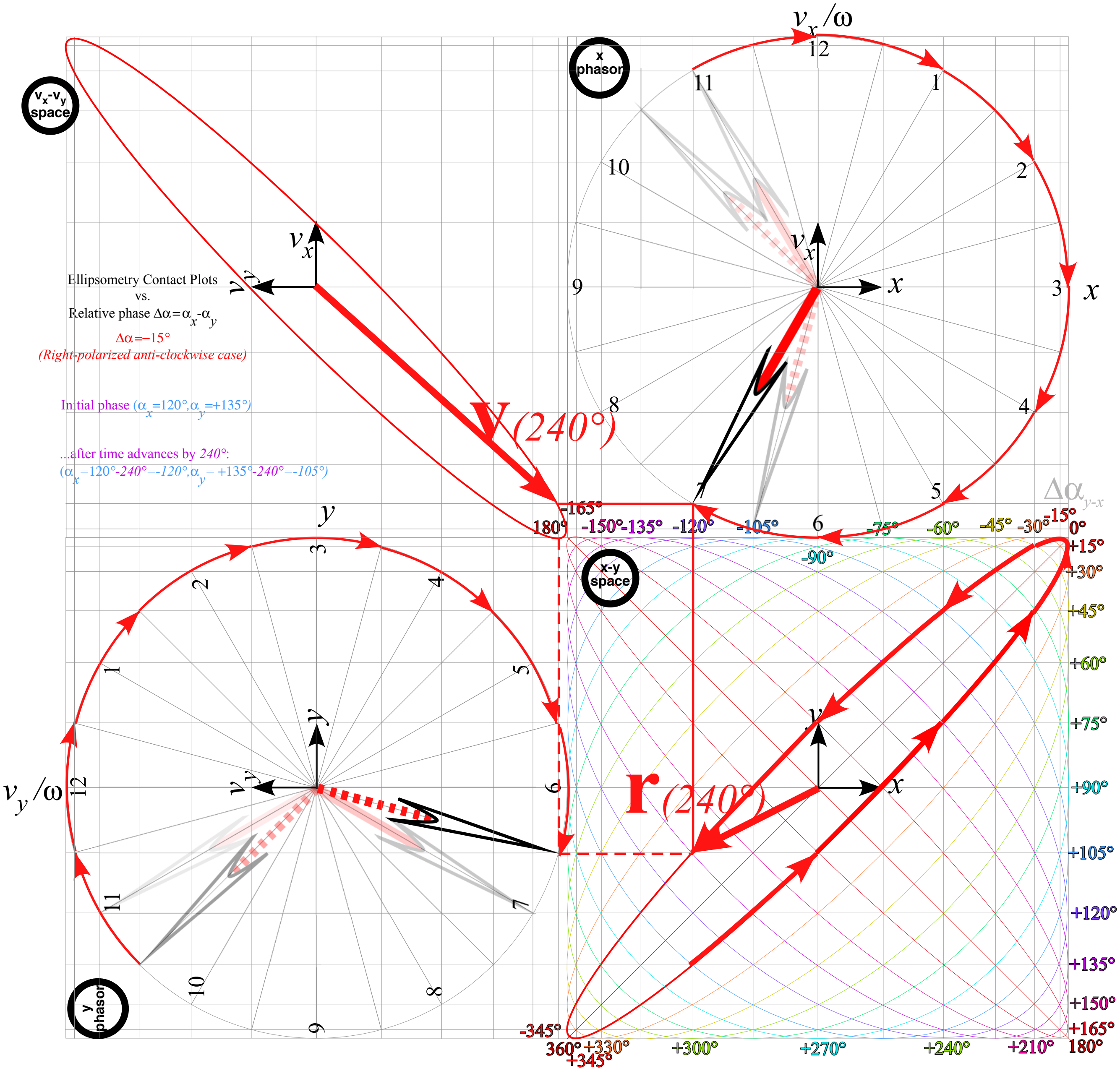


Honors Physics Colloquium

A Classical Mechanical Road to Relativity and Quantum Theory

Part 1

Classical axioms, geometry, and dynamics



Physics 3922H Instructors and Developers
Professor William G. Harter - Text and lecture developer
Dr. Tyle C. Reimer - Computer graphics and animation production
Al Calabrese - Video direction and geometry instruction

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Preface: A back-to-the-future look at the classics

Before beginning a book attempting to merge *classical* mechanics with *quantum* mechanics it should be noted that *classical* mechanics is out of date. For centuries, following work by Galileo and Newton, mechanics *was* physics. No *classical* descriptor was needed. Then along came the *quantum* and *relativity* revolutions started principally by Max Planck (1900) and Albert Einstein (1905). After that we have to distinguish classical mechanics from quantum mechanics.

While classical mechanics may be out of date, it's not obsolete and never will be for things that go *Bang!* or *Click!* or any other acoustical *onomatopoeias*. (Google if unfamiliar.) Our first examples, involving banging cars and balls, are easy *classical* problems but *very* difficult *quantum* problems. Detailed *21st*-century quantum mechanical solutions at even a Joule of energy would require impossible 10^{40} byte computers. Classical mechanics, on the other hand, permits solution by classical Greek computers, that is, a ruler and compass. Quantum mechanics may be more *fundamental* and *elementary* but it is not *easier* since it involves an astronomical increase in number of variables.

This text will begin with material excerpted from the introductory review Unit 1 of our graduate text on classical mechanics. (*Classical Mechanics with a BANG!* is web-available (as are all our texts/ lectures including, eventually, this one) at our own 2016 Honors Physics Colloquium web site. Much effort is going toward making this text and lectures more accessible to undergraduate students, still you may refer to *CMwB* for more detailed treatments.)

The approach herein and by *CMwB* combines Euclidian geometry with Newtonian calculus in ways that Newton did in his *Principia*. However, *21st*-century computer graphics are much better at exposing hidden power of geometry than Newton's tediously engraved *18th*-century figures. With old fashioned engraving, authors had to overlap multiple geometric steps into indecipherable spider-webs. Thus we get a modern impression that the logic of algebra and calculus *always* trumps that of geometry.

Of the many physics books only a few attempt to use analytic geometry to gain derivations, solutions, and most important, *understanding* of physics. Among a few that begin to revive this ancient art are famous CalTech lectures by Richard Feynman and lesser known (*Six Easy Pieces (Persius 1997)*). Also, works by Vladimir Moser and Frank Crawford use geometric developments.

The weapons of math instruction

When your physics fails (as in String theory) it could be you have lousy axioms. If so, it's back to the drawing board. That's how we start this course. It goes *wa-aaay* back to geometry of Thales *(600BCE)* and Euclid *(300BCE)*. You should always ask what tools have survived the test of time and check them out.

Toolbox 1: Euclidian plane geometry (Rule and compass)

Note that Toolbox 1 has a *rule* not the rul*er*. That's in Toolbox 2. A *rule* is just a *straightedge*, a ruler without its inch or *mm* scale. Euclid's pretty strict about this. Formal plane geometry is kind of a game to see how much you can do drawing lines and circles with just these tools. Toolbox 1 has limitations, at least by formal rules of Mr. Euclid. You may have heard that you can't trisect an angle as Mr. Euclid wants it done, formally and *exactly* in a *finite* number of steps. That won't stop us. We'll do that and other "illegal" moves *approximately* and in as few steps as possible using tools below.

Toolbox 2: Navigational geometry (Set 1+ protractor, ruler, divider, parallel rule)

These were the tools used by the Portuguese, Spanish, Dutch, French, and English navigators who were at least indirectly responsible for many of us living where we are. These tools were also used by weekend sailors until the Global Positioning System made obsolete all but six-packs of beer.

Toolbox 3: Analytic geometry (Set 2+ graph paper, algebra, calculus, calculator)

The idea is not to discard algebra and other such formalisms but to *understand* them better. So one of the first things we do with each geometric graph is figure it out using algebra. This is called *analytic geometry* and is one of the quickest ways to understand calculus as applied to physics. This leads to complex algebra and geometry also important to physics. Certainly we include scientific calculators. (Most of these have complex algebra capability.)

Toolbox 4: Computer geometry (Set 3+ high resolution graphics, C++ etc.)

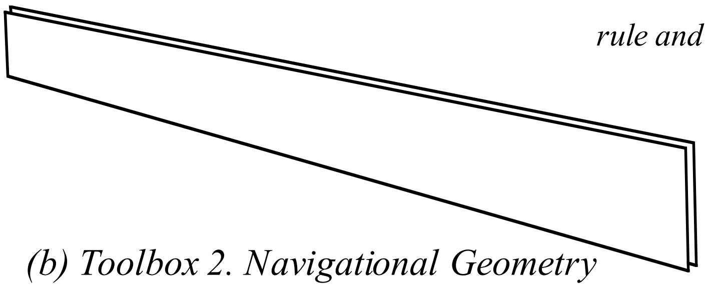
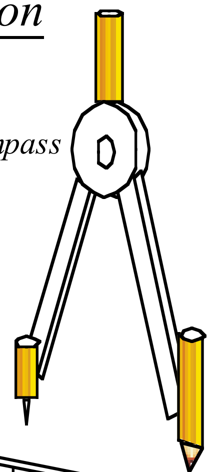
This is the "open" class of geometric analysis, and anything goes. A modern scientist without graphics programming is at a disadvantage. Current languages of greatest general usage, speed, and power are C^{++} and *Objective C* used to write simulations *BounceIt*, *BandIt*, etc. for this book and now root-level HTML to run them on anybody's browser. High-level languages such as Maple™, Mathematica™ are fine, too, though sometimes pricey.

Toolbox 5: You, the algebraic geometer

This is challenging stuff. Doing it will seem hard sometimes. Rome was not built in a day and neither was any understanding of Nature. So this book depends most on how much *you* like thinking and *doing*. Ignorance about science is not a burden you must accept. It is a challenge you should overcome.

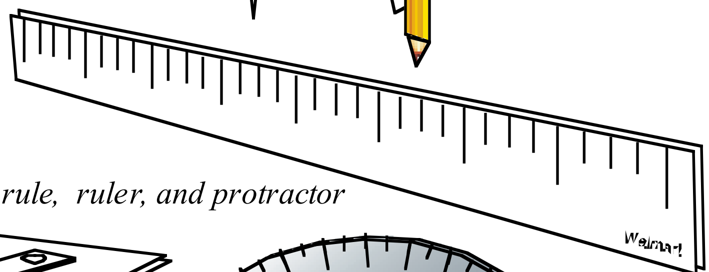
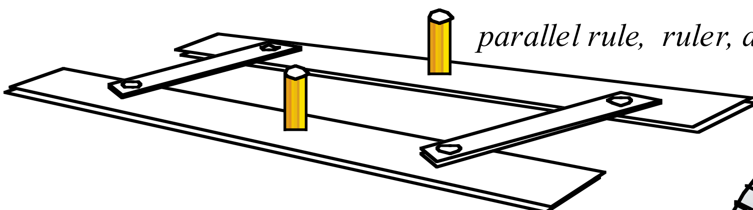
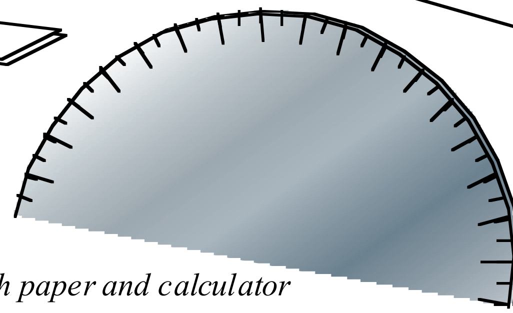
The Weapons of Math Instruction

(a) Toolbox 1. Euclidian Geometry

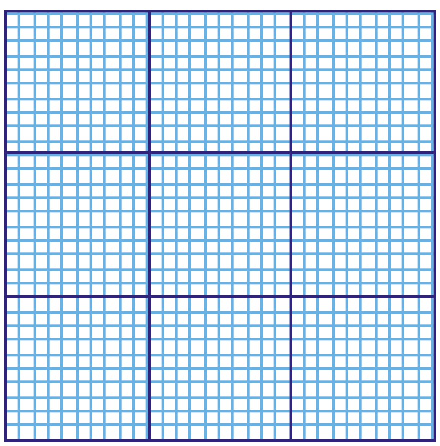
rule and compass

(b) Toolbox 2. Navigational Geometry

parallel rule, ruler, and protractor

(c) Toolbox 3. Analytical geometry

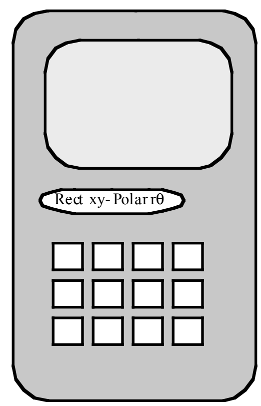


Graph paper and calculator

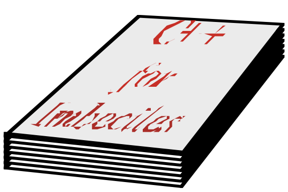
Complex algebra and calculus

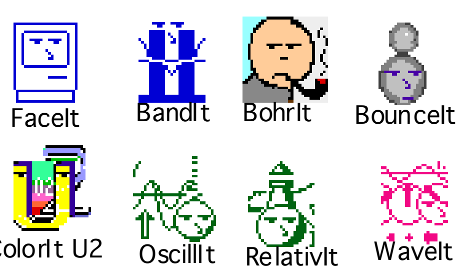
$$1/z = r^{-1} e^{-i\theta}$$

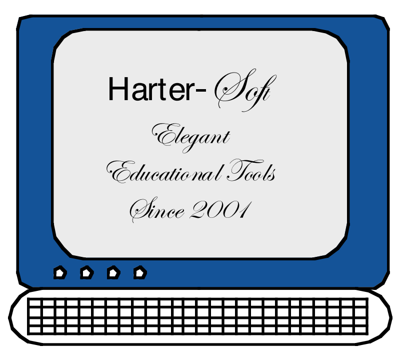
$$\int 1/z \, dz = \ln z$$



(d) Toolbox 4. Computer geometry...Anything goes!







About the computer simulations and future capabilities

The first tier of computer programs used to make figures in this book and provide animated visualizations of physical phenomena or analogies thereof in this book is *LearnIt* series consisting of *BounceIt*, *OscillIt*, *QuantIt*, *WaveIt*, etc. listed in the tables and links below. The idea was to make them like analog computers that allow text figures to become dynamic thought experiments.

The suffix *"It"* attached to most of these programs is derived from the *FaceIt* interface invented by Dan Kampemier of *FaceWare* in Urbana, IL a worldwide programming project I joined in 1985 to 1993. A lot has changed since then! Now with T.C. Reimer begins re-application using *X-Code*, IOS, HTML5, *Mathematica*, and others. One needs a graphical user/programmer interface (GUI or GPI) that can be easily updated with controls, text editors, OpenGL, 3D stereo windows and so forth.

Academic application needs GPI to keep model, control, and view separate to avoid wasting time reinventing the wheel or debugging buttons in class. Teaching *useful* root-level object oriented programming along with physics is possible. Mixing serious academics with coding is coming of age.

GPI's facilitate a tree of programming projects for a given course. Such project trees make up a *CodeIt* system. Eventually, students can use one or more branches of *CodeIt* trees to build their own applications as homework or lab projects, leading to applications of sufficient complexity to aid in their thesis or dissertation research projects. Also, select *CodeIt* applications may be added to the *LearnIt*. Ideally, each *LearnIt* program has an accompanying expository text and/or on-line help hypertext. Listed below are Units 1-3 with some *LearnIt* and *CodeIt* programs that apply to each.

Unit 1 Review of elementary classical mechanics of velocity, momentum, energy, and fields.
BounceIt , *AnalyIt* , *BoxIt* , *CoulIt,* Trebuchet, and *ColorU(2)*.

Unit 2 Oscillations and waves.
OscillIt, *WaveIt, ColorU(2), JerkIt*, *Pendulum*, *Cycloidulum*, and *BoxIt*.

Unit 3 Relativistic mechanics and advanced topics. *RelativIt, Relawavity, BohrIt, and QuantIt*

To make the applications as widely available as possible, the old Fortran, Pascal, C++ *FaceIt* applications for Classic Mac, listed above, were ported to the HTML5 (HyperText Markup Language Version 5), Javascript, and CSS (Cascading Styles Sheets) programming languages. The user interfaces was changed, from those of the *Classic* versions, to accommodate the native hypertext markup language tags; yet much of the functionality previously available only to those running Macintosh computers is now available, at yet higher resolution, to most devices that connect to the internet with a modern web browser.

The following is a list of the LearnIt applications that have been ported to the web as of Jan. 15, 2016.

Links to select LearnIt Web Applications for Physics

BohrIt at *http://www.uark.edu/ua/modphys/markup/BohrItWeb.html*
BounceIt at *http://www.uark.edu/ua/modphys/markup/BounceItWeb.html*
BoxIt at *http://www.uark.edu/ua/modphys/markup/BoxItWeb.html*
CoulIt at *http://www.uark.edu/ua/modphys/markup/CoulItWeb.html*
Cycloidulum at *http://www.uark.edu/ua/modphys/markup/CycloidulumWeb.html*
JerkIt at *http://www.uark.edu/ua/modphys/markup/JerkItWeb.html*
MolVibes at *http://www.uark.edu/ua/modphys/markup/MolVibesWeb.html*
Pendulum at *http://www.uark.edu/ua/modphys/markup/PendulumWeb.html*
QuantIt at *http://www.uark.edu/ua/modphys/markup/QuantItWeb.html*
RelativIt at *http://www.uark.edu/ua/modphys/markup/RelativItWeb.html*
Relativity - 2005 Pirelli Entrant at *http://www.uark.edu/ua/pirelli* Title page Site map
RelaWavity at *http://www.uark.edu/ua/modphys/markup/RelaWavityWeb.html*
Trebuchet at *http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html*
WaveIt at *http://www.uark.edu/ua/modphys/markup/WaveItWeb.html*

For the latest apps available online,

LearnIt Portal Page at *http://www.uark.edu/ua/modphys/markup/LearnItTitlePage.html*
LearnIt Resource Listing at *http://www.uark.edu/ua/modphys*

For the original *2005 Pirelli Multimedia Challenge site for relativity and quantum theory*
URL: https://www.uark.edu/ua/pirelli Title page Site map

Chapter 1. Introducing classical momentum and collisions

Momentum and *energy* are the currency of physics, the "*Bang!*" for the "$*Buck*$" as an old American expression goes (along with all its terrible historical connotations). We'll start with classical momentum (the "*Bang*") and let that discussion lead us to classical energy. (Right now ten *kiloWatt·Hours* of energy costs you about one "Buck" or $1.00 at the local electric power grid.) If you're in a fireworks store near the 4th of July (Independence Day in the USA) you may hear patriot patrons requesting more "*Bang-for-the-Buck*." Similar requests will be heard year-around in the US Congress as DoD lobbyists plead endlessly (and quite successfully) to spend more *GigaBucks-for-the-TeraBang*.

Without some understanding of momentum and energy you can't intelligently discuss either the old classical physics or the modern quantum physics we introduce later on. (And, to intelligently discuss modern and future world politics you are going to need all of the above!)

Some terminology needed to define Momentum

The word *momentum* shares its first two letters "*mo*" with the word *motion.* Both words share first letter "*m*" with the word *mass.* Momentum is defined as a product of the other two. It's our first equation:

$$(\text{momentum } p) = (\text{mass } m)\cdot(\text{motion } v) \quad \text{or:} \quad p=m\cdot v \tag{1.1}$$

Most people can quantify motion as *velocity* in miles per hour (*mi./hr.*) or feet per second (*ft./sec.*) or, in metric units of meters per second (*m/s*) or kilometers per hour (*km/hr*) if one lives outside the US. However, most people outside of the STEM crowd have trouble quantifying *mass* and even some of the STEMites may goof on this.

The metric system of *mks*-units makes it easy with its *kilogram* (*kg*) unit of mass. (*mks* is an abbreviation of *meter-kilogram-second*.) Perhaps, it is ironic that drug dealers of *"kilos"* are little ahead of most in defining mass correctly. (However, a "kilo" of anything weighs 2.2 pounds. That most likely would be a serious overdose!) Sixty or so years ago scientists favored the *cgs*-system of *centimeter-gram-second* units for which *1-gram* is the mass of a cubic-centimeter (*1cc*) of water, a lot smaller dose.

In the British system of units (that US and Cameroon use) the official unit of mass is called the *Slug*. It is <u>not</u> a *pound*, that's a unit of *force* that we'll deal with later. In the meantime, it is ok to say that a slug <u>weighs</u> *32.2lbs*, approximately, but to speak of pounds of mass is a definite *faux pas* in the politely correct STEM society!

Now, if only we could explain what mass <u>is</u> or how something gets massive. It is not helpful to just claim that mass results from eating Higgs-Boson cookies. Instead, a classical theory posits *axioms* (also known as *laws*) that describe what mass <u>does</u>, or more precisely, what it does <u>not</u> do.

The great momentum axiom

Here is a restatement of Newton's *1st Law* or *Axiom* for mass and momentum.

Axiom-1: All mass or masses keep their total momentum until it is changed by some outsider.

The clause before "***until***" sounds like a powerful conservation law. But, then it's undone by any "***outsider***" that comes along to give or take part of this precious "***total momentum***." A little historical context helps to see the law's power. It's actually due to Galileo about a century before Newton. He saw large vessels gliding in ports like Venice or Pisa. So he was able to disabuse the Aristotelian notion that all motion required an outside force to keep dragging it along. Instead, motion (momentum or just plain velocity in this case) is constant until an outside force alters it. If only there had been frictionless air tracks or magnetic levitation gliders for Aristotle to study, then ***Axiom-1*** might have been seen sooner.

Geometry makes momentum axiom-1 more powerful

Let's use ***Axiom 1*** to calculate outcomes of a freeway collision. Normally, you wouldn't try such a calculation until several more weeks of formulas, axioms, or laws. (There are three laws of Newton.) But, we can let geometry simplify this and then re-derive these laws including those for ***energy***.

The stage is set by a distance vs. time graph in Fig. 1.1a showing a *4-ton* SUV going *60 mph* (a mile per minute) toward a rear-end collision with a *1-ton* VW putting along at *10 mph*. The SUV driver is busy with his I-phone text message during the countdown *-60sec.*, *-48sec.*,...,*-12sec.*, *-6sec.*,... until just before collision at *0 sec.* in upper right of the plot. (By then it's way too late to avoid a *Bang!*)

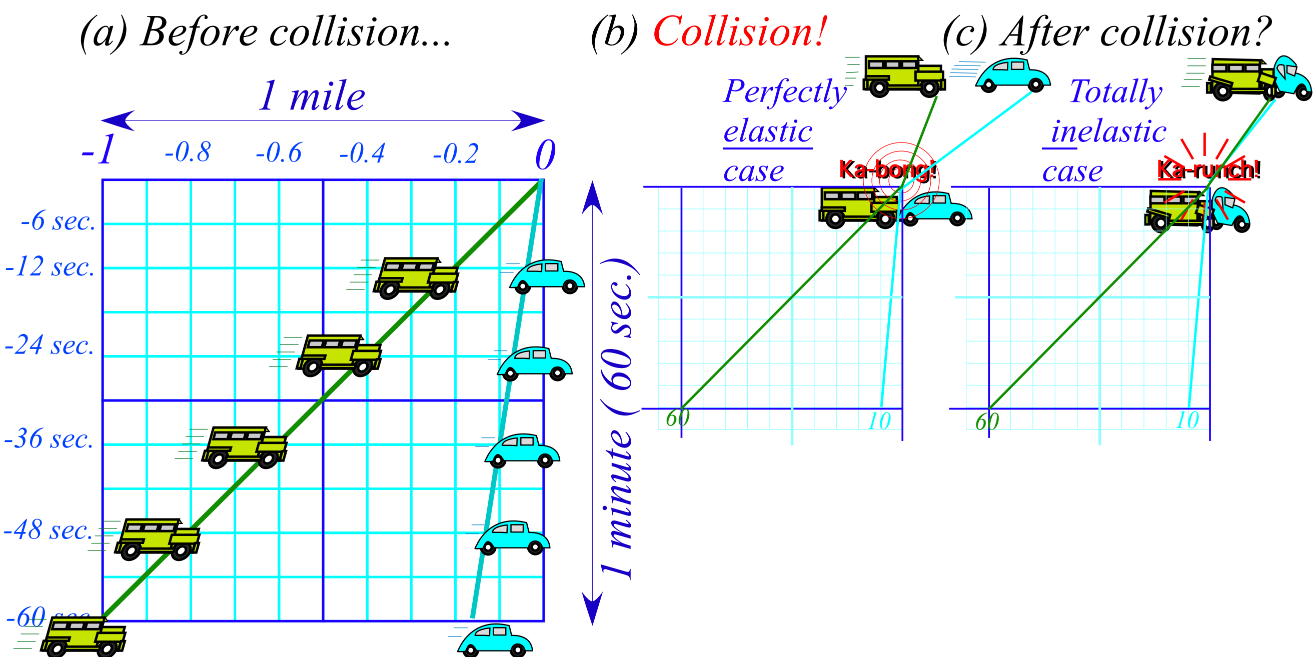


Fig. 1.1 Time vs. space graphs of (a) SUV (going 60mph) and VW (going10mph), (b) collision, and (c) possible outcomes of two extreme cases: the inelastic "ka-runch!" and perfectly elastic "ka-bong!"

To calculate outcomes we need to decide if this collision is a *"ka-runch!"* that welds cars into a single mass (See top right of Fig. 1.1c.) or a *"ka-bong!"* an opposite extreme that bounces them with no damage (very unlikely) as in center Fig. 1.1b. More likely is an intermediate *"ka-whump!"* collision to be detailed later on. The technical term for *ka-runch* is a *totally inelastic collision* that we'll study first followed by the *ka-bong* (technically a *perfectly elastic collision*) and later a generic range of *ka-whumps* or *partially inelastic collisions* that lie between the first two extremes.

However, regardless of whether this collision is a *ka-runch* or a *ka-bong* or even a *ka-whump*, it obeys ***Axiom-1*** with constant total momentum. The sum of SUV and VW momenta is a constant P_{TOTAL}.

$$P_{TOTAL} = M_{SUV}V_{SUV} + M_{VW}V_{VW} = 4\cdot 60 + 1\cdot 10 = 250 = constant \tag{1.2}$$

In the instant of collision, the velocities V_{SUV} and V_{VW} each change very suddenly, but the weighted sum P_{TOTAL} in (1.2) cannot vary. Initial bottom-right point (V_{SUV}, V_{VW})=(*60,10*) on the plot of V_{VW} versus V_{SUV} in *Fig. 1.2a* will move (very suddenly) along a sloping line that points up and to the left as the *ka-runch!* occurs. Whatever momentum the SUV loses, the VW gains so that (1.2) has zero change (ΔP_{TOTAL}=0). Then whatever velocity SUV loses (say, ΔV_{SUV} = -10), the VW picks up *times-four* (ΔV_{VW} = +40).

$$0 \equiv \Delta P_{TOTAL} = M_{SUV}\Delta V_{SUV} + M_{VW}\Delta V_{VW} \quad implies: \quad \Delta V_{VW} = -\frac{M_{SUV}}{M_{VW}}\Delta V_{SUV} = -4\Delta V_{SUV} \tag{1.3}$$

Thus the collision line in *Fig. 1.2a* has a slope that is minus the mass ratio: $-M_{SUV}/M_{VW}$ =-4 . But, where on the collision line in *Fig. 1.2b* does FINAL velocity point stop? It cannot keep rising forever, can it?

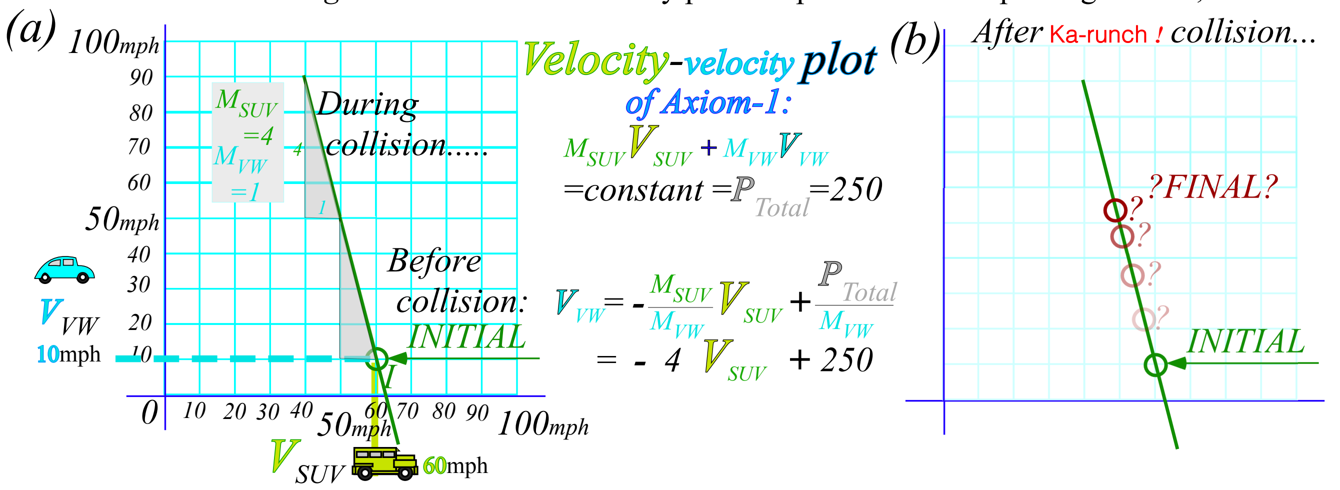


Fig. 1.2(a) (V_{VW}, V_{SUV}) *plot, INITIAL (10,60)-point, and* P_{TOTAL}*-Line (b) Find FINAL* **Ka-runch**! *point??*

If VW and SUV *Ka-runch!* into a single mass their final velocities must be equal ($V_{SUV}^{FINAL}=V_{VW}^{FINAL}$). A *45°* *(y =x)*-line from origin hits the collision line at FINAL *Ka-runch!*-point ($V_{SUV}^{FINAL}= 50 =V_{VW}^{FINAL}$) in Fig. 1.3a. That point is the *Center of Momentum (COM)*. It also locates the FINAL *Ka-Bong!*-point in Fig. 1.3b. You simply strike a compass arc from INITIAL point around COM to hit it at ($V_{SUV}^{FINAL}=40, V_{VW}^{FINAL}=90$). A perfect elastic *Ka-Bong!* has to *Un-krunch* as much coming out of COM as it *krunched* going into COM.

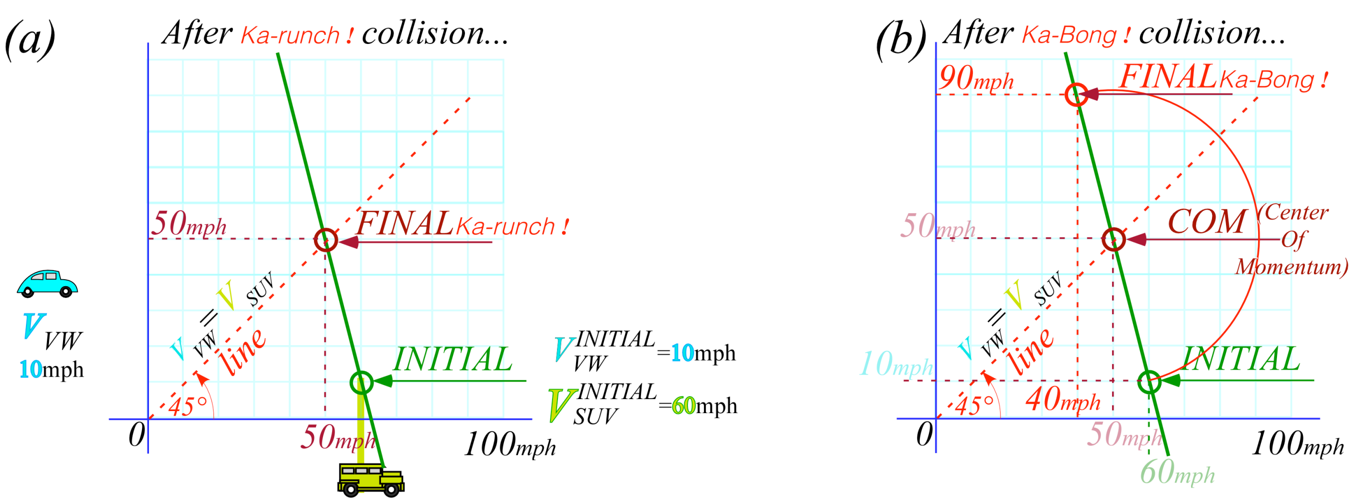


Fig. 1.3(a)Find FINAL **Ka-runch**! *point at 45°. (b)Find FINAL* **Ka-Bong**! *point using circle around COM.*

SUV de-accelerates from *60mph* to *50mph* in the *ka-runch!* and then from *50mph* to *40mph* if it could do a perfectly elastic *ka-Bong!* VW accelerates by 4 times that; from *10mph* to *50mph* in a *ka-runch!* and then from *50mph* to *90mph* if it does a *ka-Bong!* Cars built to crumple and finish with a total *ka-runch!* at COM will suffer half the acceleration that a *ka-Bong!* would entail. A 50% reduction of ***acceleration***, as we'll see, is a 50% reduction of **force** and less passenger injury. By crumpling or *ka-runching*, modern cars waste **energy** that would injure passengers. Our momentum-conservation analysis will let us define and quantify energy-conservation and then see ways to *not* conserve it.

Galilean relativity and space-time symmetry

Galileo grew up near a sea port and observed ships gliding along wharves. We can imagine that he noted how apparent velocity of a ship decreases as you catch up with it. This may seem to be a trivial observation, but replacing the word "apparent" with *relative* begins a theory of *relativity*.

In any case, what we now call *Galilean relativity* posits that if you add a velocity v to yourself (say, by riding on a ship having velocity v) then you subtract that velocity v from (or add $-v$ to) all other objects or phenomena in the universe. Later, when we take up *Einstein relativity*, our first task will be to explain how Galileo's axiom starts to fail for velocity near that of light (c=*299,792,458m/s*), and how chasing a lightwave using a high-speed-v ship never yields the slightest change of the wave's speed c.

Application of Galilean relativity to classical collisions is simpler than that of the Einstein theory particularly if you apply velocity-velocity geometry shown in Fig. 1.4. (Later we find geometric tricks that simplify Einstein relativity, too.) Fig. 1.4a shows a "slide-rule" to transform a *ka-Bong* collision from the initial (V_{SUV}, V_{VW})-or *Earth*-frame view in Fig. 1.3b to that seen in COM-frame. Galileo does it by subtracting velocity vector $\mathbf{V}_{COM}$=(V_{COM}, V_{COM})=(*50,50*) from each *Earth*-frame point. In fig. 1.4a abbreviations F_{Earth} for *FINAL* point (*40,90*), M_{Earth} for *COM* point (*50,50*) and I_{Earth} for *INITIAL* point (*60,10*) apply to those points in Fig. 1.3b. Subtraction gives COM-frame points in Fig.1.4a below.

$$\begin{aligned} F_{COM} &= F_{Earth} - \mathbf{V}_{COM} = (40,90)-(50,50) = (-10,40) \\ M_{COM} &= M_{Earth} - \mathbf{V}_{COM} = (50,50)-(50,50) = (0,0) \\ I_{COM} &= I_{Earth} - \mathbf{V}_{COM} = (60,10)-(50,50) = (10,-40) \end{aligned} \tag{1.4}$$

In COM view final vector $F_{COM}=(-10,40)$ is inversion of initial vector $I_{COM}=(10,-40)=-F_{COM}$ thru M_{COM}.

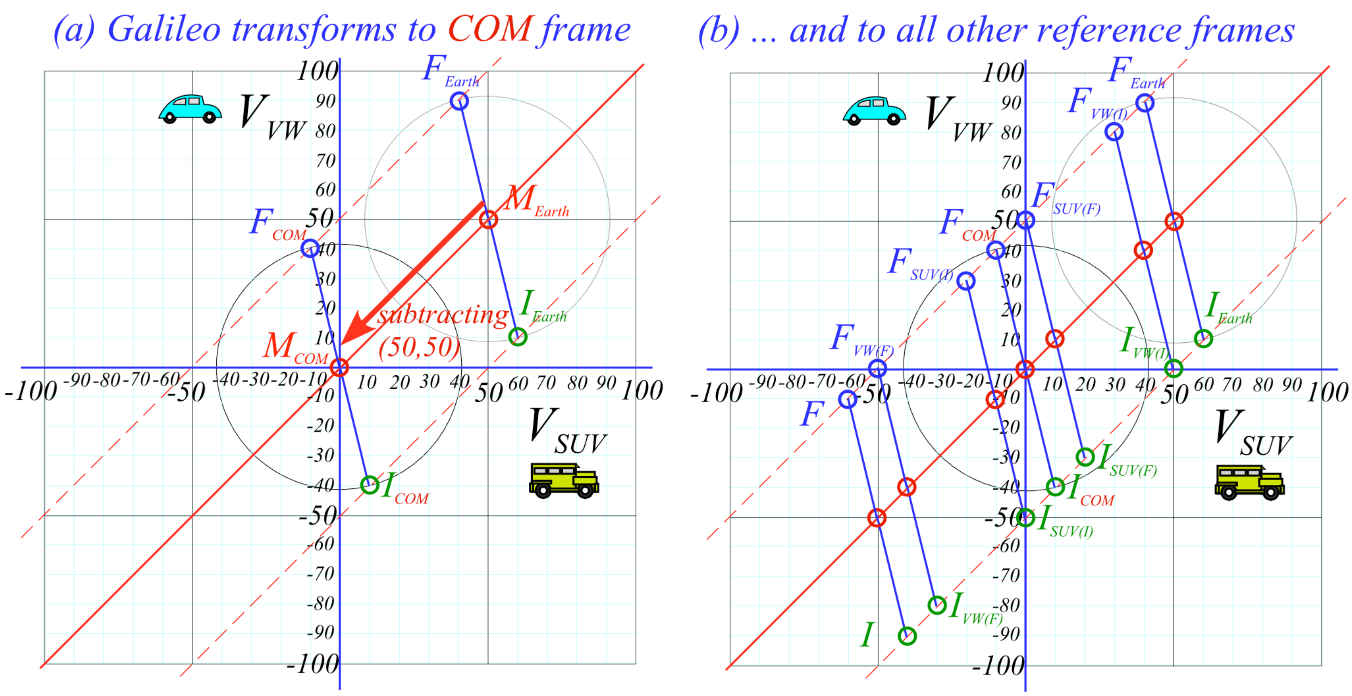


Fig. 1.4 Galilean transform of "KaBong" in Fig. 1.3 to (a) COM-frame and (b) to other frame views

In COM view middle vector M_{COM} (that is the COM point) is reduced from M_{Earth}=(*50,50*) to M_{COM}=(0,0).

Geometry of Balance: Center of Momentum (COM) and Center of Gravity (COG)

The uniqueness and constancy of a *COM* for the SUV and VW is connected with underlying space-time symmetry or geometry of spatial balance in Newton's ***Axiom-1*** (1.2) repeated below in different forms.

$$P_{Total} = P_{SUV} + P_{VW} = M_{SUV}\cdot V_{SUV} + m_{VW}\cdot V_{VW} = M_{TOTAL}\cdot V_{COM} = constant \qquad (1.5a)$$

Total momentum is a product of V_{COM} and total mass $M_{TOTAL}=M_{SUV}+m_{VW}$ of a 5-ton SUV-VW "hunk". This holds whether the "hunk" parts stick in a *Ka-Runch* or the cars bounce off in a *Ka-Bong* or *Ka-whump*. Both $P_{Total} = M_{TOTAL}\cdot V_{COM}$ and V_{COM} are constant throughout the collision regardless of "auto-body-elasticity."

$$V_{COM} = \frac{M_{SUV}\cdot V_{SUV} + m_{VW}\cdot V_{VW}}{M_{SUV}+m_{VW}} = \underset{\text{of } V_{SUV} \text{ and } V_{VW}}{\overset{M_{SUV}:m_{VW}}{\text{weighted average}}} = \frac{constant}{M_{TOTAL}} \qquad (1.5b)$$

Weighted average V_{COM} of (V_{SUV}, V_{VW}) is fixed as V goes from *initial* to *middle* to *final* values. Collisions in Earth frame Fig. 1.3 have $V_{COM}=50$. The *4:1*-weighted average of each coordinate pair *(40,90)*, *(50,50)*, *(60,10)*, *(70,-30)*,*etc.* on the slope-*(-1:4)*-line (in Fig. 1.5a below) is $V_{COM}=50$. In COM view $V_{COM}=0$.

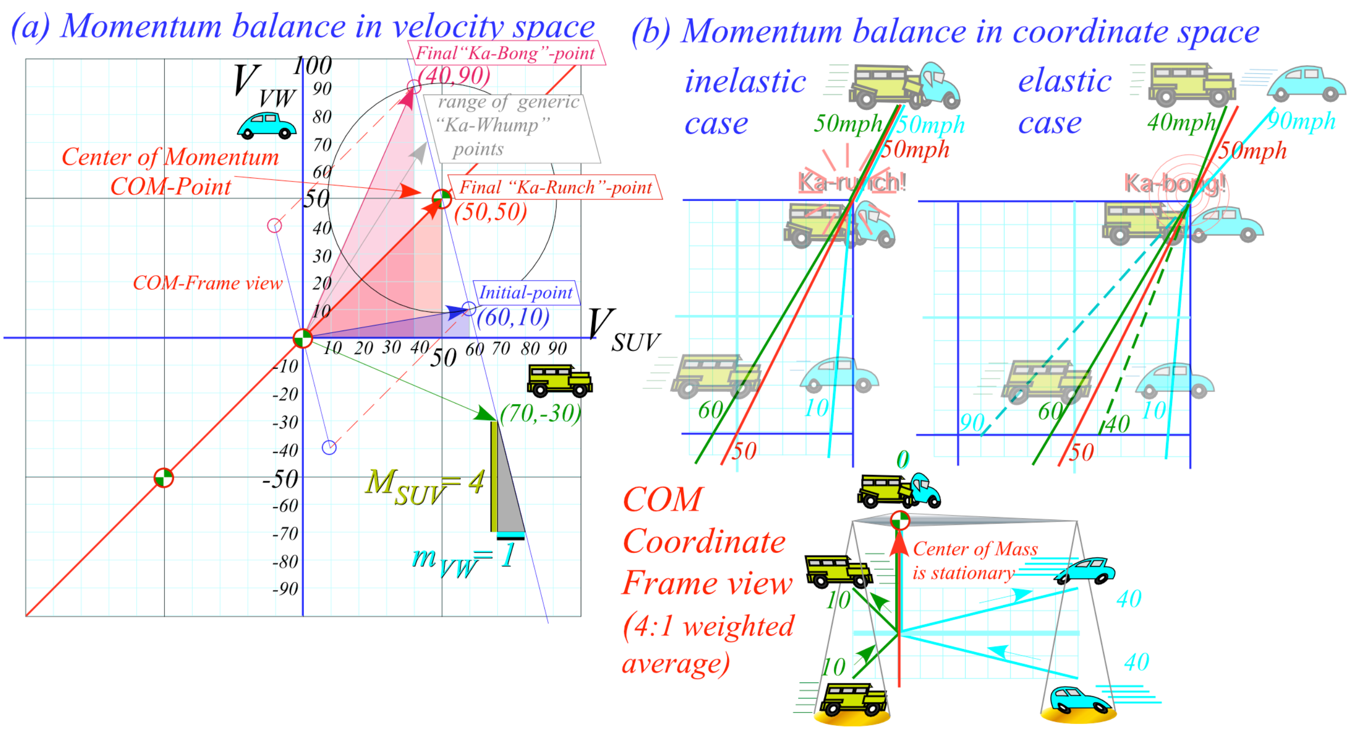


Fig. 1.5 Geometry of (a) 4:1-weighted velocity average (b) 4:1-weighted coordinate average.

Weighted average of *velocity* V_{SUV} and V_{VW} in (2.5b) implies similar balance of *position* x_{SUV} and x_{VW}.

$$x_{COM} = \frac{M_{SUV}\cdot x_{SUV} + m_{VW}\cdot x_{VW}}{M_{SUV}+m_{VW}} = \underset{\text{of } x_{SUV} \text{ and } x_{VW}}{\overset{M_{SUV}:m_{VW}}{\text{weighted average}}} \qquad (1.6)$$

As SUV and VW close, collide, bounce, or stick, the *Center of Mass* x_{COM} maintains a constant velocity V_{COM}. In the *COM* frame V_{COM} is zero as sketched in lower right of Fig. 1.5b. Weighted average x_{COM} in (1.6) of coordinates x_{SUV} and x_{VW} is also called a *Center of Gravity* and is cartooned by a 4:1 balance. Fig. 1.5 sums up momentum analysis of the collision problem posed in Fig. 1.1. Next: energy analyses.

Exercise 1.1 Abstract reasoning about symmetry of collisions

Fig. 1.4b shows the SUV-VW collision in Fig. 1.3 from the viewpoint of six different reference frames (not counting the original Earth-relative frame). Each of the frames belong to three points on a sloping line, two of which are labeled *I*(INITIAL) and *F*(FINAL) with a *COM* point midway between *I* and *F*. (First copy plot in Fig. 1.4b. Then write or diagram answers to the following questions onto plot.)

(a) Each frame moves faster than one to its right. Give direction of motion and speed of each frame.

(b) One of the frames is the IN frame of the VW *before* it was hit. Another frame is the FIN frame for the VW *after* it was hit. Show how *I*, *COM*, and *F* points on one of these frames is related to *F*, *COM*, and *I* points, respectively, on the other frame by a *vector inversion operator* $\mathscr{I}$.

($\mathscr{I}$ is defined by: $\mathscr{I}\cdot(V_{SUV},V_{VW}) = (-V_{SUV},-V_{VW})$. or by: $\mathscr{I}\cdot\mathbf{V}= -\mathbf{V}$)

(c) Do (b) with VW replaced by SUV.

(d) Explain how the COM-frame is the only one that has *inversion symmetry*, that is, is unmoved by $\mathscr{I}$.

(e) In some sense the $\mathscr{I}$-operator acts like a time-reversal operator. Explain.

Exercise 1.2

Plot a $(V_{SUV\text{-}1},V_{SUV\text{-}2})=(60,10)$ collision like Fig. 1.5 but with an identical $M=4$ SUV replacing the VW.

Chapter 2. Vector-tensor algebra of momentum and energy conservation

Classical momentum theory is aided by vector-tensor-matrix notation as is quantum theory. Let us condense $P=M\cdot V$ forms (1.5) into matrix notation. We store mass values M_1, M_2… in a *mass* **M**-*tensor* or *M-matrix*, and define vector $\mathbf{V}=(V_1, V_2,\ldots)$ for velocity and $\mathbf{P}=(P_1, P_2,\ldots)$ for momentum.

$$\left.\begin{array}{l} P_{SUV}=M_{SUV}V_{SUV} \\ P_{VW}=M_{VW}V_{VW} \end{array}\right\} denoted:\vec{\mathbf{P}}=\vec{\vec{\mathbf{M}}}\bullet\vec{\mathbf{V}} \quad or: \begin{pmatrix} P_{SUV} \\ P_{VW} \end{pmatrix}=\begin{pmatrix} M_{SUV} & 0 \\ 0 & M_{VW} \end{pmatrix}\begin{pmatrix} V_{SUV} \\ V_{VW} \end{pmatrix} \tag{2.0a}$$

This generalizes later from 2D $\{1{=}SUV, 2{=}VW\}$ to n-dimensional matrices of n^2 inertial coefficients M_{jk}.

$$\left.\begin{array}{l} P_1=M_{11}V_1+M_{12}V_2+\ldots \\ P_2=M_{21}V_1+M_{22}V_2+\ldots \\ \vdots \quad\quad \vdots \quad\quad\quad \vdots \end{array}\right\} \; denoted:\vec{\mathbf{P}}=\vec{\vec{\mathbf{M}}}\bullet\vec{\mathbf{V}} \quad or: \begin{pmatrix} P_1 \\ P_2 \\ \vdots \end{pmatrix}=\begin{pmatrix} M_{11} & M_{12} & \cdots \\ M_{21} & M_{22} & \cdots \\ \vdots & \vdots & \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \\ \vdots \end{pmatrix} \tag{2.0b}$$

Tensor expressions for Axiom-1

The following is a summary of ***Axiom-1*** given first by (1.2) and later by (1.5). Fig. 2.1 plots (2.1) below. Recall Fig. 1.5a plot of (1.5) with 45° diagonal $\mathbf{V}^{COM}$ vector with equal components: $V_1^{COM}=V_2^{COM}=V^{COM}$.

$$P_{Total}=M_1V_1^{IN}+M_2V_2^{IN}=M_1V_1^{FIN}+M_2V_2^{FIN}=M_1V^{COM}+M_2V^{COM}=M_{Total}V^{COM} \tag{2.1a}$$

A product of total momentum P_{Total} and V^{COM} is expressed by *tensor quadratic forms* **v•M•u** as follows.

$$V^{COM}P_{Total}=\vec{\mathbf{V}}^{COM}\bullet\vec{\vec{\mathbf{M}}}\bullet\vec{\mathbf{V}}^{IN}=\vec{\mathbf{V}}^{COM}\bullet\vec{\vec{\mathbf{M}}}\bullet\vec{\mathbf{V}}^{FIN}=\vec{\mathbf{V}}^{COM}\bullet\vec{\vec{\mathbf{M}}}\bullet\vec{\mathbf{V}}^{COM}=V^{COM}M_{Total}V^{COM} \tag{2.1b}$$

It helps to write this out with the numbers appearing in the original Fig. 1.5 starting with $V^{COM}=50$.

$$\begin{aligned} 50P_{Total} &= \begin{pmatrix} 50 & 50 \end{pmatrix}\bullet\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}\bullet\begin{pmatrix} 60 \\ 10 \end{pmatrix}=\begin{pmatrix} 50 & 50 \end{pmatrix}\bullet\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}\bullet\begin{pmatrix} 40 \\ 90 \end{pmatrix}=50M_{Total}50=12{,}500 \\ &= \begin{pmatrix} 50 & 50 \end{pmatrix}\bullet\begin{pmatrix} 4\cdot 60 \\ 1\cdot 10 \end{pmatrix} \qquad\quad =\begin{pmatrix} 50 & 50 \end{pmatrix}\bullet\begin{pmatrix} 4\cdot 40 \\ 1\cdot 90 \end{pmatrix} \\ &= \quad 50\cdot(240+10) \qquad\quad = \quad 50\cdot(160+90) \qquad =2500\cdot 5=250\cdot 50 \end{aligned} \tag{2.1c}$$

We use a tricky COM vector notation: $\vec{\mathbf{V}}^{COM}=(V^{COM}\,V^{COM})=(50\;\;50)$ or: $\vec{\mathbf{V}}^{COM}=\begin{pmatrix} V^{COM} \\ V^{COM} \end{pmatrix}=\begin{pmatrix} 50 \\ 50 \end{pmatrix}$

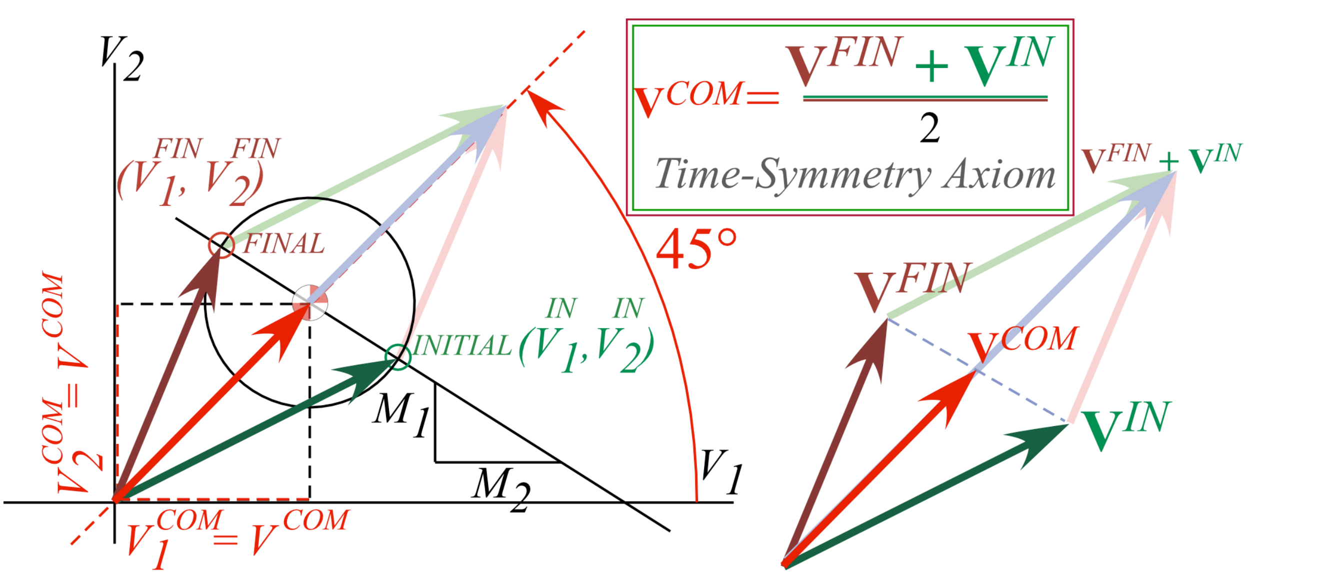


Fig. 2.1 Generic time-symmetric collision geometry. (Recall Fig. 1.5 or Fig. 1.3b.)

Tensor expressions for Axiom-2

Relations (2.1) require that momentum $P_{Total}=250$ is the same at *all* points on line of slope $-M_1/M_2$. That includes IN, FIN, and COM. Now we invoke ***Axiom-2*** a.k.a. the ***T*-symmetry** or ***Ka-Bong! Axiom***:

$$\vec{\mathbf{V}}^{COM}=(\vec{\mathbf{V}}^{FIN}+\vec{\mathbf{V}}^{IN})/2 \tag{2.2}$$

Axiom-2 has *ka-runch!* from $\vec{\mathbf{V}}^{IN}$ to $\vec{\mathbf{V}}^{COM}$ precede an ideal ***time-reversed*** *!hcnur-ak* from $\vec{\mathbf{V}}^{COM}$ to $\vec{\mathbf{V}}^{FIN}$ that completes a perfectly elastic *ka-Bong!* That's only possible in quantum worlds, but we'll follow a time-honored practice of approximating *T*-symmetry in most of classical mechanics theory. With that caveat, we substitute ***Axiom-2*** *T*-symmetry relation (2.2) for $\vec{\mathbf{V}}^{COM}$ into ***Axiom-1*** relations (2.1b).

$$V^{COM}P_{Total}=\frac{\vec{\mathbf{V}}^{FIN}+\vec{\mathbf{V}}^{IN}}{2}\bullet\vec{\vec{\mathbf{M}}}\bullet\vec{\mathbf{V}}^{IN}=\frac{\vec{\mathbf{V}}^{FIN}+\vec{\mathbf{V}}^{IN}}{2}\bullet\vec{\vec{\mathbf{M}}}\bullet\vec{\mathbf{V}}^{FIN}=\frac{\vec{\mathbf{V}}^{FIN}+\vec{\mathbf{V}}^{IN}}{2}\bullet\vec{\vec{\mathbf{M}}}\bullet\frac{\vec{\mathbf{V}}^{FIN}+\vec{\mathbf{V}}^{IN}}{2}=\vec{\mathbf{V}}^{COM}\bullet\vec{\vec{\mathbf{M}}}\bullet\vec{\mathbf{V}}^{COM} \tag{2.2a}$$

The 2nd and 3rd parts of (2.2a) each split into pairs, a symmetric quadratic form plus a lop-sided one.

$$V^{COM}P_{Total}=\frac{1}{2}\vec{\mathbf{V}}^{IN}\bullet\vec{\vec{\mathbf{M}}}\bullet\vec{\mathbf{V}}^{IN}+\frac{1}{2}\vec{\mathbf{V}}^{FIN}\bullet\vec{\vec{\mathbf{M}}}\bullet\vec{\mathbf{V}}^{IN} \quad = \quad \frac{1}{2}\vec{\mathbf{V}}^{FIN}\bullet\vec{\vec{\mathbf{M}}}\bullet\vec{\mathbf{V}}^{FIN}+\frac{1}{2}\vec{\mathbf{V}}^{IN}\bullet\vec{\vec{\mathbf{M}}}\bullet\vec{\mathbf{V}}^{FIN} \tag{2.2b}$$

Transpose symmetry ($M_{jk}=M_{kj}$) of matrix **M** makes the two lopsided terms equal. (Here $M_{12}=0=M_{21}$.)

$$\begin{array}{ccc}\vec{\mathbf{V}}^{FIN}\bullet\vec{\vec{\mathbf{M}}}\bullet\vec{\mathbf{V}}^{IN} & = & \vec{\mathbf{V}}^{IN}\bullet\vec{\vec{\mathbf{M}}}\bullet\vec{\mathbf{V}}^{FIN}\\ \begin{pmatrix}40 & 90\end{pmatrix}\bullet\begin{pmatrix}4&0\\0&1\end{pmatrix}\bullet\begin{pmatrix}60\\10\end{pmatrix} & = & \begin{pmatrix}60 & 10\end{pmatrix}\bullet\begin{pmatrix}4&0\\0&1\end{pmatrix}\bullet\begin{pmatrix}40\\90\end{pmatrix}\\ =\quad 100\cdot105 & = & 100\cdot105 \quad =10{,}500\end{array} \tag{2.2c}$$

Subtracting lopsided term shows that symmetric terms are also equal. A new conservation law appears!

$$V^{COM}P_{Total}-\frac{\vec{\mathbf{V}}^{FIN}\bullet\vec{\vec{\mathbf{M}}}\bullet\vec{\mathbf{V}}^{IN}}{2}=\frac{\vec{\mathbf{V}}^{IN}\bullet\vec{\vec{\mathbf{M}}}\bullet\vec{\mathbf{V}}^{IN}}{2}=\frac{\vec{\mathbf{V}}^{FIN}\bullet\vec{\vec{\mathbf{M}}}\bullet\vec{\mathbf{V}}^{FIN}}{2} \tag{2.2d}$$

$$V^{COM}P_{Total}-\frac{1}{2}\left(M_1V_1^{FIN}V_1^{IN}+M_2V_2^{FIN}V_2^{IN}\right)=\frac{1}{2}M_1\left(V_1^{IN}\right)^2+\frac{1}{2}M_2\left(V_2^{IN}\right)^2=\frac{1}{2}M_1\left(V_1^{FIN}\right)^2+\frac{1}{2}M_2\left(V_2^{FIN}\right)^2 \tag{2.2e}$$

Conservation of kinetic-energy (KE)

This is ***conservation of kinetic energy*** ($KE=\frac{1}{2}M_1V_1^2+\frac{1}{2}M_2V_2^2$): *KE* is the same at IN and FIN as long as *both* ***Axiom-1*** (conservation of momentum P_{Total}) *and* ***Axiom-2*** (time-reversal *T*-symmetry) holds true.

It was noted after momentum conservation equations (1.2) and (1.3) that ***Axiom-1*** is a *linear* relation whose (V_1,V_2)-plot is a straight line in Fig. 1.2 and Fig. 1.3. ***Axiom-2*** or *KE* conservation is a *quadratic* relation whose (V_1,V_2)-plot is an ellipse. To see this we rearrange *KE* conservation relation (2.2e) by placing *KE* and masses in the denominator. (Numeric labels (V_1,V_2) replace (V_{SUV},V_{VW}) here.)

$$\tfrac{1}{2}M_1\cdot V_1^{\,2}+\tfrac{1}{2}M_2\cdot V_2^{\,2}=KE \quad \textit{becomes:} \quad \frac{V_1^{\,2}}{\left(\dfrac{2\cdot KE}{M_1}\right)}+\frac{V_2^{\,2}}{\left(\dfrac{2\cdot KE}{M_2}\right)}=1 \tag{2.3a}$$

Fig. 2.2a shows (V_{SUV},V_{VW})-plot of elastic *ka-Bong!*-ellipse (2.2e) of (x,y)-radii-(a,b) matching (2.3b).

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{where:} \begin{cases}(x,y)= & (V_1,V_2)\\ (a,b)= & \left(\sqrt{\dfrac{2\cdot KE}{M_1}},\sqrt{\dfrac{2\cdot KE}{M_2}}\right)\end{cases} \tag{2.3b}$$

Just inside the elastic ellipse is the inelastic *ka-Runch*-ellipse. A smaller *COM*-frame-ellipse is in Fig. 2.2b.

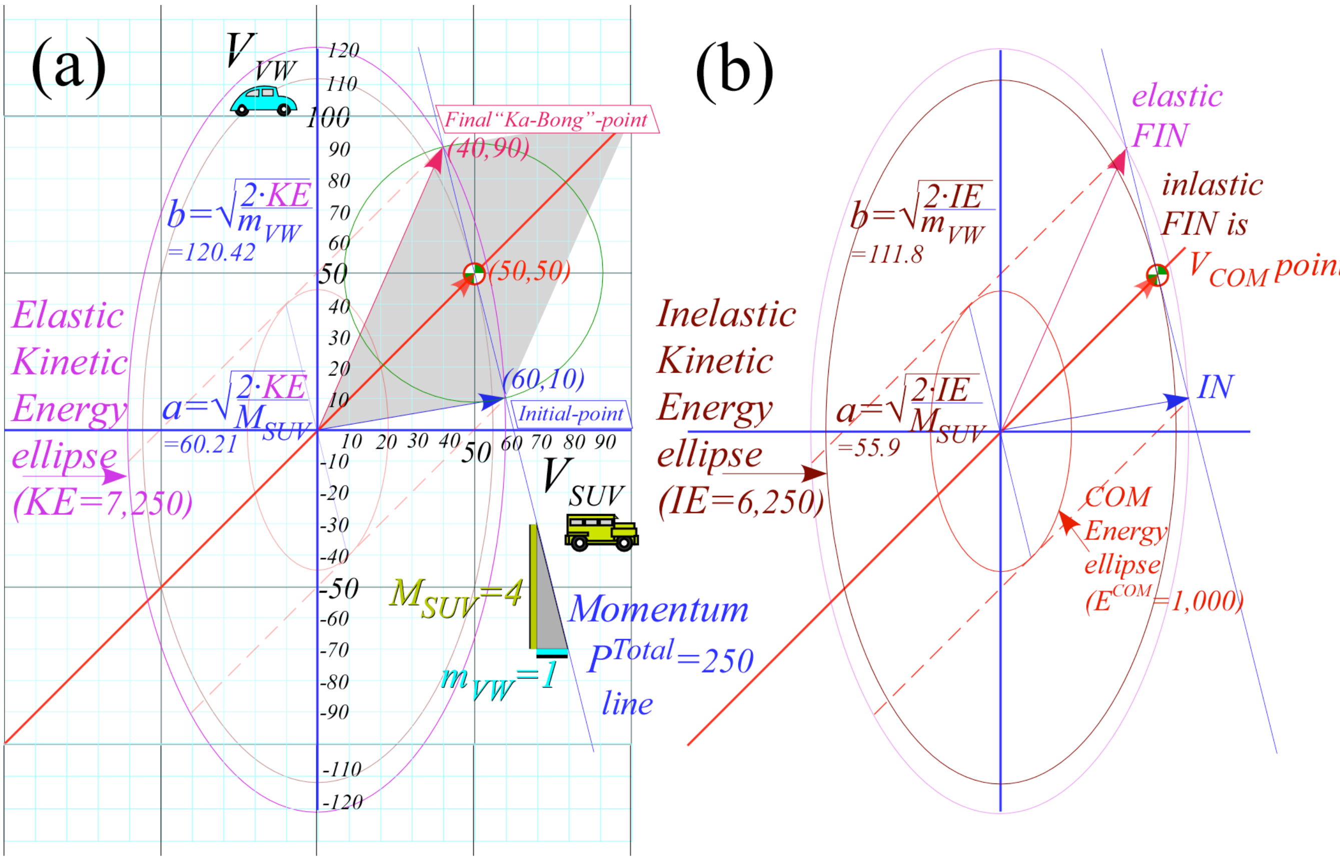


Fig. 2.2a Elastic KE-ellipse hits (P^{Total})-line at IN and FIN pts. b. Inelastic IE-ellipse hits only at V_{COM} pt.

Kinetic energy at the V^{COM}-point involves first and last parts of the ***Axiom 1&2*** equation (2.2a) times ½.

$$\frac{1}{2}V^{COM}P_{Total}=\frac{1}{2}\frac{\vec{V}^{FIN}+\vec{V}^{IN}}{2}\bullet\vec{M}\bullet\frac{\vec{V}^{FIN}+\vec{V}^{IN}}{2}=\frac{1}{2}\vec{V}^{COM}\bullet\vec{M}\bullet\vec{V}^{COM}=\frac{1}{2}(M_1+M_2)(V^{COM})^2 \tag{2.4a}$$

It is expanded like (2.2b) then reduced by equality of lopsided terms and of symmetric terms (2.2c-d).

$$\frac{1}{2}V^{COM}P_{Total}=\frac{1}{2}\frac{\vec{V}^{FIN}\bullet\vec{M}\bullet\vec{V}^{FIN}+\vec{V}^{IN}\bullet\vec{M}\bullet\vec{V}^{IN}+\vec{V}^{FIN}\bullet\vec{M}\bullet\vec{V}^{IN}+\vec{V}^{FIN}\bullet\vec{M}\bullet\vec{V}^{IN}}{4}=\frac{1}{2}(M_1+M_2)(V^{COM})^2 \tag{2.4b}$$

$$\frac{1}{2}V^{COM}P_{Total}=\frac{1}{4}\left(\vec{V}^{IN}\bullet\vec{M}\bullet\vec{V}^{IN}+\vec{V}^{FIN}\bullet\vec{M}\bullet\vec{V}^{IN}\right)=\frac{1}{2}(M_1+M_2)(V^{COM})^2\equiv KE^{COM} \tag{2.4c}$$

This reduces to a relation between KE^{COM}, KE^{IN}, and the lopsided term. Again, it helps check the numbers.

$$\tfrac{1}{2}V^{COM}P_{Total}=\tfrac{1}{2}KE^{IN}+\tfrac{1}{4}\vec{V}^{FIN}\bullet\vec{M}\bullet\vec{V}^{IN}=\tfrac{1}{2}(M_1+M_2)(V^{COM})^2\equiv KE^{COM}=\tfrac{1}{4}\vec{V}^{IN}\bullet\vec{M}\bullet\vec{V}^{IN}+\tfrac{1}{4}\vec{V}^{FIN}\bullet\vec{M}\bullet\vec{V}^{IN} \tag{2.4d}$$

$$\tfrac{1}{2}12{,}500=\tfrac{1}{2}\,(60\ 10)\bullet\begin{bmatrix}4&0\\0&1\end{bmatrix}\bullet\begin{pmatrix}60\\10\end{pmatrix}+\tfrac{1}{4}\,(50\ 90)\bullet\begin{bmatrix}4&0\\0&1\end{bmatrix}\bullet\begin{pmatrix}60\\10\end{pmatrix}=\tfrac{1}{2}(5)(50)^2 \qquad \text{Initial energy}=KE^{IN}$$

$$6{,}250\ =\ \tfrac{1}{4}(4\cdot 60^2+1\cdot 10^2)+\tfrac{1}{4}(50\cdot 4\cdot 60+90\cdot 1\cdot 10)=6{,}250. \qquad KE^{IN}=\tfrac{1}{2}\vec{V}^{IN}\bullet\vec{M}\bullet\vec{V}^{IN} \tag{2.4e}$$

$$6{,}250\ =\ 3{,}625\ +\ 2{,}625\ =6{,}250. \qquad KE^{IN}=2\cdot 3{,}625=7{,}250$$

Consider difference KE^{IN}-KE^{COM} of total initial energy KE^{IN} and kinetic energy KE^{COM} left at V^{COM}-point.

$$KE^{IN}-KE^{COM}=\tfrac{1}{2}KE^{IN}-\tfrac{1}{4}\vec{V}^{FIN}\bullet\vec{M}\bullet\vec{V}^{IN}=\tfrac{1}{4}\vec{V}^{IN}\bullet\vec{M}\bullet\vec{V}^{IN}-\tfrac{1}{4}\vec{V}^{FIN}\bullet\vec{M}\bullet\vec{V}^{IN}=3{,}625\text{-}2{,}625=1{,}000 \tag{2.5}$$

KE^{COM} in (2.4d) is the half-sum of symmetric and lopsided terms. KE^{IN}-KE^{COM} in (2.5) is the half-difference of those terms and equal to *1,000* energy units. That is a small fraction of the total $KE^{IN} = 7,250$ units. That difference $KE^{IN}-KE^{COM} = 1,000$ is *ka-Runch!* energy lost to heat in a totally *inelastic ka-Runch!* case that ends at $\mathbf{V}^{COM}$ point. Initial energy $KE^{IN} = 7,250$ drops by *1,000* to $KE^{COM} = 6,250$ in (2.4e). The totally elastic *ka-Bong!* is the opposite extreme. It springs back from $\mathbf{V}^{COM}$ point to a final $\mathbf{V}^{FIN}$ point having recovered *1,000* units of energy to end with initial kinetic energy $KE^{FIN} = 7,250 = KE^{IN}$. In this case we say the $KE^{IN}-KE^{COM} = 1,000$ was stored as *potential energy* of compression at the $\mathbf{V}^{COM}$ point and (in this ideal case) none gets lost to heat and all the original KE^{IN} is recovered.

Energy $KE^{IN}(V_{SUV}=60, V_{VW}=10)$ or $KE^{COM}(V_{SUV}=50, V_{VW}=50)$ in Fig. 2.2a use Earth-relative velocity.

$$KE^{IN} = \tfrac{1}{2}4\cdot 60^2 + \tfrac{1}{2}1\cdot 10^2 = 7,250 \qquad KE^{COM} = \tfrac{1}{2}\,4\cdot 50^2 + \tfrac{1}{2}\,1\cdot 50^2 = 6,250 \tag{2.6a}$$

Energy $KE^{IN}(V_{SUV}=10, V_{VW}=-40)$ in Fig. 2.2b uses *COM*-relative velocity obtained by Galilean subtraction.

$$\begin{gathered} KE^{IN}_{COM\text{-}relative} = \tfrac{1}{2}4\cdot 10^2 + \tfrac{1}{2}1\cdot(-40)^2 = 1,000 \\ \text{where: } \mathbf{V}^{IN}_{COM\text{-}relative} = \mathbf{V}^{IN}_{Earth\text{-}relative} - \mathbf{V}^{COM} = (60,10)-(50,50) = (10,-40) \end{gathered} \tag{2.6b}$$

Difference in energy between the two extreme types of collision, *Ka-Bong* and *Ka-runch*, is *1,000* units in the Earth frame and the same in *all* frames including the *COM* frame. But, only in the *COM* frame does the *Ka-runch!* take *all* their kinetic energy so both cars end up stationary at origin $\mathbf{V}^{FIN}=(0,0)$ in Fig. 2.2b.

Head-on collisions $(V^{IN}_{SUV}=3, V^{IN}_{VW}=-4)$ are plotted in Fig. 2.3 with increasing inelastic frictional loss shown in parts (b) and (c). (Here the *SUV* is $M_1=6ton$.) The elastic KE-*Ka-Bong* ellipse (Energy $KE^{ka\text{-}Bong}=35$ in Fig. 3.2a) shrinks to a smaller inelastic *ka-Whump* ellipse ($KE^{whump}=23^1/_3$ in Fig. 3.2b has loss $11⅔$ to heat chosen arbitrarily). The smallest ellipse ($KE^{ka\text{-}Runch}=14$ in Fig. 2.3c) is the totally inelastic *ka-Runch*-case. Generic *ka-whump* cases are "in-between-ideals" and more like the real world. Each *ka-whump* case has *two* possible final *F*-points where its momentum conservation line intersects its *ka-whump* ellipse. The top F_{whump} point shows an incomplete rebound from $\mathbf{V}^{COM}(2,2)$ to $(^4/_3,6)$. The nature of the lower F_{flump} point is left as a thought-exercise. Indeed, it takes more thought to tease real-world physics from simple axioms.

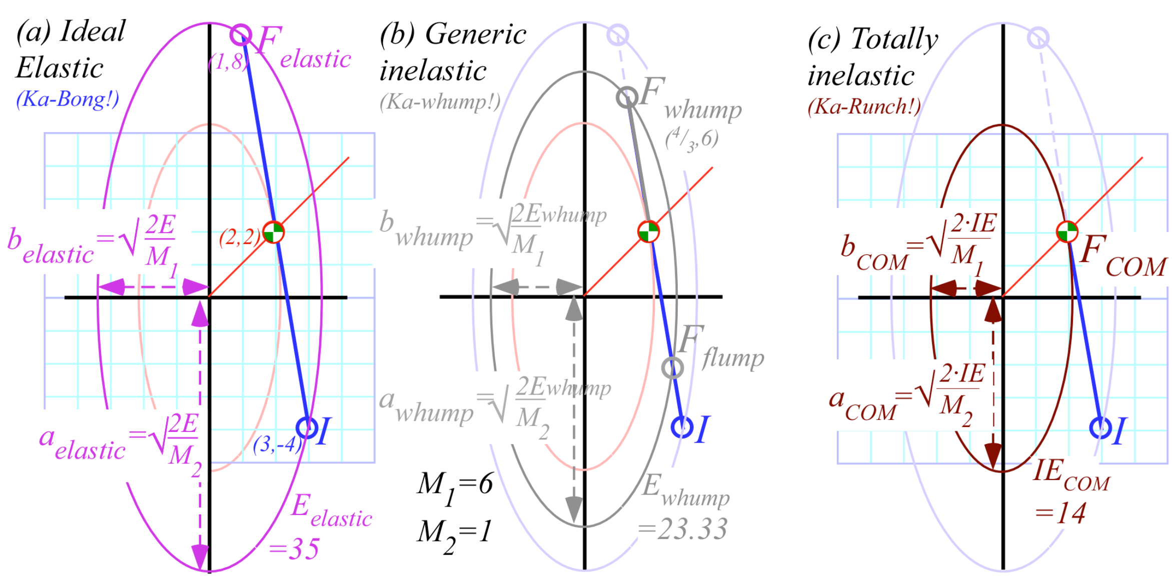


Fig. 2.3 $(V_1=3, V_2=-4)$ *collisions. (a) Elastic* $(E^{loss}=0)$. *(a) Generic* $(E^{loss}=11^2/_3)$. *(a) Inelastic* $(E^{loss}=21=E^{COM})$.

Gaps between "idealized" models and "real-world" physics are always open to criticism. The present model of point-masses on a frictionless 1-dimensional rail ignores "outsider" effects of air and road friction and multi-dimensional mechanical structures of cars composed of thousands of parts able to fly in any of 3-dimensions. ***Axiom-1*** (momentum conservation) is a good approximation even with internal friction, but ***axiom-2*** (energy-conservation) may apply only for collisions of less than 5 mph (and even then not perfectly). To better understand effects of axioms 1 and 2 let us simulate an ideal rail to study multiple collisions of a pair of masses M_1 and M_2 and opposite ends of the rail. This simulation is by a web-app called *BounceIt* available with the lecture notes for this text.

Introducing chained collisions and KE ellipse geometry

A top view of the two masses on their rail is shown in Fig. 2.4a. A (V_1, V_2)-plot with initial velocity $\mathbf{V}^{IN}=(V_1=60, V_2=10)$ is shown in Fig. 2.4b. It traces a line between $\mathbf{V}^{IN}$ and $\mathbf{V}^{FIN}=(40,90)$ that is the momentum-conservation line in Fig. 1.3b. Then Fig. 2.4b shows the lighter mass ($M_2=1$) bouncing elastically off the top end of the track. This changes the velocity vector from $\mathbf{V}^{FIN}=(40,90)$ to reflected vector $\mathbf{V}^{FIN\text{-}2}=(40,-90)$ so ($M_2=1$) has a 2nd collision with the heavier ($M_1=4$). After that is a second reflection and a 3rd collision ending at $\mathbf{V}^{FIN\text{-}3}$. Next M_1 reflects to just past $\mathbf{V}^{COM}=(50,50)$ by bouncing off lower end of track. Fig. 2.4c shows many such bounces. All pause on an energy ellipse like Fig. 2.2a.

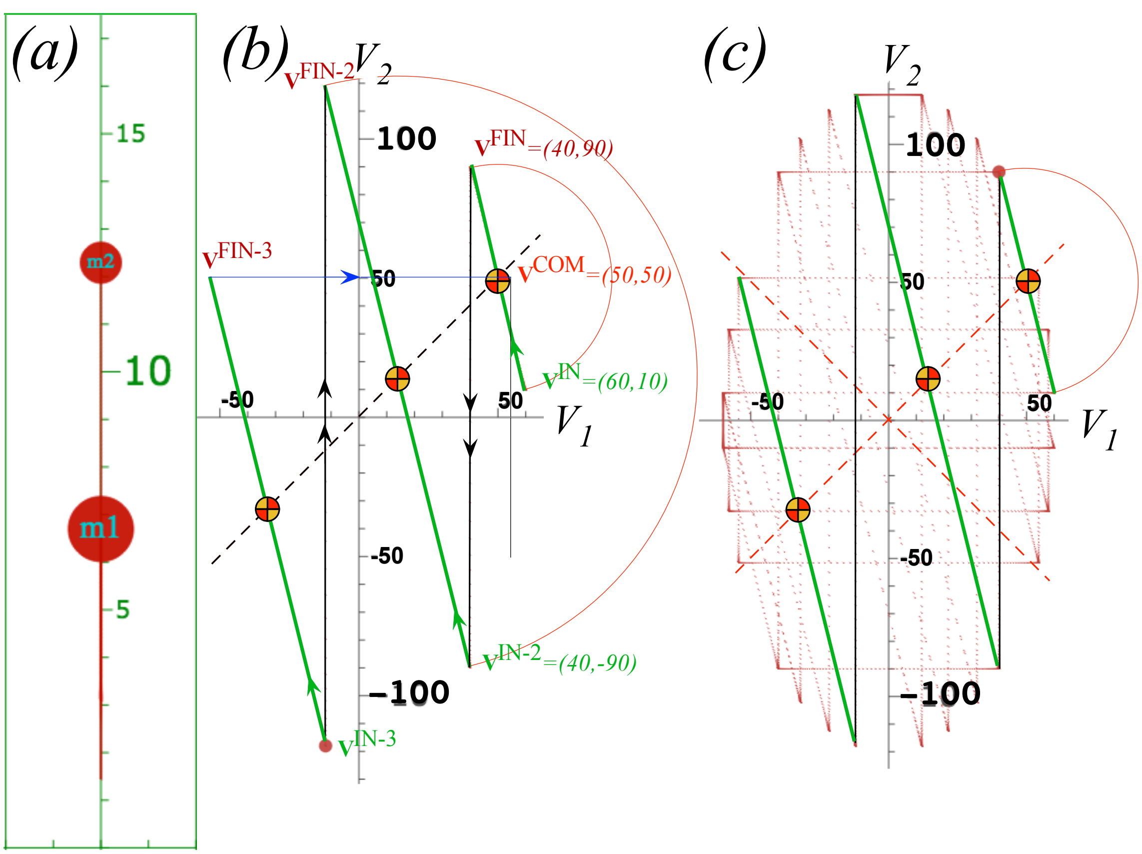


Fig. 2.4(a) Multiple ($M_1=4, M_2=1$) collisions. (b) 5 collisions starting at ($V_1=60$, $V_2=10$). c. Many more.

Exercise 2.1 (A critical-thinking problem)

The linear-plus-quadratic equations of momentum (1.5) and kinetic energy (2.3a) have two algebraic solutions corresponding to geometric intersections of a momentum-line with kinetic energy ellipse or ellipses. For perfectly elastic (*ka-Bong!*) collisions the solutions are just *INITIAL*(IN) and *FINAL*(FIN), and similarly for perfectly inelastic (*ka-Runch!*) collisions, as shown in Fig. 2.3a and Fig. 2.3c, respectively.

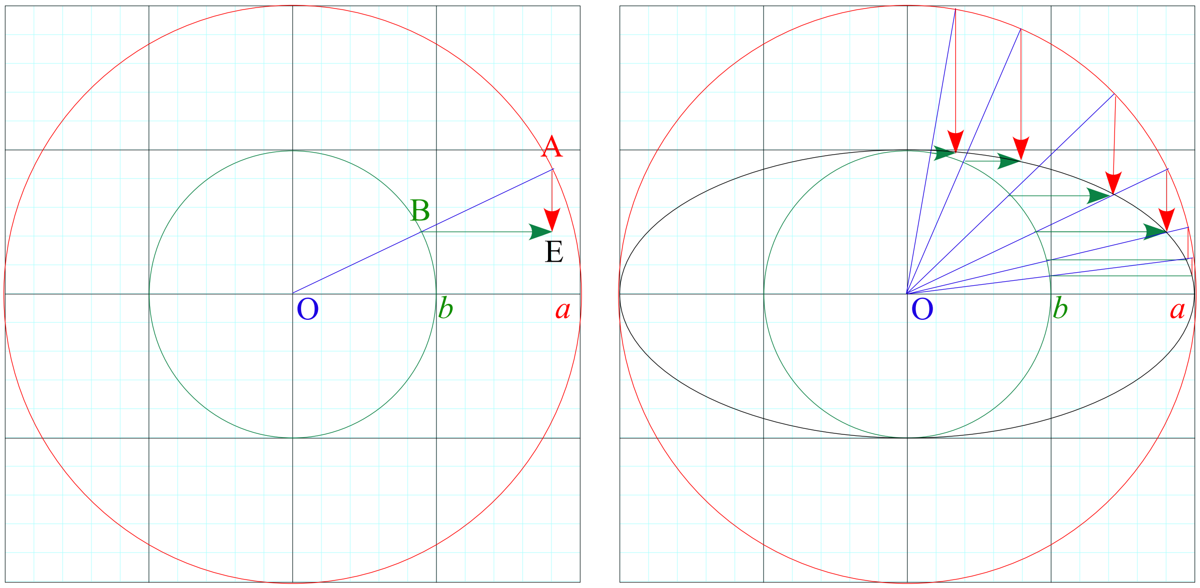
However, for partially elastic (*ka-whump!*) collisions, as shown in Fig. 2.3b, there seem to be four possible intersections. One in particular is labeled as F_{flump} and lies below primary solution point F_{whump}.

(a) Discuss F_{flump} and compare to F_{whump}. Are both physically possible? What situation could possibly give rise to an F_{flump}? Think outside of the box(es) of cars and more in the realm of molecular, atomic, nuclear, and quantum particle collisions.

(b) Actually, you can simulate an F_{flump} on *BounceIt*. (That would pretty well take care of this problem!) See if you can divine what is going on before doing the simulation.

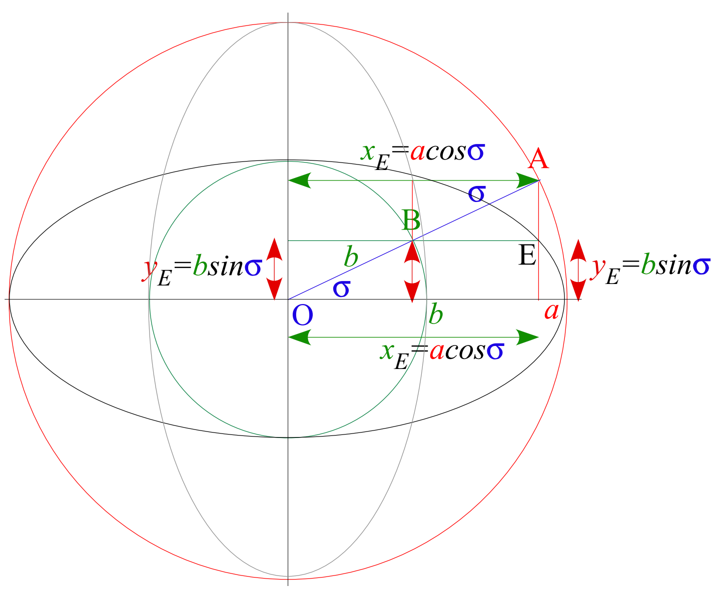
Quick construction of Energy ellipses

Graph paper facilitates construction of energy ellipses given the two radii a and b in (2.3). First step: draw concentric circles of radius a and b. Then any radial line OBA "points" to point E on the ellipse. Ellipse point E lies at the intersection of a vertical line AE thru radial intersection A with circle a and a horizontal line BE thru radial intersection B with circle b. Graph grid helps locate E for a radius OBA, and usually there is no need to draw AE or BE. You can pick x and find y or else *vice-versa*.



Exercise Fig. 2.5 Ellipse construction

Ellipse coordinates (x_E=a·cosσ, y_E=b·sinσ) are rescaled base and altitude (x_r=r·cosσ, y_r=r·sinσ) of Fig. 1.4.



Exercise Fig. 2.6 Complimentary analytic ellipse geometry

Verify that the values (x =a·cosσ, y =b·sinσ) satisfy an ellipse equation (3.7b).

A dual or complimentary (gray) ellipse results if compliment angle σ_c=$\pi/2-\sigma$ is used so x and y values switch.

Chapter 3. Dynamics and geometry of successive collisions

One-dimensional two-mass (1D-2-body) collisions occupy Ch. 2-3. Now they become more dangerous. Introducing the X2-super bouncer from *Project Ball*, a 1969 pre-med class project at the university of Southern California published in *Am. J. Phys.* **39**, 656 (1971). See product liability disclaimer in Fig. 3.1.

Caution: Product Liablility Disclaimer
This ballpoint pen could be hazardous to your health!
The experiments which are the subject of this discussion are both spectacular and potentially dangerous, and care to protect one's eyes should be taken. The simplest experiment involves sticking a ball point pen into a superball or other hard rubber ball and dropping the two onto a hard floor. If done correctly the pen will eject the ball with such force it may stick in the ceiling of the room. Obviously you want to be careful with this weapon. And, this goes doubly and triply for the more advanced models that may be developed in the course of studying this stuff. It is recommended that experimenters wear safety glasses when doing these experiments with pens. (We could just say don't use pens, but that's an easy way to do this experiment and probably the way most people will go about it.) Some of the tangential experiments associated with this development are less hazardous. To measure the potential force function of a ball one may simply paint the ball and measure the spot size as a function of drop height h.

The saggital approximation $d=r^2/2R$ allows one to quickly convert spot radius r to penetration depth x for a superball of radius R as shown in the figure. Equating this to Mgh gives the ball potential energy function $V(x)$.

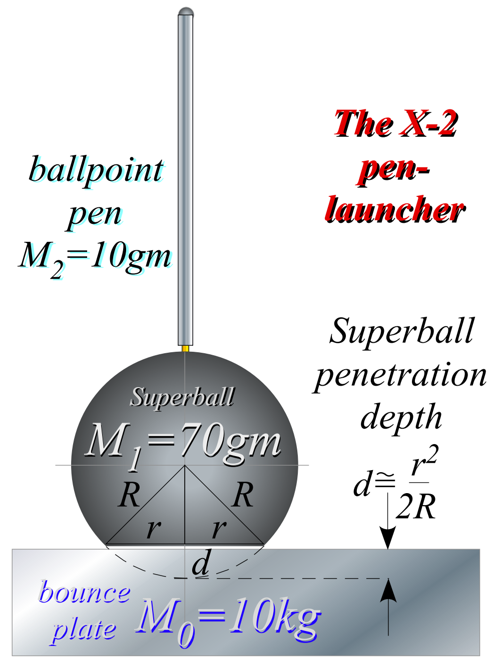


Fig. 3.1 The X2-pen launcher with product liability disclaimer.

At first, X2 looks like a 2-body device. A mass M_1 *=70gm superball*(©™*Whammo Corp.*) launches a ballpoint pen of mass M_2 *=10gm*. But, bounce plate mass-M_O*=10kg* (the rectangle in Fig. 3.1) is a 3rd body sitting on a 4th body, good old Mother Earth of mass $M_\oplus = 6 \cdot 10^{24} kg$. Earth mass $M_\oplus$ and solar mass $M_\odot = 2 \cdot 10^{30} kg$ are 2-figure approximations to $M_\oplus = 5.9742 \cdot 10^{24} kg$ and $M_\odot = 1.9891 \cdot 10^{30} kg$.

Collisions of tiny masses with huge ones can be simple. Mass ratio $M_1/M_\oplus$ is momentum P-line slope $-\Delta V_1/\Delta V_\oplus$ in $(V_1, V_\oplus)$-space. It drops to quasi-horizontal for large $M_\oplus$ in Fig. 3.2a where that slope is $^{-1}/_{100}$. Fig. 3.2b has quasi-vertical slope $\Delta V_\oplus/\Delta V_2 = -M_\oplus/M_2 = -100$ in $(V_\oplus, V_2)$-space. *(Compare to Fig. 2.2.)*

To find $\mathbf{V}^{FIN}$ for an elastic collision in Fig. 3.2a we do a (nearly) horizontal reflection of $\mathbf{V}^{IN}$ thru the *COM* point ($\mathbf{V}^{COM}$) along the P-line. It is a (nearly) vertical reflection in Fig. 3.2b with $M_\oplus = 100\, M_2$. (Inset sketches exaggerate P-line slopes beyond the *100:1* ratios barely showing in their exact plots.)

Let us see how a large mass $M_\oplus$ may give large momentum to a smaller M_1 but have KE loss be tiny or none. In *COM* frame view Fig. 3.2a, $M_1=1$ is bounced from (approximately) $V_1^{IN} = -1$ to $V_1^{FIN} = +1$ (approximately) off a plate of mass $M_\oplus = 100$ rising slowly at speed $V_1^{FIN} M_1/M_\oplus = {}^{1}/_{100}$. $M_\oplus$ then recoils just as slowly. Now M_1 in its frame (where V_1^{IN} is initially zero) sees a big $M_\oplus$ plate rising at full speed $+1$ (approximately) and knocking M_1 up to (approximately) twice that speed while velocity lost by $M_\oplus$ is (approximately) zero. Now if the plate is the Earth with *much* greater mass $M_\oplus = 6 \cdot 10^{24} kg$ then all our (*approximately*) modifiers may be replaced by (*exactly*) for all practical purposes.

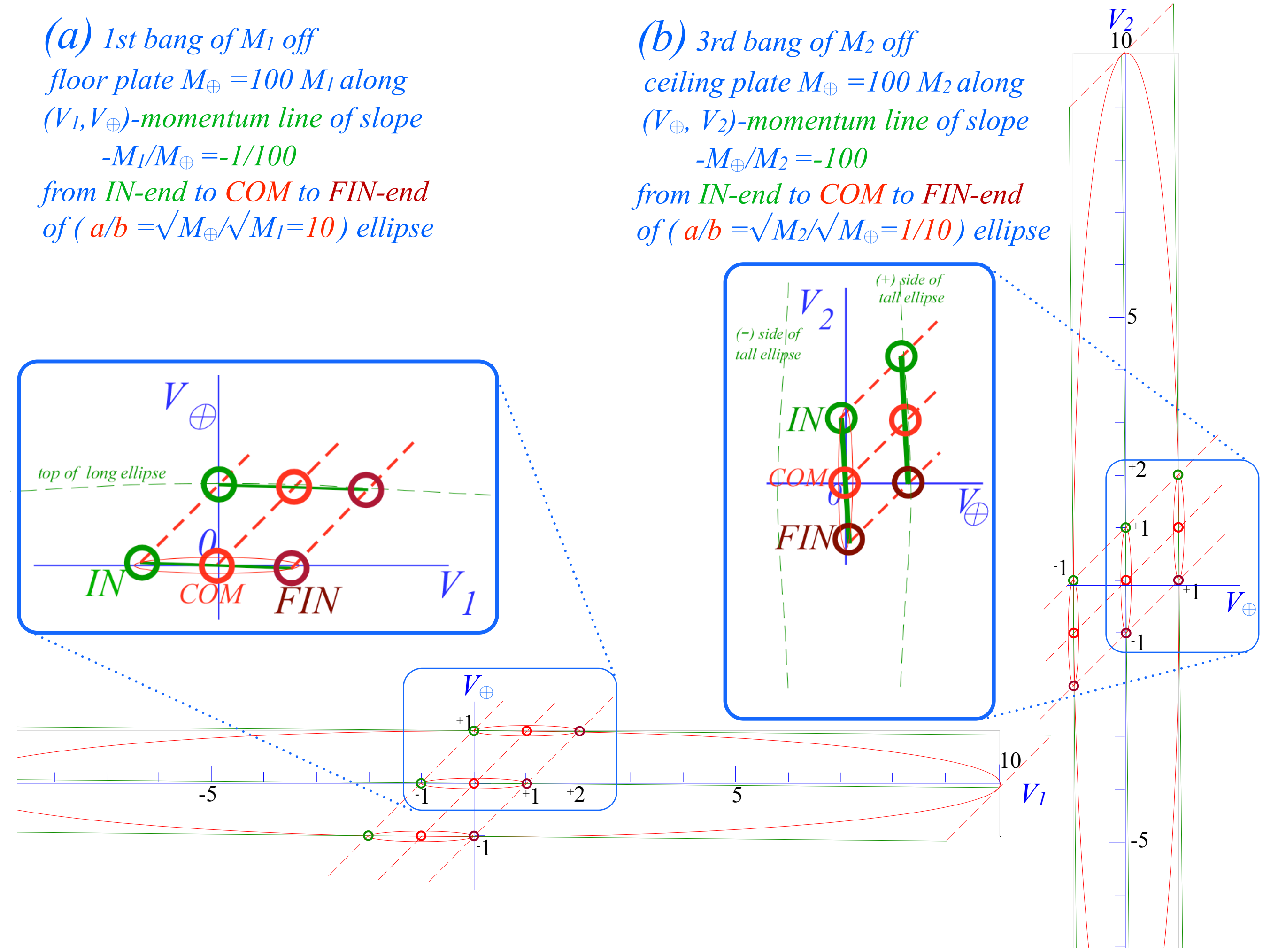


Fig. 3.2 Extreme mass-ratio collisions (a) M_1/ M_2 approaches infinity. (b) M_1/ M_2 approaches zero.

Fig. 3.2a reflects our common ideal of a bouncy ball of mass M_1 hitting the Earth of mass $M_\oplus$ with velocity $V^{IN}=-1$ and being reflected to velocity $V^{FIN}=+1$. By standing in the Earth frame, one is very nearly in the COM frame since Earth's COM velocity is a tiny fraction $M_2/M_\oplus$ of ball speed $|V^{IN}|$. For super-balls of mass $M_2=60gm$, the fraction $M_2/M_\oplus$ is $0.06/(6 \cdot 10^{24})=10^{-26}$ or *1/(100 Trillion trillion)!*

Bounce momentum absorbed by Earth is $2M_2V_0$ (or M_2V_0 if the ball goes *"Ka-runch!"*), but Earth absorbs at most a tiny *KE* of $\frac{1}{2}M_\oplus(V^{IN}M_2/M_\oplus)^2$, a fraction 10^{-26} of ball $KE\frac{1}{2}M_2(V^{IN})^2$. Ma-Earth returns virtually *all KE* to M_2 in elastic (*ka-Bong!*) collisions all while absorbing *double* momentum $P=2M_2V_0$. So our common ideal of balls with 100% rebound off of Earth has a solid logical and quantitative basis.

However, common experience does not prepare us for the amazing X-2 in Fig. 3.3. As shown in Fig. 3.3a the X-2 experiments from *Project Ball* can (as in 1997 movie *Flubber* or 1961 *Absent-Minded Professor*) easily rebound mass M_2 with more than *twice* its drop velocity. As we'll see that means M_2 may rise to more than *four* times its drop height! Later we will also reveal X-3 and X-4 experiments that do many times that. If you are looking for an appealing and spectacular way to teach classical momentum and energy physics, it's hard to beat the discoveries from *Project Ball*.

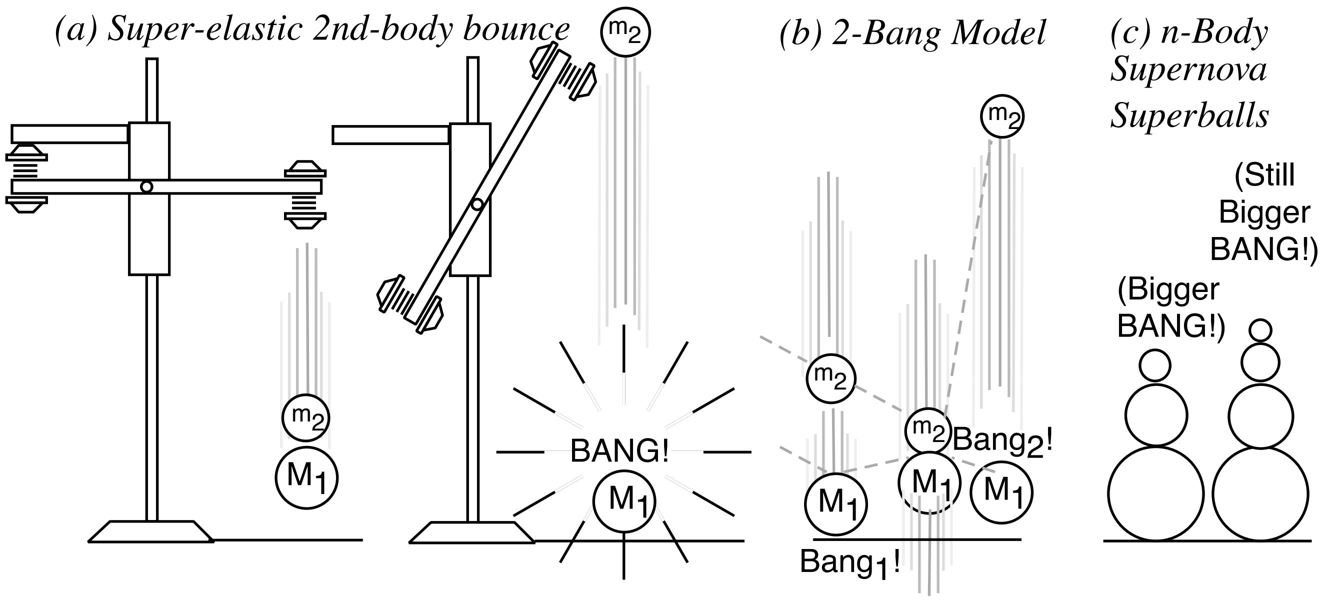


Fig. 3.3 n-Body collision experiments. (a) X-2 drop. (b) Independent collision model. (c) Ball towers.

Independent bang models (IBM)

To compute final velocities of M_1 and M_2 it helps to idealize the collision of three bodies M_1, M_2, and $M_\oplus$ as a sequence of two separate 2-body collisions that are completely determined by *P* and *KE* conservation. First M_1 bounces off Earth $M_\oplus$. Only then does M_1 knock M_2 to a faster speed as in Fig. 3.3b. The first collision is labeled $Bang\text{-}1_{(01)}$ in Fig. 3.4a followed by $Bang\text{-}2_{(12)}$ in Fig. 3.4b. The first $Bang\text{-}1_{(01)}$ between Earth $M_\oplus$ and M_1 has a horizontal line like the *IN-FIN* line in Fig. 3.2a. The second $Bang\text{-}2_{(12)}$ between mass M_1 and M_2 has a line of slope $-M_1/M_2$ =-7 for a $M_1 = 70gm$ and $M_2 = 10gm$ (that of a superball and pen, respectively). The $Bang\text{-}2_{(12)}$ line is like the *IF* lines in Fig. 3.1 or Fig. 3.2.

This approximation is called an *independent bang model* (IBM) and is one secret to analyzing such a 1D-3-body bang-up that otherwise has too many unknown velocities to be solved by just two equations $\Delta P=0$ and $\Delta KE=0$ alone. IBM is exactly true if we initially separate M_1 and M_2 so three M_1, M_2, and $M_\oplus$ never *collectively* bargain for available momentum and energy. IBM also applies to *n*-ball towers in Fig. 3.3c. They give very high-energy ejections and serve as classical models for supernovae. (*N*-body bangs will be treated later in Ch.8.)

Velocity geometry suggests a family of X2 solutions as shown in Fig. 3.5 for a range of mass ratio M_1/M_2. This is an advantage of geometric solutions. Just a few points in Fig. 3.5a show all elastic $(V_1\text{-}V_2)$ points lie on the 45°-line *CPL*. Extreme or optimal cases are located in Fig. 3.5b.

Extreme and optimal cases

An upper limit for elastic final velocity is $V_2=3\cdot V_0$ at pt-***I*** for infinite mass ratio $M_1/M_2 \rightarrow \infty$. A particle of dust on a superball may be ejected three times as fast as the ball hits the floor and, it could go *nine* ($9=3^2$) times the drop height. However, elastic IBM models usually fail for tiny M_2 due to friction and/or molecular forces so bouncing balls don't put dust in ceilings. (Fortunately! *But in a vacuum...?*)

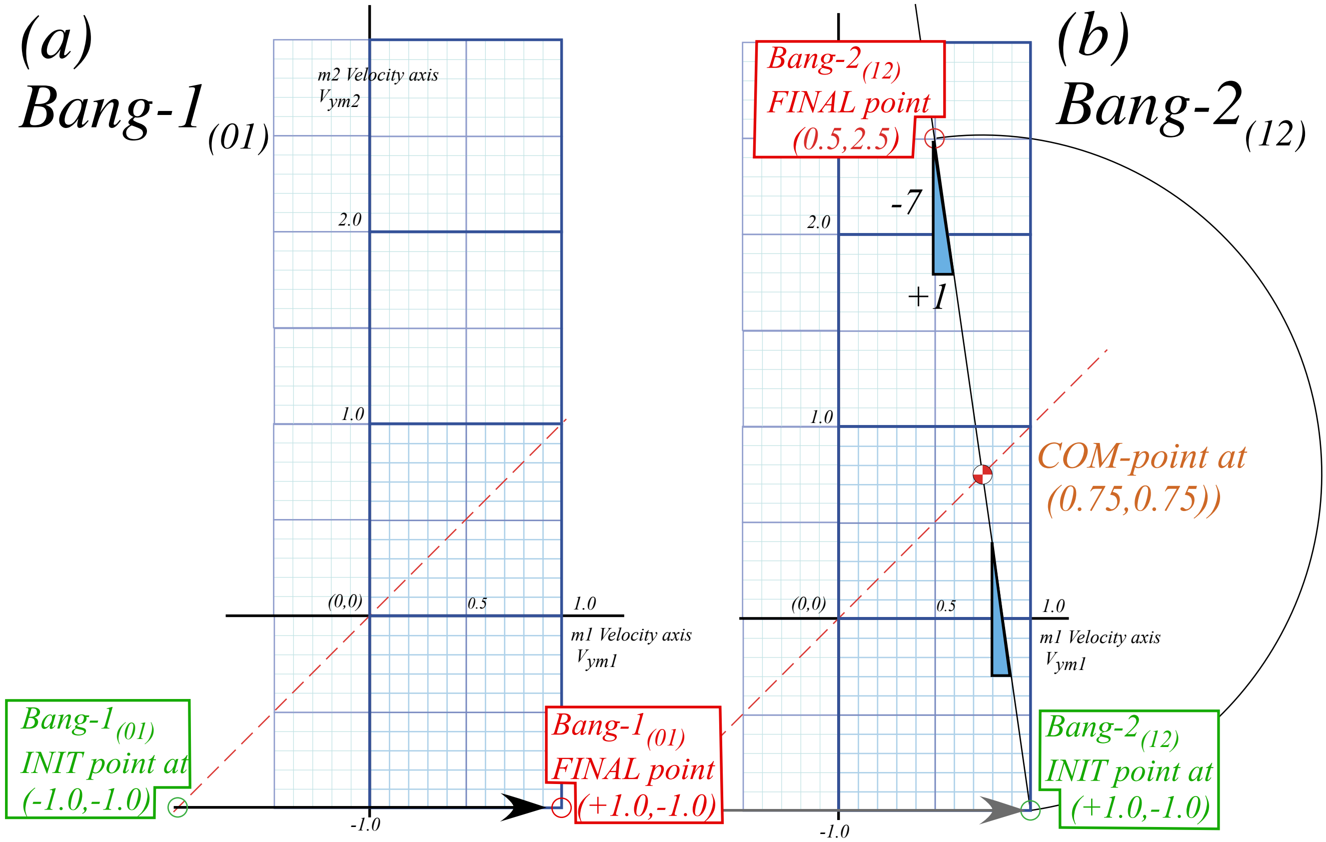


Fig. 3.4 (V_1-V_2)-plot of 2-Bang collision. (a) M_1 bounces off floor. (b) M_1 hits M_2 head-on.

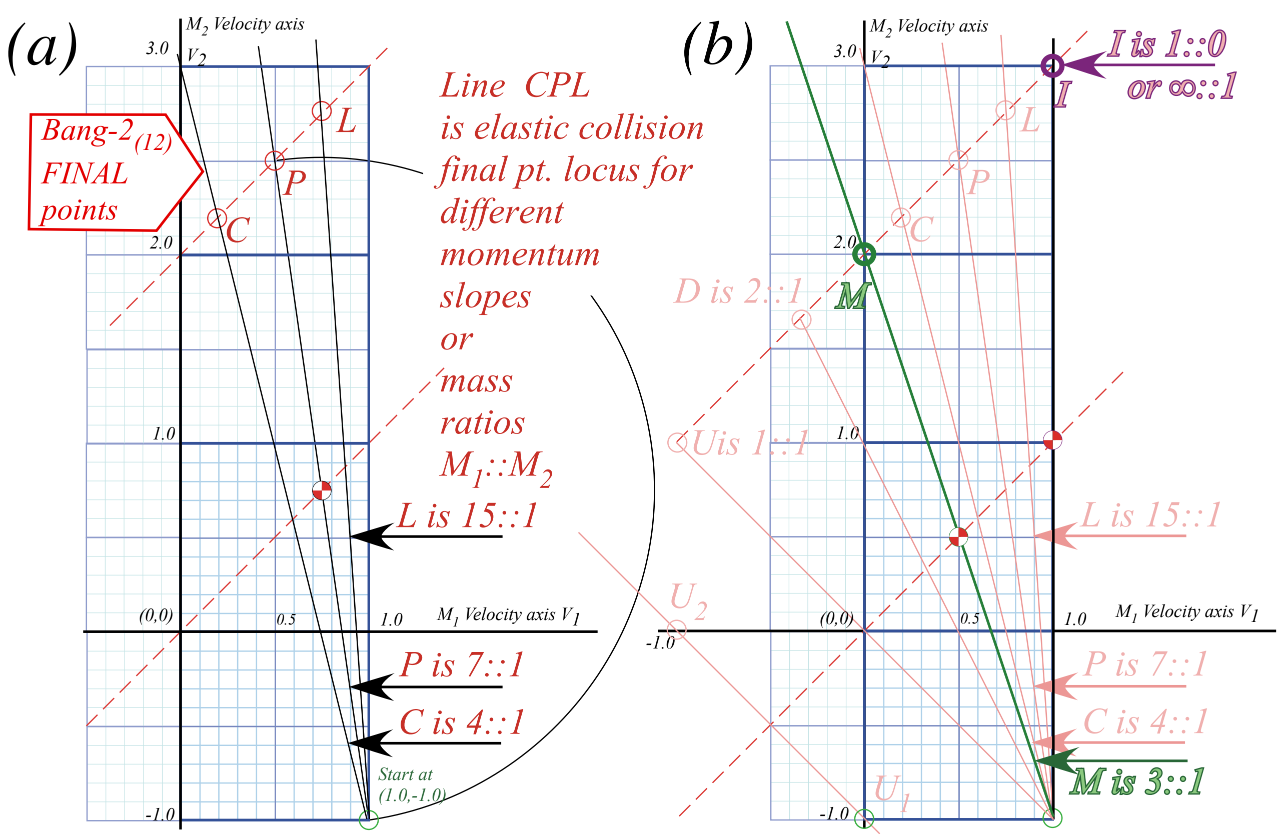


Fig. 3.5 X2-Final (V_1,V_2) (a) Final point locus. (b) Infinite ratio pt. I and maximum transfer pt. M.

An optimal performance case is at pt-$\boldsymbol{M}$ in Fig. 3.5b where the collision achieves a 100% transfer of energy to projectile M_2. The $\boldsymbol{M}$-point is the intersection of the *CPL* line with the V_2-axis on which the M_1-ball velocity is zero. ($V_1=0$) There mass ratio and (-)slope of the $\boldsymbol{M}$-line is $M_1/M_2=3.0$.

Another interesting point $\boldsymbol{U}$ is for unit ratio $M_1/M_2=1$, a familiar ratio for players of billiards or pool. $\boldsymbol{U}$ undergoes inversion of velocities $(+1,-1)\text{->}(-1,+1)$. (Its COM point lies at origin.) If the $\boldsymbol{U}$-line is boosted by (-1) to $(0,-2)\text{->}(-2,0)$ it is like a straight elastic pool shot. A 100% of *KE* transfers from a moving ball to an equal sized ball that was stationary. The same process at half that speed is $(0,-1)\text{->}(-1,0)$ shown by the Galileo-shifted line $\boldsymbol{U}_1\text{->}\boldsymbol{U}_2$ in the lower left hand side of Fig. 3.5b.

Points $\boldsymbol{D}$ between $\boldsymbol{U}$ and $\boldsymbol{M}$ have ball M_1 knocked to negative velocity by the down-coming M_2. Then M_1 hits the floor (Earth) at velocity $-v$ to rebound at $+v$. For unit ratio case $\boldsymbol{U}$, M_1 and M_2 rebound quite like a rigid body. Below $\boldsymbol{U}$, ball M_1 rebounds at a speed faster than M_2 to hit M_2 again. In cases of low mass ratio, $(M_1/M_2<<1)$ mass M_1 must hit M_2 many times to turn it around as will be seen later.

Integrating velocity plots to find position

It is important to see how velocity values of Fig. 3.4b are turned into space-time position plot lines. Consider the first collision (*Bang-1*$_{(10)}$) in Fig. 3.6a and corresponding space-time paths in Fig. 3.6b. Initial velocity $V_{y1}(0)=-1.0$ gives a slope *(distance)/(time)* of an M_1 path but doesn't tell *where* is the path or particle. The same for velocity $V_{y2}(0)=-1$ of M_2 in Fig. 3.6a. The paths need to be *positioned*.

Initial position values such as ($y_1(0)=1$, $y_2(0)=3$) locate the paths as shown in Fig. 3.6b. Each path keeps its slope until a collision (*Bang-1*$_{(10)}$) between M_1 and the floor occurs at $y_1(t=1)$ where its path and the floor intersect. Then, as seen in Fig. 3.6a, M_1 bounces its slope from $V_{y1}=-1$ up to $V_{y1}=+1$. Meanwhile, upper path (M_2) maintains its down slope of $V_{y2}=-1$ until it intersects rising M_1's path.

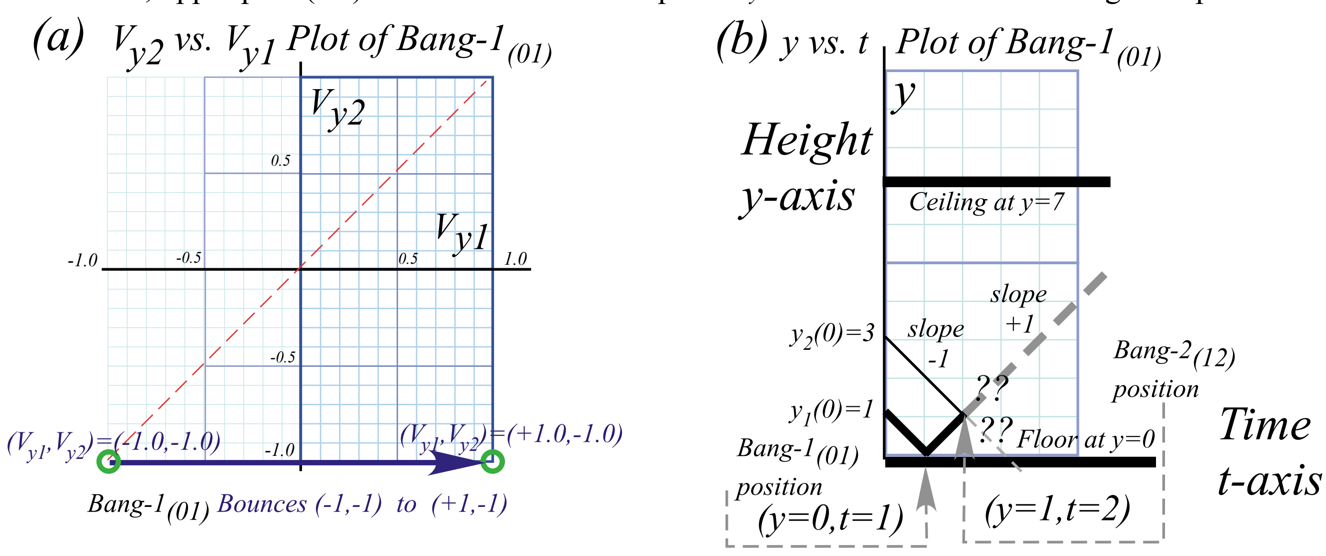


Fig. 3.6 Plots of 1st collision (Bang-1$_{(10)}$*). (a) Velocity-velocity plot. (b) Space-time plot.*

At time $(t=2)$ there is an intersection of paths and the 2nd collision (*Bang-2*$_{(12)}$) between M_1 and M_2 at space-time point ($y_1(2)=1$, $y_2(2)=3$). This gives $V_{y1}=0.5$ and $V_{y2}=2.5$ in Fig. 3.4b or in Fig. 3.7a-b below. Then to keep M_2 from flying away we install an elastic ceiling at $y=7$.

The game becomes interesting as *Bang-3*$_{(20)}$ between the ceiling (part of Earth $M_\oplus$) is shown in Fig. 3.7b by a vertical arrow (like an *IF* line in Fig. 3.2b) reflecting M_2 to speed $V_{y2}=-2.5$. Then M_2 has *Bang-4*$_{(12)}$ between M_1 and itself that sends it back to the ceiling at a blistering speed of $V_{y2}=+2.7$ as M_1 descends toward the floor with a more modest velocity $V_{y1}=-0.5$.

The high speed of M_2 lets it go to the ceiling for *Bang-5*$_{(20)}$ and return to knock M_1 down once more (*Bang-6*$_{(12)}$) before M_1 hits the floor at $V_{y1}=-0.9$. (*Bang-7*$_{(10)}$) Then M_2 having lost speed to $V_{y2}=+1.5$ hits the ceiling (*Bang-8*$_{(02)}$) and returns for *Bang-9*$_{(12)}$ with M_1 rising at $V_{y1}=+0.9$.

Masses are treated as ***point-masses*** moving along straight lines between collisions in space-time plots. This is an ideal gravity-free IBM approximation with only straight lines in (V_1,V_2)-plots. It lets us derive motion without integrating kinetic equations (2.1) thru (2.2) as *BounceIt* does in Fig. 2.4. If the masses have finite size, say a minimum center-to-center separation radius r_{12}, then the M_2 position graph is drawn that much higher than that of M_1.

Fig. 3.7c and *BounceIt* V_1-V_2 simulations in Fig. 2.4 build an ellipse out of multiple *IN-FIN* line endpoints. Ellipse radii (a,b) follow from *KE conservation* equation (3.7b).

$$KE(unit V_1,V_2)=\tfrac{1}{2}M_1 1^2+\tfrac{1}{2}M_2 1^2=\tfrac{1}{2}\cdot 8 \begin{cases} M_1=7 & minor\ radius\ a=\sqrt{2\cdot KE/M_1}=\sqrt{8}=2.828 \\ M_2=1 & major\ radius\ b=\sqrt{2\cdot KE/M_2}=\sqrt{8/7}=1.069 \end{cases}$$

(This is a quite *non*-traditional ellipse construction! A more traditional construction is given in the exercise section at end of the preceding Chapter 2.)

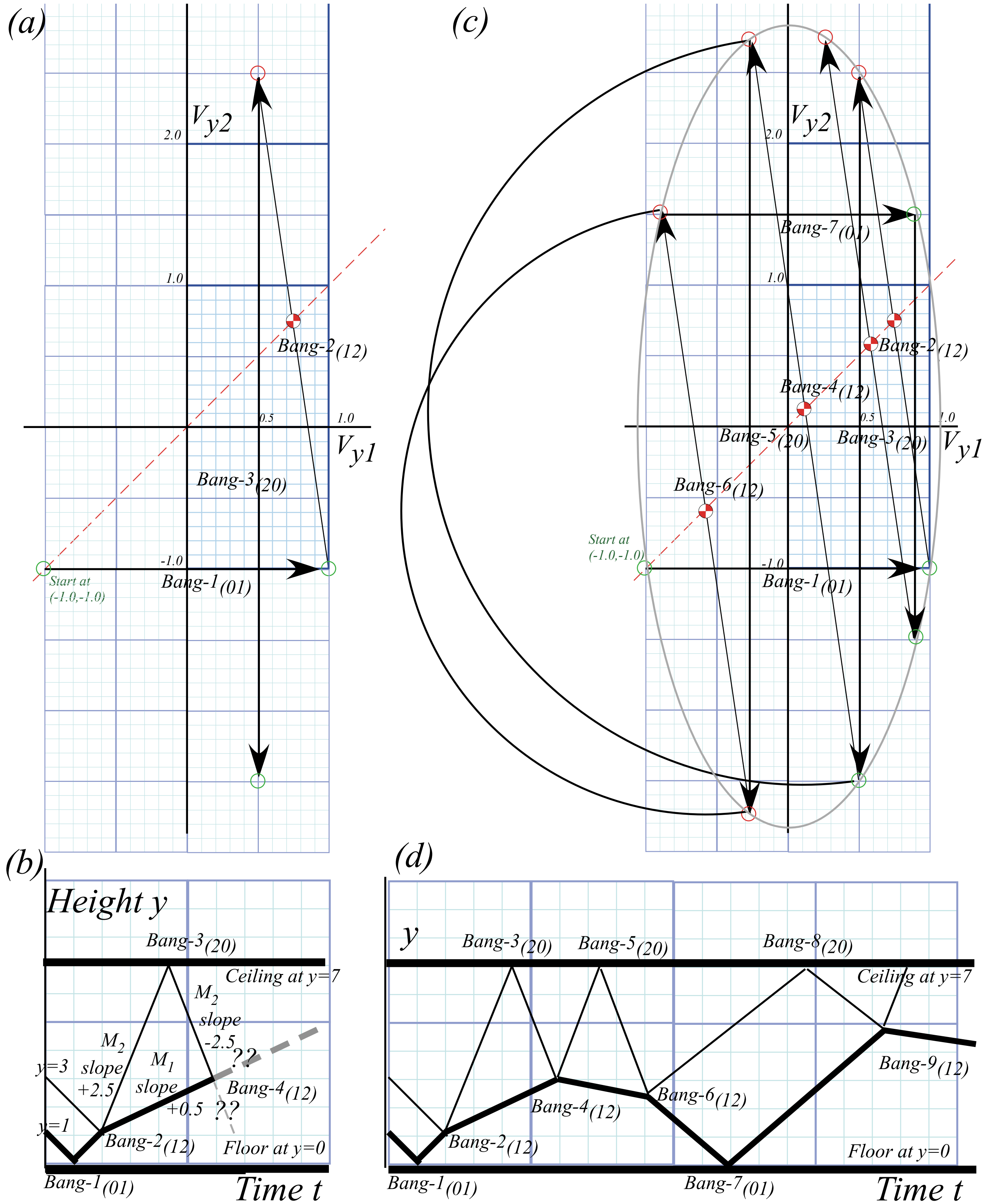


Fig. 3.7 X-2 (M_1=70gm,M_2=10gm)Collision sequence. (a-b) Up to Bang-4$_{(12)}$. (c-d) Up to Bang-9$_{(12)}$.

Using vector notation in space-space plots

Balance equation (2.2) concisely sums up preceding constructions or plots of elastic collisions.

$$\begin{matrix}\left(V_1^{FIN}+V_1^{IN}\right)/2=V^{COM}\\ \left(V_2^{FIN}+V_2^{IN}\right)/2=V^{COM}\end{matrix} \quad \text{or:} \quad \begin{matrix}V_1^{FIN}=2V^{COM}-V_1^{IN}\\ V_2^{FIN}=2V^{COM}-V_2^{IN}\end{matrix} \qquad (2.2)_{redone} \ \text{or:} \ (3.0)$$

This a more concise notation uses *vector* equations or column arrays.

$$\begin{matrix}v_1^{FIN}=2V^{COM}-v_1^{IN}\\ v_2^{FIN}=2V^{COM}-v_2^{IN}\end{matrix} \quad \text{is written:} \quad \begin{pmatrix}v_1^{FIN}\\ v_2^{FIN}\end{pmatrix}=\begin{pmatrix}2V^{COM}-v_1^{IN}\\ 2V^{COM}-v_2^{IN}\end{pmatrix}=2\begin{pmatrix}V^{COM}\\ V^{COM}\end{pmatrix}-\begin{pmatrix}v_1^{IN}\\ v_2^{IN}\end{pmatrix} \qquad (3.1)$$

It saves writing two (=)'s and two (-)'s. Also, each *column vector* may be labeled by a "fat" letter.

$$\mathbf{v}^{FIN}=\begin{pmatrix}v_1^{FIN}\\ v_2^{FIN}\end{pmatrix}=\vec{v}^{FIN}, \qquad \mathbf{V}^{COM}=\begin{pmatrix}V^{COM}\\ V^{COM}\end{pmatrix}=\vec{V}^{COM}, \qquad \mathbf{v}^{IN}=\begin{pmatrix}v_1^{IN}\\ v_2^{IN}\end{pmatrix}=\vec{v}^{IN}. \qquad (3.2)$$

The *Gibbs vector form* of equation (3.4) or (4.1) uses fat-**v** and/or over-arrow-$\vec{v}$ for column vectors.

$$\mathbf{v}^{FIN}=2\,\mathbf{V}^{COM}-\mathbf{v}^{IN}, \qquad \text{or:} \qquad \mathbf{V}^{COM}=\frac{\mathbf{v}^{IN}+\mathbf{v}^{FIN}}{2}. \qquad (3.3)$$

Note vector $\mathbf{V}^{COM}$ bisecting the ($\mathbf{v}^{IN}+\mathbf{v}^{FIN}$)-parallelogram diagonal as per T-symmetry relation from (2.2) and Fig. 2.1. Here vectors $\mathbf{v}=(v_1, v_2)$ denote *two* particles each in *one*-dimension. More common is vector $\mathbf{v}=(v_x, v_y)$ (or $\mathbf{v}=(v_x, v_y, v_z)$) for *one* particle in *two*-dimensions (or three dimensions).

Fig. 3.8 shows how velocity **v(n)** vectors find results of *Bang-1*$_{(01)}$ and *Bang-2*$_{(12)}$ collisions in Fig. 3.7. What's new is a *space-space* y_2 *vs.* y_1 or *position-vector* **y(n)**-plot whose paths are *spatial-trajectories* or just plain *trajectories*. Space-time paths are found in Fig. 3.6 and Fig. 3.7 by transferring velocity slopes over to the space-space or space-time plot, but vectors in Fig. 3.8 simplify this process. Again, ideally small masses called *point masses* are assumed.

Construction steps in Fig. 3.8 show how to transfer direction of each velocity vector **v(n)** from the V_2 *vs.* V_1 plot so it points away from start point **y(n)** in the y_2 *vs.* y_1 plot. *Step-0* does this by drawing initial velocity **v(0)**=*(-1,-1)* to point away from our given initial position **y(0)**=*(1,3)*. Then you extend that **v**-vector until it hits the floor (as **v(0)** does at **y(1)**=*(0,2)*), or else hits the collision line *(y_2=y_1)* (as **v(1)** does at **y(2)**=*(1,1)*), or else hits the ceiling (as **v(2)** does at **y(3)**=*(2.2,7)*.). Each such "hit" is a Bang, *Bang-1*$_{(01)}$ at **y(1)**, *Bang-2*$_{(12)}$ at **y(2)**, or *Bang-3*$_{(20)}$ at **y(3)**. Then from each *Bang-n* position point **y(n)** is drawn the next **v(n)**-velocity vector from the V_2 *vs.* V_1 plots. This process continues in exercises that lead to Fig. 3.9 and beyond.

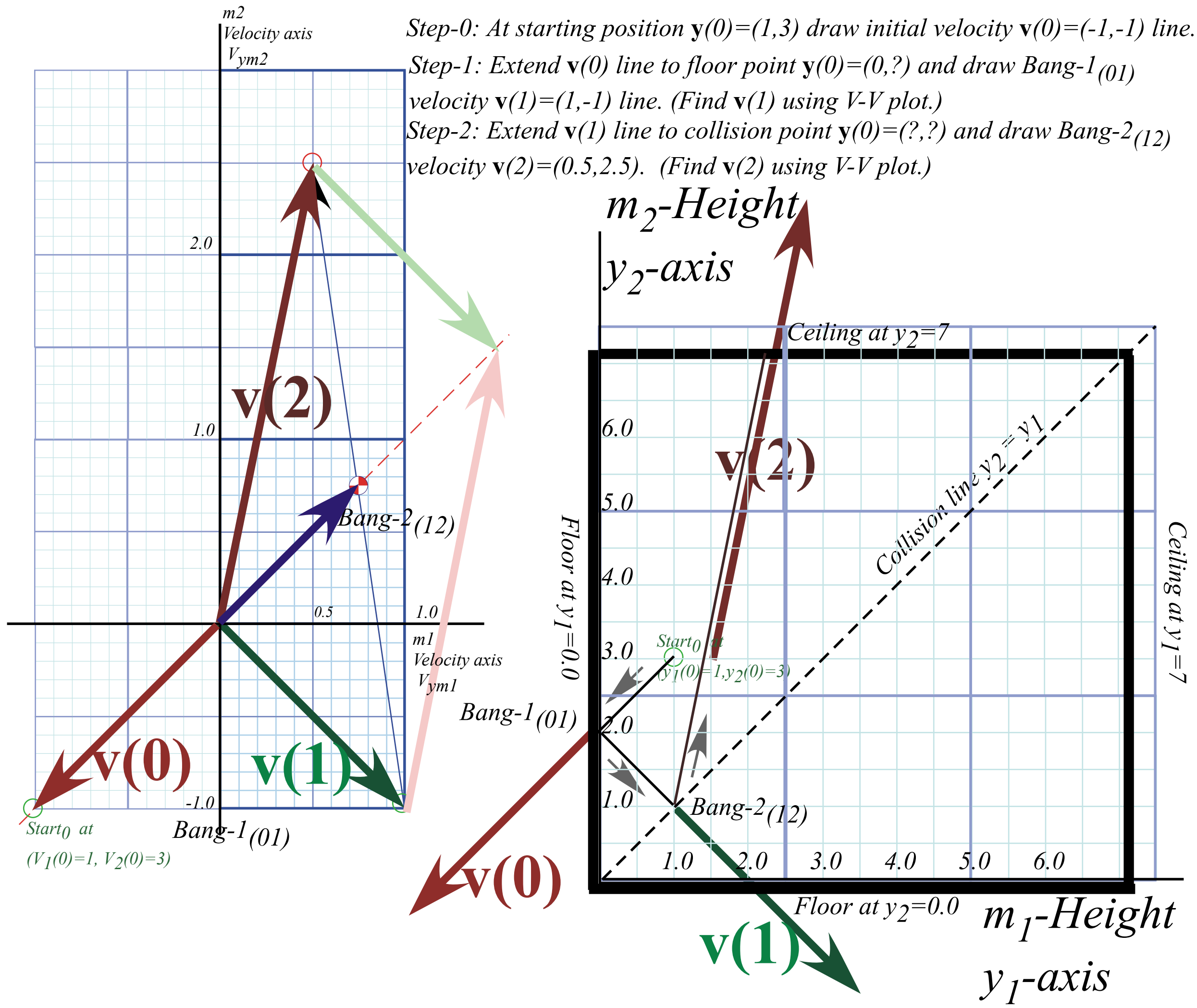


Fig. 3.8 Vector collision velocity diagrams with Velocity-Velocity space and space-space.

Help! I'm trapped in a triangle.

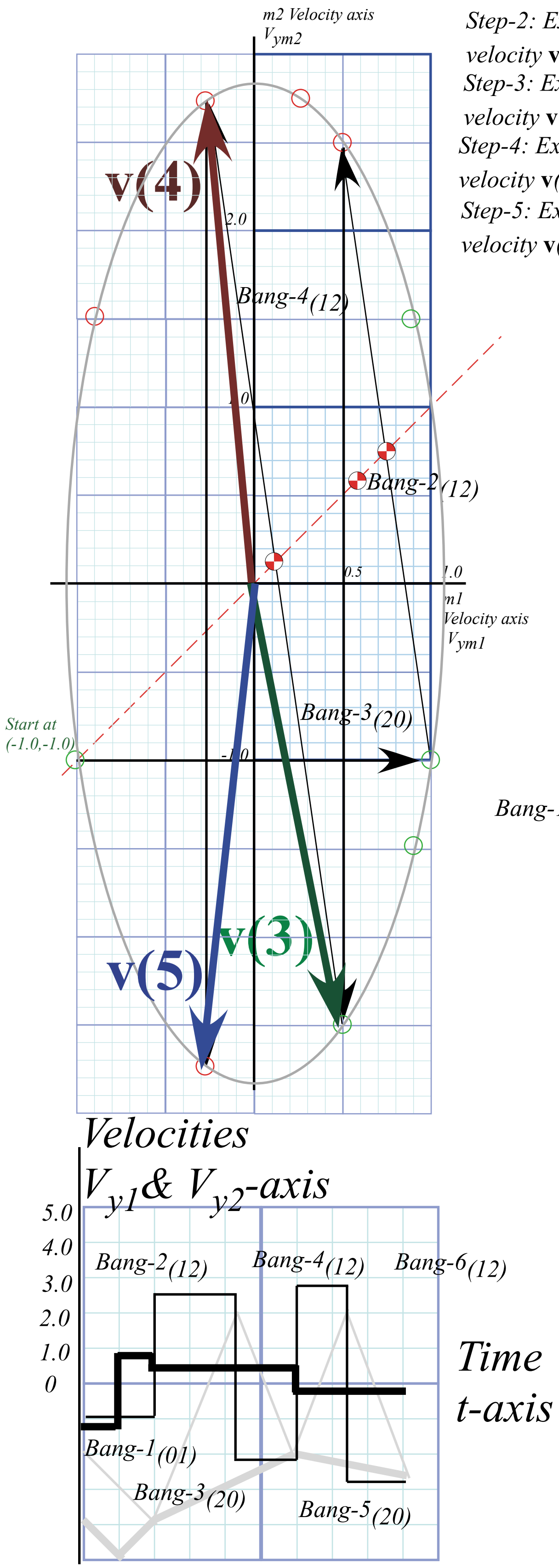
Trajectories in (y_1, y_2)-plots are confined to the triangle above the *45°*-collision line. Our model keeps m_2 above m_1. The right-hand "ceiling" in the figures never is hit because m_1 always is knocked down by m_2 before it touches the ceiling, and m_2 never sees the floor because m_1 is in the way. Modern physicists beware! Quantum theory doesn't encourage this feature. Quantum objects are wavelike and may, depending on inter-particle potentials, pass right through each other!

Step-2: Extend **v**(2) *line to ceiling point* **y**(3)=(?,7.1) *and draw Bang-3*$_{(20)}$ *velocity* **v**(3)=(1,-1) *line. (Find* **v**(3) *using V-V plot.)*

Step-3: Extend **v**(3) *line to collision point* **y**(4)=(?,?) *and draw Bang-4*$_{(12)}$ *velocity* **v**(4)=(0.5,2.5). *(Find* **v**(4) *using V-V plot.)*

Step-4: Extend **v**(4) *line to ceiling point* **y**(4)=(?,7.1) *and draw Bang-5*$_{(20)}$ *velocity* **v**(5)=(1,-1) *line. (Find* **v**(5) *using V-V plot.)*

Step-5: Extend **v**(5) *line to collision point* **y**(6)=(?,?) *and draw Bang-6*$_{(12)}$ *velocity* **v**(6)=(0.5,2.5). *(Find* **v**(6) *using V-V plot.)*



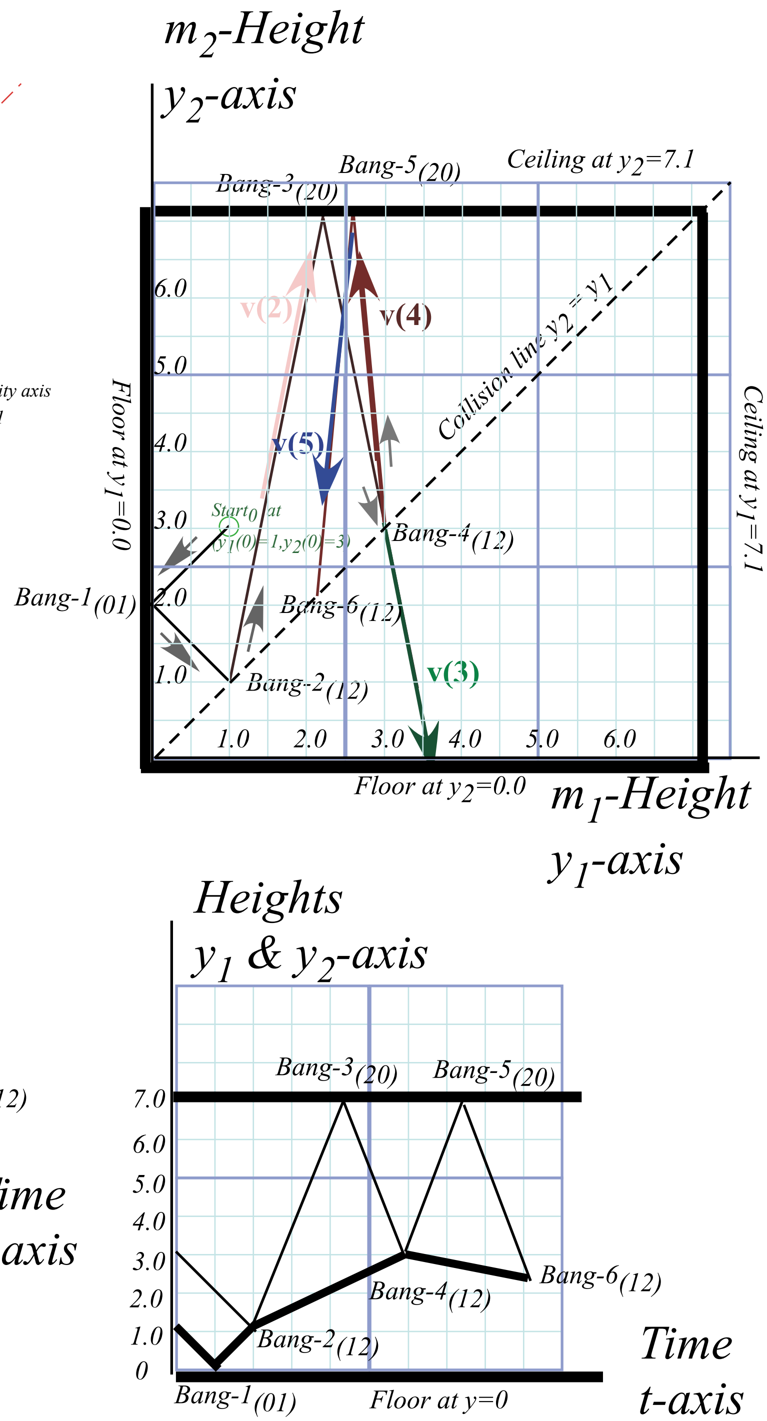


Fig. 3.9 Vector collision diagrams continued with velocity-time and space-time plots added.

Two balls in 1D vs. one ball in 2D

For ball-Earth collisions involving ceiling or floor, the paths bounce in the space-space plot like they're inside a box. One component V_1 or V_2 changes each time by changing ±sign. Off the floor: (V_1, V_2) changes to $(-V_1, V_2)$, off ceiling: (V_1, V_2) changes to $(V_1, -V_2)$. It is like a *single* particle bouncing around a pool table. Here (V_1, V_2) acts like (V_X, V_Y) in *two* dimensions, so *two* particles in *one*-dimension use graphs similar to *one* particle in *two* dimensions, an interesting analogy for quantum theory.

Angle of incidence=Angle of reflection (or NOT)

When paths bounce off the floor and ceiling in the space-space plot, the angle of incidence equals the angle of reflection just as light rays reflect off mirrors. (Newton imagined little light *corpuscles* bouncing around.) It is customary to measure path angles from the *normal* or perpendicular to a mirror so a normal bisects the angle between the incident and reflected paths.

For m_1-m_2 *Bangs* off the $45°$-collision line, the bisecting line has the slope $-M_1/M_2=-7$. It is like having mirror facets at slope $M_2/M_1=1/7$ along the $45°$-collision line. For equal-mass-$(M_1=M_2)$ balls, or *one* ball in *two* dimensions, the bisector line slope at the $45°$-collision line is -1 or $-45°$ and the collision line acts like a unit-slope mirror on a triangular billiard table. It is not quite that simple if $M_1/M_2 \neq 1$.

Consider the two collisions *Bang-3*$_{(20)}$ and *Bang-4*$_{(12)}$ in Fig. 4.12. Velocity **v(2)** bounces off the ceiling in *Bang-3*$_{(20)}$ into **v(3)**, whose velocity slope is close to the mass-ratio M_1/M_2 which is *7:1* here. So the next collision *Bang-4*$_{(12)}$ bounces **v(3)** off the diagonal into **v(4)** which is close to –**v(3)**. It's followed by another ceiling bounce *Bang-5*$_{(20)}$ into **v(5)** heading down for another collision *Bang-6*$_{(12)}$.

Bang force

Lower Fig. 3.9 has a *velocity vs. time* plot next to a space-time plot. (A *y-t* plot in gray is by the *V-t* plot.) Each *Bang* means a change in velocity for any particle involved in the collision. By *Newton's 2nd law* each change in *momentum*, $m\mathbf{v}$ to $m(\mathbf{v}+\Delta\mathbf{v})$, requires a force impulse $\mathbf{F}\cdot\Delta t= m(\Delta\mathbf{v})$ on each mass that changes. Shortly, we study ways to deal with this **F**.

Kinematics versus Dynamics

The velocity-velocity (v_1,v_2) plots, such as the left side of Fig. 4.12, fall in a category known as *kinematics*, or *momentum analysis*, which is concerned with how things are <u>going</u>, where they're headed, or what is their *velocity* or *momentum* and *energy*. (*kinos* means movement.)

In contrast, the space-time plots, such as the right side of Fig. 4.12, fall in a category known as *dynamics*, or *coordinate analysis*, which is concerned with how things are <u>located</u>, where they <u>are</u>, or what are their *coordinate* or *position* and *time* schedules. (*dynos* means change.) We introduced the *space-space* (x_1,x_2) *plot*, another geometric or *trajectory* representation of dynamics.

Before going on, let's compare how *kinos* and *dynos* play out in classical Newtonian physics versus their corresponding roles in quantum physics. This is a preview for later Units.

Dynos and Kinos: Classical vs. quantum theory

In Newtonian physics, a precise position plot (y_k *vs.* time) lets you find a precise velocity plot, too, and, a velocity plot (V_k *vs.* time) lets you find a position plot if you know starting position values. (We did just that in Fig. 4.7 and Fig. 4.11.) In calculus, finding position from velocity values is called *integration*, and finding velocity from position values is called *differentiation*. Of the two, the latter is formally easier but numerically and experimentally more sensitive to imprecision and noise.

In quantum physics, having a precise velocity plot renders a position plot meaningless and *vice-versa*! Werner Heisenberg was first to state this quantum idea, now known as Heisenberg's Principle. If you know momentum exactly, that means a uniform wave is everywhere, and all positions are equally possible. If you know position exactly, that means every momentum is possible, implying a "wave-bomb" about to blow up the universe!

All this sounds crazy to most of us who are born-and-bred Aristotelean-to-Newtonian students. It is difficult enough to go from Aristotle's what-you-see-is-what-you-get *(WYSIWYG)* universe to Newton's corpuscular one. A quantum universe is yet another step removed on the *WYSIWYG* scale.

A way to see the quantum universe (Perhaps, it is *the* way.) is to learn about *wave* kinematics and dynamics without Newtonian corpuscles and see how waves *mimic* corpuscles and do so quite cleverly. The quantum universe is a W*YDA*W*YG* (*waves*-you-*don't* see-*are*-what-you-get) world!

So our plan is to cast classical Newtonian kinematics and dynamics in a form that carries over into vibration and wave kinematics or dynamics. It is done by *analogy* with classical waves such as sound waves, water waves, and (most important) light waves. Many classical wave analyses invoke corpuscles (including, for Newton, light waves) so these analogies, like any analogy, need critical use of a well sharpened Occam's razor. Above all, symmetry (and *same*-try) principles must be taken seriously.

IF-ellipse geometry of Ch. 2 related velocity, momentum and energy, and Ch. 4 derived space-time paths. This relates Lagrangian and Hamiltonian mechanics and leads to geometries of relativity and quantum mechanics where space-space and space-time plots relate to modern physics in subtle ways involving *inverse* space-time.

Exercise 1.4.1: Construct a history of a *4:1* mass ratio bounce. $x_1(0)=1.5$, $x_2(0)=3.0$, $v_1(0)=-1$, $v_2(0)=-1$ *Ceiling height=7.0*.(For bottom row: *Ceiling height=6.0*) The *4:1* mass ratio case is surprisingly periodic. Note, position **y(n)**-vectors of the *Bang-n* points in Fig. 3.9 are not drawn to reduce clutter.

Exercise 1.4.2: Continue Fig. 3.7 and Fig. 3.8 with more steps using same ceiling *height=7.0*. Continue until you reach the "gameover" point of last possible M_1-M_2 collision assuming the floor is open after *Bang-1* so both masses fall thru indefinitely. When and where do they last collide?

Chapter 4 Matrix operator analysis of collisions

Analysis of collision chain dynamics can be done by *matrix algebra* and *symmetry operator* geometry that is used in quantum theory. This provides an opportunity to learn about these techniques in a more "down-to-Earth" setting of classical bang physics while discovering some surprising effects.

Doing collisions with matrix products

Fig. 4.1 shows a big mass $m_1=49$ hittting a little mass $m_2=1$ about ten times off the ceiling before being halted. This tests our collision precision! To check our results we use our previous vector equation (3.1) to make a *matrix equation* in (4.1) with $V^{COM}=(m_1v_1+m_2v_2)/M$ and total mass $M = m_1+ m_2$.

$$\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} = \begin{pmatrix} 2V^{COM}-v_1^{IN} \\ 2V^{COM}-v_2^{IN} \end{pmatrix} (4.1)_{repeated} \begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} = \begin{pmatrix} 2\dfrac{m_1v_1+m_2v_2}{m_1+m_2}-v_1 \\ 2\dfrac{m_1v_1+m_2v_2}{m_1+m_2}-v_2 \end{pmatrix} = \frac{1}{M}\begin{pmatrix} m_1v_1-m_2v_1+2m_2v_2 \\ 2m_1v_1+m_2v_2-m_1v_2 \end{pmatrix} \quad (4.1a)$$

(Let $v_1^{IN}=v_1$ and $v_2^{IN}=v_2$ here.) Vector equation (4.1a) is converted to matrix equation $\mathbf{v}^{FIN}=\mathbf{M}\bullet\mathbf{v}$ in (4.1b).

$$\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} = \frac{1}{M}\begin{pmatrix} m_1-m_2 & 2m_2 \\ 2m_1 & m_2-m_1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad (4.1b)$$

Each *IN*-to-*FIN* bang is a $\mathbf{v}^{FIN}=\mathbf{M}\bullet\mathbf{v}^{IN}$ operation (4.2a). Matrix product $\mathbf{M}\bullet\mathbf{N}$ (4.2b) is bang-**M** *after* bang-**N**.

$$\mathbf{M}\bullet\mathbf{v} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} Aa+Bb \\ Ca+Db \end{pmatrix} (4.2a) \qquad \mathbf{M}\bullet\mathbf{N} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} Aa+Bb & Ac+Bd \\ Ca+Db & Cc+Dd \end{pmatrix} \quad (4.2b)$$

Matrix **M** acts column-by-column on another matrix **N** as it does on vector **v**. Off-ceiling bang matrix **C** $=\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ changes (v_1, v_2) to (v_1, $-v_2$). $\mathbf{C}\bullet\mathbf{M}$ is a ceiling bang **C** *following* a 2-ball collision matrix **M**.

$$\mathbf{C}\bullet\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{M}\begin{pmatrix} m_1-m_2 & 2m_2 \\ 2m_1 & m_2-m_1 \end{pmatrix} = \frac{1}{M}\begin{pmatrix} m_1-m_2 & 2m_2 \\ -2m_1 & m_1-m_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} = \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \quad (4.3)$$

A chain of p factors acts on $\mathbf{v}^{IN}$ to give $\mathbf{v}^{FIN-p}=(\mathbf{C}\bullet\mathbf{M})^p\bullet\mathbf{v}=(\mathbf{C}\bullet\mathbf{M})\bullet(\mathbf{C}\bullet\mathbf{M})\bullet(\mathbf{C}\bullet\mathbf{M})\bullet\ldots(\mathbf{C}\bullet\mathbf{M})\bullet\mathbf{v}$ in (4.4) with *(p=5)* double-bangs $\mathbf{C}\bullet\mathbf{M}=\begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix}$ *after* a floor-bounce $\mathbf{F}=\begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$ or the *11* bangs plotted in Fig. 4.1.

$$\begin{pmatrix} v_1^{FIN-11} \\ v_2^{FIN-11} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot\begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot\begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot\begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot\begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}\cdot\begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}\begin{pmatrix} v_1^{IN}=-1 \\ v_2^{IN}=-1 \end{pmatrix}_{\text{(INITIAL (0))}}$$

$$\begin{pmatrix} v_1^{FIN-11} \\ v_2^{FIN-11} \end{pmatrix} = \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix}\cdot\begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix}\cdot\begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix}\cdot\begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix}\cdot\begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix}\begin{pmatrix} v_1=1 \\ v_2=-1 \end{pmatrix}_{\text{(after Bang-1)}} \quad (4.4)$$

$$\begin{pmatrix} v_1^{FIN-11} \\ v_2^{FIN-11} \end{pmatrix} = \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix}\cdot\begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix}\cdot\begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix}\cdot\begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix}\begin{pmatrix} v_1=0.92 \\ v_2=-2.92 \end{pmatrix}_{\text{(after Bang-3)}} \quad \text{Note: } \begin{pmatrix} 0.92 \\ -2.92 \end{pmatrix} = \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} v_1^{FIN-11} \\ v_2^{FIN-11} \end{pmatrix} = \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix}\cdot\begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix}\cdot\begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix}\begin{pmatrix} v_1=0.7664 \\ v_2=-4.606 \end{pmatrix}_{\text{(after Bang-5)}} \quad \text{Note: } \begin{pmatrix} 0.7664 \\ -4.606 \end{pmatrix} = \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix}\begin{pmatrix} 0.92 \\ -2.92 \end{pmatrix}$$

$$\begin{pmatrix} v_1^{FIN-11} \\ v_2^{FIN-11} \end{pmatrix} = \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix}\cdot\begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix}\cdot\begin{pmatrix} v_1=0.5515 \\ v_2=-5.924 \end{pmatrix}_{\text{(after Bang-7)}} \quad \text{Note: } \begin{pmatrix} 0.5515 \\ -5.924 \end{pmatrix} = \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix}\begin{pmatrix} 0.7664 \\ -4.606 \end{pmatrix}$$

$$\begin{pmatrix} v_1^{FIN-11} \\ v_2^{FIN-11} \end{pmatrix} = \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix}\cdot\begin{pmatrix} v_1=0.2925 \\ v_2=-6.768 \end{pmatrix}_{\text{(after Bang-9)}}$$

Even after 9 bangs, the larger m_1 is still rising at velocity $v_1=0.2925$. V after *Bang-11*$_{(02)}$ is in (4.5). Now big m_1 is nearly stopped and little m_2 is coming down at $v_2=-7.071$ with practically *all the initial energy*!

$$\begin{pmatrix} v_1^{FIN-11} \\ v_2^{FIN-11} \end{pmatrix} = \begin{pmatrix} v_1 = 0.0100 \\ v_2 = -7.071 \end{pmatrix}_{(\text{after Bang-11})} \tag{4.5}$$

Look out below! As m_1 is turned back it crosses v_2-axis (where $v_1=0$) in Fig. 4.1a. The greatest curvature (acceleration or force) for the path of m_1 is between *Bang-8* or *9* and *Bang-14* or *15* in Fig. 5.1b because that is when m_2 is busiest in its apparently furious effort to beat back poor old m_1.

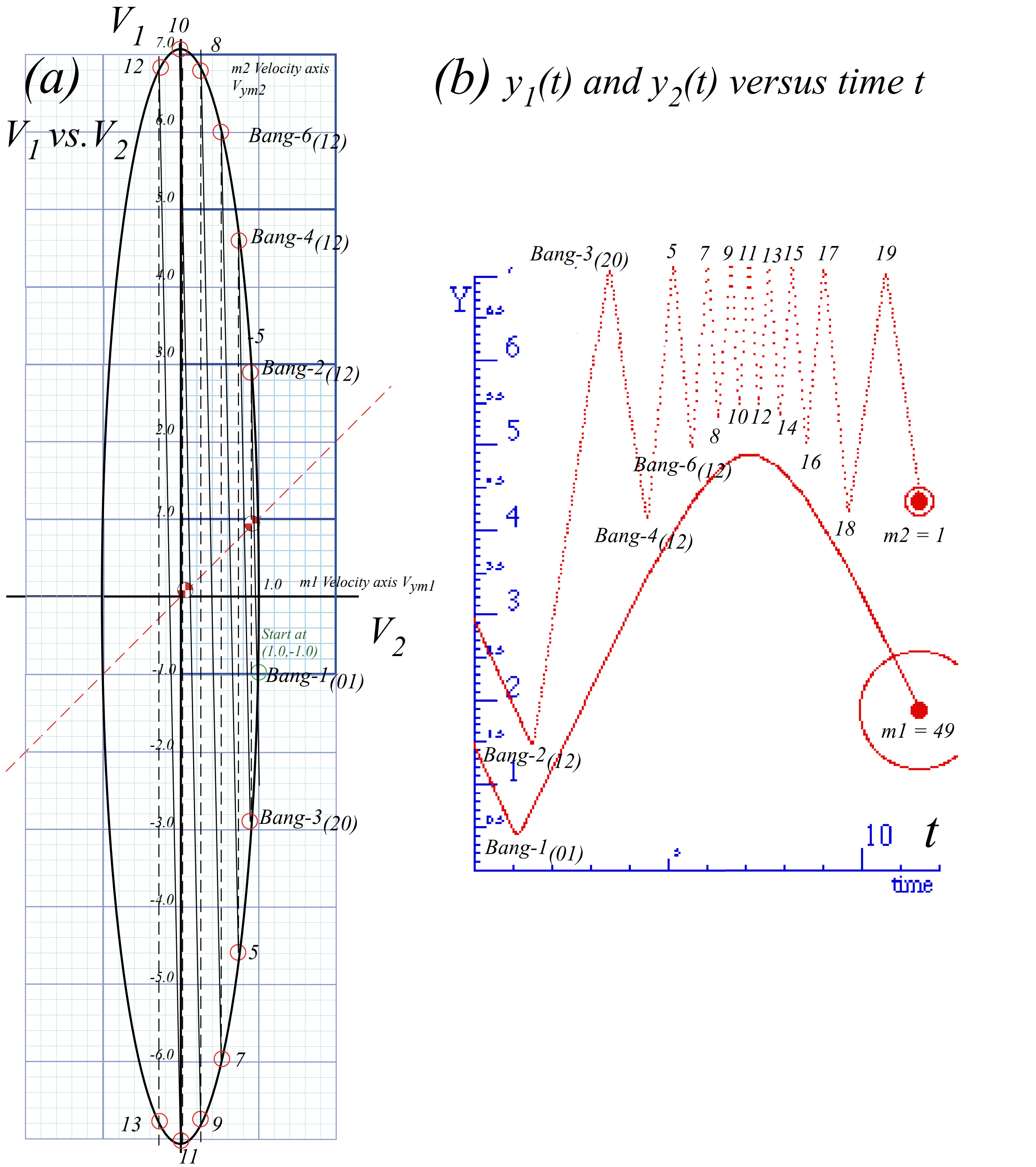


Fig. 4.1 Multiple Bangs of the m_1=49 and m_2=1 superball system. (a) V vs V plot. (b) Y vs time.

Big m_1 is repelled down by repeated m_2 hits and gains speed as m_2 loses it. If no floor intervenes to rebound m_1 there comes a final bang that leaves m_2 slower than m_1 who falls away so m_2 can't hit it again. (Exercises 4.1 and 4.2, ask you to find this a *game-over* point for various cases.)

However, if a floor intervenes, then a 2nd floor-bounce matrix $\mathbf{F}=\begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$ bangs (v_1, v_2) to $(-v_1, v_2)$ and bounces ball-m_1 back up to start the whole process over again. Ball-m_1 does a similar up-down trip but not exactly the one shown in Fig. 4.1. Next we see how to predict the periodicity of such processes.

Except for floor bounces, the m_1-ball in Fig. 4.1 experiences a smoother flight than in Fig. 3.9 where a more massive m_2-ball jerks it quite severely. A smaller mass m_2 has less momentum-per-bang and gives a quasi-continuous force field for m_1. Later we will use this to derive a funny kind of force and potential field theory from this.

Rotating in velocity space: Ticking around the collision clock

Here is an example of geometry of slope ratios. If you view the ellipse in Fig. 4.1a lower-edge-on (After exercise to finish it!) you may see it as a circular clock with each double-bang (*odd*-bangs *1,3,5,…*) rotating the **v**-vector like a clock hand ticking equal-angle jumps around a dial.

You can make an energy ellipse $(2E=m_1v_1^2+m_2v_2^2)$ like Fig. 4.1(a) or Fig. 4.2(a) into an energy *circle* $(2E=\mathsf{V}_1^2+\mathsf{V}_2^2)$ like Fig. 4.2(b) by rescaling velocity (v_1, v_2) to $(\mathsf{V}_1=v_1\cdot\sqrt{m_1},\ \mathsf{V}_2=v_2\cdot\sqrt{m_2})$.

$$\mathsf{V}_1=v_1\cdot\sqrt{m_1},\qquad \mathsf{V}_2=v_2\cdot\sqrt{m_2}\qquad \text{where: } 2E=m_1v_1^2+m_2v_2^2=\mathsf{V}_1^2+\mathsf{V}_2^2 \tag{4.6}$$

Big-V variables replace little-v's by setting $(v_1=\mathsf{V}_1/\sqrt{m_1},\ v_2=\mathsf{V}_2/\sqrt{m_2})$ in matrix relation (4.1).

$$\begin{pmatrix} v_1^{FIN_1} \\ v_2^{FIN_1} \end{pmatrix}=\frac{1}{M}\begin{pmatrix} m_1-m_2 & 2m_2 \\ 2m_1 & m_2-m_1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}(5.1)_{repeated}\qquad \begin{pmatrix} \mathsf{V}_1^{FIN_1}/\sqrt{m_1} \\ \mathsf{V}_2^{FIN_1}/\sqrt{m_2} \end{pmatrix}=\frac{1}{M}\begin{pmatrix} m_1-m_2 & 2m_2 \\ 2m_1 & m_2-m_1 \end{pmatrix}\begin{pmatrix} \mathsf{V}_1/\sqrt{m_1} \\ \mathsf{V}_2/\sqrt{m_2} \end{pmatrix}(5.7)$$

Clearing scale factors $\sqrt{m_k}$ gives the following big-V matrix relations to replace (4.1) above.

$$\mathsf{V}^{FIN_1}=\begin{pmatrix} \mathsf{V}_1^{FIN_1} \\ \mathsf{V}_2^{FIN_1} \end{pmatrix}=\frac{1}{M}\begin{pmatrix} m_1-m_2 & 2\sqrt{m_1m_2} \\ 2\sqrt{m_1m_2} & m_2-m_1 \end{pmatrix}\begin{pmatrix} \mathsf{V}_1 \\ \mathsf{V}_2 \end{pmatrix}=\mathsf{M}\bullet\mathsf{V}\ (4.8)\quad \mathbf{V}^{FIN_2}=\begin{pmatrix} \mathbf{V}_1^{FIN_2} \\ \mathbf{V}_2^{FIN_2} \end{pmatrix}=\frac{1}{M}\begin{pmatrix} m_1-m_2 & 2\sqrt{m_1m_2} \\ -2\sqrt{m_1m_2} & m_1-m_2 \end{pmatrix}\begin{pmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{pmatrix}=\mathbf{C}\bullet\mathbf{M}\bullet\mathbf{V} \tag{4.9}$$

The trick is to notice a Pythagorean relation $x^2+y^2=1$ for the circular bang-matrix components.

$$\left(\frac{m_1-m_2}{M}\right)^2+\left(\frac{2\sqrt{m_1m_2}}{M}\right)^2=\frac{m_1+m_2}{m_1+m_2}=1 \tag{4.10a}$$

The matrix is defined using $\sin\theta$ and $\cos\theta$ shown for $m_1=49$ and $m_2=1$ and angle $\theta=16.26°$ in Fig. 4.2(c).

$$Define:\ \cos\theta\equiv\left(\frac{m_1-m_2}{M}\right)\quad and:\ \sin\theta\equiv\left(\frac{2\sqrt{m_1m_2}}{M}\right) \tag{4.10b}$$

A *1-Bang* matrix is a V space *reflection* at θ. A *2-Bang* matrix is clockwise *rotation* by angle $-\theta=-16.26°$.

$$\begin{pmatrix} \mathsf{V}_1^{FIN_1} \\ \mathsf{V}_2^{FIN_1} \end{pmatrix}=\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \mathsf{V}_1 \\ \mathsf{V}_2 \end{pmatrix}(4.11)\qquad \begin{pmatrix} \mathbf{V}_1^{FIN_2} \\ \mathbf{V}_2^{FIN_2} \end{pmatrix}=\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{pmatrix}=\begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix}\begin{pmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{pmatrix}(4.12)$$

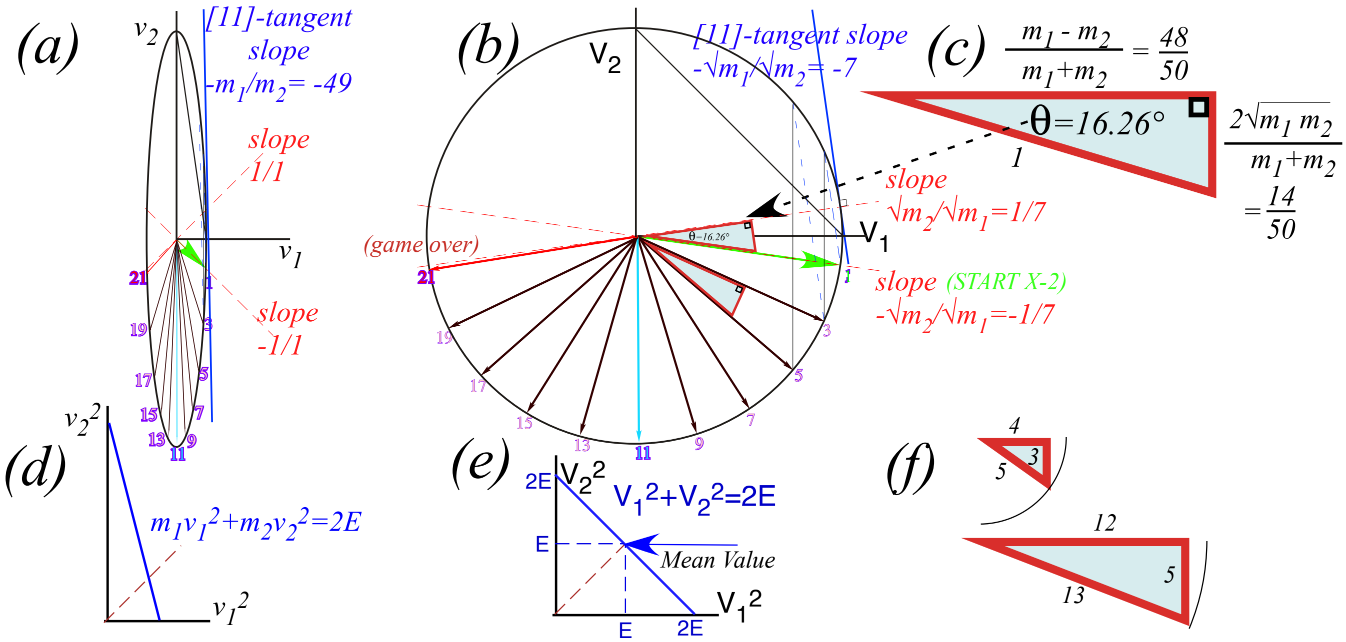


Fig. 4.2 Velocity-velocity clocks. (a) Energy ellipse (As in Fig. 5.1) (b-c) Energy bang-clock angles (d) *Velocity-squared E-plot. (e) Mass-scaled V-squared E-plot. (f) Integral right triangles*

Matrix (4.12) reduces N-double-bang chains like (4.4) to one formula! If $\theta = 16.26°$ in (4.12) is replaced by $N\theta = 81.30°$ (for $N=5$ double-bangs) then (4.13a) results. Relation ($\mathsf{V}_1=v_1\sqrt{m_1}$,$\mathsf{V}_2=v_2\sqrt{m_2}$) gives (4.13b).

$$\begin{pmatrix} \mathbf{V}_1^{FIN_{2N}} \\ \mathbf{V}_2^{FIN_{2N}} \end{pmatrix} = (\mathbf{C}\bullet\mathbf{M})^N \bullet \mathbf{V} = \begin{pmatrix} \cos N\theta & \sin N\theta \\ -\sin N\theta & \cos N\theta \end{pmatrix} \begin{pmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{pmatrix} = \begin{pmatrix} \cos 5\theta & \sin 5\theta \\ -\sin 5\theta & \cos 5\theta \end{pmatrix} \begin{pmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{pmatrix} = \begin{pmatrix} 0.1512 & 0.9885 \\ -0.9885 & 0.1512 \end{pmatrix} \begin{pmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{pmatrix} (for: N=5) \qquad (4.13a)$$

Then we see that (4.13b) easily gives (4.5). Recall: $(\mathbf{C}\bullet\mathbf{M})^N$ *follows* initial floor $\mathbf{F}$ reflection $(v_1, v_2)=(1,-1)$.

$$\begin{pmatrix} v_1^{FIN_{2N}} \\ v_2^{FIN_{2N}} \end{pmatrix} = \begin{pmatrix} \cos N\theta & \sqrt{\frac{m_2}{m_1}}\sin N\theta \\ -\sqrt{\frac{m_1}{m_2}}\sin N\theta & \cos N\theta \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \cos 5\theta & \frac{1}{7}\sin 5\theta \\ -7\sin 5\theta & \cos 5\theta \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0.1512 & 0.1412 \\ -6.9194 & 0.1512 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0.010 \\ -7.071 \end{pmatrix} for: \begin{cases} N=5 \\ \frac{m_1}{m_2}=49 \end{cases} \qquad (4.13b)$$

Without a 2nd floor-bounce-back operation $\mathbf{F}$, this sequence ends at the "game-over" point near bang-*21*. (See exercise 5.1.) Matrix group products allow us to "engineer" collision sequences.

Statistical mechanics: Average energy

If two balls of mass $m_2=1$ and $m_1=7$ do many bangs it happens that the small ball goes faster on the average than the bigger one. How much faster? The arrows on the scaled velocity clock in Fig. 4.2(b) are uniformly distributed around its circle, and after another floor bounce will be uniformly distributed again. (Fig. 5.2(b) shows only m_1-m_2-bounce arrows. m_2-ceiling-bounce-arrows fill up the upper half.) A ball's velocity and momentum must sum and average to zero otherwise it is not remaining bounded in the region between the floor and the ceiling. But, what is average squared-velocity v^2 of each ball?

An energy plot in the space $(V_1)^2$ *vs* $(V_2)^2$ of scaled velocity-squared helps to answer this. The result is a $45°$ line shown in Fig. 4.2(e) that corresponds to total kinetic energy conservation ($KE=E$). Points on the circle in Fig. 4.2(b) get mapped onto that $45°$ line in Fig. 4.2(e) by KE conservation.

$$(V_1)^2 + (V_2)^2 = 2\ KE = m_1(v_1)^2 + m_2(v_2)^2 \tag{4.14}$$

The average of all points on the $45°$ line is its bisector.

$$(V_1)^2 = KE = (V_2)^2 \qquad \text{or:} \qquad m_1(v_1)^2 = KE = m_2(v_2)^2$$

This gives the average velocities or *root-mean-square-speeds* v_1^{rms} and v_1^{rms} of m_1 and m_2.

$$v_1^{rms} = \sqrt{KE/m_1} \qquad\qquad v_2^{rms} = \sqrt{KE/m_2} \tag{4.15}$$

Each ball, regardless of mass, averages an equal share (50% if there are just two) of the total energy. So, if m_1 is 7 times m_2 then the mean speed of m_2 is $\sqrt{7}=2.65$ times faster than that of m_1. The 1st bang in Fig. 3.4b has m_2 only 2.5 times faster than m_1 was before collision. But after that $v_1=0.5$ so then $v_2/v_1=5.0$ and root ratio is $\sqrt{v_2/v_1} = \sqrt{5} = 2.236..$

Bonus: Rational right triangles

Geometry often offers interesting numerics. In this case, the general right triangle in Fig. 4.2(c) makes integer or rational fraction solutions to the Pythagorean sum $a^2+b^2=c^2$ such as the famous $(a=3,b=4,c=5)$ right triangle. Perfect-square mass values (m_1 and $m_2=1, 4, 9, 16, 25, 36, 49, 81, 100,\ldots$) give integral valued right triangle altitude $a=\sqrt{(4\, m_1 \cdot m_2)}$, base m_1-m_2, and hypotenuse m_1+m_2. Examples in Fig. 4.2 are $(a=14,b=48,c=50)$ for ($m_1=49$, $m_2=1$) and $(a=12,b=5,c=13)$ for ($m_1=9$, $m_2=4$).

Reflections about rotations: It's all done with mirrors

In 1843 Hamilton discovered his *quaternion algebra* $\{\mathbf{1},\mathbf{i},\mathbf{j},\mathbf{k}\}$, a mathematical jewel. In 1930 Pauli used related *spinor matrices* $\{1,\sigma_X,\sigma_Y,\sigma_Z\}$. We label Pauli matrix σ_Z as *sigma-A*$=\sigma_A$ (*A for Asymmetric*) and σ_X as *sigma-B*$=\sigma_B$ (*B for Balanced*). They are Hamilton's $\mathbf{k}$ and $\mathbf{i}$ with an imaginary factor $i=\sqrt{-1}$ attached.

$$\boldsymbol{\sigma}_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_Z = i\mathbf{k} \tag{4.15a}$$

$$\boldsymbol{\sigma}_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_X = i\mathbf{i} \tag{4.15b}$$

Other matrices, *sigma-C*$=\sigma_C$ (*C for Circular*) and *sigma-0*$=\sigma_0$ (*0 for "Origin"*) are products like $\sigma_A\sigma_B$ or σ_A^2.

$$\boldsymbol{\sigma}_A\boldsymbol{\sigma}_B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_C = i\sigma_Y = -\mathbf{j} \tag{4.15c}$$

$$\boldsymbol{\sigma}_A\boldsymbol{\sigma}_A = \boldsymbol{\sigma}_B\boldsymbol{\sigma}_B = \boldsymbol{\sigma}_C\boldsymbol{\sigma}_C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \boldsymbol{\sigma}_0 = 1 = \mathbf{1} \tag{4.15d}$$

Hamilton's $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ square to $\mathbf{-1}$. ($\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=\mathbf{-1}$) That is like $i^2=-1$. But, Pauli-σ's square to $+1$. ($1=\sigma_X^2=\sigma_Y^2=\sigma_Z^2$.)

We now relate σ-matrices to simple super-ball collision reflections and rotations shown in Fig. 4.2. For example, the σ_A is our "ceiling bounce" **C** in (4.3) and our "floor bounce" **F** in (4.3) is just $-\sigma_A$.

$$\boldsymbol{\sigma}_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \mathbf{C} \tag{4.15e}$$

$$-\boldsymbol{\sigma}_A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{F} \tag{4.15f}$$

A geometric view of σ_A (or $-\sigma_A$) is *mirror reflection* thru Cartesian x (or y) axes in Fig. 4.3a while σ_B (or $-\sigma_B$) is reflection thru mirror planes tilted at angle $\pi/4$ (or $-\pi/4$) *between* x-y axes in Fig. 4.3b. General reflection σ_ϕ thru a mirror plane tilted at angle $\phi/2$ (Fig. 4.3c) is a sum (4.15c) of $\sigma_A \cos\phi$ and $\sigma_B \sin\phi$.

$$\boldsymbol{\sigma}_\phi = \boldsymbol{\sigma}_A \cos\phi + \boldsymbol{\sigma}_B \sin\phi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cos\phi + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} \qquad (4.15c)$$

Like all reflections, σ_ϕ must square-to-one. *($\sigma_\phi^2=\mathbf{1}$)* It does so because $\sigma_A^2=\mathbf{1}=\sigma_B^2$ and $\sigma_A\sigma_B = -\sigma_B\sigma_A$. We test σ_ϕ on unit vectors $\hat{\mathbf{x}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\hat{\mathbf{y}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and see that matrix algebra checks with geometry in Fig.4.3c.

$$\boldsymbol{\sigma}_\phi \bullet \hat{\mathbf{x}} = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \quad (4.16a) \qquad \boldsymbol{\sigma}_\phi \bullet \hat{\mathbf{y}} = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\phi \\ -\cos\phi \end{pmatrix} \quad (4.16b)$$

Fig. 4.3d geometry shows a product $\sigma_2\sigma_1$ of any two reflection matrices is a *rotation matrix R*. In Fig. 4.3d $\sigma_\phi\sigma_A$ is right-hand rotation $\mathbf{R}_{+\phi}$ but $\sigma_A\sigma_\phi=\mathbf{R}_{-\phi}$ in Fig. 4.3e is left-handed. Rotation angle ϕ is *twice* the angle $^{\phi}/_{2}$ between mirrors. Direction of rotation $\sigma_2\sigma_1$ is from 1st mirror (of σ_1) to 2nd mirror (of σ_2).

$$\boldsymbol{\sigma}_\phi \bullet \boldsymbol{\sigma}_A = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} \bullet \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \quad (4.17a) \qquad \boldsymbol{\sigma}_A \bullet \boldsymbol{\sigma}_\phi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \bullet \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \quad (4.17a)$$

For example, rotation $\sigma_B\sigma_A$ is by *+90°* and $\sigma_A\sigma_B$ is by *-90°*. Rotation $\sigma_A(-\sigma_A)=(-\sigma_A)\sigma_A$ is by *±180°*.

Through the clothing store looking glass

The rotation in V_1 *vs* V_2 space of Fig. 4.2b is a product of ceiling bounce and m_1-m_2 collision that are each a reflection. An even simpler example of paired-reflection rotation is a clothing store mirror in Fig. 4.4a. It lets you swing two mirrors like doors to view multiple images of yourself. If you set the angle between mirrors to $\phi/2=30°$ as in Fig. 4.3 d-e or to *60°* as in Fig. 4.4a then you see yourself rotated by twice that angle. Images are turned *120°* counter-clockwise in the right mirror and clockwise (*-120°)* in the left mirror of the latter.

The sketches in Fig. 4.4a oversimplify the actual images shown by photos of a real mirror pair. The single reflections for σ_A are not shown in the sketch but clearly visible in photos where the σ_A and σ_ϕ images both have backwards text and a left hand image of the original right hand. This is corrected in the *(-120°)*-rotated $\sigma_A\sigma_\phi$ image and the *(+120°)*-rotated $\sigma_\phi\sigma_A$ image.

A special case is rotation $\sigma_A(-\sigma_A)=(-\sigma_A)\sigma_A$ by *±180°* due to setting mirrors at exactly $\phi/2=90°$ as in Fig. 5.4b. The result is known as a *corner-reflector image*. Wherever you stand while viewing a *90°* corner you see your image centered and rotated±*180°* to face you but it is *not reflected*. A *90°* corner image is as others see you, complete with a readable monogram on your jacket and your right hand on the right side.

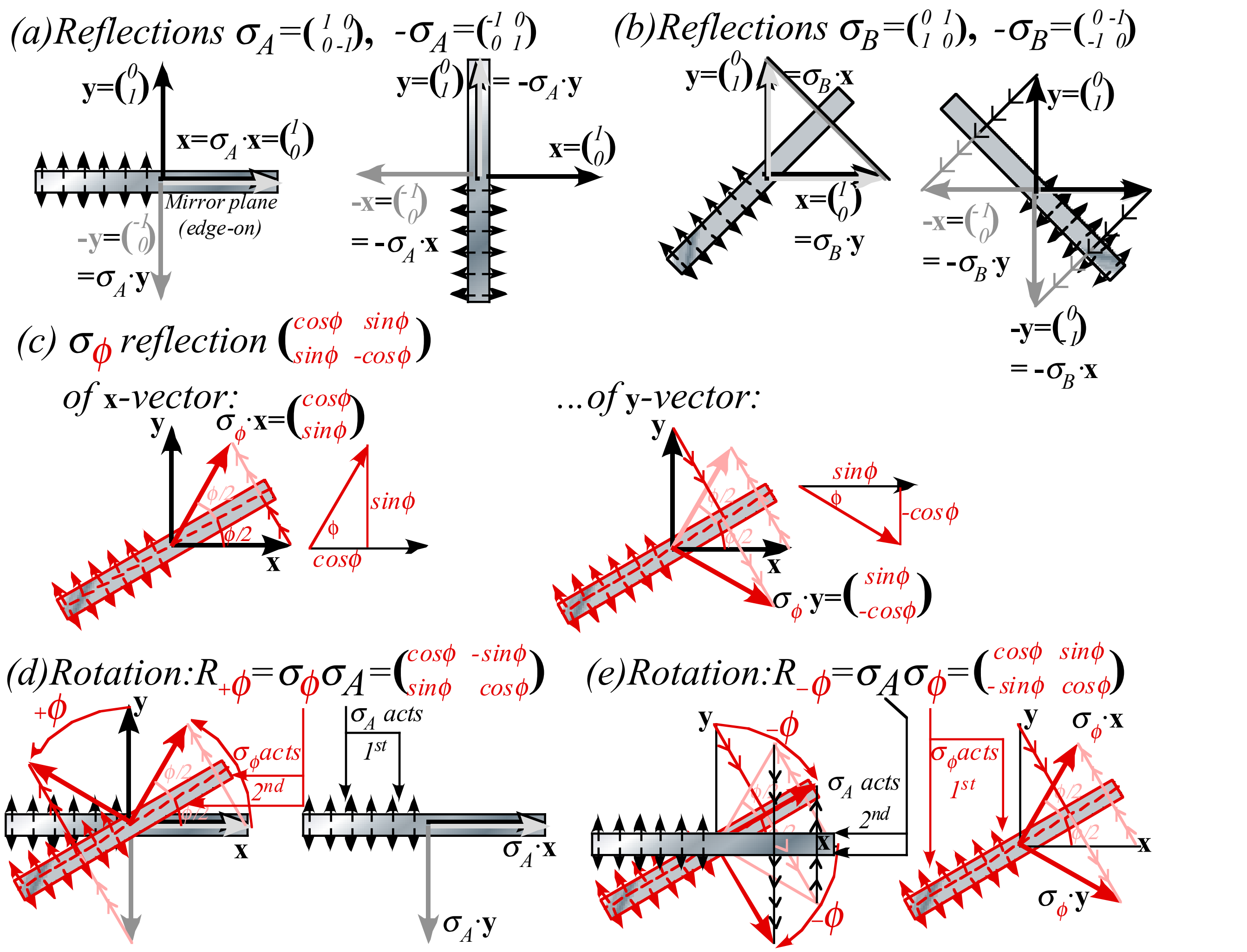


Fig. 5.3 Mirror geometry (a)$\pm\sigma_A$, (b) $\pm\sigma_B$, (c) σ_ϕ. Right-vs-left-handed rotation (e) $\sigma_\phi\sigma_A$ (f) $\sigma_A\sigma_\phi$.

How fundamental are reflections?

A product of two reflections is a rotation $\mathbf{R}_\phi=\sigma_2\sigma_1$, but two rotations just give another rotation $\mathbf{R}_{\phi+\theta}= \mathbf{R}_\phi\mathbf{R}_\theta$ and never a reflection. This makes reflections more basic and *productive* than rotations. On the other hand, you cannot do a reflection of a real solid object without entering an Alice-in-Wonderland looking-glass-world. Moving every atom in a classical object to a reflected position (without destroying it) is unthinkable! Yet, we easily rotate semi-solid objects (like your eyeballs while reading this).

Waves, on the other hand, are very *un-solid* and do reflection effortlessly. Rotation takes twice the effort as seen in the looking glass images of Fig. 4.4. This is one reason reflection operations are so basic to the study of wave mechanics, quantum theory, and relativistic symmetry as we will see later. They are elementary symmetry generators in a 1D world. A 1D translation by distance *a* is two reflections by 1D mirrors separated by distance *a/2*.

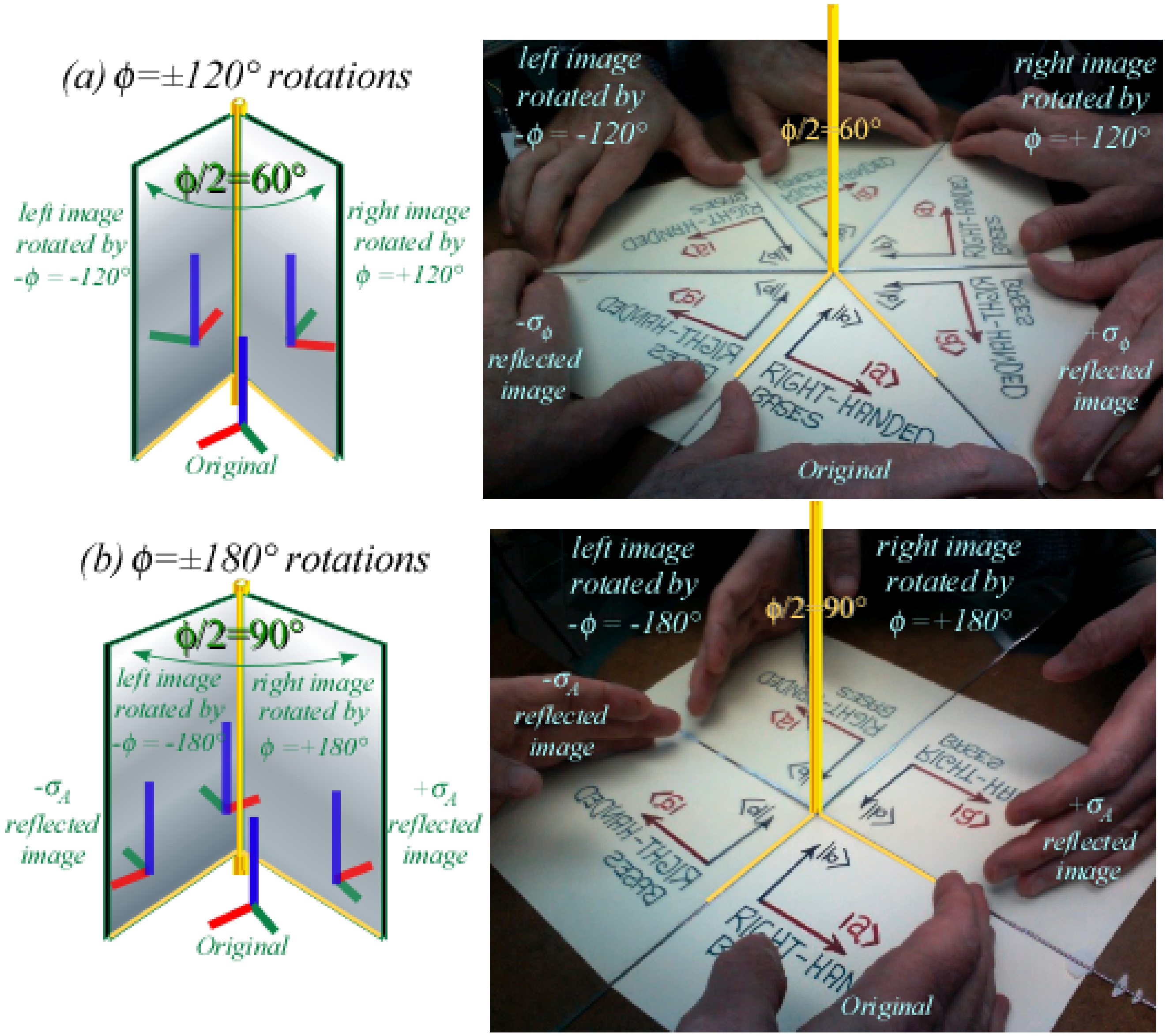


Fig. 5.4 Mirror reflections and rotations with relative angle: (a) 60° (b) 90° (corner reflector images).

Symmetry operation **R** or σ is defined by what it does to unit vectors $\hat{\mathbf{x}}=\binom{1}{0}$ and $\hat{\mathbf{y}}=\binom{0}{1}$ as σ_ϕ (4.16) is done in Fig. 5.3c. That matrix does that same operation to any and all vectors $\mathbf{v}=\binom{v_1}{v_2}=v_1\hat{\mathbf{x}}+v_2\hat{\mathbf{y}}$ in the space.

$$\boldsymbol{\sigma}_\phi \bullet \mathbf{v} = v_1\boldsymbol{\sigma}_\phi \bullet \hat{\mathbf{x}} + v_2\boldsymbol{\sigma}_\phi \bullet \hat{\mathbf{y}} = v_1\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix} + v_2\begin{pmatrix}\sin\phi\\ -\cos\phi\end{pmatrix} = \begin{pmatrix}\cos\phi & \sin\phi\\ \sin\phi & -\cos\phi\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} \tag{4.18}$$

A way to distinguish rotation and reflection operators is by the *determinant* det|M| of their matrices.

$$\det|\mathbf{M}| = \det\left|\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right| = a\cdot d - b\cdot c \qquad\qquad \det\left|\begin{pmatrix}u_x & v_x\\ u_y & v_y\end{pmatrix}\right| = u_x\cdot v_y - v_x\cdot u_y = |\mathbf{u}||\mathbf{v}|\sin\angle_{\mathbf{u}}^{\mathbf{v}}$$

A determinant of matrix M quantifies the space (area in this case) enclosed by vectors in M's rows or columns (**u** and **v** enclose a parallelogram in this case).

Determinant of a rotation (or reflection) is *+1* (or *–1*). Reflected area or angle in Fig. 1.3 is negative.

$$\det\left|\mathbf{R}_\phi\right| = \det\left|\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}\right| = \cos^2\phi + \sin^2\phi = +1 \qquad \det\left|\boldsymbol{\sigma}_\phi\right| = \det\left|\begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}\right| = -\cos^2\phi - \sin^2\phi = -1$$

Determinants track matrix multiplication. The determinant of a product is a product of determinants.

$$\det|\mathrm{M}\cdot\mathrm{N}| = (\det|\mathrm{M}|)(\det|\mathrm{N}|) = \det|\mathrm{N}\cdot\mathrm{M}|$$

Thus, two reflections each with $det|\sigma|=-1$ form a product of $det|\sigma_1\,\sigma_2|=(-1)(-1)=+1$, that of a rotation. This also shows a product of rotations cannot make a negative-det-matrix and so cannot be a reflection.

Chapter 5 Introducing Force, Potential Energy, and Action

Analysis of *force* is one of the trickier parts of Newtonian mechanics and one that Aristotle seems to have not done so well. We, like Aristotle, feel we know force after being pushed and pulled around by it most of our conscious lives. Aristotle related force directly to mass and its motion. If he ever wrote equations then, perhaps, Aristotle's equation would be $F=MV$.

NOT! MV is *momentum*, not force. Galileo and Newton may be the first to realize that force should be equated to a *change* in momentum. A famous equation $F=Ma$ equates force to mass or *inertia* M times *acceleration a*, the *rate of change* of velocity. It is called *Newton's 2nd law* or *NEWTON-TWO*.

$$F = \frac{dP}{dt} = M\frac{dV}{dt} = M \cdot a \tag{5.0}$$

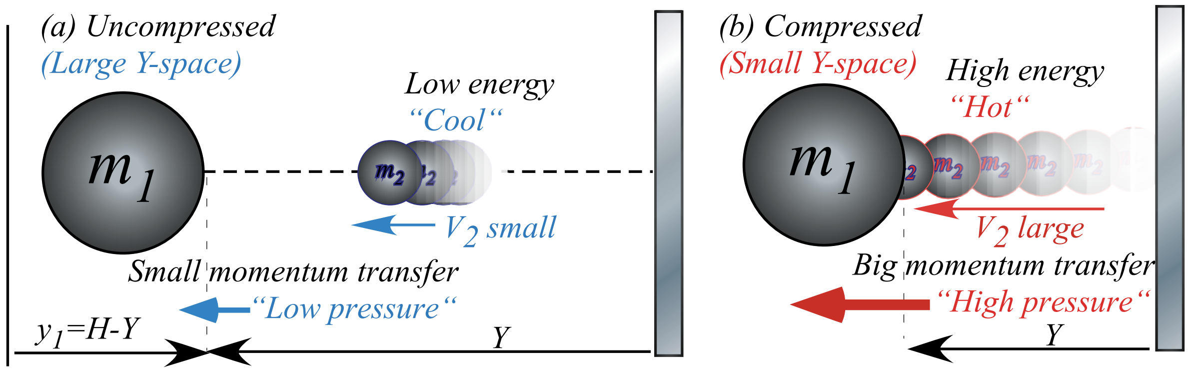


Fig. 5.1 Big mass m_1 feels "force field" or "pressure" of small ball rapidly bouncing to and fro.

MBM force fields and potentials

Motion of m_1 in Fig. 4.1b suggests a *kinetic model* and a *potential force field*. Boltzman uses this to derive gas force laws for volume, temperature, and pressure. As a big m_1-ball squeezes space (*volume*) for a tiny m_2-ball in Fig. 5.1, the speed v_2 and energy $\frac{1}{2} m_2 v_2^2$ of m_2 increases. So does the *momentum transfer rate* or *bang-force* on m_1. Energy is related to *temperature* and bang-force is related to *pressure*. A madly bouncing m_2 is like a 1-atom gas getting hot when its Y-space is compressed as in Fig. 5.1b.

A "double-whammy" hits the m_1-ball as it closes in with velocity v_1 toward m_2 and wall $(Y=0)$:

(1) Bang rate B with m_2 increases with shrinking distance $2Y$ traveled by m_2 between m_1 and wall.

(2) Increased velocity v_2 (due to v_1) increases momentum $m_2 v_2$ and ΔP transferred to m_1 by each bang.

(3) Increased velocity v_2 (due to v_1) increases bang rate even more. It's really a *triple* whammy!

If m_1 is huge (say $1kg$) compared to atom or molecule m_2 (say $(2/3)\cdot 10^{-27}kg$ for an H-atom), the speed v_1 of the macro-mass m_1 may be negligible compared to typical atomic speeds v_2 of 10^3 m/s. Then we ignore (2) and (3) due to tiny v_1 in a so-called *isothermal* model. An *adiabatic* model includes them.

Isothermal model force laws

For each bang of m_1, atom m_2 travels distance $2Y$ back & forth between m_1 and ceiling at Y. If v_1 is slow, the time Δt between bangs is $2Y$ divided by velocity v_2 of m_2. *Bang rate B* is the inverse: $B=1/\Delta t$.

$$\Delta t = 2Y/v_2 \text{ (seconds per bang)} \quad (5.1a) \qquad B = 1/\Delta t = v_2/2Y \text{ (bangs per sec)} \quad (5.1b)$$

Each head-on bang of big m_1 on small m_2 changes velocity of m_2 from $-v_2$ to $+v_2^{FIN}$ as shown in Fig. 5.2.

$$(for: m_1>>m_2): \qquad v_2^{FIN} = v_2+2v_1 \qquad (\approx v_2 \text{ for: } v_2>>v_1) \qquad (5.2)$$

Added speed for m_2 is $2v_1$, twice that of incoming m_1. (V-V-plot Fig. 5.2 assumes large-m_1.) The change ΔP of momentum m_2v_2 is the difference between FIN value $+m_2v_2^{FIN}$ and IN value $-m_2v_2$.

$$\Delta P = (+m_2v_2^{FIN})-(-m_2v_2)=2m_2v_2+2m_2v_1 \qquad (\approx 2m_2v_2 \text{ for: } v_2>>v_1) \qquad (5.3)$$

So, if "atomic" velocity v_2 is large compared to v_1 it gives a *bang-force* $F=B\cdot\Delta P = \Delta P/\Delta t$ on m_1.

$$BP=\Delta P/\Delta t = F = 2m_2v_2(v_2/2Y) = m_2v_2^2/Y \qquad (5.4)$$

So a *force field* $F=2\cdot KE/Y$ on m_1 due to m_2 is proportional to $KE={}^1/_2m_2v_2^2$ or *temperature* T of m_2. *Boltzman's constant k* of proportionality ($KE=kT$) gives an *isothermal* force law $FY=2kT$. It is a 1-D version of *Boyle's ideal gas law: PV=2kT*. Here a ceiling tries to keep energy or "temperature" of m_2 constant in spite of m_1 constantly trying to anger poor little m_2.

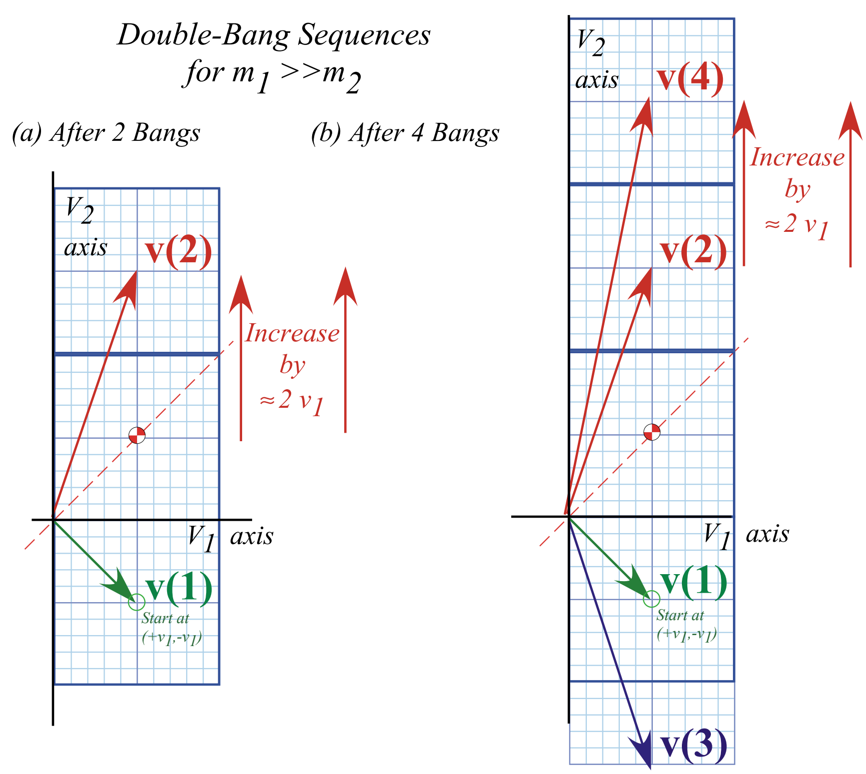


Fig. 5.2 Large mass-ratio ($m_1/m_2>>1$) bounce sequence. (Compare to Fig. 3.2a.)

Adiabatic model force laws

An elastic ceiling can't give or take energy so each m_1 bang adds velocity $2v_1$ to v_2 at rate $B=v_2/2Y$ (5.1). As m_1 closes at speed v_1 it reduces distance $2Y$ that m_2 travels. Rate B grows due to more v_2 *and* less Y.

$$\frac{dv_2}{dt}=2v_1B \qquad =2v_1\frac{v_2}{2Y}, \qquad y=v_1t=H-Y, \qquad \frac{dy}{dt}=v_1=-\frac{dY}{dt} \qquad (5.5a)$$

We cancel time and v_1 to show this force is inverse-Y- *cubed*, a lot "harder" than inverse-Y in (5.4).

$$\frac{dv_2}{dt}=\left(\frac{dY}{dt}\frac{dv_2}{dY}=-v_1\frac{dv_2}{dY}\right)=2v_1\frac{v_2}{2Y}, \quad \frac{dv_2}{v_2}=-\frac{dY}{Y}, \quad v_2=\frac{const.}{Y}=\frac{v_2^{IN}Y(t=0)}{Y}, \quad F=\frac{m_2v_2^2}{Y}=m_2\frac{(const.)^2}{Y^3} \quad (5.5b)$$

This is an adiabatic or "fast" force law. Collisions are so fast that an isothermal-seeking "Robin Hood" in the ceiling hasn't time to steal m_2's energy if it's judged too energy-rich or give energy back if m_2 becomes energy-poor. So m_2 can get hotter and hit m_1 harder and more often as gap Y shrinks.

Conservative forces and potential energy functions

Either force law (5.4) and (5.5b) actually conserves the energy of the big-m_1 ball in the long run. By that we mean that m_1 will come out with practically the same energy that it had when it went in.

The adiabatic case is easier to see. Each bang conserves energy as demanded by the kinetic energy *(KE)* conservation relation (3.5a). Little-ball velocity $v_2=const./Y$ from (5.5b) is used here.

$$E=\frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2=\frac{1}{2}m_1v_1^2+\frac{1}{2}m_2\left(\frac{const.}{Y}\right)^2=const. \qquad (5.6)$$

The first term is m_1's kinetic energy KE_1. The second term, which is really m_2's kinetic energy, is called m_1's potential energy PE_1 or just plain PE. It is labeled $U(Y)$ and varies only according to height Y of m_1.

$$E=KE_1+PE=\frac{1}{2}m_1{v_1}^2+U(Y) \qquad \text{where:} \quad PE=U(Y)=\frac{1}{2}m_2\left(\frac{const.}{Y}\right)^2 \qquad (5.7)$$

The PE is energy that m_1 lends to m_2 each time m_1 moves a distance ΔY closer so m_1 does a little bit of *work* ΔW on m_2. *Work* is defined as *force times distance*. $(\Delta W=F\cdot\Delta Y)$ *Power*, the rate of work done, is defined as *force times velocity*. Here distance is a small ΔY and the force F in (5.5b) is $m_2\,const.^2/Y^3$. But "work" force might be plus-or-minus $(\pm)m_2\,const.^2/Y^3$. Which sign? (+) or (–)? Conflicting sign conventions make force-physics confusing. The sign depends on how *force* and *direction* are defined. (It's all relative!)

Is it +or-? Physicist vs. mathematician and the 3rd law

A physicist's force F^{phys} is what is felt by a free object (Here that's m_1.) whose motion is driven by force field $F=F^{phys}$. A mathematician's force F^{math} is what is needed to *hold back* that object in the force field. (How *apropos*! A physicist lets it *go* but a constipated mathematician holds it *back*!) They differ by (±) sign only, that is, $F^{math}=-F^{phys}$, and F^{math} is the equal-but-opposite force by an object (m_1 here) on its field or force agent(s) (m_2 here). (This is essentially *Newton's 3rd law*. *(NEWTON-THREE)*)

Force is momentum flow. Momentum is stuff that's conserved, so the flow rate F^{phys} of this stuff *into* an object m_1 must be balanced by an equal-but-opposite negative flow, $F^{math}=-F^{phys}$, *out of* the forcing agent(s) (m_2 here), and, *vice versa*, whatever flows *out of* m_1 flows *into* m_2. Momentum $\mathbf{p}=m\mathbf{v}$ and force **F** are both vector quantities and a ±sign gives direction to-or-fro, another confusing (±) sign to bother us. But, whatever the flow rate F^{phys} seen by m_1, then m_2 sees the opposite rate $F^{math}=-F^{phys}$.

Let's define positive Y and F direction to be *away* from the wall in Fig. 5.1. So incoming m_1 has negative velocity $v_1=-\Delta Y/\Delta t$, but after m_1 reverses $V=\Delta Y/\Delta t$ is positive. Positive $V=-v_1$ (increasing Y) and positive F^{phys} means both momentum and energy of m_1 are being *increased* by force F^{phys}. Each bit of energy or work $\Delta W=F^{phys}\Delta Y$ gained by m_1 is energy lost by the force-field's potential "bank" that is m_2. *($\Delta U=-\Delta W$)*

$$\Delta W=F^{phys}\cdot\Delta Y=-\Delta U \qquad where:\; F^{phys}=F(Y)=m_2\frac{(const.)^2}{Y^3} \tag{5.8}$$

In other words, power $\Pi=F^{phys}\cdot V$ *into* m_1 is power ($-\Delta U/\Delta t$) *out of* the field. ($V=\Delta Y/\Delta t$ is velocity of m_1.)

$$\Pi=F^{phys}\cdot V=-\frac{\Delta U}{\Delta t}=-\frac{\Delta U}{\Delta Y}\frac{\Delta Y}{\Delta t}=-\frac{\Delta U}{\Delta Y}V \qquad where:\; F^{phys}=-\frac{\Delta U}{\Delta Y} \tag{5.9}$$

But is this consistent? Does force F^{phys} in (5.8) really equal minus the slope of potential (5.7)?

$$F^{phys}=m_2\frac{(const.)^2}{Y^3} \quad \underset{with:}{consistent} \quad F^{phys}=-\frac{\Delta U}{\Delta Y}=-\frac{d}{dY}\frac{1}{2}m_2\left(\frac{const.}{Y}\right)^2=m_2\frac{(const.)^2}{Y^3} \tag{5.10}$$

It *checks!!* Note that $F=-\Delta U/\Delta Y$ needs that $^1/_2$ on kinetic energy $^1/_2\, m_2v_2^2$. (Recall discussion of (3.5).)

Isothermal "Robin Hood"and "Fed rules"

The isothermal case is a weird one. The little "force-field agent" m_2 maintains it kinetic energy at around the same initial value $^1/_2\, m_2v_2^2$ no matter how much the big mass m_1 loses or gains kinetic energy.

It's as though a "Robin-Hood" in the ceiling acts like a big Federal Reserve Bank. (*"The Fed."*) Whatever energy m_2 earns from m_1 is taken and stored away if its over initial deposit $\frac{1}{2}(m_2v_2^2)=T$, but if m_2 deposits falls below that value, the Fed makes up the difference. This energy or deposit limit is determined by a prevailing allowed "temperature" of the ceiling or the current money supply. (I'm not making this up. It's what happens in nature and very roughly what happens in our economy. It becomes a problem if the Fed stops being a Robin Hood and becomes a robbing hood!)

Under ideal conditions, force agent m_2 makes a much "softer" $1/Y$ force field $F=m_2v_2^2/Y$ given by (5.9). Definition (5.9) of force F as negative-U-slope $-\Delta U/\Delta Y$ then gives a $log_eY=lnY$ potential.

$$F^{phys} = m_2 \frac{v_2^2}{Y} = -\frac{\Delta U}{\Delta Y} \qquad \textit{implies:} \qquad U = -m_2 v_2^2 \ln(Y) \tag{5.11}$$

It may seem weird that we can define a useful potential while energy-funds are being siphoned in and out. Nevertheless, the ceiling "Robin Hood" is true to his word. (Analogy with "The Fed" ends here!) He puts back all the energy that m_1 gave up to m_2 (the potential U) on the way in, so that, except for small-change or "tips" left with m_2 after the final parting collision, m_1 recovers the energy it originally had. Such a force field, if determined by such a reliable potential, is also a *conservative* one. We discuss later the details of what is needed for general multi-dimensional fields to be labeled "conservative."

Oscillator force field and potential

Consider a mass m_1 between two walls and two little speeding m_2 masses as in Fig. 5.5. m_1 feels a force like that of an oscillator. As m_1 moves distance x off center the left wall space expands to $Y+x$ and the right wall space shrinks to $Y-x$. Two opposing forces (5.11) then are unbalanced. (Only $x^2, x^4, \ldots$ terms cancel.)

$$F^{total} = \frac{f}{1+x} - \frac{f}{1-x} = f\left[1-x+x^2-x^3\ldots\right] - f\left[1+x+x^2+x^3\ldots\right] = -2f\cdot x - 2f\cdot x^3 -$$

Here we let $Y=1$ be a unit interval and assume an isothermal kinetic constant $k \equiv 2f = 2m_2v_2^2$ for each side. For small x $(x<<1)$ the force F^{total} has a *linear* or *Hooke's law* form, and the potential U^{total} is *quadratic*.

$$F^{total} \simeq -k\cdot x = -\frac{\partial U^{total}}{\partial x} \qquad\qquad U^{total} \simeq \frac{1}{2}k\cdot x^2 = -\int F^{total}dx \tag{5.12}$$

Harmonic oscillator (HO) linear forces and *quadratic* potentials are, perhaps, the most useful ones in AMO physics because they approximate any stable system. Normally, they are analogized by a mass on a spring, rubber band, or pendulum, only rarely (if ever) in a context like Fig. 5.3. *HO* motion is sinusoidal $y(t)=A\sin(\omega t+\varphi)$ with *angular frequency* $\omega=\sqrt{k/m_1}$ and period $\tau=2\pi/\omega$ independent of the oscillator amplitude A or phase φ. The calculation of period for Fig. 5.3c is left as an exercise.

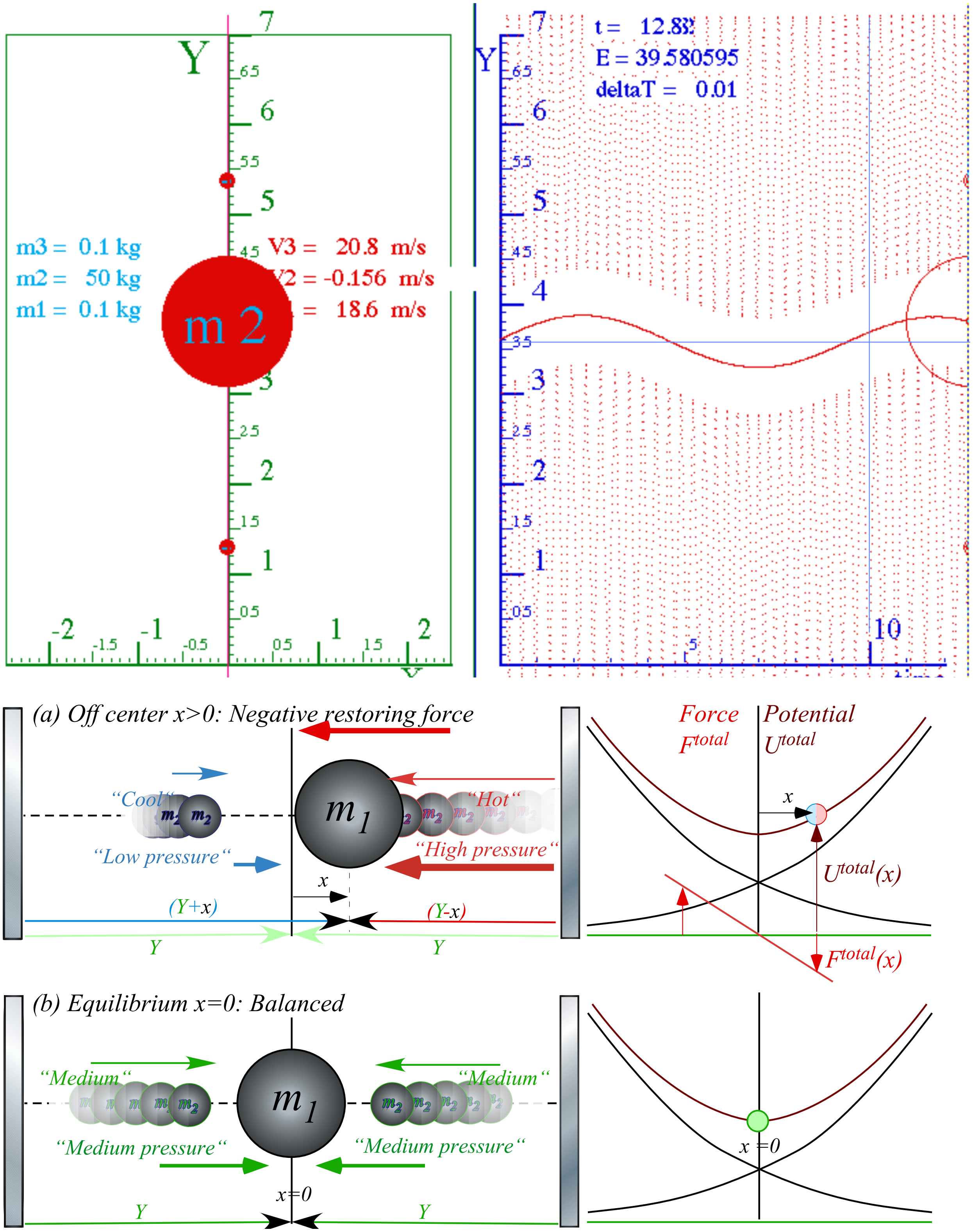


Fig. 5.3 Oscillator potential (a) Off center with (-)force (b) On center at equilibrium. (c) M=50 Quasi-harmonic oscillation of in adiabatic force of two m=0.1 masses of speed v_0=20 and range Y_0=3.

The 2nd most useful field is probably the Coulomb potential $U=-k/r$ and force $F=k/r^2$. (See Ch. 7 for electrostatics and Earth gravity, which also have 2D *HO* potentials at their cores.) After that, the 2D Coulomb $U=k\cdot ln(r)$ and $F=k/r$ is an important field shown in Unit 10. (The latter is like (5.11). A pair of them underlies Fig. 5.3 for the isothermal case.)

You should be warned that an oscillator like Fig. 5.3 is not as simple as it might appear, and as we will see, neither are springs, rubber bands, or pendulums. Also, balls bouncing against moving objects are particularly dicey devices. A simple model with one ball and one oscillating wall is called a *Fermi oscillator*, and is quite chaotic. The thing in Fig. 5.3 can be even more devilish if m_2 is not very small. *Caveat emptor!*

The simplest force field F=const.

We have mentioned power-law forces $F_{adiab}=k/y^3=ky^{-3}$ (5.5), $F_{Coul}=k/y^2=ky^{-2}$, $F_{isoT}=k/y=ky^{-1}$ (5.4), and lastly $F_{osc}=-ky$ (5.12), but have forgotten the simplest, namely *zero* power law $F_{const}=k=ky^0$. This last one is like a constant near-Earth-surface gravity force $F_{\odot}=-\frac{\partial U}{\partial y}=mg=-m|g|$ on a mass m. ((-) sign for *down*ward.) Acceleration of gravity at Earth's surface is nearly *-10* meters per second per second or *very* nearly *–9.8*. $(g=-9.7997m/s^2)$ Terrestrial objects experience this whether bundled together or not.

All power-law forces $F=ky^p$ have power-law potentials $U=-\int F\cdot dy=-ky^p/(p+1)$, except for $p=-1$ where $F_{isoT}=k/y$ has a logarithmic $U_{isoT}=-k\ ln(y)$. (5.11) Earth-surface potential $U_{\odot}=mgh$ is *linear* in height $y=h$. This we use to compute height of a superball toss by equating its floor level $KE=1/2mV^2$ to maximum $PE=mgh$.

$$gh_{max}=\frac{1}{2}V_{floor}^2 \quad (5.13a) \qquad\qquad V_{floor}=\sqrt{2gh_{max}} \quad (5.13b)$$

Ejection height goes as *square* of ejection velocity. A *3*-fold velocity gain means $3^2=9$-fold height gain.

Introducing Action. It's conserved (sort of)

It is remarkable that a bouncing mass has a physical property called *action* $S = \oint P\, dx$ that is more or less constant even if its position x momentum P and kinetic energy KE are not. Action is defined by the area of a one-cycle loop swept out in a momentum *vs* position *phase-plot (P vs x)*. That is analogous to an energy or *power-plot* of force *vs* position *(F vs x)* whose loop area $\oint F\, dx$ is work per cycle.

Conservation of momentum and conservation of energy are each a rigorously obeyed axiom or theorem for an isolated classical system. However, conservation of action is "more or less" or "sort of" and "it depends" *for a driven system*. The concept of action is both subtle and deep and it lies at the heart of quantum theory and accounts for a lot of how we affect and are affected by the world around us.

Here we use a geometric construction of a bouncing ball trajectory to quantify action conservation or lack thereof. We suppose the little mass m_2 is caught as before in Fig. 5.1 and Fig. 5.2 between a rock and a hard place, that is, bouncing between a big mass m_1 (moving in at a constant velocity $v_1 = 1$ from the left) and a hard elastic wall. The big ball path is indicated in Fig. 5.4 by a line of slope=1= v_1 that hits an initially fixed m_2 following a vertical line (slope=0=v_2) that then gets knocked up to a line of slope=2=v_2 (after Bang(1)). Throughout the imagined collision sequence we suppose the big ball is so much more massive that its change in velocity is not noticeable. This is in spite of the fact that it is absorbing more and more momentum from the little ball with each bang. (Surely, *something* in it is going to break eventually!)

Each time the small ball is banged elastically by the big one it picks up two more units of velocity v_1 that it maintains, apart from change in sign, through its subsequent bang with the elastic wall. Each time it returns for more, is banged again, and increases its speed by two units. (Recall Fig. 5.2.)

The horizontal dashed lines in Fig. 5.4 indicate the range Δx available to the small ball at each instant of its bang with the wall. Note that the product of the range Δx and the speed v_2 is a constant three units even as spatial range Δx rapidly decreases and the velocity range $\Delta v = 2|v_2|$ increases just as rapidly.

$$\Delta x\ v_2 = 3.0 = \Delta x\ \Delta v/2 \tag{5.14}$$

This is an example of conservation of action mentioned before. If we define the small ball's "range of velocity" by $\Delta v=2|v_2|$ then this relation takes the form of a weird kind of uncertainty relation, that is, it looks like Heisenberg's famous *minimum uncertainty relation* $\Delta x\ \Delta p=\hbar=(constant)$ for position and momentum. It happens that the two are related even though the constant used by Heisenberg is an unimaginably tiny Planck constant $(\hbar\sim 10^{-34}Js)$ compared to a constant 3.0 appearing above. (Ours has *gadzillions* of wave quanta!)

The geometry behind this relation is exposed in Fig. 5.4 (b). It is obtained by considering intersections between lines of integral speeds or slopes $v_2=\pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 7,\ldots$ that are relevant to the bang sequence. They are also relevant to quantum theory where the speeds of a particle in a box are indeed quantized to integers times a tiny number. (This is where that tiny $\hbar$ comes in.) That is simply a reflection (pun intended) of the fact that mutually reflecting waves require that an integral (or half-integral) number of the wavelengths fit perfectly between mirroring containment walls or cavities.

Now we might ask if the action area $\Delta x\ \Delta v$ in Fig. 5.4c-e stays the same if the big-ball speed v_1 varies. Action variance was argued hotly by Einstein and the "quantum gang" at the1920 Solvay Conference. They imagined a hotel chandelier being dragged up or down by a clerk holding its support cable upstairs. They concluded that if the clerk could not detect the swinging pendulum phase or frequency, then he would seldom be able to change its action. However, if he could synchronize his oscillations then he could drive the chandelier exponentially to destruction. We shall review this important and explosive process known as *parametric resonance* in later units. It is fundamental to mechanics and particularly quantum wave mechanics. Action and its wiggly antics deserve our attention.

Monster mass M_1 and Galilean symmetry (It's deja vu all over, again.)

"Monster mass" M_1 bongs hapless m_2-atoms in Fig. 5.4 using Galilean symmetry. To show symmetry we imagine two head-on monster M_1's going at $\pm V_1=\pm 1$ in Fig. 5.5. A mirror image of Fig. 5.4 lies in extended m_2-path lines. The red paths of even integral velocity $v_2=0, \pm 2, \pm 4,\ldots$ are copies of Fig. 5.4 paths. Odd integral velocity $v_2=\pm 1, \pm 3,\ldots$ paths mesh with even ones to make a full grid. Any initial v_2 between $\pm V_1$ has a path on the grid. A blue path is drawn thru a series of bongs with $v_2=-0.2,+2.2,-4.2,+6.2,\ldots$in Fig. 5.5.

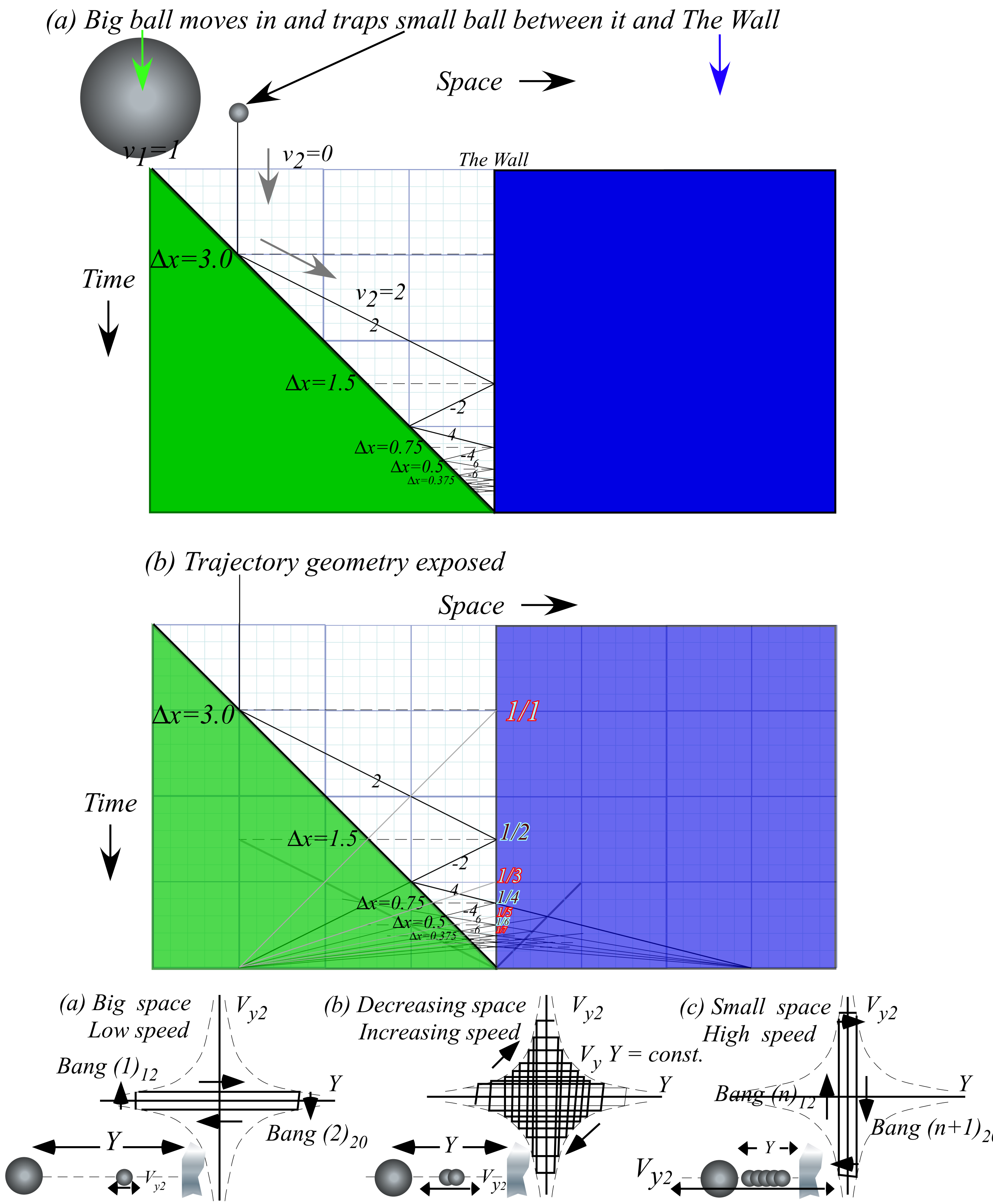


Fig. 5.4 Bang sequence for small ball between big ball and wall. (a) Spacetime paths. (b-c) Geometry of constant product $Y{\cdot}V_Y$ of velocity and coordinate ranges.

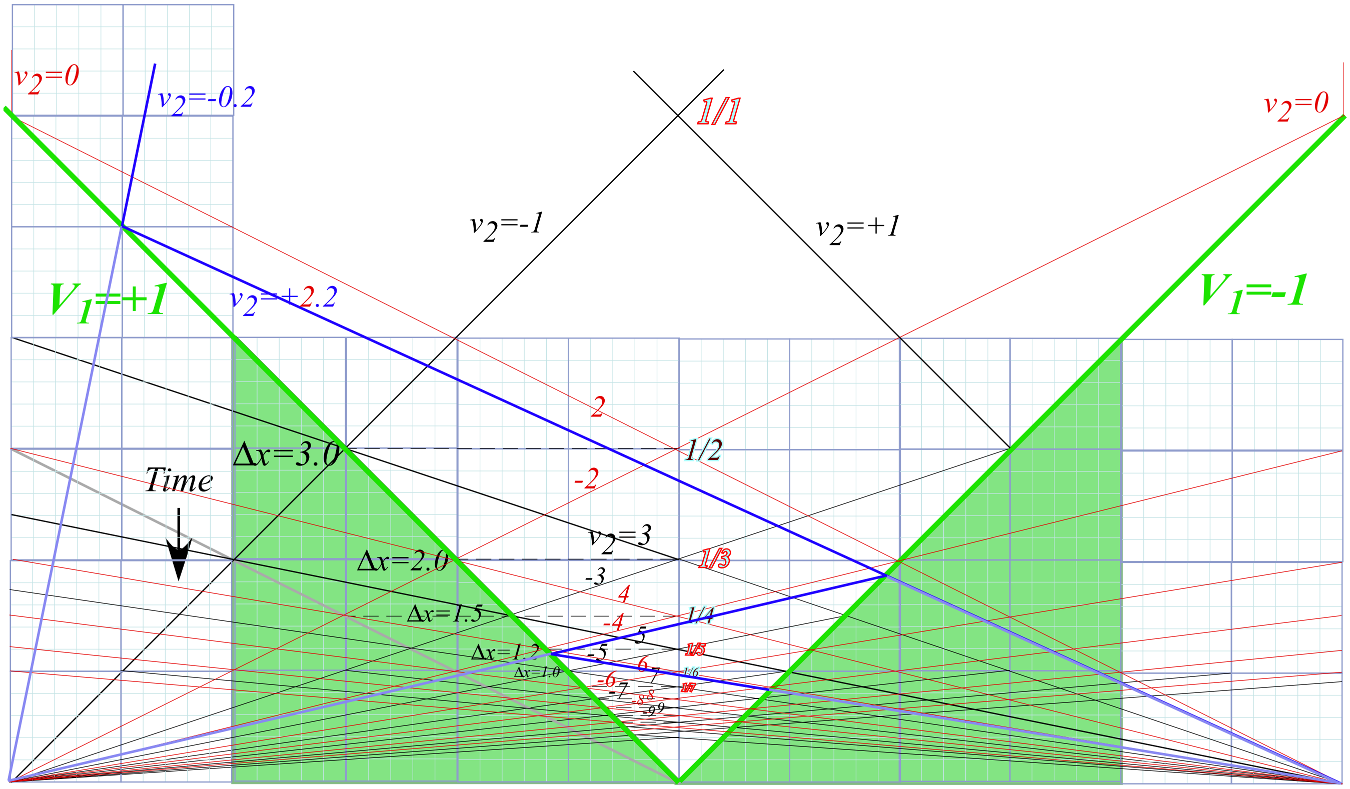


Fig. 5.5 Symmetric pair of head-on $V_1=\pm 1$ monster-m_1-masses pong tiny-m_2-atoms to higher speeds.

Monster M_1/m_2-ratios have simple V_1-v_2-plots shown in Fig. 5.6a. (Recall Fig. 5.2.) Wall M_1 simply adds twice its speed ($2V_1$) to incoming speed v_2 of atom m_2 as M_1 bounces m_2 out at that speed $v^{FIN}{}_2=v^{IN}{}_2+2V_1$. Monster M_1 *is* the COM so its path bisects in-and-out paths as it balances v^{IN} and v^{FIN} paths of atom m_2. (In its COM frame each bong is simply a change of sign for velocity. Recall balance in Fig. 2.6.)

The geometry of adding slope $2V_1$ to speed v_2 is shown if Fig. 5.6a. It is based on the unit square and unit velocity $V_1=1$. Incoming $-v^{IN}{}_2$ is an altitude of a right triangle with vertical base $V_1=1$, and it is reflected thru the square diagonal to $+v^{IN}{}_2$ then added to $2V_1$ to give sum $v^{FIN}{}_2=v^{IN}{}_2+2V_1$ as long side of the triangle with right side vertical base $V_1=1$ in Fig. 5.6a. The hypotenuse is the final path with final slope $v^{FIN}{}_2$. Each m_2-path and slope originates or terminates at base *pt*-B_- or else *pt*-B_+ . These are ends of the double-unit square bisected by unit slope path of M_1 terminating at B_0. Fig. 5.6.c shows quadrilateral $B_-B_+A_+A_-$ bisected by M_1 path B_0CA_0. Similar triangles explain multiple coincidences.

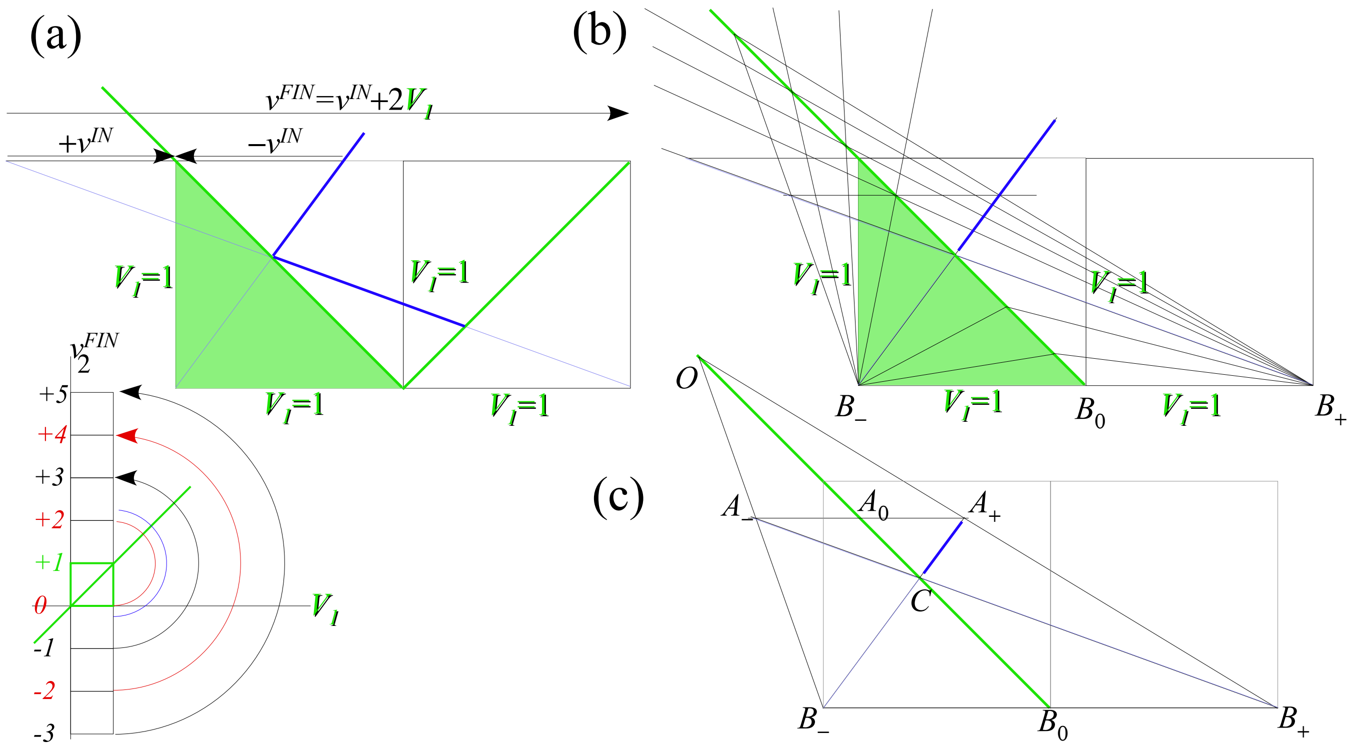


Fig. 5.6 Bisection geometry of Fig. 5.5.

Fig. 5.7 contains time plots for paths in different Galilean reference frames. An excerpt plot in Fig. 5.7a shows how Fig. 5.4 (copied in Fig. 5.7b) appears to a frame traveling at $V=1$ with each velocity in Fig. 5.7b reduced by $V=1$ in Fig. 5.7a. Also shown in Fig. 5.7a is the extension of lines connecting the two plots and this highlights this remarkable symmetry. All collision times in Fig. 5.7a match perfectly with ones in Fig. 5.7b though all velocities are shifted. Galileo's symmetry wouldn't have it any other way.

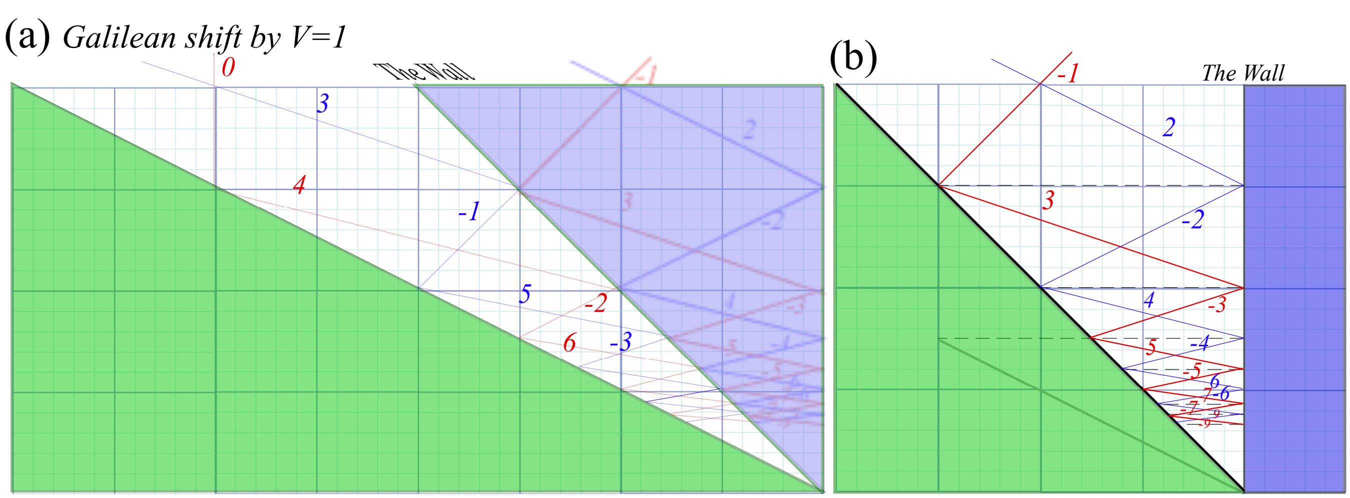


Fig. 5.7 (a) Galilean frame shift by frame velocity V=1 of collision sequence in Fig. 5.4 (shown in (b)).

Exercise 1.5.1 Suppose Fig. 5.3 shows a mass $m_1=1kg$ ball trapped between two smaller mass $m_2=1gm$ balls of high speed ($v_2(0)=1000m/s$ for $x=0$) that provide m_1 with an effective force law $F(x)$ based on isothermal approximation (5.11) while assuming m_1 moves only moderately far or fast from equilibrium at $x=0$.

(a) A further approximation is the one-Dimensional Harmonic Oscillator (1D-HO) force and PE in (5.12). If each mass m_2 start in an interval $Y_0=1m$, derive approximate 1D-HO frequency and period for mass m_1.

(b) What if the adiabatic approximation is used instead? Does the frequency decrease, increase, or just become anharmonic? Compare isothermal and adiabatic quantitative results for $m_1=1kg$ ball being hit by two $m_2=1gm$ balls each having speed of $v_2(0)=1000m/s$ as each starts bouncing in a space of $Y_0=1m$ on either side of the equilibrium point $x=0$ for the $1kg$ ball.

(c) How does the frequency decrease or increase in isothermal case *versus* the adiabatic case if we shorten the run interval $Y_0=1m$ to one-quarter meter?…What if we reduce the mass ratio m_1/ m_2 by one-quarter?

(d) Derive the adiabatic frequency for the case $M=50kg$ in adiabatic force of two $m=0.1kg$ masses of initial speed $v_0=20m/s$ and range $Y_0=3m$. Compare with Fig. 1.5.3c.

Exercise 1.5.2 The moving ballwall-trapped-ball constructions in Fig. 5.4 involves a plot of a ballwall coming in with unit slope (velocity). Consider a construction where it has a velocity of $1/2$ and intercepts a trapped ball of velocity -1 at space-time point $(x=-2, t=4)$ that is 2 units from the fixed wall. Construct five or more back-and-forth collisions and comment on what, if any, differences exist with Fig. 5.4. If you can, also construct one or two *prior* collisions (before $t=4$).
Evaluate approximate or average action values as described in class or after Fig. 5.4 in Unit 1.

Chapter 6 Interaction Forces and Potentials in Collisions

Derivation of force field potentials in Ch. 5 used elementary bangs by tiny m_2's on a big M_1. We predicted elementary bangs between a ball and floor, ceiling, or another ball without knowing potentials. However, three (or more) objects having a *ménage a trois* may involve *3-body interactions* that depend even more sensitively on whatever interaction potential or force law couples the participants.

Geometry of superball force law

When a superball or any elastic sphere hits the floor or ceiling it dents itself and, maybe it dents the surface it's hitting a little bit, too. But, if the floor, wall, or ceiling is much harder than the ball, we might assume only the ball develops a "flat-tire" as shown in the Figure 6.1a below.

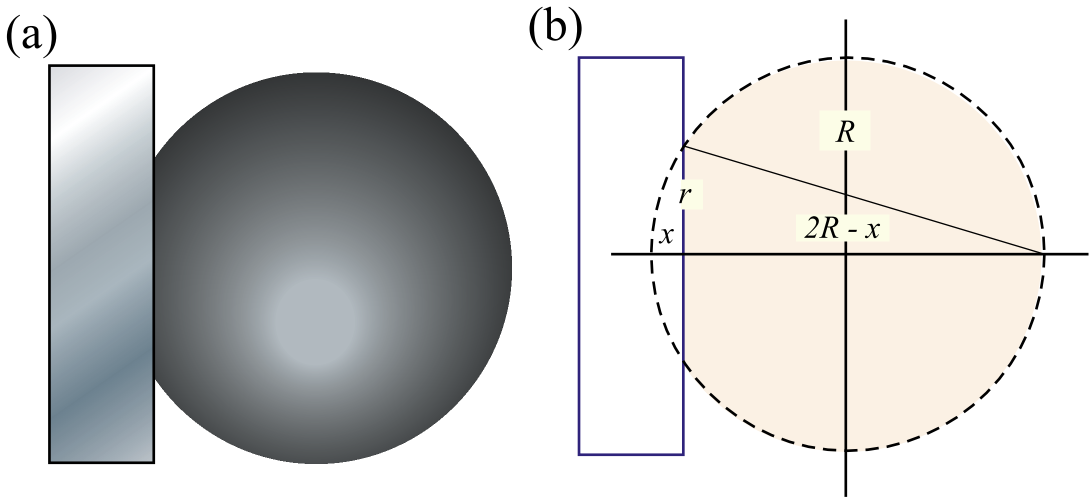


Fig. 6.1 Superball collides with solid wall. (a) "flat" (b) Saggital ("Bow") mean geometry

The radius r of the ball's "flat" is indicated by an altitude in Fig. 6.1b and is the geometric mean of the depression distance x and the remainder $2R$-x of the ball diameter. This is Thales geometry.

$$r = \sqrt{x(2R - x)} \quad \left(\approx \sqrt{2Rx} \quad for: \quad x << R\right) \tag{6.1a}$$

Solving approximately for depression x gives the Saggital ("bow") formula. (It's used for thin lense arc.)

$$x \approx \frac{r^2}{2R} \qquad for: \quad x << R \tag{6.1b}$$

How much force $F(x)$ is needed to depress the ball by distance x?

Well, "It depends." A hollow rubber ball or balloon with pressure P pushes back with force equal to product $P \cdot A$ of pressure and area of contact $A = \pi r^2$. It's a *linear (Hooke) force law* like an ideal spring.

$$F_{balloon}(x) = P \cdot A = P\,\pi r^2 \approx 2\pi PRx \tag{6.2}$$

(Recall (5.12) and Fig. 5.3.) Another example is gravity inside the Earth. (See later Chapter 8.)

However, the pressure and force in a solid ball varies non-linearly with x. Even if force varies only linearly with volume of the x-dent in Fig. 6.1b, it's still non-linear in x. As seen in (6.4) below, sector volume varies roughly as quadratic x^2 function. Superballs involve even higher power laws. (Superpower!)

$$Volume(X) = \int_0^X \pi r^2 dx = \int_0^X \pi x\left(2R - x\right)dx$$

$$= \int_0^X 2R\pi x dx - \int_0^X \pi x^2 dx = R\pi X^2 - \frac{\pi X^3}{3} \approx \begin{cases} R\pi X^2 & \left(for: X << R\right) \\ \frac{4}{3}\pi R^3 & \left(for: X = 2R\right) \end{cases} \tag{6.4}$$

(Here we check that our integral gives the whole ball volume $4\pi r^3/3$ for $x=2R$. That's the equivalent of crushing the superball into a black hole (or black spot). It's likely to complain before we get that far!)

Dynamics of superball force: The Project-Ball story

One of the interesting things to come out of *Project Ball* was the superball's peculiar force law behavior. The USC mechanical engineering department took an interest in this crazy project when it showed up on NBC News "Ray Duncan Reports." They offered to measure the superball force curve on a precise tension meter. But, that curve never worked. It didn't predict the bounces the students were observing. Nothing was making any sense even though we had a big analog computer working it all out.

That was a low point in the project. Even with all this fancy experiment, computers, and theory, I looked like I didn't know what the heck I was doing. So, what's new? That's science most of the time! But, to make things worse we got kicked out of the Project Ballroom, the old basement Lab 69 that we'd squatted in. It was up to be repainted so we had to drag all our stuff out and store it down the hall.

Well, after that I had to do something with the students so I arranged for a visit to Whammo Mfg. Co. in San Gabriel, California, where superballs and other goofy stuff was made. The Whammo man said maybe we could talk business about selling our super-elastic toy. So, a day or so later, with $$-signs in our eyes, we piled into our cars and drove down to the plant.

The trip to Whammo

By the time we got there, the inventors were on an all-day "alpha-wave break." That's a 60's fad where you try to increase your creativity by looking at your brain waves. I said, "Maybe, I could use some of that stuff!" But, the company lawyer wanted to show us around. After awhile, he said our invention was cool, but its product liability potential looked too high to make a commercial toy.

We all must have looked pretty sad after hearing that. So he went in a back room and dragged out a big collection of superballs that had been rejected for one reason or another. "Here, take as many as you want!" We thanked him and loaded the balls into some boxes and headed back to USC.

When we got back to Rm 69, the painters were done but the paint wasn't quite dry. So I said, "Let's drop off our new balls so we're ready for tomorrow." The students took "drop" to mean literally and dumped them out of the boxes into the empty room. Right away the balls bounced into the wet paint and made lots of little polka-dot spots all over the floor and wall. What fun! What a mess.

Eureka! Polka-dots save Project Ball

But, suddenly, it occurred to me what was wrong with our force analysis and how we might fix it. The engineers had carefully and slowly produced a static or *isothermal* force curve, but what we *really* needed was a fast-response or *adiabatic* force curve. I thought, *"Maybe that force law can be told by the polka-dots!"*

From a polka-dot radius r made by a superball of mass M and radius R dropped from a height h we could relate gravitational potential energy Mgh to an *adiabatic* superball potential energy U, and then find a $U(x)$ curve for each value of $x=r^2/2R$ in formula (6.1b) by plotting height h against x given by dot radius r. Then the adiabatic force curve $F(x)$ can be found from the slope $dU(x)/dx$ of a $U(x)$ curve.

Just as the adiabatic $F=1/Y^3$ in (6.5) force curve is steeper and curvier than the isothermal $F=1/Y$ in (5.4) so was the polka-dot bounce curve steeper than what we had been using. We stuck our new $F(x)$ on the analog computer's diode function generator and started getting good predictions. Now we could work out the deadly *Model-X3*, a *3*-ball super tower! (This is described later in Chapter 7.)

The "polka-dot" potential

First, let's look carefully at this "polka-dot" potential theory. What we did, like most of physics, was an approximation. Using gravitational potential to estimate superball $U(x)$ is a neat trick only if the superball forces are large and quick compared to the gravitational force or weight mg of the ball.

Fig. 6.2a shows a massive (Bowling-ball sized) superball at its $(V=0)$ drop point h, where potential energy is mgh. Kinetic energy rises from zero as the ball falls and flattens on the floor until it passes a point where the upward floor force cancels the ball's downward weight mg. That point-x_{static} of *static equilibrium* is at the bottom of the total potential energy curve in Fig. 6.2b. The ball would sit still if put gently at x_{static} with no kinetic energy. It's a point of zero slope since total force $F(x_{static})$ is zero there.

After passing x_{static} the ball slows down due to upward force. (That's positive $F(x)$ for $x<x_{static}$.) Finally it stops at its maximum penetration point x_{max} where the total energy line intersects the total potential line in Fig. 6.2c. Now the ball's initial gravity potential mgh_0 has been converted completely into potential energy $U(x_{max})$ due to compressing rubber a distance x_{max}. (We're ignoring tiny frictional heat.)

In the example, the ball's weight is almost as large as the inertial bang-force driving the ball into the floor. An indication of this is how flat the ball is in Fig. 6.2 b when its weight and compressive force are equal. A standard superball sits stiffly on a table with no noticeable depression, and mg is a tiny part of the total force. It's so stiff that its bang force is several times its weight and lasts only a tiny fraction of a second. Very stiff rebounding potentials are shown in the later Fig. 6.3 and Fig. 6.4 b in which gravity is a negligible force and stiff rebound forces dominate during the collision.

By comparison, the ball in Fig. 6.2 is heavy and its potential is not so stiff. Instead it is so soft it has a big "flat" if sits still with zero KE at x_{static} just as it does when passing that point in Fig. 6.2 b. The collision shown in Fig. 6.2 a-c is less like a bang and more like a lingering smooch! Similarly soft collision energy for a linear rebound force and quadratic potential is shown in parts (d) and (e) of Fig. 6.4.

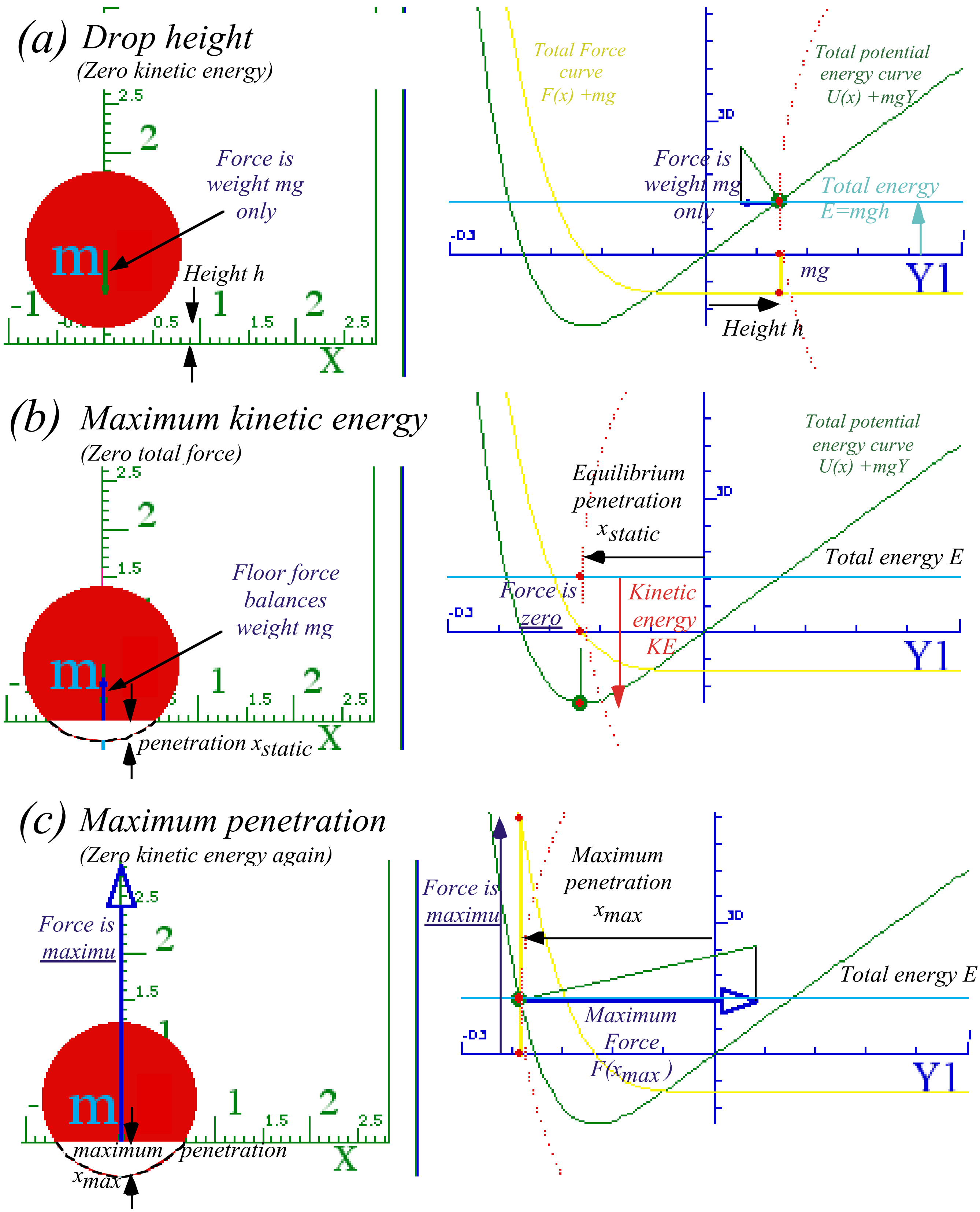


Fig. 6.2 Geometry of ball hitting floor (a) Ball is dropped. (b) Ball at max speed. (c) Ball at low point.

Force geometry: Work and impulse vs. energy and momentum

TV daredevils jump off *30*-meter towers and belly-flop into kiddy-pools that are less than 1 meter deep. What a way to earn a buck! And, how do they ever survive such stunts?

Two important physical quantities tell about survival chances. The first is the product $F{\cdot}x$ of force-times-distance, or, more precisely, the integral $\int F dx$ of force over distance. The second is the product $F{\cdot}t$ of force-times-time, or, more precisely, the integral $\int F dt$ of force over time. These are the fundamental Galileo-Newton relations.

The first quantity $\int F dx$ is *work* done or *energy* $-U(x)$ acquired. $U(x)$ is area under an $-F$ *vs.* x plot.

$$Work = W = \int F(x)\,dx = Energy\ acquired = Area\ of\ F(x) = -U(x) \tag{6.5a}$$

If energy is stored as potential energy $U(x)$, then force $-F(x)$ is the slope of a $U(x)$ plot at point x.

$$F(x) = -\frac{dU(x)}{dx} \tag{6.5b}$$

(Recall the discussion of force and potential leading up to (6.10).)

A second quantity $\int F dt$ is *impulse* done or *momentum* $P(t)$ acquired and area under an F *vs.t* plot.

$$Impulse = P = \int F(t)\,dt = Momentum\ acquired = Area\ of\ F(t) = P(t) \tag{6.5c}$$

If momentum is stored in kinetic velocity $V(t)= P(t)/M$ then force $F(t)$ is slope of the $P(t)$ plot at time t.

$$F(t) = \frac{dP(t)}{dt} \tag{6.5d}$$

The time equation (6.5c-d) is just Newton's 2nd law. The space force law (6.5a-b) is just the slope rule first stated (with the physicist's minus-sign) in (5.9). Both laws deal with conserved stuff. If you, a daredevil, acquire x of this stuff (energy or momentum) sooner or later you are going to have to find something or someone help you get rid of x. Or else!

A daredevil falling *30* meters acquires energy equal to gravity force (body weight Mg) times *thirty* meters. Fig. 6.3a-b plots a constant $F=-Mg$ and a linear potential $U(y)=Mg\,y$ from $y=30$ to $y=0$. The *1m* kiddy-pool must get rid of the *30Mg* (*Newton meters*) of energy in *one* meter, by applying a force of *30Mg* (*Newtons*) *steadily* over the *entire meter* from $y=0$ to $y=-1$. (That's a $30g{\sim}300ms^{-2}$ deceleration. Human survivability is somewhere around *50g*.) An alternative is to get rid of that energy in the concrete below the pool in about *1millimeter*, a *30 thousand* g deceleration. (That is *not* survivable!)

Kiddy-pool versus trampoline

Suppose the daredevil falls onto a special trampoline that applies exactly the same constant force as the kiddy-pool, but stores the energy as potential instead of dissipating it all by dousing the audience with a huge splash. (Recall *Ka-Bong!* versus *Ka-Runch!* in Ch. 1.) The trampoline could then toss the daredevil back up to the *30 m* tower to do the fall over again. (My gosh! What a daredevil has to do to satisfy a sated TV audience these days!) Such a potential is plotted by a steep-slope line $U(y)=-30y$ in Fig. 6.3b.

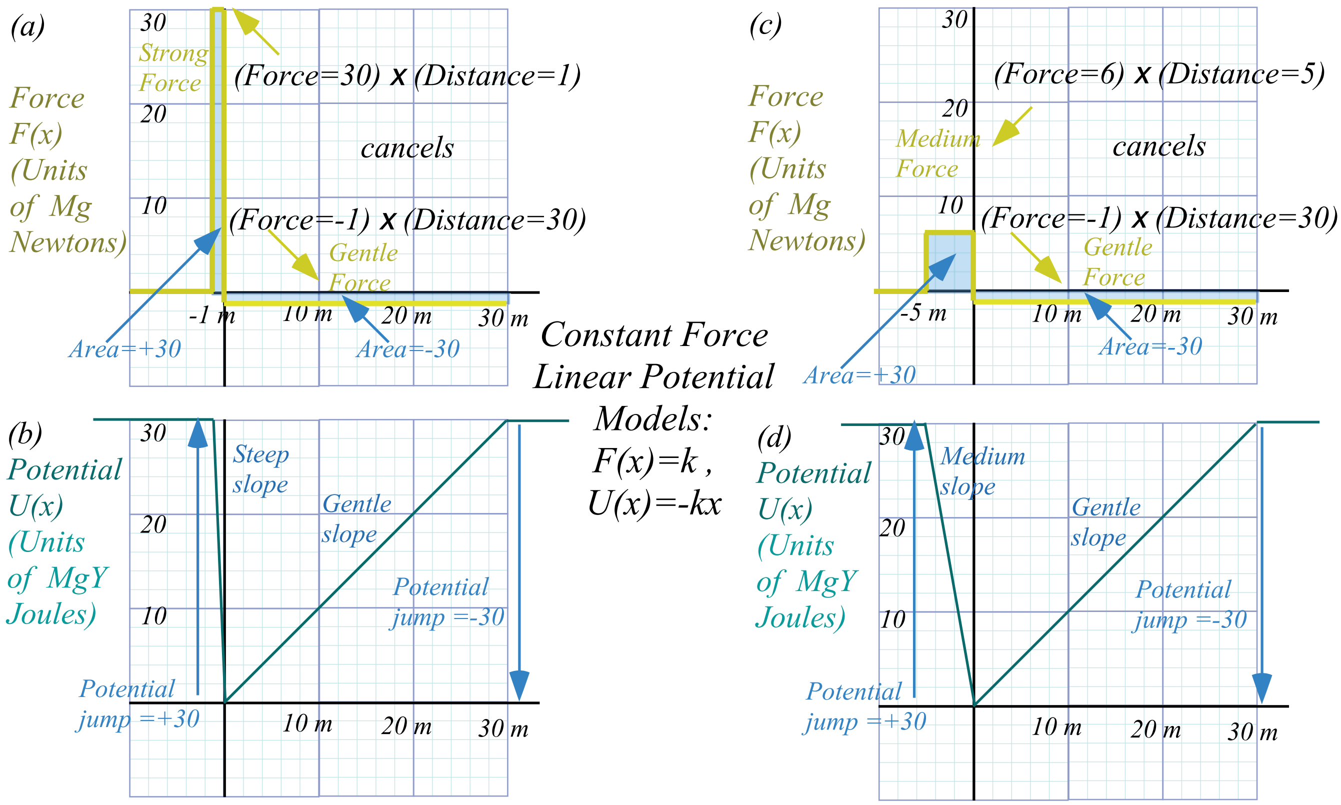


Fig. 6.3 Force and potential plots. (a-b) Strong (30g) deceleration. (c-d) Medium (6g) deceleration.

Suppose the Americans for Humane Daredevilry (AHD) demand that the deceleration distance be increased from 1 meter to 5 meters. (That's what Olympic divers get for a *10 m* fall.) As shown in Fig. 6.3c this reduces the deceleration by a factor of 5 from *30g* to only *6g*. (A walk in the park!) The sloping *U(x)* lines are tallying the area-accumulation under the *F(x)* lines. Starting on the right hand side, *U(x)* drops by *30* units in *30* meters in Fig. 6.3b to correspond to the *–30* units of area under the gravitational *F=-1* unit line for the same distance in Fig. 6.3a. The daredevil's kinetic energy must increase by *30* units to conserve total energy. So trampoline or pool is hit at *24* meters per sec. or 55 mph. (Recall (5.13).)

$$\tfrac{1}{2} M V^2 = 30\, Mg \qquad \text{or: } V = \sqrt{(60g)} = \sqrt{588} = 24.2 m/sec.$$

Getting rid of this *30 J* potential deficit means climbing a steep *30 J* high slope between $y=0$ and -1 in Fig. 6.3b or a medium slope of the same height between $y=0$ and -5 in Fig. 6.3d. Both cases have the same *+30 J* area under a force line, but having 5 meters instead of just one reduces the force to $^{30}/_{5}$=6.

Time functions *F(t)* and *MV(t)=P(t)* relate to *F(x)* and *U(x)* using *Newton II:* $F=M^{dV}/_{dt}$ in (6.5d).

$$-U(x) = \int F(x)\,dx = \int M\frac{dV}{dt}dx = \int M\frac{dx}{dt}dV = \int MVdV = M\frac{V^2}{2} - const. \quad \text{or: } M\frac{V^2}{2} + U(x) = const. \qquad (6.6a)$$

$$P(t) = \int F(t)\,dt = \int M\frac{dV}{dt}dt = \int MdV = MV + const. \qquad \text{or: } P(t) - MV(t) = const. \qquad (6.6b)$$

The first relation is *total energy conservation* (*KE+PE=const.*) first stated in (5.6) and (5.7).

Linear force law, again (But, with constant gravity, too)

Let's imagine the AHD demands further protection of daredevils from themselves by outlawing constant-force targets that turn on a full force suddenly upon entry. Claiming that "high-jerk" is bad, the AHD requires *linear-force* targets, instead. Physicists like this since a harmonic-oscillator linear-force-quadratic-potential (5.12) is a favorite force law that is also the inside-Earth potential. (Later in Ch.8.)

Plots of linear-force-quadratic-potentials are shown in Fig. 6.4. Just like the preceding Fig. 6.3, a constant gravitational force $F_{grav}=-Mg$ is present both in and out of the $(y<0)$-region where the linear $F=-ky$ force and the $U(y)=\frac{1}{2}ky^2$ potential exist as a sum of constant and linear forces for $(y<0)$.

$$F^{Total}=F^{grav}+F^{target}=\begin{cases}-Mg & (y\geq 0)\\ -Mg-ky & (y<0)\end{cases} \qquad U^{Total}=U^{grav}+U^{target}=\begin{cases}Mg\ y & (y\geq 0)\\ Mg\ y+\frac{1}{2}ky^2 & (y<0)\end{cases}$$

(6.7a) (6.7b)

If a linear potential $b{\cdot}y$ is added to a quadratic $a{\cdot}y^2$ potential we get the same parabolic curve $U=a{\cdot}y^2$, but that curve is shifted to the left by $y_{shift}=-b/2a$ and down by $U_{shift}=-b^2/4a$ as follows.

$$U^{Total}(y)=ay^2+by=a\left(y+\frac{b}{2a}\right)^2-\frac{b^2}{4a}=a\left(y-y_{shift}\right)^2+U_{shift} \tag{6.8a}$$

$$y_{shift}=-\frac{b}{2a},\qquad U_{shift}=-\frac{b^2}{4a}=-a\left(\frac{b}{2a}\right)^2=-U\left(y_{shift}\right) \tag{6.8b}$$

The nose or tip of the parabola, which is the equilibrium resting point, follows an upside-down copy of the U-parabola itself! This important geometric fact is shown in Fig. 6.4. The geometry does not reveal itself until we look in Fig. 6.4e at a "soft ball" that is soft enough to clearly show its gravitational shifts. A hard superball is more like Fig. 6.4b that barely shows such a small shift.

Hardball total potential is $u(y)=8y^2+y$ with a total force function $f(y)=-16y-1$ in graph units of Fig. 6.4(a-b). A medium total potential is $u(y)=y^2+y$ with a total force function $f(y)=-2y-1$ is plotted in Fig. 6.4(c-d). The latter clearly shows the equilibrium or lowest "sag" point of zero force. The softball total potential is $u(y)=(1/4)y^2+y$ with a total force function $f(y)=-(1/2)y-1$ in Fig. 6.4e. The hardball potential requires about 6 meters ($Y=-6$ or $y=-0.6$) to cancel the energy from the 30 meter fall (from $Y=30$ or $y=3$) and maximum force of about $F=10$. This is much more than the constant $F=6$ that stopped the same daredevil in 5 meters in Fig. 6.3c because a linear force has only the area under a triangle which has a factor of $\frac{1}{2}$. Here $\frac{1}{2}(F=10)(Y=-6)$ gives the necessary energy of *30 Joules*. So the AHD ruling has actually *increased* the maximum force on the daredevil! (But, only during the final milliseconds is F large.)

Note that the focus of the $U(y)$ parabola is on the y-axis because we plot gravity with *slope=1*. Can you find a geometrical a way to locate that focus given some allowed stopping distance?

Parabolic geometry of an oscillator potential subject to a uniform (or nearly uniform) force field is an important one in physics. Electronic charges pinned to an atomic potential well behave like oscillators in an electric field of a passing light wave. Generally the light wavelength of 0.5 micron

(*0.5E-6m*) is many thousand times as long as the atomic radius of a few Angstrom (*1E-10m*). So the effective potential is a rigid parabola Fig. 6.4e going both to-fro and up-down at optical frequency.

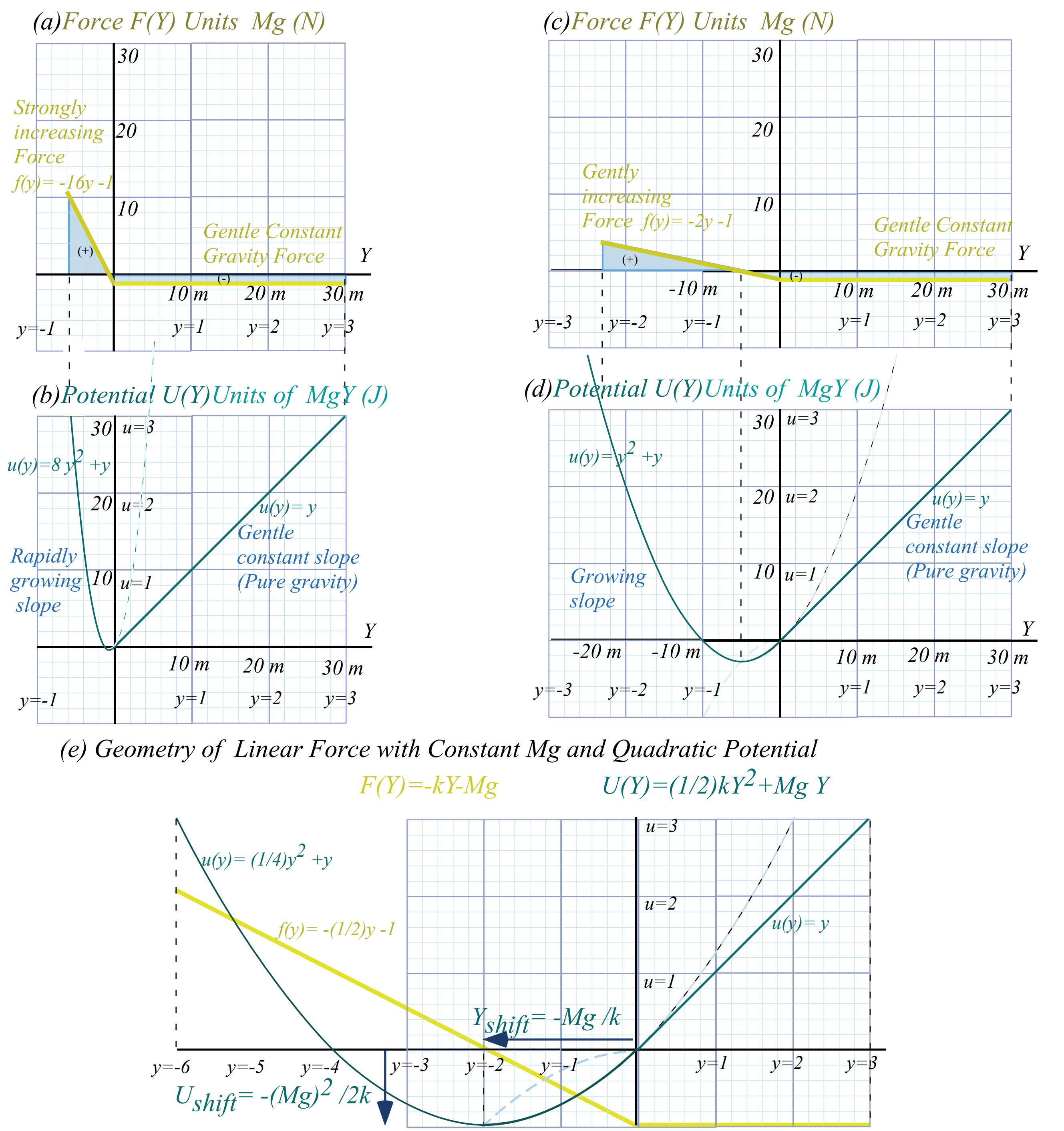


Fig. 6.4 Linear deceleration force after constant falling force. (a-b) Hard (c-d) Medium (e)Soft

As we mentioned, superball force function is non-linear; approximately $F_{ball}(y)$~y^4 plotted in Fig. 6.2 and Fig. 6.5 below. Compare this to the linear balloon-like force curve $F_{balloon}(y)$~y^1 in Fig. 6.4e above. (Recall (6.2).) $F_{balloon}(y)$ is a pair of straight lines bent at contact point $y=0$, while $F_{ball}(y)$ has a long flat region below $y=0$. For either case, the force integrals $\int F^{total}(y)dy$ and the areas they represent cancel between any two points $y=h$ and $y=y_{max}$ that have the same potential energy $U(h)=E=U(y_{max})$. If that energy is the total energy E then these points $y=h$ and $y=y_{max}$ are *classical turning points*. The mass M stops with zero *KE* (no speed) to turn around and fall backward or forward, respectively, into the potential valley lying between h and y_{max}. *PE* curve $U^{total}(y)$ near bottom (y_{static}) in Fig. 6.2-5 is nearly parabolic as is $U(x)$ in Fig. 6.3. The difference for Fig. 6.4 is that *all* of the $U^{total}(y)$ curves are *perfectly* parabolic for $y<0$. (See exercise 1.6.1.)

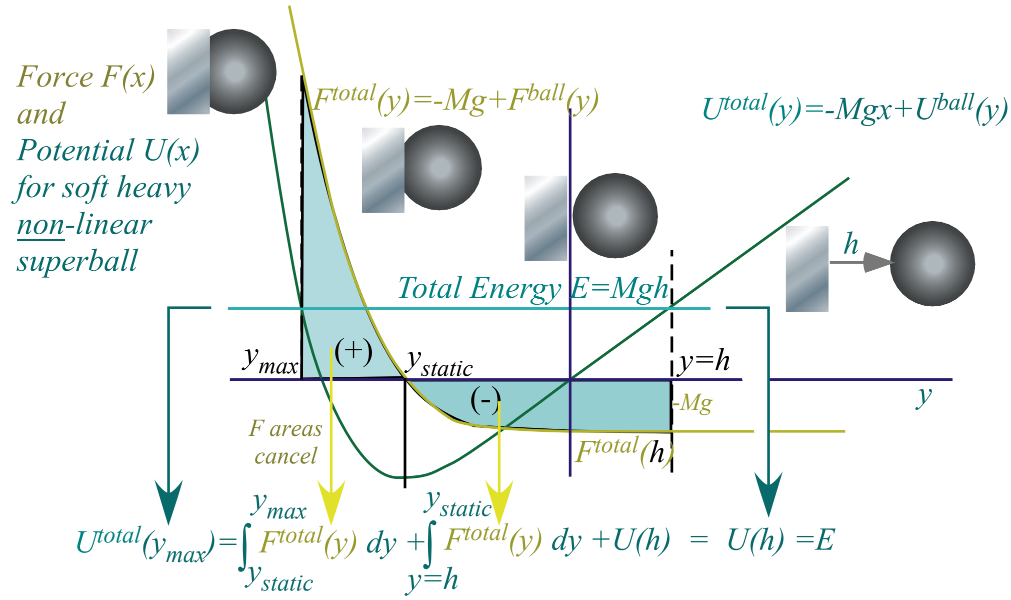


Fig. 6.5 Force and potential for soft nonlinear ($F=ky^4$) superball dropped from height h

Why super-elastic bounce?

Super-elastic bounce involving two balls was introduced way back in Fig. 4.5 and "explained" by the 2-Bang model sketched there. Is that the only explanation? Certainly not! Is it even right? Well, yes and no. Here is a chance to discuss how science works or doesn't work. It is, after all, a human endeavor. *(To err is…)*

RumpCo VS *Crap Corp*

Let's imagine a big scientific fight between two research groups something like real ones I've seen. We'll imagine it's about superball dynamics. On one side is a small but creative group working for the Rumpany Company® that first discovers the effect and explains it with the 2-Bang model. But their small budget limits them to things you can do cheaply with a ruler and compass.

On the other side is the huge *Crap Corporation*®. With unlimited military contracts, *CrapCorp* can afford any kind of computer or lab equipment. They hear about RumpCo's discovery and decide to develop and sell it to the Army as a bomb detonation system.

I hope you'll excuse a scatological nomenclature and contempt for shortsighted and mindless goals often associated with post-modern cash-flow-science. My allegorical objective is to encourage curiosity-driven-science that is now becoming regarded as quaint. I do believe that humans are capable of creating much more than fertilizer and should be strongly encouraged to do better. If earning gets in the way of learning, then humans do poorly. I have watched big labs in government, industry, and university die of a pernicious groupthink fueled by the *a*cquisitive rather than the *in*quisitive human drives. People lose their ability to reflect and become happy to merely genuflect. A novel *Radiance* by Carter Scholz *(Picador 2002)* is a "Star Wars" *romaine a'clef* exposing foibles of scientists at Livermore and Los Alamos.

On one side of our allegory is poor but resourceful little RumpCo full of ideas but nowhere to go. Their 2-Bang model of super-elastic bounce is simple, elegant, but appears wrong. The powerful *CrapCorp*, on the other hand, knows where it's going and what's right. It has every resource imaginable. Except wisdom.

CrapCorp's first move is to discredit RumpCo's work. They set up a computer that uses lab observed potential functions to fully analyze a *2*-ball bounce. Let's compare two competing vu-graphs side-by-side.

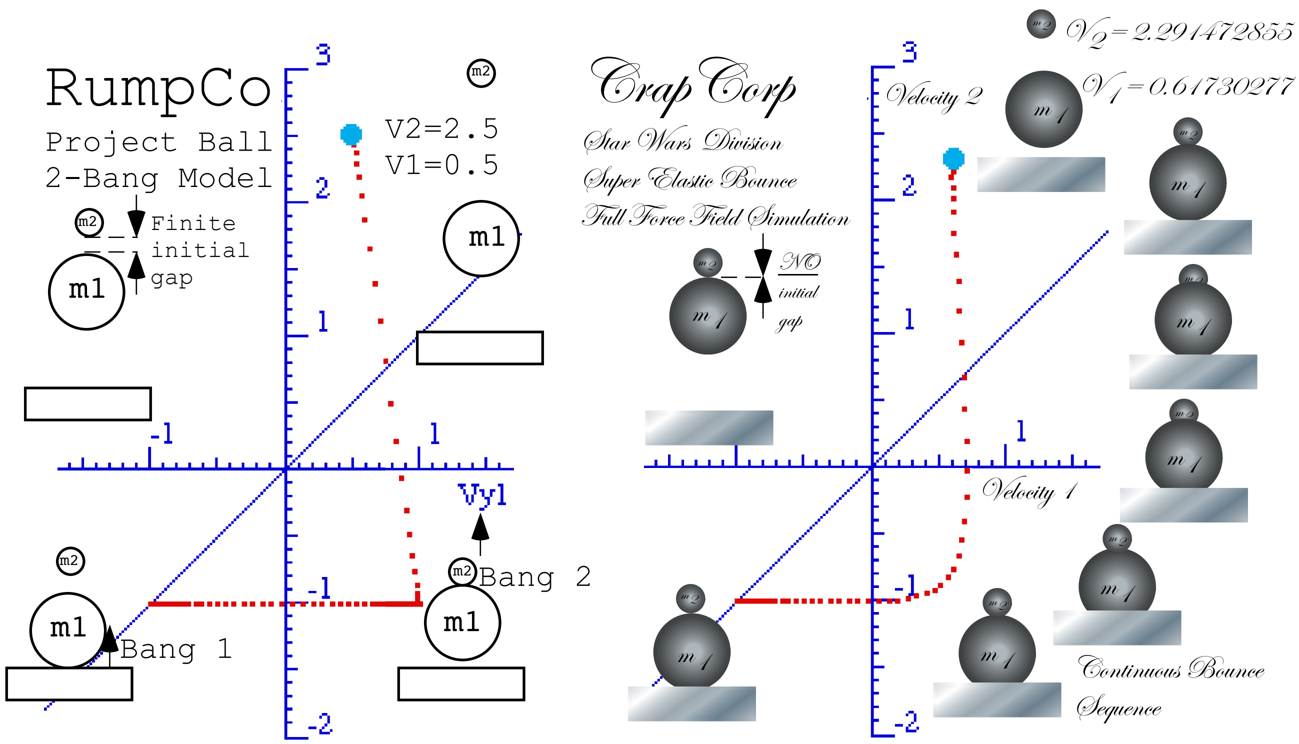


Fig. 6.6 RumpCo *theory versus* *CrapCorp*'s *simulation. (*RumpCo*) Finite initial gap (*CrapCorp*) NO gap*

One thing is clear. *CrapCorp* does fancy-schmancy vu-graphs! They resemble wedding invitations. And, while *CrapCorp*'s 10-figure precision is dubious, we note their V_1=0.62 and V_2=2.29 disagree with RumpCo's predictions (Recall Fig. 3.4.) of final V_1=0.5 and V_2=2.5 by a little. Furthermore, RumpCo uses an independent 2-ball bang model. They assume or *idealize* an initial gap separating mass m_1 from m_2 so *Bang-1* of m_1 with the floor is independent of *Bang-2* between m_1 and m_2. So V_1 and V_2 result from 2-body energy-momentum conservation. RumpCo's results are not sensitive to force functions.

CrapCorp can compute the difficult 3-body collision between m_2 , m_1, and m_0 (the Earth) all together just like what's really happening on the floor. *CrapCorp* 's curvy V_1 *vs.* V_2 plot in Fig. 6.6 is very sensitive to each force function $F(y)$ between each pair of colliding bodies. When (and *if*) *CrapCorp* values check out with experiment, they'll happily sneer at the primitive pair of straight lines in the `RumpCo` velocity plot.

Does `RumpCo` have nearly the right (V_1, V_2) for wrong reasons? Not entirely. The reason a 2-Bang model works at all is that the force function for these balls is highly *non*-linear. A quartic function $F(y)=y^4$ has a flat bottom as noted before Fig. 6.5. That allows the floor-m_1 collision to nearly finish before the m_1-m_2 bang really gets going even though the balls are in contact all during the collision.

Realizing this, the `RumpCo` researchers suggest that *CrapCorp* try a *linear* force $F(y)=y^1$ simulation to see if super-elastic bounce disappears. They do, it does, and the rest is history. As seen in Fig. 6.7, m_1 and m_2 bounce up in unison. It's a *pax de deux*. Super-elastic bounce goes away!

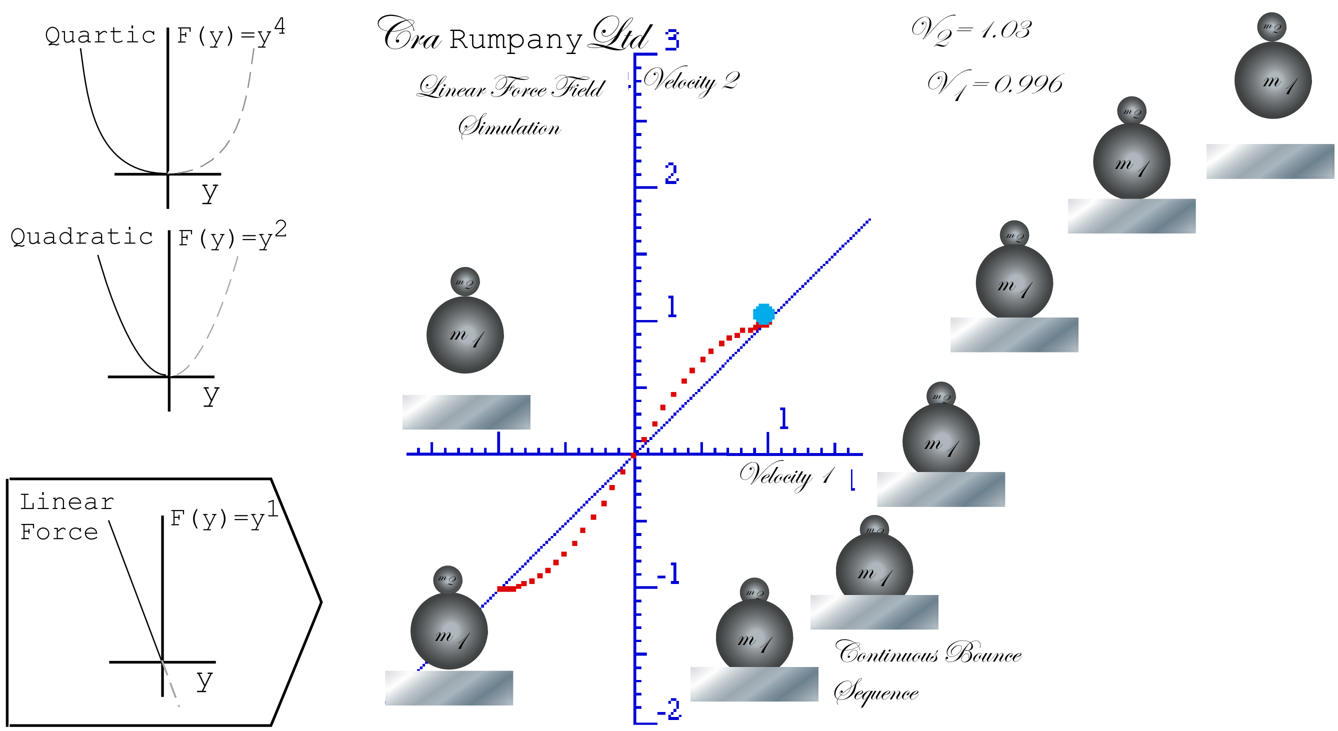


Fig. 6.7 Linear force kills super-elastic bounce. (Collaborative effort.)

The two groups decide to stop feuding and join forces. A corporate merger results in a multi-national conglomerate *Carumpany Ltd.* based in the Caymans. They lived happily ever after. (Well, sort of.)

Seatbelts and buckboards

Another important physics lesson from this section is, "Fasten your seatbelts…tightly!" To avoid great and damaging force you need to avoid non-linear force functions and fasten yourself with linear ones that can start working off your kinetic energy and momentum most immediately after a collision. The non-linear force with its "flat" region applies little or no force at first but then has to make up for its procrastination with deadly high force after it's too late. Note how nonlinear force in Fig. 6.5 finishes much higher than the linear force in Fig. 6.4. Even worse is having no seatbelt at all. That's like a very non-linear force of, say, $F(x)=kx^{100}$. It's a flat gap with a practically vertical wall waiting to crush you!

One of the most dangerous vehicles in the Wild West of the early US was the *buckboard*, a wagon with no suspension except for a set of springs right under the rider's seat. When the buckboard hit a bump it generally lived up to its name. Unfortunate riders ended up like a little m_1 superball knocked skyward by a big m_2 wagon. A safer and more comfortable ride is had in a car with a body as much heavier than the wheels and suspension as possible. So-called "Monster trucks" have the worst kind of ratio possible for stability.

Friction and all that "dirty" stuff

Slowly we have put back some of the "real-world" features of the superball collisions that our idealized "Bang-Bang" models of Ch.4 ignored in order to make the problems more easily solvable. The effects of gravity during collision have been introduced and applied to interacting zero-gap superballs. More such effects will be studied in what follows since interacting linear forces are very common in nature and there are ways to make them easily solvable, too. An oscillating neutron star (Later Ch. 8) provides a taste of what is to come in the study of waves and oscillation.

But even the neutron star model neglects what is the bane of the purist physicist, the dreaded *frictional forces*. These are among the most neglected and poorly treated physical effects in physics. If anything goes wrong with a theory, we just blame it on friction! Often we have little choice in this matter.

Friction is a result of having more particles than we'd like to admit. Consider one m_1*=72 gram* superball. That's about a *mole* of Carbon C_6 rings and a mole has *6.02E23* (That's *Avogodro's number*.) of these C_6 molecules. So we're dealing with not *one* mass m_1 particle but an enormous heap with an unimaginably huge number *60,200,000,000,0000,000,000,000* of particles that individually are (mostly) friction-free and well behaved, but their mob-behavior is just plain abominable!

You've got to get down to at least the individual molecular level before "internal-friction" is pretty much a non-existent phenomena and pure quantum wave mechanics rules. So what we call "frictional loss" is simply the best accounting we can do of 60.2 gazillion chiseling thieves stealing bits of energy that turn up later as "heat." In conservative economics the effect is known as "supply side" or "trickle-down." Let's see if we can account for energy chiseled by just *three* thieves. (And, then we'll hire more thieves until we bankrupt the whole operation!)

Important atomic and molecular force geometry

6.1 A most important mechanics problems is that of atomic oscillators affected by electric fields since it is basic to all spectroscopy. A useful approximate model is potential $V^{atom}(x)=k\,x^2/2$ function of center x of charge Q where k is a spring constant of atomic polarizability. A uniform electric field E is assumed to apply a force $F=Q\cdot E$ to the charge by adding a potential $V^E(x)$ to $V^{atom}(x)$. (Give $V^E(x)$=______ and $F^E(x)$=_____)
Consider the resulting potential $V^{total}(x)$ for an atom for unit constants $k=1$ and $Q=1$. Derive and plot the new values for equilibrium position $x^{equil}(E)$, energy $V^{equil}(E)$, dipole moment $p^{equil}(E)$. Plot $V^{total}(x)$ for field values of $E=-2,-1, 0, 1,$ *and* 2. Does charge oscillation frequency $\omega^{equil}(E)$ change? If so express in terms of $\omega^{equil}(0)$ and E?

Chapter 7 N-Body Collisions: Two's company but three's a crowd

Without knowing force and potential effects on superball collisions, it is often impossible to even approximately predict the outcome for $N=3, 4,$ or more balls. But, if all N masses have independent one-on-one collisions with the floor, the ceiling, and each other, prediction can be done "Bang-by-Bang" as in Ch.4. Difficulty arises when three or more collide at once. Then prediction may need precise and detailed treatment of their interactive force laws. Elastic binary or one-on-one collisions in *one* dimension are solved completely by momentum conservation alone as we've done since Ch. 3. But, as we'll see, anything more complicated may require more work, and often it requires a *lot* more work!

The X3: Three-ball towers

One of the goals of *Project Ball* at USC was to optimize final velocity for superball towers with three or more balls stacked up like a pyramid as in a multi-stage rocket. One dumb idea was a cheap satellite launcher. It's dumb because, even if you could achieve *8 km/s* (See discussion in Ch. 9.), you'd burn it up in the atmosphere. (Well, OK, but on the *moon*…?)

Actually we were happy just to break the theoretical 2-ball limit of *3.0*-times-initial. (Recall discussion of the INF limit in and after Fig. 3.5.) As seen in Fig. 7.1a that is done quite easily by a 3-stage tower which achieves a velocity that is $V_3=3.41$ times initial drop-speed $(V_n(0)=1$ *for* $n=1,2,3)$.

An even better final speed of $V_3=3.62$ is had in independent collisions caused by setting initial gaps between the falling balls as shown in Fig. 7.1(b) so each collision can be completed before the next one begins. Then the result becomes independent of the force law governing the detailed trajectory within each collision, and a geometric construction in Fig. 7.1(b), based on momentum conservation, finds velocity accurately if collisions are independent. This requires force non-linearity or large initial gaps that are enough to reduce or eliminate N-body contact effects for $N>2$.

Conversely, zero initial gaps often reduce the final velocity maximum below independent collision values. This is particularly true if the force law is linear as shown in Fig. 7.1(c). The 3-ball linear case comes out very much like the linear case for a 2-ball tower in Fig. 6.7. No single mass gains much speed over its neighbors. Super-elastic bounce is essentially squelched.

The American Journal of Physics[†] paper produced by *Project Ball* contains a discussion of attempts to optimize super-elastic bounce in towers of 3 or 4 balls. Progress was made but the theory needs work. As we will see later, this dynamics is somewhat analogous to wave motion in a varying channel. An early AJP paper[††] has an analogy between a trumpet and a chain of sliding balls whose masses increase geometrically. It's also analogous to tsunami wave build-up. A rule-of-thumb is that optimum-velocity chains satisfy a geometric-mean mass relation $m_2=\sqrt{(m_1 m_3)}$ as is approximately so in Fig. 7.1. Later on, some of this technology was developed into a toy by Stirling Colgate (astrophysicist and toothpaste heir) and his company that got a patent in 1990 for an idea published in 1971!

[†] Class of WGH, *Am. J. Phys.* **39**, 656 (1971).

[††] J. B. Hart and R. B. Herrmann *Am. J. Phys.* **36**, 46 (1968).

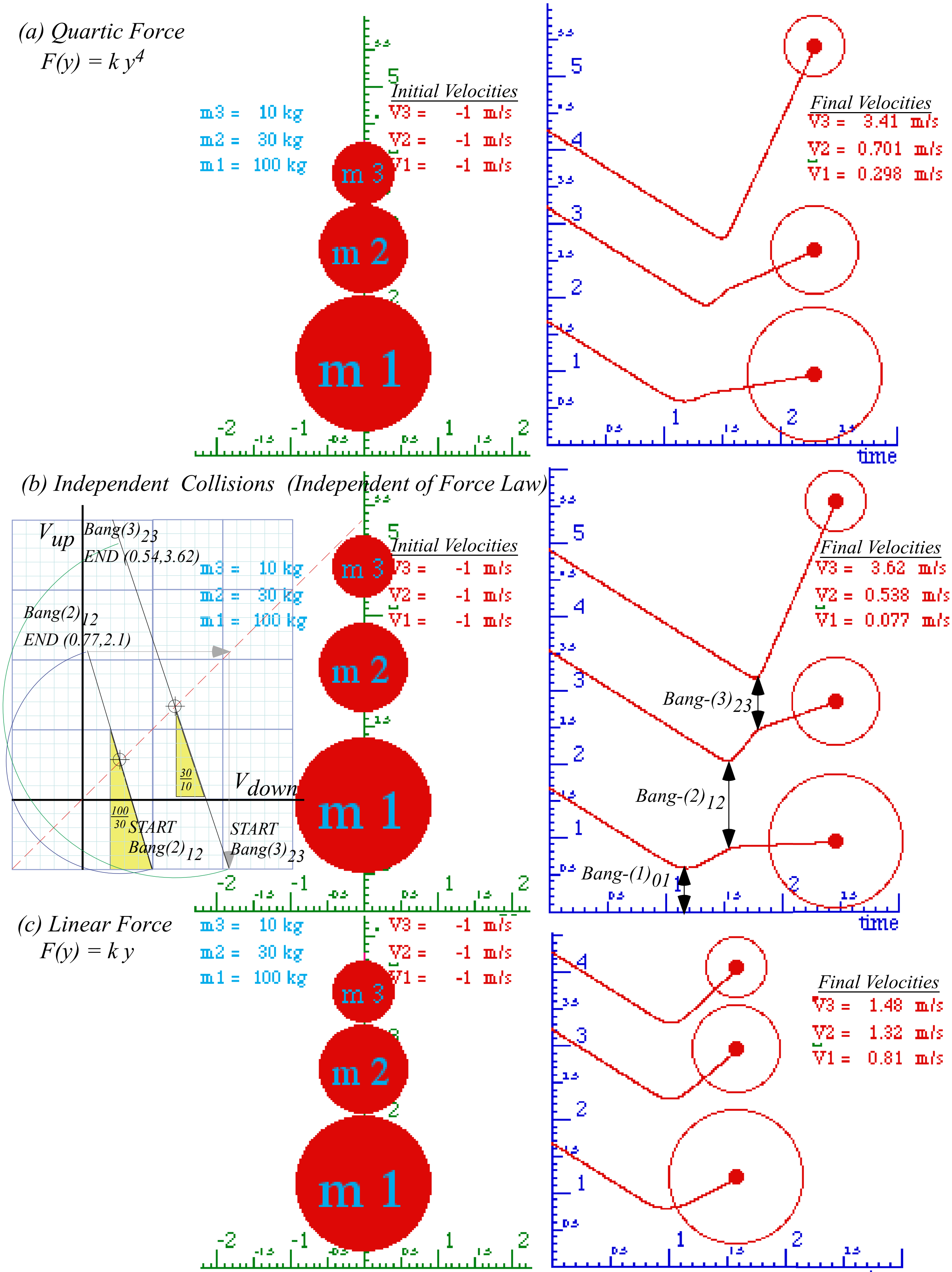


Fig. 7.1 Dropped 3-ball tower. (a) Quartic force (b) Independent (Finite gap) (c) Linear force.

Geometric properties of N-stage collisions

The 3-stage collision construction in Fig. 7.1b uses earlier construction of Fig. 3.4. It begins after the lowest mass $m_1=100$ has rebounded from the floor to the *Bang(2)*$_{12}$ *START* point $(V_1=1, V_2=-1)$ where it meets mass $m_2=30$ and bangs up to *Bang(2)*$_{12}$ *END* point $(V_1=0.77, V_2=2.1)$ on a slope $^{100}/_{30}$ line.

The second velocity $(V_2=2.1)$ of mass $m_2=30$ is then transferred (See gray arrows.) to the first component of *Bang(3)*$_{23}$ *START* point $(V_2=2.1, V_3=-1)$. There m_2 meets mass $m_3=10$ and bangs it up to *Bang(3)*$_{23}$ *END* point $(V_2=0.54, V_3=3.62)$ on a slope $^{30}/_{10}$ line, giving final top m_3 velocity $V_3=3.62$.

A 4-stage collision tower sequence with nearly the same mass ratios is constructed in Fig. 7.2(a). Here each mass m_1, m_2, and m_3, is exactly 3-times the one above it, and the top mass m_4 gets the biggest boost of nearly 5.8. Recall *Maximum Energy Transfer* (*MET*) case in Fig. 3.5 where a mass ratio of three $(m_1/m_2=3)$ leaves the lowest ball stopped $(V_1=0)$. In Fig. 7.1b m_1 is nearly stopped. $(V_1=0.077)$.

The same arrangement with a higher mass ratio $m_k/m_{k+1}=7$ is constructed in Fig. 7.2b. Here the top mass m_4 gets a boost of over 9.0. That is a kinetic energy boost factor of $(V_4)^2=81$ and an altitude bounce of four or five hundred feet if dropped from arm's length. (Friction is being seriously neglected!)

Supernovae super-duper-elastic bounce (SSDEB)

Imagine dropping two towers like the ones in Fig. 7.2a-b from either side of a tunnel through the Earth so the two lowest m_1-masses run into each other at the center. If the resulting collisions were elastic, they could send the other masses to infinity with energy to spare! Later we see escape from Earth's surface takes only three times the energy it takes to sit there. *(Starlet escapes!)* Energy factors for a conservative $3{:}1$-tower are $2^2=4$, $3.5^2=12.3$, and $5.8^2=34.8$ and more than enough for a free ride to kingdom come. Astrophysical modeling of Type-II supernovae reveals just such a high speed SSDEB when a star, like a spherical layer-cake with lighter elements above heavier ones, collapses. *Boom!* It appears that most of our Earth and bodily stuff has come along on such a ride! As Carl Sagan remarked, we are of blown-up stars.

Newton's balls

Novelty stores have simple examples of multistage collisions made by hanging identical ball bearings in line as sketched in Fig. 7.2c-d. These are also common lecture demos, and they have been called "Newton's balls." That can at least elicit some giggles from otherwise boring lectures.

Few teachers explain the details of the cool pop-up-single in Fig. 7.2d. In fact, it won't work unless all the collisions are *independent*, and this requires *non*-linearity of the sphere-on-sphere force function, as we saw in Fig. 7.1. Cooler still, is an elastic *4*-ball column-bounce in Fig. 7.3c. *N*-balls need $N(N+1)/2(=10$ *if* $N=4)$ independent bangs to get all *N* balls back with the same speed. Given this, it seems a wonder that solid objects can bounce elastically. (In fact, they cannot, quite!)

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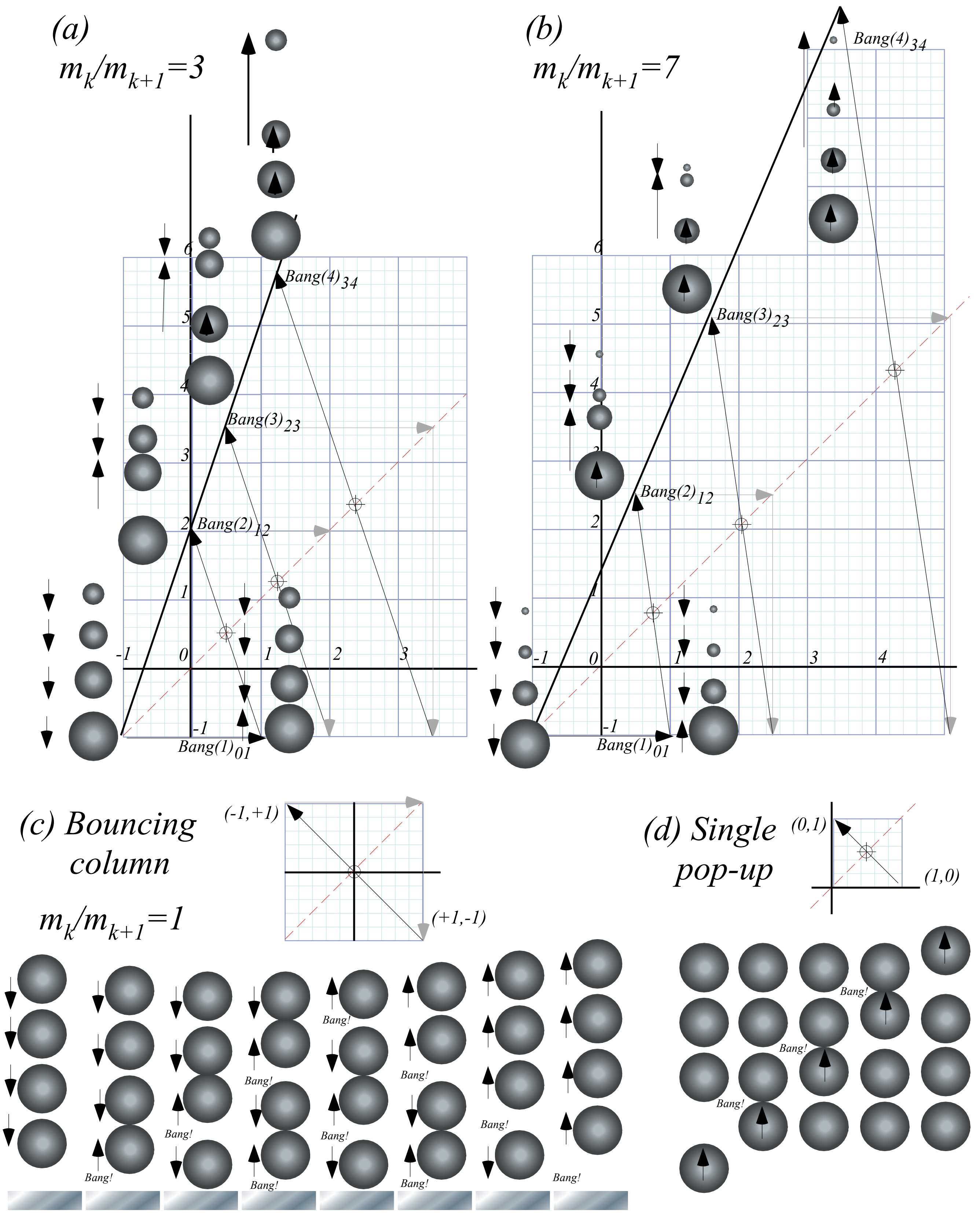

Fig. 7.2 4-ball towers. Mass-ratios m_k/m_{k+1} (a) 3, (b) 7, (c-d) 1. Independent bangs used for all.

Friction, again: Inelastic energy-momentum quadratic equations

Perhaps, you noticed that *FINAL* velocity values could be found from *INITIAL* values by two different ways. Back in Fig. 2.1 we noted an easy way using a momentum conserving straight line and a circle through $\mathbf{V}^{COM}$ from $\mathbf{v}^{IN}$ to the answer $\mathbf{v}^{FIN}$. But, Fig. 3.1 showed another way using an energy-conserving ellipse to connect $\mathbf{v}^{IN}$ to the answer $\mathbf{v}^{FIN}$. The first way uses simple linear equations and the second way uses more complex quadratic equations.

Why are there two ways? Often this means that situations exist where both are needed. Here friction or *inelastic* collisions make total kinetic energy decrease. (Recall our *60.2*-gazillion thieves? They're *baa-ck*!) Such a situation is plotted in Fig. 7.3b with the energy decrease indicated by a smaller ellipse inside the initial ellipse in Fig. 7.3a. This similar to an earlier Fig. 2.2 or Fig. 2.3.

The idea is that momentum conservation is still true even if the two masses are exerting sticky, energy-wasteful, forces on each other. No matter how wasteful those inter-particle forces may be, they still must obey Newton's 3rd axiom demanding equal-and-opposite forces on each other. So the final answer for $\mathbf{v}^{FIN}$ must be at an intersection of the *old* momentum line with a *new and smaller* ellipse.

However, intersecting an ellipse and a line uses a *quadratic* equation. And, in Fig. 7.3, there appear *two* solutions to the quadratic equation. One $\mathbf{u}^{FIN}$ we want is near the old energy-conserving $\mathbf{v}^{FIN}$. But, the other one that we now *don't* want is a $\mathbf{u}^{IN}$, which is nearer to the old $\mathbf{v}^{IN}$.

Let's look at a quadratic equation for u_1^{FIN}. There are two given constants $KE(u)$ and MV^{COM}.

$$m_1 u_1 + m_2 u_2 = MV^{COM} = p_u = const. \quad (7.1) \qquad \frac{1}{2}m_1 u_1^2 + \frac{1}{2}m_2 u_2^2 = KE(u) = k_u \quad (7.2)$$

The COM momentum p_u in (7.1) is a constant during the entire collision. Not so for the kinetic energy k_u in (7.2). It's just a given loss parameter that is quite difficult to predict. We first solve p_u for u_2.

$$u_2 = \frac{p_u - m_1 u_1}{m_2} \tag{7.4a}$$

Then we insert the u_2 result into k_u equation (7.2) to get the needed quadratic equation for just u_1.

$$\frac{1}{2}m_1 u_1^2 + \frac{1}{2}m_2\left(\frac{p_u - m_1 u_1}{m_2}\right)^2 = k_u \text{ or: } m_1\left(\frac{m_1 + m_2}{m_2}\right)u_1^2 - 2p_u\frac{m_1}{m_2}u_1 + \frac{p_u{}^2}{m_2} - 2k_u = 0 \tag{7.4b}$$

The solution isn't pretty but its ± gives both u_1^{FIN} and u_1^{IN} shown in Fig. 7.3b.

$$u_1 = \frac{2p_u\left(m_1/m_2\right) \pm \sqrt{\left(2p_u\right)^2 - 4\left(m_1/m_2\right)\left(m_1+m_2\right)\left[\left(p_u{}^2/m_2\right) - 2k_u\right]}}{2\left(m_1/m_2\right)\left(m_1+m_2\right)} = V^{COM} \pm \frac{\sqrt{p_u{}^2 - \left(m_1/m_2\right)\left(m_1+m_2\right)\left[\left(p_u{}^2/m_2\right) - 2k_u\right]}}{\left(m_1/m_2\right)\left(m_1+m_2\right)}$$

(7.5a) (7.5b)

The unwanted (+) solution u_1^{IN} (given that we started with v_1^{IN}) means the two balls "wiffle" through each other. In classical physics, only u_1^{FIN} makes sense starting with v_1^{IN} and only u_1^{IN} makes sense starting with v_1^{FIN}. In quantum theory, masses can "wiffle." Then *both* solutions make sense (sort of).

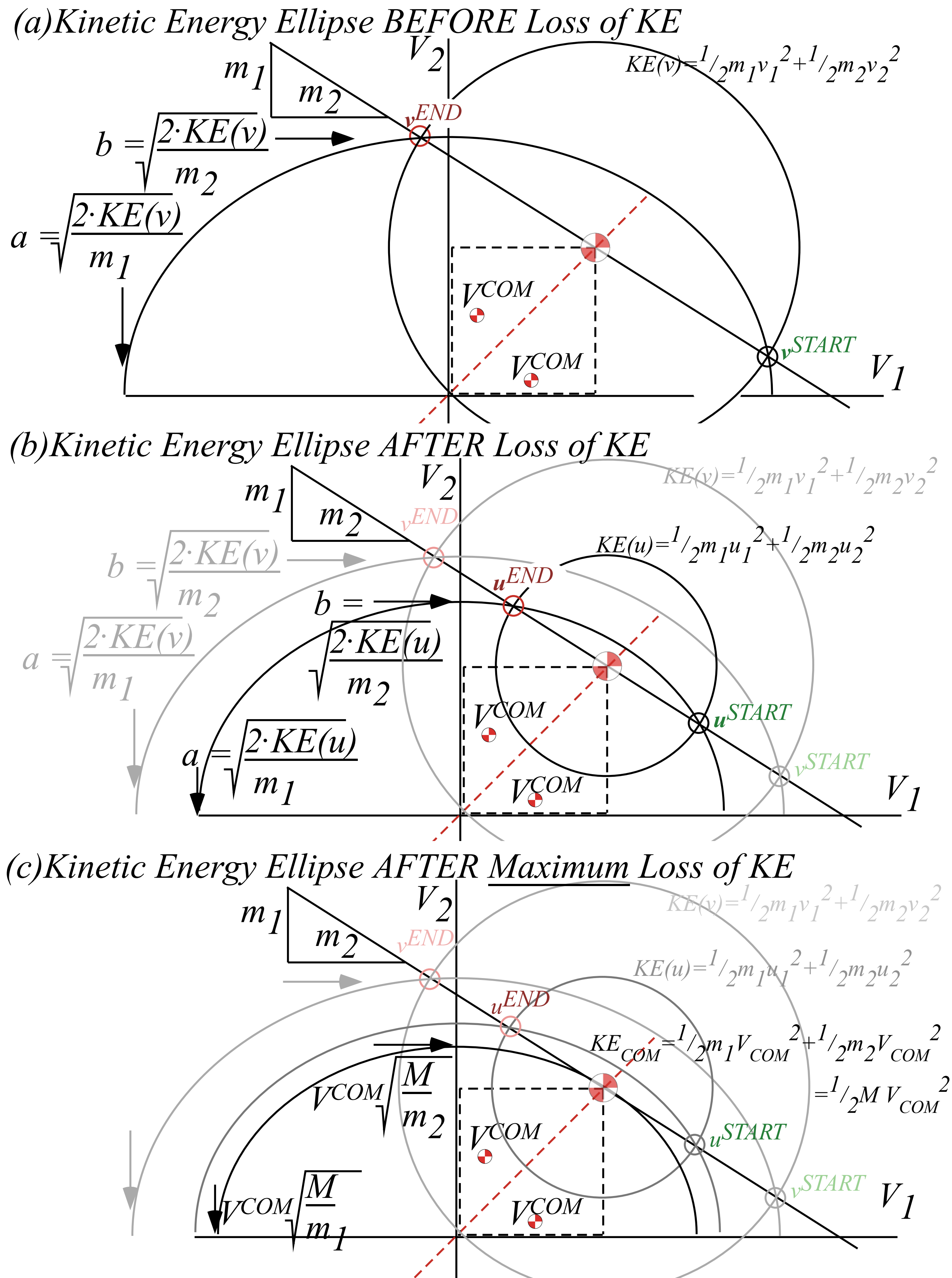

Fig. 7.3 KE-Ellipse shrinks by frictional loss. (a) Elastic (No loss). (b) Inelastic. (c) Totally inelastic.

Geometric derivation of elastic and inelastic energy ellipses

Can you do quadratic solutions (7.5) with a ruler and compass? At first it seems hard, but energy ellipse construction in Fig. 2.5 and Thales geo-mean square root construction in Fig. 6.1 are used.

As shown in Fig. 2.6, an ellipse has two radii, a *major radius a* giving x-coordinate $x=a\cos\theta$, and a *minor radius b* giving y-coordinate $y=b\sin\theta$. The Cartesian ellipse equation (2.3) is satisfied by these x and y, and polar angle parameter θ is eliminated. (x and y may switch places.)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1=\frac{m_1}{2\cdot KE}(V_1)^2+\frac{m_2}{2\cdot KE}(V_2)^2 \qquad (2.3)_{\text{repeated}}$$

Velocity values $x=V_1$ and $y=V_2$ have equal magnitude for initial *Bang(0)* ($V_1=-V^{IN}$, $V_2=-V^{IN}$) or *Bang(1)* (V^{IN},-V^{IN}), and for a totally inelastic final state ($V_1=V^{COM}$, $V_2=V^{COM}$). The geometry needed to solve for the initial elliptic radii (a^{IN}, b^{IN}) in Fig. 7.3a or totally inelastic radii (a^{COM}, b^{COM}) in Fig 7.3c is described in Fig. 7.4. Then an energy ellipse in (V_1, V_2)–space such as in Fig. 7.3b may be derived for any radii ($a^{FIN}\sqrt{R}$, $b^{FIN}\sqrt{R}$) where the *energy retention ratio* $R= KE^{FIN}/ KE^{IN}$ ranges from $R=1$ down to $R_{min}=(a^{COM}/a)^2=(b^{COM}/b)^2$ as (a^{FIN}, b^{FIN}) range from initial radii (a^{IN}, b^{IN}) to totally inelastic (a^{COM}, b^{COM}) at minimum KE allowed by momentum conservation.

The roots (7.5) are two points where energy ellipse and momentum line intersect. For totally inelastic collision they coalesce and the momentum line is tangent at (V^{COM}, V^{COM}) as in Fig. 7.3c. The slope $m_1/m_2=a^2/b^2$ of the momentum line is fixed no matter how much energy is wasted. So is ellipse aspect ratio $a/b=\sqrt{(m_1/m_2)}$. Square root construction (from Thales) finds a/b from a^2/b^2 in Fig. 7.4a-c.

The construction begins by boxing the momentum line in the 1st quadrant and doubling it using a semi-circular arc around its upper left hand corner. An extended box including the arc is drawn in Fig. 7.4b. The center of the extended box is the center of a second arc that finds the square root $\sqrt{(m_1/m_2)}$ of the momentum line slope in Fig. 7.4c that is the desired ellipse aspect ratio a/b of all possible energy ellipses for the masses m_1 and m_2. The basis of this construction is the mean geometry of Fig. 6.1a.

Location of radii a^{COM} and b^{COM} in Fig. 7.4d uses vertical and horizontal projections of *pt*-(V^{COM}, V^{COM}) to the ($\sqrt{(m_1/m_2)}=a/b$)-line. This is helped by the fact that *pt*-(V^{COM}, V^{COM}) lies on the ellipse *and* on the $45°$ line so that its x-coordinate ($x=a\cos\theta$) and y-coordinate ($y=b\sin\theta$) are equal. Thus angle parameter is $\tan^{-1}a/b=\theta$, the a/b line slope. So x and y projections of (V^{COM}, V^{COM}) onto the θ–line yield hypotenuse lengths a^{COM} and b^{COM} in Fig. 7.4d. Concentric circles of radii a^{COM} and b^{COM} let us construct the ellipse as in Fig. 3.7.

Initial *pt*-(V^{IN}, V^{IN}) gives initial elliptic radii a^{IN} and b^{IN} in Fig. 7.4e. Square-radii ratio $(a^{COM}/ a^{IN})^2=(b^{COM}/b^{IN})^2$ or ratio $(a^{COM}b^{COM})/(a^{IN} b^{IN})$ of the two ellipse areas lets us find the lowest possible kinetic energy retention ratio R_{min}. You should prove (geometrically *and* algebraically) that minimum ratio is given as follows.

$$\sqrt{R_{\min}}=\frac{V^{COM}}{V^{IN}}=\frac{m_1-m_2}{m_1+m_2} \qquad (7.6a)$$

$$\frac{m_2}{m_1}=\frac{V^{IN}-V^{COM}}{V^{IN}+V^{COM}}=\sqrt{\frac{1-\sqrt{R_{\min}}}{1+\sqrt{R_{\min}}}} \qquad (7.6b)$$

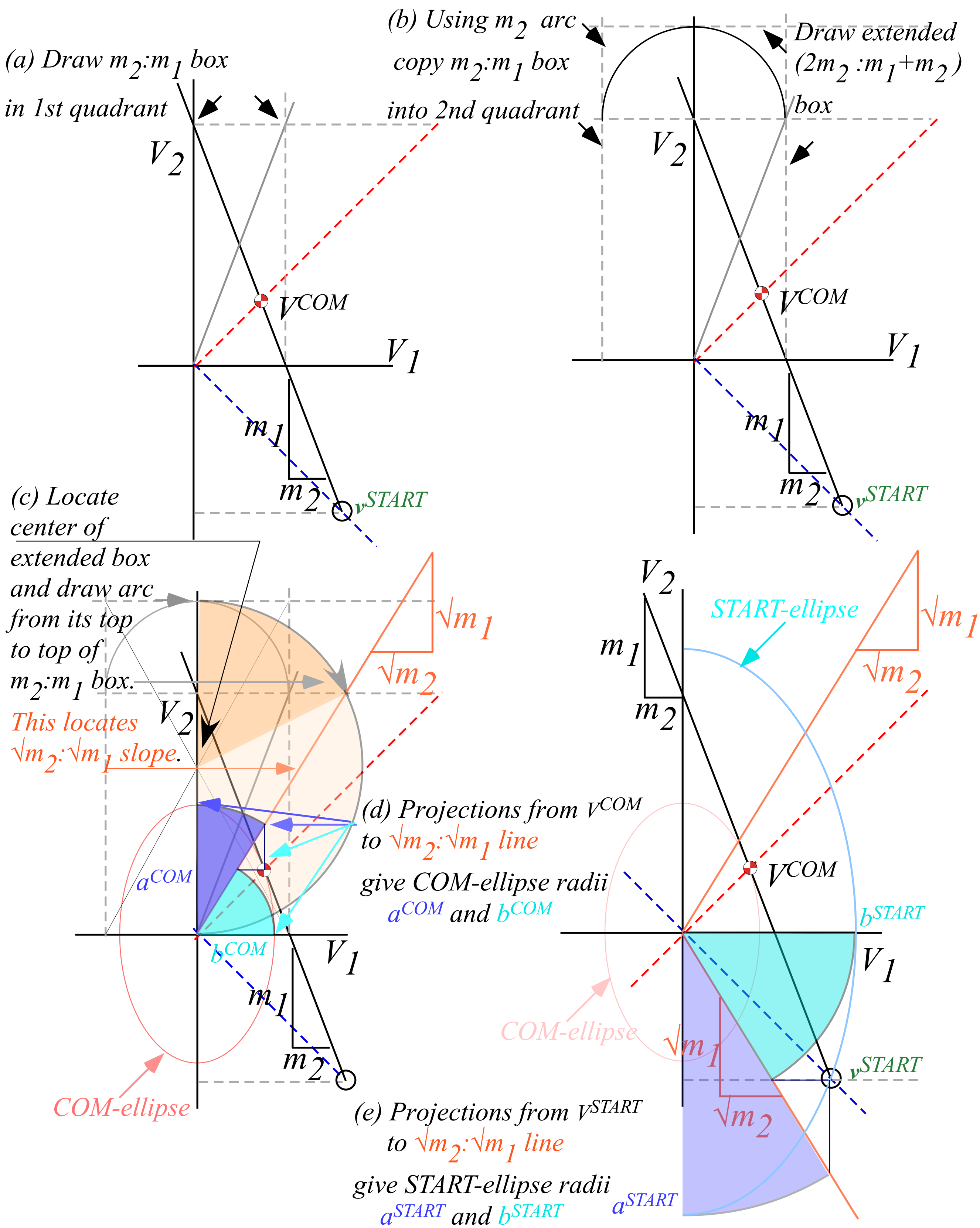

Fig. 7.4 Energy ellipse geometry. (a-c) Axes ratio $\sqrt{m_2}:\sqrt{m_1}$. (d) a^{COM} and b^{COM}. (e) a^{START} and b^{START}.

Ka-Runch-Ka-Runch-Ka-Runch-Ka-Runch-…:Inelastic pile-ups

N-body collisions described so far have been mostly elastic. That's not true for California freeway pile-ups. California pile-up chains start when a cell-phony driver enters a fog at 60 mph and rear-ends a vehicle or vehicles that have slowed down or stopped. Cars drive bumper-to-bumper so dozens may be involved.

Pile-up mass grows with each car added to it by a series of inelastic *"Ka-runch"* collisions like Fig. 1.1 of Ch. 1. Cars may be added to a pile-up's rear or to its front or even to both ends. Fig. 7.5 shows a single *60 mph* car piling up a line of five stationary cars and, *vice versa*, Fig. 7.6 shows a line of five *60 mph* cars piling up on a single stationary car. Each pile-up collision loses as much energy as it can while keeping momentum constant. It makes the smallest ellipse that touches the momentum line in Fig. 3.2c and Fig. 7.3c.

In each case the sequence of velocity-velocity slopes is an *arithmetic progression* *1:1*, *2:1*, *3:1*, *4:1*,… similar to velocity sequences in Fig. 6.4 and Fig. 6.5. Both have lines that intersect on a single point and inverse or complimentary slope sequence *1/1*, *1/2*, *1/3*, *1/4*,…, known as a *harmonic progression*.

The incoming car in Fig. 7.5 has momentum $P^{IN}=mv=60$ and energy $KE^{IN}=\frac{1}{2}mv^2=1800$ with $v=v^{IN}=60$. The final pile-up mass $M=6$ has the same momentum $P^{FIN}=MV=60$ but reduced velocity $V=v^{FIN}=10$ and energy $KE^{FIN}=\frac{1}{2}MV^2=300$ down by *1500* units. (These are (very) Old English units with unit mass *(m=1 ton)* cars.)

The incoming cars in Fig. 7.6 together have momentum $P^{IN}=5mv=300$ and energy $KE^{IN}=5\frac{1}{2}mv^2=9000$. The final pile-up mass $M=6$ has the same momentum $P^{FIN}=MV=300$ with increased velocity $V=v^{FIN}=50$ but reduced energy $KE^{IN}=\frac{1}{2}MV^2=7500$. The same energy deficit of *1500* units is seen in Fig. 7.5 and Fig. 7.6.

Of these two equal-energy-loss nightmares the latter is worse since it began with five times the kinetic energy and still has *7500* units to dissipate. Worse nightmares combine the two as shown in Fig. 7.7. This a particularly troubling set of nightmares since there are many possible outcomes that have different orders of combination with differing results.

How would you like to be an insurance adjustor for that one?

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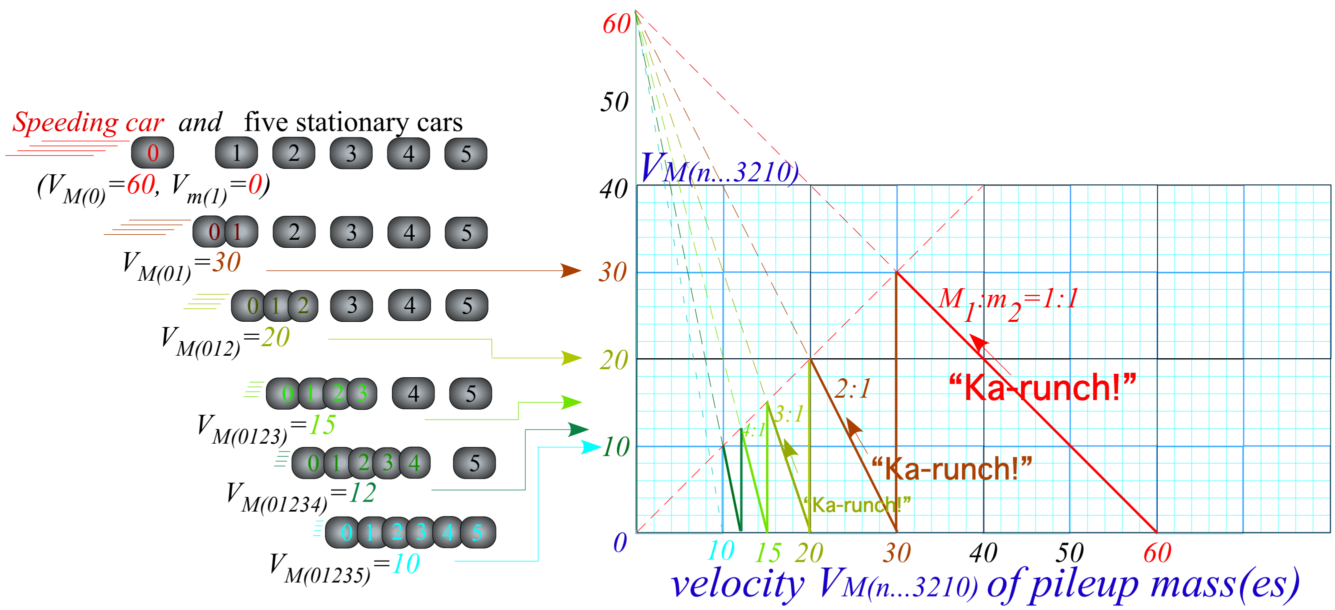


Fig. 7.5 Pile-up due to one 60 mph car hitting stationary line of five cars

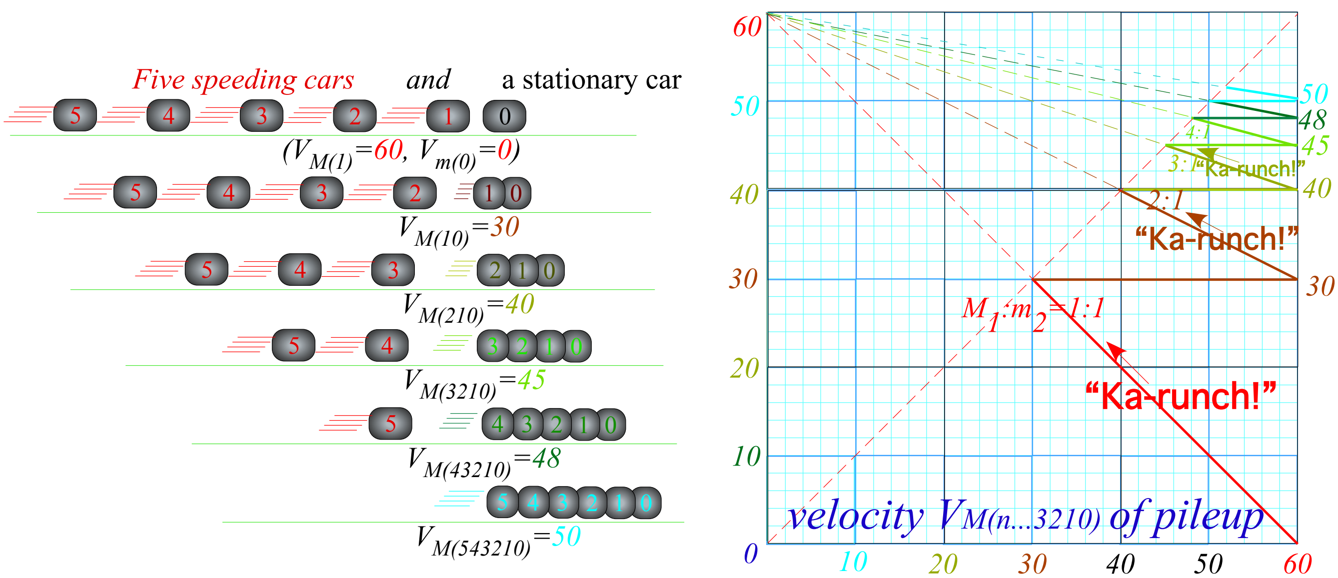


Fig. 7.6 Pile-up due to a line of five 60 mph cars hitting one stationary car

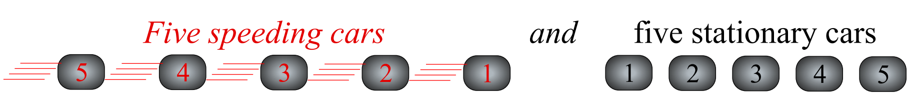


Fig. 7.7 A worse nightmare: Line of five 60 mph cars hitting five stationary cars.

Ka-pow-Ka-pow-Ka-pow-Ka-pow-...:Rocket science

An N-body model of rocket propulsion is made by "time-reversing" pile-ups. Let us imagine a line of $N=11$ equal $(m=1)$-masses separated by explosive charges (*"pow!"*) to blow one fuel-pellet at a time backwards off the rear end of a rocket and propel the remaining rocket mass forward.

Fig. 7.8 is a velocity-velocity plot of seven such *"pow!"*-blasts after which a rocket with just three masses numbered *8*, *9*, and *10* speeds off the page to the right. The *payload* of this rocket is the ball labeled *10* at the head of the line. For $N=11$ balls, there are ten *pow(b)*-blasts numbered by $b=0$ *to* 9.

The velocity unit in Fig. 7.8 is the relative exhaust velocity $\Delta v_e=-1$ of each *pow(b)*-blast. The 0^{th}-blast at the bottom of Fig. 7.8a starts with eleven stationary balls and blows ball-*0* off the line of ten balls *1-2-3...8-9-10*. To conserve momentum (initially 0) the *10*-ball rocket of mass $(M=10m=10)$ has final velocity $\Delta V_M=+1/10$ to cancel momentum $\Delta P_0=m\cdot\Delta v_0=-1$ of fuel-pellet ball-*0* in a zero-sum *pow(0)*-blast.

$$m\cdot\Delta v_0+10m\cdot\Delta V_M(0)=0 \qquad (7.7a)$$

The 0^{th}-blast line begins at the origin $(V_M=0,v_e=0)$ of the V_M-v_e-plot in Fig. 7.8b and extends one unit down and $1/10^{th}$ unit right to point $(V_M(0)=1/10,v_e=-1)$. *Pow(0)*-line slope is mass ratio $(-m/M=-1/10)$. It is a COM line of a *time reversed* totally *in*elastic collision. (You might call it a *super-elastic* collision.)

The 0^{th},1^{st},2^{nd},3^{rd},..., or 9^{th} blast blows off fuel pellet-ball $b=0$, *1*, *2*, *3*..., or *9*, respectively. Each blast gives a larger rocket velocity boost $\Delta V_M(1)=1/9$, $\Delta V_M(2)=1/8$, $\Delta V_M(3)=1/7\ldots\Delta V_M(b)=1/(10-b)$ since rocket mass is less by $m=1$ after each blast but exhaust momentum impulse $m\cdot\Delta v_e=-1$ is equal each time.

$$m\cdot\Delta v_1+9m\cdot\Delta V_M(1)=0 \qquad m\cdot\Delta v_2+8m\cdot\Delta V_M(2)=0 \quad \ldots \quad m\cdot\Delta v_b+(10-b)m\cdot\Delta V_M(b)=0 \qquad (7.7b)$$

The *harmonic progression* $1/10,1/9,1/8\ldots1/5,1/4,1/3,1/2,1$ in Fig. 7.8a contains momentum impulse terms $\Delta V_M(b)$ in a *10*-term *harmonic series* $1/10+1/9+1/8\ldots1/5+1/4+1/3+1/2+1$. Rocket velocity after b^{th} *pow(b)*-blast is a partial sum of the first $b+1$ harmonic terms. $(V_M\,,v_e)$-plots in Fig. 7.8b show this.

0^{th}: $V(0)=1/10=0.1$ $\quad$ 1^{st}: $V(1)=1/10+1/9=0.211$ $\quad$ 2^{nd}: $V(2)=1/10+1/9+1/8=0.336$

3^{rd}: $V(3)=V(2)+1/7=0.478$ $\quad$ 4^{th}: $V(4)=V(3)+1/6=0.646$ $\quad$ 5^{th}: $V(5)=V(4)+1/5=0.846$

6^{th}: $V(6)=V(5)+1/4=1.096$ $\quad$ 7^{th}: $V(7)= V(6)+1/3=1.429$ $\quad$ 8^{th}: $V(8)=V(7)+1/2=1.929$

On its 9^{th} and final *pow(9)* the rocket is boosted by a whole unit exhaust velocity to $V(9)=V(8)+1=2.929$.

A *10*-blast rocket exceeds exhaust velocity $(|v_e|=1)$ on its 6^{th} *pow(6)*-blast with $V(6)=1.096$. This is labeled in extreme lower right hand side of Fig. 7.8b. In COM frame, exhaust mass *6* thru *9* end up moving *forward* but in rocket frame each exhaust mass leaves moving *backward* at exactly $v_e=-1$ until another blast-boost hits the rocket. Exhaust masses numbered *0-9* separate from each other and from payload mass-*10*. Total COM momentum stays zero. All eleven balls "balance" at COM origin.

N-blast velocity is a *logarithm* function if N is large. Momentum is still conserved for each blast.

$$M\cdot\Delta V=-v_e\cdot\Delta M \qquad \text{becomes:} \qquad M\cdot dV=-v_e\cdot dM \quad \text{or:} \quad dV=-v_e\frac{dM}{M} \qquad (7.8a)$$

We integrate this from initial rocket mass M_{IN} to final payload M_{FIN} and from rocket V_{IN} to final V_{FIN}.

$$\int_{V_{IN}}^{V_{FIN}} dV=-v_e\int_{V_{IN}}^{M_{FIN}}\frac{dM}{M} \quad \text{becomes:} \qquad V_{FIN}-V_{IN}=-v_e\left[\ln M_{FIN}-\ln M_{IN}\right]=v_e\left[\ln\frac{M_{IN}}{M_{FIN}}\right] \quad (7.8b)$$

This is the famous *rocket equation*. (Its predictions discourage interstellar travel. See exercises.)

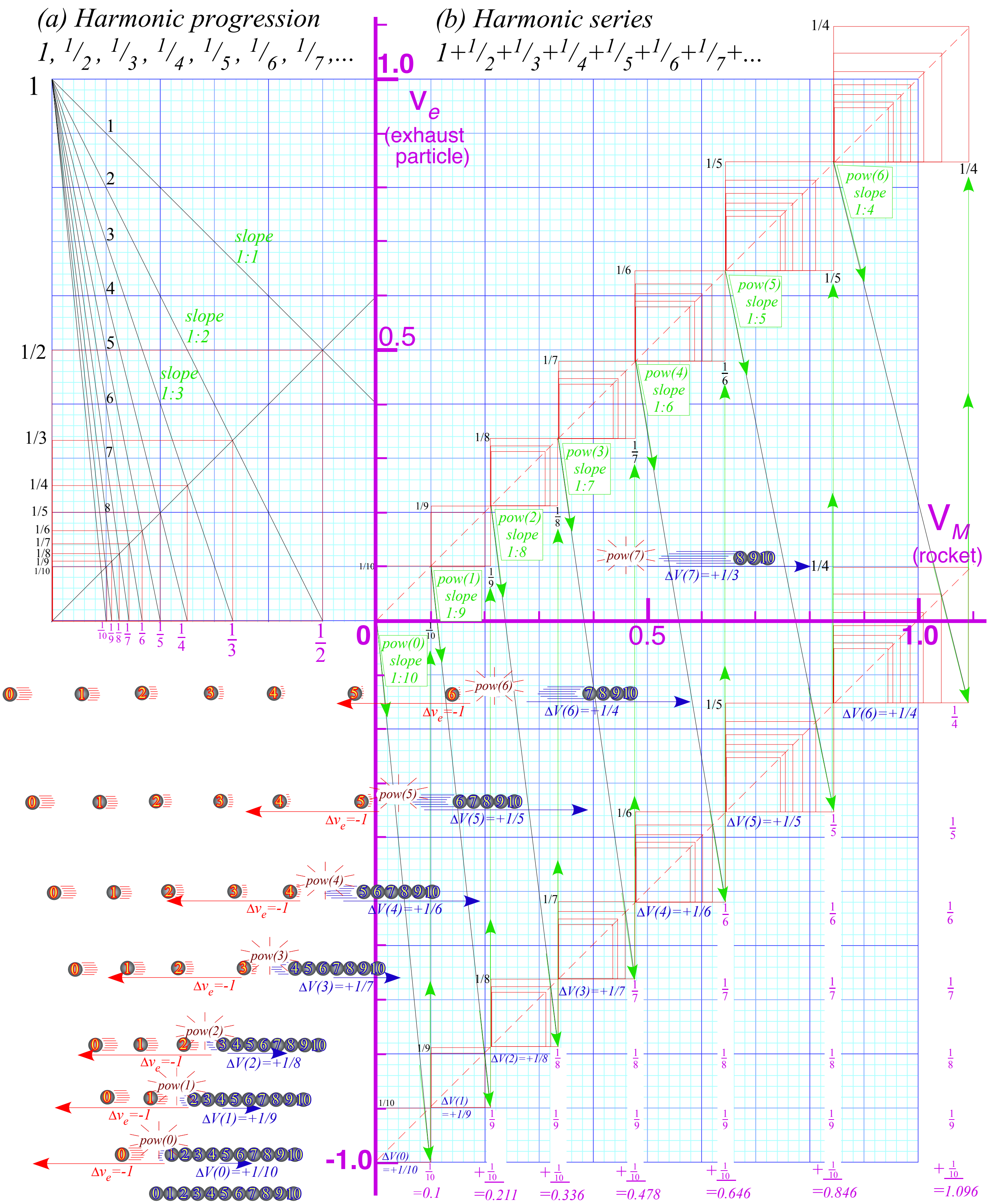


Fig. 7.8 Rocket science by harmonic series geometry.

Exercise 1.7.1 Maximum Energy Transfer (MET Limit)
Suppose each ball has just the right mass ratio with the one above it to pass on all its energy to the next in line. Construct v-v diagrams, velocity at each stage, and mass values for
(a) $N=2$, (b) $N=3$, (c) $N=4$, (d) Give algebraic formulas for general N.

Exercise 1.7.2 Absolute Maximum Velocity Limit (INF Limit)
Suppose each ball is very much larger than the one above so as to approach upper limit. Construct v-v diagrams , limiting intermediate velocity values and limiting top value for (a) $N=2$, (b) $N=3$, (c) $N=4$, (d) Give algebraic formulas for general N.

Exercise 1.7.3 Rocket Science and Backside of exponentials
Compare discrete-blast rocketry in eq.(7.7) or Fig. 7.8 with continuous-blast "rocket science" of eq.(7.8) and study logarithmic-exponential geometry of the latter.
(a) In particular, when do blasted exhaust particles end up going in the same direction as the rocket in the initial (lab) frame where the rocket starts out with zero velocity?
(b) Plot exponential $y=e^x$ and $y=\log_e x$ functions on same graph and draw tangent-triangle whose hypotenuse is tangent to a curves and intercepts x or y axes at -2, -1, 0, 1, 2,.. Give the base and altitude coordinates of the tangent point in each case.

Chapter 8 Geometry and physics of common potential fields

Physical and geometric aspects of elementary force and potential fields are introduced in this section. Most important are oscillator and Coulomb fields that are so important for resonance and orbit theory.

Geometric multiplication and power sequences

The most common *power-law potentials* are $U(x) = Ax^2$ (*Oscillator potential*) in Fig. 8.1, $U(x) = Ax$ (*Uniform field potential*) , and $U(x) = Ax^{-1}$ (*Coulomb potential*) Fig. 8.5. Power-law potentials and force laws have exact geometric constructions, exponential or logarithmic fields only approximately.

Multiplicative power operations are done using a staircase of similar triangles as shown in Fig. 8.2. A geometric progression $\{1=s^0, s=s^1, s^2, s^3,\ldots\}$ and an inverse progression $\{1=s^0, 1/s=s^{-1}, s^{-2}, s^{-3},\ldots\}$ lie on either side of the unit stair step $1=s^0$. A slope or scale factor $s=2$ or $s=1/2$ is used in Fig. 8.2a or Fig. 8.2b. They resemble perspective drawings of school hallways. (Elementary School is (a) and High School is (b).) Each stair zigzags between slope-1 line-$(y=x)$ and slope-s *line*-$(y=s{\cdot}x)$ or between *line*-$(y=-x)$ and *line*-$(y=x/s)$. The *line*-$(y=s{\cdot}x)$ and *line*-$(y=x/s)$ are perpendicular or *normal* to each other. So are *line*-$(y=x)$ and *line*-$(y=-x)$.

A two-step triangle in Fig. 8.1a gives each point on the oscillator potential, a parabola $y=x^2$. To find where the parabola hits vertical *line*-$(x=2.2)$, for example, we go up that line to the $45°$ *line*-$(y=x)$ and then go across to vertical *line*-$(x=1)$. A dashed blue line is drawn from origin thru that point to an arrow intersecting *line*-$(x=2.2)$ at *pt*-$(x=2.2, y=2.2^2)$ on parabola-$(y=x^2)$. A similar zigzag gives *pt*-$(x=-2, y=4)$ or any point on the parabola $(y=U(x)=x^2)$ below.

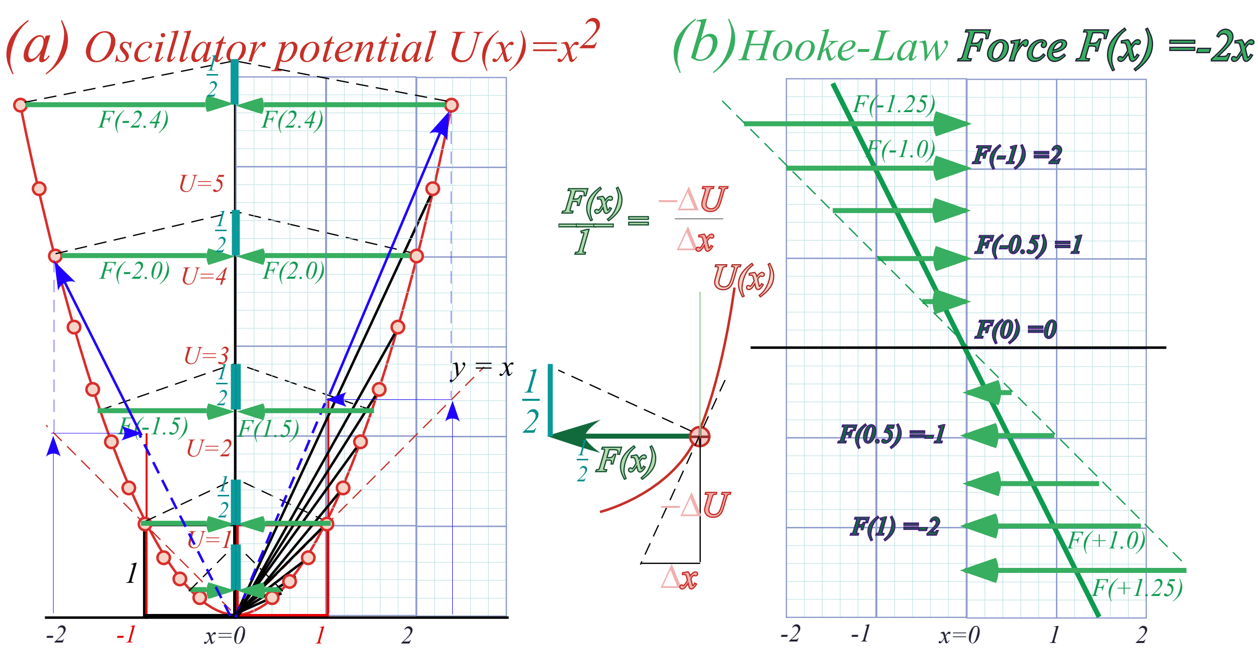


Fig. 8.1 Geometric construction of $U(x)=x^2$ potential and Hooke's force law $F(x)=-2x$.

The physicist *Force =-Slope* rule (5.9) is drawn using force triangles in Fig. 8.1a. Force is linear in x, that is, $F=-2x$, and that is minus the slope of x^2. A line of slope -2 in Fig. 8.1b plots $F(x)$. Force vector **F** scaled by $1/2$ gives a force vector in Fig. 8.1a equal and opposite to coordinate x. Each force triangle has base **F**$/2$, an altitude that is constant $1/2$, and a hypotenuse normal to the parabola tangent. It is like the tangent triangle with base ΔU and altitude Δx (center of Fig.8.1) that shows *force=-slope* ($F(x)=-\frac{\Delta U}{\Delta x}$).

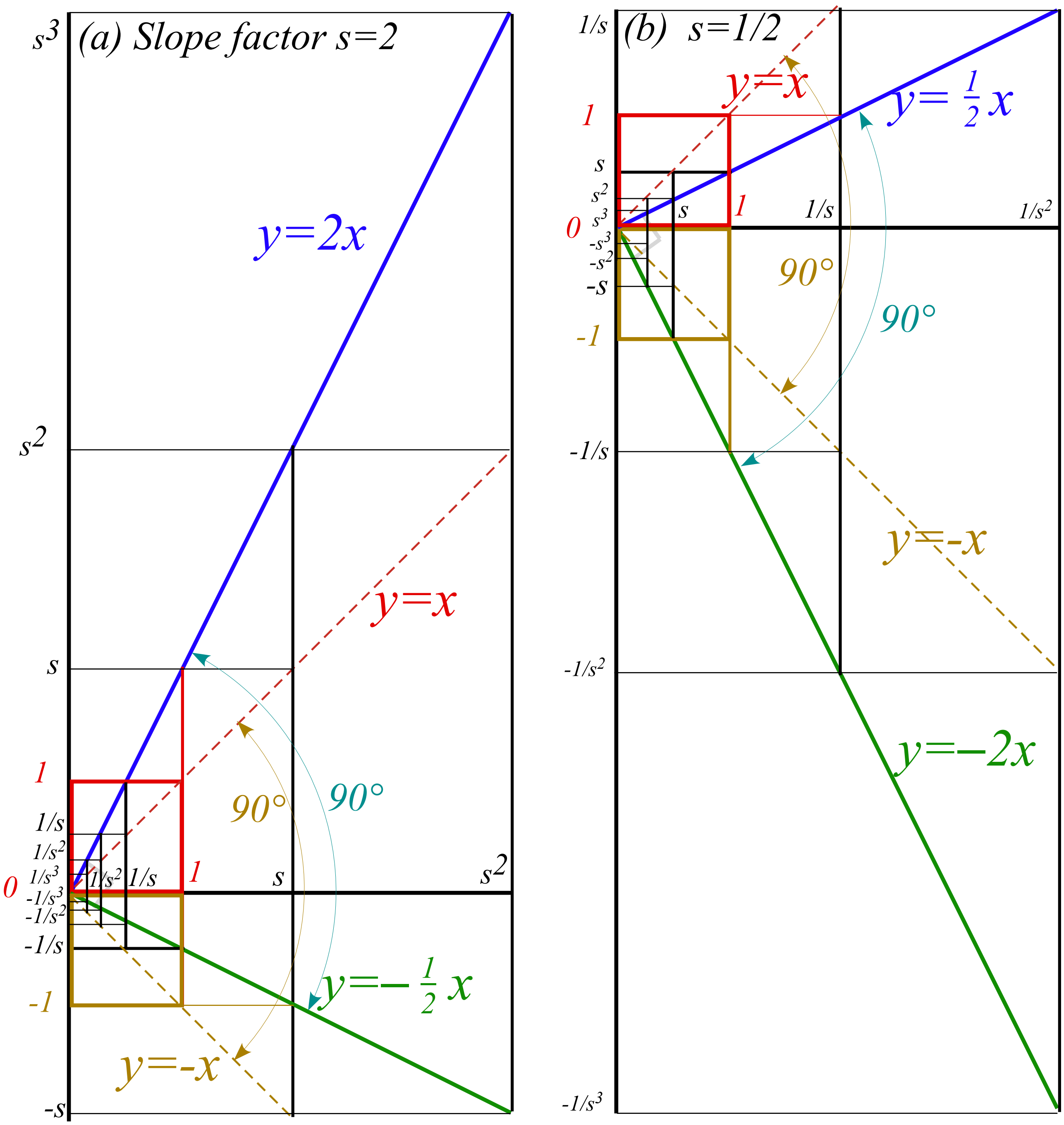


Fig. 8.2 Geometric sequences and "staircases" for slope or scale factor (a) s=2, and (b) s=1/2 .

Parabolic geometry

A parabola $U(x)=Ax^2$ has a *focal point* at $y=U=A/4$ where vertical rays meet if reflected by parabola tangents in Fig. 8.3b. A parabolic radius is its half-width λ at the focus. For $y=x^2$ it is $\lambda=1/2$. (Note $F(\pm 0.5)$ vectors point at focus in Fig. 8.1a.) An old name for λ is *latus rectum*. A circle through the focus about any parabolic point will be tangent to a line called the *directrix* located at a distance λ from the focus. Focus and directrix define a parabola that passes midway between them thru the tip-point M of the parabola where its focal radius and equal distance-to-directrix both reach their minimum value λ/2.

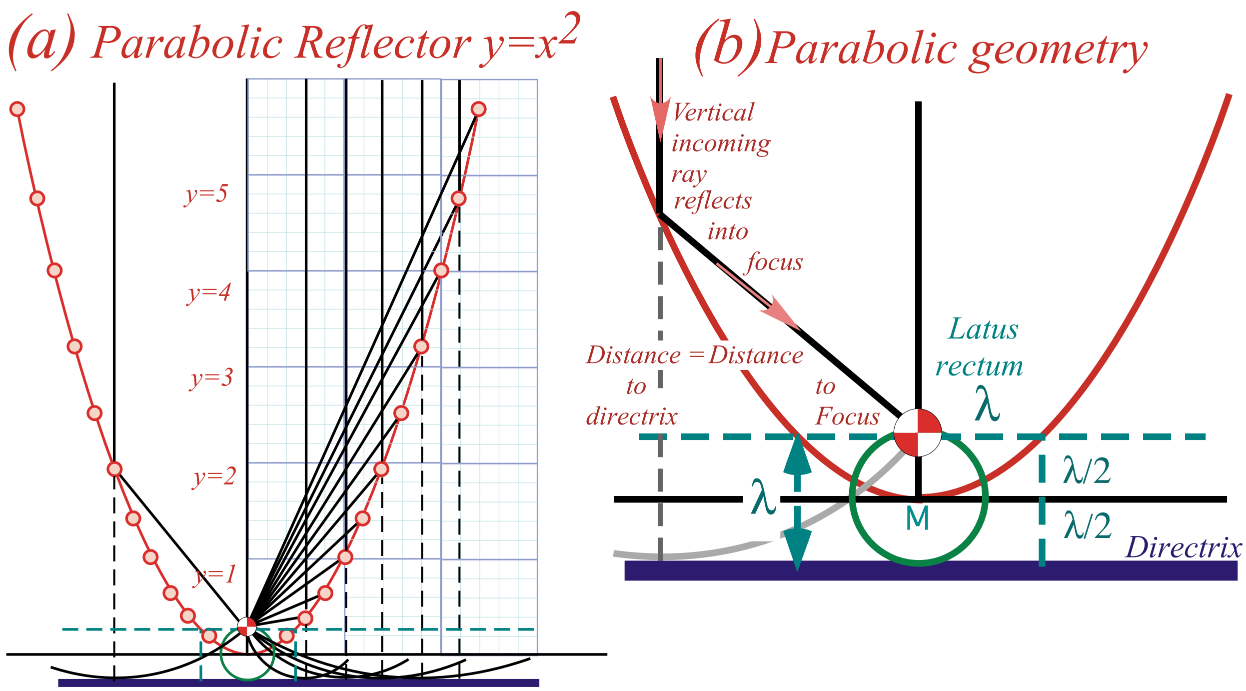


Fig. 8.3 Parabola and geometry (a) Rays converging on focus. (b) λ-geometry of tangent reflection.

Directrix is a so named because it "directs" both the rays and wave phase of an optical reflector. Since the *focal radius* (length of each sloping ray line in Fig. 8.3a) equals the perpendicular *directrix distance* (length of corresponding dashed vertical line), waves are guaranteed to be plane waves. Also, the equality of angle of incidence and reflection off the parabola bisecting the dashed and solid lines, guarantees vertical parallel rays for all which leave the focus and bounce off the inside of the parabola. It also guarantees that parallel vertical rays bouncing off the *outside* will go away from the focus. Either side of a parabolic surface converts plane waves to spherical ones or *vice-versa*.

Parabolic geometric optics suggests the *tangent-kite* for varying tangent slope values. A blue kite of *slope=2* in Fig. 8.4a and yellow kite of *slope=5/2* in Fig. 8.4b have equal focal radius equal to normal distance-to-directrix forming the major iscosoles triangle of the kite. A minor iscosoles triangle (upside down in Fig. 8.4) shares a base with the major one. Their perpendicular bisector is the tangent line. The bisection point is slope $\frac{dy}{dx}=\frac{x}{\lambda}=\frac{x}{2p}$ in units of λ as indicated by vertical arrows.

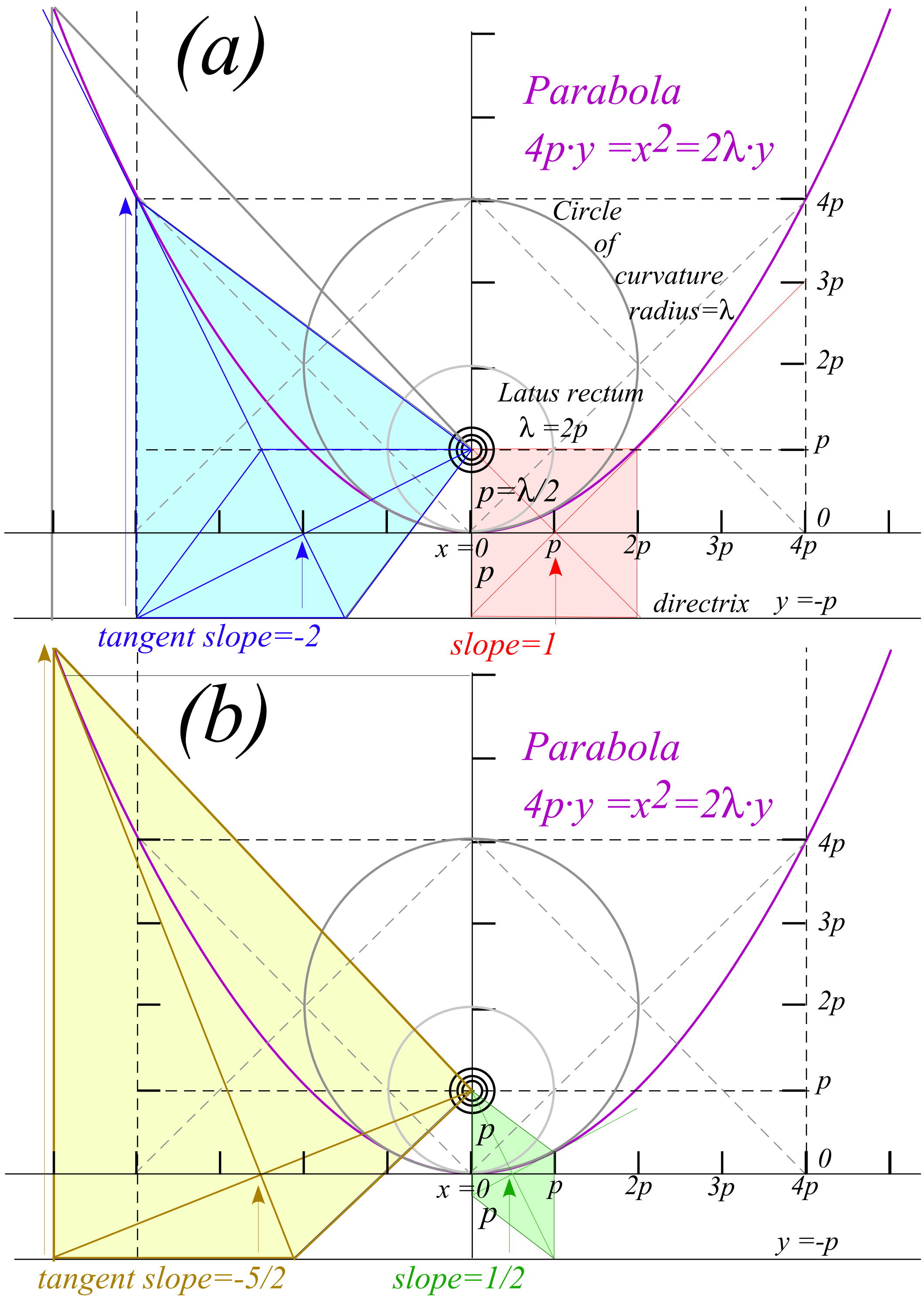


Fig. 8.4 Parabola and geometry of curvature and slope of tangent-kites.

A singular case is the red kite of *slope=1* that is square. Lesser *slope=1/2* gives a rhomboidal green kite with one side on the vertical parabolic axis instead of on the horizontal directrix. Points of *slope=±1* on the ($4py=x^2=2\lambda y$)-parabola lie on either side of its focus at distance $\lambda=2p$ from it. $\lambda=2p$ is also the (minimum) *radius of curvature* of the parabola at its tip (minimum y at $x=0$) that lies a distance $\lambda/2=p$ below the focus.

Coulomb and oscillator force fields

Our atoms and molecules depend on the electrostatic Coulomb field to have stable chemistry and biology. Like charges repel and opposites attract with a force that varies inversely with the square of distance r between them. A simple version of the electric Coulomb force law (axiom) is:

$$F(r) = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2} \quad where: \frac{1}{4\pi\varepsilon_0} = 9{,}000{,}000{,}000\frac{Newtons \cdot meter \cdot square}{per\ square\ Coulomb} \tag{8.1}$$

Units are standard *mks* but magnitudes are *mind boggling*. It's nine *billion* Newtons for just two charge-units a meter apart. (To be precise: $8.99 \cdot 10^9\ Nm^2/C^2$.) Now $1N$ is only about $\frac{1}{4}lb$, but imagine a billion sticks of butter? Also, you have *thousands* of Coulomb charge units in each fingertip with only a centimeter separation so add another factor of *(100)-squared*. Make that ninety *trillion* Newtons for each Coulomb or about a *million trillion* Newtons trying their darndest to blow your pinkie to bits!

But, still we're underestimating this monster force. Most of the electronic charge in the world is crammed into atoms and molecules with at most a nanometer (10^{-9} meter) across and some are an *Angstrom* (10^{-10} meter) or a tenth of a *nano*. So put on another factor of (10^{-9})-squared or million-billion trying to undo your pinkie, that's a *trillion-trillion-billion*. Does your manicurist know about this?

These forces get loose in a TNT blast, but usually, tiny nuclei with an equal positive charge hold down potentially rebellious electrons. Still, what's holding nuclei together? Nuclear radii are *femto*-meters (10^{-15} meter) or *Fermi*. (Note: both *fm* and *Fm* are abbreviations for $10^{-15}m=10^{-13}cm$.)

Oops! That's a factor of $(10^{-15})^2$ or million-trillion-trillion to increase our stress level. Nuclear charge is 10^5 times more pent-up than its atomic electronic counterpart , a grand total of about a *trillion-trillion-trillion-trillion* Newtons hankering to blow up your fingertip nuclei. Cancel the manicure!

When nuclei do blow up, the result is more than 10^5 times more devastating than TNT bangs. We don't use *force* to estimate the devastation of a nuclear fission bomb or the yield of nuclear fuel. Rather we use electric *potential energy*, that varies as $1/r$ not $1/r^2$. (Slope of a $U(r)=1/r$-curve is $F(r)=1/r^2$.)

$$U(r) = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r} \quad where: \frac{1}{4\pi\varepsilon_0} = 9{,}000{,}000{,}000\frac{Joule}{per\ square\ Coulomb} \tag{8.2a}$$

Energy or *(Force)-times-(distance)*-unit is *Joule* or *Newton meter* ($N \cdot m$). Like superball PE field $U(r)$ in (5.9), force $F(r)$ (8.1) is a *(-)*derivative of potential $U(r)$ that in turn is *(-)*integral of force $F(r)$ in (6.5)

$$F(r) = -\frac{dU(r)}{dr} = -\frac{qQ}{4\pi\varepsilon_0}\frac{d}{dr}r^{-1} = \frac{qQ}{4\pi\varepsilon_0}r^{-2} \tag{8.2b}$$

$$U(R) = -\int_\infty^R F(r) \cdot dr = \frac{qQ}{4\pi\varepsilon_0}r^{-1}\Big|_\infty^R = \frac{qQ}{4\pi\varepsilon_0}R^{-1} \tag{8.2c}$$

Nuclear PE yield is about a million times greater than for the same number of chemical PE sources since *femto*-meter nuclei are a million times smaller $(R_{NUC} \sim 10^{-15})$ than *nano*-meter molecules $(R_{MOL} \sim 10^{-9})$. Nuclear forces would then be a trillion times greater than typical atomic and molecular forces.

Fig. 8.5 plots attractive Coulomb force $F(r)=-1/r^2$ and potential $U(r)=-1/r$ of *negative* charge $-q$ to a *positve* $+Q$ nucleus. (Negative force points toward the $+Q$ origin $(x=0)$.) It uses the zigzags of Fig. 8.4.

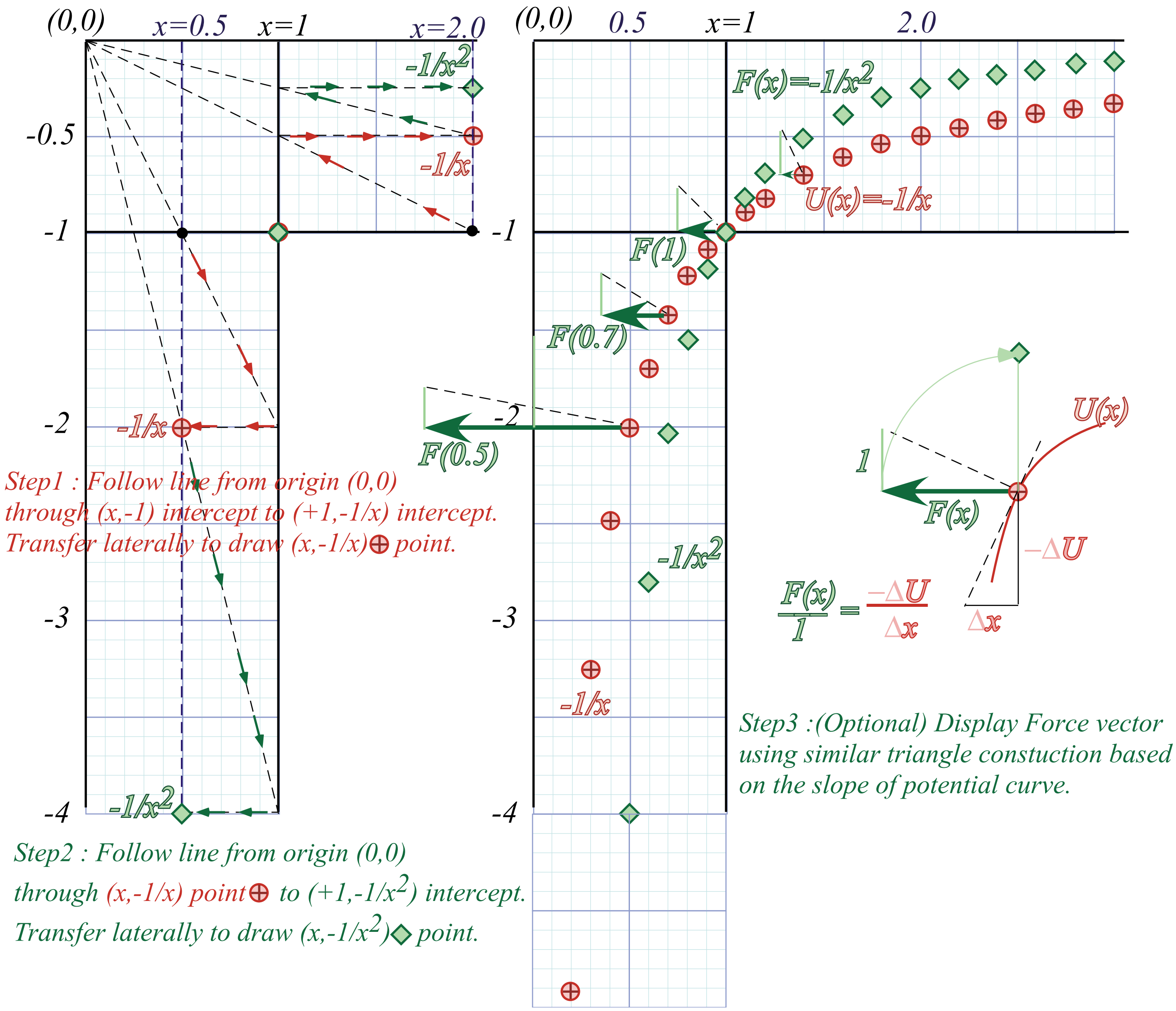


*Fig. 8.5 Attractive Coulomb force F(x) and potential U(x) curves. (**F**(x) vectors drawn at 1/4-scale.)*

Could the Coulomb $F(r)\sim 1/r^2$ force field be derived like the superball force $F(Y)\sim 1/Y^3$ in (6.10) by counting momentum bangs? Indeed, if a charge ejected a cloud of little "bang-balls" then the number of bangs scored at distance r would vary inversely with area $4\pi r^2$ of a radius r sphere. But, this doesn't explain so well *attraction* of a charge $+Q$ to a $-q$ or of a mass M to a mass m in Newton's gravity law.

$$F_{grav}(r) = -GMm / r^2 \text{ , where: } G=0.000000000067 \text{ N m/kg}^2 \tag{8.3}$$

Gravity is universally attractive (no "negative" matter readily available) but much weaker than electric pull since G constant $6.672E\text{-}11$ ($\frac{2}{3}\cdot 10^{-10}$ in *mks* units) is smaller (*by* 10^{20} *times!*) than the $9\cdot 10^{+9}$ in (8.2).

As of this writing it is still a mystery why these are so different. We really do not yet understand either of these forces at a fundamental level. They are still very much in the axiom box.

Tunneling to Australia: Earth gravity inside and out

Imagine $x=1$ in Fig. 8.5 is the Earth radius $R_\oplus=6.4E6m$. The $F(r)$ plot shows gravity falling off for $r>R_\oplus$ or $x>1$. But it's wrong for subterranean radii $(r<R_\oplus)$ unless Earth is compressed. $F(r)=-1/r^2$ is not true everywhere unless Earth is crushed to a 10 millimeter radius **black hole**.(More on this later.)

If you could be at sub-$R_\oplus$ levels all Earth mass at radii above your radius r may be ignored in figuring your weight! It is easy to see you're weightless at the center ($r=0$) since the pull of all Earth's mass exactly cancels there. But, so also does your attraction to a spherical mass shell cancel anywhere inside it. One could float weightlessly anywhere therein regardless of the shell's size or weight.

Such cancellation is a geometric peculiarity of inverse square law. (It also underlies a Gauss law explanation of why you're safe inside a car struck by lightning.) Any direction you look inside a uniform mass shell has a mass element m whose force is cancelled by another element M behind. (See Fig. 8.6.)

The shell tangent to the m-point you're facing intersects the tangent to the M-point behind you to make an isosceles triangle whose sides make an angle Θ with your line of sight along the base. This means a narrow cone of sight will include shell mass $m=Ad^2$ at a distance d in front of you and shell mass $M=AD^2$ at a distance D directly behind you, where the angular factor $A\sim 1/\sin\Theta$ is the same for both giving perfect cancellation of gravity m/d^2 in front with $-M/D^2$ behind youfor *all directions*.

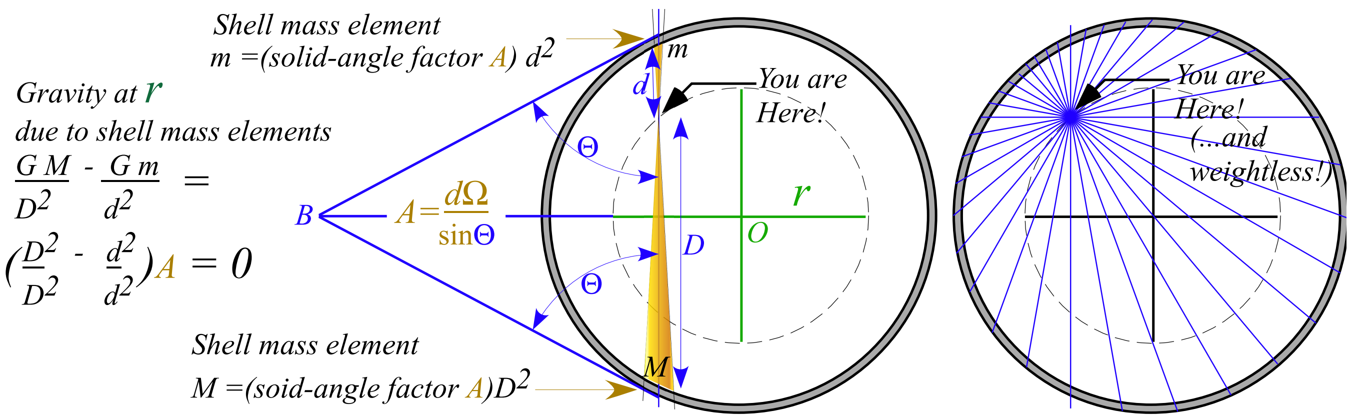


Fig. 8.6 Equal-opposite attraction. Geometry for you floating weightless inside a spherical shell.

A mass m at radius r inside Earth feels gravity attraction $GmM_</r^2$ where $M_<$ is Earth mass *inside* the radius r indicated by the dashed circle in Fig. 8.6. If Earth is uniform density ρ, then that inside-mass is $M_<=4\rho\pi r^3/3$. Force law r^{-2} cancels all but one r of the r^3 in mass $M_<$. We then get a *linear* force law.

$$F_{inside}(r)=GmM_</r^2=m(G4\pi\rho\ /3)\ r=mg(r/R_\oplus)=mgx \tag{8.4a}$$

$$\text{(Earth surface gravity: } g= G\,R_\oplus 4\pi\rho\ /3=9.8ms^{-2}) \tag{8.4b}$$

The linear force law (8.4) is like that of a harmonic oscillator in Fig. 8.1b and so the inside-Earth potential must be a parabola like Fig. 8.1a. Force $F(1)=-1$ is continuous as we cross $x=1$ and so must be the slope of potential $U(x)$ as U changes from $-1/x^2$ to parabola. Terrestrial beings such as ourselves live in a nearly-constant-field ($\frac{dF}{dx}\sim 0$)-region near $x=1$. In Fig. 8.7 we find the potential parabola geometrically by its focal point and directrix using the tangent at $x=1$. Recall a tangent at $x=\lambda=2p$ in Fig. 8.4a has $slope=1$ or $45°$. So does the parabola at $x=1$ in Fig. 8.7 below have a slope of $(+1)$ and a force of (-1) (That's $-mg$ in mks units.)

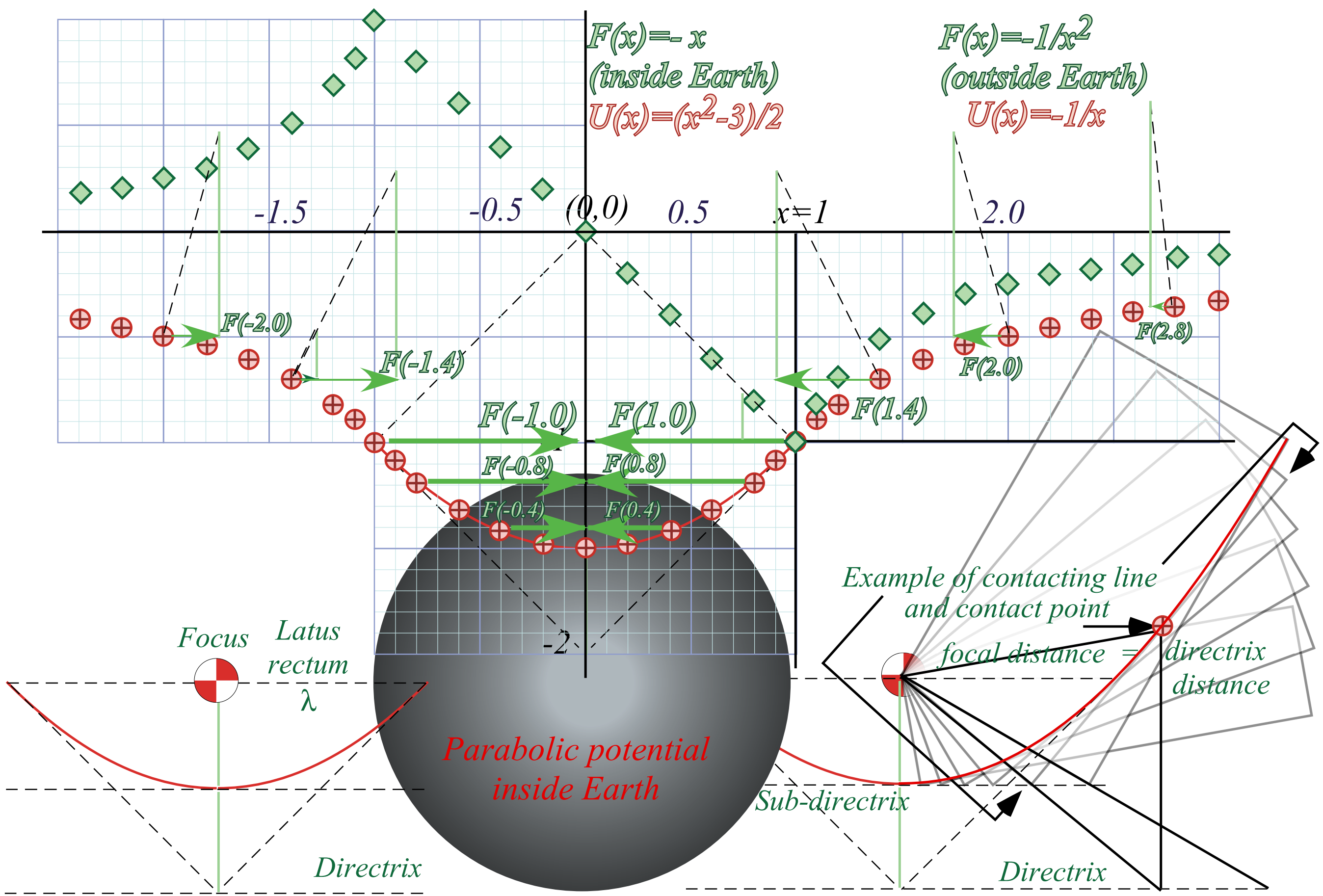


Fig. 8.7 Construction of Earth gravitational fields inside and outside.(units of x: $R_\oplus$,; F: mg; U: $mgR_\oplus$)

A parabola tangent bisects the angle between the line to the focus and the directrix drop-line as in Fig. 8.4. Twice *45°* gives *90°*. The focus is $\lambda=1.0$ units straight across and the directrix is $\lambda=1.0$ units below as shown in Fig. 8.7 (lower-left). Using this we may construct the parabola by rotating a square corner of a piece of graph paper around the focus so the corner touches a line halfway to the directrix. (We can call this half-way line the *sub-directrix*. It is the line of tangent intersections indicated by arrows in Fig. 8.4.)

The parabola so constructed is $y=x^2/2-3/2$. That is the *interior* potential $U^{IN}(x)$ $(-1<x<1)$. It meets the curve $y=-1/x$ that is the *exterior* potential $U^{EX}(x)$ $(1<x<\infty)$ at $x=1$ where they are equal ($U^{IN}(1)=-1=U^{EX}(1)$) as is slope, which is the force ($F^{IN}(1)=-1=F^{EX}(1)$). (However, the slope of the *force* curve takes a *big* jump!)

Adding a constant to a potential won't alter slope or force. We added $\frac{-3}{2}$ to $\frac{x^2}{2}$ to make it equal $\frac{-1}{x}$ at $x=1$.

To catch a falling neutron starlet

The “glue” that holds in the rebellious nuclear proton charge is called *nuclear matter*, a mix of neutrons, mesons, and their ingredients. Let’s imagine a fingertip *(1cc)* of neutrons as densely packed as they are in a nucleus or neutron star and estimate how such a neutron starlet might travel through Earth. First, we find the density of nuclear matter. Let’s say a nucleus of atomic weight *50* has a radius of *3 fm*, so it has *50* nucleons each with a mass $2{\cdot}10^{-27}kg$. (It’s actually more like $1.67{\cdot}10^{-27}$, but roughly $2{\cdot}10^{-27}$.)

That is $100{\cdot}10^{-27}=10^{-25}$ *kg* packed into a volume of ${}^{4\pi}/_{3}r^3= {}^{4\pi}/_{3}\,(3{\cdot}10^{-15})^3\ m^3$ or about $10^{-43}\ m^3$. That gives a huge density of $10^{-25+43} = 10^{18}kg$ *per* m^3 or a *trillion* kilograms in the size of a fingertip.

That’s a weighty fingertip! *mg* is ten trillion Newtons. (Well, actually 8.8 trillion Newtons. No need to exaggerate here!) In spite of this, its gravitational attraction to nearby rocks on the Earth is comparatively moderate. A (10cm)^3 1kg rock would cling to the starlet with a force of about

$$F_{rock}=Gm(1kg)/r^2= 100Gm = 100(6.7E\text{-}11)1E12 = 6{,}700\ N, \qquad (m=M_{starlet}=10^{12}kg)$$

or less than a ton that is tiny for a starlet weighing billions of tons and cutting into the Earth like a bullet going through cotton candy. Let’s see what speed it might gain falling from the surface.

Starlet energy is assumed constant since friction would be tiny compared to its enormous weight.

$$E = KE + PE = {}^1/_2\, m\, v^2 + U(x) = {}^1/_2\, m\, v^2 + {}^1/_2\, mg\, (x^2 - 3)=const. \qquad (8.5)$$

Let it be released at Earth surface *(x=1)* with zero velocity. This sets the energy constant.

$$E = {}^1/_2\, m0^2 + {}^1/_2\, mg\, (1^2 - 3)=const.=-\ mg \qquad (8.6)$$

At Earth center *(x=0)* we solve for the velocity there. (The starlet mass *m* cancels out.)

$$E = {}^1/_2\, mv^2 + {}^1/_2\, mg\, (0^2 - 3)=const.=-\ mg \quad \text{or: } v^2 = g\ , \qquad (8.7a)$$

$$v = \sqrt{g} \quad (\text{In } mks \text{ units: } v^2 = gR_\oplus\ , \ \text{ or : } v_0 = \sqrt{(g\,R_\oplus)}=8\ km/s) \qquad (8.7b)$$

$v_0 = 8\ km/s$ is also Earth’s minimum *orbital insertion speed*. A mass dropped down the tunnel flies with the same *x*-coordinate as one shot with the speed v_0 into circular orbit. One flies above the other and they meet each other on the other side *42* minutes later as shown in Fig. 8.8. We now show this synchrony of orbital timing holds for all pairs of starlets sent from anywhere inside the Earth!

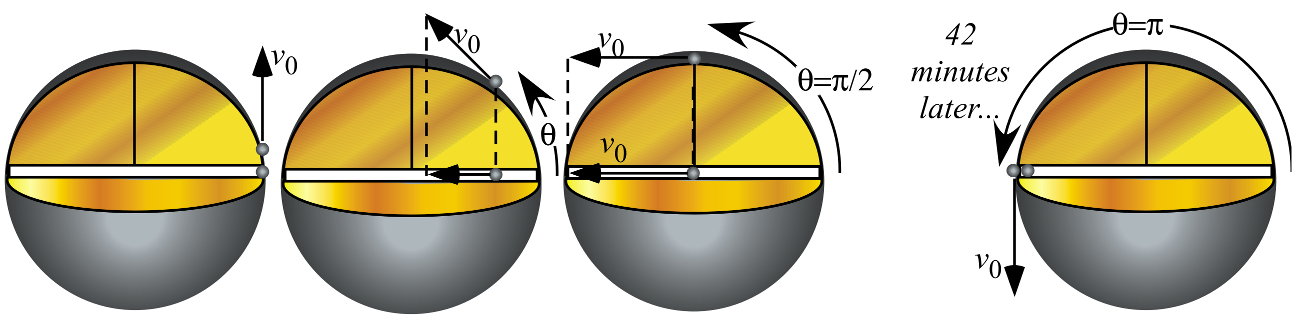


Fig. 8.8 Neutron starlet penetrates Earth as satellite orbits to meet 1/2-way around in 42 minutes.

This synchrony involves a physicist's most favored type of potential energy $U=\frac{1}{2}kx^2$. When $PE=U$ is a square like kinetic energy $KE=\frac{1}{2}mv^2$ it has a symmetry between position x and velocity v.

$$E=KE+PE=const.=\tfrac{1}{2}mv^2+\tfrac{1}{2}kx^2$$

We make any constant-sum-of-squares into a *Pythagorian relation* $1=\sin^2\theta+\cos^2\theta$ as we did to analyze the sum (4.10) of super-ball *KE*. Here (8.5) is a sum $E=KE+PE$ and the constant k is starlet weight mg.

$$1=(m\,v^2/2E)+(k\,x^2/2E)=\sin^2\theta+\cos^2\theta \qquad (8.8a)$$

Position x and velocity v are then expressed in terms sine and cosine of a *phase angle* θ .

$$x=\sqrt{(2E/k)}\,\sin\theta \qquad (8.8b) \qquad\qquad v=\sqrt{(2E/m)}\,\cos\theta\ . \qquad (8.8c)$$

Velocity v is proportional to $\cos\theta$ and θ has a constant *angular velocity* $\omega=\frac{d\theta}{dt}$ so that $\theta=\omega\cdot t+\alpha$. ($\alpha=\theta_0=const.$)

$$v=\frac{dx}{dt}=\frac{dx}{d\theta}\frac{d\theta}{dt}=\frac{dx}{d\theta}\omega=\omega\sqrt{\frac{2E}{k}}\cos\theta=\sqrt{\frac{2E}{m}}\cos\theta \quad (8.9a) \qquad where: \qquad \omega=\frac{d\theta}{dt}=\sqrt{\frac{k}{m}} \qquad (8.9)$$

Angle θ is polar angle in *$(x,v/\omega)$-phasor-space* of Fig. 8.10a. $(x,v/\omega)$-orbits are circular-*clockwise* ($\omega=-|\omega|$) if positive phasor v-axis is *up* and positive-x axis is to the *right*. Earth xy-orbits may be elliptical with a polar angle ϕ that can orbit ± in Fig. 8.10. Each spatial dimension x and y has a constant sub-total energy.

$$KE_{Total}=e_y+e_y \quad where: \quad e_x=const.=\tfrac{1}{2}mv_x^2+\tfrac{1}{2}kx^2 \qquad and: \quad e_y=const.=\tfrac{1}{2}mv_y^2+\tfrac{1}{2}ky^2 \qquad (8.10)$$

Equal constants $(e_x=e_y)$ give a circular orbit in Fig. 8.8. Frequency ω (8.9) is independent of energy value e_x or e_y and orbit and x-tunnel motion share frequency $\omega=\sqrt{g}$. Tunnel motion with same e_x but zero e_y, has half the energy. All motions of the starlet inside the Earth have the same *84*-minute period. That is a *fundamental harmonic period* of a uniform Earth and approximates behavior of the real Earth.

To depict the force vector **F** on the starlet simply draw an arrow from it to origin as in Fig. 8.9a since **F** is proportional to coordinate vector **-r**. (In Fig. 8.7, F is equal to $-r$.) It's projection on x or y-axes are the forces components driving the *84*-minute oscillations along x or y-axes. Perhaps, there is a starlet deep below us swishing out *84*-minute elliptical orbits as in Fig. 8.9b.

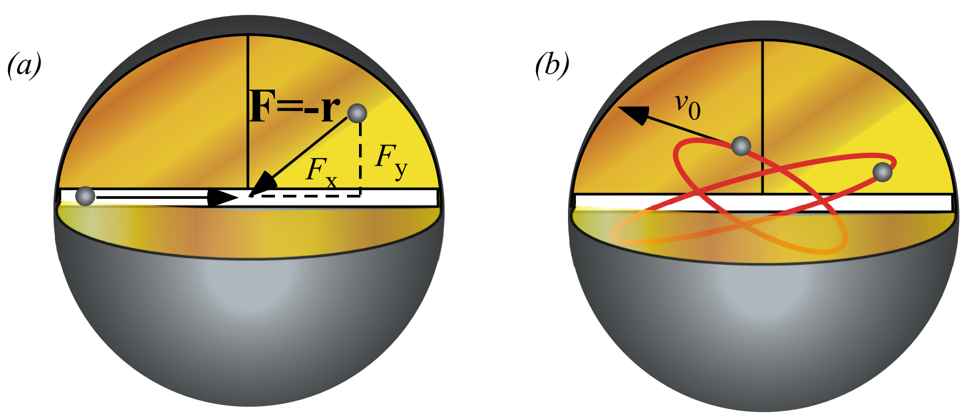


Fig. 8.9 Force and orbits inside Earth. (a) ***F*** *is minus the coordinate vector (b) Typical orbits.*

Starlet escapes! (In 3 equal steps)

Imagine starlet-*m* has decayed to the bottom of the $U(x)={}^1/_2mg(x^2–3)$ curve in Fig. 8.7. How much energy E_∞ does it take for it to escape from Earth center and go back home to ∞? The plot of $U(x)$ in Fig. 8.7 suggest three equal steps of $½E_?$ that bring energy $-{}^3/_2E_\infty$ at $x=0$ up to *zero* at $x=\infty$

Step-1 is to drag or shoot the starlet-*m* to the Earth's surface. That takes energy $\Delta E_1={}^1/_2$. (That's ${}^1/_2mgR_\oplus$ in *mks* units.) Shooting radially at velocity $v_0 = \sqrt{(gR_\oplus)}$ given by (8.7b) would do this first step. It would then come to rest (momentarily) at the Earth surface at $r=R_\oplus$.

Step-2 is to launch starlet-*m* into a minimal circular orbit from the Earth's surface. That takes dollop of energy $\Delta E_2={}^1/_2$ equal to the first. (Again, that's ${}^1/_2mgR_\oplus$ in *mks* units.) Shooting tangentially with minimum orbital insertion velocity $v_0 = \sqrt{(gR_\oplus)}$ given by (8.7b) does this second step.

Step-3 involves a final energy jump $\Delta E_3={}^1/_2$ equal to each of the first two by increasing from the orbital insertion velocity $v_0 = \sqrt{(gR_\oplus)}$ to the *escape velocity* V_e from Earth's surface $r=R_\oplus$.

$$V_e = v_0\sqrt{2}= \sqrt{(2gR_\oplus)} =11.3\ km/s=7\ mile/s \qquad (8.11a)$$

In terms of fundamental potential $U_{grav}(R_\oplus)= -GMm\,/R_\oplus$ at a planets surface $r=R_\oplus$ the escape velocity is

$$V_e = v_0\sqrt{2}=\sqrt{(2GM/R_\oplus)}\ . \qquad (8.11b)$$

Orbital threshold velocity v_0 of radius $R_\oplus$ is $\sqrt{2}=0.707$ or about *71%* of the escape velocity V_e from there.

No escape: A black-hole Earth!

By uniformly compressing Earth, we imagine extending the region of the Coulomb potential $–1/r$ in Fig. 8.5 to lower values of *r* while making the harmonic potential $U(r)= {}^1/_2kr^2$ inside the body occupy a smaller and smaller radius $R_\oplus$ and take on narrower, deeper, and more negative energy values.

The plot in Fig. 8.5 maintains its shape but we rescale to accommodate a squashed Earth. The escape velocity in (8.11b) grows as we consider a decreasing squashed-planet radius $R_\otimes$. Finally there comes a particular radius $R_\otimes$ where the escape velocity (8.11b) is the speed *c* of light.

$$c =\sqrt{(2GM/R_\otimes)} \qquad (8.12a)$$

That radius is called the *Schwarschild radius* or "black hole" radius since light cannot escape.

$$R_\otimes = 2GM/c^2 \qquad (8.12b)$$

For the earth of mass $M_\oplus = 6\cdot 10^{24}\ kg$ the radius $R_\otimes$ is about nine *mm*, or the size of a fingertip. It is hard to imagine our world so squashed! Things may be collapsing all around, but please, not *that* much.

Oscillator phasor plots and elliptic orbits

Oscillator functions in (8.8) suggest a coordinate-velocity plot or *phase-space* plot. By (8.9) the phase angle $\theta=\omega\cdot t+\alpha$ is a product of angular frequency ω and time. A circle starting on $+x$-axis has initial phase to $\alpha=\theta_0=\pi/2$ and plot $(x= X\cos\omega t,\ v/\omega= -X\sin\omega t)$ for its "clock" or *phasor plot* in Fig. 8.10a.

So that positive v *versus* x defines its 1st quadrant, a phasor rotates *clockwise* like a clock hand so angle $\theta=-|\omega|t$ has a minus sign. (This is quite *apropos* since our clocks now *are* waves and harmonic oscillators. Each dimension x and y has its phasor plot as indicated by Fig. 8.10b. In other words there are four phase-space or phasor dimensions $(x\ ,\ v_x/\omega\ ,\ y\ ,\ v_y/\omega)$ being plotted. Here the frequency ω for each dimension x and y is identical due to symmetry or *isotropy* of the Earth model. But, initial phases α_x and α_y of x and y are independent. In Fig. 8.10b we set x-oscillator phase to *2* o'clock (on a 16-hour clock) and y-oscillator 2 hours ahead to *4* o'clock so the ellipse orbit is *clockwise* and have a *left-handed* symmetry. Setting x to be 2 hours ahead of y makes the same orbit but it will go *counter*-clockwise and have a *right-handed* symmetry.

The x *versus* y plot with x always two hours or *45°* behind y, is an inclined elliptical xy-orbit path in Fig. 8.10b. It might represent a typical neutron starlet path in the Earth. Or else, it might represent an optical polarization ellipse described later. Below is a discussion of some special cases of orbit ellipses.

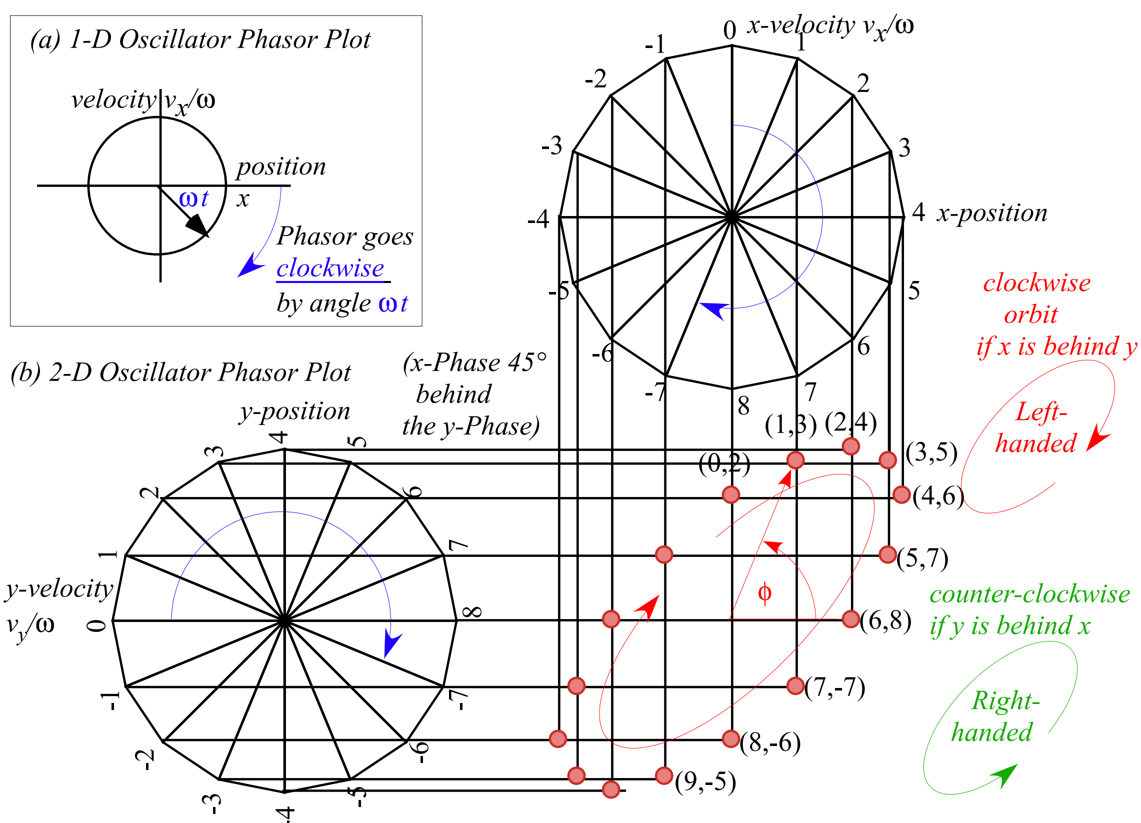


Fig. 8.10 Oscillator plots. (a) 1D-HO phasor plot. (b) Isotropic 2D-oscillator phasors and xy-path.

First we verify that orbits in Fig. 8.10-11 are ellipses. Fig. 8.11a has x running $90°$ behind y with a *relative phase lag* $\Delta\alpha=\alpha_x-\alpha_y=\pi/2$ that is 4 hours or $1/4$-period behind in phase on a 16-hour clock. We say such a $90°$-lagging-x-motion is *in-quadrature* to y-motion. It gives an un-tilted ellipse with a left-handed orbit, and if $e_x=a=b=e_y$ then it gives a circular orbit or *left-circular polarization*. (See Fig. 8.11a on right.) For right-handed orbits x-motion and x-motion switch leads to $\Delta\alpha=\alpha_x-\alpha_y=-\pi/2$.

In-quadrature xy -motion is a *cosine* and *sine* projection on a-side and b-side of an ellipse, respectively, based on expressions (8.8).

$$x = a \cos \omega t , \quad (8.13a) \qquad y = b \cos(\pi/2-\omega t) = b \sin \omega t . \quad (8.13b)$$

Squaring and adding cosine and sine expressions gives a standard xy-ellipse equation.

$$x^2 / a^2 + y^2 / b^2 = 1 \quad (8.13c)$$

Zero phase lag $\Delta\alpha=0$ or *in-phase* motion gives *linear* polarization in Fig. 8.11b. In the case of Fig. 8.11b where x and y-motions are *in-phase* we have

$$x = a \cos \omega\cdot t , \quad (8.14a) \qquad y = b \cos \omega\cdot t . \quad (8.14b)$$

Combining these two gives a trajectory that follows a straight line of slope (b/a) seen in the figure.

$$y = (b/a)\, x \quad (8.14c)$$

Lag $\Delta\alpha=\pm\pi$ or *pi-out-of-phase* is a linear polarized motion, too.

$$x = a \cos \omega\cdot t , \quad (8.15a) \qquad y = -b \cos \omega\cdot t . \quad (8.15b)$$

It is simply a horizontal mirror reflection of the in-phase path.

$$y =-(b/a)\, x \quad (8.15c)$$

In each of the figures we could imagine *three* starlets going in unison. The first starlet obeys the y-equation (8.13b) with $x=0$. The second starlet obeys the x-equation (8.13a) with $y=0$ and tunnels as in Fig. 8.8. A third starlet obeys *both* the x *and* y equations like the starlet orbiting above tunneling one(s). In a *linear force field* $\mathbf{F}=-k\mathbf{r}$ all Cartesian components oscillate sinusoidally at the same frequency.

$$\mathbf{F}=-k\mathbf{r} \quad \text{implies : } F_x=-kx , \quad F_y=-ky , \quad F_z=-kz \quad (8.15)$$

Neither the coulomb field $\mathbf{F}=-k\mathbf{r}/r^3$ nor any other power-law field $\mathbf{F}=-k\mathbf{r}r^p$ is so convenient!

As shown later, negative energy orbits in Coulomb fields are also elliptic, and elegant ruler & compass geometry applies to them, as well. However, Coulomb ellipses are symmetric about origin only for circular orbits. All other Coulomb orbits are *eccentric* since they orbit about an off-center *focal point* instead of the on-center ellipse point of symmetry that lies at origin (**r=0**) for any Hooke's law oscillator orbit of a starlet.

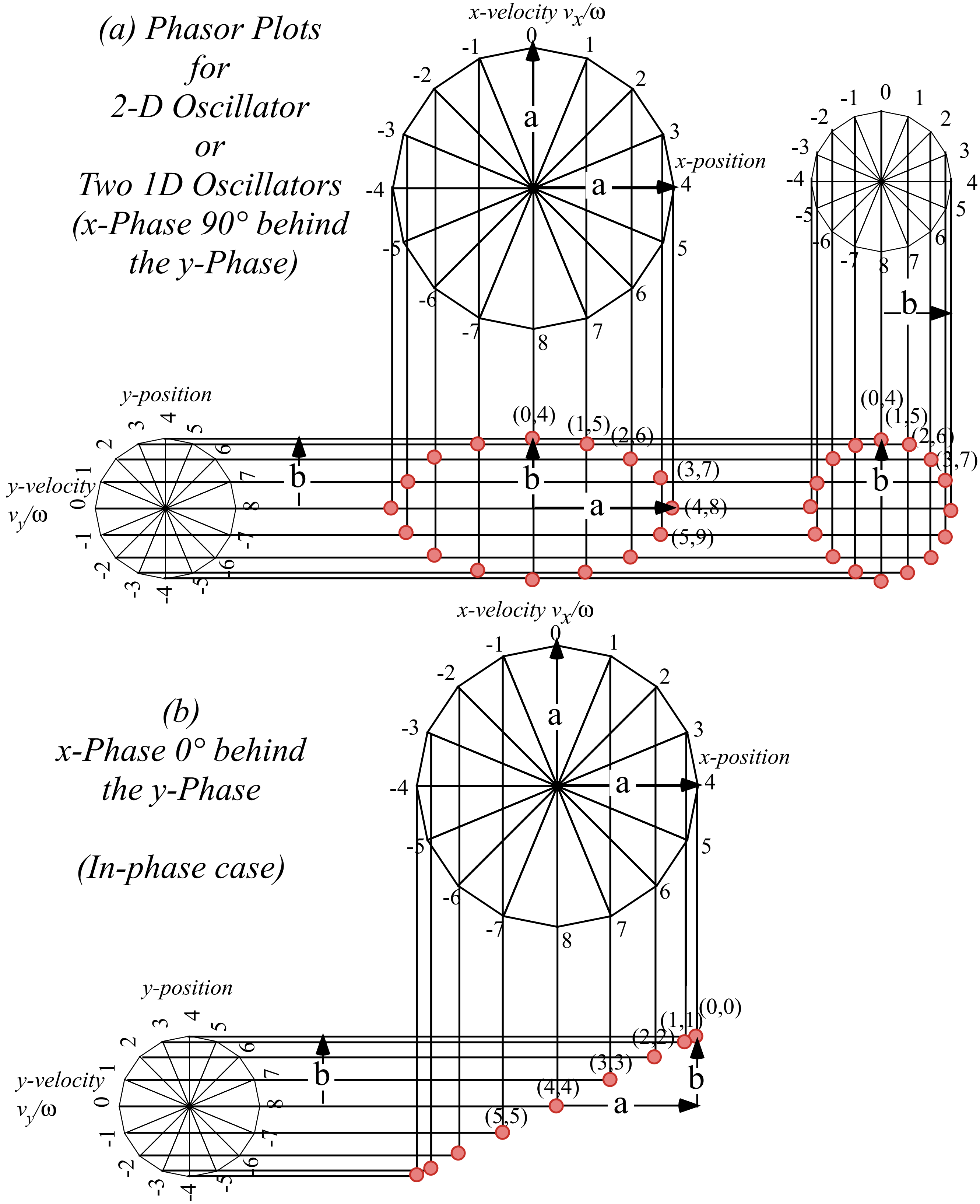
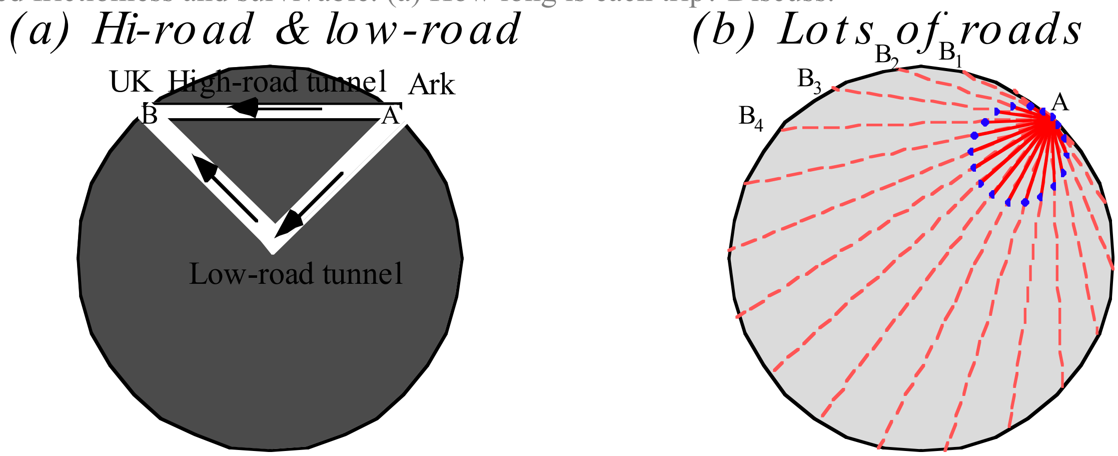


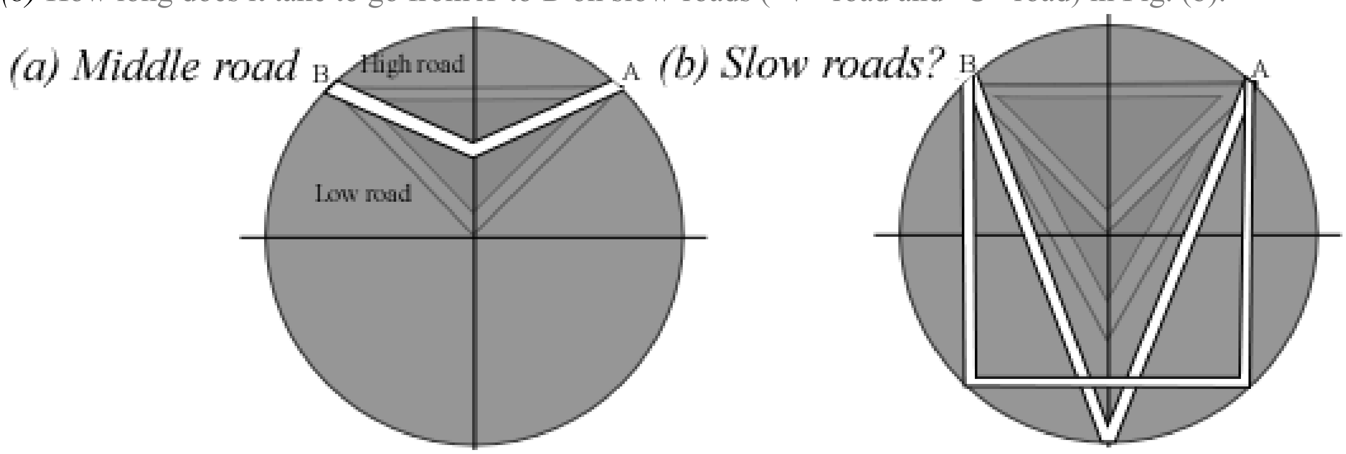
Fig. 8.11 Two 1-D oscillator phasor plots combine to give 2D-oscillator xy-trajectory.

Exercise 1.8.3. Tunnels to UK (5600 miles away as an earthworm crawls) are shown below. One high-road is a direct route. The other low-road turns around at the Earth center. Travel and turn-around are assumed frictionless and survivable. (a) How long is each trip? Discuss.



(b) A network of subways leaving Ark. at time t=0. What curve (like the dots) describe each moment?

Exercise 1.8.4. Consider competing tunnels between points *A*-to-*B* separated by $R\sqrt{2}\sim 5600$ miles (thru Earth) or $\Delta\phi=90°$ of longitude and 6 Time Zones. The preceding problem asked you to compare the high-road or direct-route to the low-road or via-Earth-center-and-back-route. Here we consider middle-road routes such as in Fig *(a)* below. *(a)* Find the fastest 2-straight-section middle road *A*-to-*B* by geometry or algebra. How much faster is it? (Give answer for local travel:$\Delta\phi=1°$, long distance:$\Delta\phi=90°$ and for general $\Delta\phi$.)
(b) How long does it take to go from *A*-to-*B* on slow-roads ("V"-road and "U"-road) in Fig. (b).



Exercise 1.8.5. Construct 24-point neutron-starlet orbits (One point for every hour assuming a 24-hour orbital period.) inside a uniform asteroid with x-component oscillation amplitude exactly *equal* to that of y and the x-component phase fixed relative to that of y as follows:
(a) x is *in phase* with y. *(b)* x is *behind* y by 1 hour. *(c)* 2 hours. *(d)* 3 hours. *(e)* 4 hours. *(f)* 5 hours. *(g)* 6 hours. *(h)* 7 hours.
Do the orbits change if we replace *behind* by *ahead* in *(a)* to *(h)*? Discuss or describe.