Lecture 22 C_N Wave Modes
Thursday 3.31.2016

C_N-Symmetric Wave Modes and 2-CW Algebra and Geometry

(Ch. 5 of Unit 4 3.31.16)

Wave resonance in cyclic C_n symmetry (REVIEW)
- *C_6 symmetric mode model:Distant neighbor coupling*
  - *C_6 moving waves and degenerate standing waves*
  - *C_6 dispersion functions for 1st, 2nd, and 3rd-neighbor coupling*
  - *C_6 dispersion functions split by C-type symmetry(complex, chiral, ...)*

C_{12} and higher symmetry mode models: Archetypes of dispersion functions and 1-CW phase velocity
- *½-Sum-½-Diff-theory of 2-CW group and phase velocity*

Given two 1-CW phases: Find 2-CW phase velocity $\mathbf{V}_{phase}^{(2\text{-}CW)}$ and group velocity $\mathbf{V}_{group}^{(2\text{-}CW)}$
- *Example: Bohr Dispersion 2-CW made of 1-CW(m=-1) + 1-CW(m=2)*
- *Find 2-CW space-time (x,t) lattice from per-space-time (κ,υ) by matrix-algebra/geometry*
  - *Same 1-CW(m=-1) + 1-CW(m=2) Example*

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Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigen-bra-vectors $\langle(m)|$

$$\mathbf{P}^{(0)}=\tfrac{1}{3}(\mathbf{r}^0+\mathbf{r}^1+\mathbf{r}^2)=\tfrac{1}{3}(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2)$$

$$\mathbf{P}^{(1)}=\tfrac{1}{3}(\mathbf{r}^0+\rho_1^*\mathbf{r}^1+\rho_2^*\mathbf{r}^2)=\tfrac{1}{3}(\mathbf{1}+e^{-i2\pi/3}\mathbf{r}^1+e^{+i2\pi/3}\mathbf{r}^2)$$

$$\mathbf{P}^{(2)}=\tfrac{1}{3}(\mathbf{r}^0+\rho_2^*\mathbf{r}^1+\rho_1^*\mathbf{r}^2)=\tfrac{1}{3}(\mathbf{1}+e^{+i2\pi/3}\mathbf{r}^1+e^{-i2\pi/3}\mathbf{r}^2)$$

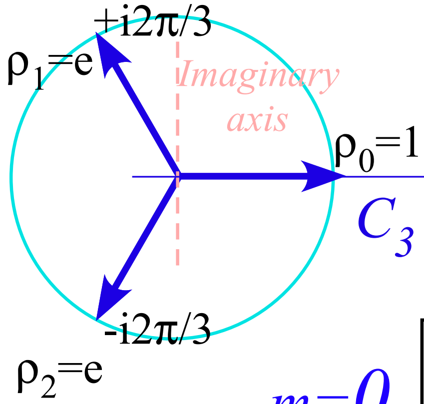
$$\langle(0_3)|=\langle 0|\mathbf{P}^{(0)}\sqrt{3}=\sqrt{\tfrac{1}{3}}(\,1\quad 1\quad 1\,)$$

$$\langle(1_3)|=\langle 0|\mathbf{P}^{(1)}\sqrt{3}=\sqrt{\tfrac{1}{3}}(\,1\quad e^{-i2\pi/3}\quad e^{+i2\pi/3})$$

$$\langle(2_3)|=\langle 0|\mathbf{P}^{(2)}\sqrt{3}=\sqrt{\tfrac{1}{3}}(1\quad e^{+i2\pi/3}\quad e^{-i2\pi/3})$$

Basic "trinary" wavefunction: $\psi^{k_m}(x_p)=e^{ik_m x_p}=e^{i\frac{2\pi mp}{3}}$ $\quad\rho_{mp}=$

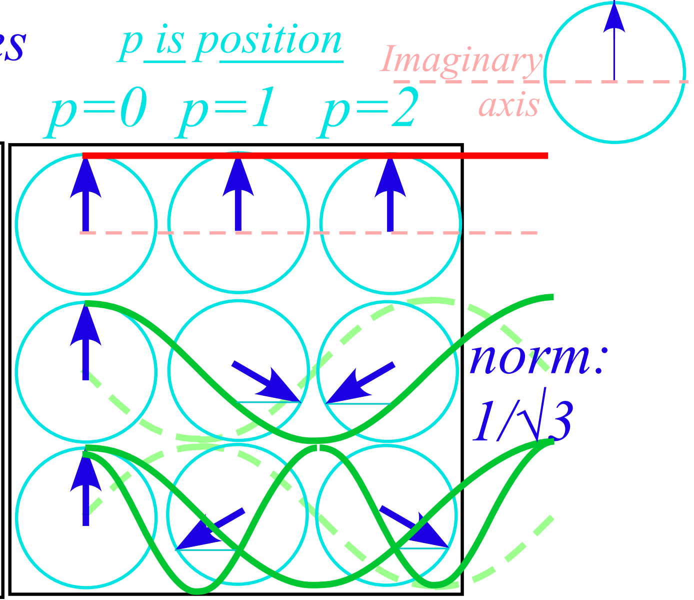
(m_3) means: *m-modulo-3* (Details follow)

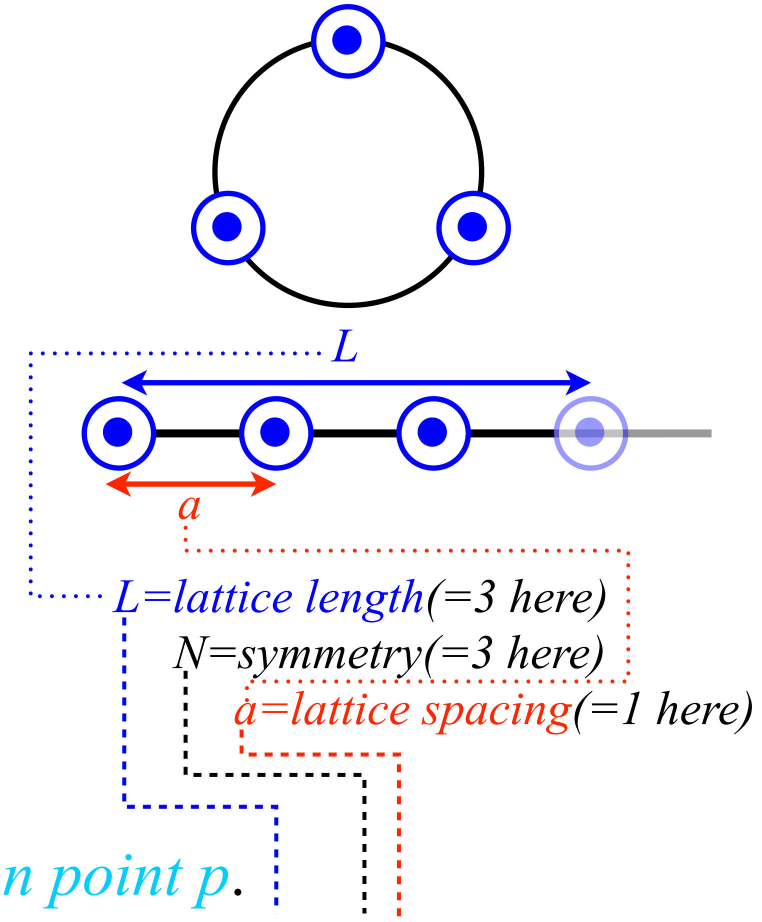


C_3 mode phase character tables

wave-number m= “momentum”

	p=0	*p=1*	*p=2*
$m=0_3$	$\rho^*_{00}=1$	$\rho^*_{01}=1$	$\rho^*_{02}=1$
$m=1_3$	$\rho^*_{10}=1$	$\rho^*_{11}=e^{-i2\pi/3}$	$\rho^*_{12}=e^{i2\pi/3}$
$m=2_3$	$\rho^*_{20}=1$	$\rho^*_{21}=e^{i2\pi/3}$	$\rho^*_{22}=e^{-i2\pi/3}$





Two distinct types of “quantum” numbers.

$p=0,1,$or 2 is *power p* of operator $\mathbf{r}^p$ and defines each oscillator’s *position point p*.

$m=0,1,$or 2 is *mode momentum m* of the waves or wavevector $k_m=2\pi/\lambda_m=2\pi m/L$. $(L=Na=3)$

wavelength $\lambda_m=2\pi/k_m=L/m$

Each quantum number follows *modular arithmetic:* sums or products are an *integer-modulo-3*, that is, always $0,1,$or $2,$ or else $-1,0,$or $1,$ or else $-2,-1,$or $0,$ *etc.,* depending on choice of origin.

For example, for $m=2$ and $p=2$ the number $(\rho_m)^p=(e^{im2\pi/3})^p$ is $e^{imp\cdot 2\pi/3}=e^{i4\cdot 2\pi/3}=e^{i1\cdot 2\pi/3}\,e^{i2\pi}=e^{i2\pi/3}=\rho_1$.

That is, (*2-times-2*) mod *3* is not *4* but *1* (*4* mod *3=1*, the remainder of *4* divided by *3*.)

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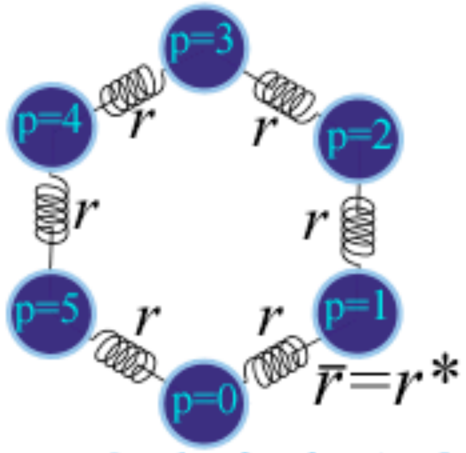
C_6 Symmetric Mode Model: 1st neighbor coupling

We usually assume Real $r=\bar{r}$
Stability only requires $(r)^*=\bar{r}$

(a) 1st Neighbor C_6



$$\mathbf{H}^{B1(6)}=\begin{array}{cccccc|c} p=0 & 1 & 2 & 3 & 4 & 5 & \\ \end{array}$$

$$\mathbf{H}^{B1(6)}=\left(\begin{array}{cccccc} H_1 & -r & \cdot & \cdot & \cdot & -\bar{r} \\ -\bar{r} & H_1 & -r & \cdot & \cdot & \cdot \\ \cdot & -\bar{r} & H_1 & -r & \cdot & \cdot \\ \cdot & \cdot & -\bar{r} & H_1 & -r & \cdot \\ \cdot & \cdot & \cdot & -\bar{r} & H_1 & -r \\ -r & \cdot & \cdot & \cdot & -\bar{r} & H_1 \end{array}\right)\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}$$

$$= H_1\mathbf{1} - r\mathbf{r} - \bar{r}\mathbf{r}^{-1}$$

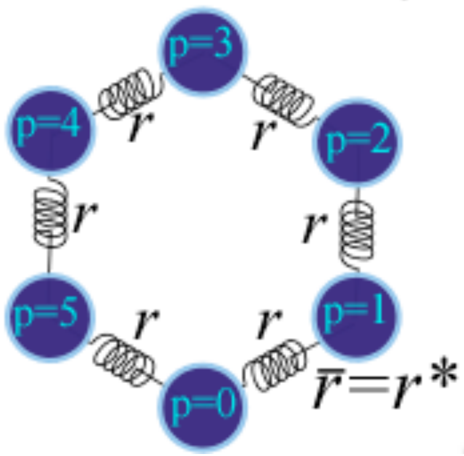


Fig. 12 International Journal of Molecular Science 14, 749 (2013)

C_6 Symmetric Mode Model: 1st and 2nd neighbor coupling

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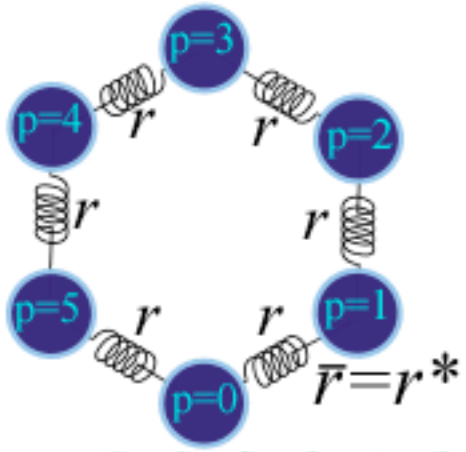
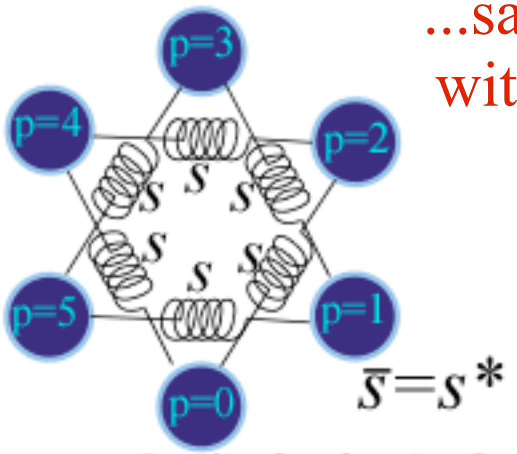
(a) 1st Neighbor C_6



$$\mathbf{H}^{B1(6)}=\begin{array}{c}\begin{array}{cccccc} p=0 & 1 & 2 & 3 & 4 & 5 \end{array}\\ \left(\begin{array}{cccccc} H_1 & -r & \cdot & \cdot & \cdot & -\bar{r} \\ -\bar{r} & H_1 & -r & \cdot & \cdot & \cdot \\ \cdot & -\bar{r} & H_1 & -r & \cdot & \cdot \\ \cdot & \cdot & -\bar{r} & H_1 & -r & \cdot \\ \cdot & \cdot & \cdot & -\bar{r} & H_1 & -r \\ -r & \cdot & \cdot & \cdot & -\bar{r} & H_1 \end{array}\right)\end{array}\begin{array}{c} \\ 0\\1\\2\\3\\4\\5\end{array}$$

$$= H_1\mathbf{1} - r\mathbf{r} - \bar{r}\mathbf{r}^{-1}$$

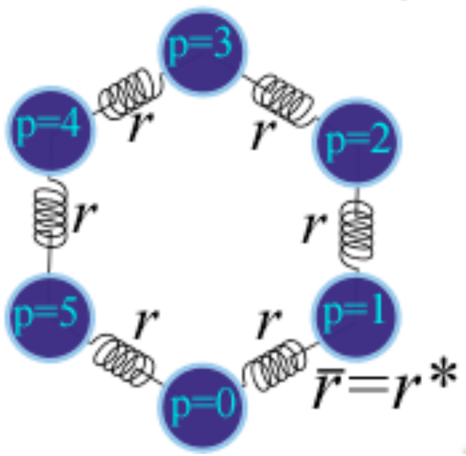
(b) 2nd Neighbor C_6



...same with s

$$\mathbf{H}^{B2(6)}=\begin{array}{c}\begin{array}{cccccc} p=0 & 1 & 2 & 3 & 4 & 5 \end{array}\\ \left(\begin{array}{cccccc} H_2 & \cdot & -s & \cdot & -\bar{s} & \cdot \\ \cdot & H_2 & \cdot & -s & \cdot & -\bar{s} \\ -\bar{s} & \cdot & H_2 & \cdot & -s & \cdot \\ \cdot & -\bar{s} & \cdot & H_2 & \cdot & -s \\ -s & \cdot & -\bar{s} & \cdot & H_2 & \cdot \\ \cdot & -s & \cdot & -\bar{s} & \cdot & H_2 \end{array}\right)\end{array}\begin{array}{c} \\ 0\\1\\2\\3\\4\\5\end{array}$$

$$= H_2\mathbf{1} - s\mathbf{r}^2 - \bar{s}\mathbf{r}^{-2}$$



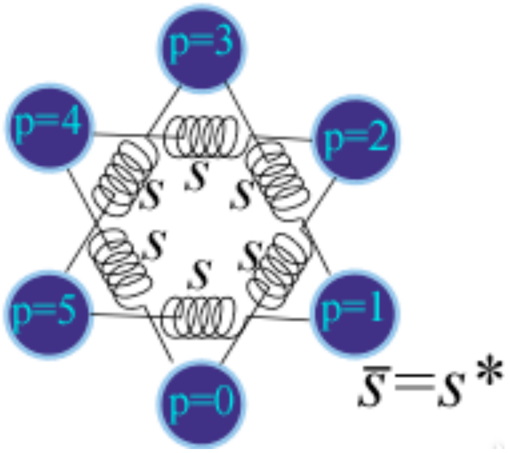
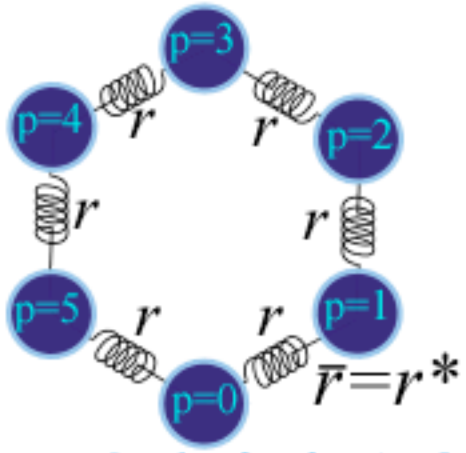


Fig. 12 International Journal of Molecular Science 14, 749 (2013)

C_6 Symmetric Mode Model: 1st , 2nd and 3rd neighbor coupling

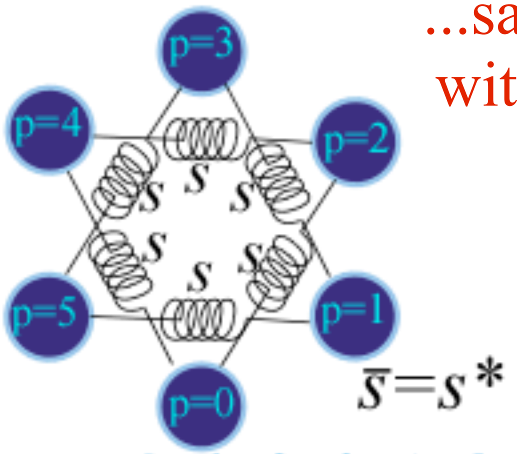
(a) 1st Neighbor C_6



$$\mathbf{H}^{B1(6)}=\begin{array}{c} \begin{array}{cccccc} p=0 & 1 & 2 & 3 & 4 & 5 \end{array} & \\ \left(\begin{array}{cccccc} H_1 & -r & \cdot & \cdot & \cdot & -\bar{r} \\ -\bar{r} & H_1 & -r & \cdot & \cdot & \cdot \\ \cdot & -\bar{r} & H_1 & -r & \cdot & \cdot \\ \cdot & \cdot & -\bar{r} & H_1 & -r & \cdot \\ \cdot & \cdot & \cdot & -\bar{r} & H_1 & -r \\ -r & \cdot & \cdot & \cdot & -\bar{r} & H_1 \end{array}\right) & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \end{array}$$

$$= H_1\mathbf{1} - r\mathbf{r} - \bar{r}\mathbf{r}^{-1}$$

(b) 2nd Neighbor C_6



...same with s

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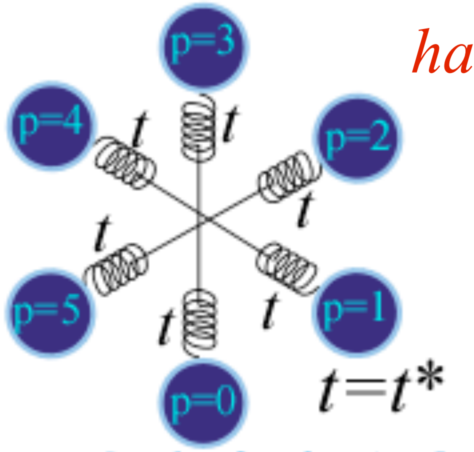
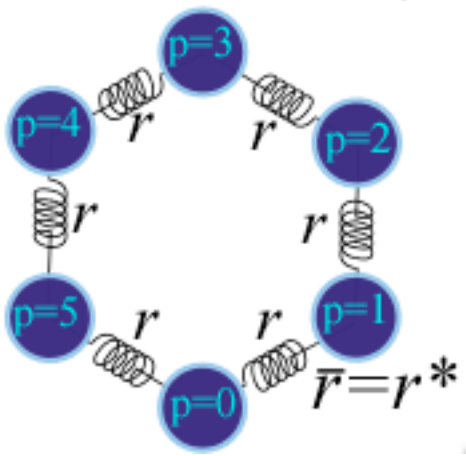
(c) 3rd Neighbor C_6

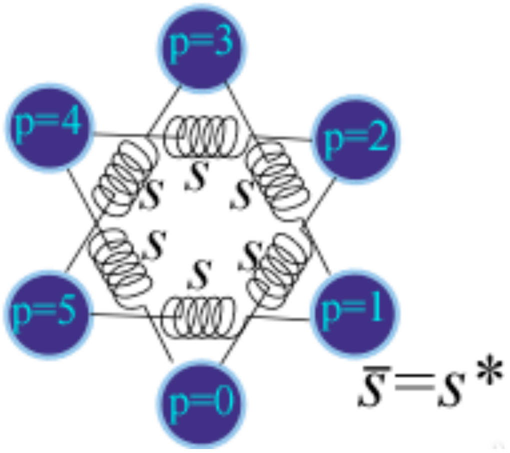


...but t *has* to be real

$$\mathbf{H}^{B3(6)}=\begin{array}{c} \begin{array}{cccccc} p=0 & 1 & 2 & 3 & 4 & 5 \end{array} & \\ \left(\begin{array}{cccccc} H_3 & \cdot & \cdot & -t & \cdot & \cdot \\ \cdot & H_3 & \cdot & \cdot & -t & \cdot \\ \cdot & \cdot & H_3 & \cdot & \cdot & -t \\ -t & \cdot & \cdot & H_3 & \cdot & \cdot \\ \cdot & -t & \cdot & \cdot & H_3 & \cdot \\ \cdot & \cdot & -t & \cdot & \cdot & H_3 \end{array}\right) & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \end{array}$$

$$= H_3\mathbf{1} - t\mathbf{r}^3 - t\mathbf{r}^{-3}$$





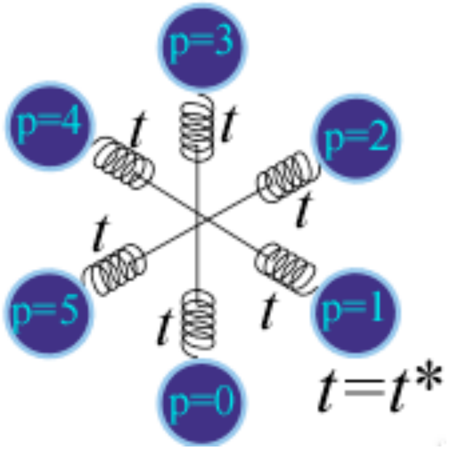


Fig. 12 International Journal of Molecular Science 14, 749 (2013)

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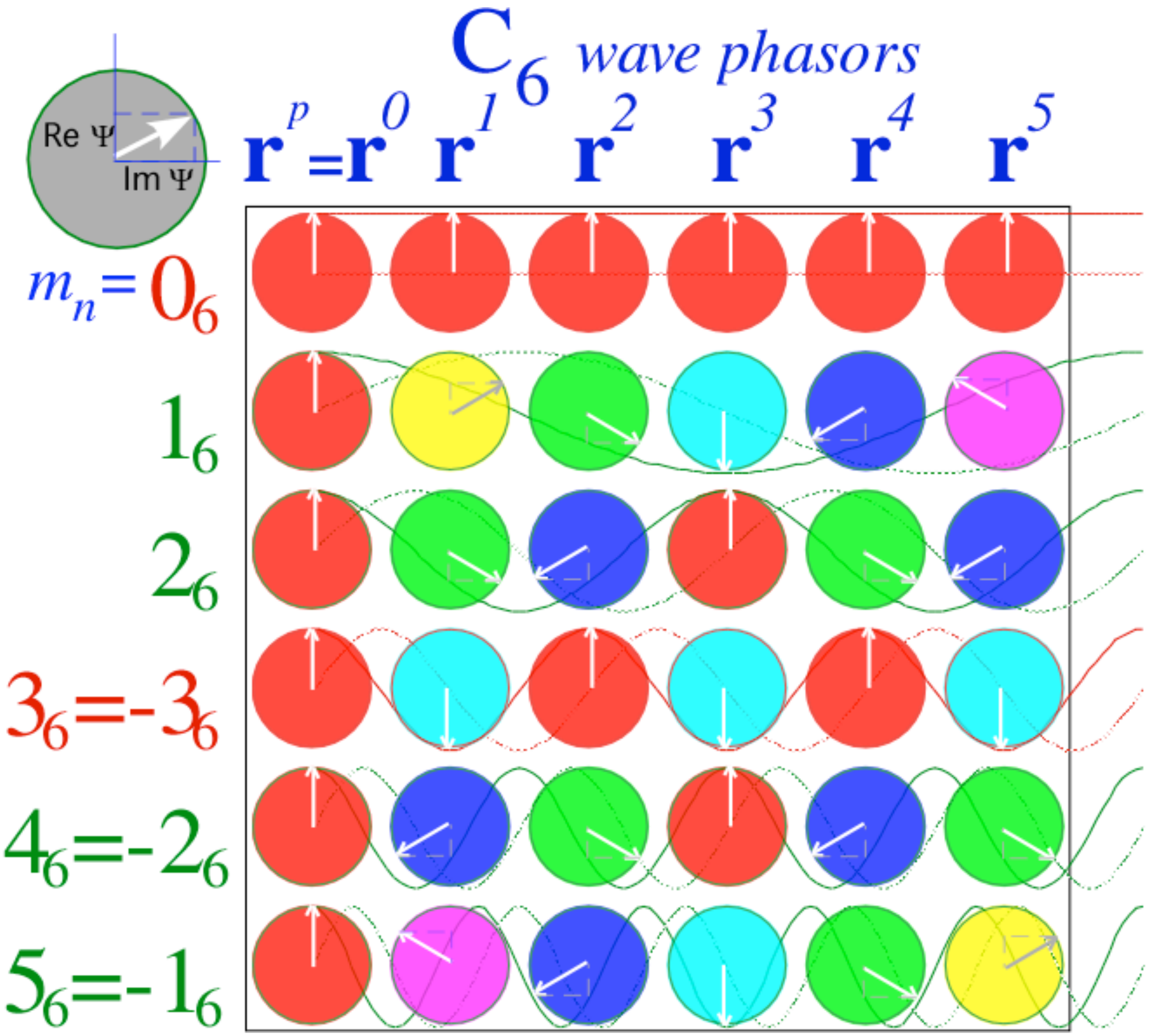
Same 1-CW(m=-1) + 1-CW(m=2) Example

C_6 Spectral resolution: 6th roots of unity

$\chi_p^{m*}(C_6)$	$\mathbf{r}^{p=0}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{r}^3$	$\mathbf{r}^4$	$\mathbf{r}^5$
$m=0_6$	1	1	1	1	1	1
1_6	1	ε^*	ε^{2*}	-1	ε^2	ε
2_6	1	ε^{2*}	ε^2	1	ε^{2*}	ε^2
$3_6=-3_6$	1	-1	1	-1	1	-1
$4_6=-2_6$	1	ε^2	ε^{2*}	1	ε^2	ε^{2*}
$5_6=-1_6$	1	ε	ε^2	-1	ε^{2*}	ε^*

$$\textit{Wavefunction}:\ \Psi^m(x_p)=\chi_p^{m*}=D^{m*}(\mathbf{r}^p)$$



$$\chi_p^m = e^{ik_m r^p} = e^{\frac{2\pi imp}{6}}$$

WaveIt C_6 Character Phasors Web Simulation

Fig. 13 International Journal of Molecular Science 14, 752 (2013)

C_6 Spectral resolution: 6th roots of unity

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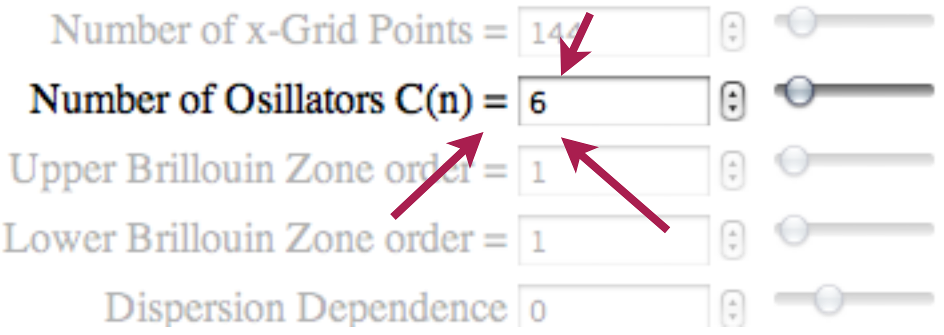
WaveIt
Local Controls



Number of x-Grid Points = 14

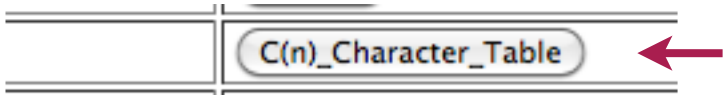
Number of Oscillators C(n) = 6

Upper Brillouin Zone order = 1

Lower Brillouin Zone order = 1

Dispersion Dependence 0

WaveIt Scenarios



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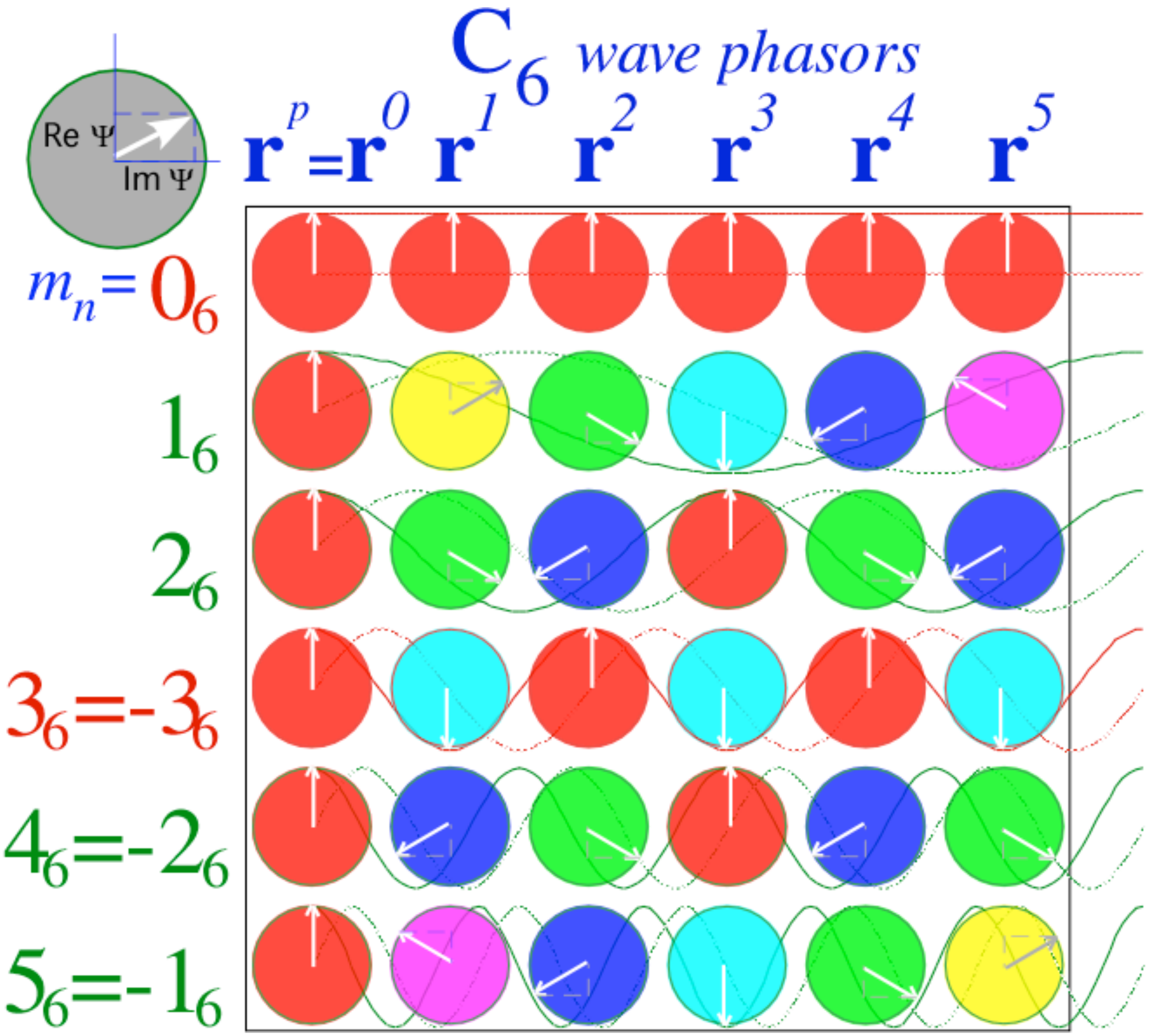

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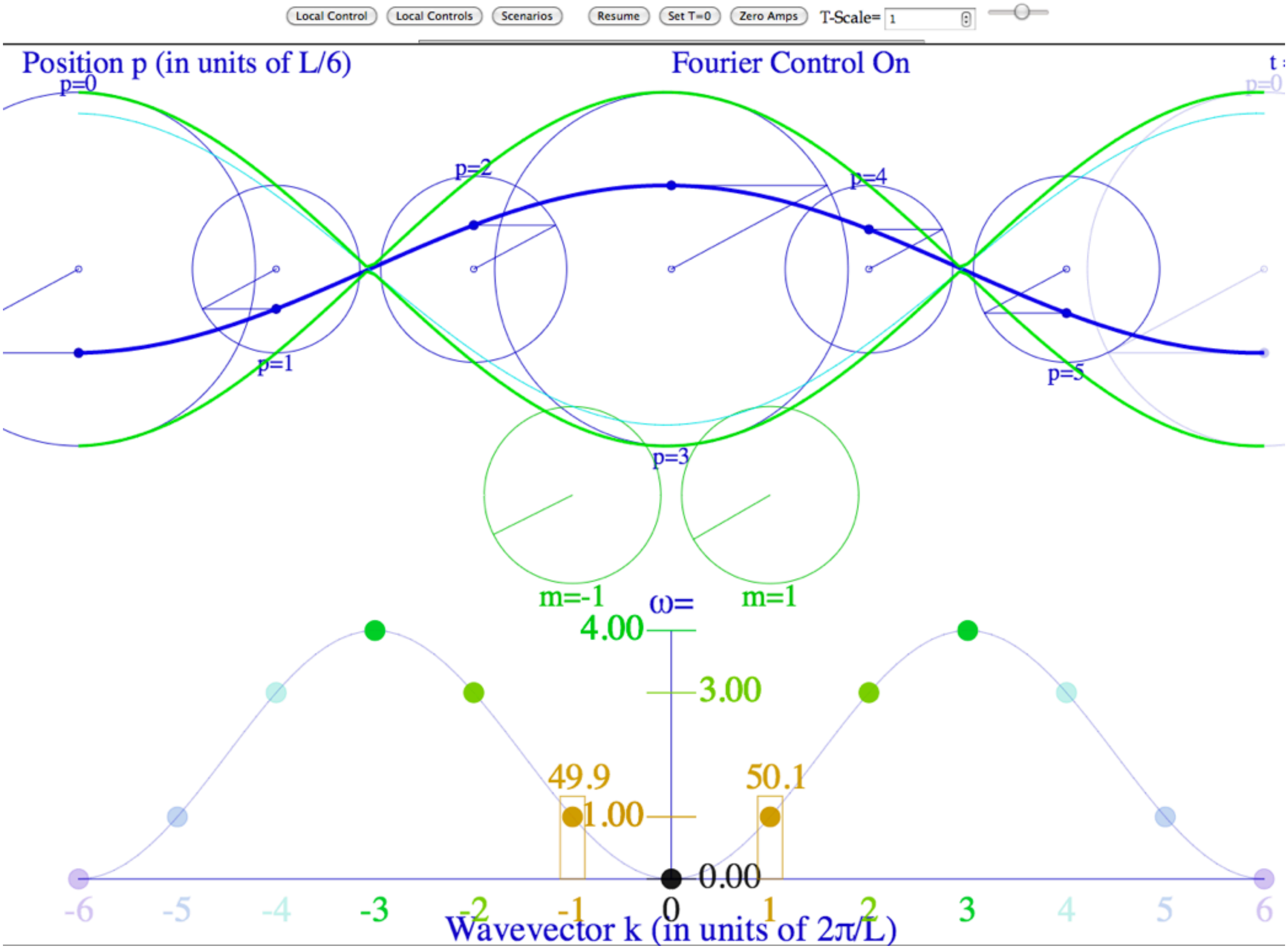

[WaveIt Web Simulation - *Standing Wave (N=6)*]

[WaveIt Web Simulation - *Galloping Wave (N=6)*]

[WaveIt Web Simulation - *Galloping Wave (N=12)*]

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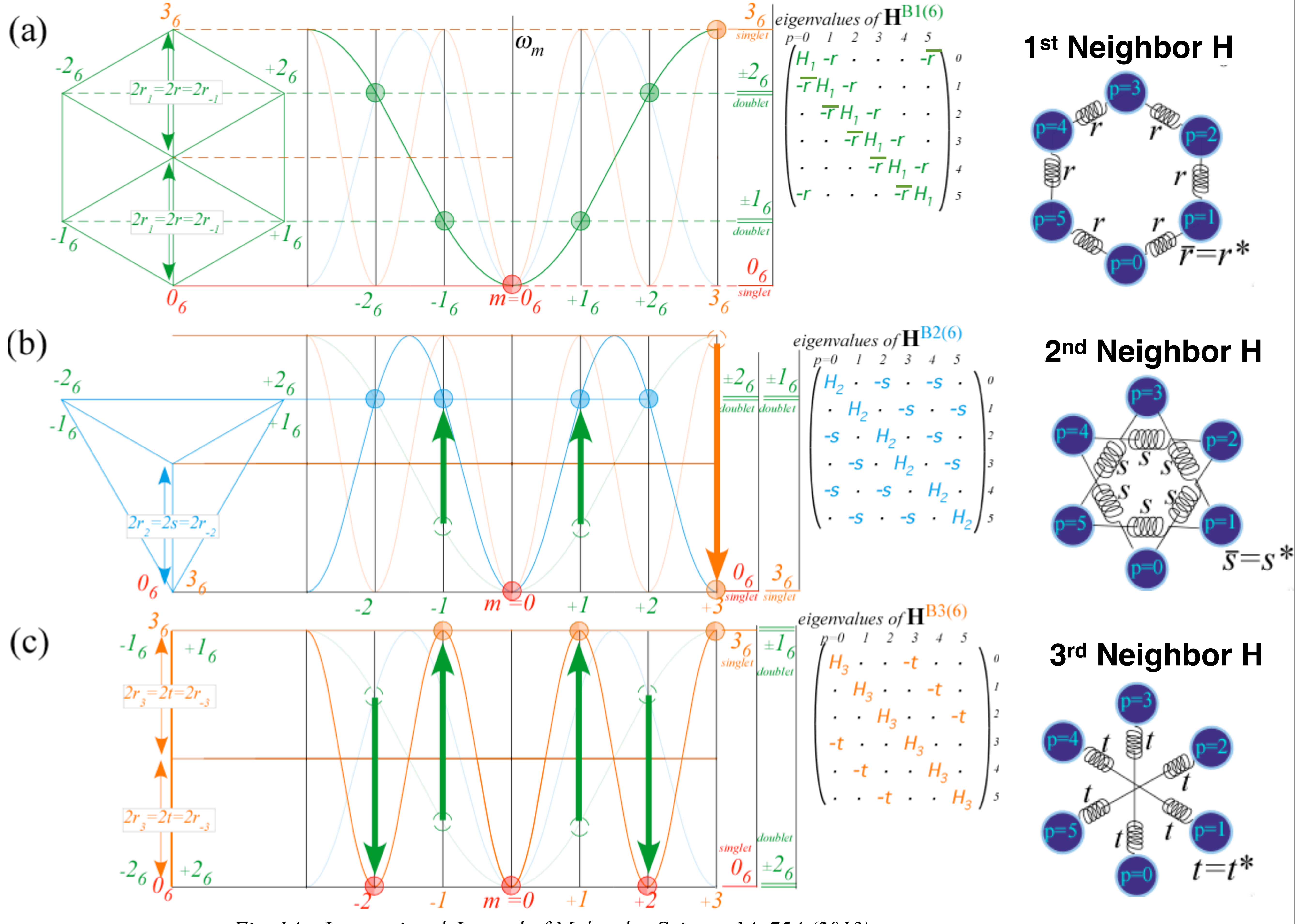
C_6 Spectral resolution of n^{th} Neighbor H: Same modes but different dispersion



eigenvalues of $\mathbf{H}^{B1(6)}$

$$\begin{array}{c|cccccc}
 & p=0 & 1 & 2 & 3 & 4 & 5 \\
\hline
0 & H_1 & -r & \cdot & \cdot & \cdot & -\bar{r} \\
1 & -\bar{r} & H_1 & -r & \cdot & \cdot & \cdot \\
2 & \cdot & -\bar{r} & H_1 & -r & \cdot & \cdot \\
3 & \cdot & \cdot & -\bar{r} & H_1 & -r & \cdot \\
4 & \cdot & \cdot & \cdot & -\bar{r} & H_1 & -r \\
5 & -r & \cdot & \cdot & \cdot & -\bar{r} & H_1
\end{array}$$

eigenvalues of $\mathbf{H}^{B2(6)}$

$$\begin{array}{c|cccccc}
 & p=0 & 1 & 2 & 3 & 4 & 5 \\
\hline
0 & H_2 & \cdot & -s & \cdot & -s & \cdot \\
1 & \cdot & H_2 & \cdot & -s & \cdot & -s \\
2 & -s & \cdot & H_2 & \cdot & -s & \cdot \\
3 & \cdot & -s & \cdot & H_2 & \cdot & -s \\
4 & -s & \cdot & -s & \cdot & H_2 & \cdot \\
5 & \cdot & -s & \cdot & -s & \cdot & H_2
\end{array}$$

eigenvalues of $\mathbf{H}^{B3(6)}$

$$\begin{array}{c|cccccc}
 & p=0 & 1 & 2 & 3 & 4 & 5 \\
\hline
0 & H_3 & \cdot & \cdot & -t & \cdot & \cdot \\
1 & \cdot & H_3 & \cdot & \cdot & -t & \cdot \\
2 & \cdot & \cdot & H_3 & \cdot & \cdot & -t \\
3 & -t & \cdot & \cdot & H_3 & \cdot & \cdot \\
4 & \cdot & -t & \cdot & \cdot & H_3 & \cdot \\
5 & \cdot & \cdot & -t & \cdot & \cdot & H_3
\end{array}$$

Fig. 14 International Journal of Molecular Science 14, 754 (2013)

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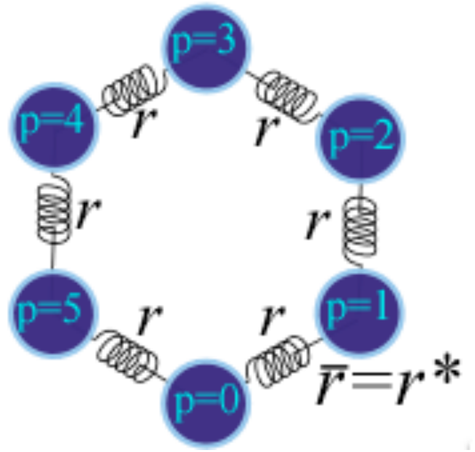
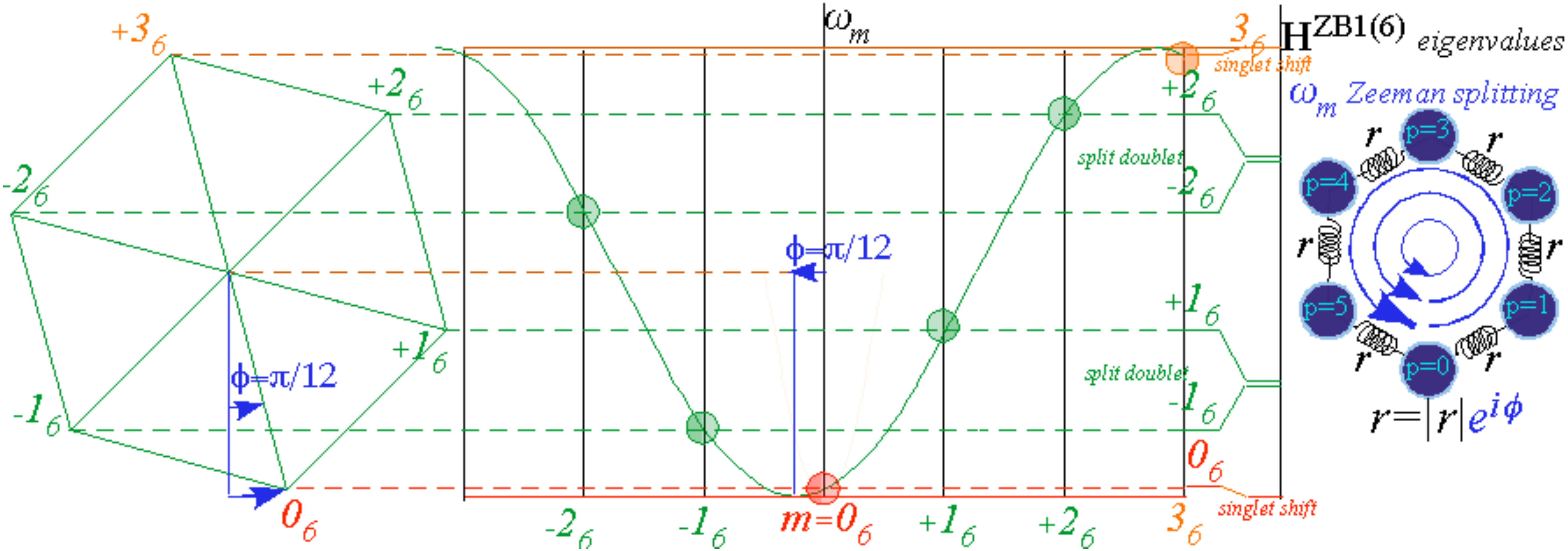
C_6 Spectra of 1st neighbor gauge splitting by C-type (Chiral, Coriolis,…,

1st Neighbor H





Standing wave combinations like $\cos kx=(e^{+ikx}+e^{-ikx})/2$
are not eigenmodes unless $\phi=0$.

Fig. 15 International Journal of Molecular Science 14, 755 (2013)

Wave resonance in cyclic C_n symmetry (REVIEW)
C_6 symmetric mode model:Distant neighbor coupling
- *C_6 moving waves and degenerate standing waves*
- *C_6 dispersion functions for 1st, 2nd, and 3rd-neighbor coupling*
- *C_6 dispersion functions split by C-type symmetry(complex, chiral, ...)*

➔ *C_{12} and higher symmetry mode models: Archetypes of dispersion functions and 1-CW phase velocity* ⬅
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Given two 1-CW phases: Find 2-CW phase velocity $\mathbf{V}_{phase}^{(2\text{-}CW)}$ and group velocity $\mathbf{V}_{group}^{(2\text{-}CW)}$
- *Example: Bohr Dispersion 2-CW made of 1-CW(m=-1) + 1-CW(m=2)*

Find 2-CW space-time (x,t) lattice from per-space-time (κ,υ) by matrix-algebra/geometry
- *Same 1-CW(m=-1) + 1-CW(m=2) Example*

C_N Symmetric Mode Models:

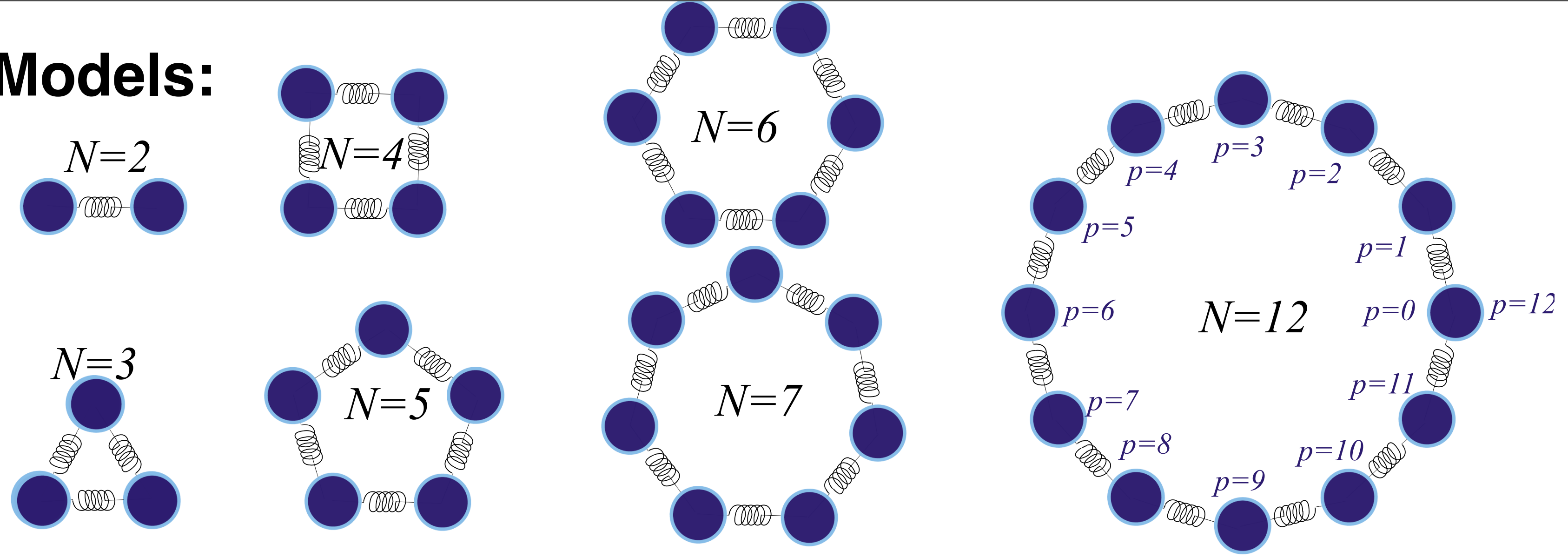

Fig. 4.8.4
Unit 4
CMwBang

C_N Symmetric Mode Models:

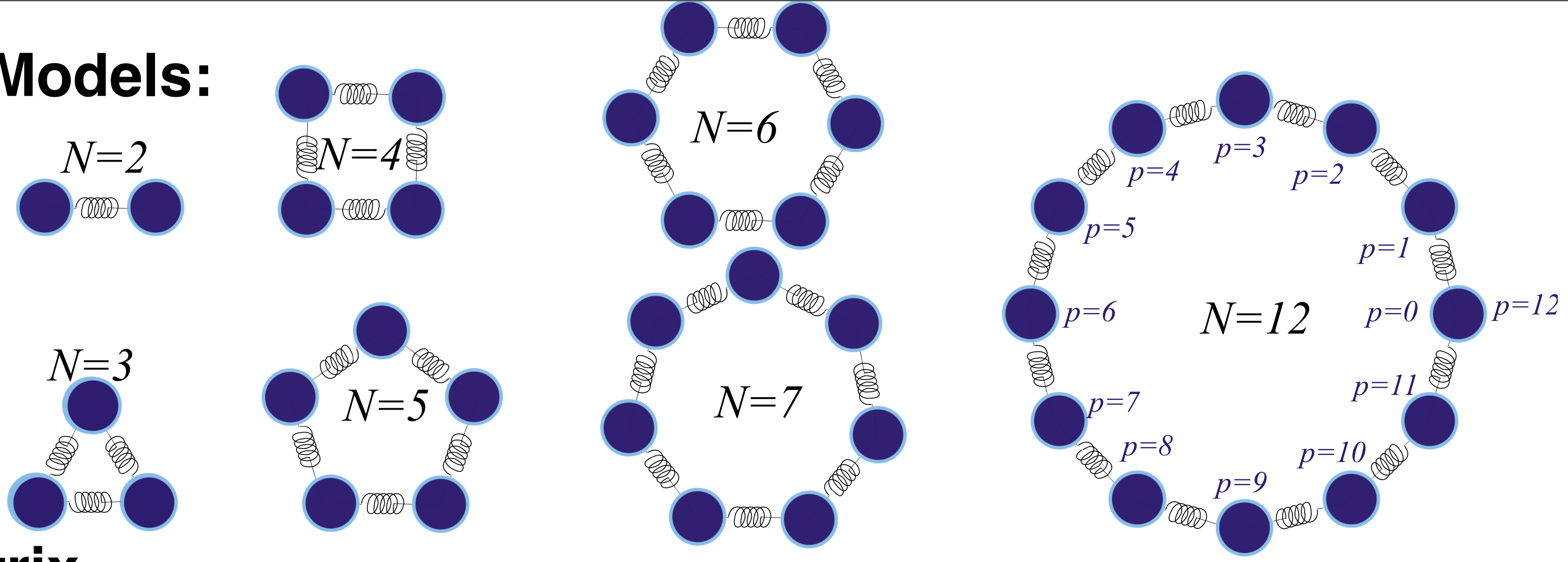

Fig. 4.8.4
Unit 4
CMwBang

1st Neighbor K-matrix

$$-\begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ \vdots \\ F_{N-1} \end{pmatrix} = \begin{pmatrix} K & -k_{12} & . & . & . & \cdots & -k_{12} \\ -k_{12} & K & -k_{12} & . & . & \cdots & . \\ . & -k_{12} & K & -k_{12} & . & \cdots & . \\ . & . & -k_{12} & K & -k_{12} & \cdots & . \\ . & . & . & -k_{12} & K & \cdots & . \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & -k_{12} \\ -k_{12} & . & . & . & . & -k_{12} & K \end{pmatrix} \bullet \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{N-1} \end{pmatrix}$$

where:

$$K = k + 2k_{12}$$

$$k = \frac{Mg}{\ell}$$

$$(\cdot) = 0$$

C_N Symmetric Mode Models:

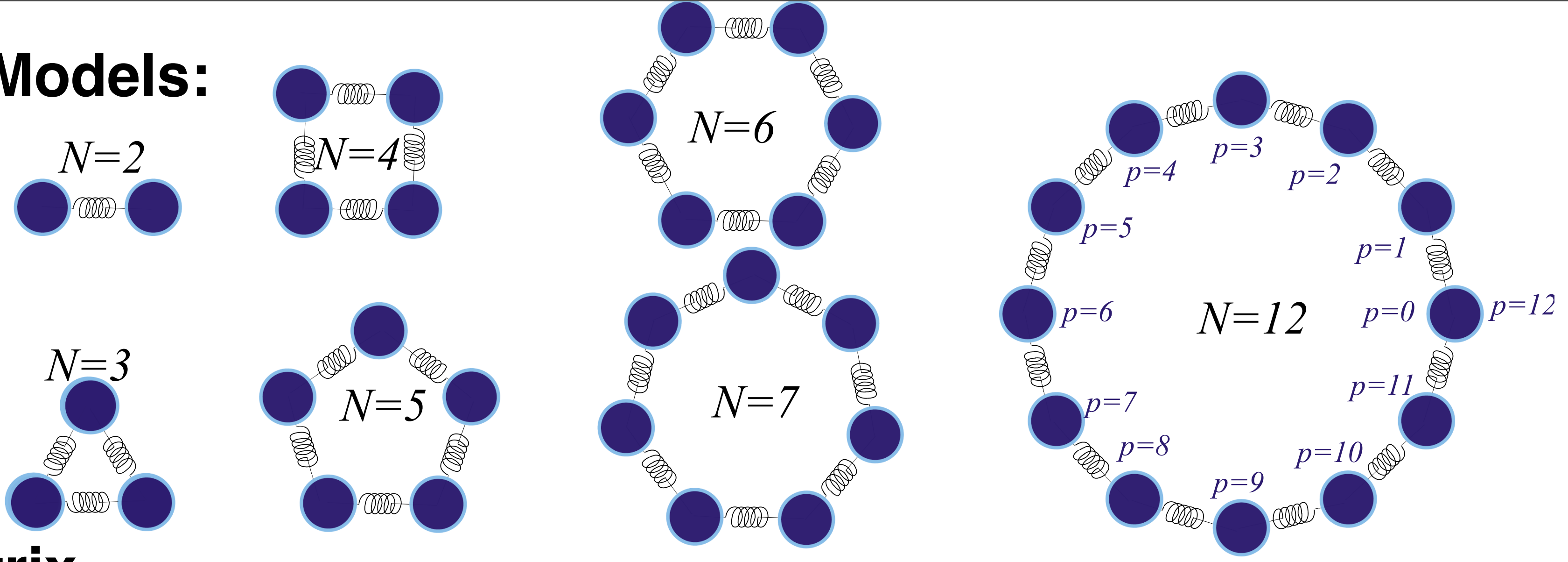

Fig. 4.8.4
Unit 4
CMwBang

1st Neighbor K-matrix

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Nth roots of 1 $e^{i\,m\cdot p\,2\pi/N} = \langle m|\, \mathbf{r}^p\, |m\rangle$ *serving as e-values, eigenfunctions, transformation matrices, dispersion relations, Group reps. etc.*

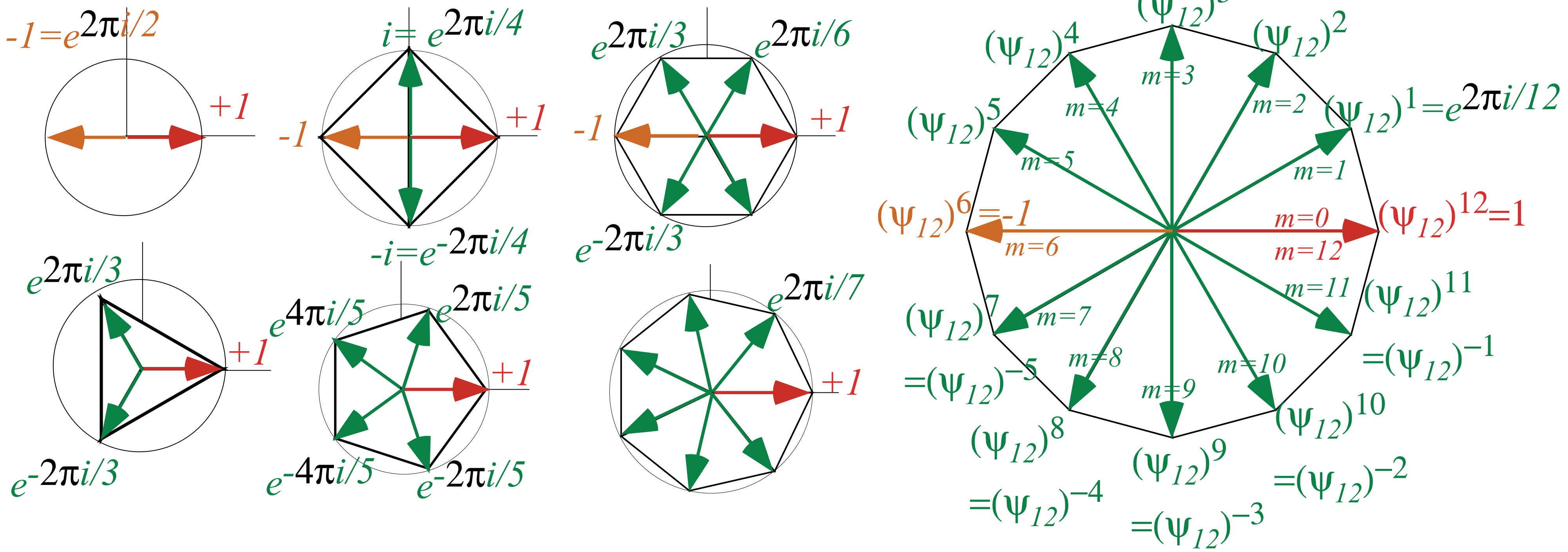
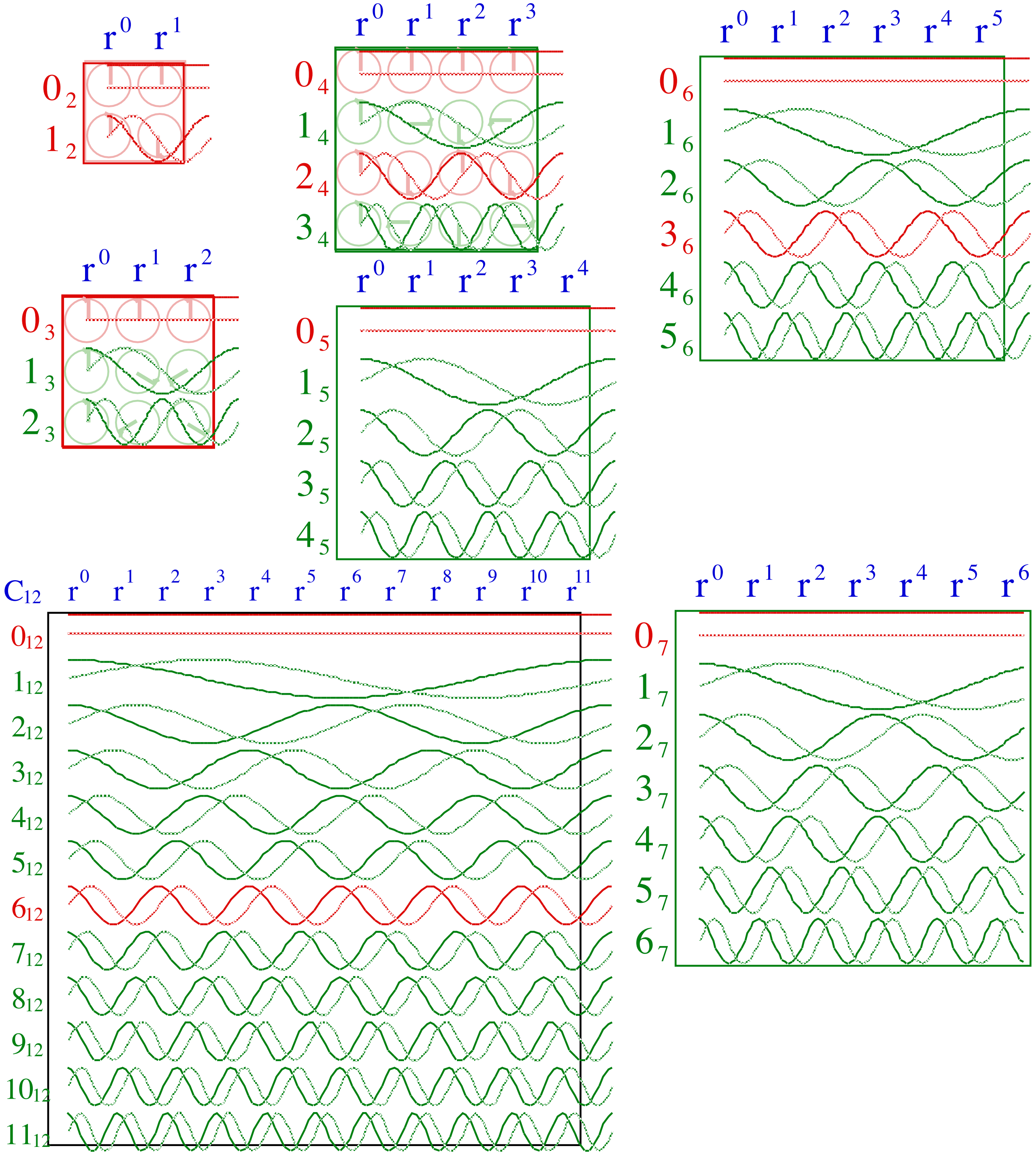


Fig. 4.8.5
Unit 4
CMwBang

C_N Symmetric Mode Models:

Nth roots of 1 $e^{i\, m\cdot p\, 2\pi/N} = \langle m|\, \mathbf{r}^p\, |m\rangle$ *serving as e-values, eigenfunctions, transformation matrices, dispersion relations, Group reps. etc.*

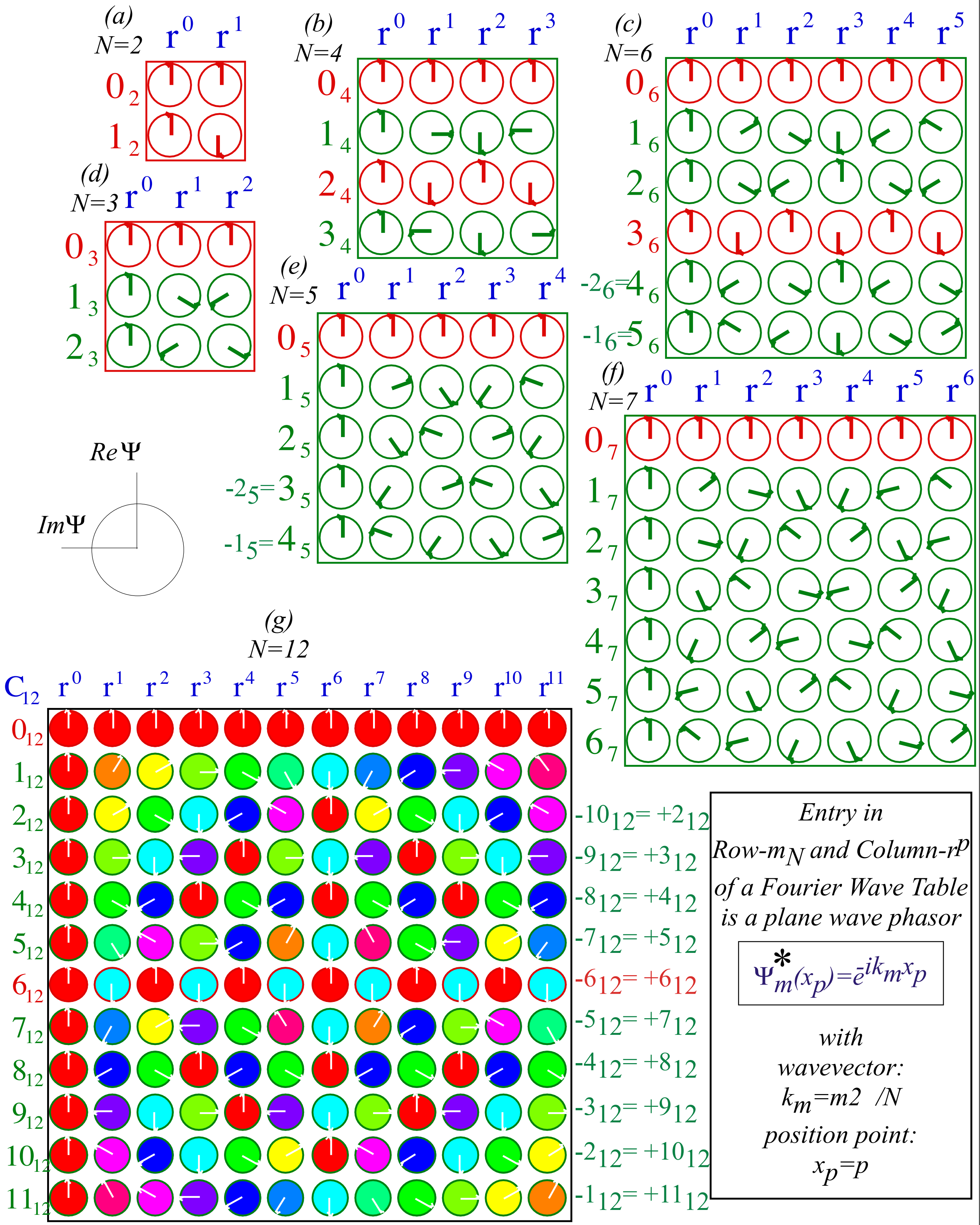


WaveIt C12 Web Simulation

Orig. Fig. 4.8.6-7
Unit 4
CMwBang

Fourier

transformation matrices



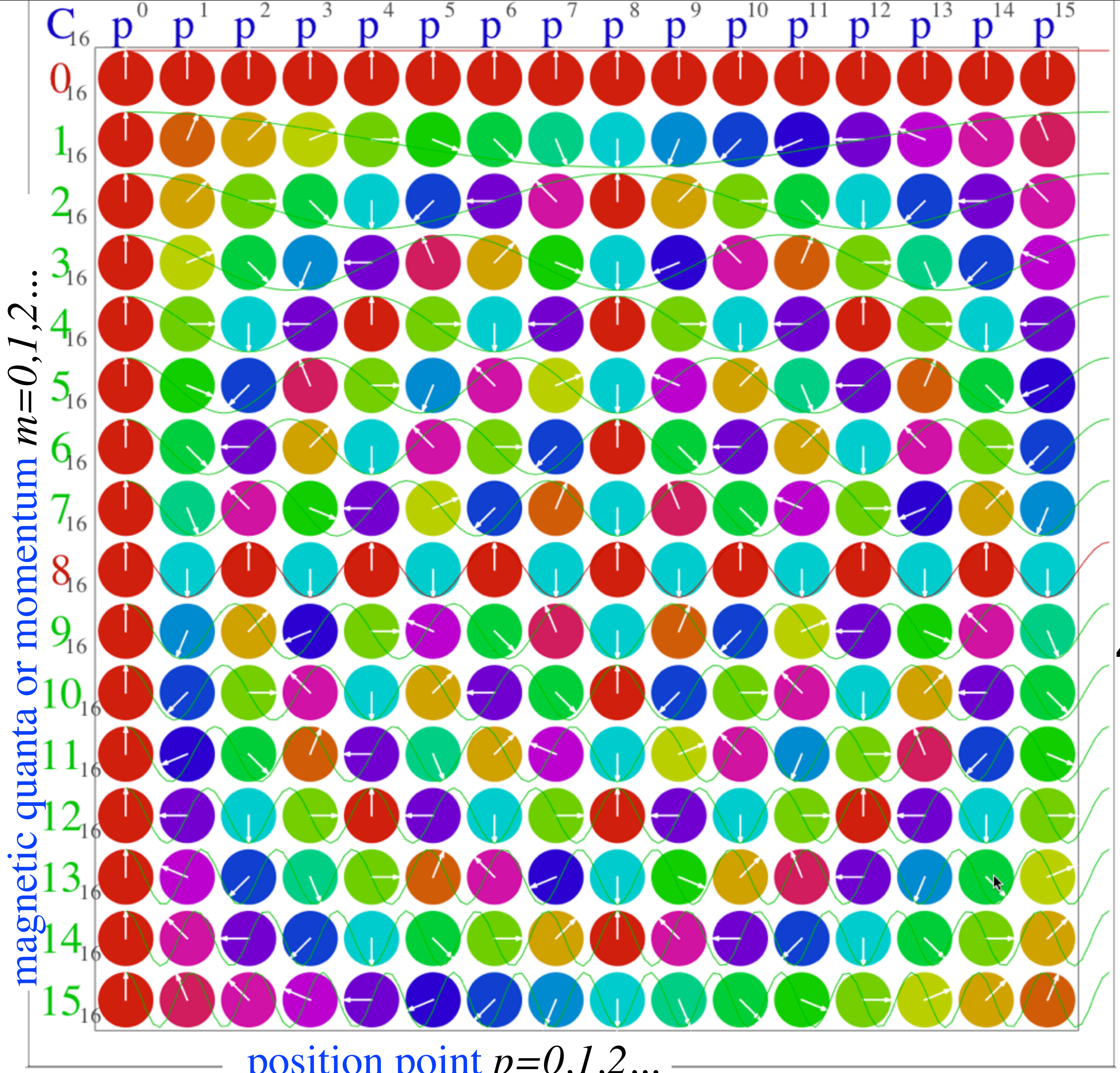
Entry in Row-m_N and Column-r^p of a Fourier Wave Table is a plane wave phasor

$$\Psi^*_m(x_p) = e^{-ik_m x_p}$$

with wavevector: $k_m = m2\ \ /N$

position point: $x_p = p$

WaveIt C12 Character Phasors Web Simulation



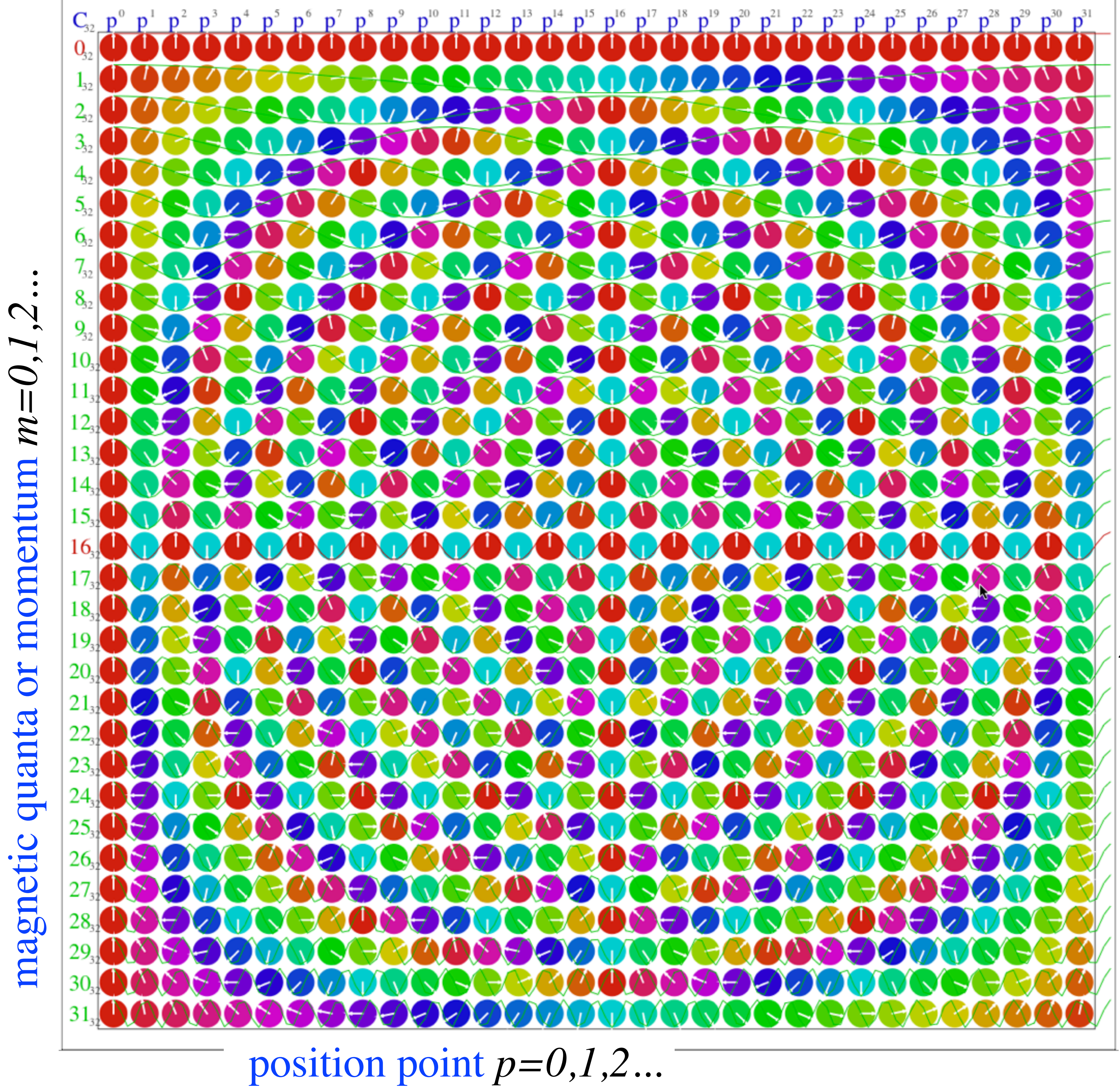
C_{16}

phasor character table

$$\chi_p^m = e^{ik_m r^p}$$

$$= e^{\frac{2\pi imp}{16}}$$

WaveIt C_{16} Character Phasors
Web Simulation



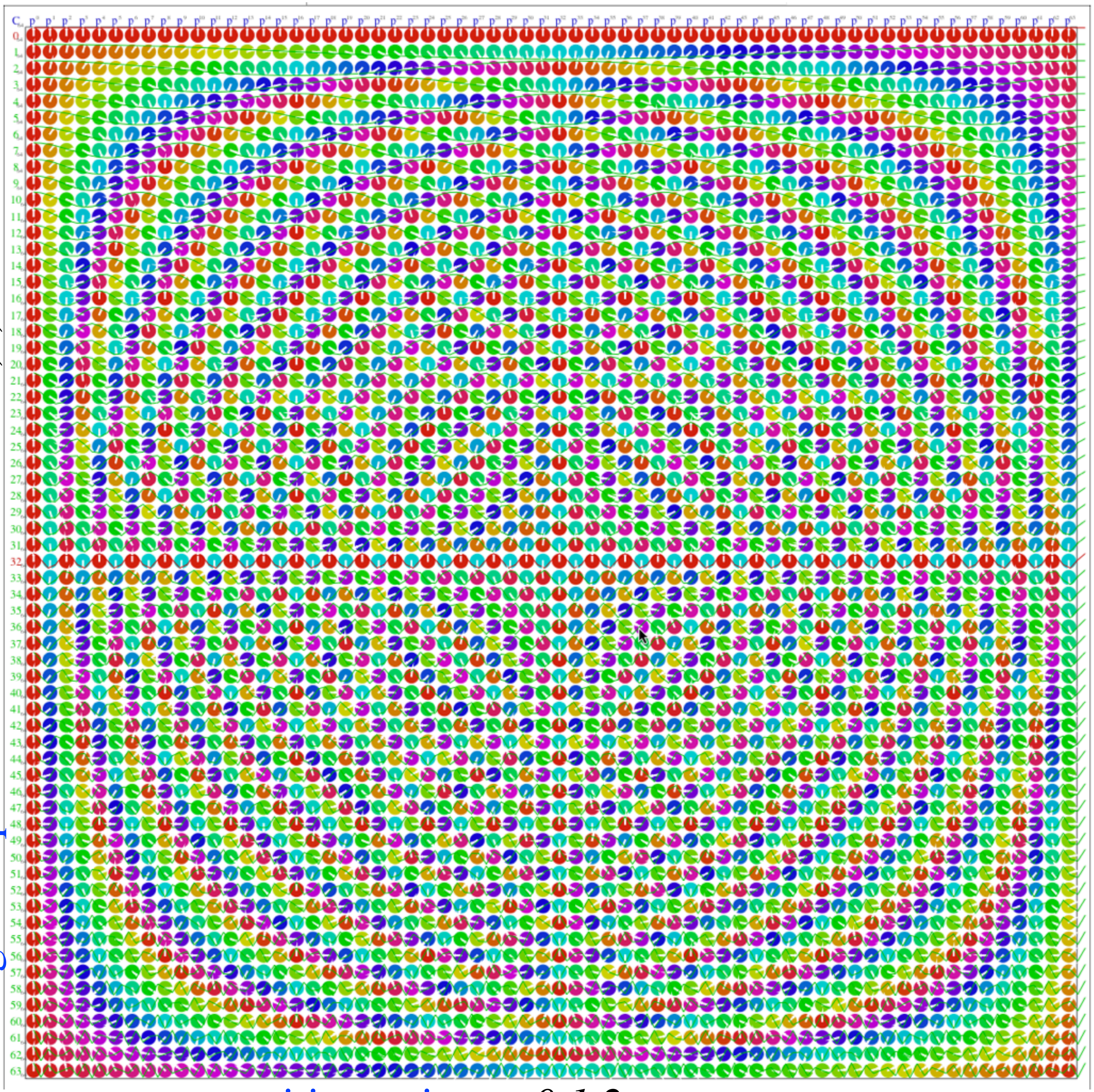
C_{32}

phasor character table

$$\chi_p^m = e^{ik_m r^p}$$

$$= e^{\frac{2\pi imp}{32}}$$

WaveIt C_{32} Character Phasors Web Simulation



magnetic quanta or momentum *m=0,1,2…*

position point *p=0,1,2…*

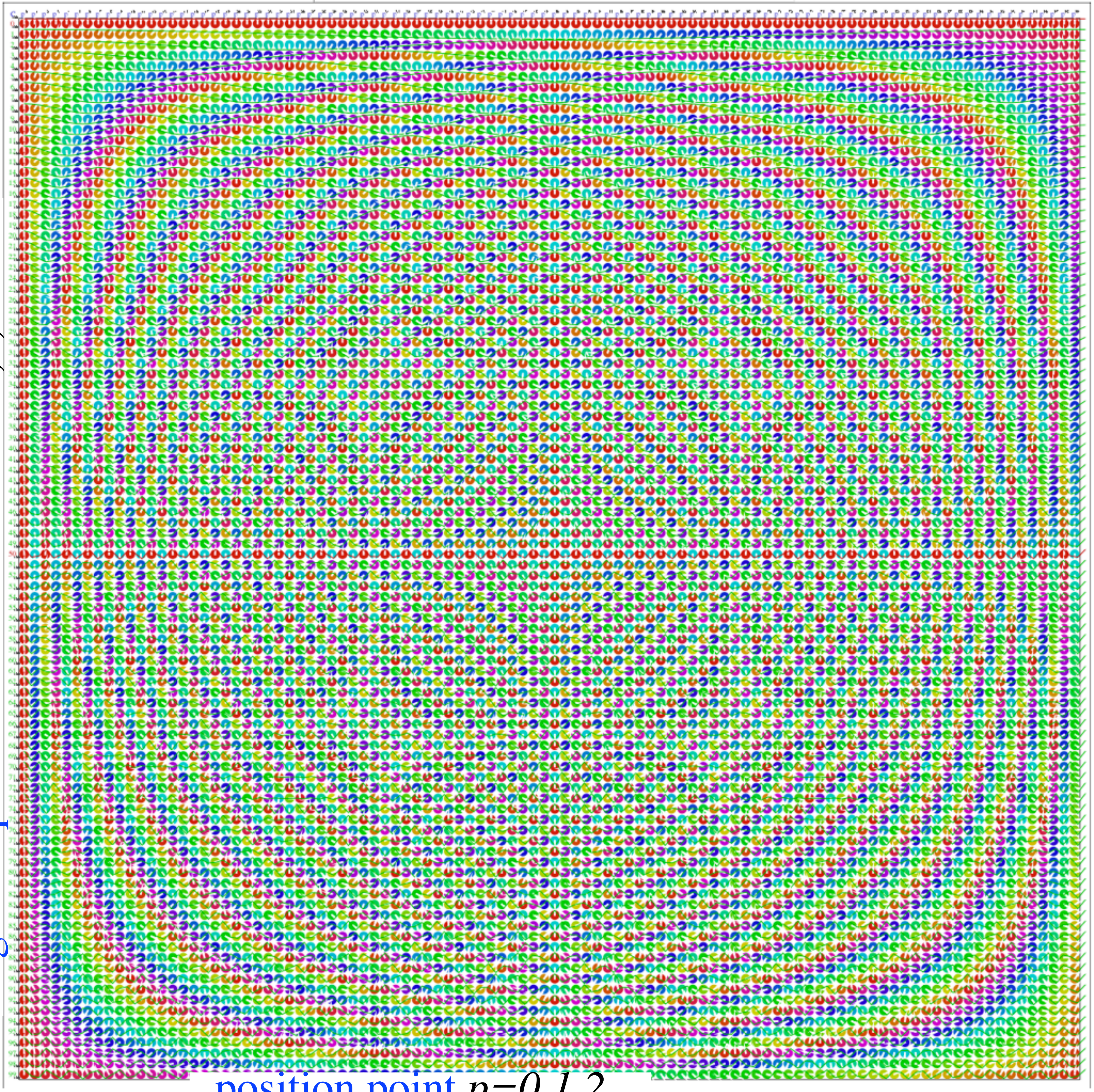
C_{64}

phasor character table

$$\chi_p^m = e^{ik_m r^p}$$

$$= e^{\frac{2\pi imp}{64}}$$

Invariant phase
“Uncertainty”
hyperbolas:
m·p=const.



magnetic quanta or momentum $m=0,1,2\ldots$

position point $p=0,1,2\ldots$

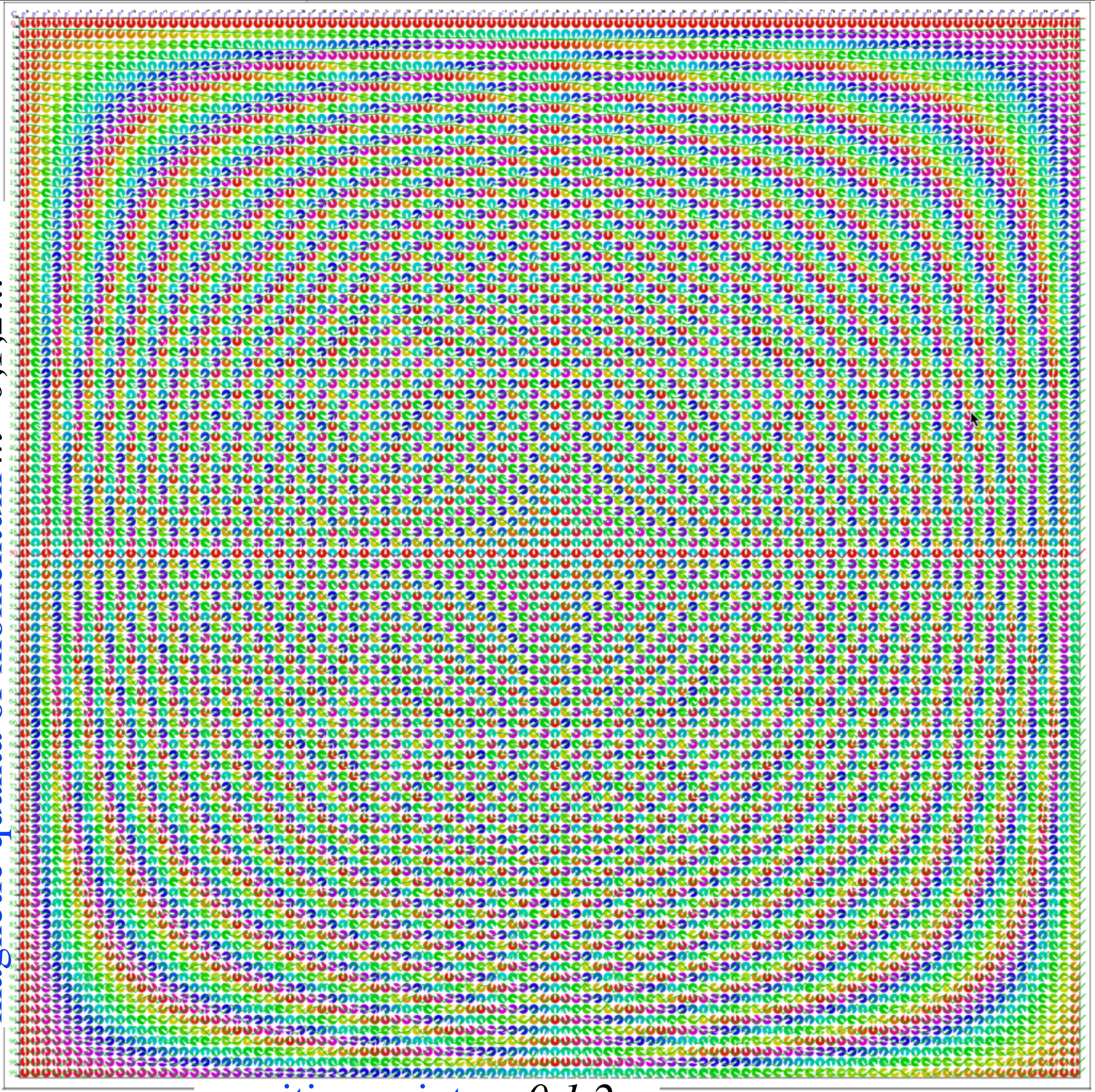
C_{100}

phasor
character
table

$$\chi_p^m = e^{ik_m r^p}$$

$$= e^{\frac{2\pi imp}{100}}$$

Invariant phase
"Uncertainty"
hyperbolas:
$m\cdot p = const.$



magnetic quanta or momentum $m=0,1,2\ldots$

position point $p=0,1,2\ldots$

C_{256}

phasor
character
table

$$\chi_p^m = e^{ik_m r^p}$$

$$= e^{\frac{2\pi imp}{256}}$$

Invariant phase
"Uncertainty"
hyperbolas:
$m \cdot p = const.$

WaveIt C_{256} Character Phasors
Web Simulation

Wave resonance in cyclic C_n symmetry (REVIEW)

C_6 symmetric mode model:Distant neighbor coupling

- *C_6 moving waves and degenerate standing waves*
- *C_6 dispersion functions for 1st, 2nd, and 3rd-neighbor coupling*
- *C_6 dispersion functions split by C-type symmetry(complex, chiral, ...)*

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Given two 1-CW phases: Find 2-CW phase velocity $\mathbf{V}_{phase}^{(2\text{-}CW)}$ and group velocity $\mathbf{V}_{group}^{(2\text{-}CW)}$

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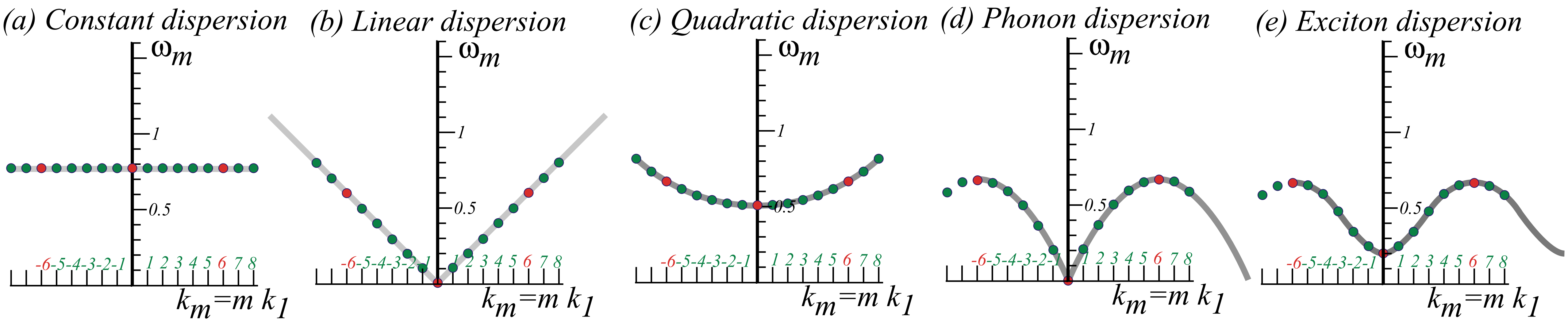
Archetypical Examples of C_{12} Dispersion Functions



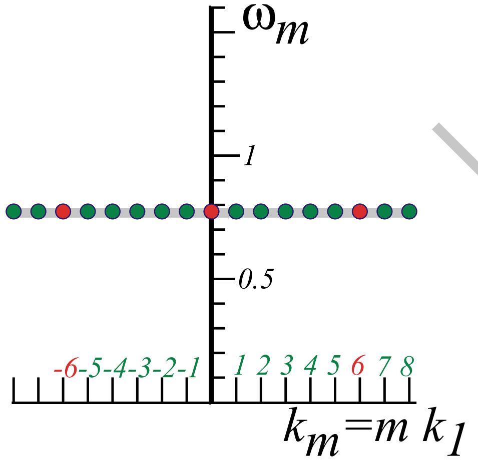
(a) Constant dispersion (b) Linear dispersion (c) Quadratic dispersion (d) Phonon dispersion (e) Exciton dispersion

Applications:

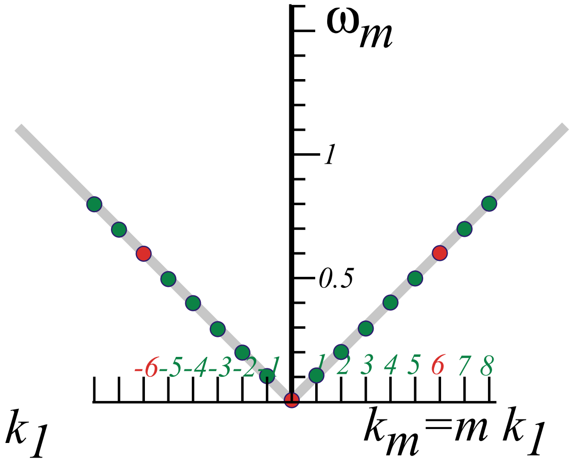
Uncoupled pendulums	Weakly coupled pendulums (No gravity)	Weakly coupled pendulums (With gravity)	Strongly coupled pendulums (No gravity)	Strongly coupled pendulums (With gravity)
Movie marquis Xmas lights	Light in vacuum (Exactly) Sound (Approximately)	Light in fiber (Approx) Non-relativistic Schrodinger matter wave	Acoustic mode in solids	Optical mode in solids Relativistic matter (If exact hyperbola)

Archetypical Examples of C_{12} Dispersion Functions

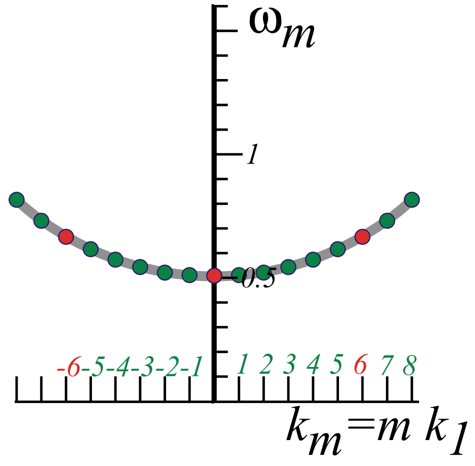
(a) Constant dispersion



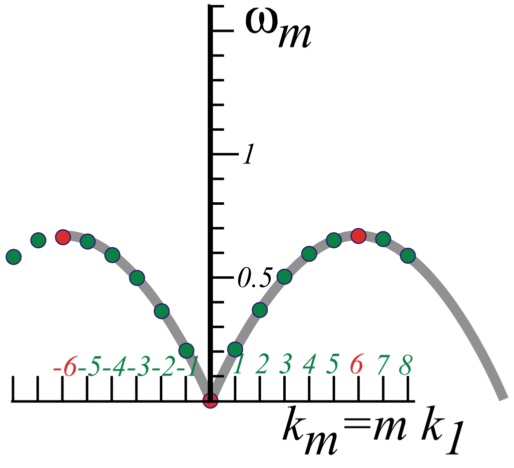
(b) Linear dispersion



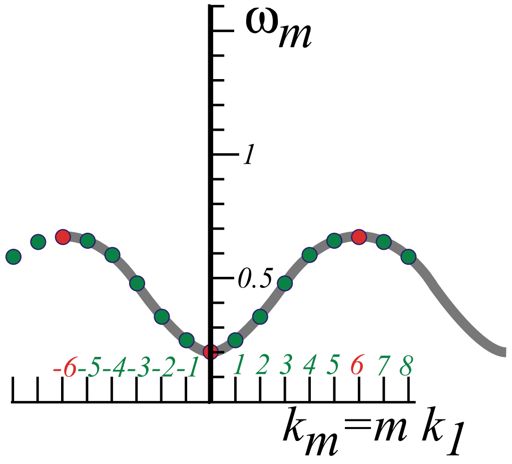
(c) Quadratic dispersion



(d) Phonon dispersion

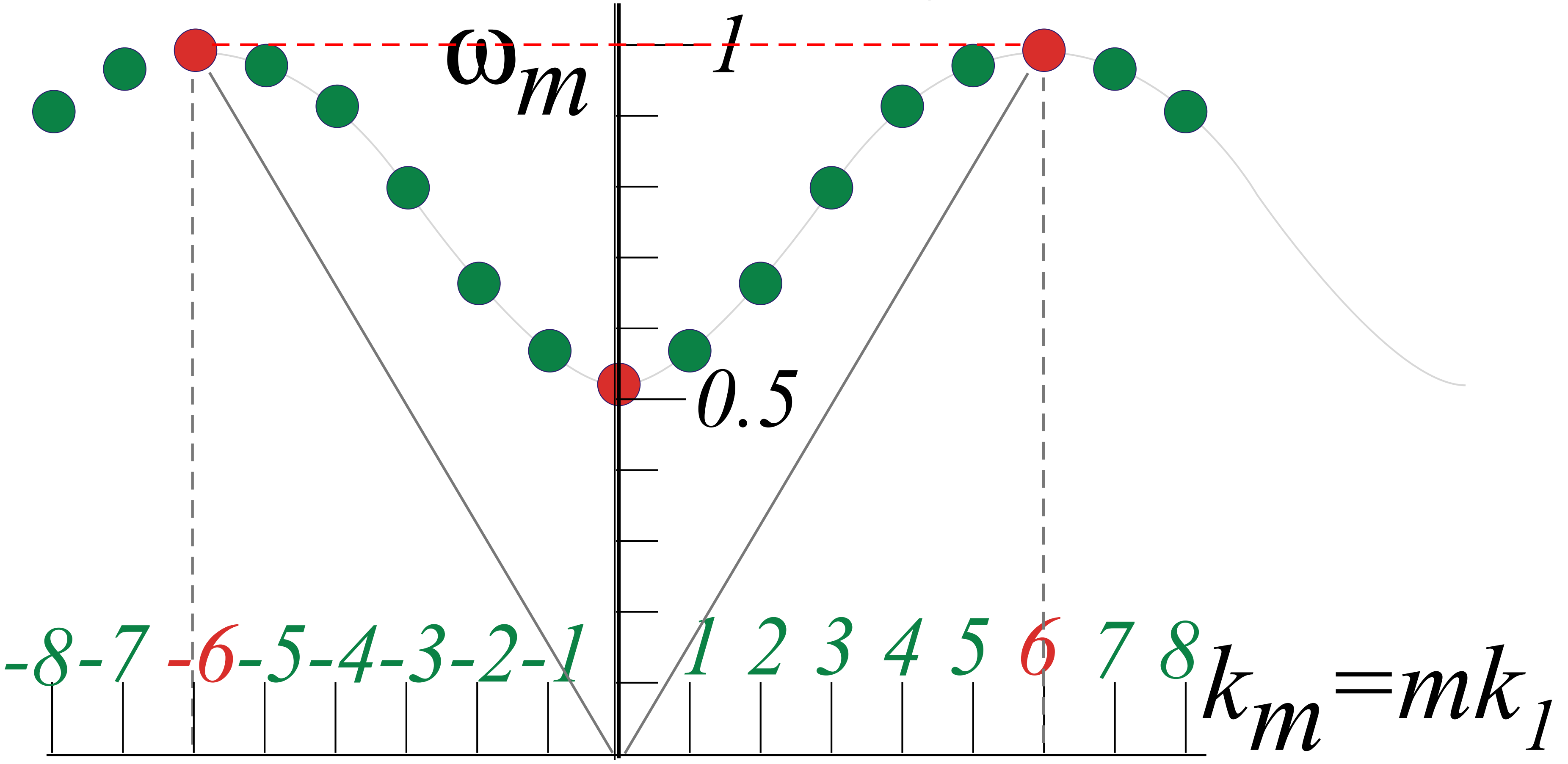


(e) Exciton dispersion

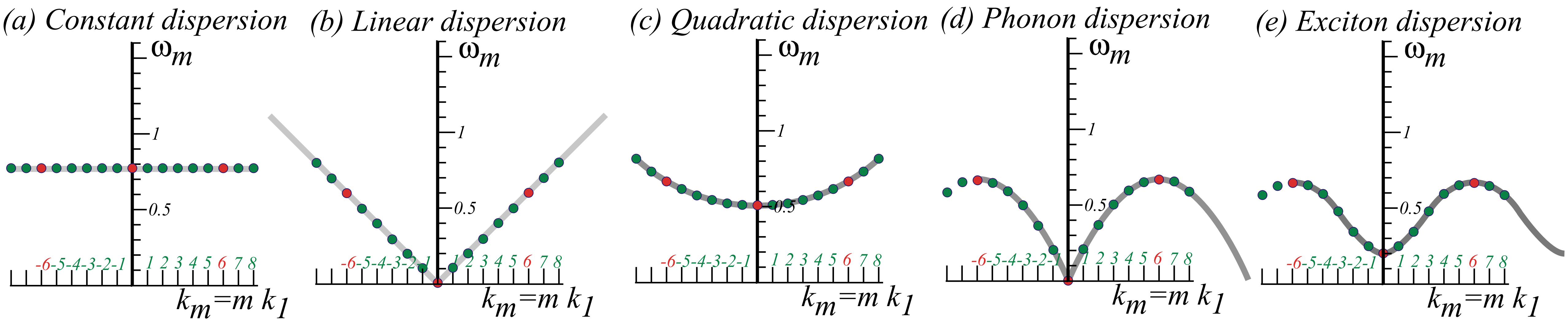


Applications:

Uncoupled pendulums	*Weakly coupled pendulums (No gravity)*	*Weakly coupled pendulums (With gravity)*	*Strongly coupled pendulums (No gravity)*	*Strongly coupled pendulums (With gravity)*
Movie marquis *Xmas lights*	*Light in vacuum (Exactly)* *Sound (Approximately)*	*Light in fiber (Approx)* *Non-relativistic Schrodinger matter wave*	*Acoustic mode in solids*	*Optical mode in solids* *Relativistic matter* *(If exact hyperbola)*



Archetypical Examples of C_{12} Dispersion Functions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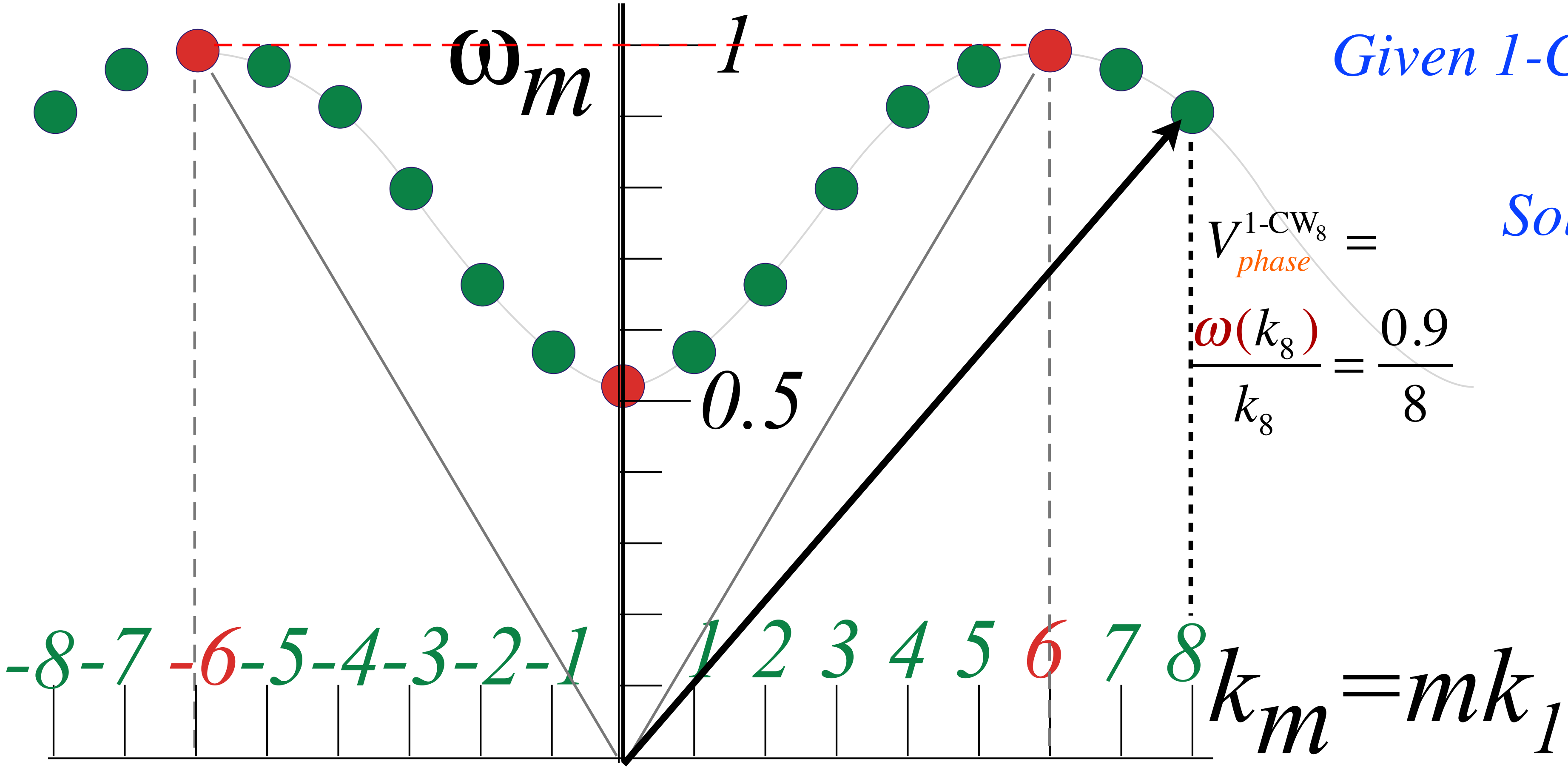

$$V^{1\text{-}CW_8}_{phase} = \frac{\omega(k_8)}{k_8} = \frac{0.9}{8}$$

Given 1-CW phase of wave $e^{\overbrace{i(kx-\omega t)}}$:

$$a = \overbrace{k\cdot x - \omega\cdot t}$$

Solve for 1-CW $\underbrace{\text{phase velocity}}$

$$x = \frac{\omega}{k}\cdot t + \frac{a}{k}$$

Wave velocities depend on
Dispersion function
$\omega = \omega(k)$

(a) 1-CW *phase* velocity:

$$V^{1\text{-}CW}_{phase} = \frac{\omega(k)}{k}$$

Wave resonance in cyclic C_n symmetry (REVIEW)

C_6 symmetric mode model:Distant neighbor coupling

- *C_6 moving waves and degenerate standing waves*
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C_{12} and higher symmetry mode models: Archetypes of dispersion functions and 1-CW phase velocity

➡ *½-Sum-½-Diff-theory of 2-CW group and phase velocity* ⬅

Given two 1-CW phases: Find 2-CW phase velocity $\mathbf{V}_{phase}{}^{(2\text{-}CW)}$ and group velocity $\mathbf{V}_{group}{}^{(2\text{-}CW)}$

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Find 2-CW space-time (x,t) lattice from per-space-time (κ,υ) by matrix-algebra/geometry

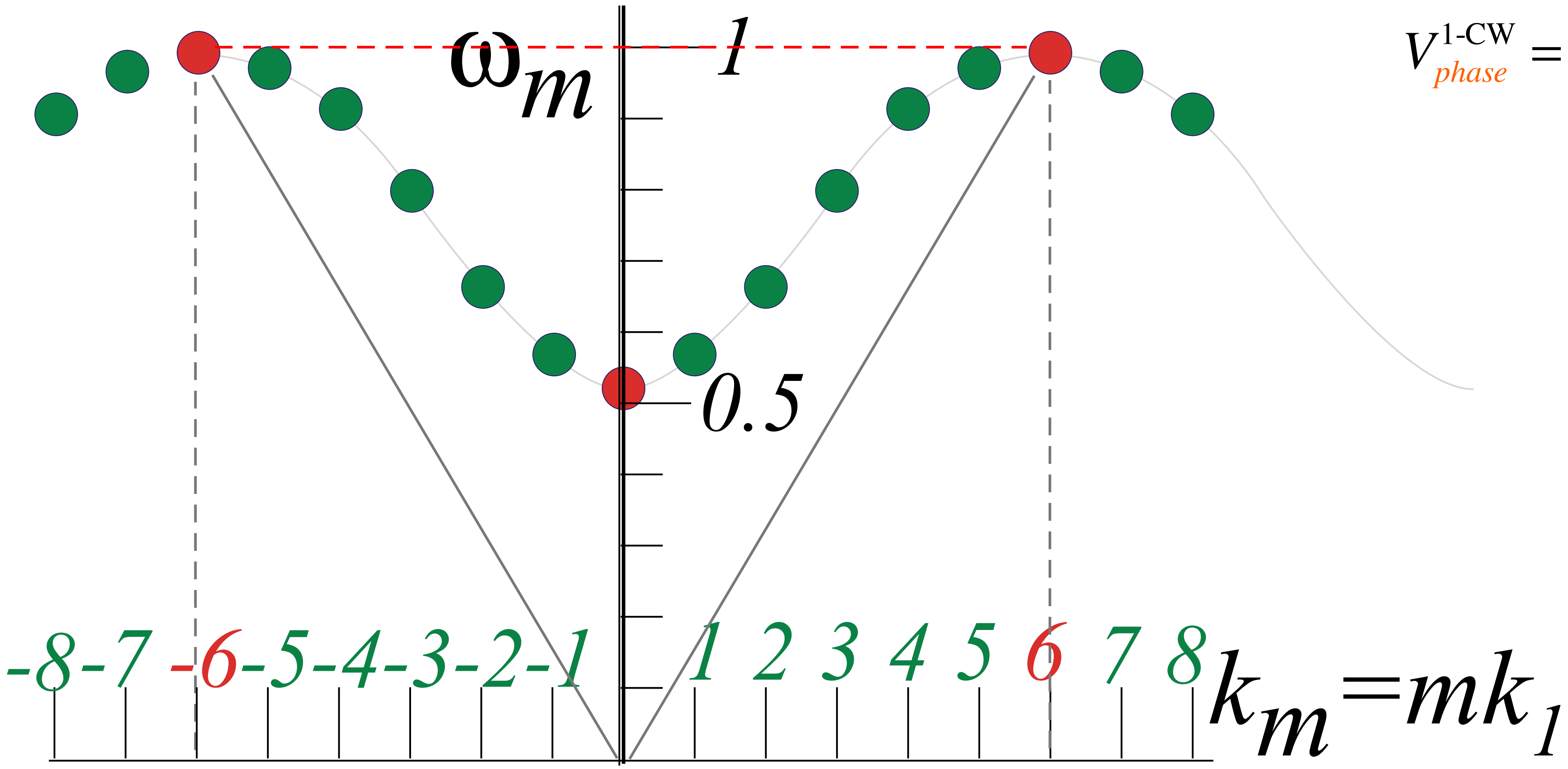
- *Same 1-CW(m=-1) + 1-CW(m=2) Example*

The ½-Sum-½-Diff-Identity and 2-CW phase and group velocity

Given 2-CW phases:

$a = k_a \cdot x - \omega_a \cdot t$ and $b = k_b \cdot x - \omega_b \cdot t$

...find 2-CW phase velocity $V_{phase}^{\text{2-CW}}$ and group velocity $V_{group}^{\text{2-CW}}$



Velocities depend upon
Dispersion function
$\omega = \omega(k)$

(*a*) 1-CW *phase* velocity:

$$V_{phase}^{\text{1-CW}} = \frac{\omega(k)}{k}$$

The ½-Sum-½-Diff-Identity and 2-CW phase and group velocity

Given 2-CW phases:

...find 2-CW phase velocity $V^{2\text{-CW}}_{phase}$ and group velocity $V^{2\text{-CW}}_{group}$

$$a = k_a \cdot x - \omega_a \cdot t \quad \text{and} \quad b = k_b \cdot x - \omega_b \cdot t$$

$$\frac{e^{ia} + e^{ib}}{2} = e^{i\frac{a+b}{2}} \left(\frac{e^{i\frac{a-b}{2}} + e^{-i\frac{a-b}{2}}}{2} \right) = e^{i\frac{a+b}{2}} \cos\left(\frac{a-b}{2} \right)$$

Velocities depend upon

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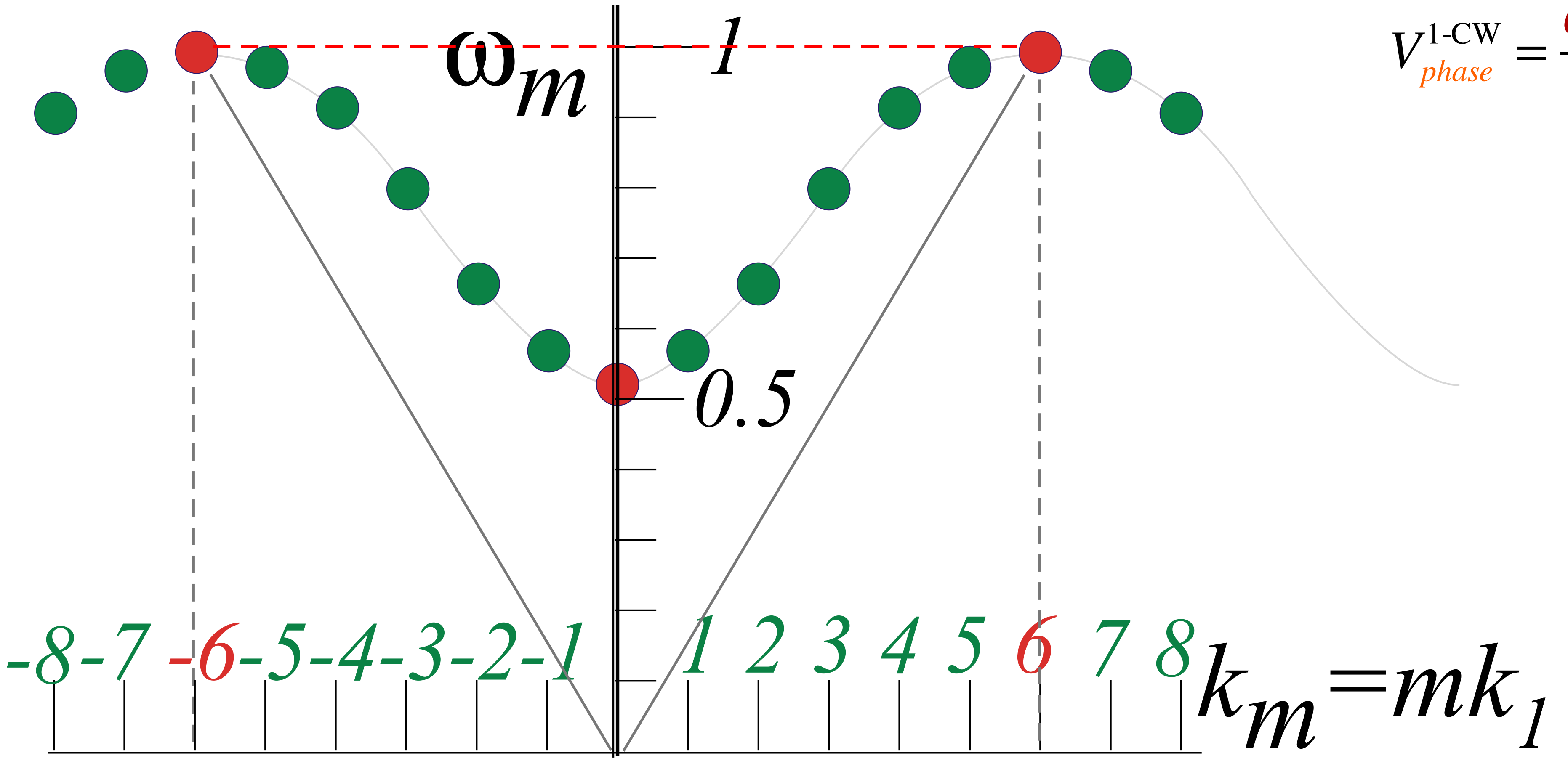
$$V^{1\text{-CW}}_{phase} = \frac{\omega(k)}{k}$$



The ½-Sum-½-Diff-Identity and 2-CW phase and group velocity

Given two 1-CW phases:

...find 2-CW phase velocity $V^{2\text{-CW}}_{phase}$ *and group velocity* $V^{2\text{-CW}}_{group}$

$a = k_a \cdot x - \omega_a \cdot t$ *and* $b = k_b \cdot x - \omega_b \cdot t$

$$\frac{e^{ia}+e^{ib}}{2} = e^{i\frac{a+b}{2}}\left(\frac{e^{i\frac{a-b}{2}}+e^{-i\frac{a-b}{2}}}{2}\right) = e^{i\frac{a+b}{2}}\cos\left(\frac{a-b}{2}\right)$$

$$= e^{i\frac{(k_a+k_b)}{2}x-\frac{(\omega_a+\omega_b)}{2}t}\cos\left(\frac{(k_a-k_b)}{2}x-\frac{(\omega_a-\omega_b)}{2}t\right)$$

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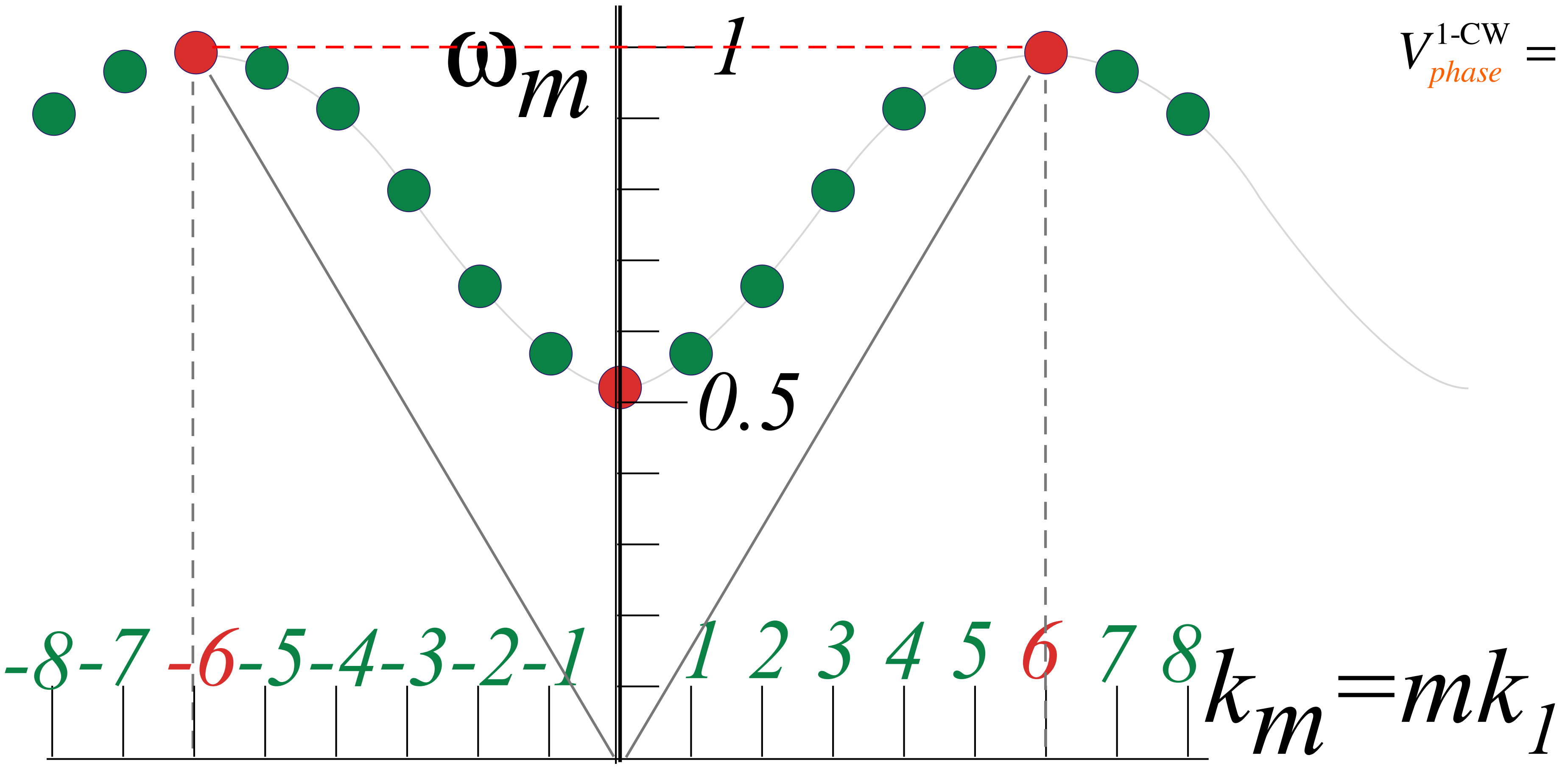

The ½-Sum-½-Diff-Identity and 2-CW phase and group velocity

Given 2-CW phases:

...find 2-CW phase velocity $V_{phase}^{\text{2-CW}}$ and group velocity $V_{group}^{\text{2-CW}}$

$$a = k_a \cdot x - \omega_a \cdot t \quad \text{and} \quad b = k_b \cdot x - \omega_b \cdot t$$

$$\frac{e^{ia} + e^{ib}}{2} = e^{i\frac{a+b}{2}} \left(\frac{e^{i\frac{a-b}{2}} + e^{-i\frac{a-b}{2}}}{2} \right) = e^{i\frac{a+b}{2}} \cos\left(\frac{a-b}{2} \right)$$

$$= e^{i\frac{(k_a+k_b)}{2}x - \frac{(\omega_a+\omega_b)}{2}t} \cos\left(\frac{(k_a - k_b)}{2} x - \frac{(\omega_a - \omega_b)}{2} t \right)$$

$$V_{phase}^{\text{2-CW}} = \frac{(\omega_a + \omega_b)}{(k_a + k_b)} \qquad V_{group}^{\text{2-CW}} = \frac{(\omega_a - \omega_b)}{(k_a - k_b)}$$

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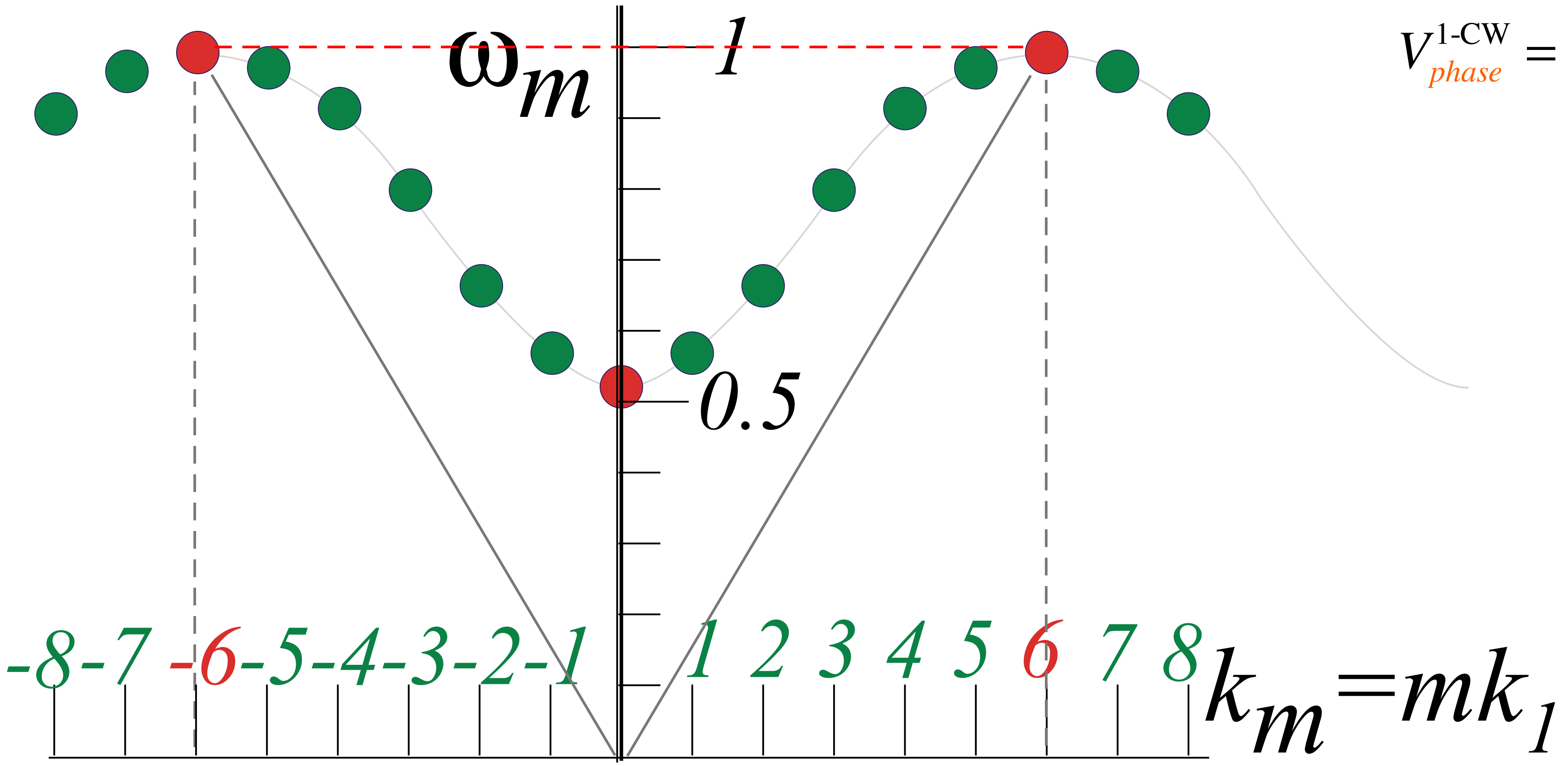

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C_{12} and higher symmetry mode models: Archetypes of dispersion functions and 1-CW phase velocity
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➔ *Given two 1-CW phases: Find 2-CW phase velocity $\mathbf{V}_{phase}^{(2\text{-}CW)}$ and group velocity $\mathbf{V}_{group}^{(2\text{-}CW)}$* ⬅
- *Example: Bohr Dispersion 2-CW made of 1-CW(m=-1) + 1-CW(m=2)*
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  - *Same 1-CW(m=-1) + 1-CW(m=2) Example*

The ½-Sum-½-Diff-Identity and 2-CW phase and group velocity

Given 2-CW phases:

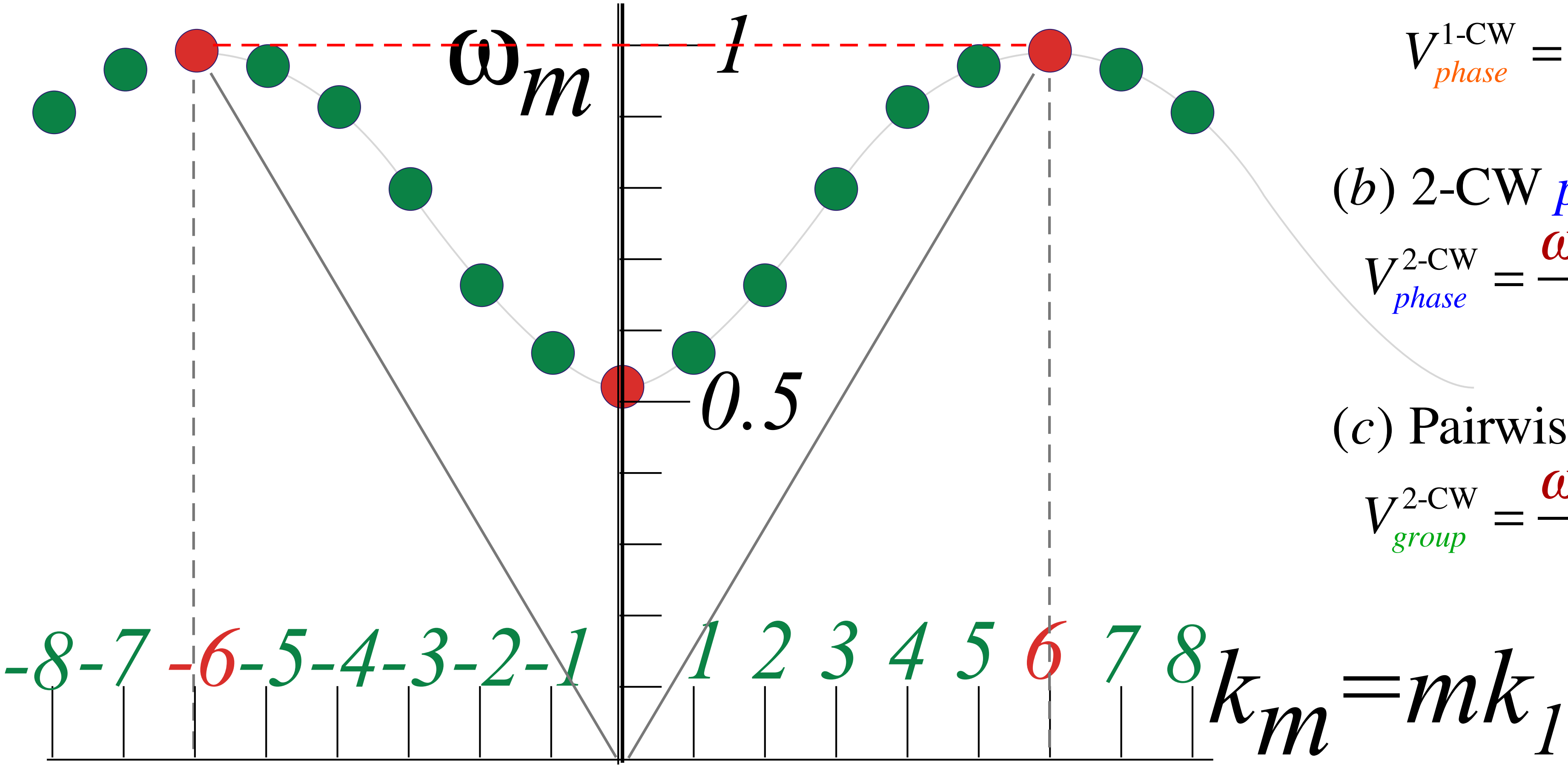
...find 2-CW phase velocity $V_{phase}^{2\text{-CW}}$ and group velocity $V_{group}^{2\text{-CW}}$

$a = k_a \cdot x - \omega_a \cdot t$ and $b = k_b \cdot x - \omega_b \cdot t$

$$\frac{e^{ia}+e^{ib}}{2} = e^{i\frac{a+b}{2}}\left(\frac{e^{i\frac{a-b}{2}}+e^{-i\frac{a-b}{2}}}{2}\right) = e^{i\frac{a+b}{2}}\cos\left(\frac{a-b}{2}\right)$$

$$= e^{i\frac{(k_a+k_b)}{2}x-\frac{(\omega_a+\omega_b)}{2}t}\cos\left(\frac{(k_a-k_b)}{2}x-\frac{(\omega_a-\omega_b)}{2}t\right)$$

$$V_{phase}^{2\text{-CW}} = \frac{(\omega_a+\omega_b)}{(k_a+k_b)} \qquad V_{group}^{2\text{-CW}} = \frac{(\omega_a-\omega_b)}{(k_a-k_b)}$$



Velocities depend upon
Dispersion function
$\omega = \omega(k)$

(a) 1-CW *phase* velocity:

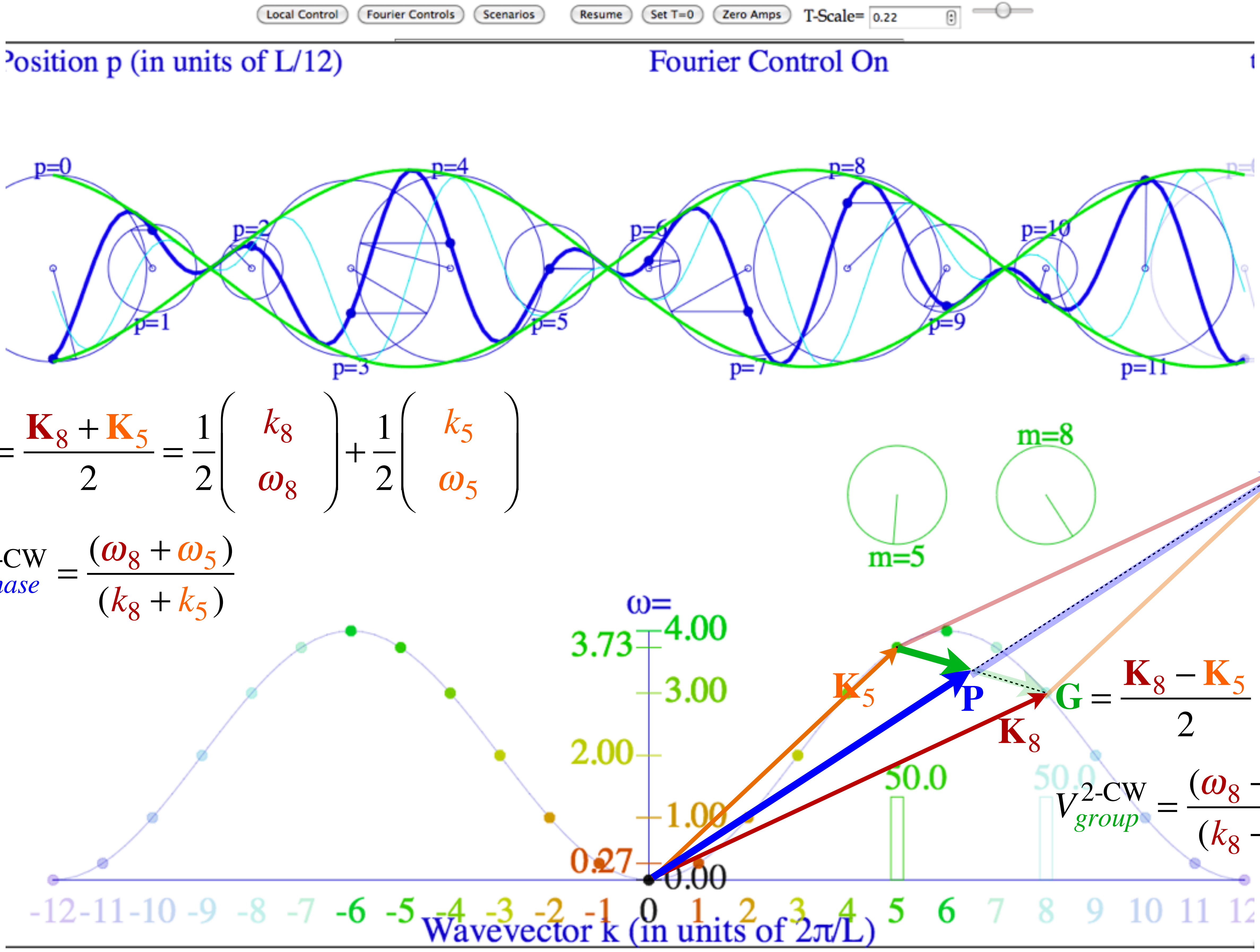
$$V_{phase}^{1\text{-CW}} = \frac{\omega(k)}{k}$$

(b) 2-CW *phase* velocity:

$$V_{phase}^{2\text{-CW}} = \frac{\omega(k_1)+\omega(k_2)}{k_1+k_2}$$

(c) Pairwise *group* velocity:

$$V_{group}^{2\text{-CW}} = \frac{\omega(k_1)-\omega(k_2)}{k_1-k_2}$$



$$\mathbf{P} = \frac{\mathbf{K}_8 + \mathbf{K}_5}{2} = \frac{1}{2}\begin{pmatrix} k_8 \\ \omega_8 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} k_5 \\ \omega_5 \end{pmatrix}$$

$$V^{2\text{-CW}}_{phase} = \frac{(\omega_8 + \omega_5)}{(k_8 + k_5)}$$

$$\mathbf{G} = \frac{\mathbf{K}_8 - \mathbf{K}_5}{2}$$

$$V^{2\text{-CW}}_{group} = \frac{(\omega_8 - \omega_5)}{(k_8 - k_5)}$$

WaveIt Web Simulation - *Wave Mixing (N=12; k=5, 8)*

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- *C_6 dispersion functions split by C-type symmetry(complex, chiral, ...)*

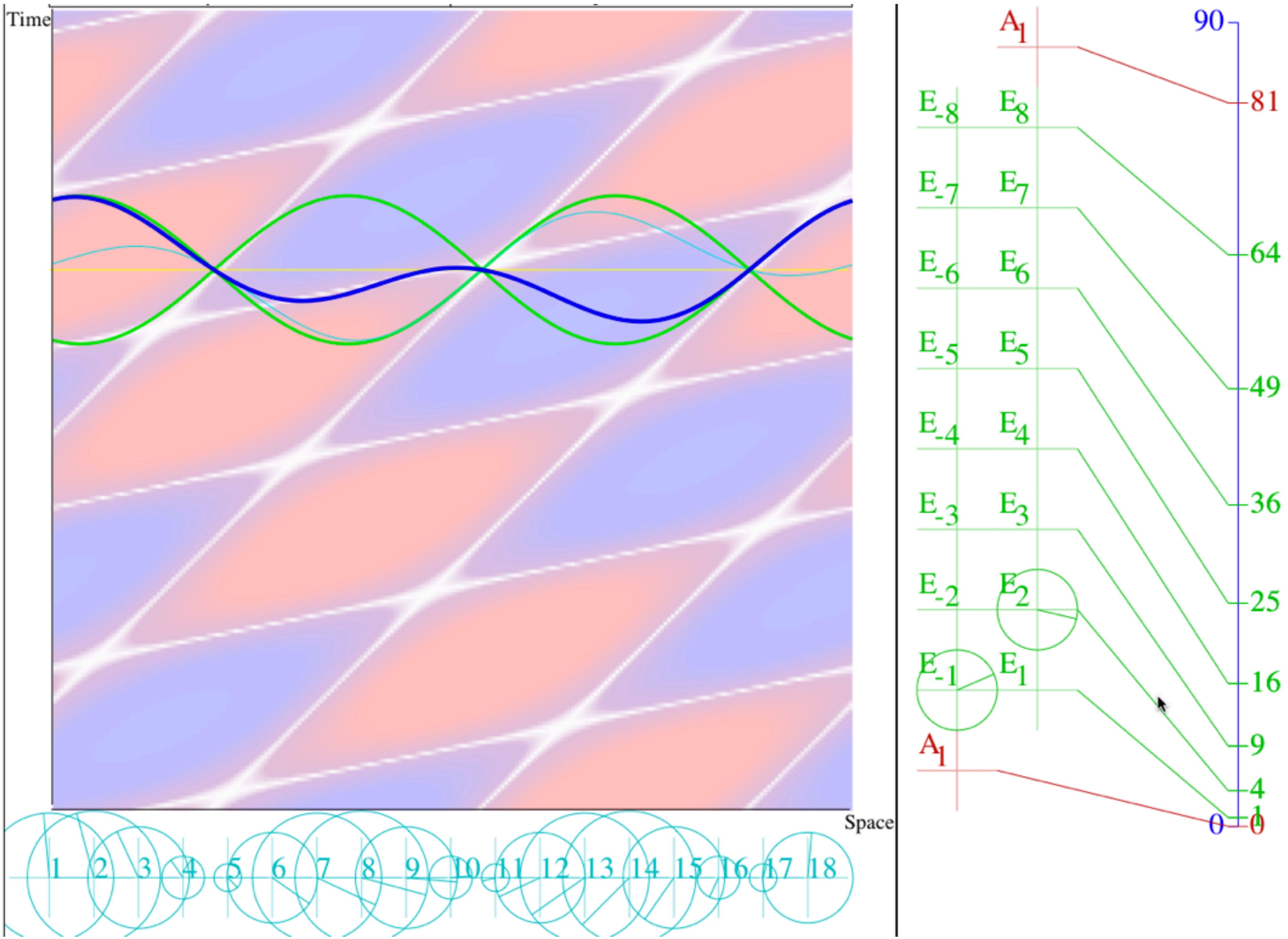
C_{12} and higher symmetry mode models: Archetypes of dispersion functions and 1-CW phase velocity

- *½-Sum-½-Diff-theory of 2-CW group and phase velocity*

Given two 1-CW phases: Find 2-CW phase velocity $\mathbf{V}_{phase}^{(2\text{-}CW)}$ and group velocity $\mathbf{V}_{group}^{(2\text{-}CW)}$

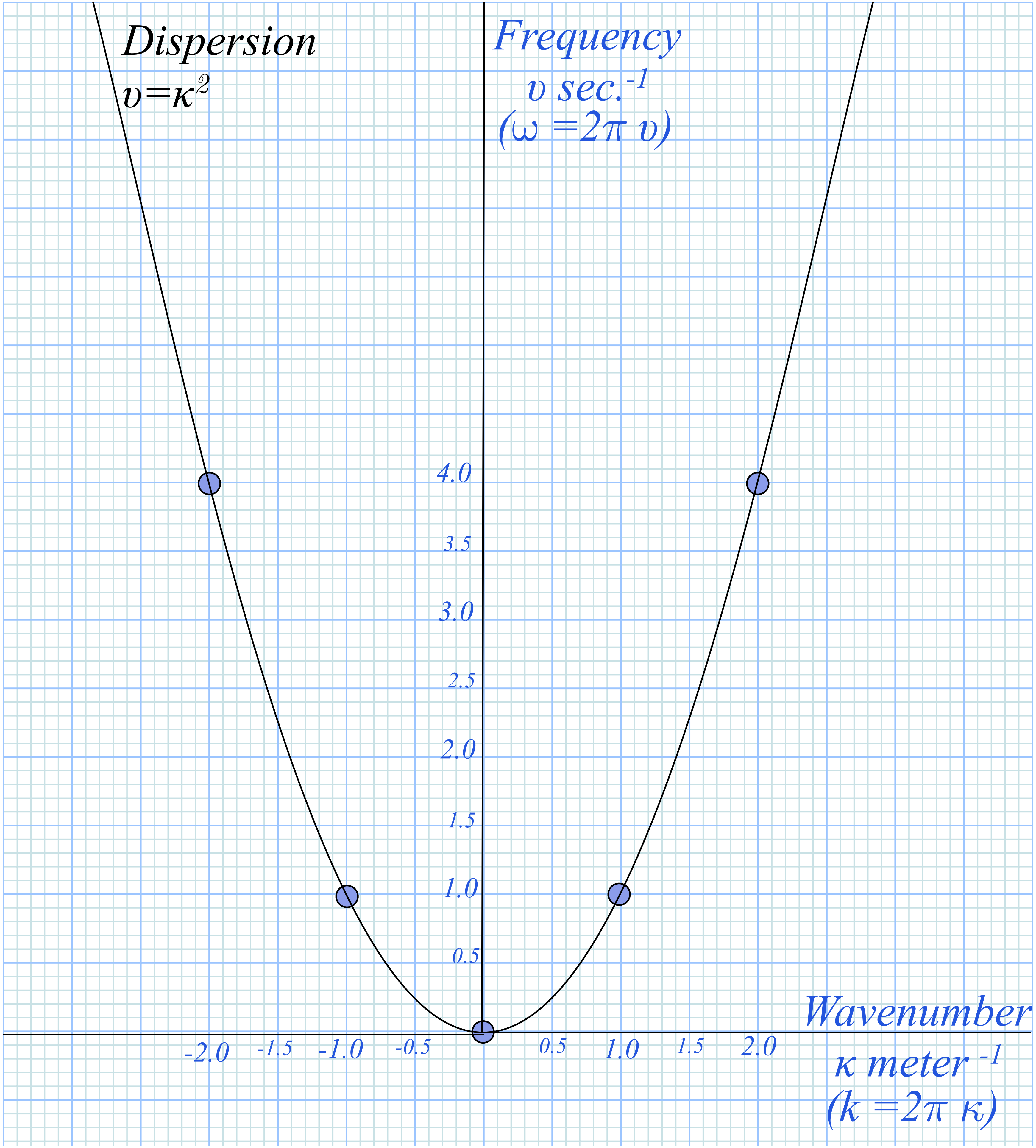
➡ *Example: Bohr Dispersion 2-CW made of 1-CW(m=-1) + 1-CW(m=2)* ⬅

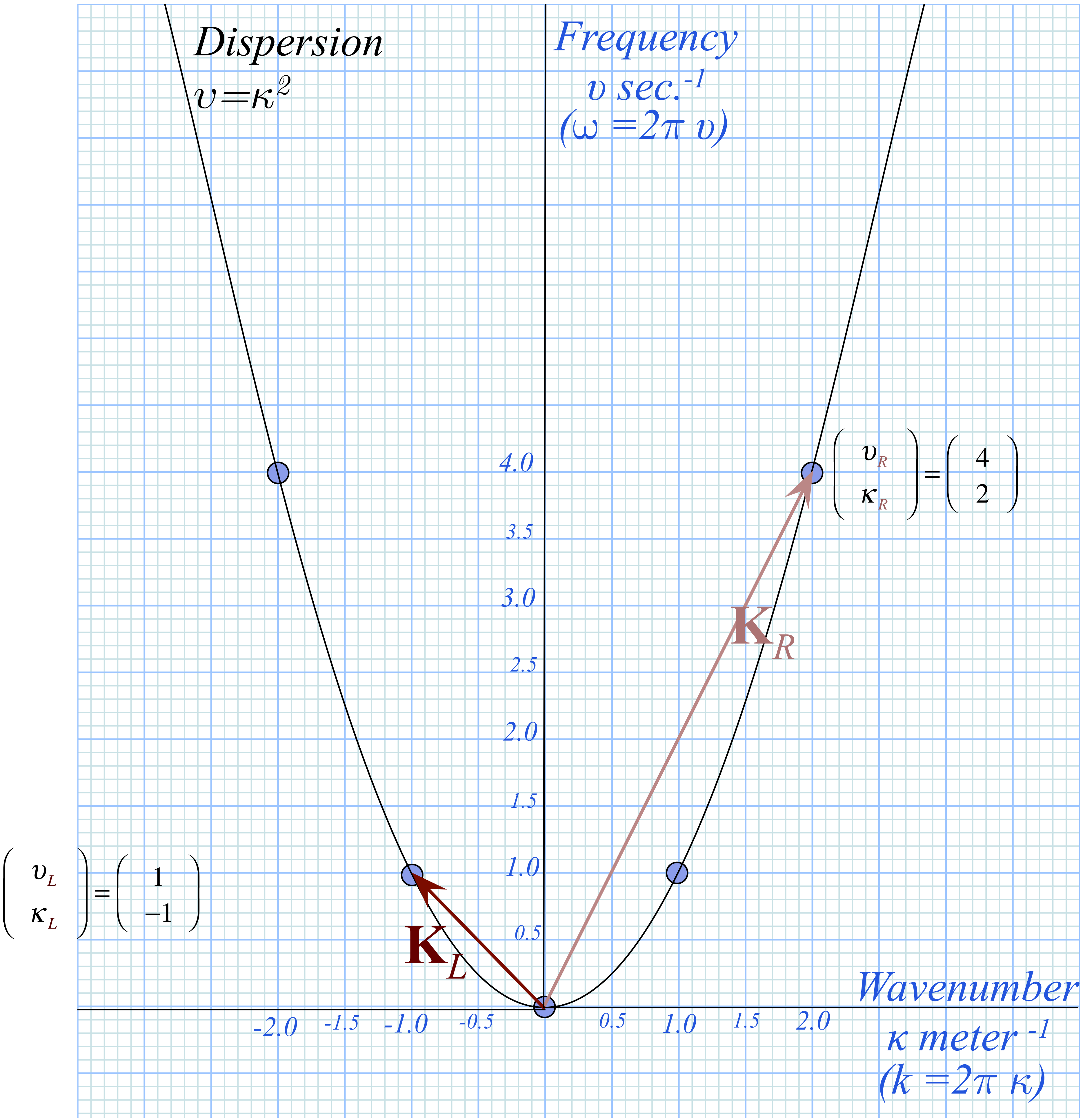
- *Find 2-CW space-time (x,t) lattice from per-space-time (κ,υ) by matrix-algebra/geometry*
  - *Same 1-CW(m=-1) + 1-CW(m=2) Example*

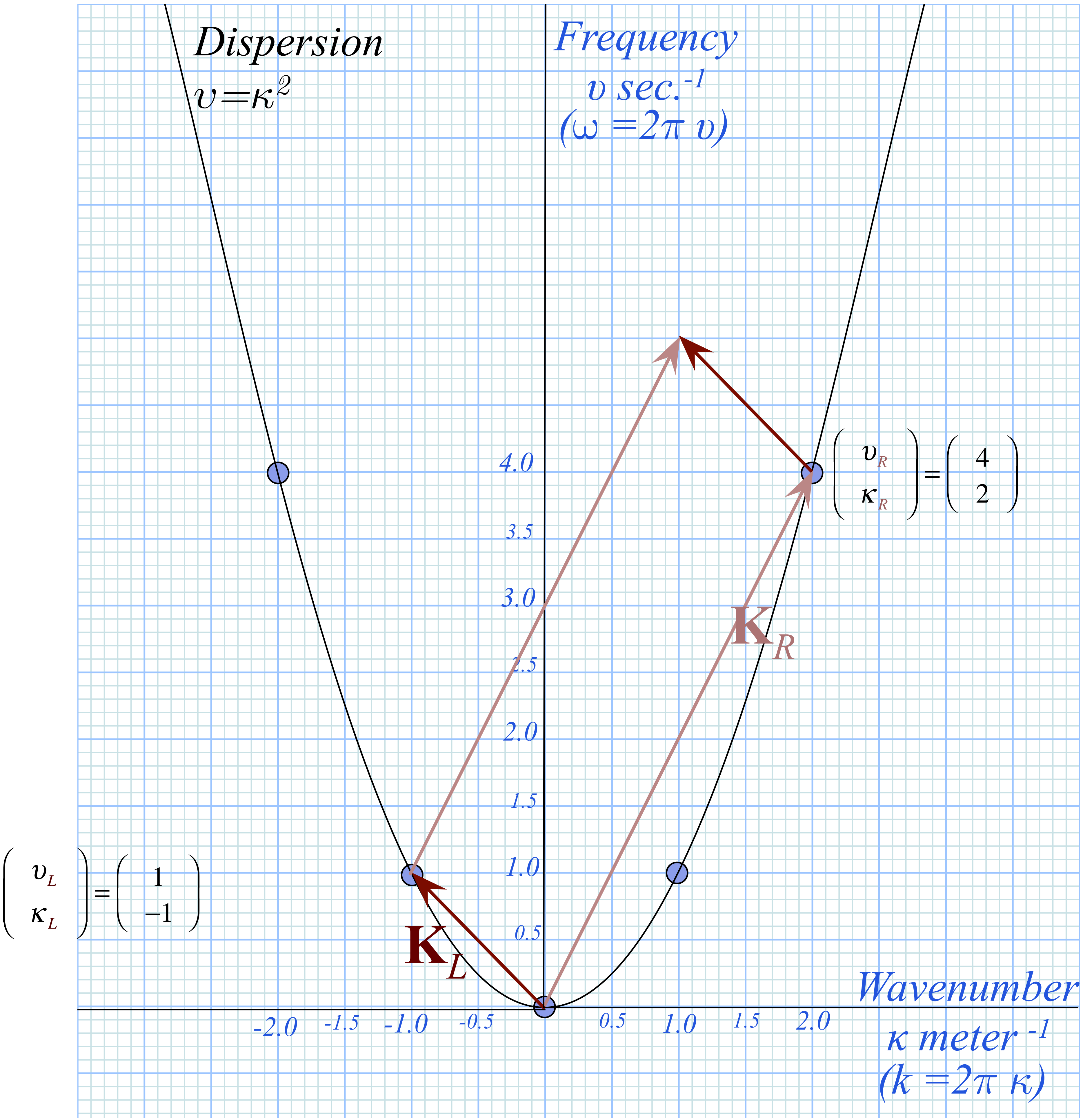


BohrIt Web Simulation

Bohr-Schrödinger {Quadratic dispersion} Wave Mixing

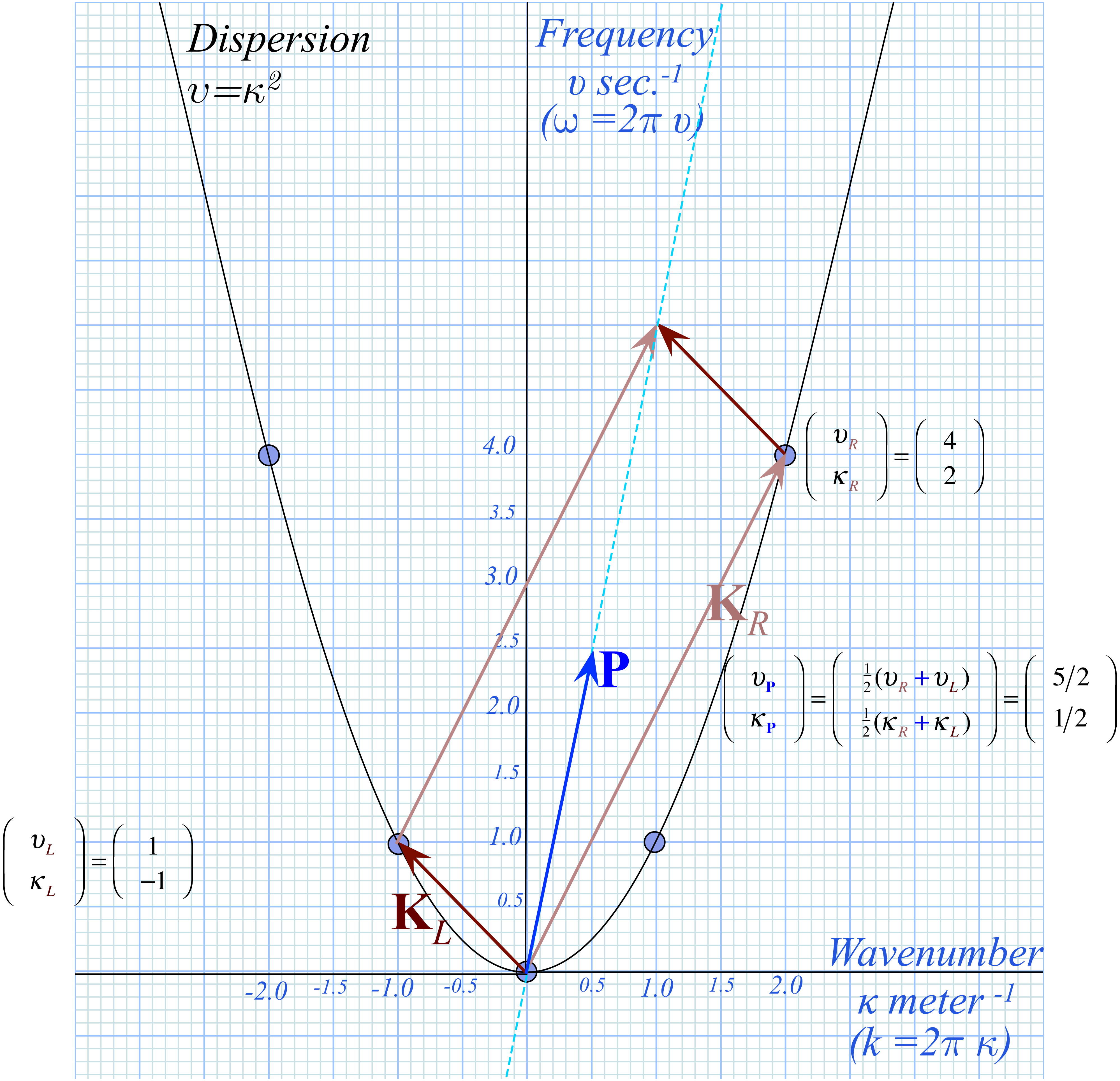

Dispersion
$\upsilon=\kappa^2$
Frequency
υ sec.$^{-1}$
($\omega=2\pi\,\upsilon$)
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
Wavenumber
κ meter $^{-1}$
($k=2\pi\,\kappa$)
-2.0
-1.5
-1.0
-0.5
0.5
1.0
1.5
2.0


Dispersion
$\upsilon=\kappa^2$
Frequency
υ sec.$^{-1}$
$(\omega=2\pi\upsilon)$
Wavenumber
κ meter $^{-1}$
$(k=2\pi\kappa)$
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
-2.0
-1.5
-1.0
-0.5
0.5
1.0
1.5
2.0
$\mathbf{K}_R$
$\mathbf{K}_L$
$\begin{pmatrix}\upsilon_R\\ \kappa_R\end{pmatrix}=\begin{pmatrix}4\\ 2\end{pmatrix}$
$\begin{pmatrix}\upsilon_L\\ \kappa_L\end{pmatrix}=\begin{pmatrix}1\\ -1\end{pmatrix}$



$$\begin{pmatrix}\upsilon_R\\ \kappa_R\end{pmatrix}=\begin{pmatrix}4\\ 2\end{pmatrix}$$

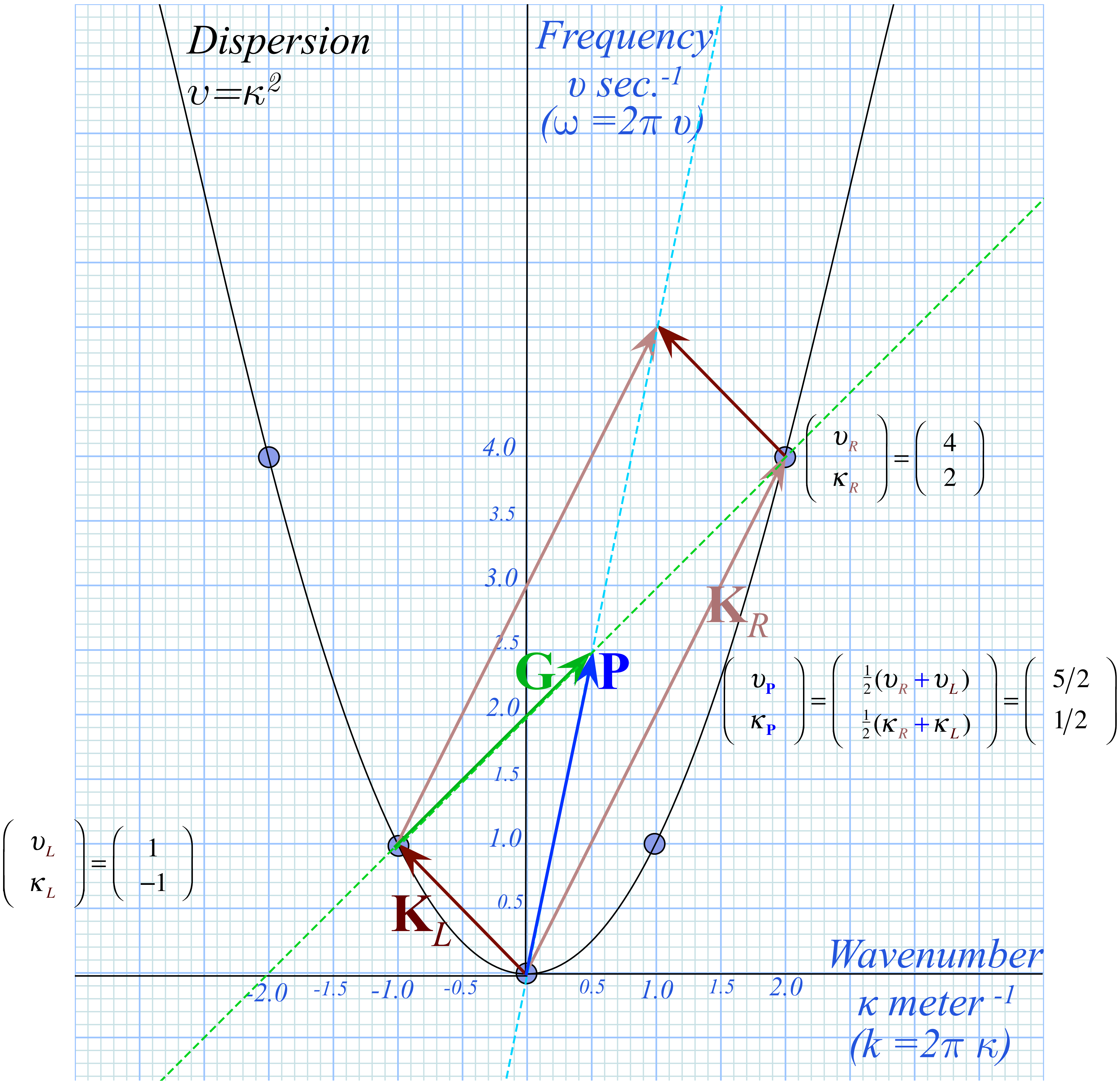
$$\begin{pmatrix}\upsilon_L\\ \kappa_L\end{pmatrix}=\begin{pmatrix}1\\ -1\end{pmatrix}$$



$$\begin{pmatrix}\upsilon_R\\ \kappa_R\end{pmatrix}=\begin{pmatrix}4\\ 2\end{pmatrix}$$

$$\begin{pmatrix}\upsilon_{\mathbf{P}}\\ \kappa_{\mathbf{P}}\end{pmatrix}=\begin{pmatrix}\frac{1}{2}(\upsilon_R+\upsilon_L)\\ \frac{1}{2}(\kappa_R+\kappa_L)\end{pmatrix}=\begin{pmatrix}5/2\\ 1/2\end{pmatrix}$$

$$\begin{pmatrix}\upsilon_L\\ \kappa_L\end{pmatrix}=\begin{pmatrix}1\\ -1\end{pmatrix}$$



$$\begin{pmatrix}\upsilon_R\\ \kappa_R\end{pmatrix}=\begin{pmatrix}4\\ 2\end{pmatrix}$$

$$\begin{pmatrix}\upsilon_{\mathbf{P}}\\ \kappa_{\mathbf{P}}\end{pmatrix}=\begin{pmatrix}\frac{1}{2}(\upsilon_R+\upsilon_L)\\ \frac{1}{2}(\kappa_R+\kappa_L)\end{pmatrix}=\begin{pmatrix}5/2\\ 1/2\end{pmatrix}$$

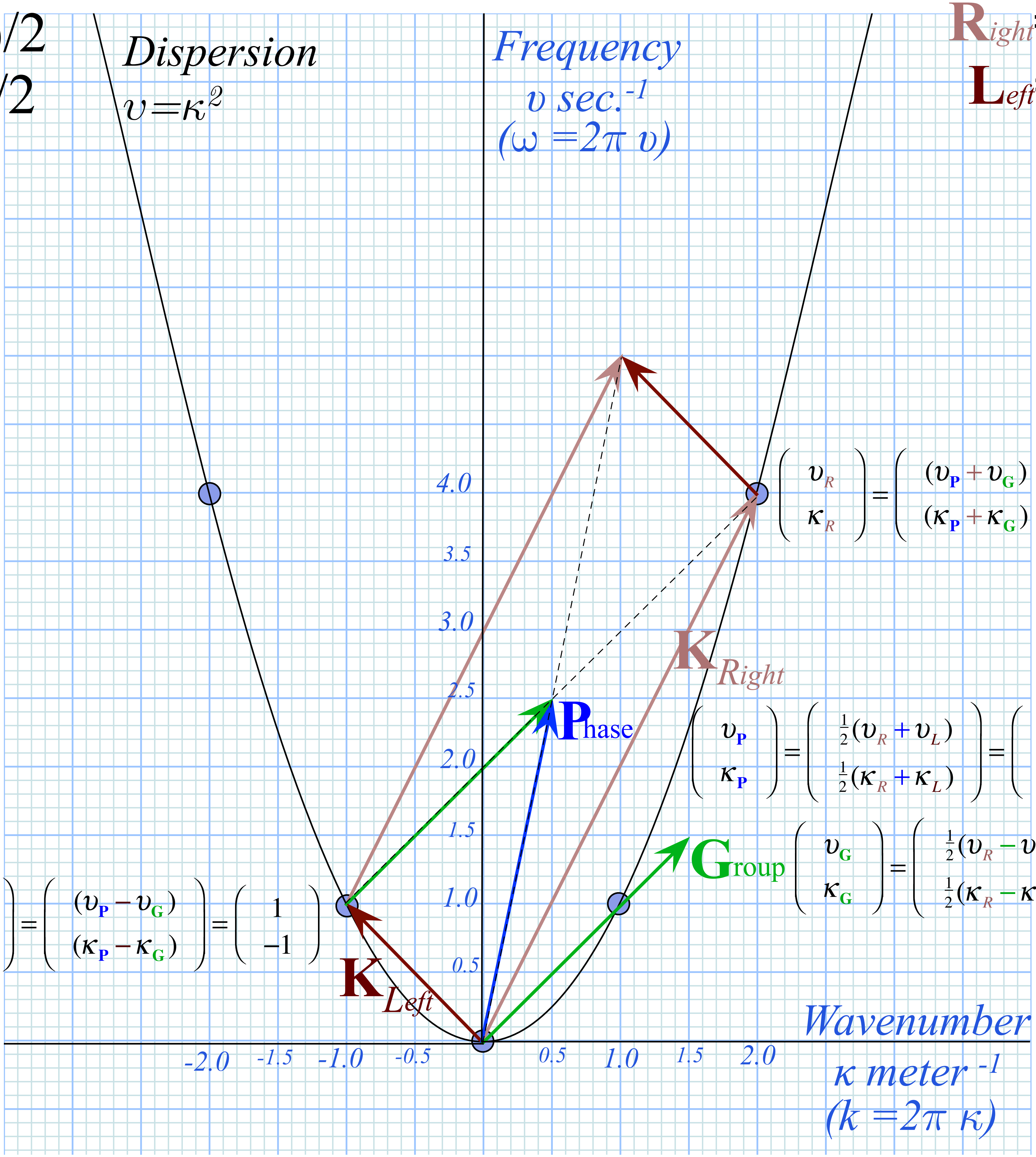
$$\begin{pmatrix}\upsilon_L\\ \kappa_L\end{pmatrix}=\begin{pmatrix}1\\ -1\end{pmatrix}$$

$\mathbf{P}_{\text{hase}}=(\mathbf{R}+\mathbf{L})/2$

$\mathbf{G}_{\text{roup}}=(\mathbf{R}-\mathbf{L})/2$

$\mathbf{R}_{ight}=\mathbf{P}_{\text{hase}}+\mathbf{G}_{\text{roup}}$

$\mathbf{L}_{eft}=\mathbf{P}_{\text{hase}}-\mathbf{G}_{\text{roup}}$



$$\begin{pmatrix}\upsilon_R\\ \kappa_R\end{pmatrix}=\begin{pmatrix}(\upsilon_{\mathbf{P}}+\upsilon_{\mathbf{G}})\\ (\kappa_{\mathbf{P}}+\kappa_{\mathbf{G}})\end{pmatrix}=\begin{pmatrix}4\\ 2\end{pmatrix}$$

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$$\begin{pmatrix}\upsilon_{\mathbf{G}}\\ \kappa_{\mathbf{G}}\end{pmatrix}=\begin{pmatrix}\frac{1}{2}(\upsilon_R-\upsilon_L)\\ \frac{1}{2}(\kappa_R-\kappa_L)\end{pmatrix}=\begin{pmatrix}3/2\\ 3/2\end{pmatrix}$$

$$\begin{pmatrix}\upsilon_L\\ \kappa_L\end{pmatrix}=\begin{pmatrix}(\upsilon_{\mathbf{P}}-\upsilon_{\mathbf{G}})\\ (\kappa_{\mathbf{P}}-\kappa_{\mathbf{G}})\end{pmatrix}=\begin{pmatrix}1\\ -1\end{pmatrix}$$

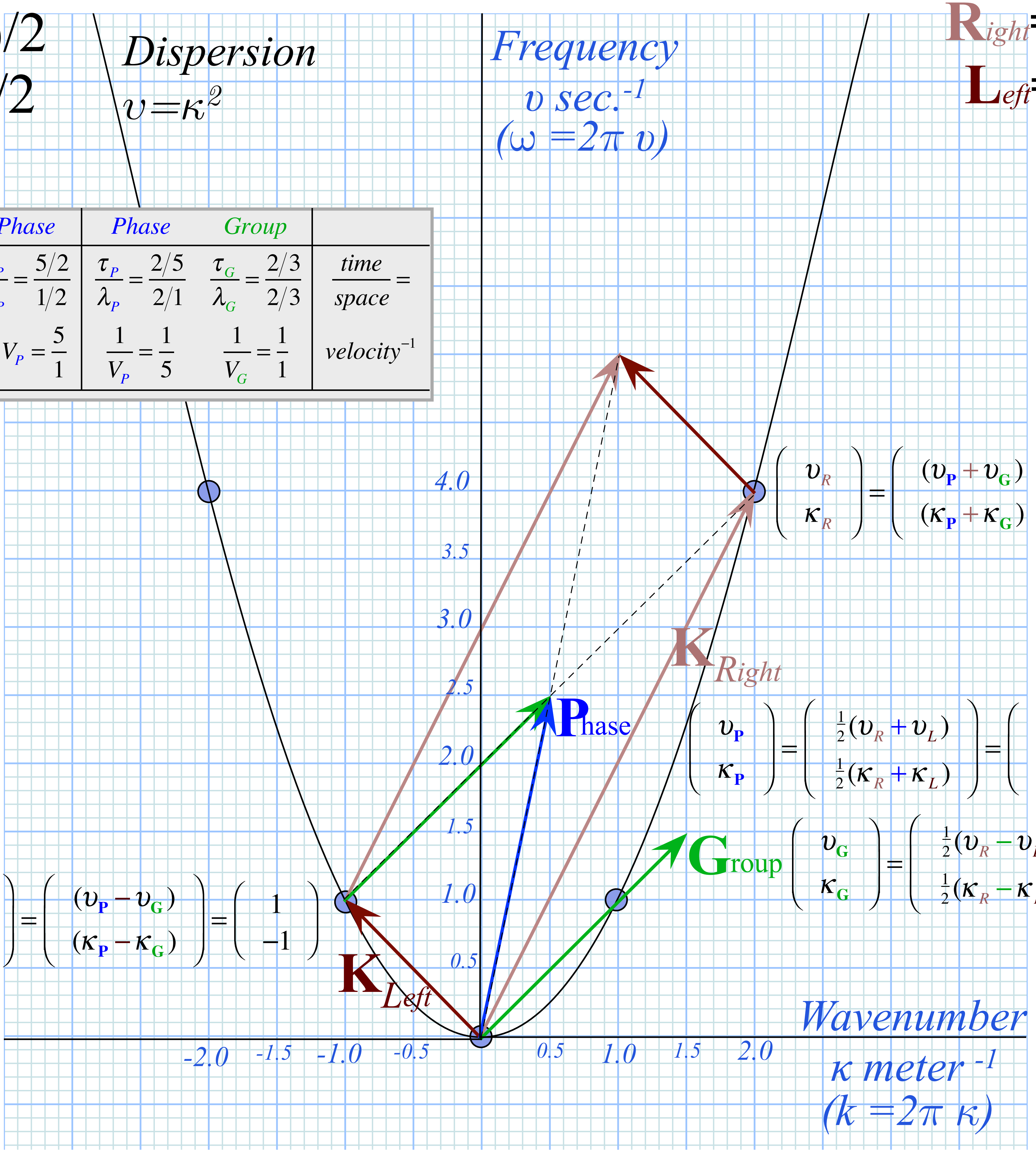
Phase=(**R**+**L**)/2

Group=(**R**-**L**)/2

Right=**P**hase+**G**roup

Left=**P**hase-**G**roup

	Group	*Phase*	*Phase*	*Group*	
$\frac{\textit{per-time}}{\textit{per-space}}$	$\frac{\upsilon_G}{\kappa_G}=\frac{3/2}{3/2}$	$\frac{\upsilon_P}{\kappa_P}=\frac{5/2}{1/2}$	$\frac{\tau_P}{\lambda_P}=\frac{2/5}{2/1}$	$\frac{\tau_G}{\lambda_G}=\frac{2/3}{2/3}$	$\frac{\textit{time}}{\textit{space}}=$
$=\textit{velocity}$	$=V_G=\frac{1}{1}$	$=V_P=\frac{5}{1}$	$\frac{1}{V_P}=\frac{1}{5}$	$\frac{1}{V_G}=\frac{1}{1}$	$\textit{velocity}^{-1}$



$$\begin{pmatrix}\upsilon_R\\\kappa_R\end{pmatrix}=\begin{pmatrix}(\upsilon_{\mathbf{P}}+\upsilon_{\mathbf{G}})\\(\kappa_{\mathbf{P}}+\kappa_{\mathbf{G}})\end{pmatrix}=\begin{pmatrix}4\\2\end{pmatrix}$$

$$\begin{pmatrix}\upsilon_{\mathbf{P}}\\\kappa_{\mathbf{P}}\end{pmatrix}=\begin{pmatrix}\frac{1}{2}(\upsilon_R+\upsilon_L)\\\frac{1}{2}(\kappa_R+\kappa_L)\end{pmatrix}=\begin{pmatrix}5/2\\1/2\end{pmatrix}$$

$$\begin{pmatrix}\upsilon_{\mathbf{G}}\\\kappa_{\mathbf{G}}\end{pmatrix}=\begin{pmatrix}\frac{1}{2}(\upsilon_R-\upsilon_L)\\\frac{1}{2}(\kappa_R-\kappa_L)\end{pmatrix}=\begin{pmatrix}3/2\\3/2\end{pmatrix}$$

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$\mathbf{R}_{ight}=\mathbf{P}_{\text{hase}}+\mathbf{G}_{\text{roup}}$

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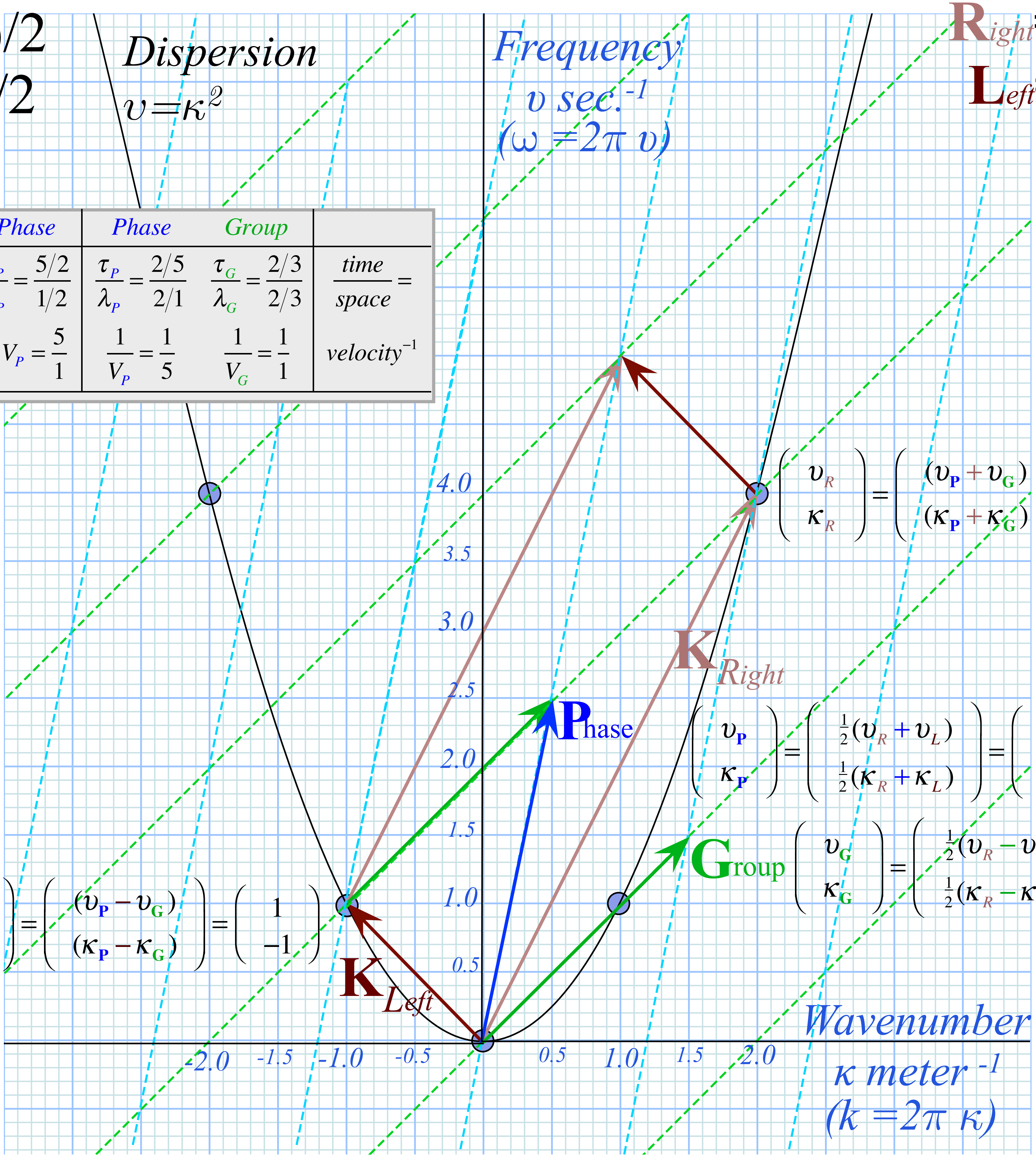

	Group	*Phase*	*Phase*	*Group*	
$\frac{per\text{-}time}{per\text{-}space}$	$\frac{\upsilon_G}{\kappa_G}=\frac{3/2}{3/2}$	$\frac{\upsilon_P}{\kappa_P}=\frac{5/2}{1/2}$	$\frac{\tau_P}{\lambda_P}=\frac{2/5}{2/1}$	$\frac{\tau_G}{\lambda_G}=\frac{2/3}{2/3}$	$\frac{time}{space}=$
$=velocity$	$=V_G=\frac{1}{1}$	$=V_P=\frac{5}{1}$	$\frac{1}{V_P}=\frac{1}{5}$	$\frac{1}{V_G}=\frac{1}{1}$	$velocity^{-1}$

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$$\begin{pmatrix}\upsilon_L\\ \kappa_L\end{pmatrix}=\begin{pmatrix}(\upsilon_{\mathbf{P}}-\upsilon_{\mathbf{G}})\\ (\kappa_{\mathbf{P}}-\kappa_{\mathbf{G}})\end{pmatrix}=\begin{pmatrix}1\\ -1\end{pmatrix}$$

Wave resonance in cyclic C_n symmetry (REVIEW)

C_6 symmetric mode model:Distant neighbor coupling

- *C_6 moving waves and degenerate standing waves*
- *C_6 dispersion functions for 1st, 2nd, and 3rd-neighbor coupling*
- *C_6 dispersion functions split by C-type symmetry(complex, chiral, ...)*

C_{12} and higher symmetry mode models: Archetypes of dispersion functions and 1-CW phase velocity

- *½-Sum-½-Diff-theory of 2-CW group and phase velocity*

Given two 1-CW phases: Find 2-CW phase velocity $\mathbf{V}_{phase}^{(2\text{-}CW)}$ and group velocity $\mathbf{V}_{group}^{(2\text{-}CW)}$

- *Example: Bohr Dispersion 2-CW made of 1-CW(m=-1) + 1-CW(m=2)*

➔ *Find 2-CW space-time (x,t) lattice from per-space-time (κ,υ) by matrix-algebra/geometry*

- *Same 1-CW(m=-1) + 1-CW(m=2) Example*

2-CW space-time (x,t) lattice from per-space-time (κ,υ) by algebra

Real wave-zeros ($\cos\phi=0$) need $\phi=kx-\omega t=N\pi/2$ for odd N.

2-CW space-time (x,t) lattice from per-space-time (κ,υ) by algebra

Real wave-zeros ($\cos\phi=0$) need $\phi=kx-\omega t=N\pi/2$ for odd N.

Real ψ_{phase}-zeros ($\cos\phi_P=0$) need $\phi_P= k_P x-\omega_P t =N_P\pi/2$ for odd $N_P=...\pm3, \pm1, \pm3, \pm5,...$

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Real ψ_{group}-zeros ($\cos\phi_G=0$) need $\phi_G=k_G x-\omega_G t=N_G\pi/2$ for odd $N_G=\ldots\pm 3, \pm 1, \pm 3, \pm 5,\ldots$

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Real (x,t) lattice zero-points need BOTH:

$$k_P x-\omega_P t = N_P \tfrac{\pi}{2}$$

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$$k_G x-\omega_G t = N_G\tfrac{\pi}{2}$$

...becomes
matrix equation:
$$\begin{pmatrix} k_P & -\omega_P \\ k_G & -\omega_G \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix}=\begin{pmatrix} N_P \\ N_G \end{pmatrix}\tfrac{\pi}{2}$$

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...with inverted matrix solution:

$$\begin{pmatrix} x \\ t \end{pmatrix}_{\substack{P\\G}}=\begin{pmatrix} k_P & -\omega_P \\ k_G & -\omega_G \end{pmatrix}^{-1}\begin{pmatrix} N_P \\ N_G \end{pmatrix}\tfrac{\pi}{2}$$

2-CW space-time (x,t) lattice from per-space-time (κ,υ) by algebra

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Real (x,t) lattice zero-points need BOTH:

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$$\begin{pmatrix} x \\ t \end{pmatrix}_{\substack{P\\G}}=\frac{\begin{pmatrix} -\omega_G & \omega_P \\ -k_G & k_P \end{pmatrix}\begin{pmatrix} N_P \\ N_G \end{pmatrix}\tfrac{\pi}{2}}{k_G\omega_P-k_P\omega_G}$$

2-CW space-time (x,t) lattice from per-space-time (κ,υ) by algebra

Real wave-zeros ($\cos\phi=0$) need $\phi=kx-\omega t=N\pi/2$ for odd N.

Real ψ_{phase}-zeros ($\cos\phi_P=0$) need $\phi_P= k_P x-\omega_P t =N_P\pi/2$ for odd N_P=...±3, ±1, ±3, ±5,...

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Real (x,t) lattice zero-points need BOTH:

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$$k_G x-\omega_G t = N_G\tfrac{\pi}{2}$$

...becomes matrix equation:

$$\begin{pmatrix} k_P & -\omega_P \\ k_G & -\omega_G \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix}=\begin{pmatrix} N_P \\ N_G \end{pmatrix}\tfrac{\pi}{2}$$

...with inverted matrix solution:

$$\begin{pmatrix} x \\ t \end{pmatrix}_{\substack{P\\G}}=\begin{pmatrix} k_P & -\omega_P \\ k_G & -\omega_G \end{pmatrix}^{-1}\begin{pmatrix} N_P \\ N_G \end{pmatrix}\tfrac{\pi}{2}$$

$$\begin{pmatrix} x \\ t \end{pmatrix}_{\substack{P\\G}}=\frac{\begin{pmatrix} -\omega_G & \omega_P \\ -k_G & k_P \end{pmatrix}\begin{pmatrix} N_P \\ N_G \end{pmatrix}\tfrac{\pi}{2}}{k_G\omega_P\text{-}k_P\omega_G}=\frac{\begin{pmatrix} -\omega_G N_P+\omega_P N_G \\ -k_G N_P+k_P N_G \end{pmatrix}\tfrac{\pi}{2}}{k_G\omega_P\text{-}k_P\omega_G}$$

2-CW space-time (x,t) lattice from per-space-time (κ,υ) by algebra

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$$\begin{pmatrix} x \\ t \end{pmatrix}_{\substack{P\\G}}=\frac{\begin{pmatrix} -\omega_G & \omega_P \\ -k_G & k_P \end{pmatrix}\begin{pmatrix} N_P \\ N_G \end{pmatrix}\tfrac{\pi}{2}}{k_G\omega_P\text{-}k_P\omega_G}=\frac{\begin{pmatrix} -\omega_G N_P+\omega_P N_G \\ -k_G N_P+k_P N_G \end{pmatrix}\tfrac{\pi}{2}}{k_G\omega_P\text{-}k_P\omega_G}$$

$$\begin{pmatrix} x \\ t \end{pmatrix}_{\substack{P\\G}}=\frac{-N_P\begin{pmatrix} \omega_G \\ k_G \end{pmatrix}\tfrac{\pi}{2}+N_G\begin{pmatrix} \omega_P \\ k_P \end{pmatrix}\tfrac{\pi}{2}}{k_G\omega_P\text{-}k_P\omega_G}$$

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$$\begin{pmatrix} x \\ t \end{pmatrix}_{\substack{P\\G}}=\begin{pmatrix} k_P & -\omega_P \\ k_G & -\omega_G \end{pmatrix}^{-1}\begin{pmatrix} N_P \\ N_G \end{pmatrix}\tfrac{\pi}{2}$$

$$\begin{pmatrix} x \\ t \end{pmatrix}_{\substack{P\\G}}=\frac{\begin{pmatrix} -\omega_G & \omega_P \\ -k_G & k_P \end{pmatrix}\begin{pmatrix} N_P \\ N_G \end{pmatrix}\tfrac{\pi}{2}}{k_G\omega_P-k_P\omega_G}=\frac{\begin{pmatrix} -\omega_G N_P+\omega_P N_G \\ -k_G N_P+k_P N_G \end{pmatrix}\tfrac{\pi}{2}}{k_G\omega_P-k_P\omega_G}$$

Replace the *reduced*, π-laden variables with Hertz-Kaiser wave parameters.
$$\omega_G=2\pi\,\upsilon_G\ ,\ k_G=2\pi\,\kappa_G\ ,\ \omega_P=2\pi\,\upsilon_P\ ,\ k_P=2\pi\,\kappa_P\ \ldots$$

$$\begin{pmatrix} x \\ t \end{pmatrix}_{\substack{P\\G}}=\frac{-N_P\begin{pmatrix} \omega_G \\ k_G \end{pmatrix}\tfrac{\pi}{2}+N_G\begin{pmatrix} \omega_P \\ k_P \end{pmatrix}\tfrac{\pi}{2}}{k_G\omega_P-k_P\omega_G}$$

2-CW space-time (x,t) lattice from per-space-time (κ,υ) by algebra

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Real ψ_{phase}-zeros ($\cos\phi_P=0$) need $\phi_P= k_P x-\omega_P t =N_P\pi/2$ for odd $N_P=\ldots\pm 3, \pm 1, \pm 3, \pm 5,\ldots$

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Real (x,t) lattice zero-points need BOTH:

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$$\begin{pmatrix} x \\ t \end{pmatrix}_{\substack{P\\G}} = \frac{\begin{pmatrix} -\omega_G & \omega_P \\ -k_G & k_P \end{pmatrix}\begin{pmatrix} N_P \\ N_G \end{pmatrix}\tfrac{\pi}{2}}{k_G\omega_P - k_P\omega_G} = \frac{\begin{pmatrix} -\omega_G N_P + \omega_P N_G \\ -k_G N_P + k_P N_G \end{pmatrix}\tfrac{\pi}{2}}{k_G\omega_P - k_P\omega_G}$$

Replace the *reduced*, π-laden variables with Hertz-Kaiser wave parameters.

$$\omega_G = 2\pi\upsilon_G \;,\; k_G = 2\pi\kappa_G \;,\; \omega_P = 2\pi\upsilon_P \;,\; k_P = 2\pi\kappa_P \;\ldots$$

$$\begin{pmatrix} x \\ t \end{pmatrix}_{\substack{P\\G}} = \frac{-N_P\begin{pmatrix} \omega_G \\ k_G \end{pmatrix}\tfrac{\pi}{2} + N_G\begin{pmatrix} \omega_P \\ k_P \end{pmatrix}\tfrac{\pi}{2}}{k_G\omega_P - k_P\omega_G} \quad = \frac{-N_P\begin{pmatrix} \upsilon_G \\ \kappa_G \end{pmatrix}2\pi\tfrac{\pi}{2} + N_G\begin{pmatrix} \upsilon_P \\ \kappa_P \end{pmatrix}2\pi\tfrac{\pi}{2}}{(\kappa_G\upsilon_P - \kappa_P\upsilon_G)(2\pi)(2\pi)}$$

2-CW space-time (x,t) lattice from per-space-time (κ,υ) by algebra

Real wave-zeros ($\cos\phi=0$) need $\phi=kx-\omega t=N\pi/2$ for odd N.

Real ψ_{phase}-zeros ($\cos\phi_P=0$) need $\phi_P= k_P x-\omega_P t =N_P\pi/2$ for odd $N_P=\ldots\pm3, \pm1, \pm3, \pm5,\ldots$

Real ψ_{group}-zeros ($\cos\phi_G=0$) need $\phi_G=k_G x-\omega_G t=N_G\pi/2$ for odd $N_G=\ldots\pm3, \pm1, \pm3, \pm5,\ldots$

Real (x,t) lattice zero-points need BOTH:

$$k_P x-\omega_P t = N_P \tfrac{\pi}{2}$$

$$k_G x-\omega_G t = N_G \tfrac{\pi}{2}$$

...becomes matrix equation:

$$\begin{pmatrix} k_P & -\omega_P \\ k_G & -\omega_G \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} N_P \\ N_G \end{pmatrix}\tfrac{\pi}{2}$$

...with inverted matrix solution:

$$\begin{pmatrix} x \\ t \end{pmatrix}_{\substack{P\\G}} = \begin{pmatrix} k_P & -\omega_P \\ k_G & -\omega_G \end{pmatrix}^{-1}\begin{pmatrix} N_P \\ N_G \end{pmatrix}\tfrac{\pi}{2}$$

$$\begin{pmatrix} x \\ t \end{pmatrix}_{\substack{P\\G}} = \frac{\begin{pmatrix} -\omega_G & \omega_P \\ -k_G & k_P \end{pmatrix}\begin{pmatrix} N_P \\ N_G \end{pmatrix}\tfrac{\pi}{2}}{k_G\omega_P - k_P\omega_G} = \frac{\begin{pmatrix} -\omega_G N_P + \omega_P N_G \\ -k_G N_P + k_P N_G \end{pmatrix}\tfrac{\pi}{2}}{k_G\omega_P - k_P\omega_G}$$

Replace the *reduced*, π-laden variables with Hertz-Kaiser wave parameters.

$$\omega_G = 2\pi\upsilon_G \;,\; k_G = 2\pi\kappa_G \;,\; \omega_P = 2\pi\upsilon_P \;,\; k_P = 2\pi\kappa_P \;\ldots$$

$$\begin{pmatrix} x \\ t \end{pmatrix}_{\substack{P\\G}} = \frac{-N_P\begin{pmatrix} \omega_G \\ k_G \end{pmatrix}\tfrac{\pi}{2} + N_G\begin{pmatrix} \omega_P \\ k_P \end{pmatrix}\tfrac{\pi}{2}}{k_G\omega_P - k_P\omega_G} = \frac{-N_P\begin{pmatrix} \upsilon_G \\ \kappa_G \end{pmatrix}2\pi\tfrac{\pi}{2} + N_G\begin{pmatrix} \upsilon_P \\ \kappa_P \end{pmatrix}2\pi\tfrac{\pi}{2}}{(\kappa_G\upsilon_P - \kappa_P\upsilon_G)(2\pi)(2\pi)}$$

$$\begin{pmatrix} x \\ t \end{pmatrix}_{\substack{P\\G}} = \frac{-N_P\begin{pmatrix} \upsilon_G \\ \kappa_G \end{pmatrix} + N_G\begin{pmatrix} \upsilon_P \\ \kappa_P \end{pmatrix}}{4(\kappa_G\upsilon_P - \kappa_P\upsilon_G)}$$

2-CW space-time (x,t) lattice from per-space-time (κ,υ) by algebra

Real wave-zeros ($\cos\phi=0$) need $\phi=kx-\omega t=N\pi/2$ for odd N.

Real ψ_{phase}-zeros ($\cos\phi_P=0$) need $\phi_P= k_P x-\omega_P t =N_P\pi/2$ for odd $N_P=\ldots\pm3, \pm1, \pm3, \pm5,\ldots$

Real ψ_{group}-zeros ($\cos\phi_G=0$) need $\phi_G=k_G x-\omega_G t=N_G\pi/2$ for odd $N_G=\ldots\pm3, \pm1, \pm3, \pm5,\ldots$

Real (x,t) lattice zero-points need BOTH:
$$k_P x-\omega_P t = N_P\tfrac{\pi}{2}$$
$$k_G x-\omega_G t = N_G\tfrac{\pi}{2}$$

...becomes matrix equation:
$$\begin{pmatrix} k_P & \text{-}\omega_P \\ k_G & \text{-}\omega_G \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix}=\begin{pmatrix} N_P \\ N_G \end{pmatrix}\tfrac{\pi}{2}$$

...with inverted matrix solution:
$$\begin{pmatrix} x \\ t \end{pmatrix}_{\substack{P\\G}}=\begin{pmatrix} k_P & \text{-}\omega_P \\ k_G & \text{-}\omega_G \end{pmatrix}^{-1}\begin{pmatrix} N_P \\ N_G \end{pmatrix}\tfrac{\pi}{2}$$

$$\begin{pmatrix} x \\ t \end{pmatrix}_{\substack{P\\G}}=\frac{\begin{pmatrix} \text{-}\omega_G & \omega_P \\ \text{-}k_G & k_P \end{pmatrix}\begin{pmatrix} N_P \\ N_G \end{pmatrix}\tfrac{\pi}{2}}{k_G\omega_P\text{-}k_P\omega_G}=\frac{\begin{pmatrix} \text{-}\omega_G N_P+\omega_P N_G \\ \text{-}k_G N_P+k_P N_G \end{pmatrix}\tfrac{\pi}{2}}{k_G\omega_P\text{-}k_P\omega_G}$$

	Group	*Phase*	*Phase*	*Group*
$\dfrac{\textit{per-time}}{\textit{per-space}}$	$\dfrac{\upsilon_G}{\kappa_G}=\dfrac{3/2}{3/2}$	$\dfrac{\upsilon_P}{\kappa_P}=\dfrac{5/2}{1/2}$	$\dfrac{\tau_P}{\lambda_P}=\dfrac{2/5}{2/1}$	$\dfrac{\tau_G}{\lambda_G}=\dfrac{2/3}{2/3}$
$=\textit{velocity}$	$=V_G=\dfrac{1}{1}$	$=V_P=\dfrac{5}{1}$	$\dfrac{1}{V_P}=\dfrac{1}{5}$	$\dfrac{1}{V_G}=\dfrac{1}{1}$

Replace π-laden variables with Hertz-Kaiser wave parameters.
$$\omega_G=2\pi\,\upsilon_G\ ,\ k_G=2\pi\,\kappa_G\ ,\ \omega_P=2\pi\,\upsilon_P\ ,\ k_P=2\pi\,\kappa_P\ \ldots$$

$$\begin{pmatrix} x \\ t \end{pmatrix}_{\substack{P\\G}}=\frac{\text{-}N_P\begin{pmatrix} \omega_G \\ k_G \end{pmatrix}\tfrac{\pi}{2}+N_G\begin{pmatrix} \omega_P \\ k_P \end{pmatrix}\tfrac{\pi}{2}}{k_G\omega_P\text{-}k_P\omega_G}\qquad=\frac{\text{-}N_P\begin{pmatrix} \upsilon_G \\ \kappa_G \end{pmatrix}2\pi\tfrac{\pi}{2}+N_G\begin{pmatrix} \upsilon_P \\ \kappa_P \end{pmatrix}2\pi\tfrac{\pi}{2}}{(\kappa_G\upsilon_P\text{-}\kappa_P\upsilon_G)(2\pi)(2\pi)}$$

$$\begin{pmatrix} x \\ t \end{pmatrix}_{\substack{P\\G}}=\frac{\text{-}N_P\begin{pmatrix} \upsilon_G \\ \kappa_G \end{pmatrix}+N_G\begin{pmatrix} \upsilon_P \\ \kappa_P \end{pmatrix}}{4(\kappa_G\upsilon_P\text{-}\kappa_P\upsilon_G)}$$

2-CW space-time (x,t) lattice from per-space-time (κ,υ) by algebra

Real wave-zeros ($\cos\phi=0$) need $\phi=kx\text{-}\omega t=N\pi/2$ for odd N.

Real ψ_{phase}-zeros ($\cos\phi_P=0$) need $\phi_P= k_Px\text{-}\omega_Pt =N_P\pi/2$ for odd N_P=...±3, ±1, ±3, ±5,...

Real ψ_{group}-zeros ($\cos\phi_G=0$) need $\phi_G=k_Gx\text{-}\omega_Gt=N_G\pi/2$ for odd N_G=...±3, ±1, ±3, ±5,...

Real (x,t) lattice zero-points need BOTH:

$$k_Px-\omega_Pt=N_P\tfrac{\pi}{2}$$

$$k_Gx-\omega_Gt=N_G\tfrac{\pi}{2}$$

...becomes matrix equation:

$$\begin{pmatrix} k_P & \text{-}\omega_P \\ k_G & \text{-}\omega_G \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix}=\begin{pmatrix} N_P \\ N_G \end{pmatrix}\tfrac{\pi}{2}$$

...with inverted matrix solution:

$$\begin{pmatrix} x \\ t \end{pmatrix}_{\substack{P\\G}}=\begin{pmatrix} k_P & \text{-}\omega_P \\ k_G & \text{-}\omega_G \end{pmatrix}^{-1}\begin{pmatrix} N_P \\ N_G \end{pmatrix}\tfrac{\pi}{2}$$

$$\begin{pmatrix} x \\ t \end{pmatrix}_{\substack{P\\G}}=\frac{\begin{pmatrix} \text{-}\omega_G & \omega_P \\ \text{-}k_G & k_P \end{pmatrix}\begin{pmatrix} N_P \\ N_G \end{pmatrix}\tfrac{\pi}{2}}{k_G\omega_P\text{-}k_P\omega_G}=\frac{\begin{pmatrix} \text{-}\omega_GN_P+\omega_PN_G \\ \text{-}k_GN_P+k_PN_G \end{pmatrix}\tfrac{\pi}{2}}{k_G\omega_P\text{-}k_P\omega_G}$$

	Group	*Phase*	*Phase*	*Group*
$\frac{per\text{-}time}{per\text{-}space}$	$\frac{\upsilon_G}{\kappa_G}=\frac{3/2}{3/2}$	$\frac{\upsilon_P}{\kappa_P}=\frac{5/2}{1/2}$	$\frac{\tau_P}{\lambda_P}=\frac{2/5}{2/1}$	$\frac{\tau_G}{\lambda_G}=\frac{2/3}{2/3}$
$=velocity$	$=V_G=\frac{1}{1}$	$=V_P=\frac{5}{1}$	$\frac{1}{V_P}=\frac{1}{5}$	$\frac{1}{V_G}=\frac{1}{1}$

Replace π-laden variables with Hertz-Kaiser wave parameters.

$$\omega_G=2\pi\,\upsilon_G\ ,\ k_G=2\pi\,\kappa_G\ ,\ \omega_P=2\pi\,\upsilon_P\ ,\ k_P=2\pi\,\kappa_P\ \ldots$$

$$\begin{pmatrix} x \\ t \end{pmatrix}_{\substack{P\\G}}=\frac{\text{-}N_P\begin{pmatrix} \omega_G \\ k_G \end{pmatrix}\tfrac{\pi}{2}+N_G\begin{pmatrix} \omega_P \\ k_P \end{pmatrix}\tfrac{\pi}{2}}{k_G\omega_P\text{-}k_P\omega_G}=\frac{\text{-}N_P\begin{pmatrix} \upsilon_G \\ \kappa_G \end{pmatrix}2\pi\tfrac{\pi}{2}+N_G\begin{pmatrix} \upsilon_P \\ \kappa_P \end{pmatrix}2\pi\tfrac{\pi}{2}}{(\kappa_G\upsilon_P\text{-}\kappa_P\upsilon_G)(2\pi)(2\pi)}$$

$$\begin{pmatrix} x \\ t \end{pmatrix}_{\substack{P\\G}}=\frac{\text{-}N_P\begin{pmatrix} \upsilon_G \\ \kappa_G \end{pmatrix}+N_G\begin{pmatrix} \upsilon_P \\ \kappa_P \end{pmatrix}}{4(\kappa_G\upsilon_P\text{-}\kappa_P\upsilon_G)}=\frac{\text{-}N_P\begin{pmatrix} 3/2 \\ 3/2 \end{pmatrix}+N_G\begin{pmatrix} 5/2 \\ 1/2 \end{pmatrix}}{4(\frac{3}{2}\frac{5}{2}\text{-}\frac{1}{2}\frac{3}{2})}$$

2-CW space-time (x,t) lattice from per-space-time (κ,υ) by algebra

Real wave-zeros ($\cos\phi=0$) need $\phi=kx-\omega t=N\pi/2$ for odd N.

Real ψ_{phase}-zeros ($\cos\phi_P=0$) need $\phi_P= k_P x-\omega_P t =N_P\pi/2$ for odd $N_P=\ldots\pm3, \pm1, \pm3, \pm5,\ldots$

Real ψ_{group}-zeros ($\cos\phi_G=0$) need $\phi_G=k_G x-\omega_G t=N_G\pi/2$ for odd $N_G=\ldots\pm3, \pm1, \pm3, \pm5,\ldots$

Real (x,t) lattice zero-points need BOTH:

$$k_P x-\omega_P t=N_P\tfrac{\pi}{2}$$

$$k_G x-\omega_G t=N_G\tfrac{\pi}{2}$$

...becomes matrix equation:

$$\begin{pmatrix} k_P & -\omega_P \\ k_G & -\omega_G \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix}=\begin{pmatrix} N_P \\ N_G \end{pmatrix}\tfrac{\pi}{2}$$

...with inverted matrix solution:

$$\begin{pmatrix} x \\ t \end{pmatrix}_{\substack{P\\G}}=\begin{pmatrix} k_P & -\omega_P \\ k_G & -\omega_G \end{pmatrix}^{-1}\begin{pmatrix} N_P \\ N_G \end{pmatrix}\tfrac{\pi}{2}$$

$$\begin{pmatrix} x \\ t \end{pmatrix}_{\substack{P\\G}}=\frac{\begin{pmatrix} -\omega_G & \omega_P \\ -k_G & k_P \end{pmatrix}\begin{pmatrix} N_P \\ N_G \end{pmatrix}\tfrac{\pi}{2}}{k_G\omega_P-k_P\omega_G}=\frac{\begin{pmatrix} -\omega_G N_P+\omega_P N_G \\ -k_G N_P+k_P N_G \end{pmatrix}\tfrac{\pi}{2}}{k_G\omega_P-k_P\omega_G}$$

	Group	*Phase*	*Phase*	*Group*
$\frac{per\text{-}time}{per\text{-}space}$	$\frac{\upsilon_G}{\kappa_G}=\frac{3/2}{3/2}$	$\frac{\upsilon_P}{\kappa_P}=\frac{5/2}{1/2}$	$\frac{\tau_P}{\lambda_P}=\frac{2/5}{2/1}$	$\frac{\tau_G}{\lambda_G}=\frac{2/3}{2/3}$
$=velocity$	$=V_G=\frac{1}{1}$	$=V_P=\frac{5}{1}$	$\frac{1}{V_P}=\frac{1}{5}$	$\frac{1}{V_G}=\frac{1}{1}$

Replace π-laden variables with Hertz-Kaiser wave parameters.

$$\omega_G=2\pi\upsilon_G\ ,\ k_G=2\pi\kappa_G\ ,\ \omega_P=2\pi\upsilon_P\ ,\ k_P=2\pi\kappa_P\ \ldots$$

$$\begin{pmatrix} x \\ t \end{pmatrix}_{\substack{P\\G}}=\frac{-N_P\begin{pmatrix} \omega_G \\ k_G \end{pmatrix}\tfrac{\pi}{2}+N_G\begin{pmatrix} \omega_P \\ k_P \end{pmatrix}\tfrac{\pi}{2}}{k_G\omega_P-k_P\omega_G}=\frac{-N_P\begin{pmatrix} \upsilon_G \\ \kappa_G \end{pmatrix}2\pi\tfrac{\pi}{2}+N_G\begin{pmatrix} \upsilon_P \\ \kappa_P \end{pmatrix}2\pi\tfrac{\pi}{2}}{(\kappa_G\upsilon_P-\kappa_P\upsilon_G)(2\pi)(2\pi)}$$

$$\begin{pmatrix} x \\ t \end{pmatrix}_{\substack{P\\G}}=\frac{-N_P\begin{pmatrix} \upsilon_G \\ \kappa_G \end{pmatrix}+N_G\begin{pmatrix} \upsilon_P \\ \kappa_P \end{pmatrix}}{4(\kappa_G\upsilon_P-\kappa_P\upsilon_G)}=\frac{-N_P\begin{pmatrix} 3/2 \\ 3/2 \end{pmatrix}+N_G\begin{pmatrix} 5/2 \\ 1/2 \end{pmatrix}}{4(\frac{3}{2}\frac{5}{2}-\frac{1}{2}\frac{3}{2})}=\frac{1}{24}\begin{pmatrix} 5N_G-3N_P \\ 1N_G-3N_P \end{pmatrix}$$

Wave resonance in cyclic C_n symmetry (REVIEW)
C_6 symmetric mode model:Distant neighbor coupling
- *C_6 moving waves and degenerate standing waves*
- *C_6 dispersion functions for 1st, 2nd, and 3rd-neighbor coupling*
- *C_6 dispersion functions split by C-type symmetry(complex, chiral, ...)*

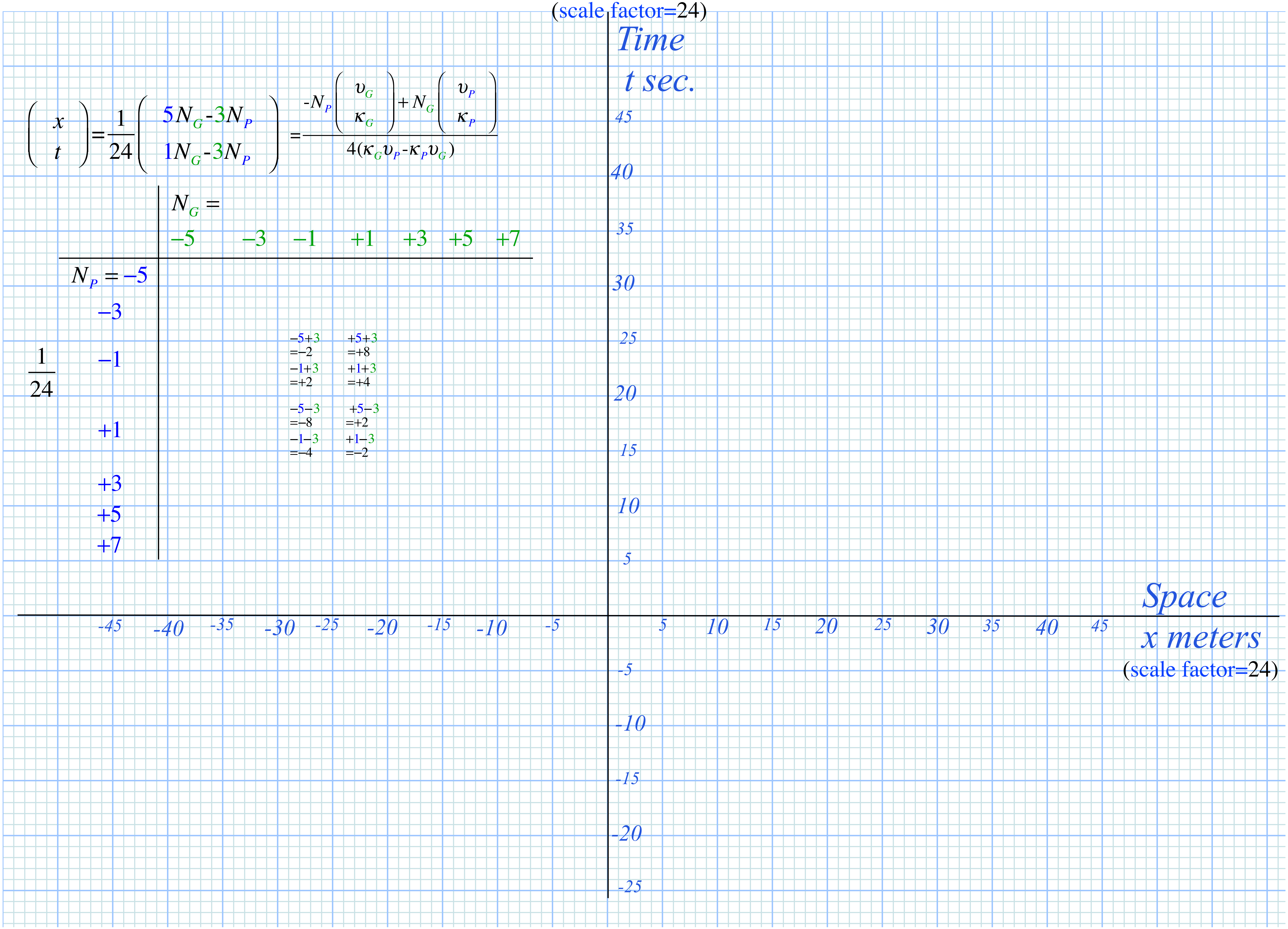
C_{12} and higher symmetry mode models: Archetypes of dispersion functions and 1-CW phase velocity
- *½-Sum-½-Diff-theory of 2-CW group and phase velocity*

Given two 1-CW phases: Find 2-CW phase velocity $\mathbf{V}_{phase}^{(2\text{-}CW)}$ and group velocity $\mathbf{V}_{group}^{(2\text{-}CW)}$
- *Example: Bohr Dispersion 2-CW made of 1-CW(m=-1) + 1-CW(m=2)*
- *Find 2-CW space-time (x,t) lattice from per-space-time (κ,υ) by matrix-algebra/geometry*
  - ➔ *Same 1-CW(m=-1) + 1-CW(m=2) Example* ↑

$$\begin{pmatrix} x \\ t \end{pmatrix} = \frac{1}{24}\begin{pmatrix} 5N_G - 3N_P \\ 1N_G - 3N_P \end{pmatrix} = \frac{-N_P\begin{pmatrix}\upsilon_G \\ \kappa_G\end{pmatrix} + N_G\begin{pmatrix}\upsilon_P \\ \kappa_P\end{pmatrix}}{4(\kappa_G\upsilon_P - \kappa_P\upsilon_G)}$$

$\frac{1}{24}$

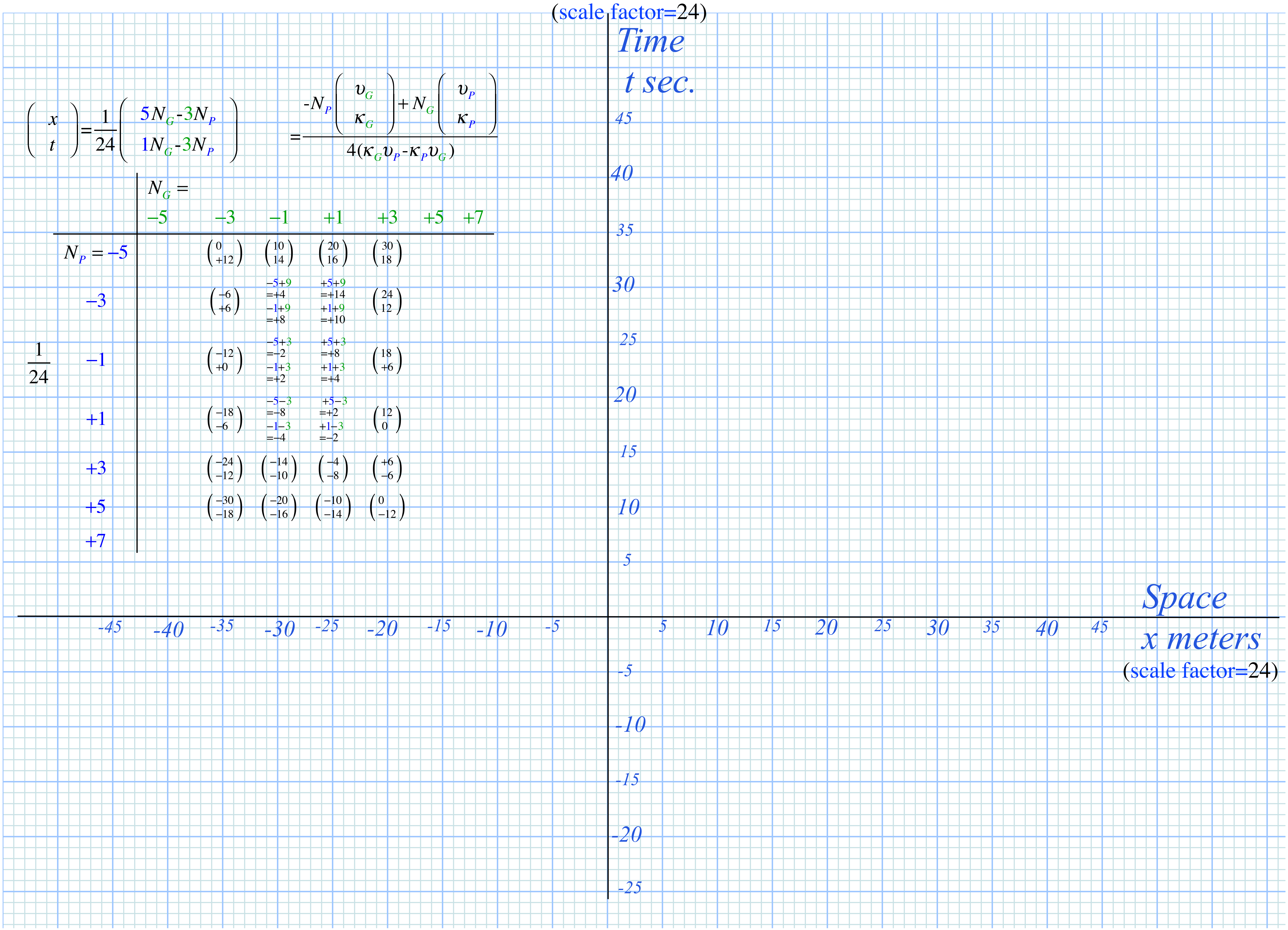
	$N_G =$						
	−5	−3	−1	+1	+3	+5	+7
$N_P = -5$							
−3							
−1			−5+3 =−2 −1+3 =+2	+5+3 =+8 +1+3 =+4			
+1			−5−3 =−8 −1−3 =−4	+5−3 =+2 +1−3 =−2			
+3							
+5							
+7							

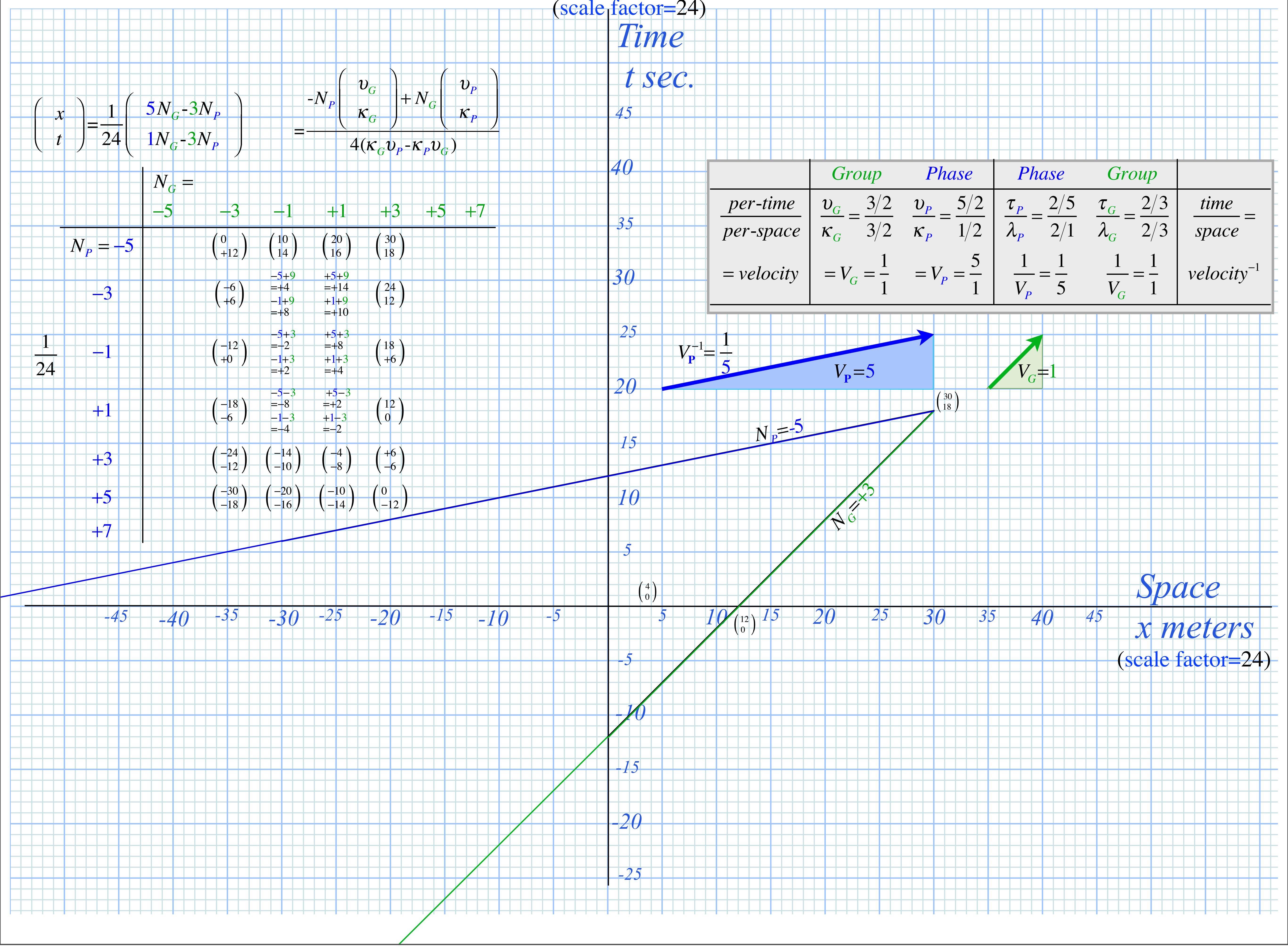


$$\begin{pmatrix} x \\ t \end{pmatrix} = \frac{1}{24}\begin{pmatrix} 5N_G - 3N_P \\ 1N_G - 3N_P \end{pmatrix} = \frac{-N_P\begin{pmatrix} \upsilon_G \\ \kappa_G \end{pmatrix} + N_G\begin{pmatrix} \upsilon_P \\ \kappa_P \end{pmatrix}}{4(\kappa_G \upsilon_P - \kappa_P \upsilon_G)}$$

$\frac{1}{24}$

	$N_G =$						
	−5	−3	−1	+1	+3	+5	+7
$N_P = -5$		$\begin{pmatrix} 0 \\ +12 \end{pmatrix}$	$\begin{pmatrix} 10 \\ 14 \end{pmatrix}$	$\begin{pmatrix} 20 \\ 16 \end{pmatrix}$	$\begin{pmatrix} 30 \\ 18 \end{pmatrix}$		
−3		$\begin{pmatrix} -6 \\ +6 \end{pmatrix}$	−5+9 =+4 −1+9 =+8	+5+9 =+14 +1+9 =+10	$\begin{pmatrix} 24 \\ 12 \end{pmatrix}$		
−1		$\begin{pmatrix} -12 \\ +0 \end{pmatrix}$	−5+3 =−2 −1+3 =+2	+5+3 =+8 +1+3 =+4	$\begin{pmatrix} 18 \\ +6 \end{pmatrix}$		
+1		$\begin{pmatrix} -18 \\ -6 \end{pmatrix}$	−5−3 =−8 −1−3 =−4	+5−3 =+2 +1−3 =−2	$\begin{pmatrix} 12 \\ 0 \end{pmatrix}$		
+3		$\begin{pmatrix} -24 \\ -12 \end{pmatrix}$	$\begin{pmatrix} -14 \\ -10 \end{pmatrix}$	$\begin{pmatrix} -4 \\ -8 \end{pmatrix}$	$\begin{pmatrix} +6 \\ -6 \end{pmatrix}$		
+5		$\begin{pmatrix} -30 \\ -18 \end{pmatrix}$	$\begin{pmatrix} -20 \\ -16 \end{pmatrix}$	$\begin{pmatrix} -10 \\ -14 \end{pmatrix}$	$\begin{pmatrix} 0 \\ -12 \end{pmatrix}$		
+7							





$$\begin{pmatrix}x\\t\end{pmatrix}=\frac{1}{24}\begin{pmatrix}5N_G-3N_P\\1N_G-3N_P\end{pmatrix}=\frac{-N_P\begin{pmatrix}\upsilon_G\\\kappa_G\end{pmatrix}+N_G\begin{pmatrix}\upsilon_P\\\kappa_P\end{pmatrix}}{4(\kappa_G\upsilon_P-\kappa_P\upsilon_G)}$$

$\frac{1}{24}$

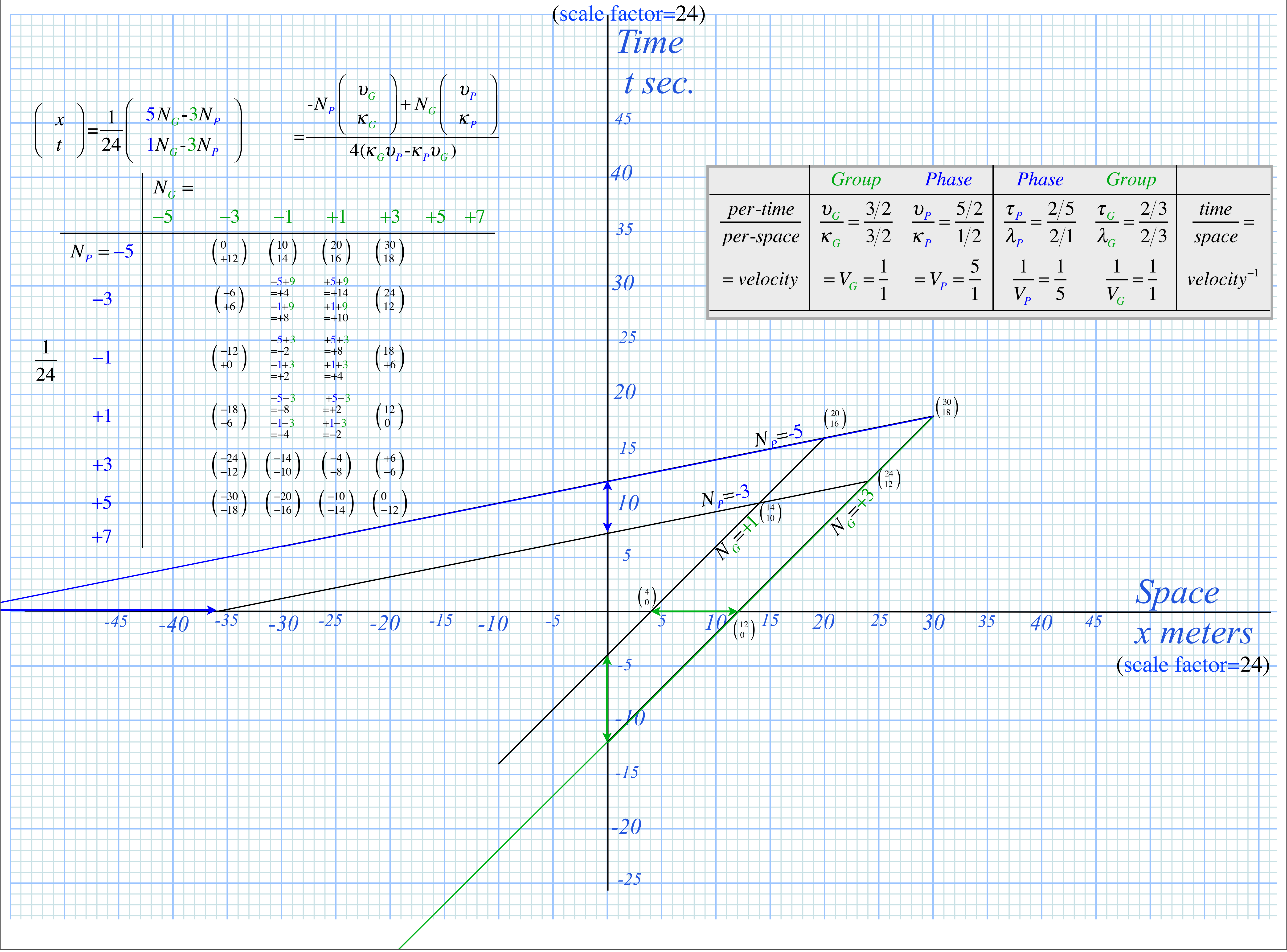
	$N_G=$ -5	-3	-1	+1	+3	+5	+7
$N_P=-5$		$\begin{pmatrix}0\\+12\end{pmatrix}$	$\begin{pmatrix}10\\14\end{pmatrix}$	$\begin{pmatrix}20\\16\end{pmatrix}$	$\begin{pmatrix}30\\18\end{pmatrix}$		
-3		$\begin{pmatrix}-6\\+6\end{pmatrix}$	$-5+9$ $=+4$ $-1+9$ $=+8$	$+5+9$ $=+14$ $+1+9$ $=+10$	$\begin{pmatrix}24\\12\end{pmatrix}$		
-1		$\begin{pmatrix}-12\\+0\end{pmatrix}$	$-5+3$ $=-2$ $-1+3$ $=+2$	$+5+3$ $=+8$ $+1+3$ $=+4$	$\begin{pmatrix}18\\+6\end{pmatrix}$		
+1		$\begin{pmatrix}-18\\-6\end{pmatrix}$	$-5-3$ $=-8$ $-1-3$ $=-4$	$+5-3$ $=+2$ $+1-3$ $=-2$	$\begin{pmatrix}12\\0\end{pmatrix}$		
+3		$\begin{pmatrix}-24\\-12\end{pmatrix}$	$\begin{pmatrix}-14\\-10\end{pmatrix}$	$\begin{pmatrix}-4\\-8\end{pmatrix}$	$\begin{pmatrix}+6\\-6\end{pmatrix}$		
+5		$\begin{pmatrix}-30\\-18\end{pmatrix}$	$\begin{pmatrix}-20\\-16\end{pmatrix}$	$\begin{pmatrix}-10\\-14\end{pmatrix}$	$\begin{pmatrix}0\\-12\end{pmatrix}$		
+7							

	Group	Phase	Phase	Group	
$\frac{\text{per-time}}{\text{per-space}}$ $=$ velocity	$\frac{\upsilon_G}{\kappa_G}=\frac{3/2}{3/2}$ $=V_G=\frac{1}{1}$	$\frac{\upsilon_P}{\kappa_P}=\frac{5/2}{1/2}$ $=V_P=\frac{5}{1}$	$\frac{\tau_P}{\lambda_P}=\frac{2/5}{2/1}$ $\frac{1}{V_P}=\frac{1}{5}$	$\frac{\tau_G}{\lambda_G}=\frac{2/3}{2/3}$ $\frac{1}{V_G}=\frac{1}{1}$	$\frac{\text{time}}{\text{space}}=$ velocity^{-1}

$$\begin{pmatrix} x \\ t \end{pmatrix} = \frac{1}{24}\begin{pmatrix} 5N_G\text{-}3N_P \\ 1N_G\text{-}3N_P \end{pmatrix} = \frac{-N_P\begin{pmatrix} \upsilon_G \\ \kappa_G \end{pmatrix} + N_G\begin{pmatrix} \upsilon_P \\ \kappa_P \end{pmatrix}}{4(\kappa_G\upsilon_P\text{-}\kappa_P\upsilon_G)}$$

		$N_G = -5$	-3	-1	$+1$	$+3$	$+5$	$+7$
	$N_P = -5$		$\begin{pmatrix} 0 \\ +12 \end{pmatrix}$	$\begin{pmatrix} 10 \\ 14 \end{pmatrix}$	$\begin{pmatrix} 20 \\ 16 \end{pmatrix}$	$\begin{pmatrix} 30 \\ 18 \end{pmatrix}$		
	-3		$\begin{pmatrix} -6 \\ +6 \end{pmatrix}$	$-5+9$ $=+4$ $-1+9$ $=+8$	$+5+9$ $=+14$ $+1+9$ $=+10$	$\begin{pmatrix} 24 \\ 12 \end{pmatrix}$		
$\frac{1}{24}$	-1		$\begin{pmatrix} -12 \\ +0 \end{pmatrix}$	$-5+3$ $=-2$ $-1+3$ $=+2$	$+5+3$ $=+8$ $+1+3$ $=+4$	$\begin{pmatrix} 18 \\ +6 \end{pmatrix}$		
	$+1$		$\begin{pmatrix} -18 \\ -6 \end{pmatrix}$	$-5-3$ $=-8$ $-1-3$ $=-4$	$+5-3$ $=+2$ $+1-3$ $=-2$	$\begin{pmatrix} 12 \\ 0 \end{pmatrix}$		
	$+3$		$\begin{pmatrix} -24 \\ -12 \end{pmatrix}$	$\begin{pmatrix} -14 \\ -10 \end{pmatrix}$	$\begin{pmatrix} -4 \\ -8 \end{pmatrix}$	$\begin{pmatrix} +6 \\ -6 \end{pmatrix}$		
	$+5$		$\begin{pmatrix} -30 \\ -18 \end{pmatrix}$	$\begin{pmatrix} -20 \\ -16 \end{pmatrix}$	$\begin{pmatrix} -10 \\ -14 \end{pmatrix}$	$\begin{pmatrix} 0 \\ -12 \end{pmatrix}$		
	$+7$							

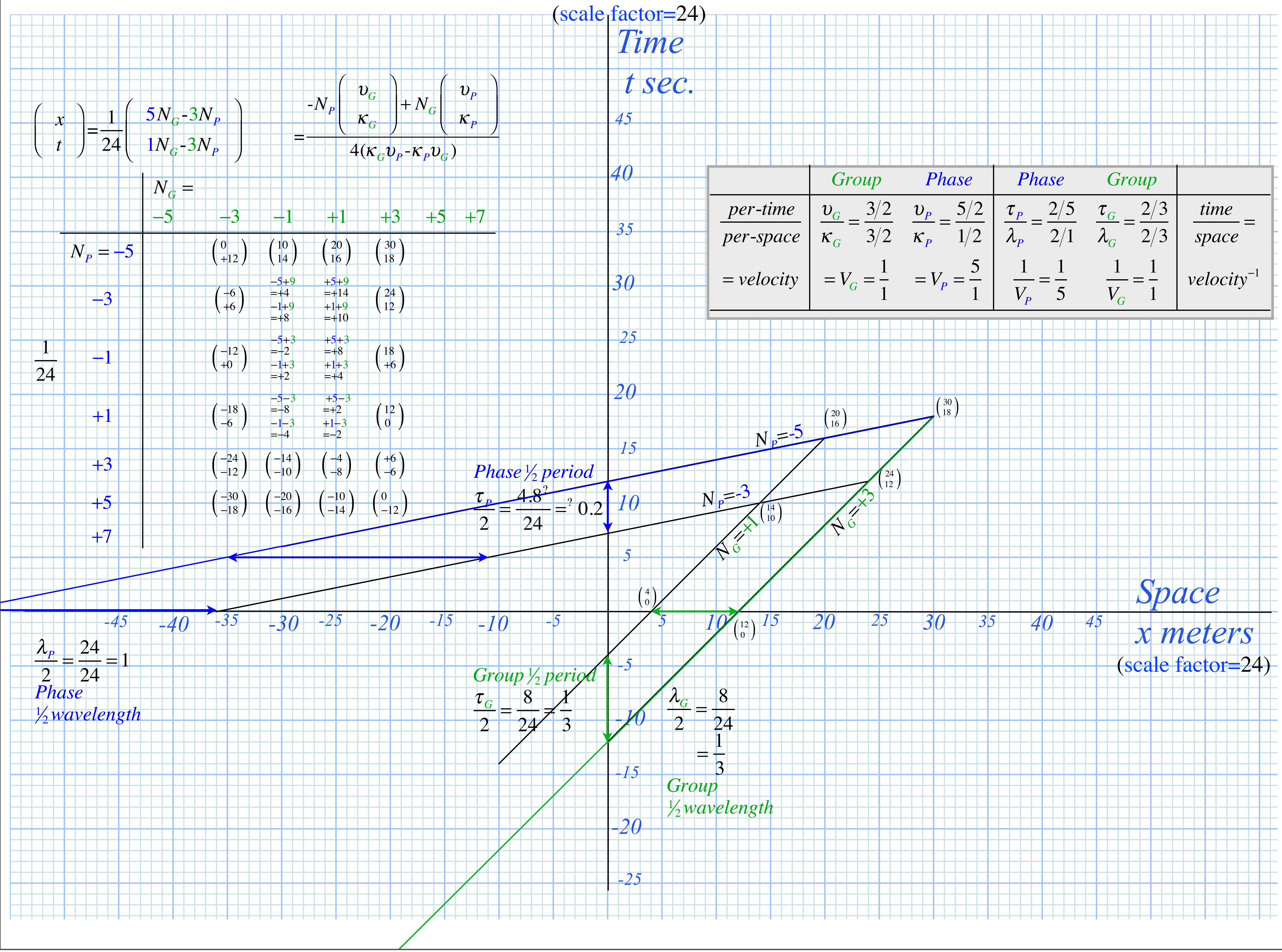
	Group	Phase	Phase	Group	
$\frac{\text{per-time}}{\text{per-space}}$	$\frac{\upsilon_G}{\kappa_G} = \frac{3/2}{3/2}$	$\frac{\upsilon_P}{\kappa_P} = \frac{5/2}{1/2}$	$\frac{\tau_P}{\lambda_P} = \frac{2/5}{2/1}$	$\frac{\tau_G}{\lambda_G} = \frac{2/3}{2/3}$	$\frac{\text{time}}{\text{space}} =$
$= \text{velocity}$	$= V_G = \frac{1}{1}$	$= V_P = \frac{5}{1}$	$\frac{1}{V_P} = \frac{1}{5}$	$\frac{1}{V_G} = \frac{1}{1}$	velocity^{-1}



$$\begin{pmatrix} x \\ t \end{pmatrix} = \frac{1}{24}\begin{pmatrix} 5N_G - 3N_P \\ 1N_G - 3N_P \end{pmatrix} = \frac{-N_P\begin{pmatrix} \upsilon_G \\ \kappa_G \end{pmatrix} + N_G\begin{pmatrix} \upsilon_P \\ \kappa_P \end{pmatrix}}{4(\kappa_G\upsilon_P - \kappa_P\upsilon_G)}$$

		$N_G =$						
		-5	-3	-1	$+1$	$+3$	$+5$	$+7$
	$N_P = -5$		$\begin{pmatrix} 0 \\ +12 \end{pmatrix}$	$\begin{pmatrix} 10 \\ 14 \end{pmatrix}$	$\begin{pmatrix} 20 \\ 16 \end{pmatrix}$	$\begin{pmatrix} 30 \\ 18 \end{pmatrix}$		
	-3		$\begin{pmatrix} -6 \\ +6 \end{pmatrix}$	$-5+9$ $=+4$ $-1+9$ $=+8$	$+5+9$ $=+14$ $+1+9$ $=+10$	$\begin{pmatrix} 24 \\ 12 \end{pmatrix}$		
$\frac{1}{24}$	-1		$\begin{pmatrix} -12 \\ +0 \end{pmatrix}$	$-5+3$ $=-2$ $-1+3$ $=+2$	$+5+3$ $=+8$ $+1+3$ $=+4$	$\begin{pmatrix} 18 \\ +6 \end{pmatrix}$		
	$+1$		$\begin{pmatrix} -18 \\ -6 \end{pmatrix}$	$-5-3$ $=-8$ $-1-3$ $=-4$	$+5-3$ $=+2$ $+1-3$ $=-2$	$\begin{pmatrix} 12 \\ 0 \end{pmatrix}$		
	$+3$		$\begin{pmatrix} -24 \\ -12 \end{pmatrix}$	$\begin{pmatrix} -14 \\ -10 \end{pmatrix}$	$\begin{pmatrix} -4 \\ -8 \end{pmatrix}$	$\begin{pmatrix} +6 \\ -6 \end{pmatrix}$		
	$+5$		$\begin{pmatrix} -30 \\ -18 \end{pmatrix}$	$\begin{pmatrix} -20 \\ -16 \end{pmatrix}$	$\begin{pmatrix} -10 \\ -14 \end{pmatrix}$	$\begin{pmatrix} 0 \\ -12 \end{pmatrix}$		
	$+7$							

	Group	Phase	Phase	Group	
$\frac{per\text{-}time}{per\text{-}space}$	$\frac{\upsilon_G}{\kappa_G} = \frac{3/2}{3/2}$	$\frac{\upsilon_P}{\kappa_P} = \frac{5/2}{1/2}$	$\frac{\tau_P}{\lambda_P} = \frac{2/5}{2/1}$	$\frac{\tau_G}{\lambda_G} = \frac{2/3}{2/3}$	$\frac{time}{space} =$
$= velocity$	$= V_G = \frac{1}{1}$	$= V_P = \frac{5}{1}$	$\frac{1}{V_P} = \frac{1}{5}$	$\frac{1}{V_G} = \frac{1}{1}$	$velocity^{-1}$



Phase ½ period

$$\frac{\tau_P}{2} = \frac{4.8^?}{24} =^? 0.2$$

$$\frac{\lambda_P}{2} = \frac{24}{24} = 1$$

Phase
½wavelength

Group ½ period

$$\frac{\tau_G}{2} = \frac{8}{24} = \frac{1}{3}$$

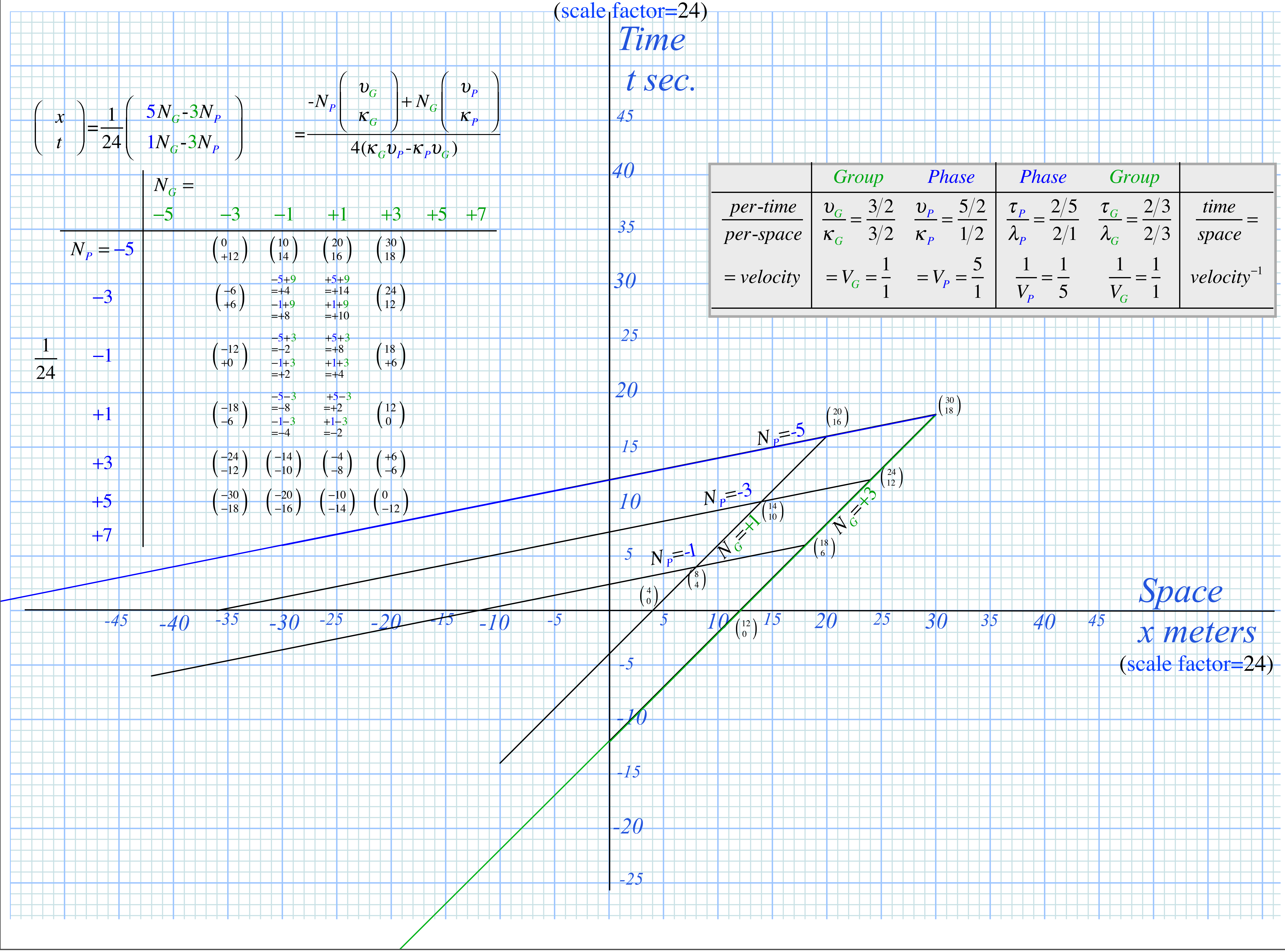
$$\frac{\lambda_G}{2} = \frac{8}{24} = \frac{1}{3}$$

Group
½wavelength

$$\begin{pmatrix}x\\t\end{pmatrix}=\frac{1}{24}\begin{pmatrix}5N_G-3N_P\\1N_G-3N_P\end{pmatrix}=\frac{-N_P\begin{pmatrix}\upsilon_G\\\kappa_G\end{pmatrix}+N_G\begin{pmatrix}\upsilon_P\\\kappa_P\end{pmatrix}}{4(\kappa_G\upsilon_P-\kappa_P\upsilon_G)}$$

		$N_G =$						
		−5	−3	−1	+1	+3	+5	+7
	$N_P = -5$		$\begin{pmatrix}0\\+12\end{pmatrix}$	$\begin{pmatrix}10\\14\end{pmatrix}$	$\begin{pmatrix}20\\16\end{pmatrix}$	$\begin{pmatrix}30\\18\end{pmatrix}$		
	−3		$\begin{pmatrix}-6\\+6\end{pmatrix}$	−5+9 =+4 −1+9 =+8	+5+9 =+14 +1+9 =+10	$\begin{pmatrix}24\\12\end{pmatrix}$		
$\frac{1}{24}$	−1		$\begin{pmatrix}-12\\+0\end{pmatrix}$	−5+3 =−2 −1+3 =+2	+5+3 =+8 +1+3 =+4	$\begin{pmatrix}18\\+6\end{pmatrix}$		
	+1		$\begin{pmatrix}-18\\-6\end{pmatrix}$	−5−3 =−8 −1−3 =−4	+5−3 =+2 +1−3 =−2	$\begin{pmatrix}12\\0\end{pmatrix}$		
	+3		$\begin{pmatrix}-24\\-12\end{pmatrix}$	$\begin{pmatrix}-14\\-10\end{pmatrix}$	$\begin{pmatrix}-4\\-8\end{pmatrix}$	$\begin{pmatrix}+6\\-6\end{pmatrix}$		
	+5		$\begin{pmatrix}-30\\-18\end{pmatrix}$	$\begin{pmatrix}-20\\-16\end{pmatrix}$	$\begin{pmatrix}-10\\-14\end{pmatrix}$	$\begin{pmatrix}0\\-12\end{pmatrix}$		
	+7							

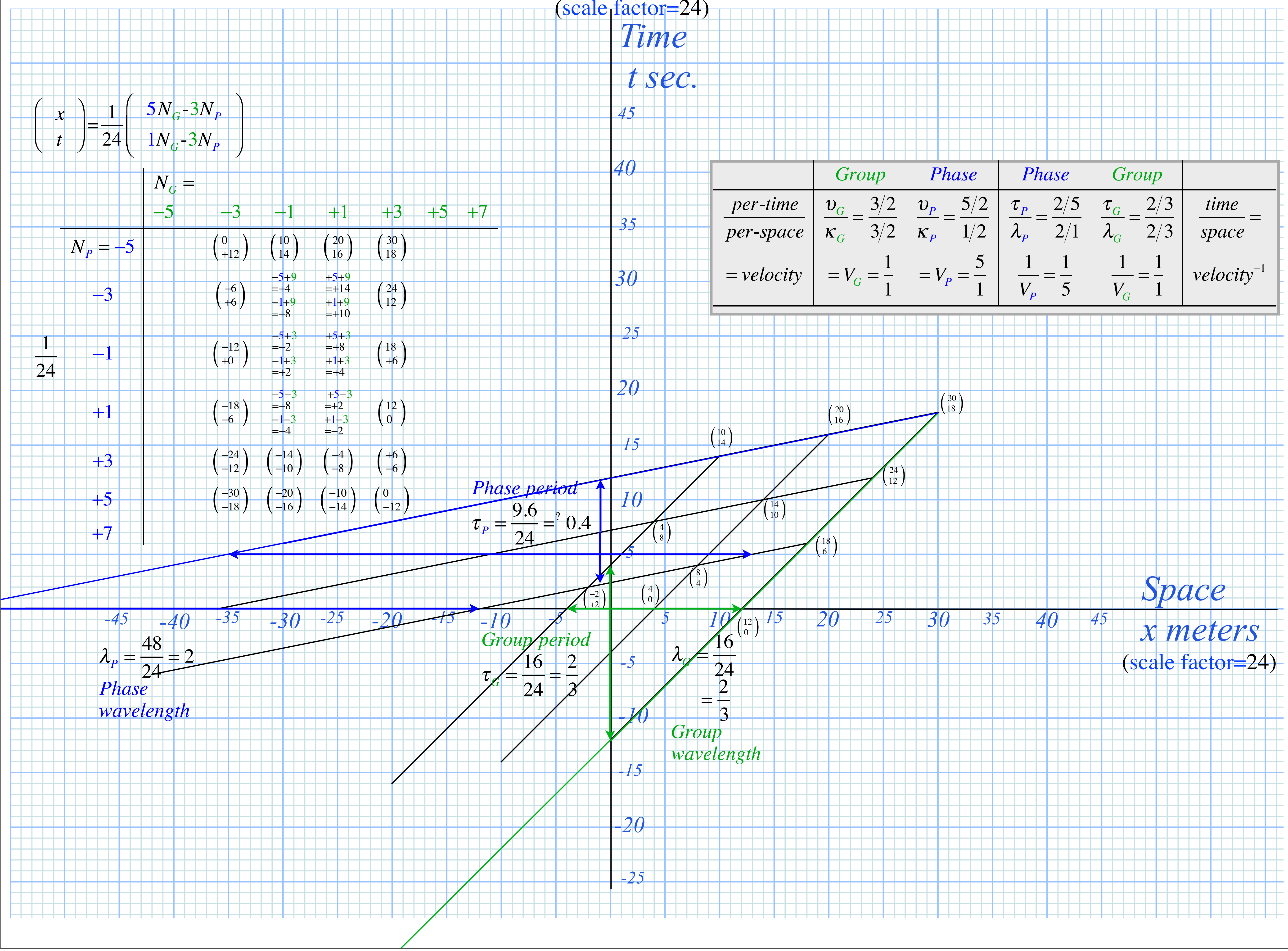
	Group	Phase	Phase	Group	
$\frac{\text{per-time}}{\text{per-space}}$	$\frac{\upsilon_G}{\kappa_G}=\frac{3/2}{3/2}$	$\frac{\upsilon_P}{\kappa_P}=\frac{5/2}{1/2}$	$\frac{\tau_P}{\lambda_P}=\frac{2/5}{2/1}$	$\frac{\tau_G}{\lambda_G}=\frac{2/3}{2/3}$	$\frac{\text{time}}{\text{space}}=$
= velocity	$=V_G=\frac{1}{1}$	$=V_P=\frac{5}{1}$	$\frac{1}{V_P}=\frac{1}{5}$	$\frac{1}{V_G}=\frac{1}{1}$	velocity^{-1}



$$\begin{pmatrix} x \\ t \end{pmatrix} = \frac{1}{24} \begin{pmatrix} 5N_G\text{-}3N_P \\ 1N_G\text{-}3N_P \end{pmatrix}$$

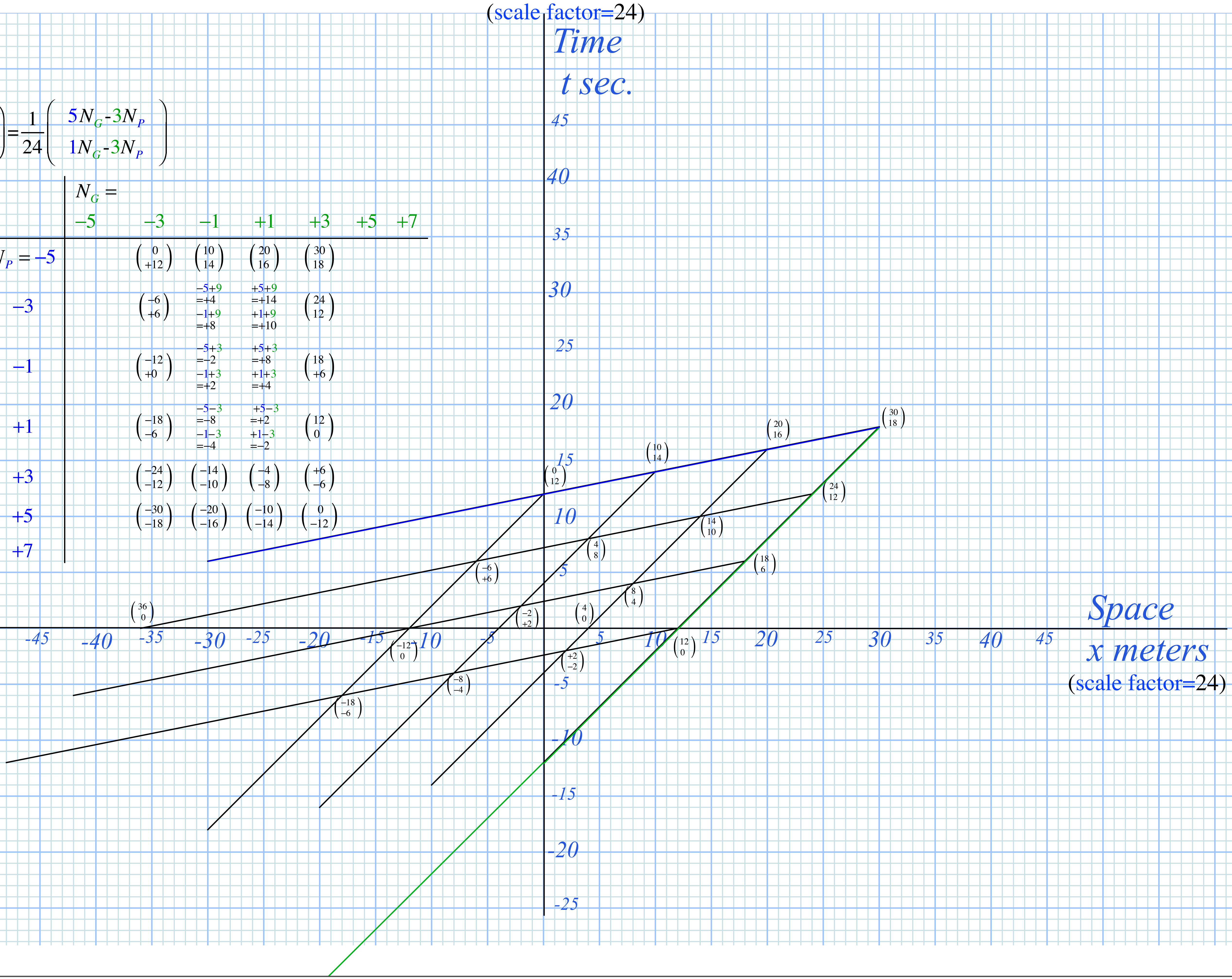
		$N_G =$						
		-5	-3	-1	$+1$	$+3$	$+5$	$+7$
	$N_P = -5$		$\begin{pmatrix}0\\+12\end{pmatrix}$	$\begin{pmatrix}10\\14\end{pmatrix}$	$\begin{pmatrix}20\\16\end{pmatrix}$	$\begin{pmatrix}30\\18\end{pmatrix}$		
	-3		$\begin{pmatrix}-6\\+6\end{pmatrix}$	$-5+9$ $=+4$ $-1+9$ $=+8$	$+5+9$ $=+14$ $+1+9$ $=+10$	$\begin{pmatrix}24\\12\end{pmatrix}$		
$\frac{1}{24}$	-1		$\begin{pmatrix}-12\\+0\end{pmatrix}$	$-5+3$ $=-2$ $-1+3$ $=+2$	$+5+3$ $=+8$ $+1+3$ $=+4$	$\begin{pmatrix}18\\+6\end{pmatrix}$		
	$+1$		$\begin{pmatrix}-18\\-6\end{pmatrix}$	$-5-3$ $=-8$ $-1-3$ $=-4$	$+5-3$ $=+2$ $+1-3$ $=-2$	$\begin{pmatrix}12\\0\end{pmatrix}$		
	$+3$		$\begin{pmatrix}-24\\-12\end{pmatrix}$	$\begin{pmatrix}-14\\-10\end{pmatrix}$	$\begin{pmatrix}-4\\-8\end{pmatrix}$	$\begin{pmatrix}+6\\-6\end{pmatrix}$		
	$+5$		$\begin{pmatrix}-30\\-18\end{pmatrix}$	$\begin{pmatrix}-20\\-16\end{pmatrix}$	$\begin{pmatrix}-10\\-14\end{pmatrix}$	$\begin{pmatrix}0\\-12\end{pmatrix}$		
	$+7$							

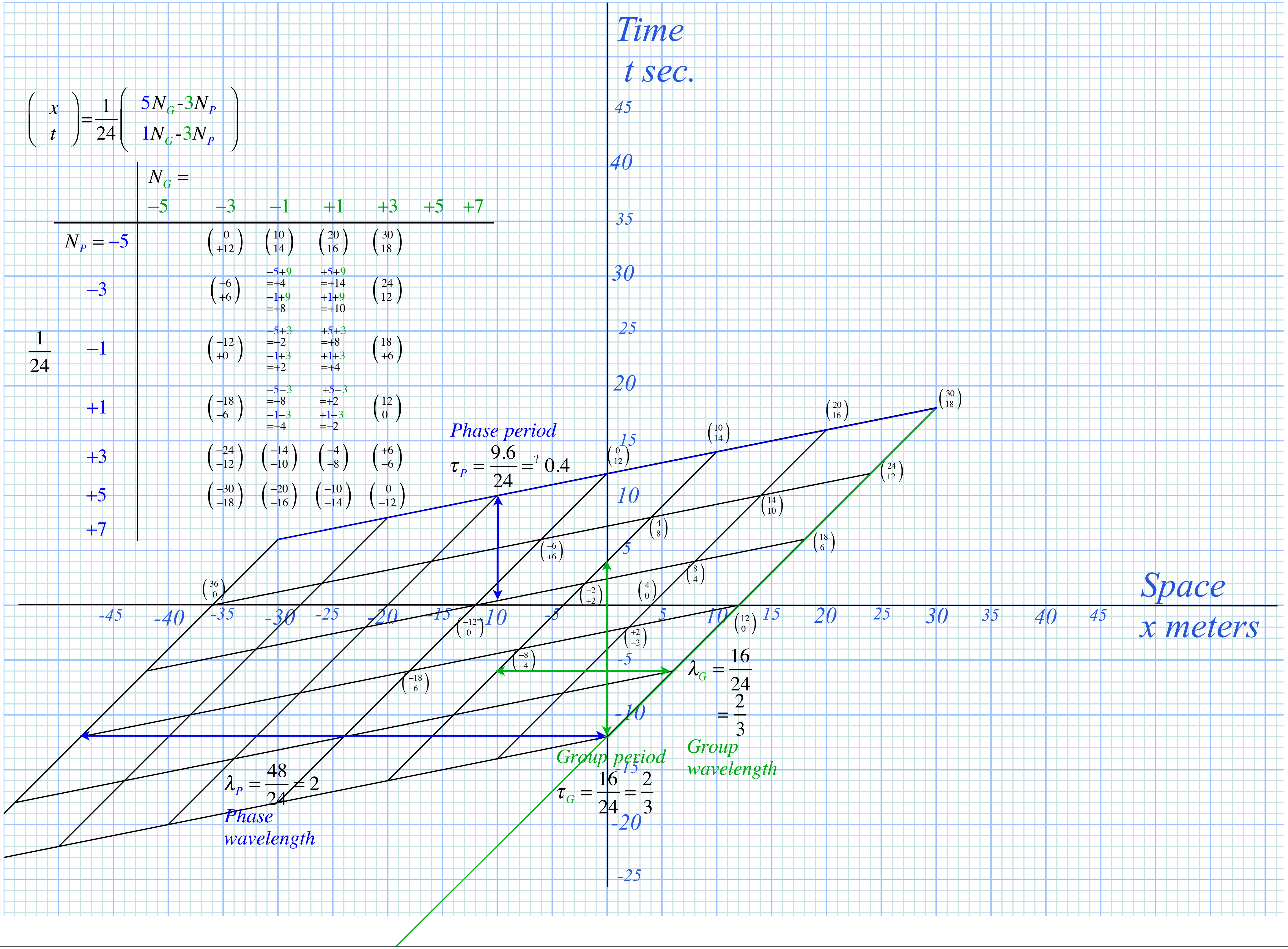
	Group	Phase	Phase	Group	
$\frac{\textit{per-time}}{\textit{per-space}}$	$\frac{\upsilon_G}{\kappa_G} = \frac{3/2}{3/2}$	$\frac{\upsilon_P}{\kappa_P} = \frac{5/2}{1/2}$	$\frac{\tau_P}{\lambda_P} = \frac{2/5}{2/1}$	$\frac{\tau_G}{\lambda_G} = \frac{2/3}{2/3}$	$\frac{\textit{time}}{\textit{space}} =$
$= \textit{velocity}$	$= V_G = \frac{1}{1}$	$= V_P = \frac{5}{1}$	$\frac{1}{V_P} = \frac{1}{5}$	$\frac{1}{V_G} = \frac{1}{1}$	$\textit{velocity}^{-1}$



$$\begin{pmatrix} x \\ t \end{pmatrix} = \frac{1}{24} \begin{pmatrix} 5N_G - 3N_P \\ 1N_G - 3N_P \end{pmatrix}$$

		$N_G =$ -5	-3	-1	$+1$	$+3$	$+5$	$+7$
	$N_P = -5$		$\begin{pmatrix} 0 \\ +12 \end{pmatrix}$	$\begin{pmatrix} 10 \\ 14 \end{pmatrix}$	$\begin{pmatrix} 20 \\ 16 \end{pmatrix}$	$\begin{pmatrix} 30 \\ 18 \end{pmatrix}$		
	-3		$\begin{pmatrix} -6 \\ +6 \end{pmatrix}$	$-5+9$ $=+4$ $-1+9$ $=+8$	$+5+9$ $=+14$ $+1+9$ $=+10$	$\begin{pmatrix} 24 \\ 12 \end{pmatrix}$		
$\frac{1}{24}$	-1		$\begin{pmatrix} -12 \\ +0 \end{pmatrix}$	$-5+3$ $=-2$ $-1+3$ $=+2$	$+5+3$ $=+8$ $+1+3$ $=+4$	$\begin{pmatrix} 18 \\ +6 \end{pmatrix}$		
	$+1$		$\begin{pmatrix} -18 \\ -6 \end{pmatrix}$	$-5-3$ $=-8$ $-1-3$ $=-4$	$+5-3$ $=+2$ $+1-3$ $=-2$	$\begin{pmatrix} 12 \\ 0 \end{pmatrix}$		
	$+3$		$\begin{pmatrix} -24 \\ -12 \end{pmatrix}$	$\begin{pmatrix} -14 \\ -10 \end{pmatrix}$	$\begin{pmatrix} -4 \\ -8 \end{pmatrix}$	$\begin{pmatrix} +6 \\ -6 \end{pmatrix}$		
	$+5$		$\begin{pmatrix} -30 \\ -18 \end{pmatrix}$	$\begin{pmatrix} -20 \\ -16 \end{pmatrix}$	$\begin{pmatrix} -10 \\ -14 \end{pmatrix}$	$\begin{pmatrix} 0 \\ -12 \end{pmatrix}$		
	$+7$							





$$\begin{pmatrix}x\\t\end{pmatrix}=\frac{1}{24}\begin{pmatrix}5N_G\text{-}3N_P\\1N_G\text{-}3N_P\end{pmatrix}$$

		$N_G =$						
		−5	−3	−1	+1	+3	+5	+7
	$N_P = -5$		$\begin{pmatrix}0\\+12\end{pmatrix}$	$\begin{pmatrix}10\\14\end{pmatrix}$	$\begin{pmatrix}20\\16\end{pmatrix}$	$\begin{pmatrix}30\\18\end{pmatrix}$		
	−3		$\begin{pmatrix}-6\\+6\end{pmatrix}$	−5+9 =+4 −1+9 =+8	+5+9 =+14 +1+9 =+10	$\begin{pmatrix}24\\12\end{pmatrix}$		
$\frac{1}{24}$	−1		$\begin{pmatrix}-12\\+0\end{pmatrix}$	−5+3 =−2 −1+3 =+2	+5+3 =+8 +1+3 =+4	$\begin{pmatrix}18\\+6\end{pmatrix}$		
	+1		$\begin{pmatrix}-18\\-6\end{pmatrix}$	−5−3 =−8 −1−3 =−4	+5−3 =+2 +1−3 =−2	$\begin{pmatrix}12\\0\end{pmatrix}$		
	+3		$\begin{pmatrix}-24\\-12\end{pmatrix}$	$\begin{pmatrix}-14\\-10\end{pmatrix}$	$\begin{pmatrix}-4\\-8\end{pmatrix}$	$\begin{pmatrix}+6\\-6\end{pmatrix}$		
	+5		$\begin{pmatrix}-30\\-18\end{pmatrix}$	$\begin{pmatrix}-20\\-16\end{pmatrix}$	$\begin{pmatrix}-10\\-14\end{pmatrix}$	$\begin{pmatrix}0\\-12\end{pmatrix}$		
	+7							

Phase period

$$\tau_P = \frac{9.6}{24} \overset{?}{=} 0.4$$

$$\lambda_G = \frac{16}{24} = \frac{2}{3}$$

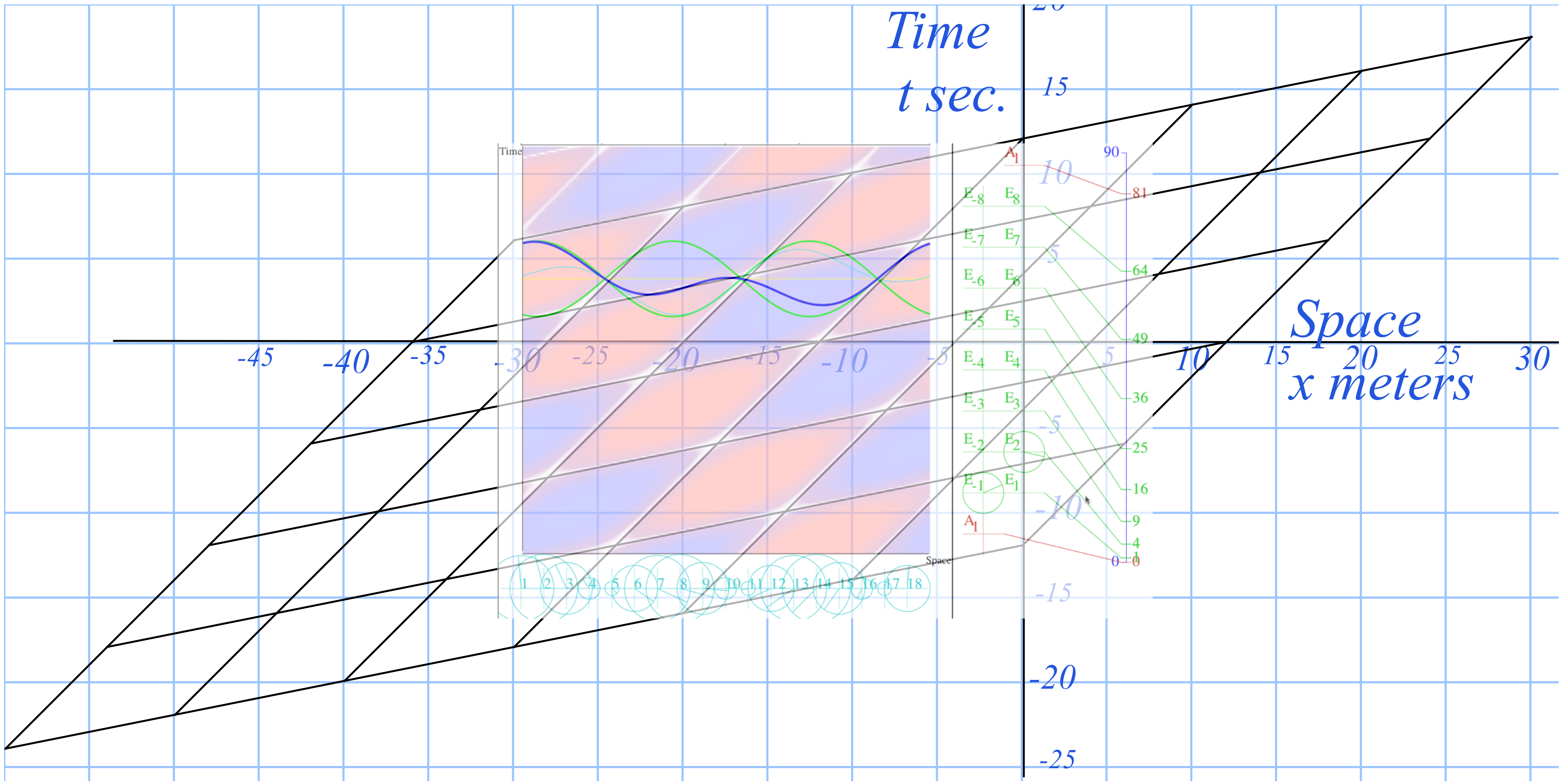
Group period

$$\tau_G = \frac{16}{24} = \frac{2}{3}$$

Group wavelength

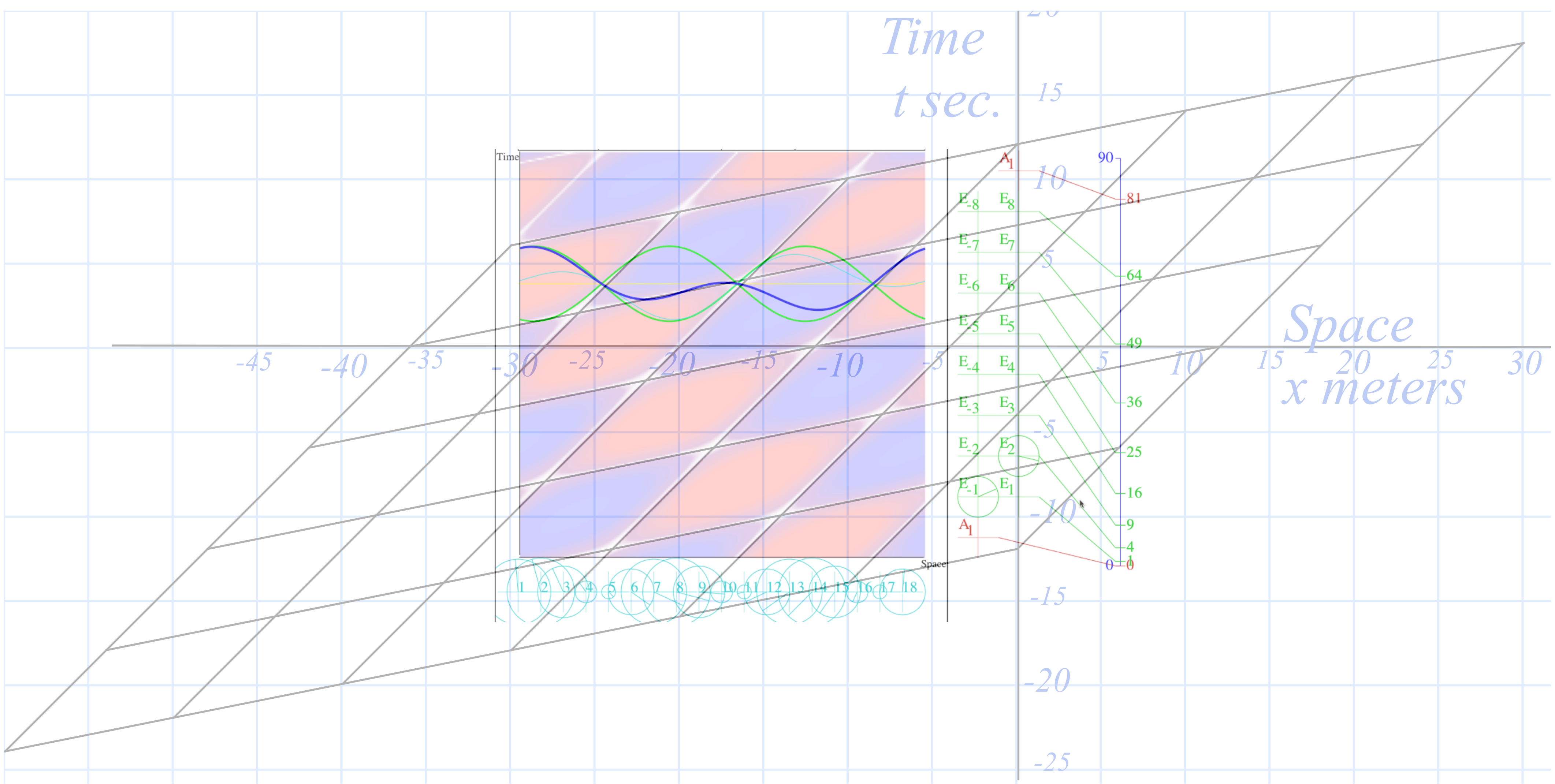
$$\lambda_P = \frac{48}{24} = 2$$

Phase wavelength



BohrIt Web Simulation

Bohr-Schrödinger {Quadratic dispersion} Wave Mixing for k=-1, 2



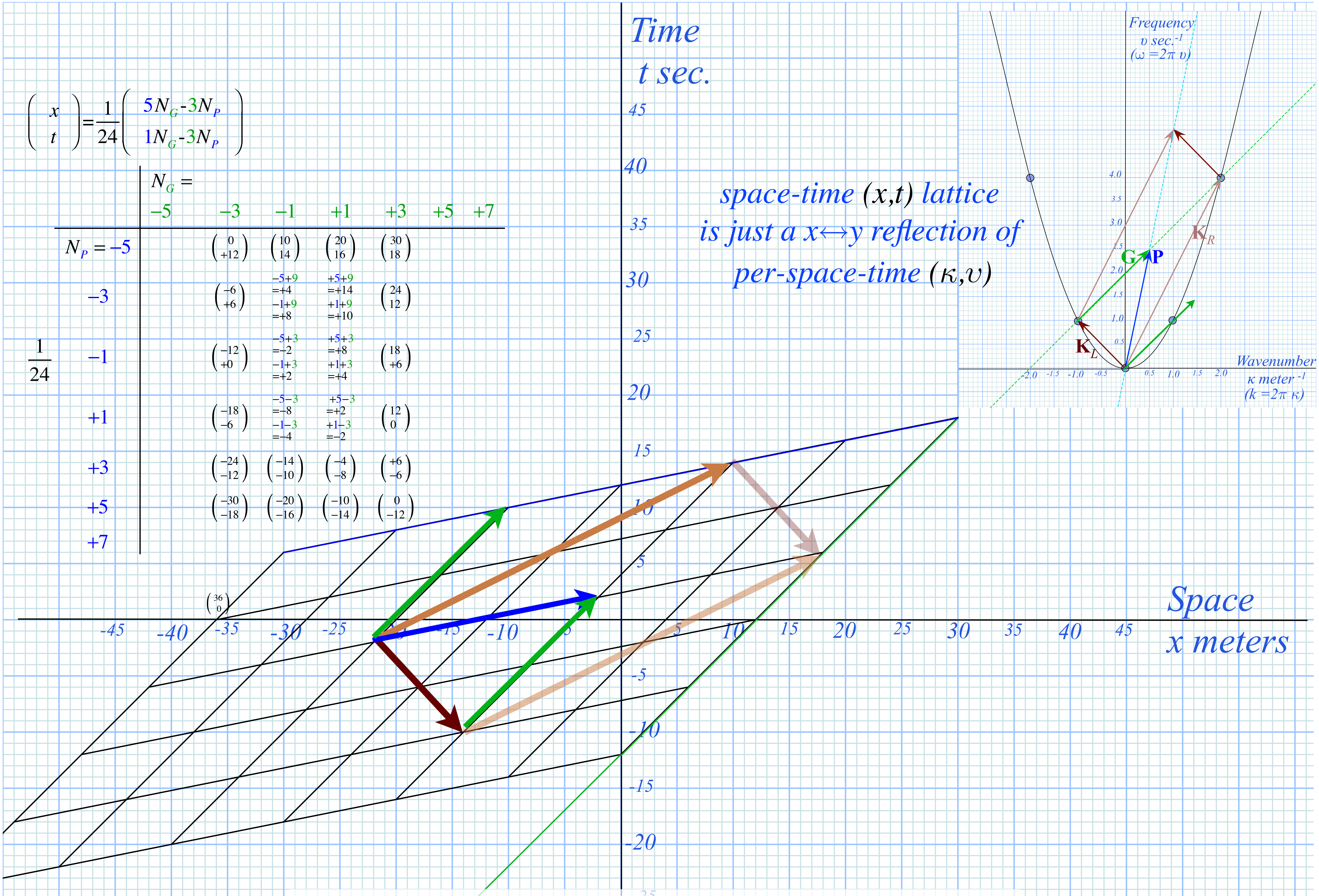
BohrIt Web Simulation

Bohr-Schrödinger {Quadratic dispersion} Wave Mixing for k=-1, 2

space-time (x,t) lattice is just a $x\leftrightarrow y$ reflection of per-space-time (κ,υ)

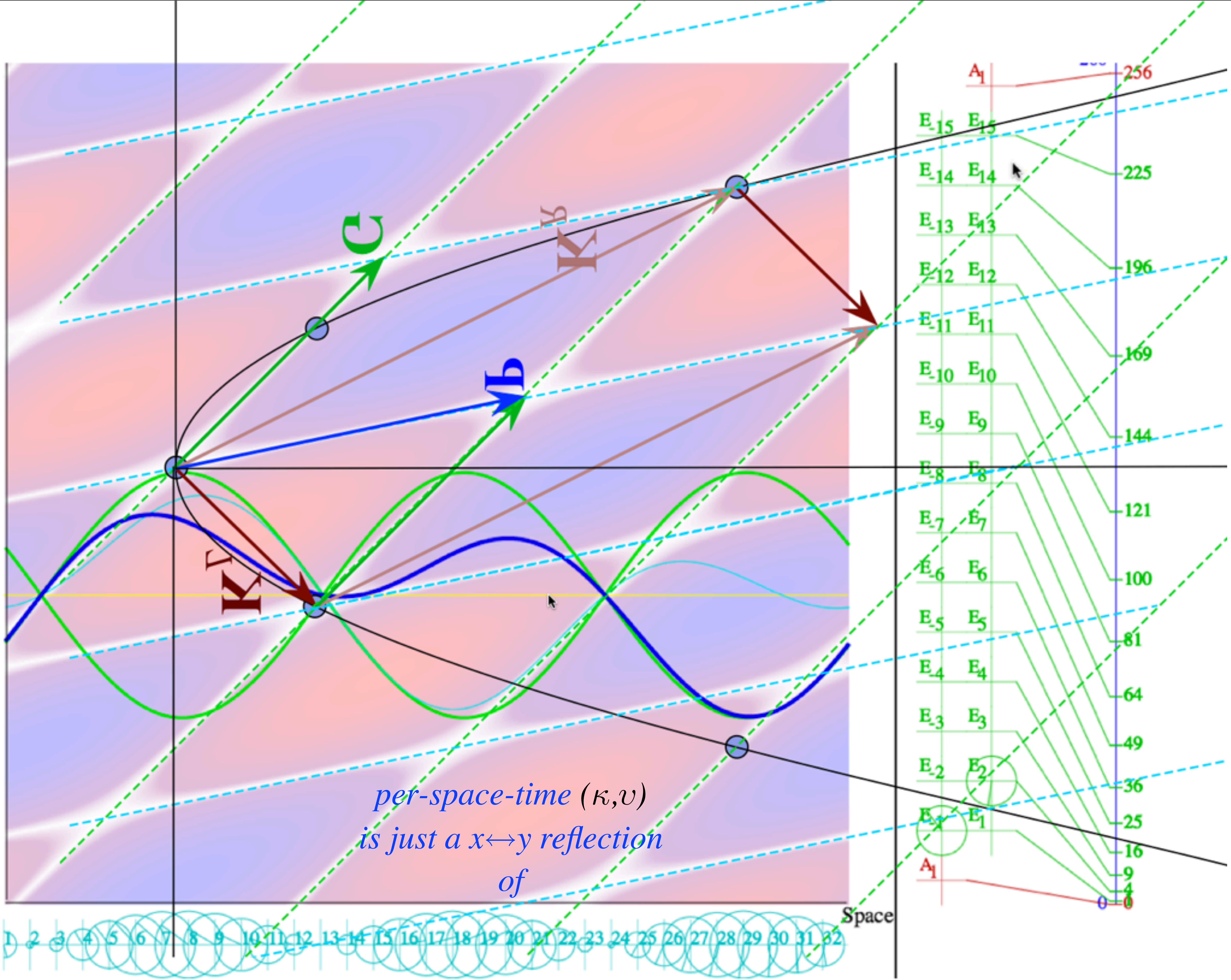
$$\begin{pmatrix} x \\ t \end{pmatrix} = \frac{1}{24}\begin{pmatrix} 5N_G - 3N_P \\ 1N_G - 3N_P \end{pmatrix}$$

		$N_G =$ -5	-3	-1	$+1$	$+3$	$+5$	$+7$
	$N_P = -5$		$\binom{0}{+12}$	$\binom{10}{14}$	$\binom{20}{16}$	$\binom{30}{18}$		
	-3		$\binom{-6}{+6}$	−5+9 =+4 −1+9 =+8	+5+9 =+14 +1+9 =+10	$\binom{24}{12}$		
$\frac{1}{24}$	-1		$\binom{-12}{+0}$	−5+3 =−2 −1+3 =+2	+5+3 =+8 +1+3 =+4	$\binom{18}{+6}$		
	$+1$		$\binom{-18}{-6}$	−5−3 =−8 −1−3 =−4	+5−3 =+2 +1−3 =−2	$\binom{12}{0}$		
	$+3$		$\binom{-24}{-12}$	$\binom{-14}{-10}$	$\binom{-4}{-8}$	$\binom{+6}{-6}$		
	$+5$		$\binom{-30}{-18}$	$\binom{-20}{-16}$	$\binom{-10}{-14}$	$\binom{0}{-12}$		
	$+7$							



BohrIt Web Simulation

Bohr-Schrödinger {Quadratic dispersion} Wave Mixing for k=-1, 2



per-space-time (κ,υ)

is just a x↔y reflection

of

BohrIt Web Simulation

Bohr-Schrödinger {Quadratic dispersion} Wave Mixing for k=-1, 2

Symmetrized finite-difference operators

$$\bar{\Delta} = \frac{1}{2}\begin{pmatrix} \ddots & \vdots & & & & \\ \cdots & 0 & 1 & & & \\ & -1 & 0 & 1 & & \\ & & -1 & 0 & 1 & \\ & & & -1 & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix}, \ \bar{\Delta}^3 = \frac{1}{2^3}\begin{pmatrix} \ddots & \vdots & 0 & -1 & & \\ \cdots & 0 & 3 & 0 & -1 & \\ 0 & -3 & 0 & 3 & 0 & -1 \\ 1 & 0 & -3 & 0 & 3 & 0 \\ & 1 & 0 & -3 & 0 & 3 \\ & & 1 & 0 & -3 & 0 \end{pmatrix}$$

$$\bar{\Delta}^2 = \frac{1}{2^2}\begin{pmatrix} \ddots & \vdots & 1 & & & \\ \cdots & -2 & 0 & 1 & & \\ 1 & 0 & -2 & 0 & 1 & \\ & 1 & 0 & -2 & 0 & 1 \\ & & 1 & 0 & -2 & 0 \\ & & & 1 & 0 & -2 \end{pmatrix}, \bar{\Delta}^4 = \frac{1}{2^4}\begin{pmatrix} \ddots & \vdots & -4 & 0 & 1 & \\ \cdots & 6 & 0 & -4 & 0 & 1 \\ -4 & 0 & 6 & 0 & -4 & 0 \\ 0 & -4 & 0 & 6 & 0 & -4 \\ 1 & 0 & -4 & 0 & 6 & 0 \\ & 1 & 0 & -4 & 0 & 6 \end{pmatrix}$$