AMOP reference links on page 2

4.18.18 class 24: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

$(S_n)^*(U(m))$ shell model of electronic spin-orbit states and interactions

Marrying spin $s=½$ and orbital $\ell=1$ together: U(3)×U(2)

  The $\ell=1$ p=shell in a nutshell

U(6)⊃U(3)×U(2) approach: Coupling spin-orbit ($s=½$, $\ell=1$) tableaus

Introducing atomic spin-orbit state assembly formula

  Slater determinants

  p-shell Spin-orbit calculations (not finished)

Clebsch Gordan coefficients. (Rev. Mod. Phys. annual gift)

S_n projection for atomic spin and orbit states

  Review of Mach-Mock (particle-state) principle

  Tableau P-operators on orbits

  Tableau P-operators on spin

Fermi-Dirac-Pauli anti-symmetric p^3-states

Boson operators and symmetric p^2-states

  Connecting to angular momentum

  Projecting to angular momentum

AMOP reference links *(Updated list given on **2nd** and 3rd pages of each class presentation)*

Web Resources - front page
UAF Physics UTube channel

Quantum Theory for the Computer Age
Principles of Symmetry, Dynamics, and Spectroscopy
Classical Mechanics with a Bang!
Modern Physics and its Classical Foundations

2014 AMOP
2017 Group Theory for QM
2018 AMOP

Representaions Of Multidimensional Symmetries In Networks - harter-jmp-1973

Alternative Basis for the Theory of Complex Spectra

- Alternative_Basis_for_the_Theory_of_Complex_Spectra_I_-_harter-pra-1973
- Alternative_Basis_for_the_Theory_of_Complex_Spectra_II_-_harter-patterson-pra-1976
- Alternative_Basis_for_the_Theory_of_Complex_Spectra_III_-_patterson-harter-pra-1977

Frame Transformation Relations And Multipole Transitions In Symmetric Polyatomic Molecules - RMP-1978
Asymptotic eigensolutions of fourth and sixth rank octahedral tensor operators - Harter-Patterson-JMP-1979
Rotational energy surfaces and high- J eigenvalue structure of polyatomic molecules - Harter - Patterson - 1984
Galloping waves and their relativistic properties - ajp-1985-Harter
Rovibrational Spectral Fine Structure Of Icosahedral Molecules - Cpl 1986 (Alt Scan)

Theory of hyperfine and superfine levels in symmetric polyatomic molecules.

I) Trigonal and tetrahedral molecules: Elementary spin-1/2 cases in vibronic ground states - PRA-1979-Harter-Patterson (Alt scan)
II) Elementary cases in octahedral hexafluoride molecules - Harter-PRA-1981 (Alt scan)

Rotation–vibration spectra of icosahedral molecules.

I) Icosahedral symmetry analysis and fine structure - harter-weeks-jcp-1989 (Alt scan)
II) Icosahedral symmetry, vibrational eigenfrequencies, and normal modes of buckminsterfullerene - weeks-harter-jcp-1989 (Alt scan)
III) Half-integral angular momentum - harter-reimer-jcp-1991

Rotation-vibration scalar coupling zeta coefficients and spectroscopic band shapes of buckminsterfullerene - Weeks-Harter-CPL-1991 (Alt scan)
Nuclear spin weights and gas phase spectral structure of 12C60 and 13C60 buckminsterfullerene -Harter-Reimer-Cpl-1992 - (Alt1, Alt2 Erratum)
Gas Phase Level Structure of C60 Buckyball and Derivatives Exhibiting Broken Icosahedral Symmetry - reimer-diss-1996
Fullerene symmetry reduction and rotational level fine structure/ the Buckyball isotopomer 12C 13C59 - jcp-Reimer-Harter-1997 (HiRez)
Wave Node Dynamics and Revival Symmetry in Quantum Rotors - harter - jms - 2001
Molecular Symmetry and Dynamics - Ch32-Springer Handbooks of Atomic, Molecular, and Optical Physics - Harter-2006

Resonance and Revivals

I) QUANTUM ROTOR AND INFINITE-WELL DYNAMICS - ISMSLi2012 (Talk) OSU knowledge Bank
II) Comparing Half-integer Spin and Integer Spin - Alva-ISMS-Ohio2013-R777 (Talks)
III) Quantum Resonant Beats and Revivals in the Morse Oscillators and Rotors - (2013-Li-Diss)

Resonance and Revivals in Quantum Rotors - Comparing Half-integer Spin and Integer Spin - Alva-ISMS-Ohio2013-R777 (Talk)
Molecular Eigensolution Symmetry Analysis and Fine Structure - IJMS-harter-mitchell-2013
Quantum Revivals of Morse Oscillators and Farey-Ford Geometry - Li-Harter-cpl-2013
QTCA Unit 10 Ch 30 - 2013
AMOP Ch 0 Space-Time Symmetry - 2019

**In development - a web based A.M.O.P. oriented reference page, with thumbnail/previews, greater control over the information display, and eventually full on* Apache-SOLR *Index and search for* nuanced, whole-site content/metadata level *searching.*

AMOP reference links *(Updated list given on 2nd and 3rd pages of each class presentation)*

(Int.J.Mol.Sci, 14, 714(2013) p.755-774 , QTCA Unit 7 Ch. 23-26), (PSDS - Ch. 5, 7)

Int.J.Mol.Sci, 14, 714(2013), QTCA Unit 8 Ch. 23-25, QTCA Unit 9 Ch. 26, PSDS Ch. 5, PSDS Ch. 7

Intro spin ½ coupling
Unit 8 Ch. 24 p3

H atom hyperfine-B-level crossing
Unit 8 Ch. 24 p15

Hyperf. theory Ch. 24 p48.

Hyperf. theory Ch. 24 p48.
Deeper theory ends p53

Intro 2p3p coupling
Unit 8 Ch. 24 p17.

Intro LS-jj coupling
Unit 8 Ch. 24 p22.

CG coupling derived (start)
Unit 8 Ch. 24 p39.

CG coupling derived (formula)
Unit 8 Ch. 24 p44.

Lande' g-factor
Unit 8 Ch. 24 p26.

Irrep Tensor building
Unit 8 Ch. 25 p5.

Irrep Tensor Tables
Unit 8 Ch. 25 p12.

Wigner-Eckart tensor Theorem.
Unit 8 Ch. 25 p17.

Tensors Applied to d,f-levels.
Unit 8 Ch. 25 p21.

Tensors Applied to high J levels.
Unit 8 Ch. 25 p63.

Intro 3-particle coupling.
Unit 8 Ch. 25 p28.

Intro 3,4-particle Young Tableaus
GrpThLect29 p42.

Young Tableau Magic Formulae
GrpThLect29 p46-48.

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AMOP reference links *(Updated list given on 2nd and 3rd and 4th pages of each class presentation)*

Predrag Cvitanovic's: Birdtrack Notation, Calculations, and Simplification

Chaos_Classical_and_Quantum_-_2018-Cvitanovic-ChaosBook

Group Theory - PUP_Lucy_Day_-_Diagrammatic_notation_-_Ch4

Simplification_Rules_for_Birdtrack_Operators_-_Alcock-Zeilinger-Weigert-zeilinger-jmp-2017

Group Theory - Birdtracks_Lies_and_Exceptional_Groups_-_Cvitanovic-2011

Simplification_rules_for_birdtrack_operators-_jmp-alcock-zeilinger-2017

Birdtracks for SU(N) - 2017-Keppeler

Frank Rioux's: UMA method of vibrational induction

Quantum_Mechanics_Group_Theory_and_C60_-_Frank_Rioux_-_Department_of_Chemistry_Saint_Johns_U

Symmetry_Analysis_for_H20-_H20GrpTheory-_Rioux

Quantum_Mechanics-Group_Theory_and_C60_-_JChemEd-Rioux-1994

Group_Theory_Problems-_Rioux-_SymmetryProblemsX

Comment_on_the_Vibrational_Analysis_for_C60_and_Other_Fullerenes_Rioux-RSP

Supplemental AMOP Techniques & Experiment

Many Correlation Tables are Molien Sequences - Klee (Draft 2016)

High-resolution_spectroscopy_and_global_analysis_of_CF4_rovibrational_bands_to_model_its_atmospheric_absorption-_carlos-Boudon-jqsrt-2017

Symmetry and Chirality - Continuous_Measures_-_Avnir

*

Special Topics & Colloquial References

r-process_nucleosynthesis_from_matter_ejected_in_binary_neutron_star_mergers-PhysRevD-Bovard-2017

**In development - a web based A.M.O.P. oriented reference page, with thumbnail/previews, greater control over the information display, and eventually full on* Apache-SOLR *Index and search for* nuanced, whole-site content/metadata level *searching.*

AMOP reference links on pages 2-4

4.16.18 class 23: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

$(S_n)^*(U(m))$ shell model of electrostatic quadrupole-quadrupole-e interactions

Marrying spin $s=½$ and orbital $\ell=1$ together: U(3)×U(2)
- The $\ell=1$ p=shell in a nutshell

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Introducing atomic spin-orbit state assembly formula
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- Tableau P-operators on spin

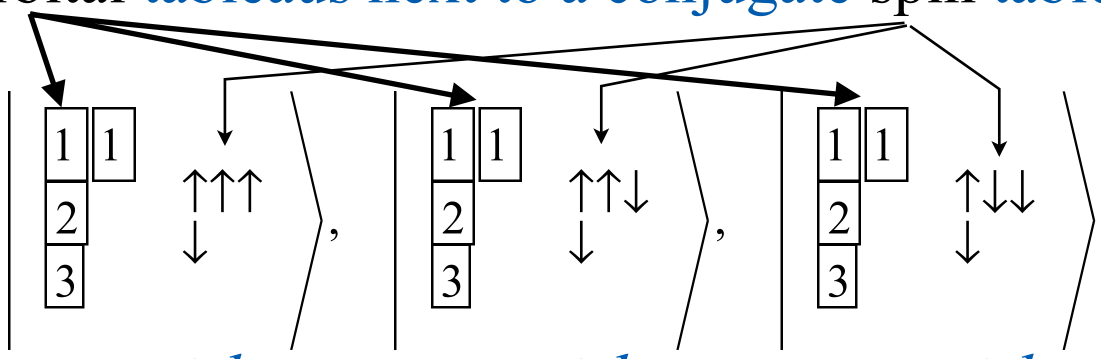
Fermi-Dirac-Pauli anti-symmetric p^3-states

Boson operators and symmetric p^2-states
- Connecting to angular momentum
- Projecting to angular momentum

Marrying spin s=½ and orbital ℓ=1 together: U(3)×U(2)

A state satisfying Pauli-antisymmetry (Exclusion principle) can be simply represented by putting an orbital tableaus next to a conjugate spin tableaus. (Rows flipped with columns)

^{3}P *spin-triplet*



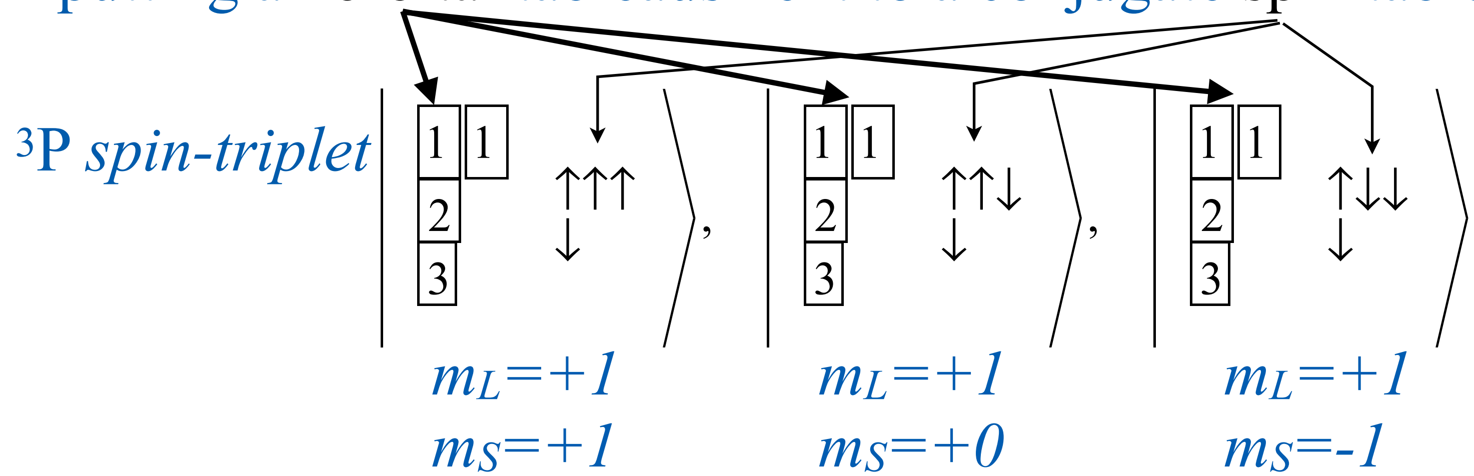
m_L=+1, m_S=+1 | m_L=+1, m_S=+0 | m_L=+1, m_S=-1

U(3): m_ℓ=+1: | 1 ⟩, m_ℓ=0: | 2 ⟩, m_ℓ=-1: | 3 ⟩

U(2): m_s=+½: | ↑ ⟩, m_s=-½: | ↓ ⟩

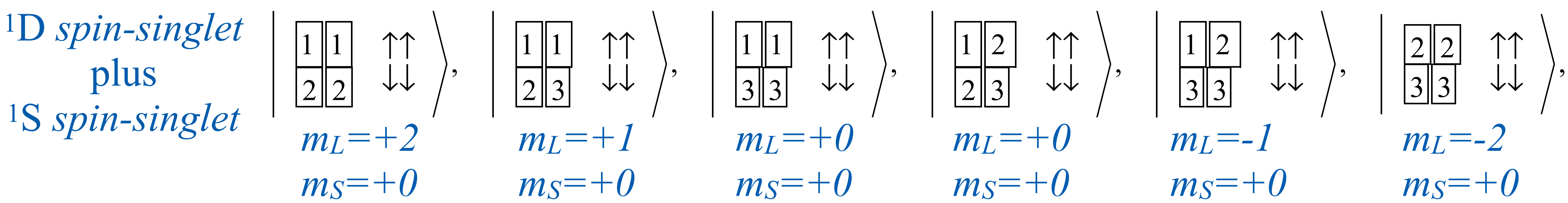
Marrying spin $s=\frac{1}{2}$ and orbital $\ell=1$ together: U(3)×U(2)

A state satisfying Pauli-antisymmetry (Exclusion principle) can be simply represented by putting an orbital tableaus next to a conjugate spin tableaus. (Rows flipped with columns)



U(3): $m_\ell=+1$: $|\,1\,\rangle$, $m_\ell=0$: $|\,2\,\rangle$, $m_\ell=-1$: $|\,3\,\rangle$

U(2): $m_s=+\frac{1}{2}$: $|\uparrow\rangle$, $m_s=-\frac{1}{2}$: $|\downarrow\rangle$



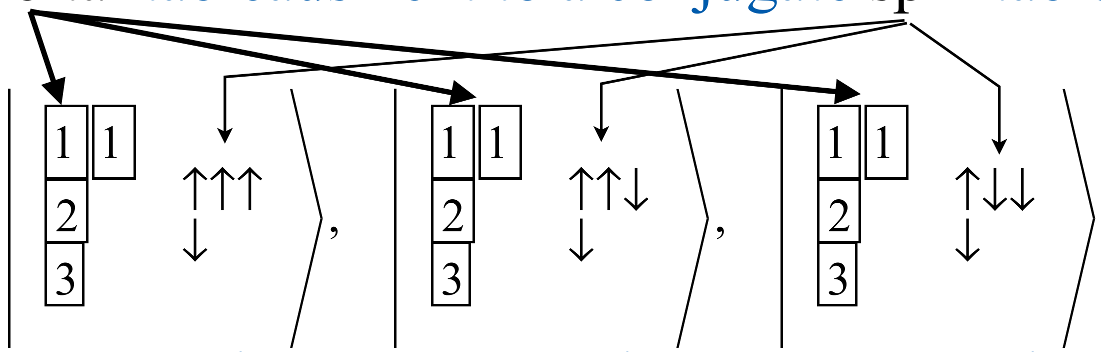
Marrying spin $s=½$ and orbital $\ell=1$ together: U(3)×U(2)

A state satisfying Pauli-antisymmetry (Exclusion principle) can be simply represented by putting an orbital tableaus next to a conjugate spin tableaus. (Rows flipped with columns)

U(3): $m_\ell=+1$: $|\,1\,\rangle$, $m_\ell=0$: $|\,2\,\rangle$, $m_\ell=-1$: $|\,3\,\rangle$

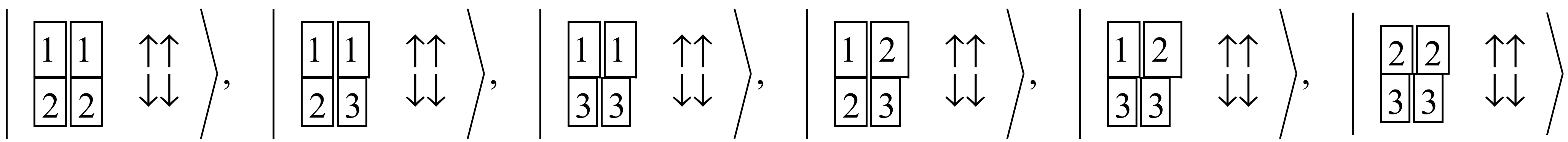
U(2): $m_s=+½$: $|\uparrow\rangle$, $m_s=-½$: $|\downarrow\rangle$

^{3}P *spin-triplet*



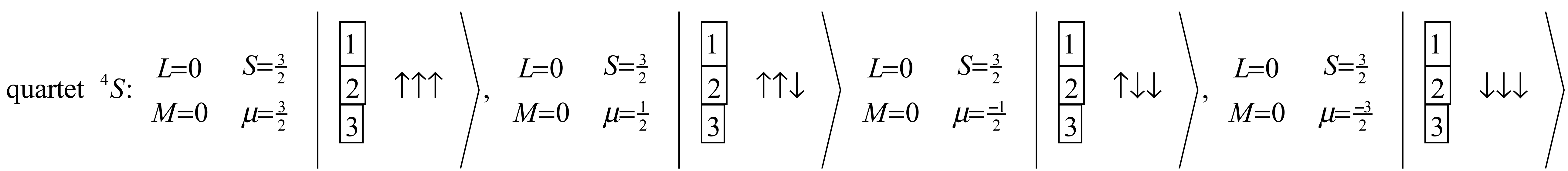
$m_L=+1$	$m_L=+1$	$m_L=+1$
$m_S=+1$	$m_S=+0$	$m_S=-1$

^{1}D *spin-singlet*
plus
^{1}S *spin-singlet*



$m_L=+2$	$m_L=+1$	$m_L=+0$	$m_L=+0$	$m_L=-1$	$m_L=-2$
$m_S=+0$	$m_S=+0$	$m_S=+0$	$m_S=+0$	$m_S=+0$	$m_S=+0$

These involve fairly complicated S_n-coupled U(3)×U(2) combinations that will be developed later.



AMOP reference links on pages 2-4

4.16.18 class 23: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

$(S_n)^*(U(m))$ shell model of electrostatic quadrupole-quadrupole-e interactions

Marrying spin *s*=½ and orbital ℓ=1 together: U(3)×U(2)

➡ The ℓ=1 *p*=shell in a nutshell

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Introducing atomic spin-orbit state assembly formula

- Slater determinants
- p-shell Spin-orbit calculations (not finished)

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- Tableau P-operators on spin

Fermi-Dirac-Pauli anti-symmetric p^3-states

Boson operators and symmetric p^2-states

- Connecting to angular momentum
- Projecting to angular momentum

The ℓ=1 p=shell in a nutshell

quartet ^4S:

$$\left|\begin{matrix}L=0 & S=\frac{3}{2}\\ M=0 & \mu=\frac{3}{2}\end{matrix}\right. \left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\ \uparrow\uparrow\uparrow\right\rangle,\quad \left|\begin{matrix}L=0 & S=\frac{3}{2}\\ M=0 & \mu=\frac{1}{2}\end{matrix}\right. \left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\ \uparrow\uparrow\downarrow\right\rangle\ \left|\begin{matrix}L=0 & S=\frac{3}{2}\\ M=0 & \mu=\frac{-1}{2}\end{matrix}\right. \left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\ \uparrow\downarrow\downarrow\right\rangle,\quad \left|\begin{matrix}L=0 & S=\frac{3}{2}\\ M=0 & \mu=\frac{-3}{2}\end{matrix}\right. \left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\ \downarrow\downarrow\downarrow\right\rangle.$$

Doublet ^2D, M=2:

$$\left|\begin{matrix}L=2, & S=\frac{1}{2}\\ M=2, & \mu=\frac{1}{2}\end{matrix}\right. \left|\begin{array}{|c|c|}\hline 1 & 1\\\hline 2\\\cline{1-1}\end{array}\ \begin{matrix}\uparrow\uparrow\\ \downarrow\end{matrix}\right\rangle,\quad \left|\begin{matrix}L=2, & S=-\frac{1}{2}\\ M=2, & \mu=\frac{1}{2}\end{matrix}\right. \left|\begin{array}{|c|c|}\hline 1 & 1\\\hline 2\\\cline{1-1}\end{array}\ \begin{matrix}\uparrow\downarrow\\ \downarrow\end{matrix}\right\rangle.$$

Doublet ^2D, M=1:

$$\left|\begin{matrix}L=2, & S=\frac{1}{2}\\ M=1, & \mu=\frac{1}{2}\end{matrix}\right\rangle = \frac{1}{\sqrt{2}}\left|\begin{array}{|c|c|}\hline 1 & 2\\\hline 2\\\cline{1-1}\end{array}\ \begin{matrix}\uparrow\uparrow\\ \downarrow\end{matrix}\right\rangle + \frac{1}{\sqrt{2}}\left|\begin{array}{|c|c|}\hline 1 & 1\\\hline 3\\\cline{1-1}\end{array}\ \begin{matrix}\uparrow\uparrow\\ \downarrow\end{matrix}\right\rangle,$$

$$\left|\begin{matrix}L=2, & S=\frac{1}{2}\\ M=1, & \mu=\frac{-1}{2}\end{matrix}\right\rangle = +\frac{1}{\sqrt{2}}\left|\begin{array}{|c|c|}\hline 1 & 2\\\hline 2\\\cline{1-1}\end{array}\ \begin{matrix}\uparrow\downarrow\\ \downarrow\end{matrix}\right\rangle + \frac{1}{\sqrt{2}}\left|\begin{array}{|c|c|}\hline 1 & 1\\\hline 3\\\cline{1-1}\end{array}\ \begin{matrix}\uparrow\downarrow\\ \downarrow\end{matrix}\right\rangle,$$

Doublet ^2P, M=1:

$$\left|\begin{matrix}L=1, & S=\frac{1}{2}\\ M=1, & \mu=\frac{1}{2}\end{matrix}\right\rangle = \frac{1}{\sqrt{2}}\left|\begin{array}{|c|c|}\hline 1 & 2\\\hline 2\\\cline{1-1}\end{array}\ \begin{matrix}\uparrow\uparrow\\ \downarrow\end{matrix}\right\rangle - \frac{1}{\sqrt{2}}\left|\begin{array}{|c|c|}\hline 1 & 1\\\hline 3\\\cline{1-1}\end{array}\ \begin{matrix}\uparrow\uparrow\\ \downarrow\end{matrix}\right\rangle,$$

$$\left|\begin{matrix}L=1, & S=\frac{1}{2}\\ M=1, & \mu=\frac{-1}{2}\end{matrix}\right\rangle = +\frac{1}{\sqrt{2}}\left|\begin{array}{|c|c|}\hline 1 & 2\\\hline 2\\\cline{1-1}\end{array}\ \begin{matrix}\uparrow\downarrow\\ \downarrow\end{matrix}\right\rangle - \frac{1}{\sqrt{2}}\left|\begin{array}{|c|c|}\hline 1 & 1\\\hline 3\\\cline{1-1}\end{array}\ \begin{matrix}\uparrow\downarrow\\ \downarrow\end{matrix}\right\rangle,$$

Doublet ^2D, M=0:

$$\left|\begin{matrix}L=2, & S=\frac{1}{2}\\ M=0, & \mu=\frac{1}{2}\end{matrix}\right\rangle = \frac{\sqrt{3}}{2}\left|\begin{array}{|c|c|}\hline 1 & 2\\\hline 3\\\cline{1-1}\end{array}\ \begin{matrix}\uparrow\uparrow\\ \downarrow\end{matrix}\right\rangle + \frac{1}{2}\left|\begin{array}{|c|c|}\hline 1 & 3\\\hline 2\\\cline{1-1}\end{array}\ \begin{matrix}\uparrow\uparrow\\ \downarrow\end{matrix}\right\rangle,$$

$$\left|\begin{matrix}L=2, & S=\frac{1}{2}\\ M=0, & \mu=\frac{-1}{2}\end{matrix}\right\rangle = +\frac{\sqrt{3}}{2}\left|\begin{array}{|c|c|}\hline 1 & 2\\\hline 3\\\cline{1-1}\end{array}\ \begin{matrix}\uparrow\downarrow\\ \downarrow\end{matrix}\right\rangle + \frac{1}{2}\left|\begin{array}{|c|c|}\hline 1 & 3\\\hline 2\\\cline{1-1}\end{array}\ \begin{matrix}\uparrow\downarrow\\ \downarrow\end{matrix}\right\rangle,$$

Doublet ^2P, M=0:

$$\left|\begin{matrix}L=1, & S=\frac{1}{2}\\ M=0, & \mu=\frac{1}{2}\end{matrix}\right\rangle = \frac{-1}{2}\left|\begin{array}{|c|c|}\hline 1 & 2\\\hline 3\\\cline{1-1}\end{array}\ \begin{matrix}\uparrow\uparrow\\ \downarrow\end{matrix}\right\rangle + \frac{\sqrt{3}}{2}\left|\begin{array}{|c|c|}\hline 1 & 1\\\hline 3\\\cline{1-1}\end{array}\ \begin{matrix}\uparrow\uparrow\\ \downarrow\end{matrix}\right\rangle,$$

$$\left|\begin{matrix}L=1, & S=\frac{1}{2}\\ M=0, & \mu=\frac{-1}{2}\end{matrix}\right\rangle = -\frac{1}{2}\left|\begin{array}{|c|c|}\hline 1 & 2\\\hline 3\\\cline{1-1}\end{array}\ \begin{matrix}\uparrow\downarrow\\ \downarrow\end{matrix}\right\rangle + \frac{\sqrt{3}}{2}\left|\begin{array}{|c|c|}\hline 1 & 3\\\hline 2\\\cline{1-1}\end{array}\ \begin{matrix}\uparrow\downarrow\\ \downarrow\end{matrix}\right\rangle,$$

⋮ (*M*=-*1* row) ⋮

Doublet ^2D, $M=-2$:

$$\left|\begin{matrix}L=2, & S=\frac{1}{2}\\ M=-2, & \mu=\frac{1}{2}\end{matrix}\right. \left|\begin{array}{|c|c|}\hline 2 & 3\\\hline 3\\\cline{1-1}\end{array}\ \begin{matrix}\uparrow\uparrow\\ \downarrow\end{matrix}\right\rangle,\quad \left|\begin{matrix}L=2, & S=\frac{1}{2}\\ M=-2, & \mu=\frac{-1}{2}\end{matrix}\right. \left|\begin{array}{|c|c|}\hline 2 & 3\\\hline 3\\\cline{1-1}\end{array}\ \begin{matrix}\uparrow\downarrow\\ \downarrow\end{matrix}\right\rangle.$$

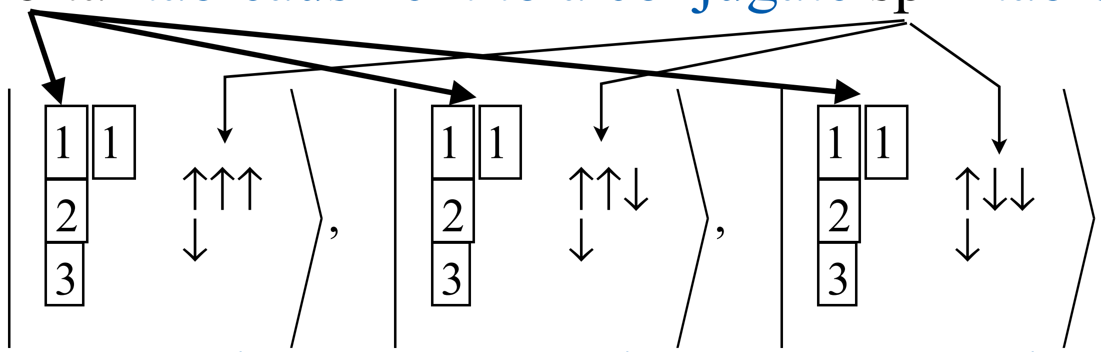
U(3)×U(2) approach: Coupling total orbit-L tableaus to total spin *S* tableaus

A state satisfying Pauli-antisymmetry (Exclusion principle) can be simply represented by putting an orbital tableaus next to a conjugate spin tableaus. (Rows flipped with columns)

U(3): $m_\ell=+1$: | 1 ⟩, $m_\ell=0$: | 2 ⟩, $m_\ell=-1$: | 3 ⟩

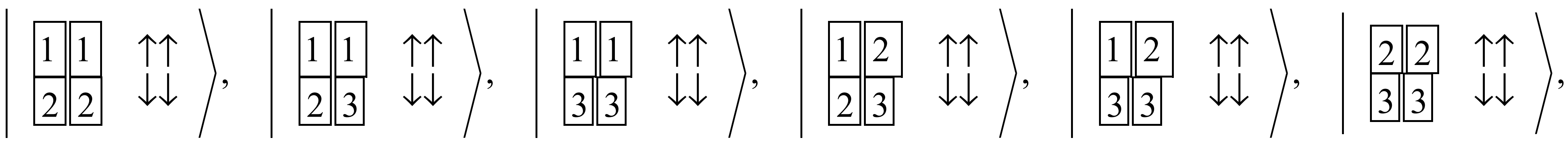
U(2): $m_s=+½$: | ↑ ⟩, $m_s=-½$: | ↓ ⟩

^{3}P *spin-triplet*



$m_L=+1$, $m_S=+1$ | $m_L=+1$, $m_S=+0$ | $m_L=+1$, $m_S=-1$

^{1}D *spin-singlet*
plus
^{1}S *spin-singlet*



$m_L=+2$, $m_S=+0$ | $m_L=+1$, $m_S=+0$ | $m_L=+0$, $m_S=+0$ | $m_L=+0$, $m_S=+0$ | $m_L=-1$, $m_S=+0$ | $m_L=-2$, $m_S=+0$

These involve fairly complicated S_n-coupled U(3)×U(2) combinations that will be developed later.

An elementary development using U(6) combinations of so called *Slater determinants* is done first.

AMOP reference links on pages 2-4

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Fermi-Dirac-Pauli anti-symmetric p^3-states

Boson operators and symmetric p^2-states

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- Projecting to angular momentum

U(6)⊃U(3)×U(2) approach: Coupling spin-orbit (s=½, ℓ=1) tableaus

Six states of a single (s=½) electron in (ℓ=1) p-shell labeled by a to f.

$$U(6)\text{ bases: }\left\{\left|a\right\rangle \equiv\left|1\uparrow\right\rangle ,\ \left|b\right\rangle \equiv\left|1\downarrow\right\rangle ,\ \left|c\right\rangle \equiv\left|2\uparrow\right\rangle ,\ \left|d\right\rangle \equiv\left|2\downarrow\right\rangle ,\ \left|e\right\rangle \equiv\left|3\uparrow\right\rangle ,\ \left|f\right\rangle \equiv\left|3\downarrow\right\rangle \right\}$$

U(6)⊃U(3)×U(2) approach: Coupling spin-orbit (s=½, ℓ=1) tableaus

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U(6) tensor operators are outer products of U(3) $\mathbf{v}^k{}_q$(*orbit*) with U(2) $\mathbf{v}^\lambda{}_\sigma$(*spin*) operators

$$\left\langle \begin{matrix} \ell & \frac{1}{2} \\ m' & \mu' \end{matrix} \middle| v^{k\ \lambda}_{q\ \sigma} \middle| \begin{matrix} \ell & \frac{1}{2} \\ m & \mu \end{matrix} \right\rangle = \left\langle \begin{matrix} \ell \\ m' \end{matrix} \middle| v^{k}_{q} \middle| \begin{matrix} \ell \\ m \end{matrix} \right\rangle \left\langle \begin{matrix} \frac{1}{2} \\ \mu' \end{matrix} \middle| v^{\lambda}_{\sigma} \middle| \begin{matrix} \frac{1}{2} \\ \mu \end{matrix} \right\rangle$$

U(6)⊃U(3)×U(2) approach: Coupling spin-orbit ($s=½$, $\ell=1$) tableaus

Six states of a single ($s=½$) electron in ($\ell=1$) p-shell labeled by a to f.

$$U(6) \text{ bases: } \left\{ \left|a\right\rangle \equiv \left|1\uparrow\right\rangle,\ \left|b\right\rangle \equiv \left|1\downarrow\right\rangle,\ \left|c\right\rangle \equiv \left|2\uparrow\right\rangle,\ \left|d\right\rangle \equiv \left|2\downarrow\right\rangle,\ \left|e\right\rangle \equiv \left|3\uparrow\right\rangle,\ \left|f\right\rangle \equiv \left|3\downarrow\right\rangle \right\}$$

U(6) tensor operators are outer products of U(3) $\mathbf{v}^k{}_q(orbit)$ with U(2) $\mathbf{v}^\lambda{}_\sigma(spin)$ operators

$$\left\langle \begin{matrix} \ell & \frac{1}{2} \\ m' & \mu' \end{matrix} \middle| v^{k\ \lambda}_{q\ \sigma} \middle| \begin{matrix} \ell & \frac{1}{2} \\ m & \mu \end{matrix} \right\rangle = \left\langle \begin{matrix} \ell \\ m' \end{matrix} \middle| v^k_q \middle| \begin{matrix} \ell \\ m \end{matrix} \right\rangle \left\langle \begin{matrix} \frac{1}{2} \\ \mu' \end{matrix} \middle| v^{\lambda}_{\sigma} \middle| \begin{matrix} \frac{1}{2} \\ \mu \end{matrix} \right\rangle$$

$$\left\langle \mathbf{v}^2_{\bar{2}} \right\rangle = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \end{pmatrix} \quad \left\langle \mathbf{v}^2_{\bar{1}} \right\rangle = \begin{pmatrix} \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \\ \cdot & \bar{1} & \cdot \end{pmatrix}\tfrac{1}{\sqrt{2}} \quad \left\langle \mathbf{v}^2_{0} \right\rangle = \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & \bar{2} & \cdot \\ \cdot & \cdot & 1 \end{pmatrix}\tfrac{1}{\sqrt{6}} \quad \left\langle \mathbf{v}^2_{1} \right\rangle = \begin{pmatrix} \cdot & \bar{1} & \cdot \\ \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \end{pmatrix}\tfrac{1}{\sqrt{2}} \quad \left\langle \mathbf{v}^2_{2} \right\rangle = \begin{pmatrix} \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

$$\left\langle \mathbf{v}^1_{\bar{1}} \right\rangle = \begin{pmatrix} \cdot & \cdot \\ 1 & \cdot \end{pmatrix} \quad \left\langle \mathbf{v}^1_{0} \right\rangle = \begin{pmatrix} 1 & \cdot \\ \cdot & \bar{1} \end{pmatrix}\tfrac{1}{\sqrt{2}} \quad \left\langle \mathbf{v}^1_{1} \right\rangle = \begin{pmatrix} \cdot & \bar{1} \\ \cdot & \cdot \end{pmatrix}$$

$$\left\langle \mathbf{v}^0_{0} \right\rangle = \begin{pmatrix} 1 & \cdot \\ \cdot & 1 \end{pmatrix}\tfrac{1}{\sqrt{3}}$$

$$\left\langle \mathbf{v}^1_{\bar{1}} \right\rangle = \begin{pmatrix} \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \end{pmatrix}\tfrac{1}{\sqrt{2}} \quad \left\langle \mathbf{v}^1_{0} \right\rangle = \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & \bar{1} \end{pmatrix}\tfrac{1}{\sqrt{2}} \quad \left\langle \mathbf{v}^1_{1} \right\rangle = \begin{pmatrix} \cdot & \bar{1} & \cdot \\ \cdot & \cdot & \bar{1} \\ \cdot & \cdot & \cdot \end{pmatrix}\tfrac{1}{\sqrt{2}}$$

Notational compaction:

$\bar{1} \equiv -1$, $\bar{2} \equiv -2$, *etc.*

$$\left\langle \mathbf{v}^0_{0} \right\rangle = \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix}\tfrac{1}{\sqrt{3}}$$

U(6)⊃U(3)×U(2) approach: Coupling spin-orbit (s=½, ℓ=1) tableaus

Six states of a single (s=½) electron in (ℓ=1) p-shell labeled by a to f.

$$U(6) \text{ bases: } \left\{ \left|a\right\rangle \equiv \left|1\uparrow\right\rangle ,\ \left|b\right\rangle \equiv \left|1\downarrow\right\rangle ,\ \left|c\right\rangle \equiv \left|2\uparrow\right\rangle ,\ \left|d\right\rangle \equiv \left|2\downarrow\right\rangle ,\ \left|e\right\rangle \equiv \left|3\uparrow\right\rangle ,\ \left|f\right\rangle \equiv \left|3\downarrow\right\rangle \right\}$$

U(6) tensor operators are outer products of U(3) $\mathbf{v}^k{}_q(orbit)$ with U(2) $\mathbf{v}^\lambda{}_\sigma(spin)$ operators

$$\left\langle \begin{matrix} \ell & \frac{1}{2} \\ m' & \mu' \end{matrix} \right| v^{k\,\lambda}_{q\,\sigma} \left| \begin{matrix} \ell & \frac{1}{2} \\ m & \mu \end{matrix} \right\rangle = \left\langle \begin{matrix} \ell \\ m' \end{matrix} \right| v^k_q \left| \begin{matrix} \ell \\ m \end{matrix} \right\rangle \left\langle \begin{matrix} \frac{1}{2} \\ \mu' \end{matrix} \right| v^{\lambda}_{\sigma} \left| \begin{matrix} \frac{1}{2} \\ \mu \end{matrix} \right\rangle$$

$$\left\langle \mathbf{v}^2_{\bar{2}} \right\rangle = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \end{pmatrix} \quad \left\langle \mathbf{v}^2_{\bar{1}} \right\rangle = \begin{pmatrix} \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \\ \cdot & \bar{1} & \cdot \end{pmatrix}\tfrac{1}{\sqrt{2}} \quad \left\langle \mathbf{v}^2_{0} \right\rangle = \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & \bar{2} & \cdot \\ \cdot & \cdot & 1 \end{pmatrix}\tfrac{1}{\sqrt{6}} \quad \left\langle \mathbf{v}^2_{1} \right\rangle = \begin{pmatrix} \cdot & \bar{1} & \cdot \\ \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \end{pmatrix}\tfrac{1}{\sqrt{2}} \quad \left\langle \mathbf{v}^2_{2} \right\rangle = \begin{pmatrix} \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

$$\left\langle \mathbf{v}^1_{\bar{1}} \right\rangle = \begin{pmatrix} \cdot & \cdot \\ 1 & \cdot \end{pmatrix} \quad \left\langle \mathbf{v}^1_{0} \right\rangle = \begin{pmatrix} 1 & \cdot \\ \cdot & \bar{1} \end{pmatrix}\tfrac{1}{\sqrt{2}} \quad \left\langle \mathbf{v}^1_{1} \right\rangle = \begin{pmatrix} \cdot & \bar{1} \\ \cdot & \cdot \end{pmatrix}$$

$$\left\langle \mathbf{v}^0_{0} \right\rangle = \begin{pmatrix} 1 & \cdot \\ \cdot & 1 \end{pmatrix}\tfrac{1}{\sqrt{3}}$$

$$\left\langle \mathbf{v}^1_{\bar{1}} \right\rangle = \begin{pmatrix} \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \end{pmatrix}\tfrac{1}{\sqrt{2}} \quad \left\langle \mathbf{v}^1_{0} \right\rangle = \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & \bar{1} \end{pmatrix}\tfrac{1}{\sqrt{2}} \quad \left\langle \mathbf{v}^1_{1} \right\rangle = \begin{pmatrix} \cdot & \bar{1} & \cdot \\ \cdot & \cdot & \bar{1} \\ \cdot & \cdot & \cdot \end{pmatrix}\tfrac{1}{\sqrt{2}}$$

$$\left\langle \mathbf{v}^0_{0} \right\rangle = \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix}\tfrac{1}{\sqrt{3}}$$

Notational compaction:
$\bar{1} \equiv -1,\ \ \bar{2} \equiv -2,\quad etc.$

$\frac{1}{\sqrt{2}}(\text{-}E_{cb}\text{-}E_{ed}) =$

$$\left\langle \mathbf{v}^{11}_{\bar{1}1} \right\rangle = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \bar{1} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \bar{1} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}\tfrac{1}{\sqrt{2}} \quad \left\langle \mathbf{v}^{11}_{00} \right\rangle = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \bar{1} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \bar{1} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}\tfrac{1}{2}$$

$$\left\langle \mathbf{v}^{11}_{1\bar{1}} \right\rangle = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \bar{1} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \bar{1} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}\tfrac{1}{\sqrt{2}} = \mathbf{v}^1_1 \otimes \mathbf{v}^1_{\bar{1}} = \tfrac{1}{\sqrt{2}}\begin{pmatrix} \cdot & \bar{1} & \cdot \\ \cdot & \cdot & \bar{1} \\ \cdot & \cdot & \cdot \end{pmatrix} \otimes \begin{pmatrix} \cdot & \cdot \\ 1 & \cdot \end{pmatrix}$$

$= \frac{1}{\sqrt{2}}(\text{-}E_{bc}\text{-}E_{de})$

$$\left\langle \mathbf{v}^{11}_{\bar{1}\bar{1}} \right\rangle = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix}\tfrac{1}{\sqrt{2}} \quad \left\langle \mathbf{v}^{00}_{00} \right\rangle = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}\tfrac{1}{\sqrt{6}} \quad \left\langle \mathbf{v}^{11}_{11} \right\rangle = \begin{pmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}\tfrac{1}{\sqrt{2}}$$

AMOP reference links on pages 2-4

4.16.18 class 23: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

$(S_n)^*(U(m))$ shell model of electrostatic quadrupole-quadrupole-e interactions

Marrying spin $s=½$ and orbital $\ell=1$ together: U(3)×U(2)

The $\ell=1$ p=shell in a nutshell

U(6)⊃U(3)×U(2) approach: Coupling spin-orbit ($s=½$, $\ell=1$) tableaus

➡ Introducing atomic spin-orbit state assembly formula

Slater determinants

p-shell Spin-orbit calculations (not finished)

Clebsch Gordan coefficients. (Rev. Mod. Phys. annual gift)

S_n projection for atomic spin and orbit states

Review of Mach-Mock (particle-state) principle

Tableau P-operators on orbits

Tableau P-operators on spin

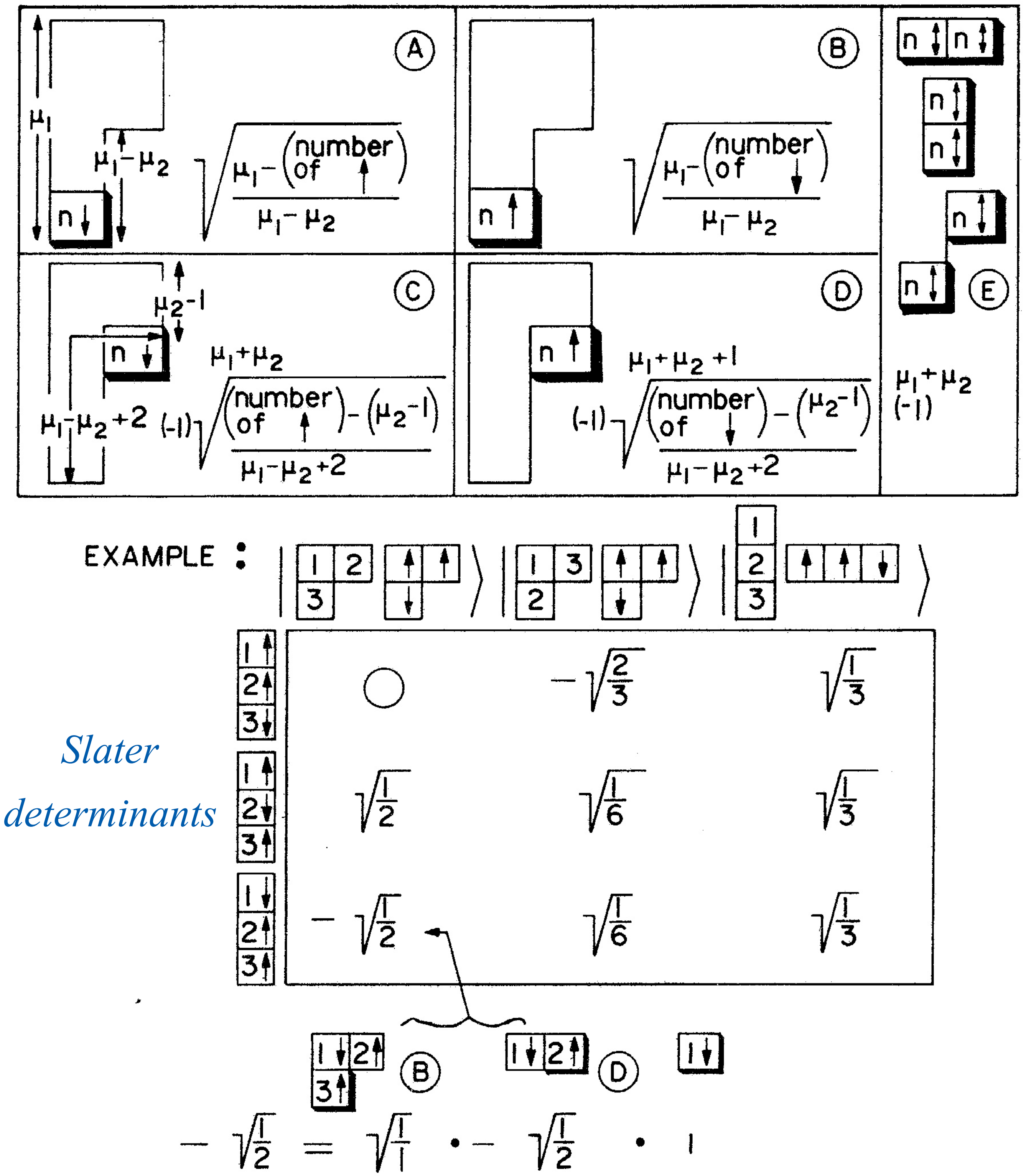
Fermi-Dirac-Pauli anti-symmetric p^3-states

Boson operators and symmetric p^2-states

Connecting to angular momentum

Projecting to angular momentum

Introducing atomic spin-orbit state assembly formula and Slater determinants



Slater determinants

FIG. 5. Assembly formula for combining orbital and spin states. Each column state (Slater determinant) on the left-hand side of the sample table has a definite spin (arrow)on each orbital state (number). The formulas will give the overlap of this Slater state with a given orbital tableau state if we first write the spins within this orbital tableau in exactly the same way. Then we proceed to remove boxes with numbered spins starting with the highest number(s). Each "removal" gives a factor depending on what is being removed and where (cases A–E). All of the numbers in the formulas refer to the condition of the tableau just before the box outlined in the figure is removed.

Introducing atomic spin-orbit state assembly formula and Slater determinants

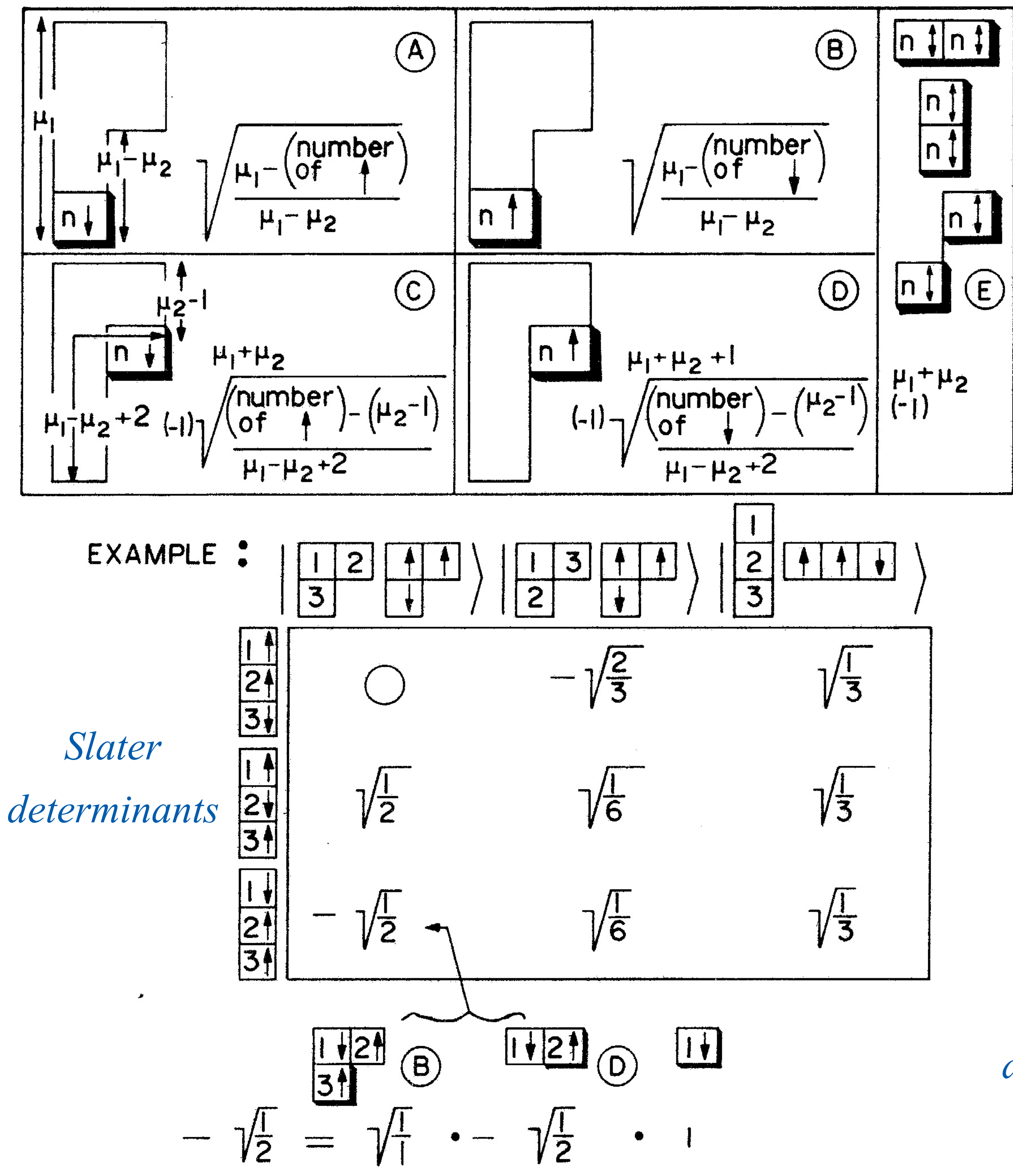
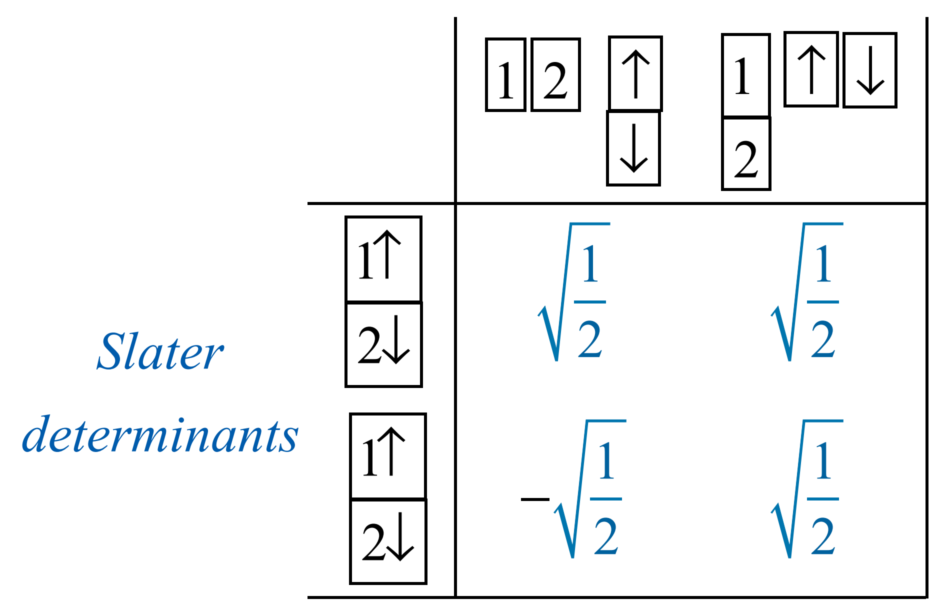


FIG. 5. Assembly formula for combining orbital and spin states. Each column state (Slater determinant) on the left-hand side of the sample table has a definite spin (arrow)on each orbital state (number). The formulas will give the overlap of this Slater state with a given orbital tableau state if we first write the spins within this orbital tableau in exactly the same way. Then we proceed to remove boxes with numbered spins starting with the highest number(s). Each "removal" gives a factor depending on what is being removed and where (cases A–E). All of the numbers in the formulas refer to the condition of the tableau just before the box outlined in the figure is removed.

The simplest assembly:



p-shell Spin-orbit calculation

AMOP reference links on pages 2-4

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The $\ell=1$ p=shell in a nutshell

U(6)⊃U(3)×U(2) approach: Coupling spin-orbit ($s=½$, $\ell=1$) tableaus

Introducing atomic spin-orbit state assembly formula

Slater determinants

p-shell Spin-orbit calculations (not finished)

➡ Clebsch Gordan coefficients. (Rev. Mod. Phys. annual gift)

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Tableau P-operators on spin

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Connecting to angular momentum

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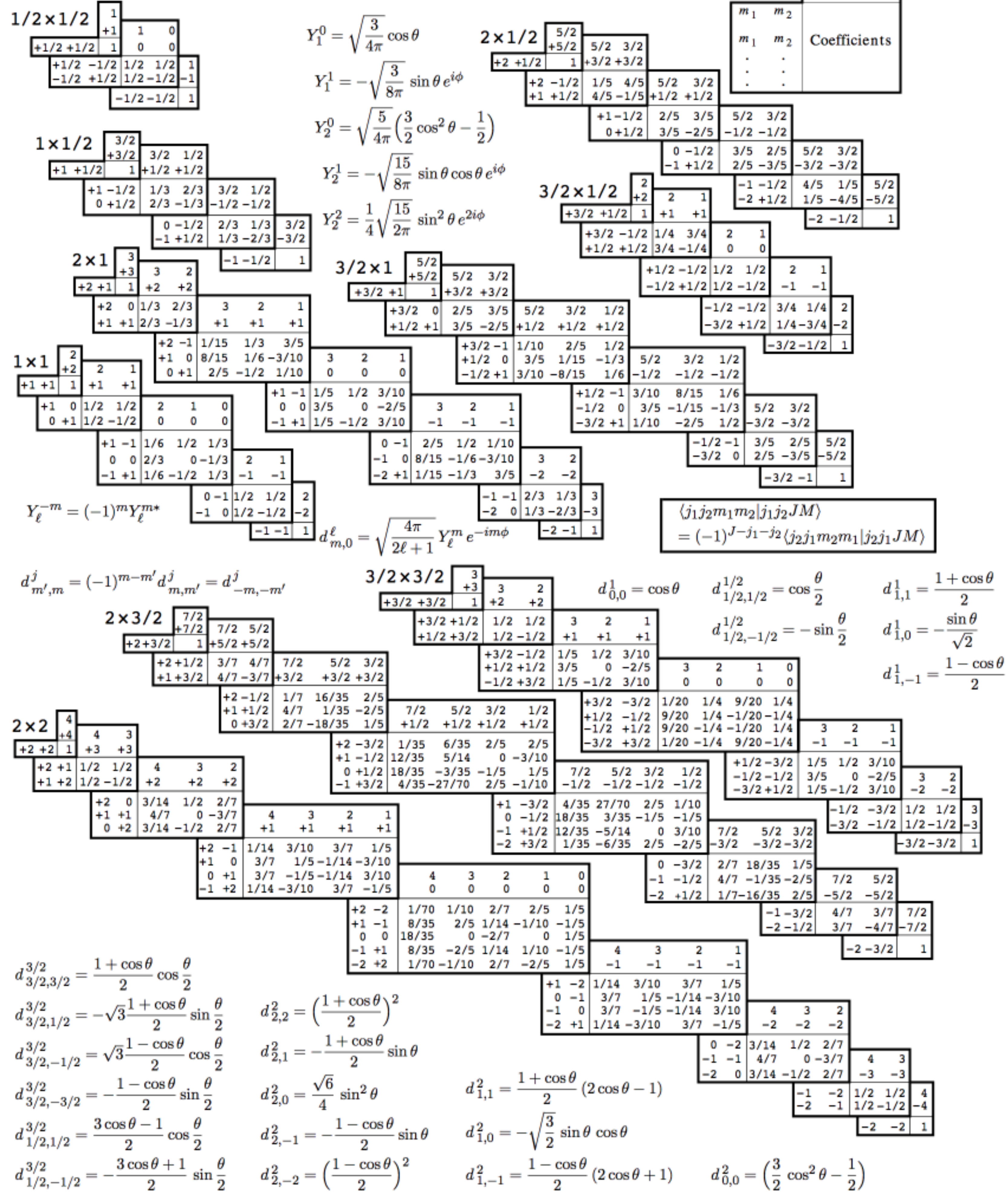

AMOP reference links on pages 2-4

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Projecting to angular momentum

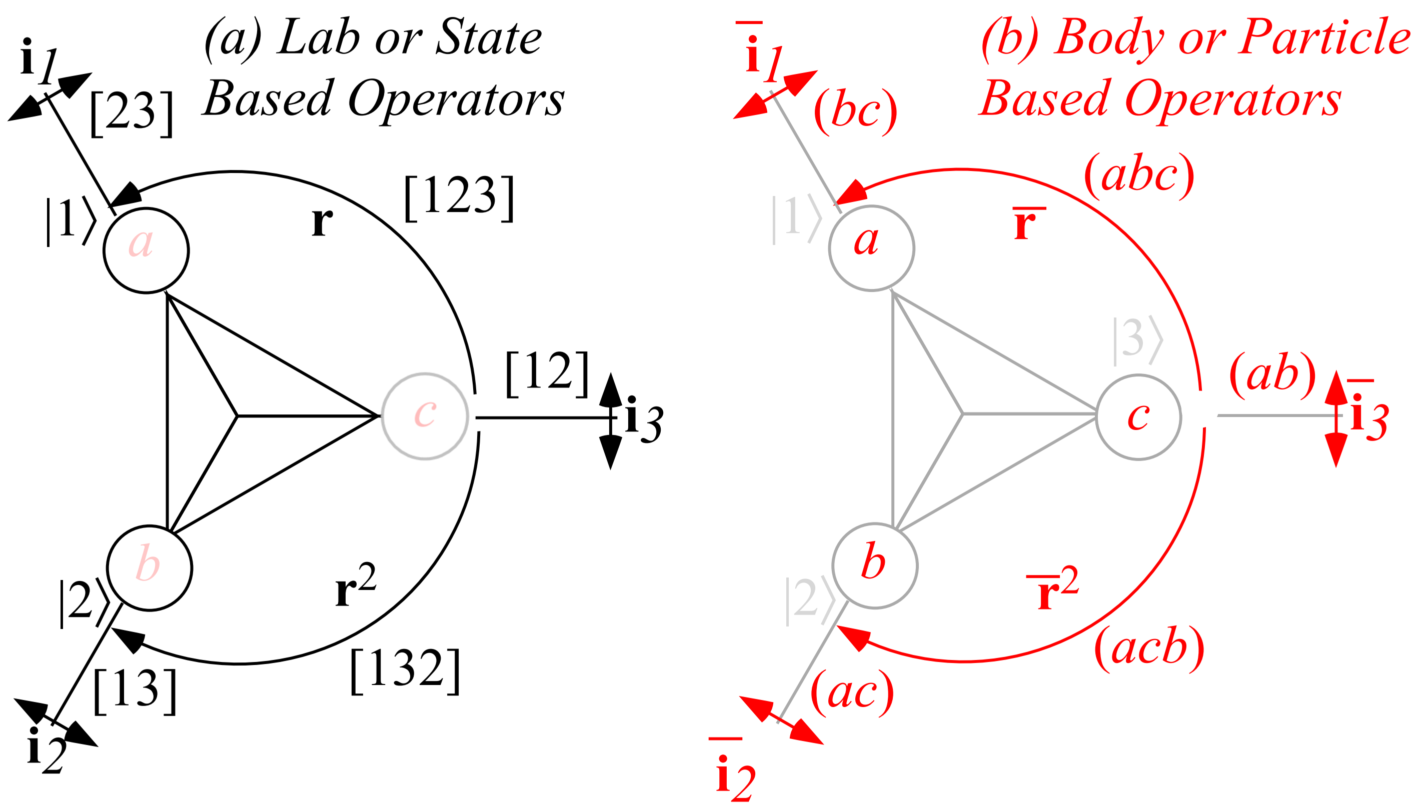
S_n projection for atomic spin and orbit states

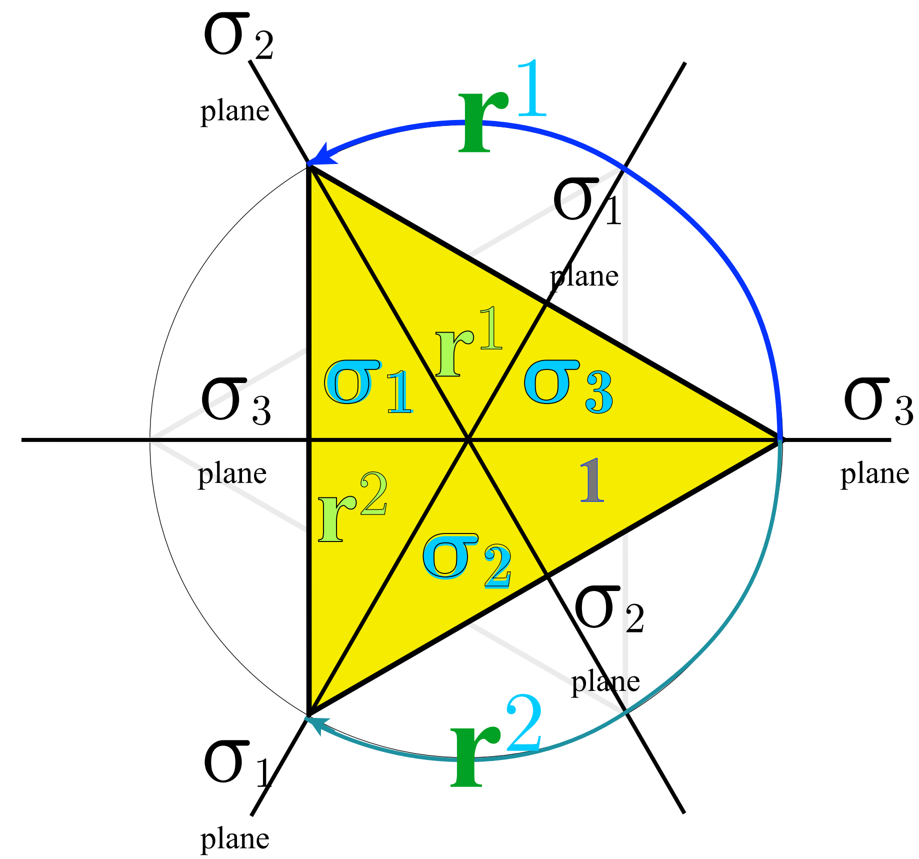
Review of Lect. 20 p.37 to 41

D3<C3v nomogram
AMOP Class 12 pdf p30

D3<D6 nomogram
AMOP Class 14 pdf p28

Fig. 25.3.0 QTforCA Unit 8 Ch.25 pdf p28



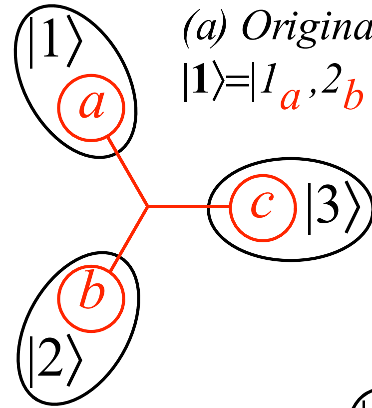


C_{3v} $\mathbf{g}\mathbf{g}^\dagger$ form	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	σ_1	σ_2	σ_3
$(a)(b)(c)=\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	σ_1	σ_2	σ_3
$(abc)=\mathbf{r}^1$	$\mathbf{r}^1$	$\mathbf{1}$	$\mathbf{r}^2$	σ_2	σ_3	σ_1
$(acb)=\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	$\mathbf{1}$	σ_3	σ_1	σ_2
$(bc)=\sigma_1$	σ_1	σ_2	σ_3	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$
$(ac)=\sigma_2$	σ_2	σ_3	σ_1	$\mathbf{r}^1$	$\mathbf{1}$	$\mathbf{r}^2$
$(ab)=\sigma_3$	σ_3	σ_1	σ_2	$\mathbf{r}^2$	$\mathbf{r}^1$	$\mathbf{1}$

S_n projection for atomic spin and orbit states

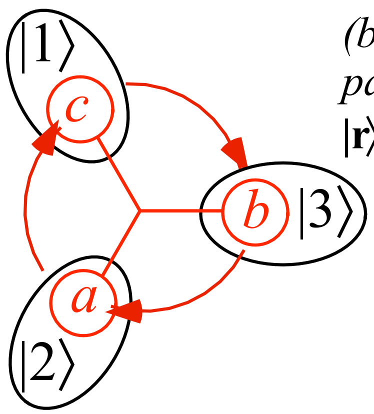
Fig. 25.3.1 QTforCA Unit 8 Ch.25 pdf p29

Review of Lect. 20 p.37 to 41



(a) Original state

$|\mathbf{1}\rangle = |1_a, 2_b, 3_c\rangle$



(b) Lab-fixed particle-3-cycle (120°rotation **r***)*

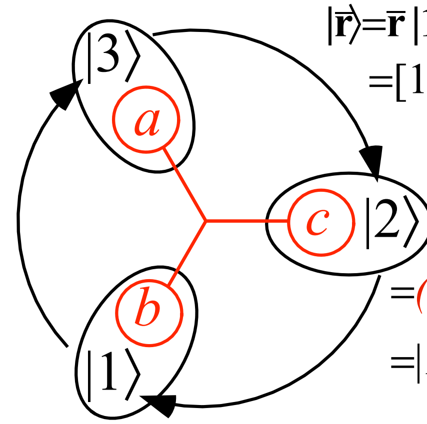
$|\mathbf{r}\rangle = \mathbf{r}\,|\mathbf{1}\rangle$

$= (abc)|1_a, 2_b, 3_c\rangle$

$= |1_c, 2_a, 3_b\rangle = |2_a, 3_b, 1_c\rangle$

$= \bar{\mathbf{r}}^2|\mathbf{1}\rangle = [132]|1_a, 2_b, 3_c\rangle$

$[132]|1_a, 2_b, 3_c\rangle = |2_a, 3_b, 1_c\rangle$



(c) Particle-fixed lab-120°rotation $\bar{\mathbf{r}}$

$|\bar{\mathbf{r}}\rangle = \bar{\mathbf{r}}\,|\mathbf{1}\rangle$

$= [123]|1_a, 2_b, 3_c\rangle$

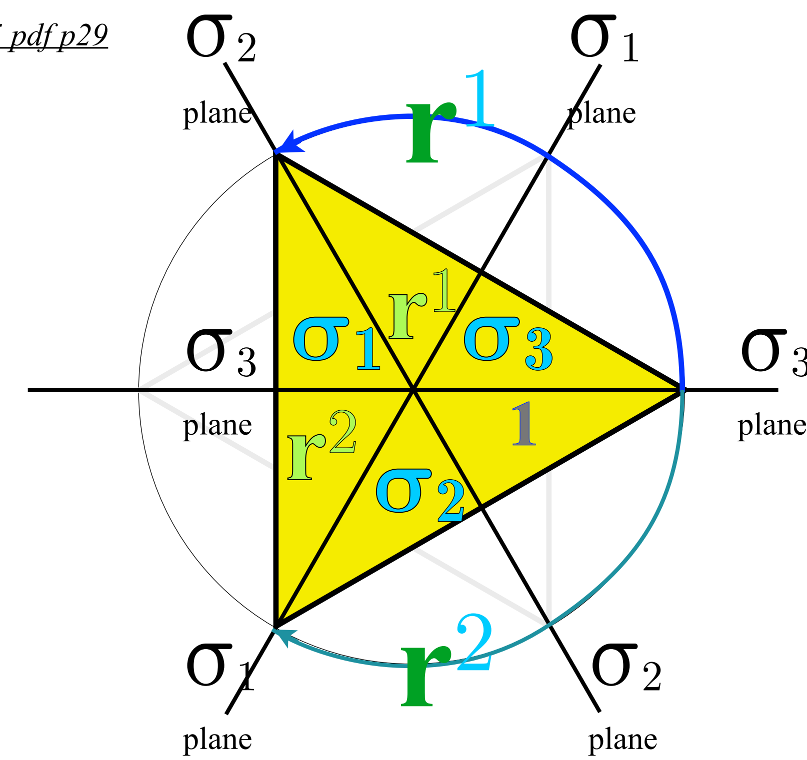
$= |3_a, 1_b, 2_c\rangle$

$= \mathbf{r}^2|\mathbf{1}\rangle$

$= (acb)|1_a, 2_b, 3_c\rangle$

$= |1_b, 2_c, 3_a\rangle$

$[123]|1_a, 2_b, 3_c\rangle = |3_a, 1_b, 2_c\rangle$



1	r^2	r	i_1	i_2	i_3
r	1	r^2	i_2	i_3	i_1
r^2	r	1	i_3	i_1	i_2
i_1	i_2	i_3	1	r^2	r
i_2	i_3	i_1	r	1	r^2
i_3	i_1	i_2	r^2	r	1

C_{3v} $\mathbf{gg}^\dagger$ form	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	σ_1	σ_2	σ_3
$(a)(b)(c) = \mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	σ_1	σ_2	σ_3
$(abc) = \mathbf{r}^1$	$\mathbf{r}^1$	$\mathbf{1}$	$\mathbf{r}^2$	σ_2	σ_3	σ_1
$(acb) = \mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	$\mathbf{1}$	σ_3	σ_1	σ_2
$(bc) = \sigma_1$	σ_1	σ_2	σ_3	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$
$(ac) = \sigma_2$	σ_2	σ_3	σ_1	$\mathbf{r}^1$	$\mathbf{1}$	$\mathbf{r}^2$
$(ab) = \sigma_3$	σ_3	σ_1	σ_2	$\mathbf{r}^2$	$\mathbf{r}^1$	$\mathbf{1}$

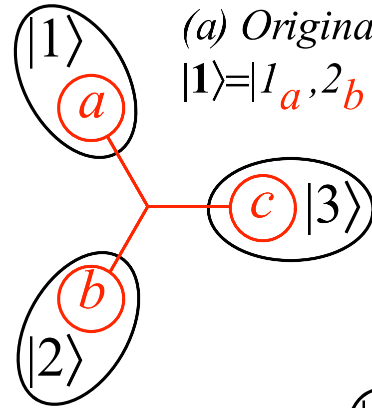
(1)	(acb)	(abc)	(bc)	(ac)	(ab)
(abc)	(1)	(acb)	(ac)	(ab)	(bc)
(acb)	(abc)	(1)	(ab)	(bc)	(ac)
(bc)	(ac)	(ab)	(1)	(acb)	(abc)
(ac)	(ab)	(bc)	(abc)	(1)	(acb)
(ab)	(bc)	(ac)	(acb)	(abc)	(1)

[1]	[132]	[123]	[23]	[13]	[12]
[123]	[1]	[132]	[13]	[12]	[23]
[132]	[123]	[1]	[12]	[23]	[13]
[23]	[13]	[12]	[1]	[132]	[123]
[13]	[12]	[23]	[123]	[1]	[132]
[12]	[23]	[13]	[132]	[123]	[1]

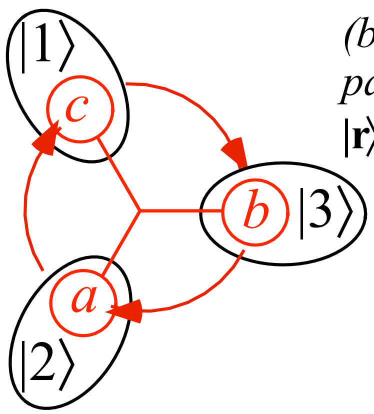
Fig. 25.3.1 Relating D_3 *and* S_3 *permutation operations*

S_n projection for atomic spin and orbit states

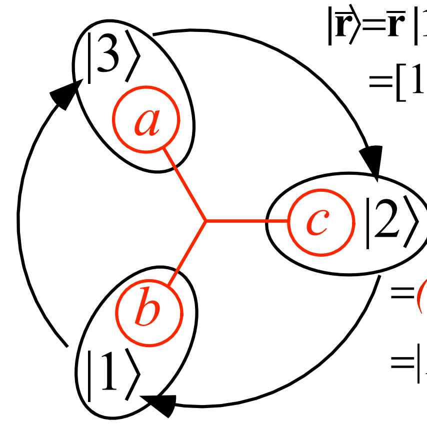
Fig. 25.3.1 QTforCA Unit 8 Ch.25 pdf p29



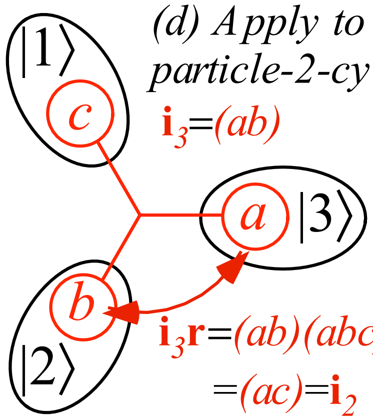
(a) Original state
$|\mathbf{1}\rangle = |1_a, 2_b, 3_c\rangle$



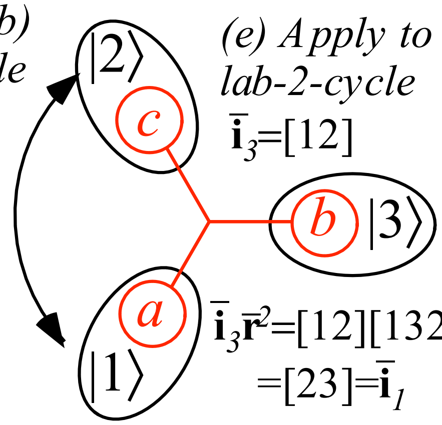
(b) Lab-fixed particle-3-cycle (120°rotation **r***)*
$|\mathbf{r}\rangle = \mathbf{r}\,|\mathbf{1}\rangle$
$= (abc)|1_a, 2_b, 3_c\rangle$
$= |1_c, 2_a, 3_b\rangle = |2_a, 3_b, 1_c\rangle$
$= \bar{\mathbf{r}}^2|\mathbf{1}\rangle = [132]|1_a, 2_b, 3_c\rangle$



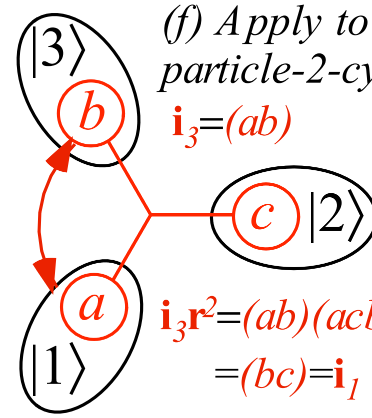
(c) Particle-fixed lab-120°rotation $\bar{\mathbf{r}}$
$|\bar{\mathbf{r}}\rangle = \bar{\mathbf{r}}\,|\mathbf{1}\rangle$
$= [123]|1_a, 2_b, 3_c\rangle$
$= |3_a, 1_b, 2_c\rangle$
$= \mathbf{r}^2|\mathbf{1}\rangle$
$= (acb)|1_a, 2_b, 3_c\rangle$
$= |1_b, 2_c, 3_a\rangle$



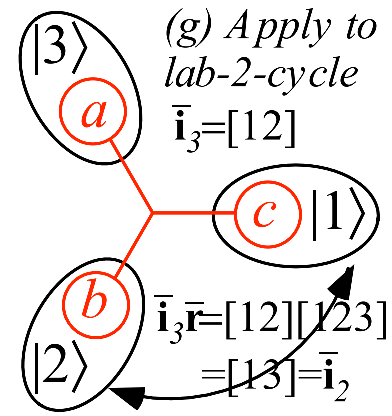
(d) Apply to (b) particle-2-cycle
$\mathbf{i}_3 = (ab)$
$\mathbf{i}_3\mathbf{r} = (ab)(abc)$
$= (ac) = \mathbf{i}_2$



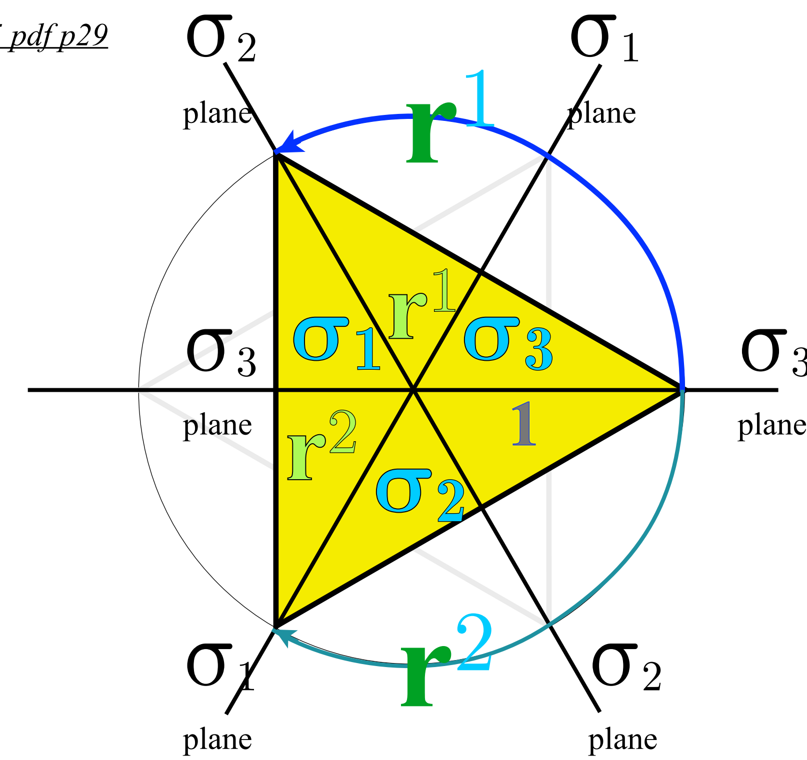
(e) Apply to (b) lab-2-cycle
$\bar{\mathbf{i}}_3 = [12]$
$\bar{\mathbf{i}}_3\bar{\mathbf{r}}^2 = [12][132]$
$= [23] = \bar{\mathbf{i}}_1$



(f) Apply to (c) particle-2-cycle
$\mathbf{i}_3 = (ab)$
$\mathbf{i}_3\mathbf{r}^2 = (ab)(acb)$
$= (bc) = \mathbf{i}_1$



(g) Apply to (c) lab-2-cycle
$\bar{\mathbf{i}}_3 = [12]$
$\bar{\mathbf{i}}_3\bar{\mathbf{r}} = [12][123]$
$= [13] = \bar{\mathbf{i}}_2$



1	r^2	r	i_1	i_2	i_3
r	1	r^2	i_2	i_3	i_1
r^2	r	1	i_3	i_1	i_2
i_1	i_2	i_3	1	r^2	r
i_2	i_3	i_1	r	1	r^2
i_3	i_1	i_2	r^2	r	1

C_{3v} $gg^\dagger$ form	**1**	$\mathbf{r}^2$	$\mathbf{r}^1$	σ_1	σ_2	σ_3
$(a)(b)(c) = \mathbf{1}$	**1**	$\mathbf{r}^2$	$\mathbf{r}^1$	σ_1	σ_2	σ_3
$(abc) = \mathbf{r}^1$	$\mathbf{r}^1$	**1**	$\mathbf{r}^2$	σ_2	σ_3	σ_1
$(acb) = \mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	**1**	σ_3	σ_1	σ_2
$(bc) = \sigma_1$	σ_1	σ_2	σ_3	**1**	$\mathbf{r}^2$	$\mathbf{r}^1$
$(ac) = \sigma_2$	σ_2	σ_3	σ_1	$\mathbf{r}^1$	**1**	$\mathbf{r}^2$
$(ab) = \sigma_3$	σ_3	σ_1	σ_2	$\mathbf{r}^2$	$\mathbf{r}^1$	**1**

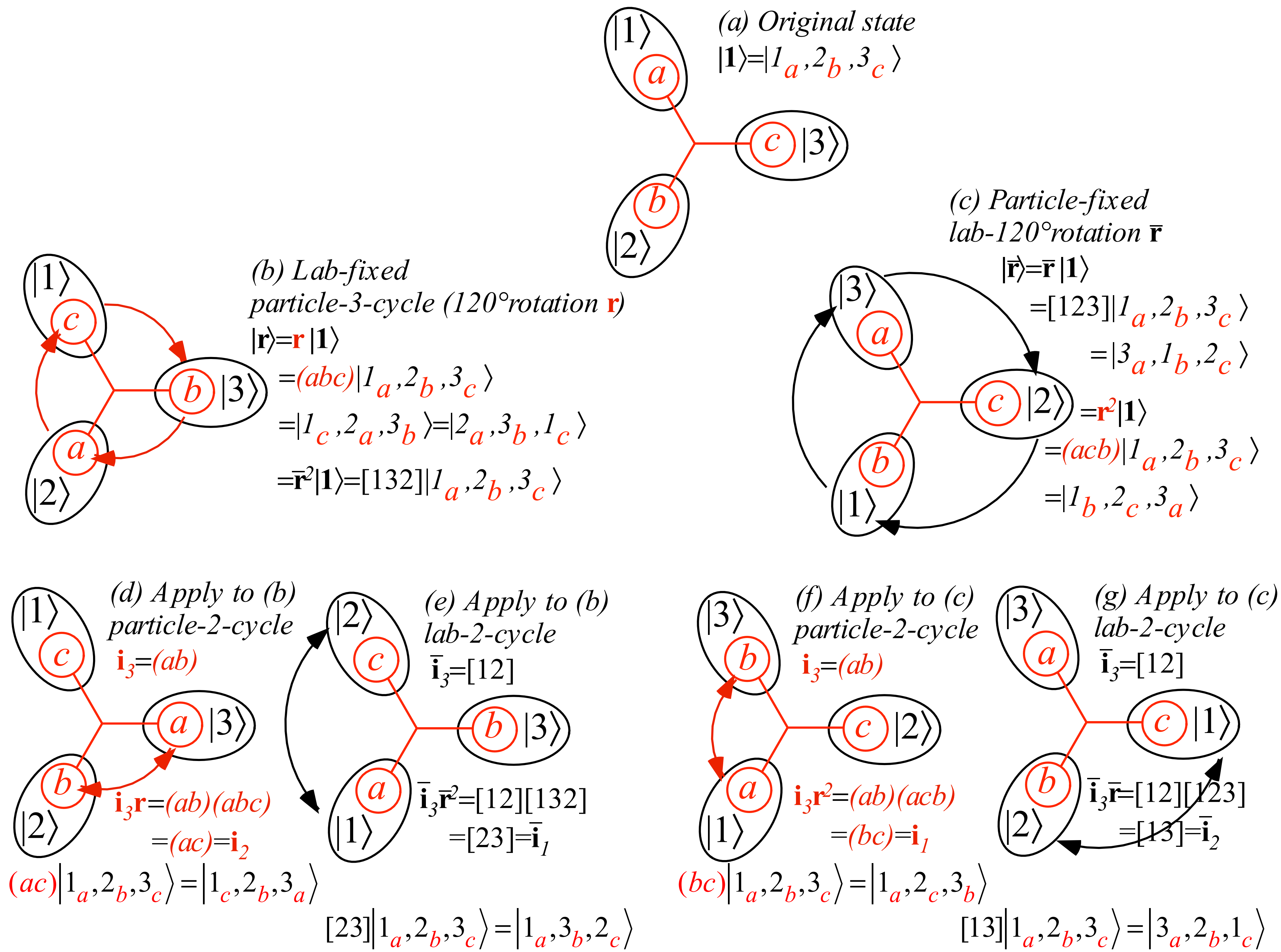
(1)	(acb)	(abc)	(bc)	(ac)	(ab)
(abc)	(1)	(acb)	(ac)	(ab)	(bc)
(acb)	(abc)	(1)	(ab)	(bc)	(ac)
(bc)	(ac)	(ab)	(1)	(acb)	(abc)
(ac)	(ab)	(bc)	(abc)	(1)	(acb)
(ab)	(bc)	(ac)	(acb)	(abc)	(1)

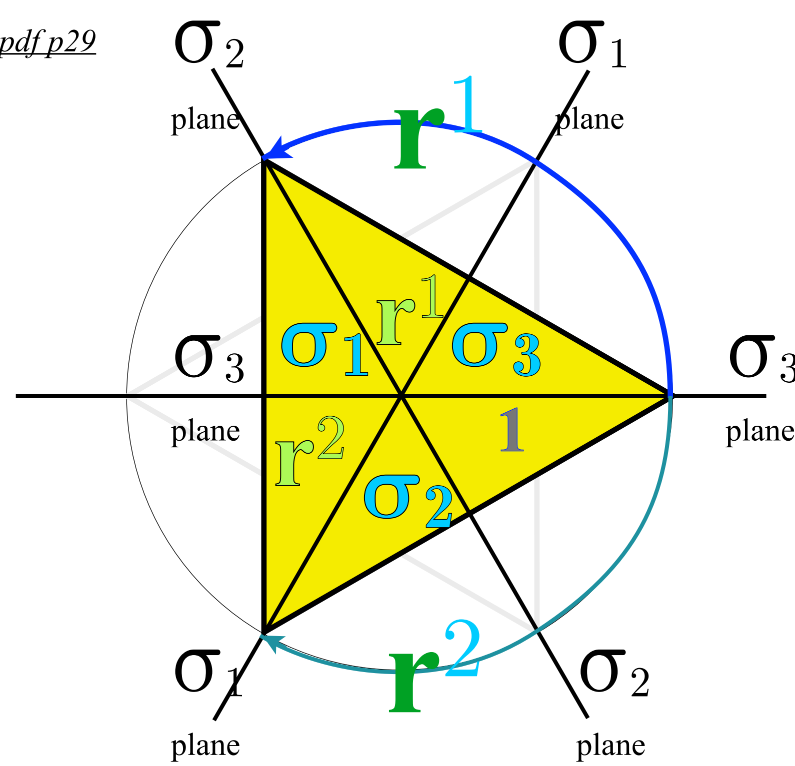
[1]	[132]	[123]	[23]	[13]	[12]
[123]	[1]	[132]	[13]	[12]	[23]
[132]	[123]	[1]	[12]	[23]	[13]
[23]	[13]	[12]	[1]	[132]	[123]
[13]	[12]	[23]	[123]	[1]	[132]
[12]	[23]	[13]	[132]	[123]	[1]

Fig. 25.3.1 Relating D_3 *and* S_3 *permutation operations*

S_n projection for atomic spin and orbit states

Fig. 25.3.1 QTforCA Unit 8 Ch.25 pdf p29





(1)	(acb)	(abc)	(bc)	(ac)	(ab)
(abc)	(1)	(acb)	(ac)	(ab)	(bc)
(acb)	(abc)	(1)	(ab)	(bc)	(ac)
(bc)	(ac)	(ab)	(1)	(acb)	(abc)
(ac)	(ab)	(bc)	(abc)	(1)	(acb)
(ab)	(bc)	(ac)	(acb)	(abc)	(1)

[1]	[132]	[123]	[23]	[13]	[12]
[123]	[1]	[132]	[13]	[12]	[23]
[132]	[123]	[1]	[12]	[23]	[13]
[23]	[13]	[12]	[1]	[132]	[123]
[13]	[12]	[23]	[123]	[1]	[132]
[12]	[23]	[13]	[132]	[123]	[1]

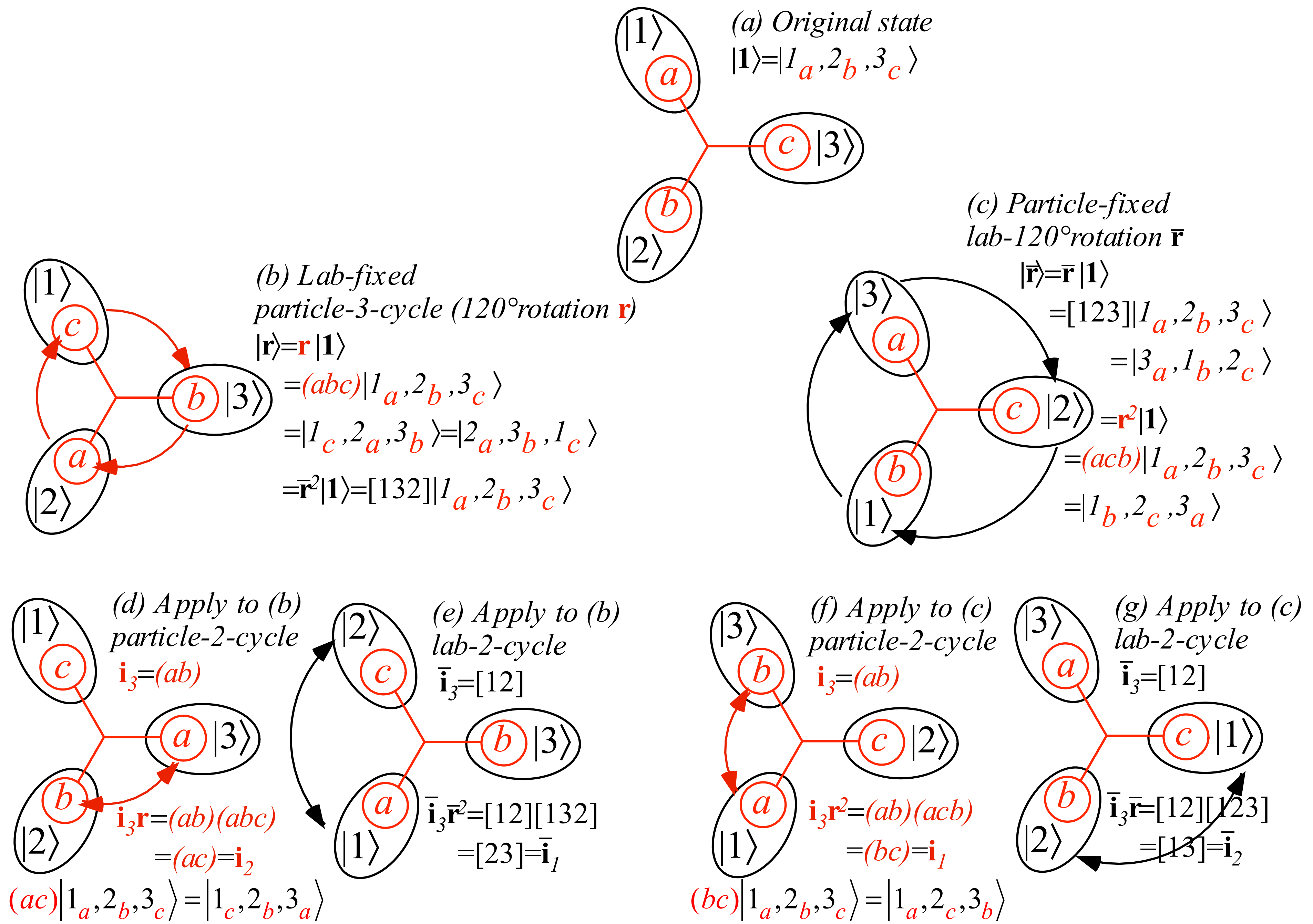
1	r^2	r	i_1	i_2	i_3
r	1	r^2	i_2	i_3	i_1
r^2	r	1	i_3	i_1	i_2
i_1	i_2	i_3	1	r^2	r
i_2	i_3	i_1	r	1	r^2
i_3	i_1	i_2	r^2	r	1

C_{3v} $\mathbf{gg}^\dagger$ form	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	σ_1	σ_2	σ_3
$(a)(b)(c)=\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	σ_1	σ_2	σ_3
$(abc)=\mathbf{r}^1$	$\mathbf{r}^1$	$\mathbf{1}$	$\mathbf{r}^2$	σ_2	σ_3	σ_1
$(acb)=\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	$\mathbf{1}$	σ_3	σ_1	σ_2
$(bc)=\sigma_1$	σ_1	σ_2	σ_3	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$
$(ac)=\sigma_2$	σ_2	σ_3	σ_1	$\mathbf{r}^1$	$\mathbf{1}$	$\mathbf{r}^2$
$(ab)=\sigma_3$	σ_3	σ_1	σ_2	$\mathbf{r}^2$	$\mathbf{r}^1$	$\mathbf{1}$

Fig. 25.3.1 Relating D_3 and S_3 permutation operations

S_n projection for atomic spin and orbit states

Fig. 25.3.1 QTforCA Unit 8 Ch.25 pdf p29

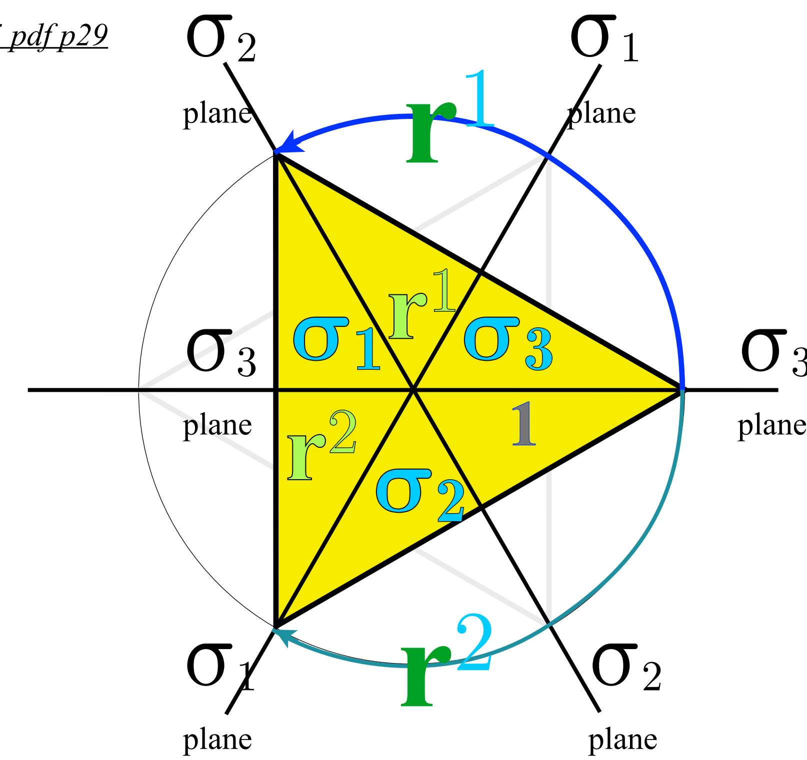


$$[23]|1_a, 2_b, 3_c\rangle = |1_a, 3_b, 2_c\rangle = |1_a, 2_c, 3_b\rangle = |2_c, 1_a, 3_b\rangle = |2_c, 3_b, 1_a\rangle = |3_b, 2_c, 1_a\rangle = |3_b, 1_a, 2_c\rangle$$

$$[13]|1_a, 2_b, 3_c\rangle = |3_a, 2_b, 1_c\rangle$$

Only *relative* position counts here!

(Mock-Mach Principle!)



1	r^2	r	i_1	i_2	i_3
r	1	r^2	i_2	i_3	i_1
r^2	r	1	i_3	i_1	i_2
i_1	i_2	i_3	1	r^2	r
i_2	i_3	i_1	r	1	r^2
i_3	i_1	i_2	r^2	r	1

C_{3v} $gg^\dagger$ form	**1**	$\mathbf{r}^2$	$\mathbf{r}^1$	σ_1	σ_2	σ_3
$(a)(b)(c) = \mathbf{1}$	**1**	$\mathbf{r}^2$	$\mathbf{r}^1$	σ_1	σ_2	σ_3
$(abc) = \mathbf{r}^1$	$\mathbf{r}^1$	**1**	$\mathbf{r}^2$	σ_2	σ_3	σ_1
$(acb) = \mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	**1**	σ_3	σ_1	σ_2
$(bc) = \sigma_1$	σ_1	σ_2	σ_3	**1**	$\mathbf{r}^2$	$\mathbf{r}^1$
$(ac) = \sigma_2$	σ_2	σ_3	σ_1	$\mathbf{r}^1$	**1**	$\mathbf{r}^2$
$(ab) = \sigma_3$	σ_3	σ_1	σ_2	$\mathbf{r}^2$	$\mathbf{r}^1$	**1**

Fig. 25.3.1 Relating D_3 and S_3 permutation operations

AMOP reference links on pages 2-4

4.16.18 class 23: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

$(S_n)^*(U(m))$ shell model of electrostatic quadrupole-quadrupole-e interactions

Marrying spin $s=½$ and orbital $\ell=1$ together: U(3)×U(2)

- The $\ell=1$ p=shell in a nutshell

U(6)⊃U(3)×U(2) approach: Coupling spin-orbit ($s=½$, $\ell=1$) tableaus

Introducing atomic spin-orbit state assembly formula

- Slater determinants
- p-shell Spin-orbit calculations (not finished)

Clebsch Gordan coefficients. (Rev. Mod. Phys. annual gift)

S_n projection for atomic spin and orbit states

- Review of Mach-Mock (particle-state) principle
- ➡ Tableau P-operators on orbits (Yamonouchi formula)
- Tableau P-operators on spin

Fermi-Dirac-Pauli anti-symmetric p^3-states

Boson operators and symmetric p^2-states

- Connecting to angular momentum
- Projecting to angular momentum

S_n projection for atomic spin and orbit states

Dirac-ket-ket-ket product represents states 1, 2, 3 that variously occupy particles *a*, *b*, and *c*,

$$\left|1,2,3\right\rangle \equiv \left|1\right\rangle_{particle-a} \left|2\right\rangle_{particle-b} \left|3\right\rangle_{particle-c} \equiv \left|1\right\rangle_{a} \left|2\right\rangle_{b} \left|3\right\rangle_{c}$$

S_n projection for atomic spin and orbit states

Dirac-ket-ket-ket product represents states 1, 2, 3 that variously occupy particles a, b, and c,

$$\left|1,2,3\right\rangle \equiv \left|1\right\rangle_{particle-a}\left|2\right\rangle_{particle-b}\left|3\right\rangle_{particle-c} \equiv \left|1\right\rangle_{a}\left|2\right\rangle_{b}\left|3\right\rangle_{c}$$

Sub-tableaus $\boxed{a}\boxed{b}$ (or $\begin{array}{c}\boxed{a}\\\boxed{b}\end{array}$) label symmetry (anti-symmetry) by single row (or single column)

$$D^{E}[12] = \begin{array}{c|cc|} \begin{array}{l}\boxed{1}\boxed{2}\\\boxed{3}\end{array} & +1 & 0 \\ \begin{array}{l}\boxed{1}\boxed{3}\\\boxed{2}\end{array} & 0 & -1 \end{array} \qquad D^{E}(ab) = \begin{array}{c|cc|} \begin{array}{l}\boxed{a}\boxed{b}\\\boxed{c}\end{array} & +1 & 0 \\ \begin{array}{l}\boxed{a}\boxed{c}\\\boxed{b}\end{array} & 0 & -1 \end{array}$$

S_n projection for atomic spin and orbit states: Tableau P-operators

Dirac-ket-ket-ket product represents states 1, 2, 3 that variously occupy particles *a*, *b*, and *c*,

$$\left|1,2,3\right\rangle \equiv \left|1\right\rangle_{particle-a}\left|2\right\rangle_{particle-b}\left|3\right\rangle_{particle-c} \equiv \left|1\right\rangle_{a}\left|2\right\rangle_{b}\left|3\right\rangle_{c}$$

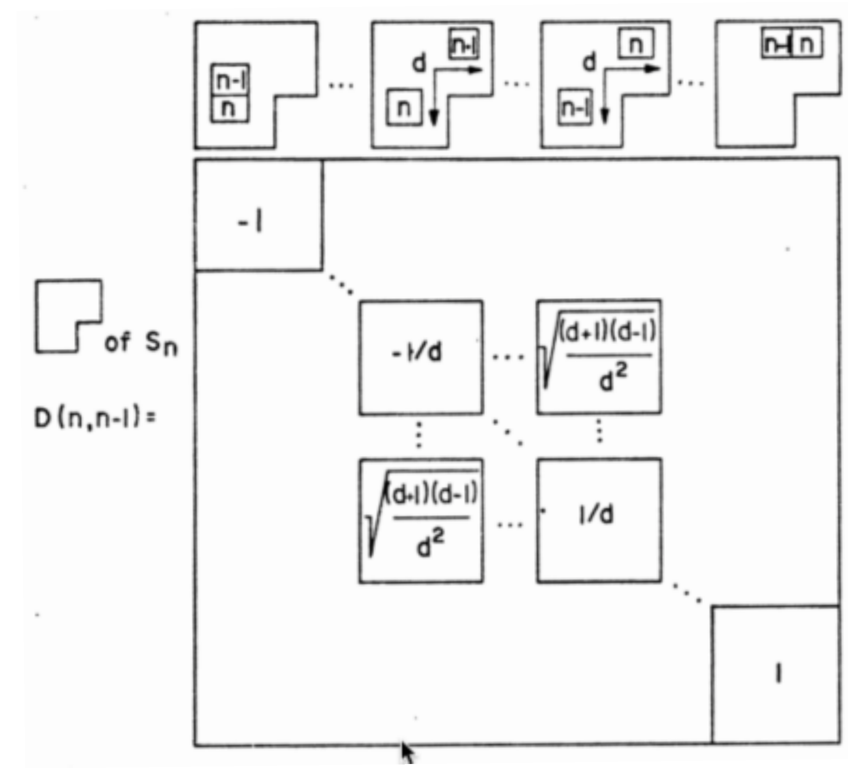
Sub-tableaus $\boxed{a}\boxed{b}$ (or $\begin{smallmatrix}\boxed{a}\\ \boxed{b}\end{smallmatrix}$) label symmetry (anti-symmetry) by single row (or single column)

$$D^E[12]=\begin{array}{c|cc|} \begin{smallmatrix}\boxed{1}\boxed{2}\\ \boxed{3}\end{smallmatrix} & +1 & 0 \\ \begin{smallmatrix}\boxed{1}\boxed{3}\\ \boxed{2}\end{smallmatrix} & 0 & -1 \end{array} \qquad D^E(ab)=\begin{array}{c|cc|} \begin{smallmatrix}\boxed{a}\boxed{b}\\ \boxed{c}\end{smallmatrix} & +1 & 0 \\ \begin{smallmatrix}\boxed{a}\boxed{c}\\ \boxed{b}\end{smallmatrix} & 0 & -1 \end{array}$$

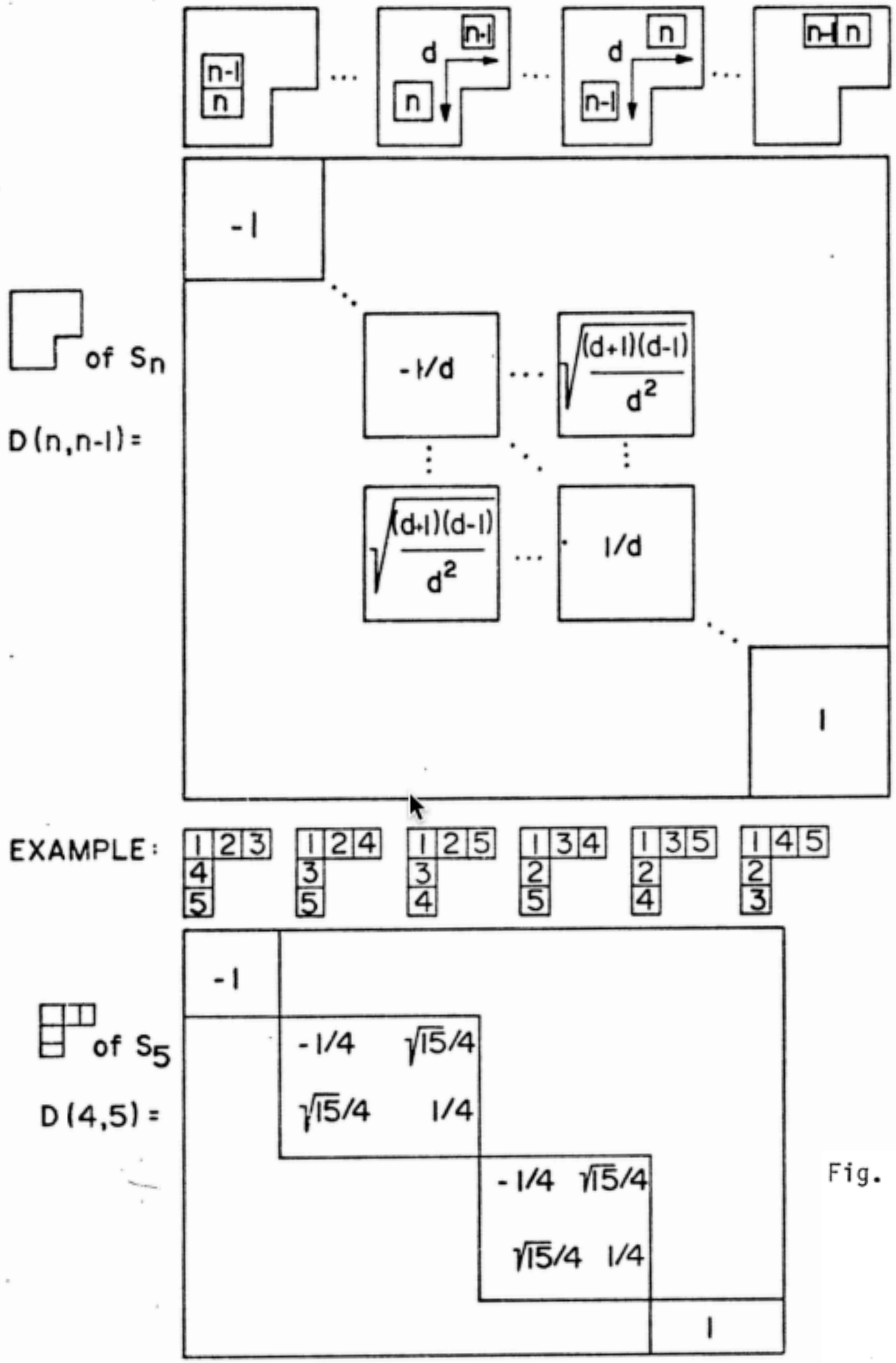
Yamanouchi formula for irrep of bicycle operation [*n,n-1*] *i.e.* [23].

(*following page)*

$$D^E[23]=\begin{array}{c|cc|} \begin{smallmatrix}\boxed{1}\boxed{2}\\ \boxed{3}\end{smallmatrix} & \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ \begin{smallmatrix}\boxed{1}\boxed{3}\\ \boxed{2}\end{smallmatrix} & \frac{\sqrt{3}}{2} & \frac{+1}{2} \end{array} \qquad D^E(bc)=\begin{array}{c|cc|} \begin{smallmatrix}\boxed{a}\boxed{b}\\ \boxed{c}\end{smallmatrix} & \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ \begin{smallmatrix}\boxed{a}\boxed{c}\\ \boxed{b}\end{smallmatrix} & \frac{\sqrt{3}}{2} & \frac{+1}{2} \end{array}$$



S_n projection for atomic spin and orbit states: Tableau P-operators



$$D^E_{(\sigma_2)} = D^{[2,1]}(bc) = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$$

(rows labelled by tableaux ab/c and ac/b)

$$D^{[2,1]}(ab) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(rows labelled by tableaux ab/c and ac/b)

From unpublished Ch.10 for
Principles of Symmetry, Dynamics & Spectroscopy

Fig. 10.1.2 Yamanouchi formulas for permutation operators. Integer d is the "city block" distance between (n) and (n-1) blocks, i.e., the minimum number of streets to be crossed when traveling from one to the other. Note that when numbers (n) and (n-1) are ordered smaller above larger, the permutation is negative (anti-symmetric if d=1), and positive (symmetric if d=1) when the smaller number is left of the larger number. [The (n-1) will never be above and left of (n) since that arrangement would be "non-standard."]

S_n projection for atomic spin and orbit states: Tableau P-operators

Dirac-ket-ket-ket product represents states 1, 2, 3 that variously occupy particles *a*, *b*, and *c*,

$$|1,2,3\rangle \equiv |1\rangle_{particle-a}|2\rangle_{particle-b}|3\rangle_{particle-c} \equiv |1\rangle_a|2\rangle_b|3\rangle_c$$

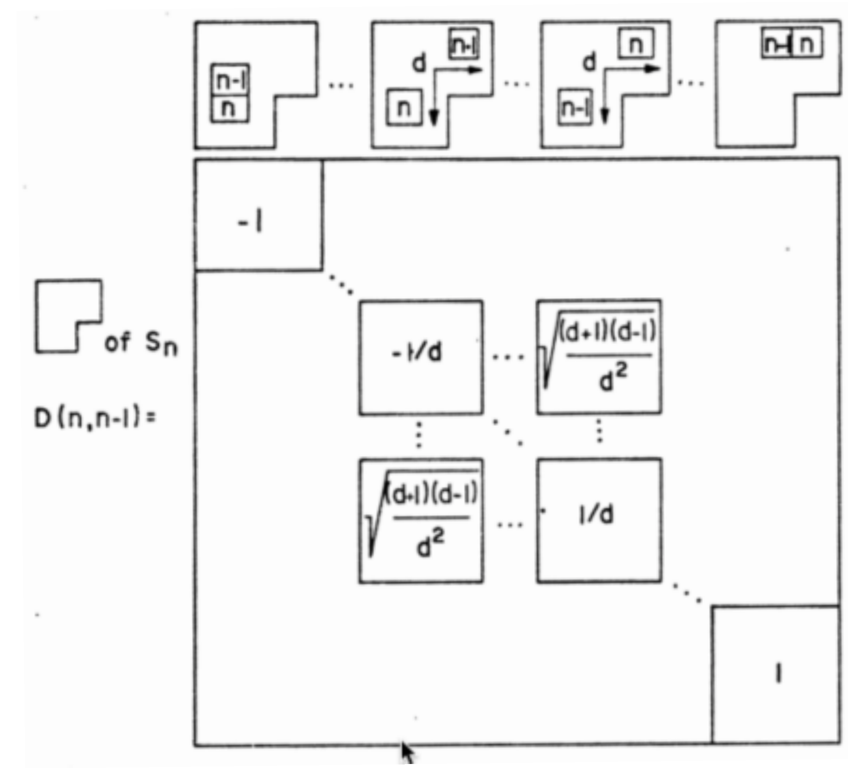
Ssub-tableaus $\boxed{a}\boxed{b}$ (or $\begin{array}{c}\boxed{a}\\\boxed{b}\end{array}$) label symmetry (anti-symmetry) by single row (or single column)

$$D^E[12]=\begin{array}{c}\boxed{1}\boxed{2}\\\boxed{3}\\\boxed{1}\boxed{3}\\\boxed{2}\end{array}\left|\begin{array}{cc}+1 & 0\\ 0 & -1\end{array}\right| \qquad D^E(ab)=\begin{array}{c}\boxed{a}\boxed{b}\\\boxed{c}\\\boxed{a}\boxed{c}\\\boxed{b}\end{array}\left|\begin{array}{cc}+1 & 0\\ 0 & -1\end{array}\right|$$

Yamanouchi formula for irrep of bicycle operation [*n,n-1*] *i.e.* [23].

(*following page)*

$$D^E[23]=\begin{array}{c}\boxed{1}\boxed{2}\\\boxed{3}\\\boxed{1}\boxed{3}\\\boxed{2}\end{array}\left|\begin{array}{cc}\frac{-1}{2} & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{+1}{2}\end{array}\right| \qquad D^E(bc)=\begin{array}{c}\boxed{a}\boxed{b}\\\boxed{c}\\\boxed{a}\boxed{c}\\\boxed{b}\end{array}\left|\begin{array}{cc}\frac{-1}{2} & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{+1}{2}\end{array}\right|$$



Gives complete set of permutation ireps and projectors.

$\mathbf{g}=$	$\mathbf{1}=(a)(b)(c)$	$\mathbf{r}=(abc)$	$\mathbf{r}^2=(acb)$	$\mathbf{i}_1=(bc)$	$\mathbf{i}_2=(ac)$	$\mathbf{i}_3=(ab)$
$D^{A_1}(\mathbf{g})=$	1	1	1	1	1	1
$D^{A_2}(\mathbf{g})=$	1	1	1	−1	−1	−1
$D^{E_1}_{x_2y_2}(\mathbf{g})=$	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}-1/2 & -\sqrt{3}/2\\\sqrt{3}/2 & -1/2\end{pmatrix}$	$\begin{pmatrix}-1/2 & \sqrt{3}/2\\-\sqrt{3}/2 & -1/2\end{pmatrix}$	$\begin{pmatrix}-1/2 & \sqrt{3}/2\\\sqrt{3}/2 & 1/2\end{pmatrix}$	$\begin{pmatrix}-1/2 & -\sqrt{3}/2\\-\sqrt{3}/2 & 1/2\end{pmatrix}$	$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$

S_n projection for atomic spin and orbit states: Tableau P-operators

Dirac-ket-ket-ket product represents states 1, 2, 3 that variously occupy particles *a*, *b*, and *c*,

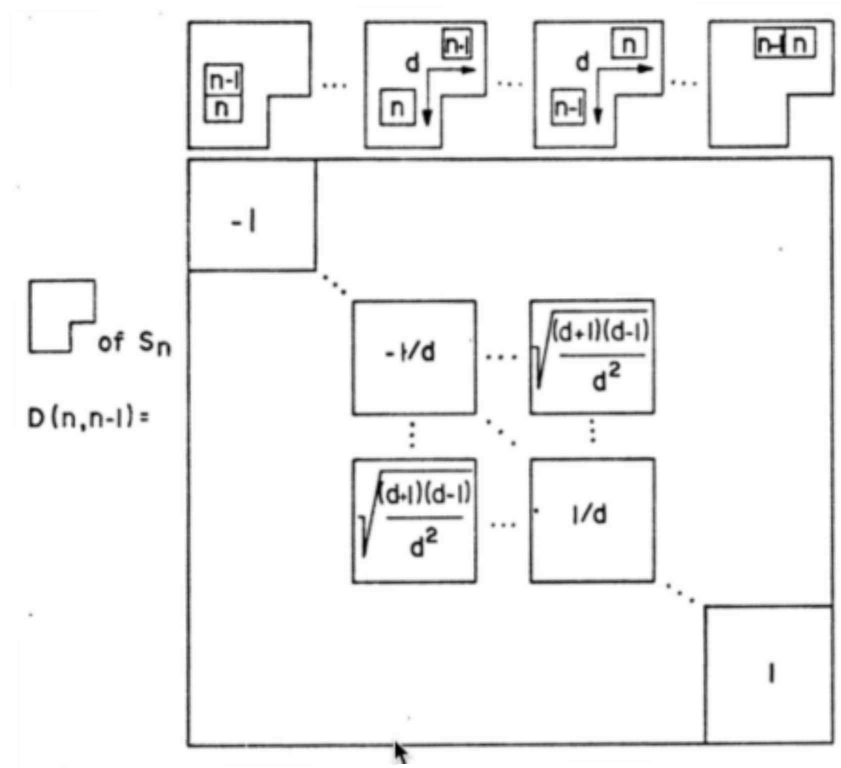
$$\left|1,2,3\right\rangle \equiv \left|1\right\rangle_{particle-a}\left|2\right\rangle_{particle-b}\left|3\right\rangle_{particle-c} \equiv \left|1\right\rangle_{a}\left|2\right\rangle_{b}\left|3\right\rangle_{c}$$

Ssub-tableaus $\boxed{a}\boxed{b}$ (or $\begin{array}{c}\boxed{a}\\\boxed{b}\end{array}$) label symmetry (anti-symmetry) by single row (or single column)

$$D^{E}[12]=\begin{array}{c|cc|} \begin{array}{c}\boxed{1}\boxed{2}\\\boxed{3}\end{array} & +1 & 0 \\ \begin{array}{c}\boxed{1}\boxed{3}\\\boxed{2}\end{array} & 0 & -1 \end{array} \qquad D^{E}(ab)=\begin{array}{c|cc|} \begin{array}{c}\boxed{a}\boxed{b}\\\boxed{c}\end{array} & +1 & 0 \\ \begin{array}{c}\boxed{a}\boxed{c}\\\boxed{b}\end{array} & 0 & -1 \end{array}$$

Yamanouchi formula for irrep of bicycle operation [*n,n-1*] *i.e.* [23].

$$D^{E}[23]=\begin{array}{c|cc|} \begin{array}{c}\boxed{1}\boxed{2}\\\boxed{3}\end{array} & \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ \begin{array}{c}\boxed{1}\boxed{3}\\\boxed{2}\end{array} & \frac{\sqrt{3}}{2} & \frac{+1}{2} \end{array} \qquad D^{E}(bc)=\begin{array}{c|cc|} \begin{array}{c}\boxed{a}\boxed{b}\\\boxed{c}\end{array} & \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ \begin{array}{c}\boxed{a}\boxed{c}\\\boxed{b}\end{array} & \frac{\sqrt{3}}{2} & \frac{+1}{2} \end{array}$$



Gives complete set of permutation ireps and projectors. (*following page)*

$\mathbf{g}=$	$\mathbf{1}=(a)(b)(c)$	$\mathbf{r}=(abc)$	$\mathbf{r}^2=(acb)$	$\mathbf{i}_1=(bc)$	$\mathbf{i}_2=(ac)$	$\mathbf{i}_3=(ab)$
$D^{A_1}(\mathbf{g})=$	1	1	1	1	1	1
$D^{A_2}(\mathbf{g})=$	1	1	1	−1	−1	−1
$D^{E_1}_{x_2y_2}(\mathbf{g})=$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}-1/2&-\sqrt{3}/2\\\sqrt{3}/2&-1/2\end{pmatrix}$	$\begin{pmatrix}-1/2&\sqrt{3}/2\\-\sqrt{3}/2&-1/2\end{pmatrix}$	$\begin{pmatrix}-1/2&\sqrt{3}/2\\\sqrt{3}/2&1/2\end{pmatrix}$	$\begin{pmatrix}-1/2&-\sqrt{3}/2\\-\sqrt{3}/2&1/2\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

$$\mathbf{P}^{[\mu]}_{j,k}\ \left|1\right\rangle\text{norm} = \sqrt{\frac{\ell^{[\mu]}}{{}^{\circ}G}}\left(D^{[\mu]}_{j,k}(1)\left|1\right\rangle + D(r)\left|\mathbf{r}\right\rangle + D(r^2)\left|\mathbf{r}^2\right\rangle + D(i_1)\left|\mathbf{i}_1\right\rangle + D(i_2)\left|\mathbf{i}_2\right\rangle + D(i_3)\left|\mathbf{i}_3\right\rangle\right)$$

S_n projection for atomic spin and orbit states: Tableau P-operators

$$\mathbf{P}^{[\mu]}_{j,k}\,|1\rangle\,\text{norm} = \sqrt{\frac{\ell^{[\mu]}}{{}^{\circ}G}}\left(D^{[\mu]}_{j,k}(1)|1\rangle + D(r)|\mathbf{r}\rangle + D(r^2)|\mathbf{r}^2\rangle + D(i_1)|\,\mathbf{i}_1\rangle + D(i_2)|\mathbf{i}_2\rangle + D(i_3)|\mathbf{i}_3\rangle\right)$$

$$\left|\begin{array}{c}\square\square\square\\ \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\end{array}\right\rangle = \mathbf{P}^{\square\square\square}_{\begin{array}{|c|c|c|}\hline a&b&c\\\hline\end{array}\;\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}}|1,2,3\rangle\sqrt{6} = \left(\frac{|1,2,3\rangle + |2,3,1\rangle + |3,1,2\rangle + |1,3,2\rangle + |3,2,1\rangle + |2,1,3\rangle}{\sqrt{6}}\right)$$

$$\left|\begin{array}{c}\begin{array}{c}\square\\\square\\\square\end{array}\\ \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\end{array}\right\rangle = \mathbf{P}^{\begin{array}{c}\square\\\square\\\square\end{array}}_{\begin{array}{|c|}\hline a\\\hline b\\\hline c\\\hline\end{array}\;\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}}|1,2,3\rangle\sqrt{6} = \left(\frac{|1,2,3\rangle + |2,3,1\rangle + |3,1,2\rangle + (-1)|1,3,2\rangle + (-1)|3,2,1\rangle + (-1)|2,1,3\rangle}{\sqrt{6}}\right)$$

$$\left|\begin{array}{c}\begin{array}{l}\square\square\\\square\end{array}\\ \begin{array}{|c|c|}\hline a&b\\\hline c\\\cline{1-1}\end{array}\;\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\end{array}\right\rangle = \mathbf{P}^{\begin{array}{l}\square\square\\\square\end{array}}_{\begin{array}{|c|c|}\hline a&b\\\hline c\\\cline{1-1}\end{array}\;\begin{array}{|c|c|}\hline a&b\\\hline c\\\cline{1-1}\end{array}}|1,2,3\rangle\sqrt{3} = \left(\frac{2|1,2,3\rangle + (-1)|2,3,1\rangle + (-1)|3,1,2\rangle + (-1)|1,3,2\rangle + (-1)|3,2,1\rangle + 2|2,1,3\rangle}{2\sqrt{3}}\right)$$

$$\left|\begin{array}{c}\begin{array}{l}\square\square\\\square\end{array}\\ \begin{array}{|c|c|}\hline a&c\\\hline b\\\cline{1-1}\end{array}\;\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\end{array}\right\rangle = \mathbf{P}^{\begin{array}{l}\square\square\\\square\end{array}}_{\begin{array}{|c|c|}\hline a&c\\\hline b\\\cline{1-1}\end{array}\;\begin{array}{|c|c|}\hline a&b\\\hline c\\\cline{1-1}\end{array}}|1,2,3\rangle\sqrt{3} = \left(\frac{0|1,2,3\rangle + (+1)|2,3,1\rangle + (-1)|3,1,2\rangle + (+1)|1,3,2\rangle + (-1)|3,2,1\rangle + 0|2,1,3\rangle}{2}\right)$$

$$\left|\begin{array}{c}\begin{array}{l}\square\square\\\square\end{array}\\ \begin{array}{|c|c|}\hline a&b\\\hline c\\\cline{1-1}\end{array}\;\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\end{array}\right\rangle = \mathbf{P}^{\begin{array}{l}\square\square\\\square\end{array}}_{\begin{array}{|c|c|}\hline a&b\\\hline c\\\cline{1-1}\end{array}\;\begin{array}{|c|c|}\hline a&c\\\hline b\\\cline{1-1}\end{array}}|1,2,3\rangle\sqrt{3} = \left(\frac{0|1,2,3\rangle + (-1)|2,3,1\rangle + (+1)|3,1,2\rangle + (+1)|1,3,2\rangle + (-1)|3,2,1\rangle + 0|2,1,3\rangle}{2}\right)$$

$$\left|\begin{array}{c}\begin{array}{l}\square\square\\\square\end{array}\\ \begin{array}{|c|c|}\hline a&c\\\hline b\\\cline{1-1}\end{array}\;\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\end{array}\right\rangle = \mathbf{P}^{\begin{array}{l}\square\square\\\square\end{array}}_{\begin{array}{|c|c|}\hline a&c\\\hline b\\\cline{1-1}\end{array}\;\begin{array}{|c|c|}\hline a&c\\\hline b\\\cline{1-1}\end{array}}|1,2,3\rangle\sqrt{3} = \left(\frac{2|1,2,3\rangle + (-1)|2,3,1\rangle + (-1)|3,1,2\rangle + (+1)|1,3,2\rangle + (+1)|3,2,1\rangle - 2|2,1,3\rangle}{2}\right)$$

particle (*abc*) labels [j] of $\mathbf{P}_{[j](k)}$ projectors face left

state (*123*) labels [k] face the state $|1,2,3\rangle$ on the right.

AMOP reference links on pages 2-4

4.16.18 class 23: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

(S_n)*(U(m)) shell model of electrostatic quadrupole-quadrupole-e interactions

Marrying spin s=½ and orbital ℓ=1 together: U(3)×U(2)

The ℓ=1 p=shell in a nutshell

U(6)⊃U(3)×U(2) approach: Coupling spin-orbit (s=½, ℓ=1) tableaus

Introducing atomic spin-orbit state assembly formula

Slater determinants

p-shell Spin-orbit calculations (not finished)

Clebsch Gordan coefficients. (Rev. Mod. Phys. annual gift)

S_n projection for atomic spin and orbit states

Review of Mach-Mock (particle-state) principle

Tableau P-operators on orbits (Yamonouchi formula)

➡ Tableau P-operators on spin

Fermi-Dirac-Pauli anti-symmetric p^3-states

Boson operators and symmetric p^2-states

Connecting to angular momentum

Projecting to angular momentum

S_n projection for atomic spin and orbit states: Tableau P-operators on spin

Projectors are applied to 3-electron spin states of which there are eight *($2^3=8$)*.

First is a single symmetric A_1 projection $\mathbf{P}^{A_1} = \mathbf{P}^{\square\square\square}$ of state $\left|\uparrow\uparrow\uparrow\right\rangle$

$$\left|\begin{matrix}\square\square\square & 3/2 \\ \boxed{\uparrow}\boxed{\uparrow}\boxed{\uparrow} & 3/2\end{matrix}\right\rangle = \mathbf{P}^{\square\square\square}_{\boxed{a}\boxed{b}\boxed{c}\ \boxed{\uparrow}\boxed{\uparrow}\boxed{\uparrow}}\left|\uparrow\uparrow\uparrow\right\rangle = \left|\uparrow\uparrow\uparrow\right\rangle$$

(Note $\mathbf{P}^{E_1} = \mathbf{P}^{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}$ acting on $\left|\uparrow\uparrow\uparrow\right\rangle$ is zero.)

S_n projection for atomic spin and orbit states: Tableau P-operators on spin

Projectors are applied to 3-electron spin states of which there are eight *($2^3=8$)*.

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(Note $\mathbf{P}^{E_1} = \mathbf{P}^{\begin{smallmatrix}\square\square\\ \square\ \end{smallmatrix}}$ acting on $\left|\uparrow\uparrow\uparrow\right\rangle$ is zero.)

Anti symmetric A_2 projection fails on *all* spin-1/2 states

$$\left| \begin{array}{c} \square \\ \square \\ \square \\ \boxed{\uparrow} \\ \boxed{\uparrow} \\ \boxed{\uparrow} \end{array} \right\rangle = \mathbf{P}^{\begin{smallmatrix}\square\\ \square\\ \square\end{smallmatrix}}_{\begin{smallmatrix}\boxed{a}\\ \boxed{b}\\ \boxed{c}\end{smallmatrix}\ \begin{smallmatrix}\boxed{\uparrow}\\ \boxed{\uparrow}\\ \boxed{\uparrow}\end{smallmatrix}} \left\| \uparrow,\uparrow,\uparrow \right\rangle\!\right\rangle = 0 \quad \textit{(Does not exist)}, \left| \begin{array}{c} \square \\ \square \\ \square \\ \boxed{\uparrow} \\ \boxed{\uparrow} \\ \boxed{\downarrow} \end{array} \right\rangle = \mathbf{P}^{\begin{smallmatrix}\square\\ \square\\ \square\end{smallmatrix}}_{\begin{smallmatrix}\boxed{a}\\ \boxed{b}\\ \boxed{c}\end{smallmatrix}\ \begin{smallmatrix}\boxed{\uparrow}\\ \boxed{\uparrow}\\ \boxed{\downarrow}\end{smallmatrix}} \left\| \uparrow,\uparrow,\downarrow \right\rangle\!\right\rangle = 0 \quad \textit{(Does not exist),...etc.}$$

Symmetric $\mathbf{P}^{A_1} = \mathbf{P}^{\square\square\square}$ or para-symmetric $\mathbf{P}^{E_1} = \mathbf{P}^{\begin{smallmatrix}\square\square\\ \square\ \end{smallmatrix}}$ projection of $\left|\uparrow\uparrow\downarrow\right\rangle$ and $\left|\uparrow\downarrow\downarrow\right\rangle$ give $\left| \begin{smallmatrix} S=3/2 \\ M=\pm 1/2 \end{smallmatrix} \right\rangle$ or $\left| \begin{smallmatrix} S=1/2 \\ M=\pm 1/2 \end{smallmatrix} \right\rangle$.

S_n projection for atomic spin and orbit states: Tableau P-operators on spin

Projectors are applied to 3-electron spin states of which there are eight $(2^3=8)$.

First is a single symmetric A_1 projection $\mathbf{P}^{A_1} = \mathbf{P}^{\square\square\square}$ of state $\left|\uparrow\uparrow\uparrow\right\rangle$

$$\left|\begin{matrix}\square\square\square & 3/2\\ \boxed{\uparrow}\boxed{\uparrow}\boxed{\uparrow} & 3/2\end{matrix}\right\rangle = \mathbf{P}^{\square\square\square}_{\boxed{a}\boxed{b}\boxed{c}\;\boxed{\uparrow}\boxed{\uparrow}\boxed{\uparrow}}\left|\uparrow\uparrow\uparrow\right\rangle = \left|\uparrow\uparrow\uparrow\right\rangle$$

(Note $\mathbf{P}^{E_1} = \mathbf{P}^{\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}}$ acting on $\left|\uparrow\uparrow\uparrow\right\rangle$ is zero.)

Anti symmetric A_2 projection fails on *all* spin-1/2 states

$$\left|\begin{matrix}\square\\ \square\\ \square\\ \boxed{\uparrow}\\ \boxed{\uparrow}\\ \boxed{\uparrow}\end{matrix}\right\rangle = \mathbf{P}^{\begin{smallmatrix}\square\\ \square\\ \square\end{smallmatrix}}_{\begin{smallmatrix}\boxed{a}\\ \boxed{b}\\ \boxed{c}\end{smallmatrix}\begin{smallmatrix}\boxed{\uparrow}\\ \boxed{\uparrow}\\ \boxed{\uparrow}\end{smallmatrix}}\left|\left|\uparrow,\uparrow,\uparrow\right\rangle\right\rangle = 0 \quad \textit{(Does not exist)}, \left|\begin{matrix}\square\\ \square\\ \square\\ \boxed{\uparrow}\\ \boxed{\uparrow}\\ \boxed{\downarrow}\end{matrix}\right\rangle = \mathbf{P}^{\begin{smallmatrix}\square\\ \square\\ \square\end{smallmatrix}}_{\begin{smallmatrix}\boxed{a}\\ \boxed{b}\\ \boxed{c}\end{smallmatrix}\begin{smallmatrix}\boxed{\uparrow}\\ \boxed{\uparrow}\\ \boxed{\downarrow}\end{smallmatrix}}\left|\left|\uparrow,\uparrow,\downarrow\right\rangle\right\rangle = 0 \quad \textit{(Does not exist),...etc.}$$

Symmetric $\mathbf{P}^{A_1} = \mathbf{P}^{\square\square\square}$ or para-symmetric $\mathbf{P}^{E_1} = \mathbf{P}^{\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}}$ projection of $\left|\uparrow\uparrow\downarrow\right\rangle$ and $\left|\uparrow\downarrow\downarrow\right\rangle$ give $\left|\begin{smallmatrix}S=3/2\\ M=\pm1/2\end{smallmatrix}\right\rangle$ or $\left|\begin{smallmatrix}S=1/2\\ M=\pm1/2\end{smallmatrix}\right\rangle$.

$$\left|\begin{matrix}\square\square\square & 3/2\\ \boxed{\uparrow}\boxed{\uparrow}\boxed{\downarrow} & 1/2\end{matrix}\right\rangle = \mathbf{P}^{\square\square\square}_{\boxed{a}\boxed{b}\boxed{c}\;\boxed{a}\boxed{b}\boxed{c}}\left|\uparrow,\uparrow,\downarrow\right\rangle\sqrt{3} = \left(\frac{\left|\uparrow,\uparrow,\downarrow\right\rangle + \left|\uparrow,\downarrow,\uparrow\right\rangle + \left|\downarrow,\uparrow,\uparrow\right\rangle + \left|\uparrow,\downarrow,\uparrow\right\rangle + \left|\downarrow,\uparrow,\uparrow\right\rangle + \left|\uparrow,\uparrow,\downarrow\right\rangle}{\sqrt{6}}\right) = \left(\frac{\left|\uparrow,\uparrow,\downarrow\right\rangle + \left|\uparrow,\downarrow,\uparrow\right\rangle + \left|\downarrow,\uparrow,\uparrow\right\rangle}{\sqrt{3}}\right)$$

$$\left|\begin{matrix}\begin{smallmatrix}\square\square\\ \square\end{smallmatrix} & 1/2\\ \begin{smallmatrix}\boxed{a}\boxed{b}\\ \boxed{c}\end{smallmatrix}\begin{smallmatrix}\boxed{\uparrow}\boxed{\uparrow}\\ \boxed{\downarrow}\end{smallmatrix} & 1/2\end{matrix}\right\rangle = \mathbf{P}^{\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}}_{\begin{smallmatrix}\boxed{a}\boxed{b}\\ \boxed{c}\end{smallmatrix}\begin{smallmatrix}\boxed{a}\boxed{b}\\ \boxed{c}\end{smallmatrix}}\left|\uparrow,\uparrow,\downarrow\right\rangle\sqrt{\frac{3}{2}} = \left(\frac{2\left|\uparrow,\uparrow,\downarrow\right\rangle+(-1)\left|\uparrow,\downarrow,\uparrow\right\rangle+(-1)\left|\downarrow,\uparrow,\uparrow\right\rangle+(-1)\left|\uparrow,\downarrow,\uparrow\right\rangle+(-1)\left|\downarrow,\uparrow,\uparrow\right\rangle+2\left|\uparrow,\uparrow,\downarrow\right\rangle}{2\sqrt{6}}\right) = \left(\frac{2\left|\uparrow,\uparrow,\downarrow\right\rangle+(-1)\left|\uparrow,\downarrow,\uparrow\right\rangle+(-1)\left|\downarrow,\uparrow,\uparrow\right\rangle}{\sqrt{6}}\right)$$

$$\left|\begin{matrix}\begin{smallmatrix}\square\square\\ \square\end{smallmatrix} & 1/2\\ \begin{smallmatrix}\boxed{a}\boxed{c}\\ \boxed{b}\end{smallmatrix}\begin{smallmatrix}\boxed{\uparrow}\boxed{\uparrow}\\ \boxed{\downarrow}\end{smallmatrix} & 1/2\end{matrix}\right\rangle = \mathbf{P}^{\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}}_{\begin{smallmatrix}\boxed{a}\boxed{c}\\ \boxed{b}\end{smallmatrix}\begin{smallmatrix}\boxed{a}\boxed{b}\\ \boxed{c}\end{smallmatrix}}\left|\uparrow,\uparrow,\downarrow\right\rangle\sqrt{\frac{3}{2}} = \left(\frac{0\left|\uparrow,\uparrow,\downarrow\right\rangle+(+1)\left|\uparrow,\downarrow,\uparrow\right\rangle+(-1)\left|\downarrow,\uparrow,\uparrow\right\rangle+(+1)\left|\uparrow,\downarrow,\uparrow\right\rangle+(-1)\left|\downarrow,\uparrow,\uparrow\right\rangle+0\left|\uparrow,\uparrow,\downarrow\right\rangle}{2\sqrt{2}}\right) = \left(\frac{(+1)\left|\uparrow,\downarrow,\uparrow\right\rangle+(-1)\left|\downarrow,\uparrow,\uparrow\right\rangle}{\sqrt{2}}\right)$$

The latter make a permutation doublet.

There are two spin-$S=1/2$ states $\left|\begin{smallmatrix}S=1/2\\ M=\pm1/2\end{smallmatrix}\right\rangle$ but only one spin-$S=3/2$ state $\left|\begin{smallmatrix}S=3/2\\ M=\pm1/2\end{smallmatrix}\right\rangle$ have z-component $M=+1/2$.

S_n projection for atomic spin and orbit states: Tableau P-operators on spin

Projectors are applied to 3-electron spin states of which there are eight *($2^3=8$)*.

First is a single symmetric A_1 projection $\mathbf{P}^{A_1} = \mathbf{P}^{\square\square\square}$ of state $\left|\uparrow\uparrow\uparrow\right\rangle$

$$\left|\begin{matrix}\square\square\square & 3/2 \\ \boxed{\uparrow}\boxed{\uparrow}\boxed{\uparrow} & 3/2\end{matrix}\right\rangle = \mathbf{P}^{\square\square\square}_{\boxed{a}\boxed{b}\boxed{c}\ \boxed{\uparrow}\boxed{\uparrow}\boxed{\uparrow}}\left|\uparrow\uparrow\uparrow\right\rangle = \left|\uparrow\uparrow\uparrow\right\rangle \qquad \text{(Note } \mathbf{P}^{E_1} = \mathbf{P}^{\substack{\square\square\\ \square}} \text{ acting on } \left|\uparrow\uparrow\uparrow\right\rangle \text{ is zero.)}$$

Anti symmetric A_2 projection fails on *all* spin-1/2 states

$$\left|\begin{matrix}\square\\ \square\\ \square\\ \\ \boxed{\uparrow}\\ \boxed{\uparrow}\\ \boxed{\uparrow}\end{matrix}\right\rangle = \mathbf{P}^{\substack{\square\\ \square\\ \square}}_{\substack{\boxed{a}\\ \boxed{b}\\ \boxed{c}}\ \substack{\boxed{\uparrow}\\ \boxed{\uparrow}\\ \boxed{\uparrow}}}\left|\left|\uparrow,\uparrow,\uparrow\right\rangle\right\rangle = 0 \quad \textit{(Does not exist)}, \left|\begin{matrix}\square\\ \square\\ \square\\ \\ \boxed{\uparrow}\\ \boxed{\uparrow}\\ \boxed{\downarrow}\end{matrix}\right\rangle = \mathbf{P}^{\substack{\square\\ \square\\ \square}}_{\substack{\boxed{a}\\ \boxed{b}\\ \boxed{c}}\ \substack{\boxed{\uparrow}\\ \boxed{\uparrow}\\ \boxed{\downarrow}}}\left|\left|\uparrow,\uparrow,\downarrow\right\rangle\right\rangle = 0 \quad \textit{(Does not exist),...etc.}$$

Symmetric $\mathbf{P}^{A_1} = \mathbf{P}^{\square\square\square}$ and para-symmetric $\mathbf{P}^{E_1} = \mathbf{P}^{\substack{\square\square\\ \square}}$ projection of $\left|\uparrow\uparrow\downarrow\right\rangle$ and $\left|\uparrow\downarrow\downarrow\right\rangle$ give $\left|\substack{S=3/2\\ M=\pm 1/2}\right\rangle$ and $\left|\substack{S=1/2\\ M=\pm 1/2}\right\rangle$.

$$\left|\begin{matrix}\square\square\square & 3/2 \\ \boxed{\uparrow}\boxed{\uparrow}\boxed{\downarrow} & 1/2\end{matrix}\right\rangle = \mathbf{P}^{\square\square\square}_{\boxed{a}\boxed{b}\boxed{c}\ \boxed{a}\boxed{b}\boxed{c}}\left|\uparrow,\uparrow,\downarrow\right\rangle\sqrt{3} = \left(\frac{\left|\uparrow,\uparrow,\downarrow\right\rangle + \left|\uparrow,\downarrow,\uparrow\right\rangle + \left|\downarrow,\uparrow,\uparrow\right\rangle + \left|\uparrow,\downarrow,\uparrow\right\rangle + \left|\downarrow,\uparrow,\uparrow\right\rangle + \left|\uparrow,\uparrow,\downarrow\right\rangle}{\sqrt{6}}\right) = \left(\frac{\left|\uparrow,\uparrow,\downarrow\right\rangle + \left|\uparrow,\downarrow,\uparrow\right\rangle + \left|\downarrow,\uparrow,\uparrow\right\rangle}{\sqrt{3}}\right)$$

$$\left|\begin{matrix}\substack{\square\square\\ \square} & 1/2 \\ \substack{\boxed{a}\boxed{b}\\ \boxed{c}}\ \substack{\boxed{\uparrow}\boxed{\uparrow}\\ \boxed{\downarrow}} & 1/2\end{matrix}\right\rangle = \mathbf{P}^{\substack{\square\square\\ \square}}_{\substack{\boxed{a}\boxed{b}\\ \boxed{c}}\ \substack{\boxed{a}\boxed{b}\\ \boxed{c}}}\left|\uparrow,\uparrow,\downarrow\right\rangle\sqrt{\frac{3}{2}} = \left(\frac{2\left|\uparrow,\uparrow,\downarrow\right\rangle + (-1)\left|\uparrow,\downarrow,\uparrow\right\rangle + (-1)\left|\downarrow,\uparrow,\uparrow\right\rangle + (-1)\left|\uparrow,\downarrow,\uparrow\right\rangle + (-1)\left|\downarrow,\uparrow,\uparrow\right\rangle + 2\left|\uparrow,\uparrow,\downarrow\right\rangle}{2\sqrt{6}}\right) = \left(\frac{2\left|\uparrow,\uparrow,\downarrow\right\rangle + (-1)\left|\uparrow,\downarrow,\uparrow\right\rangle + (-1)\left|\downarrow,\uparrow,\uparrow\right\rangle}{\sqrt{6}}\right)$$

$$\left|\begin{matrix}\substack{\square\square\\ \square} & 1/2 \\ \substack{\boxed{a}\boxed{c}\\ \boxed{b}}\ \substack{\boxed{\uparrow}\boxed{\uparrow}\\ \boxed{\downarrow}} & 1/2\end{matrix}\right\rangle = \mathbf{P}^{\substack{\square\square\\ \square}}_{\substack{\boxed{a}\boxed{c}\\ \boxed{b}}\ \substack{\boxed{a}\boxed{b}\\ \boxed{c}}}\left|\uparrow,\uparrow,\downarrow\right\rangle\sqrt{\frac{3}{2}} = \left(\frac{0\left|\uparrow,\uparrow,\downarrow\right\rangle + (+1)\left|\uparrow,\downarrow,\uparrow\right\rangle + (-1)\left|\downarrow,\uparrow,\uparrow\right\rangle + (+1)\left|\uparrow,\downarrow,\uparrow\right\rangle + (-1)\left|\downarrow,\uparrow,\uparrow\right\rangle + 0\left|\uparrow,\uparrow,\downarrow\right\rangle}{2\sqrt{2}}\right) = \left(\frac{(+1)\left|\uparrow,\downarrow,\uparrow\right\rangle + (-1)\left|\downarrow,\uparrow,\uparrow\right\rangle}{\sqrt{2}}\right)$$

The latter make a permutation doublet.

There are two spin-$S=1/2$ states $\left|\substack{S=1/2\\ M=\pm 1/2}\right\rangle$ but only one spin-$S=3/2$ state $\left|\substack{S=3/2\\ M=\pm 1/2}\right\rangle$ have z-component $M=+1/2$.

All 3 states project from $\left|\uparrow\uparrow\downarrow\right\rangle$. The left [j]-labels of the last two make a particle doublet $\left\{\substack{\boxed{a}\boxed{b}\\ \boxed{c}}\ \ \substack{\boxed{a}\boxed{c}\\ \boxed{b}}\right\}$.

S_n projection for atomic spin and orbit states: Tableau P-operators on spin

Symmetric $\mathbf{P}^{A_1} = \mathbf{P}^{\square\square\square}$ and para-symmetric $\mathbf{P}^{E_1} = \mathbf{P}^{\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}}$ projection of $|\uparrow\uparrow\downarrow\rangle$ and $|\uparrow\downarrow\downarrow\rangle$ give $\left|\begin{smallmatrix}S=3/2\\ M=\pm 1/2\end{smallmatrix}\right\rangle$ and $\left|\begin{smallmatrix}S=1/2\\ M=\pm 1/2\end{smallmatrix}\right\rangle$.

$$\left|\begin{smallmatrix}\square\square\square & 3/2\\ \boxed{\uparrow}\boxed{\uparrow}\boxed{\downarrow} & 1/2\end{smallmatrix}\right\rangle = \mathbf{P}^{\square\square\square}_{\boxed{a}\boxed{b}\boxed{c}\;\boxed{a}\boxed{b}\boxed{c}}|\uparrow,\uparrow,\downarrow\rangle\sqrt{3} = \left(\frac{|\uparrow,\uparrow,\downarrow\rangle+|\uparrow,\downarrow,\uparrow\rangle+|\downarrow,\uparrow,\uparrow\rangle+|\uparrow,\downarrow,\uparrow\rangle+|\downarrow,\uparrow,\uparrow\rangle+|\uparrow,\uparrow,\downarrow\rangle}{\sqrt{6}}\right) = \left(\frac{|\uparrow,\uparrow,\downarrow\rangle+|\uparrow,\downarrow,\uparrow\rangle+|\downarrow,\uparrow,\uparrow\rangle}{\sqrt{3}}\right)$$

$$\left|\begin{smallmatrix}\square\square & \\ \square & 1/2\\ \begin{smallmatrix}\boxed{a}\boxed{b}\\ \boxed{c}\end{smallmatrix}\begin{smallmatrix}\boxed{\uparrow}\boxed{\uparrow}\\ \boxed{\downarrow}\end{smallmatrix} & 1/2\end{smallmatrix}\right\rangle = \mathbf{P}^{\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}}_{\begin{smallmatrix}\boxed{a}\boxed{b}\\ \boxed{c}\end{smallmatrix}\;\begin{smallmatrix}\boxed{a}\boxed{b}\\ \boxed{c}\end{smallmatrix}}|\uparrow,\uparrow,\downarrow\rangle\sqrt{\frac{3}{2}} = \left(\frac{2|\uparrow,\uparrow,\downarrow\rangle+(-1)|\uparrow,\downarrow,\uparrow\rangle+(-1)|\downarrow,\uparrow,\uparrow\rangle+(-1)|\uparrow,\downarrow,\uparrow\rangle+(-1)|\downarrow,\uparrow,\uparrow\rangle+2|\uparrow,\uparrow,\downarrow\rangle}{2\sqrt{6}}\right) = \left(\frac{2|\uparrow,\uparrow,\downarrow\rangle+(-1)|\uparrow,\downarrow,\uparrow\rangle+(-1)|\downarrow,\uparrow,\uparrow\rangle}{\sqrt{6}}\right)$$

$$\left|\begin{smallmatrix}\square\square & \\ \square & 1/2\\ \begin{smallmatrix}\boxed{a}\boxed{c}\\ \boxed{b}\end{smallmatrix}\begin{smallmatrix}\boxed{\uparrow}\boxed{\uparrow}\\ \boxed{\downarrow}\end{smallmatrix} & 1/2\end{smallmatrix}\right\rangle = \mathbf{P}^{\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}}_{\begin{smallmatrix}\boxed{a}\boxed{c}\\ \boxed{b}\end{smallmatrix}\;\begin{smallmatrix}\boxed{a}\boxed{b}\\ \boxed{c}\end{smallmatrix}}|\uparrow,\uparrow,\downarrow\rangle\sqrt{\frac{3}{2}} = \left(\frac{0|\uparrow,\uparrow,\downarrow\rangle+(+1)|\uparrow,\downarrow,\uparrow\rangle+(-1)|\downarrow,\uparrow,\uparrow\rangle+(+1)|\uparrow,\downarrow,\uparrow\rangle+(-1)|\downarrow,\uparrow,\uparrow\rangle+0|\uparrow,\uparrow,\downarrow\rangle}{2\sqrt{2}}\right) = \left(\frac{(+1)|\uparrow,\downarrow,\uparrow\rangle+(-1)|\downarrow,\uparrow,\uparrow\rangle}{\sqrt{2}}\right)$$

The latter make a permutation doublet.

There are two spin-$S=1/2$ states $\left|\begin{smallmatrix}S=1/2\\ M=\pm 1/2\end{smallmatrix}\right\rangle$ but only one spin-$S=3/2$ state $\left|\begin{smallmatrix}S=3/2\\ M=\pm 1/2\end{smallmatrix}\right\rangle$ have z-component $M=+1/2$.

S_n projection for atomic spin and orbit states (Top 3 lines moved up.)

Symmetric $\mathbf{P}^{A_1} = \mathbf{P}^{\square\square\square}$ and para-symmetric $\mathbf{P}^{E_1} = \mathbf{P}^{\substack{\square\square\\ \square}}$ projection of $|\uparrow\uparrow\downarrow\rangle$ and $|\uparrow\downarrow\downarrow\rangle$ give $\left|\begin{smallmatrix} S=3/2\\ M=\pm 1/2\end{smallmatrix}\right\rangle$ and $\left|\begin{smallmatrix} S=1/2\\ M=\pm 1/2\end{smallmatrix}\right\rangle$.

$$\left|\begin{matrix}\square\square\square & 3/2\\ \boxed{\uparrow}\boxed{\uparrow}\boxed{\downarrow} & 1/2\end{matrix}\right\rangle = \mathbf{P}^{\square\square\square}_{\boxed{a}\boxed{b}\boxed{c}\ \boxed{a}\boxed{b}\boxed{c}}|\uparrow,\uparrow,\downarrow\rangle\sqrt{3} = \left(\frac{|\uparrow,\uparrow,\downarrow\rangle + |\uparrow,\downarrow,\uparrow\rangle + |\downarrow,\uparrow,\uparrow\rangle + |\uparrow,\downarrow,\uparrow\rangle + |\downarrow,\uparrow,\uparrow\rangle + |\uparrow,\uparrow,\downarrow\rangle}{\sqrt{6}}\right) = \left(\frac{|\uparrow,\uparrow,\downarrow\rangle + |\uparrow,\downarrow,\uparrow\rangle + |\downarrow,\uparrow,\uparrow\rangle}{\sqrt{3}}\right)$$

$$\left|\begin{matrix}\substack{\square\square\\ \square} & \\ & 1/2\\ \substack{\boxed{a}\boxed{b}\\ \boxed{c}}\ \substack{\boxed{\uparrow}\boxed{\uparrow}\\ \boxed{\downarrow}} & 1/2\end{matrix}\right\rangle = \mathbf{P}^{\substack{\square\square\\ \square}}_{\substack{\boxed{a}\boxed{b}\\ \boxed{c}}\ \substack{\boxed{a}\boxed{b}\\ \boxed{c}}}|\uparrow,\uparrow,\downarrow\rangle\sqrt{\frac{3}{2}} = \left(\frac{2|\uparrow,\uparrow,\downarrow\rangle + (-1)|\uparrow,\downarrow,\uparrow\rangle + (-1)|\downarrow,\uparrow,\uparrow\rangle + (-1)|\uparrow,\downarrow,\uparrow\rangle + (-1)|\downarrow,\uparrow,\uparrow\rangle + 2|\uparrow,\uparrow,\downarrow\rangle}{2\sqrt{6}}\right) = \left(\frac{2|\uparrow,\uparrow,\downarrow\rangle + (-1)|\uparrow,\downarrow,\uparrow\rangle + (-1)|\downarrow,\uparrow,\uparrow\rangle}{\sqrt{6}}\right)$$

$$\left|\begin{matrix}\substack{\square\square\\ \square} & \\ & 1/2\\ \substack{\boxed{a}\boxed{c}\\ \boxed{b}}\ \substack{\boxed{\uparrow}\boxed{\uparrow}\\ \boxed{\downarrow}} & 1/2\end{matrix}\right\rangle = \mathbf{P}^{\substack{\square\square\\ \square}}_{\substack{\boxed{a}\boxed{c}\\ \boxed{b}}\ \substack{\boxed{a}\boxed{b}\\ \boxed{c}}}|\uparrow,\uparrow,\downarrow\rangle\sqrt{\frac{3}{2}} = \left(\frac{0|\uparrow,\uparrow,\downarrow\rangle + (+1)|\uparrow,\downarrow,\uparrow\rangle + (-1)|\downarrow,\uparrow,\uparrow\rangle + (+1)|\uparrow,\downarrow,\uparrow\rangle + (-1)|\downarrow,\uparrow,\uparrow\rangle + 0|\uparrow,\uparrow,\downarrow\rangle}{2\sqrt{2}}\right) = \left(\frac{(+1)|\uparrow,\downarrow,\uparrow\rangle + (-1)|\downarrow,\uparrow,\uparrow\rangle}{\sqrt{2}}\right)$$

The latter make a permutation doublet.

There are two spin-$S=1/2$ states $\left|\begin{smallmatrix} S=1/2\\ M=\pm 1/2\end{smallmatrix}\right\rangle$ but only one spin-$S=3/2$ state $\left|\begin{smallmatrix} S=3/2\\ M=\pm 1/2\end{smallmatrix}\right\rangle$ have z-component $M=+1/2$.

All 3 states project from $|\uparrow\uparrow\downarrow\rangle$. The left [j]-labels of the last two make a particle doublet $\left\{\substack{\boxed{a}\boxed{b}\\ \boxed{c}}\quad \substack{\boxed{a}\boxed{c}\\ \boxed{b}}\right\}$.

S_n projection for atomic spin and orbit states: Tableau P-operators on spin

Symmetric $\mathbf{P}^{A_1} = \mathbf{P}^{\square\square\square}$ and para-symmetric $\mathbf{P}^{E_1} = \mathbf{P}^{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}$ projection of $|\uparrow\uparrow\downarrow\rangle$ and $|\uparrow\downarrow\downarrow\rangle$ give $\left|\begin{smallmatrix}S=3/2\\M=\pm 1/2\end{smallmatrix}\right\rangle$ and $\left|\begin{smallmatrix}S=1/2\\M=\pm 1/2\end{smallmatrix}\right\rangle$.

$$\left|\begin{smallmatrix}\square\square\square & 3/2\\ \boxed{\uparrow}\boxed{\uparrow}\boxed{\downarrow} & 1/2\end{smallmatrix}\right\rangle = \mathbf{P}^{\square\square\square}_{\boxed{a}\boxed{b}\boxed{c}\;\boxed{a}\boxed{b}\boxed{c}}|\uparrow,\uparrow,\downarrow\rangle\sqrt{3} = \left(\frac{|\uparrow,\uparrow,\downarrow\rangle + |\uparrow,\downarrow,\uparrow\rangle + |\downarrow,\uparrow,\uparrow\rangle + |\uparrow,\downarrow,\uparrow\rangle + |\downarrow,\uparrow,\uparrow\rangle + |\uparrow,\uparrow,\downarrow\rangle}{\sqrt{6}}\right) = \left(\frac{|\uparrow,\uparrow,\downarrow\rangle + |\uparrow,\downarrow,\uparrow\rangle + |\downarrow,\uparrow,\uparrow\rangle}{\sqrt{3}}\right)$$

$$\left|\begin{smallmatrix}\square\square & \\ \square & 1/2\\ \begin{smallmatrix}a&b\\c&\end{smallmatrix}\;\begin{smallmatrix}\uparrow&\uparrow\\\downarrow&\end{smallmatrix} & 1/2\end{smallmatrix}\right\rangle = \mathbf{P}^{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}_{\begin{smallmatrix}a&b\\c&\end{smallmatrix}\;\begin{smallmatrix}a&b\\c&\end{smallmatrix}}|\uparrow,\uparrow,\downarrow\rangle\sqrt{\frac{3}{2}} = \left(\frac{2|\uparrow,\uparrow,\downarrow\rangle+(-1)|\uparrow,\downarrow,\uparrow\rangle+(-1)|\downarrow,\uparrow,\uparrow\rangle+(-1)|\uparrow,\downarrow,\uparrow\rangle+(-1)|\downarrow,\uparrow,\uparrow\rangle+2|\uparrow,\uparrow,\downarrow\rangle}{2\sqrt{6}}\right) = \left(\frac{2|\uparrow,\uparrow,\downarrow\rangle+(-1)|\uparrow,\downarrow,\uparrow\rangle+(-1)|\downarrow,\uparrow,\uparrow\rangle}{\sqrt{6}}\right)$$

$$\left|\begin{smallmatrix}\square\square & \\ \square & 1/2\\ \begin{smallmatrix}a&c\\b&\end{smallmatrix}\;\begin{smallmatrix}\uparrow&\uparrow\\\downarrow&\end{smallmatrix} & 1/2\end{smallmatrix}\right\rangle = \mathbf{P}^{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}_{\begin{smallmatrix}a&c\\b&\end{smallmatrix}\;\begin{smallmatrix}a&b\\c&\end{smallmatrix}}|\uparrow,\uparrow,\downarrow\rangle\sqrt{\frac{3}{2}} = \left(\frac{0|\uparrow,\uparrow,\downarrow\rangle+(+1)|\uparrow,\downarrow,\uparrow\rangle+(-1)|\downarrow,\uparrow,\uparrow\rangle+(+1)|\uparrow,\downarrow,\uparrow\rangle+(-1)|\downarrow,\uparrow,\uparrow\rangle+0|\uparrow,\uparrow,\downarrow\rangle}{2\sqrt{2}}\right) = \left(\frac{(+1)|\uparrow,\downarrow,\uparrow\rangle+(-1)|\downarrow,\uparrow,\uparrow\rangle}{\sqrt{2}}\right)$$

The latter make a permutation doublet.

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All 3 states project from $|\uparrow\uparrow\downarrow\rangle$. The left [j]-labels of the last two make a particle doublet $\left\{\begin{smallmatrix}a&b\\c&\end{smallmatrix}\;\;\begin{smallmatrix}a&c\\b&\end{smallmatrix}\right\}$.

State $|\uparrow\uparrow\downarrow\rangle = \mathbf{P}^{\square\square}|\uparrow\uparrow\downarrow\rangle$ is invariant to symmetric subgroup projector $\mathbf{P}^{\square\square} = [1+(ab)]/2$ but $\mathbf{P}^{\square\square}$ zeros $\begin{smallmatrix}a&c\\b&\end{smallmatrix}$.

$$\left.\begin{array}{l}(ab)|\uparrow,\uparrow,\downarrow\rangle = |\uparrow,\uparrow,\downarrow\rangle\\[2ex] \mathbf{P}^{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}_{\begin{smallmatrix}a&b\\c&\end{smallmatrix}\;\begin{smallmatrix}a&c\\b&\end{smallmatrix}}(ab) = -\mathbf{P}^{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}_{\begin{smallmatrix}a&b\\c&\end{smallmatrix}\;\begin{smallmatrix}a&c\\b&\end{smallmatrix}}\end{array}\right\} implies: \mathbf{P}^{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}_{\begin{smallmatrix}a&b\\c&\end{smallmatrix}\;\begin{smallmatrix}a&c\\b&\end{smallmatrix}}|\uparrow,\uparrow,\downarrow\rangle = -\mathbf{P}^{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}_{\begin{smallmatrix}a&b\\c&\end{smallmatrix}\;\begin{smallmatrix}a&c\\b&\end{smallmatrix}}|\uparrow,\uparrow,\downarrow\rangle = 0$$

S_n projection for atomic spin and orbit states: Tableau P-operators on spin

Symmetric $\mathbf{P}^{A_1}=\mathbf{P}^{\square\square\square}$ and para-symmetric $\mathbf{P}^{E_1}=\mathbf{P}^{\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}}$ projection of $\left|\uparrow\uparrow\downarrow\right\rangle$ and $\left|\uparrow\downarrow\downarrow\right\rangle$ give $\left|\begin{smallmatrix}S=3/2\\M=\pm1/2\end{smallmatrix}\right\rangle$ and $\left|\begin{smallmatrix}S=1/2\\M=\pm1/2\end{smallmatrix}\right\rangle$.

$$\left|\begin{smallmatrix}\square\square\square & 3/2\\ \boxed{\uparrow}\boxed{\uparrow}\boxed{\downarrow} & 1/2\end{smallmatrix}\right\rangle=\mathbf{P}^{\square\square\square}_{\boxed{a}\boxed{b}\boxed{c}\;\boxed{a}\boxed{b}\boxed{c}}\left|\uparrow,\uparrow,\downarrow\right\rangle\sqrt{3}=\left(\frac{\left|\uparrow,\uparrow,\downarrow\right\rangle+\left|\uparrow,\downarrow,\uparrow\right\rangle+\left|\downarrow,\uparrow,\uparrow\right\rangle+\left|\uparrow,\downarrow,\uparrow\right\rangle+\left|\downarrow,\uparrow,\uparrow\right\rangle+\left|\uparrow,\uparrow,\downarrow\right\rangle}{\sqrt{6}}\right)=\left(\frac{\left|\uparrow,\uparrow,\downarrow\right\rangle+\left|\uparrow,\downarrow,\uparrow\right\rangle+\left|\downarrow,\uparrow,\uparrow\right\rangle}{\sqrt{3}}\right)$$

$$\left|\begin{smallmatrix}\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix} & 1/2\\ \begin{smallmatrix}\boxed{a}&\boxed{b}\\\boxed{c}&\end{smallmatrix}\begin{smallmatrix}\boxed{\uparrow}&\boxed{\uparrow}\\\boxed{\downarrow}&\end{smallmatrix} & 1/2\end{smallmatrix}\right\rangle=\mathbf{P}^{\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}}_{\begin{smallmatrix}\boxed{a}&\boxed{b}\\\boxed{c}&\end{smallmatrix}\begin{smallmatrix}\boxed{a}&\boxed{b}\\\boxed{c}&\end{smallmatrix}}\left|\uparrow,\uparrow,\downarrow\right\rangle\sqrt{\frac{3}{2}}=\left(\frac{2\left|\uparrow,\uparrow,\downarrow\right\rangle+(-1)\left|\uparrow,\downarrow,\uparrow\right\rangle+(-1)\left|\downarrow,\uparrow,\uparrow\right\rangle+(-1)\left|\uparrow,\downarrow,\uparrow\right\rangle+(-1)\left|\downarrow,\uparrow,\uparrow\right\rangle+2\left|\uparrow,\uparrow,\downarrow\right\rangle}{2\sqrt{6}}\right)=\left(\frac{2\left|\uparrow,\uparrow,\downarrow\right\rangle+(-1)\left|\uparrow,\downarrow,\uparrow\right\rangle+(-1)\left|\downarrow,\uparrow,\uparrow\right\rangle}{\sqrt{6}}\right)$$

$$\left|\begin{smallmatrix}\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix} & 1/2\\ \begin{smallmatrix}\boxed{a}&\boxed{c}\\\boxed{b}&\end{smallmatrix}\begin{smallmatrix}\boxed{\uparrow}&\boxed{\uparrow}\\\boxed{\downarrow}&\end{smallmatrix} & 1/2\end{smallmatrix}\right\rangle=\mathbf{P}^{\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}}_{\begin{smallmatrix}\boxed{a}&\boxed{c}\\\boxed{b}&\end{smallmatrix}\begin{smallmatrix}\boxed{a}&\boxed{b}\\\boxed{c}&\end{smallmatrix}}\left|\uparrow,\uparrow,\downarrow\right\rangle\sqrt{\frac{3}{2}}=\left(\frac{0\left|\uparrow,\uparrow,\downarrow\right\rangle+(+1)\left|\uparrow,\downarrow,\uparrow\right\rangle+(-1)\left|\downarrow,\uparrow,\uparrow\right\rangle+(+1)\left|\uparrow,\downarrow,\uparrow\right\rangle+(-1)\left|\downarrow,\uparrow,\uparrow\right\rangle+0\left|\uparrow,\uparrow,\downarrow\right\rangle}{2\sqrt{2}}\right)=\left(\frac{(+1)\left|\uparrow,\downarrow,\uparrow\right\rangle+(-1)\left|\downarrow,\uparrow,\uparrow\right\rangle}{\sqrt{2}}\right)$$

The latter make a permutation doublet. Similarly, projections of $\left|\uparrow\downarrow\downarrow\right\rangle$ give three M=*-1/2* states.

$$\left|\begin{smallmatrix}\square\square\square & 3/2\\ \boxed{\uparrow}\boxed{\uparrow}\boxed{\uparrow} & 1/2\end{smallmatrix}\right\rangle=\mathbf{P}^{\square\square\square}_{\boxed{a}\boxed{b}\boxed{c}\;\boxed{a}\boxed{b}\boxed{c}}\left|\uparrow,\downarrow,\downarrow\right\rangle\sqrt{3}=\left(\frac{\left|\uparrow,\downarrow,\downarrow\right\rangle+\left|\downarrow,\downarrow,\uparrow\right\rangle+\left|\downarrow,\uparrow,\downarrow\right\rangle+\left|\uparrow,\downarrow,\downarrow\right\rangle+\left|\downarrow,\downarrow,\uparrow\right\rangle+\left|\downarrow,\uparrow,\downarrow\right\rangle}{2\sqrt{3}}\right)=\left(\frac{\left|\uparrow,\downarrow,\downarrow\right\rangle+\left|\downarrow,\uparrow,\downarrow\right\rangle+\left|\uparrow,\downarrow,\downarrow\right\rangle}{\sqrt{3}}\right)$$

$$\left|\begin{smallmatrix}\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix} & 1/2\\ \begin{smallmatrix}\boxed{a}&\boxed{b}\\\boxed{c}&\end{smallmatrix}\begin{smallmatrix}\boxed{\uparrow}&\boxed{\downarrow}\\\boxed{\downarrow}&\end{smallmatrix} & -1/2\end{smallmatrix}\right\rangle=\mathbf{P}^{\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}}_{\begin{smallmatrix}\boxed{a}&\boxed{b}\\\boxed{c}&\end{smallmatrix}\begin{smallmatrix}\boxed{a}&\boxed{b}\\\boxed{c}&\end{smallmatrix}}\left|\uparrow,\downarrow,\downarrow\right\rangle\sqrt{6}=\left(\frac{2\left|\uparrow,\downarrow,\downarrow\right\rangle+(-1)\left|\downarrow,\downarrow,\uparrow\right\rangle+(-1)\left|\downarrow,\uparrow,\downarrow\right\rangle+(-1)\left|\uparrow,\downarrow,\downarrow\right\rangle+(-1)\left|\downarrow,\downarrow,\uparrow\right\rangle+2\left|\downarrow,\uparrow,\downarrow\right\rangle}{\sqrt{6}}\right)=\left(\frac{1\left|\uparrow,\downarrow,\downarrow\right\rangle+(+1)\left|\downarrow,\uparrow,\downarrow\right\rangle+(-2)\left|\downarrow,\downarrow,\uparrow\right\rangle}{\sqrt{6}}\right)$$

$$\left|\begin{smallmatrix}\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix} & 1/2\\ \begin{smallmatrix}\boxed{a}&\boxed{c}\\\boxed{b}&\end{smallmatrix}\begin{smallmatrix}\boxed{\uparrow}&\boxed{\downarrow}\\\boxed{\downarrow}&\end{smallmatrix} & -1/2\end{smallmatrix}\right\rangle=\mathbf{P}^{\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}}_{\begin{smallmatrix}\boxed{a}&\boxed{c}\\\boxed{b}&\end{smallmatrix}\begin{smallmatrix}\boxed{a}&\boxed{b}\\\boxed{c}&\end{smallmatrix}}\left|\uparrow,\downarrow,\downarrow\right\rangle\sqrt{6}=\left(\frac{0\left|\uparrow,\downarrow,\downarrow\right\rangle+(+1)\left|\downarrow,\downarrow,\uparrow\right\rangle+(-1)\left|\downarrow,\uparrow,\downarrow\right\rangle+(+1)\left|\uparrow,\downarrow,\downarrow\right\rangle+(-1)\left|\downarrow,\downarrow,\uparrow\right\rangle+0\left|\downarrow,\uparrow,\downarrow\right\rangle}{\sqrt{2}}\right)=\left(\frac{(+1)\left|\uparrow,\downarrow,\downarrow\right\rangle+(-1)\left|\downarrow,\uparrow,\downarrow\right\rangle}{\sqrt{2}}\right)$$

S_n projection for atomic spin and orbit states: Tableau P-operators on spin

Symmetric $\mathbf{P}^{A_1} = \mathbf{P}^{\square\square\square}$ and para-symmetric $\mathbf{P}^{E_1} = \mathbf{P}^{\begin{smallmatrix}\square&\square\\\square\end{smallmatrix}}$ projection of $\left|\uparrow\uparrow\downarrow\right\rangle$ and $\left|\uparrow\downarrow\downarrow\right\rangle$ give $\left|\begin{smallmatrix}S=3/2\\M=\pm1/2\end{smallmatrix}\right\rangle$ and $\left|\begin{smallmatrix}S=1/2\\M=\pm1/2\end{smallmatrix}\right\rangle$.

$$\left|\begin{smallmatrix}\square\square\square & 3/2\\ \boxed{\uparrow}\boxed{\uparrow}\boxed{\downarrow} & 1/2\end{smallmatrix}\right\rangle = \mathbf{P}^{\square\square\square}_{\boxed{a}\boxed{b}\boxed{c}\ \boxed{a}\boxed{b}\boxed{c}}\left|\uparrow,\uparrow,\downarrow\right\rangle\sqrt{3} = \left(\frac{\left|\uparrow,\uparrow,\downarrow\right\rangle+\left|\uparrow,\downarrow,\uparrow\right\rangle+\left|\downarrow,\uparrow,\uparrow\right\rangle+\left|\uparrow,\downarrow,\uparrow\right\rangle+\left|\downarrow,\uparrow,\uparrow\right\rangle+\left|\uparrow,\uparrow,\downarrow\right\rangle}{\sqrt{6}}\right) = \left(\frac{\left|\uparrow,\uparrow,\downarrow\right\rangle+\left|\uparrow,\downarrow,\uparrow\right\rangle+\left|\downarrow,\uparrow,\uparrow\right\rangle}{\sqrt{3}}\right)$$

$$\left|\begin{smallmatrix}\square\square & \\ \square & 1/2\\ \begin{smallmatrix}a&b\\c\end{smallmatrix}\ \begin{smallmatrix}\uparrow&\uparrow\\\downarrow\end{smallmatrix} & 1/2\end{smallmatrix}\right\rangle = \mathbf{P}^{\begin{smallmatrix}\square&\square\\\square\end{smallmatrix}}_{\begin{smallmatrix}a&b\\c\end{smallmatrix}\ \begin{smallmatrix}a&b\\c\end{smallmatrix}}\left|\uparrow,\uparrow,\downarrow\right\rangle\sqrt{\frac{3}{2}} = \left(\frac{2\left|\uparrow,\uparrow,\downarrow\right\rangle+(-1)\left|\uparrow,\downarrow,\uparrow\right\rangle+(-1)\left|\downarrow,\uparrow,\uparrow\right\rangle+(-1)\left|\uparrow,\downarrow,\uparrow\right\rangle+(-1)\left|\downarrow,\uparrow,\uparrow\right\rangle+2\left|\uparrow,\uparrow,\downarrow\right\rangle}{2\sqrt{6}}\right) = \left(\frac{2\left|\uparrow,\uparrow,\downarrow\right\rangle+(-1)\left|\uparrow,\downarrow,\uparrow\right\rangle+(-1)\left|\downarrow,\uparrow,\uparrow\right\rangle}{\sqrt{6}}\right)$$

$$\left|\begin{smallmatrix}\square\square & \\ \square & 1/2\\ \begin{smallmatrix}a&c\\b\end{smallmatrix}\ \begin{smallmatrix}\uparrow&\uparrow\\\downarrow\end{smallmatrix} & 1/2\end{smallmatrix}\right\rangle = \mathbf{P}^{\begin{smallmatrix}\square&\square\\\square\end{smallmatrix}}_{\begin{smallmatrix}a&c\\b\end{smallmatrix}\ \begin{smallmatrix}a&b\\c\end{smallmatrix}}\left|\uparrow,\uparrow,\downarrow\right\rangle\sqrt{\frac{3}{2}} = \left(\frac{0\left|\uparrow,\uparrow,\downarrow\right\rangle+(+1)\left|\uparrow,\downarrow,\uparrow\right\rangle+(-1)\left|\downarrow,\uparrow,\uparrow\right\rangle+(+1)\left|\uparrow,\downarrow,\uparrow\right\rangle+(-1)\left|\downarrow,\uparrow,\uparrow\right\rangle+0\left|\uparrow,\uparrow,\downarrow\right\rangle}{2\sqrt{2}}\right) = \left(\frac{(+1)\left|\uparrow,\downarrow,\uparrow\right\rangle+(-1)\left|\downarrow,\uparrow,\uparrow\right\rangle}{\sqrt{2}}\right)$$

The latter make a permutation doublet. Similarly, projections of $\left|\uparrow\downarrow\downarrow\right\rangle$ give three M=-1/2 states.

$$\left|\begin{smallmatrix}\square\square\square & 3/2\\ \boxed{\uparrow}\boxed{\uparrow}\boxed{\uparrow} & 1/2\end{smallmatrix}\right\rangle = \mathbf{P}^{\square\square\square}_{\boxed{a}\boxed{b}\boxed{c}\ \boxed{a}\boxed{b}\boxed{c}}\left|\uparrow,\downarrow,\downarrow\right\rangle\sqrt{3} = \left(\frac{\left|\uparrow,\downarrow,\downarrow\right\rangle+\left|\downarrow,\downarrow,\uparrow\right\rangle+\left|\downarrow,\uparrow,\downarrow\right\rangle+\left|\uparrow,\downarrow,\downarrow\right\rangle+\left|\downarrow,\downarrow,\uparrow\right\rangle+\left|\downarrow,\uparrow,\downarrow\right\rangle}{2\sqrt{3}}\right) = \left(\frac{\left|\uparrow,\downarrow,\downarrow\right\rangle+\left|\downarrow,\uparrow,\downarrow\right\rangle+\left|\uparrow,\downarrow,\downarrow\right\rangle}{\sqrt{3}}\right)$$

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$$\left|\begin{smallmatrix}\square\square & \\ \square & 1/2\\ \begin{smallmatrix}a&c\\b\end{smallmatrix}\ \begin{smallmatrix}\uparrow&\downarrow\\\downarrow\end{smallmatrix} & -1/2\end{smallmatrix}\right\rangle = \mathbf{P}^{\begin{smallmatrix}\square&\square\\\square\end{smallmatrix}}_{\begin{smallmatrix}a&c\\b\end{smallmatrix}\ \begin{smallmatrix}a&b\\c\end{smallmatrix}}\left|\uparrow,\downarrow,\downarrow\right\rangle\sqrt{6} = \left(\frac{0\left|\uparrow,\downarrow,\downarrow\right\rangle+(+1)\left|\downarrow,\downarrow,\uparrow\right\rangle+(-1)\left|\downarrow,\uparrow,\downarrow\right\rangle+(+1)\left|\uparrow,\downarrow,\downarrow\right\rangle+(-1)\left|\downarrow,\downarrow,\uparrow\right\rangle+0\left|\downarrow,\uparrow,\downarrow\right\rangle}{\sqrt{2}}\right) = \left(\frac{(+1)\left|\uparrow,\downarrow,\downarrow\right\rangle+(-1)\left|\downarrow,\uparrow,\downarrow\right\rangle}{\sqrt{2}}\right)$$

Finally, the fourth state of the spin-S=3/2 quartet is the following M=-3/2 .

$$\left|\begin{smallmatrix}\square\square\square & 3/2\\ \boxed{\downarrow}\boxed{\downarrow}\boxed{\downarrow} & -3/2\end{smallmatrix}\right\rangle = \mathbf{P}^{\square\square\square}_{\boxed{a}\boxed{b}\boxed{c}\ \boxed{\downarrow}\boxed{\downarrow}\boxed{\downarrow}}\left|\downarrow,\downarrow,\downarrow\right\rangle = \left|\downarrow,\downarrow,\downarrow\right\rangle$$

S_n projection for atomic spin and orbit states: Tableau P-operators on spin

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$$\left|\begin{smallmatrix}\square\square\square & 3/2\\ \boxed{\uparrow}\boxed{\uparrow}\boxed{\downarrow} & 1/2\end{smallmatrix}\right\rangle = \mathbf{P}^{\square\square\square}_{\boxed{a}\boxed{b}\boxed{c}\ \boxed{a}\boxed{b}\boxed{c}}|\uparrow,\uparrow,\downarrow\rangle\sqrt{3} = \left(\frac{|\uparrow,\uparrow,\downarrow\rangle + |\uparrow,\downarrow,\uparrow\rangle + |\downarrow,\uparrow,\uparrow\rangle + |\uparrow,\downarrow,\uparrow\rangle + |\downarrow,\uparrow,\uparrow\rangle + |\uparrow,\uparrow,\downarrow\rangle}{\sqrt{6}}\right) = \left(\frac{|\uparrow,\uparrow,\downarrow\rangle + |\uparrow,\downarrow,\uparrow\rangle + |\downarrow,\uparrow,\uparrow\rangle}{\sqrt{3}}\right)$$

$$\left|\begin{smallmatrix}\square\square & & \\ \square & & 1/2\\ \boxed{a}\boxed{b} & \boxed{\uparrow}\boxed{\uparrow} & 1/2\\ \boxed{c} & \boxed{\downarrow} & \end{smallmatrix}\right\rangle = \mathbf{P}^{\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}}_{\begin{smallmatrix}\boxed{a}\boxed{b}\\ \boxed{c}\end{smallmatrix}\ \begin{smallmatrix}\boxed{a}\boxed{b}\\ \boxed{c}\end{smallmatrix}}|\uparrow,\uparrow,\downarrow\rangle\sqrt{\frac{3}{2}} = \left(\frac{2|\uparrow,\uparrow,\downarrow\rangle + (-1)|\uparrow,\downarrow,\uparrow\rangle + (-1)|\downarrow,\uparrow,\uparrow\rangle + (-1)|\uparrow,\downarrow,\uparrow\rangle + (-1)|\downarrow,\uparrow,\uparrow\rangle + 2|\uparrow,\uparrow,\downarrow\rangle}{2\sqrt{6}}\right) = \left(\frac{2|\uparrow,\uparrow,\downarrow\rangle + (-1)|\uparrow,\downarrow,\uparrow\rangle + (-1)|\downarrow,\uparrow,\uparrow\rangle}{\sqrt{6}}\right)$$

$$\left|\begin{smallmatrix}\square\square & & \\ \square & & 1/2\\ \boxed{a}\boxed{c} & \boxed{\uparrow}\boxed{\uparrow} & 1/2\\ \boxed{b} & \boxed{\downarrow} & \end{smallmatrix}\right\rangle = \mathbf{P}^{\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}}_{\begin{smallmatrix}\boxed{a}\boxed{c}\\ \boxed{b}\end{smallmatrix}\ \begin{smallmatrix}\boxed{a}\boxed{b}\\ \boxed{c}\end{smallmatrix}}|\uparrow,\uparrow,\downarrow\rangle\sqrt{\frac{3}{2}} = \left(\frac{0|\uparrow,\uparrow,\downarrow\rangle + (+1)|\uparrow,\downarrow,\uparrow\rangle + (-1)|\downarrow,\uparrow,\uparrow\rangle + (+1)|\uparrow,\downarrow,\uparrow\rangle + (-1)|\downarrow,\uparrow,\uparrow\rangle + 0|\uparrow,\uparrow,\downarrow\rangle}{2\sqrt{2}}\right) = \left(\frac{(+1)|\uparrow,\downarrow,\uparrow\rangle + (-1)|\downarrow,\uparrow,\uparrow\rangle}{\sqrt{2}}\right)$$

The latter make a permutation doublet. Similarly, projections of $|\uparrow\downarrow\downarrow\rangle$ give three M=-1/2 states.

$$\left|\begin{smallmatrix}\square\square\square & 3/2\\ \boxed{\uparrow}\boxed{\uparrow}\boxed{\uparrow} & 1/2\end{smallmatrix}\right\rangle = \mathbf{P}^{\square\square\square}_{\boxed{a}\boxed{b}\boxed{c}\ \boxed{a}\boxed{b}\boxed{c}}|\uparrow,\downarrow,\downarrow\rangle\sqrt{3} = \left(\frac{|\uparrow,\downarrow,\downarrow\rangle + |\downarrow,\downarrow,\uparrow\rangle + |\downarrow,\uparrow,\downarrow\rangle + |\uparrow,\downarrow,\downarrow\rangle + |\downarrow,\downarrow,\uparrow\rangle + |\downarrow,\uparrow,\downarrow\rangle}{2\sqrt{3}}\right) = \left(\frac{|\uparrow,\downarrow,\downarrow\rangle + |\downarrow,\uparrow,\downarrow\rangle + |\uparrow,\downarrow,\downarrow\rangle}{\sqrt{3}}\right)$$

$$\left|\begin{smallmatrix}\square\square & & \\ \square & & 1/2\\ \boxed{a}\boxed{b} & \boxed{\uparrow}\boxed{\downarrow} & -1/2\\ \boxed{c} & \boxed{\downarrow} & \end{smallmatrix}\right\rangle = \mathbf{P}^{\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}}_{\begin{smallmatrix}\boxed{a}\boxed{b}\\ \boxed{c}\end{smallmatrix}\ \begin{smallmatrix}\boxed{a}\boxed{b}\\ \boxed{c}\end{smallmatrix}}|\uparrow,\downarrow,\downarrow\rangle\sqrt{6} = \left(\frac{2|\uparrow,\downarrow,\downarrow\rangle + (-1)|\downarrow,\downarrow,\uparrow\rangle + (-1)|\downarrow,\uparrow,\downarrow\rangle + (-1)|\uparrow,\downarrow,\downarrow\rangle + (-1)|\downarrow,\downarrow,\uparrow\rangle + 2|\downarrow,\uparrow,\downarrow\rangle}{\sqrt{6}}\right) = \left(\frac{1|\uparrow,\downarrow,\downarrow\rangle + (+1)|\downarrow,\uparrow,\downarrow\rangle + (-2)|\downarrow,\downarrow,\uparrow\rangle}{\sqrt{6}}\right)$$

$$\left|\begin{smallmatrix}\square\square & & \\ \square & & 1/2\\ \boxed{a}\boxed{c} & \boxed{\uparrow}\boxed{\downarrow} & -1/2\\ \boxed{b} & \boxed{\downarrow} & \end{smallmatrix}\right\rangle = \mathbf{P}^{\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}}_{\begin{smallmatrix}\boxed{a}\boxed{c}\\ \boxed{b}\end{smallmatrix}\ \begin{smallmatrix}\boxed{a}\boxed{b}\\ \boxed{c}\end{smallmatrix}}|\uparrow,\downarrow,\downarrow\rangle\sqrt{6} = \left(\frac{0|\uparrow,\downarrow,\downarrow\rangle + (+1)|\downarrow,\downarrow,\uparrow\rangle + (-1)|\downarrow,\uparrow,\downarrow\rangle + (+1)|\uparrow,\downarrow,\downarrow\rangle + (-1)|\downarrow,\downarrow,\uparrow\rangle + 0|\downarrow,\uparrow,\downarrow\rangle}{\sqrt{2}}\right) = \left(\frac{(+1)|\uparrow,\downarrow,\downarrow\rangle + (-1)|\downarrow,\uparrow,\downarrow\rangle}{\sqrt{2}}\right)$$

Finally, the fourth state of the spin-S=3/2 quartet is the following M=-3/2 .

$$\left|\begin{smallmatrix}\square\square\square & 3/2\\ \boxed{\downarrow}\boxed{\downarrow}\boxed{\downarrow} & -3/2\end{smallmatrix}\right\rangle = \mathbf{P}^{\square\square\square}_{\boxed{a}\boxed{b}\boxed{c}\ \boxed{\downarrow}\boxed{\downarrow}\boxed{\downarrow}}|\downarrow,\downarrow,\downarrow\rangle = |\downarrow,\downarrow,\downarrow\rangle$$

Right index correlates *state-permutaion*-symmetry, that is, whether two spins are equal.

Left index correlates *particle-permutaion*-symmetry, that is, whether two particles are the same or not.

AMOP reference links on pages 2-4

4.16.18 class 23: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

$(S_n)^*(U(m))$ shell model of electrostatic quadrupole-quadrupole-e interactions

Marrying spin $s=½$ and orbital $\ell=1$ together: U(3)×U(2)

The $\ell=1$ p=shell in a nutshell

U(6)⊃U(3)×U(2) approach: Coupling spin-orbit ($s=½$, $\ell=1$) tableaus

Introducing atomic spin-orbit state assembly formula

Slater determinants

p-shell Spin-orbit calculations (not finished)

Clebsch Gordan coefficients. (Rev. Mod. Phys. annual gift)

S_n projection for atomic spin and orbit states

Review of Mach-Mock (particle-state) principle

Tableau P-operators on orbits (Yamonouchi formula)

Tableau P-operators on spin

➡ Fermi-Dirac-Pauli anti-symmetric p^3-states

Boson operators and symmetric p^2-states

Connecting to angular momentum

Projecting to angular momentum

Fermi-Dirac-Pauli anti-symmetric p^3-states

Orbital-tableau states (10 pages above) are combined using S_N-Clebsch-Gordan coefficients (S_NCGC) with spin-tableau states (1 page above) to make Pauli-allowed spin-orbit states.

In the following simplest case the (S_3CGC) sum is a single term for each state in the ^{4}S quartet.

$$\left| p^3 \; {}^4S_{\boxed{\uparrow}\boxed{\uparrow}\boxed{\uparrow}\; \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}} \begin{matrix} S{=}3/2 \\ M_S{=}3/2 \end{matrix} \right\rangle = \left| \boxed{\uparrow}\boxed{\uparrow}\boxed{\uparrow} \right\rangle \left| \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array} \right\rangle, \quad \left| \begin{matrix} 3/2 \\ 1/2 \end{matrix} \right\rangle = \left| \boxed{\uparrow}\boxed{\uparrow}\boxed{\downarrow} \right\rangle \left| \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array} \right\rangle, \quad \left| \begin{matrix} 3/2 \\ -1/2 \end{matrix} \right\rangle = \left| \boxed{\uparrow}\boxed{\downarrow}\boxed{\downarrow} \right\rangle \left| \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array} \right\rangle, \quad \left| \begin{matrix} 3/2 \\ -3/2 \end{matrix} \right\rangle = \left| \boxed{\downarrow}\boxed{\downarrow}\boxed{\downarrow} \right\rangle \left| \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array} \right\rangle$$

Fermi-Dirac-Pauli anti-symmetric p^3-states

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So are *eight* orbital doublet pairs: a *tableau octet* of Pauli-ok unitary *U(3)* $\ell^{E_1}=8$ *multiplicity* E_1-orbitals.

$$U(3)\text{ octet tableau basis:}\left\{ \begin{array}{l}\boxed{1}\boxed{1}\\\boxed{2}\end{array}, \begin{array}{l}\boxed{1}\boxed{2}\\\boxed{2}\end{array}, \begin{array}{l}\boxed{1}\boxed{2}\\\boxed{3}\end{array}, \begin{array}{l}\boxed{1}\boxed{3}\\\boxed{2}\end{array}, \begin{array}{l}\boxed{1}\boxed{1}\\\boxed{3}\end{array}, \begin{array}{l}\boxed{1}\boxed{3}\\\boxed{3}\end{array}, \begin{array}{l}\boxed{2}\boxed{2}\\\boxed{3}\end{array}, \begin{array}{l}\boxed{2}\boxed{3}\\\boxed{3}\end{array}, \right\}$$

AMOP reference links on pages 2-4

4.16.18 class 23: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

$(S_n)*(U(m))$ shell model of electrostatic quadrupole-quadrupole-e interactions

Marrying spin $s=\frac{1}{2}$ and orbital $\ell=1$ together: U(3)×U(2)
- The $\ell=1$ p=shell in a nutshell

U(6)⊃U(3)×U(2) approach: Coupling spin-orbit ($s=\frac{1}{2}$, $\ell=1$) tableaus

Introducing atomic spin-orbit state assembly formula
- Slater determinants
- p-shell Spin-orbit calculations (not finished)

Clebsch Gordan coefficients. (Rev. Mod. Phys. annual gift)

S_n projection for atomic spin and orbit states
- Review of Mach-Mock (particle-state) principle
- Tableau P-operators on orbits (Yamonouchi formula)
- Tableau P-operators on spin

Fermi-Dirac-Pauli anti-symmetric p^3-states

➡ Boson operators and symmetric p^2-states
- Connecting to angular momentum
- Projecting to angular momentum

Boson operators and symmetric p^2-states

First non-trivial application of elementary creation-destruction pairs is to the [2,0] sextet states

$$\left\{ \left| \boxed{1}\boxed{1} \right\rangle, \left| \boxed{1}\boxed{2} \right\rangle, \left| \boxed{1}\boxed{3} \right\rangle, \left| \boxed{2}\boxed{2} \right\rangle, \left| \boxed{2}\boxed{3} \right\rangle, \left| \boxed{3}\boxed{3} \right\rangle, \right\}$$

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$$E_{12}\left|n_1,n_2\right\rangle = a_1\bar{a}_2\left|n_1,n_2\right\rangle = a_1\sqrt{n_2}\left|n_1,n_2-1\right\rangle = \sqrt{n_1+1}\sqrt{n_2}\left|n_1+1,n_2-1\right\rangle$$

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Elementary operations e_{jk} apply to each particle a, b, c, and so forth in turn.

$$E_{23}\left|3_a 3_b 3_c\right\rangle = \left|2_a 3_b 3_c\right\rangle + \left|3_a 2_b 3_c\right\rangle + \left|3_a 3_b 2_c\right\rangle = \sqrt{3}\,\frac{\left|2_a 3_b 3_c\right\rangle + \left|3_a 2_b 3_c\right\rangle + \left|3_a 3_b 2\right\rangle}{\sqrt{3}} = \sqrt{3}\left|\boxed{2}\boxed{3}\boxed{3}\right\rangle$$

$$a_2\bar{a}_3\left|n_1 = 0, n_2 = 0, n_3 = 3\right\rangle = a_2\sqrt{3}\left|0,0,2\right\rangle = \sqrt{1}\sqrt{3}\left|0,1,2\right\rangle = E_{23}\left|\boxed{3}\boxed{3}\boxed{3}\right\rangle = \sqrt{3}\left|\boxed{2}\boxed{3}\boxed{3}\right\rangle$$

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$$E_{23}\left|n_1,n_2,n_3\right\rangle = a_2\bar{a}_3\left|n_1,n_2,n_3\right\rangle = a_2\sqrt{n_3}\left|n_1,n_2,n_3-1\right\rangle = \sqrt{n_2+1}\sqrt{n_3}\left|n_1,n_2+1,n_3-1\right\rangle$$

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$$E_{23}\left|3_a3_b3_c\right\rangle = \left|2_a3_b3_c\right\rangle + \left|3_a2_b3_c\right\rangle + \left|3_a3_b2_c\right\rangle = \sqrt{3}\frac{\left|2_a3_b3_c\right\rangle + \left|3_a2_b3_c\right\rangle + \left|3_a3_b2\right\rangle}{\sqrt{3}} = \sqrt{3}\left|\boxed{2}\boxed{3}\boxed{3}\right\rangle$$

$$a_2\bar{a}_3\left|n_1=0,n_2=0,n_3=3\right\rangle = a_2\sqrt{3}\left|0,0,2\right\rangle = \sqrt{1}\sqrt{3}\left|0,1,2\right\rangle = E_{23}\left|\boxed{3}\boxed{3}\boxed{3}\right\rangle = \sqrt{3}\left|\boxed{2}\boxed{3}\boxed{3}\right\rangle$$

The e_{jk} procedure shows $a = \mathbf{a}^\dagger$ or $\bar{a} = \mathbf{a}$ factors $\sqrt{n_k}$ or $\sqrt{n_k+1}$ arise by adjusting norms

Boson operators and symmetric p^2-states

First non-trivial application of elementary creation-destruction pairs is to the [2,0] sextet states

$$\left\{\left|\boxed{1}\boxed{1}\right\rangle,\left|\boxed{1}\boxed{2}\right\rangle,\left|\boxed{1}\boxed{3}\right\rangle,\left|\boxed{2}\boxed{2}\right\rangle,\left|\boxed{2}\boxed{3}\right\rangle,\left|\boxed{3}\boxed{3}\right\rangle,\right\}$$

$$E_{12}\left|n_1,n_2\right\rangle = a_1\bar{a}_2\left|n_1,n_2\right\rangle = a_1\sqrt{n_2}\left|n_1,n_2-1\right\rangle = \sqrt{n_1+1}\sqrt{n_2}\left|n_1+1,n_2-1\right\rangle$$

$$E_{23}\left|n_1,n_2,n_3\right\rangle = a_2\bar{a}_3\left|n_1,n_2,n_3\right\rangle = a_2\sqrt{n_3}\left|n_1,n_2,n_3-1\right\rangle = \sqrt{n_2+1}\sqrt{n_3}\left|n_1,n_2+1,n_3-1\right\rangle$$

Elementary operations e_{jk} apply to each particle a, b, c, and so forth in turn.

$$E_{23}\left|3_a3_b3_c\right\rangle = \left|2_a3_b3_c\right\rangle + \left|3_a2_b3_c\right\rangle + \left|3_a3_b2_c\right\rangle = \sqrt{3}\,\frac{\left|2_a3_b3_c\right\rangle + \left|3_a2_b3_c\right\rangle + \left|3_a3_b2\right\rangle}{\sqrt{3}} = \sqrt{3}\left|\boxed{2}\boxed{3}\boxed{3}\right\rangle$$

$$a_2\bar{a}_3\left|n_1=0,n_2=0,n_3=3\right\rangle = a_2\sqrt{3}\left|0,0,2\right\rangle = \sqrt{1}\sqrt{3}\left|0,1,2\right\rangle = E_{23}\left|\boxed{3}\boxed{3}\boxed{3}\right\rangle = \sqrt{3}\left|\boxed{2}\boxed{3}\boxed{3}\right\rangle$$

The e_{jk} procedure shows $a=\mathbf{a}^\dagger$ or $\bar{a}=\mathbf{a}$ factors $\sqrt{n_k}$ or $\sqrt{n_k+1}$ arise by adjusting norms

$$E_{23}\frac{\left|2_a3_b3_c3_d\right\rangle + \left|3_a2_b3_c3_d\right\rangle + \left|3_a3_b2_c3_d\right\rangle + \left|3_a3_b3_c2_d\right\rangle}{2} \qquad = E_{23}\left|\boxed{2}\boxed{3}\boxed{3}\boxed{3}\right\rangle$$

$$= \frac{\left|2_a2_b3_c3_d\right\rangle + \left|2_a2_b3_c3_d\right\rangle + \left|2_a3_b2_c3_d\right\rangle + \left|2_a3_b3_c2_d\right\rangle}{2} \qquad = \sqrt{6}\left[\frac{\left|2_a2_b3_c3_d\right\rangle + \left|2_a3_b2_c3_d\right\rangle + \left|2_a3_b3_c2_d\right\rangle}{\sqrt{6}}\right.$$

$$+ \frac{\left|2_a3_b2_c3_d\right\rangle + \left|3_a2_b2_c3_d\right\rangle + \left|3_a2_b2_c3_d\right\rangle + \left|3_a2_b3_c2_d\right\rangle}{2} \qquad \left. + \frac{\left|3_a2_b2_c3_d\right\rangle + \left|3_a2_b3_c2_d\right\rangle + \left|3_a3_b2_c2_d\right\rangle}{\sqrt{6}}\right]$$

$$+ \frac{\left|2_a3_b3_c2_d\right\rangle + \left|3_a2_b3_c2_d\right\rangle + \left|3_a3_b2_c2_d\right\rangle + \left|3_a3_b2_c2_d\right\rangle}{2} \qquad = \sqrt{6}\left|\boxed{2}\boxed{2}\boxed{3}\boxed{3}\right\rangle$$

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Boson operators and symmetric p^2-states: Connecting to angular momentum

Creation operator $\left(a\bar{a}\right)$ formulas give the same result in more compact notation.

$$E_{23}\left|\boxed{2}\boxed{3}\boxed{3}\boxed{3}\right\rangle = a_2\bar{a}_3\left|n_1=0,n_2=1,n_3=3\right\rangle = a_2\sqrt{3}\left|0,1,2\right\rangle = \sqrt{2}\sqrt{3}\left|0,2,2\right\rangle = \sqrt{6}\left|\boxed{2}\boxed{2}\boxed{3}\boxed{3}\right\rangle$$

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Matrix elements for [2,0] sextet states involve the following forms.

$$E_{11}\left|\boxed{1}\boxed{1}\right\rangle = 2\left|\boxed{1}\boxed{1}\right\rangle,\ E_{21}\left|\boxed{1}\boxed{1}\right\rangle = \sqrt{2}\left|\boxed{1}\boxed{2}\right\rangle,\ E_{21}\left|\boxed{1}\boxed{2}\right\rangle = \sqrt{2}\left|\boxed{2}\boxed{2}\right\rangle,\ E_{21}\left|\boxed{1}\boxed{3}\right\rangle = \left|\boxed{2}\boxed{3}\right\rangle,\ E_{21}\left|\boxed{2}\boxed{3}\right\rangle = 0$$

Boson operators and symmetric p^2-states: Connecting to angular momentum

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$$E_{23}\left|\boxed{2}\boxed{3}\boxed{3}\boxed{3}\right\rangle = a_2\bar{a}_3\left|n_1=0,n_2=1,n_3=3\right\rangle = a_2\sqrt{3}\left|0,1,2\right\rangle = \sqrt{2}\sqrt{3}\left|0,2,2\right\rangle = \sqrt{6}\left|\boxed{2}\boxed{2}\boxed{3}\boxed{3}\right\rangle$$

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Elementary operator representations are then found. (same as earlier cases by other means)

$E_{12} = E_{21}^{\dagger} =$

	11	12	22	13	23	33
11	·	$\sqrt{2}$	·	·	·	·
12		·	$\sqrt{2}$	·	·	·
22			·	·	·	·
13				·	1	·
23					·	·
33						·

$E_{23} = E_{32}^{\dagger} =$

	11	12	22	13	23	33
11	·	·	·	·	·	·
12		·	·	1	·	·
22			·	·	$\sqrt{2}$	·
13				·	·	·
23					·	$\sqrt{2}$
33						·

$E_{13} = E_{31}^{\dagger} =$

	11	12	22	13	23	33
11	·	·	·	$\sqrt{2}$	·	·
12		·	·	·	1	·
22			·	·	·	·
13				·	·	$\sqrt{2}$
23					·	·
33						·

…earlier cases
in *Lect.22p17-26.*

Boson operators and symmetric p^2-states: Connecting to angular momentum

Creation operator $\left(a\bar{a}\right)$ formulas give the same result in more compact notation.

$$E_{23}\left|\boxed{2}\boxed{3}\boxed{3}\boxed{3}\right\rangle = a_2\bar{a}_3\left|n_1=0,n_2=1,n_3=3\right\rangle = a_2\sqrt{3}\left|0,1,2\right\rangle = \sqrt{2}\sqrt{3}\left|0,2,2\right\rangle = \sqrt{6}\left|\boxed{2}\boxed{2}\boxed{3}\boxed{3}\right\rangle$$

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Elementary operator representations are then found. (same as earlier cases by other means)

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	11	12	22	13	23	33
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12		·	$\sqrt{2}$	·	·	·
22			·	·	·	·
13				·	1	·
23					·	·
33						·

$E_{23} = E_{32}^{\dagger} =$

	11	12	22	13	23	33
11	·	·	·	·	·	·
12		·	·	1	·	·
22			·	·	$\sqrt{2}$	·
13				·	·	·
23					·	$\sqrt{2}$
33						·

$E_{13} = E_{31}^{\dagger} =$

	11	12	22	13	23	33
11	·	·	·	$\sqrt{2}$	·	·
12		·	·	·	1	·
22			·	·	·	·
13				·	·	$\sqrt{2}$
23					·	·
33						·

…earlier cases in *Lect.22p17-26.*

36 "super-elementary" operators made by products of E_{23} and E_{12} and conjugates $E_{21} = E_{12}^{\dagger}$ and $E_{32} = E_{23}^{\dagger}$

$E_{13} = [E_{12}, E_{23}]$

$L_+ = L_x + i\,L_y = \sqrt{2}\left(E_{12} + E_{23}\right)$

	11	12	22	13	23	33
11	·	2	·	·	·	·
12	·	·	2	$\sqrt{2}$	·	·
22	·	·	·	·	2	·
13	·	·	·	·	$\sqrt{2}$	·
23	·	·	·	·	·	2
33	·	·	·	·	·	·

$L_- = L_+^{\dagger} =$

	11	12	22	13	23	33
11	·	·	·	·	·	·
12	2	·	·	·	·	·
22	·	2	·	·	·	·
13	·	$\sqrt{2}$	·	·	·	·
23	·	·	2	$\sqrt{2}$	·	·
33	·	·	·	·	2	·

$L^2 = L_+L_- + L_z(L_z - 1)$

	11	12	22	13	23	33
11	$4+2$	·	·	·	·	·
12	·	6	·	·	·	·
22	·	·	4	$2\sqrt{2}$	·	·
13	·	·	$2\sqrt{2}$	2	·	·
23	·	·	·	·	$4+2$	·
33	·	·	·	·	·	$0+6$

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- Connecting to angular momentum
- ➡ Projecting to angular momentum

Boson operators and symmetric p^2-states: Projecting to angular momentum

36 "super-elementary" operators made by products of E_{23} and E_{12} and conjugates $E_{21} = E_{12}^{\dagger}$ and $E_{32} = E_{23}^{\dagger}$

$$E_{13} = [E_{12}, E_{23}]$$

$L_+ = L_x + i\,L_y = \sqrt{2}\left(E_{12} + E_{23}\right)$

	11	12	22	13	23	33
11	·	2	·	·	·	·
12	·	·	2	$\sqrt{2}$	·	·
22	·	·	·	·	2	·
13	·	·	·	·	$\sqrt{2}$	·
23	·	·	·	·	·	2
33	·	·	·	·	·	·

$L_- = L_+^{\dagger} =$

	11	12	22	13	23	33
11	·	·	·	·	·	·
12	2	·	·	·	·	·
22	·	2	·	·	·	·
13	·	$\sqrt{2}$	·	·	·	·
23	·	·	2	$\sqrt{2}$	·	·
33	·	·	·	·	2	·

$L^2 = L_+ L_- + L_z(L_z - 1)$

	11	12	22	13	23	33
11	$4+2$	·	·	·	·	·
12	·	6	·	·	·	·
22	·	·	4	$2\sqrt{2}$	·	·
13	·	·	$2\sqrt{2}$	2	·	·
23	·	·	·	·	$4+2$	·
33	·	·	·	·	·	$0+6$

Angular-momentum-squared operator $\left\langle L^2 \right\rangle = L(L+1)$ tells what L-values are present

$$L_+ L_- = \left(L_x + i\,L_y\right)\left(L_x - i\,L_y\right) = L_x{}^2 + L_y{}^2 - iL_x L_y + iL_y L_x = L_x{}^2 + L_y{}^2 + L_z$$

$$L_x{}^2 + L_y{}^2 + L_z{}^2 = L_+ L_- + L_z{}^2 - L_z$$

Boson operators and symmetric p^2-states: Projecting to angular momentum

36 "super-elementary" operators made by products of E_{23} and E_{12} and conjugates $E_{21} = E_{12}^{\dagger}$ and $E_{32} = E_{23}^{\dagger}$

$$E_{13} = [E_{12}, E_{23}]$$

$L_+ = L_x + i L_y = \sqrt{2}\left(E_{12} + E_{23}\right)$

	11	12	22	13	23	33
11	·	2	·	·	·	·
12	·	·	2	$\sqrt{2}$	·	·
22	·	·	·	·	2	·
13	·	·	·	·	$\sqrt{2}$	·
23	·	·	·	·	·	2
33	·	·	·	·	·	·

$L_- = L_+^{\dagger} =$

	11	12	22	13	23	33
11	·	·	·	·	·	·
12	2	·	·	·	·	·
22	·	2	·	·	·	·
13	·	$\sqrt{2}$	·	·	·	·
23	·	·	2	$\sqrt{2}$	·	·
33	·	·	·	·	2	·

$\mathrm{L}^2 = L_+ L_- + L_z(L_z - 1)$

	11	12	22	13	23	33
11	$4+2$	·	·	·	·	·
12	·	6	·	·	·	·
22	·	·	4	$2\sqrt{2}$	·	·
13	·	·	$2\sqrt{2}$	2	·	·
23	·	·	·	·	$4+2$	·
33	·	·	·	·	·	$0+6$

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$$L_+ L_- = \left(L_x + i L_y\right)\left(L_x - i L_y\right) = L_x^{\,2} + L_y^{\,2} - iL_x L_y + iL_y L_x = L_x^{\,2} + L_y^{\,2} + L_z$$

$$L_x^{\,2} + L_y^{\,2} + L_z^{\,2} = L_+ L_- + L_z^{\,2} - L_z$$

Commutation $[L_x, L_y] = L_x L_y - L_y L_x = i L_z$ helps find L^2 matrices. Of 6 e-values, 5 are $L(L+1) = 6$ (*(L=2)* or D-orbital)

The 6th L-value *(L=0)* implies an S-orbital. Both are projected.

$$P(L=0) = \frac{\begin{pmatrix} 4 - 2(2+1) & 2\sqrt{2} \\ 2\sqrt{2} & 2 - 2(2+1) \end{pmatrix}}{0(0+1) - 2(2+1)} = \frac{1}{3}\begin{pmatrix} 1 & -\sqrt{2} \\ -\sqrt{2} & 2 \end{pmatrix}$$

$$P(L=2) = \frac{1}{3}\begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}$$

Boson operators and symmetric p^2-states: Projecting to angular momentum

36 "super-elementary" operators made by products of E_{23} and E_{12} and conjugates $E_{21} = E_{12}^\dagger$ and $E_{32} = E_{23}^\dagger$

$$E_{13} = [E_{12}, E_{23}]$$

$L_+ = L_x + iL_y = \sqrt{2}(E_{12} + E_{23})$

	11	12	22	13	23	33
11	·	2	·	·	·	·
12	·	·	2	$\sqrt{2}$	·	·
22	·	·	·	·	2	·
13	·	·	·	·	$\sqrt{2}$	·
23	·	·	·	·	·	2
33	·	·	·	·	·	·

$L_- = L_+^\dagger =$

	11	12	22	13	23	33
11	·	·	·	·	·	·
12	2	·	·	·	·	·
22	·	2	·	·	·	·
13	·	$\sqrt{2}$	·	·	·	·
23	·	·	2	$\sqrt{2}$	·	·
33	·	·	·	·	2	·

$L^2 = L_+L_- + L_z(L_z - 1)$

	11	12	22	13	23	33
11	4+2	·	·	·	·	·
12	·	6	·	·	·	·
22	·	·	4	$2\sqrt{2}$	·	·
13	·	·	$2\sqrt{2}$	2	·	·
23	·	·	·	·	4+2	·
33	·	·	·	·	·	0+6

Angular-momentum-squared operator $\langle L^2 \rangle = L(L+1)$ tells what L-values are present

$$L_+L_- = (L_x + iL_y)(L_x - iL_y) = L_x{}^2 + L_y{}^2 - iL_xL_y + iL_yL_x = L_x{}^2 + L_y{}^2 + L_z$$

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$$P(L=2) = \frac{1}{3}\begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}$$

Resulting transformation results for sextet tableau $\left|\boxed{2}\boxed{2}\right\rangle$ and $\left|\boxed{1}\boxed{3}\right\rangle$ to L-orbitals with M=*0*.

$$\begin{pmatrix} \left\langle \boxed{2}\boxed{2} \middle| \begin{matrix} L=0 \\ M=0 \end{matrix} \right\rangle & \left\langle \boxed{2}\boxed{2} \middle| \begin{matrix} L=2 \\ M=0 \end{matrix} \right\rangle \\ \left\langle \boxed{1}\boxed{3} \middle| \begin{matrix} L=0 \\ M=0 \end{matrix} \right\rangle & \left\langle \boxed{1}\boxed{3} \middle| \begin{matrix} L=2 \\ M=0 \end{matrix} \right\rangle \end{pmatrix} = \begin{pmatrix} \left\langle \boxed{0}\boxed{0} \middle| \begin{matrix} 0 \\ 0 \end{matrix} \right\rangle & \left\langle \boxed{0}\boxed{0} \middle| \begin{matrix} 2 \\ 0 \end{matrix} \right\rangle \\ \left\langle \boxed{+1}\boxed{-1} \middle| \begin{matrix} 0 \\ 0 \end{matrix} \right\rangle & \left\langle \boxed{+1}\boxed{-1} \middle| \begin{matrix} 2 \\ 0 \end{matrix} \right\rangle \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} \\ -\sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} \end{pmatrix}$$

(push-to-top)

Boson operators and symmetric p^2-states: Projecting to angular momentum

Commutation $[L_x, L_y] = L_x L_y - L_y L_x = i L_z$ helps find matrices.
The 6th L-value *(L=0)* implies an S-orbital. Both are projected.

$$P(L=0) = \frac{\begin{pmatrix} 4-2(2+1) & 2\sqrt{2} \\ 2\sqrt{2} & 2-2(2+1) \end{pmatrix}}{0(0+1)-2(2+1)} = \frac{1}{3}\begin{pmatrix} 1 & -\sqrt{2} \\ -\sqrt{2} & 2 \end{pmatrix}$$

Resulting transformation results for sextet tableau $\left|\boxed{2}\boxed{2}\right\rangle$ and $\left|\boxed{1}\boxed{3}\right\rangle$ to *L*-orbitals with *M=0*.

$$\begin{pmatrix} \left\langle \boxed{2}\boxed{2} \middle| \begin{smallmatrix} L=0 \\ M=0 \end{smallmatrix} \right\rangle & \left\langle \boxed{2}\boxed{2} \middle| \begin{smallmatrix} L=2 \\ M=0 \end{smallmatrix} \right\rangle \\ \left\langle \boxed{1}\boxed{3} \middle| \begin{smallmatrix} L=0 \\ M=0 \end{smallmatrix} \right\rangle & \left\langle \boxed{1}\boxed{3} \middle| \begin{smallmatrix} L=2 \\ M=0 \end{smallmatrix} \right\rangle \end{pmatrix} = \begin{pmatrix} \left\langle \boxed{0}\boxed{0} \middle| \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right\rangle & \left\langle \boxed{0}\boxed{0} \middle| \begin{smallmatrix} 2 \\ 0 \end{smallmatrix} \right\rangle \\ \left\langle \boxed{+1}\boxed{-1} \middle| \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right\rangle & \left\langle \boxed{+1}\boxed{-1} \middle| \begin{smallmatrix} 2 \\ 0 \end{smallmatrix} \right\rangle \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} \\ -\sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} \end{pmatrix}$$

(pushed-to-top)

Boson operators and symmetric p^2-states: Projecting to angular momentum

Commutation $[L_x, L_y] = L_x L_y - L_y L_x = i L_z$ helps find matrices.

The 6th L-value $(L=0)$ implies an S-orbital. Both are projected.

$$P(L=0) = \frac{\begin{pmatrix} 4-2(2+1) & 2\sqrt{2} \\ 2\sqrt{2} & 2-2(2+1) \end{pmatrix}}{0(0+1)-2(2+1)} = \frac{1}{3}\begin{pmatrix} 1 & -\sqrt{2} \\ -\sqrt{2} & 2 \end{pmatrix}$$

Resulting transformation results for sextet tableau $\left|\boxed{2}\boxed{2}\right\rangle$ and $\left|\boxed{1}\boxed{3}\right\rangle$ to L-orbitals with $M=0$.

$$\begin{pmatrix} \left\langle \boxed{2}\boxed{2} \middle| {L=0 \atop M=0} \right\rangle & \left\langle \boxed{2}\boxed{2} \middle| {L=2 \atop M=0} \right\rangle \\ \left\langle \boxed{1}\boxed{3} \middle| {L=0 \atop M=0} \right\rangle & \left\langle \boxed{1}\boxed{3} \middle| {L=2 \atop M=0} \right\rangle \end{pmatrix} = \begin{pmatrix} \left\langle \boxed{0}\boxed{0} \middle| {0 \atop 0} \right\rangle & \left\langle \boxed{0}\boxed{0} \middle| {2 \atop 0} \right\rangle \\ \left\langle \boxed{+1}\boxed{-1} \middle| {0 \atop 0} \right\rangle & \left\langle \boxed{+1}\boxed{-1} \middle| {2 \atop 0} \right\rangle \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} \\ -\sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} \end{pmatrix}$$

Compare this to $(M=0)$-Clebsch-Gordan coefficients under $\left|{2 \atop 0}\right\rangle$ and $\left|{0 \atop 0}\right\rangle$ columns:

$$\left|1\otimes 1 {L=0 \atop M=0}\right\rangle = \sum C^{1\,1\,0}_{mm'0} \left|{1 \atop 0}\right\rangle \left|{1 \atop 0}\right\rangle$$
$$= C^{110}_{000}\left|{1 \atop 0}\right\rangle\left|{1 \atop 0}\right\rangle + C^{1\,1\,0}_{+1-1\,0}\left|{1 \atop +1}\right\rangle\left|{1 \atop -1}\right\rangle + C^{1\,1\,0}_{-1+1\,0}\left|{1 \atop -1}\right\rangle\left|{1 \atop +1}\right\rangle$$
$$= -\sqrt{\frac{1}{3}}\left|{1 \atop 0}\right\rangle\left|{1 \atop 0}\right\rangle + \sqrt{\frac{1}{3}}\left|{1 \atop +1}\right\rangle\left|{1 \atop -1}\right\rangle + \sqrt{\frac{1}{3}}\left|{1 \atop -1}\right\rangle\left|{1 \atop +1}\right\rangle$$
$$= -\sqrt{\frac{1}{3}}\left|\boxed{0}\boxed{0}\right\rangle + \sqrt{\frac{2}{3}}\left|\boxed{+1}\boxed{-1}\right\rangle$$

$$\left|1\otimes 1 {L=2 \atop M=0}\right\rangle = \sum C^{1\,1\,2}_{mm'0} \left|{1 \atop 0}\right\rangle \left|{1 \atop 0}\right\rangle$$
$$= C^{112}_{000}\left|{1 \atop 0}\right\rangle\left|{1 \atop 0}\right\rangle + C^{1\,1\,1}_{+1-1\,0}\left|{1 \atop +1}\right\rangle\left|{1 \atop -1}\right\rangle + C^{1\,1\,2}_{-1+1\,0}\left|{1 \atop -1}\right\rangle\left|{1 \atop +1}\right\rangle$$
$$= \sqrt{\frac{2}{3}}\left|{1 \atop 0}\right\rangle\left|{1 \atop 0}\right\rangle + \sqrt{\frac{1}{6}}\left|{1 \atop +1}\right\rangle\left|{1 \atop -1}\right\rangle + \sqrt{\frac{1}{6}}\left|{1 \atop -1}\right\rangle\left|{1 \atop +1}\right\rangle$$
$$= \sqrt{\frac{2}{3}}\left|\boxed{0}\boxed{0}\right\rangle + \sqrt{\frac{1}{3}}\left|\boxed{+1}\boxed{-1}\right\rangle$$

(end for 4.18)

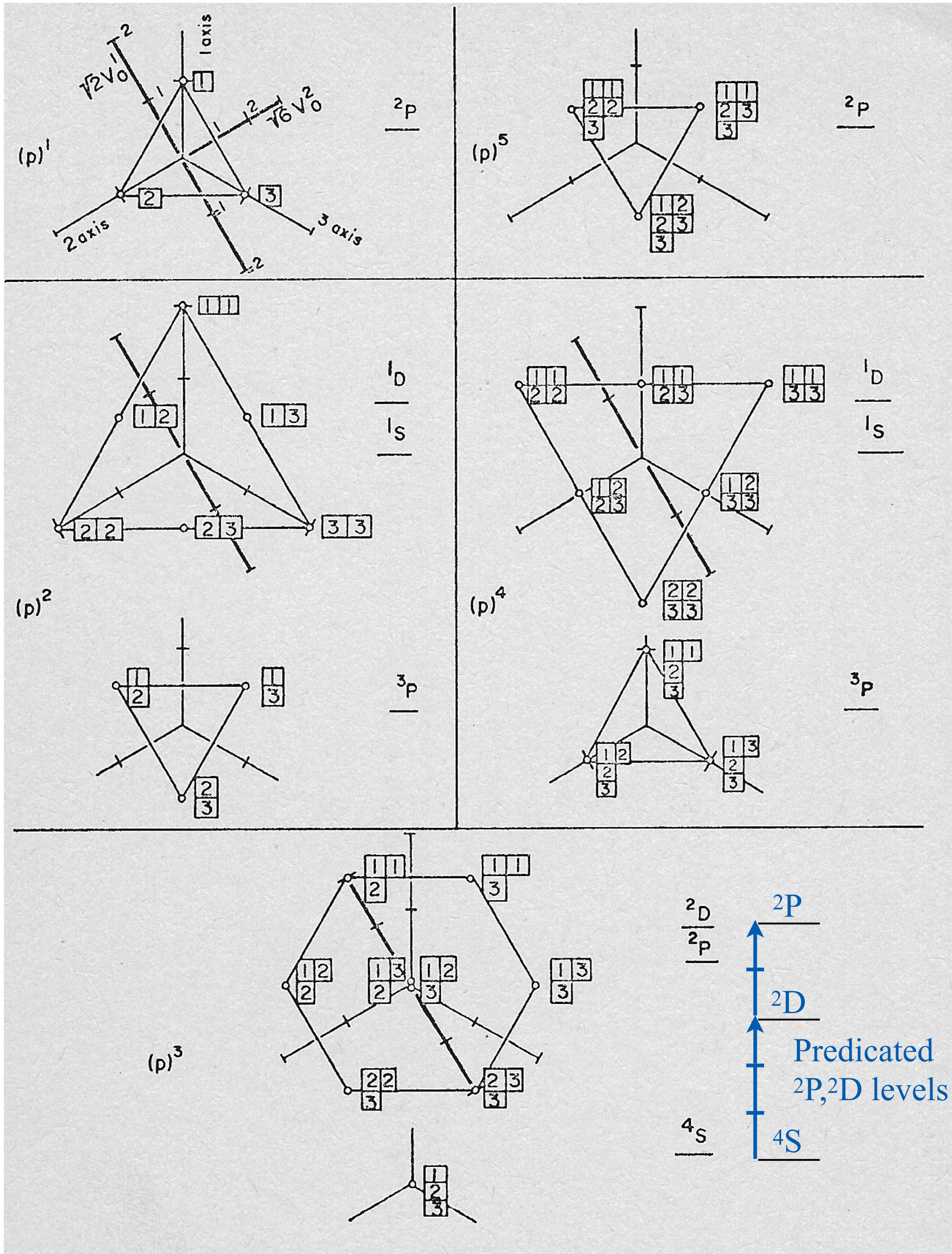

Fig.8 Weight or Moment Diagrams of Atomic $(p)^n$ States
Each tableau is located at point $(x_1\ x_2\ x_3)$ in a cartesian co-ordinate system for which x_n is the number of n's in the tableau. An alternative co-ordinate system is $(v_0^2,\ v_0^1,\ v_0^0)$ defined by Eq.16 which gives the zz-quadrupole moment, z-magnetic dipole moment, and number of particles, respectively. The last axis (v_0^0) would be pointing straight out of the figure, and each family of states lies in a plane perpendicular to it.

A Unitary Calculus for Electronic Orbitals
William G. Harter and Christopher W. Patterson
Springer-Verlag Lectures in Physics 49 1976

Alternative basis for the theory of complex spectra I
William G. Harter
Physical Review A 8 3 p2819 (1973)

Alternative basis for the theory of complex spectra II
William G. Harter and Christopher W. Patterson
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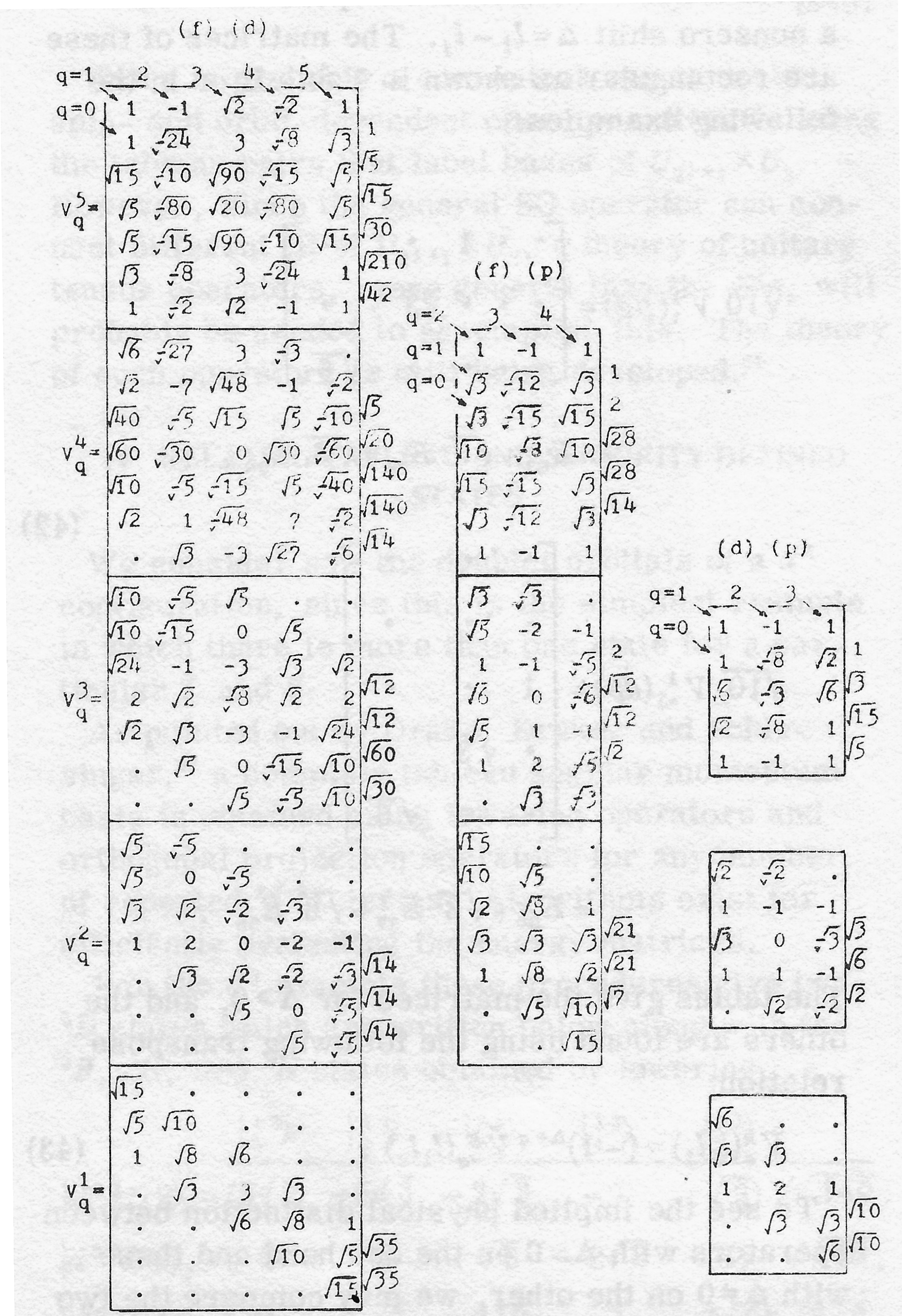
Alternative basis for the theory of complex spectra III
William G. Harter and Christopher W. Patterson
Physical Review A ??



Alternative basis for the theory of complex spectra II
William G. Harter and Christopher W. Patterson
Physical Review A 13 3 p1076-1082 (1976)

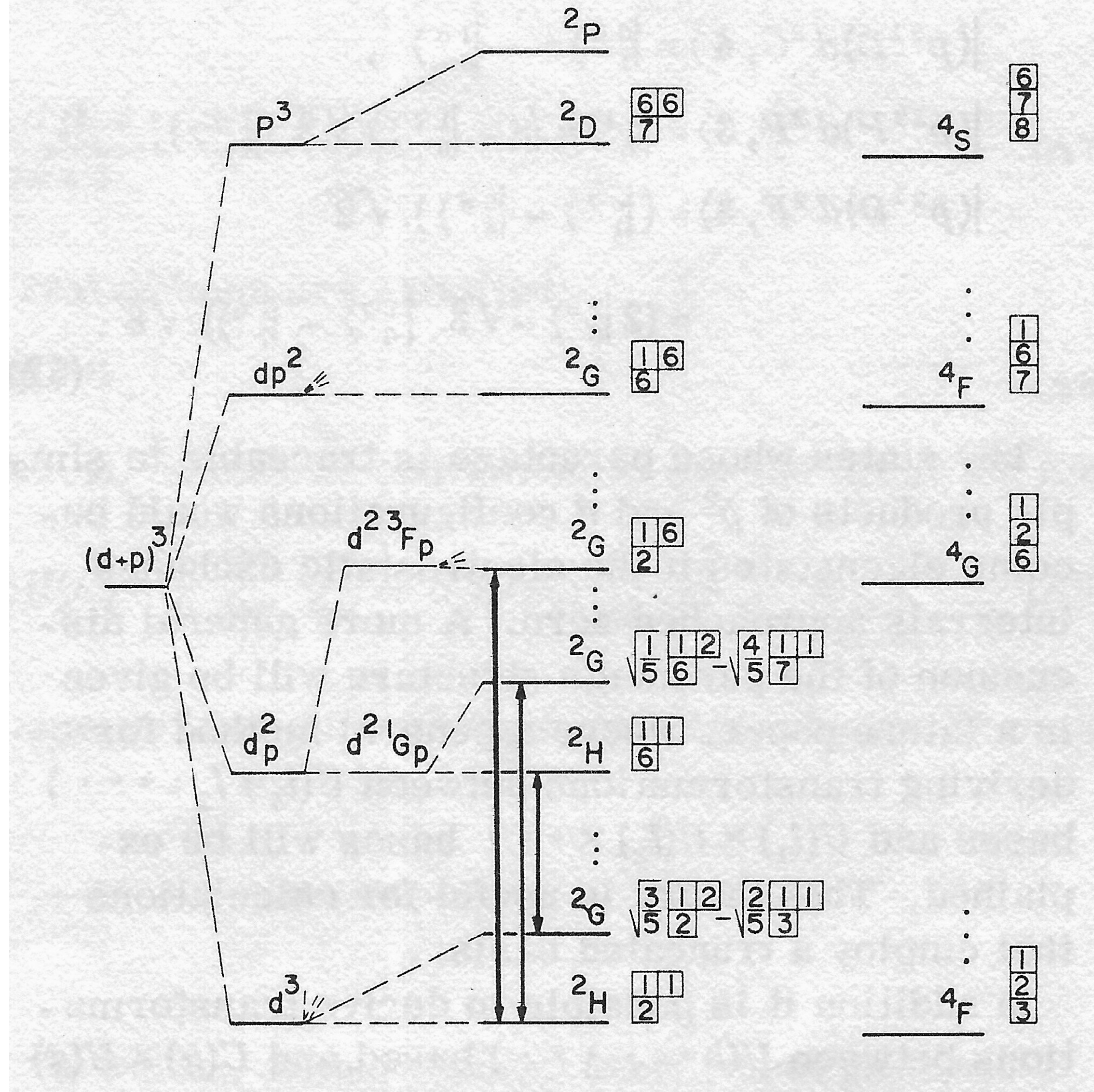
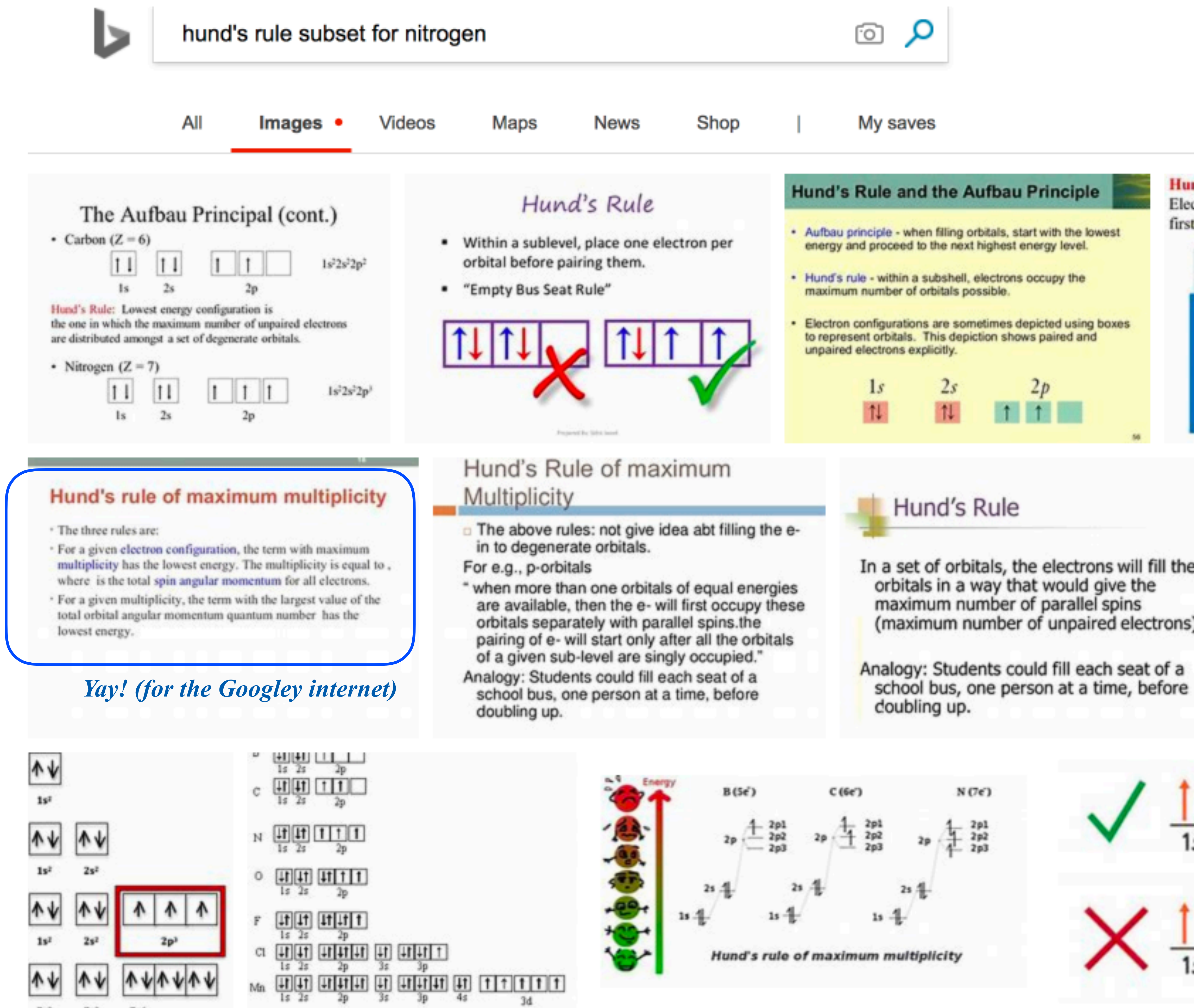


FIG. 6. Example of unitary tableau notation for multiple-shell states. The calculation of the dipole operator using the jawbone formula between states of definite spin and orbit as shown is given in Eq. (48).



Yay! (for the Googley internet)

Complete set of E_{jk} matrix elements for the doublet (spin-½) p^3 orbits

	$\vert{}^{1\,1}_{2}\rangle$:	$\vert{}^{1\,2}_{2}\rangle$	$\vert{}^{1\,1}_{3}\rangle$:	$\vert{}^{1\,2}_{3}\rangle$	$\vert{}^{1\,3}_{2}\rangle$:	$\vert{}^{1\,3}_{3}\rangle$	$\vert{}^{2\,2}_{3}\rangle$:	$\vert{}^{2\,3}_{3}\rangle$
	$M=2$	$M=1$		$M=0$		$M=-1$		$M=-2$
$\langle{}^{1\,1}_{2}\vert$	$2^{(11)}+1^{(22)}$	$1^{(12)}$	$1^{(23)}$	$-\sqrt{\tfrac{1}{2}}^{(13)}$	$\sqrt{\tfrac{3}{2}}^{(13)}$			
$\langle{}^{1\,2}_{2}\vert$		$1^{(11)}+2^{(22)}$		$\sqrt{\tfrac{1}{2}}^{(23)}$	$\sqrt{\tfrac{3}{2}}^{(23)}$		$-1^{(13)}$	
$\langle{}^{1\,1}_{3}\vert$			$2^{(11)}+1^{(33)}$	$\sqrt{2}^{(12)}$		$1^{(13)}$		
$\langle{}^{1\,2}_{3}\vert$				$1^{(11)}+1^{(22)}+1^{(33)}$		$\sqrt{\tfrac{1}{2}}^{(23)}$	$\sqrt{2}^{(12)}$	$\sqrt{\tfrac{1}{2}}^{(13)}$
$\langle{}^{1\,3}_{2}\vert$					$1^{(11)}+1^{(22)}+1^{(33)}$	$\sqrt{\tfrac{3}{2}}^{(23)}$		$\sqrt{\tfrac{3}{2}}^{(13)}$
$\langle{}^{1\,3}_{3}\vert$						$1^{(11)}+2^{(33)}$		$1^{(12)}$
$\langle{}^{2\,2}_{3}\vert$							$2^{(22)}+1^{(33)}$	$1^{(23)}$
$\langle{}^{2\,3}_{3}\vert$								$1^{(22)}+2^{(33)}$

$=\langle E_{ij}\rangle$

notation:
(jk) numbers tell
which E_{jk} gave that entry

Diagonal examples in *n-particle* notation:

$$\sqrt{3}\mathbf{V}^0_0 = E_{11}+E_{22}+E_{33}$$

$$\sqrt{2}\mathbf{V}^1_0 = E_{11} \qquad - E_{33} \equiv L_z$$

$$\sqrt{6}\mathbf{V}^2_0 = E_{11}-2E_{22}+E_{33}$$

Off-Diagonal examples in *n-particle* notation:

$$\mathbf{V}^2_2 = E_{13}\ , \quad -2\mathbf{V}^2_1 = \sqrt{2}(E_{12}-E_{23})\ , \quad 2\mathbf{V}^2_{-1} = \sqrt{2}(E_{21}-E_{32})\ , \quad 2\mathbf{V}^2_{-2} = E_{31}\ ,$$

$$-2\mathbf{V}^1_1 = \sqrt{2}(E_{12}+E_{23}) \equiv L_+, \quad 2\mathbf{V}^1_{-1} = \sqrt{2}(E_{21}+E_{32}) \equiv L_-\ .$$

Tableau calculation of 3-electron ℓ=1 orbital p^3-states and their $\mathbf{V}^k{}_q$ matrices

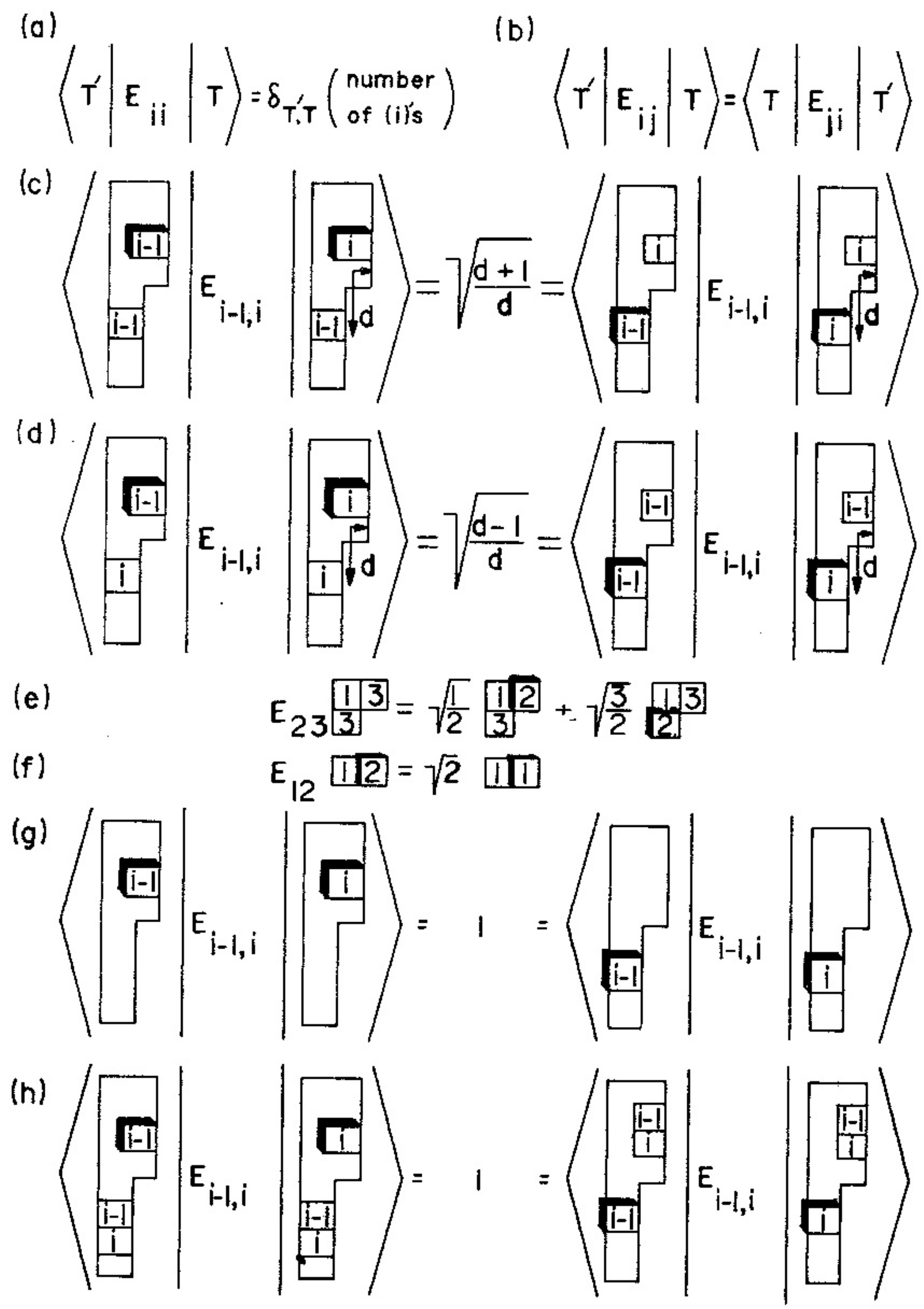
Start with highest angular momentum (L=2) p^3 state: $\left|{}^2D,{}^{L=2}_{M=2}\right\rangle = \begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}$ (Fermi spin-mate $\begin{array}{|c|c|}\hline \uparrow&\uparrow\\\hline \downarrow\\\cline{1-1}\end{array}$)

Then apply lowering operator $L_- \equiv \sqrt{2}(E_{21}+E_{32})$

$$\left|{}^2D^{L=2}_{M=1}\right\rangle = \tfrac{1}{2}L_-\left|{}^2D^{L=2}_{M=2}\right\rangle = \tfrac{1}{2}\sqrt{2}(E_{21}+E_{32})\left|\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\right\rangle$$

Here this is done using Tableau "Jawbone" formula.

$$= = \frac{1}{\sqrt{2}}\left(\left|\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\right\rangle + \left|\begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\right\rangle\right)$$



Orthogonal to this is a ^{2}P (M=1) state

$$\left|{}^2P^{L=1}_{M=1}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\right\rangle - \left|\begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\right\rangle\right)$$

Next we calculate 2^n-pole moments the pair:

$$\left\langle {}^2P^{L=1}_{M=1}\right|V^k_0\left|{}^2D^{L=2}_{M=1}\right\rangle =$$

$$\frac{1}{\sqrt{2}}\left(\left\langle\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\right| + \left\langle\begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\right|\right)\left[\binom{k}{11}E_{11}+\binom{k}{22}E_{22}+\binom{k}{33}E_{33}\right]\left(\left|\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\right\rangle - \left|\begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\right\rangle\right)$$

$$= \tfrac{1}{2}\left[-\binom{2}{11}E_{11}+2\binom{2}{22}E_{22}-\binom{2}{33}\right] = -\sqrt{\tfrac{3}{2}} \quad \text{for}: \; k=2$$

$$= \tfrac{1}{2}\left[-\binom{1}{11}E_{11}+2\binom{1}{22}E_{22}-\binom{1}{33}\right] = 0 \quad \text{for}: \; k=1$$

$$= \tfrac{1}{2}\left[-\binom{0}{11}E_{11}+2\binom{0}{22}E_{22}-\binom{0}{33}\right] = 0 \quad \text{for}: \; k=0$$

$$\left|1,2,3\right\rangle \equiv \left|1\right\rangle_{particle-a}\left|2\right\rangle_{particle-b}\left|3\right\rangle_{particle-c} \equiv \left|1\right\rangle_{a}\left|2\right\rangle_{b}\left|3\right\rangle_{c}$$

Single particle p^1-orbitals: U(3) triplet $\left|p^1 \square\right\rangle$

$$e_{11}=\begin{pmatrix}1&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}, e_{12}=\begin{pmatrix}\cdot&1&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}, e_{13}=\begin{pmatrix}\cdot&\cdot&1\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}, e_{21}=\begin{pmatrix}\cdot&\cdot&\cdot\\1&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}, \ldots e_{33}=\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\\1&\cdot&\cdot\end{pmatrix}.$$

$e_{12}e_{21}=e_{11}$ $\quad |1\rangle\langle 2\|2\rangle\langle 1|=|1\rangle\langle 1|$

$e_{12}e_{22}=e_{12}$ $\quad |1\rangle\langle 2\|2\rangle\langle 2|=|1\rangle\langle 2|$

Elementary matrix algebra $\quad \vdots$

$e_{jk}e_{pq}=\delta_{pk}e_{jq}$ $\quad |j\rangle\langle k\|p\rangle\langle q|=\delta_{pk}|j\rangle\langle q|$

General elementary operator commutation $[E_{jk}, E_{pq}]= \delta_{kp}E_{jq} - \delta_{qj}E_{pk}$
has same form as 1-particle commutation: $[e_{jk}, e_{pq}]= \delta_{kp}e_{jq} - \delta_{qj}e_{pk}$

Elementary-elementary operator commutation algebra

This applies to all of multi-particle representations of E_{jk} and to momentum operators L_x, L_y, and L_z.

Single particle p-orbit ($\ell=1$) representation of L_x, L_y, and L_z

$$D^1_{mn}(L_x)=\frac{1}{\sqrt{2}}\begin{pmatrix}\cdot&1&\cdot\\1&\cdot&1\\\cdot&1&\cdot\end{pmatrix},\quad D^1_{mn}(L_y)=\frac{-i}{\sqrt{2}}\begin{pmatrix}\cdot&1&\cdot\\-1&\cdot&1\\\cdot&-1&\cdot\end{pmatrix},\quad D^1_{mn}(L_z)=\begin{pmatrix}1&\cdot&\cdot\\\cdot&0&\cdot\\\cdot&\cdot&-1\end{pmatrix}$$

Elementary operator form of L_x, L_y, and L_z

$$L_x=\left(E_{12}+E_{23}+E_{21}+E_{32}\right)/\sqrt{2},\quad L_y=-i\left(E_{12}+E_{23}-E_{21}-E_{32}\right)/\sqrt{2},\quad L_z=E_{11}-E_{33}$$

…and of raise-lower operators L_+ and L_-

$$L_+=L_x+iL_y=\sqrt{2}\left(E_{12}+E_{23}\right),\quad L_-=L_x-iL_y=\sqrt{2}\left(E_{21}+E_{32}\right)=L_+^\dagger,\quad L_z=[L_+,L_-]$$