Lecture 9

Revised 12.21.2012 9.18.2012

Geometry of Dual Quadratic Forms: Lagrange vs Hamilton

(Ch. 11 and Ch. 12 of Unit 1)

Introduction to dual matrix operator geometry

- *Review of dual IHO elliptic orbits (Lecture 7-8)*
  - *Construction by Phasor-pair projection*
  - *Construction by Kepler anomaly projection*
- *Operator geometric sequences and eigenvectors*
- *Rescaled description of matrix operator geometry*
  - *Vector calculus of tensor operation*

Introduction to Lagrangian-Hamiltonian duality

- *Review of partial differential relations*
  - *Chain rule and order symmetry*
- *Duality relations of Lagrangian and Hamiltonian ellipse*
- *Introducing the 1st (partial* $\frac{\partial ?}{\partial ?}$ *) differential equations of mechanics*

Introduction to dual matrix operator geometry

Review of dual IHO elliptic orbits (Lecture 7-8)

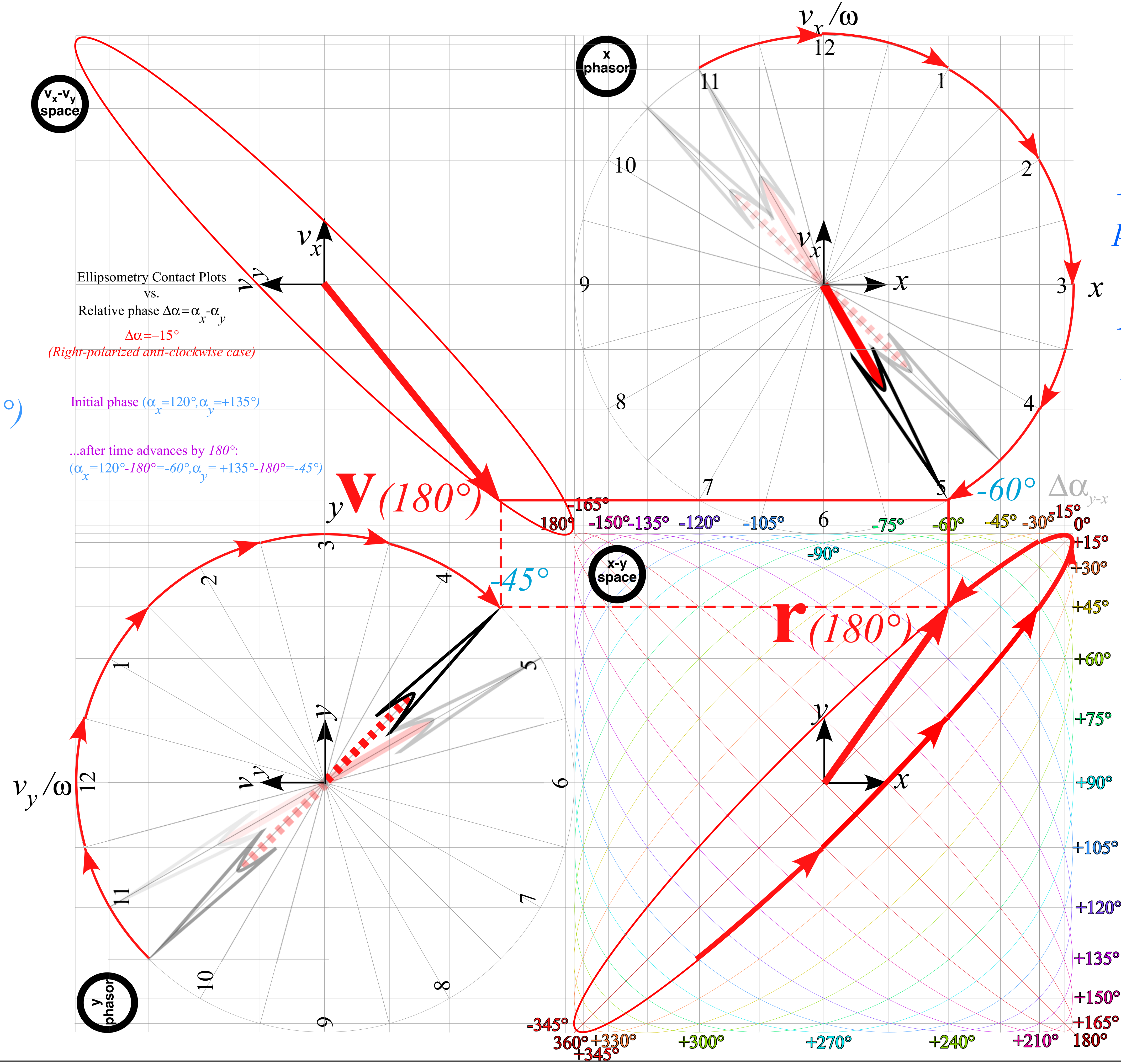
→ *Construction by Phasor-pair projection*

Construction by Kepler anomaly projection

Operator geometric sequences and eigenvectors

Rescaled description of matrix operator geometry

Vector calculus of tensor operation



See

Lecture 7

pages 37 to 49

and

Lecture 8

pages 7 to 15

Introduction to dual matrix operator geometry

Review of dual IHO elliptic orbits (Lecture 7-8)

Construction by Phasor-pair projection

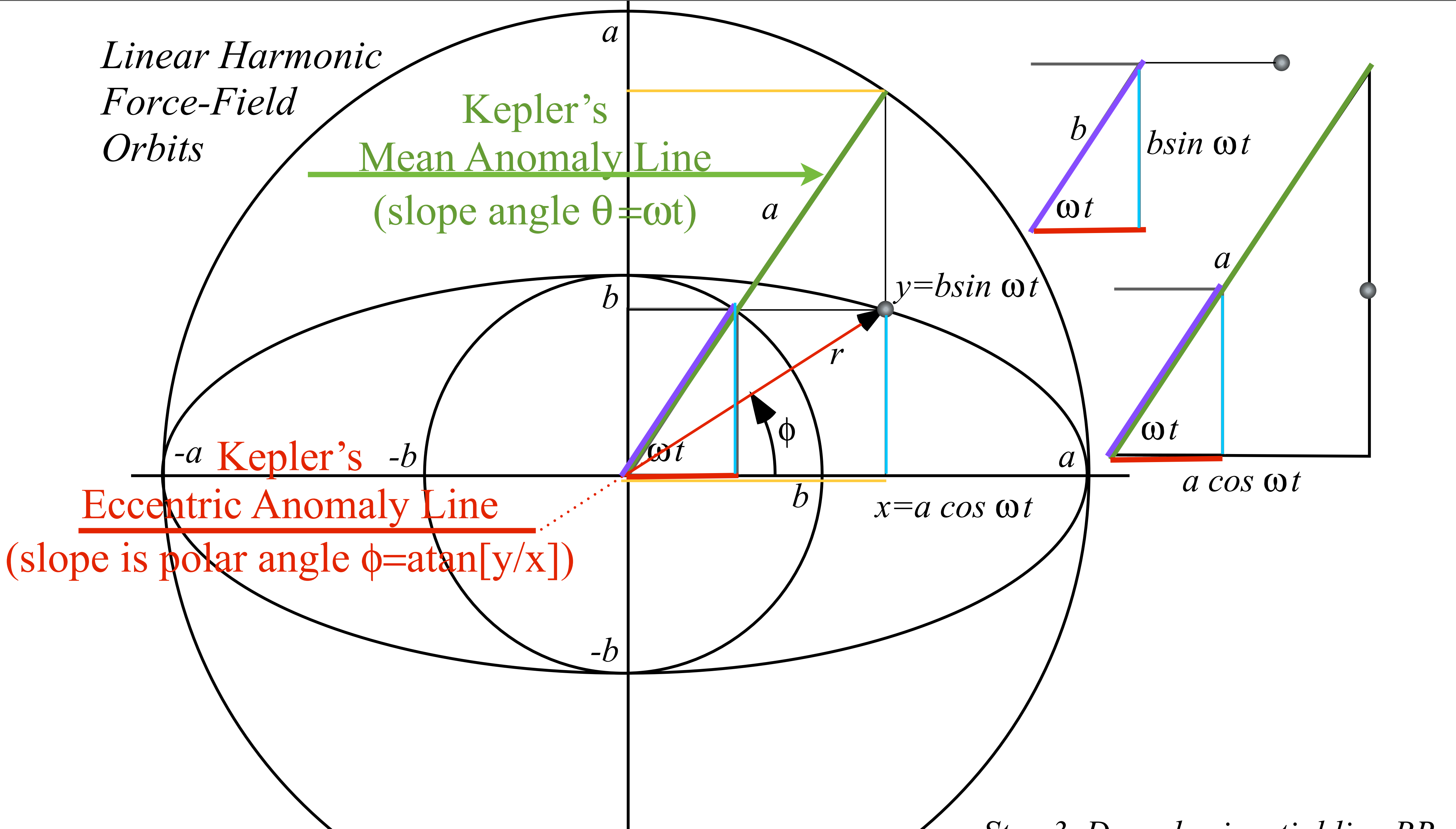
→ *Construction by Kepler anomaly projection*

Operator geometric sequences and eigenvectors

Rescaled description of matrix operator geometry

Vector calculus of tensor operation

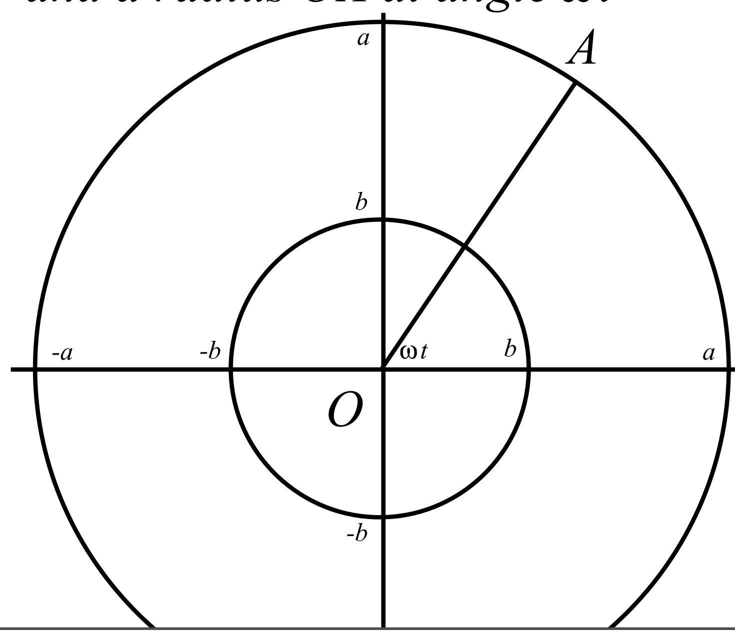
Linear Harmonic Force-Field Orbits



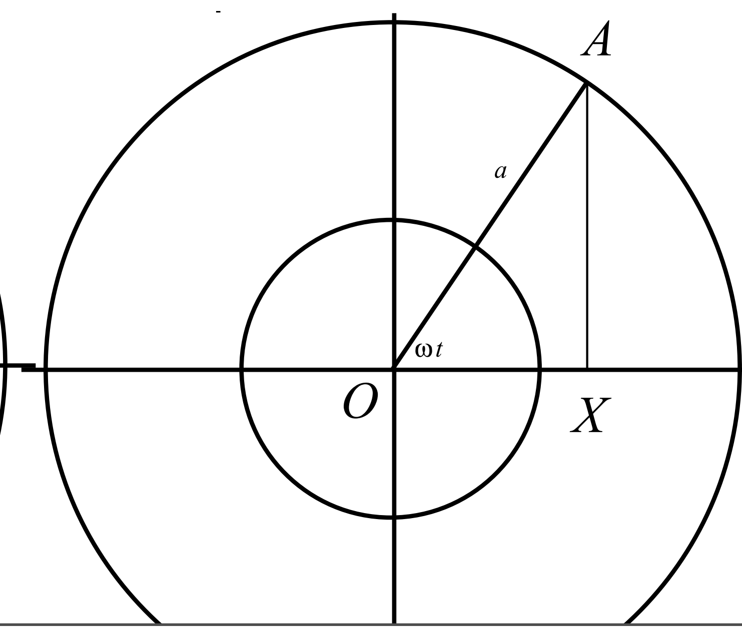
Unit 1
Fig. 11.1
(top 2/3's)

See
Lecture 8
pages 17 to 25

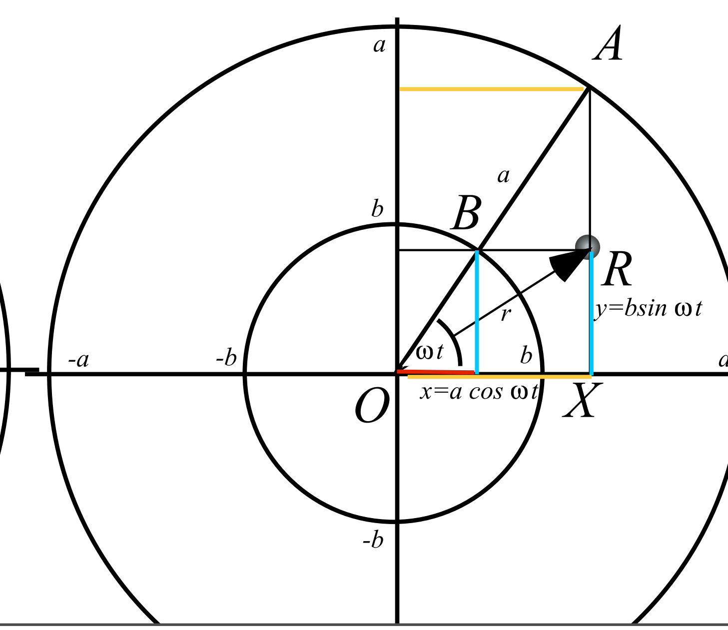
Step 1. Draw concentric circles of radius a and b and a radius OA at angle ωt



Step 2. Draw vertical line AX from a-circle at ωt to x-axis



Step 3. Draw horizontial line BR from b-circle at ωt to line AX. Intersection is orbit point R.



Quadratic forms and tangent contact geometry of their ellipses

A matrix Q that generates an ellipse by **r**•*Q*•**r**=*1 is called* positive-definite (*if* **r**•*Q*•**r** *always >0*)

$$\mathbf{r}\bullet\mathbf{Q}\bullet\mathbf{r} \qquad = 1$$

$$\begin{pmatrix} x & y \end{pmatrix}\bullet\begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix}\bullet\begin{pmatrix} x \\ y \end{pmatrix} = 1 = \begin{pmatrix} x & y \end{pmatrix}\bullet\begin{pmatrix} \frac{x}{a^2} \\ \frac{y}{b^2} \end{pmatrix} = \frac{x^2}{a^2}+\frac{y^2}{b^2}$$

A inverse matrix Q^{-1} generates an ellipse by **p**•Q^{-1}•**p**=*1 called* inverse *or* dual *ellipse:*

$$\mathbf{p}\bullet\mathbf{Q}^{-1}\bullet\mathbf{p} \qquad = 1$$

$$\begin{pmatrix} p_x & p_y \end{pmatrix}\bullet\begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}\bullet\begin{pmatrix} p_x \\ p_y \end{pmatrix} = 1 = \begin{pmatrix} p_x & p_y \end{pmatrix}\bullet\begin{pmatrix} a^2 p_x \\ b^2 p_y \end{pmatrix} = a^2 p_x^2 + b^2 p_y^2$$

Quadratic forms and tangent contact geometry of their ellipses

A matrix Q that generates an ellipse by **r**•*Q*•**r**=*1 is called* positive-definite (*if* **r**•*Q*•**r** *always >0*)

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A inverse matrix Q^{-1} *generates an ellipse by* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$ *called* inverse *or* dual *ellipse:*

$$\overbrace{\begin{pmatrix} p_x & p_y \end{pmatrix} \bullet \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} \bullet \begin{pmatrix} p_x \\ p_y \end{pmatrix}}^{\mathbf{p}\bullet\mathbf{Q}^{-1}\bullet\mathbf{p}\ =1} = 1 = \overbrace{\begin{pmatrix} p_x & p_y \end{pmatrix}}^{\mathbf{p}} \bullet \overbrace{\begin{pmatrix} a^2 p_x \\ b^2 p_y \end{pmatrix}}^{\mathbf{Q}^{-1}\bullet\mathbf{p}} = a^2 p_x^2 + b^2 p_y^2$$

Quadratic forms and tangent contact geometry of their ellipses

A matrix Q that generates an ellipse by **r**•*Q*•**r**=*1 is called* positive-definite (*if* **r**•*Q*•**r** *always >0*)

$$\overbrace{\begin{pmatrix} x & y \end{pmatrix} \bullet \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix} \bullet \begin{pmatrix} x \\ y \end{pmatrix}}^{\mathbf{r}\bullet\mathbf{Q}\bullet\mathbf{r}=1} = 1 = \overbrace{\begin{pmatrix} x & y \end{pmatrix}}^{\mathbf{r}} \bullet \overbrace{\begin{pmatrix} \frac{x}{a^2} \\ \frac{y}{b^2} \end{pmatrix}}^{\mathbf{Q}\bullet\mathbf{r}=\mathbf{p}} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

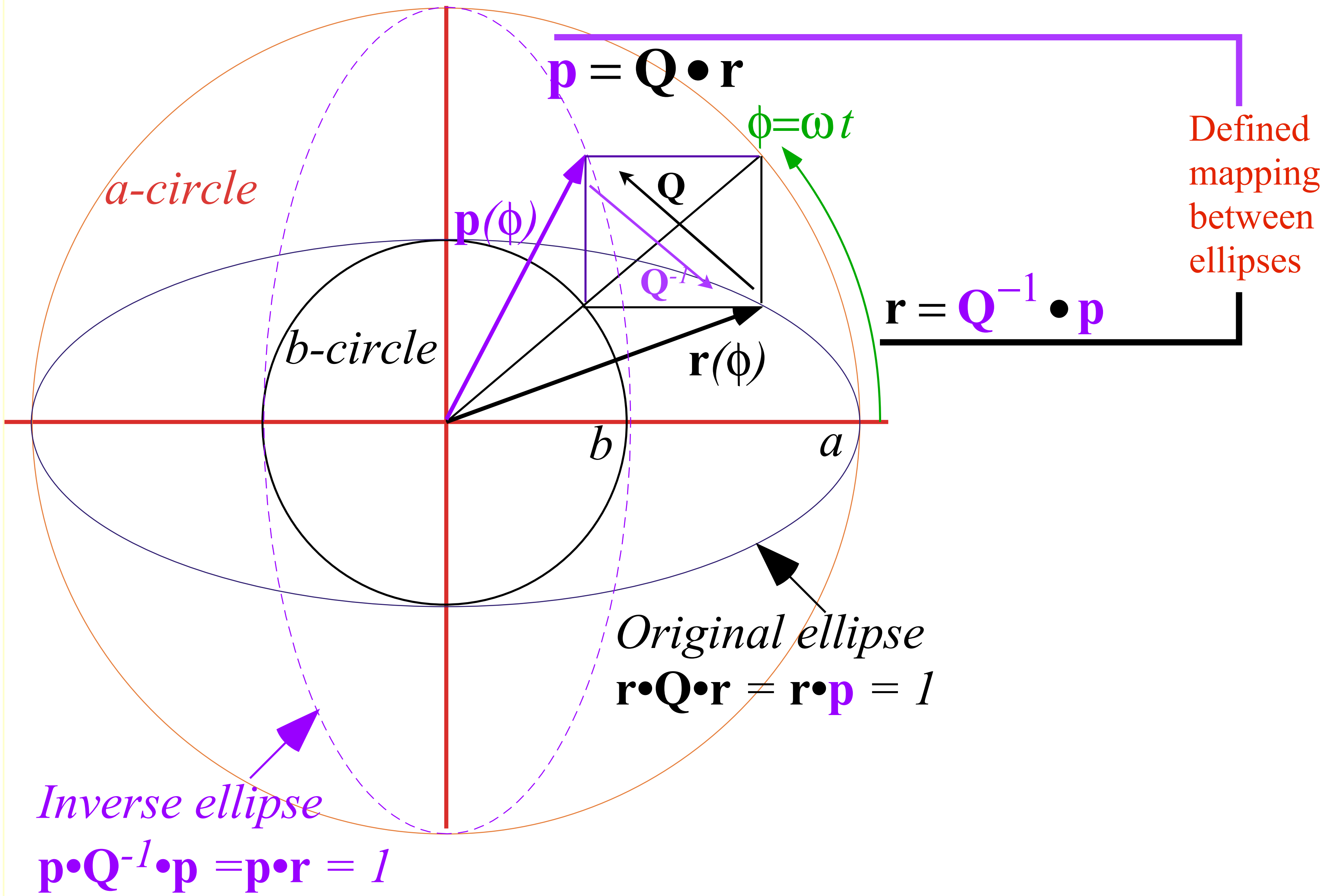
Defined mapping between ellipses

A inverse matrix Q^{-1} generates an ellipse by **p**•Q^{-1}•**p**=*1 called* inverse *or* dual *ellipse:*

$$\overbrace{\begin{pmatrix} p_x & p_y \end{pmatrix} \bullet \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} \bullet \begin{pmatrix} p_x \\ p_y \end{pmatrix}}^{\mathbf{p}\bullet\mathbf{Q}^{-1}\bullet\mathbf{p}=1} = 1 = \overbrace{\begin{pmatrix} p_x & p_y \end{pmatrix}}^{\mathbf{p}} \bullet \overbrace{\begin{pmatrix} a^2 p_x \\ b^2 p_y \end{pmatrix}}^{\mathbf{Q}^{-1}\bullet\mathbf{p}=\mathbf{r}} = a^2 p_x^2 + b^2 p_y^2$$

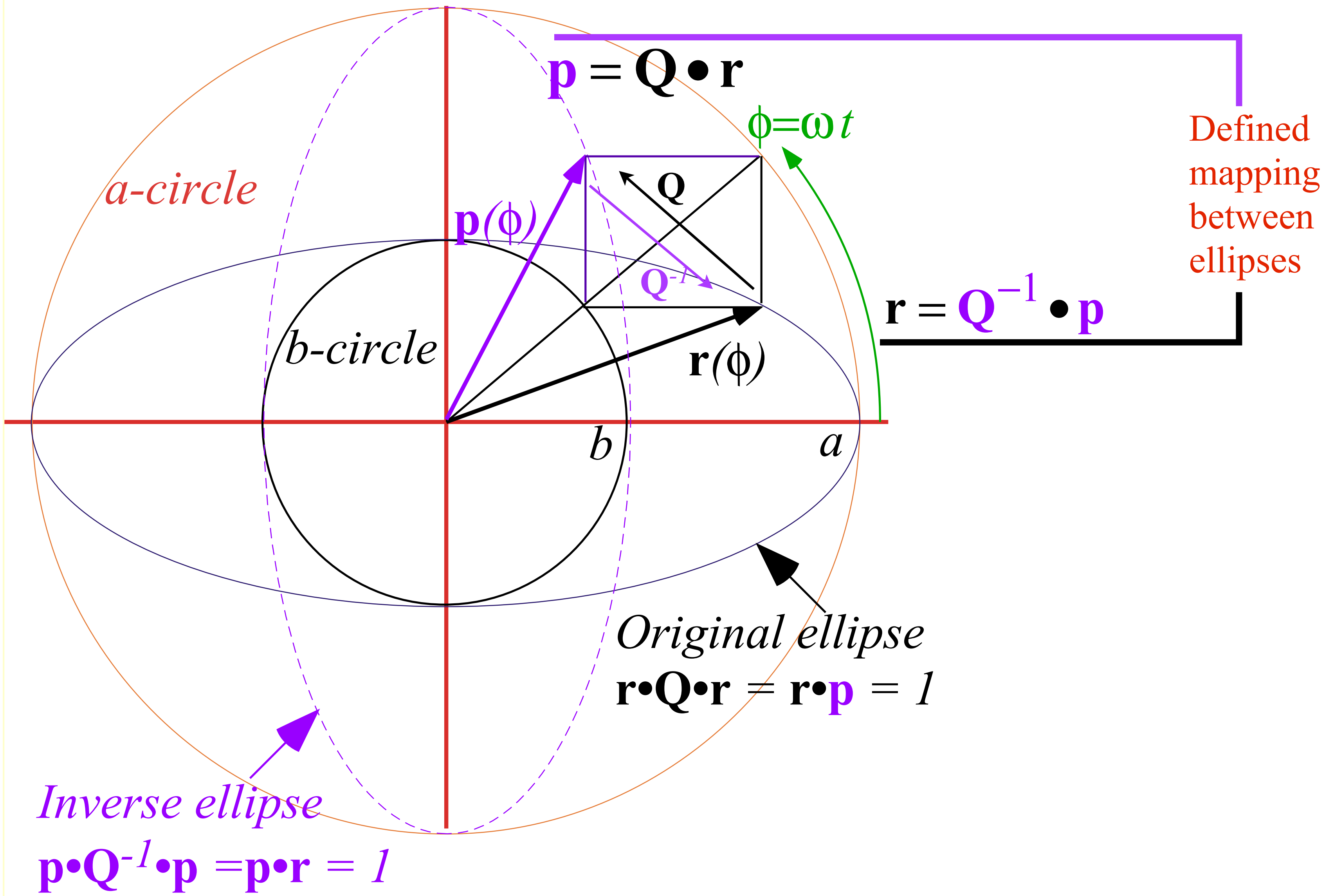
(a) Quadratic form ellipse and Inverse quadratic form ellipse

based on
Unit 1
Fig. 11.6



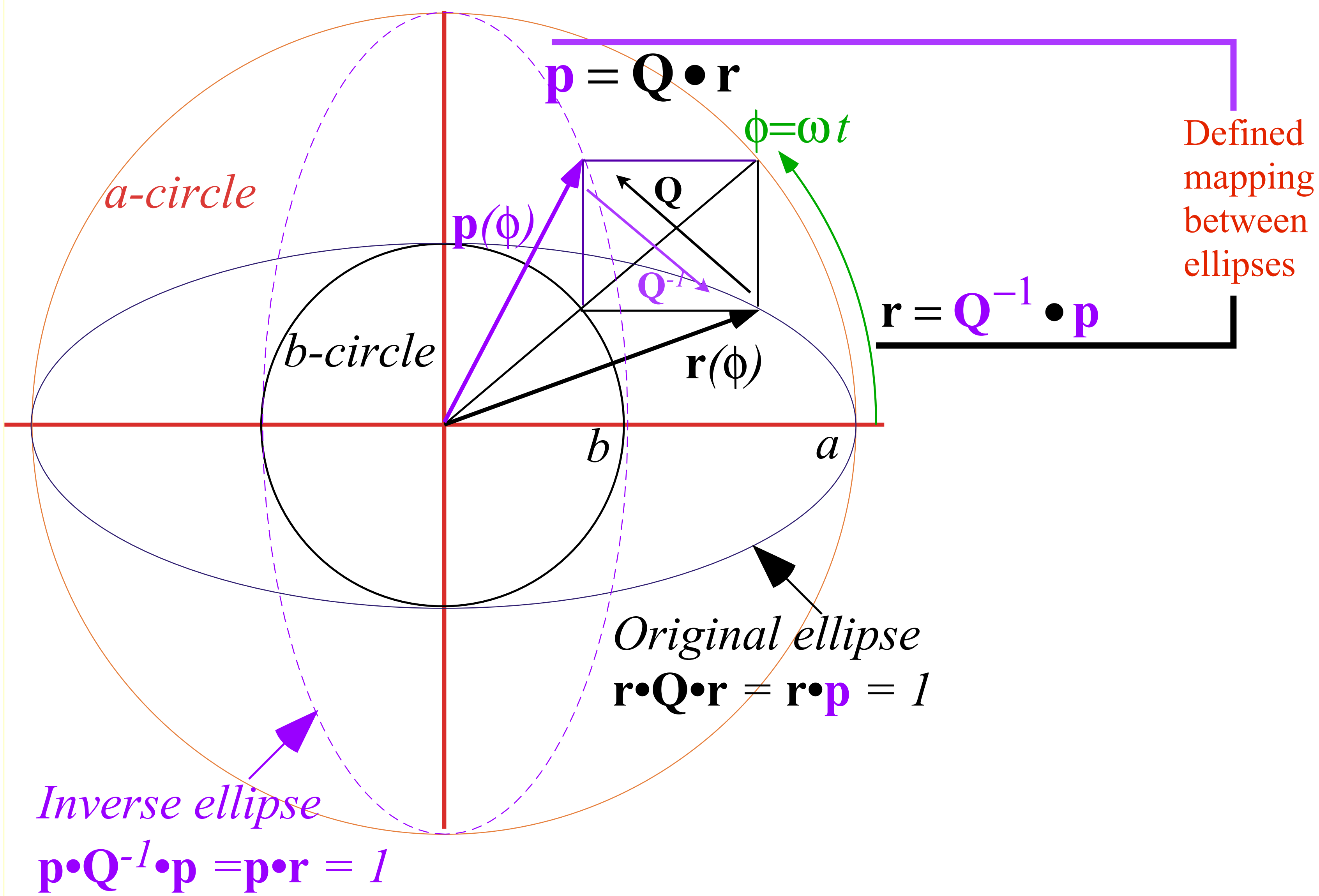
(a) Quadratic form ellipse and *Inverse quadratic form ellipse*

based on
Unit 1
Fig. 11.6



Quadratic form **r•Q•r** $=1$ *has mutual duality relations with inverse form* **p•Q**$^{-1}$**•p** $=1=$ **p•r**

(a) Quadratic form ellipse and *Inverse quadratic form ellipse*

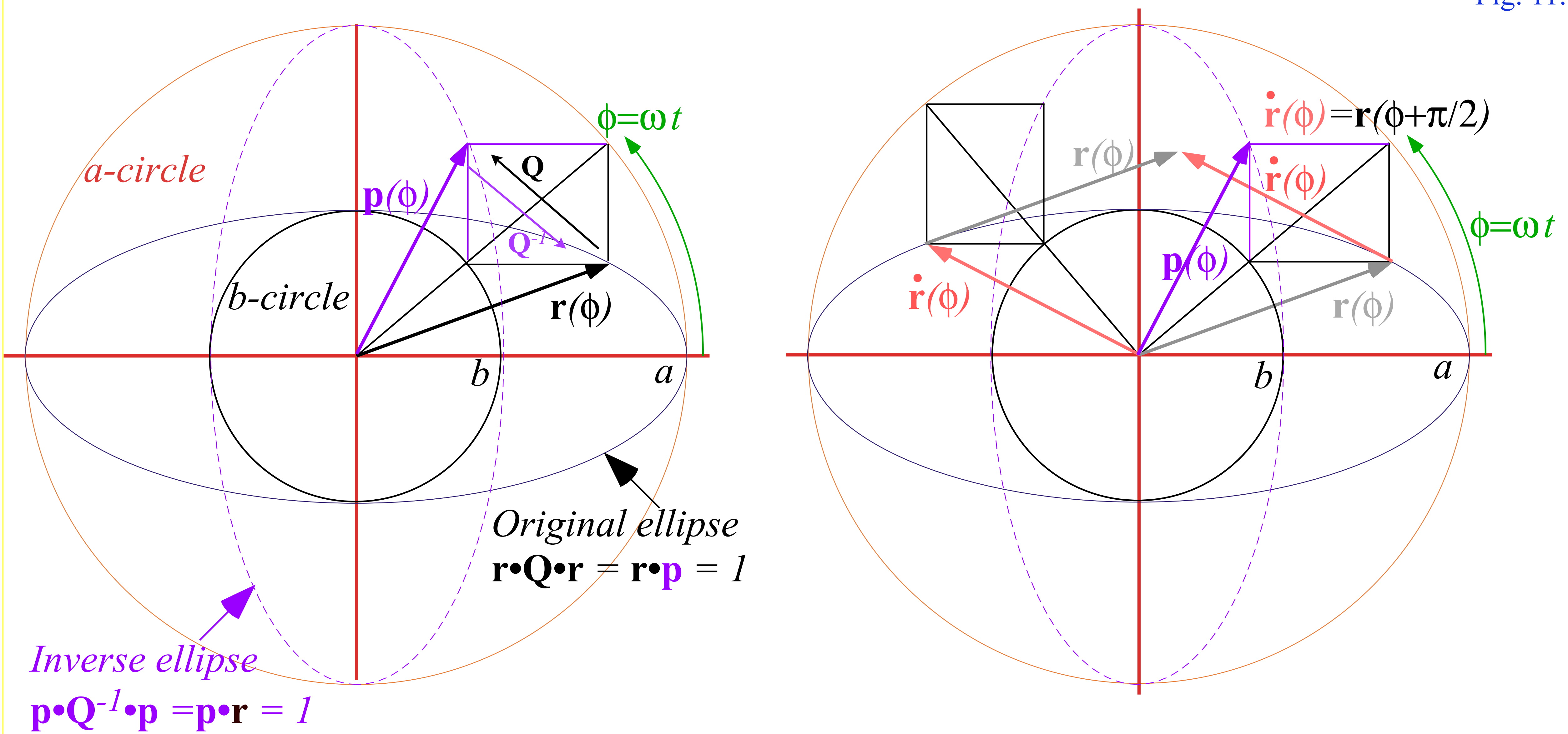


Quadratic form $\mathbf{r}\bullet\mathbf{Q}\bullet\mathbf{r} = 1$ *has mutual duality relations with inverse form* $\mathbf{p}\bullet\mathbf{Q}^{-1}\bullet\mathbf{p} = 1 = \mathbf{p}\bullet\mathbf{r}$

$$\mathbf{p} = \mathbf{Q}\bullet\mathbf{r} = \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix} \bullet \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/a^2 \\ y/b^2 \end{pmatrix} = \begin{pmatrix} (1/a)\cos\phi \\ (1/b)\sin\phi \end{pmatrix} \quad \text{where: } \begin{array}{l} x = r_x = a\cos\phi = a\cos\omega t \\ y = r_y = b\sin\phi = b\sin\omega t \end{array} \quad \text{so: } \boxed{\mathbf{p}\bullet\mathbf{r} = 1}$$

(a) Quadratic form ellipse and Inverse quadratic form ellipse *(b) Ellipse tangents*

based on Unit 1 Fig. 11.6



Quadratic form $\mathbf{r}\bullet\mathbf{Q}\bullet\mathbf{r} = 1$ *has mutual duality relations with inverse form* $\mathbf{p}\bullet\mathbf{Q}^{-1}\bullet\mathbf{p} = 1 = \mathbf{p}\bullet\mathbf{r}$

$$\mathbf{p} = \mathbf{Q}\bullet\mathbf{r} = \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix} \bullet \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/a^2 \\ y/b^2 \end{pmatrix} = \begin{pmatrix} (1/a)\cos\phi \\ (1/b)\sin\phi \end{pmatrix}$$

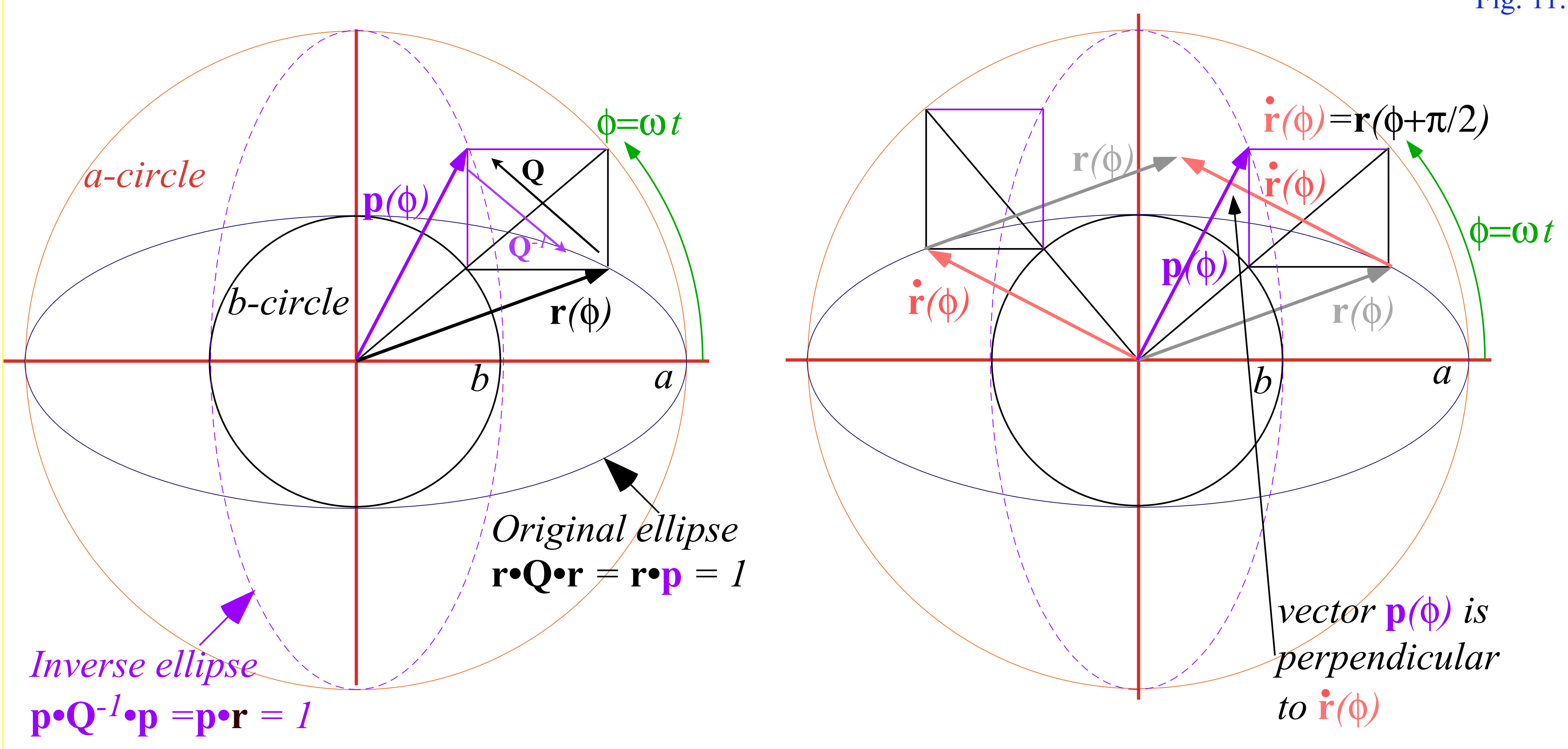
where: $x = r_x = a\cos\phi = a\cos\omega t$, $y = r_y = b\sin\phi = b\sin\omega t$

so: $\boxed{\mathbf{p}\bullet\mathbf{r} = 1}$

(a) Quadratic form ellipse and *Inverse quadratic form ellipse*

(b) Ellipse tangents

based on Unit 1 Fig. 11.6



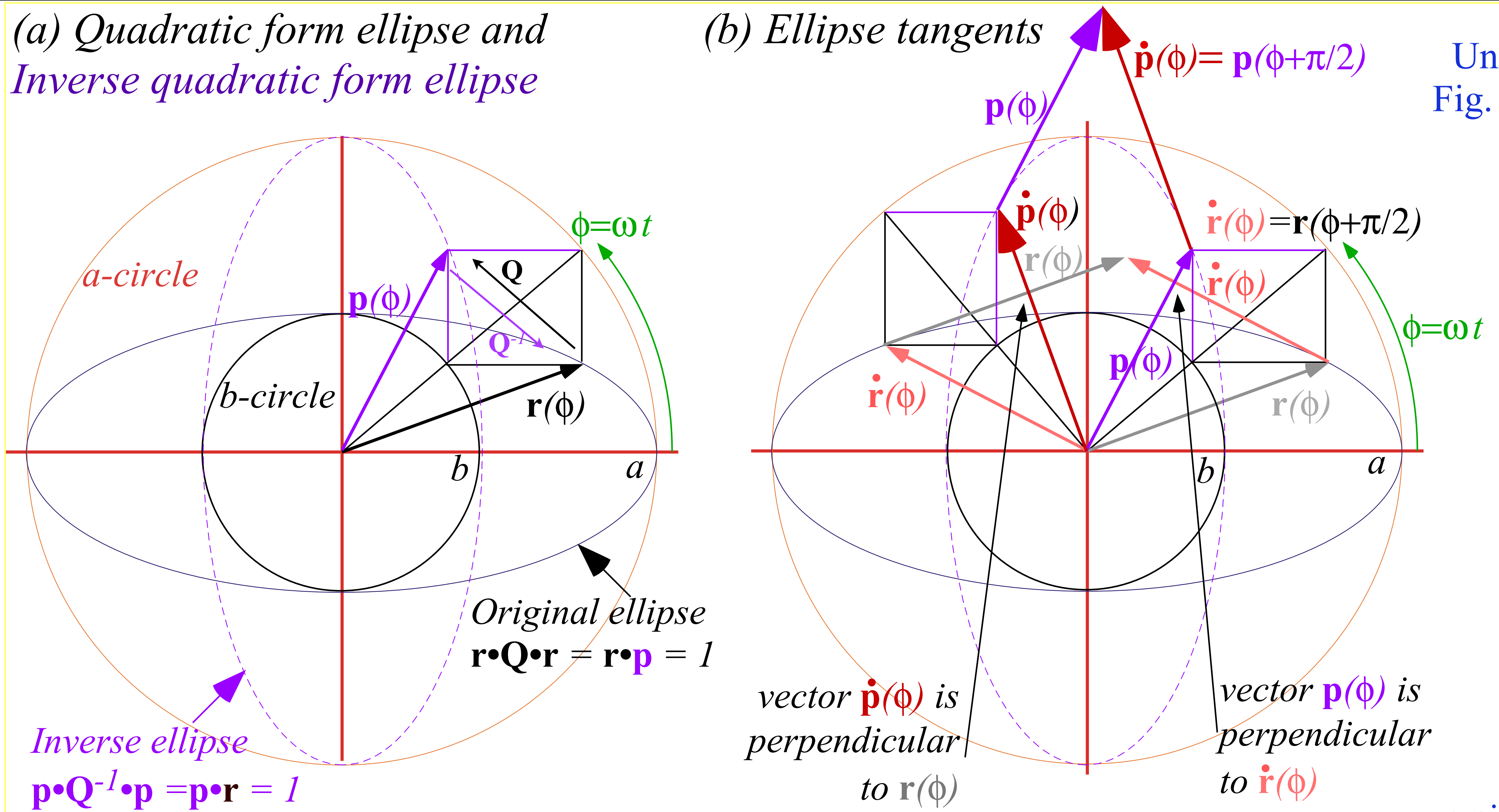
Quadratic form $\mathbf{r}\bullet\mathbf{Q}\bullet\mathbf{r}=1$ *has mutual duality relations with inverse form* $\mathbf{p}\bullet\mathbf{Q}^{-1}\bullet\mathbf{p}=1=\mathbf{p}\bullet\mathbf{r}$

$$\mathbf{p}=\mathbf{Q}\bullet\mathbf{r}=\begin{pmatrix}1/a^2 & 0\\ 0 & 1/b^2\end{pmatrix}\bullet\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x/a^2\\ y/b^2\end{pmatrix}=\begin{pmatrix}(1/a)\cos\phi\\ (1/b)\sin\phi\end{pmatrix}\quad \text{where: } \begin{matrix}x=r_x=a\cos\phi=a\cos\omega t\\ y=r_y=b\sin\phi=b\sin\omega t\end{matrix}\quad \text{so: } \boxed{\mathbf{p}\bullet\mathbf{r}=1}$$

$\mathbf{p}$ *is perpendicular to velocity* $\mathbf{v}=\dot{\mathbf{r}}$ *, a mutual orthogonality*

$$\boxed{\dot{\mathbf{r}}\bullet\mathbf{p}=0}=\begin{pmatrix}\dot{r}_x & \dot{r}_y\end{pmatrix}\bullet\begin{pmatrix}p_x\\ p_y\end{pmatrix}=\begin{pmatrix}-a\sin\phi & b\cos\phi\end{pmatrix}\bullet\begin{pmatrix}(1/a)\cos\phi\\ (1/b)\sin\phi\end{pmatrix}\quad \text{where: } \begin{matrix}\dot{r}_x=-a\sin\phi\\ \dot{r}_y=b\cos\phi\end{matrix}\quad \text{and: } \begin{matrix}p_x=(1/a)\cos\phi\\ p_y=(1/b)\sin\phi\end{matrix}$$

Unit 1
Fig. 11.6



unit mutual projection

Quadratic form **r•Q•r** *=1 has mutual duality relations with inverse form* **p•Q⁻¹•p** *=1*

$$\mathbf{p}=\mathbf{Q}\bullet\mathbf{r}=\begin{pmatrix}1/a^2 & 0\\ 0 & 1/b^2\end{pmatrix}\bullet\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x/a^2\\ y/b^2\end{pmatrix}=\begin{pmatrix}(1/a)\cos\phi\\ (1/b)\sin\phi\end{pmatrix}$$

where: $x=r_x=a\cos\phi=a\cos\omega t$, $y=r_y=b\sin\phi=b\sin\omega t$ so: $\mathbf{p}\bullet\mathbf{r}=1$

***p** is perpendicular to velocity **v**=**ṙ**, a mutual orthogonality. So is **r** perpendicular to **ṗ**:* $\dot{\mathbf{p}}\bullet\mathbf{r}=0$

$$\dot{\mathbf{r}}\bullet\mathbf{p}=0=\begin{pmatrix}\dot r_x & \dot r_y\end{pmatrix}\bullet\begin{pmatrix}p_x\\ p_y\end{pmatrix}=\begin{pmatrix}-a\sin\phi & b\cos\phi\end{pmatrix}\bullet\begin{pmatrix}(1/a)\cos\phi\\ (1/b)\sin\phi\end{pmatrix}$$

where: $\dot r_x=-a\sin\phi$, $\dot r_y=b\cos\phi$ and: $p_x=(1/a)\cos\phi$, $p_y=(1/b)\sin\phi$

Introduction to dual matrix operator geometry

Review of dual IHO elliptic orbits (Lecture 7-8)

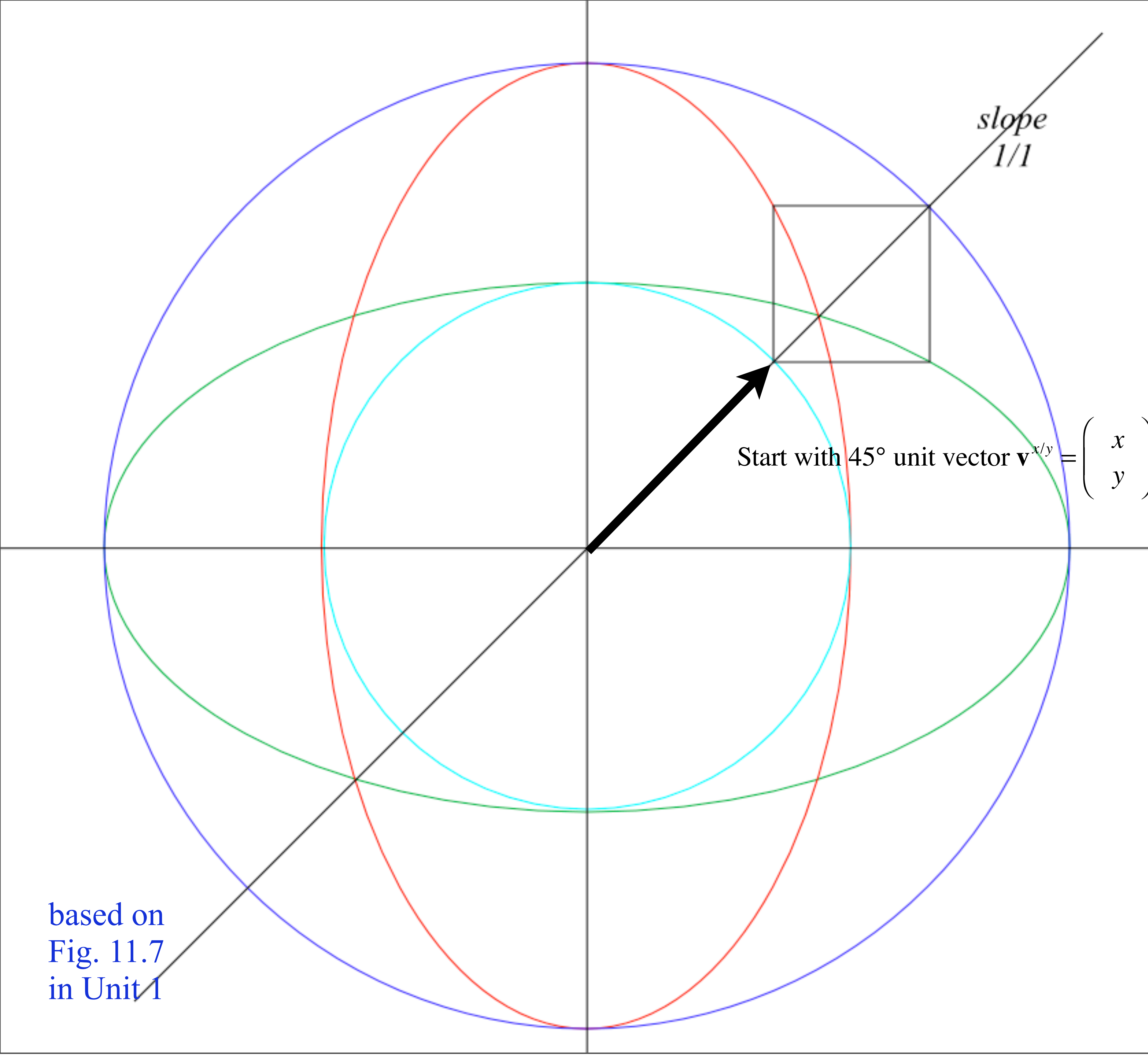
Construction by Phasor-pair projection

Construction by Kepler anomaly projection

→ *Operator geometric sequences and eigenvectors*

Rescaled description of matrix operator geometry

Vector calculus of tensor operation



Start with 45° unit vector $\mathbf{v}^{x/y} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$.

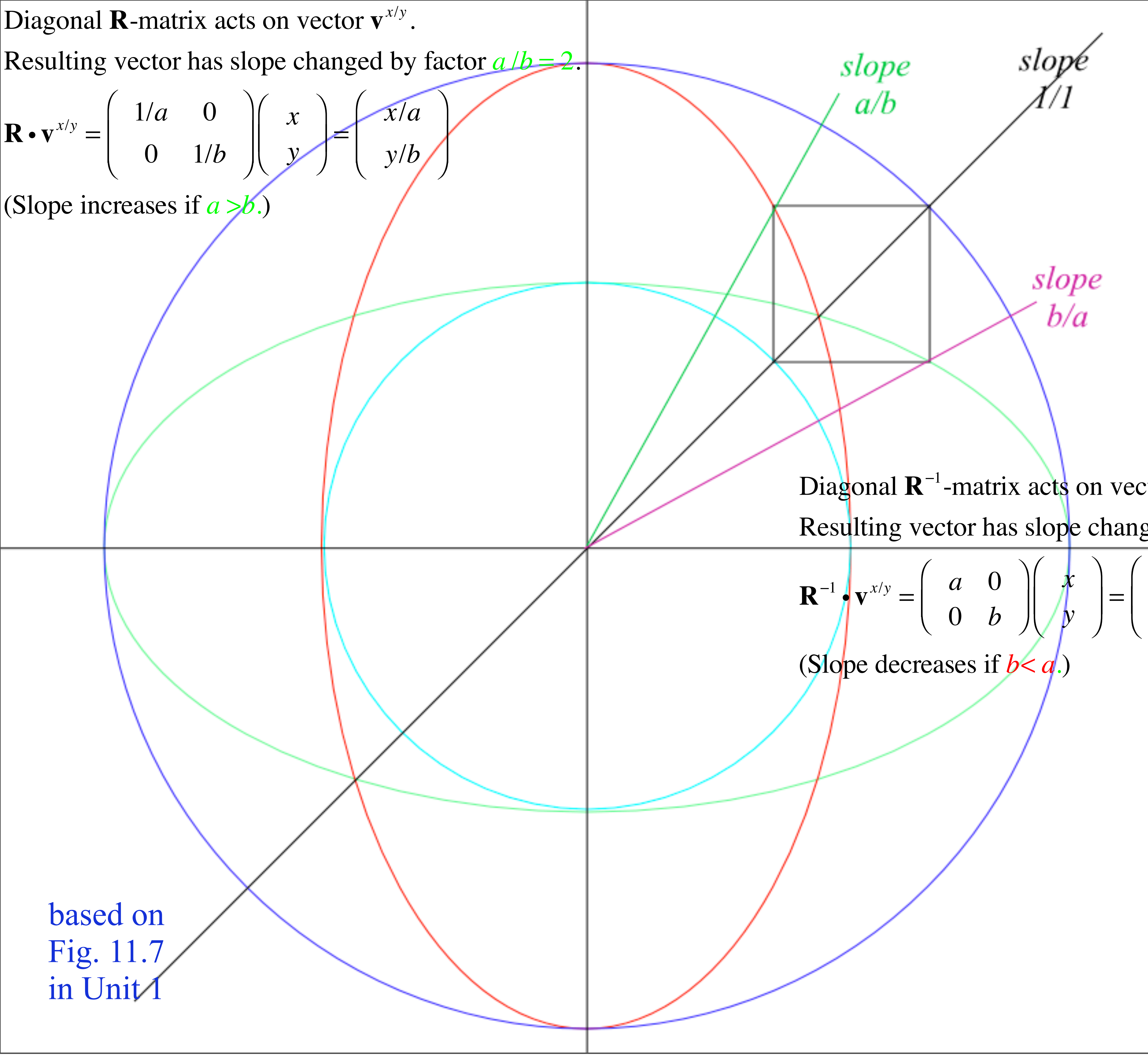
based on Fig. 11.7 in Unit 1

Diagonal **R**-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a/b=2$.

$$\mathbf{R}\bullet\mathbf{v}^{x/y}=\begin{pmatrix}1/a&0\\0&1/b\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x/a\\y/b\end{pmatrix}$$

(Slope increases if $a>b$.)



Diagonal $\mathbf{R}^{-1}$-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor b/a.

$$\mathbf{R}^{-1}\bullet\mathbf{v}^{x/y}=\begin{pmatrix}a&0\\0&b\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\cdot a\\y\cdot b\end{pmatrix}$$

(Slope decreases if $b<a$.)

based on Fig. 11.7 in Unit 1

Diagonal **R**-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a/b = 2$.

$$\mathbf{R}\bullet\mathbf{v}^{x/y} = \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/a \\ y/b \end{pmatrix}$$

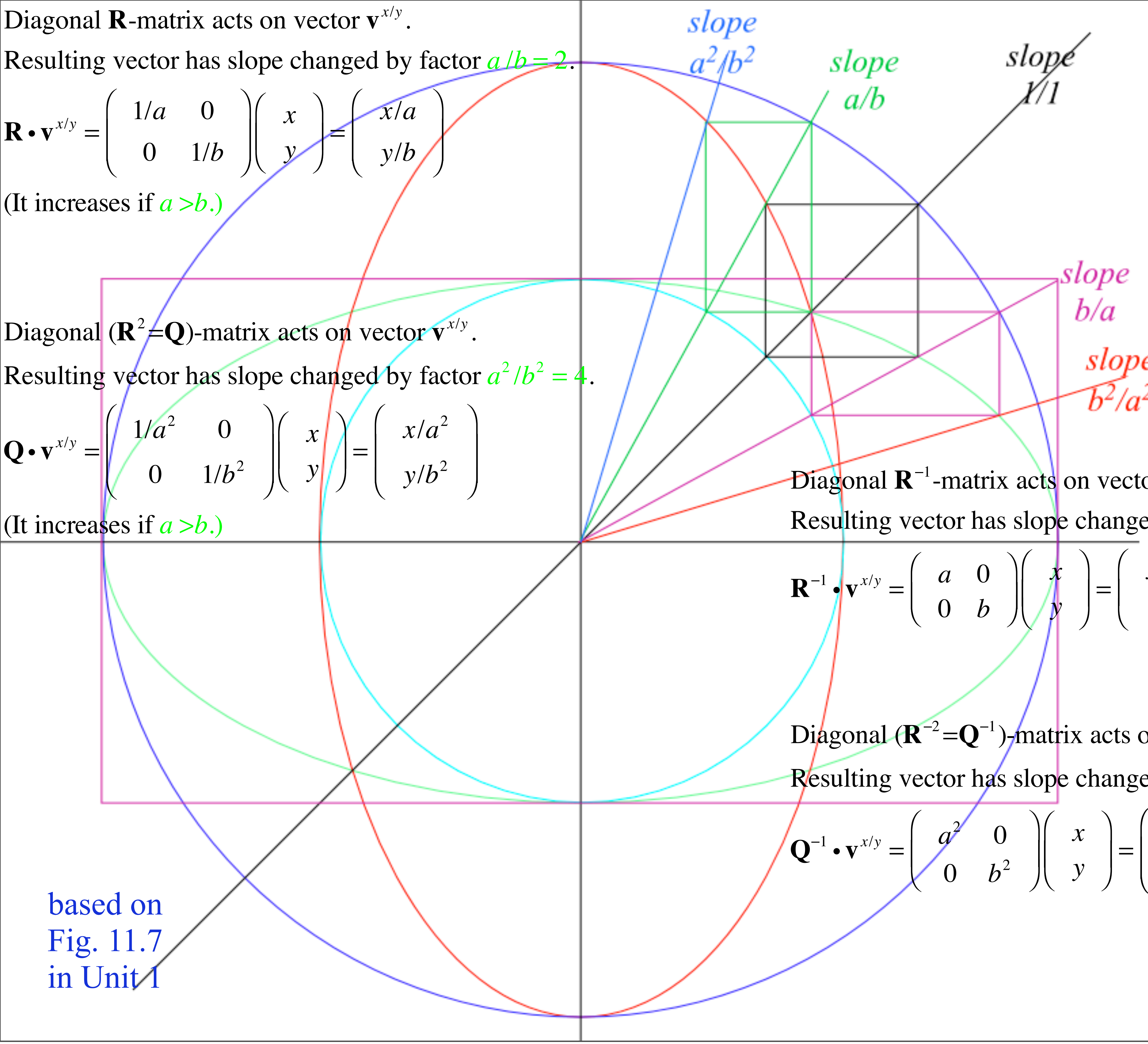
(It increases if $a > b$.)

Diagonal ($\mathbf{R}^2 = \mathbf{Q}$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a^2/b^2 = 4$.

$$\mathbf{Q}\bullet\mathbf{v}^{x/y} = \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/a^2 \\ y/b^2 \end{pmatrix}$$

(It increases if $a > b$.)



Diagonal $\mathbf{R}^{-1}$-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $b/a = 1/2$.

$$\mathbf{R}^{-1}\bullet\mathbf{v}^{x/y} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cdot a \\ y\cdot b \end{pmatrix}$$

Diagonal ($\mathbf{R}^{-2} = \mathbf{Q}^{-1}$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $b^2/a^2 = 1/4$.

$$\mathbf{Q}^{-1}\bullet\mathbf{v}^{x/y} = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cdot a^2 \\ y\cdot b^2 \end{pmatrix}$$

based on Fig. 11.7 in Unit 1

Diagonal **R**-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a/b = 2$.

$$\mathbf{R}\bullet\mathbf{v}^{x/y} = \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/a \\ y/b \end{pmatrix}$$

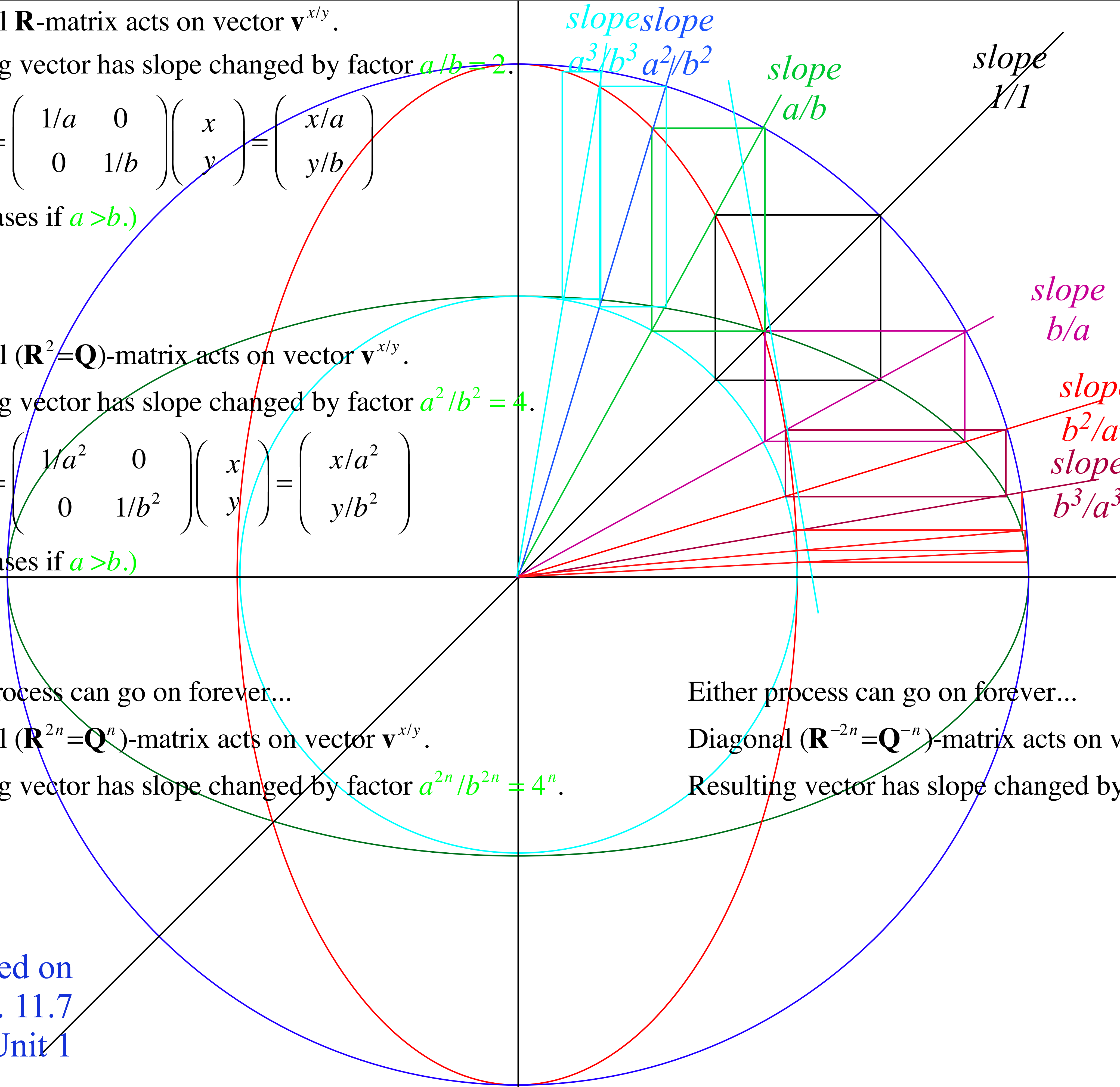
(It increases if $a > b$.)

Diagonal ($\mathbf{R}^2 = \mathbf{Q}$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a^2/b^2 = 4$.

$$\mathbf{Q}\bullet\mathbf{v}^{x/y} = \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/a^2 \\ y/b^2 \end{pmatrix}$$

(It increases if $a > b$.)



Either process can go on forever...

Diagonal ($\mathbf{R}^{2n} = \mathbf{Q}^n$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a^{2n}/b^{2n} = 4^n$.

Either process can go on forever...

Diagonal ($\mathbf{R}^{-2n} = \mathbf{Q}^{-n}$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $b^{2n}/a^{2n} = 4^{-n}$.

based on
Fig. 11.7
in Unit 1

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Resulting vector has slope changed by factor $a/b = 2$.

$$\mathbf{R}\bullet\mathbf{v}^{x/y} = \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/a \\ y/b \end{pmatrix}$$

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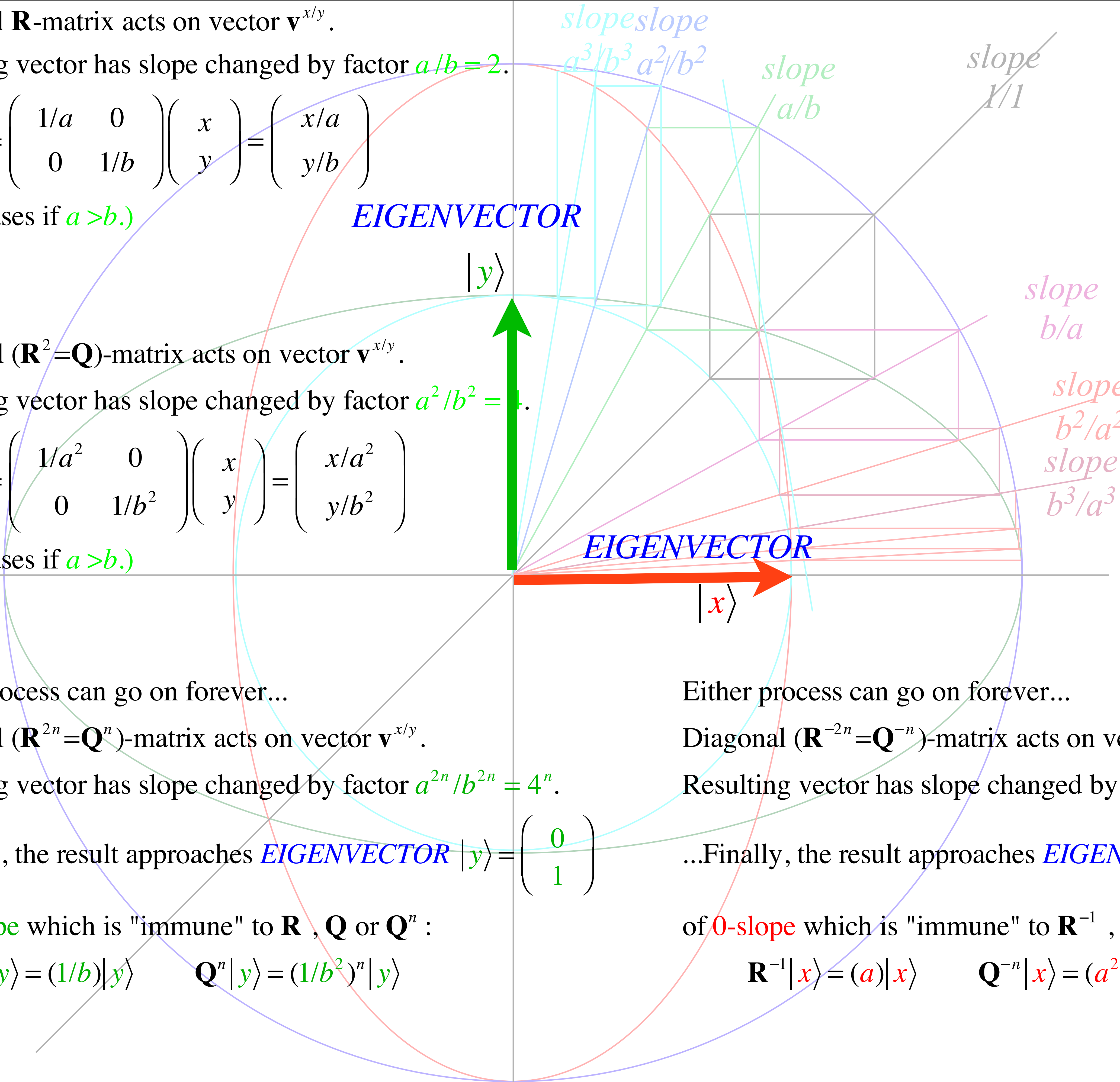


Either process can go on forever...

Diagonal ($\mathbf{R}^{2n} = \mathbf{Q}^n$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a^{2n}/b^{2n} = 4^n$.

...Finally, the result approaches *EIGENVECTOR* $|y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

of ∞-slope which is "immune" to **R** , **Q** or $\mathbf{Q}^n$:

$$\mathbf{R}|y\rangle = (1/b)|y\rangle \qquad \mathbf{Q}^n|y\rangle = (1/b^2)^n|y\rangle$$

Either process can go on forever...

Diagonal ($\mathbf{R}^{-2n} = \mathbf{Q}^{-n}$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $b^{2n}/a^{2n} = 4^{-n}$.

...Finally, the result approaches *EIGENVECTOR* $|x\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

of 0-slope which is "immune" to $\mathbf{R}^{-1}$, $\mathbf{Q}^{-1}$ or $\mathbf{Q}^{-n}$:

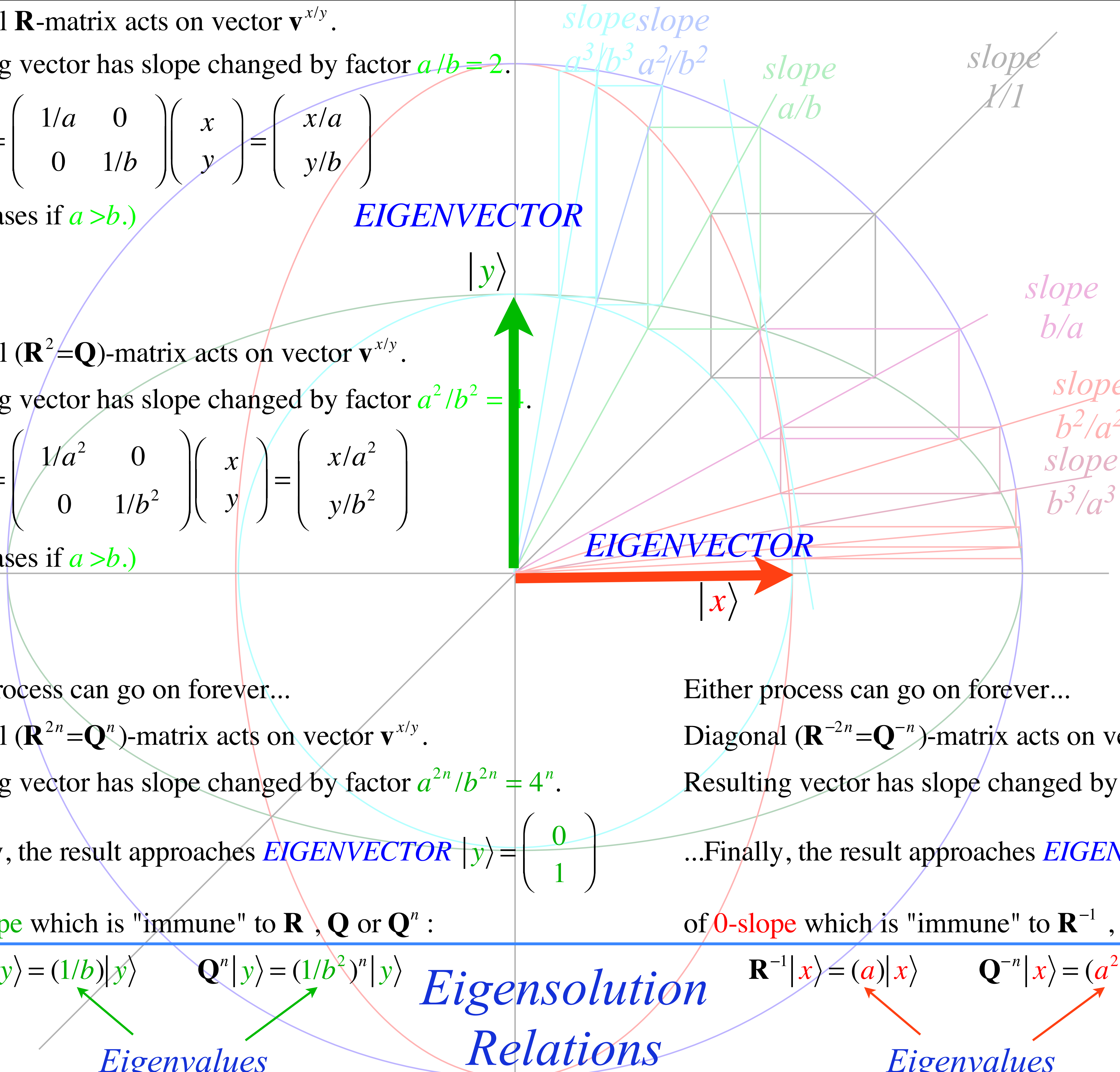
$$\mathbf{R}^{-1}|x\rangle = (a)|x\rangle \qquad \mathbf{Q}^{-n}|x\rangle = (a^2)^n|x\rangle$$

Diagonal **R**-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a/b = 2$.

$$\mathbf{R}\bullet\mathbf{v}^{x/y} = \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/a \\ y/b \end{pmatrix}$$

(It increases if $a > b$.)



Diagonal ($\mathbf{R}^2 = \mathbf{Q}$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a^2/b^2 = 4$.

$$\mathbf{Q}\bullet\mathbf{v}^{x/y} = \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/a^2 \\ y/b^2 \end{pmatrix}$$

(It increases if $a > b$.)

Either process can go on forever...

Diagonal ($\mathbf{R}^{2n} = \mathbf{Q}^n$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a^{2n}/b^{2n} = 4^n$.

...Finally, the result approaches *EIGENVECTOR* $|y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

of ∞-slope which is "immune" to $\mathbf{R}$, $\mathbf{Q}$ or $\mathbf{Q}^n$:

Either process can go on forever...

Diagonal ($\mathbf{R}^{-2n} = \mathbf{Q}^{-n}$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $b^{2n}/a^{2n} = 4^{-n}$.

...Finally, the result approaches *EIGENVECTOR* $|x\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

of 0-slope which is "immune" to $\mathbf{R}^{-1}$, $\mathbf{Q}^{-1}$ or $\mathbf{Q}^{-n}$:

$\mathbf{R}|y\rangle = (1/b)|y\rangle$ $\quad$ $\mathbf{Q}^n|y\rangle = (1/b^2)^n|y\rangle$

Eigenvalues

Eigensolution Relations

$\mathbf{R}^{-1}|x\rangle = (a)|x\rangle$ $\quad$ $\mathbf{Q}^{-n}|x\rangle = (a^2)^n|x\rangle$

Eigenvalues

Introduction to dual matrix operator geometry

Review of dual IHO elliptic orbits (Lecture 7-8)

Construction by Phasor-pair projection

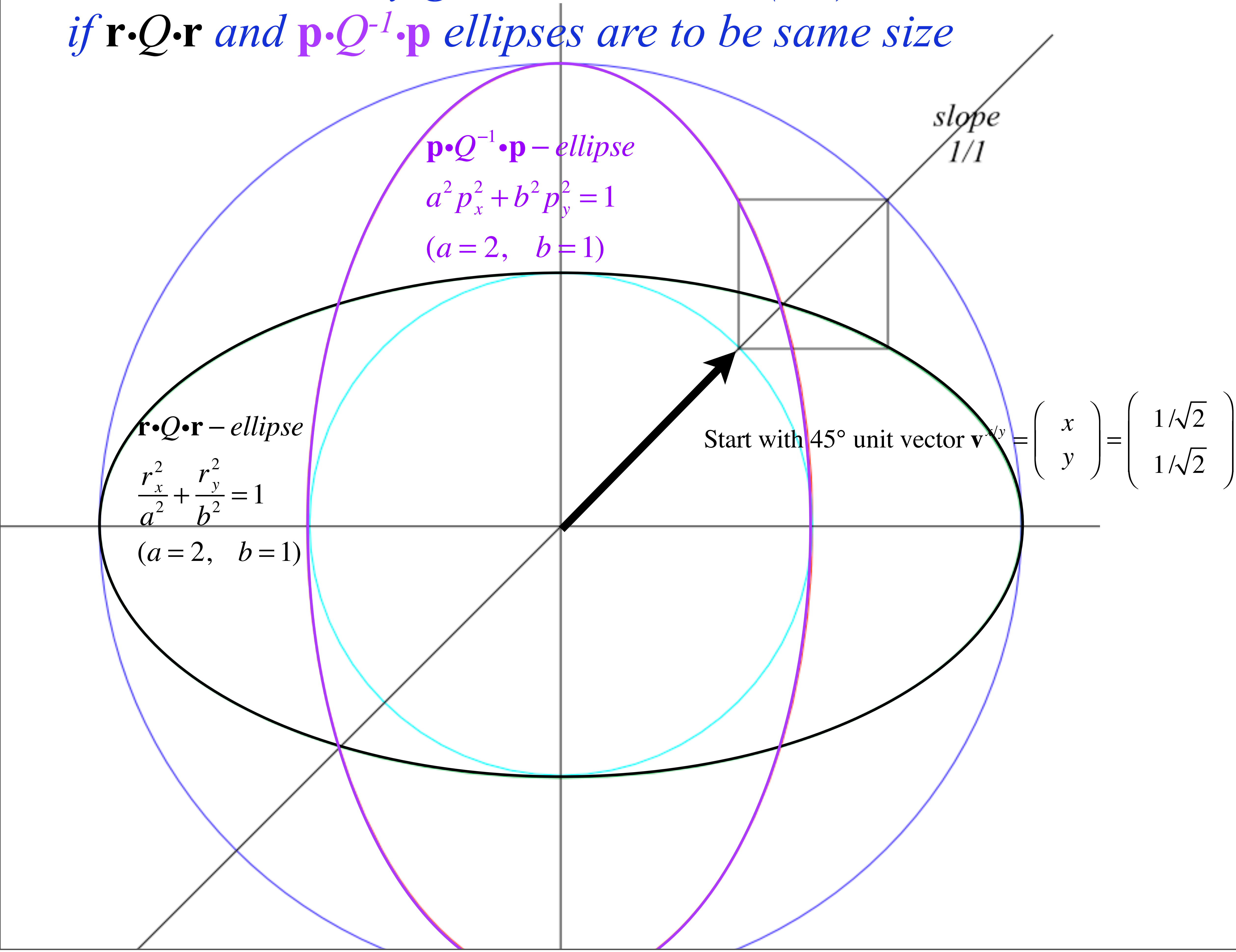
Construction by Kepler anomaly projection

Operator geometric sequences and eigenvectors

→ *Rescaled description of matrix operator geometry*

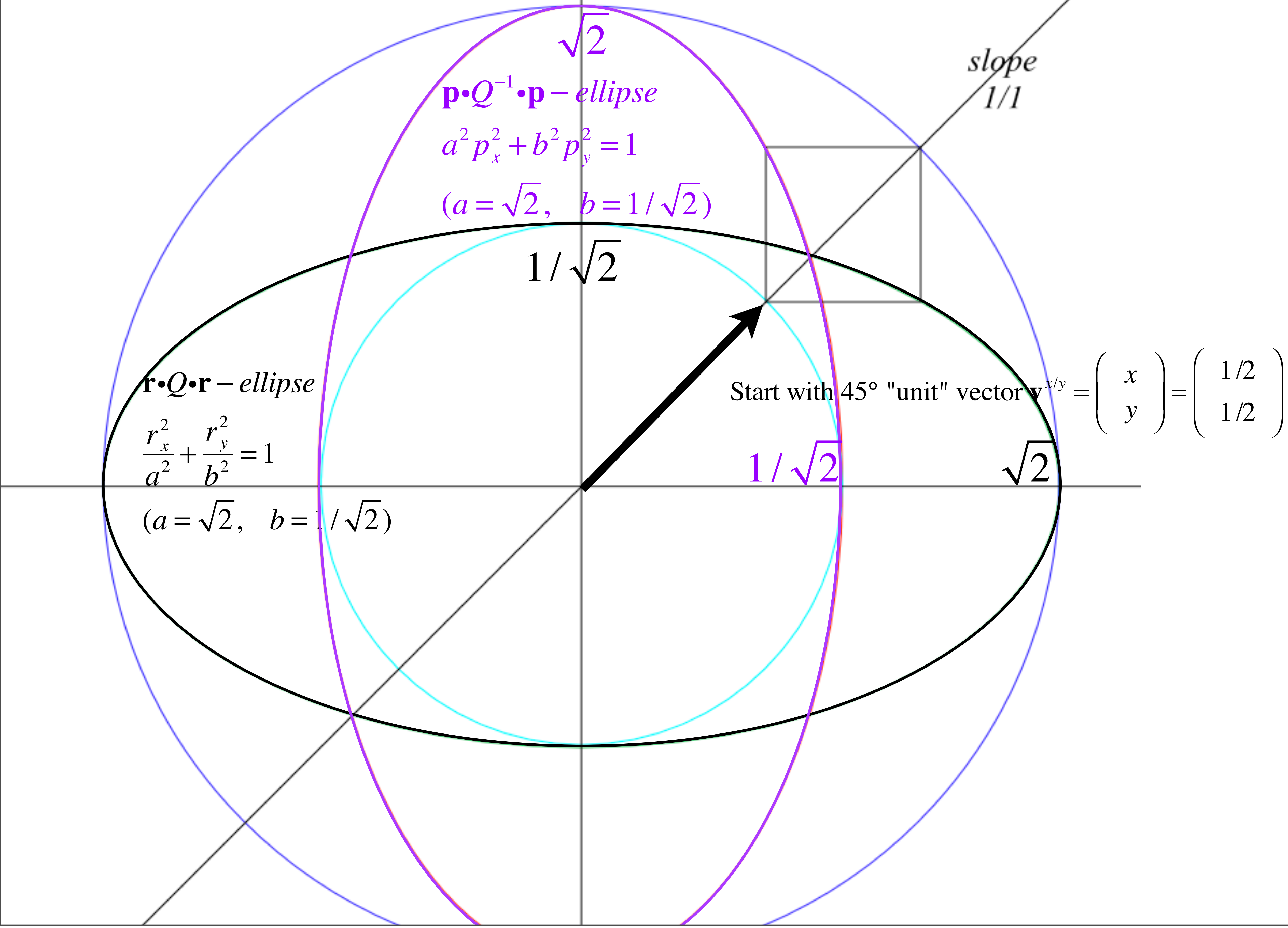
Vector calculus of tensor operation

Need to rescale by geometric mean √(a·b)
if **r·*Q*·r** *and* **p·*Q*⁻¹·p** *ellipses are to be same size*

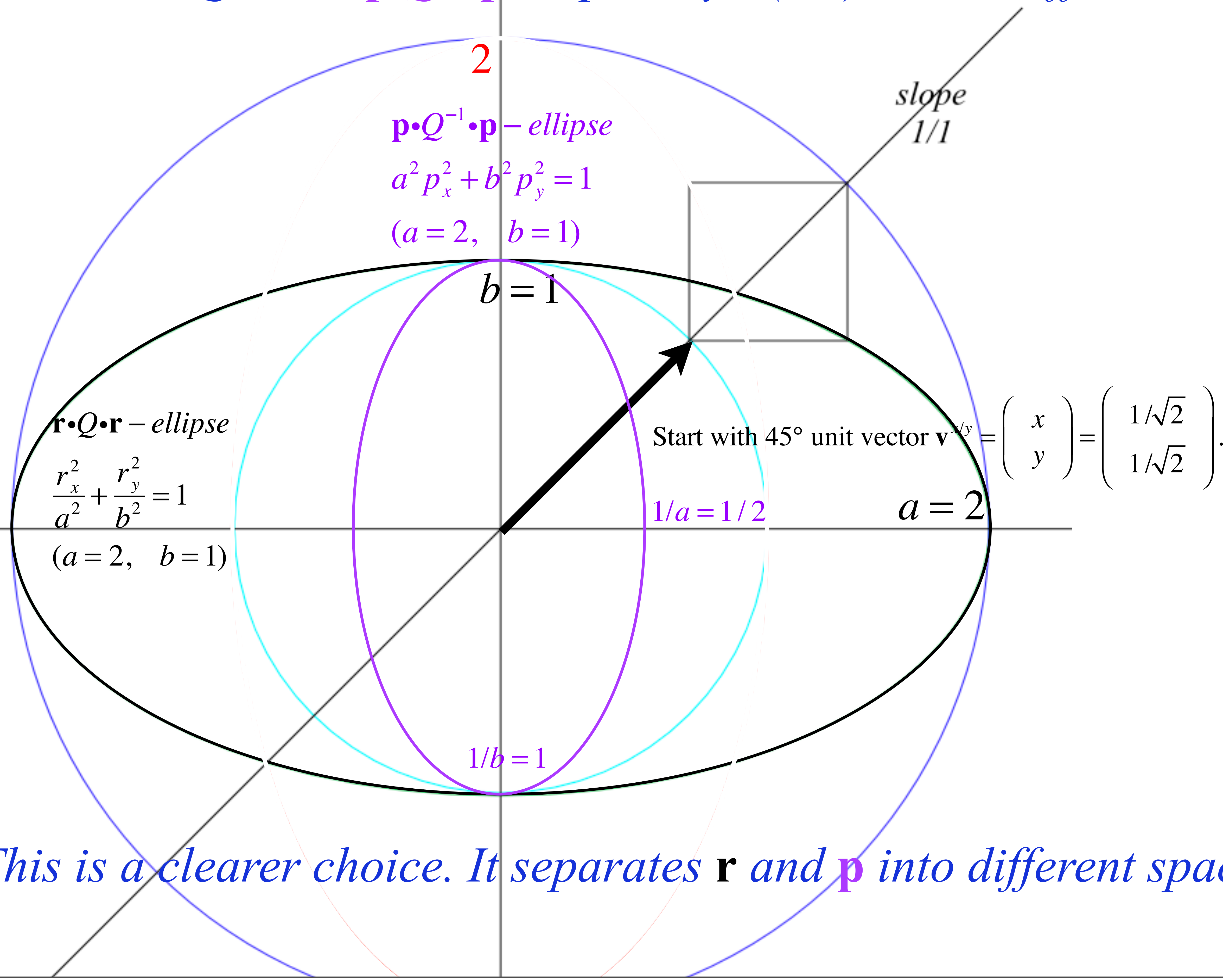


Need to rescale by geometric mean √(a·b) *(so a·b=1)*

if **r·*Q*·r** *and* **p·*Q*$^{-1}$·p** *ellipses are to be same size*



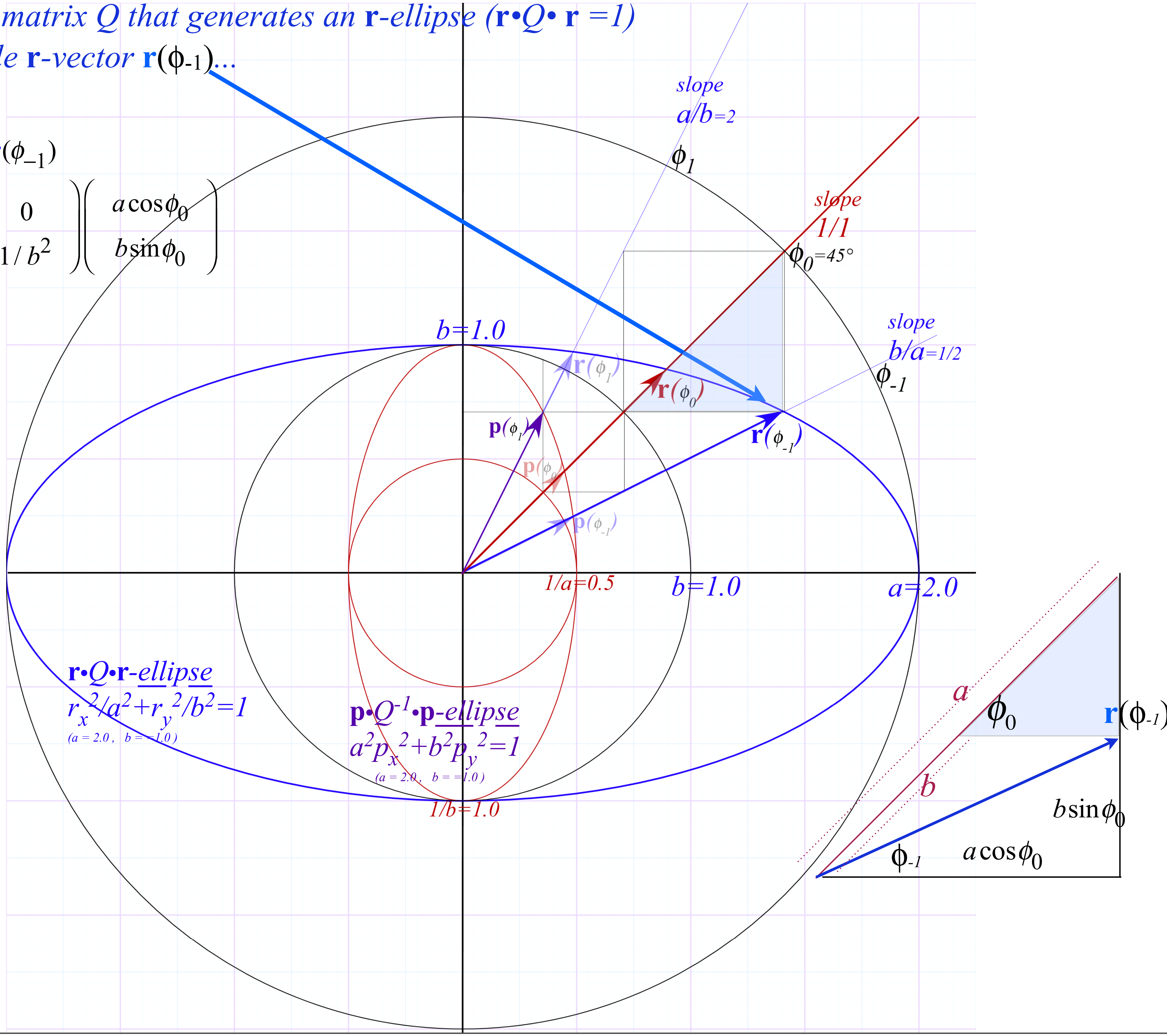
...or rescale **r**•*Q*•**r** and **p**•Q^{-1}•**p** ellipses by $\sqrt{(a\cdot b)}=\sqrt{2}$ to different size



This is a clearer choice. It separates **r** *and* **p** *into different spaces*

Action of matrix Q that generates an **r**-*ellipse* (**r**•*Q*• **r** =*1)* *on a single* **r**-*vector* **r**(ϕ_{-1})...

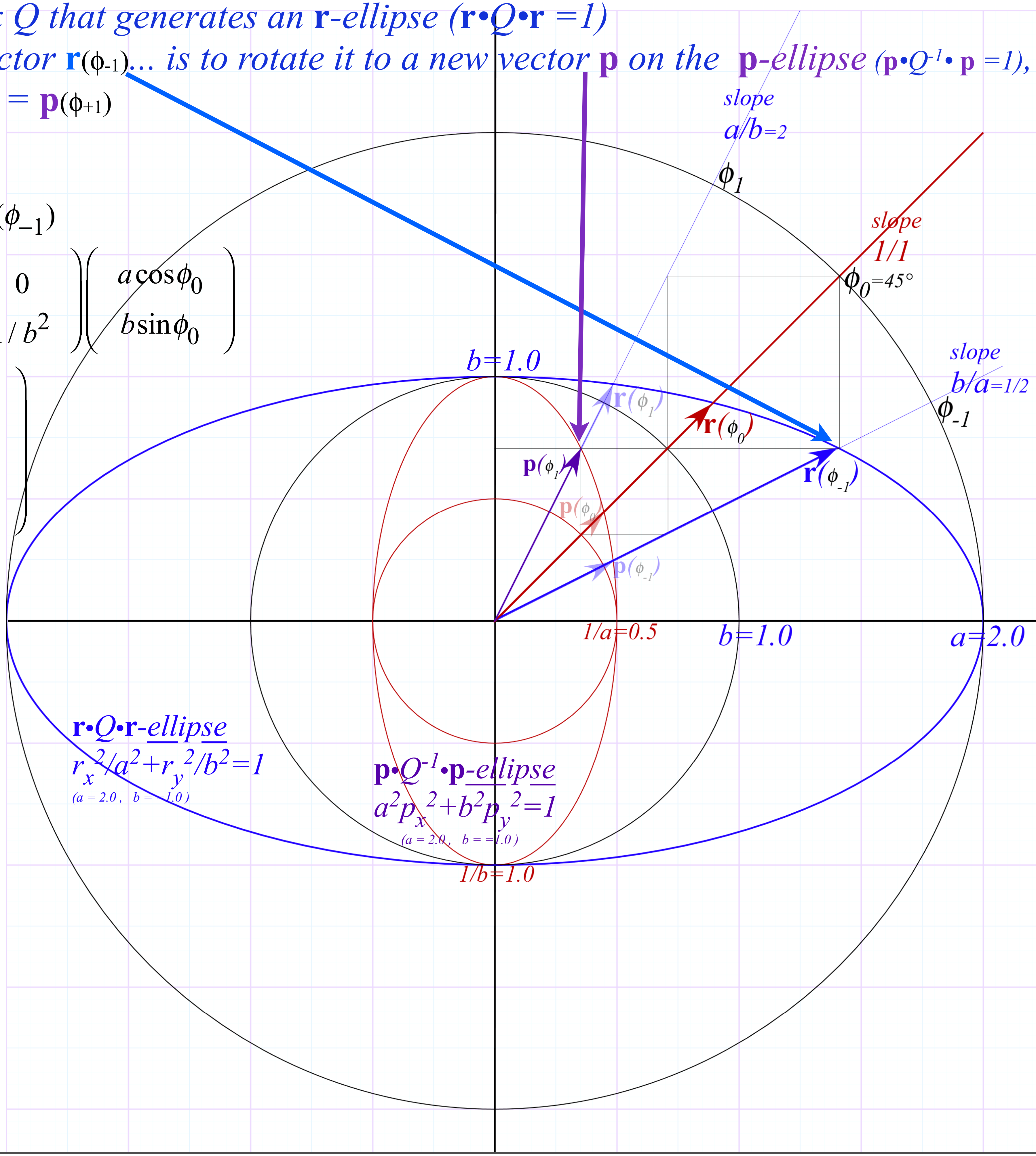
$$\mathbf{p}(\phi_1) = \mathbf{Q} \bullet \mathbf{r}(\phi_{-1}) = \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix} \begin{pmatrix} a\cos\phi_0 \\ b\sin\phi_0 \end{pmatrix}$$



based on Fig. 11.7 in Unit 1

Action of matrix Q that generates an **r***-ellipse* (**r**•*Q*•**r** *=1)*
on a single **r***-vector* **r**(ϕ-1)*... is to rotate it to a new vector* **p** *on the* **p***-ellipse* (**p**•*Q*$^{-1}$• **p** *=1),*
that is, *Q*•**r**(ϕ-1) = **p**(ϕ+1)

$$\mathbf{p}(\phi_1) = \mathbf{Q}\bullet\mathbf{r}(\phi_{-1})$$
$$= \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix}\begin{pmatrix} a\cos\phi_0 \\ b\sin\phi_0 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{a}\cos\phi_0 \\ \frac{1}{b}\sin\phi_0 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{2}\frac{1}{\sqrt{2}} \\ \frac{1}{1}\frac{1}{\sqrt{2}} \end{pmatrix}$$



based on
Fig. 11.7
in Unit 1

Action of matrix Q that generates an **r***-ellipse (***r**•*Q*•**r** *=1)*
on a single **r***-vector* **r**(ϕ_{-1})*... is to rotate it to a new vector* **p** *on the* **p***-ellipse* $(\mathbf{p}\bullet Q^{-1}\bullet \mathbf{p} = 1)$*,*
that is, $Q\bullet\mathbf{r}(\phi_{-1}) = \mathbf{p}(\phi_{+1})$

$$\mathbf{p}(\phi_1) = \mathbf{Q}\bullet\mathbf{r}(\phi_{-1})$$

$$= \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix}\begin{pmatrix} a\cos\phi_0 \\ b\sin\phi_0 \end{pmatrix}$$

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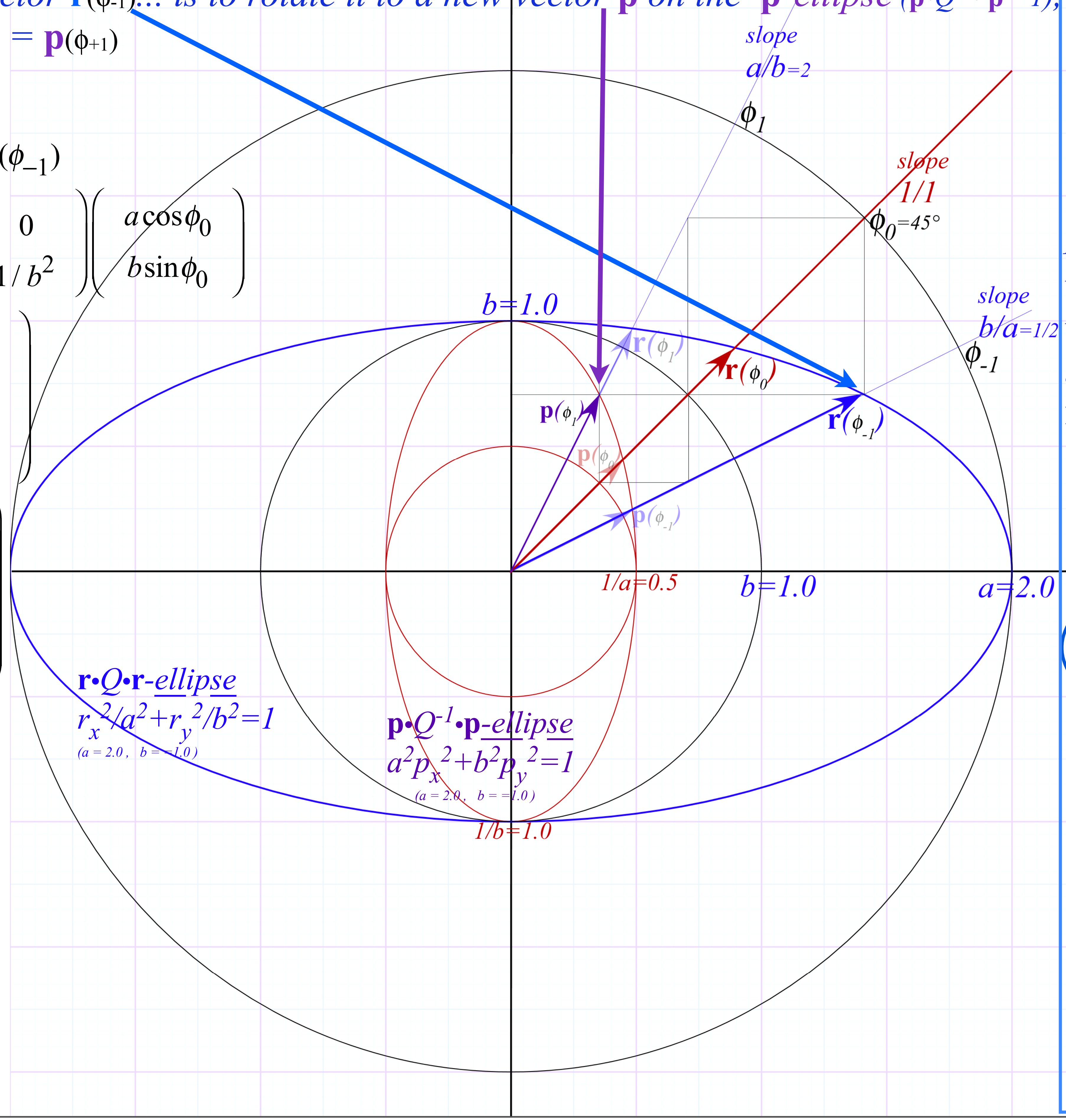
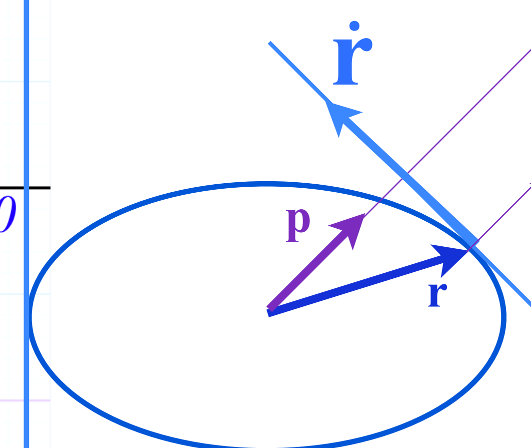


based on
Fig. 11.7
in Unit 1

Key points of matrix geometry:

Matrix Q maps any vector **r** *to a new vector* **p** *normal to the tangent* $\dot{\mathbf{r}}$ *to its* **r**•*Q*•**r***-ellipse.*



Action of matrix Q that generates an **r***-ellipse* (**r**•*Q*•**r** *=1)* *on a single* **r***-vector* **r**(ϕ-1)*... is to rotate it to a new vector* **p** *on the* **p***-ellipse* (**p**•Q^{-1}• **p** *=1),* *that is,* *Q*•**r**(ϕ-1) = **p**(ϕ+1)

$$\mathbf{p}(\phi_1) = \mathbf{Q}\cdot\mathbf{r}(\phi_{-1})$$

$$= \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix} \begin{pmatrix} a\cos\phi_0 \\ b\sin\phi_0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{a}\cos\phi_0 \\ \frac{1}{b}\sin\phi_0 \end{pmatrix}$$

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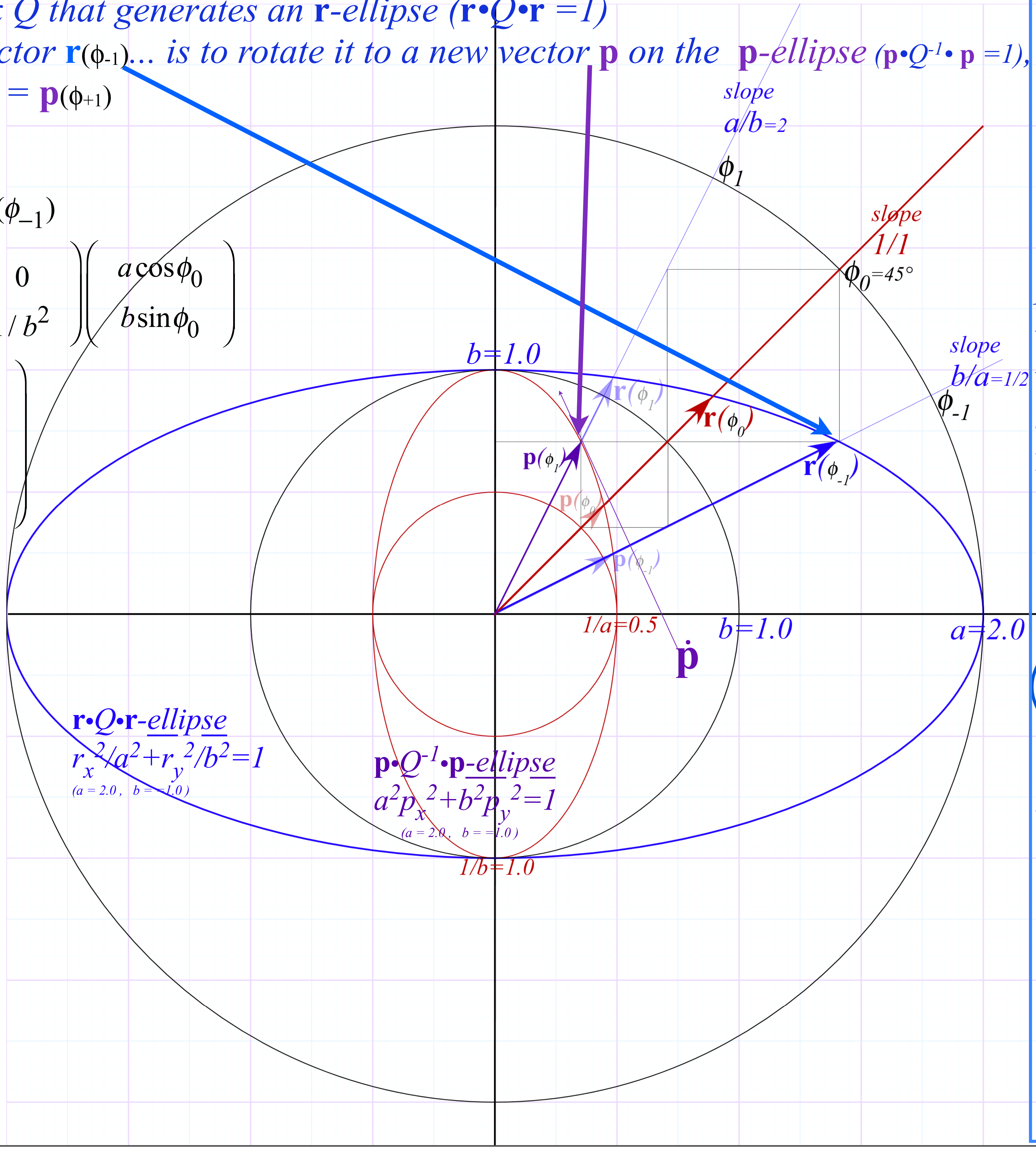
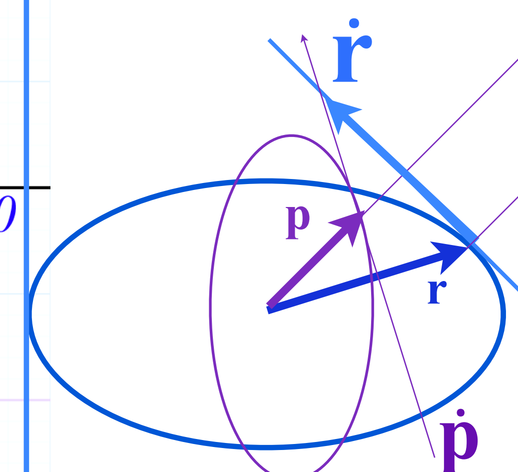


Key points of matrix geometry:

Matrix Q maps any vector **r** *to a new vector* **p** *normal to the tangent* **ṙ** *to its* **r**•*Q*•**r***-ellipse.*



Matrix Q^{-1} *maps* **p** *back to* **r** *that is normal to the tangent* **ṗ** *to its* **p**• Q^{-1}• **p***-ellipse.*

based on
Fig. 11.7
in Unit 1

Introduction to dual matrix operator geometry

Review of dual IHO elliptic orbits (Lecture 7-8)

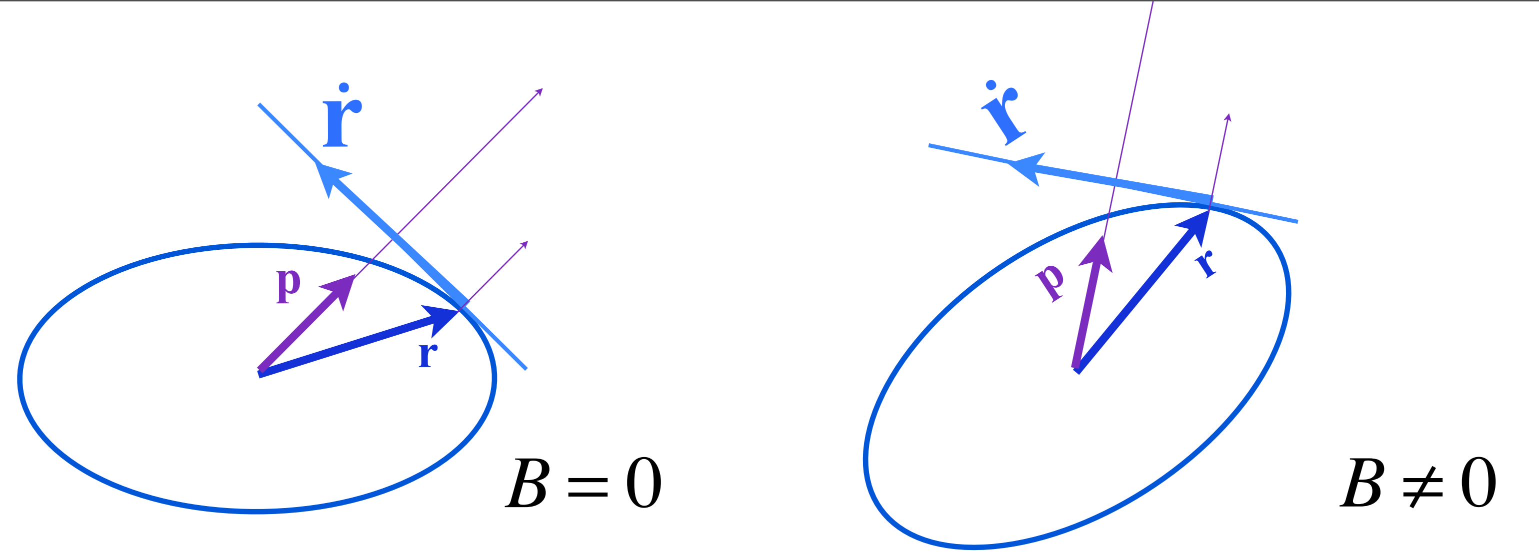
Construction by Phasor-pair projection

Construction by Kepler anomaly projection

Operator geometric sequences and eigenvectors

Rescaled description of matrix operator geometry

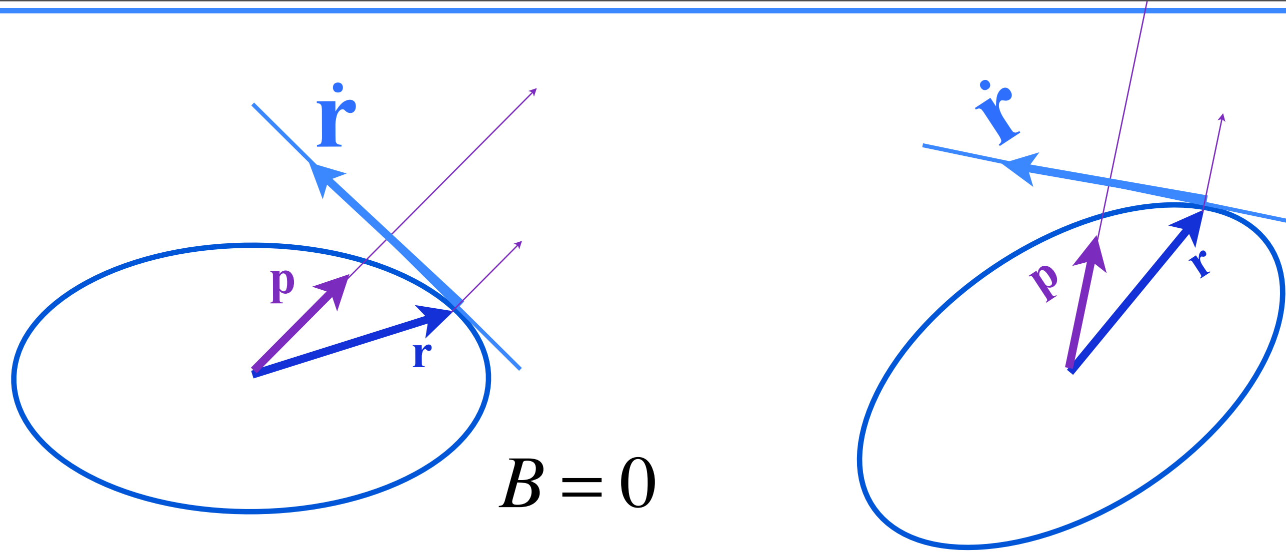
→ *Vector calculus of tensor operation*



Derive matrix "normal-to-ellipse" geometry by vector calculus:

Let matrix $Q = \begin{pmatrix} A & B \\ B & D \end{pmatrix}$

define the ellipse $1=\mathbf{r}\bullet Q\bullet\mathbf{r} = \begin{pmatrix} x & y \end{pmatrix}\bullet\begin{pmatrix} A & B \\ B & D \end{pmatrix}\bullet\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix}\bullet\begin{pmatrix} A\cdot x + B\cdot y \\ B\cdot x + D\cdot y \end{pmatrix} = A\cdot x^2 + 2B\cdot xy + D\cdot y^2 = 1$



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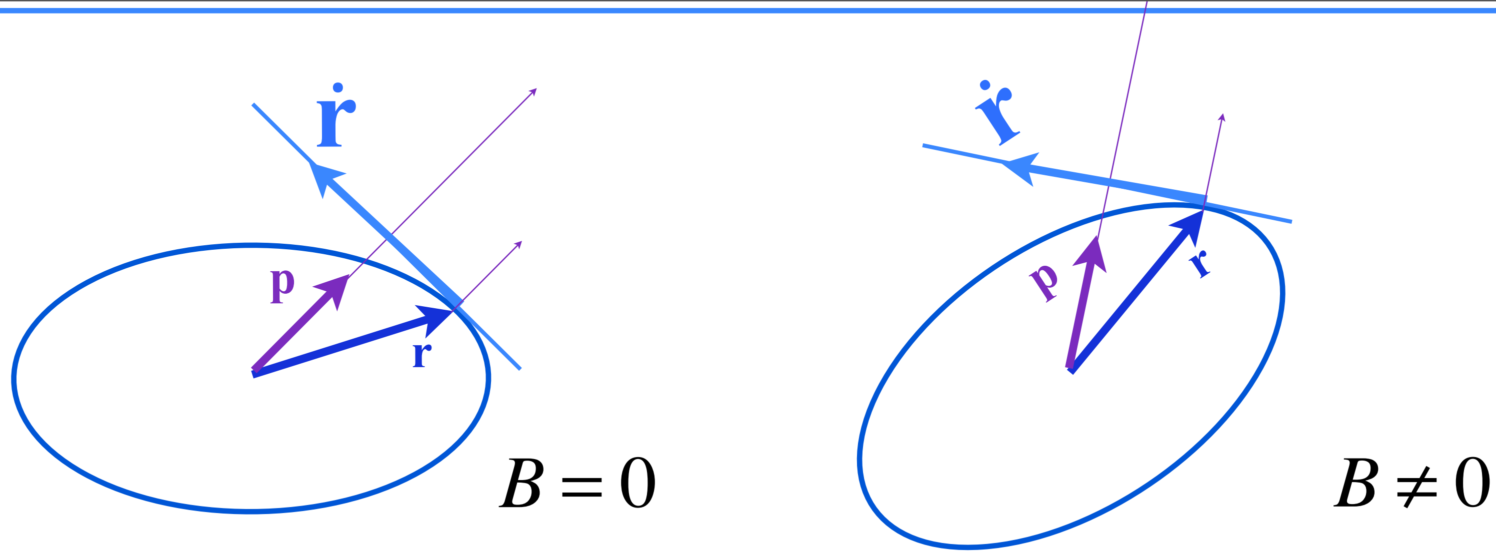
Compare operation by Q on vector **r** *with* *vector derivative or gradient of* $\mathbf{r}\bullet Q\bullet\mathbf{r}$

$$\begin{pmatrix} A & B \\ B & D \end{pmatrix}\bullet\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A\cdot x + B\cdot y \\ B\cdot x + D\cdot y \end{pmatrix}$$

$$\frac{\partial}{\partial\mathbf{r}}(\mathbf{r}\bullet Q\bullet\mathbf{r}) = \nabla(\mathbf{r}\bullet Q\bullet\mathbf{r})$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}\left(A\cdot x^2 + 2B\cdot xy + D\cdot y^2\right) = \begin{pmatrix} 2A\cdot x + 2B\cdot y \\ 2B\cdot x + 2D\cdot y \end{pmatrix}$$



Derive matrix "normal-to-ellipse" geometry by vector calculus:

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Compare operation by Q on vector **r** *with* *vector derivative or gradient of* $\mathbf{r}\bullet Q\bullet\mathbf{r}$

$$\begin{pmatrix} A & B \\ B & D \end{pmatrix}\bullet\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A\cdot x + B\cdot y \\ B\cdot x + D\cdot y \end{pmatrix}$$

$$\frac{\partial}{\partial\mathbf{r}}(\mathbf{r}\bullet Q\bullet\mathbf{r}) = \nabla(\mathbf{r}\bullet Q\bullet\mathbf{r})$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}\left(A\cdot x^2 + 2B\cdot xy + D\cdot y^2\right) = \begin{pmatrix} 2A\cdot x + 2B\cdot y \\ 2B\cdot x + 2D\cdot y \end{pmatrix}$$

Very simple result:

$$\frac{\partial}{\partial\mathbf{r}}\left(\frac{\mathbf{r}\bullet Q\bullet\mathbf{r}}{2}\right) = \nabla\left(\frac{\mathbf{r}\bullet Q\bullet\mathbf{r}}{2}\right) = Q\bullet\mathbf{r}$$

Action of "sqrt-"matrix $R=\sqrt{Q}$ *(R generates another ellipse **r**•R•**r** =1 not shown)*

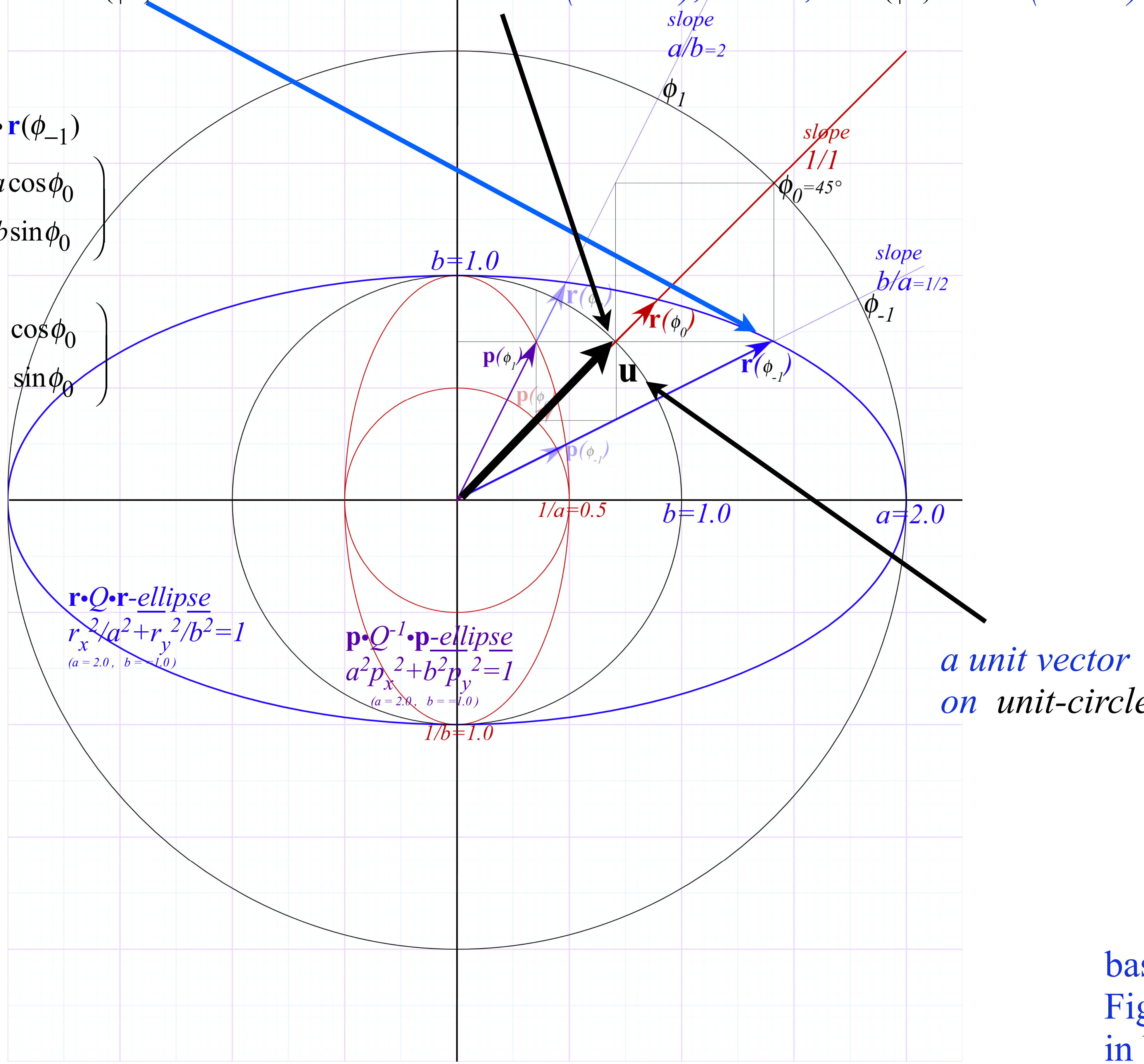
*on a single **r**-vector **r**(ϕ_{-1})... is to rotate it to **u**-circle (**u**•**u** =1), that is, R•**r**(ϕ_{-1}) = **u** =(const.)**r**(ϕ_0)*

$$\mathbf{u} = \sqrt{\mathbf{Q}} \bullet \mathbf{r}(\phi_{-1}) = \mathbf{R} \bullet \mathbf{r}(\phi_{-1})$$

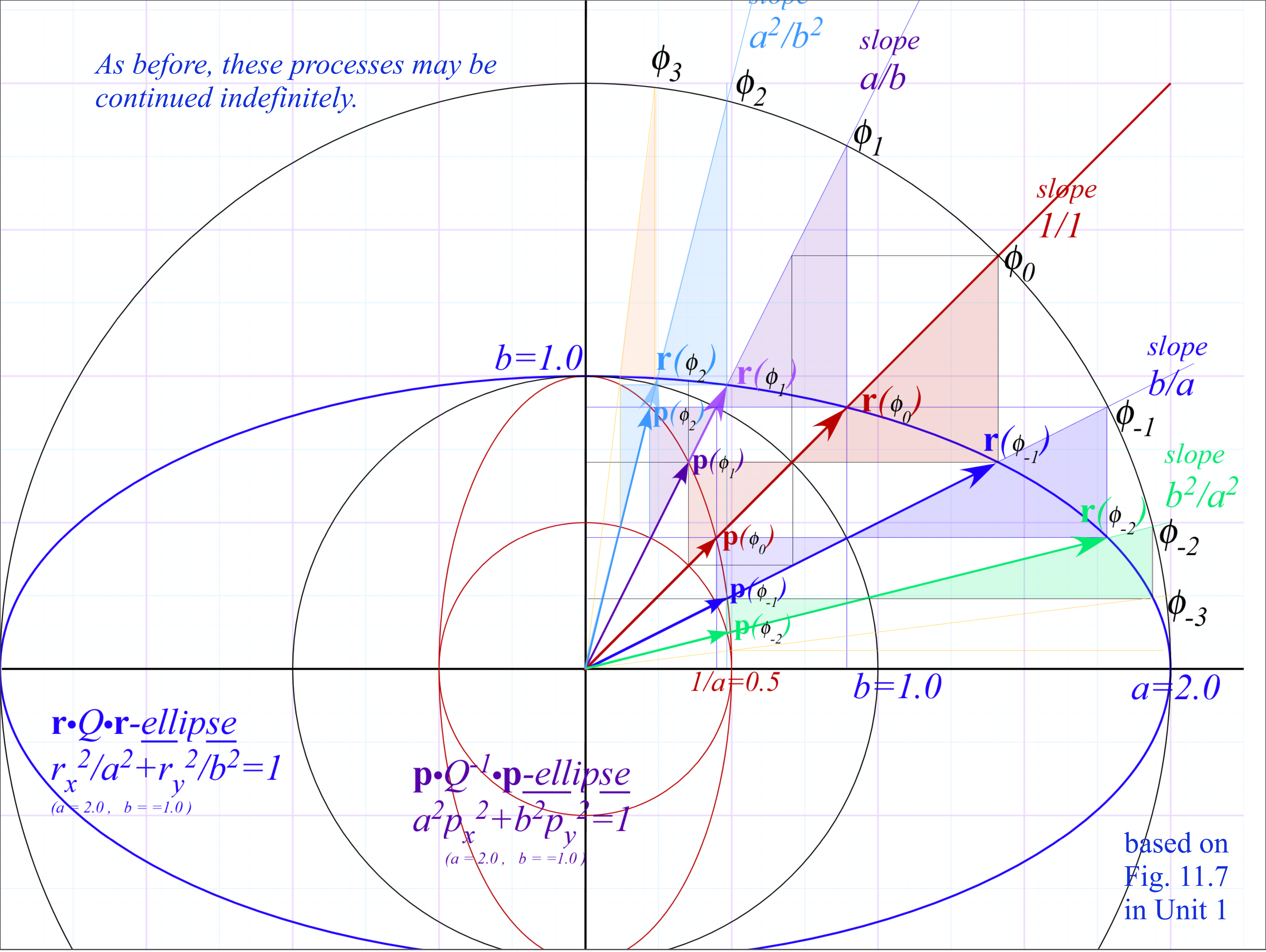
$$= \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix} \begin{pmatrix} a\cos\phi_0 \\ b\sin\phi_0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{a} a\cos\phi_0 \\ \frac{1}{b} b\sin\phi_0 \end{pmatrix} = \begin{pmatrix} \cos\phi_0 \\ \sin\phi_0 \end{pmatrix}$$

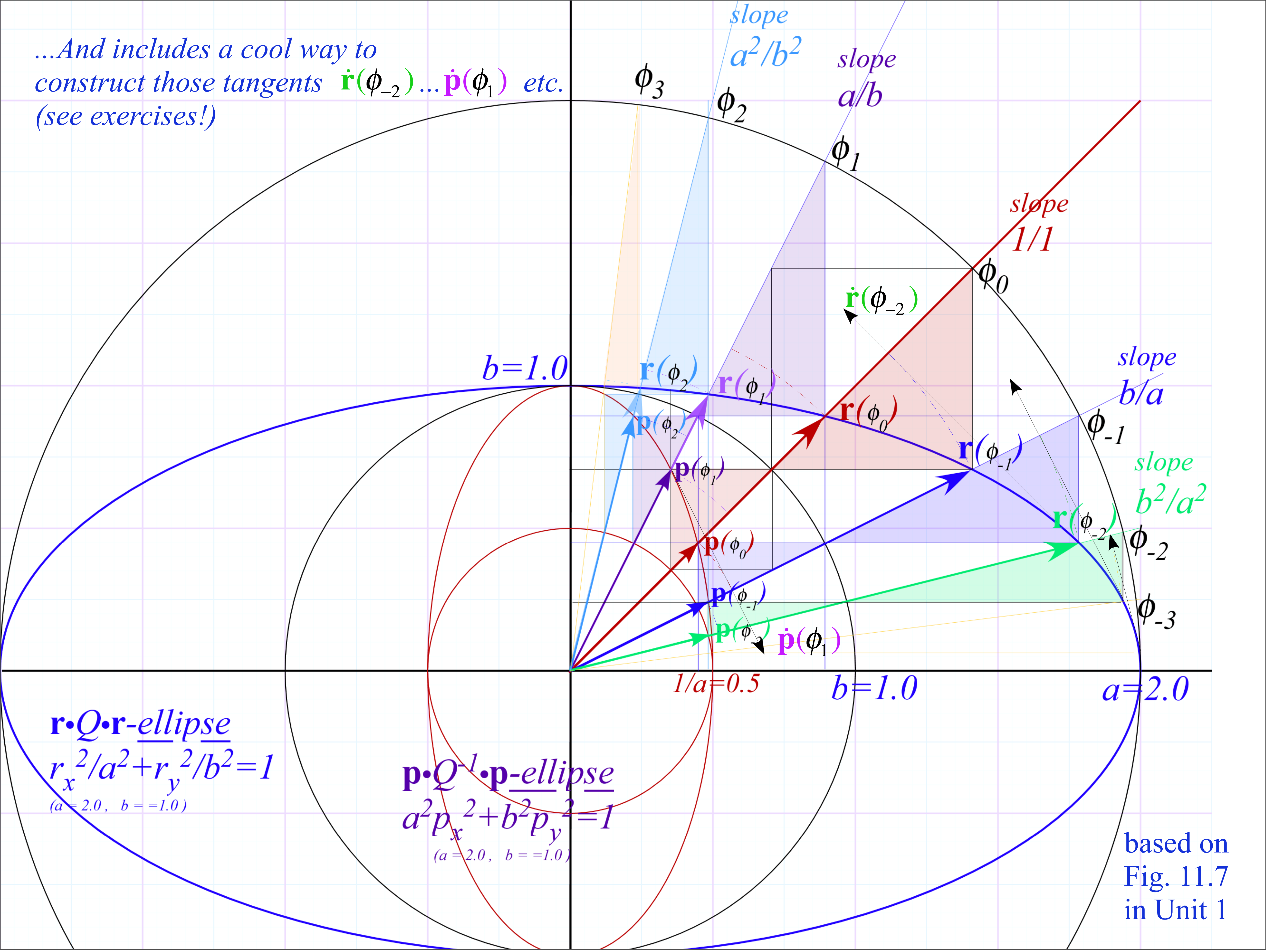
$$= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$



based on
Fig. 11.7
in Unit 1



As before, these processes may be continued indefinitely.
slope
a²/b²
slope
a/b
slope
1/1
slope
b/a
slope
b²/a²
ϕ3
ϕ2
ϕ1
ϕ0
ϕ-1
ϕ-2
ϕ-3
b=1.0
r(ϕ2)
r(ϕ1)
r(ϕ0)
r(ϕ-1)
r(ϕ-2)
p(ϕ2)
p(ϕ1)
p(ϕ0)
p(ϕ-1)
p(ϕ-2)
1/a=0.5
b=1.0
a=2.0
r•Q•r-ellipse
rx²/a²+ry²/b²=1
(a = 2.0 , b = =1.0)
p•Q⁻¹•p-ellipse
a²px²+b²py²=1
(a = 2.0 , b = =1.0)
based on
Fig. 11.7
in Unit 1



...And includes a cool way to construct those tangents $\dot{\mathbf{r}}(\phi_{-2})\ldots\dot{\mathbf{p}}(\phi_1)$ etc. (see exercises!)
slope a^2/b^2
slope a/b
slope 1/1
slope b/a
slope b^2/a^2
ϕ_3
ϕ_2
ϕ_1
ϕ_0
ϕ_{-1}
ϕ_{-2}
ϕ_{-3}
$\dot{\mathbf{r}}(\phi_{-2})$
$b=1.0$
$\mathbf{r}(\phi_2)$
$\mathbf{r}(\phi_1)$
$\mathbf{r}(\phi_0)$
$\mathbf{r}(\phi_{-1})$
$\mathbf{r}(\phi_{-2})$
$\mathbf{p}(\phi_2)$
$\mathbf{p}(\phi_1)$
$\mathbf{p}(\phi_0)$
$\mathbf{p}(\phi_{-1})$
$\mathbf{p}(\phi_{-2})$
$\dot{\mathbf{p}}(\phi_1)$
$1/a=0.5$
$b=1.0$
$a=2.0$
$\mathbf{r}\bullet Q\bullet\mathbf{r}$-ellipse
$r_x^2/a^2+r_y^2/b^2=1$
$(a=2.0,\ b==1.0)$
$\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}$-ellipse
$a^2p_x^2+b^2p_y^2=1$
$(a=2.0,\ b==1.0)$
based on Fig. 11.7 in Unit 1

Introduction to Lagrangian-Hamiltonian duality

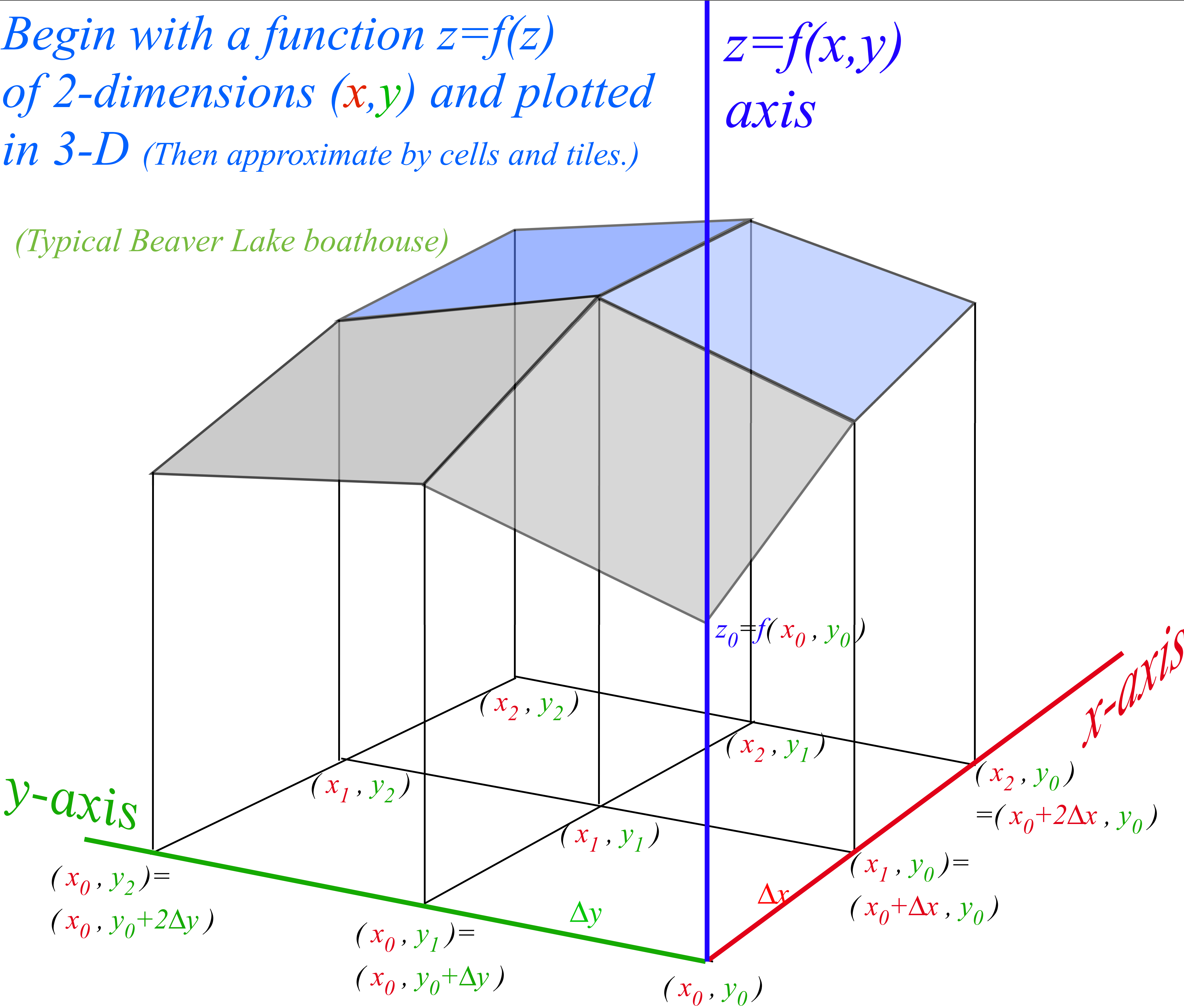
→ *Review of partial differential relations*

Chain rule and order symmetry

Duality relations of Lagrangian and Hamiltonian ellipse

Introducing the 1st (partial $\frac{\partial ?}{\partial ?}$) differential equations of mechanics

Begin with a function z=f(z) of 2-dimensions (x,y) and plotted in 3-D *(Then approximate by cells and tiles.)*



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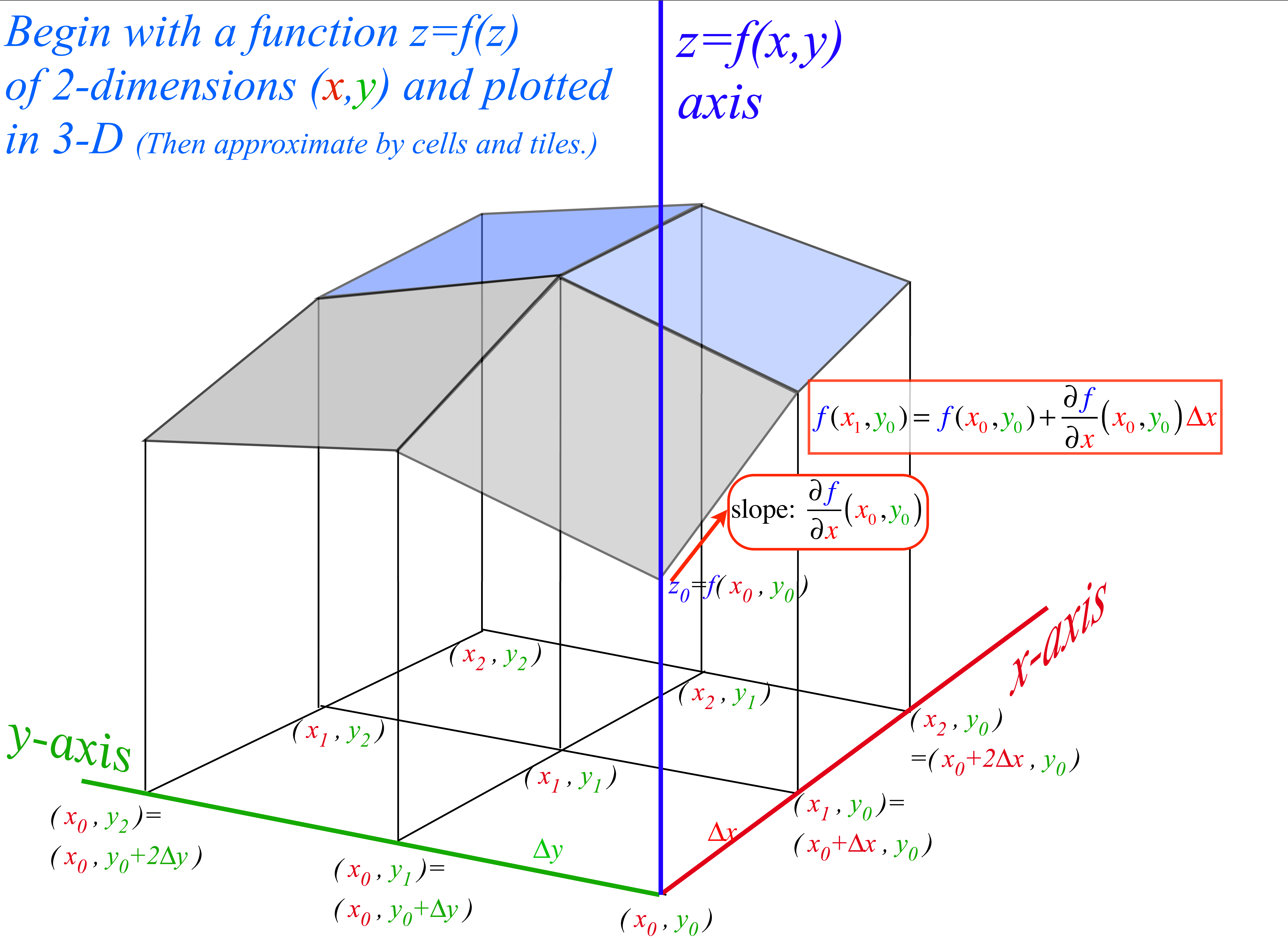

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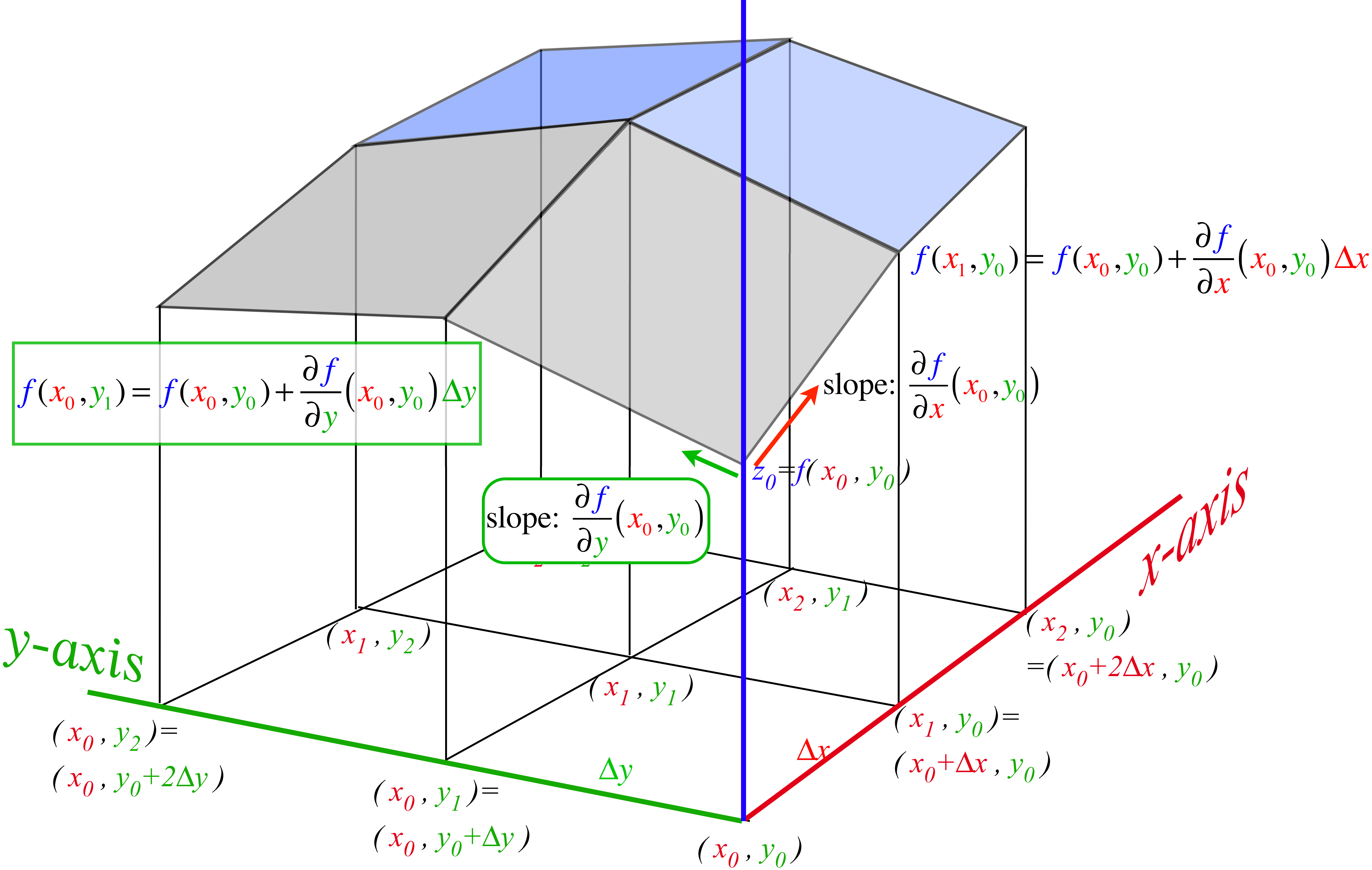

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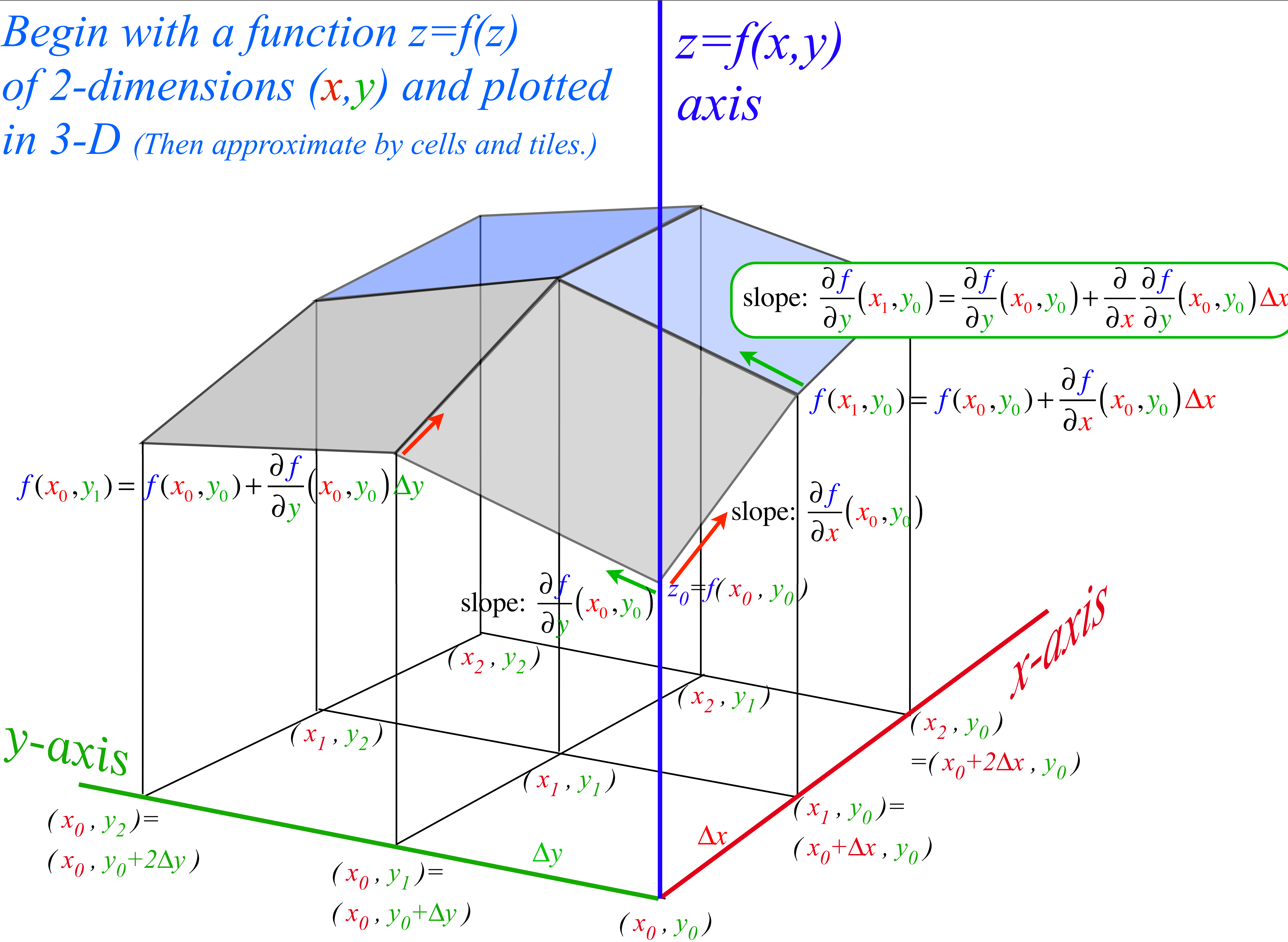

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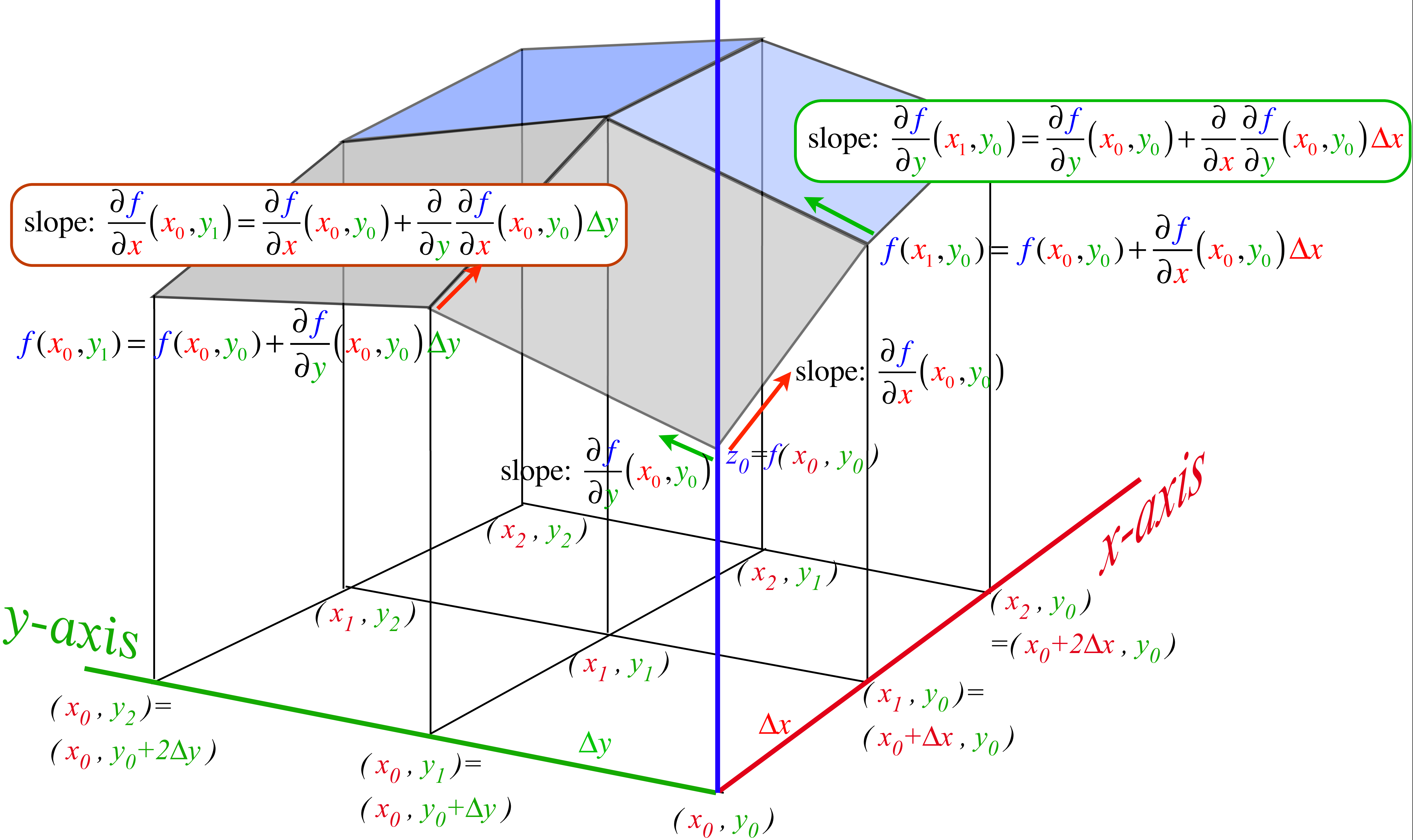
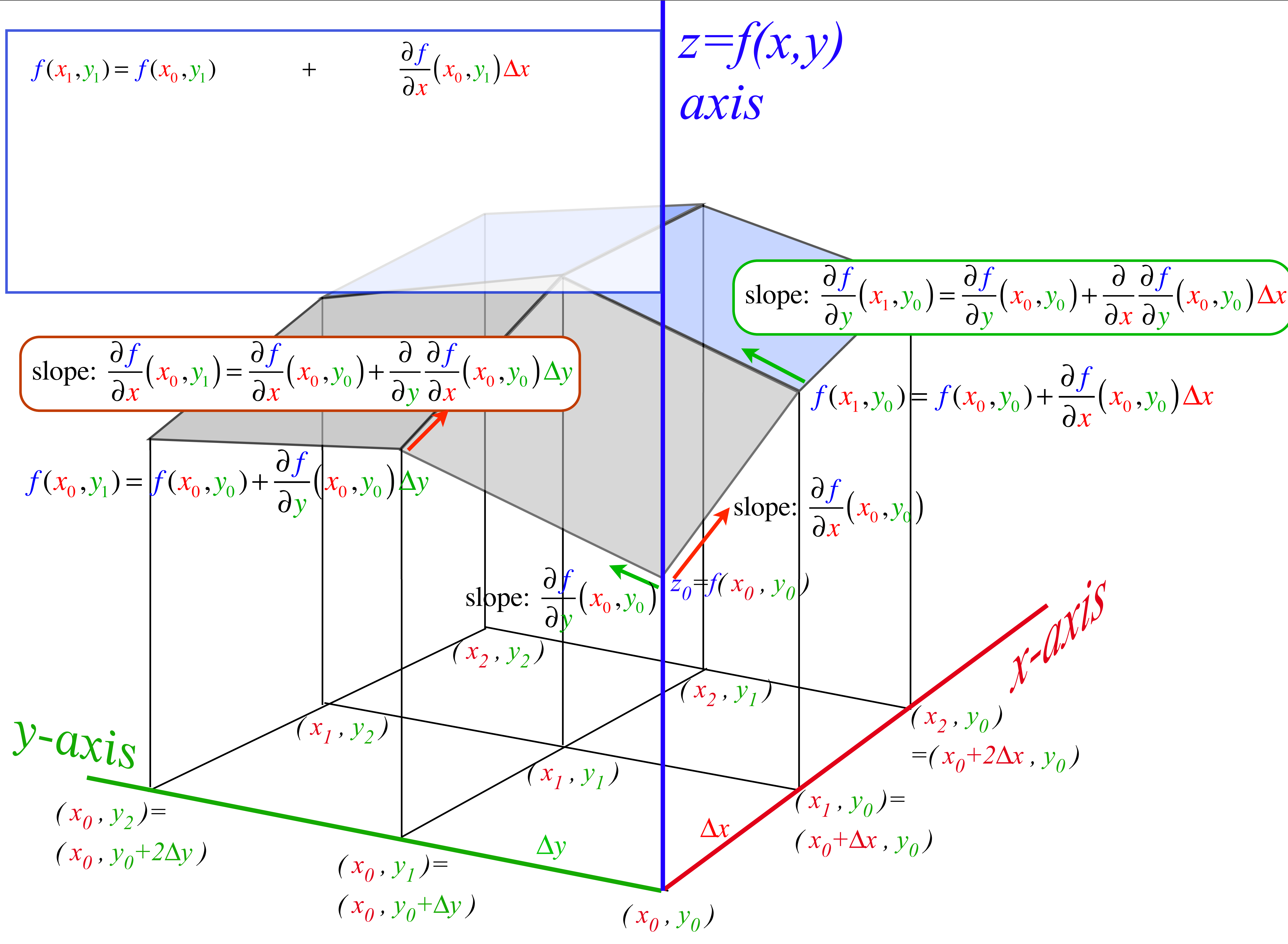
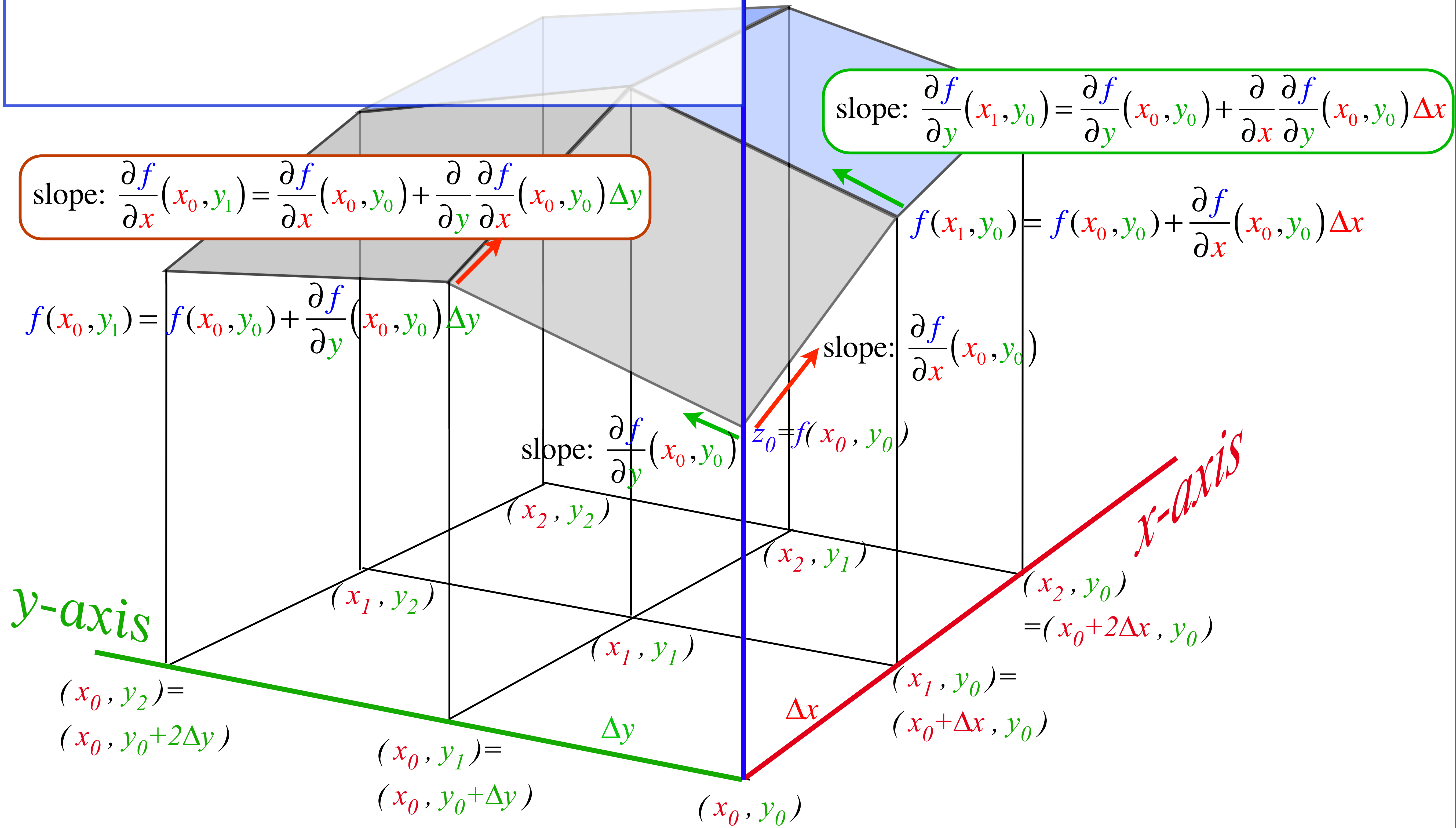




$f(x_1,y_1)=f(x_0,y_1) + \frac{\partial f}{\partial x}(x_0,y_1)\Delta x$
z=f(x,y)
axis
slope: $\frac{\partial f}{\partial y}(x_1,y_0)=\frac{\partial f}{\partial y}(x_0,y_0)+\frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x_0,y_0)\Delta x$
slope: $\frac{\partial f}{\partial x}(x_0,y_1)=\frac{\partial f}{\partial x}(x_0,y_0)+\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta y$
$f(x_1,y_0)=f(x_0,y_0)+\frac{\partial f}{\partial x}(x_0,y_0)\Delta x$
$f(x_0,y_1)=f(x_0,y_0)+\frac{\partial f}{\partial y}(x_0,y_0)\Delta y$
slope: $\frac{\partial f}{\partial x}(x_0,y_0)$
slope: $\frac{\partial f}{\partial y}(x_0,y_0)$
$z_0=f(x_0,y_0)$
x-axis
y-axis
(x_2,y_2)
(x_2,y_1)
(x_1,y_2)
(x_1,y_1)
(x_2,y_0)
$=(x_0+2\Delta x,y_0)$
$(x_1,y_0)=$
$(x_0+\Delta x,y_0)$
Δx
Δy
$(x_0,y_2)=$
$(x_0,y_0+2\Delta y)$
$(x_0,y_1)=$
$(x_0,y_0+\Delta y)$
(x_0,y_0)

$$f(x_1,y_1) = f(x_0,y_1) + \frac{\partial f}{\partial x}(x_0,y_1)\Delta x$$

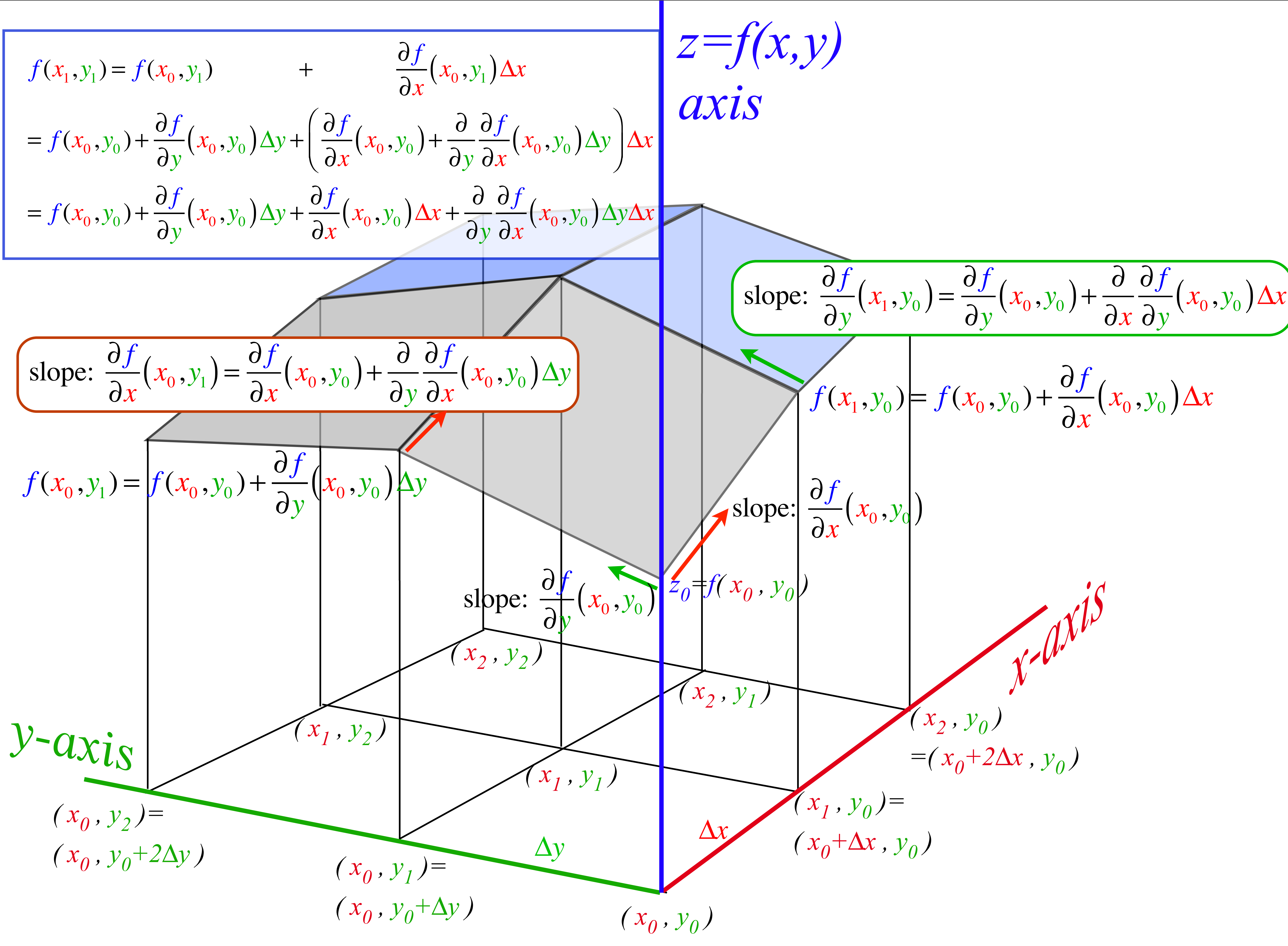
$$= f(x_0,y_0) + \frac{\partial f}{\partial y}(x_0,y_0)\Delta y + \left(\frac{\partial f}{\partial x}(x_0,y_0) + \frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta y\right)\Delta x$$



$$f(x_1,y_1)=f(x_0,y_1) + \frac{\partial f}{\partial x}(x_0,y_1)\Delta x$$

$$= f(x_0,y_0)+\frac{\partial f}{\partial y}(x_0,y_0)\Delta y+\left(\frac{\partial f}{\partial x}(x_0,y_0)+\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta y\right)\Delta x$$

$$= f(x_0,y_0)+\frac{\partial f}{\partial y}(x_0,y_0)\Delta y+\frac{\partial f}{\partial x}(x_0,y_0)\Delta x+\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta y\Delta x$$



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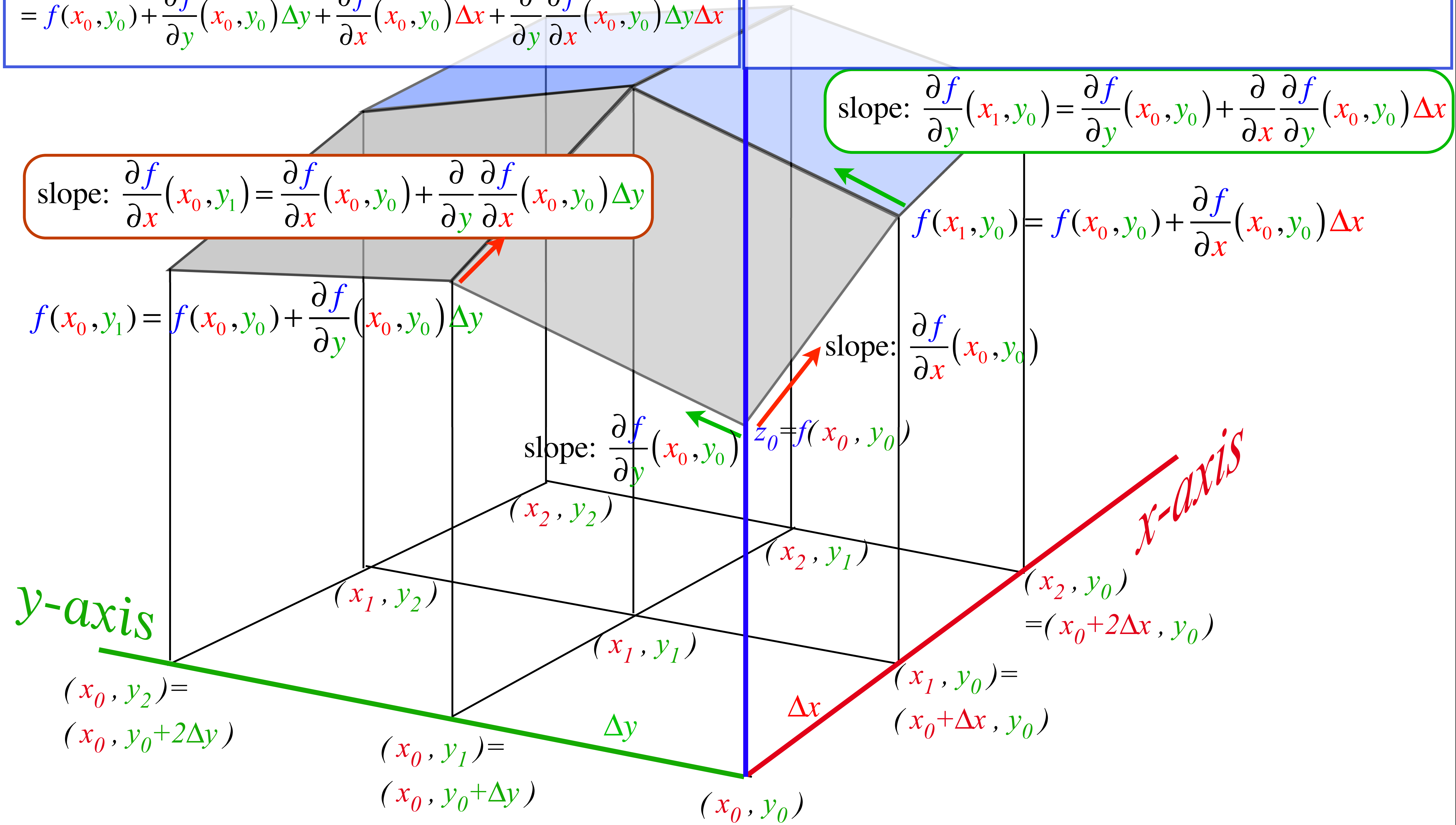
$$=f(x_0,y_0)+\frac{\partial f}{\partial y}(x_0,y_0)\Delta y+\left(\frac{\partial f}{\partial x}(x_0,y_0)+\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta y\right)\Delta x$$

$$=f(x_0,y_0)+\frac{\partial f}{\partial y}(x_0,y_0)\Delta y+\frac{\partial f}{\partial x}(x_0,y_0)\Delta x+\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta y\Delta x$$

$$f(x_1,y_1)=f(x_1,y_0)+\frac{\partial f}{\partial y}(x_1,y_0)\Delta y$$



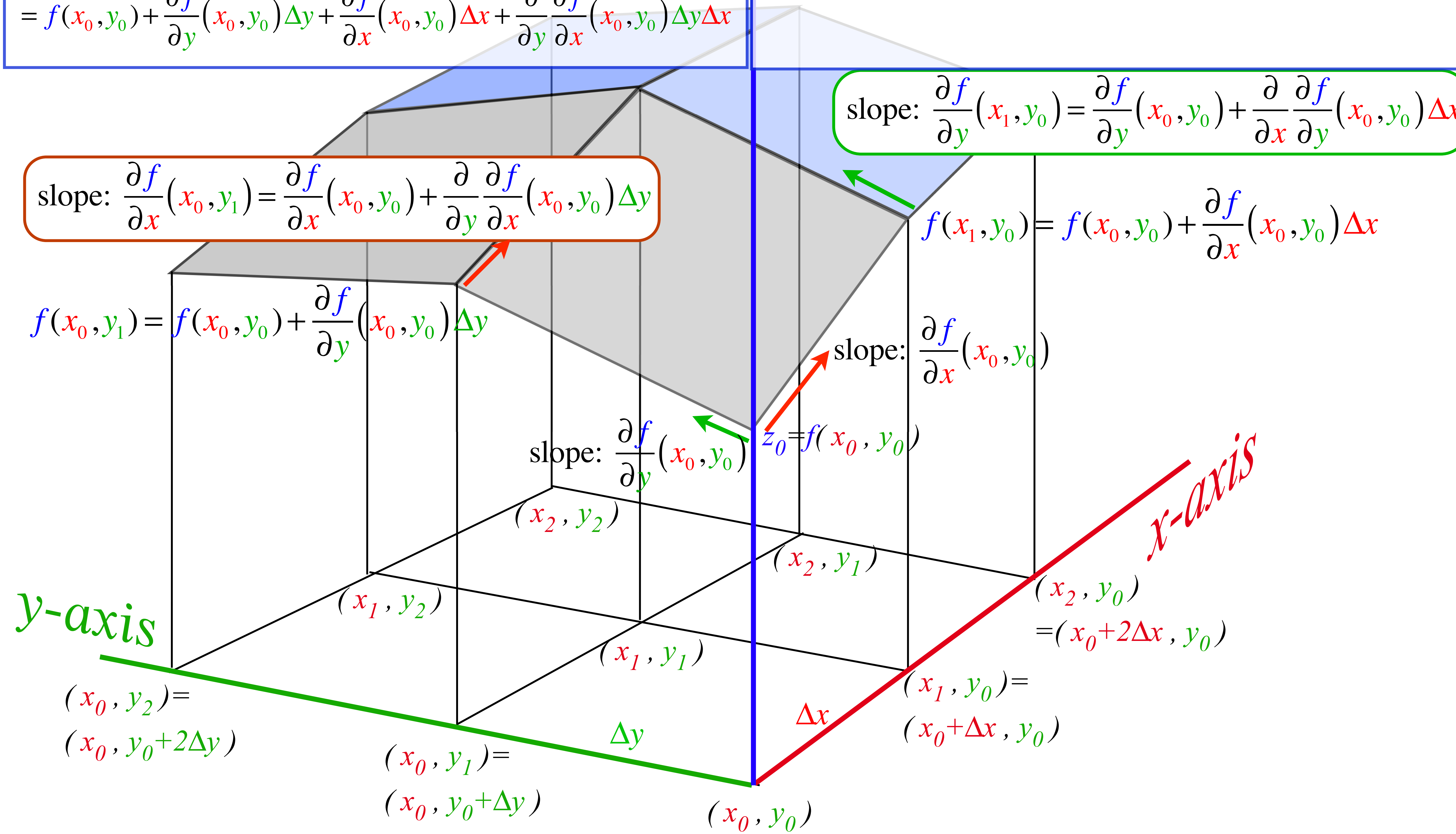
$$f(x_1,y_1) = f(x_0,y_1) + \frac{\partial f}{\partial x}(x_0,y_1)\Delta x$$

$$= f(x_0,y_0) + \frac{\partial f}{\partial y}(x_0,y_0)\Delta y + \left(\frac{\partial f}{\partial x}(x_0,y_0) + \frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta y\right)\Delta x$$

$$= f(x_0,y_0) + \frac{\partial f}{\partial y}(x_0,y_0)\Delta y + \frac{\partial f}{\partial x}(x_0,y_0)\Delta x + \frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta y\Delta x$$

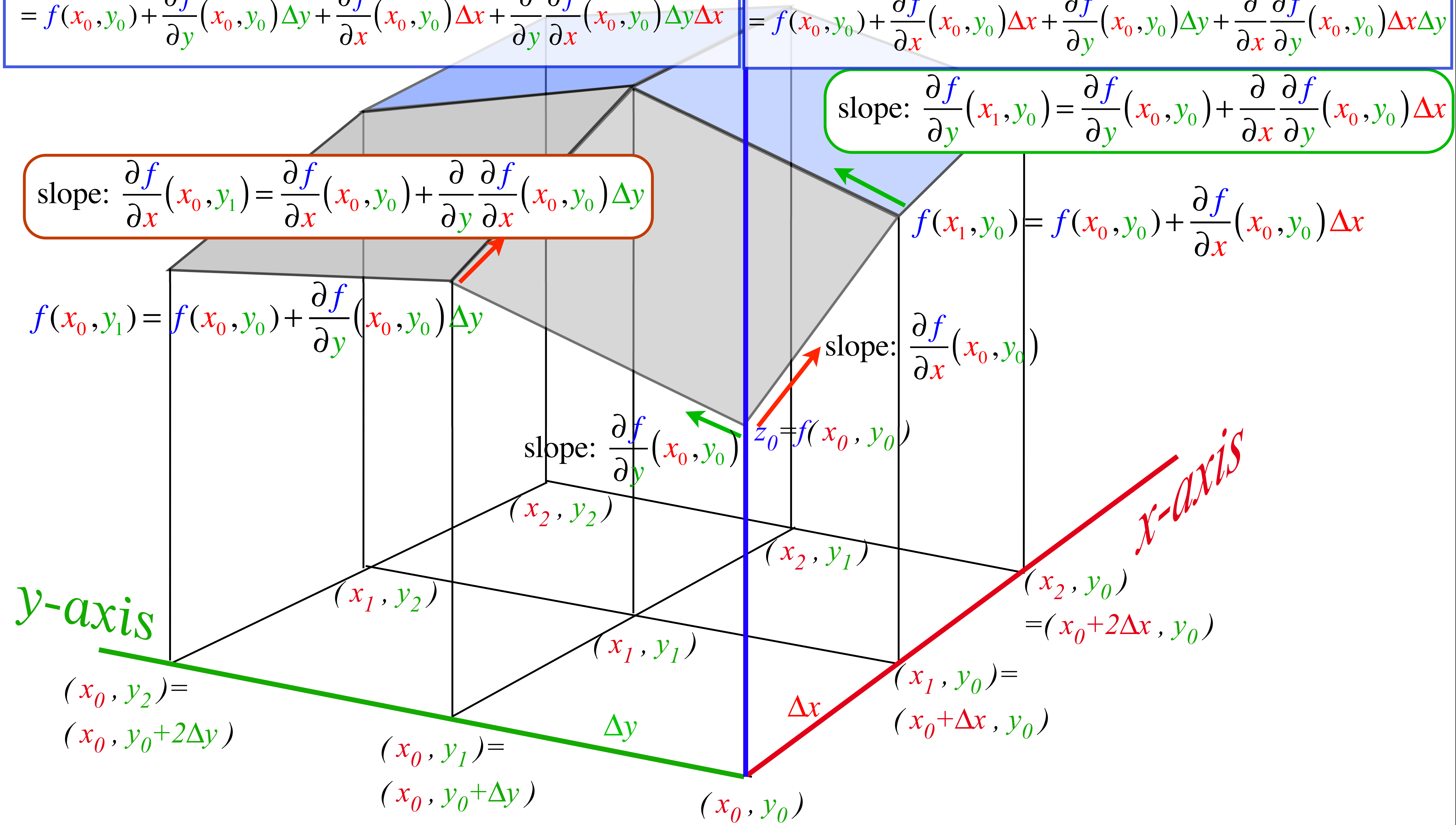
$$f(x_1,y_1) = f(x_1,y_0) + \frac{\partial f}{\partial y}(x_1,y_0)\Delta y$$

$$= f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)\Delta x + \left(\frac{\partial f}{\partial y}(x_0,y_0) + \frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x_0,y_0)\Delta x\right)\Delta y$$



$$f(x_1,y_1)=f(x_0,y_1)+\frac{\partial f}{\partial x}(x_0,y_1)\Delta x$$
$$=f(x_0,y_0)+\frac{\partial f}{\partial y}(x_0,y_0)\Delta y+\left(\frac{\partial f}{\partial x}(x_0,y_0)+\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta y\right)\Delta x$$
$$=f(x_0,y_0)+\frac{\partial f}{\partial y}(x_0,y_0)\Delta y+\frac{\partial f}{\partial x}(x_0,y_0)\Delta x+\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta y\Delta x$$

$$f(x_1,y_1)=f(x_1,y_0)+\frac{\partial f}{\partial y}(x_1,y_0)\Delta y$$
$$=f(x_0,y_0)+\frac{\partial f}{\partial x}(x_0,y_0)\Delta x+\left(\frac{\partial f}{\partial y}(x_0,y_0)+\frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x_0,y_0)\Delta x\right)\Delta y$$
$$=f(x_0,y_0)+\frac{\partial f}{\partial x}(x_0,y_0)\Delta x+\frac{\partial f}{\partial y}(x_0,y_0)\Delta y+\frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x_0,y_0)\Delta x\Delta y$$



Introduction to Lagrangian-Hamiltonian duality

Review of partial differential relations

→ *Chain rule and order symmetry*

Duality relations of Lagrangian and Hamiltonian ellipse

Introducing the 1st (partial $\frac{\partial ?}{\partial ?}$ *) differential equations of mechanics*

What the geometry indicates....(Two important results)

$$f(x_1,y_1) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)\Delta x + \frac{\partial f}{\partial y}(x_0,y_0)\Delta y + \frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta x \Delta y$$

$$= f(x_0,y_0) + \frac{\partial f}{\partial y}(x_0,y_0)\Delta y + \frac{\partial f}{\partial x}(x_0,y_0)\Delta x + \frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x_0,y_0)\Delta y \Delta x$$

What the geometry indicates....(Two important results)

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1. Chain rules

$$\left[f(x_1,y_1)-f(x_0,y_0)\right]=df=\frac{\partial f}{\partial x}(x_0,y_0)dx+\frac{\partial f}{\partial y}(x_0,y_0)dy\ldots_{(keep\ 1^{st}-order\ terms\ only!)}$$

$$\frac{df}{dt}=\frac{\partial f}{\partial x}(x_0,y_0)\frac{dx}{dt}+\frac{\partial f}{\partial y}(x_0,y_0)\frac{dy}{dt}$$

$$\dot{f}=\frac{\partial f}{\partial x}\dot{x}+\frac{\partial f}{\partial y}\dot{y}\qquad(shorthand\ notation)$$

What the geometry indicates....(Two important results)

$$f(x_1,y_1) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)\Delta x + \frac{\partial f}{\partial y}(x_0,y_0)\Delta y + \frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta x\Delta y$$

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1. Chain rules

$$[f(x_1,y_1) - f(x_0,y_0)] = df = \frac{\partial f}{\partial x}(x_0,y_0)dx + \frac{\partial f}{\partial y}(x_0,y_0)dy \ldots_{(keep\ 1^{st}-order\ terms\ only!)}$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}(x_0,y_0)\frac{dx}{dt} + \frac{\partial f}{\partial y}(x_0,y_0)\frac{dy}{dt}$$

$$\dot{f} = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} \quad (shorthand\ notation) \quad = \partial_x f\,\dot{x} + \partial_y f\,\dot{y}$$

2. Symmetry of partial deriv. ordering

(pay attention to the 2^{nd} −order terms, too!)

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} \quad \text{or:} \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} \quad \text{or:} \quad \partial_y\partial_x f = \partial_x\partial_y f$$

(shorthand notation)

What the geometry indicates....(Two important results)

$$f(x_1,y_1)=f(x_0,y_0)+\frac{\partial f}{\partial x}(x_0,y_0)\Delta x+\frac{\partial f}{\partial y}(x_0,y_0)\Delta y+\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta x\Delta y$$
$$=f(x_0,y_0)+\frac{\partial f}{\partial y}(x_0,y_0)\Delta y+\frac{\partial f}{\partial x}(x_0,y_0)\Delta x+\frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x_0,y_0)\Delta y\Delta x$$

1. Chain rules

$$[f(x_1,y_1)-f(x_0,y_0)]=df=\frac{\partial f}{\partial x}(x_0,y_0)dx+\frac{\partial f}{\partial y}(x_0,y_0)dy\ldots_{(keep\ 1^{st}-order\ terms\ only!)}$$
$$\frac{df}{dt}=\frac{\partial f}{\partial x}(x_0,y_0)\frac{dx}{dt}+\frac{\partial f}{\partial y}(x_0,y_0)\frac{dy}{dt}$$
$$\dot f=\frac{\partial f}{\partial x}\dot x+\frac{\partial f}{\partial y}\dot y \quad (shorthand\ notation) \quad =\partial_x f\,\dot x+\partial_y f\,\dot y$$

2. Symmetry of partial deriv. ordering

(*pay attention to the* 2^{nd} *-order terms, too!*)

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\frac{\partial f}{\partial y} \quad \text{or:} \quad \frac{\partial^2 f}{\partial y\partial x}=\frac{\partial^2 f}{\partial x\partial y} \quad \text{or:} \quad \partial_y\partial_x f=\partial_x\partial_y f$$

(*shorthand notation*)

$$Let:\vec\nabla=\begin{pmatrix}\partial_x & \partial_y\end{pmatrix} \quad so:\vec\nabla f\bullet\mathbf{dr}=\begin{pmatrix}\partial_x f & \partial_y f\end{pmatrix}\bullet\begin{pmatrix}dx\\dy\end{pmatrix}=\partial_x f\,dx+\partial_y f\,dy=df$$

Introduction to Lagrangian-Hamiltonian duality

Review of partial differential relations

Chain rule and order symmetry

→ *Duality relations of Lagrangian and Hamiltonian ellipse*

Introducing the 1st (partial $\frac{\partial ?}{\partial ?}$) differential equations of mechanics

Three ways to express energy: Consider kinetic energy (KE) first

1. **Lagrangian** is explicit function of **velocity**: $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$$L(v_k \ldots) = \tfrac{1}{2}(m_1 v_1^2 + m_2 v_2^2 + \ldots) = L(\mathbf{v}\ldots) = \tfrac{1}{2}\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v} + \ldots = \tfrac{1}{2}\begin{pmatrix} v_1 & v_2 \end{pmatrix}\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \ldots$$

2. **"Estrangian"** is explicit function of **R**-rescaled velocity:

(or l'Estrangian

or: **"speedinum"** V $\mathbf{V} = \mathbf{R}\bullet\mathbf{v}$ or: $\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} \sqrt{m_1} & 0 \\ 0 & \sqrt{m_2} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$$E(V_k \ldots) = \tfrac{1}{2}(V_1^2 + V_2^2 + \ldots) = E(\mathbf{V}\ldots) = \tfrac{1}{2}\mathbf{V}\bullet\mathbf{1}\bullet\mathbf{V} + \ldots = \tfrac{1}{2}\begin{pmatrix} V_1 & V_2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} + \ldots$$

3. **Hamiltonian** is explicit function of **M=R**^{2}-rescaled velocity:

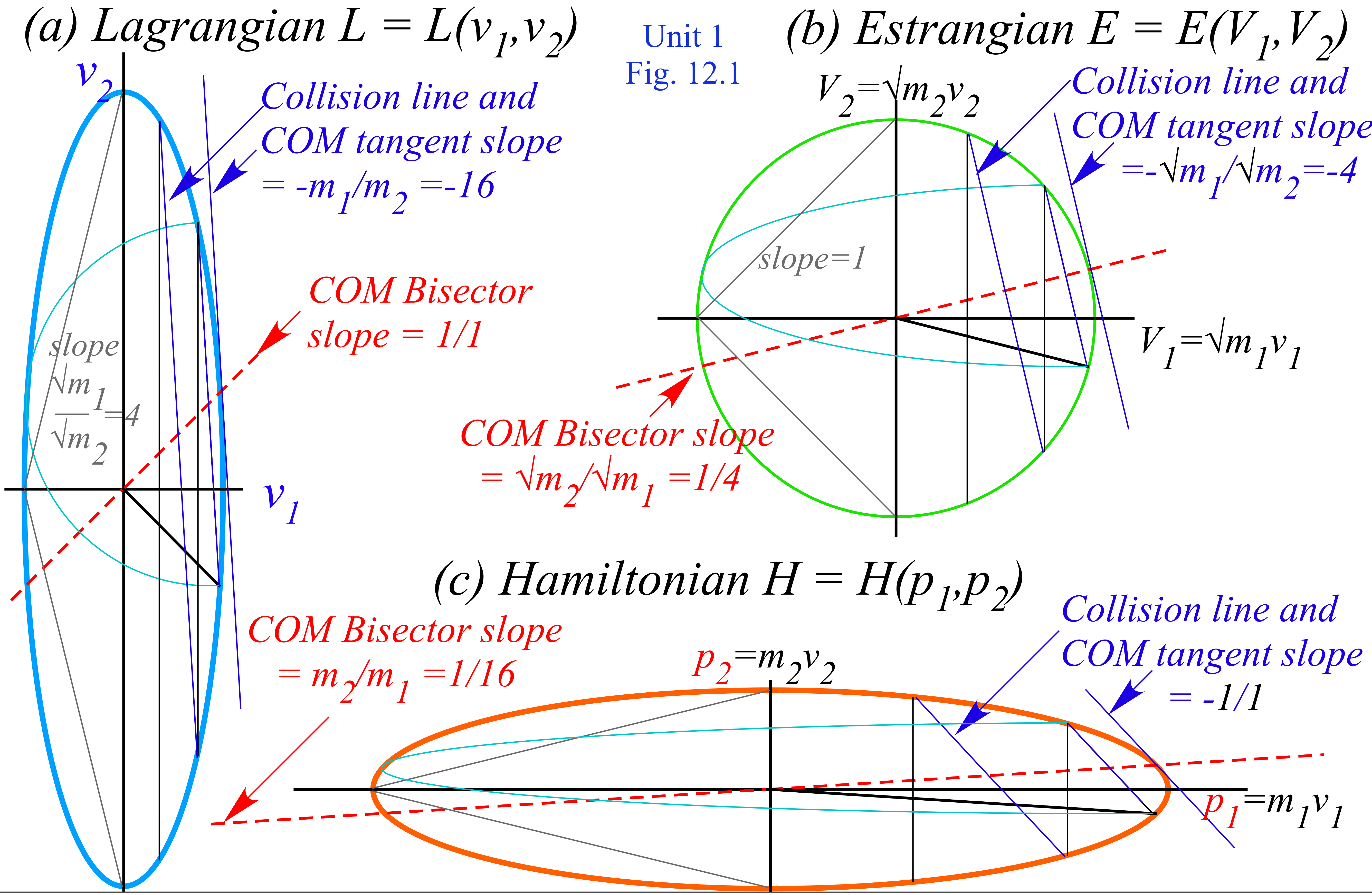
or: **momentum p** $\mathbf{p} = \mathbf{M}\bullet\mathbf{v}$ or: $\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} m_1 v_1 \\ m_2 v_2 \end{pmatrix}$

$$H(p_k \ldots) = \tfrac{1}{2}\left(\frac{p_1^2}{m_1} + \frac{p_2^2}{m_2} + \ldots\right) = H(\mathbf{p}\ldots) = \tfrac{1}{2}\mathbf{p}\bullet\mathbf{M}^{-1}\bullet\mathbf{p} + \ldots = \tfrac{1}{2}\begin{pmatrix} p_1 & p_2 \end{pmatrix}\begin{pmatrix} 1/m_1 & 0 \\ 0 & 1/m_2 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + \ldots$$

The R and Q matrix transformations are like the mechanics rescaling matrices $\sqrt{\mathbf{M}}$ *and* $\mathbf{M}$*:*

Like Q=R²: $\mathbf{M} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} = \mathbf{R}^2$ *Like √Q=R:* $\sqrt{\mathbf{M}} = \begin{pmatrix} \sqrt{m_1} & 0 \\ 0 & \sqrt{m_2} \end{pmatrix} = \mathbf{R}$ *Like* $Q^{-1}=R^{-2}$*:* $\mathbf{M}^{-1} = \begin{pmatrix} 1/m_1 & 0 \\ 0 & 1/m_2 \end{pmatrix} = \mathbf{R}^{-2}$

Unit 1
Fig. 12.1



Introduction to Lagrangian-Hamiltonian duality

Review of partial differential relations

Chain rule and order symmetry

Duality relations of Lagrangian and Hamiltonian ellipse

→ *Introducing the 1st (partial $\frac{\partial ?}{\partial ?}$) differential equations of mechanics*

Introducing the (partial $\frac{\partial ?}{\partial ?}$) differential equations of mechanics

Starts out with simple demands for explicit-dependence, "loyalty" or "fealty to the colors"

Lagrangian *and* ***Estrangian*** *have* *__no__* *explicit dependence on* ***momentum*** **p**

$$\frac{\partial L}{\partial p_k} \equiv 0 \equiv \frac{\partial E}{\partial p_k}$$

Hamiltonian *and* ***Estrangian*** *have* *__no__* *explicit dependence on* ***velocity*** **v**

$$\frac{\partial H}{\partial v_k} \equiv 0 \equiv \frac{\partial E}{\partial v_k}$$

Lagrangian *and* ***Hamiltonian*** *have* *__no__* *explicit dependence on* ***speedinum*** **V**

$$\frac{\partial L}{\partial V_k} \equiv 0 \equiv \frac{\partial H}{\partial V_k}$$

Introducing the (partial$\frac{\partial ?}{\partial ?}$) differential equations of mechanics

Starts out with simple demands for explicit-dependence, "loyalty" or "fealty to the colors"

Lagrangian *and* ***Estrangian*** *have no explicit dependence on* ***momentum p***

$$\frac{\partial L}{\partial p_k} \equiv 0 \equiv \frac{\partial E}{\partial p_k}$$

Hamiltonian *and* ***Estrangian*** *have no explicit dependence on* ***velocity v***

$$\frac{\partial H}{\partial v_k} \equiv 0 \equiv \frac{\partial E}{\partial v_k}$$

Lagrangian *and* ***Hamiltonian*** *have no explicit dependence on* ***speedinum V***

$$\frac{\partial L}{\partial V_k} \equiv 0 \equiv \frac{\partial H}{\partial V_k}$$

Such non-dependencies hold in spite of "under-the-table" matrix and partial-differential connections

$$\nabla_v L = \frac{\partial L}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \frac{\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}}{2} = \mathbf{M}\bullet\mathbf{v} = \mathbf{p}$$

$$\begin{pmatrix} \frac{\partial L}{\partial v_1} \\ \frac{\partial L}{\partial v_2} \end{pmatrix} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

$$\nabla_p H = \mathbf{v} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\partial}{\partial \mathbf{p}} \frac{\mathbf{p}\bullet\mathbf{M}^{-1}\bullet\mathbf{p}}{2} = \mathbf{M}^{-1}\bullet\mathbf{p} = \mathbf{v}$$

$$\begin{pmatrix} \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \end{pmatrix} = \begin{pmatrix} m_1^{-1} & 0 \\ 0 & m_2^{-1} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Introducing the (partial $\frac{\partial ?}{\partial ?}$) differential equations of mechanics

Starts out with simple demands for explicit-dependence, "loyalty" or "fealty to the colors"

Lagrangian *and* ***Estrangian*** *have no explicit dependence on* ***momentum* p**

$$\frac{\partial L}{\partial p_k} \equiv 0 \equiv \frac{\partial E}{\partial p_k}$$

Hamiltonian *and* ***Estrangian*** *have no explicit dependence on* ***velocity* v**

$$\frac{\partial H}{\partial v_k} \equiv 0 \equiv \frac{\partial E}{\partial v_k}$$

Lagrangian *and* ***Hamiltonian*** *have no explicit dependence on* ***speedinum* V**

$$\frac{\partial L}{\partial V_k} \equiv 0 \equiv \frac{\partial H}{\partial V_k}$$

Such non-dependencies hold in spite of "under-the-table" matrix and partial-differential connections

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$$\begin{pmatrix} \frac{\partial L}{\partial v_1} \\ \frac{\partial L}{\partial v_2} \end{pmatrix} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

Lagrange's 1st equation(s)

$$\frac{\partial L}{\partial v_k} = p_k \quad \text{or:} \quad \frac{\partial L}{\partial \mathbf{v}} = \mathbf{p}$$

$$\nabla_p H = \mathbf{v} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\partial}{\partial \mathbf{p}} \frac{\mathbf{p} \bullet \mathbf{M}^{-1} \bullet \mathbf{p}}{2} = \mathbf{M}^{-1} \bullet \mathbf{p} = \mathbf{v}$$

$$\begin{pmatrix} \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \end{pmatrix} = \begin{pmatrix} m_1^{-1} & 0 \\ 0 & m_2^{-1} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Hamilton's 1st equation(s)

$$\frac{\partial H}{\partial p_k} = v_k \quad \text{or:} \quad \frac{\partial H}{\partial \mathbf{p}} = \mathbf{v}$$

(Forget Estrangian for now)

Unit 1
Fig. 12.2