Lecture 29.

Relativity of interfering and galloping waves: SWR and SWQ I.

(Ch. 4-6 of Unit 2 4.10.12)

Wave guide and cavity dynamics in space-time (x_0,x_1, x_2, x_3) and per-space-time (ω_0, ck_1, ck_2, ck_3)
- *Above cut-off:Group vs. phase velocity*
- *Below cut-off:Evanescent waves*
- *Cavity eigenfunctions and eigenvalues*

Galloping waves due to unmatched amplitudes
- *Standing Wave Ratio (SWR) and Standing Wave Quotient (SWQ)*

2-Dimensional wave mechanics: guided waves and dispersion in the "Hall of Mirrors"

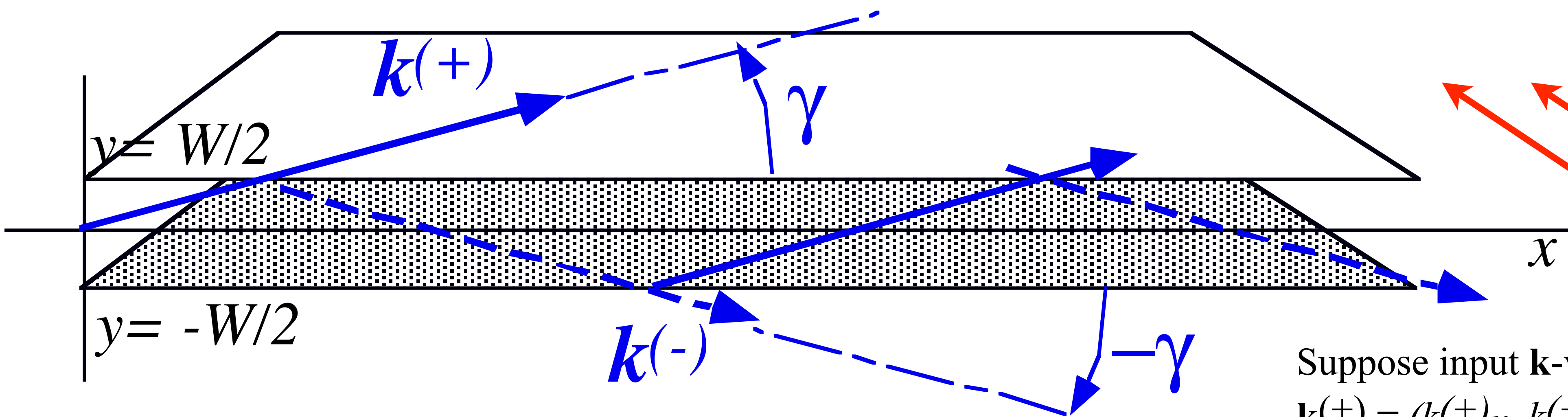
Any two or three-dimensional wave will be seen to exceed the *c*-limit when it approaches an axis obliquely. It happens for plane waves. The phase velocities along coordinate axes are given by

$$v_x = \omega / k_x , \qquad v_y = \omega / k_y , \qquad v_z = \omega / k_z .$$

Each of the components (k_x , k_y , k_z) must be less than or equal to magnitude $k = \sqrt{(k_x^2 + k_y^2 + k_z^2)}$.
Thus, all the component phase velocities equal or exceed the phase velocity ω / k which is *c* for light!
A water waves exceeds *c* if it breaks parallel to shore so 'break-line" moves infinitely fast with $k_x = 0$.

Consider 'Hall of Mirrors" with two parallel mirrors on either side of the *x*-axis be separated by a distance *y=W*.
The South wall will be at *y=-W/2* and the North wall at *y=W/2*. (*z*-axis or "up" is into the page here.)
The Hall should have a floor and ceiling at $z=\pm H/2$ as discussed later. Here waves move in *xy*-plane only.

Assume TransverseElectric-mode.
It always has **E** polarized parallel to *xz* plane



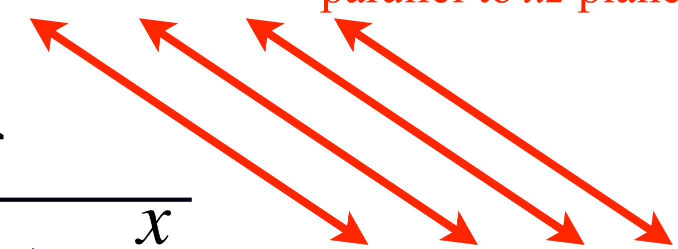


Fig. 6B.1 A "hall of mirrors" model for an optical wave guide of width W.

Suppose input **k**-vector $\mathbf{k}^{(+)}$ enters at angle $+\gamma$.

$$\mathbf{k}^{(+)} = (k^{(+)}_x, k^{(+)}_y, 0) = (k \cos\gamma , k \sin\gamma, 0)$$

y-reflected mirror image has **k**-vector $\mathbf{k}^{(-)}$ at angle $-\gamma$.

$$\mathbf{k}^{(-)} = (k^{(-)}_x, k^{(-)}_y, 0) = (k \cos\gamma , -k \sin\gamma, 0).$$

$$\mathbf{E}(\mathbf{r},t) = \exp i(\mathbf{k}^{(+)}\bullet\mathbf{r} - \omega t) + \exp i(\mathbf{k}^{(-)}\bullet\mathbf{r} - \omega t)$$
$$= \exp i(kx \cos\gamma + ky \sin\gamma - \omega t) + \exp i(kx \cos\gamma - ky \sin\gamma - \omega t)$$
$$= \exp i(kx \cos\gamma - \omega t) [\exp i(ky \sin\gamma) + \exp i(-ky \sin\gamma)]$$
$$= e^{i(kx \cos\gamma - \omega t)} [2\cos(ky \sin\gamma)]$$

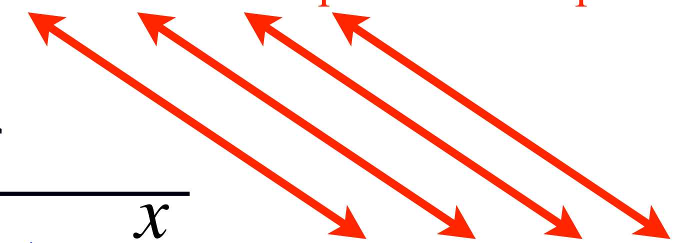
guide phase wave and group wave

TE boundary conditions make group be zero on metal walls $y=\pm W/2$.

$0=2 \cos(k(W/2) \sin\gamma)$, or: $k(W/2) \sin\gamma = \pi/2$, or: $\sin\gamma = \pi/(kW)$

Waveguide dispersion and geometry

Assume $T_{\text{ransverse}}E_{\text{lectric}}$-mode.
It always has **E** polarized parallel to *xz* plane



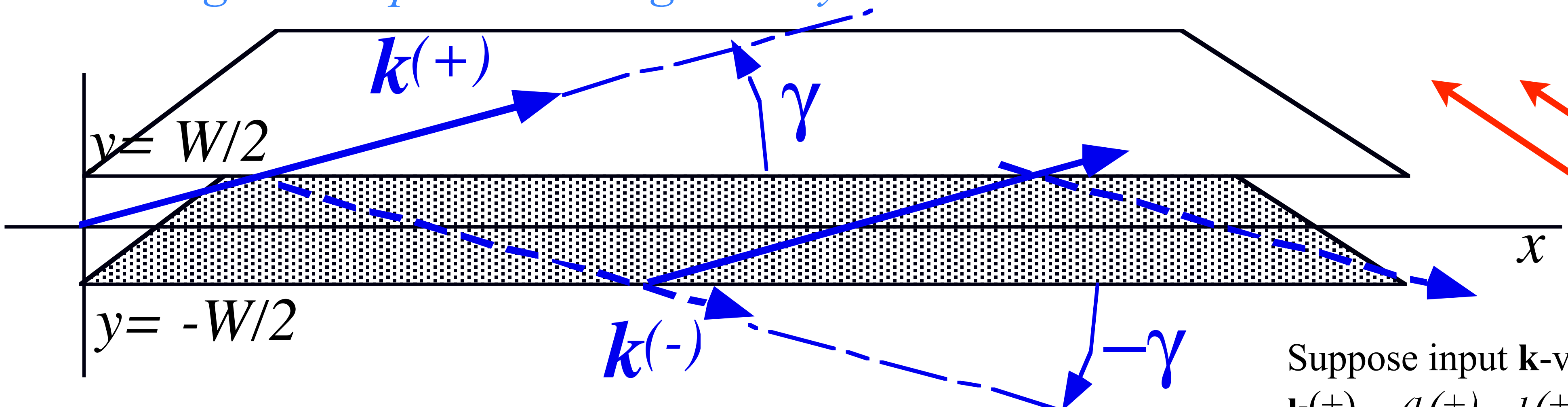


Fig. 6B.1 A "hall of mirrors" model for an optical wave guide of width W.

Suppose input **k**-vector $\mathbf{k}^{(-)}$ enters at angle +γ.
$\mathbf{k}^{(+)} = (k^{(+)}_x, k^{(+)}_y, 0) = (k\cos\gamma, k\sin\gamma, 0)$

y-reflected mirror image has **k**-vector $\mathbf{k}^{(-)}$ at angle -γ.
$\mathbf{k}^{(-)} = (k^{(-)}_x, k^{(-)}_y, 0) = (k\cos\gamma, -k\sin\gamma, 0)$.

$$\mathbf{E}(\mathbf{r},t) = \exp i(\mathbf{k}^{(+)}\bullet\mathbf{r} - \omega t) + \exp i(\mathbf{k}^{(-)}\bullet\mathbf{r} - \omega t)$$
$$= \exp i(kx\cos\gamma + ky\sin\gamma - \omega t) + \exp i(kx\cos\gamma - ky\sin\gamma - \omega t)$$
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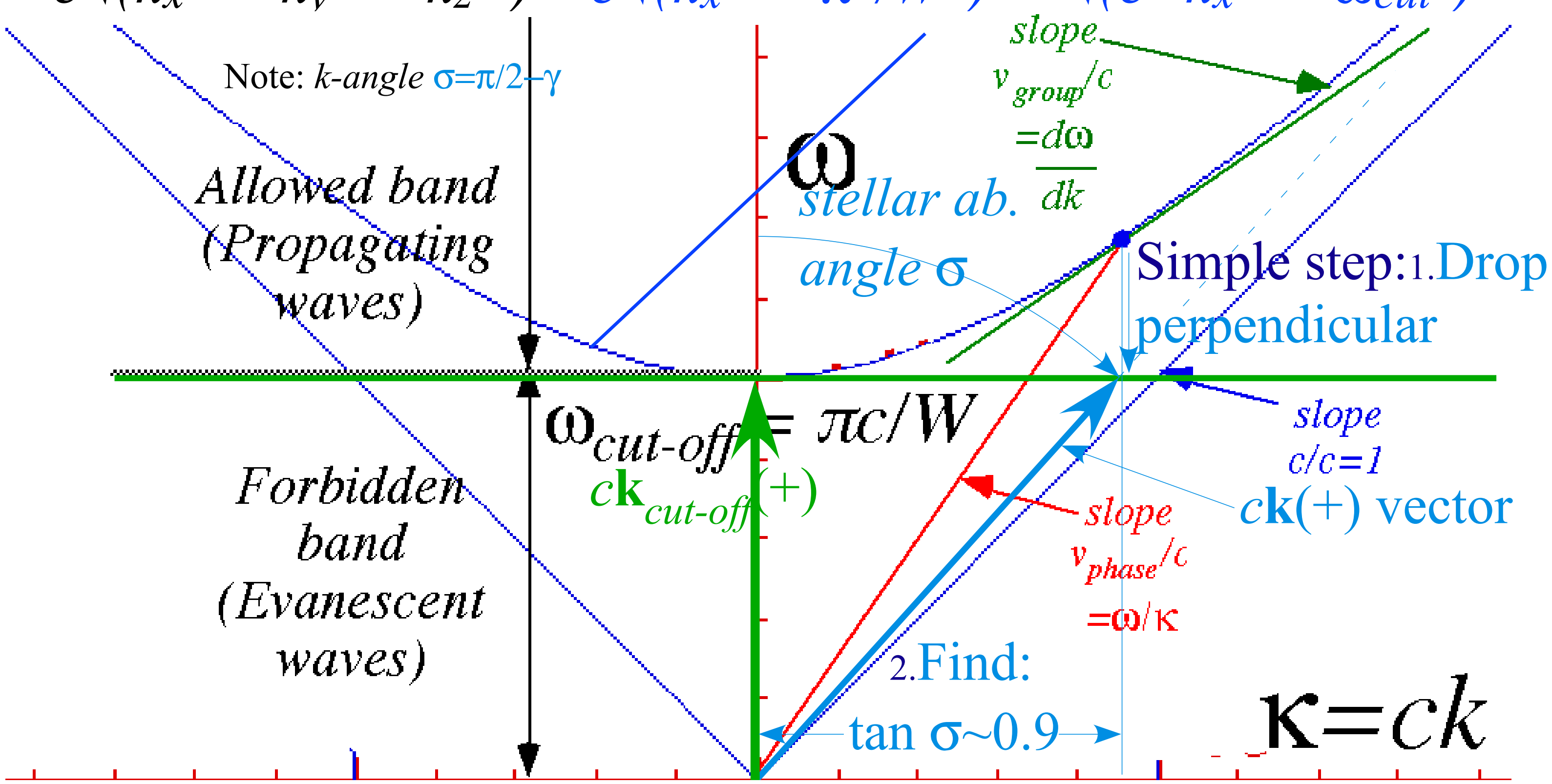
$$= e^{\,i(kx\cos\gamma - \omega t)}\,[2\cos(ky\sin\gamma)]$$

guide *phase wave* and *group wave*

TE boundary conditions make group be zero on metal walls $y=\pm W/2$.
$0=2\cos(k(W/2)\sin\gamma)$, or: $k(W/2)\sin\gamma = \pi/2$, or: $\sin\gamma = \pi/(kW)$

Condition $k^{(+)}_y = k\sin\gamma = \pi/W$ gives *dispersion function* $\omega(k_x)$ or ω *vs.* k_x *relation*

$$\omega = kc = c\sqrt{(k_x^2 + k_y^2 + k_z^2)} = c\sqrt{(k_x^2 + \pi^2/W^2)} = \sqrt{(c^2 k_x^2 + \omega_{cut}^2)}$$ where: $\omega_{cut} = \pi c/W$.



$$k_x = \sqrt{(\omega^2/c^2 - \pi^2/W^2)}$$

Fig. 6B.2 Dispersion function for a fundamental TE wave guide mode

Waveguide dispersion and geometry

$$\omega = kc = c\sqrt{(k_x^2 + k_y^2 + k_z^2)} = c\sqrt{(k_x^2 + \pi^2/W^2)} = \sqrt{(c^2 k_x^2 + \omega_{cut}^2)}$$

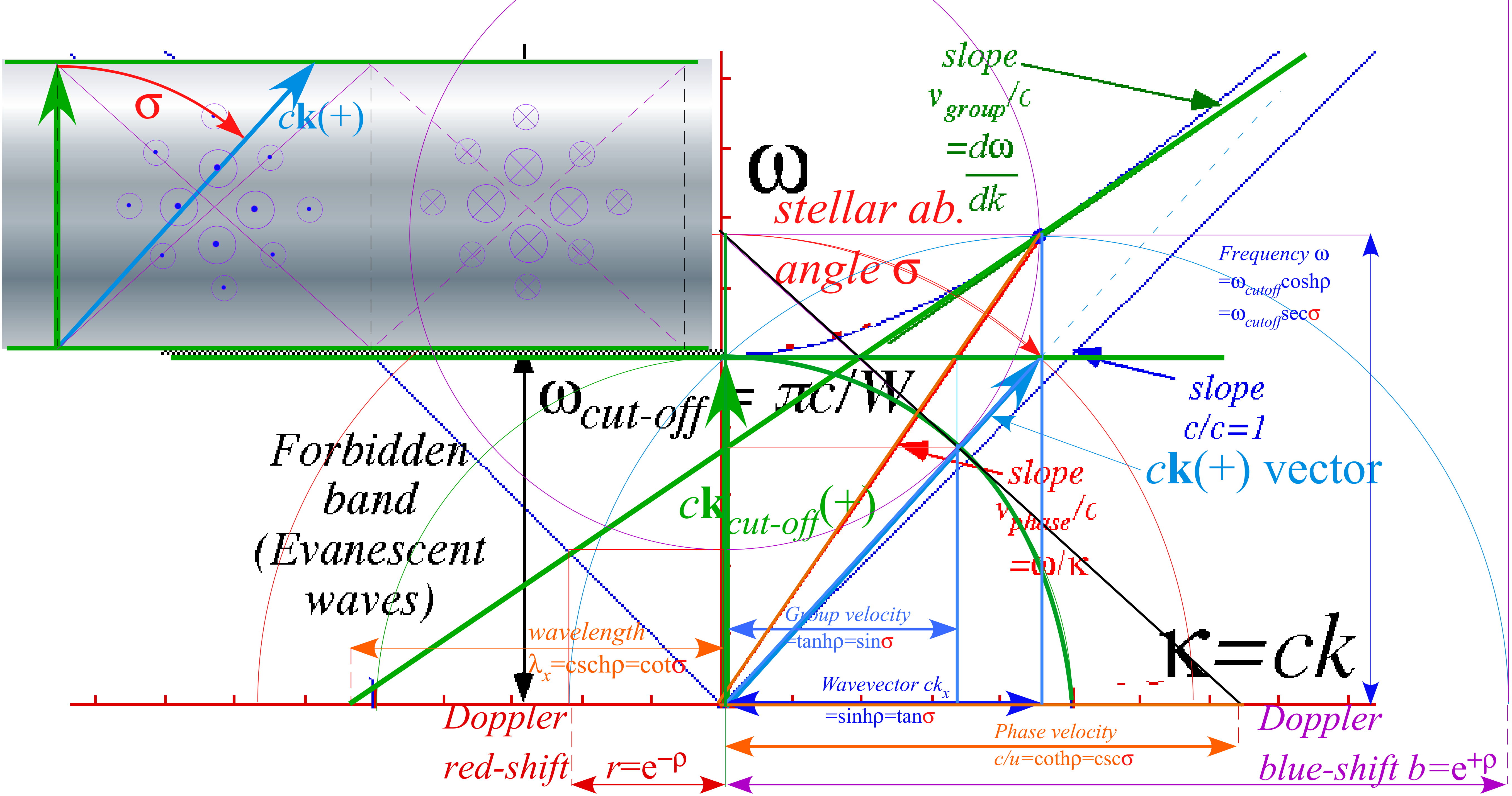


Fig. 6B.8 Thales geometry of cavity or waveguide mode

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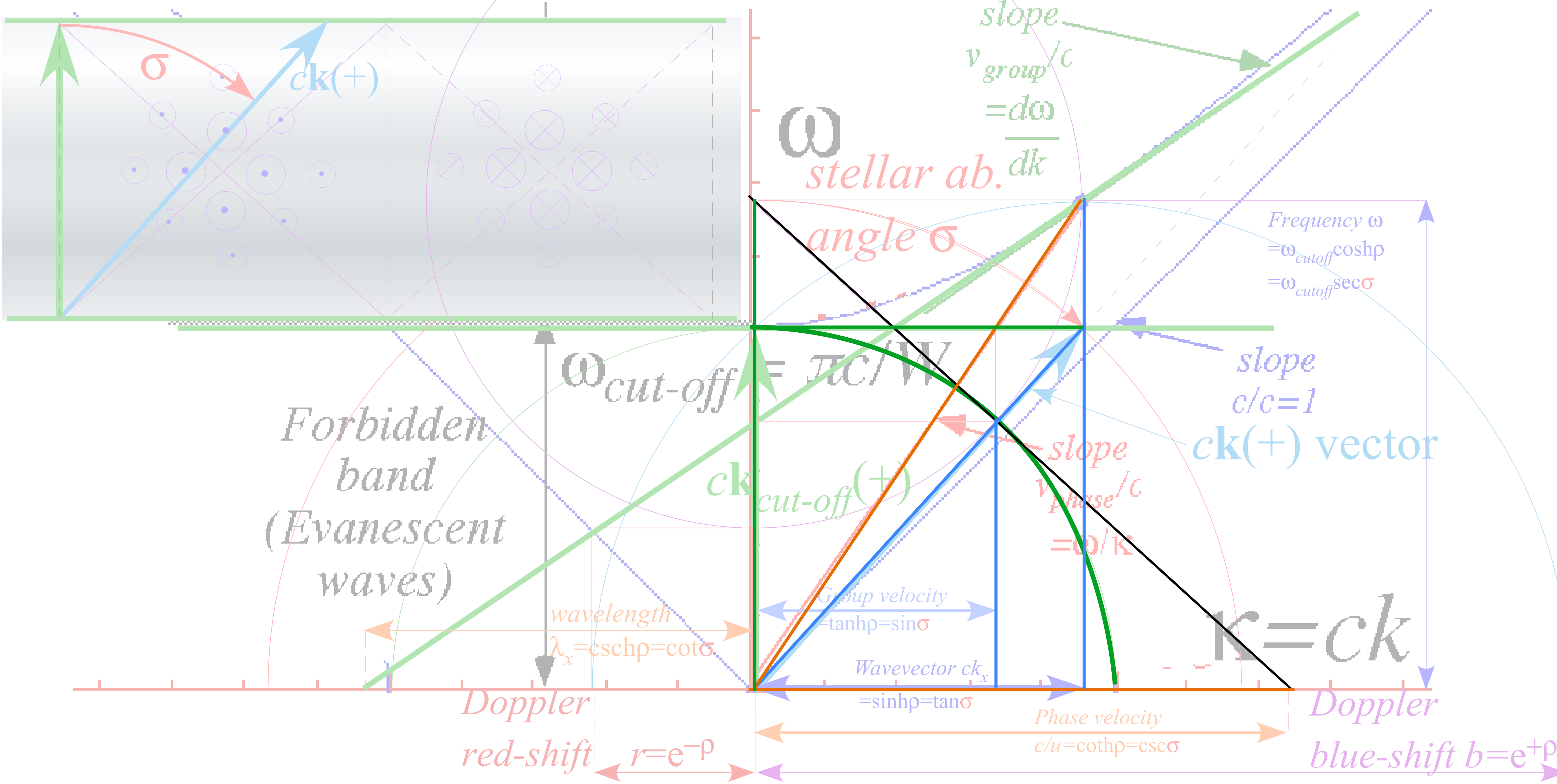
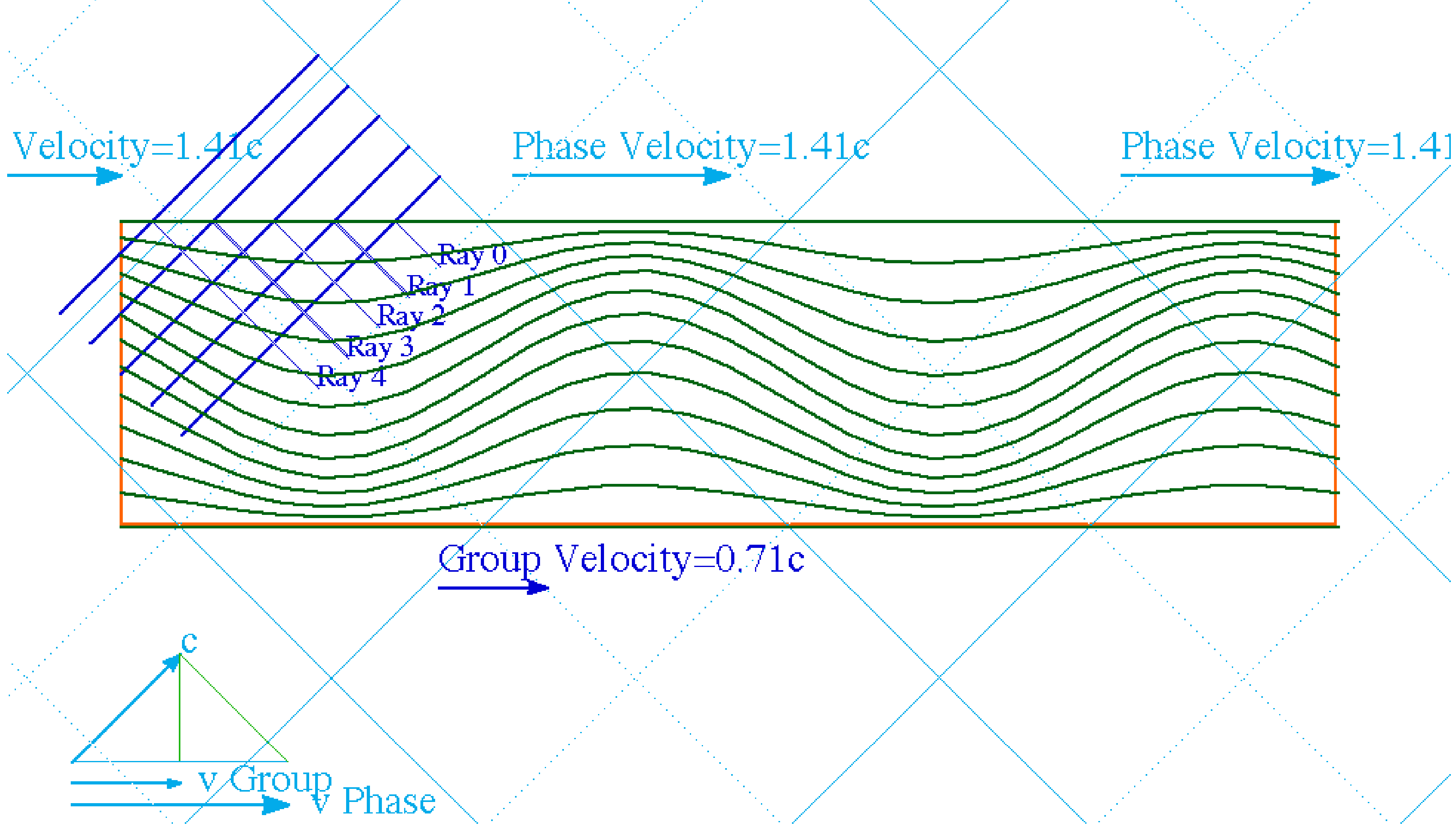


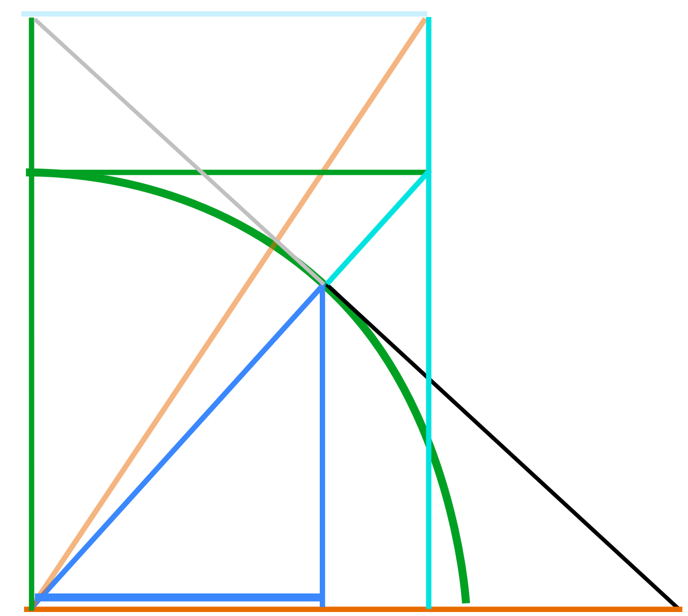
Fig. 6B.8 Thales geometry of cavity or waveguide mode

(Lecture 28 ends here)

Waveguide dispersion and geometry

$$\omega = kc = \sqrt{(c^2 k_x^2 + \omega_{cut}^2)}$$





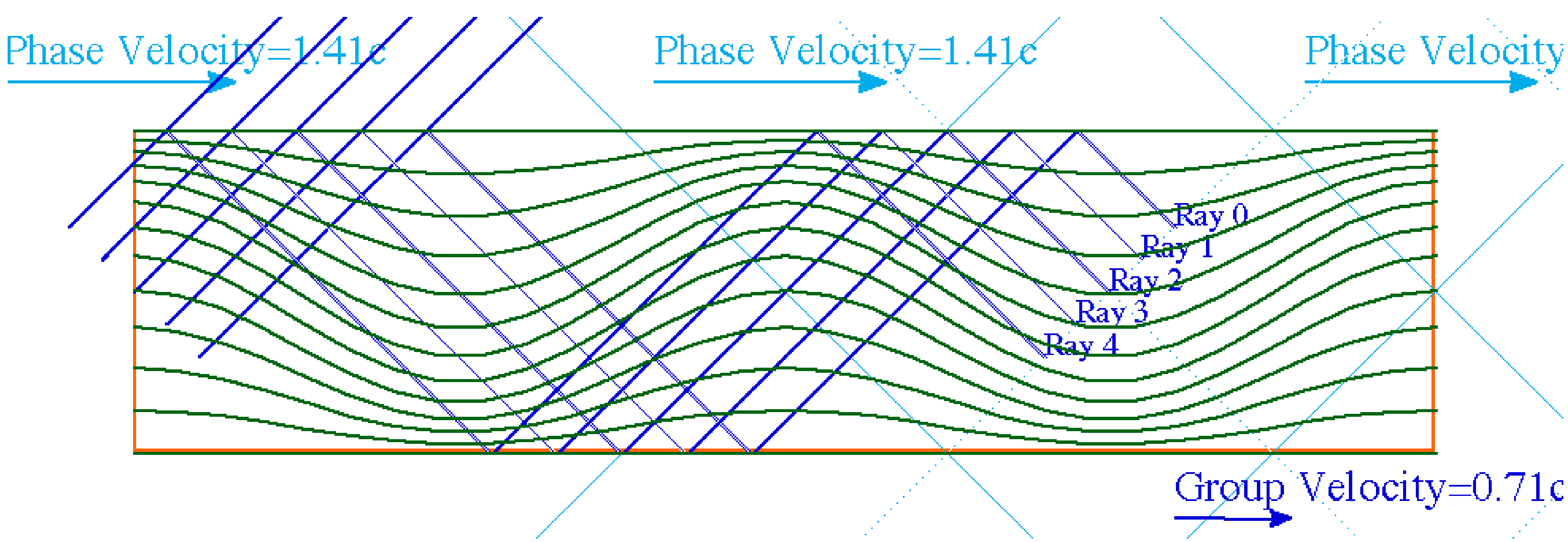


Fig. 6B.3 Right moving guide wave with $\gamma = 45°$, $V_{phase}=\sqrt{2}c$, $V_{group}=c/\sqrt{2}$.

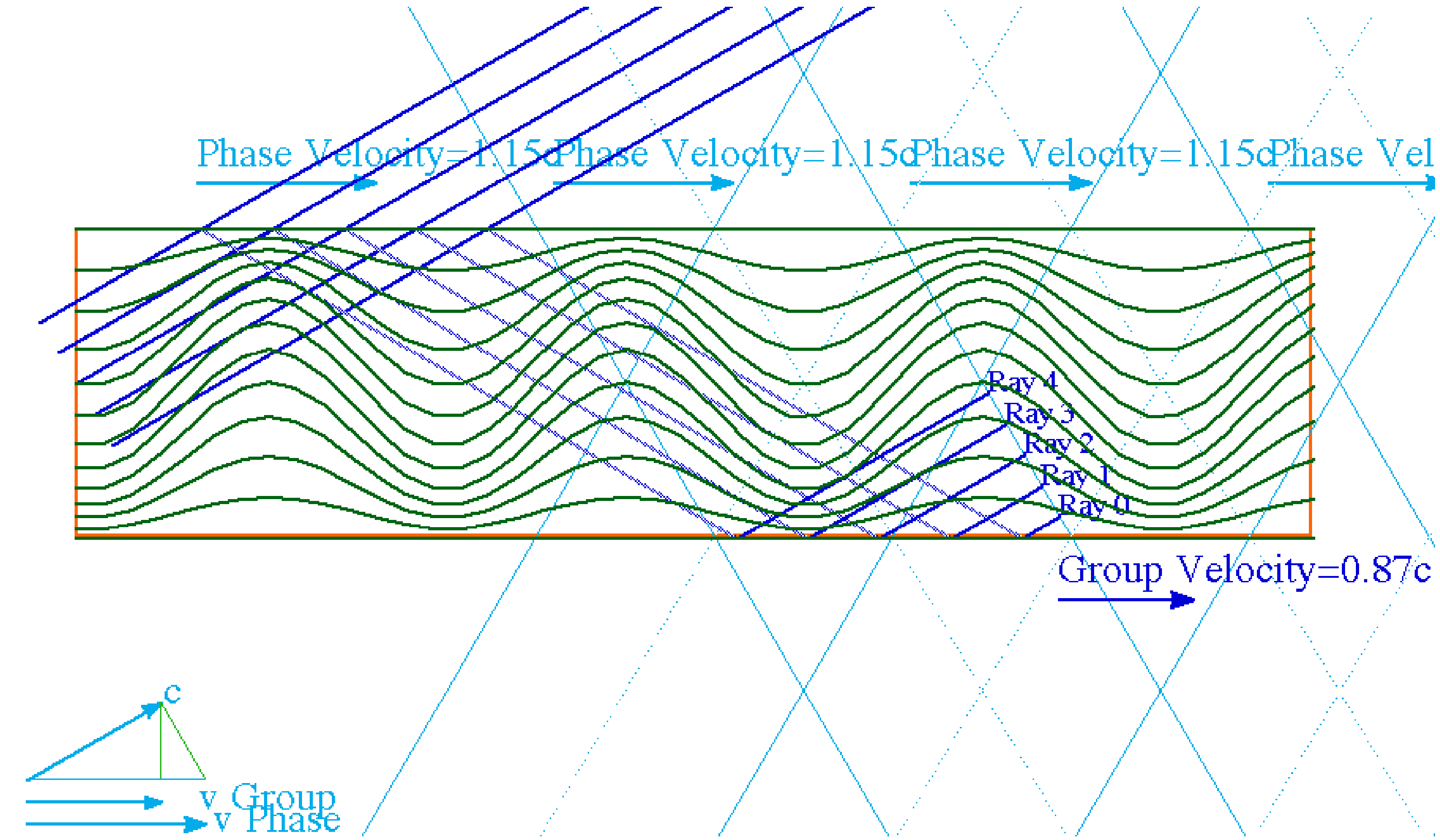
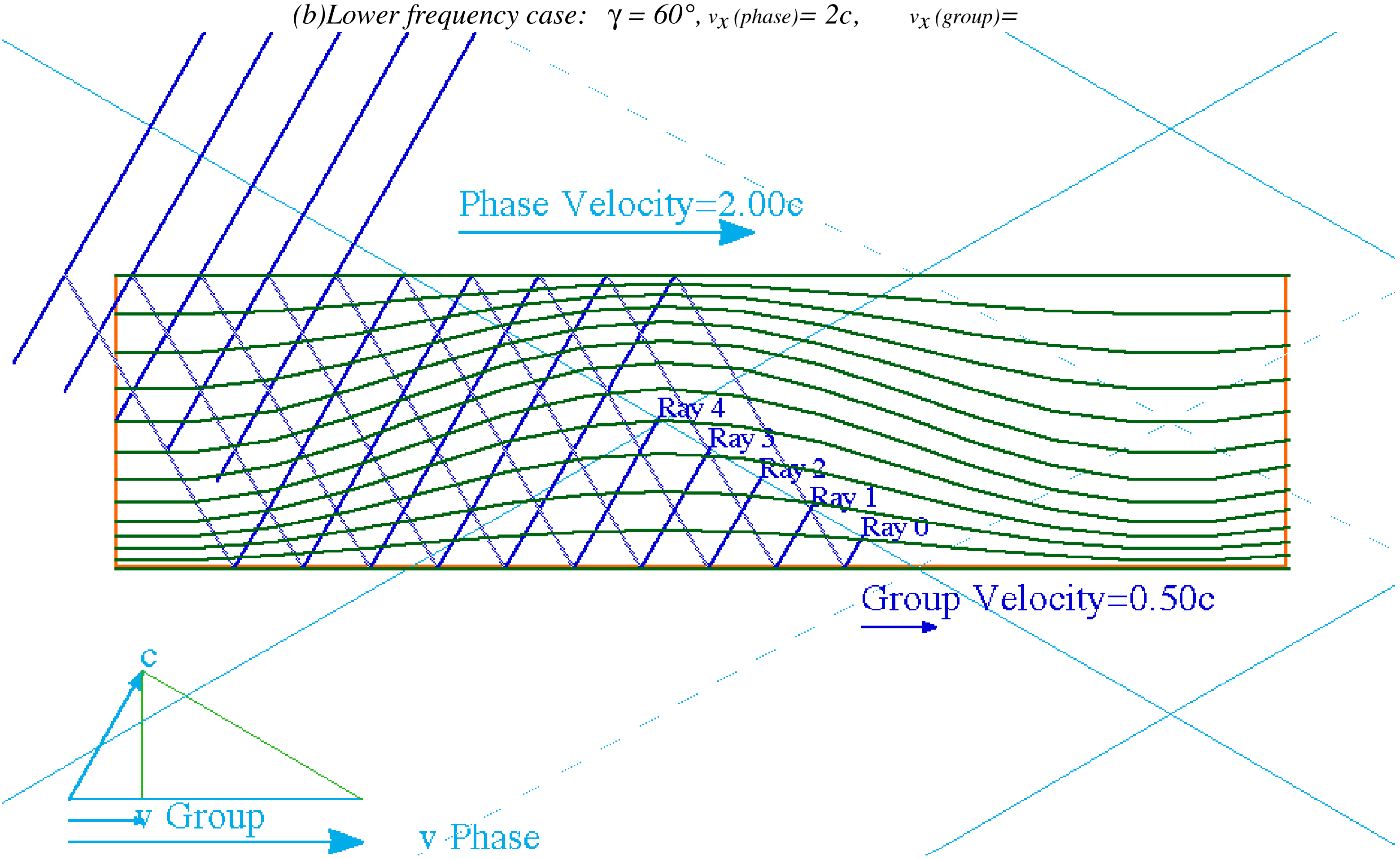


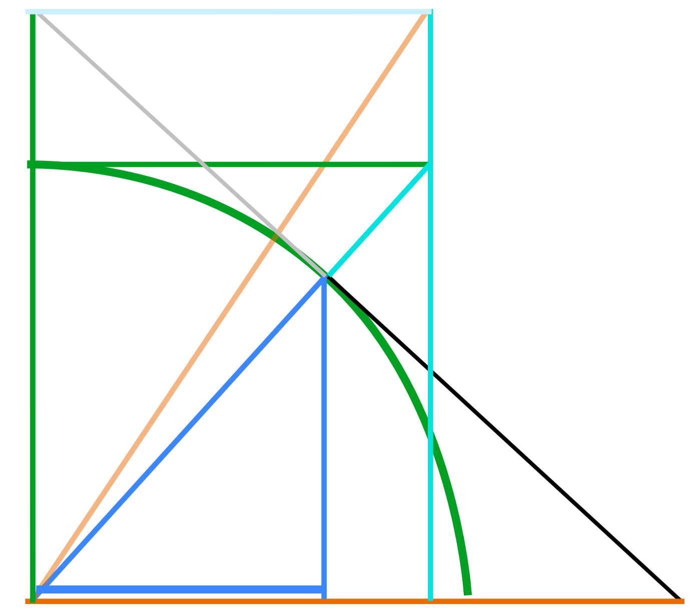
Fig. 6B.5 Guide waves. (a)Higher frequency case: $\gamma = 30°$, $v_{x\,(phase)}=c\sqrt{3}/2c$, $v_{x\,(group)}=c2/\sqrt{3}$.

(b)Lower frequency case: $\gamma = 60°$, $v_{x\,(phase)}= 2c$, $v_{x\,(group)}=$



$k_x = \sqrt{(\omega^2/c^2 - \pi^2/W^2)}$

$\omega = kc = \sqrt{(c^2 k_x^2 + \omega_{cut}^2)}$

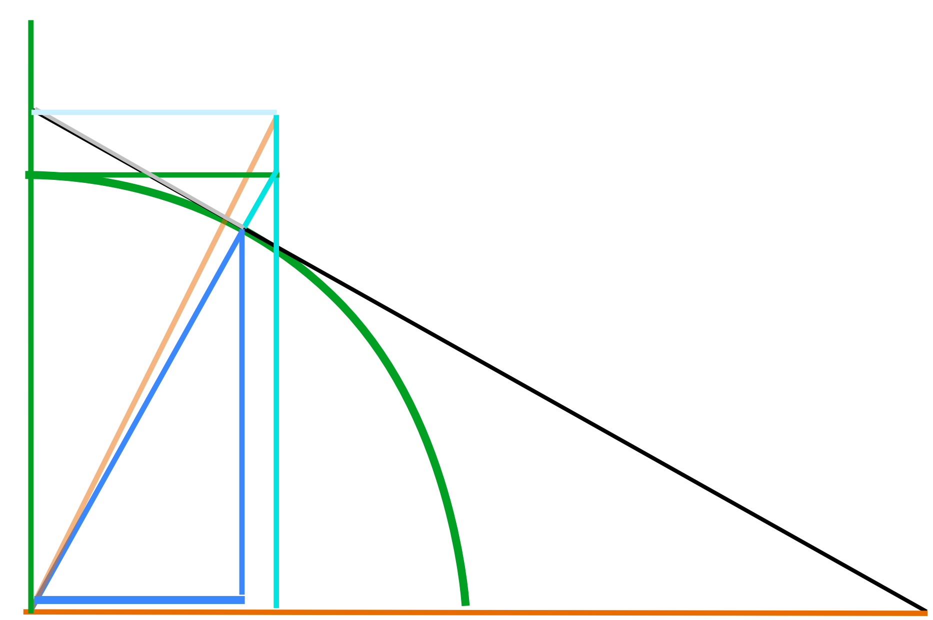


$v_x\,(phase) = \omega/k_x \;=\; c\omega/\sqrt{(\omega^2 - \pi^2 c^2/W^2)}$

$= c/\cos\gamma = c/\sin\sigma = c \csc\sigma$

$v_x\,(group) = d\omega/dk_x \;=\; ck_x/\sqrt{(k_x^2 + \pi^2/W^2)}$

$= c\,(\omega^2 - \pi^2 c^2/W^2)^{1/2}/\omega = c\cos\gamma = c\sin\sigma$



Below cut-off:Evanescent waves

Consider angular frequency below the so-called *cut-off value* ω_{cut} from (6B.5b).

$$\omega_{cut} = \pi\, c/W$$

Then the wave vector k_x will go thru zero to becomes imaginary.

$$k_x = \sqrt{(\omega^2 - \pi^2 c^2/W^2)}$$

This affects the the usual *propagating* wave $\Psi = exp\ i(k_x x - \omega t)$ rather severely.

Instead of propagating nicely, we get a so-called *evanescent wave* $\Psi = exp(-\mu_x x) exp\ i(-\omega t)$

It decays exponentially with distance x inside wave guide with decay rate constant $\mu_x = \sqrt{(\pi^2 c^2/W^2 - \omega^2)} = ik_x$

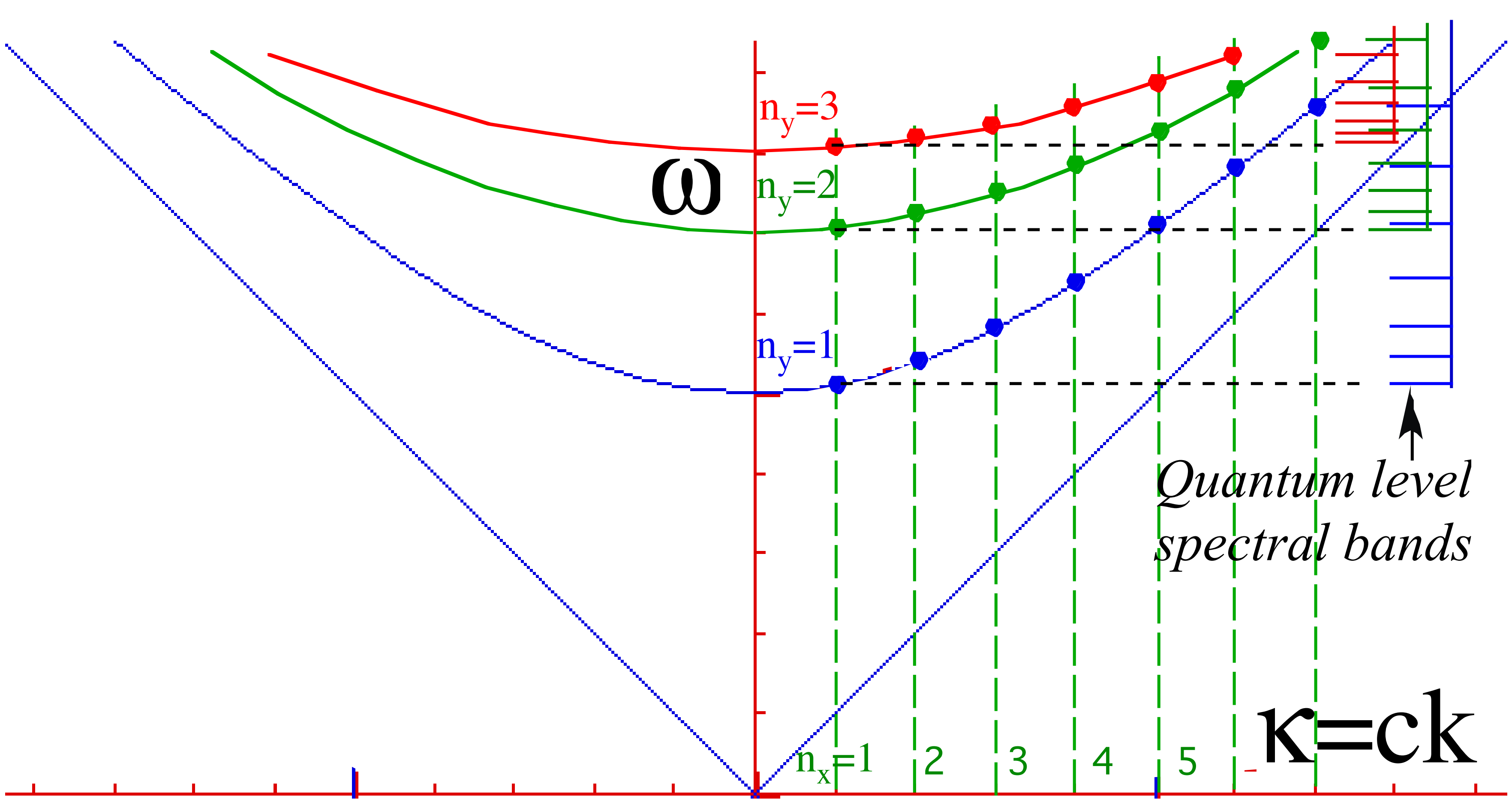
Cavity eigenfunctions and eigenvalues

Hall of Mirrors capped by a pair of doors at $x=0$ and $x=L$ becomes a *wave cavity* of length L.
The doors demand the wave electric field be zero at x-boundaries as well as along the walls. New boundary conditions:

$$k_x = k \cos\gamma = n_x \pi / L \quad (n_x = 1, 2, \ldots)$$

Frequency bands are broken into discrete "quantized" values $\omega_{n_x n_y}$, one for each pair of integers or "quantum numbers" n_x and n_y.

$$\omega_{n_x n_y} = kc = c\sqrt{(k_x^2 + k_y^2 + k_z^2)} = c\sqrt{(n_x^2 \pi^2/L^2 + n_y^2 \pi^2/W^2)}$$



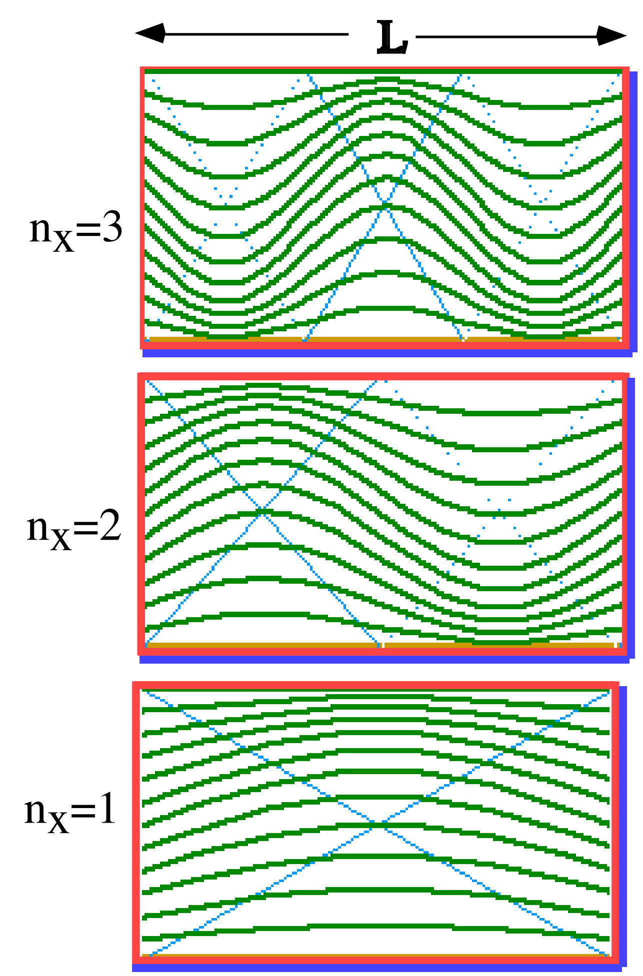


Fig. 6B.7 Cavity modes for three lowest quantum numbers

Fig. 6B.6 Cavity mode dispersion diagram showing overlapping and discrete ω and k values.

Galloping waves due to unmatched amplitudes

2-CW dynamics has two 1-CW amplitudes $A_{\rightarrow}$ and $A_{\leftarrow}$ that we now allow to be *un*matched. $(A_{\rightarrow} \neq A_{\leftarrow})$

$$A_{\rightarrow} e^{i(k_{\rightarrow} x - \omega_{\rightarrow} t)} + A_{\leftarrow} e^{i(k_{\leftarrow} x - \omega_{\leftarrow} t)} = e^{i(k_{\Sigma} x - \omega_{\Sigma} t)} [A_{\rightarrow} e^{i(k_{\Delta} x - \omega_{\Delta} t)} + A_{\leftarrow} e^{-i(k_{\Delta} x - \omega_{\Delta} t)}]$$

Waves have half-sum mean-phase rates $(k_{\Sigma}, \omega_{\Sigma})$ and half-difference group rates $(k_{\Delta}, \omega_{\Delta})$.

$$k_{\Sigma} = (k_{\rightarrow} + k_{\leftarrow})/2 \qquad k_{\Delta} = (k_{\rightarrow} - k_{\leftarrow})/2$$

$$\omega_{\Sigma} = (\omega_{\rightarrow} + \omega_{\leftarrow})/2 \qquad \omega_{\Delta} = (\omega_{\rightarrow} - \omega_{\leftarrow})/2$$

Also important is amplitude mean $A_{\Sigma} = (A_{\rightarrow} + A_{\leftarrow})/2$ and half-difference $A_{\Delta} = (A_{\rightarrow} - A_{\leftarrow})/2$.

Detailed wave motion depends on standing-wave-ratio *SWR* or the inverse standing-wave-quotient *SWQ*.

$$SWR = \frac{(A_{\rightarrow} - A_{\leftarrow})}{(A_{\rightarrow} + A_{\leftarrow})} \qquad SWQ = \frac{(A_{\rightarrow} + A_{\leftarrow})}{(A_{\rightarrow} - A_{\leftarrow})}$$

These are analogous mean frequency ratios for group velocity and its inverse that is phase velocity.

$$V_{group} = \frac{\omega_{\Delta}}{k_{\Delta}} = c\frac{(\omega_{\rightarrow} - \omega_{\leftarrow})}{(\omega_{\rightarrow} + \omega_{\leftarrow})} \qquad V_{phase} = \frac{\omega_{\Sigma}}{k_{\Sigma}} = c\frac{(\omega_{\rightarrow} + \omega_{\leftarrow})}{(\omega_{\rightarrow} - \omega_{\leftarrow})}$$

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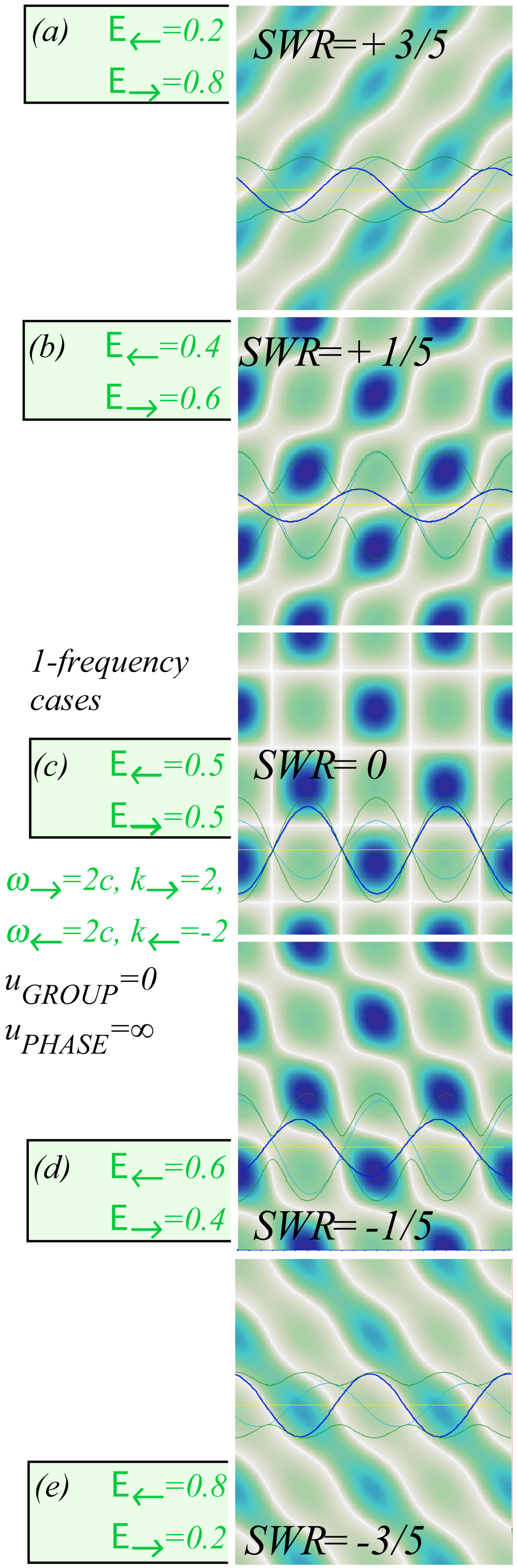

Fig. 6.1 Monochromatic (1-frequency) 2-CW wave space-time patterns.

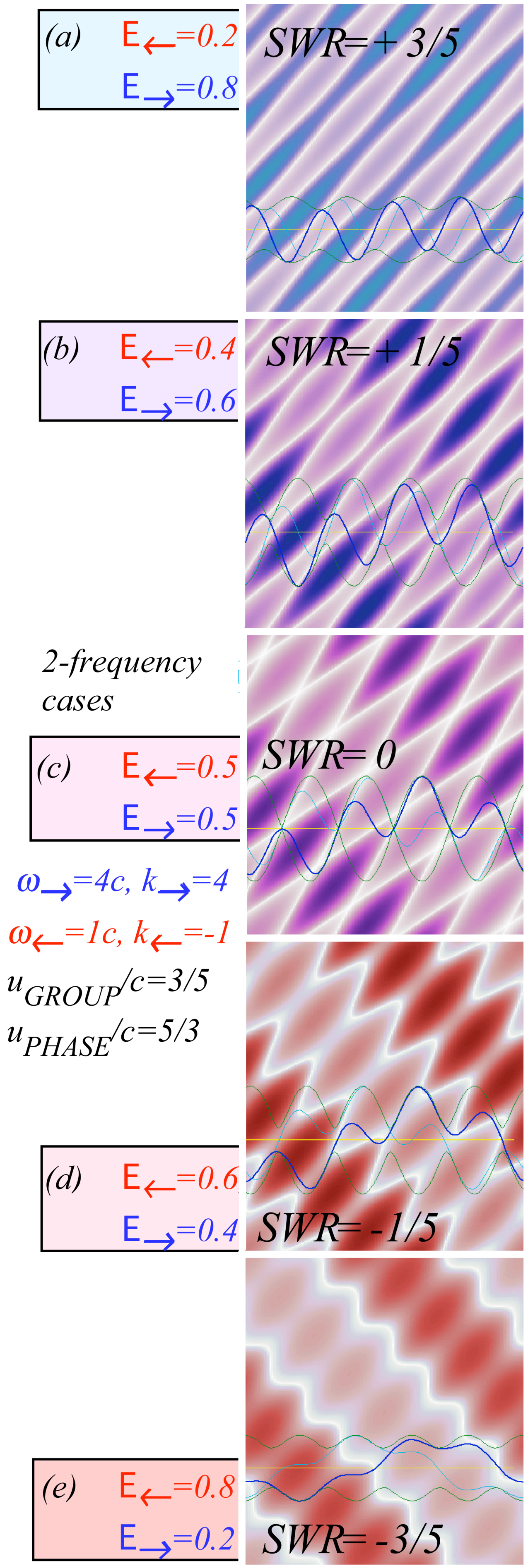

Fig. 6.2 Dichromatic (2-frequency) 2-CW wave space-time patterns.

Fig. 6.3 (a-g) Elliptic polarization ellipses relate to galloping waves in Fig. 6.1. (h-i) Kepler anomalies.

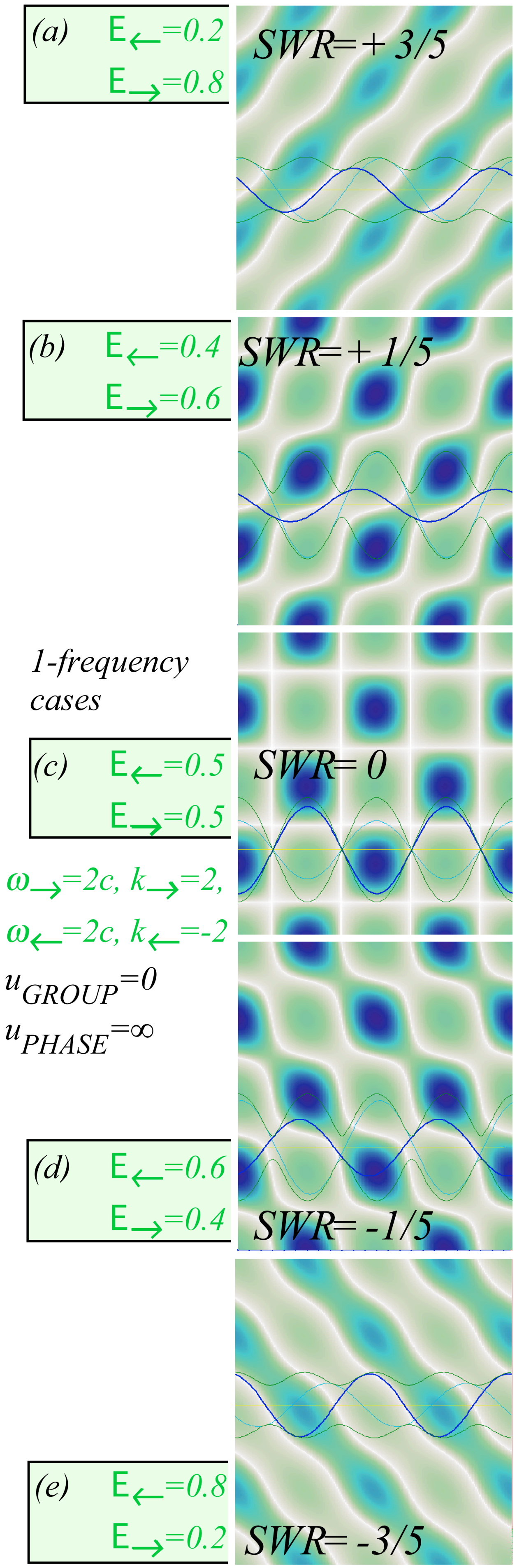

Fig. 6.1 Monochromatic (1-frequency) 2-CW wave space-time patterns.

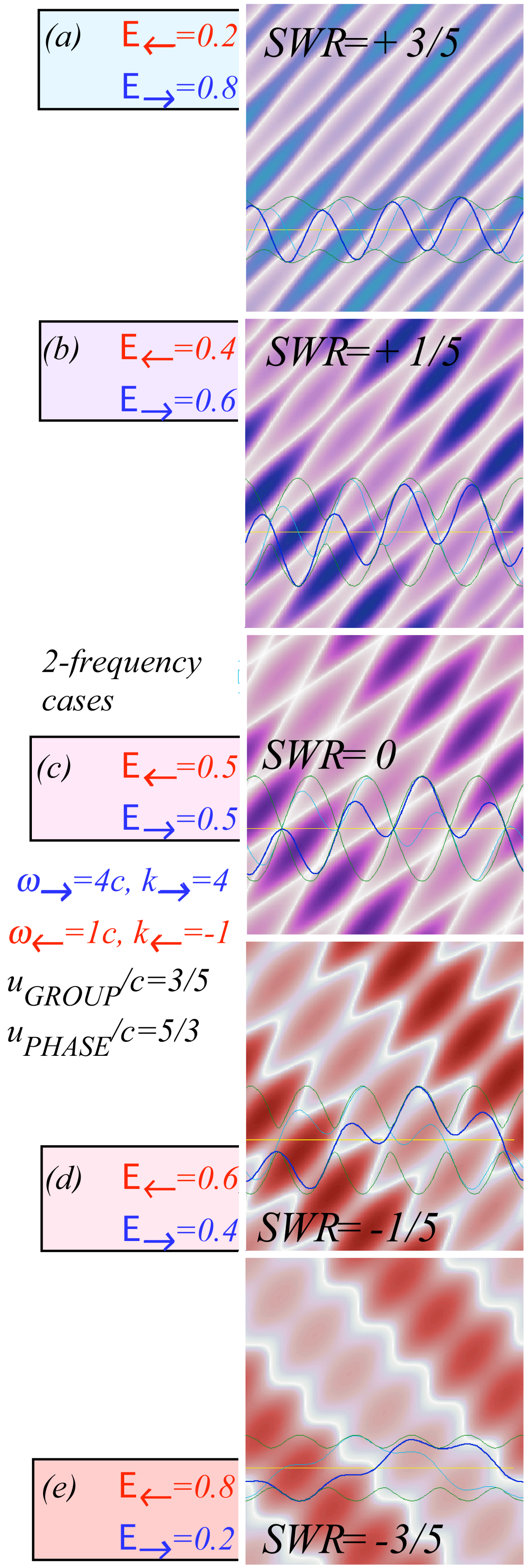

Fig. 6.2 Dichromatic (2-frequency) 2-CW wave space-time patterns.

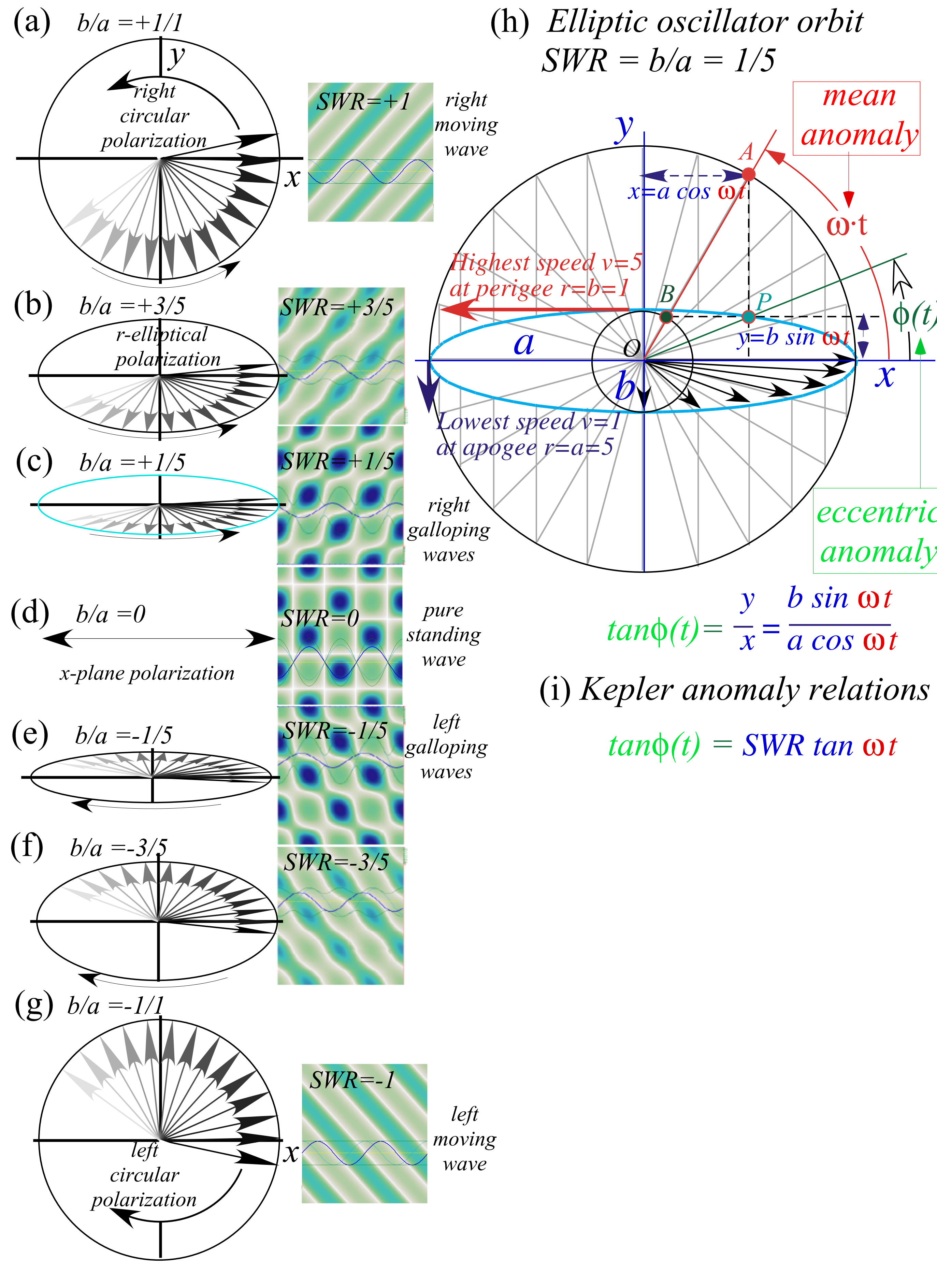

Fig. 6.3 (a-g) Elliptic polarization ellipses relate to galloping waves in Fig. 6.1. (h-i) Kepler anomalies.