Lecture 17
Tue. 2.28.2012

Lagrangian and Hamiltonian dynamics: Living with duality in GCC cells and vectors Part II.

(Ch. 12 of Unit 1)

0. Review of Hamilton equations 1 and 2

1. Hamilton prefers Contravariant g^{mn} with Covariant momentum p_m
Deriving Hamilton's equations in GCC form
How to finesse centrifugal and Coriolis energy and other things like phase space.

2. Examples of Hamiltonian dynamics and phase plots
$I_{sotropic}$ $H_{armonic}$ $O_{scillator}$ in polar coordinates and "effective potential" (Simulation)
Coulomb orbits in polar coordinates and "effective potential" (Simulation)
1D Pendulum and phase plot (Simulation) Lecture 17 ended here
Phase control (Simulation)

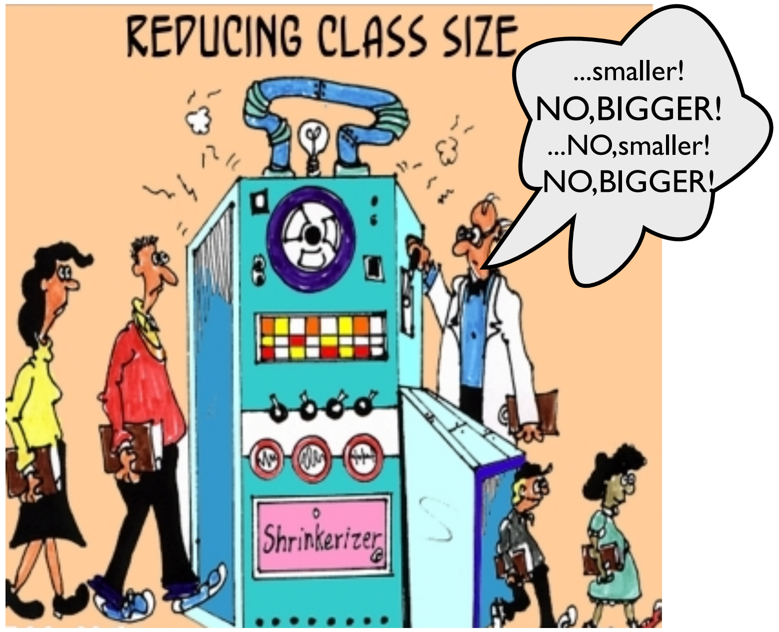
3. Exploring phase space and Lagrangian mechanics more deeply
A weird "derivation" of Lagrange's equations
Poincare identity and Action

Deriving Hamilton's equations

Consider total time derivative of Lagrangian L=T-U
that is explicit function of coordinates and velocity $\dot{q}$...

$$\dot{L}(q,\dot{q},t)=\frac{dL}{dt}=\frac{\partial L}{\partial q^m}\frac{dq^m}{dt}+\frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt}$$

...of coordinates and velocity and ***time****, too. (Imagine Mad Scientist turning U-dial.)*



$$\dot{L}(q,\dot{q},t)=\frac{dL}{dt}=\frac{\partial L}{\partial q^m}\frac{dq^m}{dt}+\frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt}+\frac{\partial L}{\partial t}$$

Recall Lagrange equations: $\boxed{\dot{p}_m=\frac{\partial L}{\partial q^m}}$ $\boxed{p_m=\frac{\partial L}{\partial \dot{q}^m}}$

$$\dot{L}(q,\dot{q},t)=\frac{dL}{dt}=\dot{p}_m\frac{dq^m}{dt}+p_m\frac{d\dot{q}^m}{dt}+\frac{\partial L}{\partial t}$$

Use product rule:
$$\dot{u}\frac{dv}{dt}+u\frac{d\dot{v}}{dt}=\frac{d}{dt}(u\dot{v})$$

$$-\frac{\partial L}{\partial t}=\frac{d}{dt}\left(p_m\dot{q}^m\right)-\frac{dL}{dt}$$

Define the Hamiltonian function $H(\mathbf{p})=\mathbf{p}\bullet\mathbf{v}-L(\mathbf{v})$ *(That's the old Legendre transform)*

$$\frac{d}{dt}\left(p_m\dot{q}^m-L\right)=\boxed{-\frac{\partial L}{\partial t}=\frac{dH}{dt}}\quad where: H=p_m\dot{q}^m-L$$

(Recall: $\frac{\partial L}{\partial p_m}\equiv 0$
and: $\frac{\partial H}{\partial \dot{q}^m}\equiv 0$ *)*

a most ***peculiar*** *relation*
involving partial vs total

Hamilton's 1st GCC equation

$$\frac{\partial H}{\partial p_m}=\dot{q}^m$$

Hamilton's 2nd GCC equation

$$\frac{\partial H}{\partial q^m}=-\dot{p}_m$$

1. Hamilton prefers <u>Contra</u>*variant* g^{mn} *with* <u>Co</u>*variant* ***momentum*** p_m

Hamilton prefers Contravariant g^{mn} with Covariant momentum p_m

Using Legendre transform of Lagrangian $L=T-U$ with covariant metric definitions of L and p_m

We already have: $H = p_m \dot{q}^m - L$ *and:* $L(\dot{q}) = \frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n - U$ *and:* $p_m = \frac{\partial L}{\partial \dot{q}^m} = M g_{mn} \dot{q}^n$

Now we combine all these:

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$$= M g_{mn} \dot{q}^m \dot{q}^n - \frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n + U$$

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This gives an "illegal dependence" for the Hamiltonian (It musn't be "explicit" in velocity $\dot{q}^m$.)

$$H = \frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n + U = T + U$$

(Numerically correct ONLY!)

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An inverse metric relation $\dot{q}^m = \frac{1}{M} g^{mn} p_n$ *gives correct form that depends on momentum p_m.*

$$\boxed{H = \frac{1}{2M} g^{mn} p_m p_n + U = T + U \equiv E}$$

(Formally **and** Numerically correct)

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(Formally **and** Numerically correct)

Polar coordinate Lagrangian was given as:

$$L(\dot{r}, \dot{\phi}, r, \phi) = \tfrac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2) - U(r,\phi) = \tfrac{1}{2} M (\dot{r}^2 + r^2 \cdot \dot{\phi}^2) - U(r,\phi)$$

Polar coordinate Hamiltonian is given here:

$$H(p_r, p_\phi, r, \phi) = \frac{1}{2M} (g^{rr} p_r^2 + g^{\phi\phi} p_\phi^2) + U(r,\phi) = \frac{1}{2M} (p_r^2 + \frac{1}{r^2} \cdot p_\phi^2) + U(r,\phi)$$

2. Examples of Hamiltonian dynamics and phase plots

Isotropic Harmonic Oscillator in polar coordinates and "effective potential" (Simulation)
Coulomb orbits in polar coordinates and "effective potential"
1D Pendulum and phase plot
Phase control

Effective potential analysis (Reducing 2D-problem to 1D-problem)

Polar coordinate Hamiltonian can take advantage of H-conservation and p_m-conservation

Consider polar coordinate Hamiltonian for $I_{sotropic}$ $H_{armonic}$ $O_{scillator}$ potential $U(r) = kr^2/2$:

$$H(p_r, p_\phi, r, \phi) = \frac{1}{2M}(g^{rr} p_r{}^2 + g^{\phi\phi} p_\phi{}^2) + k \cdot r^2/2 = \frac{1}{2M}(p_r{}^2 + \frac{1}{r^2} \cdot p_\phi{}^2) + \frac{k \cdot r^2}{2} = E = const.$$

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H is not explicit function of ϕ, and so Hamilton's 2nd says: $\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0$

Thus momentum p_ϕ is conserved constant: $p_\phi = \ell = const.$

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$$p_r{}^2 = 2ME - \frac{\ell^2}{r^2} - Mk \cdot r^2$$

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Radial KE is $\frac{M\dot{r}^2}{2} = E - \frac{\ell^2}{2Mr^2} - \frac{k}{2}\cdot r^2$

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$$\frac{dr}{dt} = \sqrt{\frac{2E}{M} - \frac{\ell^2}{M^2 r^2} - \frac{k}{M} \cdot r^2}$$

Solution: $t = \int_{r<}^{r>} \frac{dr}{\sqrt{\frac{2E}{M} - \frac{\ell^2}{M^2 r^2} - \frac{k}{M} \cdot r^2}}$

Effective potential analysis (Reducing 2D-problem to 1D-problem)

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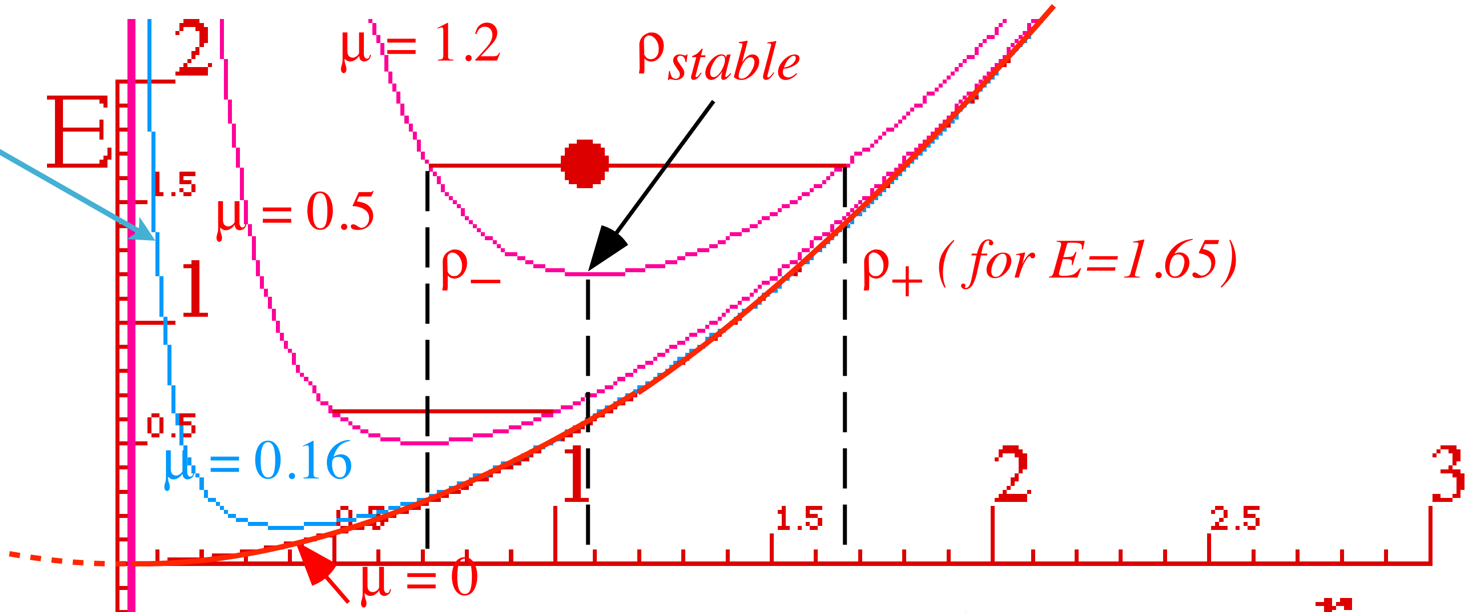
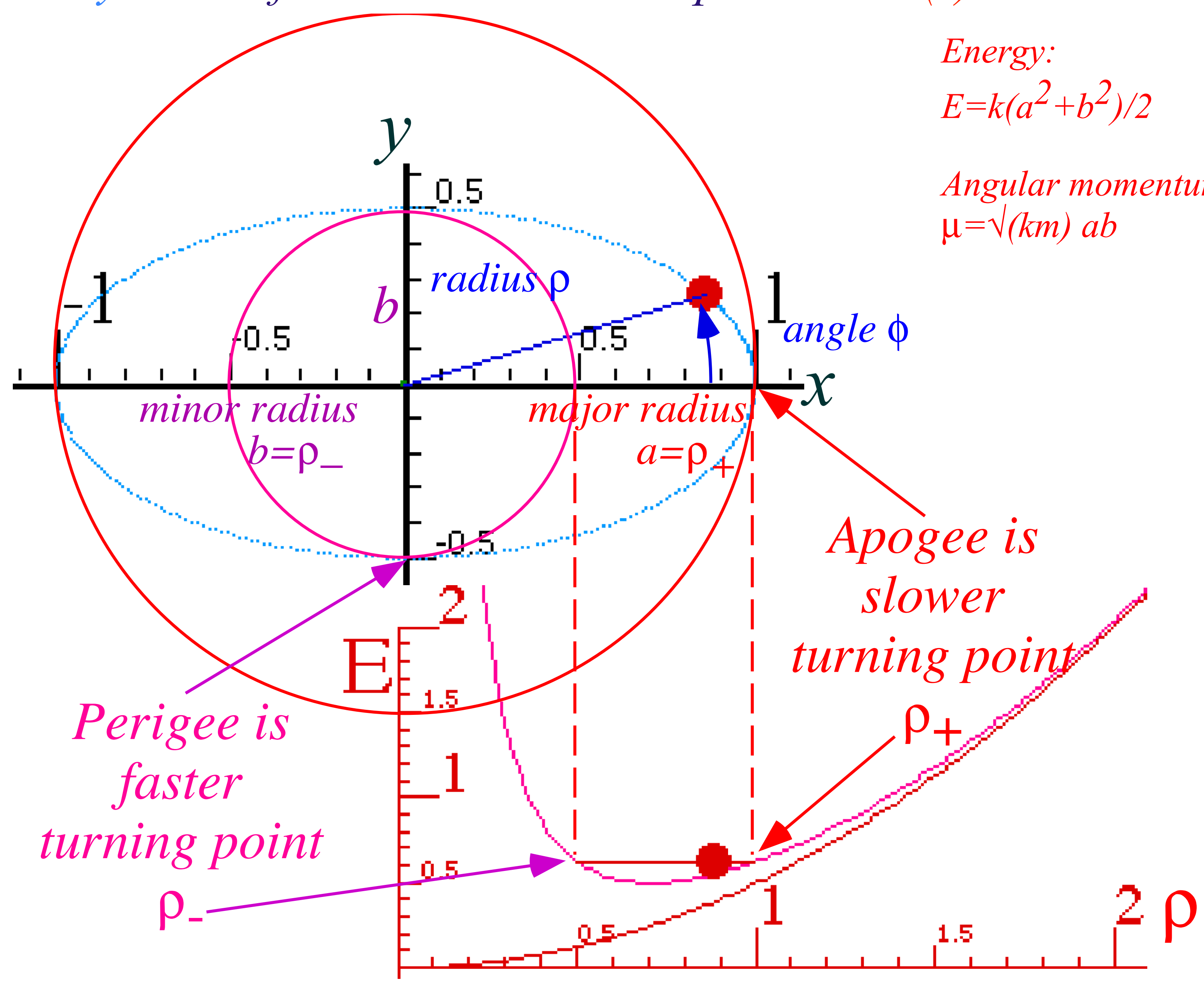
$$t = \int_{r<}^{r>} \frac{dr}{\sqrt{\frac{2E}{M} - \frac{\ell^2}{M^2 r^2} - \frac{2U(r)}{M}}}$$

$$E = \frac{p_r^{\ 2}}{2M} + \overbrace{\underbrace{\frac{\ell^2}{2Mr^2}}_{\text{“centifugal-barrier” PE}} + \overbrace{\frac{k}{2}r^2}^{\text{“real” PE}}}^{\text{“effective” PE}}$$



Hamiltonian dynamics for Isotropic Harmonic Oscillator potential $U(r) = kr^2/2$



Energy:
$E=k(a^2+b^2)/2$

Angular momentum:
$\mu=\sqrt{(km)}\,ab$

2. Examples of Hamiltonian dynamics and phase plots

*I*sotropic *H*armonic *O*scillator in polar coordinates and "effective potential"

Coulomb orbits in polar coordinates and "effective potential" (Simulation)

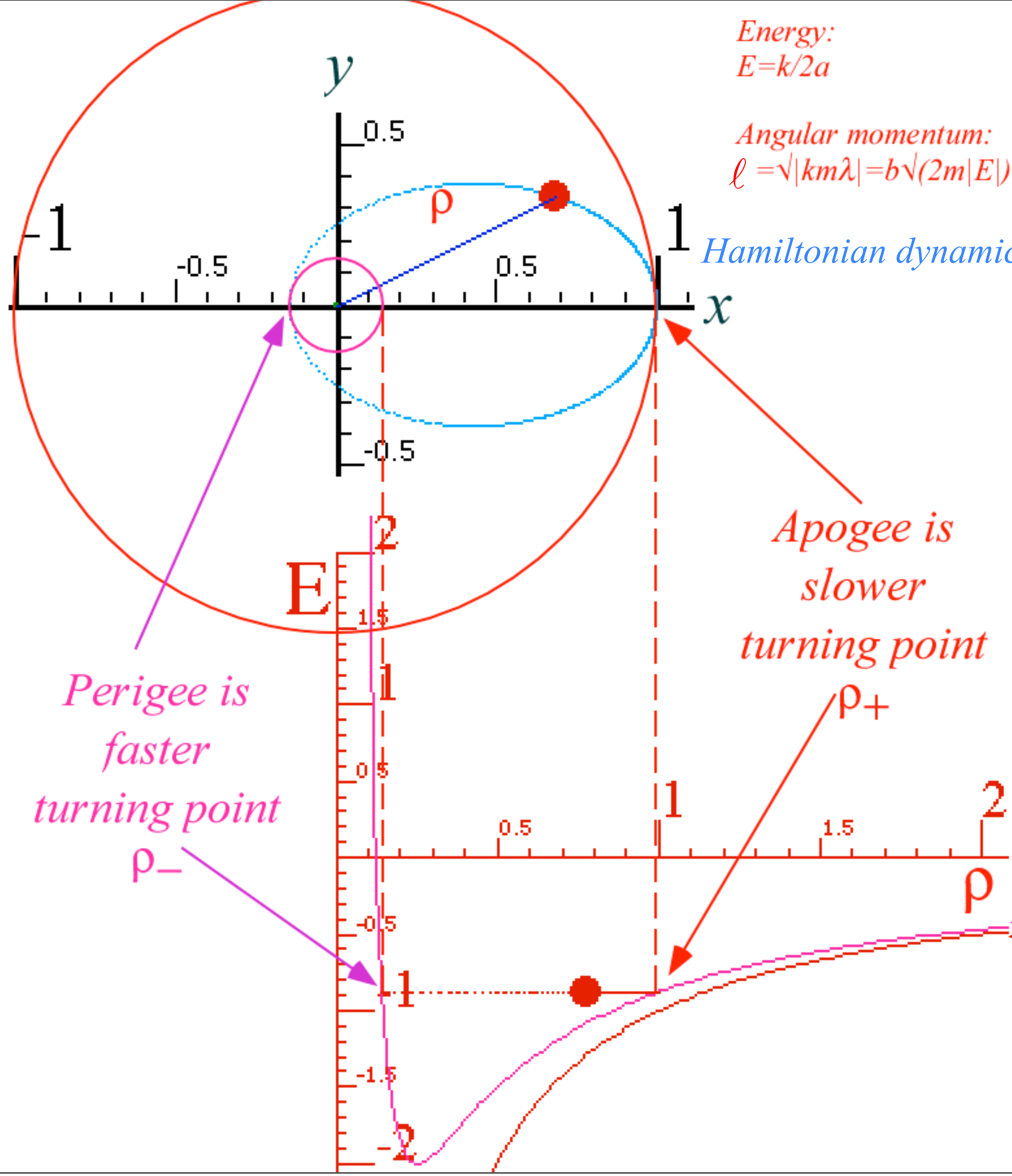
1D Pendulum and phase plot (Simulation)

Phase control (Simulation)

Energy:
E=k/2a

Angular momentum:
ℓ =√|kmλ|=b√(2m|E|)

Hamiltonian dynamics for Coulomb potential U(r) =-k/r



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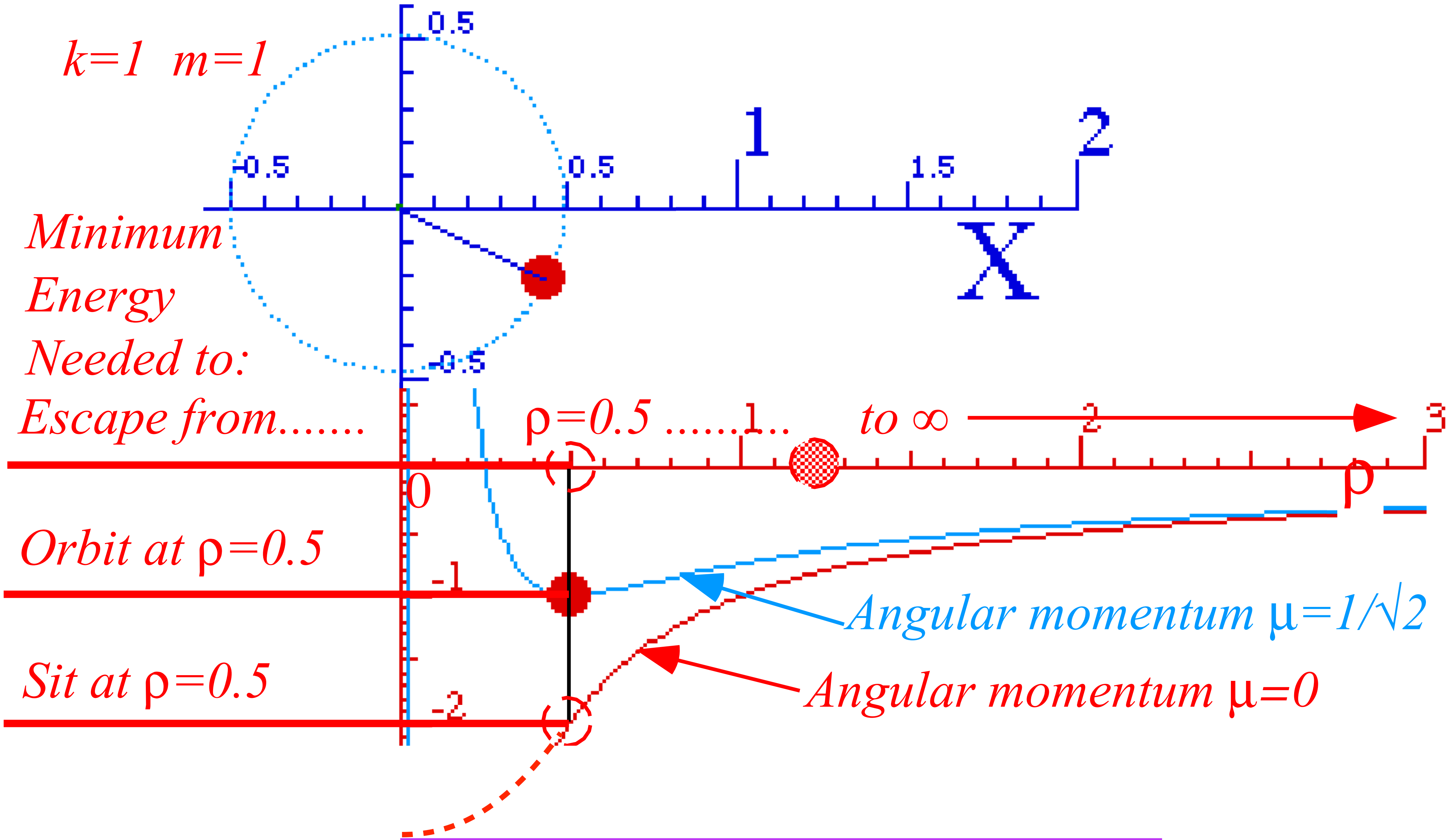


Lecture 17 ends here
Tue. 2.28.2012

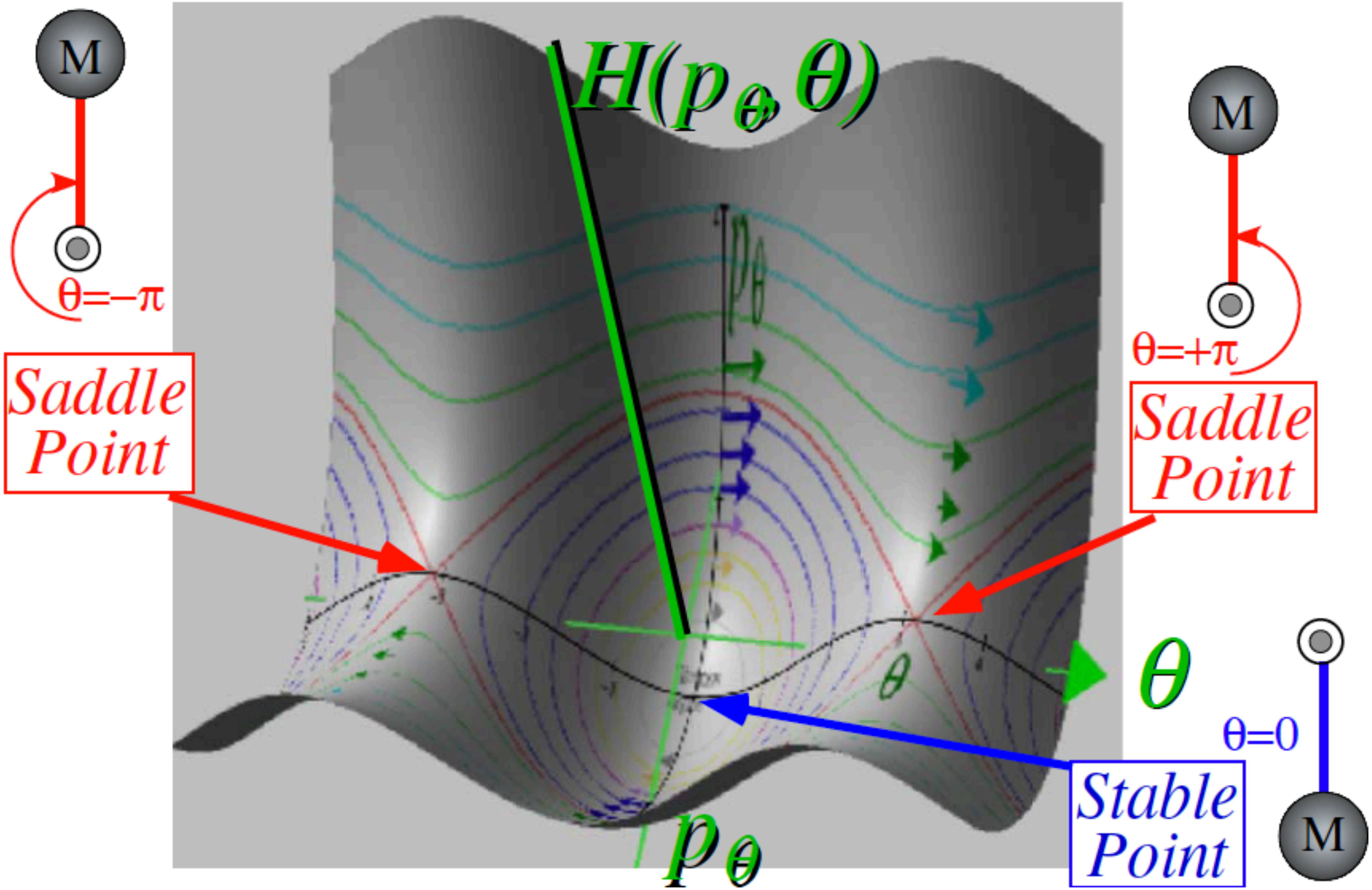
2. Examples of Hamiltonian dynamics and phase plots

Isotropic Harmonic Oscillator in polar coordinates and "effective potential"

Coulomb orbits in polar coordinates and "effective potential"

1D Pendulum and phase plot (Simulation)

Phase control (Simulation)



Example of plot of Hamilton for 1D-solid pendulum in its Phase Space (θ,p_θ)

$$H(p_\theta,\theta)=E=\frac{1}{2I}p_\theta{}^2-MgR\cos\theta\ ,\quad \text{or:}\quad p_\theta=\sqrt{2I\left(E+MgR\cos\theta\right)}$$

$$\begin{pmatrix}\dot{q}\\ \dot{p}\end{pmatrix}=\begin{pmatrix}\partial_p H\\ -\partial_q H\end{pmatrix}=\mathbf{e_H}\times\left(-\nabla H\right)=\text{(H-axis)}\times\text{(fall line), where:}\begin{cases}\text{(H-axis)}=\mathbf{e_H}=\mathbf{e_q}\times\mathbf{e_p}\\ \text{(fall line)}=-\nabla H\end{cases}$$