Lecture 28.

Relativistic geometry of 2-and-3-dimensional waves

(Ch. 4-5 of Unit 2   4.09.12)

Plane wave 4-vector ($\omega_0,\omega_x,\omega_y,\omega_z$) and space-time ($x_0,x_1, x_2, x_3$) geometry

Pattern recognition: "Occam's Sword"

Reviewing the stellar aberration angle σ vs. rapidity ρ geometry

Reviewing "Sin-Tan Rosetta" geometry

Lorentz boost of North-South-East-West plane-wave 4-vectors ($\omega_0,\omega_x,\omega_y,\omega_z$)

Thales-like construction of Lorentz boost in 2D and 3D

The spectral ellipsoid

Combination and interference of 4-vector plane waves (Idealized polarization case)

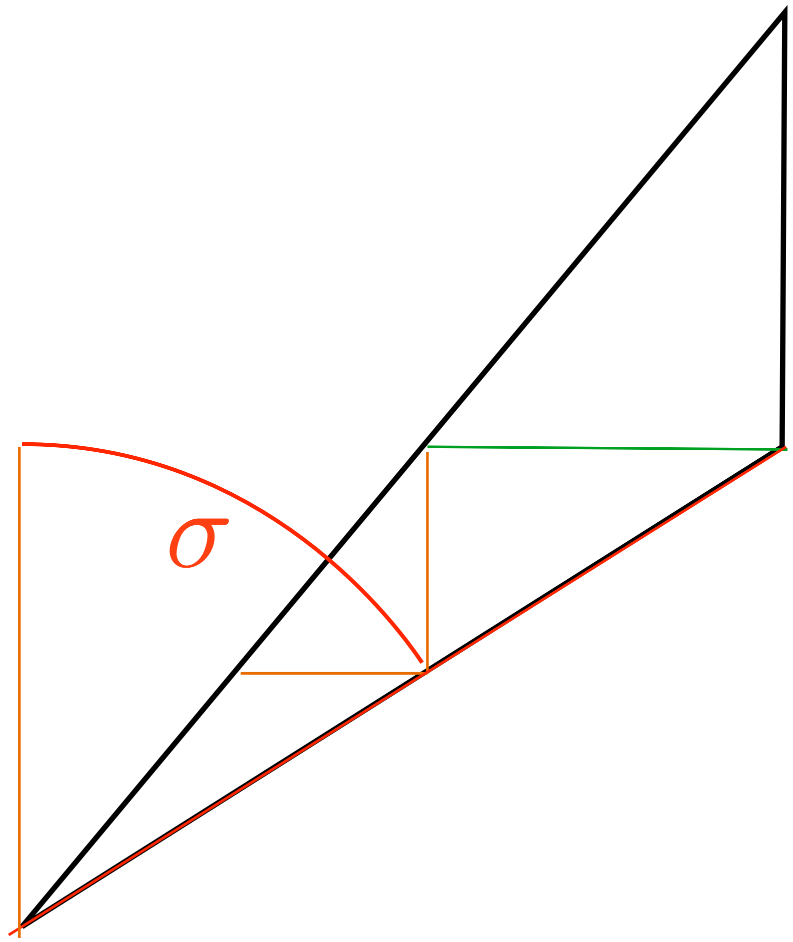
Combination group and phase waves define 4D Minkowski coordinates

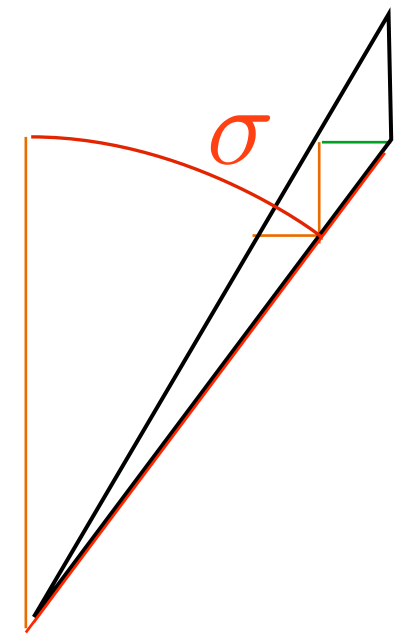
Combination group and phase waves define wave guide dynamics

Waveguide dispersion and geometry

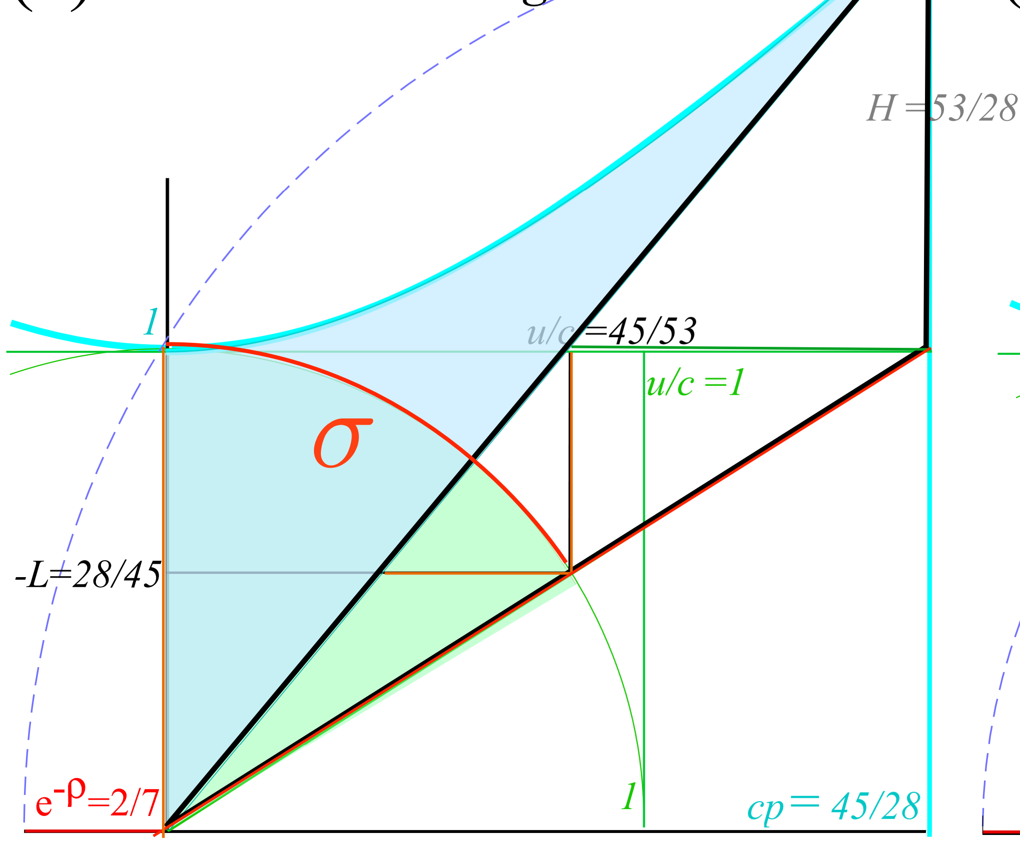
1st-quantized cavity modes

Pattern recognition: "Occam's Sword"





(c) *Basic construction given u/c= 45/53*



(d) *u/c=3/5*

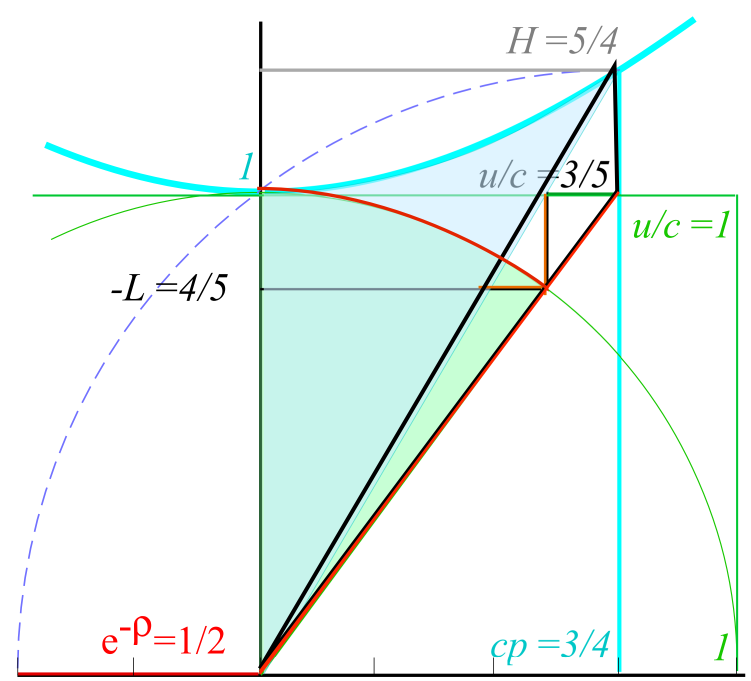
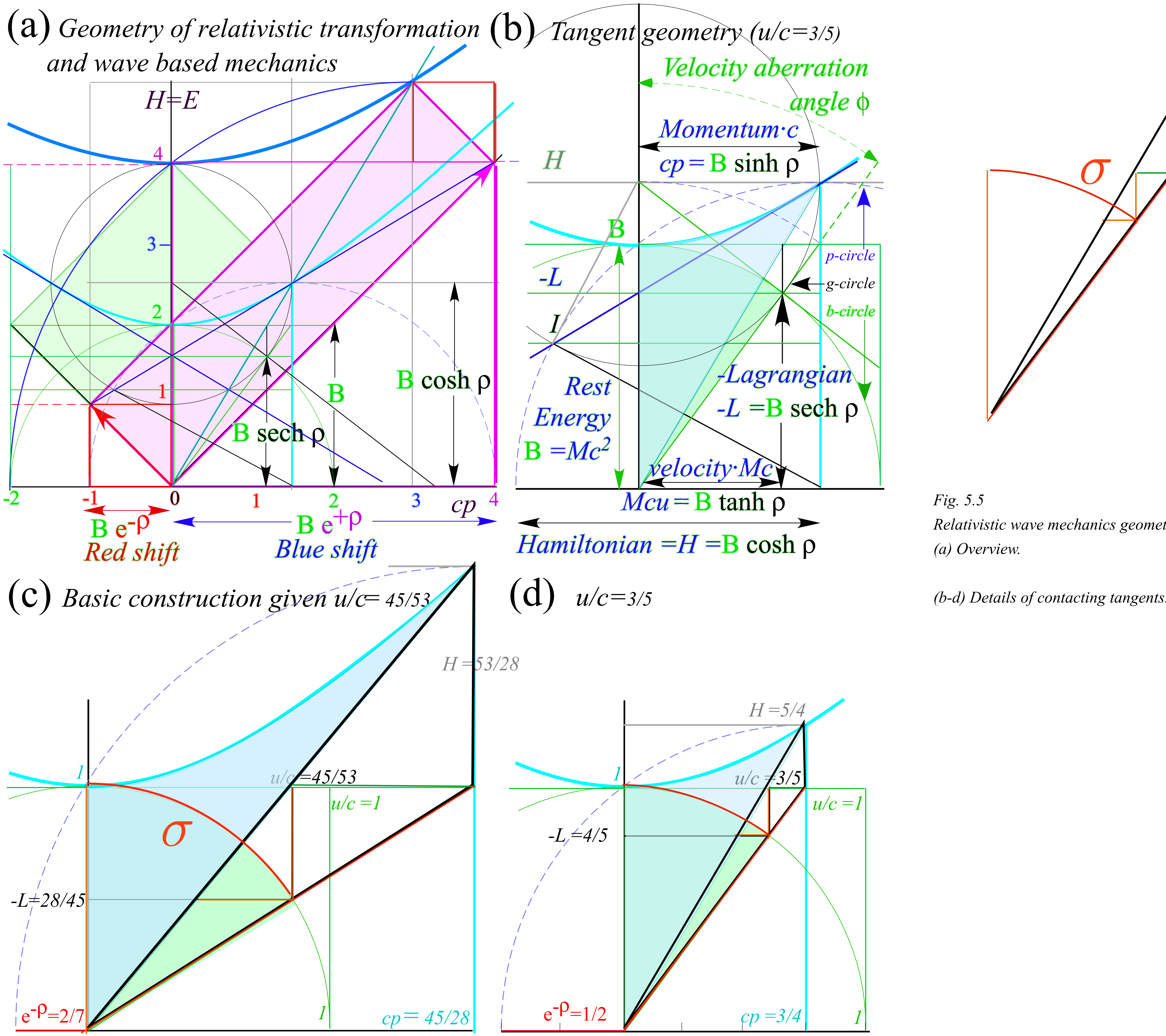


Fig. 5.5
Relativistic wave mechanics geometry.
(a) Overview.

(b-d) Details of contacting tangents.



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(a) Overview.

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Pattern recognition: "Occam's Sword"

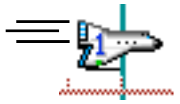
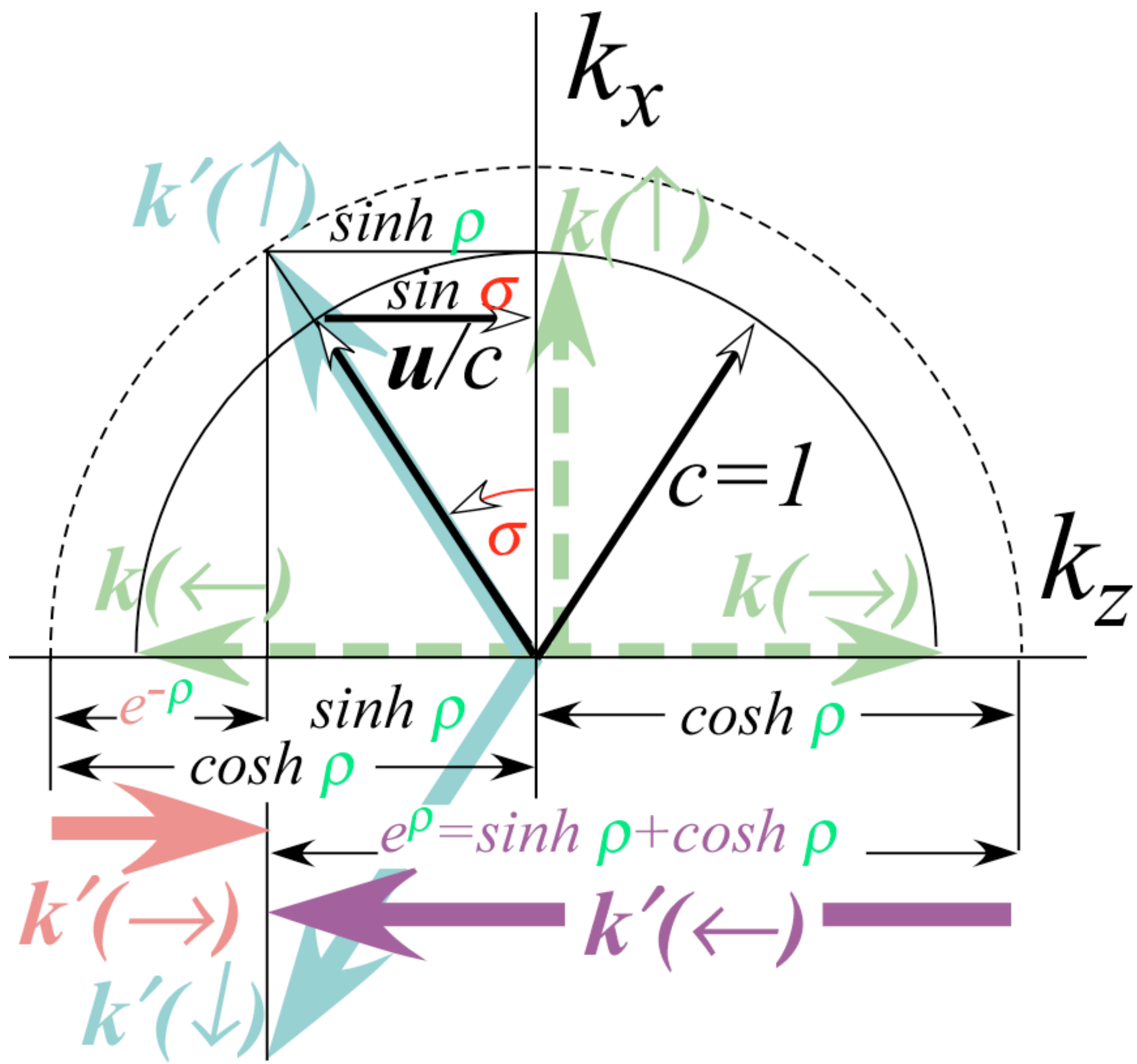


Fig. 5.10 CW cosmic speedometer.

Geometry of Lorentz boost of counter-propagating waves.



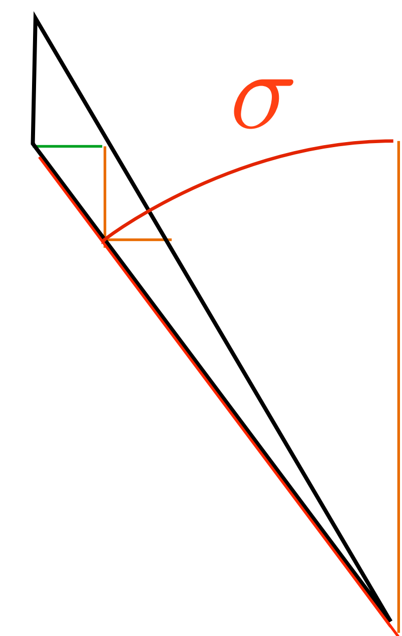
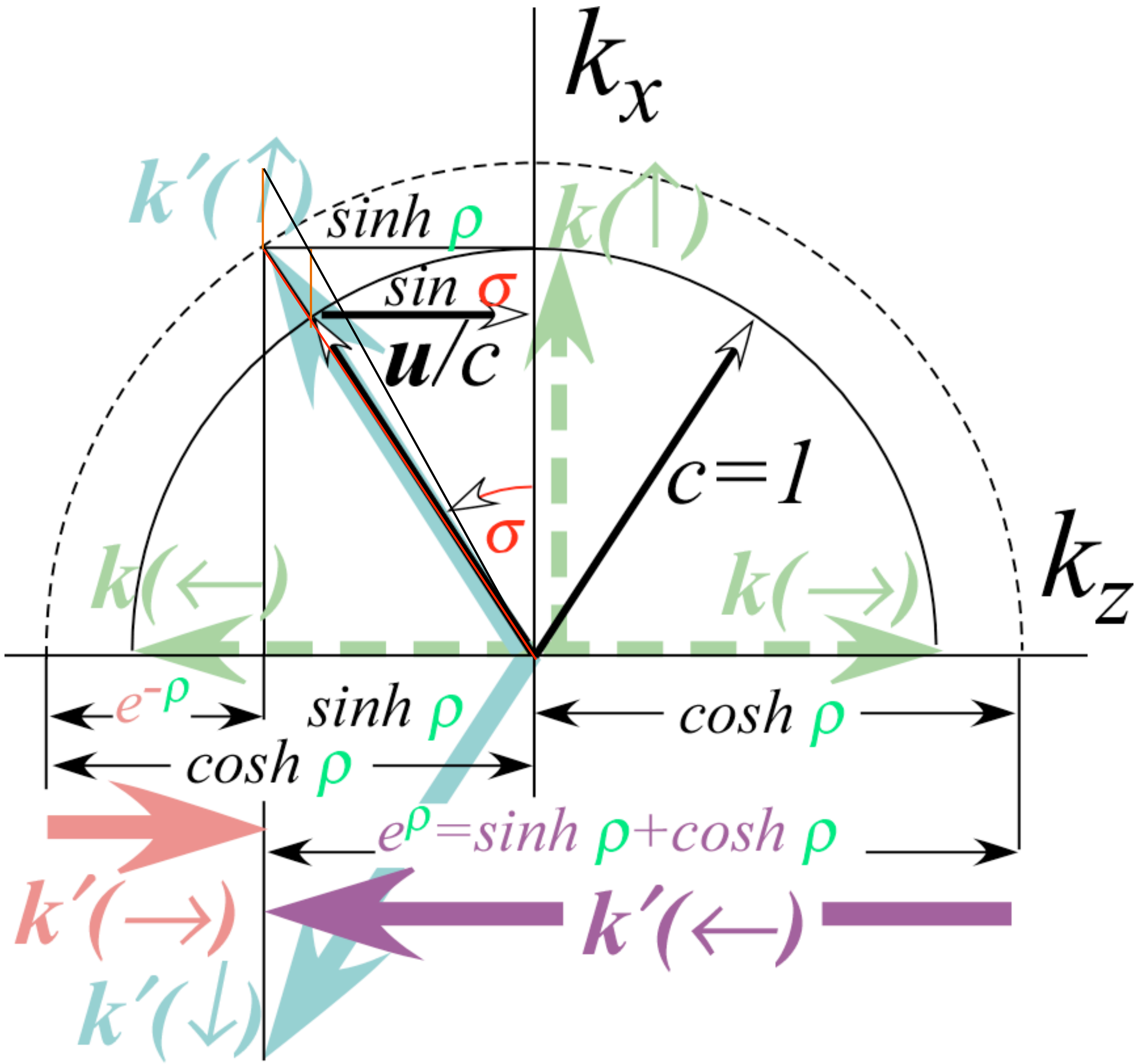
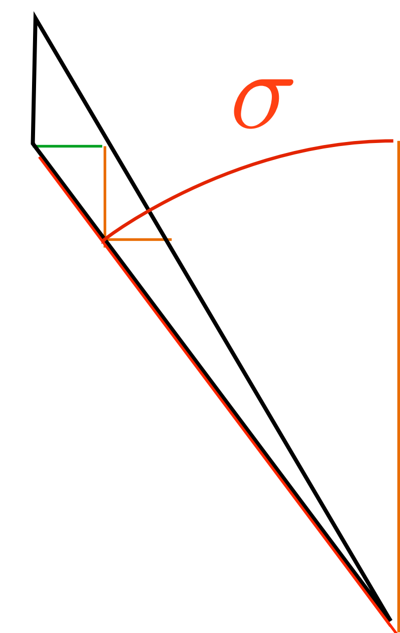


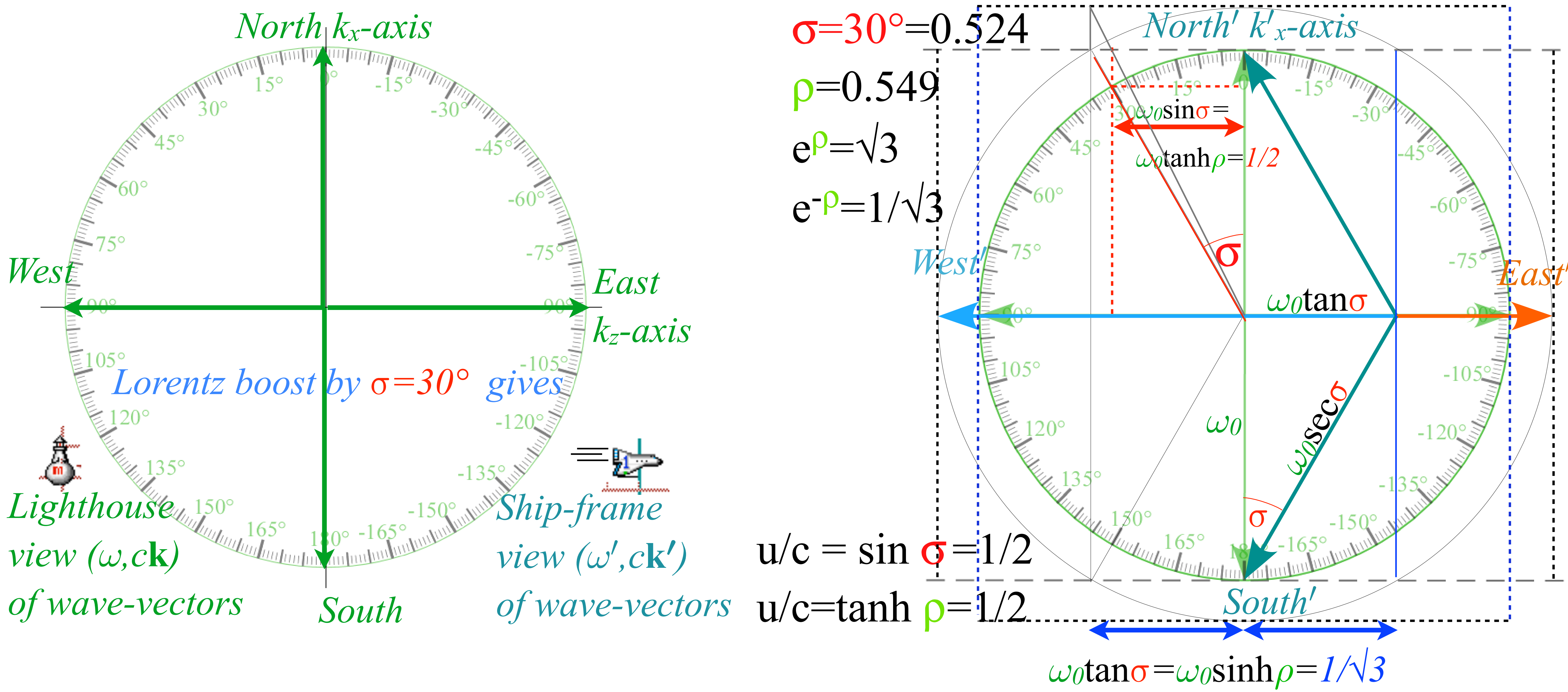
Fig. 5.10 CW cosmic speedometer.

Geometry of Lorentz boost of counter-propagating waves.





Lecture 27 discusses Lorentz boost of North-South-East-West plane-wave 4-vectors $(\omega_0,\omega_x,\omega_y,\omega_z)$

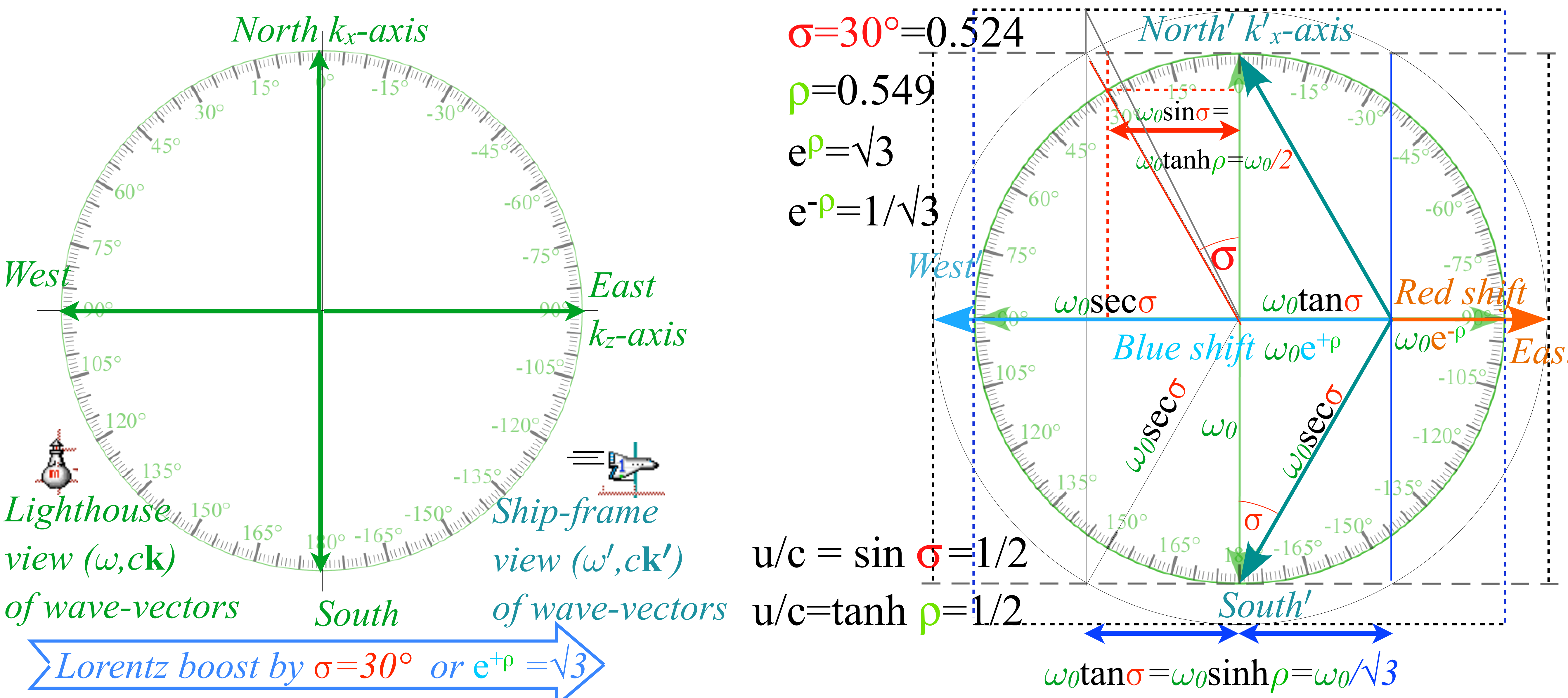


South starlight in lighthouse frame is straight down x-axis : $\left(\omega_{\downarrow},ck_{x\downarrow},ck_{y\downarrow},ck_{z\downarrow}\right)=\left(\omega_0,-\omega_0,0,0\right)$

$+\rho_z$-rapidity ship frame sees starlight Lorentz transformed to :$\left(\omega'_{\downarrow},ck'_{x\downarrow},ck'_{y\downarrow},ck'_{z\downarrow}\right)=\left(\omega_0\cosh\rho_z,-\omega_0,0,-\omega_0\sinh\rho_z\right)$

$$\begin{pmatrix}\omega'_{\downarrow}\\ ck'_{x\downarrow}\\ ck'_{y\downarrow}\\ ck'_{z\downarrow}\end{pmatrix}=\begin{pmatrix}\cosh\rho_z & \cdot & \cdot & -\sinh\rho_z\\ \cdot & 1 & \cdot & \cdot\\ \cdot & \cdot & 1 & \cdot\\ -\sinh\rho_z & \cdot & \cdot & \cosh\rho_z\end{pmatrix}\begin{pmatrix}\omega_{\downarrow}\\ ck_{x\downarrow}\\ ck_{y\downarrow}\\ ck_{z\downarrow}\end{pmatrix}=\begin{pmatrix}\cosh\rho_z & \cdot & \cdot & -\sinh\rho_z\\ \cdot & 1 & \cdot & \cdot\\ \cdot & \cdot & 1 & \cdot\\ -\sinh\rho_z & \cdot & \cdot & \cosh\rho_z\end{pmatrix}\begin{pmatrix}\omega_0\\ -\omega_0\\ 0\\ 0\end{pmatrix}=\begin{pmatrix}\omega_0\cosh\rho_z\\ -\omega_0\\ 0\\ -\omega_0\sinh\rho_z\end{pmatrix}=\begin{pmatrix}\omega_0\sec\sigma\\ -\omega_0\\ 0\\ -\omega_0\tan\sigma\end{pmatrix}$$

Lecture 27 discusses Lorentz boost of North-South-East-West plane-wave 4-vectors $(\omega_0,\omega_x,\omega_y,\omega_z)$



For ship going $u=c \tanh\rho$ along z-axis

West starlight $(\omega_0,0,0,-\omega_0)$ is *blue shifted by* $e^{+\rho}=\cosh\rho+\sinh\rho$

$$\begin{pmatrix} \omega'_{\leftarrow} \\ ck'_{x\leftarrow} \\ ck'_{y\leftarrow} \\ ck'_{z\leftarrow} \end{pmatrix} = \omega_0 \begin{pmatrix} \cosh\rho_z + \sinh\rho_z \\ 0 \\ 0 \\ -\sinh\rho_z - \cosh\rho_z \end{pmatrix} = \begin{pmatrix} \omega_0 e^{+\rho_z} \\ 0 \\ 0 \\ -\omega_0 e^{+\rho_z} \end{pmatrix}$$

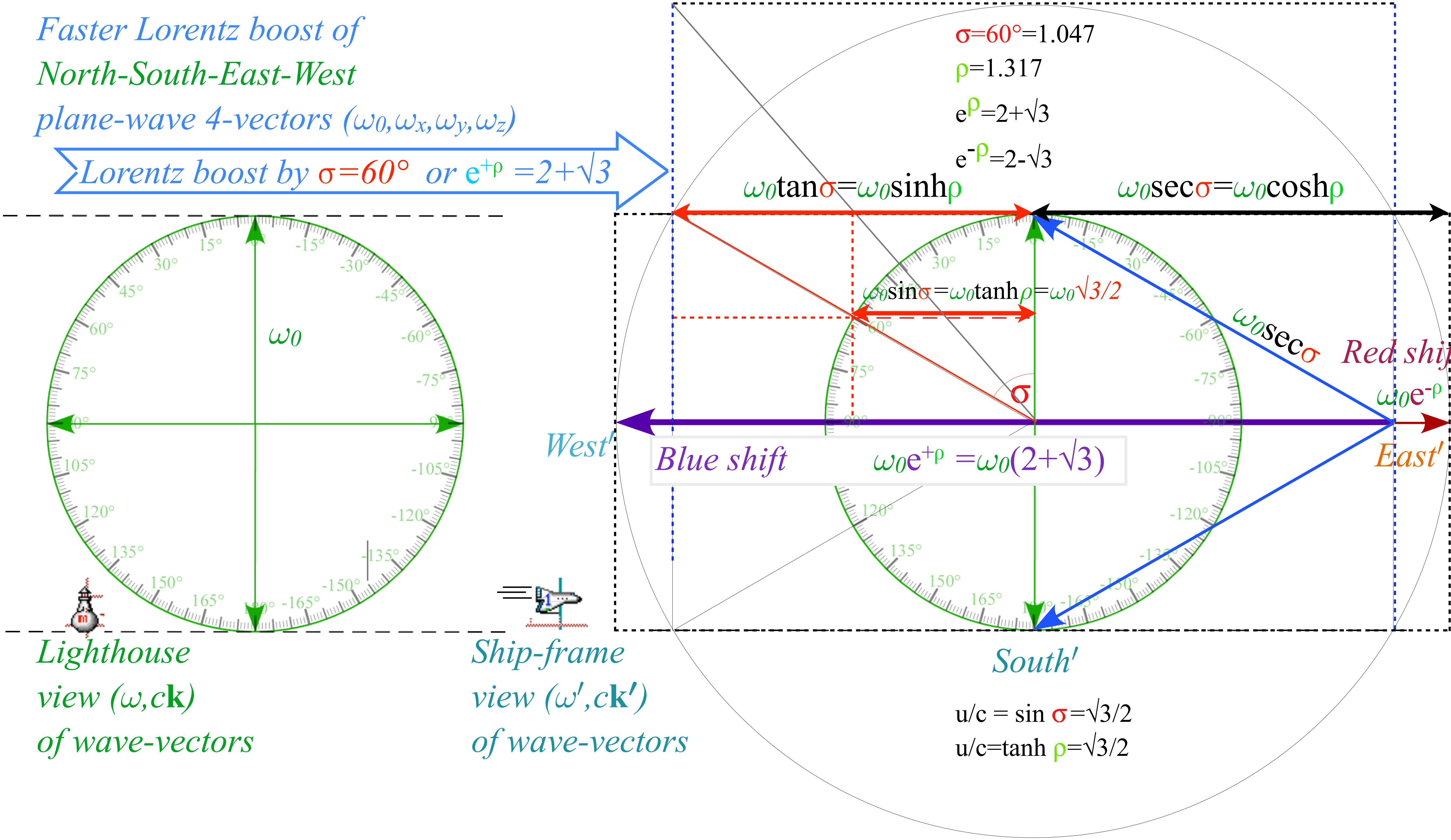
and *East* starlight $(\omega_0,0,0,+\omega_0)$ is *red shifted by* $e^{-\rho}=\cosh\rho-\sinh\rho$

$$\begin{pmatrix} \omega'_{\rightarrow} \\ ck'_{x\rightarrow} \\ ck'_{y\rightarrow} \\ ck'_{z\rightarrow} \end{pmatrix} = \omega_0 \begin{pmatrix} \cosh\rho_z - \sinh\rho_z \\ 0 \\ 0 \\ -\sinh\rho_z + \cosh\rho_z \end{pmatrix} = \begin{pmatrix} \omega_0 e^{-\rho_z} \\ 0 \\ 0 \\ -\omega_0 e^{-\rho_z} \end{pmatrix}$$

Blue shift factor is $e^{+\rho}=\cosh\rho+\sinh\rho=\sec\sigma+\tan\sigma$

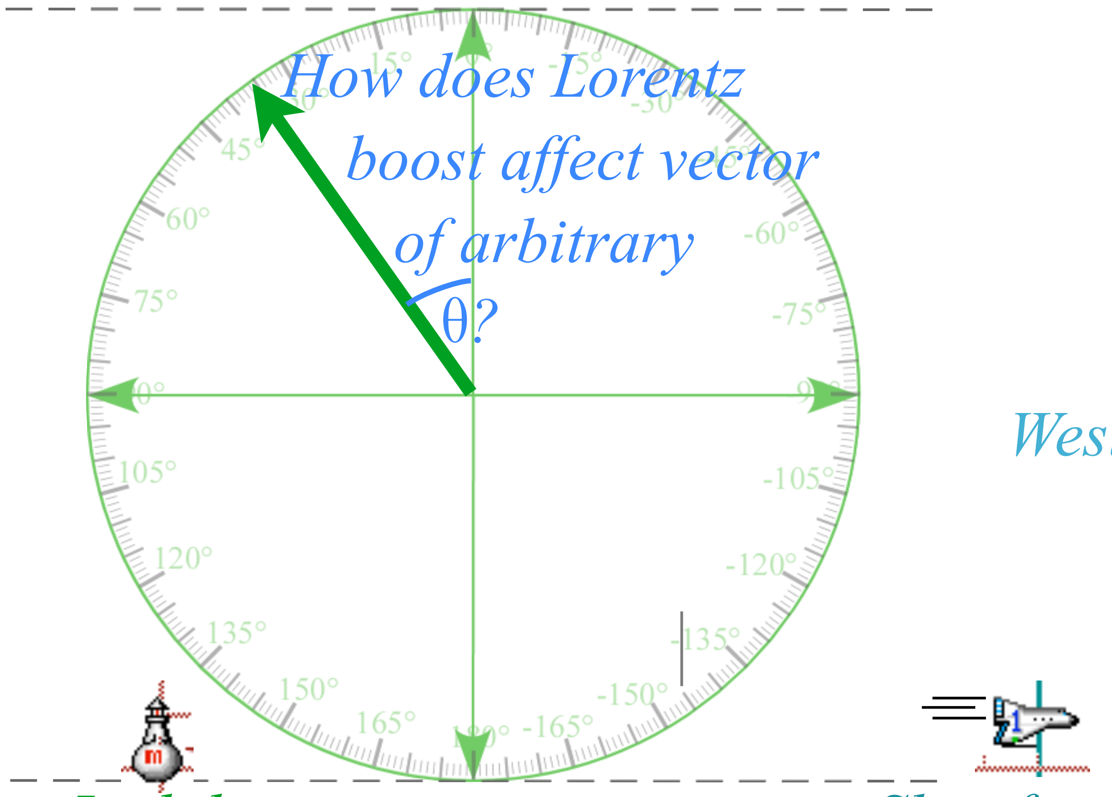
Red shift factor is $e^{-\rho}=\cosh\rho-\sinh\rho=\sec\sigma-\tan\sigma$

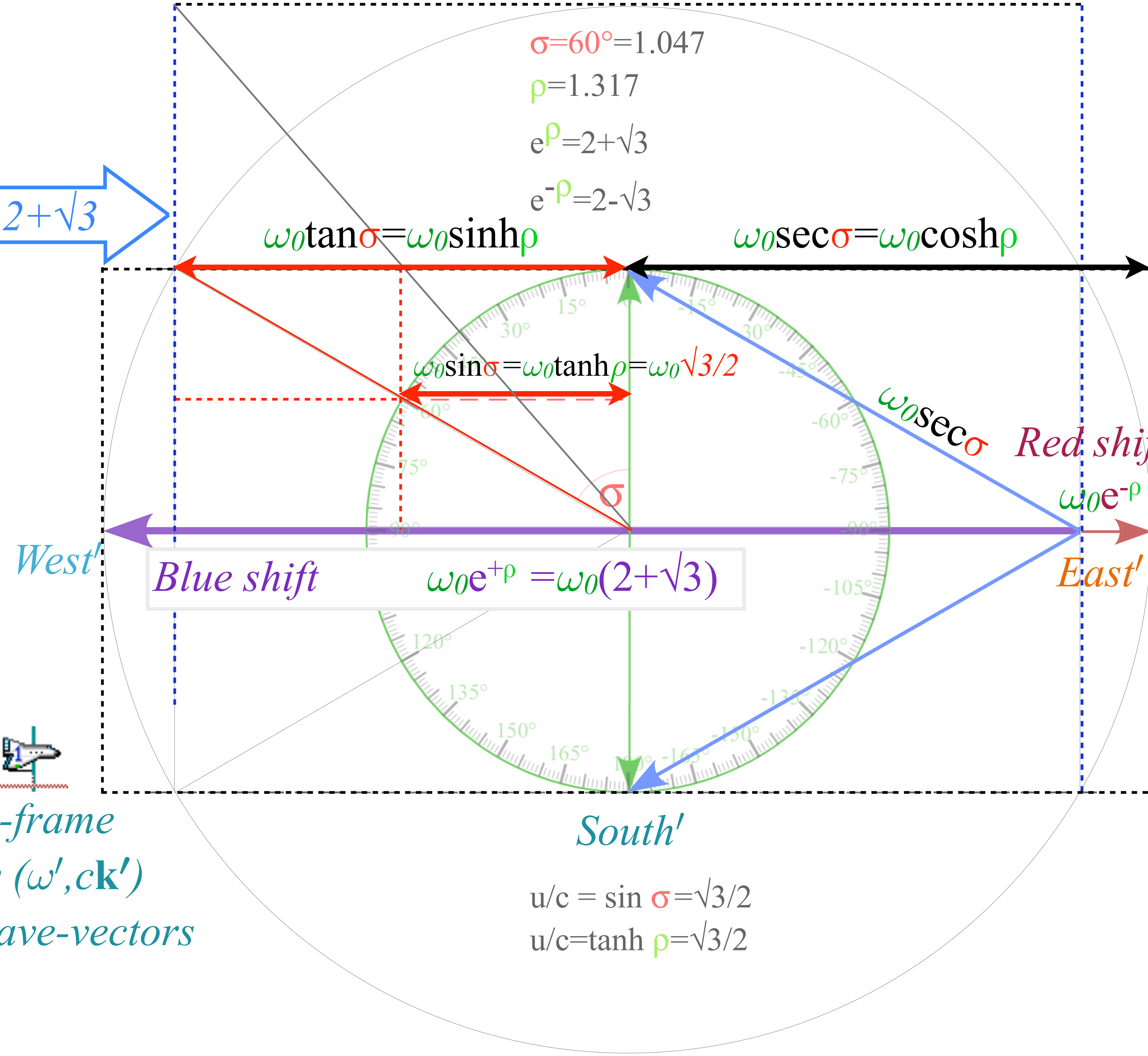
Faster Lorentz boost of North-South-East-West plane-wave 4-vectors $(\omega_0,\omega_x,\omega_y,\omega_z)$



Faster Lorentz boost of
North-South-East-West
plane-wave 4-vectors $(\omega_0,\omega_x,\omega_y,\omega_z)$

Lorentz boost by $\sigma=60°$ *or* $e^{+\rho}=2+\sqrt{3}$

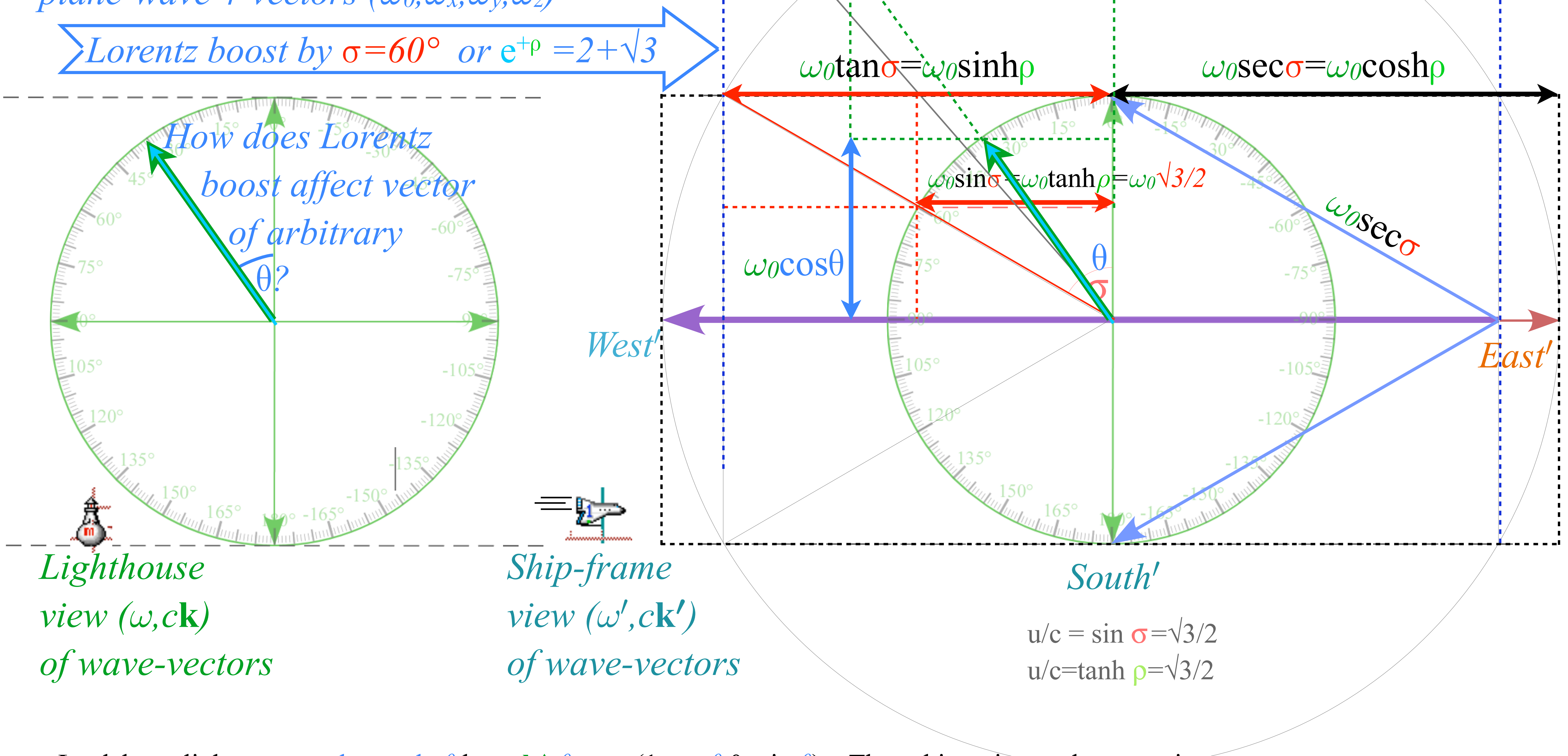




Lighthouse view $(\omega,c\mathbf{k})$ *of wave-vectors*

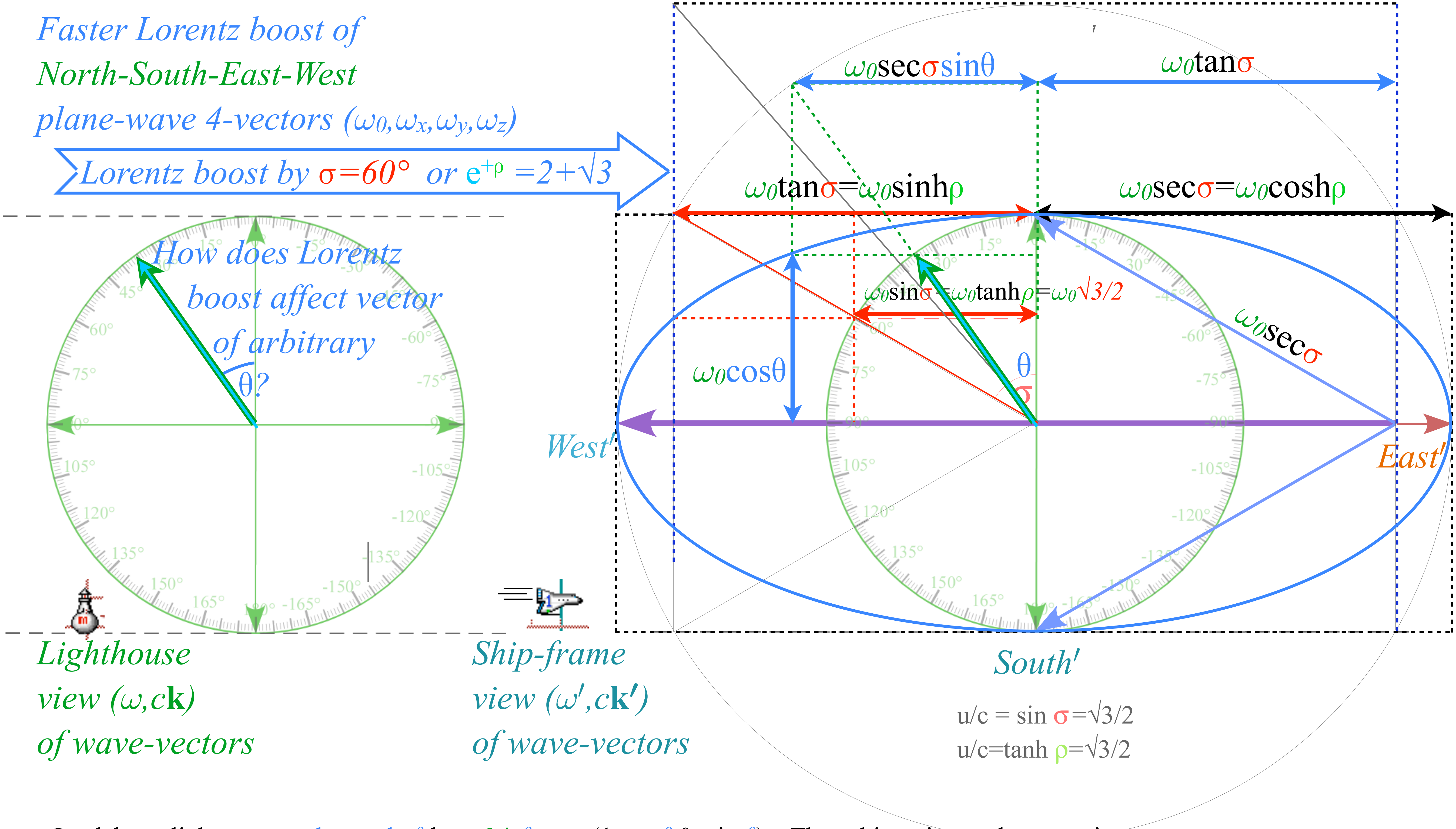
Ship-frame view $(\omega',c\mathbf{k}')$ *of wave-vectors*

Faster Lorentz boost of
North-South-East-West
plane-wave 4-vectors ($\omega_0,\omega_x,\omega_y,\omega_z$)



Let lab starlight ray at polar angle θ have $\mathbf{k}\uparrow\theta = \omega_0\,(1,\cos\theta,0,-\sin\theta)$. Then ship going **u** along z-axis sees :

$$\begin{pmatrix}\omega'_{\uparrow\theta}\\ ck'_{x\uparrow\theta}\\ ck'_{y\uparrow\theta}\\ ck'_{z\uparrow\theta}\end{pmatrix} = \begin{pmatrix}\cosh\rho_z & \cdot & \cdot & -\sinh\rho_z\\ \cdot & 1 & \cdot & \cdot\\ \cdot & \cdot & 1 & \cdot\\ -\sinh\rho_z & \cdot & \cdot & \cosh\rho_z\end{pmatrix}\begin{pmatrix}\omega_0\\ \omega_0\cos\theta\\ 0\\ -\omega_0\sin\theta\end{pmatrix} = \omega_0\begin{pmatrix}\cosh\rho_z+\sinh\rho_z\sin\theta\\ \cos\theta\\ 0\\ -\sinh\rho_z-\cosh\rho_z\sin\theta\end{pmatrix} = \omega_0\begin{pmatrix}\sec\sigma+\tan\sigma\sin\theta\\ \cos\theta\\ 0\\ -\tan\sigma-\sec\sigma\sin\theta\end{pmatrix}$$



Let lab starlight ray at polar angle θ have $\mathbf{k}\!\uparrow\!\theta = \omega_0\,(1,\cos\theta,0,-\sin\theta)$. Then ship going **u** along z-axis sees :

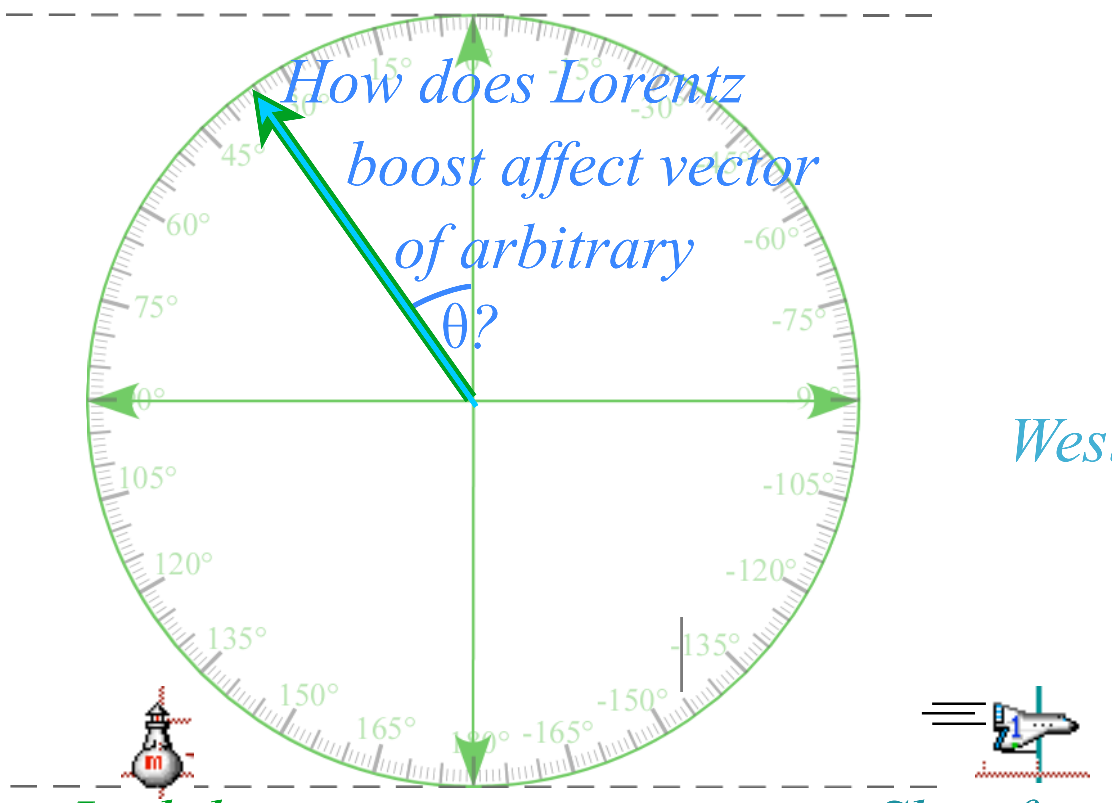
$$\begin{pmatrix}\omega'_{\uparrow\theta}\\ ck'_{x\uparrow\theta}\\ ck'_{y\uparrow\theta}\\ ck'_{z\uparrow\theta}\end{pmatrix} = \begin{pmatrix}\cosh\rho_z & \cdot & \cdot & -\sinh\rho_z\\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ -\sinh\rho_z & \cdot & \cdot & \cosh\rho_z\end{pmatrix}\begin{pmatrix}\omega_0\\ \omega_0\cos\theta\\ 0\\ -\omega_0\sin\theta\end{pmatrix} = \omega_0\begin{pmatrix}\cosh\rho_z+\sinh\rho_z\sin\theta\\ \cos\theta\\ 0\\ -\sinh\rho_z-\cosh\rho_z\sin\theta\end{pmatrix} = \omega_0\begin{pmatrix}\sec\sigma+\tan\sigma\sin\theta\\ \cos\theta\\ 0\\ -\tan\sigma-\sec\sigma\sin\theta\end{pmatrix}$$

Faster Lorentz boost of

North-South-East-West

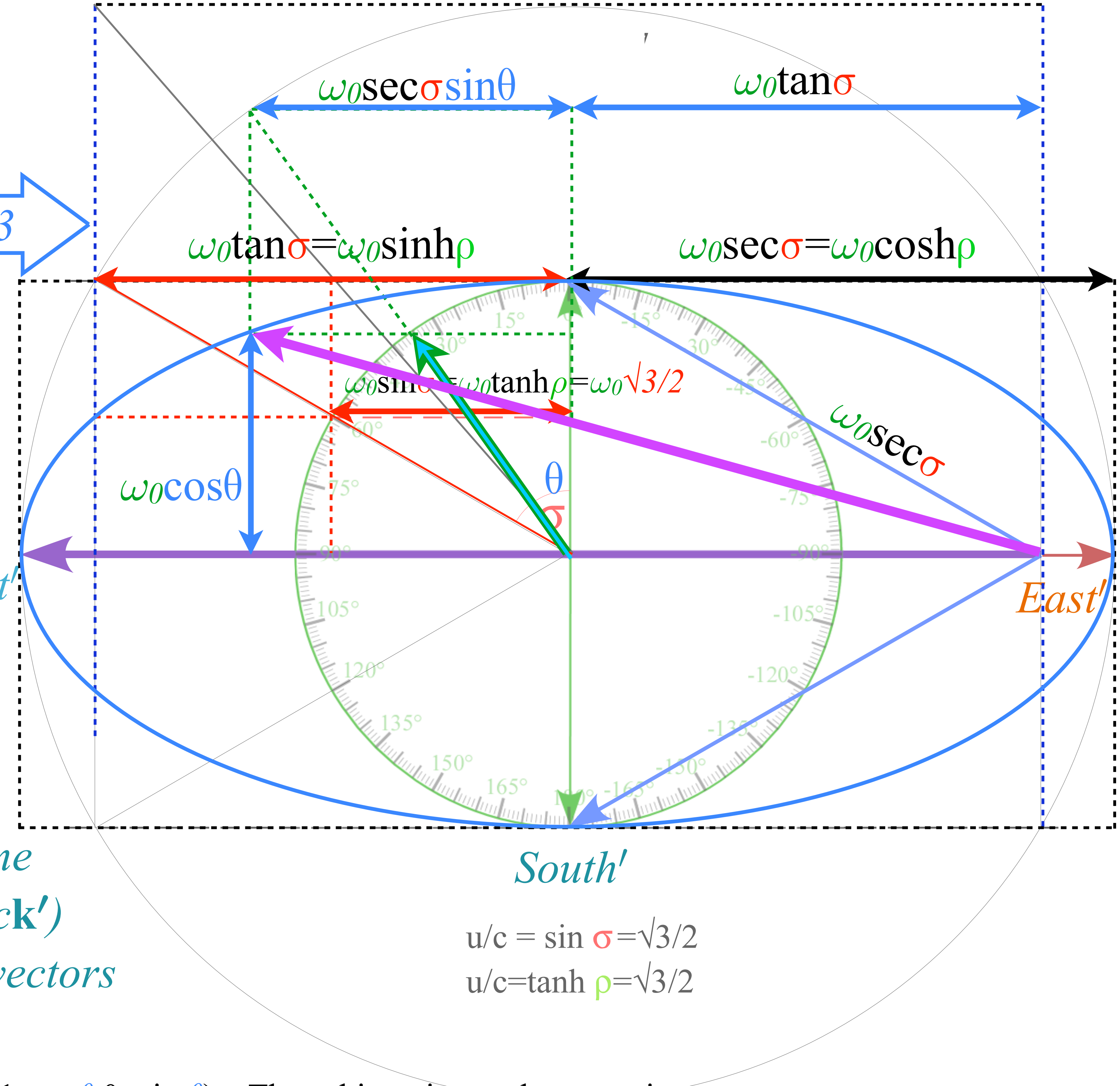
plane-wave 4-vectors $(\omega_0,\omega_x,\omega_y,\omega_z)$

Lorentz boost by $\sigma=60°$ *or* $e^{+\rho}=2+\sqrt{3}$



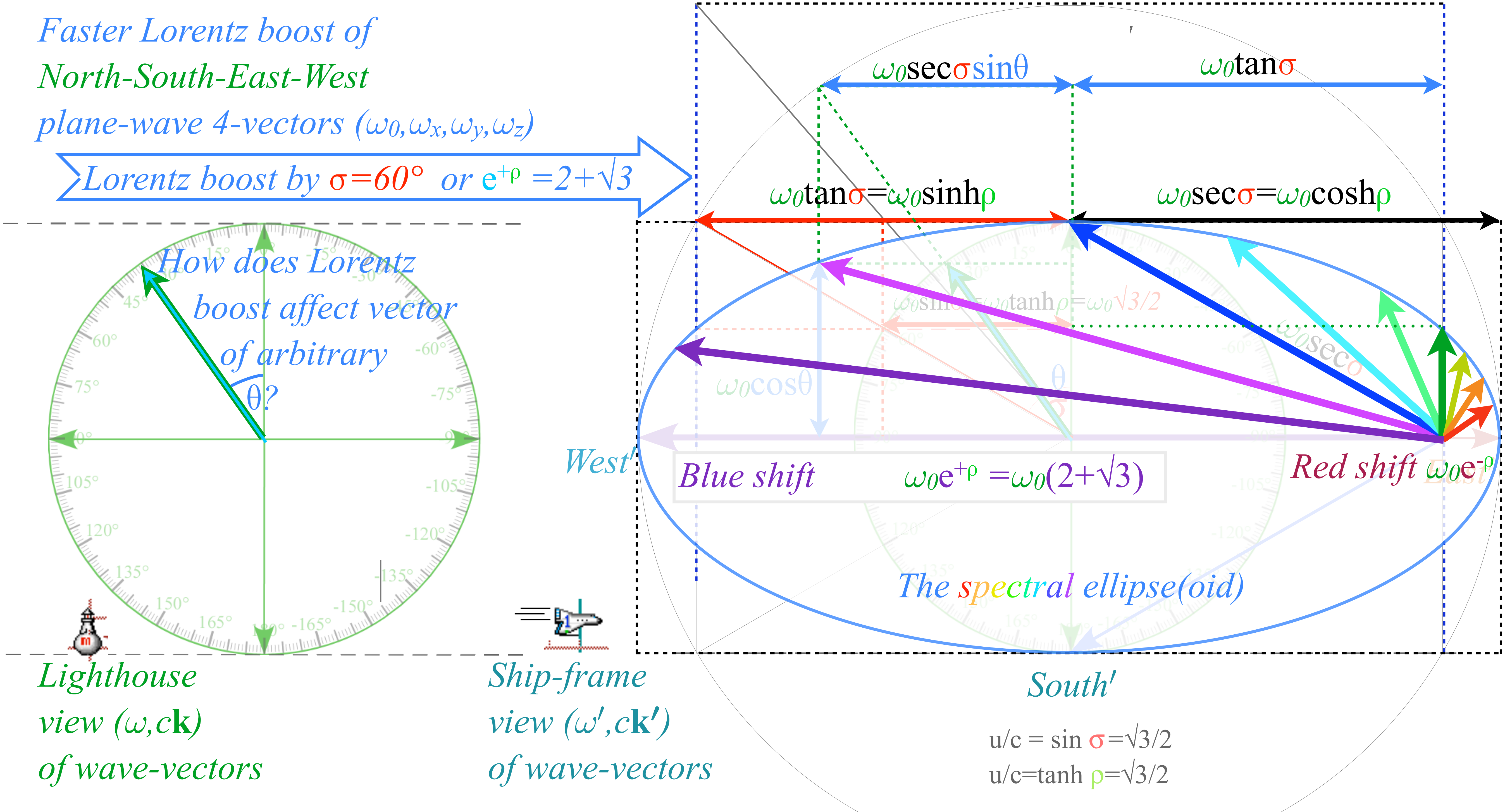
Lighthouse view $(\omega,c\mathbf{k})$ *of wave-vectors*

Ship-frame view $(\omega',c\mathbf{k}')$ *of wave-vectors*



Let lab starlight ray at polar angle θ have $\mathbf{k}\uparrow\theta=\omega_0(1,\cos\theta,0,-\sin\theta)$. Then ship going $\mathbf{u}$ along z-axis sees :

$$\begin{pmatrix}\omega'_{\uparrow\theta}\\ ck'_{x\uparrow\theta}\\ ck'_{y\uparrow\theta}\\ ck'_{z\uparrow\theta}\end{pmatrix}=\begin{pmatrix}\cosh\rho_z & \cdot & \cdot & -\sinh\rho_z\\ \cdot & 1 & \cdot & \cdot\\ \cdot & \cdot & 1 & \cdot\\ -\sinh\rho_z & \cdot & \cdot & \cosh\rho_z\end{pmatrix}\begin{pmatrix}\omega_0\\ \omega_0\cos\theta\\ 0\\ -\omega_0\sin\theta\end{pmatrix}=\omega_0\begin{pmatrix}\cosh\rho_z+\sinh\rho_z\sin\theta\\ \cos\theta\\ 0\\ -\sinh\rho_z-\cosh\rho_z\sin\theta\end{pmatrix}=\omega_0\begin{pmatrix}\sec\sigma+\tan\sigma\sin\theta\\ \cos\theta\\ 0\\ -\tan\sigma-\sec\sigma\sin\theta\end{pmatrix}$$

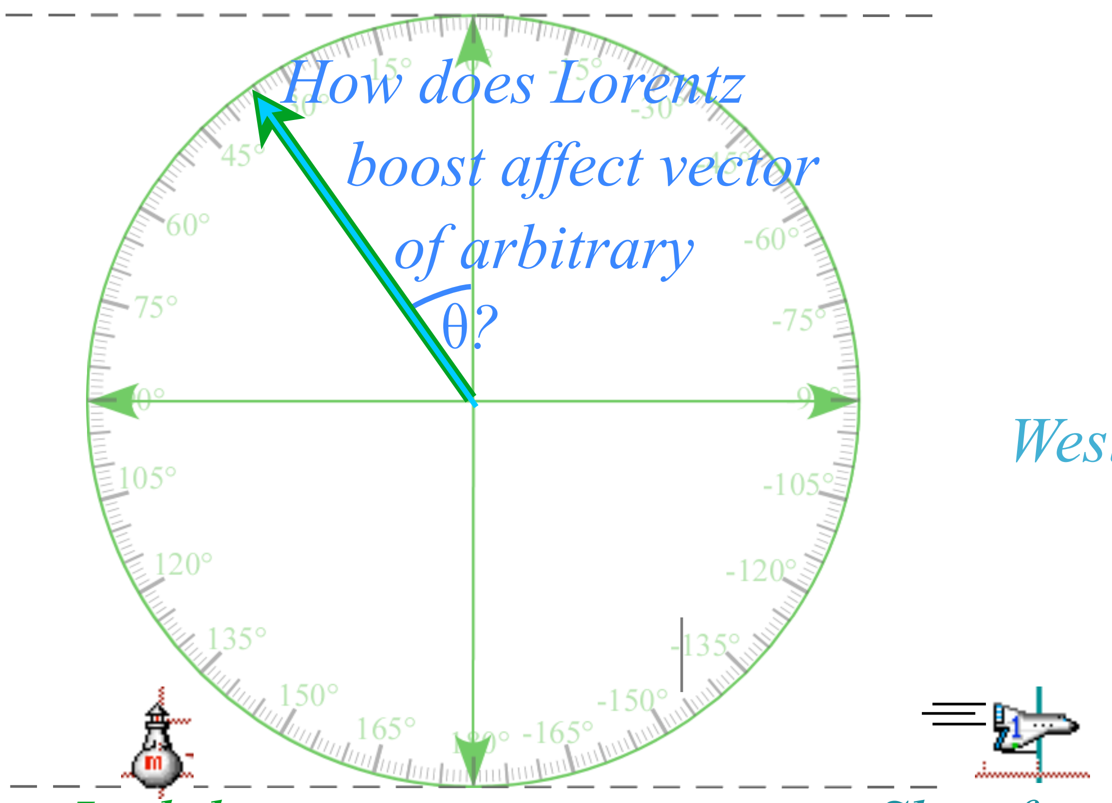


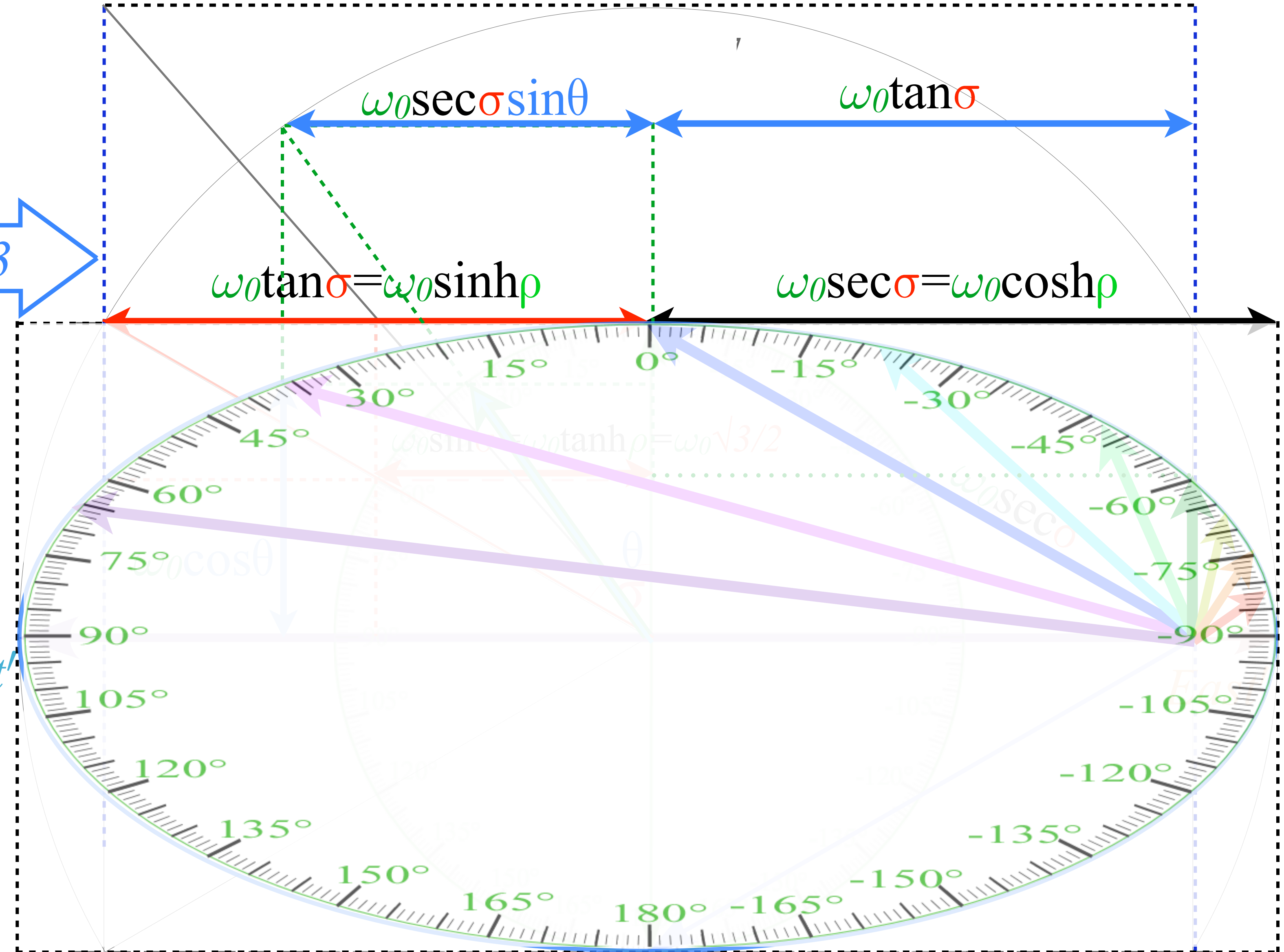
Let lab starlight ray at polar angle θ have $\mathbf{k}_{\uparrow\theta}=\omega_0(1,\cos\theta,0,-\sin\theta)$. Then ship going **u** along *z*-axis sees :

$$\begin{pmatrix}\omega'_{\uparrow\theta}\\ ck'_{x\uparrow\theta}\\ ck'_{y\uparrow\theta}\\ ck'_{z\uparrow\theta}\end{pmatrix}=\begin{pmatrix}\cosh\rho_z & \cdot & \cdot & -\sinh\rho_z\\ \cdot & 1 & \cdot & \cdot\\ \cdot & \cdot & 1 & \cdot\\ -\sinh\rho_z & \cdot & \cdot & \cosh\rho_z\end{pmatrix}\begin{pmatrix}\omega_0\\ \omega_0\cos\theta\\ 0\\ -\omega_0\sin\theta\end{pmatrix}=\omega_0\begin{pmatrix}\cosh\rho_z+\sinh\rho_z\sin\theta\\ \cos\theta\\ 0\\ -\sinh\rho_z-\cosh\rho_z\sin\theta\end{pmatrix}=\omega_0\begin{pmatrix}\sec\sigma+\tan\sigma\sin\theta\\ \cos\theta\\ 0\\ -\tan\sigma-\sec\sigma\sin\theta\end{pmatrix}$$

Faster Lorentz boost of
North-South-East-West
plane-wave 4-vectors $(\omega_0,\omega_x,\omega_y,\omega_z)$

Lorentz boost by $\sigma=60°$ or $e^{+\rho}=2+\sqrt{3}$





Lighthouse view $(\omega,c\mathbf{k})$ of wave-vectors

Ship-frame view $(\omega',c\mathbf{k}')$ of wave-vectors

South′

$u/c = \sin\sigma = \sqrt{3}/2$
$u/c = \tanh\rho = \sqrt{3}/2$

Let lab starlight ray at polar angle θ have $\mathbf{k}\uparrow\theta = \omega_0(1,\cos\theta,0,-\sin\theta)$. Then ship going $\mathbf{u}$ along z-axis sees :

$$\begin{pmatrix}\omega'_{\uparrow\theta}\\ ck'_{x\uparrow\theta}\\ ck'_{y\uparrow\theta}\\ ck'_{z\uparrow\theta}\end{pmatrix}=\begin{pmatrix}\cosh\rho_z & \cdot & \cdot & -\sinh\rho_z\\ \cdot & 1 & \cdot & \cdot\\ \cdot & \cdot & 1 & \cdot\\ -\sinh\rho_z & \cdot & \cdot & \cosh\rho_z\end{pmatrix}\begin{pmatrix}\omega_0\\ \omega_0\cos\theta\\ 0\\ -\omega_0\sin\theta\end{pmatrix}=\omega_0\begin{pmatrix}\cosh\rho_z+\sinh\rho_z\sin\theta\\ \cos\theta\\ 0\\ -\sinh\rho_z-\cosh\rho_z\sin\theta\end{pmatrix}=\omega_0\begin{pmatrix}\sec\sigma+\tan\sigma\sin\theta\\ \cos\theta\\ 0\\ -\tan\sigma-\sec\sigma\sin\theta\end{pmatrix}$$

Combination and interference of 4-vector plane waves (Idealized amplitude case)

$$\Psi_{A_\rightarrow,\omega_\rightarrow,\mathbf{k}_\rightarrow;A_\leftarrow,\omega_\leftarrow,\mathbf{k}_\leftarrow}(\mathbf{r},t) = A_\rightarrow e^{i(\mathbf{k}_\rightarrow\bullet\mathbf{r} - \omega_\rightarrow t)} + A_\leftarrow e^{i(\mathbf{k}_\leftarrow\bullet\mathbf{r} - \omega_\leftarrow t)}$$

2-CW-single-plane-polarized case: $\Psi_{\mathbf{k}}(\mathbf{r},t) = e^{i(\mathbf{k}_\rightarrow\bullet\mathbf{r}-\omega_\rightarrow t)} + e^{i(\mathbf{k}_\leftarrow\bullet\mathbf{r}-\omega_\leftarrow t)}$ *Idealized: Equal amplitudes and single plane polarization*

Factored into phase and group factors:

$$= e^{i\frac{(\mathbf{k}_\rightarrow+\mathbf{k}_\leftarrow)\bullet\mathbf{r}-(\omega_\rightarrow+\omega_\leftarrow)t}{2}}\, 2\cos\frac{(\mathbf{k}_\rightarrow-\mathbf{k}_\leftarrow)\bullet\mathbf{r}-(\omega_\rightarrow-\omega_\leftarrow)t}{2} = e^{i(\bar{\mathbf{K}}\bullet\mathbf{r}-\bar{\Omega}t)}\, 2\cos(\bar{\mathbf{k}}\bullet\mathbf{r}-\bar{\omega}t)$$

Phase (k,ω)

$$\frac{(\mathbf{k}_\rightarrow+\mathbf{k}_\leftarrow)}{2} = \bar{\mathbf{K}}\,,$$

$$\frac{(\omega_\rightarrow+\omega_\leftarrow)}{2} = \bar{\Omega}\,,$$

Group (k,ω)

$$\bar{\mathbf{k}} = \frac{(\mathbf{k}_\rightarrow-\mathbf{k}_\leftarrow)}{2}\,,$$

$$\bar{\omega} = \frac{(\omega_\rightarrow-\omega_\leftarrow)}{2}.$$

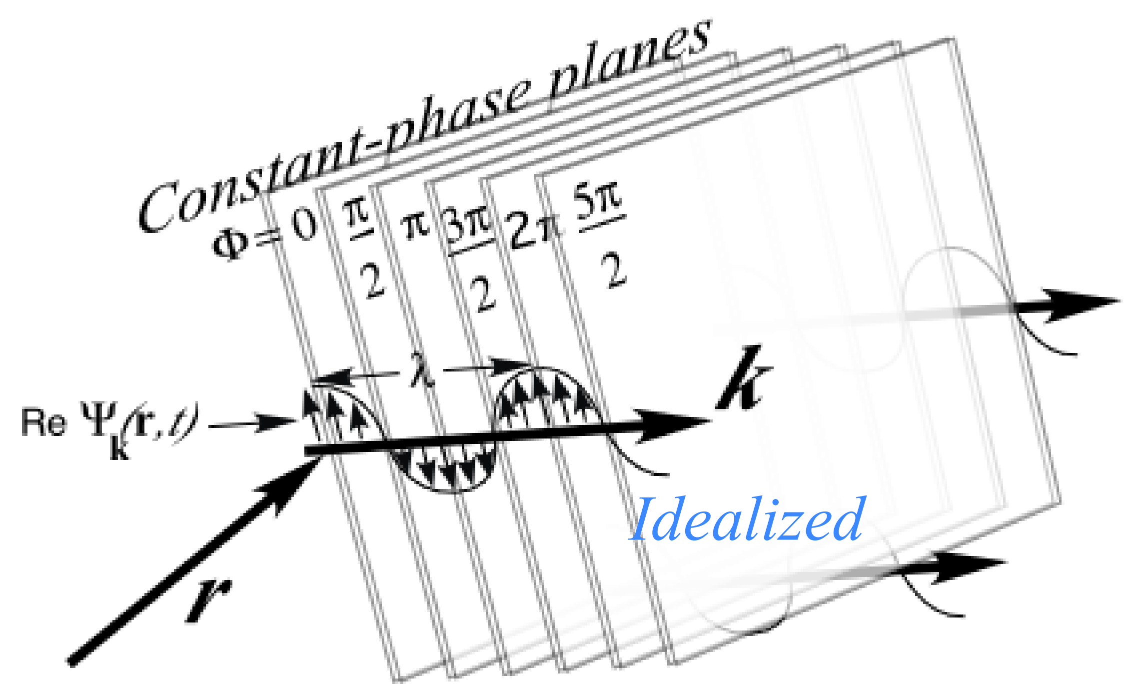


Fig. 6A.0 Sketch of a 1-CW-single-plane-polarized plane wavefunction $\Psi_{\mathbf{k}}(\mathbf{r},t) = Ae^{i\Phi} = Ae^{i(\mathbf{k}\bullet\mathbf{r}-\omega t)}$ *with wavevector* **k**.

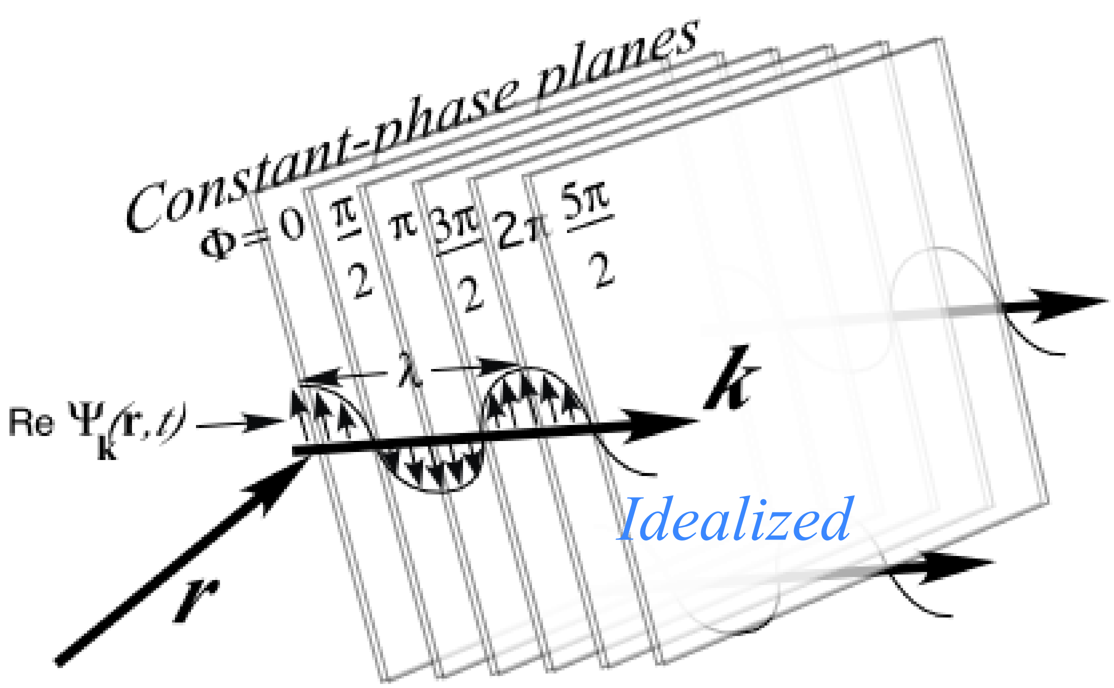
Combination and interference of 4-vector plane waves (Idealized amplitude case)

$$\Psi_{A_\rightarrow,\omega_\rightarrow,\mathbf{k}_\rightarrow;A_\leftarrow,\omega_\leftarrow,\mathbf{k}_\leftarrow}(\mathbf{r},t) = A_\rightarrow e^{i(\mathbf{k}_\rightarrow\bullet\mathbf{r} - \omega_\rightarrow t)} + A_\leftarrow e^{i(\mathbf{k}_\leftarrow\bullet\mathbf{r} - \omega_\leftarrow t)}$$

2-CW-single-plane-polarized case: $\Psi_{\mathbf{k}}(\mathbf{r},t) = e^{i(\mathbf{k}_\rightarrow\bullet\mathbf{r}-\omega_\rightarrow t)} + e^{i(\mathbf{k}_\leftarrow\bullet\mathbf{r}-\omega_\leftarrow t)}$ *Idealized: Equal amplitudes and single plane polarization*

Factored into phase and group factors:

$$= e^{i\frac{(\mathbf{k}_\rightarrow+\mathbf{k}_\leftarrow)\bullet\mathbf{r}-(\omega_\rightarrow+\omega_\leftarrow)t}{2}}\,2\cos\frac{(\mathbf{k}_\rightarrow-\mathbf{k}_\leftarrow)\bullet\mathbf{r}-(\omega_\rightarrow-\omega_\leftarrow)t}{2} = e^{i(\bar{\mathbf{K}}\bullet\mathbf{r}-\bar{\Omega}t)}\,2\cos(\bar{\mathbf{k}}\bullet\mathbf{r}-\bar{\varpi}t)$$



Phase (k,ω)

$$\frac{(\mathbf{k}_\rightarrow+\mathbf{k}_\leftarrow)}{2} = \bar{\mathbf{K}}\,,$$

$$\frac{(\omega_\rightarrow+\omega_\leftarrow)}{2} = \bar{\Omega}\,,$$

Group (k,ω)

$$\bar{\mathbf{k}} = \frac{(\mathbf{k}_\rightarrow-\mathbf{k}_\leftarrow)}{2}\,,$$

$$\bar{\varpi} = \frac{(\omega_\rightarrow-\omega_\leftarrow)}{2}\,.$$

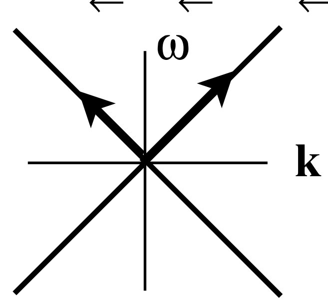
Fig. 6A.0 Sketch of a 1-CW-single-plane-polarized plane wavefunction $\Psi_k(\mathbf{r},t) = Ae^{i\Phi} = Ae^{i(\mathbf{k}\bullet\mathbf{r}-\omega t)}$ *with wavevector* **k**.

Individual laser 4-vectors reside
on light cone or null-invariant.

Ship — Lighthouse — Laser lab

$$c^2\mathbf{k}'_\rightarrow\bullet\mathbf{k}'_\rightarrow - \omega'^2_\rightarrow = c^2\mathbf{k}_\rightarrow\bullet\mathbf{k}_\rightarrow - \omega^2_\rightarrow = c^2k_0^2 - \omega_0^2 = 0$$

$$c^2\mathbf{k}'_\leftarrow\bullet\mathbf{k}'_\leftarrow - \omega'^2_\leftarrow = c^2\mathbf{k}_\leftarrow\bullet\mathbf{k}_\leftarrow - \omega^2_\leftarrow = c^2k_0^2 - \omega_0^2 = 0$$



Combination and interference of 4-vector plane waves (Idealized amplitude case)

$$\Psi_{A_{\rightarrow},\omega_{\rightarrow},\mathbf{k}_{\rightarrow};A_{\leftarrow},\omega_{\leftarrow},\mathbf{k}_{\leftarrow}}(\mathbf{r},t) = A_{\rightarrow}e^{i(\mathbf{k}_{\rightarrow}\bullet\mathbf{r}\,-\,\omega_{\rightarrow}t)} + A_{\leftarrow}e^{i(\mathbf{k}_{\leftarrow}\bullet\mathbf{r}\,-\,\omega_{\leftarrow}t)}$$

2-CW-single-plane-polarized case: $\Psi_{\mathbf{k}}(\mathbf{r},t) = e^{i(\mathbf{k}_{\rightarrow}\bullet\mathbf{r}-\omega_{\rightarrow}t)} + e^{i(\mathbf{k}_{\leftarrow}\bullet\mathbf{r}-\omega_{\leftarrow}t)}$ *Idealized: Equal amplitudes and single plane polarization*

Factored into phase and group factors:

$$= e^{i\frac{(\mathbf{k}_{\rightarrow}+\mathbf{k}_{\leftarrow})\bullet\mathbf{r}-(\omega_{\rightarrow}+\omega_{\leftarrow})t}{2}}\,2\cos\frac{(\mathbf{k}_{\rightarrow}-\mathbf{k}_{\leftarrow})\bullet\mathbf{r}-(\omega_{\rightarrow}-\omega_{\leftarrow})t}{2} = e^{i(\bar{\mathbf{K}}\bullet\mathbf{r}-\bar{\Omega}t)}\,2\cos(\bar{\mathbf{k}}\bullet\mathbf{r}-\bar{\omega}t)$$

Phase (k,ω)

$$\frac{(\mathbf{k}_{\rightarrow}+\mathbf{k}_{\leftarrow})}{2} = \bar{\mathbf{K}}\,,\qquad \frac{(\omega_{\rightarrow}+\omega_{\leftarrow})}{2} = \bar{\Omega}\,,$$

Group (k,ω)

$$\bar{\mathbf{k}} = \frac{(\mathbf{k}_{\rightarrow}-\mathbf{k}_{\leftarrow})}{2}\,,\qquad \bar{\omega} = \frac{(\omega_{\rightarrow}-\omega_{\leftarrow})}{2}\,.$$

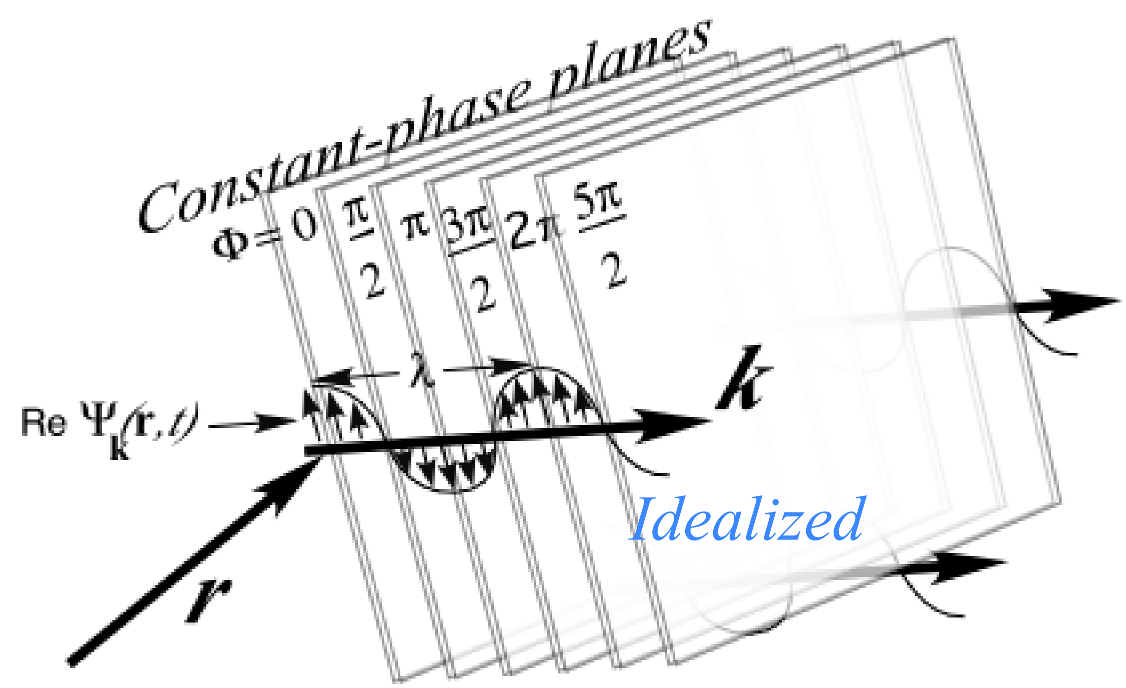


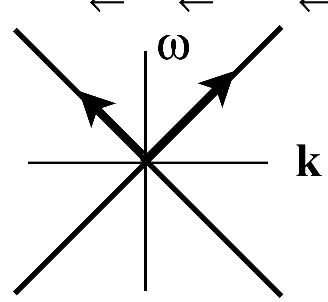
Fig. 6A.0 Sketch of a 1-CW-single-plane-polarized plane wavefunction $\Psi_k(\mathbf{r},t) = Ae^{i\Phi} = Ae^{i(\mathbf{k}\bullet\mathbf{r}-\omega t)}$ *with wavevector* $\mathbf{k}$.

Individual laser 4-vectors reside on light cone or null-invariant.

Ship	Lighthouse	Laser lab

$$c^2\mathbf{k}'_{\rightarrow}\bullet\mathbf{k}'_{\rightarrow} - \omega'^2_{\rightarrow} = c^2\mathbf{k}_{\rightarrow}\bullet\mathbf{k}_{\rightarrow} - \omega^2_{\rightarrow} = c^2k_0^2 - \omega_0^2 = 0$$

$$c^2\mathbf{k}'_{\leftarrow}\bullet\mathbf{k}'_{\leftarrow} - \omega'^2_{\leftarrow} = c^2\mathbf{k}_{\leftarrow}\bullet\mathbf{k}_{\leftarrow} - \omega^2_{\leftarrow} = c^2k_0^2 - \omega_0^2 = 0$$

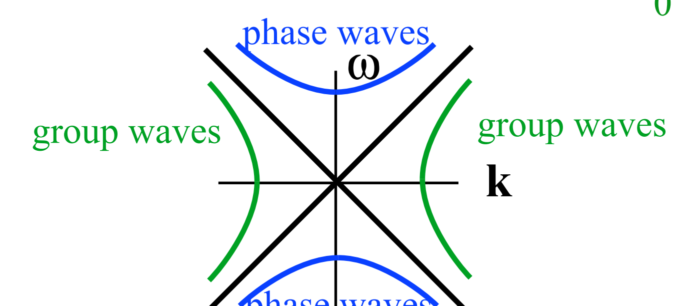


Sum and difference vectors are not on the light cone.

Ship	Lighthouse	Laser lab

$$\bar{\Omega}'^2 - c^2\bar{\mathbf{K}}'\bullet\bar{\mathbf{K}}' = \bar{\Omega}^2 - c^2\bar{\mathbf{K}}\bullet\bar{\mathbf{K}} = \omega_0^2 - 0 = c^2k_0^2$$

$$\bar{\omega}'^2 - c^2\bar{\mathbf{k}}'\bullet\bar{\mathbf{k}}' = \bar{\omega}^2 - c^2\bar{\mathbf{k}}\bullet\bar{\mathbf{k}} = 0 - c^2\mathbf{k}_0\bullet\mathbf{k}_0 = -c^2k_0^2$$



Combination group and phase define 4D Minkowski coordinates (Idealized amplitude case)

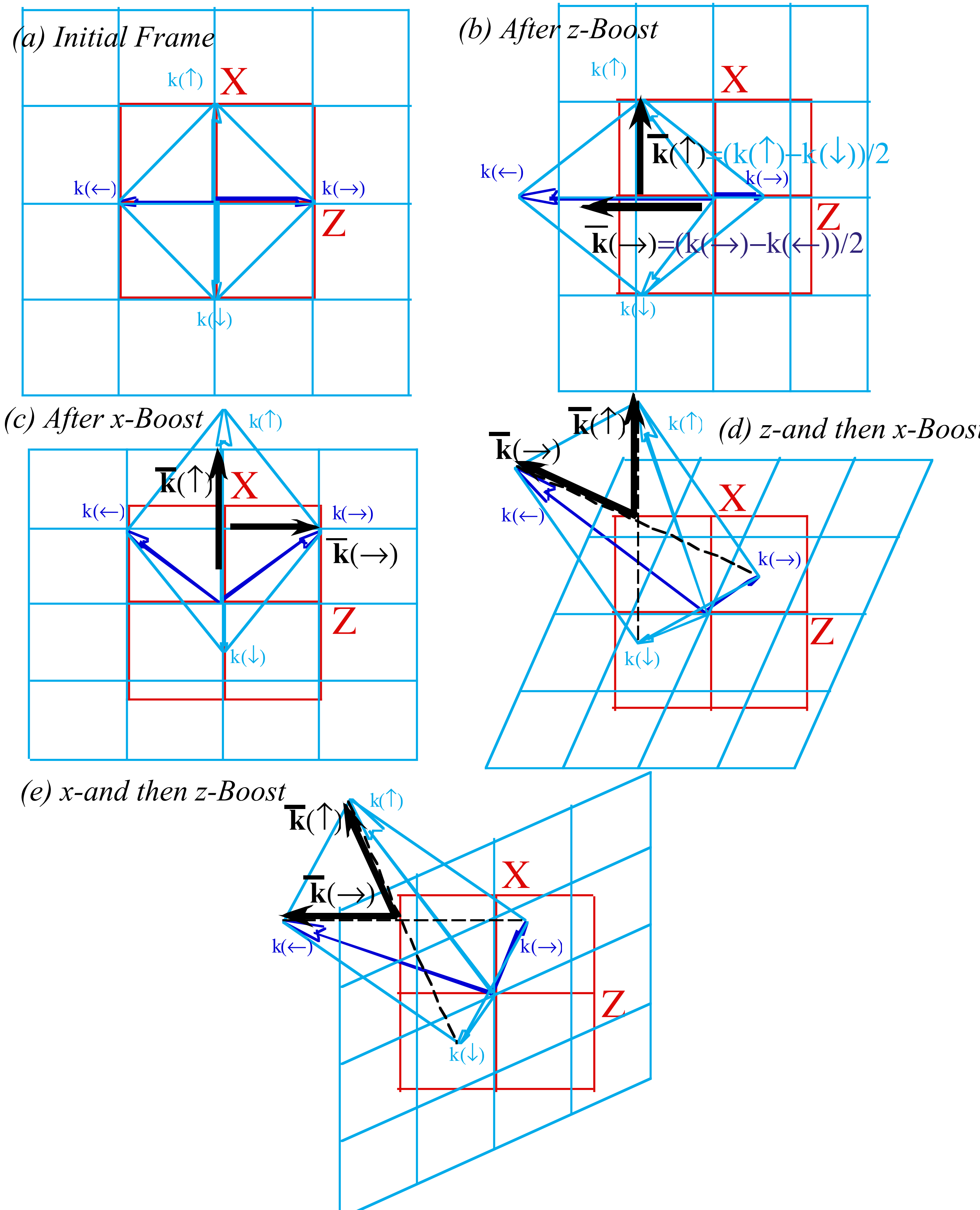


Fig. 6A.1 Examples of sequential relativistic transformations of a tetrad of light wavevectors.

2-Dimensional wave mechanics: guided waves and dispersion in the "Hall of Mirrors"

Any two or three-dimensional wave will be seen to exceed the c-limit when it approaches an axis obliquely. It happens for plane waves. The phase velocities along coordinate axes are given by

$$v_x = \omega / k_x , \qquad v_y = \omega / k_y , \qquad v_z = \omega / k_z .$$

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A water waves exceeds c if it breaks parallel to shore so 'break-line" moves infinitely fast with $k_x = 0$.

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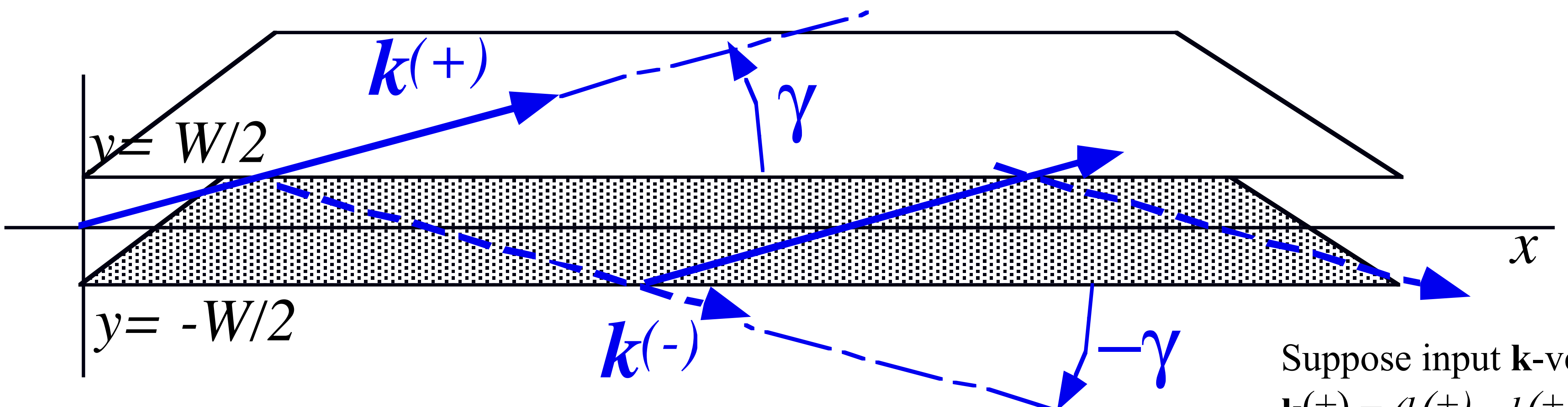

Fig. 6B.1 A "hall of mirrors" model for an optical wave guide of width W.

Suppose input **k**-vector $\mathbf{k}^{(+)}$ enters at angle $+\gamma$.
$\mathbf{k}^{(+)} = (k^{(+)}_x, k^{(+)}_y, 0) = (k\cos\gamma, k\sin\gamma, 0)$

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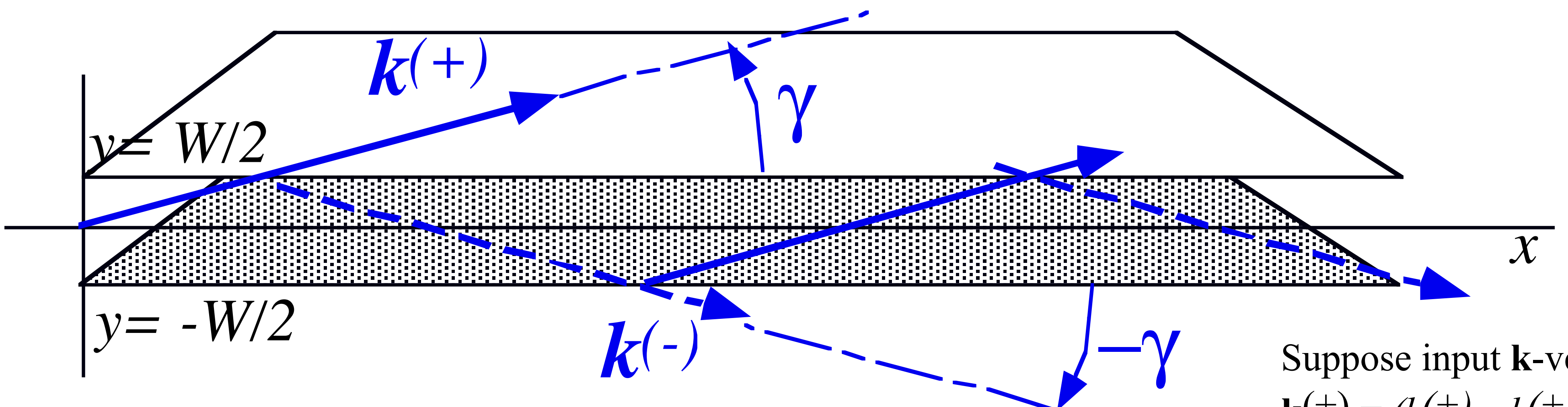

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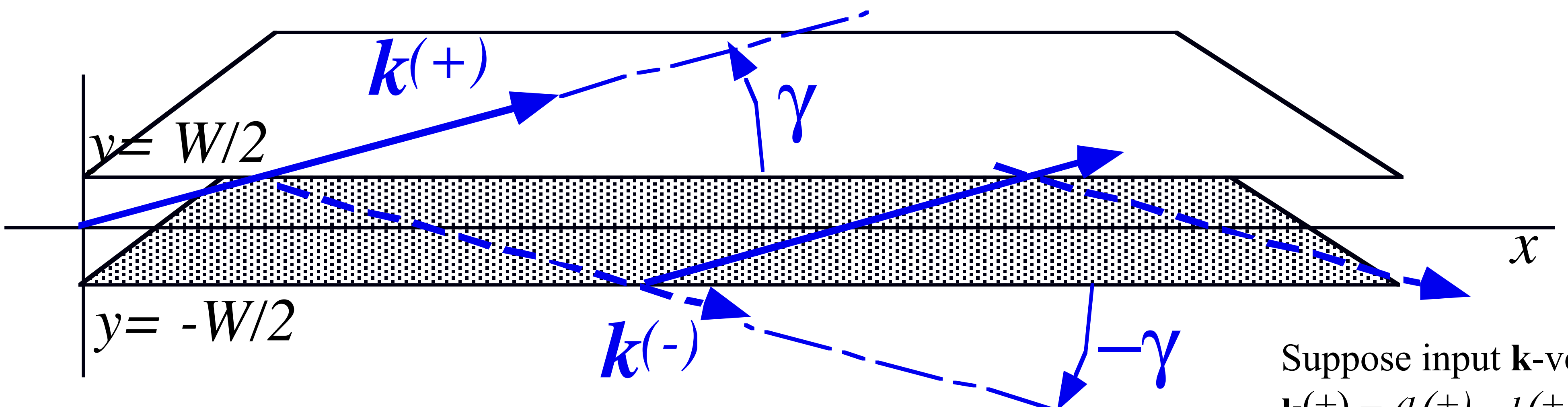

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guide phase wave and group wave

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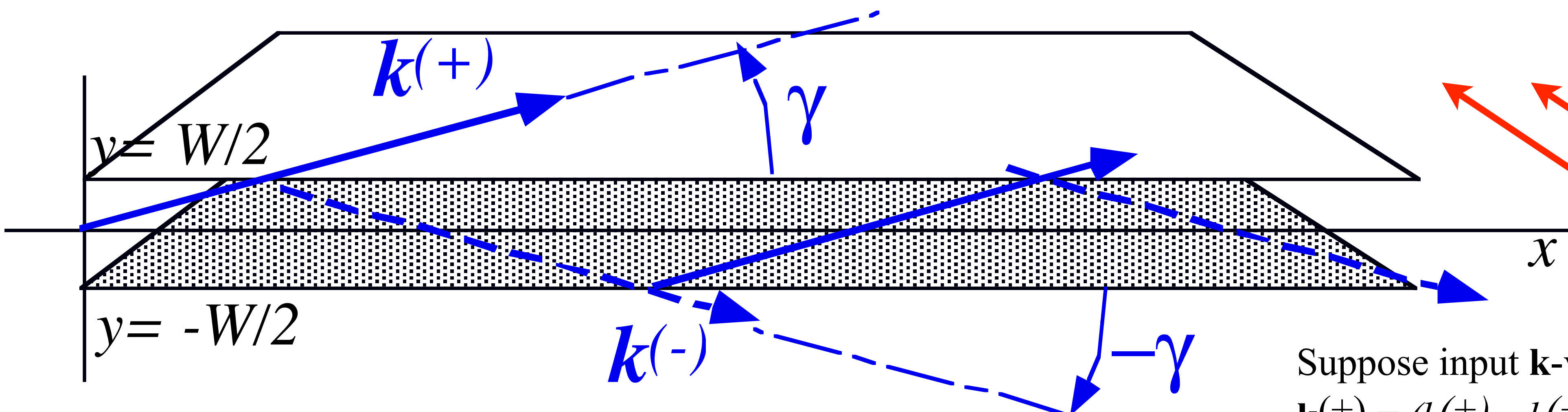
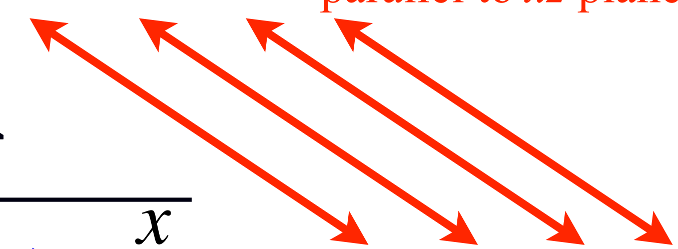
Assume $\mathrm{T_{ransverse}E_{lectric}}$-mode.
It always has **E** polarized parallel to *xz* plane





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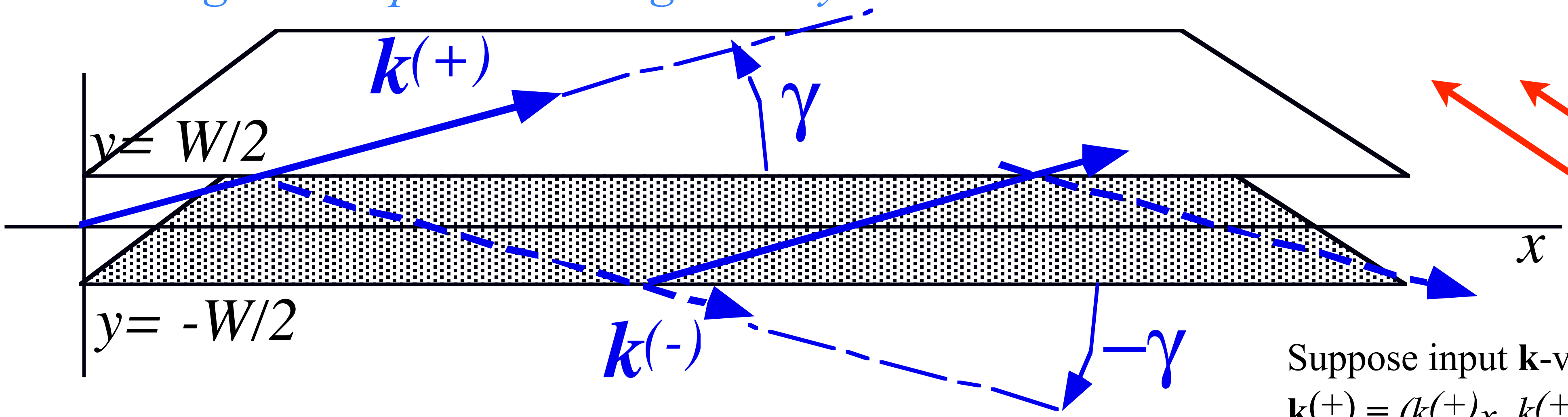
\end{aligned}$$

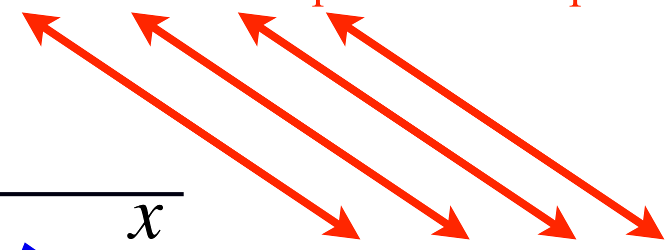
guide phase wave and group wave

TE boundary conditions make group be zero on metal walls $y=\pm W/2$.
$0 = 2\cos(k(W/2) \sin\gamma)$, or: $k(W/2) \sin\gamma = \pi/2$, or: $\sin\gamma = \pi/(kW)$

Waveguide dispersion and geometry

Assume TransverseElectric-mode.
It always has **E** polarized parallel to *xz* plane





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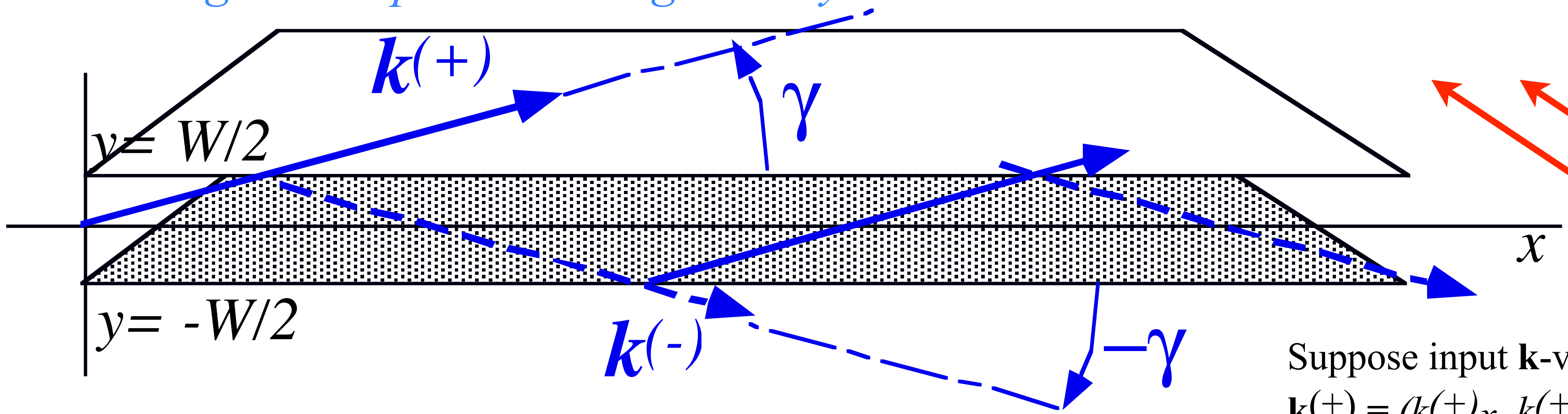

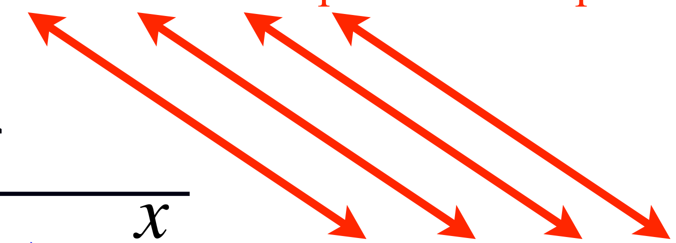

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where: $\omega_{cut} = \pi c/W$.

Waveguide dispersion and geometry

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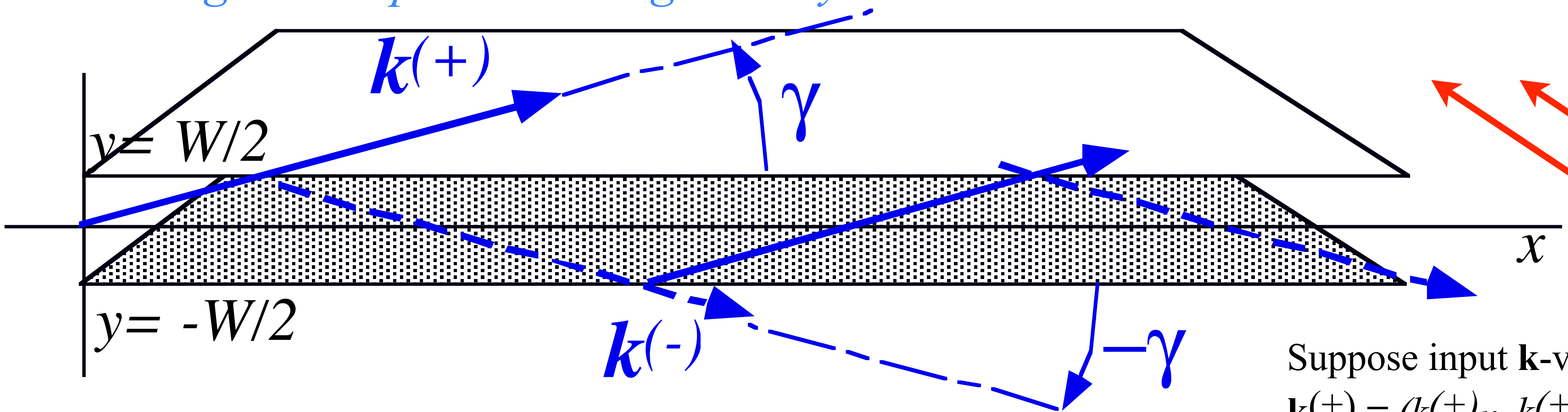

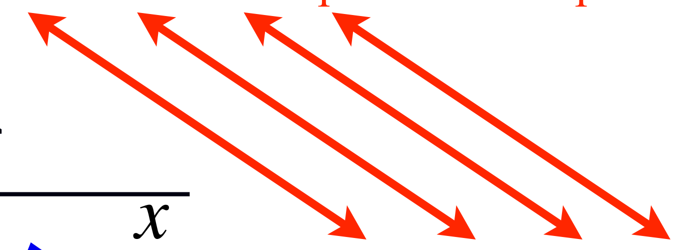

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Suppose input **k**-vector $\mathbf{k}^{(-)}$ enters at angle +γ.

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guide *phase wave* and *group wave*

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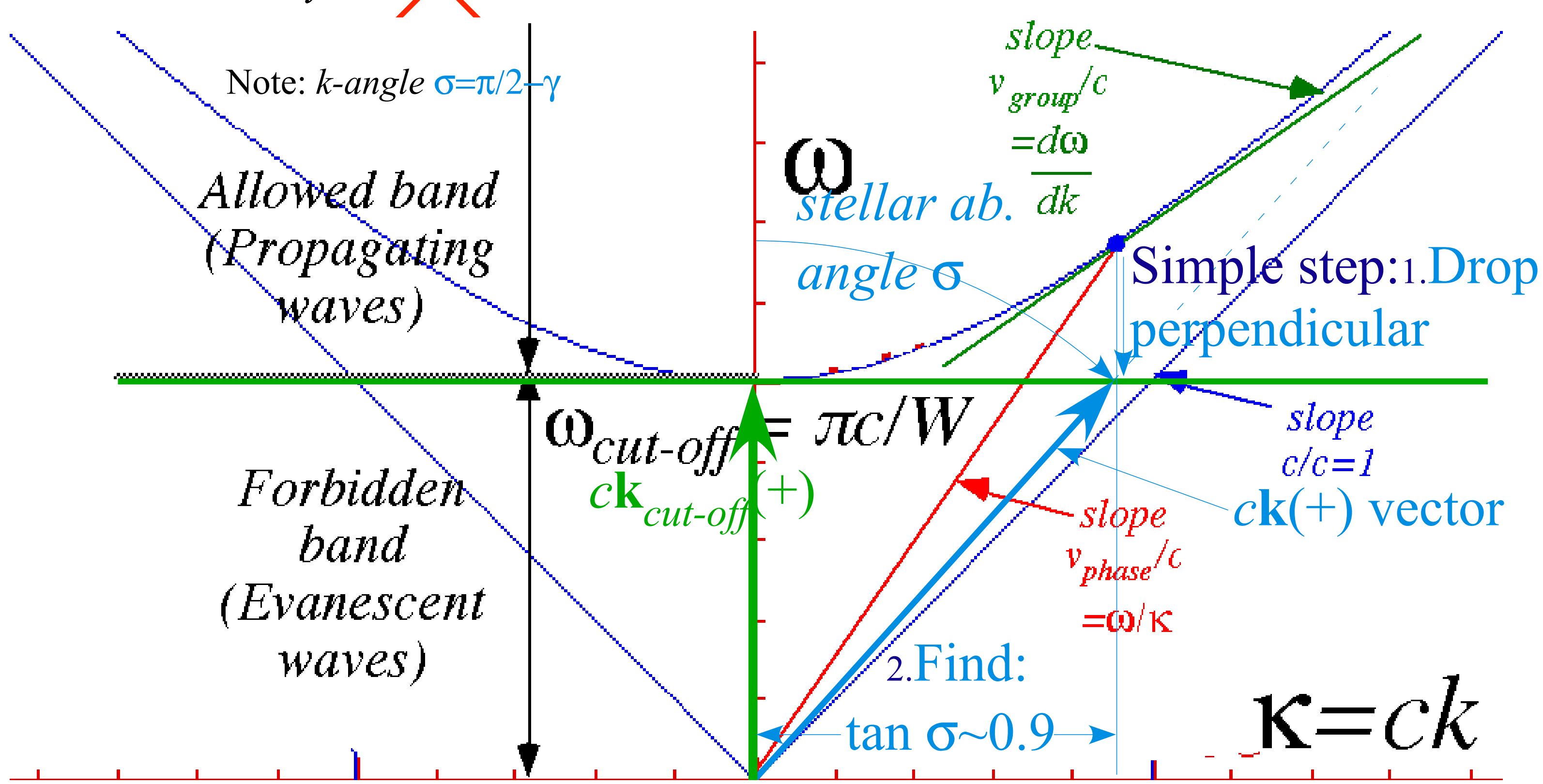


Fig. 6B.2 Dispersion function for a fundamental TE wave guide mode

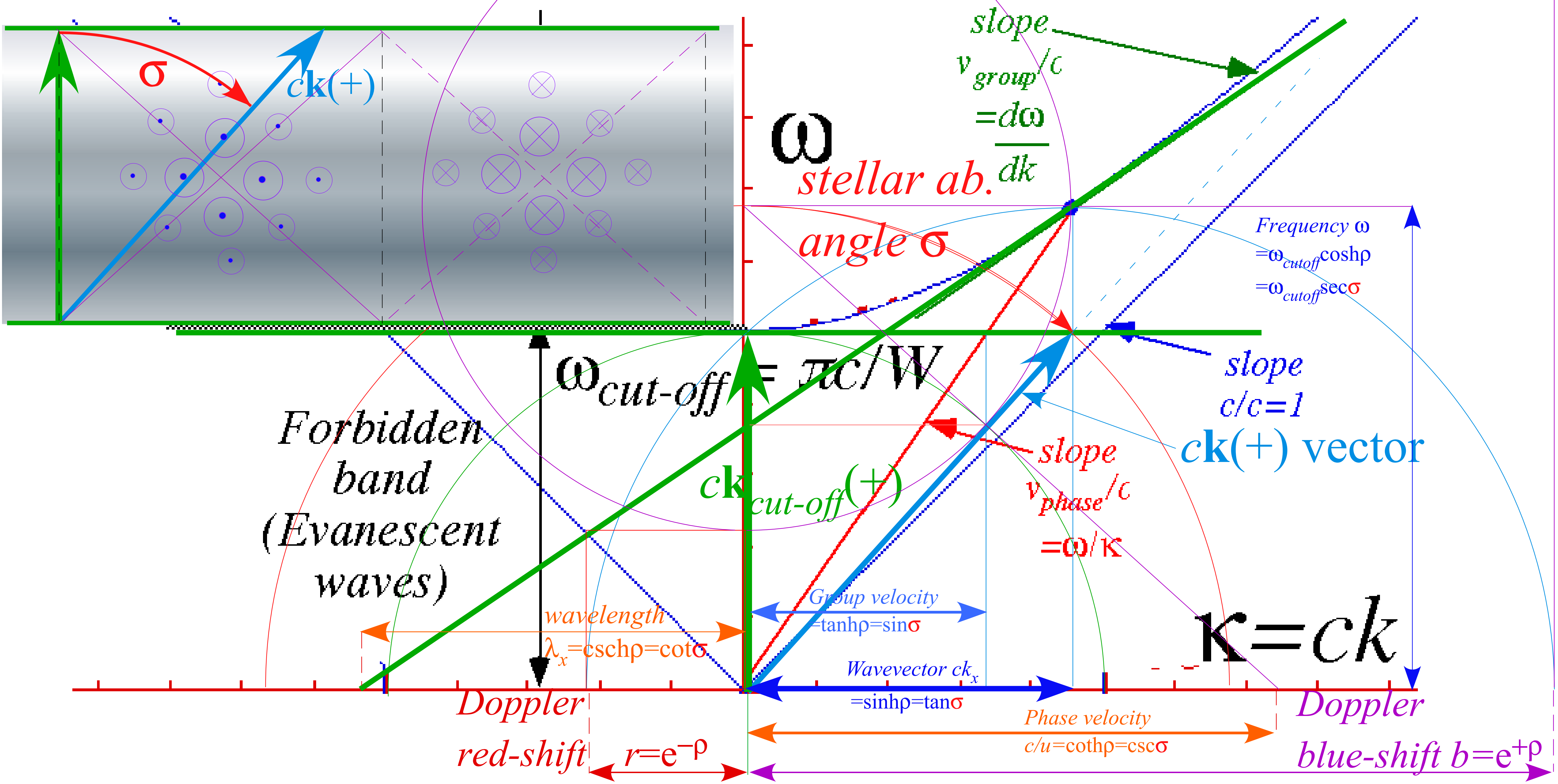


Fig. 6B.8 Thales geometry of cavity or waveguide mode

(Lecture 28 ends here)

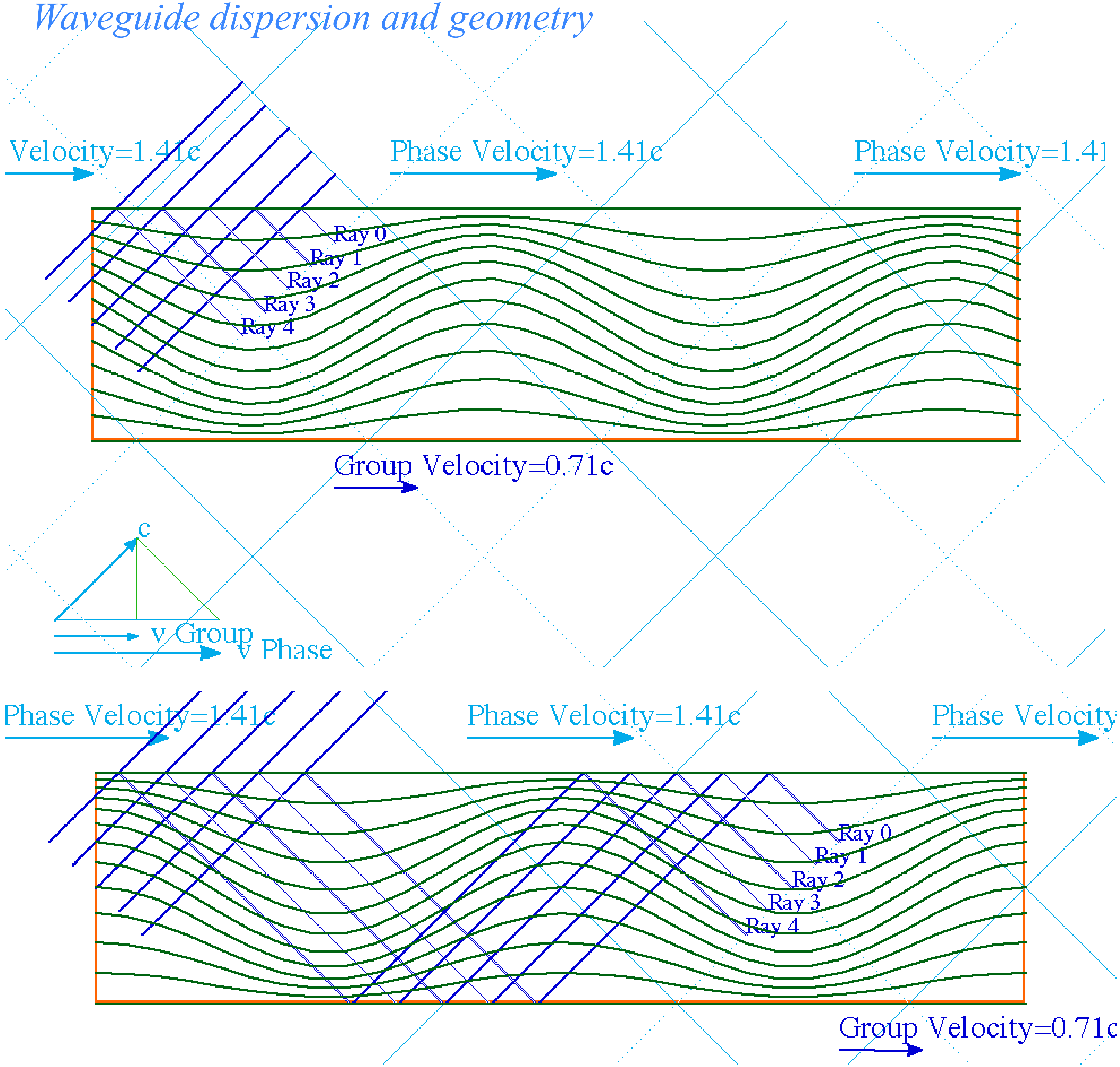


Fig. 6B.3 Right moving guide wave with $\gamma = 45°$, $V_{phase}=\sqrt{2}c$, $V_{group}=c/\sqrt{2}$.

Waveguide dispersion and geometry

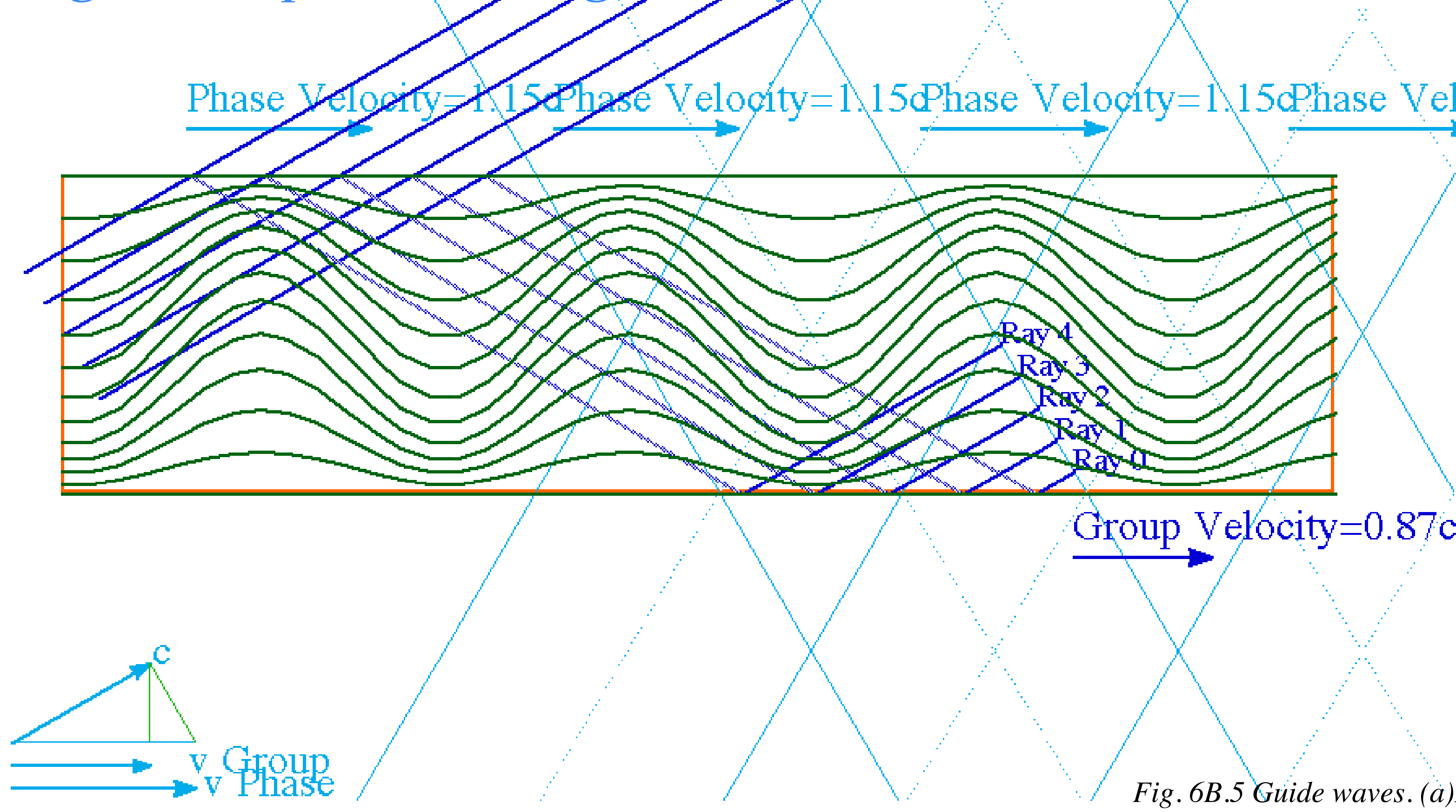
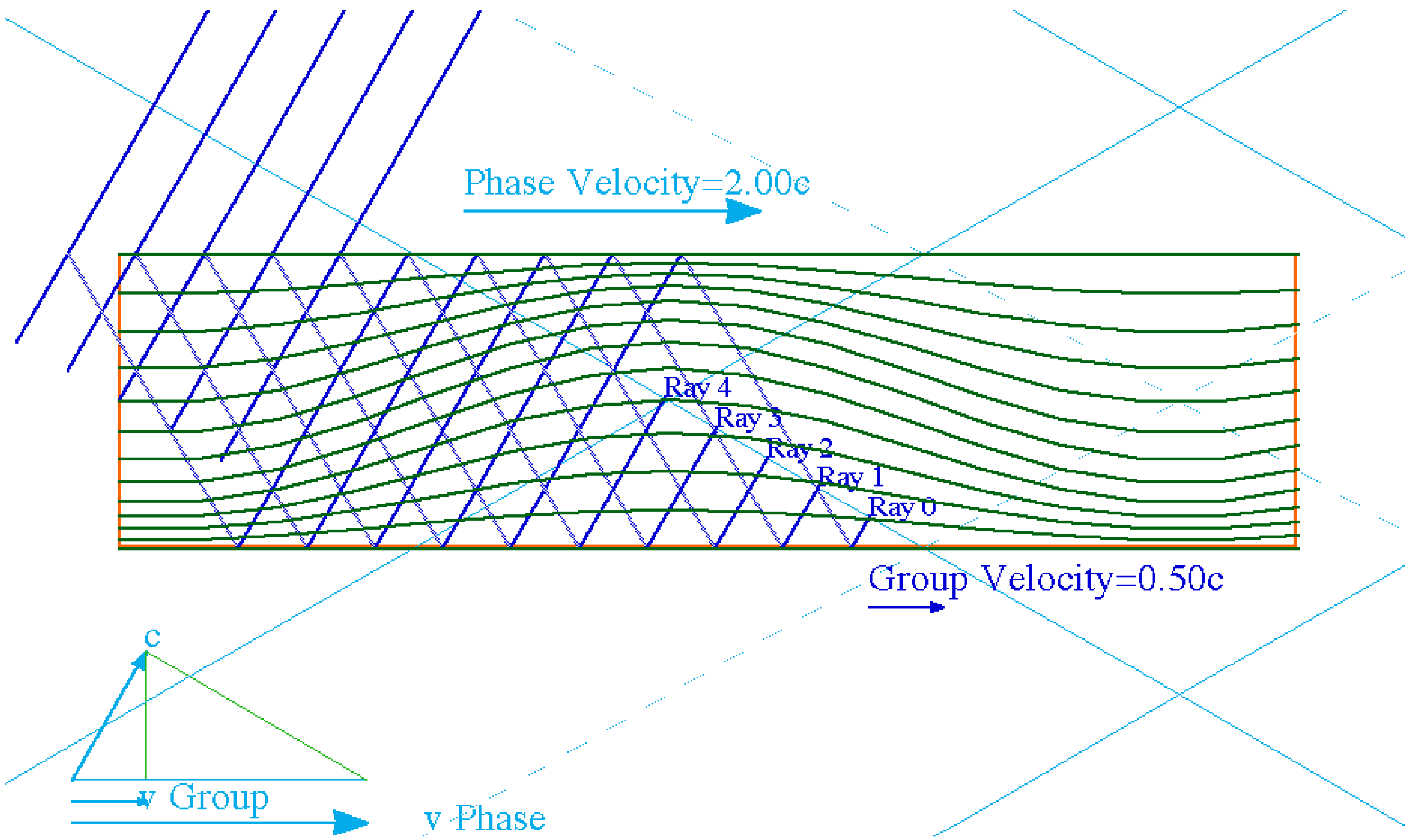
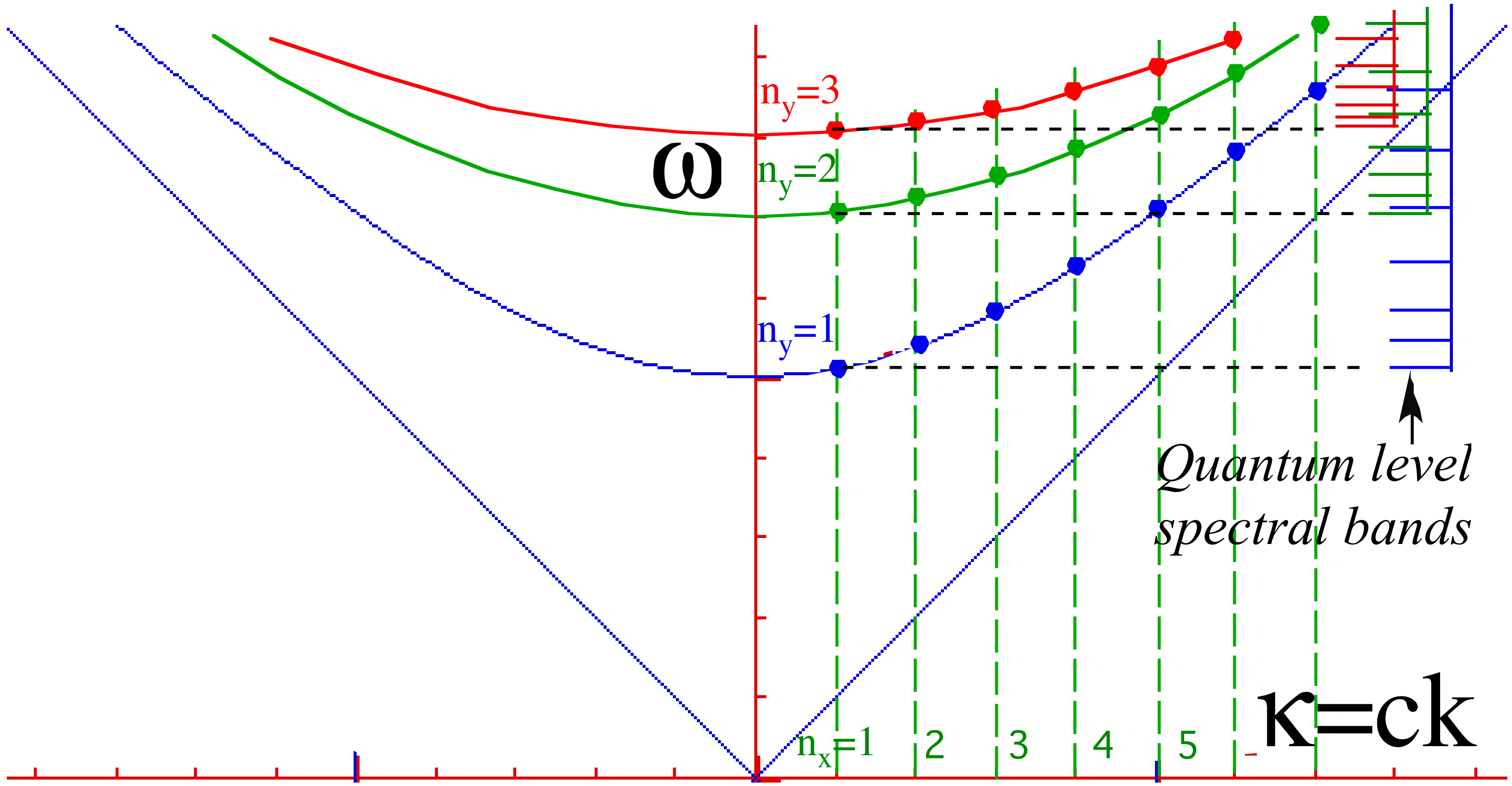


Fig. 6B.5 Guide waves. (a)Higher frequency case: $\gamma = 30°$, $v_{x\,(phase)}=c\sqrt{3}/2c$, $v_{x\,(group)}=c2/\sqrt{3}$.

(b)Lower frequency case: $\gamma = 60°$, $v_{x\,(phase)}= 2c$, $v_{x\,(group)}=$



Waveguide dispersion and geometry



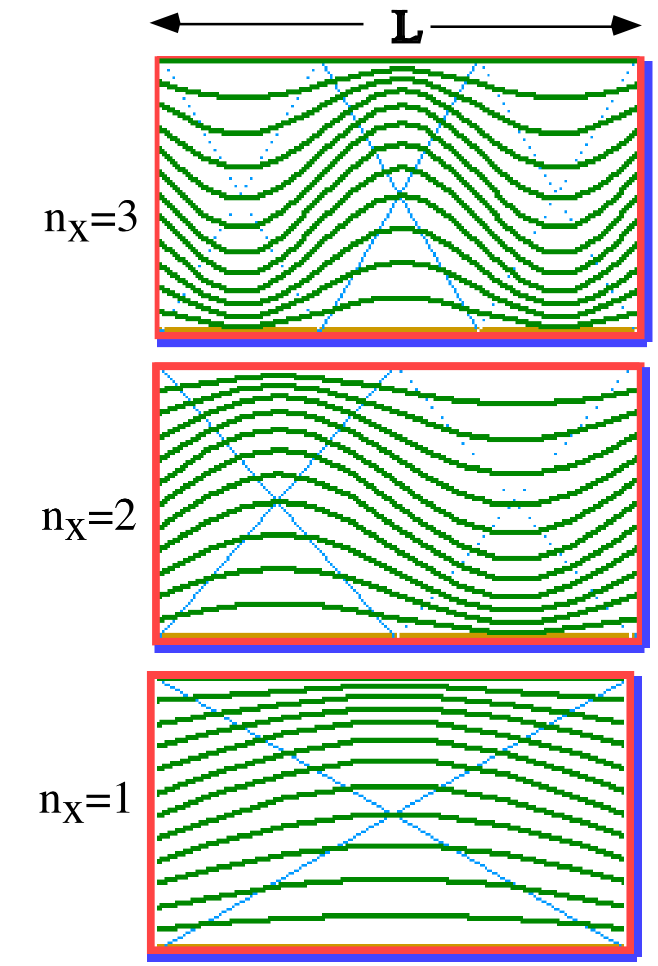


Fig. 6B.7 Cavity modes for three lowest quantum numbers

Fig. 6B.6 Cavity mode dispersion diagram showing overlapping and discrete ω *and k values.*

$\Phi = -\mu\tau = \mathbf{k}\bullet\mathbf{r} - \omega t = \mathbf{k}\bullet\mathbf{r} - (\omega/c)(ct) = \mathbf{k}'\bullet\mathbf{r}' - (\omega'/c)(ct')$.

$$\Phi_{\leftarrow} = \mathbf{k}'_{\leftarrow} \bullet \mathbf{r}' - \omega'_{\leftarrow} t' = \mathbf{k}_{\leftarrow} \bullet \mathbf{r} - \omega_{\leftarrow} t = -\mathbf{k}_0 \bullet \mathbf{r}_0 - \omega_0 t_0$$

$$\Phi_{\rightarrow} = \mathbf{k}'_{\rightarrow} \bullet \mathbf{r}' - \omega'_{\rightarrow} t' = \mathbf{k}_{\rightarrow} \bullet \mathbf{r} - \omega_{\rightarrow} t = \mathbf{k}_0 \bullet \mathbf{r}_0 - \omega_0 t_0$$

$$c^2 \mathbf{k}'_{\leftarrow} \bullet \mathbf{k}'_{\leftarrow} - {\omega'_{\leftarrow}}^2 = c^2 \mathbf{k}_{\leftarrow} \bullet \mathbf{k}_{\leftarrow} - {\omega_{\leftarrow}}^2 = c^2 k_0^2 - \omega_0^2 = 0$$

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