\rangle\}$-completeness

$$\langle x|y\rangle = \delta_{x,y} = \langle x|\mathbf{1}|y\rangle = \langle x|\varepsilon_1\rangle\langle\varepsilon_1|y\rangle + \langle x|\varepsilon_2\rangle\langle\varepsilon_2|y\rangle.$$

$$\langle x|y\rangle = \delta(x,y) = \quad \psi_1(x)\psi_1^*(y) + \psi_2(x)\psi_2^*(y) + ..$$

Dirac δ-function

$\{|\varepsilon_1\rangle, |\varepsilon_2\rangle\}$-orthonormality with $\{|x\rangle, |y\rangle\}$-completeness

$$\langle\varepsilon_i|\varepsilon_j\rangle = \delta_{i,j} = \langle\varepsilon_i|\mathbf{1}|\varepsilon_j\rangle = \langle\varepsilon_i|x\rangle\langle x|\varepsilon_j\rangle + \langle\varepsilon_i|y\rangle\langle y|\varepsilon_j\rangle$$

However Schrodinger wavefunction notation $\psi(x) = \langle x|\psi\rangle$ shows quite a difference...

Orthonormality vs. Completeness vis-a`-vis Operator vs. State

*Operator expressions for orthonormality appear quite **different** from expressions for completeness.*

$$\mathbf{P}_j\mathbf{P}_k = \delta_{jk}\mathbf{P}_k = \begin{cases} \mathbf{0} & if : j \neq k \\ \mathbf{P}_k & if : j = k \end{cases} \qquad \mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_n$$

$$|\varepsilon_j\rangle\langle\varepsilon_j|\varepsilon_k\rangle\langle\varepsilon_k| = \delta_{jk}|\varepsilon_k\rangle\langle\varepsilon_k| \text{ or: } \langle\varepsilon_j|\varepsilon_k\rangle = \delta_{jk} \qquad \mathbf{1} = |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2| + \ldots + |\varepsilon_n\rangle\langle\varepsilon_n|$$

*State vector representations of orthonormality are quite **similar** to representations of completeness.*
Like 2-sides of the same coin.

$\{|x\rangle, |y\rangle\}$-orthonormality with $\{|\varepsilon_1\rangle, |\varepsilon_2\rangle\}$-completeness

$$\langle x|y\rangle = \delta_{x,y} = \langle x|\mathbf{1}|y\rangle = \langle x|\varepsilon_1\rangle\langle\varepsilon_1|y\rangle + \langle x|\varepsilon_2\rangle\langle\varepsilon_2|y\rangle.$$

$$\langle x|y\rangle = \delta(x,y) = \qquad \psi_1(x)\psi_1^*(y) + \psi_2(x)\psi_2^*(y) + ..$$

Dirac δ-function

$\{|\varepsilon_1\rangle, |\varepsilon_2\rangle\}$-orthonormality with $\{|x\rangle, |y\rangle\}$-completeness

$$\langle\varepsilon_i|\varepsilon_j\rangle = \delta_{i,j} = \langle\varepsilon_i|\mathbf{1}|\varepsilon_j\rangle = \langle\varepsilon_i|x\rangle\langle x|\varepsilon_j\rangle + \langle\varepsilon_i|y\rangle\langle y|\varepsilon_j\rangle$$

$$\langle\varepsilon_i|\varepsilon_j\rangle = \delta_{i,j} = \qquad \ldots + \psi_i^*(x)\psi_j(x) + \psi_2(y)\psi_2^*(y) + \ldots . \rightarrow \int dx\,\psi_i^*(x)\psi_j(x)$$

However Schrodinger wavefunction notation $\psi(x) = \langle x|\psi\rangle$ shows quite a difference…

…particularly in the orthonormality integral.

2D harmonic oscillator equations

Lagrangian and matrix forms and Reciprocity symmetry

2D harmonic oscillator equation eigensolutions

Geometric method

Matrix-algebraic eigensolutions with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

Secular equation

Hamilton-Cayley equation and projectors

Idempotent projectors (how eigenvalues⇒eigenvectors)

Operator orthonormality and Completeness (Idempotent means: $\mathbf{P}\cdot\mathbf{P}=\mathbf{P}$*)*

Spectral Decompositions

Functional spectral decomposition

Orthonormality vs. Completeness vis-a`-vis Operator vs. State

Lagrange functional interpolation formula

Diagonalizing Transformations (D-Ttran) from projectors

2D-HO eigensolution example with bilateral (B-Type) symmetry

Mixed mode beat dynamics and fixed π/2 phase

2D-HO eigensolution example with asymmetric (A-Type) symmetry

Initial state projection, mixed mode beat dynamics with variable phase

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry (ABCD-Types)

A Proof of Projector Completeness *(Truer-than-true by Lagrange interpolation)*

Compare matrix *completeness relation* and *functional spectral decompositions*

$$\mathbf{1}=\mathbf{P}_1+\mathbf{P}_2+\ldots+\mathbf{P}_n=\sum_{\varepsilon_k}\mathbf{P}_k=\sum_{\varepsilon_k}\frac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$$

$$f(\mathbf{M})=f(\varepsilon_1)\mathbf{P}_1+f(\varepsilon_2)\mathbf{P}_2+\ldots+f(\varepsilon_n)\mathbf{P}_n=\sum_{\varepsilon_k}f(\varepsilon_k)\mathbf{P}_k=\sum_{\varepsilon_k}f(\varepsilon_k)\frac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$$

with *Lagrange interpolation formula* of function *f(x)* approximated by its value at N points $x_1, x_2, \ldots x_N$.

$$L\big(f(x)\big)=\sum_{k=1}^{N}f(x_k)\cdot P_k(x) \qquad \text{where: } P_k(x)=\frac{\prod_{j\neq k}^{N}\left(x-x_j\right)}{\prod_{j\neq k}^{N}\left(x_k-x_j\right)}$$

A Proof of Projector Completeness *(Truer-than-true)*

Compare matrix *completeness relation* and *functional spectral decompositions*

$$\mathbf{1}=\mathbf{P}_1+\mathbf{P}_2+\ldots+\mathbf{P}_n=\sum_{\varepsilon_k}\mathbf{P}_k=\sum_{\varepsilon_k}\frac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$$

$$f(\mathbf{M})=f(\varepsilon_1)\mathbf{P}_1+f(\varepsilon_2)\mathbf{P}_2+\ldots+f(\varepsilon_n)\mathbf{P}_n=\sum_{\varepsilon_k}f(\varepsilon_k)\mathbf{P}_k=\sum_{\varepsilon_k}f(\varepsilon_k)\frac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$$

with *Lagrange interpolation formula* of function *f(x)* approximated by its value at N points $x_1, x_2, \ldots x_N$.

$$L\big(f(x)\big)=\sum_{k=1}^{N}f(x_k)P_k(x) \qquad \text{where: } P_k(x)=\frac{\prod_{j\neq k}^{N}\left(x-x_j\right)}{\prod_{j\neq k}^{N}\left(x_k-x_j\right)}$$

Each polynomial term $P_m(x)$ has zeros at each point $x=x_j$ *except* where $x=x_m$. Then $P_m(x_m)=1$.

A Proof of Projector Completeness *(Truer-than-true)*

Compare matrix *completeness relation* and *functional spectral decompositions*

$$\mathbf{1}=\mathbf{P}_1+\mathbf{P}_2+\ldots+\mathbf{P}_n=\sum_{\varepsilon_k}\mathbf{P}_k=\sum_{\varepsilon_k}\frac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}\qquad f(\mathbf{M})=f(\varepsilon_1)\mathbf{P}_1+f(\varepsilon_2)\mathbf{P}_2+\ldots+f(\varepsilon_n)\mathbf{P}_n=\sum_{\varepsilon_k}f(\varepsilon_k)\mathbf{P}_k=\sum_{\varepsilon_k}f(\varepsilon_k)\frac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$$

with *Lagrange interpolation formula* of function $f(x)$ approximated by its value at N points $x_1, x_2, \ldots x_N$.

$$L\big(f(x)\big)=\sum_{k=1}^{N}f(x_k)\cdot P_k(x)\qquad \text{where: } P_k(x)=\frac{\prod_{j\neq k}^{N}\left(x-x_j\right)}{\prod_{j\neq k}^{N}\left(x_k-x_j\right)}$$

Each polynomial term $P_m(x)$ has zeros at each point $x=x_j$ *except* where $x=x_m$. Then $P_m(x_m)=1$.
So at each of these points this L-approximation becomes exact: $L(f(x_j))=f(x_j)$.

A Proof of Projector Completeness *(Truer-than-true)*

Compare matrix *completeness relation* and *functional spectral decompositions*

$$\mathbf{1}=\mathbf{P}_1+\mathbf{P}_2+\ldots+\mathbf{P}_n=\sum_{\varepsilon_k}\mathbf{P}_k=\sum_{\varepsilon_k}\frac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}\qquad f(\mathbf{M})=f(\varepsilon_1)\mathbf{P}_1+f(\varepsilon_2)\mathbf{P}_2+\ldots+f(\varepsilon_n)\mathbf{P}_n=\sum_{\varepsilon_k}f(\varepsilon_k)\mathbf{P}_k=\sum_{\varepsilon_k}f(\varepsilon_k)\frac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$$

with *Lagrange interpolation formula* of function $f(x)$ approximated by its value at N points $x_1, x_2, \ldots x_N$.

$$L\big(f(x)\big)=\sum_{k=1}^{N}f(x_k)P_k(x)\qquad\text{where: }P_k(x)=\frac{\prod_{j\neq k}^{N}\left(x-x_j\right)}{\prod_{j\neq k}^{N}\left(x_k-x_j\right)}$$

Each polynomial term $P_m(x)$ has zeros at each point $x=x_j$ *except* where $x=x_m$. Then $P_m(x_m)=1$.
So at each of these points this L-approximation becomes exact: $L(f(x_j))=f(x_j)$.

If $f(x)$ happens to be a polynomial of degree $N-1$ or less, then $L(f(x))=f(x)$ may be exact everywhere.

$$1=\sum_{m=1}^{N}P_m\left(x\right)\qquad x=\sum_{m=1}^{N}x_m P_m\left(x\right)\qquad x^2=\sum_{m=1}^{N}x_m^2 P_m\left(x\right)$$

A Proof of Projector Completeness *(Truer-than-true)*

Compare matrix *completeness relation* and *functional spectral decompositions*

$$\mathbf{1}=\mathbf{P}_1+\mathbf{P}_2+\ldots+\mathbf{P}_n=\sum_{\varepsilon_k}\mathbf{P}_k=\sum_{\varepsilon_k}\frac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)} \qquad f(\mathbf{M})=f(\varepsilon_1)\mathbf{P}_1+f(\varepsilon_2)\mathbf{P}_2+\ldots+f(\varepsilon_n)\mathbf{P}_n=\sum_{\varepsilon_k}f(\varepsilon_k)\mathbf{P}_k=\sum_{\varepsilon_k}f(\varepsilon_k)\frac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$$

with *Lagrange interpolation formula* of function *f(x)* approximated by its value at N points $x_1, x_2, \ldots x_N$.

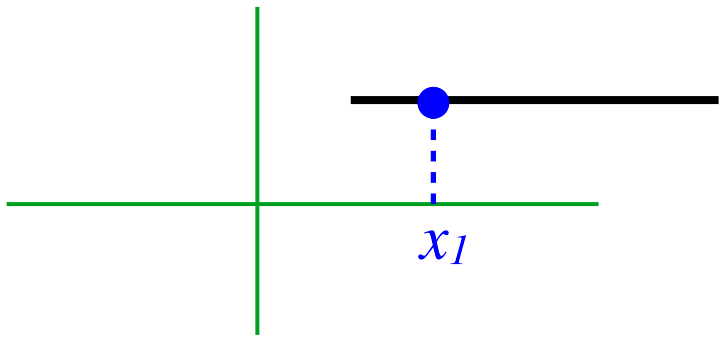
$$L\big(f(x)\big)=\sum_{k=1}^{N} f(x_k)P_k(x) \qquad \text{where: } P_k(x)=\frac{\prod_{j\neq k}^{N}\left(x-x_j\right)}{\prod_{j\neq k}^{N}\left(x_k-x_j\right)}$$

Each polynomial term $P_m(x)$ has zeros at each point $x=x_j$ *except* where $x=x_m$. Then $P_m(x_m)=1$.
So at each of these points this L-approximation becomes exact: $L(f(x_j))=f(x_j)$.

If $f(x)$ happens to be a polynomial of degree N-1 or less, then $L(f(x))=f(x)$ may be exact everywhere.

$$1=\sum_{m=1}^{N}P_m\left(x\right) \qquad x=\sum_{m=1}^{N}x_mP_m\left(x\right) \qquad x^2=\sum_{m=1}^{N}x_m^2P_m\left(x\right)$$

One point determines a constant level line,



A Proof of Projector Completeness *(Truer-than-true)*

Compare matrix *completeness relation* and *functional spectral decompositions*

$$\mathbf{1}=\mathbf{P}_1+\mathbf{P}_2+\ldots+\mathbf{P}_n=\sum_{\varepsilon_k}\mathbf{P}_k=\sum_{\varepsilon_k}\frac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}\qquad f(\mathbf{M})=f(\varepsilon_1)\mathbf{P}_1+f(\varepsilon_2)\mathbf{P}_2+\ldots+f(\varepsilon_n)\mathbf{P}_n=\sum_{\varepsilon_k}f(\varepsilon_k)\mathbf{P}_k=\sum_{\varepsilon_k}f(\varepsilon_k)\frac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$$

with *Lagrange interpolation formula* of function $f(x)$ approximated by its value at N points $x_1, x_2, \ldots x_N$.

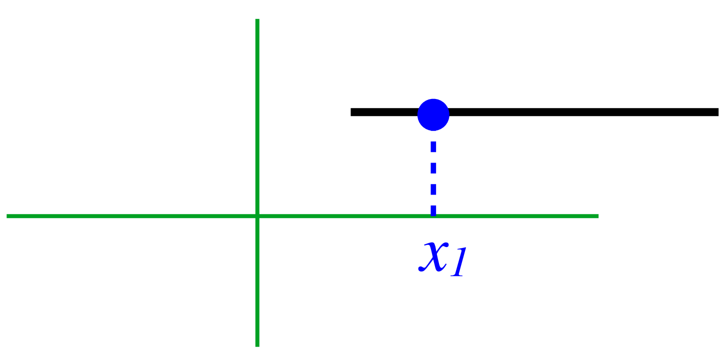
$$L\big(f(x)\big)=\sum_{k=1}^{N} f(x_k)\,P_k(x)\qquad \text{where: } P_k(x)=\frac{\prod_{j\neq k}^{N}\left(x-x_j\right)}{\prod_{j\neq k}^{N}\left(x_k-x_j\right)}$$

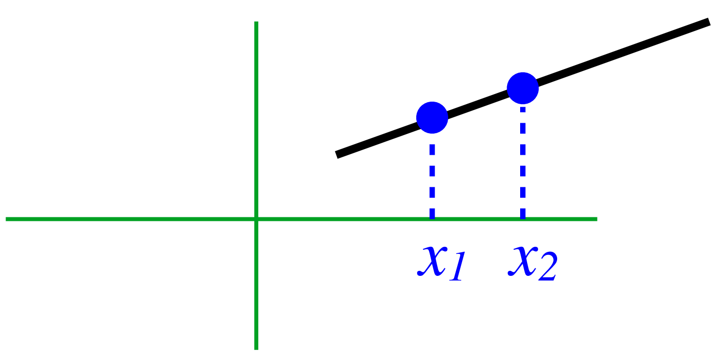
Each polynomial term $P_m(x)$ has zeros at each point $x=x_j$ *except* where $x=x_m$. Then $P_m(x_m)=1$.
So at each of these points this L-approximation becomes exact: $L(f(x_j))=f(x_j)$.

If $f(x)$ happens to be a polynomial of degree N-1 or less, then $L(f(x))=f(x)$ may be exact everywhere.

$$1=\sum_{m=1}^{N}P_m\left(x\right)\qquad x=\sum_{m=1}^{N}x_m P_m\left(x\right)\qquad x^2=\sum_{m=1}^{N}x_m^2 P_m\left(x\right)$$

One point determines a constant level line, two separate points uniquely determine a sloping line,





A Proof of Projector Completeness *(Truer-than-true)*

Compare matrix *completeness relation* and *functional spectral decompositions*

$$\mathbf{1}=\mathbf{P}_1+\mathbf{P}_2+\ldots+\mathbf{P}_n=\sum_{\varepsilon_k}\mathbf{P}_k=\sum_{\varepsilon_k}\frac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)} \qquad f(\mathbf{M})=f(\varepsilon_1)\mathbf{P}_1+f(\varepsilon_2)\mathbf{P}_2+\ldots+f(\varepsilon_n)\mathbf{P}_n=\sum_{\varepsilon_k}f(\varepsilon_k)\mathbf{P}_k=\sum_{\varepsilon_k}f(\varepsilon_k)\frac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$$

with *Lagrange interpolation formula* of function $f(x)$ approximated by its value at N points $x_1, x_2, \ldots x_N$.

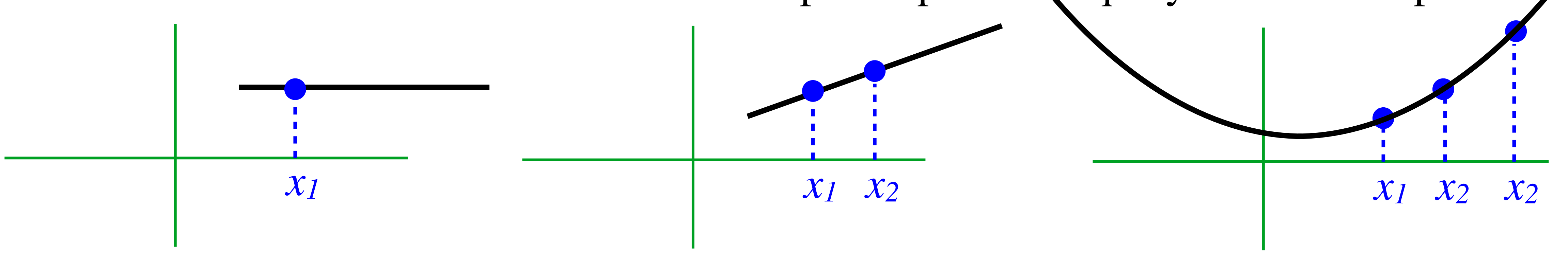
$$L\big(f(x)\big)=\sum_{k=1}^{N}f(x_k)P_k(x) \qquad \text{where: } P_k(x)=\frac{\prod_{j\neq k}^{N}\left(x-x_j\right)}{\prod_{j\neq k}^{N}\left(x_k-x_j\right)}$$

Each polynomial term $P_m(x)$ has zeros at each point $x=x_j$ *except* where $x=x_m$. Then $P_m(x_m)=1$.
So at each of these points this L-approximation becomes exact: $L(f(x_j))=f(x_j)$.

If $f(x)$ happens to be a polynomial of degree $N-1$ or less, then $L(f(x))=f(x)$ may be exact everywhere.

$$1=\sum_{m=1}^{N}P_m\left(x\right) \qquad x=\sum_{m=1}^{N}x_m P_m\left(x\right) \qquad x^2=\sum_{m=1}^{N}x_m^2 P_m\left(x\right)$$

One point determines a constant level line, two separate points uniquely determine a sloping line,
three separate points uniquely determine a parabola, etc.



A Proof of Projector Completeness *(Truer-than-true)*

Compare matrix *completeness relation* and *functional spectral decompositions*

$$\mathbf{1}=\mathbf{P}_1+\mathbf{P}_2+\ldots+\mathbf{P}_n=\sum_{\varepsilon_k}\mathbf{P}_k=\sum_{\varepsilon_k}\frac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$$

$$f(\mathbf{M})=f(\varepsilon_1)\mathbf{P}_1+f(\varepsilon_2)\mathbf{P}_2+\ldots+f(\varepsilon_n)\mathbf{P}_n=\sum_{\varepsilon_k}f(\varepsilon_k)\mathbf{P}_k=\sum_{\varepsilon_k}f(\varepsilon_k)\frac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$$

with *Lagrange interpolation formula* of function $f(x)$ approximated by its value at N points $x_1, x_2, \ldots x_N$.

$$L\big(f(x)\big)=\sum_{k=1}^{N} f(x_k)P_k(x) \qquad \text{where: } P_k(x)=\frac{\prod_{j\neq k}^{N}\left(x-x_j\right)}{\prod_{j\neq k}^{N}\left(x_k-x_j\right)}$$

Each polynomial term $P_m(x)$ has zeros at each point $x=x_j$ *except* where $x=x_m$. Then $P_m(x_m)=1$.
So at each of these points this L-approximation becomes exact: $L(f(x_j))=f(x_j)$.

If $f(x)$ happens to be a polynomial of degree N-1 or less, then $L(f(x))=f(x)$ may be exact everywhere.

$$1=\sum_{m=1}^{N}P_m(x) \qquad x=\sum_{m=1}^{N}x_m P_m(x) \qquad x^2=\sum_{m=1}^{N}x_m^2 P_m(x)$$

One point determines a constant level line, two separate points uniquely determine a sloping line, three separate points uniquely determine a parabola, etc.

Lagrange interpolation formula→*Completeness formula* as $x\rightarrow\mathbf{M}$ and as $x_k\rightarrow\varepsilon_k$ and as $P_k(x_k)\rightarrow\mathbf{P}_k$

A Proof of Projector Completeness (Truer-than-true)

Compare matrix *completeness relation* and *functional spectral decompositions*

$$\mathbf{1}=\mathbf{P}_1+\mathbf{P}_2+\ldots+\mathbf{P}_n=\sum_{\varepsilon_k}\mathbf{P}_k=\sum_{\varepsilon_k}\frac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$$

$$f(\mathbf{M})=f(\varepsilon_1)\mathbf{P}_1+f(\varepsilon_2)\mathbf{P}_2+\ldots+f(\varepsilon_n)\mathbf{P}_n=\sum_{\varepsilon_k}f(\varepsilon_k)\mathbf{P}_k=\sum_{\varepsilon_k}f(\varepsilon_k)\frac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$$

with *Lagrange interpolation formula* of function $f(x)$ approximated by its value at N points $x_1, x_2,\ldots x_N$.

$$L\big(f(x)\big)=\sum_{k=1}^{N}f(x_k)P_k(x) \qquad \text{where: } P_k(x)=\frac{\prod_{j\neq k}^{N}\left(x-x_j\right)}{\prod_{j\neq k}^{N}\left(x_k-x_j\right)}$$

Each polynomial term $P_m(x)$ has zeros at each point $x=x_j$ *except* where $x=x_m$. Then $P_m(x_m)=1$.
So at each of these points this L-approximation becomes exact: $L(f(x_j))=f(x_j)$.

If $f(x)$ happens to be a polynomial of degree N-1 or less, then $L(f(x))=f(x)$ may be exact everywhere.

$$1=\sum_{m=1}^{N}P_m\left(x\right) \qquad x=\sum_{m=1}^{N}x_mP_m\left(x\right) \qquad x^2=\sum_{m=1}^{N}x_m^2P_m\left(x\right)$$

One point determines a constant level line, two separate points uniquely determine a sloping line, three separate points uniquely determine a parabola, etc.

Lagrange interpolation formula$\to$*Completeness formula* as $x\to\mathbf{M}$ and as $x_k\to\varepsilon_k$ and as $P_k(x_k)\to\mathbf{P}_k$

All distinct values $\varepsilon_1\neq\varepsilon_2\neq\ldots\neq\varepsilon_N$ satisfy $\Sigma\mathbf{P}_k=\mathbf{1}$.

A Proof of Projector Completeness (Truer-than-true)

Compare matrix *completeness relation* and *functional spectral decompositions*

$$\mathbf{1}=\mathbf{P}_1+\mathbf{P}_2+\ldots+\mathbf{P}_n=\sum_{\varepsilon_k}\mathbf{P}_k=\sum_{\varepsilon_k}\frac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)} \qquad f(\mathbf{M})=f(\varepsilon_1)\mathbf{P}_1+f(\varepsilon_2)\mathbf{P}_2+\ldots+f(\varepsilon_n)\mathbf{P}_n=\sum_{\varepsilon_k}f(\varepsilon_k)\mathbf{P}_k=\sum_{\varepsilon_k}f(\varepsilon_k)\frac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$$

with *Lagrange interpolation formula* of function *f(x)* approximated by its value at N points $x_1, x_2,\ldots x_N$.

$$L\left(f(x)\right)=\sum_{k=1}^{N} f(x_k)\,P_k(x) \qquad \text{where: } P_k(x)=\frac{\prod_{j\neq k}^{N}\left(x-x_j\right)}{\prod_{j\neq k}^{N}\left(x_k-x_j\right)}$$

Each polynomial term $P_m(x)$ has zeros at each point $x=x_j$ *except* where $x=x_m$. Then $P_m(x_m)=1$.
So at each of these points this L-approximation becomes exact: $L(f(x_j))=f(x_j)$.

If $f(x)$ happens to be a polynomial of degree $N-1$ or less, then $L(f(x))=f(x)$ may be exact everywhere.

$$1=\sum_{m=1}^{N}P_m\left(x\right) \qquad x=\sum_{m=1}^{N}x_m P_m\left(x\right) \qquad x^2=\sum_{m=1}^{N}x_m^2 P_m\left(x\right)$$

One point determines a constant level line, two separate points uniquely determine a sloping line, three separate points uniquely determine a parabola, etc.

Lagrange interpolation formula$\rightarrow$*Completeness formula* as $x\rightarrow\mathbf{M}$ and as $x_k\rightarrow\varepsilon_k$ and as $P_k(x_k)\rightarrow\mathbf{P}_k$

All distinct values $\varepsilon_1\neq\varepsilon_2\neq\ldots\neq\varepsilon_N$ satisfy $\Sigma\mathbf{P}_k=\mathbf{1}$. Completeness is *truer than true* as is seen for $N=2$.

$$\mathbf{P}_1+\mathbf{P}_2=\frac{\prod_{j\neq 1}\left(\mathbf{M}-\varepsilon_j\mathbf{1}\right)}{\prod_{j\neq 1}\left(\varepsilon_1-\varepsilon_j\right)}+\frac{\prod_{j\neq 1}\left(\mathbf{M}-\varepsilon_j\mathbf{1}\right)}{\prod_{j\neq 1}\left(\varepsilon_2-\varepsilon_j\right)}=\frac{\left(\mathbf{M}-\varepsilon_2\mathbf{1}\right)}{\left(\varepsilon_1-\varepsilon_2\right)}+\frac{\left(\mathbf{M}-\varepsilon_1\mathbf{1}\right)}{\left(\varepsilon_2-\varepsilon_1\right)}=\frac{\left(\mathbf{M}-\varepsilon_2\mathbf{1}\right)-\left(\mathbf{M}-\varepsilon_1\mathbf{1}\right)}{\left(\varepsilon_1-\varepsilon_2\right)}=\frac{-\varepsilon_2\mathbf{1}+\varepsilon_1\mathbf{1}}{\left(\varepsilon_1-\varepsilon_2\right)}=\mathbf{1}\ (\text{for } all\ \varepsilon_j)$$

A Proof of Projector Completeness *(Truer-than-true)*

Compare matrix *completeness relation* and *functional spectral decompositions*

$$\mathbf{1}=\mathbf{P}_1+\mathbf{P}_2+\ldots+\mathbf{P}_n=\sum_{\varepsilon_k}\mathbf{P}_k=\sum_{\varepsilon_k}\frac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)} \qquad f(\mathbf{M})=f(\varepsilon_1)\mathbf{P}_1+f(\varepsilon_2)\mathbf{P}_2+\ldots+f(\varepsilon_n)\mathbf{P}_n=\sum_{\varepsilon_k}f(\varepsilon_k)\mathbf{P}_k=\sum_{\varepsilon_k}f(\varepsilon_k)\frac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$$

with *Lagrange interpolation formula* of function *f(x)* approximated by its value at N points $x_1, x_2, \ldots x_N$.

$$L\big(f(x)\big)=\sum_{k=1}^{N} f(x_k)P_k(x) \qquad \text{where: } P_k(x)=\frac{\prod_{j\neq k}^{N}\left(x-x_j\right)}{\prod_{j\neq k}^{N}\left(x_k-x_j\right)}$$

Each polynomial term $P_m(x)$ has zeros at each point $x=x_j$ *except* where $x=x_m$. Then $P_m(x_m)=1$.
So at each of these points this L-approximation becomes exact: $L(f(x_j))=f(x_j)$.

If $f(x)$ happens to be a polynomial of degree $N-1$ or less, then $L(f(x))=f(x)$ may be exact everywhere.

$$1=\sum_{m=1}^{N}P_m(x) \qquad x=\sum_{m=1}^{N}x_mP_m(x) \qquad x^2=\sum_{m=1}^{N}x_m^2P_m(x)$$

One point determines a constant level line, two separate points uniquely determine a sloping line, three separate points uniquely determine a parabola, etc.

Lagrange interpolation formula→*Completeness formula* as $x\to\mathbf{M}$ and as $x_k\to\varepsilon_k$ and as $P_k(x_k)\to\mathbf{P}_k$

All distinct values $\varepsilon_1\neq\varepsilon_2\neq\ldots\neq\varepsilon_N$ satisfy $\Sigma\mathbf{P}_k=\mathbf{1}$. Completeness is *truer than true* as is seen for $N=2$.

$$\mathbf{P}_1+\mathbf{P}_2=\frac{\prod_{j\neq1}\left(\mathbf{M}-\varepsilon_j\mathbf{1}\right)}{\prod_{j\neq1}\left(\varepsilon_1-\varepsilon_j\right)}+\frac{\prod_{j\neq1}\left(\mathbf{M}-\varepsilon_j\mathbf{1}\right)}{\prod_{j\neq1}\left(\varepsilon_2-\varepsilon_j\right)}=\frac{\left(\mathbf{M}-\varepsilon_2\mathbf{1}\right)}{\left(\varepsilon_1-\varepsilon_2\right)}+\frac{\left(\mathbf{M}-\varepsilon_1\mathbf{1}\right)}{\left(\varepsilon_2-\varepsilon_1\right)}=\frac{\left(\mathbf{M}-\varepsilon_2\mathbf{1}\right)-\left(\mathbf{M}-\varepsilon_1\mathbf{1}\right)}{\left(\varepsilon_1-\varepsilon_2\right)}=\frac{-\varepsilon_2\mathbf{1}+\varepsilon_1\mathbf{1}}{\left(\varepsilon_1-\varepsilon_2\right)}=\mathbf{1}\ (\text{for } all\ \varepsilon_j)$$

However, only *select* values ε_k work for eigen-forms $\mathbf{MP}_k=\varepsilon_k\mathbf{P}_k$ or orthonormality $\mathbf{P}_j\mathbf{P}_k=\delta_{jk}\mathbf{P}_k$.

2D harmonic oscillator equations
Lagrangian and matrix forms and Reciprocity symmetry

2D harmonic oscillator equation eigensolutions
Geometric method
Matrix-algebraic eigensolutions with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$
Secular equation
Hamilton-Cayley equation and projectors
Idempotent projectors (how eigenvalues⇒eigenvectors)
Operator orthonormality and Completeness (Idempotent means: $\mathbf{P}\cdot\mathbf{P}=\mathbf{P}$*)*
Spectral Decompositions
Functional spectral decomposition
Orthonormality vs. Completeness vis-a`-vis Operator vs. State
Lagrange functional interpolation formula
➡ *Diagonalizing Transformations (D-Ttran) from projectors* ⬅

2D-HO eigensolution example with bilateral (B-Type) symmetry
Mixed mode beat dynamics and fixed π/2 phase
2D-HO eigensolution example with asymmetric (A-Type) symmetry
Initial state projection, mixed mode beat dynamics with variable phase

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$
Hamilton-Pauli spinor symmetry (ABCD-Types)

Diagonalizing Transformations (D-Ttran) from projectors

Given our eigenvectors and their Projectors.

$$\mathbf{P}_1 = \frac{(\mathbf{M} - 5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1 \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}{k_1} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2 = \frac{(\mathbf{M} - 1\cdot\mathbf{1})}{(5-1)} = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = k_2 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}}{k_2} = |\varepsilon_2\rangle\langle\varepsilon_2|$$

Diagonalizing Transformations (D-Ttran) from projectors

Given our eigenvectors and their Projectors.

$$\mathbf{P}_1 = \frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1\begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}{k_1} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2 = \frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)} = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = k_2\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}}{k_2} = |\varepsilon_2\rangle\langle\varepsilon_2|$$

Load distinct bras $\langle\varepsilon_1|$ and $\langle\varepsilon_2|$ into d-tran rows, kets $|\varepsilon_1\rangle$ and $|\varepsilon_2\rangle$ into <u>inverse</u> d-tran columns.

Diagonalizing Transformations (D-Ttran) from projectors

Given our eigenvectors and their Projectors.

$$\mathbf{P}_1 = \frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1\begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}{k_1} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2 = \frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)} = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = k_2\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}}{k_2} = |\varepsilon_2\rangle\langle\varepsilon_2|$$

Load distinct bras $\langle\varepsilon_1|$ and $\langle\varepsilon_2|$ into d-tran rows, kets $|\varepsilon_1\rangle$ and $|\varepsilon_2\rangle$ into <u>inverse</u> d-tran columns.

$$\left\{\langle\varepsilon_1| = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \langle\varepsilon_2| = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}\right\}, \quad \left\{|\varepsilon_1\rangle = \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix}, |\varepsilon_2\rangle = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}\right\}$$

$(\varepsilon_1,\varepsilon_2) \leftarrow (1,2)$ *d–Tran matrix* $\qquad$ $(1,2) \leftarrow (\varepsilon_1,\varepsilon_2)$ INVERSE *d–Tran matrix*

$$\begin{pmatrix} \langle\varepsilon_1|x\rangle & \langle\varepsilon_1|y\rangle \\ \langle\varepsilon_2|x\rangle & \langle\varepsilon_2|y\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}, \quad \begin{pmatrix} \langle x|\varepsilon_1\rangle & \langle x|\varepsilon_2\rangle \\ \langle y|\varepsilon_1\rangle & \langle y|\varepsilon_2\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

Diagonalizing Transformations (D-Ttran) from projectors

Given our eigenvectors and their Projectors.

$$\mathbf{P}_1 = \frac{(\mathbf{M} - 5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1 \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}{k_1} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2 = \frac{(\mathbf{M} - 1\cdot\mathbf{1})}{(5-1)} = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = k_2 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}}{k_2} = |\varepsilon_2\rangle\langle\varepsilon_2|$$

Load distinct bras $\langle\varepsilon_1|$ and $\langle\varepsilon_2|$ into d-tran rows, kets $|\varepsilon_1\rangle$ and $|\varepsilon_2\rangle$ into <u>inverse</u> d-tran columns.

$$\left\{\langle\varepsilon_1| = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \langle\varepsilon_2| = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}\right\} \quad , \quad \left\{|\varepsilon_1\rangle = \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix}, |\varepsilon_2\rangle = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}\right\}$$

$(\varepsilon_1,\varepsilon_2) \leftarrow (1,2)$ *d–Tran matrix* $\qquad$ $(1,2) \leftarrow (\varepsilon_1,\varepsilon_2)$ INVERSE *d–Tran matrix*

$$\begin{pmatrix} \langle\varepsilon_1|x\rangle & \langle\varepsilon_1|y\rangle \\ \langle\varepsilon_2|x\rangle & \langle\varepsilon_2|y\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix} \quad , \quad \begin{pmatrix} \langle x|\varepsilon_1\rangle & \langle x|\varepsilon_2\rangle \\ \langle y|\varepsilon_1\rangle & \langle y|\varepsilon_2\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

Use *Dirac labeling for all components* so transformation is OK

$$\begin{pmatrix} \langle\varepsilon_1|x\rangle & \langle\varepsilon_1|y\rangle \\ \langle\varepsilon_2|x\rangle & \langle\varepsilon_2|y\rangle \end{pmatrix} \cdot \begin{pmatrix} \langle x|\mathbf{K}|x\rangle & \langle x|\mathbf{K}|y\rangle \\ \langle y|\mathbf{K}|x\rangle & \langle y|\mathbf{K}|y\rangle \end{pmatrix} \cdot \begin{pmatrix} \langle x|\varepsilon_1\rangle & \langle x|\varepsilon_2\rangle \\ \langle y|\varepsilon_1\rangle & \langle y|\varepsilon_2\rangle \end{pmatrix} = \begin{pmatrix} \langle\varepsilon_1|\mathbf{K}|\varepsilon_1\rangle & \langle\varepsilon_1|\mathbf{K}|\varepsilon_2\rangle \\ \langle\varepsilon_2|\mathbf{K}|\varepsilon_1\rangle & \langle\varepsilon_2|\mathbf{K}|\varepsilon_2\rangle \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$$

Diagonalizing Transformations (D-Ttran) from projectors

Given our eigenvectors and their Projectors.

$$\mathbf{P}_1=\frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)}=\frac{1}{4}\begin{pmatrix}1 & -1\\ -3 & 3\end{pmatrix}=k_1\begin{pmatrix}\frac{1}{2}\\ -\frac{3}{2}\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\end{pmatrix}}{k_1}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2=\frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)}=\frac{1}{4}\begin{pmatrix}3 & 1\\ 3 & 1\end{pmatrix}=k_2\begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac{3}{2} & \frac{1}{2}\end{pmatrix}}{k_2}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

Load distinct bras $\langle\varepsilon_1|$ and $\langle\varepsilon_2|$ into d-tran rows, kets $|\varepsilon_1\rangle$ and $|\varepsilon_2\rangle$ into <u>inverse</u> d-tran columns.

$$\left\{\langle\varepsilon_1|=\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\end{pmatrix},\langle\varepsilon_2|=\begin{pmatrix}\frac{3}{2} & \frac{1}{2}\end{pmatrix}\right\},\quad\left\{|\varepsilon_1\rangle=\begin{pmatrix}\frac{1}{2}\\ -\frac{3}{2}\end{pmatrix},|\varepsilon_2\rangle=\begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\end{pmatrix}\right\}$$

$(\varepsilon_1,\varepsilon_2)\leftarrow(1,2)$ *d–Tran matrix* $\qquad$ $(1,2)\leftarrow(\varepsilon_1,\varepsilon_2)$ INVERSE *d–Tran matrix*

$$\begin{pmatrix}\langle\varepsilon_1|x\rangle & \langle\varepsilon_1|y\rangle\\ \langle\varepsilon_2|x\rangle & \langle\varepsilon_2|y\rangle\end{pmatrix}=\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\ \frac{3}{2} & \frac{1}{2}\end{pmatrix},\quad\begin{pmatrix}\langle x|\varepsilon_1\rangle & \langle x|\varepsilon_2\rangle\\ \langle y|\varepsilon_1\rangle & \langle y|\varepsilon_2\rangle\end{pmatrix}=\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ -\frac{3}{2} & \frac{1}{2}\end{pmatrix}$$

Use *Dirac labeling for all components* so transformation is OK

$$\begin{pmatrix}\langle\varepsilon_1|x\rangle & \langle\varepsilon_1|y\rangle\\ \langle\varepsilon_2|x\rangle & \langle\varepsilon_2|y\rangle\end{pmatrix}\cdot\begin{pmatrix}\langle x|\mathbf{K}|x\rangle & \langle x|\mathbf{K}|y\rangle\\ \langle y|\mathbf{K}|x\rangle & \langle y|\mathbf{K}|y\rangle\end{pmatrix}\cdot\begin{pmatrix}\langle x|\varepsilon_1\rangle & \langle x|\varepsilon_2\rangle\\ \langle y|\varepsilon_1\rangle & \langle y|\varepsilon_2\rangle\end{pmatrix}=\begin{pmatrix}\langle\varepsilon_1|\mathbf{K}|\varepsilon_1\rangle & \langle\varepsilon_1|\mathbf{K}|\varepsilon_2\rangle\\ \langle\varepsilon_2|\mathbf{K}|\varepsilon_1\rangle & \langle\varepsilon_2|\mathbf{K}|\varepsilon_2\rangle\end{pmatrix}$$

$$\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\ \frac{3}{2} & \frac{1}{2}\end{pmatrix}\cdot\begin{pmatrix}4 & 1\\ 3 & 2\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ -\frac{3}{2} & \frac{1}{2}\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 5\end{pmatrix}$$

Check inverse-d-tran is really inverse of *your* d-tran.

$$\begin{pmatrix}\langle\varepsilon_1|1\rangle & \langle\varepsilon_1|2\rangle\\ \langle\varepsilon_2|1\rangle & \langle\varepsilon_2|2\rangle\end{pmatrix}\cdot\begin{pmatrix}\langle 1|\varepsilon_1\rangle & \langle 1|\varepsilon_2\rangle\\ \langle 2|\varepsilon_1\rangle & \langle 2|\varepsilon_2\rangle\end{pmatrix}=\begin{pmatrix}\langle\varepsilon_1|\mathbf{1}|\varepsilon_1\rangle & \langle\varepsilon_1|\mathbf{1}|\varepsilon_2\rangle\\ \langle\varepsilon_2|\mathbf{1}|\varepsilon_1\rangle & \langle\varepsilon_2|\mathbf{1}|\varepsilon_2\rangle\end{pmatrix}$$

$$\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\ \frac{3}{2} & \frac{1}{2}\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ -\frac{3}{2} & \frac{1}{2}\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

Diagonalizing Transformations (D-Ttran) from projectors

Given our eigenvectors and their Projectors.

$$\mathbf{P}_1=\frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)}=\frac{1}{4}\begin{pmatrix}1&-1\\-3&3\end{pmatrix}=k_1\begin{pmatrix}\frac{1}{2}\\-\frac{3}{2}\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\end{pmatrix}}{k_1}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2=\frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)}=\frac{1}{4}\begin{pmatrix}3&1\\3&1\end{pmatrix}=k_2\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac{3}{2}&\frac{1}{2}\end{pmatrix}}{k_2}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

Load distinct bras $\langle\varepsilon_1|$ and $\langle\varepsilon_2|$ into d-tran rows, kets $|\varepsilon_1\rangle$ and $|\varepsilon_2\rangle$ into <u>inverse</u> d-tran columns.

$$\left\{\langle\varepsilon_1|=\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\end{pmatrix},\langle\varepsilon_2|=\begin{pmatrix}\frac{3}{2}&\frac{1}{2}\end{pmatrix}\right\}\quad,\quad\left\{|\varepsilon_1\rangle=\begin{pmatrix}\frac{1}{2}\\-\frac{3}{2}\end{pmatrix},|\varepsilon_2\rangle=\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\end{pmatrix}\right\}$$

$(\varepsilon_1,\varepsilon_2)\leftarrow(1,2)$ *d–Tran matrix* $\qquad$ $(1,2)\leftarrow(\varepsilon_1,\varepsilon_2)$ INVERSE *d–Tran matrix*

$$\begin{pmatrix}\langle\varepsilon_1|x\rangle&\langle\varepsilon_1|y\rangle\\\langle\varepsilon_2|x\rangle&\langle\varepsilon_2|y\rangle\end{pmatrix}=\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\\\frac{3}{2}&\frac{1}{2}\end{pmatrix}\quad,\quad\begin{pmatrix}\langle x|\varepsilon_1\rangle&\langle x|\varepsilon_2\rangle\\\langle y|\varepsilon_1\rangle&\langle y|\varepsilon_2\rangle\end{pmatrix}=\begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\-\frac{3}{2}&\frac{1}{2}\end{pmatrix}$$

Use *Dirac labeling for all components* so transformation is OK

$$\begin{pmatrix}\langle\varepsilon_1|x\rangle&\langle\varepsilon_1|y\rangle\\\langle\varepsilon_2|x\rangle&\langle\varepsilon_2|y\rangle\end{pmatrix}\cdot\begin{pmatrix}\langle x|\mathbf{K}|x\rangle&\langle x|\mathbf{K}|y\rangle\\\langle y|\mathbf{K}|x\rangle&\langle y|\mathbf{K}|y\rangle\end{pmatrix}\cdot\begin{pmatrix}\langle x|\varepsilon_1\rangle&\langle x|\varepsilon_2\rangle\\\langle y|\varepsilon_1\rangle&\langle y|\varepsilon_2\rangle\end{pmatrix}=\begin{pmatrix}\langle\varepsilon_1|\mathbf{K}|\varepsilon_1\rangle&\langle\varepsilon_1|\mathbf{K}|\varepsilon_2\rangle\\\langle\varepsilon_2|\mathbf{K}|\varepsilon_1\rangle&\langle\varepsilon_2|\mathbf{K}|\varepsilon_2\rangle\end{pmatrix}$$

$$\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\\\frac{3}{2}&\frac{1}{2}\end{pmatrix}\cdot\begin{pmatrix}4&1\\3&2\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\-\frac{3}{2}&\frac{1}{2}\end{pmatrix}=\begin{pmatrix}1&0\\0&5\end{pmatrix}$$

Check inverse-d-tran is really inverse of *your* d-tran. In standard quantum matrices inverses are "easy"

$$\begin{pmatrix}\langle\varepsilon_1|x\rangle&\langle\varepsilon_1|y\rangle\\\langle\varepsilon_2|x\rangle&\langle\varepsilon_2|y\rangle\end{pmatrix}\cdot\begin{pmatrix}\langle x|\varepsilon_1\rangle&\langle x|\varepsilon_2\rangle\\\langle y|\varepsilon_1\rangle&\langle y|\varepsilon_2\rangle\end{pmatrix}=\begin{pmatrix}\langle\varepsilon_1|\mathbf{1}|\varepsilon_1\rangle&\langle\varepsilon_1|\mathbf{1}|\varepsilon_2\rangle\\\langle\varepsilon_2|\mathbf{1}|\varepsilon_1\rangle&\langle\varepsilon_2|\mathbf{1}|\varepsilon_2\rangle\end{pmatrix}$$

$$\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\\\frac{3}{2}&\frac{1}{2}\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\-\frac{3}{2}&\frac{1}{2}\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$\begin{pmatrix}\langle\varepsilon_1|x\rangle&\langle\varepsilon_1|y\rangle\\\langle\varepsilon_2|x\rangle&\langle\varepsilon_2|y\rangle\end{pmatrix}=\begin{pmatrix}\langle x|\varepsilon_1\rangle&\langle x|\varepsilon_2\rangle\\\langle y|\varepsilon_1\rangle&\langle y|\varepsilon_2\rangle\end{pmatrix}^{\dagger}=\begin{pmatrix}\langle x|\varepsilon_1\rangle^*&\langle y|\varepsilon_1\rangle^*\\\langle x|\varepsilon_2\rangle^*&\langle y|\varepsilon_2\rangle^*\end{pmatrix}=\begin{pmatrix}\langle x|\varepsilon_1\rangle&\langle x|\varepsilon_2\rangle\\\langle y|\varepsilon_1\rangle&\langle y|\varepsilon_2\rangle\end{pmatrix}^{-1}$$

2D harmonic oscillator equations

Lagrangian and matrix forms and Reciprocity symmetry

2D harmonic oscillator equation eigensolutions

Geometric method

Matrix-algebraic eigensolutions with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

Secular equation

Hamilton-Cayley equation and projectors

Idempotent projectors (how eigenvalues⇒eigenvectors)

Operator orthonormality and Completeness (Idempotent means: $\mathbf{P}\cdot\mathbf{P}=\mathbf{P}$*)*

Spectral Decompositions

Functional spectral decomposition

Orthonormality vs. Completeness vis-a`-vis Operator vs. State

Lagrange functional interpolation formula

Diagonalizing Transformations (D-Ttran) from projectors

➔ *2D-HO eigensolution example with bilateral (B-Type) symmetry* ←

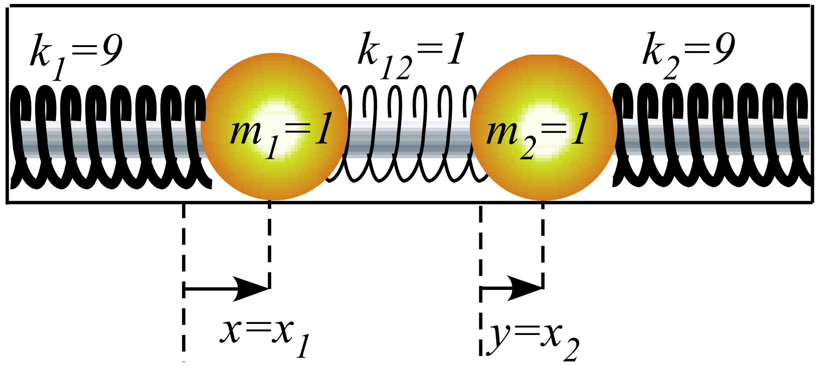
Mixed mode beat dynamics and fixed π/2 phase

2D-HO eigensolution example with asymmetric (A-Type) symmetry

Initial state projection, mixed mode beat dynamics with variable phase

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry (ABCD-Types)



Analyzing 2D-HO beats and mixed mode eigen-solutions

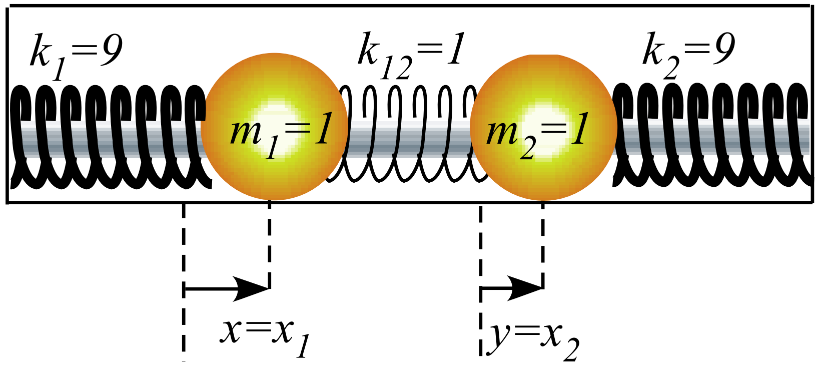
$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} = \begin{pmatrix} 10 & -1 \\ -1 & 10 \end{pmatrix}$$

$$Det(\mathbf{K}) = 10 \cdot 10 - 1 = 99$$

$$Trace(\mathbf{K}) = 10 + 10 = 20$$

The **K** secular equation $K^2 - Trace(\mathbf{K})K + Det(\mathbf{K}) = K^2 - 20K + 99 = 0 = (K-9)(K-11) = (K-K_1)(K-K_2)$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\varepsilon_k)^2$ $\quad K_1 = \omega_0^2(\varepsilon_1) = 9, \quad K_2 = \omega_0^2(\varepsilon_2) = 11,$



Analyzing 2D-HO beats and mixed mode eigen-solutions

$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} = \begin{pmatrix} 10 & -1 \\ -1 & 10 \end{pmatrix}$$

$Det(\mathbf{K}) = 10 \cdot 10 - 1 = 99$

$Trace(\mathbf{K}) = 10 + 10 = 20$

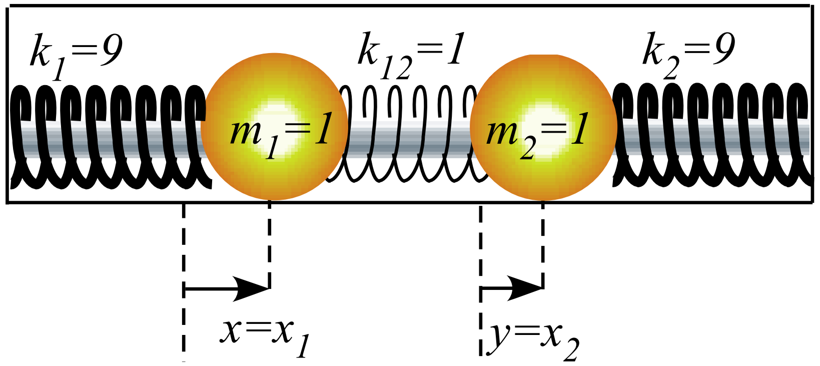
The **K** secular equation $K^2 - Trace(\mathbf{K})K + Det(\mathbf{K}) = K^2 - 20K + 99 = 0 = (K-9)(K-11) = (K-K_1)(K-K_2)$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\varepsilon_k)^2$ $\quad K_1 = \omega_0^2(\varepsilon_1) = 9, \quad K_2 = \omega_0^2(\varepsilon_2) = 11,$

Eigen-projectors $\mathbf{P}_k$

$$\mathbf{P}_1 = \frac{\begin{pmatrix} K_{11} - K_2 & K_{12} \\ K_{12} & K_{22} - K_2 \end{pmatrix}}{K_1 - K_2} = \frac{\begin{pmatrix} 10-11 & -1 \\ -1 & 10-11 \end{pmatrix}}{9-11} = \frac{\begin{pmatrix} 1 & +1 \\ +1 & 1 \end{pmatrix}}{2}$$

$$\mathbf{P}_2 = \frac{\begin{pmatrix} K_{11} - K_1 & K_{12} \\ K_{12} & K_{22} - K_1 \end{pmatrix}}{K_2 - K_1} = \frac{\begin{pmatrix} 10-9 & -1 \\ -1 & 10-9 \end{pmatrix}}{11-9} = \frac{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}{2}$$



Analyzing 2D-HO beats and mixed mode eigen-solutions

$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} = \begin{pmatrix} 10 & -1 \\ -1 & 10 \end{pmatrix}$$

$$Det(\mathbf{K}) = 10\cdot 10 - 1 = 99$$
$$Trace(\mathbf{K}) = 10 + 10 = 20$$

The $\mathbf{K}$ secular equation $\quad K^2 - Trace(\mathbf{K})K + Det(\mathbf{K}) = K^2 - 20K + 99 = 0 = (K-9)(K-11) = (K-K_1)(K-K_2)$

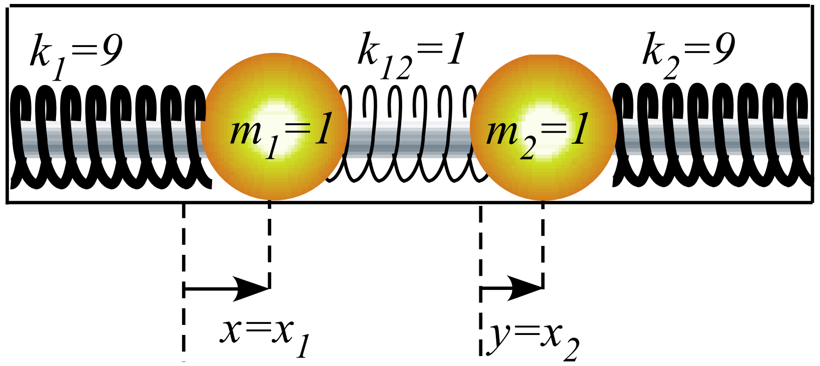
Eigenvalues K_k and squared eigenfrequencies $\omega_0(\varepsilon_k)^2$ $\quad K_1 = \omega_0^2(\varepsilon_1) = 9, \quad K_2 = \omega_0^2(\varepsilon_2) = 11,$

Eigen-projectors $\mathbf{P}_k$

$$\mathbf{P}_1 = \frac{\begin{pmatrix} K_{11}-K_2 & K_{12} \\ K_{12} & K_{22}-K_2 \end{pmatrix}}{K_1 - K_2} = \frac{\begin{pmatrix} 10-11 & -1 \\ -1 & 10-11 \end{pmatrix}}{9-11} = \frac{\begin{pmatrix} 1 & +1 \\ +1 & 1 \end{pmatrix}}{2}$$
$$= \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2 = \frac{\begin{pmatrix} K_{11}-K_1 & K_{12} \\ K_{12} & K_{22}-K_1 \end{pmatrix}}{K_2 - K_1} = \frac{\begin{pmatrix} 10-9 & -1 \\ -1 & 10-9 \end{pmatrix}}{11-9} = \frac{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}{2}$$
$$= \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} = |\varepsilon_2\rangle\langle\varepsilon_2|$$

Eigenbra vectors: $\quad \langle\varepsilon_1| = \begin{pmatrix} 1/\sqrt{2} & +1/\sqrt{2} \end{pmatrix}, \quad \langle\varepsilon_2| = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$



Analyzing 2D-HO beats and mixed mode eigen-solutions

$$\mathbf{K}=\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix}=\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix}=\begin{pmatrix} 10 & -1 \\ -1 & 10 \end{pmatrix}$$

$$Det(\mathbf{K})=10\cdot 10-1=99$$
$$Trace(\mathbf{K})=10+10=20$$

The $\mathbf{K}$ secular equation $\quad K^2-Trace(\mathbf{K})K+Det(\mathbf{K})=K^2-20K+99=0=(K-9)(K-11)=(K-K_1)(K-K_2)$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\varepsilon_k)^2$ $\quad K_1=\omega_0^2(\varepsilon_1)=9, \quad K_2=\omega_0^2(\varepsilon_2)=11,$

Eigen-projectors $\mathbf{P}_k$

$$\mathbf{P}_1=\frac{\begin{pmatrix} K_{11}-K_2 & K_{12} \\ K_{12} & K_{22}-K_2 \end{pmatrix}}{K_1-K_2}=\frac{\begin{pmatrix} 10-11 & -1 \\ -1 & 10-11 \end{pmatrix}}{9-11}=\frac{\begin{pmatrix} 1 & +1 \\ +1 & 1 \end{pmatrix}}{2}$$
$$=\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

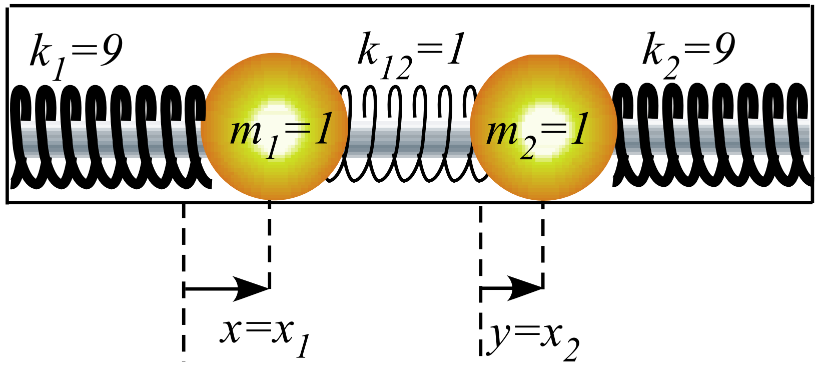
$$\mathbf{P}_2=\frac{\begin{pmatrix} K_{11}-K_1 & K_{12} \\ K_{12} & K_{22}-K_1 \end{pmatrix}}{K_2-K_1}=\frac{\begin{pmatrix} 10-9 & -1 \\ -1 & 10-9 \end{pmatrix}}{11-9}=\frac{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}{2}$$
$$=\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

Eigenbra vectors: $\langle\varepsilon_1|=\begin{pmatrix} 1/\sqrt{2} & +1/\sqrt{2} \end{pmatrix}, \quad \langle\varepsilon_2|=\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$

Mixed mode dynamics

$$|x(t)\rangle = |\varepsilon_1\rangle\langle\varepsilon_1|x(0)\rangle e^{-i\omega_1 t}+|\varepsilon_2\rangle\langle\varepsilon_2|x(0)\rangle e^{-i\omega_2 t}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}=\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}\langle\varepsilon_1|x(0)\rangle e^{-i\omega_1 t}+\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}\langle\varepsilon_2|x(0)\rangle e^{-i\omega_2 t}$$



Analyzing 2D-HO beats and mixed mode eigen-solutions

$$\mathbf{K}=\begin{pmatrix}K_{11} & K_{12}\\ K_{12} & K_{22}\end{pmatrix}=\begin{pmatrix}k_1+k_{12} & -k_{12}\\ -k_{12} & k_2+k_{12}\end{pmatrix}=\begin{pmatrix}10 & -1\\ -1 & 10\end{pmatrix}$$

$$Det(\mathbf{K})=10\cdot 10-1=99$$
$$Trace(\mathbf{K})=10+10=20$$

The **K** secular equation $K^2-Trace(\mathbf{K})K+Det(\mathbf{K})=K^2-20K+99=0=(K-9)(K-11)=(K-K_1)(K-K_2)$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\varepsilon_k)^2$ $\quad K_1=\omega_0^2(\varepsilon_1)=9,\quad K_2=\omega_0^2(\varepsilon_2)=11,$

Eigen-projectors $\mathbf{P}_k$

$$\mathbf{P}_1=\frac{\begin{pmatrix}K_{11}-K_2 & K_{12}\\ K_{12} & K_{22}-K_2\end{pmatrix}}{K_1-K_2}=\frac{\begin{pmatrix}10-11 & -1\\ -1 & 10-11\end{pmatrix}}{9-11}=\frac{\begin{pmatrix}1 & +1\\ +1 & 1\end{pmatrix}}{2}$$
$$=\begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix}\begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2=\frac{\begin{pmatrix}K_{11}-K_1 & K_{12}\\ K_{12} & K_{22}-K_1\end{pmatrix}}{K_2-K_1}=\frac{\begin{pmatrix}10-9 & -1\\ -1 & 10-9\end{pmatrix}}{11-9}=\frac{\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}}{2}$$
$$=\begin{pmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{pmatrix}\begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

Eigenbra vectors: $\langle\varepsilon_1|=\begin{pmatrix}1/\sqrt{2} & +1/\sqrt{2}\end{pmatrix},\quad \langle\varepsilon_2|=\begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix}$

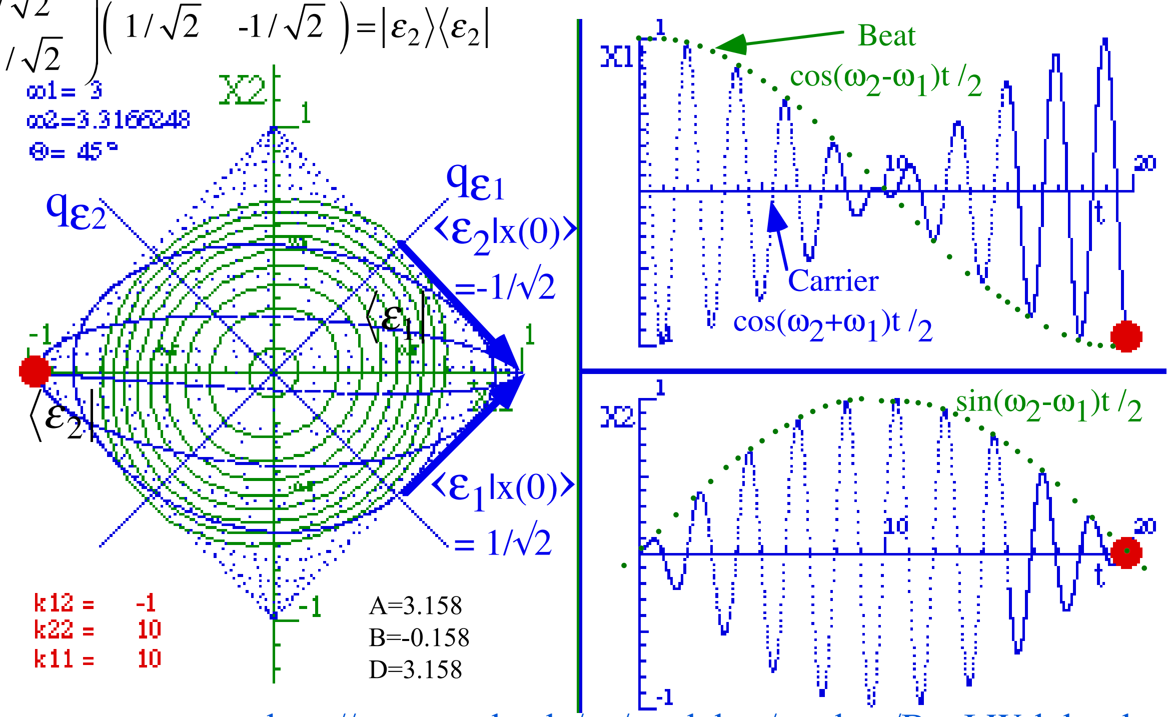
Mixed mode dynamics

$$|x(t)\rangle=|\varepsilon_1\rangle\langle\varepsilon_1|x(0)\rangle e^{-i\omega_1 t}+|\varepsilon_2\rangle\langle\varepsilon_2|x(0)\rangle e^{-i\omega_2 t}$$

$$\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}=\begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix}\langle\varepsilon_1|x(0)\rangle e^{-i\omega_1 t}+\begin{pmatrix}-1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix}\langle\varepsilon_2|x(0)\rangle e^{-i\omega_2 t}$$

100% modulation (SWR=0) $\frac{e^{ia}+e^{ib}}{2}=e^{i\frac{a+b}{2}}\frac{e^{i\frac{a-b}{2}}+e^{-i\frac{a-b}{2}}}{2}$

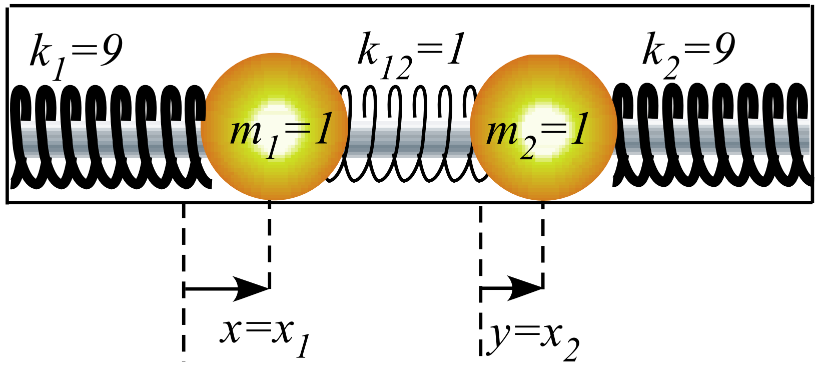
$$\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}=\begin{pmatrix}\dfrac{e^{-i\omega_1 t}+e^{-i\omega_2 t}}{2}\\ \dfrac{e^{-i\omega_1 t}-e^{-i\omega_2 t}}{2}\end{pmatrix}=\frac{e^{-i\frac{(\omega_1+\omega_2)}{2}t}}{2}\begin{pmatrix}e^{-i\frac{(\omega_1-\omega_2)}{2}t}+e^{i\frac{(\omega_1-\omega_2)}{2}t}\\ e^{-i\frac{(\omega_1-\omega_2)}{2}t}-e^{i\frac{(\omega_1-\omega_2)}{2}t}\end{pmatrix}$$



http://www.uark.edu/ua/modphys/markup/BoxItWeb.html

BoxIt (Beating) Simulation

Fig. 3.3.9 Beats in weakly coupled symmetric oscillators with equal mode magnitudes.



Analyzing 2D-HO beats and mixed mode eigen-solutions

$$\mathbf{K}=\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix}=\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix}=\begin{pmatrix} 10 & -1 \\ -1 & 10 \end{pmatrix}$$

$Det(\mathbf{K})=10\cdot 10-1=99$

$Trace(\mathbf{K})=10+10=20$

The **K** secular equation $K^2-Trace(\mathbf{K})K+Det(\mathbf{K})=K^2-20K+99=0=(K-9)(K-11)=(K-K_1)(K-K_2)$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\varepsilon_k)^2$ $\quad K_1=\omega_0^2(\varepsilon_1)=9,\quad K_2=\omega_0^2(\varepsilon_2)=11,$

Eigen-projectors $\mathbf{P}_k$

$$\mathbf{P}_1=\frac{\begin{pmatrix} K_{11}-K_2 & K_{12} \\ K_{12} & K_{22}-K_2 \end{pmatrix}}{K_1-K_2}=\frac{\begin{pmatrix} 10-11 & -1 \\ -1 & 10-11 \end{pmatrix}}{9-11}=\frac{\begin{pmatrix} 1 & +1 \\ +1 & 1 \end{pmatrix}}{2}$$

$$=\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2=\frac{\begin{pmatrix} K_{11}-K_1 & K_{12} \\ K_{12} & K_{22}-K_1 \end{pmatrix}}{K_2-K_1}=\frac{\begin{pmatrix} 10-9 & -1 \\ -1 & 10-9 \end{pmatrix}}{11-9}=\frac{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}{2}$$

$$=\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

Eigenbra vectors: $\langle\varepsilon_1|=\begin{pmatrix} 1/\sqrt{2} & +1/\sqrt{2} \end{pmatrix},\quad \langle\varepsilon_2|=\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$

Mixed mode dynamics

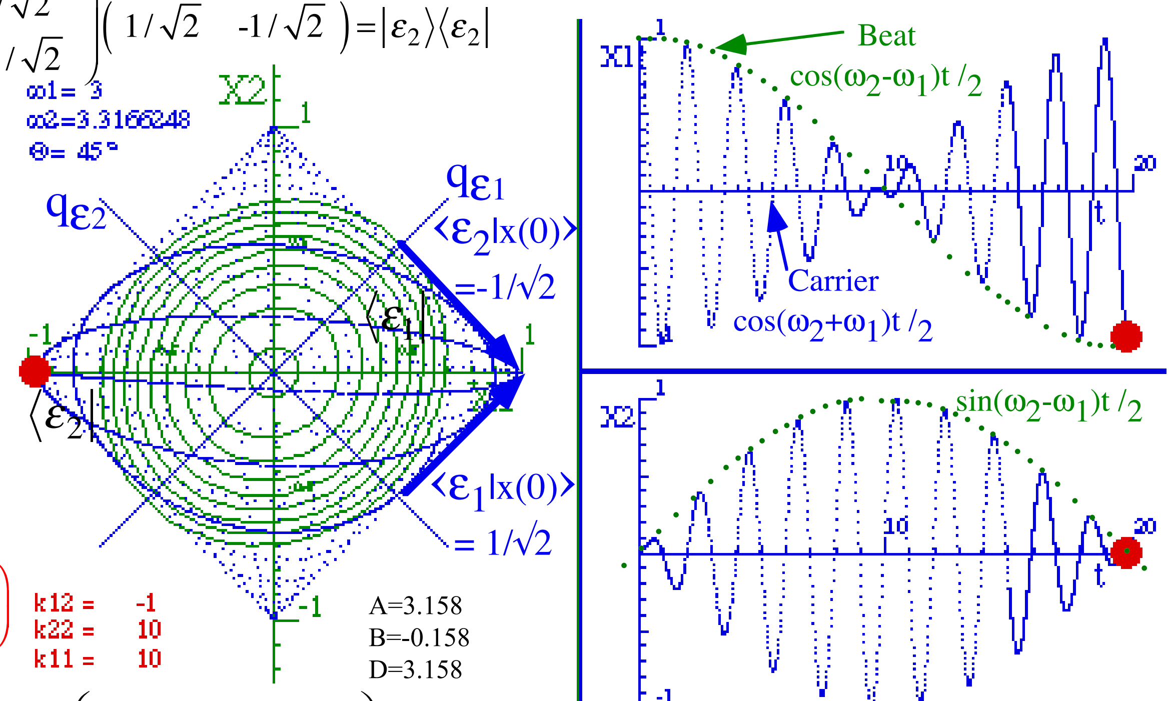
$$|x(t)\rangle = |\varepsilon_1\rangle\ \langle\varepsilon_1|x(0)\rangle e^{-i\omega_1 t} + |\varepsilon_2\rangle\ \langle\varepsilon_2|x(0)\rangle e^{-i\omega_2 t}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}=\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}\langle\varepsilon_1|x(0)\rangle e^{-i\omega_1 t}+\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}\langle\varepsilon_2|x(0)\rangle e^{-i\omega_2 t}$$

100% modulation (SWR=0) $\frac{e^{ia}+e^{ib}}{2}=e^{i\frac{a+b}{2}}\frac{e^{i\frac{a-b}{2}}+e^{-i\frac{a-b}{2}}}{2}=e^{i\frac{a+b}{2}}\cos\left(\frac{a-b}{2}\right)$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}=\begin{pmatrix} \dfrac{e^{-i\omega_1 t}+e^{-i\omega_2 t}}{2} \\ \dfrac{e^{-i\omega_1 t}-e^{-i\omega_2 t}}{2} \end{pmatrix}=\frac{e^{-i\frac{(\omega_1+\omega_2)}{2}t}}{2}\begin{pmatrix} e^{-i\frac{(\omega_1-\omega_2)}{2}t}+e^{i\frac{(\omega_1-\omega_2)}{2}t} \\ e^{-i\frac{(\omega_1-\omega_2)}{2}t}-e^{i\frac{(\omega_1-\omega_2)}{2}t} \end{pmatrix}=e^{-i\frac{(\omega_1+\omega_2)}{2}t}\begin{pmatrix} \cos\frac{(\omega_2-\omega_1)t}{2} \\ i\sin\frac{(\omega_2-\omega_1)t}{2} \end{pmatrix}$$

Note the i phase



http://www.uark.edu/ua/modphys/markup/BoxItWeb.html

BoxIt (Beating) Simulation

Fig. 3.3.9 Beats in weakly coupled symmetric oscillators with equal mode magnitudes.

2D harmonic oscillator equations
Lagrangian and matrix forms and Reciprocity symmetry

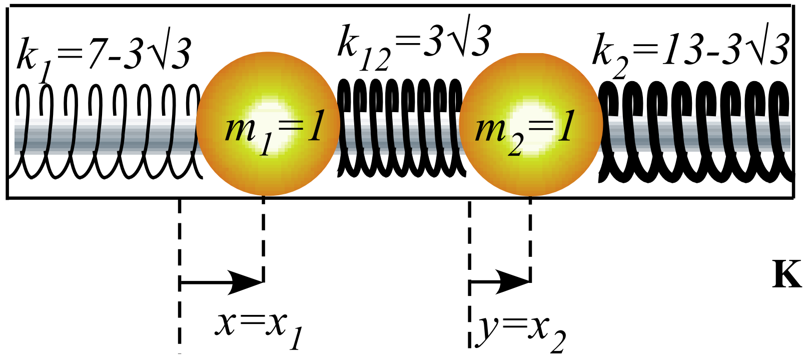
2D harmonic oscillator equation eigensolutions
Geometric method
Matrix-algebraic eigensolutions with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$
Secular equation
Hamilton-Cayley equation and projectors
Idempotent projectors (how eigenvalues⇒eigenvectors)
Operator orthonormality and Completeness (Idempotent means: $\mathbf{P}\cdot\mathbf{P}=\mathbf{P}$*)*
Spectral Decompositions
Functional spectral decomposition
Orthonormality vs. Completeness vis-a`-vis Operator vs. State
Lagrange functional interpolation formula
Diagonalizing Transformations (D-Ttran) from projectors

2D-HO eigensolution example with bilateral (B-Type) symmetry
Mixed mode beat dynamics and fixed π/2 phase

➡ *2D-HO eigensolution example with asymmetric (A-Type) symmetry* ⬅
Initial state projection, mixed mode beat dynamics with variable phase

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$
Hamilton-Pauli spinor symmetry (ABCD-Types)

Spectral decomposition of 2D-HO mode dynamics for lower symmetry

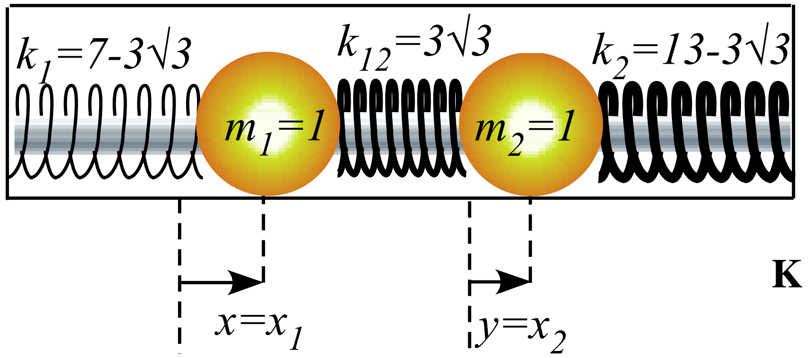


$$\mathbf{K}=\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix}=\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix}=\begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13 \end{pmatrix}$$

$$Det(\mathbf{K})=7\cdot 13-27=91-27=64$$

$$Trace(\mathbf{K})=7+13=20$$

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



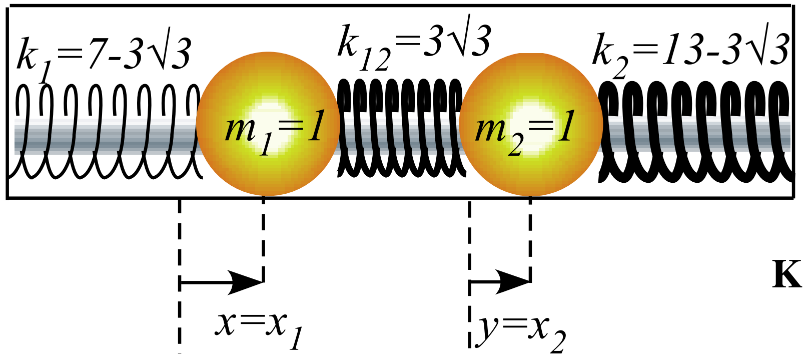
$$\mathbf{K}=\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix}=\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix}=\begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13 \end{pmatrix}$$

$Det(\mathbf{K})=7\cdot 13-27=91-27=64$

$Trace(\mathbf{K})=7+13=20$

The **K** secular equation $K^2-Trace(\mathbf{K})K+Det(\mathbf{K})=K^2-20K+64=0$

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



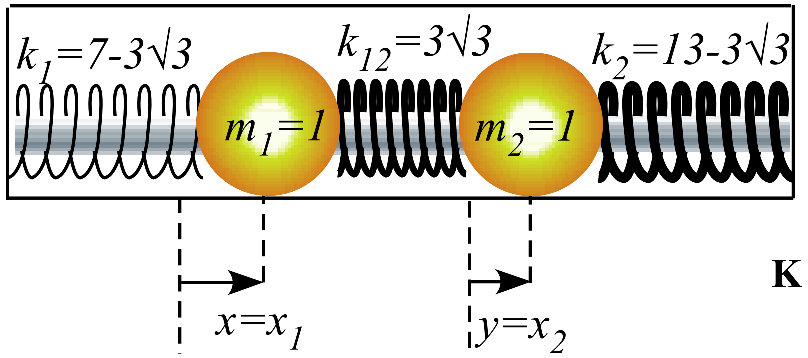
$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} = \begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13 \end{pmatrix}$$

The **K** secular equation $K^2 - Trace(\mathbf{K})K + Det(\mathbf{K}) = K^2 - 20K + 64 = 0 = (K-4)(K-16)$

$Det(\mathbf{K}) = 7 \cdot 13 - 27 = 91 - 27 = 64$

$Trace(\mathbf{K}) = 7 + 13 = 20$

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



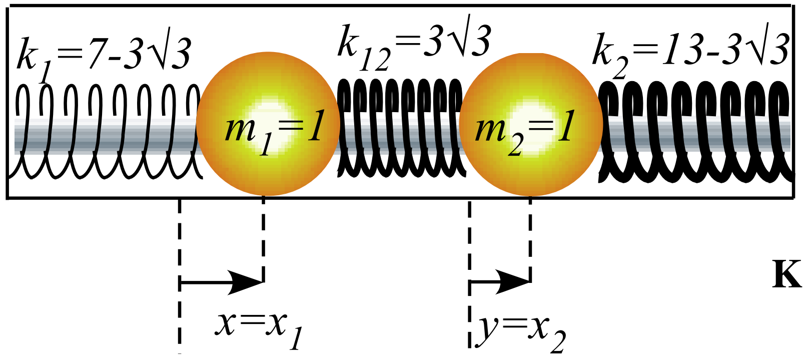
$$\mathbf{K}=\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22}\end{pmatrix}=\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12}\end{pmatrix}=\begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13\end{pmatrix}$$

$Det(\mathbf{K})=7\cdot 13-27=91-27=64$

$Trace(\mathbf{K})=7+13=20$

The **K** secular equation $K^2-Trace(\mathbf{K})K+Det(\mathbf{K})=K^2-20K+64=0=(K-4)(K-16)$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\varepsilon_k)^2$ $\quad K_1=\omega_0^2(\varepsilon_1)=4,\quad K_2=\omega_0^2(\varepsilon_2)=16,$



$$\mathbf{K}=\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix}=\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix}=\begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13 \end{pmatrix}$$

The **K** secular equation $K^2 - Trace(\mathbf{K})K + Det(\mathbf{K}) = K^2 - 20K + 64 = 0 = (K-4)(K-16)$

$Det(\mathbf{K}) = 7\cdot 13 - 27 = 91 - 27 = 64$

$Trace(\mathbf{K}) = 7 + 13 = 20$

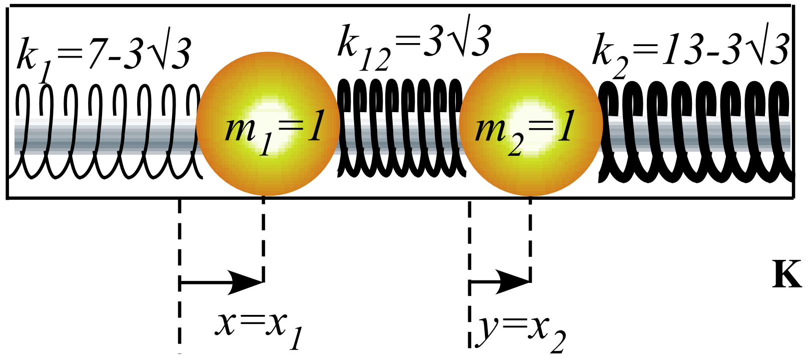
Eigenvalues K_k and squared eigenfrequencies $\omega_0(\varepsilon_k)^2$ $\quad K_1 = \omega_0^2(\varepsilon_1) = 4, \quad K_2 = \omega_0^2(\varepsilon_2) = 16,$

Eigen-projectors $\mathbf{P}_k$

$$\mathbf{P}_1 = \frac{\begin{pmatrix} K_{11}-K_2 & K_{12} \\ K_{12} & K_{22}-K_2 \end{pmatrix}}{K_1-K_2} = \frac{\begin{pmatrix} 7-16 & -3\sqrt{3} \\ -3\sqrt{3} & 13-16 \end{pmatrix}}{4-16} = \frac{\begin{pmatrix} 9 & +3\sqrt{3} \\ +3\sqrt{3} & 3 \end{pmatrix}}{12}$$

$$= \frac{\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}}{4} = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}\begin{pmatrix} \sqrt{3}/2 & 1/2 \end{pmatrix} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



$$\mathbf{K}=\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22}\end{pmatrix}=\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12}\end{pmatrix}=\begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13\end{pmatrix}$$

The **K** secular equation $K^2 - Trace(\mathbf{K})K + Det(\mathbf{K}) = K^2 - 20K + 64 = 0 = (K-4)(K-16)$

$Det(\mathbf{K}) = 7\cdot 13 - 27 = 91 - 27 = 64$

$Trace(\mathbf{K}) = 7 + 13 = 20$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\varepsilon_k)^2$ $\quad K_1 = \omega_0^2(\varepsilon_1) = 4, \quad K_2 = \omega_0^2(\varepsilon_2) = 16,$

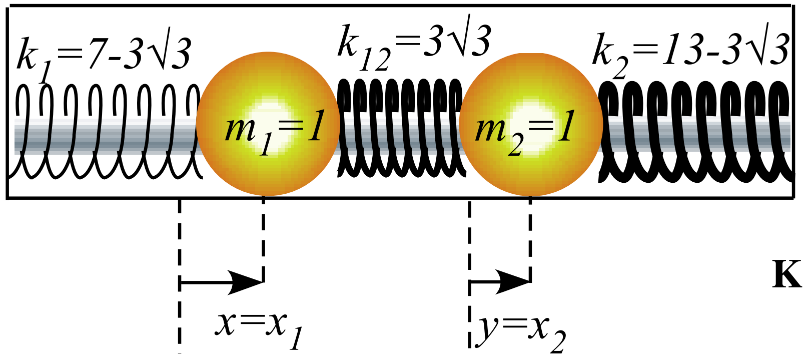
Eigen-projectors $\mathbf{P}_k$

$$\mathbf{P}_1 = \frac{\begin{pmatrix} K_{11}-K_2 & K_{12} \\ K_{12} & K_{22}-K_2\end{pmatrix}}{K_1-K_2} = \frac{\begin{pmatrix} 7-16 & -3\sqrt{3} \\ -3\sqrt{3} & 13-16\end{pmatrix}}{4-16} = \frac{\begin{pmatrix} 9 & +3\sqrt{3} \\ +3\sqrt{3} & 3\end{pmatrix}}{12}$$

$$= \frac{\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1\end{pmatrix}}{4} = \begin{pmatrix} \sqrt{3}/2 \\ 1/2\end{pmatrix}\begin{pmatrix} \sqrt{3}/2 & 1/2\end{pmatrix} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2 = \frac{\begin{pmatrix} K_{11}-K_1 & K_{12} \\ K_{12} & K_{22}-K_1\end{pmatrix}}{K_2-K_1} = \frac{\begin{pmatrix} 7-4 & -3\sqrt{3} \\ -3\sqrt{3} & 13-4\end{pmatrix}}{16-4} = \frac{\begin{pmatrix} 3 & -3\sqrt{3} \\ -3\sqrt{3} & 9\end{pmatrix}}{12}$$

$$= \frac{\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3\end{pmatrix}}{4} = \begin{pmatrix} -1/2 \\ \sqrt{3}/2\end{pmatrix}\begin{pmatrix} -1/2 & \sqrt{3}/2\end{pmatrix} = |\varepsilon_2\rangle\langle\varepsilon_2|$$



$$\mathbf{K}=\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix}=\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix}=\begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13 \end{pmatrix}$$

The **K** secular equation $K^2-Trace(\mathbf{K})K+Det(\mathbf{K})=K^2-20K+64=0=(K-4)(K-16)$

$Det(\mathbf{K})=7\cdot 13-27=91-27=64$

$Trace(\mathbf{K})=7+13=20$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\varepsilon_k)^2$ $\quad K_1=\omega_0^2(\varepsilon_1)=4,\quad K_2=\omega_0^2(\varepsilon_2)=16,$

Eigen-projectors $\mathbf{P}_k$

$$\mathbf{P}_1=\frac{\begin{pmatrix} K_{11}-K_2 & K_{12} \\ K_{12} & K_{22}-K_2 \end{pmatrix}}{K_1-K_2}=\frac{\begin{pmatrix} 7-16 & -3\sqrt{3} \\ -3\sqrt{3} & 13-16 \end{pmatrix}}{4-16}=\frac{\begin{pmatrix} 9 & +3\sqrt{3} \\ +3\sqrt{3} & 3 \end{pmatrix}}{12}$$

$$=\frac{\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}}{4}=\begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}\begin{pmatrix} \sqrt{3}/2 & 1/2 \end{pmatrix}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2=\frac{\begin{pmatrix} K_{11}-K_1 & K_{12} \\ K_{12} & K_{22}-K_1 \end{pmatrix}}{K_2-K_1}=\frac{\begin{pmatrix} 7-4 & -3\sqrt{3} \\ -3\sqrt{3} & 13-4 \end{pmatrix}}{16-4}=\frac{\begin{pmatrix} 3 & -3\sqrt{3} \\ -3\sqrt{3} & 9 \end{pmatrix}}{12}$$

$$=\frac{\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix}}{4}=\begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}\begin{pmatrix} -1/2 & \sqrt{3}/2 \end{pmatrix}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

Eigenbra vectors: $\langle\varepsilon_1|=\begin{pmatrix} \sqrt{3}/2 & 1/2 \end{pmatrix},\quad \langle\varepsilon_2|=\begin{pmatrix} -1/2 & \sqrt{3}/2 \end{pmatrix}$

2D harmonic oscillator equations

Lagrangian and matrix forms and Reciprocity symmetry

2D harmonic oscillator equation eigensolutions

Geometric method

Matrix-algebraic eigensolutions with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

Secular equation

Hamilton-Cayley equation and projectors

Idempotent projectors (how eigenvalues⇒eigenvectors)

Operator orthonormality and Completeness (Idempotent means: $\mathbf{P}\cdot\mathbf{P}=\mathbf{P}$*)*

Spectral Decompositions

Functional spectral decomposition

Orthonormality vs. Completeness vis-a`-vis Operator vs. State

Lagrange functional interpolation formula

Diagonalizing Transformations (D-Ttran) from projectors

2D-HO eigensolution example with bilateral (B-Type) symmetry

Mixed mode beat dynamics and fixed π/2 phase

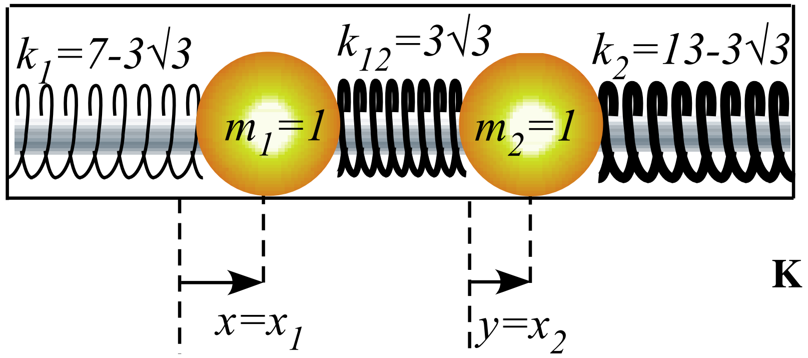
2D-HO eigensolution example with asymmetric (A-Type) symmetry

➡ *Initial state projection, mixed mode beat dynamics with variable phase* ⬅

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry (ABCD-Types)

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



$$\mathbf{K}=\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix}=\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix}=\begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13 \end{pmatrix}$$

$Det(\mathbf{K})=7\cdot 13-27=91-27=64$

$Trace(\mathbf{K})=7+13=20$

The $\mathbf{K}$ secular equation $\quad K^2-Trace(\mathbf{K})K+Det(\mathbf{K})=K^2-20K+64=0=(K-4)(K-16)$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\varepsilon_k)^2$ $\quad K_1=\omega_0^2(\varepsilon_1)=4, \quad K_2=\omega_0^2(\varepsilon_2)=16,$

Eigen-projectors $\mathbf{P}_k$

$$\mathbf{P}_1=\frac{\begin{pmatrix} K_{11}-K_2 & K_{12} \\ K_{12} & K_{22}-K_2 \end{pmatrix}}{K_1-K_2}=\frac{\begin{pmatrix} 7-16 & -3\sqrt{3} \\ -3\sqrt{3} & 13-16 \end{pmatrix}}{4-16}=\frac{\begin{pmatrix} 9 & +3\sqrt{3} \\ +3\sqrt{3} & 3 \end{pmatrix}}{12}$$

$$=\frac{\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}}{4}=\begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}\begin{pmatrix} \sqrt{3}/2 & 1/2 \end{pmatrix}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2=\frac{\begin{pmatrix} K_{11}-K_1 & K_{12} \\ K_{12} & K_{22}-K_1 \end{pmatrix}}{K_2-K_1}=\frac{\begin{pmatrix} 7-4 & -3\sqrt{3} \\ -3\sqrt{3} & 13-4 \end{pmatrix}}{16-4}=\frac{\begin{pmatrix} 3 & -3\sqrt{3} \\ -3\sqrt{3} & 9 \end{pmatrix}}{12}$$

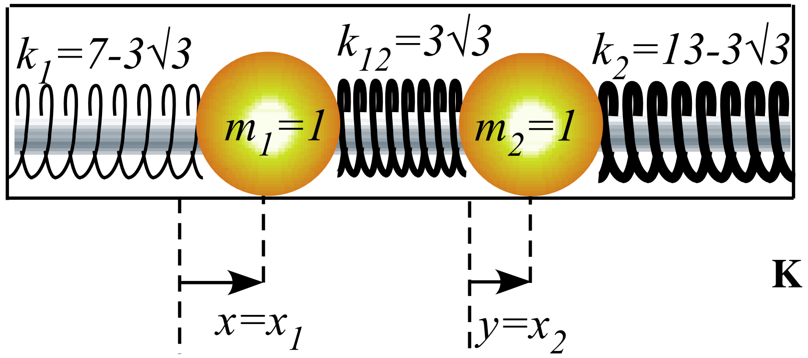
$$=\frac{\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix}}{4}=\begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}\begin{pmatrix} -1/2 & \sqrt{3}/2 \end{pmatrix}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

Eigenbra vectors: $\langle\varepsilon_1|=\begin{pmatrix} \sqrt{3}/2 & 1/2 \end{pmatrix}, \quad \langle\varepsilon_2|=\begin{pmatrix} -1/2 & \sqrt{3}/2 \end{pmatrix}$

Spectral decomposition of initial state $\mathbf{x}(0)=(1,0)$*:*

$$\mathbf{1}\cdot\mathbf{x}(0)=(\mathbf{P}_1+\mathbf{P}_2)\begin{pmatrix} 1 \\ 0 \end{pmatrix}=\begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}\otimes\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}+\begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}\otimes\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



$$\mathbf{K}=\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22}\end{pmatrix}=\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12}\end{pmatrix}=\begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13\end{pmatrix}$$

The $\mathbf{K}$ secular equation $\quad K^2-Trace(\mathbf{K})K+Det(\mathbf{K})=K^2-20K+64=0=(K-4)(K-16)$

$Det(\mathbf{K})=7\cdot 13-27=91-27=64$

$Trace(\mathbf{K})=7+13=20$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\varepsilon_k)^2$ $\qquad K_1=\omega_0^2(\varepsilon_1)=4, \quad K_2=\omega_0^2(\varepsilon_2)=16,$

Eigen-projectors $\mathbf{P}_k$

$$\mathbf{P}_1=\frac{\begin{pmatrix} K_{11}-K_2 & K_{12} \\ K_{12} & K_{22}-K_2\end{pmatrix}}{K_1-K_2}=\frac{\begin{pmatrix} 7-16 & -3\sqrt{3} \\ -3\sqrt{3} & 13-16\end{pmatrix}}{4-16}=\frac{\begin{pmatrix} 9 & +3\sqrt{3} \\ +3\sqrt{3} & 3\end{pmatrix}}{12}$$
$$=\frac{\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1\end{pmatrix}}{4}=\begin{pmatrix} \sqrt{3}/2 \\ 1/2\end{pmatrix}\begin{pmatrix} \sqrt{3}/2 & 1/2\end{pmatrix}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2=\frac{\begin{pmatrix} K_{11}-K_1 & K_{12} \\ K_{12} & K_{22}-K_1\end{pmatrix}}{K_2-K_1}=\frac{\begin{pmatrix} 7-4 & -3\sqrt{3} \\ -3\sqrt{3} & 13-4\end{pmatrix}}{16-4}=\frac{\begin{pmatrix} 3 & -3\sqrt{3} \\ -3\sqrt{3} & 9\end{pmatrix}}{12}$$
$$=\frac{\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3\end{pmatrix}}{4}=\begin{pmatrix} -1/2 \\ \sqrt{3}/2\end{pmatrix}\begin{pmatrix} -1/2 & \sqrt{3}/2\end{pmatrix}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

Eigenbra vectors: $\langle\varepsilon_1|=\begin{pmatrix} \sqrt{3}/2 & 1/2\end{pmatrix}, \quad \langle\varepsilon_2|=\begin{pmatrix} -1/2 & \sqrt{3}/2\end{pmatrix}$

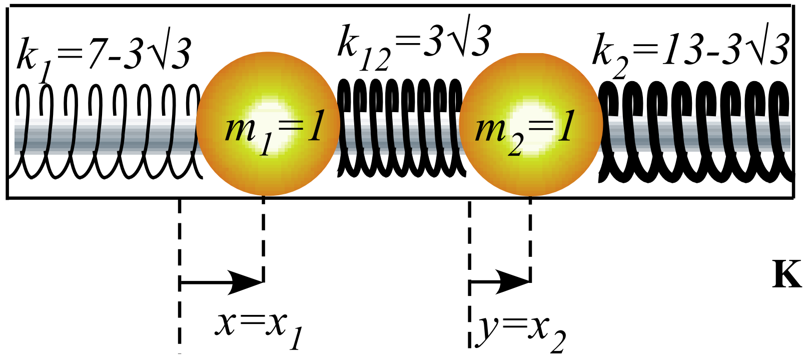
Spectral decomposition of initial state $\mathbf{x}(0)=(1,0)$:

$$\mathbf{1}\cdot\mathbf{x}(0)=(\mathbf{P}_1+\mathbf{P}_2)\begin{pmatrix} 1 \\ 0\end{pmatrix}=\begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2}\end{pmatrix}\otimes\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix} 1 \\ 0\end{pmatrix}+\begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2}\end{pmatrix}\otimes\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix} 1 \\ 0\end{pmatrix}$$

$$=\begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2}\end{pmatrix}(\tfrac{\sqrt{3}}{2})+\begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2}\end{pmatrix}(-\tfrac{1}{2})$$

(Note projection onto eigen-axes)

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



$$\mathbf{K}=\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22}\end{pmatrix}=\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12}\end{pmatrix}=\begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13\end{pmatrix}$$

The **K** secular equation $K^2 - Trace(\mathbf{K})K + Det(\mathbf{K}) = K^2 - 20K + 64 = 0 = (K-4)(K-16)$

$Det(\mathbf{K}) = 7\cdot 13 - 27 = 91 - 27 = 64$

$Trace(\mathbf{K}) = 7 + 13 = 20$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\varepsilon_k)^2$ $\quad K_1 = \omega_0^2(\varepsilon_1) = 4, \quad K_2 = \omega_0^2(\varepsilon_2) = 16,$

Eigen-projectors $\mathbf{P}_k$

$$\mathbf{P}_1 = \frac{\begin{pmatrix} K_{11}-K_2 & K_{12} \\ K_{12} & K_{22}-K_2\end{pmatrix}}{K_1-K_2} = \frac{\begin{pmatrix} 7-16 & -3\sqrt{3} \\ -3\sqrt{3} & 13-16\end{pmatrix}}{4-16} = \frac{\begin{pmatrix} 9 & +3\sqrt{3} \\ +3\sqrt{3} & 3\end{pmatrix}}{12}$$

$$= \frac{\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1\end{pmatrix}}{4} = \begin{pmatrix} \sqrt{3}/2 \\ 1/2\end{pmatrix}\begin{pmatrix} \sqrt{3}/2 & 1/2\end{pmatrix} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2 = \frac{\begin{pmatrix} K_{11}-K_1 & K_{12} \\ K_{12} & K_{22}-K_1\end{pmatrix}}{K_2-K_1} = \frac{\begin{pmatrix} 7-4 & -3\sqrt{3} \\ -3\sqrt{3} & 13-4\end{pmatrix}}{16-4} = \frac{\begin{pmatrix} 3 & -3\sqrt{3} \\ -3\sqrt{3} & 9\end{pmatrix}}{12}$$

$$= \frac{\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3\end{pmatrix}}{4} = \begin{pmatrix} -1/2 \\ \sqrt{3}/2\end{pmatrix}\begin{pmatrix} -1/2 & \sqrt{3}/2\end{pmatrix} = |\varepsilon_2\rangle\langle\varepsilon_2|$$

Eigenbra vectors: $\langle\varepsilon_1| = \begin{pmatrix} \sqrt{3}/2 & 1/2\end{pmatrix}, \quad \langle\varepsilon_2| = \begin{pmatrix} -1/2 & \sqrt{3}/2\end{pmatrix}$

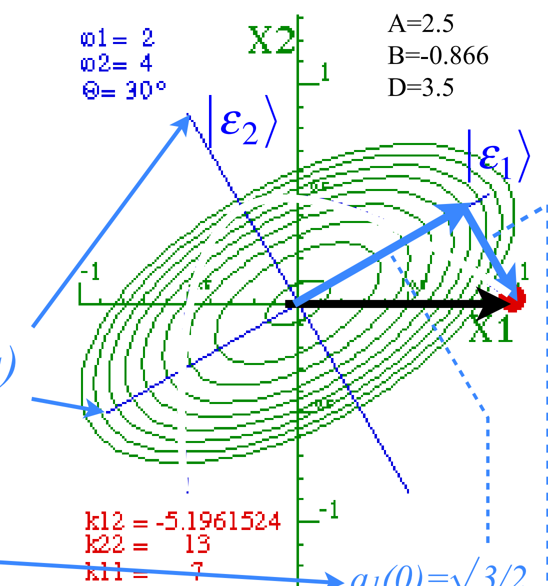
Spectral decomposition of initial state $\mathbf{x}(0)=(1,0)$:

$$\mathbf{1}\cdot\mathbf{x}(0) = (\mathbf{P}_1+\mathbf{P}_2)\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}\otimes\begin{pmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}-\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix}\otimes\begin{pmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

$$= \begin{pmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}\left(\tfrac{\sqrt{3}}{2}\right) + \begin{pmatrix}-\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix}\left(-\tfrac{1}{2}\right)$$

$$q_1(t) = \frac{\sqrt{3}}{2}\cos 2t, \quad q_2(t) = -\frac{1}{2}\cos 4t$$

(Note projection of $\mathbf{x}(0)$ *onto eigen-axes)*

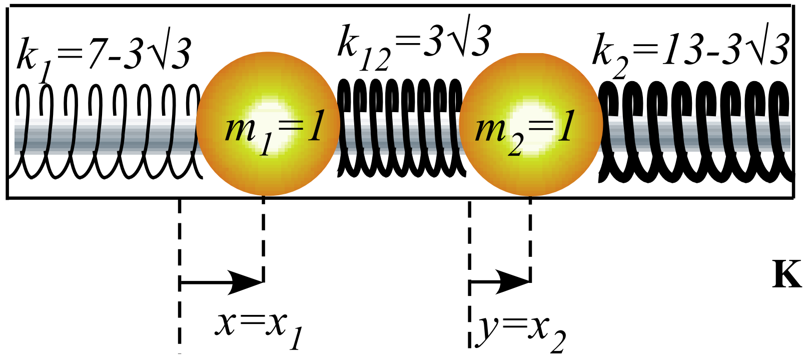


$$\mathbf{x}(0) = \begin{pmatrix}1\\0\end{pmatrix}$$

$q_1(0)=\sqrt{3}/2$

$q_2(0)=-1/2$

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



$$\mathbf{K}=\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22}\end{pmatrix}=\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12}\end{pmatrix}=\begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13\end{pmatrix}$$

The **K** secular equation $K^2 - Trace(\mathbf{K})K + Det(\mathbf{K}) = K^2 - 20K + 64 = 0 = (K-4)(K-16)$

$Det(\mathbf{K}) = 7\cdot 13 - 27 = 91 - 27 = 64$

$Trace(\mathbf{K}) = 7 + 13 = 20$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\varepsilon_k)^2$ $\quad K_1 = \omega_0^2(\varepsilon_1) = 4, \quad K_2 = \omega_0^2(\varepsilon_2) = 16,$

Eigen-projectors $\mathbf{P}_k$

$$\mathbf{P}_1 = \frac{\begin{pmatrix} K_{11}-K_2 & K_{12} \\ K_{12} & K_{22}-K_2\end{pmatrix}}{K_1-K_2} = \frac{\begin{pmatrix} 7-16 & -3\sqrt{3} \\ -3\sqrt{3} & 13-16\end{pmatrix}}{4-16} = \frac{\begin{pmatrix} 9 & +3\sqrt{3} \\ +3\sqrt{3} & 3\end{pmatrix}}{12}$$

$$= \frac{\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1\end{pmatrix}}{4} = \begin{pmatrix} \sqrt{3}/2 \\ 1/2\end{pmatrix}\begin{pmatrix} \sqrt{3}/2 & 1/2\end{pmatrix} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2 = \frac{\begin{pmatrix} K_{11}-K_1 & K_{12} \\ K_{12} & K_{22}-K_1\end{pmatrix}}{K_2-K_1} = \frac{\begin{pmatrix} 7-4 & -3\sqrt{3} \\ -3\sqrt{3} & 13-4\end{pmatrix}}{16-4} = \frac{\begin{pmatrix} 3 & -3\sqrt{3} \\ -3\sqrt{3} & 9\end{pmatrix}}{12}$$

$$= \frac{\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3\end{pmatrix}}{4} = \begin{pmatrix} -1/2 \\ \sqrt{3}/2\end{pmatrix}\begin{pmatrix} -1/2 & \sqrt{3}/2\end{pmatrix} = |\varepsilon_2\rangle\langle\varepsilon_2|$$

Eigenbra vectors: $\langle\varepsilon_1| = \begin{pmatrix} \sqrt{3}/2 & 1/2\end{pmatrix}, \quad \langle\varepsilon_2| = \begin{pmatrix} -1/2 & \sqrt{3}/2\end{pmatrix}$

Spectral decomposition of initial state $\mathbf{x}(0)=(1,0)$*:*

$$\mathbf{1}\cdot\mathbf{x}(0) = (\mathbf{P}_1+\mathbf{P}_2)\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}\otimes\begin{pmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}-\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix}\otimes\begin{pmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

$$= \begin{pmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}\left(\frac{\sqrt{3}}{2}\right) + \begin{pmatrix}-\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix}\left(-\frac{1}{2}\right)$$

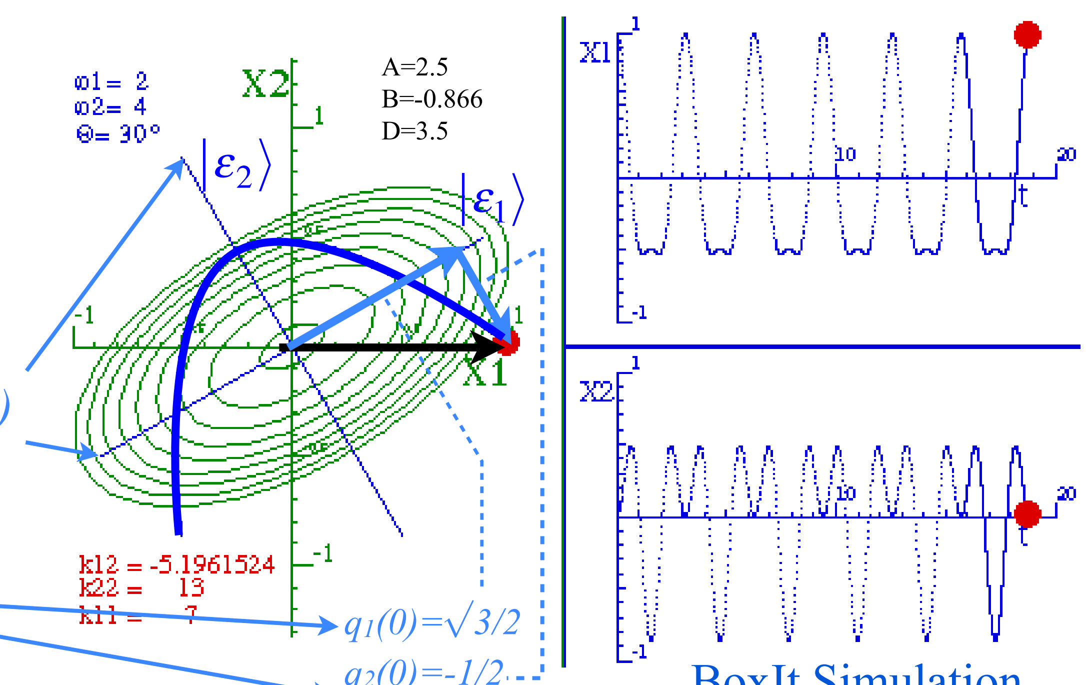
(Note projection of $\mathbf{x}(0)$ *onto eigen-axes)*

$$\left(q_1(t) = \frac{\sqrt{3}}{2}\cos 2t, \quad q_2(t) = -\frac{1}{2}\cos 4t \right)$$

$$\mathbf{x}(0) = \begin{pmatrix}1\\0\end{pmatrix}$$

Using $\cos 4t = 2\cos^2 2t - 1$ derives a parabolic trajectory!

$$q_2(t) = -\frac{1}{2}2\cos^2 2t + \frac{1}{2} = -\frac{4}{3}\left[q_1(t)\right]^2 + \frac{1}{2}$$

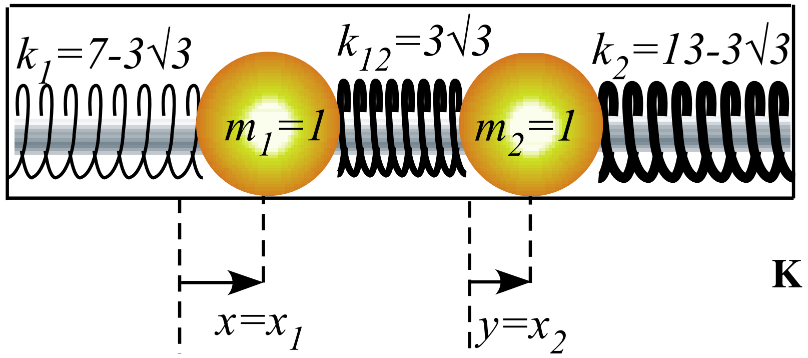


BoxIt Simulation

Fig. 3.3.6 Normal coordinate axes, coupled oscillator trajectories and equipotential (V=const.) ovals for an integral 1:2 eigenfrequency ratio ($\omega_0(\varepsilon_1)$=2.0, $\omega_0(\varepsilon_2)$= 4.0) and zero initial velocity.

http://www.uark.edu/ua/modphys/markup/BoxItWeb.html

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



$$\mathbf{K}=\begin{pmatrix}K_{11} & K_{12}\\ K_{12} & K_{22}\end{pmatrix}=\begin{pmatrix}k_1+k_{12} & -k_{12}\\ -k_{12} & k_2+k_{12}\end{pmatrix}=\begin{pmatrix}7 & -3\sqrt{3}\\ -3\sqrt{3} & 13\end{pmatrix}$$

The **K** secular equation $K^2-Trace(\mathbf{K})K+Det(\mathbf{K})=K^2-20K+64=0=(K-4)(K-16)$

$Det(\mathbf{K})=7\cdot 13-27=91-27=64$

$Trace(\mathbf{K})=7+13=20$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\varepsilon_k)^2$ $\qquad K_1=\omega_0^2(\varepsilon_1)=4,\quad K_2=\omega_0^2(\varepsilon_2)=16,$

Eigen-projectors $\mathbf{P}_k$

$$\mathbf{P}_1=\frac{\begin{pmatrix}K_{11}-K_2 & K_{12}\\ K_{12} & K_{22}-K_2\end{pmatrix}}{K_1-K_2}=\frac{\begin{pmatrix}7-16 & -3\sqrt{3}\\ -3\sqrt{3} & 13-16\end{pmatrix}}{4-16}=\frac{\begin{pmatrix}9 & +3\sqrt{3}\\ +3\sqrt{3} & 3\end{pmatrix}}{12}$$

$$=\frac{\begin{pmatrix}3 & \sqrt{3}\\ \sqrt{3} & 1\end{pmatrix}}{4}=\begin{pmatrix}\sqrt{3}/2\\ 1/2\end{pmatrix}\begin{pmatrix}\sqrt{3}/2 & 1/2\end{pmatrix}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2=\frac{\begin{pmatrix}K_{11}-K_1 & K_{12}\\ K_{12} & K_{22}-K_1\end{pmatrix}}{K_2-K_1}=\frac{\begin{pmatrix}7-4 & -3\sqrt{3}\\ -3\sqrt{3} & 13-4\end{pmatrix}}{16-4}=\frac{\begin{pmatrix}3 & -3\sqrt{3}\\ -3\sqrt{3} & 9\end{pmatrix}}{12}$$

$$=\frac{\begin{pmatrix}1 & -\sqrt{3}\\ -\sqrt{3} & 3\end{pmatrix}}{4}=\begin{pmatrix}-1/2\\ \sqrt{3}/2\end{pmatrix}\begin{pmatrix}-1/2 & \sqrt{3}/2\end{pmatrix}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

Eigenbra vectors: $\langle\varepsilon_1|=\begin{pmatrix}\sqrt{3}/2 & 1/2\end{pmatrix},\quad \langle\varepsilon_2|=\begin{pmatrix}-1/2 & \sqrt{3}/2\end{pmatrix}$

Spectral decomposition of initial state $\mathbf{x}(0)=(1,0)$:

$$\mathbf{1}\cdot\mathbf{x}(0)=(\mathbf{P}_1+\mathbf{P}_2)\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}\otimes\begin{pmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}-\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix}\otimes\begin{pmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

$$=\begin{pmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}(\tfrac{\sqrt{3}}{2})+\begin{pmatrix}-\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix}(-\tfrac{1}{2})$$

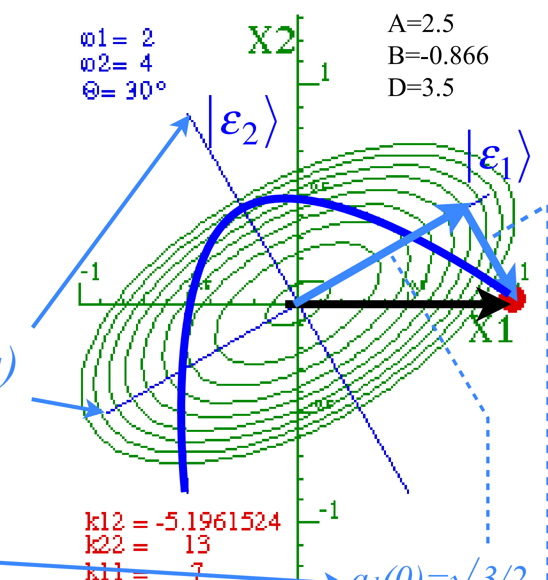
(Note projection of $\mathbf{x}(0)$ *onto eigen-axes)*

$$\left(q_1(t)=\frac{\sqrt{3}}{2}\cos 2t,\quad q_2(t)=-\frac{1}{2}\cos 4t\right)$$

Using $\cos 4t=2\cos^2 2t-1$ derives a parabolic trajectory!

$$q_2(t)=-\frac{1}{2}2\cos^2 2t+\frac{1}{2}=-\frac{4}{3}\left[q_1(t)\right]^2+\frac{1}{2}$$

$$\mathbf{x}(0)=\begin{pmatrix}1\\0\end{pmatrix}$$



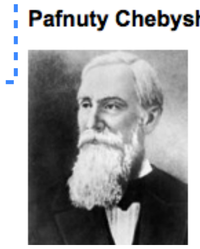
$q_1(0)=\sqrt{3/2}$

$q_2(0)=-1/2$

BoxIt Simulation

Example of a Tschebycheff Polynomial order 2

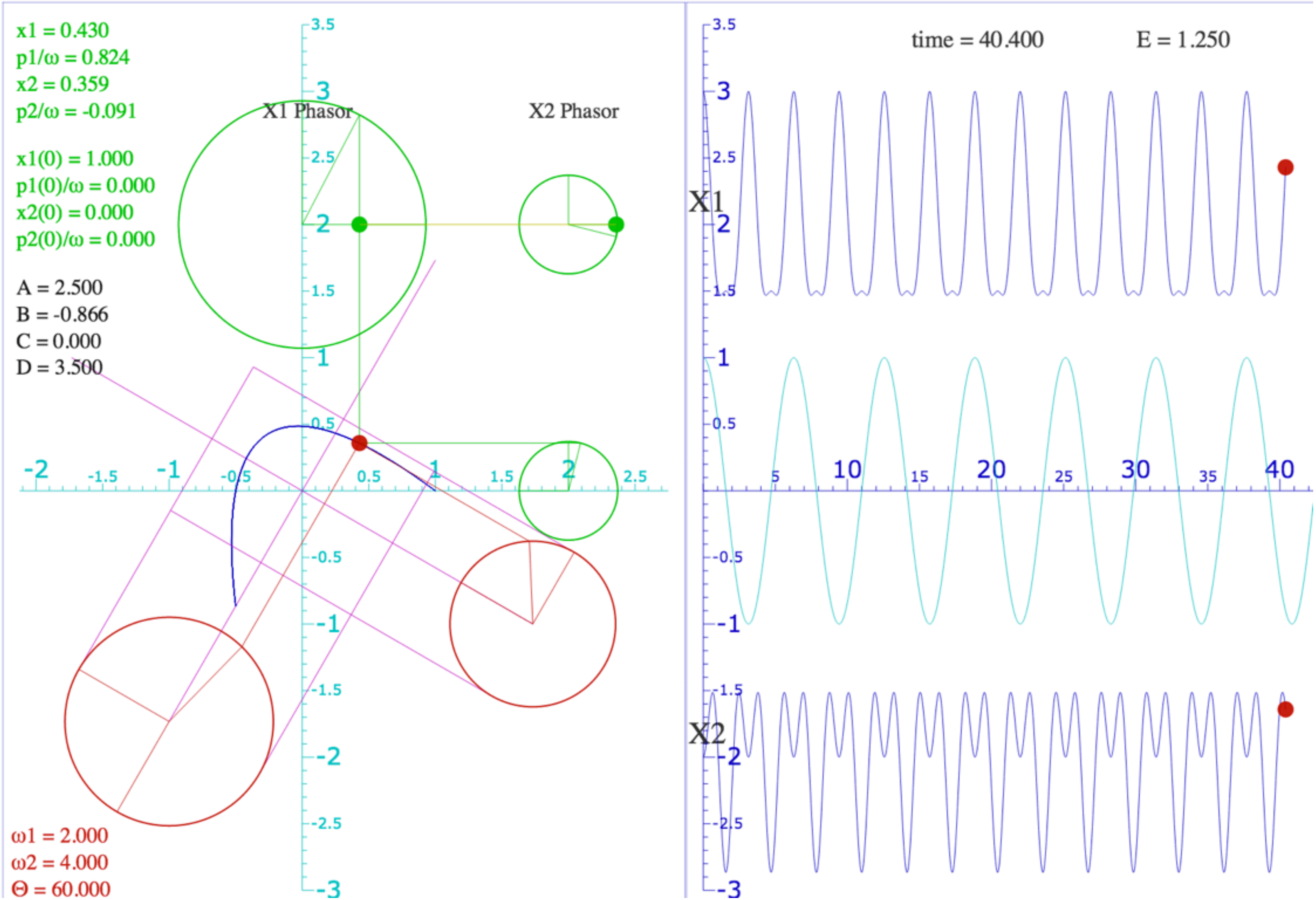
Pafnuty Chebyshev



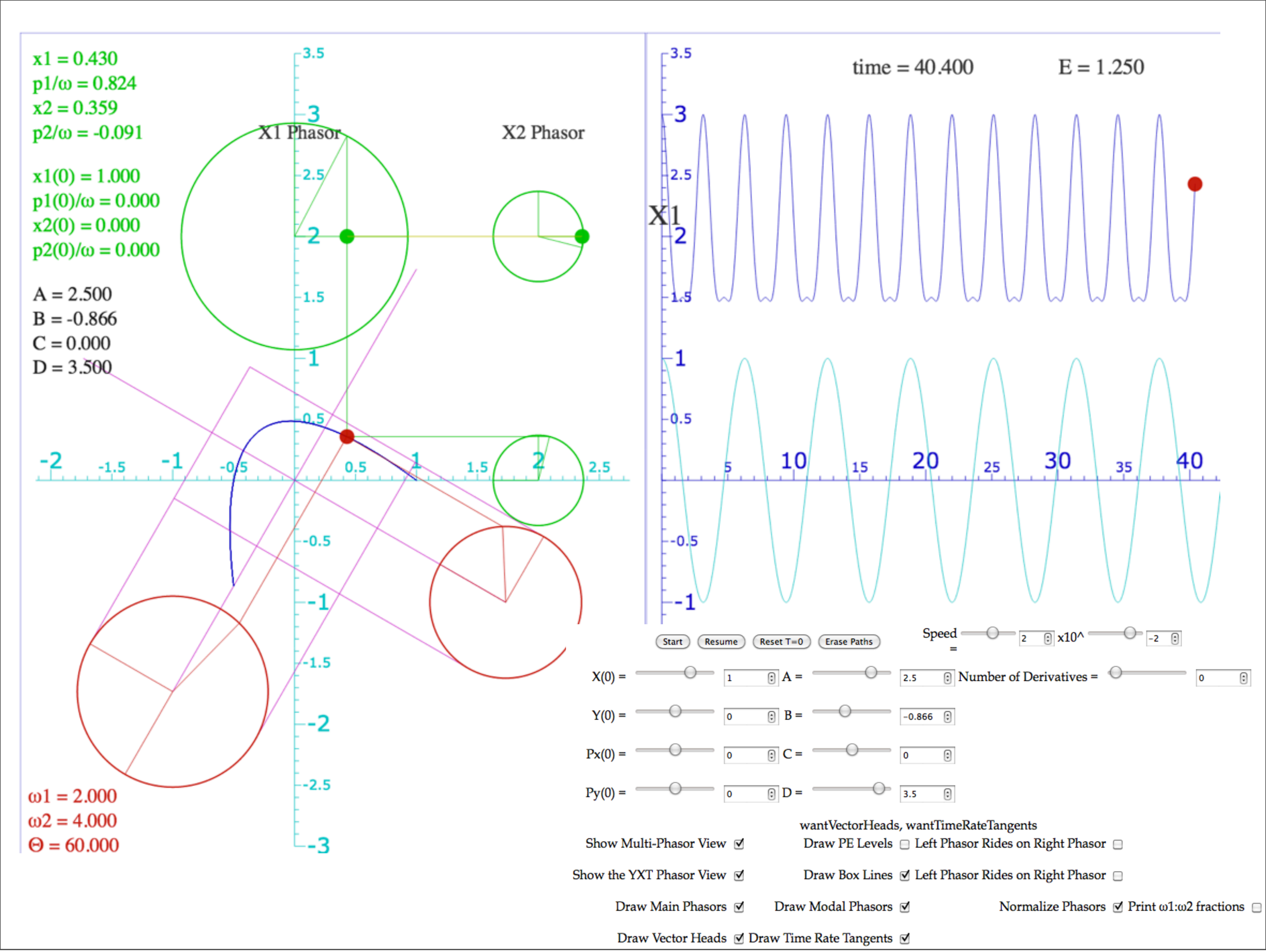
Pafnuty Lvovich Chebyshev was a Russian mathematician. His name can be alternatively transliterated as Chebychev, Chebysheff, Chebyshov, Tchebychev or Tchebycheff, or Tschebyschev or Tschebyscheff. Wikipedia

Born: May 16, 1821, Borovsk

Died: December 8, 1894, Saint Petersburg



[BoxIt Simulation]


x1 = 0.430
p1/ω = 0.824
x2 = 0.359
p2/ω = -0.091
x1(0) = 1.000
p1(0)/ω = 0.000
x2(0) = 0.000
p2(0)/ω = 0.000
A = 2.500
B = -0.866
C = 0.000
D = 3.500
ω1 = 2.000
ω2 = 4.000
Θ = 60.000
X1 Phasor
X2 Phasor
time = 40.400
E = 1.250
X1
Start
Resume
Reset T=0
Erase Paths
Speed =
x10^
X(0) =
Y(0) =
Px(0) =
Py(0) =
A =
B =
C =
D =
Number of Derivatives =
wantVectorHeads, wantTimeRateTangents
Show Multi-Phasor View
Draw PE Levels
Left Phasor Rides on Right Phasor
Show the YXT Phasor View
Draw Box Lines
Left Phasor Rides on Right Phasor
Draw Main Phasors
Draw Modal Phasors
Normalize Phasors
Print ω1:ω2 fractions
Draw Vector Heads
Draw Time Rate Tangents

2D harmonic oscillator equations
Lagrangian and matrix forms and Reciprocity symmetry

2D harmonic oscillator equation eigensolutions
Geometric method
Matrix-algebraic eigensolutions with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$
Secular equation
Hamilton-Cayley equation and projectors
Idempotent projectors (how eigenvalues⇒eigenvectors)
Operator orthonormality and Completeness (Idempotent means: $\mathbf{P}\cdot\mathbf{P}=\mathbf{P}$*)*
Spectral Decompositions
Functional spectral decomposition
Orthonormality vs. Completeness vis-a`-vis Operator vs. State
Lagrange functional interpolation formula
Diagonalizing Transformations (D-Ttran) from projectors

2D-HO eigensolution example with bilateral (B-Type) symmetry
Mixed mode beat dynamics and fixed π/2 phase
2D-HO eigensolution example with asymmetric (A-Type) symmetry
Initial state projection, mixed mode beat dynamics with variable phase

➜ *ANALOGY: 2-State Schrodinger:* $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$ ⬅
Hamilton-Pauli spinor symmetry (ABCD-Types)

ANALOGY: *2-State Schrodinger:* $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2_t \mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

$$i\hbar|\dot{\Psi}(t)\rangle = \mathbf{H}|\Psi(t)\rangle \qquad |\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$$

First start with 2-by-2 Hermitian (self-conjugate) matrix

$$\mathbf{H} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H}^\dagger$$

H_{jk} matrix must obey: $(H_{jk})^* = H_{kj}$

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2{}_t\mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

$$i\hbar|\dot{\Psi}(t)\rangle = \mathbf{H}|\Psi(t)\rangle \qquad |\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$$

First start with 2-by-2 Hermitian (self-conjugate) matrix

$$\mathbf{H} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H}^\dagger$$

H_{jk} matrix must obey: $(H_{jk})^* = H_{kj}$

that operates on 2-D complex Dirac ket vector $|\Psi\rangle$.

$$|\Psi\rangle = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Both have 4 parameters
($2^2 = 2+2$)

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2_t \mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

$$i\hbar|\dot{\Psi}(t)\rangle = \mathbf{H}|\Psi(t)\rangle \qquad |\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$$

First start with 2-by-2 Hermitian (self-conjugate) matrix

$$\mathbf{H} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H}^\dagger$$

H_{jk} matrix must obey: $(H_{jk})^* = H_{kj}$

that operates on 2-D complex Dirac ket vector $|\Psi\rangle$.

Both have 4 parameters *($2^2 = 2+2$)*

$$|\Psi\rangle = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Separate real x_k and imaginary p_k parts of Ψ_k amplitudes to convert the complex 1st-order equation $i\partial_t\Psi = \mathbf{H}\Psi$ into pairs of *real* 1st-order differential equations.

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2{}_t\mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

$$i\hbar|\dot{\Psi}(t)\rangle = \mathbf{H}|\Psi(t)\rangle \qquad |\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$$

First start with 2-by-2 Hermitian (self-conjugate) matrix

$$\mathbf{H} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H}^\dagger$$

H_{jk} matrix must obey: $(H_{jk})^* = H_{kj}$

that operates on 2-D complex Dirac ket vector $|\Psi\rangle$.

Both have 4 parameters
(2^2 = 2+2)

$$|\Psi\rangle = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Separate real x_k and imaginary p_k parts of Ψ_k amplitudes to convert the complex 1^{st}-order equation $i\partial_t\Psi = \mathbf{H}\Psi$ into pairs of *real* 1^{st}-order differential equations.

$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$$

$$i\frac{\partial}{\partial t}\begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}\begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2{}_t\mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

$$i\hbar|\dot{\Psi}(t)\rangle = \mathbf{H}|\Psi(t)\rangle \qquad |\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$$

First start with 2-by-2 Hermitian (self-conjugate) matrix

$$\mathbf{H} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H}^\dagger$$

H_{jk} matrix must obey: $(H_{jk})^* = H_{kj}$

that operates on 2-D complex Dirac ket vector $|\Psi\rangle$.

Both have 4 parameters
($2^2 = 2+2$)

$$|\Psi\rangle = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Separate real x_k and imaginary p_k parts of Ψ_k amplitudes to convert the complex 1st-order equation $i\partial_t\Psi = \mathbf{H}\Psi$ into pairs of <u>*real*</u> 1st-order differential equations.

$$\dot{x}_1 = Ap_1 + Bp_2 - Cx_2 \qquad \dot{p}_1 = -Ax_1 - Bx_2 - Cp_2$$

$$\dot{x}_2 = Bp_1 + Dp_2 + Cx_1 \qquad \dot{p}_2 = -Bx_1 - Dx_2 + Cp_1$$

$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$$

$$i\frac{\partial}{\partial t}\begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}\begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

$$\begin{pmatrix} i\dot{x}_1 - \dot{p}_1 \\ i\dot{x}_2 - \dot{p}_2 \end{pmatrix} = \begin{pmatrix} Ax_1 + Bx_2 + Cp_2 + iAp_1 + iBp_2 - iCx_2 \\ Bx_1 + Dx_2 - Cp_1 + iBp_1 + iDp_2 + iCx_1 \end{pmatrix}$$

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2_t \mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

$$i\hbar|\dot{\Psi}(t)\rangle = \mathbf{H}|\Psi(t)\rangle \qquad\qquad |\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$$

First start with 2-by-2 Hermitian (self-conjugate) matrix

$$\mathbf{H} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H}^\dagger$$

that operates on 2-D complex Dirac ket vector $|\Psi\rangle$.

$$|\Psi\rangle = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Separate real x_k and imaginary p_k parts of Ψ_k amplitudes to convert the complex 1st-order equation $i\partial_t\Psi = \mathbf{H}\Psi$ into pairs of ***real*** 1st-order differential equations.

$$\dot{x}_1 = Ap_1 + Bp_2 - Cx_2 \qquad \dot{p}_1 = -Ax_1 - Bx_2 - Cp_2$$

$$\dot{x}_2 = Bp_1 + Dp_2 + Cx_1 \qquad \dot{p}_2 = -Bx_1 - Dx_2 + Cp_1$$

Then start with classical Hamiltonian. (Designed to give same result.)

$$H_c = \frac{A}{2}\left(p_1^2 + x_1^2\right) + B\left(x_1x_2 + p_1p_2\right) + C\left(x_1p_2 - x_2p_1\right) + \frac{D}{2}\left(p_2^2 + x_2^2\right)$$

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus *Classical 2DHO:* $\partial^2_t \mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

$$i\hbar|\dot{\Psi}(t)\rangle = \mathbf{H}|\Psi(t)\rangle \qquad |\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$$

First start with 2-by-2 Hermitian (self-conjugate) matrix

$$\mathbf{H} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H}^\dagger$$

that operates on 2-D complex Dirac ket vector $|\Psi\rangle$.

$$|\Psi\rangle = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Separate real x_k and imaginary p_k parts of Ψ_k amplitudes to convert the complex 1st-order equation $i\partial_t\Psi = \mathbf{H}\Psi$ into pairs of real real 1st-order differential equations.

$$\dot{x}_1 = Ap_1 + Bp_2 - Cx_2 \qquad \dot{p}_1 = -Ax_1 - Bx_2 - Cp_2$$

$$\dot{x}_2 = Bp_1 + Dp_2 + Cx_1 \qquad \dot{p}_2 = -Bx_1 - Dx_2 + Cp_1$$

QM vs. Classical Equations are identical

Then start with classical Hamiltonian. (Designed to give same result.)

$$H_c = \frac{A}{2}\left(p_1^2 + x_1^2\right) + B\left(x_1x_2 + p_1p_2\right) + C\left(x_1p_2 - x_2p_1\right) + \frac{D}{2}\left(p_2^2 + x_2^2\right)$$

Then Hamilton's equations of motion are the following.

$$\dot{x}_1 = \frac{\partial H_c}{\partial p_1} = Ap_1 + Bp_2 - Cx_2 \qquad \dot{p}_1 = -\frac{\partial H_c}{\partial x_1} = -\left(Ax_1 + Bx_2 + Cp_2\right)$$

$$\dot{x}_2 = \frac{\partial H_c}{\partial p_2} = Bp_1 + Dp_2 + Cx_1 \qquad \dot{p}_2 = -\frac{\partial H_c}{\partial x_2} = -\left(Bx_1 + Dx_2 - Cp_1\right)$$

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ versus *Classical 2DHO:* $\partial^2_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

$$i\hbar|\dot{\Psi}(t)\rangle=\mathbf{H}|\Psi(t)\rangle \qquad |\ddot{\mathbf{x}}\rangle=-\mathbf{K}\bullet|\mathbf{x}\rangle$$

First start with 2-by-2 Hermitian (self-conjugate) matrix

$$\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}=\mathbf{H}^\dagger$$

that operates on 2-D complex Dirac ket vector $|\Psi\rangle$.

$$|\Psi\rangle=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}=\begin{pmatrix}a_1\\ a_2\end{pmatrix}$$

Separate real x_k and imaginary p_k parts of Ψ_k amplitudes to convert the complex 1^{st}-order equation $i\partial_t\Psi=\mathbf{H}\Psi$ into pairs of real real 1^{st}-order differential equations.

$$\dot{x}_1=Ap_1+Bp_2-Cx_2 \qquad \dot{p}_1=-Ax_1-Bx_2-Cp_2$$
$$\dot{x}_2=Bp_1+Dp_2+Cx_1 \qquad \dot{p}_2=-Bx_1-Dx_2+Cp_1$$

Then start with classical Hamiltonian.

$$H_c=\frac{A}{2}\left(p_1^2+x_1^2\right)+B\left(x_1x_2+p_1p_2\right)+C\left(x_1p_2-x_2p_1\right)+\frac{D}{2}\left(p_2^2+x_2^2\right)$$

Then Hamilton's equations of motion are the following.

$$\dot{x}_1=\frac{\partial H_c}{\partial p_1}=Ap_1+Bp_2-Cx_2 \qquad \dot{p}_1=-\frac{\partial H_c}{\partial x_1}=-\left(Ax_1+Bx_2+Cp_2\right)$$
$$\dot{x}_2=\frac{\partial H_c}{\partial p_2}=Bp_1+Dp_2+Cx_1 \qquad \dot{p}_2=-\frac{\partial H_c}{\partial x_2}=-\left(Bx_1+Dx_2-Cp_1\right)$$

QM vs. Classical Equations are identical

Finally a 2^{nd} time derivative (Assume *constant A, B, D,* and *let C=0*) gives 2^{nd}-order classical Newton-Hooke-like equation: $|\ddot{\mathbf{x}}\rangle=-\mathbf{K}\bullet|\mathbf{x}\rangle$

$$\begin{aligned}\ddot{x}_1&=A\dot{p}_1+B\dot{p}_2-C\dot{x}_2\\&=-A\left(Ax_1+Bx_2+Cp_2\right)-B\left(Bx_1+Dx_2-Cp_1\right)-C\left(Bp_1+Dp_2+Cx_1\right)\\&=-\left(A^2+B^2+C^2\right)x_1-\left(AB+BD\right)x_2-C\left(A+D\right)p_2\end{aligned}$$

$$\begin{aligned}\ddot{x}_2&=B\dot{p}_1+D\dot{p}_2+C\dot{x}_1\\&=-B\left(Ax_1+Bx_2+Cp_2\right)-D\left(Bx_1+Dx_2-Cp_1\right)+C\left(Ap_1+Bp_2-Cx_2\right)\\&=-\left(AB+BD\right)x_1-\left(B^2+D^2+C^2\right)x_2+C\left(A+D\right)p_1\end{aligned}$$

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus *Classical 2DHO:* $\partial^2{}_t\mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

$$i\hbar|\dot{\Psi}(t)\rangle = \mathbf{H}|\Psi(t)\rangle \qquad\qquad |\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$$

First start with 2-by-2 Hermitian (self-conjugate) matrix

$$\mathbf{H} = \begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix} = \mathbf{H}^\dagger$$

that operates on 2-D complex Dirac ket vector $|\Psi\rangle$.

$$|\Psi\rangle = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Separate real x_k and imaginary p_k parts of Ψ_k amplitudes to convert the complex 1st-order equation $i\partial_t\Psi = \mathbf{H}\Psi$ into pairs of real real 1st-order differential equations.

$$\dot{x}_1 = Ap_1 + Bp_2 - Cx_2 \qquad \dot{p}_1 = -Ax_1 - Bx_2 - Cp_2$$

$$\dot{x}_2 = Bp_1 + Dp_2 + Cx_1 \qquad \dot{p}_2 = -Bx_1 - Dx_2 + Cp_1$$

QM vs. Classical Equations are identical

Then start with classical Hamiltonian.

$$H_c = \frac{A}{2}\left(p_1^2 + x_1^2\right) + B\left(x_1x_2 + p_1p_2\right) + C\left(x_1p_2 - x_2p_1\right) + \frac{D}{2}\left(p_2^2 + x_2^2\right)$$

Then Hamilton's equations of motion are the following.

$$\dot{x}_1 = \frac{\partial H_c}{\partial p_1} = Ap_1 + Bp_2 - Cx_2 \qquad \dot{p}_1 = -\frac{\partial H_c}{\partial x_1} = -\left(Ax_1 + Bx_2 + Cp_2\right)$$

$$\dot{x}_2 = \frac{\partial H_c}{\partial p_2} = Bp_1 + Dp_2 + Cx_1 \qquad \dot{p}_2 = -\frac{\partial H_c}{\partial x_2} = -\left(Bx_1 + Dx_2 - Cp_1\right)$$

Finally a 2nd time derivative (Assume *constant A, B, D,* and *let C=0*) gives 2nd-order classical Newton-Hooke-like equation: $|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$

$$\begin{aligned}\ddot{x}_1 &= A\dot{p}_1 + B\dot{p}_2 - C\dot{x}_2 \\ &= -A\left(Ax_1 + Bx_2 + Cp_2\right) - B\left(Bx_1 + Dx_2 - Cp_1\right) - C\left(Bp_1 + Dp_2 + Cx_1\right) \\ &= -\left(A^2 + B^2 + C^2\right)x_1 - \left(AB + BD\right)x_2 - C\left(A + D\right)p_2\end{aligned}$$

$$\begin{aligned}\ddot{x}_2 &= B\dot{p}_1 + D\dot{p}_2 + C\dot{x}_1 \\ &= -B\left(Ax_1 + Bx_2 + Cp_2\right) - D\left(Bx_1 + Dx_2 - Cp_1\right) + C\left(Ap_1 + Bp_2 - Cx_2\right) \\ &= -\left(AB + BD\right)x_1 - \left(B^2 + D^2 + C^2\right)x_2 + C\left(A + D\right)p_1\end{aligned}$$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} A^2 + B^2 & AB + BD \\ AB + BD & B^2 + D^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

For C=0
Is form of 2D Hooke harmonic oscillator

$$\frac{\partial^2}{\partial t^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \equiv \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus *Classical 2DHO:* $\partial^2{}_t\mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

$$i\hbar|\dot{\Psi}(t)\rangle = \mathbf{H}|\Psi(t)\rangle \qquad |\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$$

First start with 2-by-2 Hermitian (self-conjugate) matrix

$$\mathbf{H} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H}^\dagger$$

that operates on 2-D complex Dirac ket vector $|\Psi\rangle$.

$$|\Psi\rangle = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Separate real x_k and imaginary p_k parts of Ψ_k amplitudes to convert the complex 1st-order equation $i\partial_t\Psi = \mathbf{H}\Psi$ into pairs of real real 1st-order differential equations.

$$\dot{x}_1 = Ap_1 + Bp_2 - Cx_2 \qquad \dot{p}_1 = -Ax_1 - Bx_2 - Cp_2$$

$$\dot{x}_2 = Bp_1 + Dp_2 + Cx_1 \qquad \dot{p}_2 = -Bx_1 - Dx_2 + Cp_1$$

QM vs. Classical Equations are identical

Then start with classical Hamiltonian.

$$H_c = \frac{A}{2}\left(p_1^2 + x_1^2\right) + B\left(x_1x_2 + p_1p_2\right) + C\left(x_1p_2 - x_2p_1\right) + \frac{D}{2}\left(p_2^2 + x_2^2\right)$$

Then Hamilton's equations of motion are the following.

$$\dot{x}_1 = \frac{\partial H_c}{\partial p_1} = Ap_1 + Bp_2 - Cx_2 \qquad \dot{p}_1 = -\frac{\partial H_c}{\partial x_1} = -\left(Ax_1 + Bx_2 + Cp_2\right)$$

$$\dot{x}_2 = \frac{\partial H_c}{\partial p_2} = Bp_1 + Dp_2 + Cx_1 \qquad \dot{p}_2 = -\frac{\partial H_c}{\partial x_2} = -\left(Bx_1 + Dx_2 - Cp_1\right)$$

Finally a 2nd time derivative (Assume *constant A, B, D,* and *let C=0*) gives 2nd-order classical Newton-Hooke-like equation: $|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$

$$\begin{aligned}\ddot{x}_1 &= A\dot{p}_1 + B\dot{p}_2 - C\dot{x}_2 \\ &= -A\left(Ax_1 + Bx_2 + Cp_2\right) - B\left(Bx_1 + Dx_2 - Cp_1\right) - C\left(Bp_1 + Dp_2 + Cx_1\right) \\ &= -\left(A^2 + B^2 + C^2\right)x_1 - \left(AB + BD\right)x_2 - C\left(A + D\right)p_2\end{aligned}$$

$$\begin{aligned}\ddot{x}_2 &= B\dot{p}_1 + D\dot{p}_2 + C\dot{x}_1 \\ &= -B\left(Ax_1 + Bx_2 + Cp_2\right) - D\left(Bx_1 + Dx_2 - Cp_1\right) + C\left(Ap_1 + Bp_2 - Cx_2\right) \\ &= -\left(AB + BD\right)x_1 - \left(B^2 + D^2 + C^2\right)x_2 + C\left(A + D\right)p_1\end{aligned}$$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} A^2 + B^2 & AB + BD \\ AB + BD & B^2 + D^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

For C=0 Is form of 2D Hooke harmonic oscillator

$$\frac{\partial^2}{\partial t^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \equiv \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Here is an operator view of the QM-Classical connection: Take Schrodinger operator $i\partial_t = \mathbf{H}$ (with $C=0$) and square it!

$$i\frac{\partial}{\partial t} = \begin{pmatrix} A & B \\ B & D \end{pmatrix} \Rightarrow \left(i\frac{\partial}{\partial t}\right)^2 = \begin{pmatrix} A & B \\ B & D \end{pmatrix}^2 \Rightarrow -\frac{\partial^2}{\partial t^2} = \begin{pmatrix} A^2 + B^2 & AB + BD \\ AB + BD & B^2 + D^2 \end{pmatrix}$$

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus *Classical 2DHO:* $\partial^2_t \mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

$$i\hbar|\dot{\Psi}(t)\rangle = \mathbf{H}|\Psi(t)\rangle \qquad |\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$$

First start with 2-by-2 Hermitian (self-conjugate) matrix

$$\mathbf{H} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H}^\dagger$$

that operates on 2-D complex Dirac ket vector $|\Psi\rangle$.

$$|\Psi\rangle = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Separate real x_k and imaginary p_k parts of Ψ_k amplitudes to convert the complex 1st-order equation $i\partial_t\Psi = \mathbf{H}\Psi$ into pairs of real real 1st-order differential equations.

$$\dot{x}_1 = Ap_1 + Bp_2 - Cx_2 \qquad \dot{p}_1 = -Ax_1 - Bx_2 - Cp_2$$

$$\dot{x}_2 = Bp_1 + Dp_2 + Cx_1 \qquad \dot{p}_2 = -Bx_1 - Dx_2 + Cp_1$$

QM vs. Classical Equations are identical

Then start with classical Hamiltonian.

$$H_c = \frac{A}{2}\left(p_1^2 + x_1^2\right) + B\left(x_1x_2 + p_1p_2\right) + C\left(x_1p_2 - x_2p_1\right) + \frac{D}{2}\left(p_2^2 + x_2^2\right)$$

Then Hamilton's equations of motion are the following.

$$\dot{x}_1 = \frac{\partial H_c}{\partial p_1} = Ap_1 + Bp_2 - Cx_2 \qquad \dot{p}_1 = -\frac{\partial H_c}{\partial x_1} = -\left(Ax_1 + Bx_2 + Cp_2\right)$$

$$\dot{x}_2 = \frac{\partial H_c}{\partial p_2} = Bp_1 + Dp_2 + Cx_1 \qquad \dot{p}_2 = -\frac{\partial H_c}{\partial x_2} = -\left(Bx_1 + Dx_2 - Cp_1\right)$$

Finally a 2nd time derivative (Assume *constant A, B, D,* and *let C=0*) gives 2nd-order classical Newton-Hooke-like equation: $|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$

$$\begin{aligned}\ddot{x}_1 &= A\dot{p}_1 + B\dot{p}_2 - C\dot{x}_2 \\ &= -A\left(Ax_1 + Bx_2 + Cp_2\right) - B\left(Bx_1 + Dx_2 - Cp_1\right) - C\left(Bp_1 + Dp_2 + Cx_1\right) \\ &= -\left(A^2 + B^2 + C^2\right)x_1 - \left(AB + BD\right)x_2 - C\left(A + D\right)p_2\end{aligned}$$

$$\begin{aligned}\ddot{x}_2 &= B\dot{p}_1 + D\dot{p}_2 + C\dot{x}_1 \\ &= -B\left(Ax_1 + Bx_2 + Cp_2\right) - D\left(Bx_1 + Dx_2 - Cp_1\right) + C\left(Ap_1 + Bp_2 - Cx_2\right) \\ &= -\left(AB + BD\right)x_1 - \left(B^2 + D^2 + C^2\right)x_2 + C\left(A + D\right)p_1\end{aligned}$$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} A^2 + B^2 & AB + BD \\ AB + BD & B^2 + D^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

For C=0
Is form of 2D Hooke harmonic oscillator

$$\frac{\partial^2}{\partial t^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \equiv \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Here is an operator view of the QM-Classical connection: Take Schrodinger operator $i\partial_t = \mathbf{H}$ (with $C=0$) and square it!

$$i\frac{\partial}{\partial t} = \begin{pmatrix} A & B \\ B & D \end{pmatrix} \Rightarrow \left(i\frac{\partial}{\partial t}\right)^2 = \begin{pmatrix} A & B \\ B & D \end{pmatrix}^2 \Rightarrow -\frac{\partial^2}{\partial t^2} = \begin{pmatrix} A^2 + B^2 & AB + BD \\ AB + BD & B^2 + D^2 \end{pmatrix}$$

Conclusion: 2-state Schro-equation $i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ is like "square-root" of Newton-Hooke. $\sqrt{|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle}$

ANALOGY: *2-State Schrodinger:* $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus *Classical 2DHO:* $\partial^2_t \mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

$$i\hbar|\dot{\Psi}(t)\rangle = \mathbf{H}|\Psi(t)\rangle \qquad |\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$$

First start with 2-by-2 Hermitian (self-conjugate) matrix

$$\mathbf{H} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H}^\dagger$$

that operates on 2-D complex Dirac ket vector $|\Psi\rangle$.

$$|\Psi\rangle = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Separate real x_k and imaginary p_k parts of Ψ_k amplitudes to convert the complex 1st-order equation $i\partial_t\Psi = \mathbf{H}\Psi$ into pairs of real real 1st-order differential equations.

$$\dot{x}_1 = Ap_1 + Bp_2 - Cx_2 \qquad \dot{p}_1 = -Ax_1 - Bx_2 - Cp_2$$
$$\dot{x}_2 = Bp_1 + Dp_2 + Cx_1 \qquad \dot{p}_2 = -Bx_1 - Dx_2 + Cp_1$$

QM vs. Classical Equations are identical

Then start with classical Hamiltonian.

$$H_c = \frac{A}{2}\left(p_1^2 + x_1^2\right) + B\left(x_1x_2 + p_1p_2\right) + C\left(x_1p_2 - x_2p_1\right) + \frac{D}{2}\left(p_2^2 + x_2^2\right)$$

Then Hamilton's equations of motion are the following.

$$\dot{x}_1 = \frac{\partial H_c}{\partial p_1} = Ap_1 + Bp_2 - Cx_2 \qquad \dot{p}_1 = -\frac{\partial H_c}{\partial x_1} = -\left(Ax_1 + Bx_2 + Cp_2\right)$$
$$\dot{x}_2 = \frac{\partial H_c}{\partial p_2} = Bp_1 + Dp_2 + Cx_1 \qquad \dot{p}_2 = -\frac{\partial H_c}{\partial x_2} = -\left(Bx_1 + Dx_2 - Cp_1\right)$$

Finally a 2nd time derivative (Assume *constant A, B, D,* and *let C=0*) gives 2nd-order classical Newton-Hooke-like equation: $|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$

$$\begin{aligned}\ddot{x}_1 &= A\dot{p}_1 + B\dot{p}_2 - C\dot{x}_2 \\ &= -A\left(Ax_1 + Bx_2 + Cp_2\right) - B\left(Bx_1 + Dx_2 - Cp_1\right) - C\left(Bp_1 + Dp_2 + Cx_1\right) \\ &= -\left(A^2 + B^2 + C^2\right)x_1 - \left(AB + BD\right)x_2 - C\left(A + D\right)p_2\end{aligned}$$

$$\begin{aligned}\ddot{x}_2 &= B\dot{p}_1 + D\dot{p}_2 + C\dot{x}_1 \\ &= -B\left(Ax_1 + Bx_2 + Cp_2\right) - D\left(Bx_1 + Dx_2 - Cp_1\right) + C\left(Ap_1 + Bp_2 - Cx_2\right) \\ &= -\left(AB + BD\right)x_1 - \left(B^2 + D^2 + C^2\right)x_2 + C\left(A + D\right)p_1\end{aligned}$$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} A^2 + B^2 & AB + BD \\ AB + BD & B^2 + D^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

For C=0 Is form of 2D Hooke harmonic oscillator

$$\frac{\partial^2}{\partial t^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \equiv \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Here is an operator view of the QM-Classical connection: Take Schrodinger operator $i\partial_t = \mathbf{H}$ (with $C\neq 0$) and square it!

$$i\frac{\partial}{\partial t} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} \Rightarrow \left(i\frac{\partial}{\partial t}\right)^2 = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}^2 \Rightarrow -\frac{\partial^2}{\partial t^2} = \begin{pmatrix} A^2 + B^2 + C^2 & AB + BD - i(AC + CD) \\ AB + BD + i(AC + CD) & B^2 + D^2 + C^2 \end{pmatrix}$$

Conclusion: 2-state Schro-equation $i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ is like "square-root" of Newton-Hooke. $\sqrt{\;|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle\;}$