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4.04.18 class 21: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

Characters of intertwining $(S_n)*(U(m))$ algebras and quantum applications

Generic U(3)⊃R(3) transformations: *p*-triplet in U(3) shell model
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Applying S_3 character theory
Frequency formula for $D^{[\mu]}$ with tensor trace values
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Fundamental "quark" irep. "anti-quark". "di-quark".
The octet "eightfold way" The decapalet and Ω^-
The p-shell in U(3) tableau plots
Hooklength formulas

AMOP reference links *(Updated list given on 2nd page of each class presentation)*

Web Resources - front page
UAF Physics UTube channel

Quantum Theory for the Computer Age
Principles of Symmetry, Dynamics, and Spectroscopy
Classical Mechanics with a Bang!
Modern Physics and its Classical Foundations

2014 AMOP
2017 Group Theory for QM
2018 AMOP

Frame Transformation Relations And Multipole Transitions In Symmetric Polyatomic Molecules - RMP-1978
Rotational energy surfaces and high- J eigenvalue structure of polyatomic molecules - Harter - Patterson - 1984
Galloping waves and their relativistic properties - ajp-1985-Harter
Asymptotic eigensolutions of fourth and sixth rank octahedral tensor operators - Harter-Patterson-JMP-1979
Nuclear spin weights and gas phase spectral structure of 12C60 and 13C60 buckminsterfullerene -Harter-Reimer-Cpl-1992 - (Alt1, Alt2 Erratum)

Theory of hyperfine and superfine levels in symmetric polyatomic molecules.

- I) Trigonal and tetrahedral molecules: Elementary spin-1/2 cases in vibronic ground states - PRA-1979-Harter-Patterson (Alt scan)
- II) Elementary cases in octahedral hexafluoride molecules - Harter-PRA-1981 (Alt scan)

Rotation-vibration scalar coupling zeta coefficients and spectroscopic band shapes of buckminsterfullerene - Weeks-Harter-CPL-1991 (Alt scan)
Fullerene symmetry reduction and rotational level fine structure/ the Buckyball isotopomer 12C 13C59 - jcp-Reimer-Harter-1997 (HiRez)
Molecular Eigensolution Symmetry Analysis and Fine Structure - IJMS-harter-mitchell-2013

Rotation–vibration spectra of icosahedral molecules.

- I) Icosahedral symmetry analysis and fine structure - harter-weeks-jcp-1989
- II) Icosahedral symmetry, vibrational eigenfrequencies, and normal modes of buckminsterfullerene - weeks-harter-jcp-1989
- III) Half-integral angular momentum - harter-reimer-jcp-1991

QTCA Unit 10 Ch 30 - 2013
Molecular Symmetry and Dynamics - Ch32-Springer Handbooks of Atomic, Molecular, and Optical Physics - Harter-2006
AMOP Ch 0 Space-Time Symmetry - 2019

RESONANCE AND REVIVALS

- I) QUANTUM ROTOR AND INFINITE-WELL DYNAMICS - ISMSLi2012 (Talk) OSU knowledge Bank
- II) Comparing Half-integer Spin and Integer Spin - Alva-ISMS-Ohio2013-R777 (Talks)
- III) Quantum Resonant Beats and Revivals in the Morse Oscillators and Rotors - (2013-Li-Diss)

Rovibrational Spectral Fine Structure Of Icosahedral Molecules - Cpl 1986 (Alt Scan)
Gas Phase Level Structure of C60 Buckyball and Derivatives Exhibiting Broken Icosahedral Symmetry - reimer-diss-1996
Resonance and Revivals in Quantum Rotors - Comparing Half-integer Spin and Integer Spin - Alva-ISMS-Ohio2013-R777 (Talk)
Quantum Revivals of Morse Oscillators and Farey-Ford Geometry - Li-Harter-cpl-2013
Wave Node Dynamics and Revival Symmetry in Quantum Rotors - harter - jms - 2001
Representaions Of Multidimensional Symmetries In Networks - harter-jmp-1973

**In development - a web based A.M.O.P. oriented reference page, with thumbnail/previews, greater control over the information display, and eventually full on* Apache-SOLR *Index and search for* nuanced, whole-site content/metadata level *searching. This bad boy will be a sure force multiplier.*

(Int.J.Mol.Sci, 14, 714(2013) p.755-774 , QTCA Unit 7 Ch. 23-26)
(PSDS - Ch. 5, 7)

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Rank-1 vector in R(3) or "quark"-triplet in U(3) or *p*-triplet in U(3) shell model

$$\phi_1' = \mathbf{u}\phi_1 = \phi_1 D_{11} + \phi_2 D_{21} + \phi_3 D_{31}$$
$$\phi_2' = \mathbf{u}\phi_2 = \phi_1 D_{12} + \phi_2 D_{22} + \phi_3 D_{32}$$
$$\phi_3' = \mathbf{u}\phi_3 = \phi_1 D_{13} + \phi_2 D_{23} + \phi_3 D_{33}$$

where: $D_{jk} = (\phi_j^*, \phi_k') = (\phi_j^*, \mathbf{u}\phi_k)$

$$\left|1\right\rangle = \phi_1 = \begin{pmatrix} 1 \\ \cdot \\ \cdot \end{pmatrix} \quad \left|2\right\rangle = \phi_2 = \begin{pmatrix} \cdot \\ 1 \\ \cdot \end{pmatrix} \quad \left|3\right\rangle = \phi_3 = \begin{pmatrix} \cdot \\ \cdot \\ 1 \end{pmatrix}$$

Dirac notation: where: $D_{jk}(\mathbf{u}) = \left\langle j \middle| k' \right\rangle = \left\langle j \middle| \mathbf{u} \middle| k \right\rangle$

$$\left|1'\right\rangle = \mathbf{u}\left|1\right\rangle = \left|1\right\rangle D_{11} + \left|2\right\rangle D_{21} + \left|3\right\rangle D_{31}$$
$$\left|2'\right\rangle = \mathbf{u}\left|2\right\rangle = \left|1\right\rangle D_{12} + \left|2\right\rangle D_{22} + \left|3\right\rangle D_{32}$$
$$\left|3'\right\rangle = \mathbf{u}\left|3\right\rangle = \left|1\right\rangle D_{13} + \left|2\right\rangle D_{23} + \left|3\right\rangle D_{33}$$

$$D_{jk}(\mathbf{u}) = \begin{pmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{pmatrix}$$

U(3) tensor product states and S_n permutation symmetry

Typical U(3)⊃R(3) transformations (Just like *ℓ=1* vector basis {*1=x , 2=y, 3=z*})

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$$\left|1'\right\rangle = \mathbf{u}\left|1\right\rangle = \left|1\right\rangle D_{11} + \left|2\right\rangle D_{21} + \left|3\right\rangle D_{31}$$
$$\left|2'\right\rangle = \mathbf{u}\left|2\right\rangle = \left|1\right\rangle D_{12} + \left|2\right\rangle D_{22} + \left|3\right\rangle D_{32}$$
$$\left|3'\right\rangle = \mathbf{u}\left|3\right\rangle = \left|1\right\rangle D_{13} + \left|2\right\rangle D_{23} + \left|3\right\rangle D_{33}$$

$$D_{jk}(\mathbf{u}) = \begin{pmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{pmatrix}$$

Rank-2 tensor (2 particles each with U(3) state space)

$$\left|j'\right\rangle\left|k'\right\rangle = \mathbf{u}\left|j\right\rangle\mathbf{u}\left|k\right\rangle$$
$$= \sum_{j,k} \left|j\right\rangle\left|k\right\rangle D_{jj'} D_{kk'}$$
$$= \sum_{j,k} \left|j\right\rangle\left|k\right\rangle D\otimes D_{jk:j'k'}$$

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where: $D_{jk} = (\phi_j^*, \phi_k') = (\phi_j^*, \mathbf{u}\phi_k)$

$$\left|1\right\rangle = \phi_1 = \begin{pmatrix} 1 \\ \cdot \\ \cdot \end{pmatrix} \quad \left|2\right\rangle = \phi_2 = \begin{pmatrix} \cdot \\ 1 \\ \cdot \end{pmatrix} \quad \left|3\right\rangle = \phi_3 = \begin{pmatrix} \cdot \\ \cdot \\ 1 \end{pmatrix}$$

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$$D_{jk}(\mathbf{u}) = \begin{pmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{pmatrix}$$

Rank-2 tensor (2 particles each with U(3) state space)

$$\left|j'\right\rangle\left|k'\right\rangle = \mathbf{u}\left|j\right\rangle\mathbf{u}\left|k\right\rangle = \sum_{j,k}\left|j\right\rangle\left|k\right\rangle D_{jj'}D_{kk'} = \sum_{j,k}\left|j\right\rangle\left|k\right\rangle D\otimes D_{jk:j'k'}$$

$$= \begin{pmatrix} D_{11}D_{11} & D_{11}D_{12} & D_{11}D_{13} & D_{12}D_{11} & D_{12}D_{12} & D_{12}D_{13} & \cdots \\ D_{11}D_{21} & D_{11}D_{22} & D_{11}D_{23} & D_{12}D_{21} & D_{12}D_{22} & D_{12}D_{23} & \cdots \\ D_{11}D_{31} & D_{11}D_{32} & D_{11}D_{33} & D_{12}D_{21} & D_{12}D_{22} & D_{12}D_{23} & \cdots \\ D_{21}D_{11} & D_{21}D_{12} & \vdots & D_{22}D_{11} & D_{22}D_{12} & \vdots & \cdots \\ D_{21}D_{21} & D_{21}D_{22} & \vdots & D_{22}D_{21} & D_{22}D_{22} & \vdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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U(3) tensor product states and S_n permutation symmetry

2-particle U(3) transform and outer-product U(3) transform matrix $D_{j\,j'}D_{k\,k'} = D\otimes D_{j\,k:j'k'} =$

$$
= \begin{pmatrix}
D_{11}D_{11} & D_{11}D_{12} & D_{11}D_{13} & D_{12}D_{11} & D_{12}D_{12} & D_{12}D_{13} & D_{13}D_{11} & D_{13}D_{12} & D_{13}D_{13} \\
D_{11}D_{21} & D_{11}D_{22} & D_{11}D_{23} & D_{12}D_{21} & D_{12}D_{22} & D_{12}D_{23} & D_{13}D_{21} & D_{13}D_{22} & D_{13}D_{23} \\
D_{11}D_{31} & D_{11}D_{32} & D_{11}D_{33} & D_{12}D_{31} & D_{12}D_{22} & D_{12}D_{33} & D_{13}D_{31} & D_{13}D_{23} & D_{13}D_{33} \\
D_{21}D_{11} & D_{21}D_{12} & D_{21}D_{13} & D_{22}D_{11} & D_{22}D_{12} & D_{22}D_{13} & D_{23}D_{11} & D_{23}D_{12} & D_{23}D_{13} \\
D_{21}D_{21} & D_{21}D_{22} & D_{21}D_{23} & D_{22}D_{21} & D_{22}D_{22} & D_{22}D_{23} & D_{23}D_{21} & D_{23}D_{22} & D_{23}D_{23} \\
D_{21}D_{31} & D_{21}D_{32} & D_{21}D_{33} & D_{22}D_{31} & D_{22}D_{32} & D_{22}D_{33} & D_{23}D_{31} & D_{23}D_{23} & D_{23}D_{33} \\
D_{31}D_{11} & D_{31}D_{12} & D_{31}D_{13} & D_{32}D_{11} & D_{32}D_{12} & D_{32}D_{13} & D_{33}D_{11} & D_{33}D_{12} & D_{33}D_{13} \\
D_{31}D_{21} & D_{31}D_{22} & D_{31}D_{23} & D_{32}D_{21} & D_{32}D_{22} & D_{32}D_{23} & D_{33}D_{21} & D_{33}D_{22} & D_{33}D_{23} \\
D_{31}D_{31} & D_{31}D_{23} & D_{31}D_{33} & D_{32}D_{31} & D_{32}D_{32} & D_{32}D_{33} & D_{33}D_{31} & D_{33}D_{23} & D_{33}D_{33}
\end{pmatrix}
$$

U(3) tensor product states and S_2=S_n permutation symmetry

2-particle U(3) transform and outer-product U(3) transform matrix $D_{jj'}D_{kk'} = D\otimes D_{jk:j'k'} =$

$$
= \left(\begin{array}{ccc:ccc:ccc}
D_{11}D_{11} & D_{11}D_{12} & D_{11}D_{13} & D_{12}D_{11} & D_{12}D_{12} & D_{12}D_{13} & D_{13}D_{11} & D_{13}D_{12} & D_{13}D_{13} \\
D_{11}D_{21} & D_{11}D_{22} & D_{11}D_{23} & D_{12}D_{21} & D_{12}D_{22} & D_{12}D_{23} & D_{13}D_{21} & D_{13}D_{22} & D_{13}D_{23} \\
D_{11}D_{31} & D_{11}D_{32} & D_{11}D_{33} & D_{12}D_{31} & D_{12}D_{22} & D_{12}D_{33} & D_{13}D_{31} & D_{13}D_{23} & D_{13}D_{33} \\
\hdashline
D_{21}D_{11} & D_{21}D_{12} & D_{21}D_{13} & D_{22}D_{11} & D_{22}D_{12} & D_{22}D_{13} & D_{23}D_{11} & D_{23}D_{12} & D_{23}D_{13} \\
D_{21}D_{21} & D_{21}D_{22} & D_{21}D_{23} & D_{22}D_{21} & D_{22}D_{22} & D_{22}D_{23} & D_{23}D_{21} & D_{23}D_{22} & D_{23}D_{23} \\
D_{21}D_{31} & D_{21}D_{32} & D_{21}D_{33} & D_{22}D_{31} & D_{22}D_{32} & D_{22}D_{33} & D_{23}D_{31} & D_{23}D_{23} & D_{23}D_{33} \\
\hdashline
D_{31}D_{11} & D_{31}D_{12} & D_{31}D_{13} & D_{32}D_{11} & D_{32}D_{12} & D_{32}D_{13} & D_{33}D_{11} & D_{33}D_{12} & D_{33}D_{13} \\
D_{31}D_{21} & D_{31}D_{22} & D_{31}D_{23} & D_{32}D_{21} & D_{32}D_{22} & D_{32}D_{23} & D_{33}D_{21} & D_{33}D_{22} & D_{33}D_{23} \\
D_{31}D_{31} & D_{31}D_{23} & D_{31}D_{33} & D_{32}D_{31} & D_{32}D_{32} & D_{32}D_{33} & D_{33}D_{31} & D_{33}D_{23} & D_{33}D_{33}
\end{array}\right)
$$

2-particle permutation operations: $\mathbf{s}(a)(b)|j\rangle_a|k\rangle_b = |j\rangle_a|k\rangle_b$, $\mathbf{s}(ab)|j\rangle_a|k\rangle_b = |k\rangle_a|j\rangle_b$

Represented by matrices:

$\mathbf{s}(a)(b) =$

	11	12	13	21	22	23	31	32	33
11	1	·	·	·	·	·	·	·	·
12	·	1	·	·	·	·	·	·	·
13	·	·	1	·	·	·	·	·	·
21	·	·	·	1	·	·	·	·	·
22	·	·	·	·	1	·	·	·	·
23	·	·	·	·	·	1	·	·	·
31	·	·	·	·	·	·	1	·	·
32	·	·	·	·	·	·	·	1	·
33	·	·	·	·	·	·	·	·	1

$\mathbf{s}(ab) =$

	11	12	13	21	22	23	31	32	33
11	1	·	·	·	·	·	·	·	·
12	·	·	·	1	·	·	·	·	·
13	·	·	·	·	·	·	1	·	·
21	·	1	·	·	·	·	·	·	·
22	·	·	·	·	1	·	·	·	·
23	·	·	·	·	·	·	·	1	·
31	·	·	1	·	·	·	·	·	·
32	·	·	·	·	·	1	·	·	·
33	·	·	·	·	·	·	·	·	1

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S_2 *symmetry* of U(3): *Applying S_2 projection*

S_2 matrix eigen-solution found by projectors: Minimal eq. $\mathbf{(ab)}^2-\mathbf{1}=0=(\mathbf{(ab)}+\mathbf{1})(\mathbf{(ab)}+\mathbf{1})$ yields:

Symmetric ($\square\square$): $\mathbf{P}^{\square\square}=\frac{1}{2}\left[\mathbf{1}+\mathbf{(ab)}\right]$

Anti-Symmetric ($\begin{smallmatrix}\square\\\square\end{smallmatrix}$): $\mathbf{P}^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}=\frac{1}{2}\left[\mathbf{1}-\mathbf{(ab)}\right]$

$\mathbf{P}^{\square\square}=$

	11	12	13	21	22	23	31	32	33
11	1	·	·	·	·	·	·	·	·
12	·	$\frac{1}{2}$	·	$\frac{1}{2}$	·	·	·	·	·
13	·	·	$\frac{1}{2}$	·	·	·	$\frac{1}{2}$	·	·
21	·	$\frac{1}{2}$	·	$\frac{1}{2}$	·	·	·	·	·
22	·	·	·	·	1	·	·	·	·
23	·	·	·	·	·	$\frac{1}{2}$	·	$\frac{1}{2}$	·
31	·	·	$\frac{1}{2}$	·	·	·	$\frac{1}{2}$	·	·
32	·	·	·	·	·	$\frac{1}{2}$	·	$\frac{1}{2}$	·
33	·	·	·	·	·	·	·	·	1

$\mathbf{P}^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}=$

	11	12	13	21	22	23	31	32	33
11	0	·	·	·	·	·	·	·	·
12	·	$\frac{1}{2}$	·	$\frac{-1}{2}$	·	·	·	·	·
13	·	·	$\frac{1}{2}$	·	·	·	$\frac{-1}{2}$	·	·
21	·	$\frac{-1}{2}$	·	$\frac{1}{2}$	·	·	·	·	·
22	·	·	·	·	0	·	·	·	·
23	·	·	·	·	·	$\frac{1}{2}$	·	$\frac{-1}{2}$	·
31	·	·	$\frac{-1}{2}$	·	·	·	$\frac{1}{2}$	·	·
32	·	·	·	·	·	$\frac{-1}{2}$	·	$\frac{1}{2}$	·
33	·	·	·	·	·	·	·	·	0

$\mathbf{(a)(b)}=$

	11	12	13	21	22	23	31	32	33
11	1	·	·	·	·	·	·	·	·
12	·	1	·	·	·	·	·	·	·
13	·	·	1	·	·	·	·	·	·
21	·	·	·	1	·	·	·	·	·
22	·	·	·	·	1	·	·	·	·
23	·	·	·	·	·	1	·	·	·
31	·	·	·	·	·	·	1	·	·
32	·	·	·	·	·	·	·	1	·
33	·	·	·	·	·	·	·	·	1

$\mathbf{(ab)}=$

	11	12	13	21	22	23	31	32	33
11	1	·	·	·	·	·	·	·	·
12	·	·	·	1	·	·	·	·	·
13	·	·	·	·	·	·	1	·	·
21	·	1	·	·	·	·	·	·	·
22	·	·	·	·	1	·	·	·	·
23	·	·	·	·	·	·	·	1	·
31	·	·	1	·	·	·	·	·	·
32	·	·	·	·	·	1	·	·	·
33	·	·	·	·	·	·	·	·	1

Matrix representation of *Diagonalizing Transform* (DTran T) is made by excerpting **P**-columns

$\mathbf{P}^{\square\square} =$

	11	12	13	21	22	23	31	32	33
11	1	·	·	·	·	·	·	·	·
12	·	$\frac{1}{2}$	·	$\frac{1}{2}$	·	·	·	·	·
13	·	·	$\frac{1}{2}$	·	·	·	$\frac{1}{2}$	·	·
21	·	$\frac{1}{2}$	·	$\frac{1}{2}$	·	·	·	·	·
22	·	·	·	·	1	·	·	·	·
23	·	·	·	·	·	$\frac{1}{2}$	·	$\frac{1}{2}$	·
31	·	·	$\frac{1}{2}$	·	·	·	$\frac{1}{2}$	·	·
32	·	·	·	·	·	$\frac{1}{2}$	·	$\frac{1}{2}$	·
33	·	·	·	·	·	·	·	·	1

$\mathbf{P}^{\begin{smallmatrix}\square\\\square\end{smallmatrix}} =$

	11	12	13	21	22	23	31	32	33
11	0	·	·	·	·	·	·	·	·
12	·	$\frac{1}{2}$	·	$\frac{-1}{2}$	·	·	·	·	·
13	·	·	$\frac{1}{2}$	·	·	·	$\frac{-1}{2}$	·	·
21	·	$\frac{-1}{2}$	·	$\frac{1}{2}$	·	·	·	·	·
22	·	·	·	·	0	·	·	·	·
23	·	·	·	·	·	$\frac{1}{2}$	·	$\frac{-1}{2}$	·
31	·	·	$\frac{-1}{2}$	·	·	·	$\frac{1}{2}$	·	·
32	·	·	·	·	·	$\frac{-1}{2}$	·	$\frac{1}{2}$	·
33	·	·	·	·	·	·	·	·	0

$T =$

	$\square\square$ x^2	$\square\square$ y^2	$\square\square$ z^2	$\square\square$ xy	$\square\square$ xz	$\square\square$ yz	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$ xp_y	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$ xp_z	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$ yp_z
11	1	·	·	·	·	·	·	·	·
12	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	$\frac{1}{\sqrt{2}}$	·	·
13	·	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	$\frac{-1}{\sqrt{2}}$	·
21	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	$\frac{-1}{\sqrt{2}}$	·	·
22	·	1	·	·	·	·	·	·	·
23	·	·	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	$\frac{1}{\sqrt{2}}$
31	·	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	$\frac{1}{\sqrt{2}}$	·
32	·	·	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	$\frac{-1}{\sqrt{2}}$
33	·	·	1	·	·	·	·	·	·

S_2 *symmetry* of U(3):
Applying S_2 projection

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S_2 *symmetry* of U(3): *Effect of S_2* DTran T *on intertwining S_2 - U(3) irep matrices*

S_2 matrix:

$$T^{\dagger}S(\mathbf{p}_{ab})T = 6D^{\square\square}_{\text{dim}=1}(\mathbf{p}) \oplus 3D^{\boxminus}_{\text{dim}=1}(\mathbf{p})$$

$D^{\square\square}(\mathbf{p})$								
	$D(\mathbf{p})$							
		$D(\mathbf{p})$						
			$D(\mathbf{p})$					
				$D(\mathbf{p})$				
					$D(\mathbf{p})$			
						$D^{\boxminus}(\mathbf{p})$		
							$D^{\boxminus}(\mathbf{p})$	
								$D^{\boxminus}(\mathbf{p})$

U(3) matrices:

$$T^{\dagger}D\otimes D(\mathbf{u})T = 1D^{\square\square}_{\text{dim}=6}(\mathbf{p}) \oplus 1D^{\boxminus}_{\text{dim}=3}(\mathbf{p})$$

$D^{\square\square}_{11}(\mathbf{u})$	$D_{12}(\mathbf{u})$	$D_{13}(\mathbf{u})$	$D_{14}(\mathbf{u})$	$D_{15}(\mathbf{u})$	$D_{16}(\mathbf{u})$			
$D_{21}(\mathbf{u})$	$D_{22}(\mathbf{u})$	$D_{23}(\mathbf{u})$	$D_{24}(\mathbf{u})$	$D_{25}(\mathbf{u})$	$D_{26}(\mathbf{u})$			
$D_{31}(\mathbf{u})$	$D_{32}(\mathbf{u})$	$D_{33}(\mathbf{u})$	$D_{34}(\mathbf{u})$	$D_{35}(\mathbf{u})$	$D_{36}(\mathbf{u})$			
$D_{41}(\mathbf{u})$	$D_{42}(\mathbf{u})$	$D_{43}(\mathbf{u})$	$D_{44}(\mathbf{u})$	$D_{45}(\mathbf{u})$	$D_{46}(\mathbf{u})$			
$D_{51}(\mathbf{u})$	$D_{52}(\mathbf{u})$	$D_{53}(\mathbf{u})$	$D_{54}(\mathbf{u})$	$D_{55}(\mathbf{u})$	$D_{56}(\mathbf{u})$			
$D_{61}(\mathbf{u})$	$D_{62}(\mathbf{u})$	$D_{63}(\mathbf{u})$	$D_{64}(\mathbf{u})$	$D_{65}(\mathbf{u})$	$D_{66}(\mathbf{u})$			
						$D^{\boxminus}_{11}(\mathbf{u})$	$D_{12}(\mathbf{u})$	$D_{13}(\mathbf{u})$
						$D_{21}(\mathbf{u})$	$D_{22}(\mathbf{u})$	$D_{23}(\mathbf{u})$
						$D_{31}(\mathbf{u})$	$D_{32}(\mathbf{u})$	$D_{33}(\mathbf{u})$

Diagonalized S_2 bicycle matrix:

$$T^{\dagger}(\mathbf{ab})T =$$

+1								
	+1							
		+1						
			+1					
				+1				
					+1			
						−1		
							−1	
								−1

Unicycle **(a)(b)** is unit matrix

S_2 *symmetry* of U(3): *Effect of S_2* DTran T *on intertwining S_2 - U(3) irep matrices*

S_2 matrix:

$$T^{\dagger}S(\mathbf{p}_{ab})T = 6D^{\square\square}_{\text{dim}=1}(\mathbf{p}) \oplus 3D^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}_{\text{dim}=1}(\mathbf{p})$$

$D^{\square\square}(\mathbf{p})$								
	$D\ (\mathbf{p})$							
		$D\ (\mathbf{p})$						
			$D\ (\mathbf{p})$					
				$D\ (\mathbf{p})$				
					$D\ (\mathbf{p})$			
						$D^{\square}(\mathbf{p})$		
							$D^{\square}(\mathbf{p})$	
								$D^{\square}(\mathbf{p})$

U(3) matrices:

$$T^{\dagger}D\otimes D(\mathbf{u})T = 1D^{\square\square}_{\text{dim}=6}(\mathbf{p}) \oplus 1D^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}_{\text{dim}=3}(\mathbf{p})$$

$D^{\square\square}_{11}(\mathbf{u})$	$D_{12}(\mathbf{u})$	$D_{13}(\mathbf{u})$	$D_{14}(\mathbf{u})$	$D_{15}(\mathbf{u})$	$D_{16}(\mathbf{u})$			
$D_{21}(\mathbf{u})$	$D_{22}(\mathbf{u})$	$D_{23}(\mathbf{u})$	$D_{24}(\mathbf{u})$	$D_{25}(\mathbf{u})$	$D_{26}(\mathbf{u})$			
$D_{31}(\mathbf{u})$	$D_{32}(\mathbf{u})$	$D_{33}(\mathbf{u})$	$D_{34}(\mathbf{u})$	$D_{35}(\mathbf{u})$	$D_{36}(\mathbf{u})$			
$D_{41}(\mathbf{u})$	$D_{42}(\mathbf{u})$	$D_{43}(\mathbf{u})$	$D_{44}(\mathbf{u})$	$D_{45}(\mathbf{u})$	$D_{46}(\mathbf{u})$			
$D_{51}(\mathbf{u})$	$D_{52}(\mathbf{u})$	$D_{53}(\mathbf{u})$	$D_{54}(\mathbf{u})$	$D_{55}(\mathbf{u})$	$D_{56}(\mathbf{u})$			
$D_{61}(\mathbf{u})$	$D_{62}(\mathbf{u})$	$D_{63}(\mathbf{u})$	$D_{64}(\mathbf{u})$	$D_{65}(\mathbf{u})$	$D_{66}(\mathbf{u})$			
						$D^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}_{11}(\mathbf{u})$	$D_{12}(\mathbf{u})$	$D_{13}(\mathbf{u})$
						$D_{21}(\mathbf{u})$	$D_{22}(\mathbf{u})$	$D_{23}(\mathbf{u})$
						$D_{31}(\mathbf{u})$	$D_{32}(\mathbf{u})$	$D_{33}(\mathbf{u})$

Diagonalized S_2 bicycle matrix:

$$T^{\dagger}(\mathbf{ab})T =$$

+1								
	+1							
		+1						
			+1					
				+1				
					+1			
						−1		
							−1	
								−1

Unicycle $(\mathbf{a})(\mathbf{b})$ is unit matrix

S_2 group hook formula

$$D^{\square\square}_{\text{dim}=1}(\mathbf{p}) \qquad \frac{2\cdot 1}{\boxed{2}\boxed{1}} = 1$$

$$D^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}_{\text{dim}=1}(\mathbf{p}) \qquad \frac{2\cdot 1}{\begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}} = 1$$

$U(3)$ group hook formula

$$D^{\square\square}_{\text{dim}=6}(\mathbf{u}) \qquad \frac{\boxed{3}\boxed{4}}{\boxed{2}\boxed{1}} = 6$$

$$D^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}_{\text{dim}=3}(\mathbf{u}) \qquad \frac{\begin{smallmatrix}\boxed{3}\\\boxed{2}\end{smallmatrix}}{\begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}} = 3$$

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S_3 *symmetry* of U(3): *Applying S_3 projection*

Rank-3 tensor basis $|ijk\rangle$ (3 particles each with U(3) state space) has dimension 3^3=27

S_3 *symmetry* of U(3): *Applying S_3 projection*

Rank-3 tensor basis $|i_a\, j_b\, k_c\rangle$ (3 particles each with U(3) state space) has dimension 3^3=27

[ab]

	111	112	113	121	122	123	131	132	133	211	212	213	221	222	223	231	232	233	311	312	313	321	322	323	331	332	333
111																											
112																											
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Rank-3 tensor basis $|i_a\, j_b\, k_c\rangle$ (3 particles each with U(3) state space) has dimension $3^3=27$

[ab]

	111	112	113	121	122	123	131	132	133	211	212	213	221	222	223	231	232	233	311	312	313	321	322	323	331	332	333
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Whoa!
That's pretty big.
So let's solve by S_3 *character theory*.
Only need traces that are sums of diagonal elements (just one per-class)

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(ab)

	111	112	113	121	122	123	131	132	133	211	212	213	221	222	223	231	232	233	311	312	313	321	322	323	331	332	333
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Only need traces that are sums of diagonal elements (just one per-class)

Bicycle character :
Trace s(**(ab)**) counts states like $|j_a\, j_b\, k_c\rangle$

S_3 *symmetry* of U(3): *Applying S_3 character theory*

Rank-3 tensor basis $|i_a\ j_b\ k_c\rangle$ (3 particles each with U(3) state space) has dimension 3^3=27

$Trace(\mathbf{ab}) = 9$

	111	112	113	121	122	123	131	132	133	211	212	213	221	222	223	231	232	233	311	312	313	321	322	323	331	332	333
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So let's solve by S_3 *character theory*.
Only need traces that are sums of diagonal elements (just one per-class)

Bicycle character :
Trace $s((\mathbf{ab}))$ counts states like $|j_a\ j_b\ k_c\rangle$
result: $Tr(\mathbf{ab})$=9

S_3 *symmetry* of U(3): *Applying S_3 character theory*

Rank-3 tensor basis $|i_a\, j_b\, k_c\rangle$ (3 particles each with U(3) state space) has dimension $3^3=27$

$Trace(\mathbf{abc}) = 3$

	111	112	113	121	122	123	131	132	133	211	212	213	221	222	223	231	232	233	311	312	313	321	322	323	331	332	333
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Bicycle character :
Trace s(**(ab)**) counts states like $|j_a\, j_b\, k_c\rangle$
result: *Tr***(ab)**=9

Tricycle character :
Trace s(**(abc)**) counts states like $|j_a\, j_b\, j_c\rangle$
result: *Tr***(abc)**=3

S_3 *symmetry* of U(3): *Applying S_3 character theory*

Rank-3 tensor basis $|i_a\, j_b\, k_c\rangle$ (3 particles each with U(3) state space) has dimension $3^3=27$

$Trace(\mathbf{a})(\mathbf{b})(\mathbf{c}) = 27 = 3^3$

	111	112	113	121	122	123	131	132	133	211	212	213	221	222	223	231	232	233	311	312	313	321	322	323	331	332	333
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112		1																									
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Only need traces that are sums of diagonal elements (just one per-class)

Bicycle character :
Trace s(**(ab)**) counts states like $|j_a\, j_b\, k_c\rangle$
result: *Tr*(**ab**)=9

Tricycle character :
Trace s(**(abc)**) counts states like $|j_a\, j_b\, j_c\rangle$
result: *Tr*(**abc**)=3

Unicycle character :
result: *Tr*(**a**)(**b**)(**c**)=27

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Irep.freq.formula
GrpThLect.15p.48.

Rank-3 tensor basis $|i_a\, j_b\, k_c\rangle$ (3 particles each with U(3) state space) has dimension $3^3=27$

Frequency formula for $D^{[\mu]}$: $f^{[\mu]} = \frac{1}{{}^{o}S_n} \sum_{classes(k)} \begin{pmatrix} \text{order of} \\ \text{class}(k) \end{pmatrix} \chi_k^{[\mu]} Trace(\mathbf{p}_k)$

Tensor traces: $Tr\mathbf{(a)(b)(c)}=27$, $Tr\mathbf{(abc)}=3$, $Tr\mathbf{(ab)}=9$,

S_3 *symmetry* of U(3): *Applying S_3 character theory*

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Frequency formula for $D^{[\mu]}$: $f^{[\mu]} = \frac{1}{{}^oS_n} \sum_{classes(k)} \left(\begin{matrix} order\ of \\ class(k) \end{matrix}\right) \chi_k^{[\mu]} Trace(\mathbf{p}_k)$

Tensor traces: *Tr*(**a)(b)(c)**=27, *Tr*(**abc**)=3, *Tr*(**ab**)=9,

and S_3 character table:

$\chi_k^{[\mu]}$	$k=(1)^3$ **(a)(b)(c)**	$k=(3)$ **(abc),(acb)**	$k=(1)(2)$ **(bc),(ac),(ab)**
μ=□□□	1	1	1
□ □ □	1	1	−1
□□ □	2	−1	0

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Irep.freq.formula
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Rank-3 tensor basis $|i_a\, j_b\, k_c\rangle$ (3 particles each with U(3) state space) has dimension 3^3=27

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Tensor traces: $Tr\,\mathbf{(a)(b)(c)}=27$, $Tr\,\mathbf{(abc)}=3$, $Tr\,\mathbf{(ab)}=9$,

and S_3 character table:

$\chi_k^{[\mu]}$	$k=(1)^3$ **(a)(b)(c)**	$k=(3)$ **(abc),(acb)**	$k=(1)(2)$ **(bc),(ac),(ab)**
$\mu=\square\square\square$	1	1	1
$\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$	1	1	-1
$\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$	2	-1	0

$$f^{\square\square\square} = \frac{1}{3!}\left(\binom{order\ of}{class(1)} \chi_{1^3}^{\square\square\square}\, Tr\,\mathbf{(a)(b)(c)} + \binom{order\ of}{class(3)} \chi_{(3)}^{\square\square\square}\, Tr\,\mathbf{(abc)} + \binom{order\ of}{class(1)(2)} \chi_{(1)(2)}^{\square\square\square}\, Tr\,\mathbf{(ab)} \right)$$

$$= \frac{1}{6}\left((\ 1\ ) \cdot 1 \cdot 27 + (\ 2\ ) \cdot 1 \cdot 3 + (\ 3\ ) \cdot 1 \cdot 9 \right) = 10$$

S_3 *symmetry* of U(3): *Applying S_3 character theory*

Irep.freq.formula
GrpThLect.15p.48.

Rank-3 tensor basis $|i_a\, j_b\, k_c\rangle$ (3 particles each with U(3) state space) has dimension $3^3=27$

Frequency formula for $D^{[\mu]}$: $f^{[\mu]} = \frac{1}{{}^{o}S_n} \sum_{classes(k)} \binom{\textit{order of}}{class(k)} \chi_k^{[\mu]}\, Trace\,(\mathbf{p}_k)$

Tensor traces: *Tr*(**a)(b)(c)**=27, *Tr*(**abc)**=3, *Tr*(**ab)**=9,

and S_3 character table:

$\chi_k^{[\mu]}$	$k=(1)^3$ **(a)(b)(c)**	$k=(3)$ **(abc),(acb)**	$k=(1)(2)$ **(bc),(ac),(ab)**
$\mu=\square\square\square$	1	1	1
$\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$	1	1	-1
$\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$	2	-1	0

$$f^{\square\square\square} = \frac{1}{3!}\left(\binom{\textit{order of}}{class(1)}\chi_{1^3}^{\square\square\square}\,Tr\,\mathbf{(a)(b)(c)} + \binom{\textit{order of}}{class(3)}\chi_{(3)}^{\square\square\square}\,Tr\,\mathbf{(abc)} + \binom{\textit{order of}}{class(1)(2)}\chi_{(1)(2)}^{\square\square\square}\,Tr\,\mathbf{(ab)}\right)$$

$$= \frac{1}{6}\left((1)\cdot 1\cdot 27 + (2)\cdot 1\cdot 3 + (3)\cdot 1\cdot 9\right) = 10$$

$$f^{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}} = \frac{1}{3!}\left(\binom{\textit{order of}}{class(1)}\chi_{1^3}^{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}\,Tr\,\mathbf{(a)(b)(c)} + \binom{\textit{order of}}{class(3)}\chi_{(3)}^{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}\,Tr\,\mathbf{(abc)} + \binom{\textit{order of}}{class(1)(2)}\chi_{(1)(2)}^{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}\,Tr\,\mathbf{(ab)}\right)$$

$$= \frac{1}{6}\left((1)\cdot 1\cdot 27 + (2)\cdot 1\cdot 3 + (3)\cdot(-1)\cdot 9\right) = 1$$

S_3 *symmetry* of U(3): *Applying S_3 character theory*

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Tensor traces: $Tr(\mathbf{a})(\mathbf{b})(\mathbf{c})=27$, $Tr(\mathbf{abc})=3$, $Tr(\mathbf{ab})=9$,

and S_3 character table:

$\chi_k^{[\mu]}$	$k=(1)^3$ **(a)(b)(c)**	$k=(3)$ **(abc),(acb)**	$k=(1)(2)$ **(bc),(ac),(ab)**
$\mu=$[3] (□□□)	1	1	1
[1,1,1] (vertical □□□)	1	1	−1
[2,1] (□□ over □)	2	−1	0

$$f^{[3]} = \frac{1}{3!}\left(\begin{pmatrix} \text{order of} \\ \text{class}(1) \end{pmatrix} \chi_{1^3}^{[3]} Tr\,(\mathbf{a})(\mathbf{b})(\mathbf{c}) + \begin{pmatrix} \text{order of} \\ \text{class}(3) \end{pmatrix} \chi_{(3)}^{[3]} Tr\,(\mathbf{abc}) + \begin{pmatrix} \text{order of} \\ \text{class}(1)(2) \end{pmatrix} \chi_{(1)(2)}^{[3]} Tr\,(\mathbf{ab}) \right)$$

$$= \frac{1}{6}\left((1) \cdot 1 \cdot 27 + (2) \cdot 1 \cdot 3 + (3) \cdot 1 \cdot 9 \right) = 10$$

$$f^{[1,1,1]} = \frac{1}{3!}\left(\begin{pmatrix} \text{order of} \\ \text{class}(1) \end{pmatrix} \chi_{1^3}^{[1,1,1]} Tr\,(\mathbf{a})(\mathbf{b})(\mathbf{c}) + \begin{pmatrix} \text{order of} \\ \text{class}(3) \end{pmatrix} \chi_{(3)}^{[1,1,1]} Tr\,(\mathbf{abc}) + \begin{pmatrix} \text{order of} \\ \text{class}(1)(2) \end{pmatrix} \chi_{(1)(2)}^{[1,1,1]} Tr\,(\mathbf{ab}) \right)$$

$$= \frac{1}{6}\left((1) \cdot 1 \cdot 27 + (2) \cdot 1 \cdot 3 + (3) \cdot (-1) \cdot 9 \right) = 1$$

$$f^{[2,1]} = \frac{1}{3!}\left(\begin{pmatrix} \text{order of} \\ \text{class}(1) \end{pmatrix} \chi_{1^3}^{[2,1]} Tr\,(\mathbf{a})(\mathbf{b})(\mathbf{c}) + \begin{pmatrix} \text{order of} \\ \text{class}(3) \end{pmatrix} \chi_{(3)}^{[2,1]} Tr\,(\mathbf{abc}) + \begin{pmatrix} \text{order of} \\ \text{class}(1)(2) \end{pmatrix} \chi_{(1)(2)}^{[2,1]} Tr\,(\mathbf{ab}) \right)$$

$$= \frac{1}{6}\left((1) \cdot 2 \cdot 27 + (2) \cdot (-1) \cdot 3 + (3) \cdot (0) \cdot 9 \right) = 8$$

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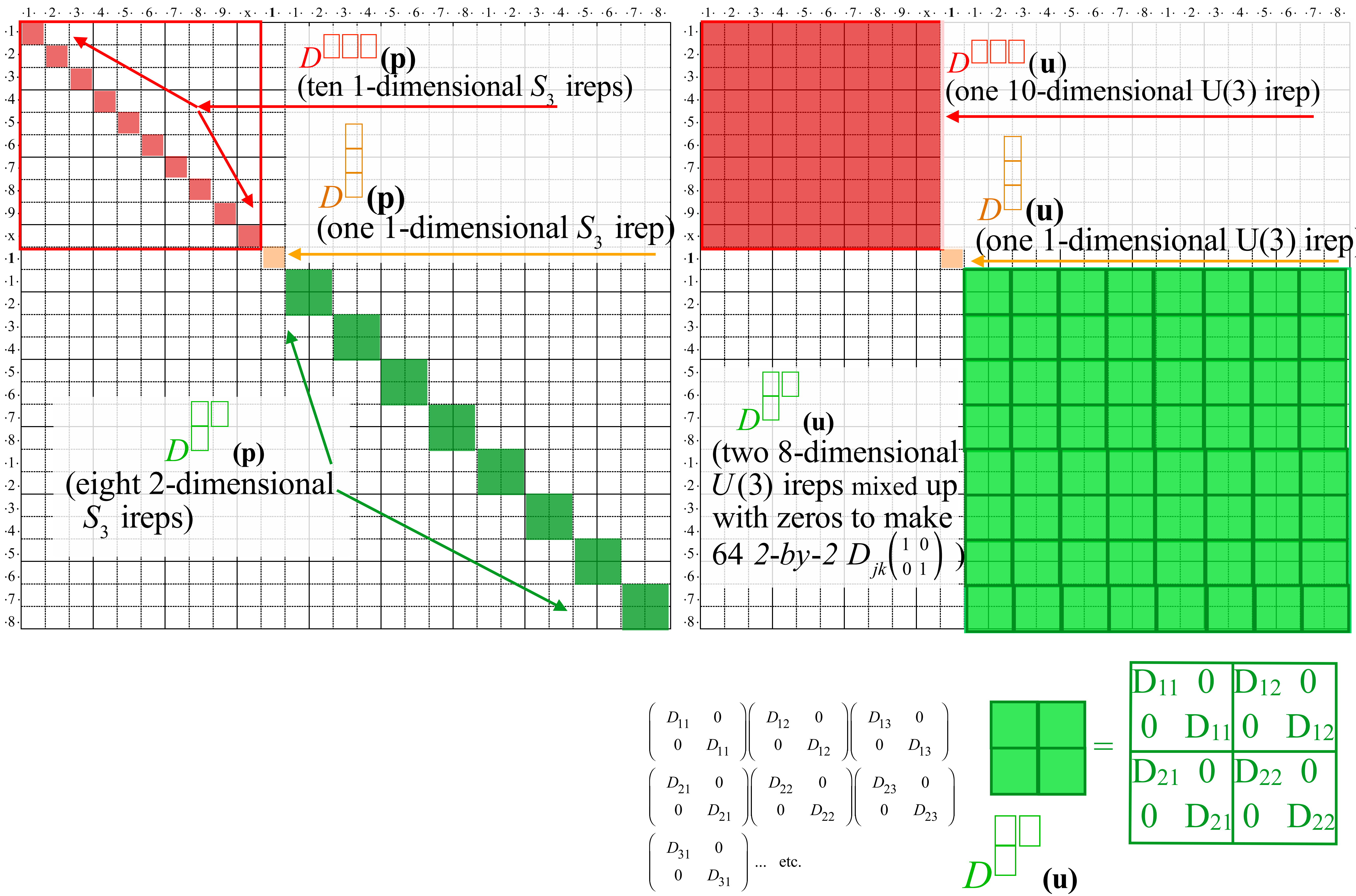
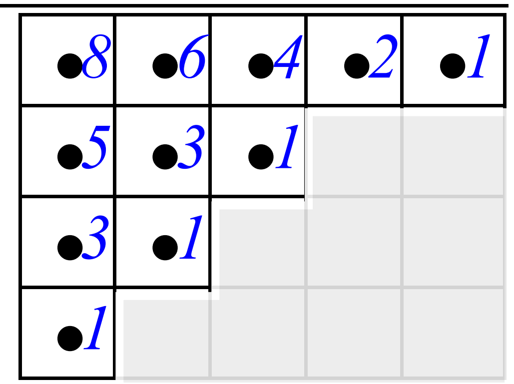
S_3 matrices: *U(3) matrices:*



S_n Young Tableaus and spin-symmetry for X_n and XY_n molecules

Tableau dimension formulae

$$\ell^{[\mu_s]}(S_n) = \text{Dimension of } S_n \text{ Tableau } [\mu_1][\mu_2]\cdots[\mu_n] = \frac{n! = n\cdot(n-1)\cdot(n-2)\cdots 3\cdot 2\cdot 1}{\text{hook-length product}}$$



Examples:

$$\ell^{A_1} = \ell^{[3,0,0]}(S_3) = \frac{3\cdot 2\cdot 1}{\begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline\end{array}} = 1$$

$$\ell^{A_2} = \ell^{[1,1,1]}(S_3) = \frac{3\cdot 2\cdot 1}{\begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array}} = 1$$

$$\ell^{E} = \ell^{[2,1,0]}(S_3) = \frac{3\cdot 2\cdot 1}{\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 & \\ \hline\end{array}} = 2$$

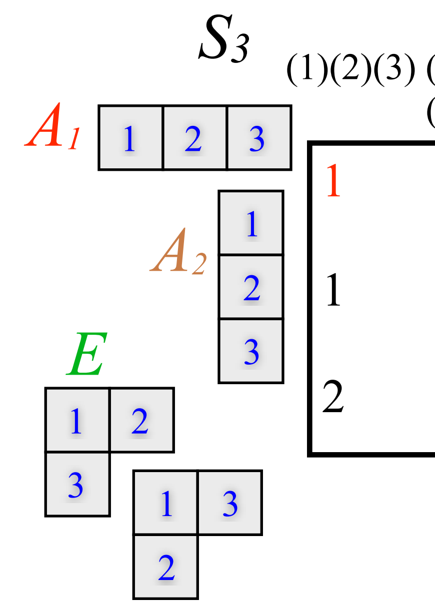
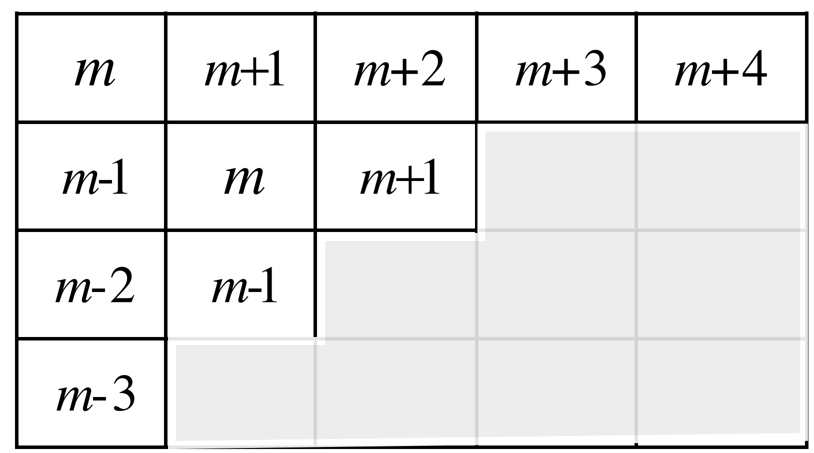
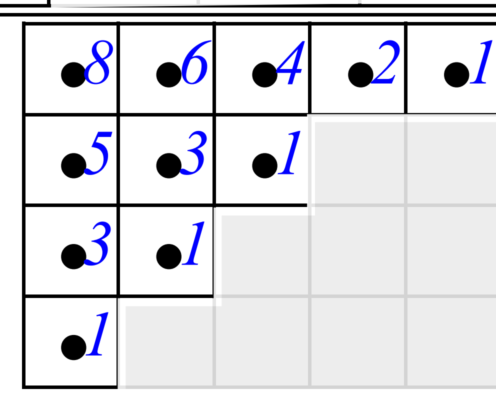


FIG. 28. Robinson formula for statistical weights. The "hook-length" of a box in the tableau is the number of boxes in a "hook" which includes that box and all boxes in the line to the right and in the column below it.

$$\ell^{[\mu_s]}(U_m) = \text{Dimension of } S_n * U_m \text{ Tableau } [\mu_1][\mu_2]\cdots[\mu_m] = \frac{m\text{-dimension product}}{\text{hook-length product}}$$





Examples:

$$\ell^{[2,1,0]}(S_3 * U(3)) = \frac{\begin{array}{|c|c|}\hline 3 & 4 \\ \hline 2 & \\ \hline\end{array}}{\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 & \\ \hline\end{array}} = 8$$

$$\ell^{[3,0,0]}(S_3 * U(3)) = \frac{\begin{array}{|c|c|c|}\hline 3 & 4 & 5 \\ \hline\end{array}}{\begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline\end{array}} = 10$$

Irep.freq.formula
GrpThLect.15p.48.

$$\ell^{[\mu_s]}(S_n) = \begin{array}{c}\textit{Dimension}\\ \textit{of } S_n \textit{ Tableau}\\ [\mu_1][\mu_2]\cdots[\mu_n]\end{array} = \frac{n! = n\cdot(n-1)\cdot(n-2)\cdots 3\cdot 2\cdot 1}{\textit{hook-length product}}$$

hook-length product

•	•	•	•	•
•	•	•		
•	•			
•				

$$\ell^{[\mu_s]}(U_m) = \begin{array}{c}\textit{Dimension}\\ \textit{of } S_n * U_m \textit{ Tableau}\\ [\mu_1][\mu_2]\cdots[\mu_m]\end{array} = \frac{m\textit{-dimension product}}{\textit{hook-length product}}$$

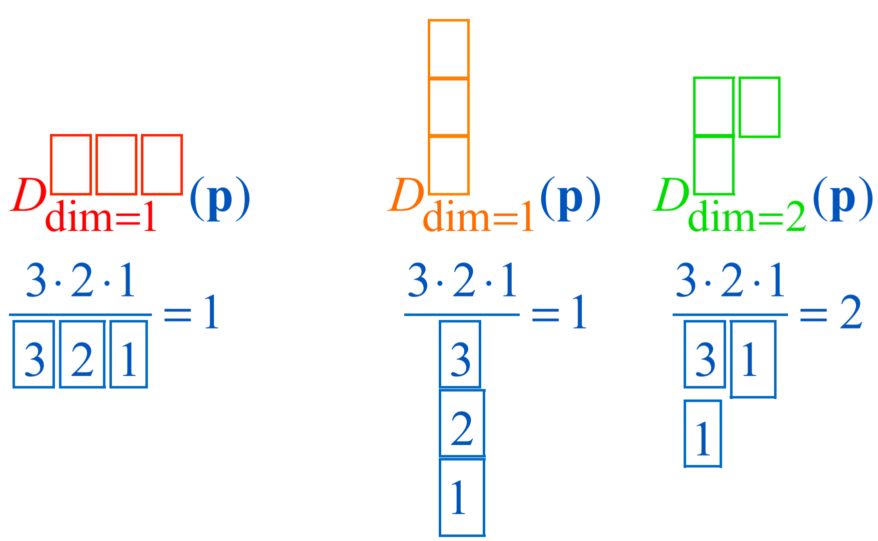
m-dimension product

m	$m+1$	$m+2$	$m+3$	$m+4$
$m-1$	m	$m+1$		
$m-2$	$m-1$			
$m-3$				

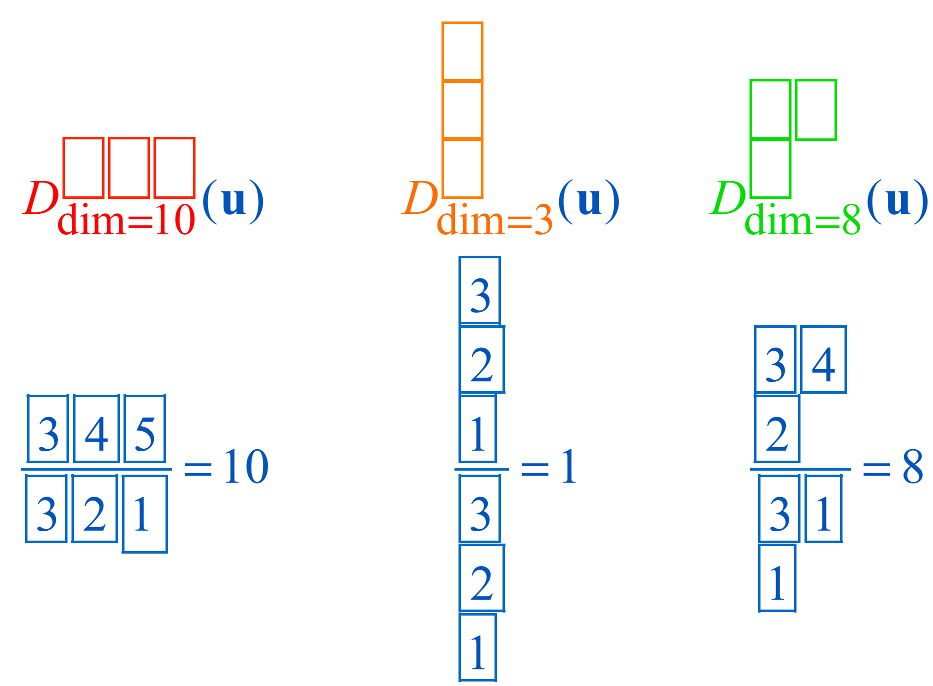
hook-length product

•	•	•	•	•
•	•	•		
•	•			
•				

S_3 group hook formula

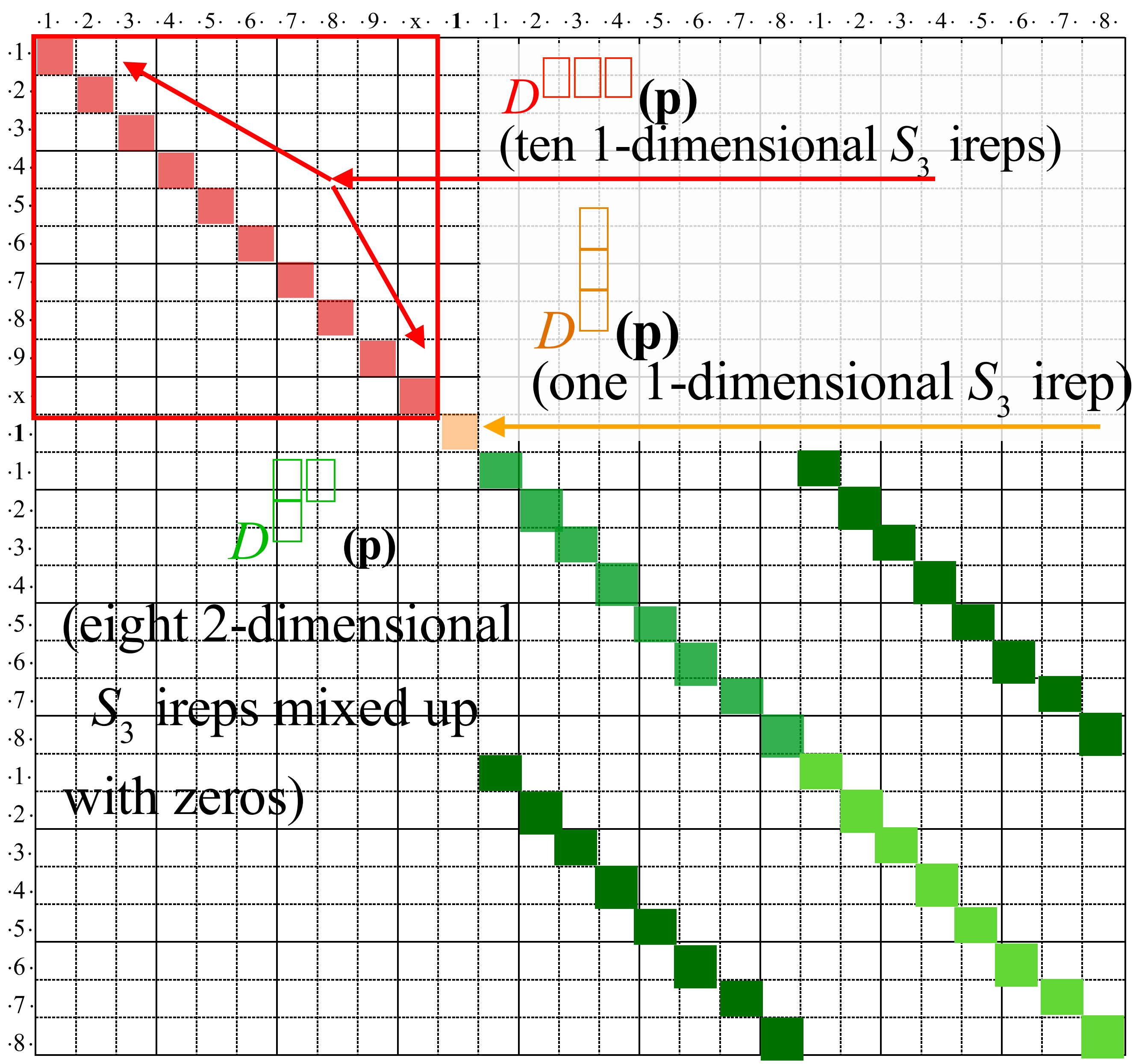


$U(3)$ group hook formula

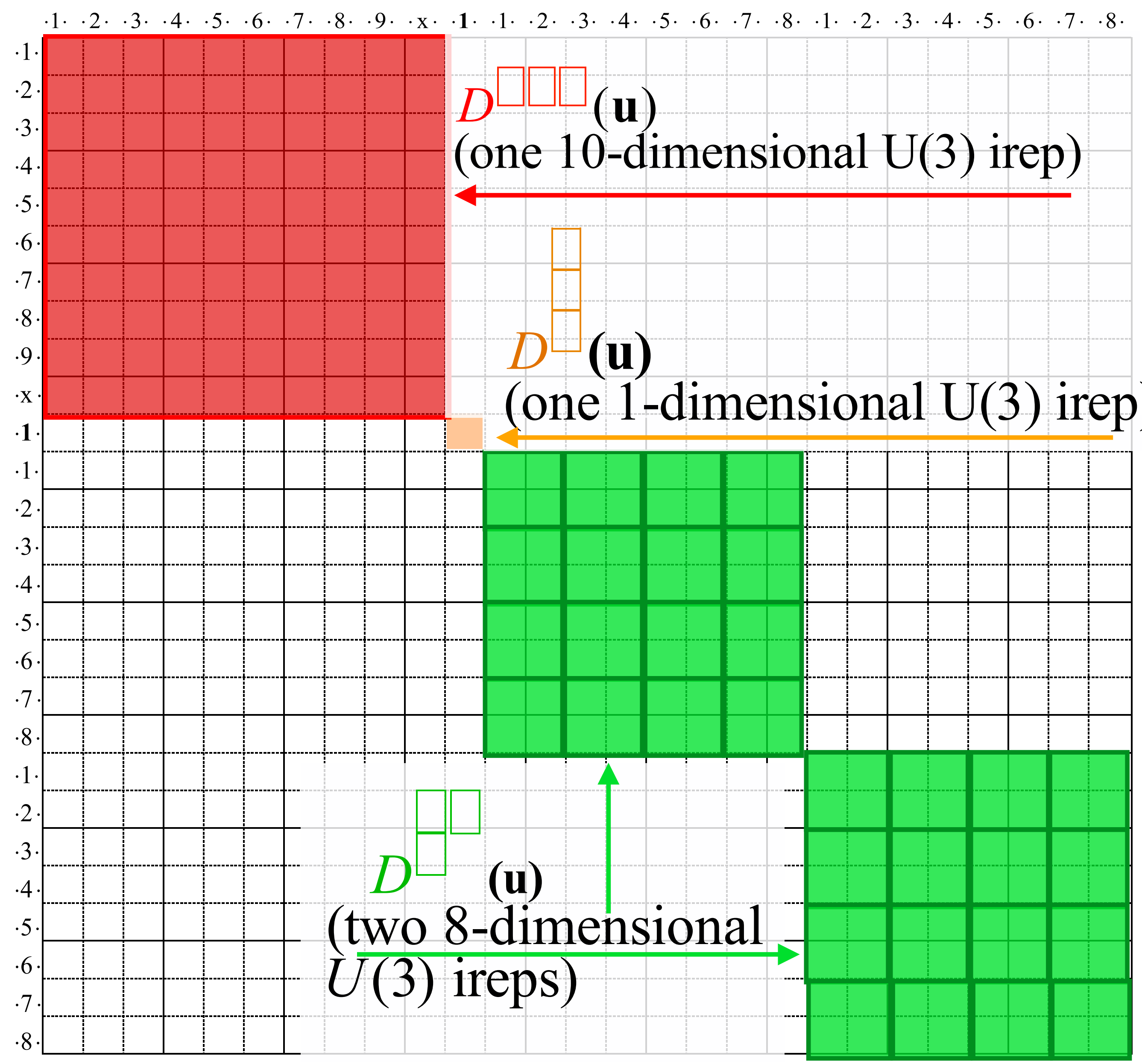


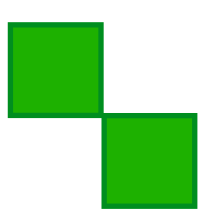
S_3 *symmetry of* U(3): *Effect of S_3* DTran T *on intertwining S_3 - U(3) irep matrices*

S_3 matrices:



U(3) matrices:





$$= \begin{matrix} D_{11} & 0 & 0 & 0 \ldots \\ 0 & D_{11} & 0 & 0 \ldots \\ 0 & 0 & D_{11} & 0 \ldots \\ 0 & 0 & 0 & D_{11} \end{matrix}$$

D (u)

$$\begin{matrix} D_{11} & D_{12} & D_{13} & D_{14} \ldots D_{18} \\ D_{21} & D_{22} & D_{23} & D_{24} \ldots D_{28} \\ D_{31} & D_{32} & D_{34} & D_{34} \ldots D_{38} \\ D_{41} & D_{42} & D_{43} & D_{44} \end{matrix}$$

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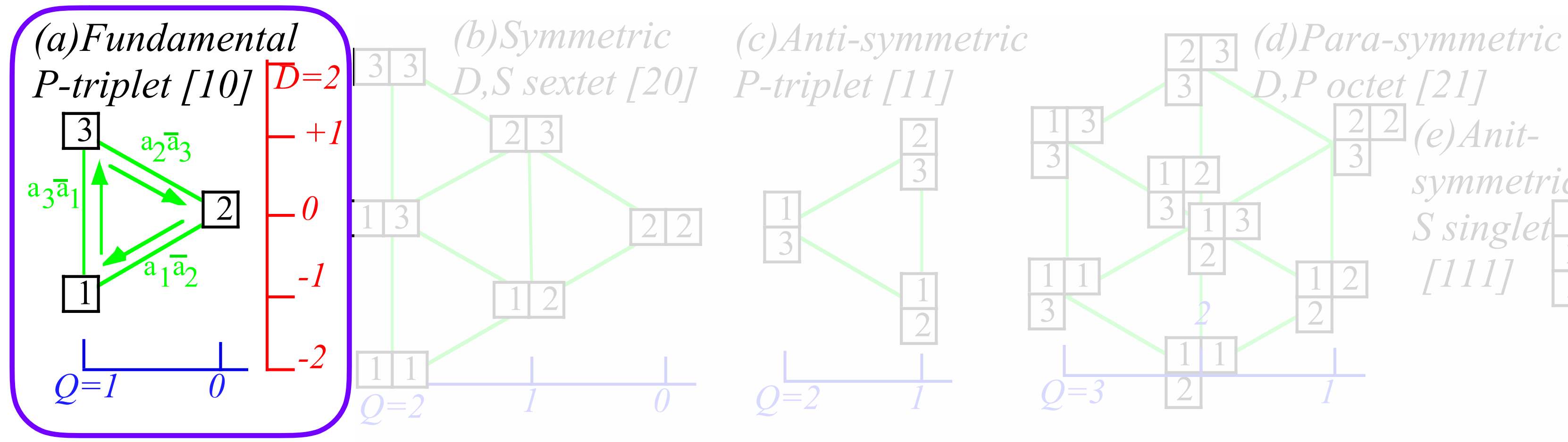
Hooklength formulas

Structure of U(3) *irep bases*

Fundamental $\ell^{\square} = 3$ "quark" irep



Each tableau has 3D Cartesian integer coordinates (n_1, n_2, n_3) determined by number operators $(a_1\bar{a}_1, a_2\bar{a}_2, a_3\bar{a}_3)$

$(\mathbf{a}_1^\dagger \mathbf{a}_1, \mathbf{a}_2^\dagger \mathbf{a}_2, \mathbf{a}_3^\dagger \mathbf{a}_3)$

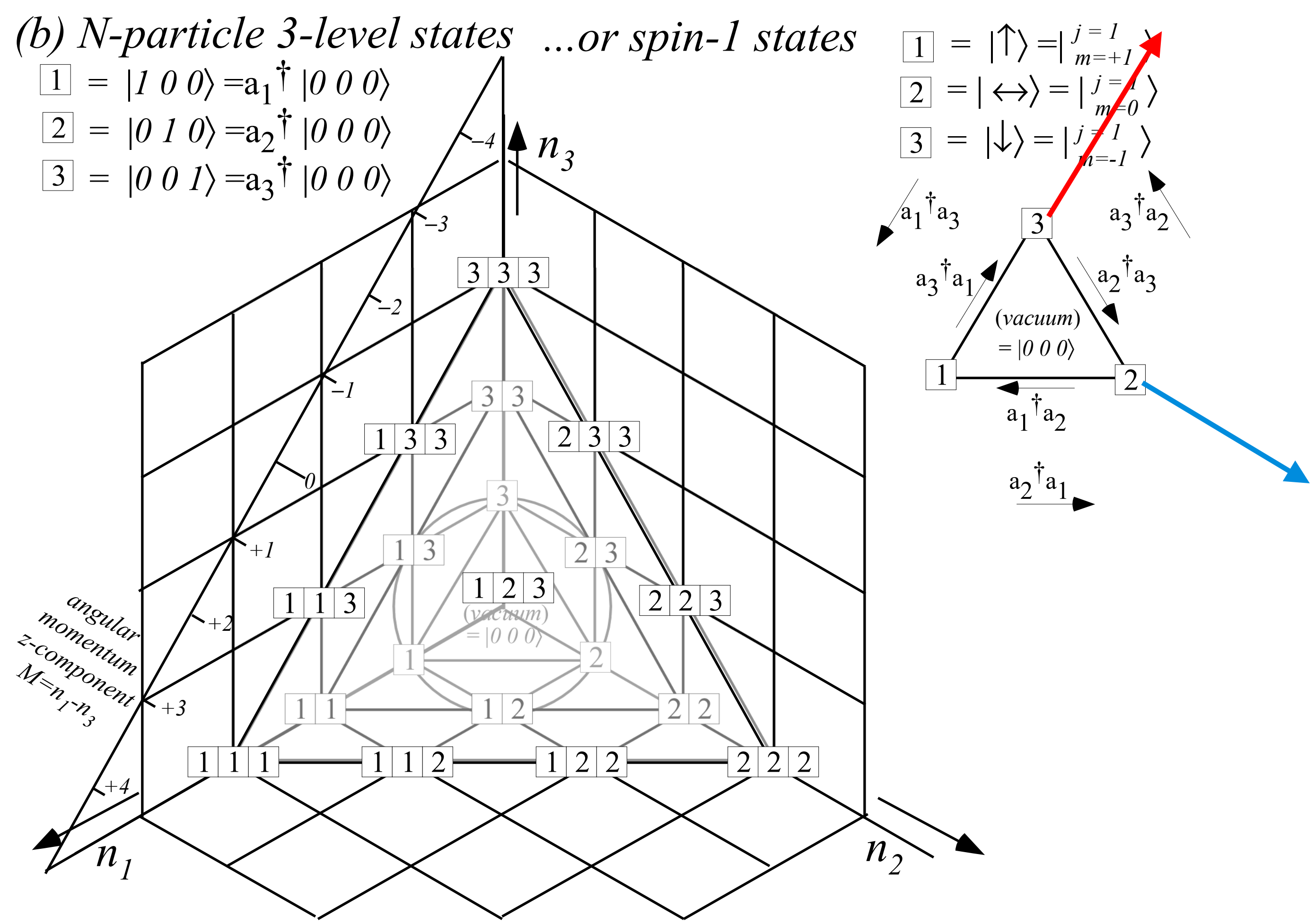
Tableaus with the same total number $N = n_1 + n_2 + n_3$ lie in the same plane normal to (1,1,1).

Plane has orthogonal D and Q axes for *dipole-sum D* of z-component momentum

$$D = \left\langle L_z \right\rangle = n_3 - n_1 = M_L$$

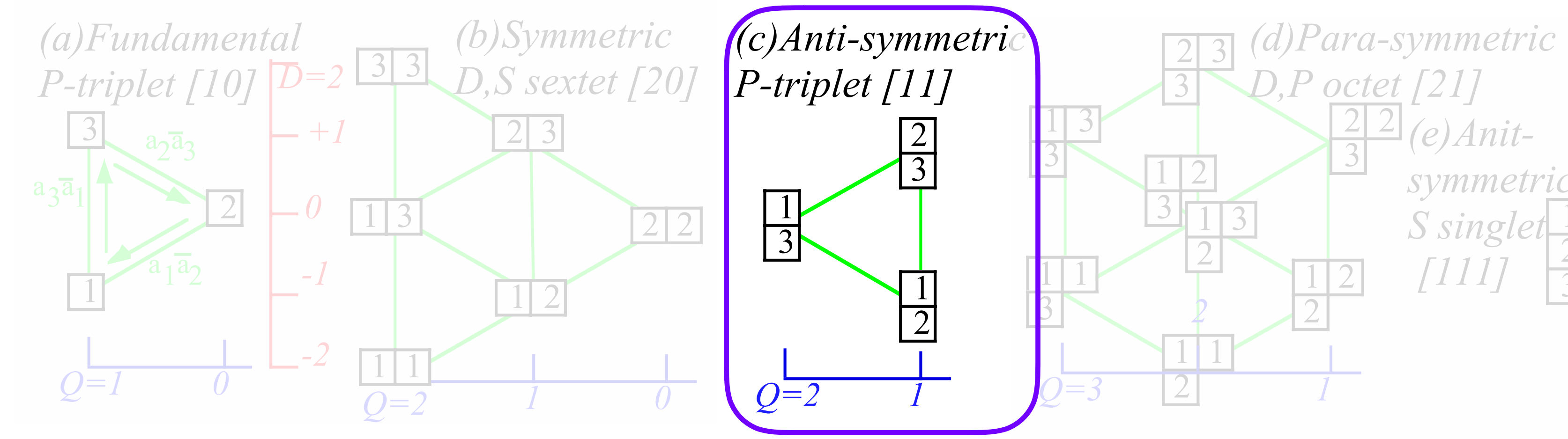
and the *quadrupole-sum Q* of squared-z-component momentum.

$$Q = \left\langle L_z^2 \right\rangle = n_3 + n_1 = N - n_2$$



Structure of U(3) *irep bases*

Fundamental $\ell^{\square}_{\square} = 3$ "anti-quark"



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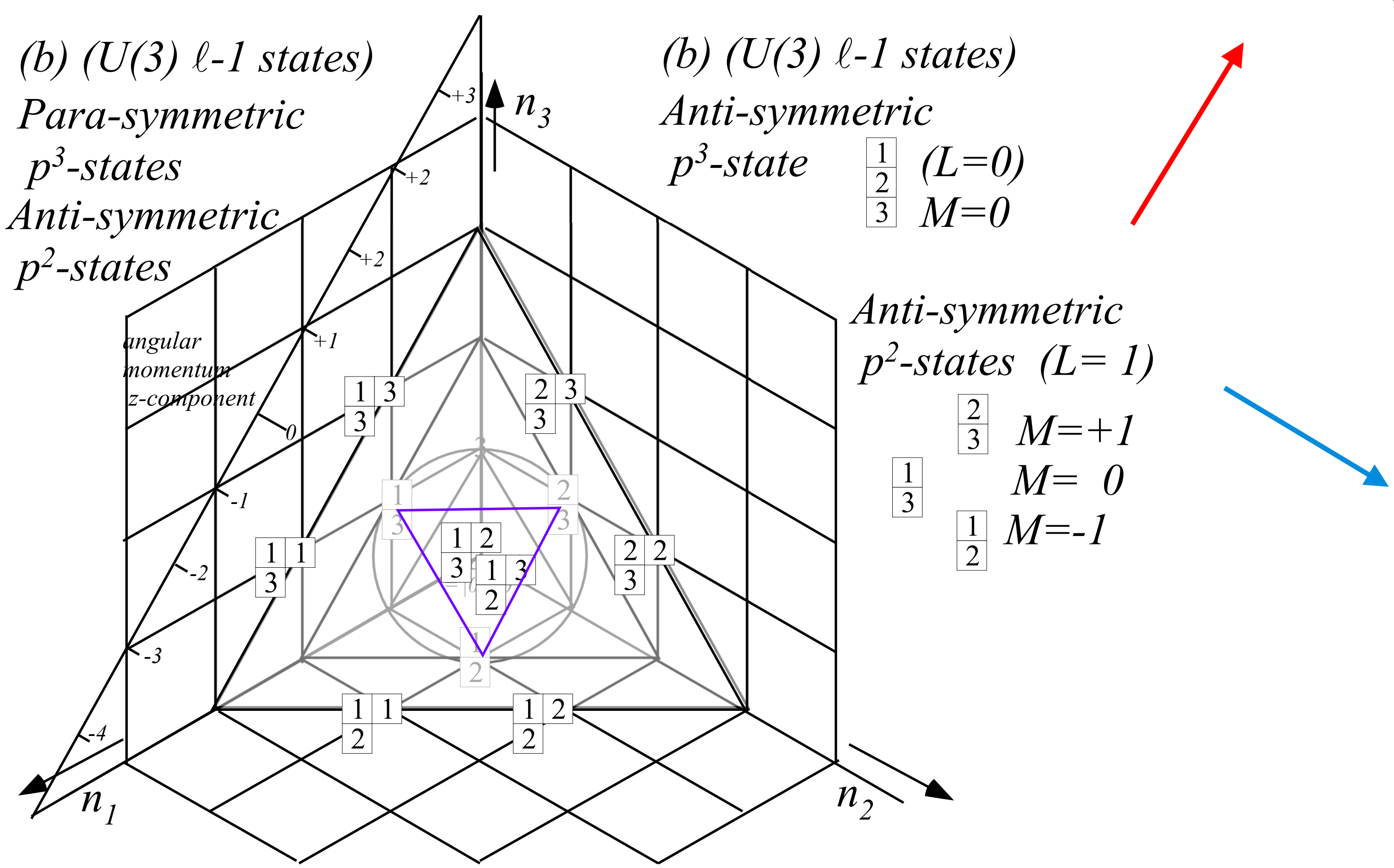
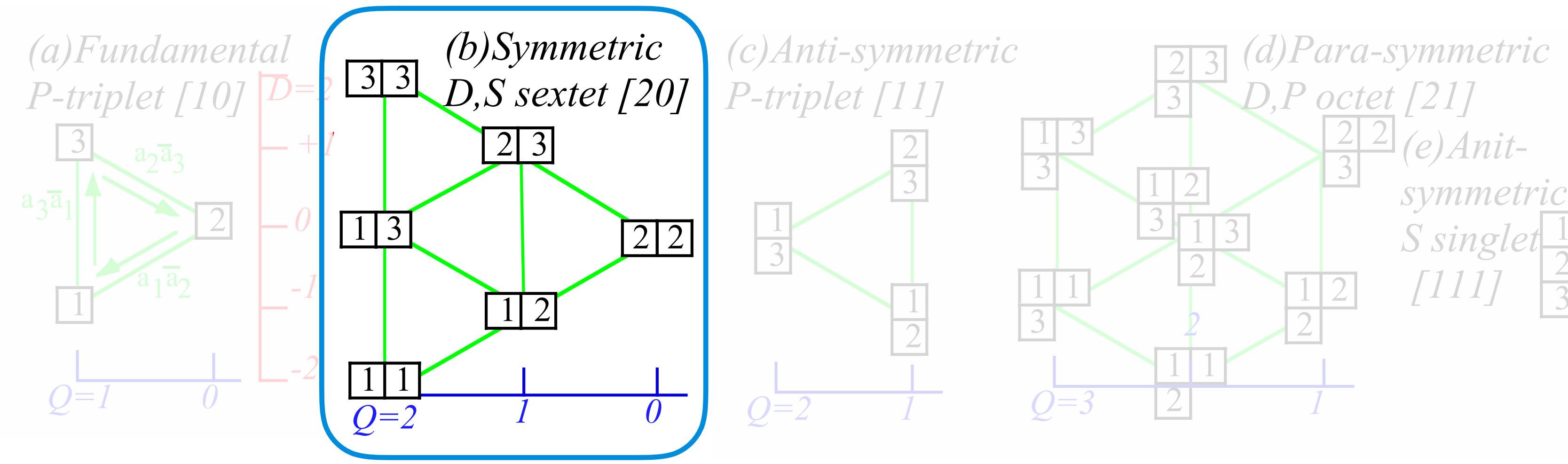


Structure of U(3) *irep bases*

Fundamental $\ell^{\square\square} = 6$ "di-quark"



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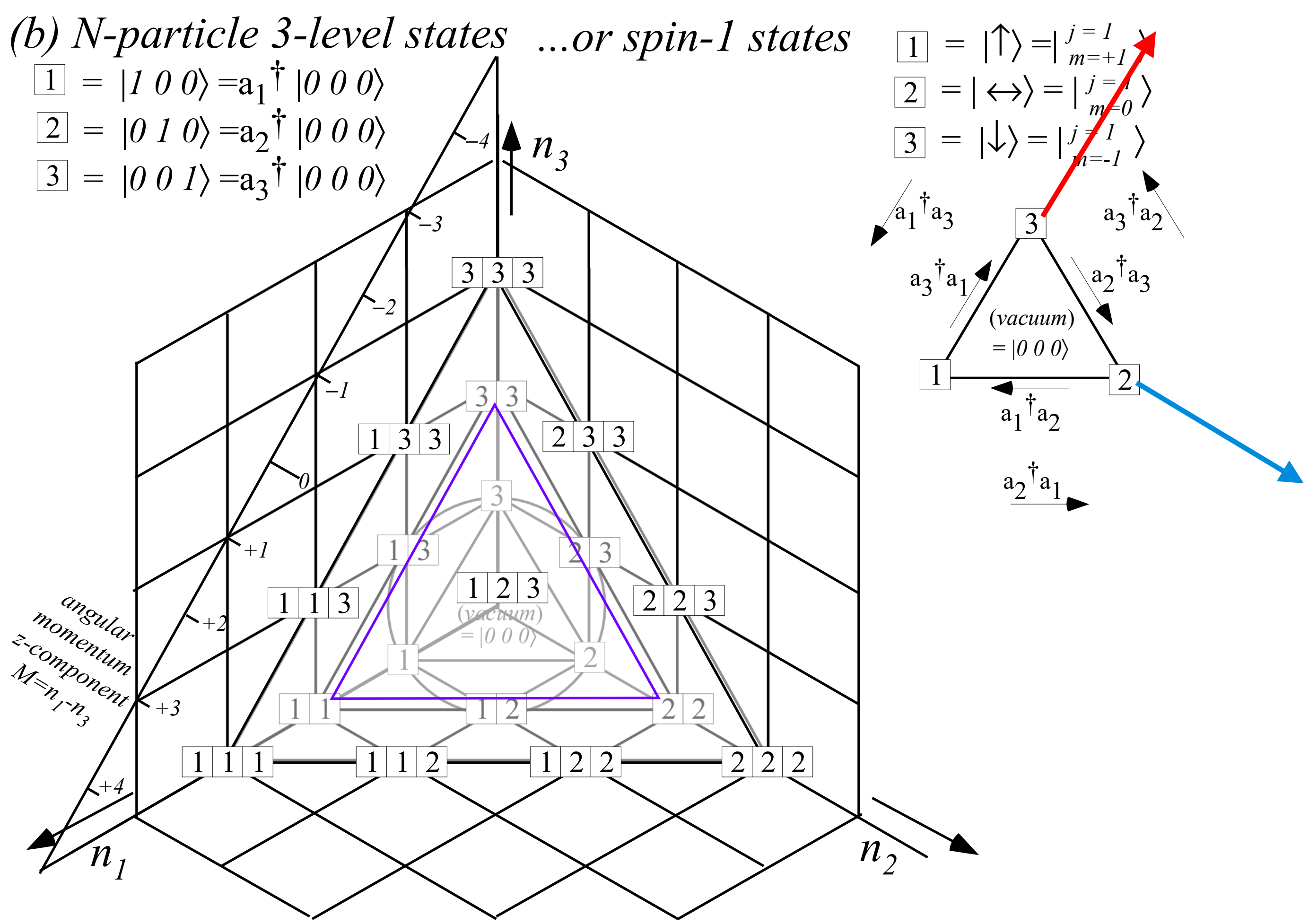

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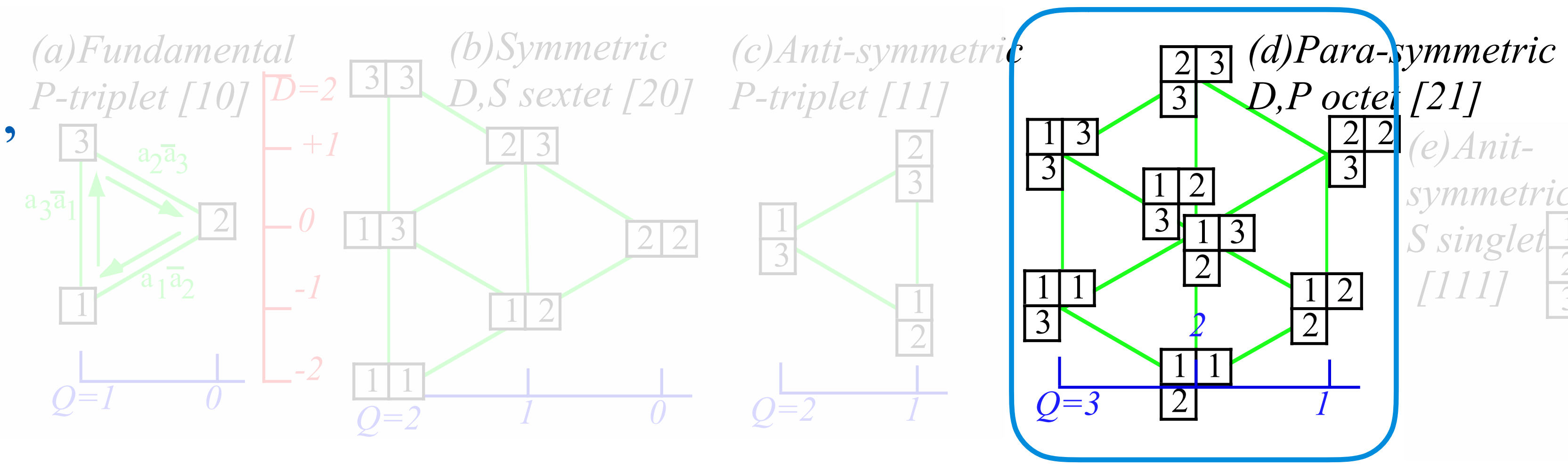
The octet $\ell^{\tiny\begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array}} = 8$ "eightfold way"



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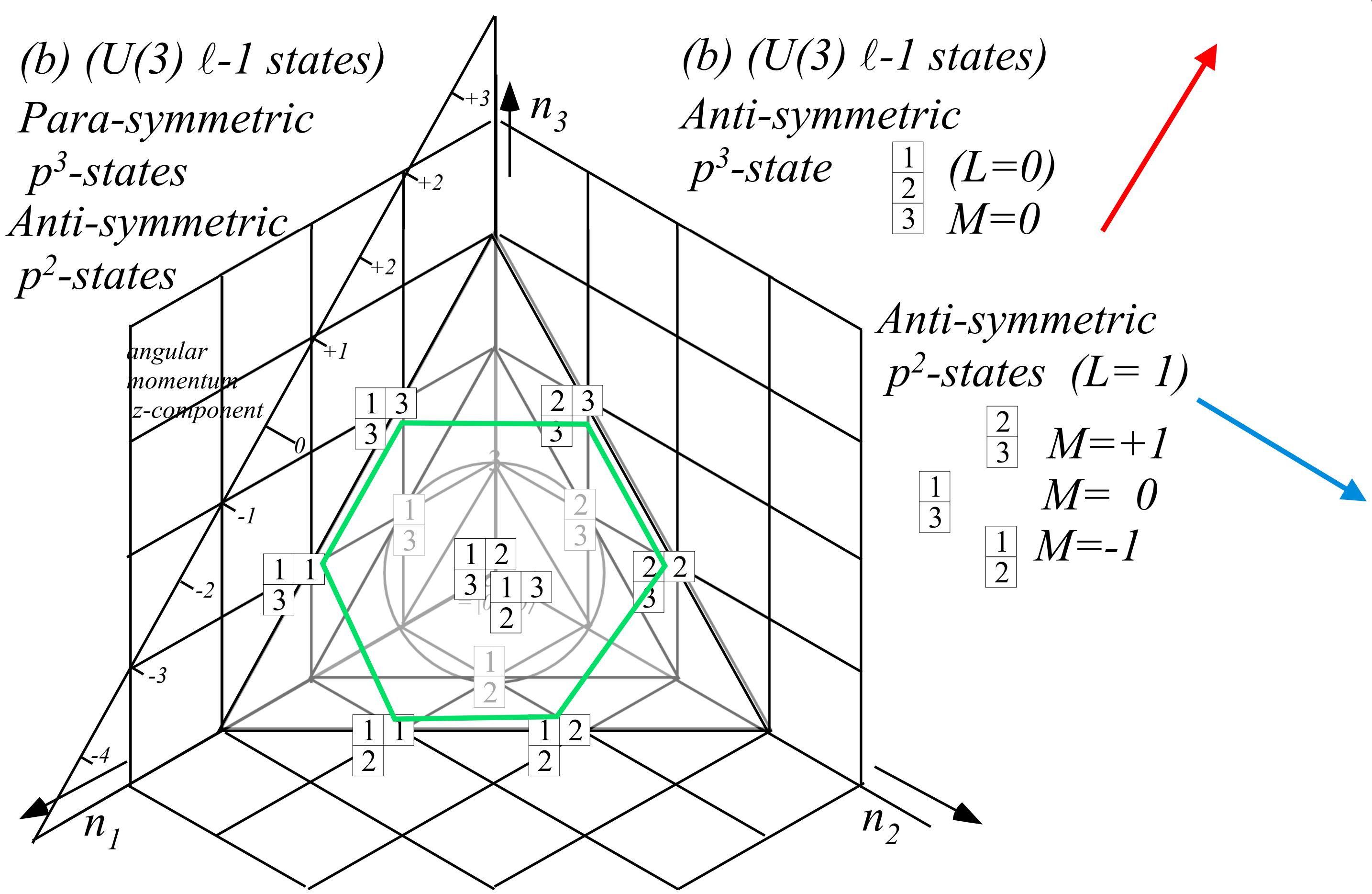
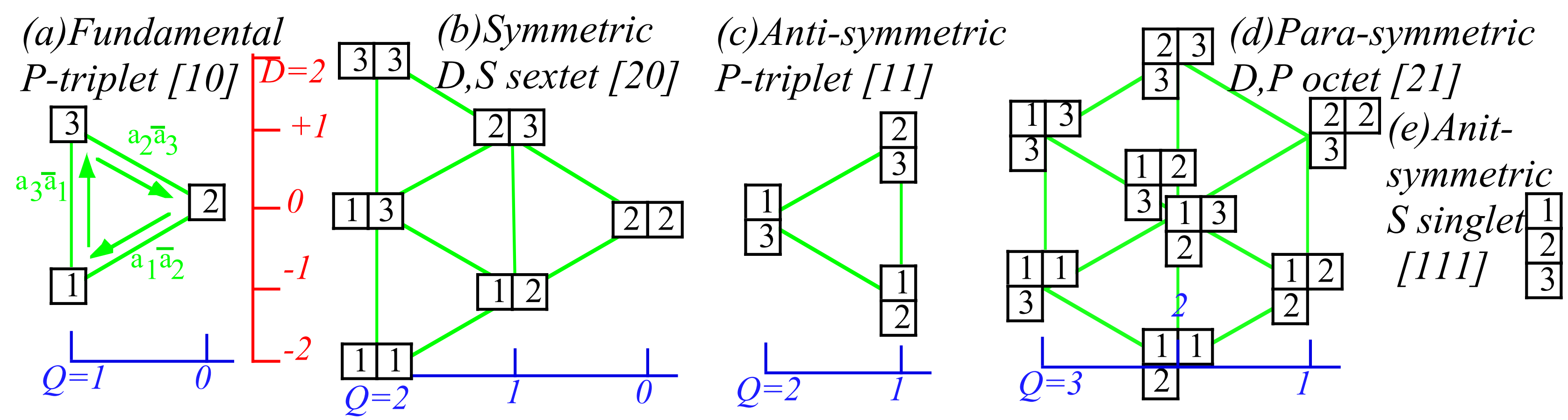


Structure of U(3) *irep bases*

The decapalet $\ell^{\square\square\square} = 10$ and Ω^-



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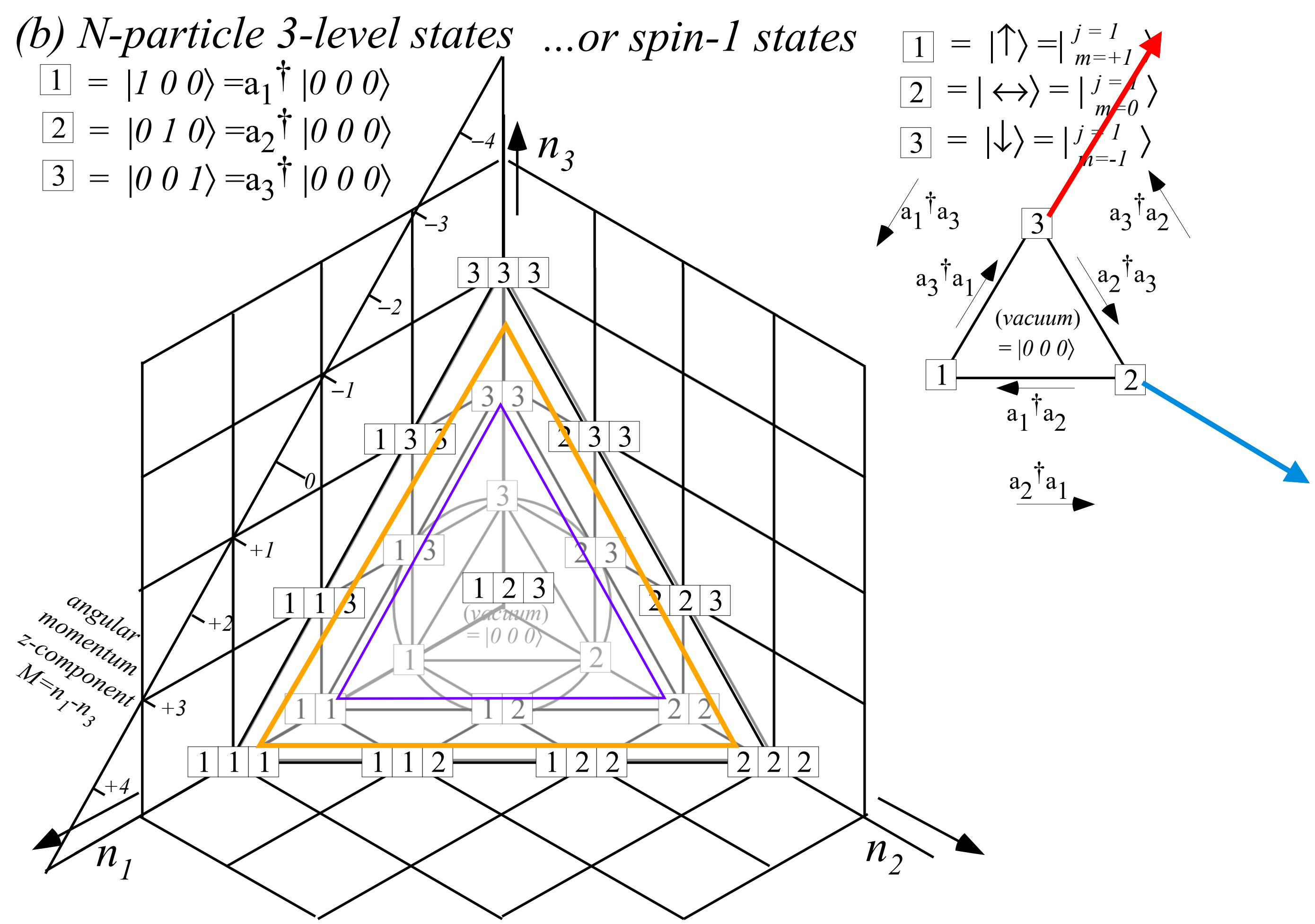

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➡ The p-shell in U(3) tableau plots

Hooklength formulas

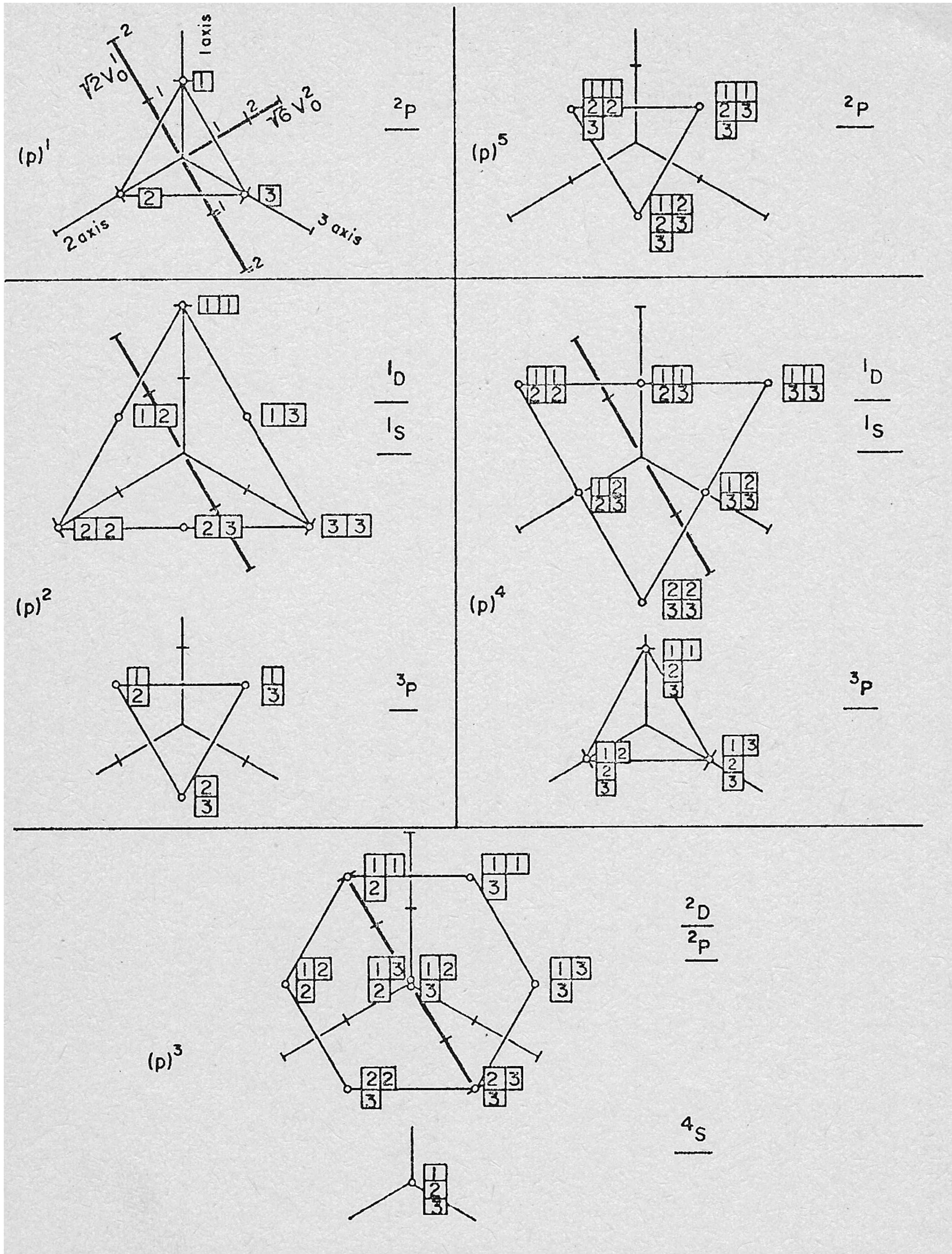


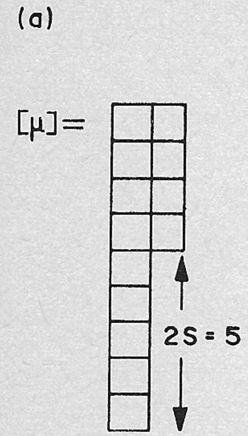
Fig.8 Weight or Moment Diagrams of Atomic $(p)^n$ States
Each tableau is located at point $(x_1\ x_2\ x_3)$ in a cartesian co-ordinate system for which x_n is the number of n's in the tableau. An alternative co-ordinate system is $(v_0^2,\ v_0^1,\ v_0^0)$ defined by Eq.16 which gives the zz-quadrupole moment, z-magnetic dipole moment, and number of particles, respectively. The last axis (v_0^0) would be pointing straight out of the figure, and each family of states lies in a plane perpendicular to it.

A Unitary Calculus for Electronic Orbitals
William G. Harter and Christopher W. Patterson
Springer-Verlag Lectures in Physics 49 1976

Alternative basis for the theory of complex spectra I
William G. Harter
Physical Review A 8 3 p2819 (1973)

Alternative basis for the theory of complex spectra II
William G. Harter and Christopher W. Patterson
Physical Review A 13 3 p1076-1082 (1976)

Alternative basis for the theory of complex spectra III
William G. Harter and Christopher W. Patterson
Physical Review A ??



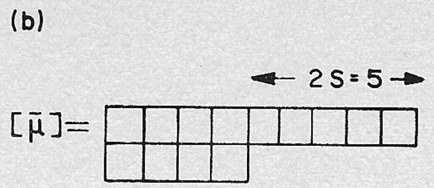


Fig.1 Young Frames

(a) A Young frame of 13 particles corresponding to all orbital states (^6L) of spin multiplicity $2S+1=6$

(b) A frame conjugate to (a) obtained by converting rows to columns, corresponds to spin states of total spin $S=5/2$, since only 5 of the 13 spins are unpaired.(These are represented by the single row of 5 boxes.)

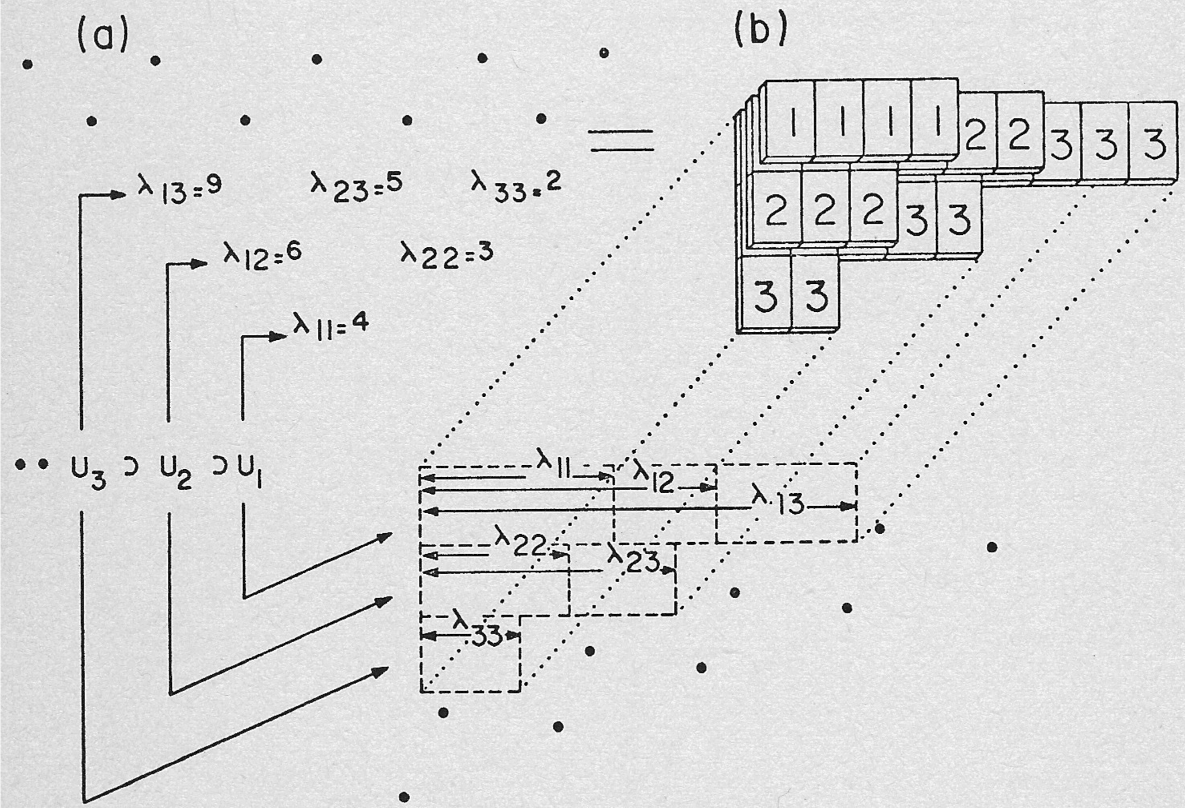


Fig.2 Unitary State Labeling

(a) Gelfand Pattern - The jth row of integers $(\lambda_{1,j}\lambda_{2,j}\cdots\lambda_{j,j})$ tells to which representation of U_j the state belongs, and similarily for the j-1th row $(\lambda_{1,j-1}\lambda_{2,j-i}\cdots\lambda_{j-1,j-1})$ which labels a unique representation of U_{j-1} contained in $(\lambda_{1,j}\lambda_{2,j}\cdots\lambda_{j,j})$. In this way each state has a unique genealogy chain and labeling.

(b) Young Tableau - Tableaus are a completely equivalent but non-algebraic "picture" of the Gelfand patterns.(When labeled algebraically, it is just an up-side-down Gelfand Pattern.)

AMOP reference links on page 2

4.04.18 class 21: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

Characters of intertwining $(S_n)*(U(m))$ algebras and quantum applications

Generic U(3)⊃R(3) transformations: *p*-triplet in U(3) shell model
Rank-1 vector in R(3) or "quark"-triplet in U(3)
Rank-2 tensor (2 particles each with U(3) state space)

U(3) tensor product states and S_n permutation symmetry
2-particle U(3) transform. 2-particle permutation operations
S_2 symmetry of U(3): Applying S_2 projection
Matrix representation of *Diagonalizing Transform* (DTran *T*)
Effect of S_2 DTran T on intertwining S_2 - U(3) irep matrices

S_3 symmetry of U(3): Applying S_3 projection
Applying S_3 character theory
Frequency formula for $D^{[\mu]}$ with tensor trace values
Effect of S_3 DTran T on intertwining S_3 - U(3) irep matrices

Structure of U(3) irep bases
Fundamental "quark" irep. "anti-quark". "di-quark".
The octet "eightfold way" The decapalet and Ω^-
➡ The p-shell in U(3) tableau plots
Hooklength formulas

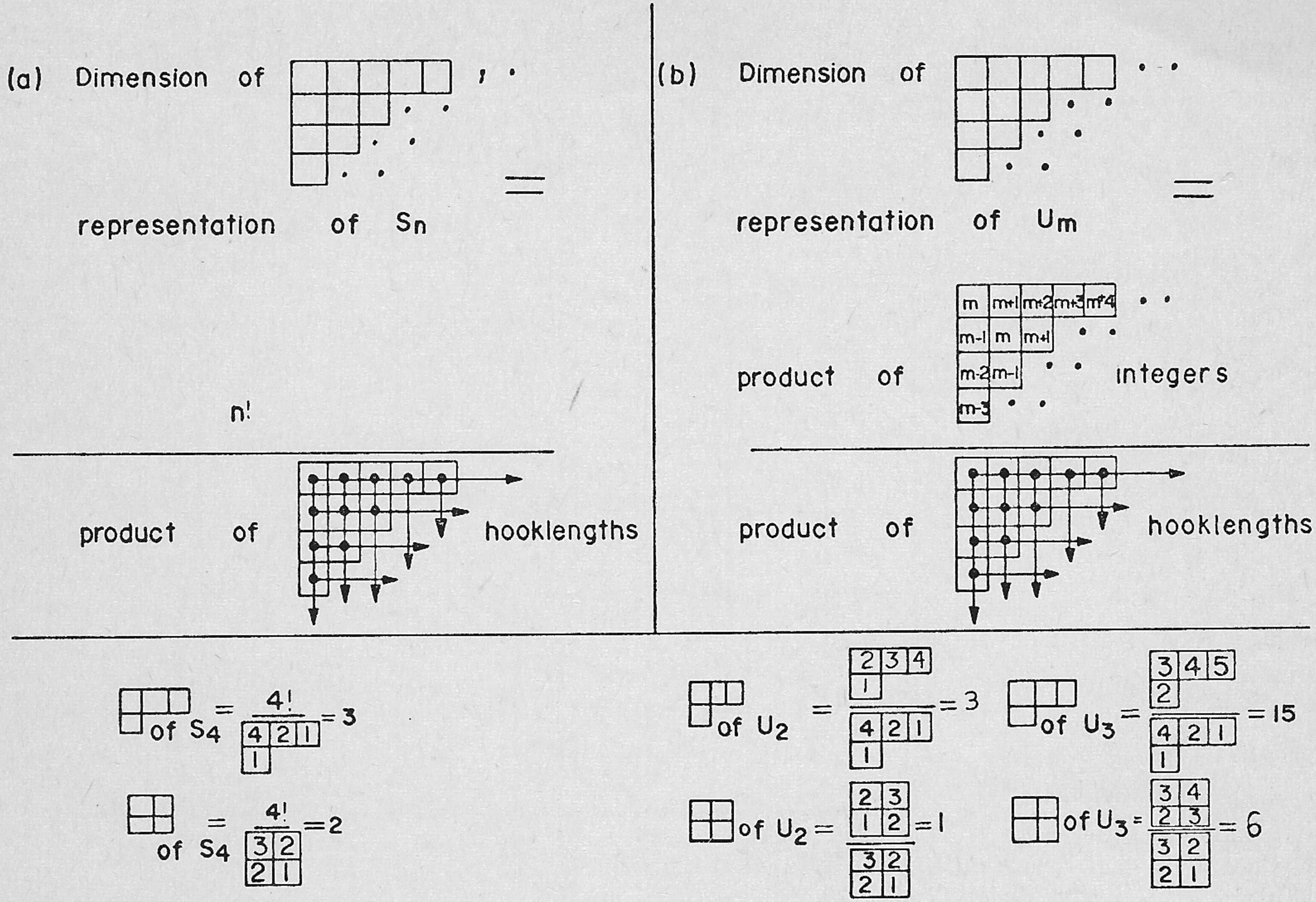


Fig.6 Hall - Robinson Hooklength Formulas

Dimension of representations of (a) S_n and (b) U_m labeled by a single tableau are given by the formulas. A hooklength of a tableau box is simply the number of boxes in a "hook" consisting of all the boxes below it, to the right of it, and itself.

Unitary raising and lowering operators E_{jk}

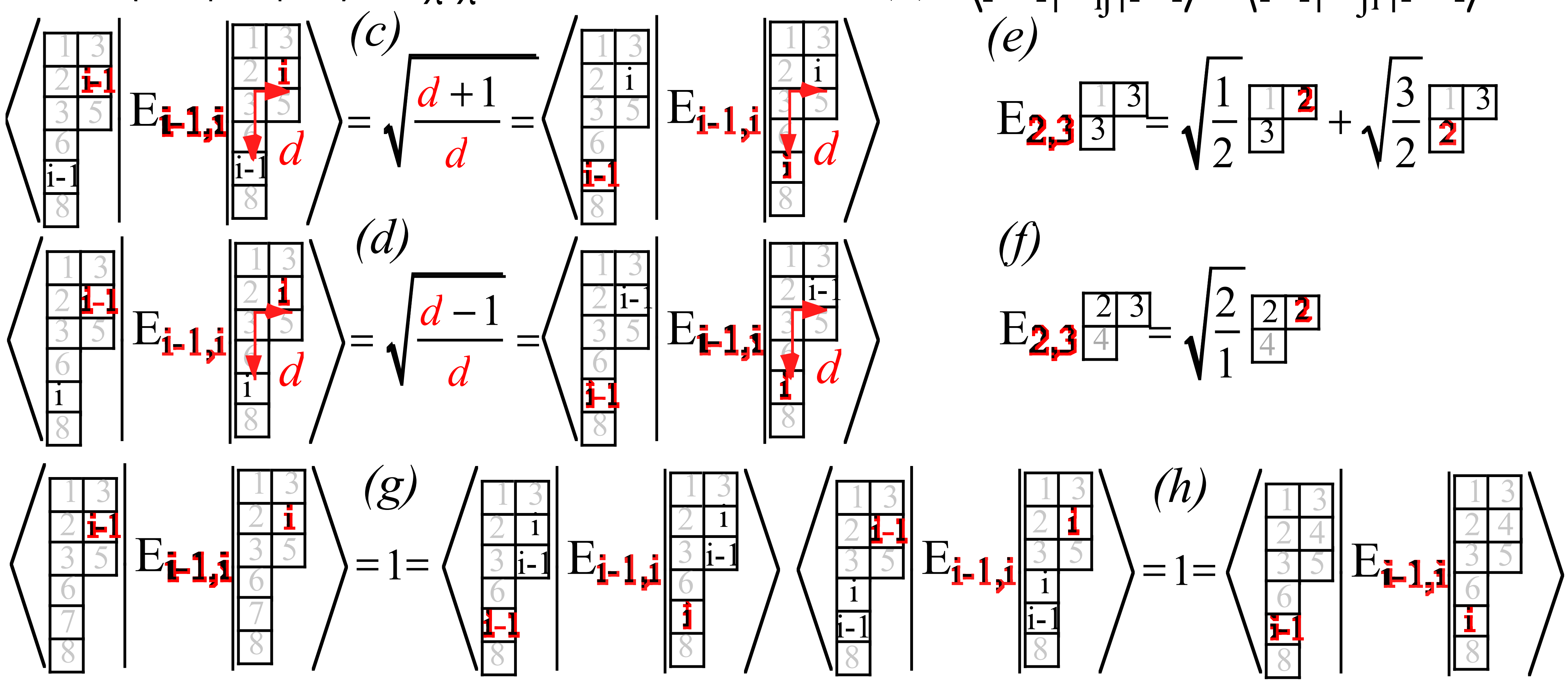
$$\left[E_{jk},E_{pq}\right]=\delta_{pk}E_{jq}-\delta_{qj}E_{pk}$$

$$\mathbf{a}_j^\dagger\mathbf{a}_k+\mathbf{b}_j^\dagger\mathbf{b}_k+\ldots=E_{jk}=a_j\bar{a}_k+b_j\bar{b}_k+\ldots$$

Hooklength formulas for E_{jk} on atomic tableau states

(a) $\langle[\lambda']|\mathrm{E}_{ii}|[\lambda]\rangle=\delta_{\lambda'\lambda}\, n_i$

(b) $\langle[\lambda']|\mathrm{E}_{ij}|[\lambda]\rangle=\langle[\lambda]|\mathrm{E}_{ji}|[\lambda']\rangle$



Multi-spin $(1/2)^N$ product states

$$2^N = \sum_{S}^{N/2} \ell^{[S]} \ell^{[\mu_1,\mu_2]}$$

$$= \sum_{S}^{N/2} (2S+1)\ell^{\left[\frac{N+2S}{2},\frac{N-2S}{2}\right]}$$

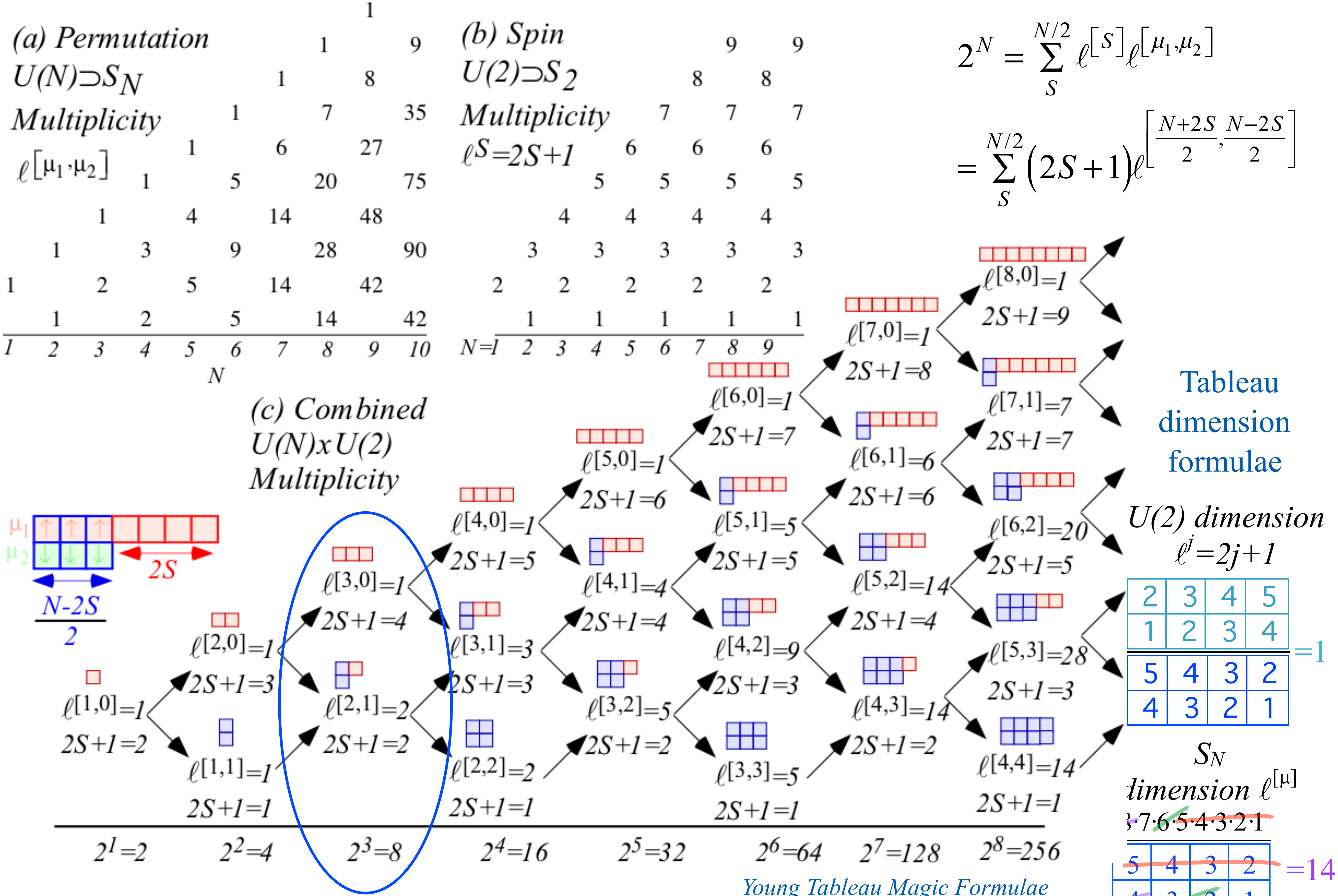


Fig. 23.3.2 *Spin-1/2 and U(2) Tableau branching diagrams*

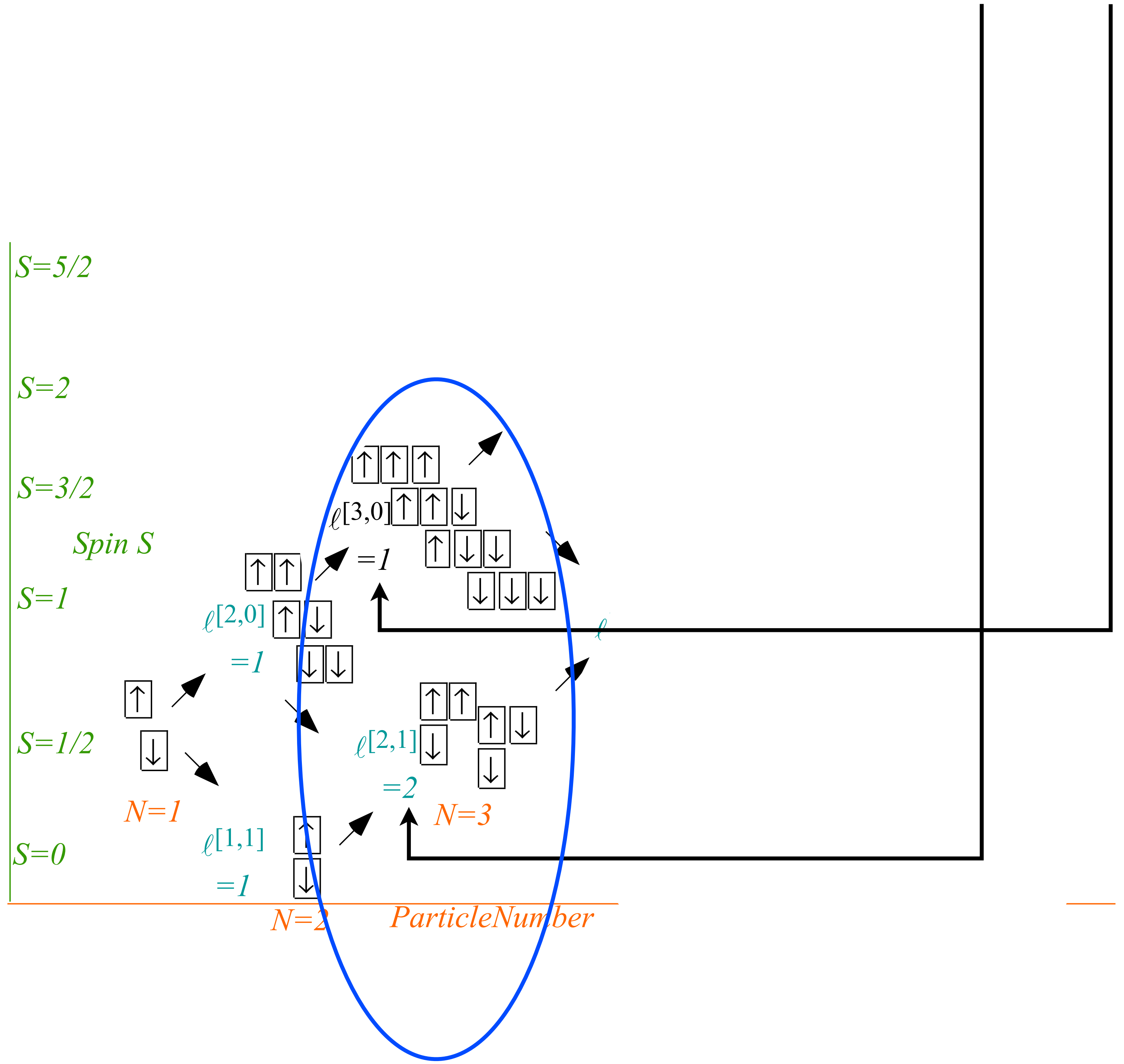
Young Tableau Magic Formulae
GrpThLect29 p46-48.

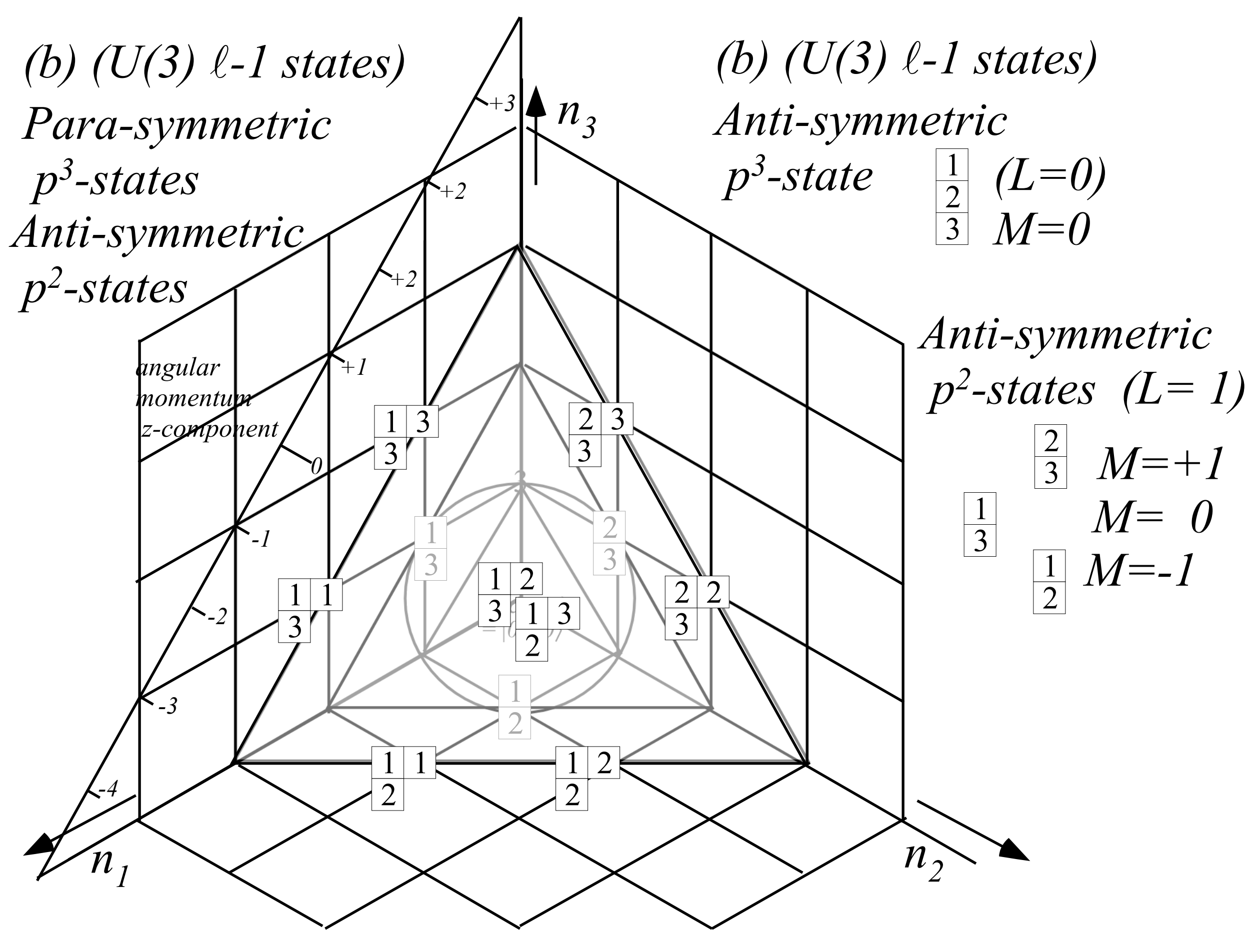
Multi-spin (1/2)^N product states

$$(d^{\frac{1}{2}} \otimes d^{\frac{1}{2}}) = d^0 + d^1$$

$$(d^{\frac{1}{2}} \otimes d^{\frac{1}{2}}) \otimes d^{\frac{1}{2}} = (d^0 + d^1) \otimes d^{\frac{1}{2}} = d^0 \otimes d^{\frac{1}{2}} + d^1 \otimes d^{\frac{1}{2}}$$

$$= \quad d^{\frac{1}{2}} + \qquad d^{\frac{1}{2}} + d^{\frac{3}{2}} = 2d^{\frac{1}{2}} + \ 1d^{\frac{3}{2}}$$




(b) (U(3) ℓ-1 states)
Para-symmetric
p3-states
Anti-symmetric
p2-states
(b) (U(3) ℓ-1 states)
Anti-symmetric
p3-state
(L=0)
M=0
Anti-symmetric
p2-states (L= 1)
M=+1
M= 0
M=-1
n3
n1
n2
angular
momentum
z-component
+3
+2
+2
+1
0
-1
-2
-3
-4

$$D^{\square\square\square}(\mathbf{p}) \qquad D^{\square\square\square}(\mathbf{u})$$

$$D^{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}(\mathbf{u}) \qquad D^{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}(\mathbf{p})$$

$$D^{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}(\mathbf{p}) \qquad D^{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}(\mathbf{u})$$

$$f^{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} = \frac{1}{3!}\left(\begin{pmatrix} \textit{order of} \\ \textit{class}(1) \end{pmatrix} \chi^{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}_{1^3} Tr\ \mathbf{(a)(b)(c)} + \begin{pmatrix} \textit{order of} \\ \textit{class}(3) \end{pmatrix} \chi^{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}_{(3)} Tr\ \mathbf{(abc)} + \begin{pmatrix} \textit{order of} \\ \textit{class}(1)(2) \end{pmatrix} \chi^{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}_{(1)(2)} Tr\ \mathbf{(ab)} \right)$$

$$= \frac{1}{6}\left((\ 1\ ) \cdot 2 \cdot 27 + (\ 2\ ) \cdot (\text{-}1) \cdot 3 + (\ 3\ ) \cdot (0) \cdot 9 \right) = 8$$

$$f^{\square\square\square} = \frac{1}{3!}\left(\begin{pmatrix} \textit{order of} \\ \textit{class}(1) \end{pmatrix} \chi^{\square\square\square}_{1^3} Tr\ \mathbf{(a)(b)(c)} + \begin{pmatrix} \textit{order of} \\ \textit{class}(3) \end{pmatrix} \chi^{\square\square\square}_{(3)} Tr\ \mathbf{(abc)} + \begin{pmatrix} \textit{order of} \\ \textit{class}(1)(2) \end{pmatrix} \chi^{\square\square\square}_{(1)(2)} Tr\ \mathbf{(ab)} \right)$$

$$= \frac{1}{6}\left((\ 1\ ) \cdot 1 \cdot 27 + (\ 2\ ) \cdot 1 \cdot 3 + (\ 3\ ) \cdot 1 \cdot 9 \right) = 10$$

$$f^{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}} = \frac{1}{3!}\left(\begin{pmatrix} \textit{order of} \\ \textit{class}(1) \end{pmatrix} \chi^{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}_{1^3} Tr\ \mathbf{(a)(b)(c)} + \begin{pmatrix} \textit{order of} \\ \textit{class}(3) \end{pmatrix} \chi^{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}_{(3)} Tr\ \mathbf{(abc)} + \begin{pmatrix} \textit{order of} \\ \textit{class}(1)(2) \end{pmatrix} \chi^{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}_{(1)(2)} Tr\ \mathbf{(ab)} \right)$$

$$= \frac{1}{6}\left((\ 1\ ) \cdot 1 \cdot 27 + (\ 2\ ) \cdot 1 \cdot 3 + (\ 3\ ) \cdot (\text{-}1) \cdot 9 \right) = 1$$

S_3 *symmetry* of U(3): *Applying S_3 projection*

Rank-3 tensor basis |ijk⟩ (3 particles each with U(3) state space) has dimension 3^3=27

[12]

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$D^{\square\square\square}(\mathbf{g}) =$	1	1	1	1	1	1
$D^{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}(\mathbf{g}) =$	1	1	1	−1	−1	−1
$D^{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}_{x_2 y_2}(\mathbf{g}) =$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$	$\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$	$\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$	$\begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

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