Lecture 36.

Introduction to classical oscillation and resonance I.

(Ch. 1 of Unit 3 4.26.12)

1D forced-damped-harmonic oscillator equations and Green's function solutions

Linear harmonic oscillator equation of motion.

Linear damped-harmonic oscillator equation of motion.

Frequency retardation and amplitude damping

Linear forced-damped-harmonic oscillator equation of motion.

Phase lag and amplitude resonance

Properties of Green's function solutions and their physical behavior

Quality factors and geometry of resonance

Complete Green's Solution for the FDHO (Forced-Damped-Harmonic Oscillator)

Beat, lifetimes, and quality factor effects

end of Lecture 36

Approximate Lorentz-Green's Function for high quality FDHO (Quantum propagator)

Linear forced-damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore} + F_{stimulus}$$

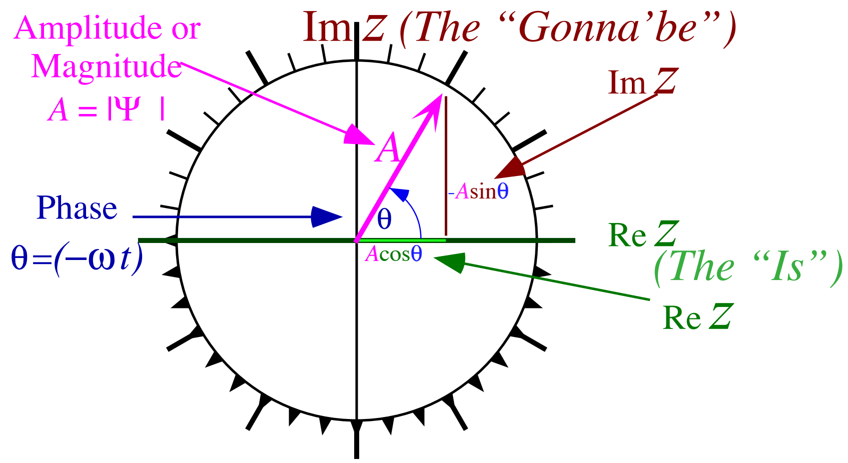
$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m} + \frac{F_{stimulus}}{m}$$

Stimulating acceleration $a_{stimulus}=a(t)$ due to stimulating force $F_{stimulus}(t)$ (Typically **E**-field)

Phasor Clock

$$Z(t) = Ae^{-i(\omega t)} = A\cos(\omega t) - i\,A\sin(\omega t)$$

Phasor clocks turn clockwise in time for positive ω



$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m}E(t)$$

Coordinate $z=z(t)$ is the response coordinate for a particle of mass m and charge e

driven by external a stimulating force → $F_{stimulus}(t) = eE(t)$

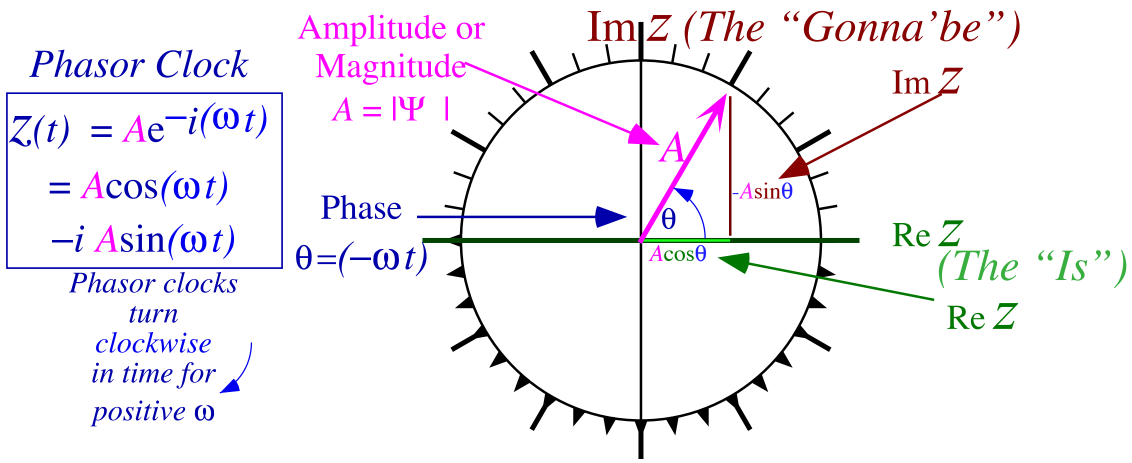
held back by a harmonic (linear) restoring force → $F_{restore} = -kz, \quad \left(k = \omega_0^2 m\right),$

retarded by frictional damping force → $F_{damping} = -b\frac{dz}{dt}, \quad \left(b = 2\Gamma m\right)$

Linear *harmonic oscillator equation of motion.*

$$F_{total}(t) = m\frac{d^2z}{dt^2} = \quad F_{restore}$$

$$\frac{d^2z}{dt^2} = \quad \frac{F_{restore}}{m}$$



$$\frac{d^2z}{dt^2} + \omega_0^2 z = 0$$

Coordinate *z=z(t)* is the response coordinate for a particle of mass *m* and charge *e*

held back by a harmonic (linear) restoring force → $F_{restore} = -kz, \quad \left(k = \omega_0^2 m\right),$

Linear *harmonic oscillator equation of motion.*

$$F_{total}(t) = m\frac{d^2z}{dt^2} = \qquad F_{restore}$$

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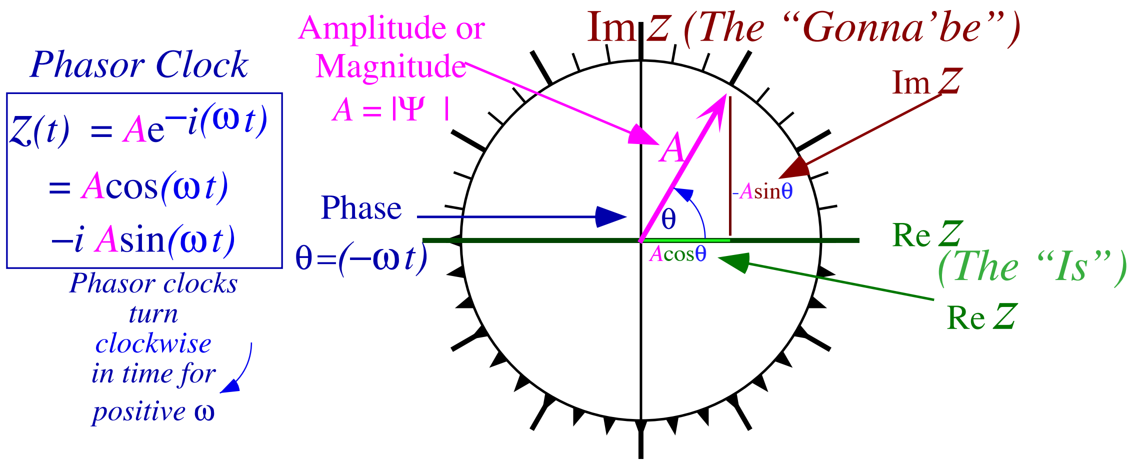


$$\frac{d^2z}{dt^2} \qquad + \omega_0^2 z = \; 0$$

Coordinate *z=z(t)* is the response coordinate for a particle of mass *m* and charge *e*

held back by a harmonic (linear) restoring force → $F_{restore} = -kz, \; \left(k = \omega_0^2 m\right),$

The famous one-Hertz (ν_0=1/s. or: $\omega_0 = 2\pi = 6.2832$rad/s.) oscillator.

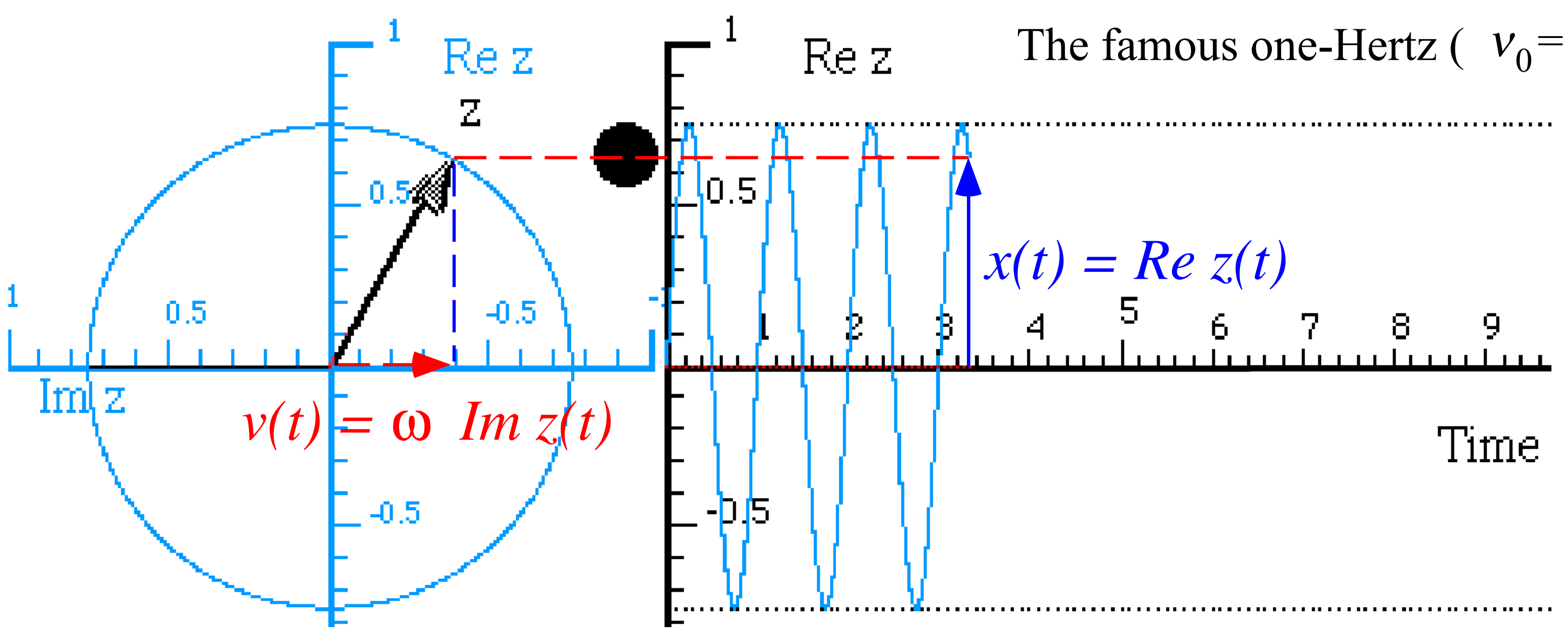
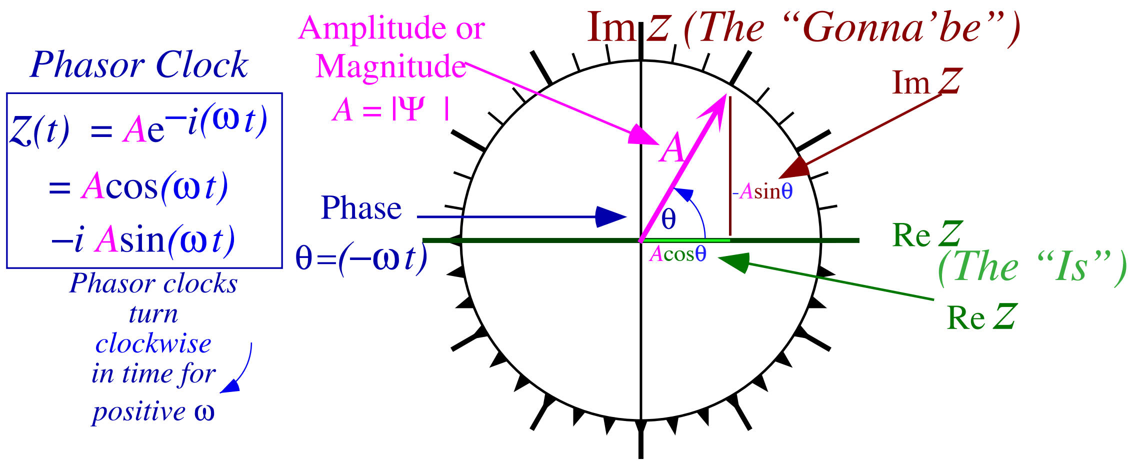


Fig. 3.2.2 Phasor z and corresponding coordinate versus time plot for $\omega_0 = 2\pi$ and $\Gamma = 0$

Linear ... *-damped-harmonic oscillator equation of motion.*

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore}$$

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$



$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = 0$$

Coordinate $z=z(t)$ is the response coordinate for a particle of mass m and charge e

held back by a harmonic (linear) restoring force → $F_{restore} = -kz, \quad \left(k = \omega_0^2 m\right),$

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Linear -damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore}$$

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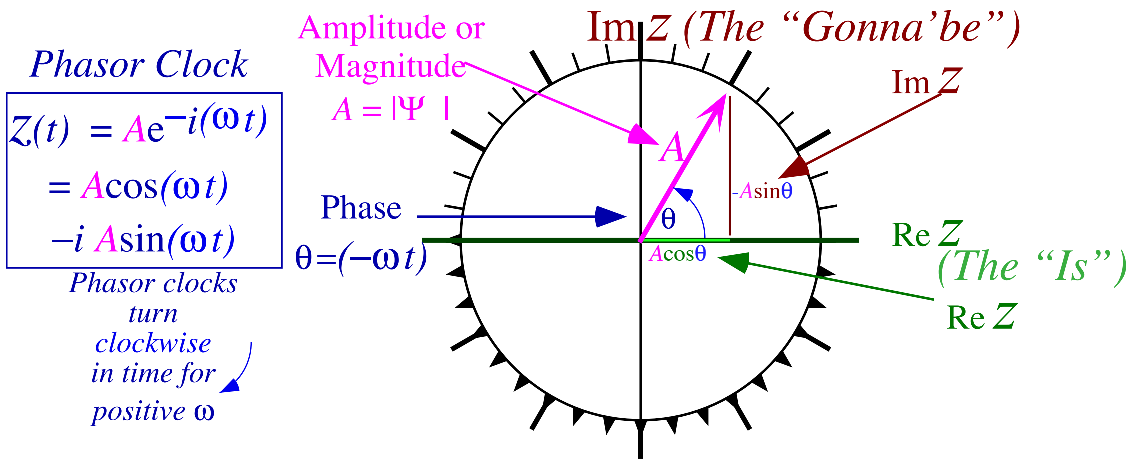

$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = 0$$

Trick:
Set: $z = z(t) = Ae^{-i\omega t}$

$$\left[(-i\omega)^2 + 2\Gamma(-i\omega) + \omega_0^2\right]e^{-i\omega t} = 0$$

$$\omega^2 + 2i\Gamma\omega - \omega_0^2 = 0$$

Coordinate $z=z(t)$ is the response coordinate for a particle of mass m and charge e

held back by a harmonic (linear) restoring force → $F_{restore} = -kz$

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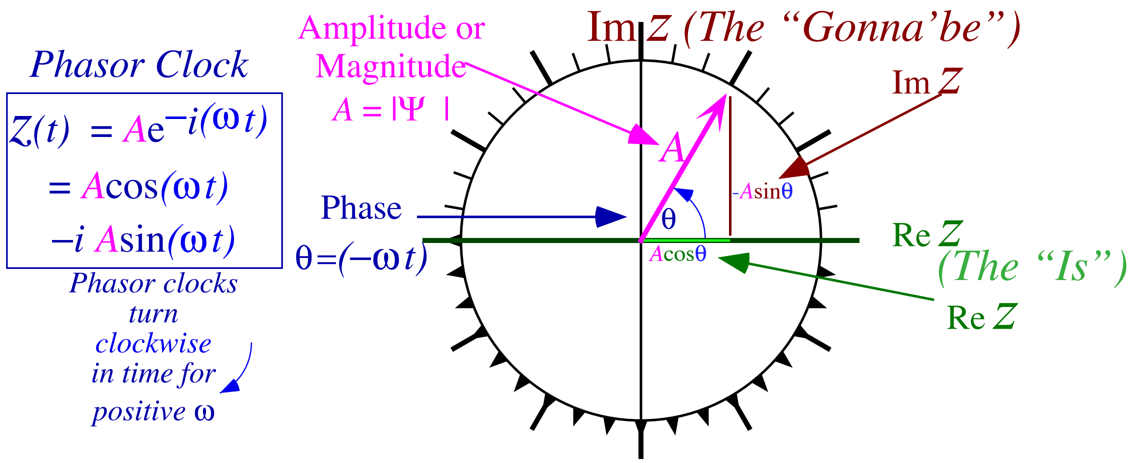
Fig. 3.2.3 Phasor z and corresponding coordinate versus time plot for $\omega_0=2\pi$ and $\Gamma=0.2$

Linear -*damped*-*harmonic* *oscillator equation of motion.*

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore}$$

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$



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Solve for: $\omega = \omega_\pm$

$$\omega_\pm = \frac{-2i\Gamma \pm \sqrt{-4\Gamma^2 + 4\omega_0^2}}{2}$$

Coordinate $z=z(t)$ is the response coordinate for a particle of mass m and charge e

held back by a harmonic (linear) restoring force → $F_{restore} = -kz$

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Fig. 3.2.3 Phasor z and corresponding coordinate versus time plot for $\omega_0=2\pi$ and $\Gamma=0.2$

Linear -damped-harmonic oscillator equation of motion.

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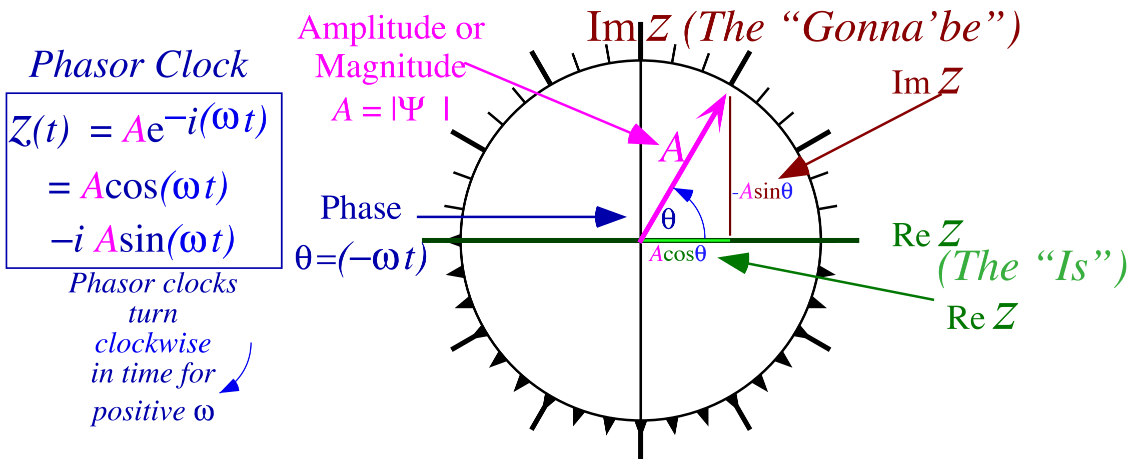

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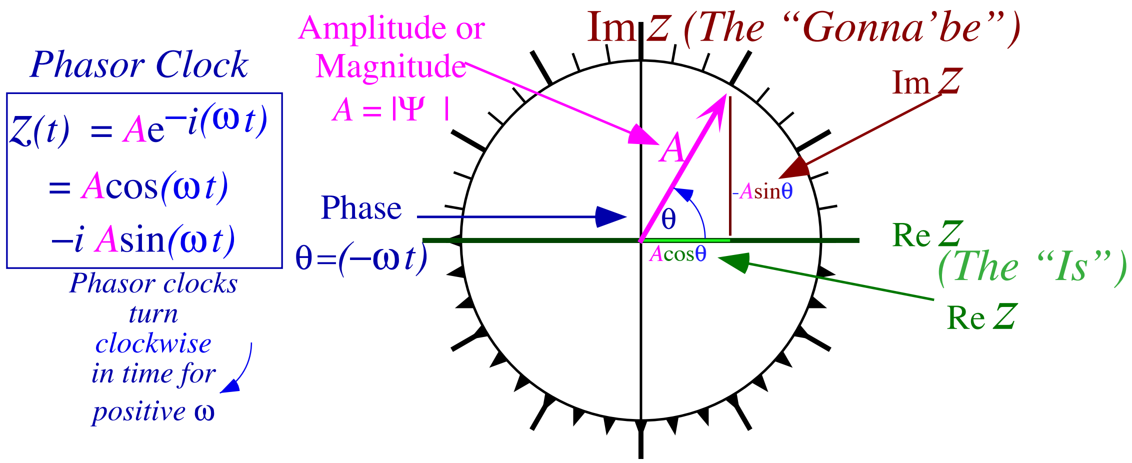

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held back by a harmonic (linear) restoring force → $F_{restore} = -kz$

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Solution:

$$z(t) = e^{-i\left(-i\Gamma \pm \sqrt{\omega_0^2 - \Gamma^2}\right)t}$$

$$= e^{\left(-\Gamma \pm i\sqrt{\omega_0^2 - \Gamma^2}\right)t}$$

$$= e^{-\Gamma t}e^{\pm i\sqrt{\omega_0^2 - \Gamma^2}\,t}$$

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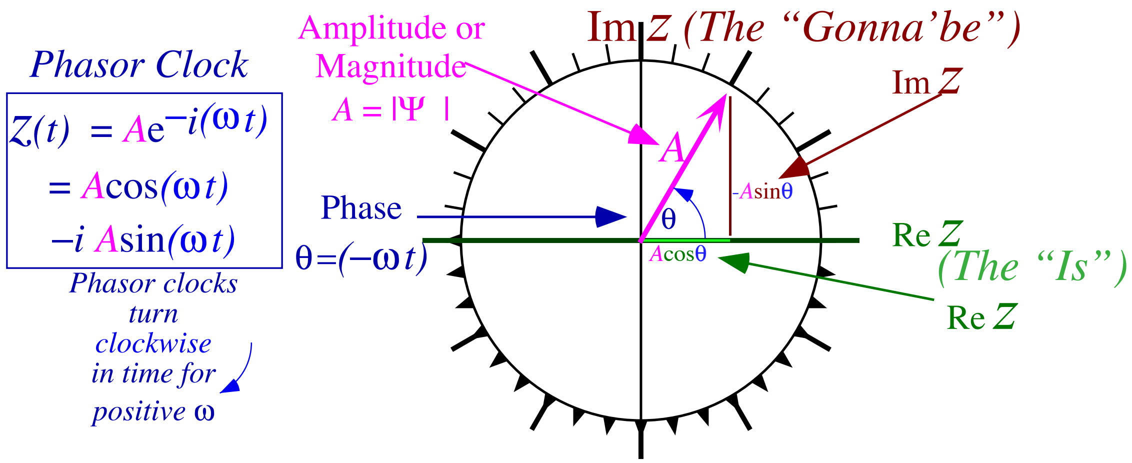

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$$= e^{-\Gamma t} e^{\pm i\omega_\Gamma t}$$

Fig. 3.2.3 Phasor z and corresponding coordinate versus time plot for $\omega_0 = 2\pi$ and $\Gamma = 0.2$

Linear -damped-harmonic oscillator equation of motion.

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Phasor Clock

$Z(t) = Ae^{-i(\omega t)}$
$= A\cos(\omega t)$
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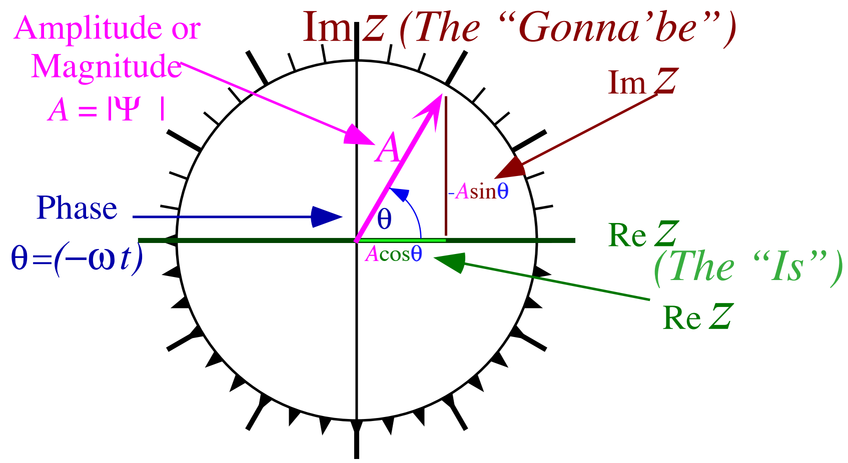
Phasor clocks turn clockwise in time for positive ω



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held back by a harmonic (linear) restoring force → $F_{restore} = -kz$

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Solution:

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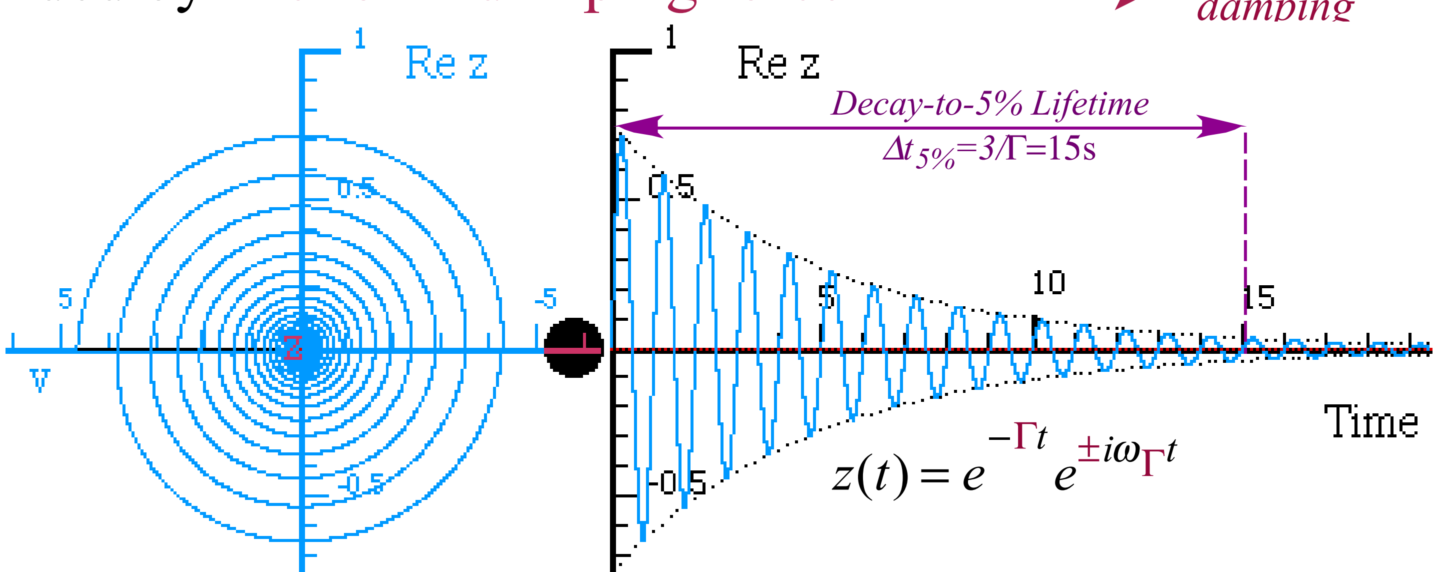

Fig. 3.2.3 Phasor z and corresponding coordinate versus time plot for $\omega_0 = 2\pi$ *and* $\Gamma = 0.2$

Linear -*damped*-*harmonic* oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore}$$

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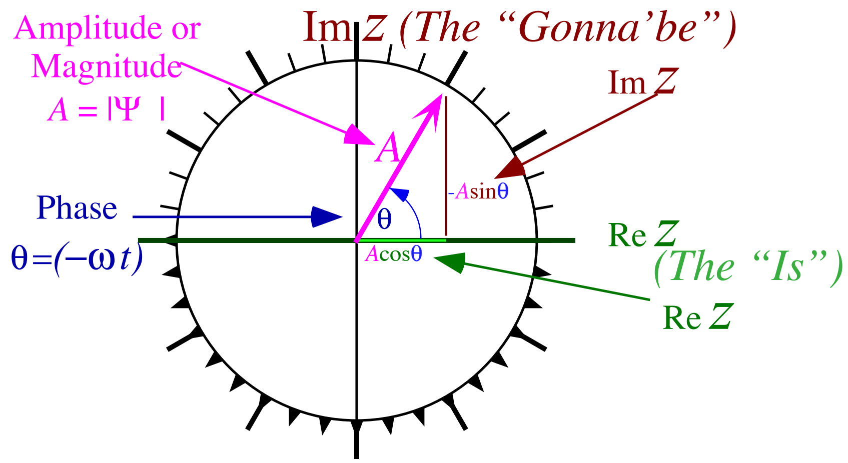

$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = 0$$

Coordinate $z=z(t)$ is the response coordinate for a particle of mass m and charge e

held back by a harmonic (linear) restoring force $\rightarrow$ $F_{restore} = -kz$

retarded by frictional damping force $\rightarrow$ $F_{damping} = -b\frac{dz}{dt}$

Oscillator Figures of Merit:

Time required to to reduce amplitude to 5%

Easy-to-recall *5%* approximation:

$$e^{-3} \cong 0.05$$

$$t_{5\%} = \frac{3}{\Gamma} = \frac{3}{0.2} = 15$$

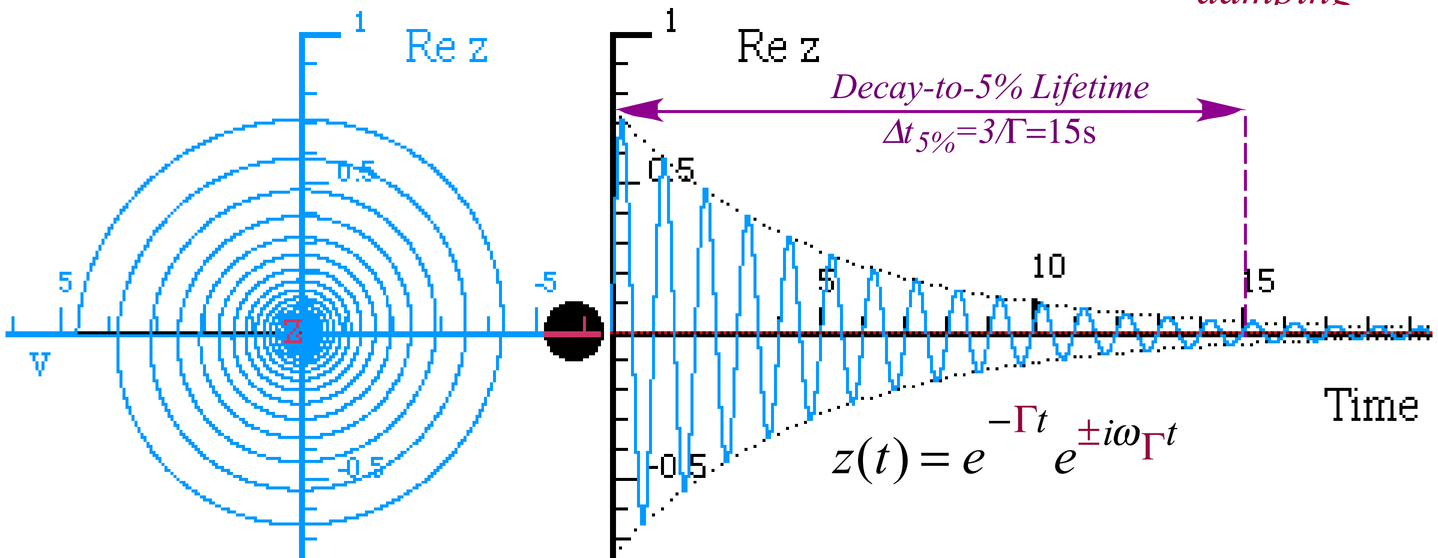

Fig. 3.2.3 Phasor z and corresponding coordinate versus time plot for $\omega_0=2\pi$ and $\Gamma=0.2$

Linear -damped-harmonic oscillator equation of motion.

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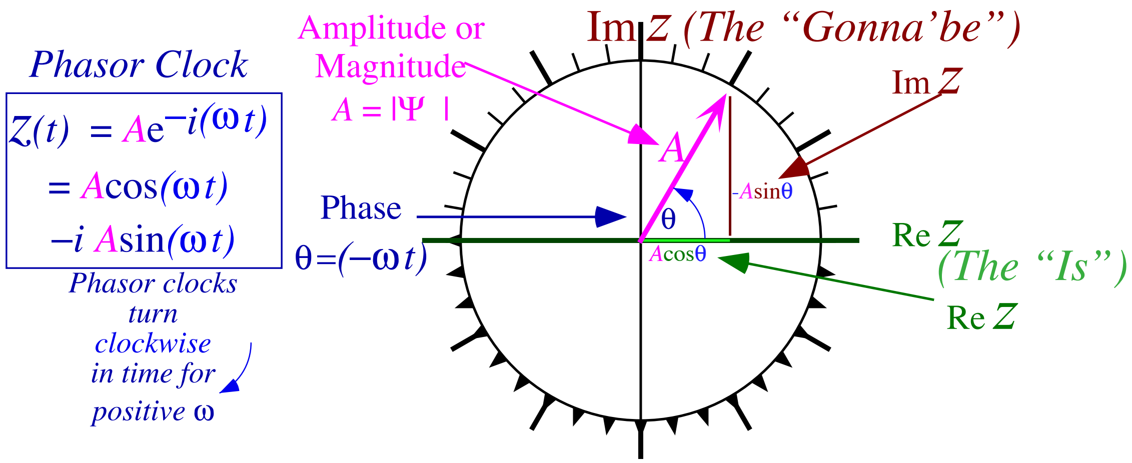


$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = 0$$

Coordinate $z=z(t)$ is the response coordinate for a particle of mass m and charge e

held back by a harmonic (linear) restoring force → $F_{restore} = -kz$

retarded by frictional damping force → $F_{damping} = -b\frac{dz}{dt}$

Oscillator Figures of Merit:

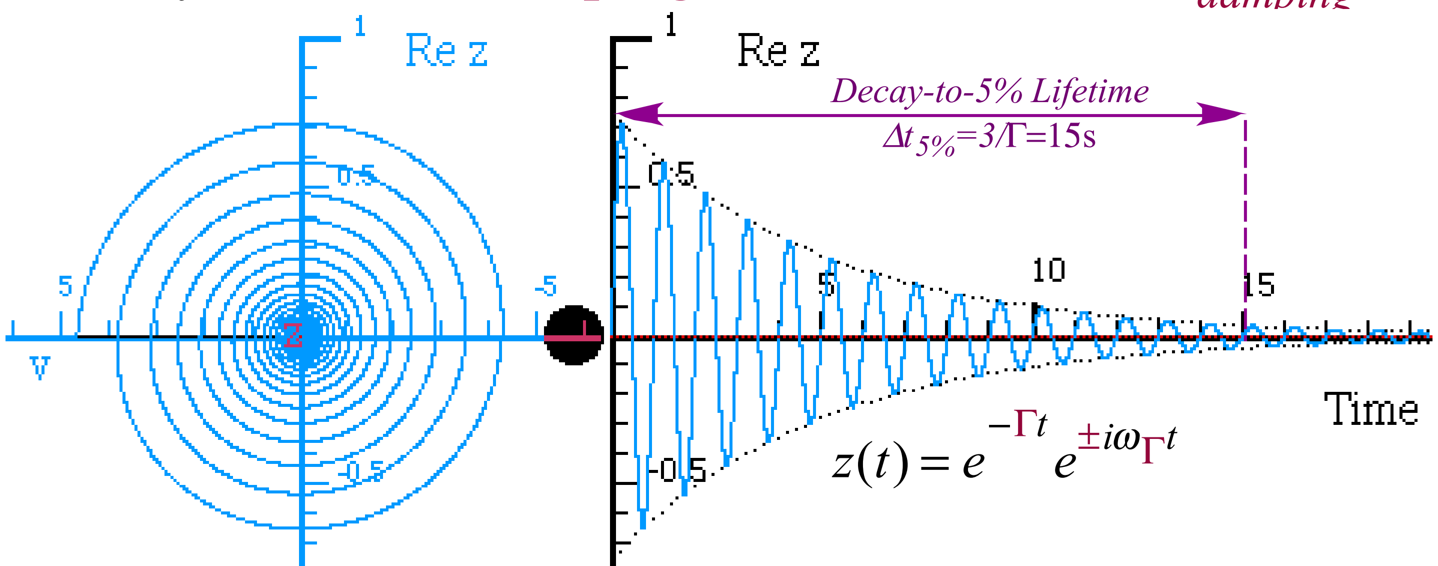
Time required to to reduce amplitude to 5% (or 4.321%)

Easy-to-recall *5%* approximation: $e^{-3} \cong 0.05$

More precise one: $e^{-\pi} \cong 0.04321$

$$t_{5\%} = \frac{3}{\Gamma} = \frac{3}{0.2} = 15$$

$$t_{4.321\%} = \frac{\pi}{\Gamma} = \frac{\pi}{0.2} = 15.708$$



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Linear -damped-harmonic oscillator equation of motion.

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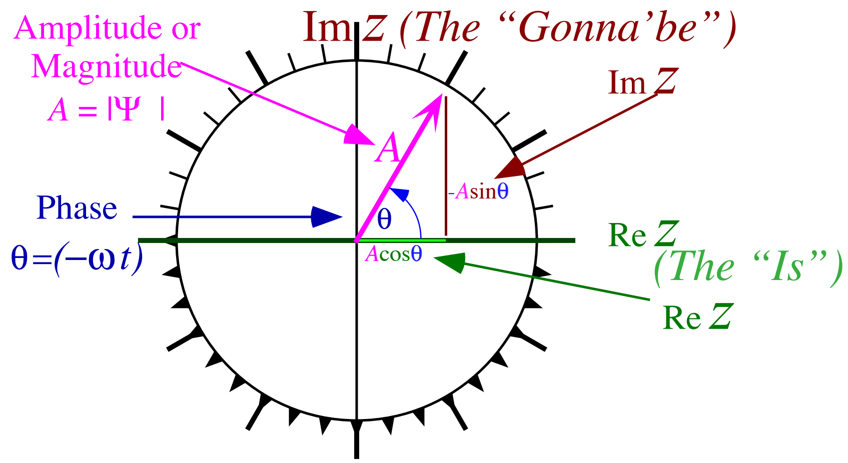

$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = 0$$

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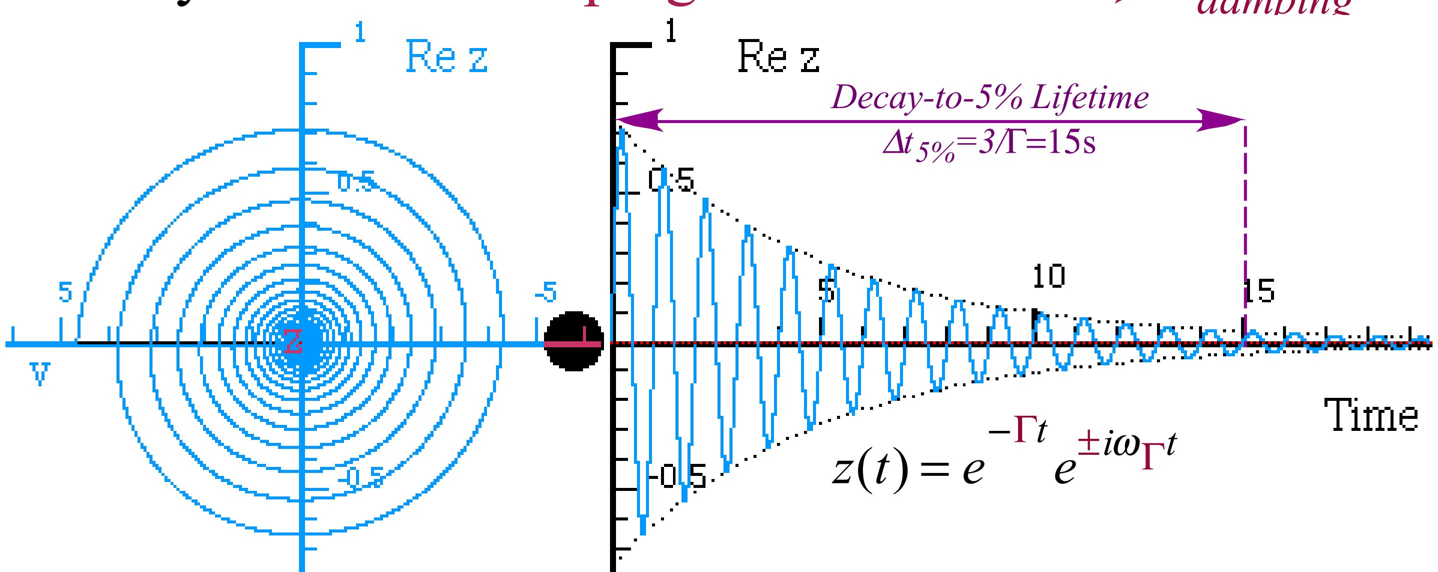
held back by a harmonic (linear) restoring force → $F_{restore} = -kz$

retarded by frictional damping force → $F_{damping} = -b\frac{dz}{dt}$

Oscillator Figures of Merit:

Number N of oscillations to reduce amplitude to 5% (or 4.321%)



$z(t) = e^{-\Gamma t} e^{\pm i\omega_\Gamma t}$

Fig. 3.2.3 Phasor z and corresponding coordinate versus time plot for $\omega_0=2\pi$ and $\Gamma=0.2$

Easy-to-recall 5% approximation: More precise one:

$$e^{-3} \cong 0.05 \qquad e^{-\pi} \cong 0.04321$$

$$N_{5\%} = \frac{\omega_\Gamma \cdot t_{5\%}}{2\pi} = \frac{3\omega_\Gamma}{2\pi\Gamma} \sim \frac{\omega_\Gamma}{2\Gamma}$$

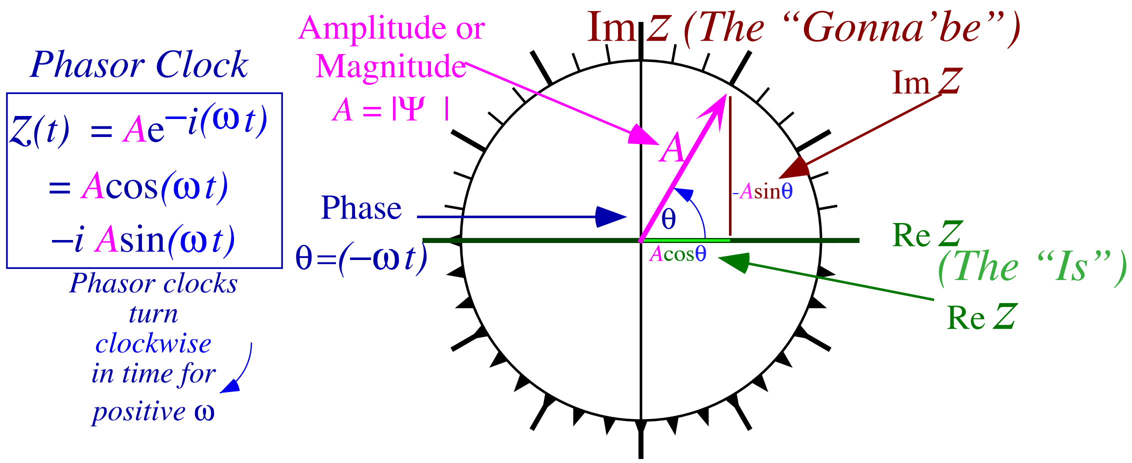
$$t_{4.321\%} = \frac{\pi}{\Gamma} = \frac{\pi}{0.2} = 15.708$$

Linear forced-damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore} + F_{stimulus}$$

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m} + \frac{F_{stimulus}}{m}$$

Stimulating acceleration $a_{stimulus}=a(t)$ due to stimulating force $F_{stimulus}(t)$ (Typically **E**-field)



$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m}E(t)$$

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driven by external a stimulating force → $F_{stimulus}(t) = eE(t)$

held back by a harmonic (linear) restoring force → $F_{restore} = -kz, \quad (k = \omega_0^2 m),$

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Linear forced-damped-harmonic oscillator equation of motion.

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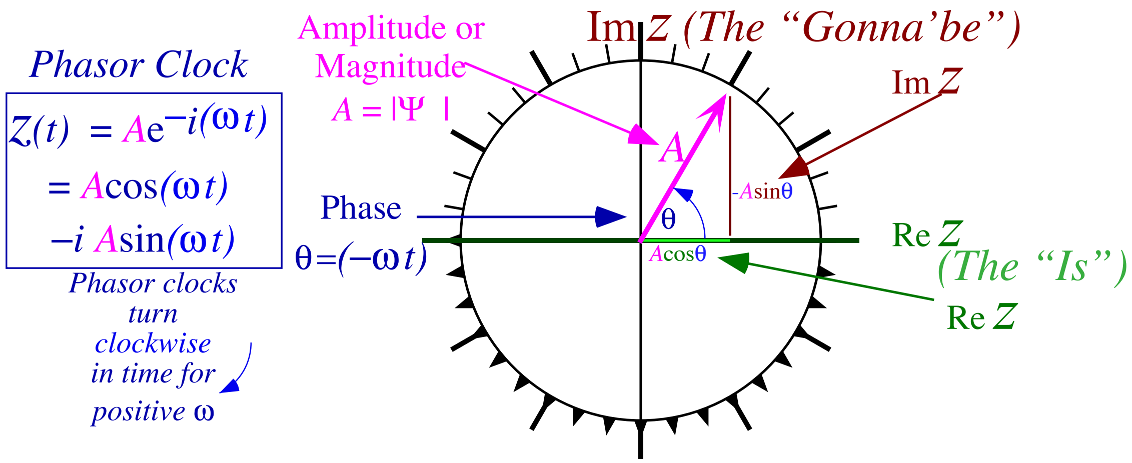

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Stimulating acceleration $a_{stimulus}=a(t)$ due to stimulating force $F_{stimulus}(t)$ (Typically **E**-field)

$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m}E(t)$$

Solving for $z_{stimulus}(t)$ given $a_{stimulus}$:

Linear forced-damped-harmonic oscillator equation of motion.

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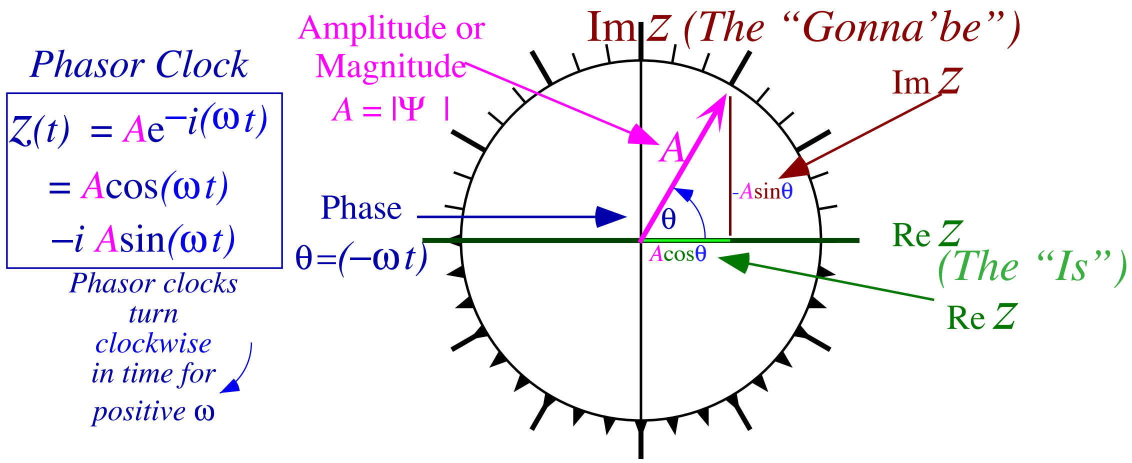

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$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m}E(t)$$

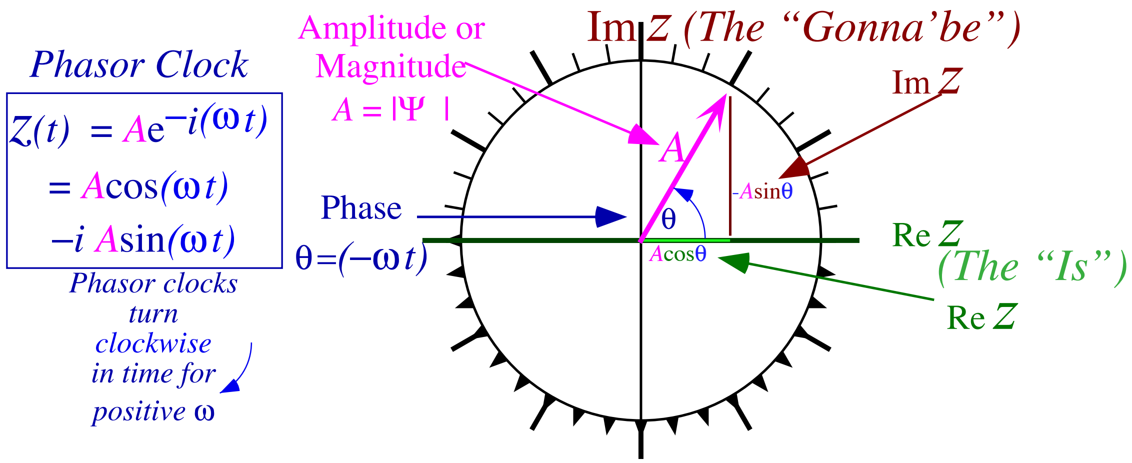
Solving for $z_{stimulus}(t)$ given $a_{stimulus}$:

$$\left(\frac{d^2}{dt^2} + 2\Gamma\frac{d}{dt} + \omega_0^2\right)z = a_{stimulus}$$

$$z = \frac{1}{\frac{d^2}{dt^2} + 2\Gamma\frac{d}{dt} + \omega_0^2}\, a_{stimulus}$$

Linear forced-damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore} + F_{stimulus}$$



$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m} + \frac{F_{stimulus}}{m}$$

Stimulating acceleration $a_{stimulus}=a(t)$ due to stimulating force $F_{stimulus}(t)$ (Typically **E**-field)

$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m}E(t)$$

Solving for $z_{stimulus}(t)$ given $a_{stimulus}$:

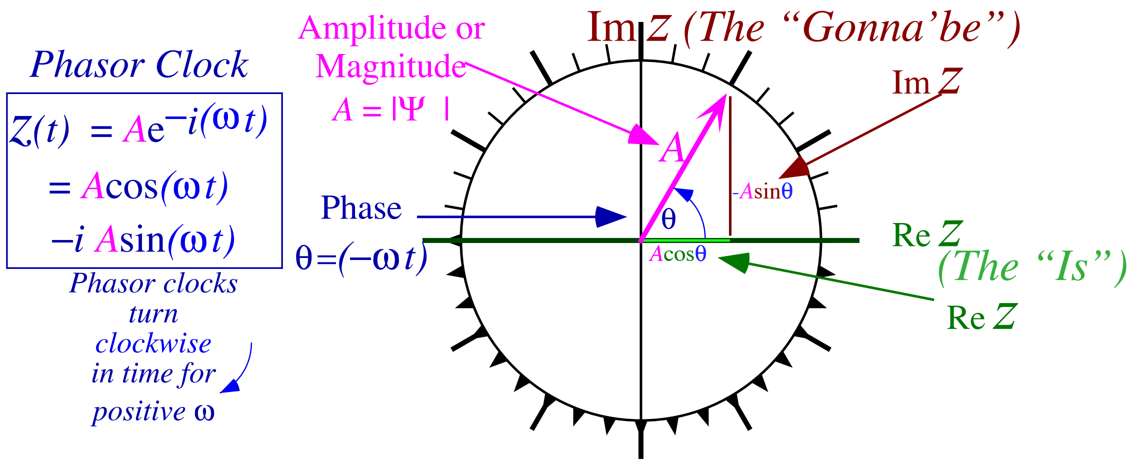
$$\left(\frac{d^2}{dt^2} + 2\Gamma\frac{d}{dt} + \omega_0^2\right)z = a_{stimulus}$$

Pretty crazy? But not so crazy if
$a_{stimulus}(t) = |a_{stimulus}|e^{-i\omega_{stimulus}t} = |a_s|e^{-i\omega_s t}$

$$z = \frac{1}{\frac{d^2}{dt^2} + 2\Gamma\frac{d}{dt} + \omega_0^2}a_{stimulus}$$

Linear forced-damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore} + F_{stimulus}$$



$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m} + \frac{F_{stimulus}}{m}$$

Stimulating acceleration $a_{stimulus} = a(t)$ due to stimulating force $F_{stimulus}(t)$ (Typically **E**-field)

$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m}E(t)$$

Solving for $z_{stimulus}(t)$ given $a_{stimulus}$:

$$\left(\frac{d^2}{dt^2} + 2\Gamma\frac{d}{dt} + \omega_0^2\right)z = a_{stimulus}$$

Pretty crazy? But not so crazy if
$a_{stimulus}(t) = |a_{stimulus}|e^{-i\omega_{stimulus}t} = |a_s|e^{-i\omega_s t}$

$$z_{stimulus} = \frac{1}{-\omega_s^2 - i2\Gamma\omega_s + \omega_0^2} a_s e^{-i\omega_s t}$$

$$z_s e^{-i\omega_s t} = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} a_s e^{-i\omega_s t}$$

$$z_s = G_{\omega_0}(\omega_s) \cdot a_s$$

Green's Function for the F-D-H Oscillator (FDHO)

George Green (14 July 1793 – 31 May 1841)

Green's Function for the FDHO (Forced-Damped-Harmonic Oscillator)

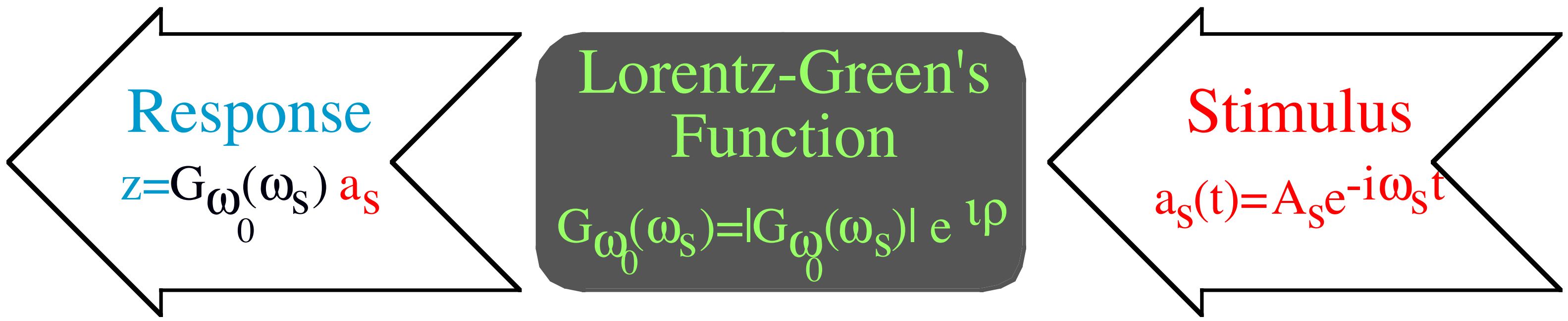


Fig. 3.2.4 Black-box diagram of oscillator response to monochromatic stimulus

$$G_{\omega_0}(\omega_s)=\frac{1}{\omega_0^2-\omega_s^2-i2\Gamma\omega_s}=\operatorname{Re}G_{\omega_0}(\omega_s)+i\operatorname{Im}G_{\omega_0}(\omega_s)$$

Real and imaginary parts of the *rectangular form* of G:

$$\operatorname{Re}G_{\omega_0}(\omega_s)=\frac{\omega_0^2-\omega_s^2}{\left(\omega_0^2-\omega_s^2\right)^2+\left(2\Gamma\omega_s\right)^2}$$

$$\operatorname{Im}G_{\omega_0}(\omega_s)=\frac{2\Gamma\omega_s}{\left(\omega_0^2-\omega_s^2\right)^2+\left(2\Gamma\omega_s\right)^2}$$

Green's Function for the FDHO (Forced-Damped-Harmonic Oscillator)

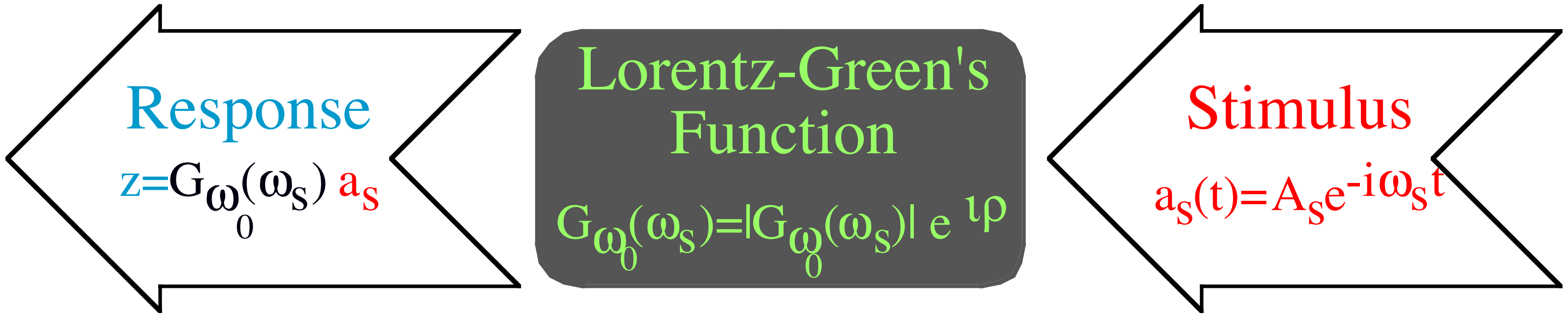


Fig. 3.2.4 Black-box diagram of oscillator response to monochromatic stimulus

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} = \operatorname{Re} G_{\omega_0}(\omega_s) + i \operatorname{Im} G_{\omega_0}(\omega_s) = \left| G_{\omega_0}(\omega_s) \right| e^{i\rho}$$

Real and imaginary parts of the *rectangular form* of G:

$$\operatorname{Re} G_{\omega_0}(\omega_s) = \frac{\omega_0^2 - \omega_s^2}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

$$\operatorname{Im} G_{\omega_0}(\omega_s) = \frac{2\Gamma\omega_s}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

Magnitude $\left| G_{\omega_0}(\omega_s) \right|$ and polar angle ρ of the *polar form* of G:

$$\left| G_{\omega_0}(\omega_s) \right| = \frac{1}{\sqrt{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}}$$

$$\rho = \tan^{-1}\left(\frac{2\Gamma\omega_s}{\omega_0^2 - \omega_s^2} \right)$$

Green's Function for the FDHO (Forced-Damped-Harmonic Oscillator)

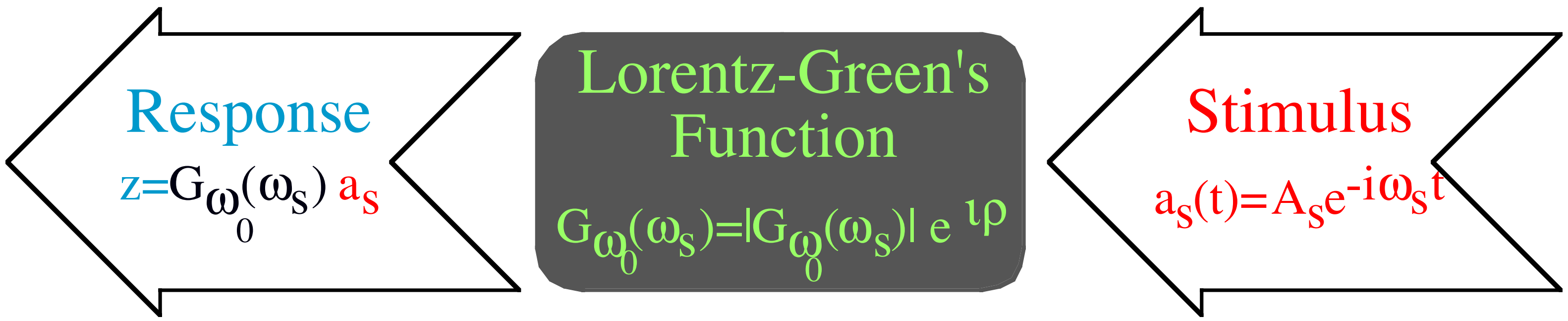


Fig. 3.2.4 Black-box diagram of oscillator response to monochromatic stimulus

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} = \mathrm{Re}\, G_{\omega_0}(\omega_s) + i\,\mathrm{Im}\, G_{\omega_0}(\omega_s) = \left|G_{\omega_0}(\omega_s)\right| e^{i\rho}$$

Real and imaginary parts of the *rectangular form* of G:

$$\mathrm{Re}\, G_{\omega_0}(\omega_s) = \frac{\omega_0^2 - \omega_s^2}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

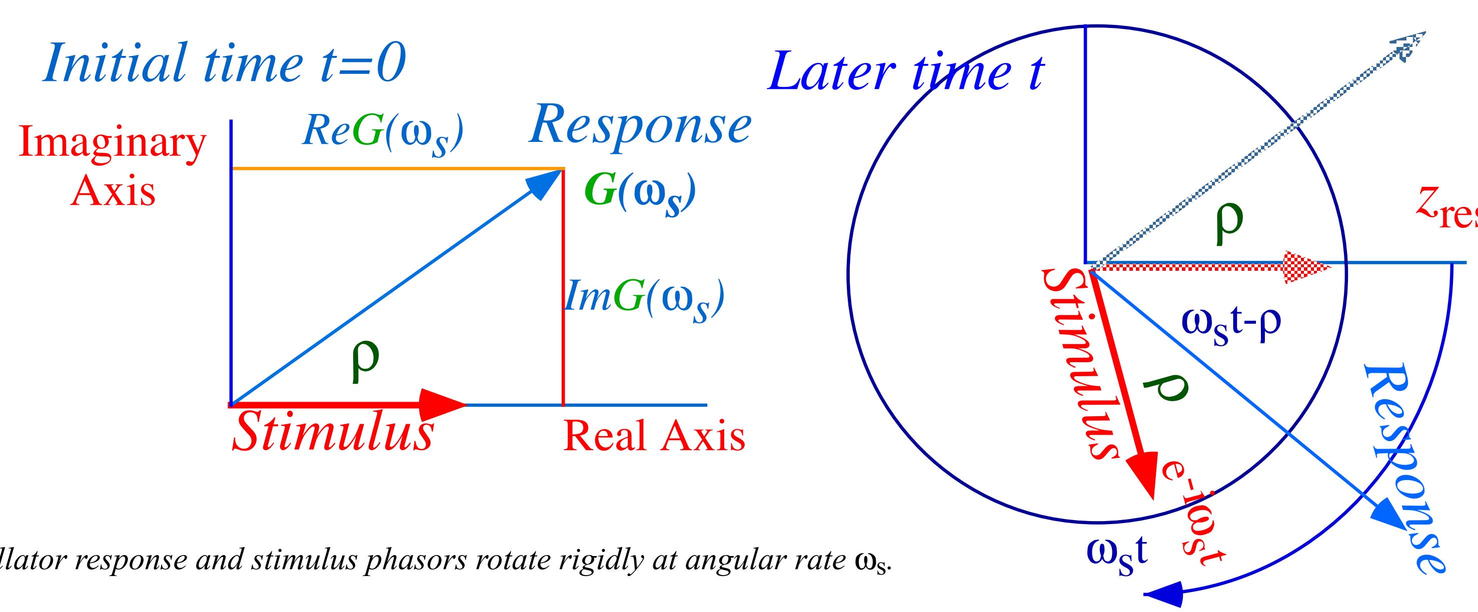
$$\mathrm{Im}\, G_{\omega_0}(\omega_s) = \frac{2\Gamma\omega_s}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

Magnitude $\left|G_{\omega_0}(\omega_s)\right|$ and *polar angle* ρ of the *polar form* of G:

$$\left|G_{\omega_0}(\omega_s)\right| = \frac{1}{\sqrt{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}}$$

$$\rho = \tan^{-1}\left(\frac{2\Gamma\omega_s}{\omega_0^2 - \omega_s^2}\right)$$

polar angle ρ is the *phase lag angle* ρ



$$z_{\text{response}}(t) = \left|G_{\omega_0}(\omega_s)\right| a(0)\, e^{-i(\omega_s t - \rho)}$$

Fig. 3.2.5 Oscillator response and stimulus phasors rotate rigidly at angular rate ω_s.

Green's Function for the FDHO (Forced-Damped-Harmonic Oscillator)

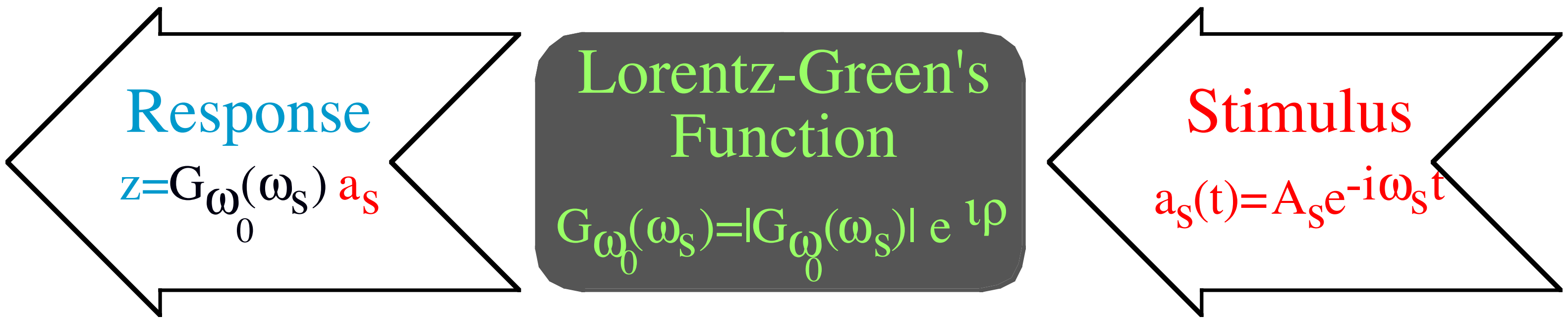


Fig. 3.2.4 Black-box diagram of oscillator response to monochromatic stimulus

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} = \mathrm{Re}\,G_{\omega_0}(\omega_s) + i\,\mathrm{Im}\,G_{\omega_0}(\omega_s) = \left|G_{\omega_0}(\omega_s)\right| e^{i\rho}$$

Real and imaginary parts of the *rectangular form* of G:

$$\mathrm{Re}\,G_{\omega_0}(\omega_s) = \frac{\omega_0^2 - \omega_s^2}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

$$\mathrm{Im}\,G_{\omega_0}(\omega_s) = \frac{2\Gamma\omega_s}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

Magnitude $\left|G_{\omega_0}(\omega_s)\right|$ and *polar angle* ρ of the *polar form* of G:

$$\left|G_{\omega_0}(\omega_s)\right| = \frac{1}{\sqrt{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}}$$

$$\rho = \tan^{-1}\left(\frac{2\Gamma\omega_s}{\omega_0^2 - \omega_s^2}\right)$$

polar angle ρ is the *phase lag angle* ρ

$$z_{\text{response}}(t) = \left|G_{\omega_0}(\omega_s)\right| a(0) e^{-i(\omega_s t - \rho)}$$

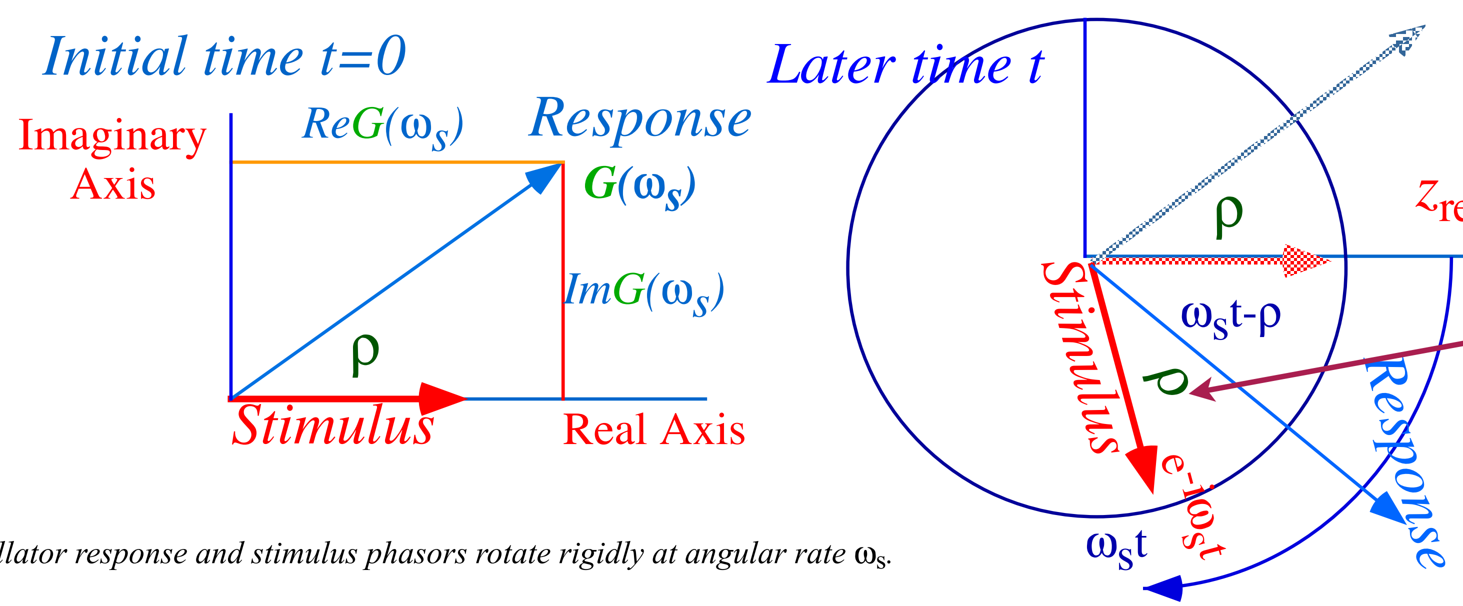


Fig. 3.2.5 Oscillator response and stimulus phasors rotate rigidly at angular rate ω_s.

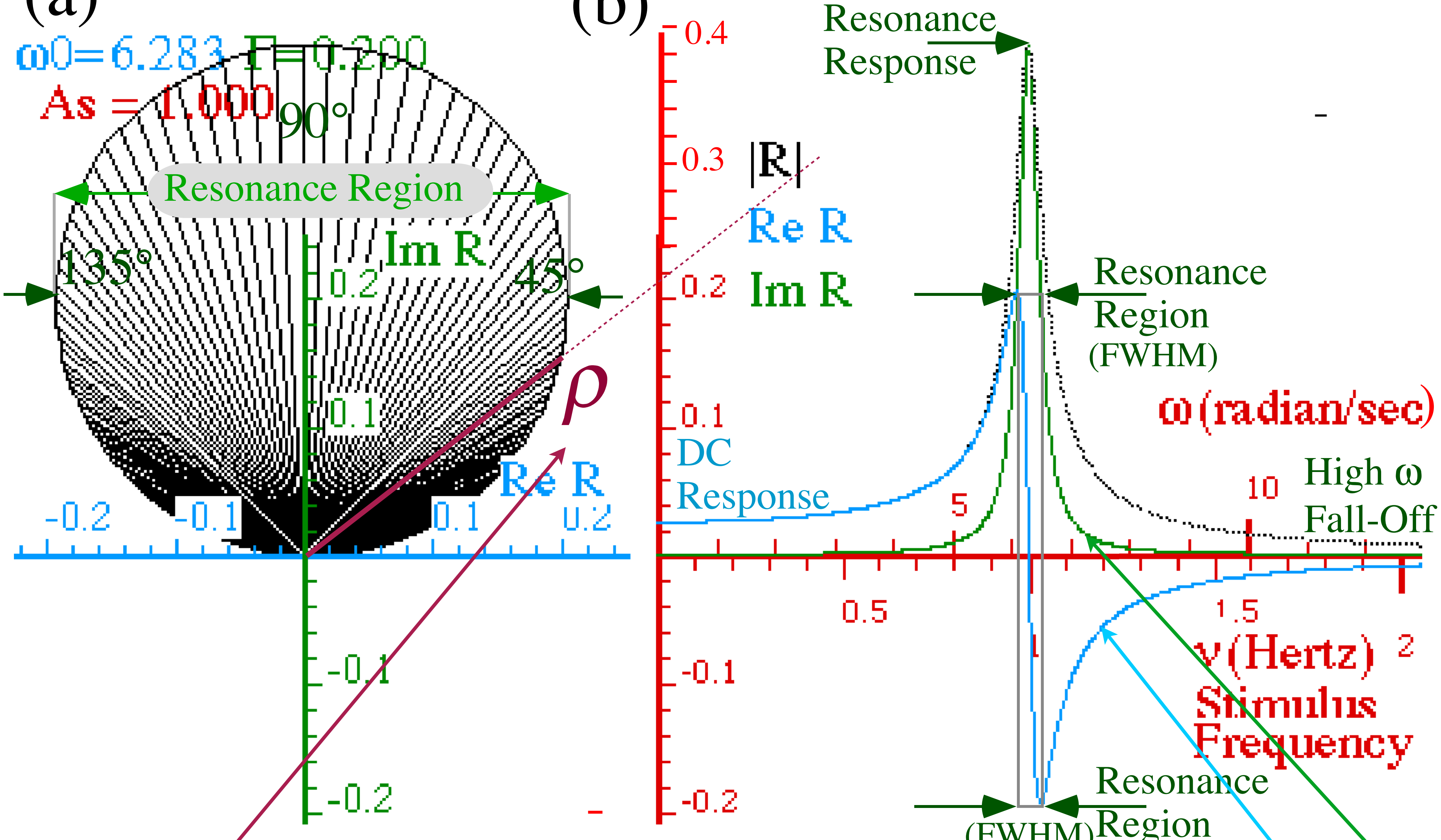


Fig. 3.2.6 Anatomy of oscillator Green-Lorentz response function plots

Phase lag angle

$$\rho = \tan^{-1}\left(\frac{2\Gamma\omega_s}{\omega_0^2 - \omega_s^2}\right)$$

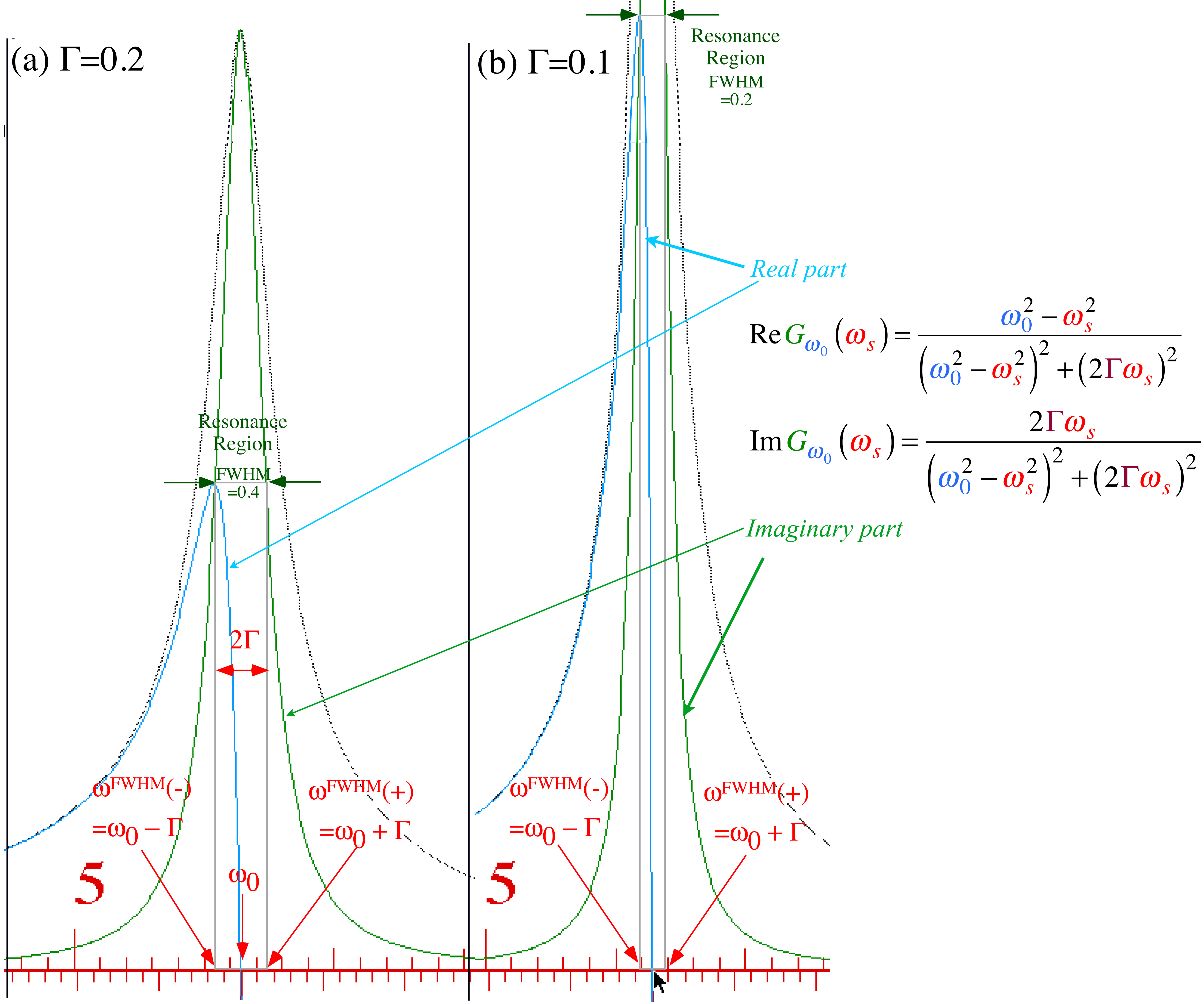
$$\mathrm{Re}\, G_{\omega_0}(\omega_s) = \frac{\omega_0^2 - \omega_s^2}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

Real part

$$\mathrm{Im}\, G_{\omega_0}(\omega_s) = \frac{2\Gamma\omega_s}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

Imaginary part

$$AAF = \frac{\text{Resonant response}}{\text{DC response}} = \frac{\left|G_{\omega_0}(\omega_s = \omega_0)\right|}{\left|G_{\omega_0}(0)\right|} = \frac{1/\left(2\Gamma\omega_0\right)}{1/\omega_0^2} = \frac{\omega_0}{2\Gamma} \equiv q \quad \text{(angular quality factor)}$$



$$\mathrm{Re}\,G_{\omega_0}(\omega_s) = \frac{\omega_0^2 - \omega_s^2}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

$$\mathrm{Im}\,G_{\omega_0}(\omega_s) = \frac{2\Gamma\omega_s}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

Fig. 3.2.7 Comparing Lorentz-Green resonance region for (a) Γ=0.2 and (b) Γ=0.1.
Maximum and minimum points of Re*G(ω) and inflection points of* Im*G(ω) are near region boundaries* $\omega^{FWHM}(\pm)=\omega_0\pm\Gamma$.

Complete Green's Solution for the FDHO (Forced-Damped-Harmonic Oscillator)

$$z(t) = z_{\text{transient}}(t) + z_{\text{response}}(t) \equiv z_{\text{decaying}}(t) + z_{\text{steady state}}(t)$$

$$= Ae^{-\Gamma t}e^{-i\omega_\Gamma t} + G_{\omega_0}(\omega_s)a(0)e^{-i\omega_s t}$$

$$= Ae^{-\Gamma t}e^{-i\omega_\Gamma t} + \left|G_{\omega_0}(\omega_s)\right|a(0)e^{-i(\omega_s t-\rho)}$$

Known as "homogeneous" solution (no force)
Let's you set initial or boundary conditions

Known as "*in*homogeneous" solution
Does not. Marches to stimulus only.

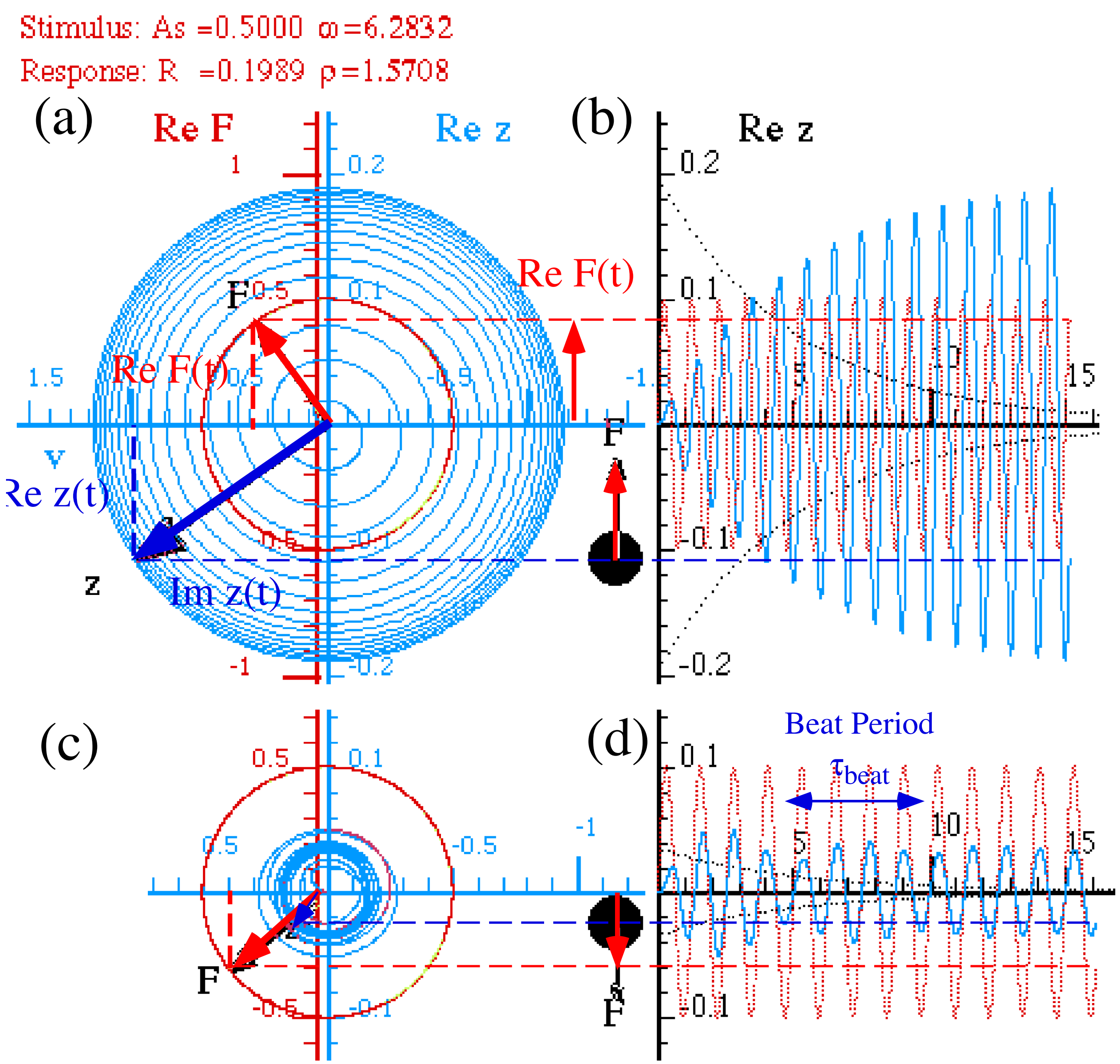


Fig. 3.2.8 On Resonance ($\omega_s=\omega_0=2\pi$ *and* $\Gamma=0.2$). *(a)Response z-phasor lags* $\rho=90°$ *behind stimulus F-phasor.* *(b) Time plots of* Re *z(t) and* Re *F(t)*

Fig. 3.2.8 Below Resonance ($\omega_s=5.03, \omega_0=2\pi, \Gamma=0.2$). *(c)Response z-phasor lags* $\rho=8.05°$ *behind stimulus F-phasor.* *(d) Time plots of* Re *z(t) and* Re *F(t). Beats are barely visible.*

end of Lecture 36

Approximate Lorentz-Green's Function for high quality FDHO (Quantum propagator)

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} \xrightarrow[\omega_s \to \omega_0]{} \frac{1}{2\omega_s}\frac{1}{\omega_0 - \omega_s - i\Gamma} \approx \frac{1}{2\omega_0}\frac{1}{\Delta - i\Gamma} = \frac{1}{2\omega_0} L(\Delta - i\Gamma)$$

Define *complex detuning-decay* $\delta = \Delta - i\Gamma$ variable δ is defined with the *real detuning* $\Delta = \omega_0 - \omega_s$

$$L(\Delta - i\Gamma) = \frac{1}{\Delta - i\Gamma} = \mathrm{Re}\,L \quad + i\,\mathrm{Im}\,L \quad = \quad \frac{\Delta}{\Delta^2 + \Gamma^2} \quad + i\frac{\Gamma}{\Delta^2 + \Gamma^2} = \quad |L|^2\,\Delta \quad + i\,|L|^2\,\Gamma$$

$$= |L|e^{i\rho} = |L|\cos\rho + i\,|L|\sin\rho = \frac{\cos\rho}{\sqrt{\Delta^2 + \Gamma^2}} + i\frac{\sin\rho}{\sqrt{\Delta^2 + \Gamma^2}} \quad \text{where: } |L| = \frac{1}{\sqrt{\Delta^2 + \Gamma^2}}$$

$$|L| = \frac{1}{\Gamma}\sin\rho$$

$$|L| = \frac{1}{\Delta}\cos\rho$$

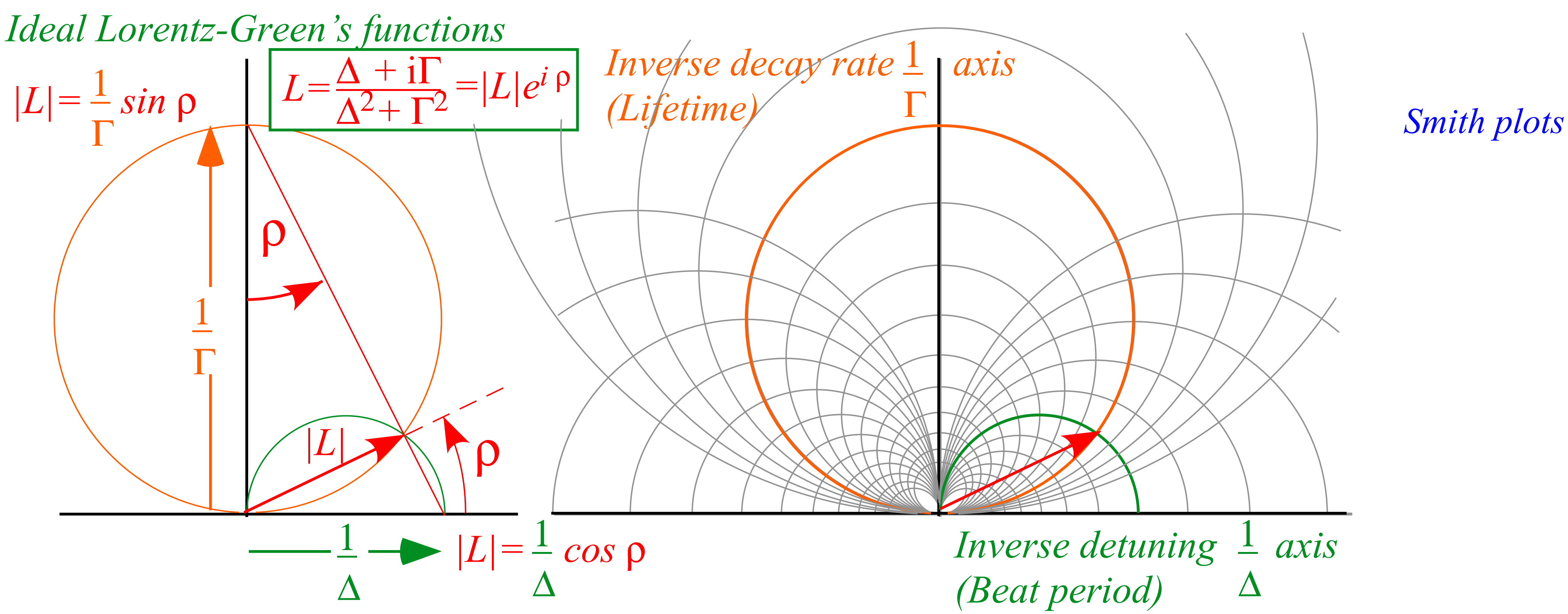
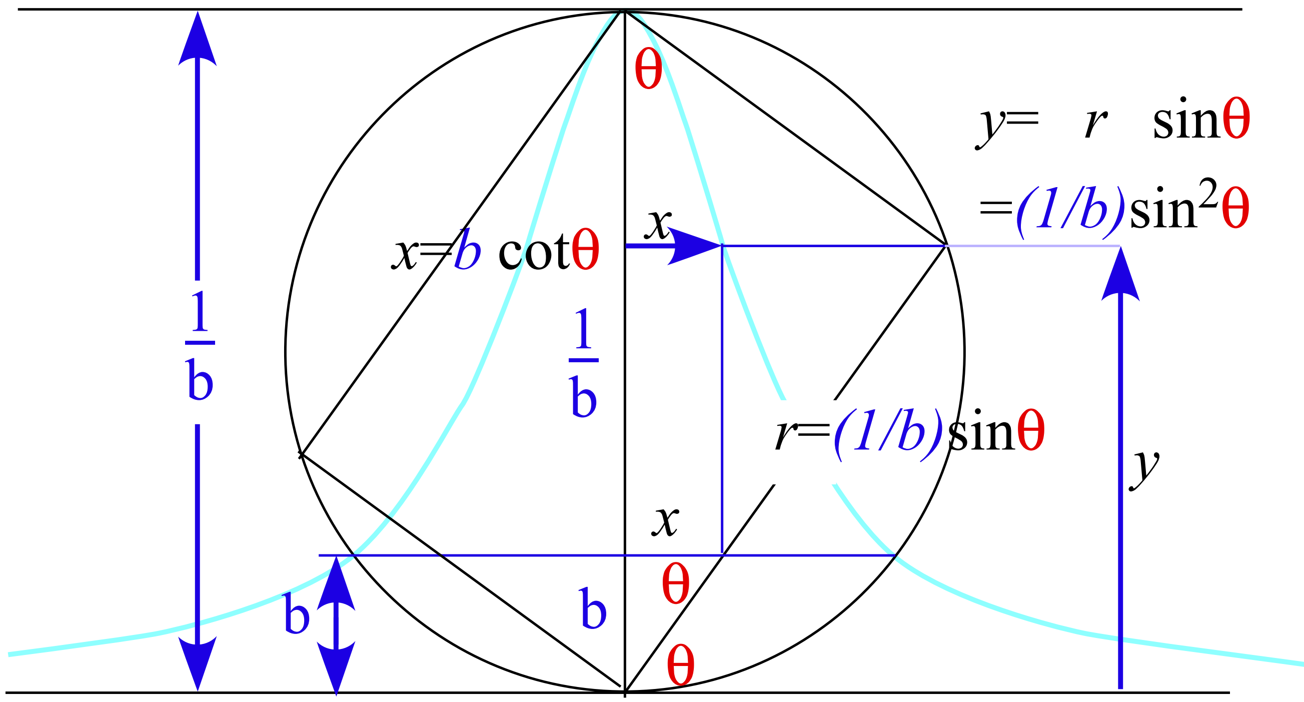
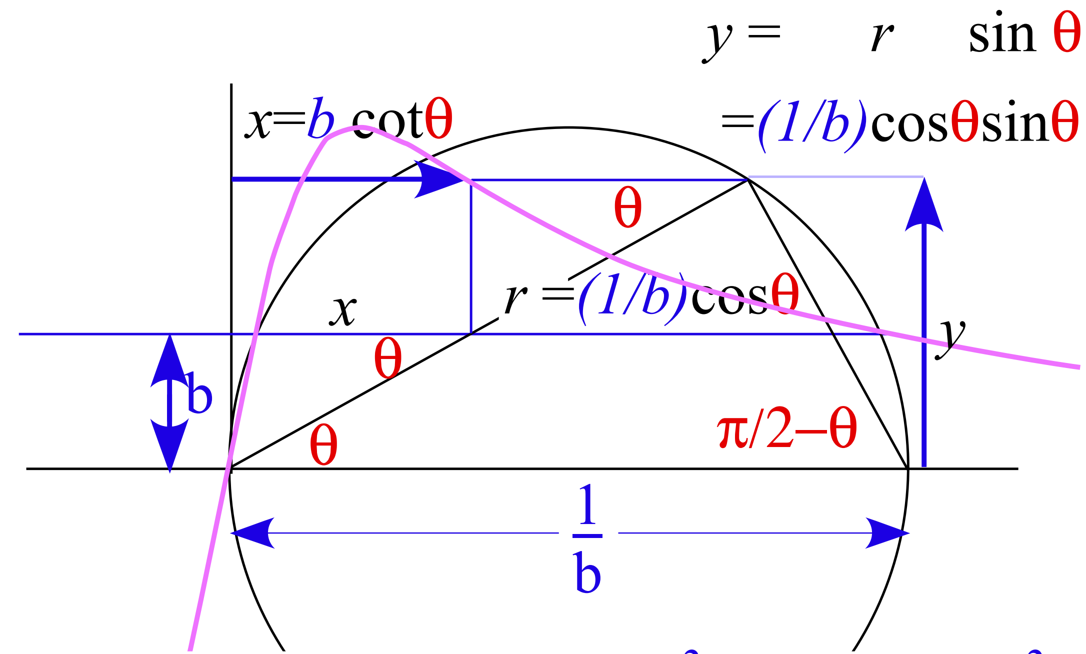


Fig. 3.2.13 Ideal Lorentzian in inverse rate space. (Smith life-time $1/\Gamma$ *vs. beat-period* $1/\Delta$ *coordinates)*

Constant Δ and Γ curves in Fig. 3.2.13 are orthogonal circles of *1/z- dipolar coordinates*. Recall Fig. 1.10.11.





$$x^2=b^2\cot^2\theta=b^2\frac{\cos^2\theta}{\sin^2\theta}=b^2\frac{1-\sin^2\theta}{\sin^2\theta}=\frac{b^2}{\sin^2\theta}b^2$$

$$\frac{x}{y}=\frac{b\cot\theta}{(1/b)\cos\theta\sin\theta}=\frac{b^2\cos\theta}{\cos\theta\sin^2\theta}=\frac{b^2}{\sin^2\theta}$$