AMOP
reference links
on page 2

4.25.18 class 26: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

$(S_3)^*(U(3))\subset U(6)$ models of p^3 electronic spin-orbit states and couplings

[2,1] tableau states lowered by $\mathbf{L}_- = \sqrt{2}(E_{21}+E_{32})$

Top-(J,M) states   to mid-level states

$\ell=1$ *p*=shell LS states combined to states of definite J

J=3/2 at L=0 (^{4}S),   J=5/2 at L=2 (^{2}D)

C-G coupling; J=3/2 at L=2 (^{2}D),   J=3/2 at L=1 (^{2}P),   J=1/2 at L=1 (^{2}P)

Spin-orbit state assembly formula and Slater determinants

Extra assembly table

$\ell=1$ *p*=shell LSJ states transformed to Slater determinants from J=3/2 (^{4}S)

Slater functions for J=5/2,   J=3/2 (^{2}D)

Slater functions for J=3/2 (^{2}P),   J=1/2 (^{2}P)

Summary of states and level connection paths

Symmetry dimension accounting

Spin-orbit Hamiltonian matrix calculation

Individual matrix components

Application to spin-orbit and entanglement break-up scattering

AMOP reference links *(Updated list given on **2nd** and 3rd pages of each class presentation)*

Web Resources - front page	Quantum Theory for the Computer Age	2014 AMOP
UAF Physics UTube channel	Principles of Symmetry, Dynamics, and Spectroscopy	2017 Group Theory for QM
	Classical Mechanics with a Bang!	2018 AMOP
	Modern Physics and its Classical Foundations	

Representaions Of Multidimensional Symmetries In Networks - harter-jmp-1973

Alternative Basis for the Theory of Complex Spectra

- Alternative_Basis_for_the_Theory_of_Complex_Spectra_I_-_harter-pra-1973
- Alternative_Basis_for_the_Theory_of_Complex_Spectra_II_-_harter-patterson-pra-1976
- Alternative_Basis_for_the_Theory_of_Complex_Spectra_III_-_patterson-harter-pra-1977

Frame Transformation Relations And Multipole Transitions In Symmetric Polyatomic Molecules - RMP-1978

Asymptotic eigensolutions of fourth and sixth rank octahedral tensor operators - Harter-Patterson-JMP-1979

Rotational energy surfaces and high- J eigenvalue structure of polyatomic molecules - Harter - Patterson - 1984

Galloping waves and their relativistic properties - ajp-1985-Harter

Rovibrational Spectral Fine Structure Of Icosahedral Molecules - Cpl 1986 (Alt Scan)

Theory of hyperfine and superfine levels in symmetric polyatomic molecules.

I) Trigonal and tetrahedral molecules: Elementary spin-1/2 cases in vibronic ground states - PRA-1979-Harter-Patterson (Alt scan)
II) Elementary cases in octahedral hexafluoride molecules - Harter-PRA-1981 (Alt scan)

Rotation–vibration spectra of icosahedral molecules.

I) Icosahedral symmetry analysis and fine structure - harter-weeks-jcp-1989 (Alt scan)
II) Icosahedral symmetry, vibrational eigenfrequencies, and normal modes of buckminsterfullerene - weeks-harter-jcp-1989 (Alt scan)
III) Half-integral angular momentum - harter-reimer-jcp-1991

Rotation-vibration scalar coupling zeta coefficients and spectroscopic band shapes of buckminsterfullerene - Weeks-Harter-CPL-1991 (Alt scan)

Nuclear spin weights and gas phase spectral structure of 12C60 and 13C60 buckminsterfullerene -Harter-Reimer-Cpl-1992 - (Alt1, Alt2 Erratum)

Gas Phase Level Structure of C60 Buckyball and Derivatives Exhibiting Broken Icosahedral Symmetry - reimer-diss-1996

Fullerene symmetry reduction and rotational level fine structure/ the Buckyball isotopomer 12C 13C59 - jcp-Reimer-Harter-1997 (HiRez)

Wave Node Dynamics and Revival Symmetry in Quantum Rotors - harter - jms - 2001

Molecular Symmetry and Dynamics - Ch32-Springer Handbooks of Atomic, Molecular, and Optical Physics - Harter-2006

Resonance and Revivals

I) QUANTUM ROTOR AND INFINITE-WELL DYNAMICS - ISMSLi2012 (Talk) OSU knowledge Bank
II) Comparing Half-integer Spin and Integer Spin - Alva-ISMS-Ohio2013-R777 (Talks)
III) Quantum Resonant Beats and Revivals in the Morse Oscillators and Rotors - (2013-Li-Diss)

Resonance and Revivals in Quantum Rotors - Comparing Half-integer Spin and Integer Spin - Alva-ISMS-Ohio2013-R777 (Talk)

Molecular Eigensolution Symmetry Analysis and Fine Structure - IJMS-harter-mitchell-2013

Quantum Revivals of Morse Oscillators and Farey-Ford Geometry - Li-Harter-cpl-2013

QTCA Unit 10 Ch 30 - 2013

AMOP Ch 0 Space-Time Symmetry - 2019

**In development - a web based A.M.O.P. oriented reference page, with thumbnail/previews, greater control over the information display, and eventually full on* Apache-SOLR *Index and search for* nuanced, whole-site content/metadata level *searching.*

AMOP reference links *(Updated list given on 2nd and 3rd pages of each class presentation)*

(Int.J.Mol.Sci, 14, 714(2013) p.755-774 , QTCA Unit 7 Ch. 23-26), (PSDS - Ch. 5, 7)

Int.J.Mol.Sci, 14, 714(2013), QTCA Unit 8 Ch. 23-25, QTCA Unit 9 Ch. 26, PSDS Ch. 5, PSDS Ch. 7

Intro spin ½ coupling
Unit 8 Ch. 24 p3

H atom hyperfine-B-level crossing
Unit 8 Ch. 24 p15

Hyperf. theory Ch. 24 p48.

Hyperf. theory Ch. 24 p48.
Deeper theory ends p53

Intro 2p3p coupling
Unit 8 Ch. 24 p17.

Intro LS-jj coupling
Unit 8 Ch. 24 p22.

CG coupling derived (start)
Unit 8 Ch. 24 p39.

CG coupling derived (formula)
Unit 8 Ch. 24 p44.

Lande' g-factor
Unit 8 Ch. 24 p26.

Irrep Tensor building
Unit 8 Ch. 25 p5.

Irrep Tensor Tables
Unit 8 Ch. 25 p12.

Wigner-Eckart tensor Theorem.
Unit 8 Ch. 25 p17.

Tensors Applied to d,f-levels.
Unit 8 Ch. 25 p21.

Tensors Applied to high J levels.
Unit 8 Ch. 25 p63.

Intro 3-particle coupling.
Unit 8 Ch. 25 p28.

Intro 3,4-particle Young Tableaus
GrpThLect29 p42.

Young Tableau Magic Formulae
GrpThLect29 p46-48.

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AMOP reference links (Updated list given on 2nd and 3rd and 4th pages of each class presentation)

Predrag Cvitanovic's: Birdtrack Notation, Calculations, and Simplification

Chaos_Classical_and_Quantum_-_2018-Cvitanovic-ChaosBook
Group Theory - PUP_Lucy_Day_-_Diagrammatic_notation_-_Ch4
Simplification_Rules_for_Birdtrack_Operators_-_Alcock-Zeilinger-Weigert-zeilinger-jmp-2017
Group Theory - Birdtracks_Lies_and_Exceptional_Groups_-_Cvitanovic-2011
Simplification_rules_for_birdtrack_operators-_jmp-alcock-zeilinger-2017
Birdtracks for SU(N) - 2017-Keppeler

Frank Rioux's: UMA method of vibrational induction

Quantum_Mechanics_Group_Theory_and_C60_-_Frank_Rioux_-_Department_of_Chemistry_Saint_Johns_U
Symmetry_Analysis_for_H20-_H20GrpTheory-_Rioux
Quantum_Mechanics-Group_Theory_and_C60_-_JChemEd-Rioux-1994
Group_Theory_Problems-_Rioux-_SymmetryProblemsX
Comment_on_the_Vibrational_Analysis_for_C60_and_Other_Fullerenes_Rioux-RSP

Supplemental AMOP Techniques & Experiment

Many Correlation Tables are Molien Sequences - Klee (Draft 2016)
High-resolution_spectroscopy_and_global_analysis_of_CF4_rovibrational_bands_to_model_its_atmospheric_absorption-_carlos-Boudon-jqsrt-2017
Symmetry and Chirality - Continuous_Measures_-_Avnir
*

Special Topics & Colloquial References

r-process_nucleosynthesis_from_matter_ejected_in_binary_neutron_star_mergers-PhysRevD-Bovard-2017

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AMOP
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Spin-orbit state assembly formula and Slater determinants

$\ell=1$ p=shell LSJ states transformed to Slater determinants from J=3/2 (^{4}S)

Slater functions for J=5/2, J=3/2 (^{2}D)

Slater functions for J=3/2 (^{2}P), J=1/2 (^{2}P)

Summary of states and level connection paths

Symmetry dimension accounting

Spin-orbit Hamiltonian matrix calculation

Application to spin-orbit and entanglement break-up scattering

[2,1] *tableau states lowered by* **L**₋$=\sqrt{2}(E_{21}+E_{32})$

	M=2	*M=1*		*M=0*		*M=-1*		*M=-2*
E_{jk}	$\left\|{11 \atop 2}\right\rangle$	$\left\|{12 \atop 2}\right\rangle$	$\left\|{11 \atop 3}\right\rangle$	$\left\|{12 \atop 3}\right\rangle$	$\left\|{13 \atop 2}\right\rangle$	$\left\|{13 \atop 3}\right\rangle$	$\left\|{22 \atop 3}\right\rangle$	$\left\|{23 \atop 3}\right\rangle$
$\left\langle{11 \atop 2}\right\|$	(11) (22) $2+1$	(12) 1	(23) 1	(13) $-\sqrt{\frac{1}{2}}$	(13) $\sqrt{\frac{3}{2}}$	·	·	·
$\left\langle{12 \atop 2}\right\|$	(21) 1	(11) (22) $1+2$	·	(23) $\sqrt{\frac{1}{2}}$	(23) $\sqrt{\frac{3}{2}}$	·	(13) -1	·
$\left\langle{11 \atop 3}\right\|$	(32) 1	·	(11) (33) $2+1$	(12) $\sqrt{2}$	·	(13) 1	·	·
$\left\langle{12 \atop 3}\right\|$	(31) $-\sqrt{\frac{1}{2}}$	(32) $\sqrt{\frac{1}{2}}$	(21) $\sqrt{2}$	(11) (22) (33) $1+1+1$	·	(23) $\sqrt{\frac{1}{2}}$	(12) $\sqrt{2}$	(13) $\sqrt{\frac{1}{2}}$
$\left\langle{13 \atop 2}\right\|$	(31) $\sqrt{\frac{3}{2}}$	(32) $\sqrt{\frac{3}{2}}$	·	·	(11) (22) (33) $1+1+1$	(23) $\sqrt{\frac{3}{2}}$	·	(13) $\sqrt{\frac{3}{2}}$
$\left\langle{13 \atop 3}\right\|$	·	·	(31) 1	(32) $\sqrt{\frac{1}{2}}$	(32) $\sqrt{\frac{3}{2}}$	(11) (33) $1+2$	·	(12) 1
$\left\langle{22 \atop 3}\right\|$	·	(31) -1	·	(21) $\sqrt{2}$	·	·	(22) (33) $2+1$	(23) 1
$\left\langle{23 \atop 3}\right\|$	·	·	·	(31) $\sqrt{\frac{1}{2}}$	(31) $\sqrt{\frac{3}{2}}$	(21) 1	(32) 1	(22) (33) $1+2$

E

$$L_z \equiv \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix} = (E_{11}\text{-}E_{33}) = \sqrt{2}\mathbf{v}^1_0$$

dipole (k=1)
∠-momentum
L*-operators*

E_{jk}-matrix
Lect.23
p.7-16
and p.74

$$L_+ \equiv \sqrt{2}\begin{pmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \end{pmatrix} = \sqrt{2}(E_{12}+E_{23}) = L_x + iL_y = \text{-}\sqrt{2}\mathbf{v}^1_1$$

$$L_- \equiv \sqrt{2}\begin{pmatrix} \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \end{pmatrix} = \sqrt{2}(E_{21}+E_{32}) = L_x - iL_y = \sqrt{2}\mathbf{v}^1_{=1}$$

$$L_-\left|{L \atop M}\right\rangle = \sqrt{(L+M)(L-M+1)}\left|{L \atop M-1}\right\rangle$$

$$L_-\left|{2 \atop 2}\right\rangle = \sqrt{(2+2)(2-2+1)}\left|{2 \atop 1}\right\rangle = 2\left|{2 \atop 1}\right\rangle$$

Start with top [2,1]-state:

$$\boxed{\left|{2 \atop 2}\right\rangle = \left|\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\right\rangle = \left|{}^2D_{M=2}\right\rangle}$$

Number of levels in fermionic spin-1/2 p^3

$$U(6) \supset U(3) \times U(2)$$

$$N = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = \frac{120}{6} = 20$$

$\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$ [2,1] *tableau states lowered by* $\mathbf{L}_{-}=\sqrt{2}(E_{21}+E_{32})$

	M=2	M=1		M=0		M=-1		M=-2
E_{jk}	$\left\lvert\begin{smallmatrix}11\\2\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}12\\2\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}11\\3\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}12\\3\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}13\\2\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}13\\3\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}22\\3\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}23\\3\end{smallmatrix}\right\rangle$
$\left\langle\begin{smallmatrix}11\\2\end{smallmatrix}\right\rvert$	$\overset{(11)\;(22)}{2+1}$	$\overset{(12)}{1}$	$\overset{(23)}{1}$	$\overset{(13)}{-\sqrt{\frac{1}{2}}}$	$\overset{(13)}{\sqrt{\frac{3}{2}}}$	·	·	·
$\left\langle\begin{smallmatrix}12\\2\end{smallmatrix}\right\rvert$	$\overset{(21)}{1}$	$\overset{(11)\;(22)}{1+2}$	·	$\overset{(23)}{\sqrt{\frac{1}{2}}}$	$\overset{(23)}{\sqrt{\frac{3}{2}}}$	·	$\overset{(13)}{-1}$	·
$\left\langle\begin{smallmatrix}11\\3\end{smallmatrix}\right\rvert$	$\overset{(32)}{1}$	·	$\overset{(11)\;(33)}{2+1}$	$\overset{(12)}{\sqrt{2}}$	·	$\overset{(13)}{1}$	·	·
$\left\langle\begin{smallmatrix}12\\3\end{smallmatrix}\right\rvert$	$\overset{(31)}{-\sqrt{\frac{1}{2}}}$	$\overset{(32)}{\sqrt{\frac{1}{2}}}$	$\overset{(21)}{\sqrt{2}}$	$\overset{(11)\;(22)\;(33)}{1+1+1}$	·	$\overset{(23)}{\sqrt{\frac{1}{2}}}$	$\overset{(12)}{\sqrt{2}}$	$\overset{(13)}{\sqrt{\frac{1}{2}}}$
$\left\langle\begin{smallmatrix}13\\2\end{smallmatrix}\right\rvert$	$\overset{(31)}{\sqrt{\frac{3}{2}}}$	$\overset{(32)}{\sqrt{\frac{3}{2}}}$	·	·	$\overset{(11)\;(22)\;(33)}{1+1+1}$	$\overset{(23)}{\sqrt{\frac{3}{2}}}$	·	$\overset{(13)}{\sqrt{\frac{3}{2}}}$
$\left\langle\begin{smallmatrix}13\\3\end{smallmatrix}\right\rvert$	·	·	$\overset{(31)}{1}$	$\overset{(32)}{\sqrt{\frac{1}{2}}}$	$\overset{(32)}{\sqrt{\frac{3}{2}}}$	$\overset{(11)\;(33)}{1+2}$	·	$\overset{(12)}{1}$
$\left\langle\begin{smallmatrix}22\\3\end{smallmatrix}\right\rvert$	·	$\overset{(31)}{-1}$	·	$\overset{(21)}{\sqrt{2}}$	·	·	$\overset{(22)\;(33)}{2+1}$	$\overset{(23)}{1}$
$\left\langle\begin{smallmatrix}23\\3\end{smallmatrix}\right\rvert$	·	·	·	$\overset{(31)}{\sqrt{\frac{1}{2}}}$	$\overset{(31)}{\sqrt{\frac{3}{2}}}$	$\overset{(21)}{1}$	$\overset{(32)}{1}$	$\overset{(22)\;(33)}{1+2}$

E_{jk}-matrix
Lect.23
p.7-16
and p.74

$$L_z \equiv \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix} = (E_{11}\text{-}E_{33}) = \sqrt{2}\mathbf{v}^1_0$$

dipole (k=1)
∠-momentum
***L**-operators*

$$L_+ \equiv \sqrt{2}\begin{pmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \end{pmatrix} = \sqrt{2}(E_{12}+E_{23}) = L_x + iL_y = \text{-}\sqrt{2}\mathbf{v}^1_1$$

$$L_- \equiv \sqrt{2}\begin{pmatrix} \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \end{pmatrix} = \sqrt{2}(E_{21}+E_{32}) = L_x - iL_y = \sqrt{2}\mathbf{v}^1_{=1}$$

$$L_-\left\lvert\begin{smallmatrix}L\\M\end{smallmatrix}\right\rangle = \sqrt{(L+M)(L-M+1)}\left\lvert\begin{smallmatrix}L\\M-1\end{smallmatrix}\right\rangle$$

$$L_-\left\lvert\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle = \sqrt{(2+2)(2-2+1)}\left\lvert\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle = 2\left\lvert\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle$$

Start with top [2,1]-state:

$$\boxed{\left\lvert\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle = \left\lvert\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{2}\end{smallmatrix}\right\rangle = \left\lvert {}^2D_{M=2}\right\rangle}$$

Number of levels in fermionic spin-1/2 p^3

$$U(6) \supset U(3)\times U(2)$$

$$N = \frac{\begin{smallmatrix}6\\5\\4\end{smallmatrix}}{\begin{smallmatrix}3\\2\\1\end{smallmatrix}} = \frac{120}{6} = 20$$

p^3 (*Nitrogen*)
^4S 4-levels
^2P 6-levels
^2D 10-levels

$\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$ [2,1] *tableau states lowered by* $\mathbf{L}_{-}=\sqrt{2}(E_{21}+E_{32})$

	$M=2$	$M=1$		$M=0$		$M=-1$		$M=-2$
E_{jk}	$\left\|\begin{smallmatrix}11\\2\end{smallmatrix}\right\rangle$	$\left\|\begin{smallmatrix}12\\2\end{smallmatrix}\right\rangle$	$\left\|\begin{smallmatrix}11\\3\end{smallmatrix}\right\rangle$	$\left\|\begin{smallmatrix}12\\3\end{smallmatrix}\right\rangle$	$\left\|\begin{smallmatrix}13\\2\end{smallmatrix}\right\rangle$	$\left\|\begin{smallmatrix}13\\3\end{smallmatrix}\right\rangle$	$\left\|\begin{smallmatrix}22\\3\end{smallmatrix}\right\rangle$	$\left\|\begin{smallmatrix}23\\3\end{smallmatrix}\right\rangle$
$\left\langle\begin{smallmatrix}11\\2\end{smallmatrix}\right\|$	$\overset{(11)\ (22)}{2+1}$	$\overset{(12)}{1}$	$\overset{(23)}{1}$	$\overset{(13)}{-\sqrt{\tfrac{1}{2}}}$	$\overset{(13)}{\sqrt{\tfrac{3}{2}}}$	$\cdot$	$\cdot$	$\cdot$
$\left\langle\begin{smallmatrix}12\\2\end{smallmatrix}\right\|$	$\overset{(21)}{1}$	$\overset{(11)\ (22)}{1+2}$	$\cdot$	$\overset{(23)}{\sqrt{\tfrac{1}{2}}}$	$\overset{(23)}{\sqrt{\tfrac{3}{2}}}$	$\cdot$	$\overset{(13)}{-1}$	$\cdot$
$\left\langle\begin{smallmatrix}11\\3\end{smallmatrix}\right\|$	$\overset{(32)}{1}$	$\cdot$	$\overset{(11)\ (33)}{2+1}$	$\overset{(12)}{\sqrt{2}}$	$\cdot$	$\overset{(13)}{1}$	$\cdot$	$\cdot$
$\left\langle\begin{smallmatrix}12\\3\end{smallmatrix}\right\|$	$\overset{(31)}{-\sqrt{\tfrac{1}{2}}}$	$\overset{(32)}{\sqrt{\tfrac{1}{2}}}$	$\overset{(21)}{\sqrt{2}}$	$\overset{(11)\ (22)\ (33)}{1+1+1}$	$\cdot$	$\overset{(23)}{\sqrt{\tfrac{1}{2}}}$	$\overset{(12)}{\sqrt{2}}$	$\overset{(13)}{\sqrt{\tfrac{1}{2}}}$
$\left\langle\begin{smallmatrix}13\\2\end{smallmatrix}\right\|$	$\overset{(31)}{\sqrt{\tfrac{3}{2}}}$	$\overset{(32)}{\sqrt{\tfrac{3}{2}}}$	$\cdot$	$\cdot$	$\overset{(11)\ (22)\ (33)}{1+1+1}$	$\overset{(23)}{\sqrt{\tfrac{3}{2}}}$	$\cdot$	$\overset{(13)}{\sqrt{\tfrac{3}{2}}}$
$\left\langle\begin{smallmatrix}13\\3\end{smallmatrix}\right\|$	$\cdot$	$\cdot$	$\overset{(31)}{1}$	$\overset{(32)}{\sqrt{\tfrac{1}{2}}}$	$\overset{(32)}{\sqrt{\tfrac{3}{2}}}$	$\overset{(11)\ (33)}{1+2}$	$\cdot$	$\overset{(12)}{1}$
$\left\langle\begin{smallmatrix}22\\3\end{smallmatrix}\right\|$	$\cdot$	$\overset{(31)}{-1}$	$\cdot$	$\overset{(21)}{\sqrt{2}}$	$\cdot$	$\cdot$	$\overset{(22)\ (33)}{2+1}$	$\overset{(23)}{1}$
$\left\langle\begin{smallmatrix}23\\3\end{smallmatrix}\right\|$	$\cdot$	$\cdot$	$\cdot$	$\overset{(31)}{\sqrt{\tfrac{1}{2}}}$	$\overset{(31)}{\sqrt{\tfrac{3}{2}}}$	$\overset{(21)}{1}$	$\overset{(32)}{1}$	$\overset{(22)\ (33)}{1+2}$

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dipole (k=1)
∠-momentum
L*-operators*

E_{jk}-matrix
Lect.23
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$$L_- \equiv \sqrt{2}\begin{pmatrix} \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \end{pmatrix} = \sqrt{2}(E_{21}+E_{32}) = L_x - iL_y = \sqrt{2}\mathbf{v}^1_{=1}$$

$$L_-\left|\begin{smallmatrix}L\\M\end{smallmatrix}\right\rangle = \sqrt{(L+M)(L-M+1)}\left|\begin{smallmatrix}L\\M-1\end{smallmatrix}\right\rangle$$

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Start with top [2,1]-state:

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Number of levels in fermionic spin-1/2 p^3

$$U(6) \supset U(3) \times U(2)$$

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p^3 (*Nitrogen*)

${}^4S$ 4-levels

${}^2P$ 6-levels

${}^2D$ 10-levels

Number of levels in bosonic spin-1 p^3

$$U(9) \supset U(3) \times U(3)$$

$$N = \frac{9\cdot 10\cdot 11}{3\cdot 2\cdot 1} = \frac{3\cdot 5\cdot 11}{1\cdot 1\cdot 1} = 165$$

$\begin{smallmatrix}\boxed{}\boxed{}\\\boxed{}\end{smallmatrix}$ = [2,1] *tableau states lowered by* $\mathbf{L}_- = \sqrt{2}(E_{21}+E_{32})$

E_{jk}	$M=2$	$M=1$		$M=0$		$M=-1$		$M=-2$
	$\left\|\begin{smallmatrix}11\\2\end{smallmatrix}\right\rangle$	$\left\|\begin{smallmatrix}12\\2\end{smallmatrix}\right\rangle$	$\left\|\begin{smallmatrix}11\\3\end{smallmatrix}\right\rangle$	$\left\|\begin{smallmatrix}12\\3\end{smallmatrix}\right\rangle$	$\left\|\begin{smallmatrix}13\\2\end{smallmatrix}\right\rangle$	$\left\|\begin{smallmatrix}13\\3\end{smallmatrix}\right\rangle$	$\left\|\begin{smallmatrix}22\\3\end{smallmatrix}\right\rangle$	$\left\|\begin{smallmatrix}23\\3\end{smallmatrix}\right\rangle$
$\left\langle\begin{smallmatrix}11\\2\end{smallmatrix}\right\|$	$\overset{(11)\ (22)}{2+1}$	$\overset{(12)}{1}$	$\overset{(23)}{1}$	$\overset{(13)}{-\sqrt{\tfrac{1}{2}}}$	$\overset{(13)}{\sqrt{\tfrac{3}{2}}}$	·	·	·
$\left\langle\begin{smallmatrix}12\\2\end{smallmatrix}\right\|$	$\overset{(21)}{1}$	$\overset{(11)\ (22)}{1+2}$	·	$\overset{(23)}{\sqrt{\tfrac{1}{2}}}$	$\overset{(23)}{\sqrt{\tfrac{3}{2}}}$	·	$\overset{(13)}{-1}$	·
$\left\langle\begin{smallmatrix}11\\3\end{smallmatrix}\right\|$	$\overset{(32)}{1}$	·	$\overset{(11)\ (33)}{2+1}$	$\overset{(12)}{\sqrt{2}}$	·	$\overset{(13)}{1}$	·	·
$\left\langle\begin{smallmatrix}12\\3\end{smallmatrix}\right\|$	$\overset{(31)}{-\sqrt{\tfrac{1}{2}}}$	$\overset{(32)}{\sqrt{\tfrac{1}{2}}}$	$\overset{(21)}{\sqrt{2}}$	$\overset{(11)\ (22)\ (33)}{1+1+1}$	·	$\overset{(23)}{\sqrt{\tfrac{1}{2}}}$	$\overset{(12)}{\sqrt{2}}$	$\overset{(13)}{\sqrt{\tfrac{1}{2}}}$
$\left\langle\begin{smallmatrix}13\\2\end{smallmatrix}\right\|$	$\overset{(31)}{\sqrt{\tfrac{3}{2}}}$	$\overset{(32)}{\sqrt{\tfrac{3}{2}}}$	·	·	$\overset{(11)\ (22)\ (33)}{1+1+1}$	$\overset{(23)}{\sqrt{\tfrac{3}{2}}}$	·	$\overset{(13)}{\sqrt{\tfrac{3}{2}}}$
$\left\langle\begin{smallmatrix}13\\3\end{smallmatrix}\right\|$	·	·	$\overset{(31)}{1}$	$\overset{(32)}{\sqrt{\tfrac{1}{2}}}$	$\overset{(32)}{\sqrt{\tfrac{3}{2}}}$	$\overset{(11)\ (33)}{1+2}$	·	$\overset{(12)}{1}$
$\left\langle\begin{smallmatrix}22\\3\end{smallmatrix}\right\|$	·	$\overset{(31)}{-1}$	·	$\overset{(21)}{\sqrt{2}}$	·	·	$\overset{(22)\ (33)}{2+1}$	$\overset{(23)}{1}$
$\left\langle\begin{smallmatrix}23\\3\end{smallmatrix}\right\|$	·	·	·	$\overset{(31)}{\sqrt{\tfrac{1}{2}}}$	$\overset{(31)}{\sqrt{\tfrac{3}{2}}}$	$\overset{(21)}{1}$	$\overset{(32)}{1}$	$\overset{(22)\ (33)}{1+2}$

E_{jk}-matrix
Lect.23
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$$L_z \equiv \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix} = (E_{11}-E_{33}) = \sqrt{2}\mathbf{v}^1_0$$

dipole (k=1)
∠-momentum
L*-operators*

$$L_+ \equiv \sqrt{2}\begin{pmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \end{pmatrix} = \sqrt{2}(E_{12}+E_{23}) = L_x + iL_y = -\sqrt{2}\mathbf{v}^1_1$$

$$L_- \equiv \sqrt{2}\begin{pmatrix} \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \end{pmatrix} = \sqrt{2}(E_{21}+E_{32}) = L_x - iL_y = \sqrt{2}\mathbf{v}^1_{=1}$$

$$L_-\left|\begin{smallmatrix}L\\M\end{smallmatrix}\right\rangle = \sqrt{(L+M)(L-M+1)}\left|\begin{smallmatrix}L\\M-1\end{smallmatrix}\right\rangle$$

$$L_-\left|\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle = \sqrt{(2+2)(2-2+1)}\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle = 2\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle$$

Start with top [2,1]-state:

$$\boxed{\left|\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle = \left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{2}\end{smallmatrix}\right\rangle = \left|{}^2D_{M=2}\right\rangle}$$

$$\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle = \tfrac{1}{2}L_-\left|\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle = \tfrac{1}{2}\sqrt{2}(E_{21}+E_{32})\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{2}\end{smallmatrix}\right\rangle = \frac{1}{\sqrt{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle + \frac{1}{\sqrt{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle = \left|{}^2D_{M=1}\right\rangle$$

$\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$ = [2,1] *tableau states lowered by* $\mathbf{L}_-=\sqrt{2}(E_{21}+E_{32})$

E_{jk}	M=2 $\left\vert{}^{11}_{2}\right\rangle$	M=1 $\left\vert{}^{12}_{2}\right\rangle$	$\left\vert{}^{11}_{3}\right\rangle$	M=0 $\left\vert{}^{12}_{3}\right\rangle$	$\left\vert{}^{13}_{2}\right\rangle$	M=-1 $\left\vert{}^{13}_{3}\right\rangle$	$\left\vert{}^{22}_{3}\right\rangle$	M=-2 $\left\vert{}^{23}_{3}\right\rangle$
$\left\langle{}^{11}_{2}\right\vert$	(11) (22) $2+1$	(12) 1	(23) 1	(13) $-\sqrt{\frac12}$	(13) $\sqrt{\frac32}$	.	.	.
$\left\langle{}^{12}_{2}\right\vert$	(21) 1	(11) (22) $1+2$	.	(23) $\sqrt{\frac12}$	(23) $\sqrt{\frac32}$	.	(13) -1	.
$\left\langle{}^{11}_{3}\right\vert$	(32) 1	.	(11) (33) $2+1$	(12) $\sqrt{2}$	.	(13) 1	.	.
$\left\langle{}^{12}_{3}\right\vert$	(31) $-\sqrt{\frac12}$	(32) $\sqrt{\frac12}$	(21) $\sqrt{2}$	(11) (22) (33) $1+1+1$	.	(23) $\sqrt{\frac12}$	(12) $\sqrt{2}$	(13) $\sqrt{\frac12}$
$\left\langle{}^{13}_{2}\right\vert$	(31) $\sqrt{\frac32}$	(32) $\sqrt{\frac32}$	.	.	(11) (22) (33) $1+1+1$	(23) $\sqrt{\frac32}$	.	(13) $\sqrt{\frac32}$
$\left\langle{}^{13}_{3}\right\vert$	.	.	(31) 1	(32) $\sqrt{\frac12}$	(32) $\sqrt{\frac32}$	(11) (33) $1+2$	.	(12) 1
$\left\langle{}^{22}_{3}\right\vert$	.	(31) -1	.	(21) $\sqrt{2}$	.	.	(22) (33) $2+1$	(23) 1
$\left\langle{}^{23}_{3}\right\vert$	.	.	.	(31) $\sqrt{\frac12}$	(31) $\sqrt{\frac32}$	(21) 1	(32) 1	(22) (33) $1+2$

E_{jk}-matrix
Lect.23
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$$L_z \equiv \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix} = (E_{11}-E_{33}) = \sqrt{2}\mathbf{v}^1_0$$

dipole (k=1)
∠-momentum
***L**-operators*

$$L_+ \equiv \sqrt{2}\begin{pmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \end{pmatrix} = \sqrt{2}(E_{12}+E_{23}) = L_x + iL_y = -\sqrt{2}\mathbf{v}^1_1$$

$$L_- \equiv \sqrt{2}\begin{pmatrix} \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \end{pmatrix} = \sqrt{2}(E_{21}+E_{32}) = L_x - iL_y = \sqrt{2}\mathbf{v}^1_{-1}$$

$$L_-\left|{}^{L}_{M}\right\rangle = \sqrt{(L+M)(L-M+1)}\left|{}^{L}_{M-1}\right\rangle$$

$$L_-\left|{}^{2}_{2}\right\rangle = \sqrt{(2+2)(2-2+1)}\left|{}^{2}_{1}\right\rangle = 2\left|{}^{2}_{1}\right\rangle$$

Start with top [2,1]-state:

$$\left|{}^{2}_{2}\right\rangle = \left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{2}\end{smallmatrix}\right\rangle = \left|{}^2D_{M=2}\right\rangle$$

$$\left|{}^{2}_{1}\right\rangle = \tfrac{1}{2}L_-\left|{}^{2}_{2}\right\rangle = \tfrac{1}{2}\sqrt{2}(E_{21}+E_{32})\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{2}\end{smallmatrix}\right\rangle = \tfrac{1}{\sqrt{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle + \tfrac{1}{\sqrt{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle = \left|{}^2D_{M=1}\right\rangle$$

Orthogonal *M=1* state: $\left|{}^2P_{M=1}\right\rangle = \left|{}^{1}_{1}\right\rangle = \tfrac{1}{\sqrt{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle - \tfrac{1}{\sqrt{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle = \left|{}^2P_{M=1}\right\rangle$

$\begin{array}{l}\square\square\\\square\end{array}=$ [2,1] *tableau states lowered by* $\mathbf{L}_-=\sqrt{2}(E_{21}+E_{32})$

	M=2	*M=1*		*M=0*		*M=-1*		*M=-2*
E_{jk}	$\left\vert{}^{11}_{2}\right\rangle$	$\left\vert{}^{12}_{2}\right\rangle$	$\left\vert{}^{11}_{3}\right\rangle$	$\left\vert{}^{12}_{3}\right\rangle$	$\left\vert{}^{13}_{2}\right\rangle$	$\left\vert{}^{13}_{3}\right\rangle$	$\left\vert{}^{22}_{3}\right\rangle$	$\left\vert{}^{23}_{3}\right\rangle$
$\left\langle{}^{11}_{2}\right\vert$	$\overset{(11)\ (22)}{2+1}$	$\overset{(12)}{1}$	$\overset{(23)}{1}$	$\overset{(13)}{-\sqrt{\tfrac{1}{2}}}$	$\overset{(13)}{\sqrt{\tfrac{3}{2}}}$	$\cdot$	$\cdot$	$\cdot$
$\left\langle{}^{12}_{2}\right\vert$	$\overset{(21)}{1}$	$\overset{(11)\ (22)}{1+2}$	$\cdot$	$\overset{(23)}{\sqrt{\tfrac{1}{2}}}$	$\overset{(23)}{\sqrt{\tfrac{3}{2}}}$	$\cdot$	$\overset{(13)}{-1}$	$\cdot$
$\left\langle{}^{11}_{3}\right\vert$	$\overset{(32)}{1}$	$\cdot$	$\overset{(11)\ (33)}{2+1}$	$\overset{(12)}{\sqrt{2}}$	$\cdot$	$\overset{(13)}{1}$	$\cdot$	$\cdot$
$\left\langle{}^{12}_{3}\right\vert$	$\overset{(31)}{-\sqrt{\tfrac{1}{2}}}$	$\overset{(32)}{\sqrt{\tfrac{1}{2}}}$	$\overset{(21)}{\sqrt{2}}$	$\overset{(11)\ (22)\ (33)}{1+1+1}$	$\cdot$	$\overset{(23)}{\sqrt{\tfrac{1}{2}}}$	$\overset{(12)}{\sqrt{2}}$	$\overset{(13)}{\sqrt{\tfrac{1}{2}}}$
$\left\langle{}^{13}_{2}\right\vert$	$\overset{(31)}{\sqrt{\tfrac{3}{2}}}$	$\overset{(32)}{\sqrt{\tfrac{3}{2}}}$	$\cdot$	$\cdot$	$\overset{(11)\ (22)\ (33)}{1+1+1}$	$\overset{(23)}{\sqrt{\tfrac{3}{2}}}$	$\cdot$	$\overset{(13)}{\sqrt{\tfrac{3}{2}}}$
$\left\langle{}^{13}_{3}\right\vert$	$\cdot$	$\cdot$	$\overset{(31)}{1}$	$\overset{(32)}{\sqrt{\tfrac{1}{2}}}$	$\overset{(32)}{\sqrt{\tfrac{3}{2}}}$	$\overset{(11)\ (33)}{1+2}$	$\cdot$	$\overset{(12)}{1}$
$\left\langle{}^{22}_{3}\right\vert$	$\cdot$	$\overset{(31)}{-1}$	$\cdot$	$\overset{(21)}{\sqrt{2}}$	$\cdot$	$\cdot$	$\overset{(22)\ (33)}{2+1}$	$\overset{(23)}{1}$
$\left\langle{}^{23}_{3}\right\vert$	$\cdot$	$\cdot$	$\cdot$	$\overset{(31)}{\sqrt{\tfrac{1}{2}}}$	$\overset{(31)}{\sqrt{\tfrac{3}{2}}}$	$\overset{(21)}{1}$	$\overset{(32)}{1}$	$\overset{(22)\ (33)}{1+2}$

E_{jk}-matrix
Lect.23
p.7-16
and p.74

$$L_z\equiv\begin{pmatrix}1&\cdot&\cdot\\\cdot&0&\cdot\\\cdot&\cdot&-1\end{pmatrix}=(E_{11}\text{-}E_{33})=\sqrt{2}\mathbf{v}^1_0$$

dipole (k=1)
∠-momentum
***L**-operators*

$$L_+\equiv\sqrt{2}\begin{pmatrix}\cdot&1&\cdot\\\cdot&\cdot&1\\\cdot&\cdot&\cdot\end{pmatrix}=\sqrt{2}(E_{12}+E_{23})=L_x+iL_y=-\sqrt{2}\mathbf{v}^1_1$$

$$L_-\equiv\sqrt{2}\begin{pmatrix}\cdot&\cdot&\cdot\\1&\cdot&\cdot\\\cdot&1&\cdot\end{pmatrix}=\sqrt{2}(E_{21}+E_{32})=L_x-iL_y=\sqrt{2}\mathbf{v}^1_{=1}$$

$$L_-\left\vert{}^{L}_{M}\right\rangle=\sqrt{(L+M)(L-M+1)}\left\vert{}^{L}_{M-1}\right\rangle$$

$$L_-\left\vert{}^{2}_{2}\right\rangle=\sqrt{(2+2)(2-2+1)}\left\vert{}^{2}_{1}\right\rangle=2\left\vert{}^{2}_{1}\right\rangle$$

Start with top [2,1]-state:

$$\boxed{\left\vert{}^{2}_{2}\right\rangle=\left\vert\begin{array}{l}\boxed{1}\boxed{1}\\\boxed{2}\end{array}\right\rangle=\left\vert{}^{2}D_{M=2}\right\rangle}$$

$$\left\vert{}^{2}_{1}\right\rangle=\tfrac{1}{2}L_-\left\vert{}^{2}_{2}\right\rangle=\tfrac{1}{2}\sqrt{2}(E_{21}+E_{32})\left\vert\begin{array}{l}\boxed{1}\boxed{1}\\\boxed{2}\end{array}\right\rangle=\frac{1}{\sqrt{2}}\left\vert\begin{array}{l}\boxed{1}\boxed{2}\\\boxed{2}\end{array}\right\rangle+\frac{1}{\sqrt{2}}\left\vert\begin{array}{l}\boxed{1}\boxed{1}\\\boxed{3}\end{array}\right\rangle=\left\vert{}^{2}D_{M=1}\right\rangle$$

Orthogonal *M=1* state: $\left\vert{}^{2}P_{M=1}\right\rangle=\left\vert{}^{1}_{1}\right\rangle=\frac{1}{\sqrt{2}}\left\vert\begin{array}{l}\boxed{1}\boxed{2}\\\boxed{2}\end{array}\right\rangle-\frac{1}{\sqrt{2}}\left\vert\begin{array}{l}\boxed{1}\boxed{1}\\\boxed{3}\end{array}\right\rangle=\left\vert{}^{2}P_{M=1}\right\rangle$

AMOP
reference links
on pages 2-4

4.25.18 class 26: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

$(S_3)^*(U(3)) \subset U(6)$ models of p^3 electronic spin-orbit states and couplings

[2,1] tableau states lowered by $\mathbf{L}_- = \sqrt{2}(E_{21}+E_{32})$

Top-(J,M) states → to mid-level states

$\ell=1$ p=shell LS states combined to states of definite J

J=3/2 at L=0 (^4S), J=5/2 at L=2 (^2D)

C-G coupling; J=3/2 at L=2 (^2D), J=3/2 at L=1 (^2P), J=1/2 at L=1 (^2P)

Spin-orbit state assembly formula and Slater determinants

$\ell=1$ p=shell LSJ states transformed to Slater determinants from J=3/2 (^4S)

Slater functions for J=5/2, J=3/2 (^2D)

Slater functions for J=3/2 (^2P), J=1/2 (^2P)

Summary of states and level connection paths

Symmetry dimension accounting

Spin-orbit Hamiltonian matrix calculation

Application to spin-orbit and entanglement break-up scattering

$\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$ = [2,1] *tableau states lowered by* $\mathbf{L}_-=\sqrt{2}(E_{21}+E_{32})$

	M=2	*M=1*		*M=0*		*M=-1*		*M=-2*
E_{jk}	$\left\lvert\begin{smallmatrix}11\\2\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}12\\2\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}11\\3\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}12\\3\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}13\\2\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}13\\3\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}22\\3\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}23\\3\end{smallmatrix}\right\rangle$
$\left\langle\begin{smallmatrix}11\\2\end{smallmatrix}\right\rvert$	$\overset{(11)\ (22)}{2+1}$	$\overset{(12)}{1}$	$\overset{(23)}{1}$	$\overset{(13)}{-\sqrt{\frac{1}{2}}}$	$\overset{(13)}{\sqrt{\frac{3}{2}}}$	·	·	·
$\left\langle\begin{smallmatrix}12\\2\end{smallmatrix}\right\rvert$	$\overset{(21)}{1}$	$\overset{(11)\ (22)}{1+2}$	·	$\overset{(23)}{\sqrt{\frac{1}{2}}}$	$\overset{(23)}{\sqrt{\frac{3}{2}}}$	·	$\overset{(13)}{-1}$	·
$\left\langle\begin{smallmatrix}11\\3\end{smallmatrix}\right\rvert$	$\overset{(32)}{1}$	·	$\overset{(11)\ (33)}{2+1}$	$\overset{(12)}{\sqrt{2}}$	·	$\overset{(13)}{1}$	·	·
$\left\langle\begin{smallmatrix}12\\3\end{smallmatrix}\right\rvert$	$\overset{(31)}{-\sqrt{\frac{1}{2}}}$	$\overset{(32)}{\sqrt{\frac{1}{2}}}$	$\overset{(21)}{\sqrt{2}}$	$\overset{(11)\ (22)\ (33)}{1+1+1}$	·	$\overset{(23)}{\sqrt{\frac{1}{2}}}$	$\overset{(12)}{\sqrt{2}}$	$\overset{(13)}{\sqrt{\frac{1}{2}}}$
$\left\langle\begin{smallmatrix}13\\2\end{smallmatrix}\right\rvert$	$\overset{(31)}{\sqrt{\frac{3}{2}}}$	$\overset{(32)}{\sqrt{\frac{3}{2}}}$	·	·	$\overset{(11)\ (22)\ (33)}{1+1+1}$	$\overset{(23)}{\sqrt{\frac{3}{2}}}$	·	$\overset{(13)}{\sqrt{\frac{3}{2}}}$
$\left\langle\begin{smallmatrix}13\\3\end{smallmatrix}\right\rvert$	·	·	$\overset{(31)}{1}$	$\overset{(32)}{\sqrt{\frac{1}{2}}}$	$\overset{(32)}{\sqrt{\frac{3}{2}}}$	$\overset{(11)\ (33)}{1+2}$	·	$\overset{(12)}{1}$
$\left\langle\begin{smallmatrix}22\\3\end{smallmatrix}\right\rvert$	·	$\overset{(31)}{-1}$	·	$\overset{(21)}{\sqrt{2}}$	·	·	$\overset{(22)\ (33)}{2+1}$	$\overset{(23)}{1}$
$\left\langle\begin{smallmatrix}23\\3\end{smallmatrix}\right\rvert$	·	·	·	$\overset{(31)}{\sqrt{\frac{1}{2}}}$	$\overset{(31)}{\sqrt{\frac{3}{2}}}$	$\overset{(21)}{1}$	$\overset{(32)}{1}$	$\overset{(22)\ (33)}{1+2}$

$$L_z \equiv \begin{pmatrix}1&\cdot&\cdot\\\cdot&0&\cdot\\\cdot&\cdot&-1\end{pmatrix} = (E_{11}-E_{33}) = \sqrt{2}\mathbf{v}^1_0$$

dipole (k=1)
∠-momentum
***L**-operators*

E_{jk}-matrix
Lect.23
p.7-16
and p.74

$$L_+ \equiv \sqrt{2}\begin{pmatrix}\cdot&1&\cdot\\\cdot&\cdot&1\\\cdot&\cdot&\cdot\end{pmatrix} = \sqrt{2}(E_{12}+E_{23}) = L_x+iL_y = -\sqrt{2}\mathbf{v}^1_1$$

$$\boxed{L_- \equiv \sqrt{2}\begin{pmatrix}\cdot&\cdot&\cdot\\1&\cdot&\cdot\\\cdot&1&\cdot\end{pmatrix} = \sqrt{2}(E_{21}+E_{32}) = L_x-iL_y = \sqrt{2}\mathbf{v}^1_{=1}}$$

$$L_-\left\lvert\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle = \sqrt{(2+1)(2-1+1)}\left\lvert\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle = \sqrt{6}\left\lvert\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle$$

$$L_-\left\lvert\begin{smallmatrix}L\\M\end{smallmatrix}\right\rangle = \sqrt{(L+M)(L-M+1)}\left\lvert\begin{smallmatrix}L\\M-1\end{smallmatrix}\right\rangle$$

$$L_-\left\lvert\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle = \sqrt{(2+2)(2-2+1)}\left\lvert\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle = 2\left\lvert\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle$$

Start with top [2,1]-state:

$$\boxed{\left\lvert\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle = \left\lvert\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{2}\end{smallmatrix}\right\rangle = \left\lvert{}^2D_{M=2}\right\rangle}$$

$$\left\lvert\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle = \tfrac{1}{2}L_-\left\lvert\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle = \tfrac{1}{2}\sqrt{2}(E_{21}+E_{32})\left\lvert\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{2}\end{smallmatrix}\right\rangle = \frac{1}{\sqrt{2}}\left\lvert\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle + \frac{1}{\sqrt{2}}\left\lvert\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle = \left\lvert{}^2D_{M=1}\right\rangle$$

Orthogonal *M=1* state: $\left\lvert{}^2P_{M=1}\right\rangle = \left\lvert\begin{smallmatrix}1\\1\end{smallmatrix}\right\rangle = \frac{1}{\sqrt{2}}\left\lvert\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle - \frac{1}{\sqrt{2}}\left\lvert\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle = \left\lvert{}^2P_{M=1}\right\rangle$

□□ = [2,1] *tableau states lowered by* $\mathbf{L}_{-}=\sqrt{2}(E_{21}+E_{32})$

E_{jk}	$M=2$	$M=1$		$M=0$		$M=-1$		$M=-2$
	$\left\lvert\begin{smallmatrix}11\\2\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}12\\2\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}11\\3\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}12\\3\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}13\\2\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}13\\3\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}22\\3\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}23\\3\end{smallmatrix}\right\rangle$
$\left\langle\begin{smallmatrix}11\\2\end{smallmatrix}\right\rvert$	$\overset{(11)\ (22)}{2+1}$	$\overset{(12)}{1}$	$\overset{(23)}{1}$	$\overset{(13)}{-\sqrt{\frac{1}{2}}}$	$\overset{(13)}{\sqrt{\frac{3}{2}}}$	·	·	·
$\left\langle\begin{smallmatrix}12\\2\end{smallmatrix}\right\rvert$	$\overset{(21)}{1}$	$\overset{(11)\ (22)}{1+2}$	·	$\overset{(23)}{\sqrt{\frac{1}{2}}}$	$\overset{(23)}{\sqrt{\frac{3}{2}}}$	·	$\overset{(13)}{-1}$	·
$\left\langle\begin{smallmatrix}11\\3\end{smallmatrix}\right\rvert$	$\overset{(32)}{1}$	·	$\overset{(11)\ (33)}{2+1}$	$\overset{(12)}{\sqrt{2}}$	·	$\overset{(13)}{1}$	·	·
$\left\langle\begin{smallmatrix}12\\3\end{smallmatrix}\right\rvert$	$\overset{(31)}{-\sqrt{\frac{1}{2}}}$	$\overset{(32)}{\sqrt{\frac{1}{2}}}$	$\overset{(21)}{\sqrt{2}}$	$\overset{(11)\ (22)\ (33)}{1+1+1}$	·	$\overset{(23)}{\sqrt{\frac{1}{2}}}$	$\overset{(12)}{\sqrt{2}}$	$\overset{(13)}{\sqrt{\frac{1}{2}}}$
$\left\langle\begin{smallmatrix}13\\2\end{smallmatrix}\right\rvert$	$\overset{(31)}{\sqrt{\frac{3}{2}}}$	$\overset{(32)}{\sqrt{\frac{3}{2}}}$	·	·	$\overset{(11)\ (22)\ (33)}{1+1+1}$	$\overset{(23)}{\sqrt{\frac{3}{2}}}$	·	$\overset{(13)}{\sqrt{\frac{3}{2}}}$
$\left\langle\begin{smallmatrix}13\\3\end{smallmatrix}\right\rvert$	·	·	$\overset{(31)}{1}$	$\overset{(32)}{\sqrt{\frac{1}{2}}}$	$\overset{(32)}{\sqrt{\frac{3}{2}}}$	$\overset{(11)\ (33)}{1+2}$	·	$\overset{(12)}{1}$
$\left\langle\begin{smallmatrix}22\\3\end{smallmatrix}\right\rvert$	·	$\overset{(31)}{-1}$	·	$\overset{(21)}{\sqrt{2}}$	·	·	$\overset{(22)\ (33)}{2+1}$	$\overset{(23)}{1}$
$\left\langle\begin{smallmatrix}23\\3\end{smallmatrix}\right\rvert$	·	·	·	$\overset{(31)}{\sqrt{\frac{1}{2}}}$	$\overset{(31)}{\sqrt{\frac{3}{2}}}$	$\overset{(21)}{1}$	$\overset{(32)}{1}$	$\overset{(22)\ (33)}{1+2}$

E_{jk}-matrix
Lect.23
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$$L_z \equiv \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix} = (E_{11}-E_{33}) = \sqrt{2}\mathbf{v}^1_0$$

dipole (k=1)
∠-momentum
L-*operators*

$$L_+ \equiv \sqrt{2}\begin{pmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \end{pmatrix} = \sqrt{2}(E_{12}+E_{23}) = L_x+iL_y = -\sqrt{2}\mathbf{v}^1_1$$

$$\boxed{L_- \equiv \sqrt{2}\begin{pmatrix} \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \end{pmatrix} = \sqrt{2}(E_{21}+E_{32}) = L_x-iL_y = \sqrt{2}\mathbf{v}^1_{=1}}$$

$$L_-\left\lvert\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle = \sqrt{(2+1)(2-1+1)}\left\lvert\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle = \sqrt{6}\left\lvert\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle$$

$$\left\lvert\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle = \tfrac{1}{\sqrt{6}}L_-\left\lvert\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle = \tfrac{1}{\sqrt{6}}\cancel{\sqrt{2}}(E_{21}+E_{32})\cancel{\tfrac{1}{\sqrt{2}}}\left(\left\lvert\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle + \left\lvert\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle\right)$$

$$L_-\left\lvert\begin{smallmatrix}L\\M\end{smallmatrix}\right\rangle = \sqrt{(L+M)(L-M+1)}\left\lvert\begin{smallmatrix}L\\M-1\end{smallmatrix}\right\rangle$$

$$L_-\left\lvert\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle = \sqrt{(2+2)(2-2+1)}\left\lvert\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle = 2\left\lvert\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle$$

Start with top [2,1]-state:

$$\boxed{\left\lvert\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle = \left\lvert\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{2}\end{smallmatrix}\right\rangle = \left\lvert {}^2D_{M=2}\right\rangle}$$

$$\left\lvert\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle = \tfrac{1}{2}L_-\left\lvert\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle = \tfrac{1}{2}\sqrt{2}(E_{21}+E_{32})\left\lvert\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{2}\end{smallmatrix}\right\rangle = \boxed{\tfrac{1}{\sqrt{2}}\left\lvert\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle + \tfrac{1}{\sqrt{2}}\left\lvert\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle = \left\lvert {}^2D_{M=1}\right\rangle}$$

Orthogonal *M=1* state: $\left\lvert {}^2P_{M=1}\right\rangle = \left\lvert\begin{smallmatrix}1\\1\end{smallmatrix}\right\rangle = \boxed{\tfrac{1}{\sqrt{2}}\left\lvert\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle - \tfrac{1}{\sqrt{2}}\left\lvert\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle = \left\lvert {}^2P_{M=1}\right\rangle}$

[2,1] *tableau states lowered by* **L**₋=√2(*E*₂₁+*E*₃₂)

E_{jk}	$M=2$ $\left\|\begin{smallmatrix}11\\2\end{smallmatrix}\right\rangle$	$M=1$ $\left\|\begin{smallmatrix}12\\2\end{smallmatrix}\right\rangle$	$\left\|\begin{smallmatrix}11\\3\end{smallmatrix}\right\rangle$	$M=0$ $\left\|\begin{smallmatrix}12\\3\end{smallmatrix}\right\rangle$	$\left\|\begin{smallmatrix}13\\2\end{smallmatrix}\right\rangle$	$M=-1$ $\left\|\begin{smallmatrix}13\\3\end{smallmatrix}\right\rangle$	$\left\|\begin{smallmatrix}22\\3\end{smallmatrix}\right\rangle$	$M=-2$ $\left\|\begin{smallmatrix}23\\3\end{smallmatrix}\right\rangle$
$\left\langle\begin{smallmatrix}11\\2\end{smallmatrix}\right\|$	(11) (22) $2+1$	(12) 1	(23) 1	(13) $-\sqrt{\frac{1}{2}}$	(13) $\sqrt{\frac{3}{2}}$	·	·	·
$\left\langle\begin{smallmatrix}12\\2\end{smallmatrix}\right\|$	(21) 1	(11) (22) $1+2$	·	(23) $\sqrt{\frac{1}{2}}$	(23) $\sqrt{\frac{3}{2}}$	·	(13) -1	·
$\left\langle\begin{smallmatrix}11\\3\end{smallmatrix}\right\|$	(32) 1	·	(11) (33) $2+1$	(12) $\sqrt{2}$	·	(13) 1	·	·
$\left\langle\begin{smallmatrix}12\\3\end{smallmatrix}\right\|$	(31) $-\sqrt{\frac{1}{2}}$	(32) $\sqrt{\frac{1}{2}}$	(21) $\sqrt{2}$	(11) (22) (33) $1+1+1$	·	(23) $\sqrt{\frac{1}{2}}$	(12) $\sqrt{2}$	(13) $\sqrt{\frac{1}{2}}$
$\left\langle\begin{smallmatrix}13\\2\end{smallmatrix}\right\|$	(31) $\sqrt{\frac{3}{2}}$	(32) $\sqrt{\frac{3}{2}}$	·	·	(11) (22) (33) $1+1+1$	(23) $\sqrt{\frac{3}{2}}$	·	(13) $\sqrt{\frac{3}{2}}$
$\left\langle\begin{smallmatrix}13\\3\end{smallmatrix}\right\|$	·	·	(31) 1	(32) $\sqrt{\frac{1}{2}}$	(32) $\sqrt{\frac{3}{2}}$	(11) (33) $1+2$	·	(12) 1
$\left\langle\begin{smallmatrix}22\\3\end{smallmatrix}\right\|$	·	(31) -1	·	(21) $\sqrt{2}$	·	·	(22) (33) $2+1$	(23) 1
$\left\langle\begin{smallmatrix}23\\3\end{smallmatrix}\right\|$	·	·	·	(31) $\sqrt{\frac{1}{2}}$	(31) $\sqrt{\frac{3}{2}}$	(21) 1	(32) 1	(22) (33) $1+2$

$$L_z \equiv \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix} = (E_{11}\text{-}E_{33}) = \sqrt{2}\mathbf{v}^1_0$$

dipole (k=1)
∠-momentum
L*-operators*

$$L_+ \equiv \sqrt{2}\begin{pmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \end{pmatrix} = \sqrt{2}(E_{12}+E_{23}) = L_x + iL_y = \text{-}\sqrt{2}\mathbf{v}^1_1$$

*E*ⱼₖ*-matrix*
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$$L_- \equiv \sqrt{2}\begin{pmatrix} \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \end{pmatrix} = \sqrt{2}(E_{21}+E_{32}) = L_x - iL_y = \sqrt{2}\mathbf{v}^1_{-1}$$

$$L_-\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle = \sqrt{(2+1)(2-1+1)}\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle = \sqrt{6}\left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle$$

$$\left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle = \tfrac{1}{\sqrt{6}}L_-\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle = \tfrac{1}{\sqrt{6}}\cancel{\sqrt{2}}(E_{21}+E_{32})\cancel{\tfrac{1}{\sqrt{2}}}\left(\left|\begin{smallmatrix}1&2\\2&\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}1&1\\3&\end{smallmatrix}\right\rangle\right)$$

$$= \tfrac{1}{\sqrt{6}}\left(E_{21}\left|\begin{smallmatrix}1&2\\2&\end{smallmatrix}\right\rangle + E_{21}\left|\begin{smallmatrix}1&1\\3&\end{smallmatrix}\right\rangle + E_{32}\left|\begin{smallmatrix}1&2\\2&\end{smallmatrix}\right\rangle + E_{32}\left|\begin{smallmatrix}1&1\\3&\end{smallmatrix}\right\rangle\right)$$

$$L_-\left|\begin{smallmatrix}L\\M\end{smallmatrix}\right\rangle = \sqrt{(L+M)(L-M+1)}\left|\begin{smallmatrix}L\\M-1\end{smallmatrix}\right\rangle$$

$$L_-\left|\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle = \sqrt{(2+2)(2-2+1)}\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle = 2\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle$$

Start with top [2,1]-state:

$$\left|\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle = \left|\begin{smallmatrix}1&1\\2&\end{smallmatrix}\right\rangle = \left|{}^2D_{M=2}\right\rangle$$

$$\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle = \tfrac{1}{2}L_-\left|\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle = \tfrac{1}{2}\sqrt{2}(E_{21}+E_{32})\left|\begin{smallmatrix}1&1\\2&\end{smallmatrix}\right\rangle = \tfrac{1}{\sqrt{2}}\left|\begin{smallmatrix}1&2\\2&\end{smallmatrix}\right\rangle + \tfrac{1}{\sqrt{2}}\left|\begin{smallmatrix}1&1\\3&\end{smallmatrix}\right\rangle = \left|{}^2D_{M=1}\right\rangle$$

Orthogonal *M=1* state: $\left|{}^2P_{M=1}\right\rangle = \left|\begin{smallmatrix}1\\1\end{smallmatrix}\right\rangle = \tfrac{1}{\sqrt{2}}\left|\begin{smallmatrix}1&2\\2&\end{smallmatrix}\right\rangle - \tfrac{1}{\sqrt{2}}\left|\begin{smallmatrix}1&1\\3&\end{smallmatrix}\right\rangle = \left|{}^2P_{M=1}\right\rangle$

$\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$ = [2,1] *tableau states lowered by* $\mathbf{L}_-=\sqrt{2}(E_{21}+E_{32})$

	$M=2$	$M=1$		$M=0$		$M=-1$		$M=-2$
E_{jk}	$\left\|\begin{smallmatrix}11\\2\end{smallmatrix}\right\rangle$	$\left\|\begin{smallmatrix}12\\2\end{smallmatrix}\right\rangle$	$\left\|\begin{smallmatrix}11\\3\end{smallmatrix}\right\rangle$	$\left\|\begin{smallmatrix}12\\3\end{smallmatrix}\right\rangle$	$\left\|\begin{smallmatrix}13\\2\end{smallmatrix}\right\rangle$	$\left\|\begin{smallmatrix}13\\3\end{smallmatrix}\right\rangle$	$\left\|\begin{smallmatrix}22\\3\end{smallmatrix}\right\rangle$	$\left\|\begin{smallmatrix}23\\3\end{smallmatrix}\right\rangle$
$\left\langle\begin{smallmatrix}11\\2\end{smallmatrix}\right\|$	$\overset{(11)\ (22)}{2+1}$	$\overset{(12)}{1}$	$\overset{(23)}{1}$	$\overset{(13)}{-\sqrt{\tfrac{1}{2}}}$	$\overset{(13)}{\sqrt{\tfrac{3}{2}}}$	$\cdot$	$\cdot$	$\cdot$
$\left\langle\begin{smallmatrix}12\\2\end{smallmatrix}\right\|$	$\overset{(21)}{1}$	$\overset{(11)\ (22)}{1+2}$	$\cdot$	$\overset{(23)}{\sqrt{\tfrac{1}{2}}}$	$\overset{(23)}{\sqrt{\tfrac{3}{2}}}$	$\cdot$	$\overset{(13)}{-1}$	$\cdot$
$\left\langle\begin{smallmatrix}11\\3\end{smallmatrix}\right\|$	$\overset{(32)}{1}$	$\cdot$	$\overset{(11)\ (33)}{2+1}$	$\overset{(12)}{\sqrt{2}}$	$\cdot$	$\overset{(13)}{1}$	$\cdot$	$\cdot$
$\left\langle\begin{smallmatrix}12\\3\end{smallmatrix}\right\|$	$\overset{(31)}{-\sqrt{\tfrac{1}{2}}}$	$\boxed{\overset{(32)}{\sqrt{\tfrac{1}{2}}}}$	$\boxed{\overset{(21)}{\sqrt{2}}}$	$\overset{(11)\ (22)\ (33)}{1+1+1}$	$\cdot$	$\overset{(23)}{\sqrt{\tfrac{1}{2}}}$	$\overset{(12)}{\sqrt{2}}$	$\overset{(13)}{\sqrt{\tfrac{1}{2}}}$
$\left\langle\begin{smallmatrix}13\\2\end{smallmatrix}\right\|$	$\overset{(31)}{\sqrt{\tfrac{3}{2}}}$	$\boxed{\overset{(32)}{\sqrt{\tfrac{3}{2}}}}$	$\cdot$	$\cdot$	$\overset{(11)\ (22)\ (33)}{1+1+1}$	$\overset{(23)}{\sqrt{\tfrac{3}{2}}}$	$\cdot$	$\overset{(13)}{\sqrt{\tfrac{3}{2}}}$
$\left\langle\begin{smallmatrix}13\\3\end{smallmatrix}\right\|$	$\cdot$	$\cdot$	$\overset{(31)}{1}$	$\overset{(32)}{\sqrt{\tfrac{1}{2}}}$	$\overset{(32)}{\sqrt{\tfrac{3}{2}}}$	$\overset{(11)\ (33)}{1+2}$	$\cdot$	$\overset{(12)}{1}$
$\left\langle\begin{smallmatrix}22\\3\end{smallmatrix}\right\|$	$\cdot$	$\overset{(31)}{-1}$	$\cdot$	$\overset{(21)}{\sqrt{2}}$	$\cdot$	$\cdot$	$\overset{(22)\ (33)}{2+1}$	$\overset{(23)}{1}$
$\left\langle\begin{smallmatrix}23\\3\end{smallmatrix}\right\|$	$\cdot$	$\cdot$	$\cdot$	$\overset{(31)}{\sqrt{\tfrac{1}{2}}}$	$\overset{(31)}{\sqrt{\tfrac{3}{2}}}$	$\overset{(21)}{1}$	$\overset{(32)}{1}$	$\overset{(22)\ (33)}{1+2}$

E

E_{jk}*-matrix*
Lect.23
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dipole (k=1)
∠-momentum
***L**-operators*

$$L_z\equiv\begin{pmatrix}1&\cdot&\cdot\\\cdot&0&\cdot\\\cdot&\cdot&-1\end{pmatrix}=(E_{11}-E_{33})=\sqrt{2}\mathbf{v}^1_0$$

$$L_+\equiv\sqrt{2}\begin{pmatrix}\cdot&1&\cdot\\\cdot&\cdot&1\\\cdot&\cdot&\cdot\end{pmatrix}=\sqrt{2}(E_{12}+E_{23})=L_x+iL_y=-\sqrt{2}\mathbf{v}^1_1$$

$$\boxed{L_-\equiv\sqrt{2}\begin{pmatrix}\cdot&\cdot&\cdot\\1&\cdot&\cdot\\\cdot&1&\cdot\end{pmatrix}=\sqrt{2}(E_{21}+E_{32})=L_x-iL_y=\sqrt{2}\mathbf{v}^1_{=1}}$$

$$L_-\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle=\sqrt{(2+1)(2-1+1)}\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle=\sqrt{6}\left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle$$

$$\left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle=\tfrac{1}{\sqrt{6}}L_-\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle=\tfrac{1}{\sqrt{6}}\cancel{\sqrt{2}}(E_{21}+E_{32})\cancel{\tfrac{1}{\sqrt{2}}}\left(\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle+\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle\right)$$

$$=\tfrac{1}{\sqrt{6}}\left(E_{21}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle+E_{21}\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle+E_{32}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle+E_{32}\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle\right)$$

$$=\tfrac{1}{\sqrt{6}}\left(0\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle+\sqrt{2}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle+\sqrt{\tfrac{1}{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle+\sqrt{\tfrac{3}{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{2}\end{smallmatrix}\right\rangle+0\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle\right)$$

$$L_-\left|\begin{smallmatrix}L\\M\end{smallmatrix}\right\rangle=\sqrt{(L+M)(L-M+1)}\left|\begin{smallmatrix}L\\M-1\end{smallmatrix}\right\rangle$$

$$L_-\left|\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle=\sqrt{(2+2)(2-2+1)}\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle=2\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle$$

Start with top [2,1]-state:

$$\boxed{\left|\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle=\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{2}\end{smallmatrix}\right\rangle=\left|{}^2D_{M=2}\right\rangle}$$

$$\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle=\tfrac{1}{2}L_-\left|\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle=\tfrac{1}{2}\sqrt{2}(E_{21}+E_{32})\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{2}\end{smallmatrix}\right\rangle=\boxed{\tfrac{1}{\sqrt{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle+\tfrac{1}{\sqrt{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle=\left|{}^2D_{M=1}\right\rangle}$$

Orthogonal $M=1$ state: $\left|{}^2P_{M=1}\right\rangle=\left|\begin{smallmatrix}1\\1\end{smallmatrix}\right\rangle=\boxed{\tfrac{1}{\sqrt{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle-\tfrac{1}{\sqrt{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle=\left|{}^2P_{M=1}\right\rangle}$

$\begin{array}{l}\square\square\\ \square\end{array}=$ [2,1] *tableau states lowered by* $\mathbf{L}_-=\sqrt{2}(E_{21}+E_{32})$

	M=2	*M=1*		*M=0*		*M=-1*		*M=-2*
E_{jk}	$\left\lvert\begin{smallmatrix}11\\2\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}12\\2\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}11\\3\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}12\\3\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}13\\2\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}13\\3\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}22\\3\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}23\\3\end{smallmatrix}\right\rangle$
$\left\langle\begin{smallmatrix}11\\2\end{smallmatrix}\right\rvert$	$\overset{(11)\ (22)}{2+1}$	$\overset{(12)}{1}$	$\overset{(23)}{1}$	$\overset{(13)}{-\sqrt{\frac{1}{2}}}$	$\overset{(13)}{\sqrt{\frac{3}{2}}}$	·	·	·
$\left\langle\begin{smallmatrix}12\\2\end{smallmatrix}\right\rvert$	$\overset{(21)}{1}$	$\overset{(11)\ (22)}{1+2}$	·	$\overset{(23)}{\sqrt{\frac{1}{2}}}$	$\overset{(23)}{\sqrt{\frac{3}{2}}}$	·	$\overset{(13)}{-1}$	·
$\left\langle\begin{smallmatrix}11\\3\end{smallmatrix}\right\rvert$	$\overset{(32)}{1}$	·	$\overset{(11)\ (33)}{2+1}$	$\overset{(12)}{\sqrt{2}}$	·	$\overset{(13)}{1}$	·	·
$\left\langle\begin{smallmatrix}12\\3\end{smallmatrix}\right\rvert$	$\overset{(31)}{-\sqrt{\frac{1}{2}}}$	$\overset{(32)}{\sqrt{\frac{1}{2}}}$	$\overset{(21)}{\sqrt{2}}$	$\overset{(11)\ (22)\ (33)}{1+1+1}$	·	$\overset{(23)}{\sqrt{\frac{1}{2}}}$	$\overset{(12)}{\sqrt{2}}$	$\overset{(13)}{\sqrt{\frac{1}{2}}}$
$\left\langle\begin{smallmatrix}13\\2\end{smallmatrix}\right\rvert$	$\overset{(31)}{\sqrt{\frac{3}{2}}}$	$\overset{(32)}{\sqrt{\frac{3}{2}}}$	·	·	$\overset{(11)\ (22)\ (33)}{1+1+1}$	$\overset{(23)}{\sqrt{\frac{3}{2}}}$	·	$\overset{(13)}{\sqrt{\frac{3}{2}}}$
$\left\langle\begin{smallmatrix}13\\3\end{smallmatrix}\right\rvert$	·	·	$\overset{(31)}{1}$	$\overset{(32)}{\sqrt{\frac{1}{2}}}$	$\overset{(32)}{\sqrt{\frac{3}{2}}}$	$\overset{(11)\ (33)}{1+2}$	·	$\overset{(12)}{1}$
$\left\langle\begin{smallmatrix}22\\3\end{smallmatrix}\right\rvert$	·	$\overset{(31)}{-1}$	·	$\overset{(21)}{\sqrt{2}}$	·	·	$\overset{(22)\ (33)}{2+1}$	$\overset{(23)}{1}$
$\left\langle\begin{smallmatrix}23\\3\end{smallmatrix}\right\rvert$	·	·	·	$\overset{(31)}{\sqrt{\frac{1}{2}}}$	$\overset{(31)}{\sqrt{\frac{3}{2}}}$	$\overset{(21)}{1}$	$\overset{(32)}{1}$	$\overset{(22)\ (33)}{1+2}$

E

E_{jk}-matrix
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$$L_z\equiv\begin{pmatrix}1&\cdot&\cdot\\ \cdot&0&\cdot\\ \cdot&\cdot&-1\end{pmatrix}=(E_{11}-E_{33})=\sqrt{2}\mathbf{v}^1_0$$

dipole (k=1)
∠-momentum
L*-operators*

$$L_+\equiv\sqrt{2}\begin{pmatrix}\cdot&1&\cdot\\ \cdot&\cdot&1\\ \cdot&\cdot&\cdot\end{pmatrix}=\sqrt{2}(E_{12}+E_{23})=L_x+iL_y=-\sqrt{2}\mathbf{v}^1_1$$

$$\boxed{L_-\equiv\sqrt{2}\begin{pmatrix}\cdot&\cdot&\cdot\\ 1&\cdot&\cdot\\ \cdot&1&\cdot\end{pmatrix}=\sqrt{2}(E_{21}+E_{32})=L_x-iL_y=\sqrt{2}\mathbf{v}^1_{=1}}$$

$$L_-\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle=\sqrt{(2+1)(2-1+1)}\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle=\sqrt{6}\left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle$$

$$\left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle=\tfrac{1}{\sqrt{6}}L_-\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle=\tfrac{1}{\sqrt{6}}\sqrt{\cancel{2}}(E_{21}+E_{32})\cancel{\tfrac{1}{\sqrt{2}}}\left(\left|\begin{array}{l}\boxed{1}\boxed{2}\\ \boxed{2}\end{array}\right\rangle+\left|\begin{array}{l}\boxed{1}\boxed{1}\\ \boxed{3}\end{array}\right\rangle\right)$$

$$=\tfrac{1}{\sqrt{6}}\left(E_{21}\left|\begin{array}{l}\boxed{1}\boxed{2}\\ \boxed{2}\end{array}\right\rangle+E_{21}\left|\begin{array}{l}\boxed{1}\boxed{1}\\ \boxed{3}\end{array}\right\rangle+E_{32}\left|\begin{array}{l}\boxed{1}\boxed{2}\\ \boxed{2}\end{array}\right\rangle+E_{32}\left|\begin{array}{l}\boxed{1}\boxed{1}\\ \boxed{3}\end{array}\right\rangle\right)$$

$$=\tfrac{1}{\sqrt{6}}\left(0\left|\begin{array}{l}\boxed{1}\boxed{2}\\ \boxed{2}\end{array}\right\rangle+\sqrt{2}\left|\begin{array}{l}\boxed{1}\boxed{2}\\ \boxed{3}\end{array}\right\rangle+\sqrt{\tfrac{1}{2}}\left|\begin{array}{l}\boxed{1}\boxed{2}\\ \boxed{3}\end{array}\right\rangle+\sqrt{\tfrac{3}{2}}\left|\begin{array}{l}\boxed{1}\boxed{3}\\ \boxed{2}\end{array}\right\rangle+0\left|\begin{array}{l}\boxed{1}\boxed{1}\\ \boxed{3}\end{array}\right\rangle\right)$$

$$=\tfrac{1}{\sqrt{6}}\left(\tfrac{3}{\sqrt{2}}\left|\begin{array}{l}\boxed{1}\boxed{2}\\ \boxed{3}\end{array}\right\rangle+\sqrt{\tfrac{3}{2}}\left|\begin{array}{l}\boxed{1}\boxed{3}\\ \boxed{2}\end{array}\right\rangle\right)=\tfrac{\sqrt{3}}{2}\left|\begin{array}{l}\boxed{1}\boxed{2}\\ \boxed{3}\end{array}\right\rangle+\tfrac{1}{2}\left|\begin{array}{l}\boxed{1}\boxed{3}\\ \boxed{2}\end{array}\right\rangle=\left|{}^2D_{M=0}\right\rangle=\left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle$$

$$L_-\left|\begin{smallmatrix}L\\M\end{smallmatrix}\right\rangle=\sqrt{(L+M)(L-M+1)}\left|\begin{smallmatrix}L\\M-1\end{smallmatrix}\right\rangle$$

$$L_-\left|\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle=\sqrt{(2+2)(2-2+1)}\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle=2\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle$$

Start with top [2,1]-state:

$$\boxed{\left|\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle=\left|\begin{array}{l}\boxed{1}\boxed{1}\\ \boxed{2}\end{array}\right\rangle=\left|{}^2D_{M=2}\right\rangle}$$

$$\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle=\tfrac{1}{2}L_-\left|\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle=\tfrac{1}{2}\sqrt{2}(E_{21}+E_{32})\left|\begin{array}{l}\boxed{1}\boxed{1}\\ \boxed{2}\end{array}\right\rangle=\boxed{\tfrac{1}{\sqrt{2}}\left|\begin{array}{l}\boxed{1}\boxed{2}\\ \boxed{2}\end{array}\right\rangle+\tfrac{1}{\sqrt{2}}\left|\begin{array}{l}\boxed{1}\boxed{1}\\ \boxed{3}\end{array}\right\rangle=\left|{}^2D_{M=1}\right\rangle}$$

Orthogonal *M=1* state: $\left|{}^2P_{M=1}\right\rangle=\left|\begin{smallmatrix}1\\1\end{smallmatrix}\right\rangle=\boxed{\tfrac{1}{\sqrt{2}}\left|\begin{array}{l}\boxed{1}\boxed{2}\\ \boxed{2}\end{array}\right\rangle-\tfrac{1}{\sqrt{2}}\left|\begin{array}{l}\boxed{1}\boxed{1}\\ \boxed{3}\end{array}\right\rangle=\left|{}^2P_{M=1}\right\rangle}$

$\begin{array}{l}\square\square\\ \square\end{array}$ = [2,1] *tableau states lowered by* $\mathbf{L}_-=\sqrt{2}(E_{21}+E_{32})$

	M=2	M=1		M=0		M=-1		M=-2
E_{jk}	$\left\vert\begin{smallmatrix}11\\2\end{smallmatrix}\right\rangle$	$\left\vert\begin{smallmatrix}12\\2\end{smallmatrix}\right\rangle$	$\left\vert\begin{smallmatrix}11\\3\end{smallmatrix}\right\rangle$	$\left\vert\begin{smallmatrix}12\\3\end{smallmatrix}\right\rangle$	$\left\vert\begin{smallmatrix}13\\2\end{smallmatrix}\right\rangle$	$\left\vert\begin{smallmatrix}13\\3\end{smallmatrix}\right\rangle$	$\left\vert\begin{smallmatrix}22\\3\end{smallmatrix}\right\rangle$	$\left\vert\begin{smallmatrix}23\\3\end{smallmatrix}\right\rangle$
$\left\langle\begin{smallmatrix}11\\2\end{smallmatrix}\right\vert$	$\overset{(11)\ (22)}{2+1}$	$\overset{(12)}{1}$	$\overset{(23)}{1}$	$\overset{(13)}{-\sqrt{\frac{1}{2}}}$	$\overset{(13)}{\sqrt{\frac{3}{2}}}$	·	·	·
$\left\langle\begin{smallmatrix}12\\2\end{smallmatrix}\right\vert$	$\overset{(21)}{1}$	$\overset{(11)\ (22)}{1+2}$	·	$\overset{(23)}{\sqrt{\frac{1}{2}}}$	$\overset{(23)}{\sqrt{\frac{3}{2}}}$	·	$\overset{(13)}{-1}$	·
$\left\langle\begin{smallmatrix}11\\3\end{smallmatrix}\right\vert$	$\overset{(32)}{1}$	·	$\overset{(11)\ (33)}{2+1}$	$\overset{(12)}{\sqrt{2}}$	·	$\overset{(13)}{1}$	·	·
$\left\langle\begin{smallmatrix}12\\3\end{smallmatrix}\right\vert$	$\overset{(31)}{-\sqrt{\frac{1}{2}}}$	$\boxed{\overset{(32)}{\sqrt{\frac{1}{2}}}}$	$\boxed{\overset{(21)}{\sqrt{2}}}$	$\overset{(11)\ (22)\ (33)}{1+1+1}$	·	$\overset{(23)}{\sqrt{\frac{1}{2}}}$	$\overset{(12)}{\sqrt{2}}$	$\overset{(13)}{\sqrt{\frac{1}{2}}}$
$\left\langle\begin{smallmatrix}13\\2\end{smallmatrix}\right\vert$	$\overset{(31)}{\sqrt{\frac{3}{2}}}$	$\boxed{\overset{(32)}{\sqrt{\frac{3}{2}}}}$	·	·	$\overset{(11)\ (22)\ (33)}{1+1+1}$	$\overset{(23)}{\sqrt{\frac{3}{2}}}$	·	$\overset{(13)}{\sqrt{\frac{3}{2}}}$
$\left\langle\begin{smallmatrix}13\\3\end{smallmatrix}\right\vert$	·	·	$\overset{(31)}{1}$	$\overset{(32)}{\sqrt{\frac{1}{2}}}$	$\overset{(32)}{\sqrt{\frac{3}{2}}}$	$\overset{(11)\ (33)}{1+2}$	·	$\overset{(12)}{1}$
$\left\langle\begin{smallmatrix}22\\3\end{smallmatrix}\right\vert$	·	$\overset{(31)}{-1}$	·	$\overset{(21)}{\sqrt{2}}$	·	·	$\overset{(22)\ (33)}{2+1}$	$\overset{(23)}{1}$
$\left\langle\begin{smallmatrix}23\\3\end{smallmatrix}\right\vert$	·	·	·	$\overset{(31)}{\sqrt{\frac{1}{2}}}$	$\overset{(31)}{\sqrt{\frac{3}{2}}}$	$\overset{(21)}{1}$	$\overset{(32)}{1}$	$\overset{(22)\ (33)}{1+2}$

$$L_z\equiv\begin{pmatrix}1&\cdot&\cdot\\ \cdot&0&\cdot\\ \cdot&\cdot&-1\end{pmatrix}=(E_{11}\text{-}E_{33})=\sqrt{2}\mathbf{v}^1_0$$

dipole (k=1)
∠-momentum
L-*operators*

E_{jk}-matrix
Lect.23
p.7-16
and p.74

$$L_+\equiv\sqrt{2}\begin{pmatrix}\cdot&1&\cdot\\ \cdot&\cdot&1\\ \cdot&\cdot&\cdot\end{pmatrix}=\sqrt{2}(E_{12}+E_{23})=L_x+iL_y=\text{-}\sqrt{2}\mathbf{v}^1_1$$

$$\boxed{L_-\equiv\sqrt{2}\begin{pmatrix}\cdot&\cdot&\cdot\\ 1&\cdot&\cdot\\ \cdot&1&\cdot\end{pmatrix}=\sqrt{2}(E_{21}+E_{32})=L_x-iL_y=\sqrt{2}\mathbf{v}^1_{=1}}$$

$$L_-\left|\begin{smallmatrix}L\\M\end{smallmatrix}\right\rangle=\sqrt{(L+M)(L-M+1)}\left|\begin{smallmatrix}L\\M-1\end{smallmatrix}\right\rangle$$

$$L_-\left|\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle=\sqrt{(2+2)(2-2+1)}\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle=2\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle$$

Start with top [2,1]-state:

$$\boxed{\left|\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle=\left|\begin{array}{l}\boxed{1}\boxed{1}\\ \boxed{2}\end{array}\right\rangle=\left|{}^2D_{M=2}\right\rangle}$$

$$\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle=\tfrac{1}{2}L_-\left|\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle=\tfrac{1}{2}\sqrt{2}(E_{21}+E_{32})\left|\begin{array}{l}\boxed{1}\boxed{1}\\ \boxed{2}\end{array}\right\rangle=\boxed{\tfrac{1}{\sqrt{2}}\left|\begin{array}{l}\boxed{1}\boxed{2}\\ \boxed{2}\end{array}\right\rangle+\tfrac{1}{\sqrt{2}}\left|\begin{array}{l}\boxed{1}\boxed{1}\\ \boxed{3}\end{array}\right\rangle=\left|{}^2D_{M=1}\right\rangle}$$

Orthogonal *M=1* state: $\left|{}^2P_{M=1}\right\rangle=\left|\begin{smallmatrix}1\\1\end{smallmatrix}\right\rangle=\boxed{\tfrac{1}{\sqrt{2}}\left|\begin{array}{l}\boxed{1}\boxed{2}\\ \boxed{2}\end{array}\right\rangle-\tfrac{1}{\sqrt{2}}\left|\begin{array}{l}\boxed{1}\boxed{1}\\ \boxed{3}\end{array}\right\rangle=\left|{}^2P_{M=1}\right\rangle}$

$$L_-\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle=\sqrt{(2+1)(2-1+1)}\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle=\sqrt{6}\left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle$$

$$\left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle=\tfrac{1}{\sqrt{6}}L_-\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle=\tfrac{1}{\sqrt{6}}\cancel{\sqrt{2}}(E_{21}+E_{32})\cancel{\tfrac{1}{\sqrt{2}}}\left(\left|\begin{array}{l}\boxed{1}\boxed{2}\\ \boxed{2}\end{array}\right\rangle+\left|\begin{array}{l}\boxed{1}\boxed{1}\\ \boxed{3}\end{array}\right\rangle\right)$$

$$=\tfrac{1}{\sqrt{6}}\left(E_{21}\left|\begin{array}{l}\boxed{1}\boxed{2}\\ \boxed{2}\end{array}\right\rangle+\boxed{E_{21}}\left|\begin{array}{l}\boxed{1}\boxed{1}\\ \boxed{3}\end{array}\right\rangle+\boxed{E_{32}}\left|\begin{array}{l}\boxed{1}\boxed{2}\\ \boxed{2}\end{array}\right\rangle+\boxed{E_{32}}\left|\begin{array}{l}\boxed{1}\boxed{1}\\ \boxed{3}\end{array}\right\rangle\right)$$

$$=\tfrac{1}{\sqrt{6}}\left(0\left|\begin{array}{l}\boxed{1}\boxed{2}\\ \boxed{2}\end{array}\right\rangle+\boxed{\sqrt{2}}\left|\begin{array}{l}\boxed{1}\boxed{2}\\ \boxed{3}\end{array}\right\rangle+\boxed{\sqrt{\tfrac{1}{2}}}\left|\begin{array}{l}\boxed{1}\boxed{2}\\ \boxed{3}\end{array}\right\rangle+\boxed{\sqrt{\tfrac{3}{2}}}\left|\begin{array}{l}\boxed{1}\boxed{3}\\ \boxed{2}\end{array}\right\rangle+0\left|\begin{array}{l}\boxed{1}\boxed{1}\\ \boxed{3}\end{array}\right\rangle\right)$$

$$=\tfrac{1}{\sqrt{6}}\left(\tfrac{3}{\sqrt{2}}\left|\begin{array}{l}\boxed{1}\boxed{2}\\ \boxed{3}\end{array}\right\rangle+\sqrt{\tfrac{3}{2}}\left|\begin{array}{l}\boxed{1}\boxed{3}\\ \boxed{2}\end{array}\right\rangle\right)=\boxed{\tfrac{\sqrt{3}}{2}\left|\begin{array}{l}\boxed{1}\boxed{2}\\ \boxed{3}\end{array}\right\rangle+\tfrac{1}{2}\left|\begin{array}{l}\boxed{1}\boxed{3}\\ \boxed{2}\end{array}\right\rangle=\left|{}^2D_{M=0}\right\rangle=\left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle}$$

Orthogonal (L=1, *M=0*) state: $\boxed{\tfrac{-1}{2}\left|\begin{array}{l}\boxed{1}\boxed{2}\\ \boxed{3}\end{array}\right\rangle+\tfrac{\sqrt{3}}{2}\left|\begin{array}{l}\boxed{1}\boxed{3}\\ \boxed{2}\end{array}\right\rangle=\left|{}^2P_{M=0}\right\rangle=\left|\begin{smallmatrix}1\\0\end{smallmatrix}\right\rangle}$

$\begin{smallmatrix}\square\square_{=}\\ \square\end{smallmatrix}$ [2,1] *tableau states lowered by* $\mathbf{L}_{-}=\sqrt{2}(E_{21}+E_{32})$

$$L_z \equiv \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix} = (E_{11}-E_{33}) = \sqrt{2}\mathbf{v}^1_0$$

dipole (k=1)
∠-momentum
***L**-operators*

$$L_+ \equiv \sqrt{2}\begin{pmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \end{pmatrix} = \sqrt{2}(E_{12}+E_{23}) = L_x + iL_y = -\sqrt{2}\mathbf{v}^1_1$$

$$\boxed{L_- \equiv \sqrt{2}\begin{pmatrix} \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \end{pmatrix} = \sqrt{2}(E_{21}+E_{32}) = L_x - iL_y = \sqrt{2}\mathbf{v}^1_{=1}}$$

E_{jk}	$M=2$	$M=1$		$M=0$		$M=-1$		$M=-2$
	$\left\|\begin{smallmatrix}11\\2\end{smallmatrix}\right\rangle$	$\left\|\begin{smallmatrix}12\\2\end{smallmatrix}\right\rangle$	$\left\|\begin{smallmatrix}11\\3\end{smallmatrix}\right\rangle$	$\left\|\begin{smallmatrix}12\\3\end{smallmatrix}\right\rangle$	$\left\|\begin{smallmatrix}13\\2\end{smallmatrix}\right\rangle$	$\left\|\begin{smallmatrix}13\\3\end{smallmatrix}\right\rangle$	$\left\|\begin{smallmatrix}22\\3\end{smallmatrix}\right\rangle$	$\left\|\begin{smallmatrix}23\\3\end{smallmatrix}\right\rangle$
$\left\langle\begin{smallmatrix}11\\2\end{smallmatrix}\right\|$	$\overset{(11)\ (22)}{2+1}$	$\overset{(12)}{1}$	$\overset{(23)}{1}$	$\overset{(13)}{-\sqrt{\tfrac{1}{2}}}$	$\overset{(13)}{\sqrt{\tfrac{3}{2}}}$	$\cdot$	$\cdot$	$\cdot$
$\left\langle\begin{smallmatrix}12\\2\end{smallmatrix}\right\|$	$\overset{(21)}{1}$	$\overset{(11)\ (22)}{1+2}$	$\cdot$	$\overset{(23)}{\sqrt{\tfrac{1}{2}}}$	$\overset{(23)}{\sqrt{\tfrac{3}{2}}}$	$\cdot$	$\overset{(13)}{-1}$	$\cdot$
$\left\langle\begin{smallmatrix}11\\3\end{smallmatrix}\right\|$	$\overset{(32)}{1}$	$\cdot$	$\overset{(11)\ (33)}{2+1}$	$\overset{(12)}{\sqrt{2}}$	$\cdot$	$\overset{(13)}{1}$	$\cdot$	$\cdot$
$\left\langle\begin{smallmatrix}12\\3\end{smallmatrix}\right\|$	$\overset{(31)}{-\sqrt{\tfrac{1}{2}}}$	$\overset{(32)}{\sqrt{\tfrac{1}{2}}}$	$\overset{(21)}{\sqrt{2}}$	$\overset{(11)\ (22)\ (33)}{1+1+1}$	$\cdot$	$\overset{(23)}{\sqrt{\tfrac{1}{2}}}$	$\overset{(12)}{\sqrt{2}}$	$\overset{(13)}{\sqrt{\tfrac{1}{2}}}$
$\left\langle\begin{smallmatrix}13\\2\end{smallmatrix}\right\|$	$\overset{(31)}{\sqrt{\tfrac{3}{2}}}$	$\overset{(32)}{\sqrt{\tfrac{3}{2}}}$	$\cdot$	$\cdot$	$\overset{(11)\ (22)\ (33)}{1+1+1}$	$\overset{(23)}{\sqrt{\tfrac{3}{2}}}$	$\cdot$	$\overset{(13)}{\sqrt{\tfrac{3}{2}}}$
$\left\langle\begin{smallmatrix}13\\3\end{smallmatrix}\right\|$	$\cdot$	$\cdot$	$\overset{(31)}{1}$	$\overset{(32)}{\sqrt{\tfrac{1}{2}}}$	$\overset{(32)}{\sqrt{\tfrac{3}{2}}}$	$\overset{(11)\ (33)}{1+2}$	$\cdot$	$\overset{(12)}{1}$
$\left\langle\begin{smallmatrix}22\\3\end{smallmatrix}\right\|$	$\cdot$	$\overset{(31)}{-1}$	$\cdot$	$\overset{(21)}{\sqrt{2}}$	$\cdot$	$\cdot$	$\overset{(22)\ (33)}{2+1}$	$\overset{(23)}{1}$
$\left\langle\begin{smallmatrix}23\\3\end{smallmatrix}\right\|$	$\cdot$	$\cdot$	$\cdot$	$\overset{(31)}{\sqrt{\tfrac{1}{2}}}$	$\overset{(31)}{\sqrt{\tfrac{3}{2}}}$	$\overset{(21)}{1}$	$\overset{(32)}{1}$	$\overset{(22)\ (33)}{1+2}$

$$L_-\left|\begin{smallmatrix}L\\M\end{smallmatrix}\right\rangle = \sqrt{(L+M)(L-M+1)}\left|\begin{smallmatrix}L\\M-1\end{smallmatrix}\right\rangle$$

$$L_-\left|\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle = \sqrt{(2+2)(2-2+1)}\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle = 2\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle$$

Start with top [2,1]-state:

$$\boxed{\left|\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle = \left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{2}\end{smallmatrix}\right\rangle = \left|{}^2D_{M=2}\right\rangle}$$

$$\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle = \tfrac{1}{2}L_-\left|\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle = \tfrac{1}{2}\sqrt{2}(E_{21}+E_{32})\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{2}\end{smallmatrix}\right\rangle = \boxed{\tfrac{1}{\sqrt{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle + \tfrac{1}{\sqrt{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle = \left|{}^2D_{M=1}\right\rangle}$$

Orthogonal *M=1* state: $\left|{}^2P_{M=1}\right\rangle = \left|\begin{smallmatrix}1\\1\end{smallmatrix}\right\rangle = \boxed{\tfrac{1}{\sqrt{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle - \tfrac{1}{\sqrt{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle = \left|{}^2P_{M=1}\right\rangle}$

$$L_-\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle = \sqrt{(2+1)(2-1+1)}\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle = \sqrt{6}\left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle$$

$$\left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle = \tfrac{1}{\sqrt{6}}L_-\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle = \tfrac{1}{\sqrt{6}}\cancel{\sqrt{2}}(E_{21}+E_{32})\cancel{\tfrac{1}{\sqrt{2}}}\left(\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle\right)$$

$$= \tfrac{1}{\sqrt{6}}\left(E_{21}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle + E_{21}\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle + E_{32}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle + E_{32}\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle\right)$$

$$= \tfrac{1}{\sqrt{6}}\left(0\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle + \sqrt{2}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle + \sqrt{\tfrac{1}{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle + \sqrt{\tfrac{3}{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{2}\end{smallmatrix}\right\rangle + 0\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle\right)$$

$$= \tfrac{1}{\sqrt{6}}\left(\tfrac{3}{\sqrt{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle + \sqrt{\tfrac{3}{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{2}\end{smallmatrix}\right\rangle\right) = \boxed{\tfrac{\sqrt{3}}{2}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle + \tfrac{1}{2}\left|\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{2}\end{smallmatrix}\right\rangle = \left|{}^2D_{M=0}\right\rangle = \left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle}$$

Orthogonal (L=1, *M=0*) state: $\boxed{\tfrac{-1}{2}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle + \tfrac{\sqrt{3}}{2}\left|\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{2}\end{smallmatrix}\right\rangle = \left|{}^2P_{M=0}\right\rangle = \left|\begin{smallmatrix}1\\0\end{smallmatrix}\right\rangle}$

$$L_-\left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle = \sqrt{(2+0)(2-0+1)}\left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle = \sqrt{6}\left|\begin{smallmatrix}2\\-1\end{smallmatrix}\right\rangle$$

$$\left|\begin{smallmatrix}2\\-1\end{smallmatrix}\right\rangle = \tfrac{1}{\sqrt{6}}L_-\left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle = \tfrac{1}{\sqrt{6}}\sqrt{2}(E_{21}+E_{32})\left(\tfrac{\sqrt{3}}{2}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle + \tfrac{1}{2}\left|\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{2}\end{smallmatrix}\right\rangle\right)$$

$\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$ = [2,1] *tableau states lowered by* $\mathbf{L}_{-}=\sqrt{2}(E_{21}+E_{32})$

E_{jk}	$M=2$	$M=1$		$M=0$		$M=-1$		$M=-2$
	$\left\|\begin{smallmatrix}11\\2\end{smallmatrix}\right\rangle$	$\left\|\begin{smallmatrix}12\\2\end{smallmatrix}\right\rangle$	$\left\|\begin{smallmatrix}11\\3\end{smallmatrix}\right\rangle$	$\left\|\begin{smallmatrix}12\\3\end{smallmatrix}\right\rangle$	$\left\|\begin{smallmatrix}13\\2\end{smallmatrix}\right\rangle$	$\left\|\begin{smallmatrix}13\\3\end{smallmatrix}\right\rangle$	$\left\|\begin{smallmatrix}22\\3\end{smallmatrix}\right\rangle$	$\left\|\begin{smallmatrix}23\\3\end{smallmatrix}\right\rangle$
$\left\langle\begin{smallmatrix}11\\2\end{smallmatrix}\right\|$	$\overset{(11)\ (22)}{2+1}$	$\overset{(12)}{1}$	$\overset{(23)}{1}$	$\overset{(13)}{-\sqrt{\tfrac{1}{2}}}$	$\overset{(13)}{\sqrt{\tfrac{3}{2}}}$	.	.	.
$\left\langle\begin{smallmatrix}12\\2\end{smallmatrix}\right\|$	$\overset{(21)}{1}$	$\overset{(11)\ (22)}{1+2}$	.	$\overset{(23)}{\sqrt{\tfrac{1}{2}}}$	$\overset{(23)}{\sqrt{\tfrac{3}{2}}}$	.	$\overset{(13)}{-1}$	.
$\left\langle\begin{smallmatrix}11\\3\end{smallmatrix}\right\|$	$\overset{(32)}{1}$	.	$\overset{(11)\ (33)}{2+1}$	$\overset{(12)}{\sqrt{2}}$	.	$\overset{(13)}{1}$	.	.
$\left\langle\begin{smallmatrix}12\\3\end{smallmatrix}\right\|$	$\overset{(31)}{-\sqrt{\tfrac{1}{2}}}$	$\overset{(32)}{\sqrt{\tfrac{1}{2}}}$	$\overset{(21)}{\sqrt{2}}$	$\overset{(11)\ (22)\ (33)}{1+1+1}$	.	$\overset{(23)}{\sqrt{\tfrac{1}{2}}}$	$\overset{(12)}{\sqrt{2}}$	$\overset{(13)}{\sqrt{\tfrac{1}{2}}}$
$\left\langle\begin{smallmatrix}13\\2\end{smallmatrix}\right\|$	$\overset{(31)}{\sqrt{\tfrac{3}{2}}}$	$\overset{(32)}{\sqrt{\tfrac{3}{2}}}$	.	.	$\overset{(11)\ (22)\ (33)}{1+1+1}$	$\overset{(23)}{\sqrt{\tfrac{3}{2}}}$	.	$\overset{(13)}{\sqrt{\tfrac{3}{2}}}$
$\left\langle\begin{smallmatrix}13\\3\end{smallmatrix}\right\|$	.	.	$\overset{(31)}{1}$	$\boxed{\overset{(32)}{\sqrt{\tfrac{1}{2}}}}$	$\boxed{\overset{(32)}{\sqrt{\tfrac{3}{2}}}}$	$\overset{(11)\ (33)}{1+2}$	.	$\overset{(12)}{1}$
$\left\langle\begin{smallmatrix}22\\3\end{smallmatrix}\right\|$	.	$\overset{(31)}{-1}$	.	$\boxed{\overset{(21)}{\sqrt{2}}}$	$\boxed{\cdot}$	.	$\overset{(22)\ (33)}{2+1}$	$\overset{(23)}{1}$
$\left\langle\begin{smallmatrix}23\\3\end{smallmatrix}\right\|$	.	.	.	$\overset{(31)}{\sqrt{\tfrac{1}{2}}}$	$\overset{(31)}{\sqrt{\tfrac{3}{2}}}$	$\overset{(21)}{1}$	$\overset{(32)}{1}$	$\overset{(22)\ (33)}{1+2}$

$$L_z \equiv \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix} = (E_{11}-E_{33}) = \sqrt{2}\mathbf{v}^1_0$$

dipole (k=1)
∠-momentum
L-*operators*

$$L_+ \equiv \sqrt{2}\begin{pmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \end{pmatrix} = \sqrt{2}(E_{12}+E_{23}) = L_x + iL_y = -\sqrt{2}\mathbf{v}^1_1$$

$$\boxed{L_- \equiv \sqrt{2}\begin{pmatrix} \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \end{pmatrix} = \sqrt{2}(E_{21}+E_{32}) = L_x - iL_y = \sqrt{2}\mathbf{v}^1_{-1}}$$

$$L_-\left|\begin{smallmatrix}L\\M\end{smallmatrix}\right\rangle = \sqrt{(L+M)(L-M+1)}\left|\begin{smallmatrix}L\\M-1\end{smallmatrix}\right\rangle$$

Start with top [2,1]-state:

$$\boxed{\left|\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle = \left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{2}\end{smallmatrix}\right\rangle = \left|{}^2D_{M=2}\right\rangle}$$

$$L_-\left|\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle = \sqrt{(2+2)(2-2+1)}\left|\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle = 2\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle$$

$$\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle = \tfrac{1}{2}L_-\left|\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle = \tfrac{1}{2}\sqrt{2}(E_{21}+E_{32})\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{2}\end{smallmatrix}\right\rangle = \boxed{\tfrac{1}{\sqrt{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle + \tfrac{1}{\sqrt{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle = \left|{}^2D_{M=1}\right\rangle}$$

Orthogonal $M=1$ state: $\left|{}^2P_{M=1}\right\rangle = \left|\begin{smallmatrix}1\\1\end{smallmatrix}\right\rangle = \boxed{\tfrac{1}{\sqrt{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle - \tfrac{1}{\sqrt{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle = \left|{}^2P_{M=1}\right\rangle}$

$$L_-\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle = \sqrt{(2+1)(2-1+1)}\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle = \sqrt{6}\left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle$$

$$\left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle = \tfrac{1}{\sqrt{6}}L_-\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle = \tfrac{1}{\sqrt{6}}\cancel{\sqrt{2}}(E_{21}+E_{32})\cancel{\tfrac{1}{\sqrt{2}}}\left(\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle\right)$$

$$= \tfrac{1}{\sqrt{6}}\left(E_{21}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle + E_{21}\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle + E_{32}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle + E_{32}\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle\right)$$

$$= \tfrac{1}{\sqrt{6}}\left(0\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle + \sqrt{2}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle + \sqrt{\tfrac{1}{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle + \sqrt{\tfrac{3}{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{2}\end{smallmatrix}\right\rangle + 0\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle\right)$$

$$= \tfrac{1}{\sqrt{6}}\left(\tfrac{3}{\sqrt{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle + \sqrt{\tfrac{3}{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{2}\end{smallmatrix}\right\rangle\right) = \boxed{\tfrac{\sqrt{3}}{2}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle + \tfrac{1}{2}\left|\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{2}\end{smallmatrix}\right\rangle = \left|{}^2D_{M=0}\right\rangle = \left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle}$$

Orthogonal (L=1, $M=0$) state: $\boxed{\tfrac{-1}{2}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle + \tfrac{\sqrt{3}}{2}\left|\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{2}\end{smallmatrix}\right\rangle = \left|{}^2P_{M=0}\right\rangle = \left|\begin{smallmatrix}1\\0\end{smallmatrix}\right\rangle}$

$$L_-\left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle = \sqrt{(2+0)(2-0+1)}\left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle = \sqrt{6}\left|\begin{smallmatrix}2\\-1\end{smallmatrix}\right\rangle$$

$$\left|\begin{smallmatrix}2\\-1\end{smallmatrix}\right\rangle = \tfrac{1}{\sqrt{6}}L_-\left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle = \tfrac{1}{\sqrt{6}}\sqrt{2}(E_{21}+E_{32})\left(\tfrac{\sqrt{3}}{2}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle + \tfrac{1}{2}\left|\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{2}\end{smallmatrix}\right\rangle\right)$$

$$= \tfrac{1}{\sqrt{3}}\left(\tfrac{\sqrt{3}}{2}\boxed{E_{21}}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle + \tfrac{1}{2}\boxed{E_{21}}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle + \tfrac{\sqrt{3}}{2}\boxed{E_{32}}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle + \tfrac{1}{2}\boxed{E_{32}}\left|\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{2}\end{smallmatrix}\right\rangle\right)$$

$\begin{smallmatrix}\square\square\\\square\end{smallmatrix}_{=}$ [2,1] *tableau states lowered by* $\mathbf{L}_-=\sqrt{2}(E_{21}+E_{32})$

	M=2	*M=1*		*M=0*		*M=-1*		*M=-2*
E_{jk}	$\left\lvert\begin{smallmatrix}11\\2\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}12\\2\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}11\\3\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}12\\3\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}13\\2\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}13\\3\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}22\\3\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}23\\3\end{smallmatrix}\right\rangle$
$\left\langle\begin{smallmatrix}11\\2\end{smallmatrix}\right\rvert$	$\overset{(11)\ (22)}{2+1}$	$\overset{(12)}{1}$	$\overset{(23)}{1}$	$\overset{(13)}{-\sqrt{\frac{1}{2}}}$	$\overset{(13)}{\sqrt{\frac{3}{2}}}$	·	·	·
$\left\langle\begin{smallmatrix}12\\2\end{smallmatrix}\right\rvert$	$\overset{(21)}{1}$	$\overset{(11)\ (22)}{1+2}$	·	$\overset{(23)}{\sqrt{\frac{1}{2}}}$	$\overset{(23)}{\sqrt{\frac{3}{2}}}$	·	$\overset{(13)}{-1}$	·
$\left\langle\begin{smallmatrix}11\\3\end{smallmatrix}\right\rvert$	$\overset{(32)}{1}$	·	$\overset{(11)\ (33)}{2+1}$	$\overset{(12)}{\sqrt{2}}$	·	$\overset{(13)}{1}$	·	·
$\left\langle\begin{smallmatrix}12\\3\end{smallmatrix}\right\rvert$	$\overset{(31)}{-\sqrt{\frac{1}{2}}}$	$\overset{(32)}{\sqrt{\frac{1}{2}}}$	$\overset{(21)}{\sqrt{2}}$	$\overset{(11)\ (22)\ (33)}{1+1+1}$	·	$\overset{(23)}{\sqrt{\frac{1}{2}}}$	$\overset{(12)}{\sqrt{2}}$	$\overset{(13)}{\sqrt{\frac{1}{2}}}$
$\left\langle\begin{smallmatrix}13\\2\end{smallmatrix}\right\rvert$	$\overset{(31)}{\sqrt{\frac{3}{2}}}$	$\overset{(32)}{\sqrt{\frac{3}{2}}}$	·	·	$\overset{(11)\ (22)\ (33)}{1+1+1}$	$\overset{(23)}{\sqrt{\frac{3}{2}}}$	·	$\overset{(13)}{\sqrt{\frac{3}{2}}}$
$\left\langle\begin{smallmatrix}13\\3\end{smallmatrix}\right\rvert$	·	·	$\overset{(31)}{1}$	$\overset{(32)}{\sqrt{\frac{1}{2}}}$	$\overset{(32)}{\sqrt{\frac{3}{2}}}$	$\overset{(11)\ (33)}{1+2}$	·	$\overset{(12)}{1}$
$\left\langle\begin{smallmatrix}22\\3\end{smallmatrix}\right\rvert$	·	$\overset{(31)}{-1}$	·	$\overset{(21)}{\sqrt{2}}$	·	·	$\overset{(22)\ (33)}{2+1}$	$\overset{(23)}{1}$
$\left\langle\begin{smallmatrix}23\\3\end{smallmatrix}\right\rvert$	·	·	·	$\overset{(31)}{\sqrt{\frac{1}{2}}}$	$\overset{(31)}{\sqrt{\frac{3}{2}}}$	$\overset{(21)}{1}$	$\overset{(32)}{1}$	$\overset{(22)\ (33)}{1+2}$

$$L_z\equiv\begin{pmatrix}1&\cdot&\cdot\\\cdot&0&\cdot\\\cdot&\cdot&-1\end{pmatrix}=(E_{11}-E_{33})=\sqrt{2}\mathbf{v}^1_0$$

dipole (k=1)
∠-momentum
L-*operators*

$$L_+\equiv\sqrt{2}\begin{pmatrix}\cdot&1&\cdot\\\cdot&\cdot&1\\\cdot&\cdot&\cdot\end{pmatrix}=\sqrt{2}(E_{12}+E_{23})=L_x+iL_y=-\sqrt{2}\mathbf{v}^1_1$$

$$L_-\equiv\sqrt{2}\begin{pmatrix}\cdot&\cdot&\cdot\\1&\cdot&\cdot\\\cdot&1&\cdot\end{pmatrix}=\sqrt{2}(E_{21}+E_{32})=L_x-iL_y=\sqrt{2}\mathbf{v}^1_{=1}$$

$$L_-\left\lvert\begin{smallmatrix}L\\M\end{smallmatrix}\right\rangle=\sqrt{(L+M)(L-M+1)}\left\lvert\begin{smallmatrix}L\\M-1\end{smallmatrix}\right\rangle$$

$$L_-\left\lvert\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle=\sqrt{(2+2)(2-2+1)}\left\lvert\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle=2\left\lvert\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle$$

Start with top [2,1]-state:

$$\left\lvert\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle=\left\lvert\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{2}\end{smallmatrix}\right\rangle=\left\lvert{}^2D_{M=2}\right\rangle$$

$$\left\lvert\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle=\tfrac{1}{2}L_-\left\lvert\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle=\tfrac{1}{2}\sqrt{2}(E_{21}+E_{32})\left\lvert\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{2}\end{smallmatrix}\right\rangle=\tfrac{1}{\sqrt{2}}\left\lvert\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle+\tfrac{1}{\sqrt{2}}\left\lvert\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle=\left\lvert{}^2D_{M=1}\right\rangle$$

Orthogonal *M=1* state: $\left\lvert{}^2P_{M=1}\right\rangle=\left\lvert\begin{smallmatrix}1\\1\end{smallmatrix}\right\rangle=\tfrac{1}{\sqrt{2}}\left\lvert\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle-\tfrac{1}{\sqrt{2}}\left\lvert\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle=\left\lvert{}^2P_{M=1}\right\rangle$

$$L_-\left\lvert\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle=\sqrt{(2+1)(2-1+1)}\left\lvert\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle=\sqrt{6}\left\lvert\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle$$

$$\left\lvert\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle=\tfrac{1}{\sqrt{6}}L_-\left\lvert\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle=\tfrac{1}{\sqrt{6}}\cancel{\sqrt{2}}(E_{21}+E_{32})\cancel{\tfrac{1}{\sqrt{2}}}\left(\left\lvert\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle+\left\lvert\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle\right)$$

$$=\tfrac{1}{\sqrt{6}}\left(E_{21}\left\lvert\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle+E_{21}\left\lvert\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle+E_{32}\left\lvert\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle+E_{32}\left\lvert\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle\right)$$

$$=\tfrac{1}{\sqrt{6}}\left(0\left\lvert\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle+\sqrt{2}\left\lvert\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle+\sqrt{\tfrac{1}{2}}\left\lvert\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle+\sqrt{\tfrac{3}{2}}\left\lvert\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{2}\end{smallmatrix}\right\rangle+0\left\lvert\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle\right)$$

$$=\tfrac{1}{\sqrt{6}}\left(\tfrac{3}{\sqrt{2}}\left\lvert\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle+\sqrt{\tfrac{3}{2}}\left\lvert\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{2}\end{smallmatrix}\right\rangle\right)=\tfrac{\sqrt{3}}{2}\left\lvert\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle+\tfrac{1}{2}\left\lvert\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{2}\end{smallmatrix}\right\rangle=\left\lvert{}^2D_{M=0}\right\rangle=\left\lvert\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle$$

Orthogonal (L=1, *M=0*) state: $\tfrac{-1}{2}\left\lvert\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle+\tfrac{\sqrt{3}}{2}\left\lvert\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{2}\end{smallmatrix}\right\rangle=\left\lvert{}^2P_{M=0}\right\rangle=\left\lvert\begin{smallmatrix}1\\0\end{smallmatrix}\right\rangle$

$$L_-\left\lvert\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle=\sqrt{(2+0)(2-0+1)}\left\lvert\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle=\sqrt{6}\left\lvert\begin{smallmatrix}2\\-1\end{smallmatrix}\right\rangle$$

$$\left\lvert\begin{smallmatrix}2\\-1\end{smallmatrix}\right\rangle=\tfrac{1}{\sqrt{6}}L_-\left\lvert\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle=\tfrac{1}{\sqrt{6}}\sqrt{2}(E_{21}+E_{32})\left(\tfrac{\sqrt{3}}{2}\left\lvert\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle+\tfrac{1}{2}\left\lvert\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{2}\end{smallmatrix}\right\rangle\right)$$

$$=\tfrac{1}{\sqrt{3}}\left(\tfrac{\sqrt{3}}{2}E_{21}\left\lvert\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle+\tfrac{1}{2}E_{21}\left\lvert\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle+\tfrac{\sqrt{3}}{2}E_{32}\left\lvert\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle+\tfrac{1}{2}E_{32}\left\lvert\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{2}\end{smallmatrix}\right\rangle\right)$$

$$=\tfrac{1}{\sqrt{3}}\left(\tfrac{\sqrt{3}}{2}\tfrac{\sqrt{2}}{1}\left\lvert\begin{smallmatrix}\boxed{2}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle+\ 0\left\lvert\begin{smallmatrix}\boxed{2}\boxed{3}\\\boxed{2}\end{smallmatrix}\right\rangle+\tfrac{\sqrt{3}}{2}\sqrt{\tfrac{1}{2}}\left\lvert\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{3}\end{smallmatrix}\right\rangle+\tfrac{1}{2}\sqrt{\tfrac{3}{2}}\left\lvert\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{3}\end{smallmatrix}\right\rangle\right)$$

☐☐ [2,1] *tableau states lowered by* $\mathbf{L}_-=\sqrt{2}(E_{21}+E_{32})$

	M=2	*M=1*		*M=0*		*M=-1*		*M=-2*
E_{jk}	$\left\vert\begin{smallmatrix}11\\2\end{smallmatrix}\right\rangle$	$\left\vert\begin{smallmatrix}12\\2\end{smallmatrix}\right\rangle$	$\left\vert\begin{smallmatrix}11\\3\end{smallmatrix}\right\rangle$	$\left\vert\begin{smallmatrix}12\\3\end{smallmatrix}\right\rangle$	$\left\vert\begin{smallmatrix}13\\2\end{smallmatrix}\right\rangle$	$\left\vert\begin{smallmatrix}13\\3\end{smallmatrix}\right\rangle$	$\left\vert\begin{smallmatrix}22\\3\end{smallmatrix}\right\rangle$	$\left\vert\begin{smallmatrix}23\\3\end{smallmatrix}\right\rangle$
$\left\langle\begin{smallmatrix}11\\2\end{smallmatrix}\right\vert$	(11) (22) $2+1$	(12) 1	(23) 1	(13) $-\sqrt{\frac{1}{2}}$	(13) $\sqrt{\frac{3}{2}}$	.	.	.
$\left\langle\begin{smallmatrix}12\\2\end{smallmatrix}\right\vert$	(21) 1	(11) (22) $1+2$	.	(23) $\sqrt{\frac{1}{2}}$	(23) $\sqrt{\frac{3}{2}}$	.	(13) -1	.
$\left\langle\begin{smallmatrix}11\\3\end{smallmatrix}\right\vert$	(32) 1	.	(11) (33) $2+1$	(12) $\sqrt{2}$	.	(13) 1	.	.
$\left\langle\begin{smallmatrix}12\\3\end{smallmatrix}\right\vert$	(31) $-\sqrt{\frac{1}{2}}$	(32) $\sqrt{\frac{1}{2}}$	(21) $\sqrt{2}$	(11) (22) (33) $1+1+1$	.	(23) $\sqrt{\frac{1}{2}}$	(12) $\sqrt{2}$	(13) $\sqrt{\frac{1}{2}}$
$\left\langle\begin{smallmatrix}13\\2\end{smallmatrix}\right\vert$	(31) $\sqrt{\frac{3}{2}}$	(32) $\sqrt{\frac{3}{2}}$	.	.	(11) (22) (33) $1+1+1$	(23) $\sqrt{\frac{3}{2}}$	.	(13) $\sqrt{\frac{3}{2}}$
$\left\langle\begin{smallmatrix}13\\3\end{smallmatrix}\right\vert$	.	.	(31) 1	(32) $\sqrt{\frac{1}{2}}$	(32) $\sqrt{\frac{3}{2}}$	(11) (33) $1+2$	.	(12) 1
$\left\langle\begin{smallmatrix}22\\3\end{smallmatrix}\right\vert$	.	(31) -1	.	(21) $\sqrt{2}$	.	.	(22) (33) $2+1$	(23) 1
$\left\langle\begin{smallmatrix}23\\3\end{smallmatrix}\right\vert$	.	.	.	(31) $\sqrt{\frac{1}{2}}$	(31) $\sqrt{\frac{3}{2}}$	(21) 1	(32) 1	(22) (33) $1+2$

$$L_-\left|\begin{smallmatrix}L\\M\end{smallmatrix}\right\rangle=\sqrt{(L+M)(L-M+1)}\left|\begin{smallmatrix}L\\M-1\end{smallmatrix}\right\rangle$$

$$L_-\left|\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle=\sqrt{(2+2)(2-2+1)}\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle=2\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle$$

Start with top [2,1]-state:

$$\boxed{\left|\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle=\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{2}\end{smallmatrix}\right\rangle=\left|{}^2D_{M=2}\right\rangle}$$

$$\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle=\tfrac{1}{2}L_-\left|\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle=\tfrac{1}{2}\sqrt{2}(E_{21}+E_{32})\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{2}\end{smallmatrix}\right\rangle=\boxed{\tfrac{1}{\sqrt{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle+\tfrac{1}{\sqrt{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle=\left|{}^2D_{M=1}\right\rangle}$$

Orthogonal *M=1* state: $\left|{}^2P_{M=1}\right\rangle=\left|\begin{smallmatrix}1\\1\end{smallmatrix}\right\rangle=\boxed{\tfrac{1}{\sqrt{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle-\tfrac{1}{\sqrt{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle=\left|{}^2P_{M=1}\right\rangle}$

$$L_z\equiv\begin{pmatrix}1&\cdot&\cdot\\\cdot&0&\cdot\\\cdot&\cdot&-1\end{pmatrix}=(E_{11}\text{-}E_{33})=\sqrt{2}\mathbf{v}^1_0$$

dipole (k=1)
∠-momentum
L*-operators*

$$L_+\equiv\sqrt{2}\begin{pmatrix}\cdot&1&\cdot\\\cdot&\cdot&1\\\cdot&\cdot&\cdot\end{pmatrix}=\sqrt{2}(E_{12}+E_{23})=L_x+iL_y=\text{-}\sqrt{2}\mathbf{v}^1_1$$

$$\boxed{L_-\equiv\sqrt{2}\begin{pmatrix}\cdot&\cdot&\cdot\\1&\cdot&\cdot\\\cdot&1&\cdot\end{pmatrix}=\sqrt{2}(E_{21}+E_{32})=L_x-iL_y=\sqrt{2}\mathbf{v}^1_{=1}}$$

$$L_-\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle=\sqrt{(2+1)(2-1+1)}\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle=\sqrt{6}\left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle$$

$$\begin{aligned}\left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle&=\tfrac{1}{\sqrt{6}}L_-\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle=\tfrac{1}{\sqrt{6}}\cancel{\sqrt{2}}(E_{21}+E_{32})\cancel{\tfrac{1}{\sqrt{2}}}\left(\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle+\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle\right)\\&=\tfrac{1}{\sqrt{6}}\left(E_{21}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle+E_{21}\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle+E_{32}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle+E_{32}\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle\right)\\&=\tfrac{1}{\sqrt{6}}\left(0\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle+\sqrt{2}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle+\sqrt{\tfrac{1}{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle+\sqrt{\tfrac{3}{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{2}\end{smallmatrix}\right\rangle+0\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle\right)\\&=\tfrac{1}{\sqrt{6}}\left(\tfrac{3}{\sqrt{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle+\sqrt{\tfrac{3}{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{2}\end{smallmatrix}\right\rangle\right)=\boxed{\tfrac{\sqrt{3}}{2}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle+\tfrac{1}{2}\left|\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{2}\end{smallmatrix}\right\rangle=\left|{}^2D_{M=0}\right\rangle=\left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle}\end{aligned}$$

Orthogonal (L=1, *M=0*) state: $\boxed{\tfrac{-1}{2}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle+\tfrac{\sqrt{3}}{2}\left|\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{2}\end{smallmatrix}\right\rangle=\left|{}^2P_{M=0}\right\rangle=\left|\begin{smallmatrix}1\\0\end{smallmatrix}\right\rangle}$

$$L_-\left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle=\sqrt{(2+0)(2-0+1)}\left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle=\sqrt{6}\left|\begin{smallmatrix}2\\-1\end{smallmatrix}\right\rangle$$

$$\begin{aligned}\left|\begin{smallmatrix}2\\-1\end{smallmatrix}\right\rangle&=\tfrac{1}{\sqrt{6}}L_-\left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle=\tfrac{1}{\sqrt{6}}\sqrt{2}(E_{21}+E_{32})\left(\tfrac{\sqrt{3}}{2}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle+\tfrac{1}{2}\left|\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{2}\end{smallmatrix}\right\rangle\right)\\&=\tfrac{1}{\sqrt{3}}\left(\tfrac{\sqrt{3}}{2}E_{21}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle+\tfrac{1}{2}E_{21}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle+\tfrac{\sqrt{3}}{2}E_{32}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle+\tfrac{1}{2}E_{32}\left|\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{2}\end{smallmatrix}\right\rangle\right)\\&=\tfrac{1}{\sqrt{3}}\left(\tfrac{\sqrt{3}}{2}\tfrac{\sqrt{2}}{1}\left|\begin{smallmatrix}\boxed{2}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle+0\left|\begin{smallmatrix}\boxed{2}\boxed{3}\\\boxed{2}\end{smallmatrix}\right\rangle+\tfrac{\sqrt{3}}{2}\sqrt{\tfrac{1}{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{3}\end{smallmatrix}\right\rangle+\tfrac{1}{2}\sqrt{\tfrac{3}{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{3}\end{smallmatrix}\right\rangle\right)\\&=\tfrac{1}{\sqrt{3}}\left(\sqrt{\tfrac{3}{2}}\left|\begin{smallmatrix}\boxed{2}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle+\sqrt{\tfrac{3}{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{3}\end{smallmatrix}\right\rangle\right)=\tfrac{1}{\sqrt{2}}\left|\begin{smallmatrix}\boxed{2}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle+\tfrac{1}{\sqrt{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{3}\end{smallmatrix}\right\rangle=\left|{}^2D_{M=-1}\right\rangle=\left|\begin{smallmatrix}2\\-1\end{smallmatrix}\right\rangle\end{aligned}$$

$\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$ = [2,1] *tableau states lowered by* $\mathbf{L}_{-}=\sqrt{2}(E_{21}+E_{32})$

	$M=2$	$M=1$		$M=0$		$M=-1$		$M=-2$
E_{jk}	$\left\lvert\begin{smallmatrix}11\\2\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}12\\2\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}11\\3\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}12\\3\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}13\\2\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}13\\3\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}22\\3\end{smallmatrix}\right\rangle$	$\left\lvert\begin{smallmatrix}23\\3\end{smallmatrix}\right\rangle$
$\left\langle\begin{smallmatrix}11\\2\end{smallmatrix}\right\rvert$	$\overset{(11)\ (22)}{2+1}$	$\overset{(12)}{1}$	$\overset{(23)}{1}$	$\overset{(13)}{-\sqrt{\frac{1}{2}}}$	$\overset{(13)}{\sqrt{\frac{3}{2}}}$	·	·	·
$\left\langle\begin{smallmatrix}12\\2\end{smallmatrix}\right\rvert$	$\overset{(21)}{1}$	$\overset{(11)\ (22)}{1+2}$	·	$\overset{(23)}{\sqrt{\frac{1}{2}}}$	$\overset{(23)}{\sqrt{\frac{3}{2}}}$	·	$\overset{(13)}{-1}$	·
$\left\langle\begin{smallmatrix}11\\3\end{smallmatrix}\right\rvert$	$\overset{(32)}{1}$	·	$\overset{(11)\ (33)}{2+1}$	$\overset{(12)}{\sqrt{2}}$	·	$\overset{(13)}{1}$	·	·
$\left\langle\begin{smallmatrix}12\\3\end{smallmatrix}\right\rvert$	$\overset{(31)}{-\sqrt{\frac{1}{2}}}$	$\overset{(32)}{\sqrt{\frac{1}{2}}}$	$\overset{(21)}{\sqrt{2}}$	$\overset{(11)\ (22)\ (33)}{1+1+1}$	·	$\overset{(23)}{\sqrt{\frac{1}{2}}}$	$\overset{(12)}{\sqrt{2}}$	$\overset{(13)}{\sqrt{\frac{1}{2}}}$
$\left\langle\begin{smallmatrix}13\\2\end{smallmatrix}\right\rvert$	$\overset{(31)}{\sqrt{\frac{3}{2}}}$	$\overset{(32)}{\sqrt{\frac{3}{2}}}$	·	·	$\overset{(11)\ (22)\ (33)}{1+1+1}$	$\overset{(23)}{\sqrt{\frac{3}{2}}}$	·	$\overset{(13)}{\sqrt{\frac{3}{2}}}$
$\left\langle\begin{smallmatrix}13\\3\end{smallmatrix}\right\rvert$	·	·	$\overset{(31)}{1}$	$\overset{(32)}{\sqrt{\frac{1}{2}}}$	$\overset{(32)}{\sqrt{\frac{3}{2}}}$	$\overset{(11)\ (33)}{1+2}$	·	$\overset{(12)}{1}$
$\left\langle\begin{smallmatrix}22\\3\end{smallmatrix}\right\rvert$	·	$\overset{(31)}{-1}$	·	$\overset{(21)}{\sqrt{2}}$	·	·	$\overset{(22)\ (33)}{2+1}$	$\overset{(23)}{1}$
$\left\langle\begin{smallmatrix}23\\3\end{smallmatrix}\right\rvert$	·	·	·	$\overset{(31)}{\sqrt{\frac{1}{2}}}$	$\overset{(31)}{\sqrt{\frac{3}{2}}}$	$\overset{(21)}{1}$	$\overset{(32)}{1}$	$\overset{(22)\ (33)}{1+2}$

E_{jk}-matrix
Lect.23
p.7-16
and p.74

$$L_z\equiv\begin{pmatrix}1&\cdot&\cdot\\\cdot&0&\cdot\\\cdot&\cdot&-1\end{pmatrix}=(E_{11}\text{-}E_{33})=\sqrt{2}\mathbf{v}^1_0$$

Dipole (k=1)
∠-momentum
***L**-operators*

$$L_+\equiv\sqrt{2}\begin{pmatrix}\cdot&1&\cdot\\\cdot&\cdot&1\\\cdot&\cdot&\cdot\end{pmatrix}=\sqrt{2}(E_{12}+E_{23})=L_x+iL_y=-\sqrt{2}\mathbf{v}^1_1$$

$$L_-\equiv\sqrt{2}\begin{pmatrix}\cdot&\cdot&\cdot\\1&\cdot&\cdot\\\cdot&1&\cdot\end{pmatrix}=\sqrt{2}(E_{21}+E_{32})=L_x-iL_y=\sqrt{2}\mathbf{v}^1_{=1}$$

$$L_-\left|\begin{smallmatrix}L\\M\end{smallmatrix}\right\rangle=\sqrt{(L+M)(L-M+1)}\left|\begin{smallmatrix}L\\M-1\end{smallmatrix}\right\rangle$$

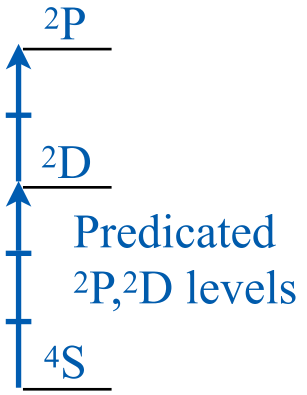
$$L_-\left|\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle=\sqrt{(2+2)(2-2+1)}\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle=2\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle$$

Start with top [2,1]-state:

$$\left|\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle=\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{2}\end{smallmatrix}\right\rangle=\left|{}^2D_{M=2}\right\rangle$$

$$\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle=\tfrac{1}{2}L_-\left|\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle=\tfrac{1}{2}\sqrt{2}(E_{21}+E_{32})\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{2}\end{smallmatrix}\right\rangle=\tfrac{1}{\sqrt{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle+\tfrac{1}{\sqrt{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle=\left|{}^2D_{M=1}\right\rangle$$

Orthogonal *M=1* state: $\left|{}^2P_{M=1}\right\rangle=\left|\begin{smallmatrix}1\\1\end{smallmatrix}\right\rangle=\tfrac{1}{\sqrt{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle-\tfrac{1}{\sqrt{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle=\left|{}^2P_{M=1}\right\rangle$



Bottom [2,1]-state:

$$\left|\begin{smallmatrix}2\\-2\end{smallmatrix}\right\rangle=\left|\begin{smallmatrix}\boxed{2}\boxed{3}\\\boxed{3}\end{smallmatrix}\right\rangle=\left|{}^2D_{M=-2}\right\rangle$$

Bottom [3,0]-state:

$$\left|\begin{smallmatrix}0\\0\end{smallmatrix}\right\rangle=\left|\begin{smallmatrix}\boxed{1}\\\boxed{2}\\\boxed{3}\end{smallmatrix}\quad {}^4S_{M=0}\right\rangle$$

$$L_-\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle=\sqrt{(2+1)(2-1+1)}\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle=\sqrt{6}\left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle$$

$$\left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle=\tfrac{1}{\sqrt{6}}L_-\left|\begin{smallmatrix}2\\1\end{smallmatrix}\right\rangle=\tfrac{1}{\sqrt{6}}\sqrt{2}(E_{21}+E_{32})\tfrac{1}{\sqrt{2}}\left(\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle+\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle\right)$$

$$=\tfrac{1}{\sqrt{6}}\left(E_{21}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle+E_{21}\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle+E_{32}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle+E_{32}\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle\right)$$

$$=\tfrac{1}{\sqrt{6}}\left(0\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle+\sqrt{2}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle+\sqrt{\tfrac{1}{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle+\sqrt{\tfrac{3}{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{2}\end{smallmatrix}\right\rangle+0\left|\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\right\rangle\right)$$

$$=\tfrac{1}{\sqrt{6}}\left(\tfrac{3}{\sqrt{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle+\sqrt{\tfrac{3}{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{2}\end{smallmatrix}\right\rangle\right)=\tfrac{\sqrt{3}}{2}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle+\tfrac{1}{2}\left|\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{2}\end{smallmatrix}\right\rangle=\left|{}^2D_{M=0}\right\rangle=\left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle$$

Orthogonal (L=1, *M=0*) state: $\tfrac{-1}{2}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle+\tfrac{\sqrt{3}}{2}\left|\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{2}\end{smallmatrix}\right\rangle=\left|{}^2P_{M=0}\right\rangle=\left|\begin{smallmatrix}1\\0\end{smallmatrix}\right\rangle$

$$L_-\left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle=\sqrt{(2+0)(2-0+1)}\left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle=\sqrt{6}\left|\begin{smallmatrix}2\\-1\end{smallmatrix}\right\rangle$$

$$\left|\begin{smallmatrix}2\\-1\end{smallmatrix}\right\rangle=\tfrac{1}{\sqrt{6}}L_-\left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle=\tfrac{1}{\sqrt{6}}\sqrt{2}(E_{21}+E_{32})\left(\tfrac{\sqrt{3}}{2}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle+\tfrac{1}{2}\left|\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{2}\end{smallmatrix}\right\rangle\right)$$

$$=\tfrac{1}{\sqrt{3}}\left(\tfrac{\sqrt{3}}{2}E_{21}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle+\tfrac{1}{2}E_{21}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\right\rangle+\tfrac{\sqrt{3}}{2}E_{32}\left|\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle+\tfrac{1}{2}E_{32}\left|\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{2}\end{smallmatrix}\right\rangle\right)$$

$$=\tfrac{1}{\sqrt{3}}\left(\tfrac{\sqrt{3}}{2}\tfrac{\sqrt{2}}{1}\left|\begin{smallmatrix}\boxed{2}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle+0\left|\begin{smallmatrix}\boxed{2}\boxed{3}\\\boxed{2}\end{smallmatrix}\right\rangle+\tfrac{\sqrt{3}}{2}\sqrt{\tfrac{1}{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{3}\end{smallmatrix}\right\rangle+\tfrac{1}{2}\sqrt{\tfrac{3}{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{3}\end{smallmatrix}\right\rangle\right)$$

$$=\tfrac{1}{\sqrt{3}}\left(\sqrt{\tfrac{3}{2}}\left|\begin{smallmatrix}\boxed{2}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle+\sqrt{\tfrac{3}{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{3}\end{smallmatrix}\right\rangle\right)=\tfrac{1}{\sqrt{2}}\left|\begin{smallmatrix}\boxed{2}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle+\tfrac{1}{\sqrt{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{3}\end{smallmatrix}\right\rangle=\left|{}^2D_{M=-1}\right\rangle=\left|\begin{smallmatrix}2\\-1\end{smallmatrix}\right\rangle$$

Orthogonal (L=1, *M=0*) state: $\tfrac{-1}{\sqrt{2}}\left|\begin{smallmatrix}\boxed{2}\boxed{2}\\\boxed{3}\end{smallmatrix}\right\rangle+\tfrac{1}{\sqrt{2}}\left|\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{3}\end{smallmatrix}\right\rangle=\left|{}^2P_{M=-1}\right\rangle=\left|\begin{smallmatrix}1\\-1\end{smallmatrix}\right\rangle$

AMOP reference links on pages 2-4

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[2,1] tableau states lowered by $\mathbf{L}_-=\sqrt{2}(E_{21}+E_{32})$

Top-(J,M) states   to mid-level states

ℓ=1 *p*=shell LS states combined to states of definite J

J=3/2 at L=0 (^{4}S),   J=5/2 at L=2 (^{2}D)

C-G coupling; J=3/2 at L=2 (^{2}D),   J=3/2 at L=1 (^{2}P),   J=1/2 at L=1 (^{2}P)

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Slater functions for J=5/2,   J=3/2 (^{2}D)

Slater functions for J=3/2 (^{2}P),   J=1/2 (^{2}P)

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Application to spin-orbit and entanglement break-up scattering

ℓ=1 *p*=shell LS states combined to states of definite J=3/2 at L=0

M_J=3/2,..

$$\left|{}^4S_{J=\frac{3}{2}}^{\;\frac{3}{2}}\right\rangle=\left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\;\uparrow\uparrow\uparrow\right\rangle,\left|{}^4S_{J=\frac{3}{2}}^{\;\frac{1}{2}}\right\rangle=\left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\;\uparrow\uparrow\downarrow\right\rangle,\left|{}^4S_{J=\frac{3}{2}}^{\;\frac{-1}{2}}\right\rangle=\left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\;\uparrow\downarrow\downarrow\right\rangle,\left|{}^4S_{J=\frac{3}{2}}^{\;\frac{-3}{2}}\right\rangle=\left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\;\downarrow\downarrow\downarrow\right\rangle$$

quartet ${}^4S$ J=$\frac{3}{2}$,

$M_J = \frac{+3}{2},\frac{+1}{2},\frac{-1}{2},\frac{-3}{2}$.

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ℓ=1 p=shell LS states combined to states of definite J =5/2 at L=2

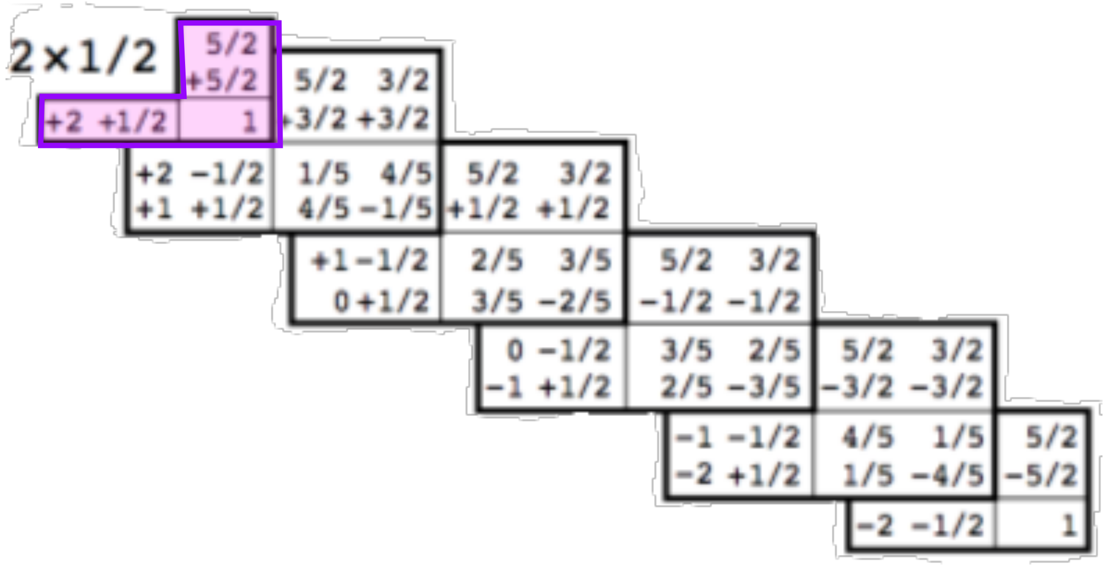
M$_J$=5/2,..

$$\left|{}^4S_{J=\frac{3}{2}}^{\;\frac{3}{2}}\right\rangle=\left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\;\uparrow\uparrow\uparrow\right\rangle,\left|{}^4S_{J=\frac{3}{2}}^{\;\frac{1}{2}}\right\rangle=\left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\;\uparrow\uparrow\downarrow\right\rangle,\left|{}^4S_{J=\frac{3}{2}}^{\;\frac{-1}{2}}\right\rangle=\left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\;\uparrow\downarrow\downarrow\right\rangle,\left|{}^4S_{J=\frac{3}{2}}^{\;\frac{-3}{2}}\right\rangle=\left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\;\downarrow\downarrow\downarrow\right\rangle$$

quartet ${}^4S$ J=$\frac{3}{2}$,
$M_J=\frac{+3}{2},\frac{+1}{2},\frac{-1}{2},\frac{-3}{2}$.

$$\left|{}^2D_{J=\frac{5}{2}}^{\;\frac{5}{2}}\right\rangle=\underline{\left|d^{L=2}_{M=2}\chi^{1/2}_{1/2}\right\rangle}$$

$\underline{\text{Doublet } {}^2D\text{, J=}\tfrac{5}{2}}$ $M_J=\frac{5}{2}$,



2×1/2	5/2	5/2	3/2	5/2	3/2	5/2	3/2	5/2	3/2	5/2
	+5/2	+3/2	+3/2	+1/2	+1/2	−1/2	−1/2	−3/2	−3/2	−5/2
+2 +1/2	1									
+2 −1/2		1/5	4/5							
+1 +1/2		4/5	−1/5							
+1 −1/2				2/5	3/5					
0 +1/2				3/5	−2/5					
0 −1/2						3/5	2/5			
−1 +1/2						2/5	−3/5			
−1 −1/2								4/5	1/5	
−2 +1/2								1/5	−4/5	
−2 −1/2										1

ℓ=1 p=shell LS states combined to states of definite J =5/2 at L=2

M_J=5/2

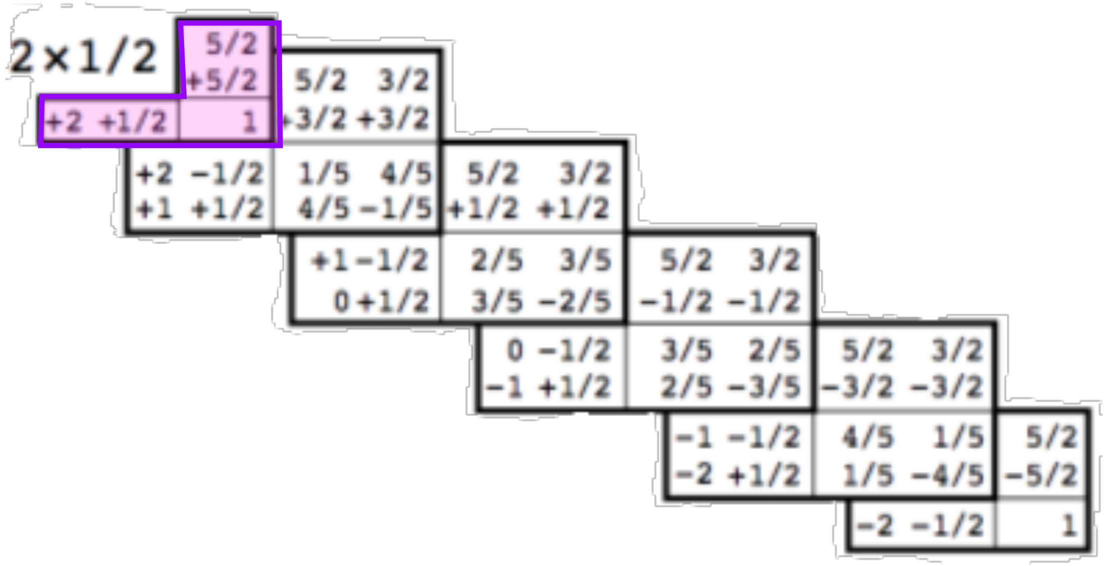
$$\boxed{\left|{}^4S_{J=\frac{3}{2}}^{\;\frac{3}{2}}\right\rangle=\left|\begin{array}{c}\boxed{1}\\\boxed{2}\\\boxed{3}\end{array}\;\uparrow\uparrow\uparrow\right\rangle,\left|{}^4S_{J=\frac{3}{2}}^{\;\frac{1}{2}}\right\rangle=\left|\begin{array}{c}\boxed{1}\\\boxed{2}\\\boxed{3}\end{array}\;\uparrow\uparrow\downarrow\right\rangle,\left|{}^4S_{J=\frac{3}{2}}^{\;\frac{-1}{2}}\right\rangle=\left|\begin{array}{c}\boxed{1}\\\boxed{2}\\\boxed{3}\end{array}\;\uparrow\downarrow\downarrow\right\rangle,\left|{}^4S_{J=\frac{3}{2}}^{\;\frac{-3}{2}}\right\rangle=\left|\begin{array}{c}\boxed{1}\\\boxed{2}\\\boxed{3}\end{array}\;\downarrow\downarrow\downarrow\right\rangle}$$

quartet ${}^4S$ J=$\frac{3}{2}$,
$M_J=\frac{+3}{2},\frac{+1}{2},\frac{-1}{2},\frac{-3}{2}$.

$$\left|{}^2D_{J=\frac{5}{2}}^{\;\frac{5}{2}}\right\rangle=\left|\underline{d^{L=2}_{M=2}\chi^{1/2}_{1/2}}\right\rangle$$

Doublet $\underline{{}^2D,\ \mathrm{J}=\frac{5}{2}\ \mathrm{M_J}=\frac{5}{2}}$,

$$=\left|\begin{array}{cc}\boxed{1}&\boxed{1}\\\boxed{2}&\end{array}\;\begin{array}{c}\uparrow\uparrow\\\downarrow\end{array}\right\rangle$$



2×1/2

5/2
+5/2
+2 +1/2 | 1

5/2 3/2
+3/2 +3/2
+2 −1/2 | 1/5 4/5
+1 +1/2 | 4/5 −1/5

5/2 3/2
+1/2 +1/2
+1 −1/2 | 2/5 3/5
0 +1/2 | 3/5 −2/5

5/2 3/2
−1/2 −1/2
0 −1/2 | 3/5 2/5
−1 +1/2 | 2/5 −3/5

5/2 3/2
−3/2 −3/2
−1 −1/2 | 4/5 1/5
−2 +1/2 | 1/5 −4/5

5/2
−5/2
−2 −1/2 | 1

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reference links
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MJ=3/2

$$\left|{}^4S^{\frac{3}{2}}_{J=\frac{3}{2}}\right\rangle=\left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\ \uparrow\uparrow\uparrow\right\rangle,\left|{}^4S^{\frac{1}{2}}_{J=\frac{3}{2}}\right\rangle=\left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\ \uparrow\uparrow\downarrow\right\rangle,\left|{}^4S^{\frac{-1}{2}}_{J=\frac{3}{2}}\right\rangle=\left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\ \uparrow\downarrow\downarrow\right\rangle,\left|{}^4S^{\frac{-3}{2}}_{J=\frac{3}{2}}\right\rangle=\left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\ \downarrow\downarrow\downarrow\right\rangle$$

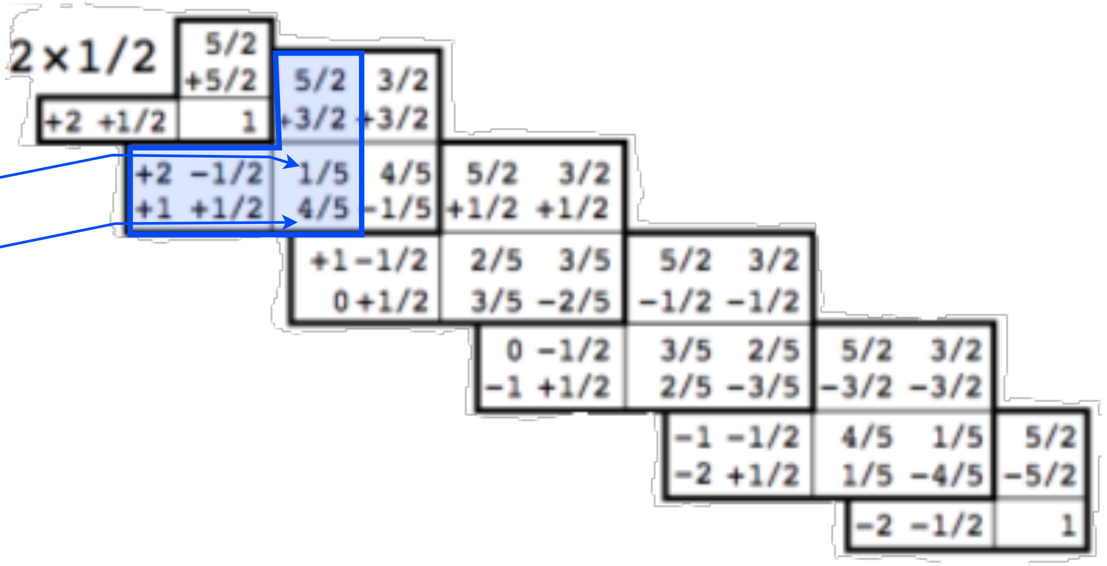
quartet ${}^4S$ J=$\frac{3}{2}$,

$M_J=\frac{+3}{2},\frac{+1}{2},\frac{-1}{2},\frac{-3}{2}$.

$$\left|{}^2D^{\frac{5}{2}}_{J=\frac{5}{2}}\right\rangle=\left|d^{L=2}_{M=2}\chi^{1/2}_{1/2}\right\rangle$$

Doublet ${}^2D$, J=$\frac{5}{2}$ $M_J=\frac{5}{2}$,

$$=\left|\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\ \begin{array}{c}\uparrow\uparrow\\\downarrow\end{array}\right\rangle$$



2×1/2	5/2 +5/2
+2 +1/2	1

	5/2 +3/2	3/2 +3/2
+2 −1/2	1/5	4/5
+1 +1/2	4/5	−1/5

	5/2 +1/2	3/2 +1/2
+1 −1/2	2/5	3/5
0 +1/2	3/5	−2/5

	5/2 −1/2	3/2 −1/2
0 −1/2	3/5	2/5
−1 +1/2	2/5	−3/5

	5/2 −3/2	3/2 −3/2
−1 −1/2	4/5	1/5
−2 +1/2	1/5	−4/5

	5/2 −5/2
−2 −1/2	1

$$\left|{}^2D^{\frac{3}{2}}_{J=\frac{5}{2}}\right\rangle=\underline{\sqrt{\tfrac{1}{5}}\left|d^{L=2}_{M=2}\chi^{1/2}_{-1/2}\right\rangle+\sqrt{\tfrac{4}{5}}\left|d^{L=2}_{M=1}\chi^{1/2}_{1/2}\right\rangle}$$

Doublet ${}^2D$, $\underline{\text{J}=\frac{5}{2}\ \text{M}_J}=\frac{3}{2}$

ℓ=1 p=shell LS states combined to states of definite J =5/2 at L=2

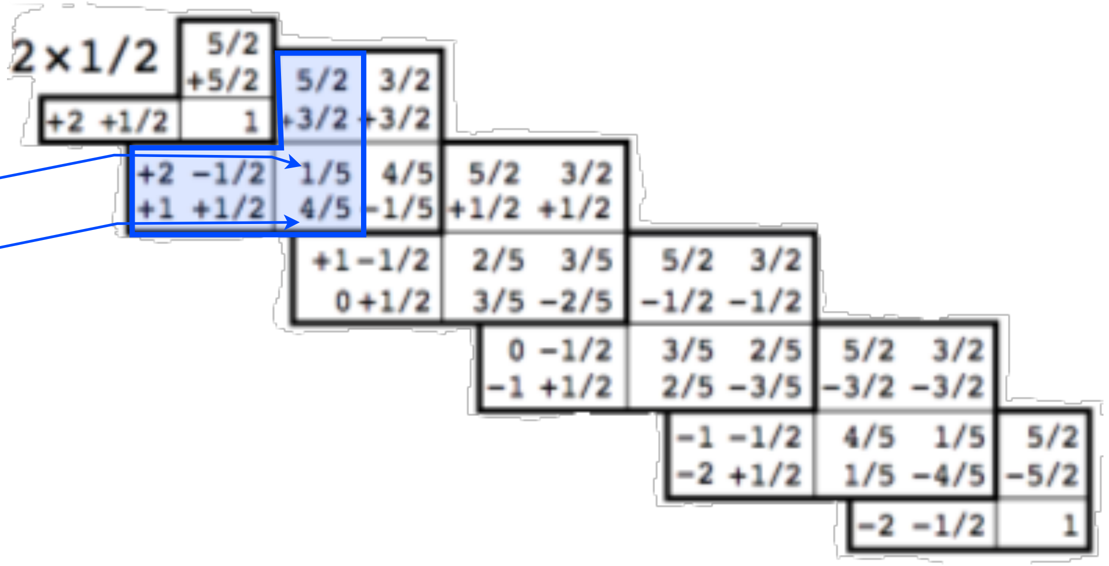
M$_J$=3/2

$$\left|{}^4S_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle=\left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\ \uparrow\uparrow\uparrow\right\rangle,\left|{}^4S_{J=\frac{3}{2}}\ \tfrac{1}{2}\right\rangle=\left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\ \uparrow\uparrow\downarrow\right\rangle,\left|{}^4S_{J=\frac{3}{2}}\ \tfrac{-1}{2}\right\rangle=\left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\ \uparrow\downarrow\downarrow\right\rangle,\left|{}^4S_{J=\frac{3}{2}}\ \tfrac{-3}{2}\right\rangle=\left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\ \downarrow\downarrow\downarrow\right\rangle$$

quartet ${}^4S$ J=$\frac{3}{2}$,
M$_J$ = $\frac{+3}{2},\frac{+1}{2},\frac{-1}{2},\frac{-3}{2}$.

$$\left|{}^2D_{J=\frac{5}{2}}\ \tfrac{5}{2}\right\rangle=\left|d^{L=2}_{M=2}\chi^{1/2}_{1/2}\right\rangle$$ Doublet ${}^2D$, J=$\frac{5}{2}$ M$_J$=$\frac{5}{2}$,

$$=\left|\begin{array}{|c|c|}\hline 1&1\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\uparrow\\\downarrow\end{smallmatrix}\right\rangle$$



2×1/2

	5/2
	+5/2
+2 +1/2	1

	5/2	3/2
	+3/2	+3/2
+2 −1/2	1/5	4/5
+1 +1/2	4/5	−1/5

	5/2	3/2
	+1/2	+1/2
+1 −1/2	2/5	3/5
0 +1/2	3/5	−2/5

	5/2	3/2
	−1/2	−1/2
0 −1/2	3/5	2/5
−1 +1/2	2/5	−3/5

	5/2	3/2
	−3/2	−3/2
−1 −1/2	4/5	1/5
−2 +1/2	1/5	−4/5

	5/2
	−5/2
−2 −1/2	1

$$\left|{}^2D_{J=\frac{5}{2}}\ \tfrac{3}{2}\right\rangle=\sqrt{\tfrac{1}{5}}\left|d^{L=2}_{M=2}\chi^{1/2}_{-1/2}\right\rangle+\sqrt{\tfrac{4}{5}}\left|d^{L=2}_{M=1}\chi^{1/2}_{1/2}\right\rangle$$ Doublet ${}^2D$, J=$\frac{5}{2}$ M$_J$=$\frac{3}{2}$

$$=\sqrt{\tfrac{1}{5}}\left|\begin{array}{|c|c|}\hline 1&1\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\downarrow\\\downarrow\end{smallmatrix}\right\rangle+\sqrt{\tfrac{4}{5}}\left|\left[\sqrt{\tfrac{1}{2}}\begin{array}{|c|c|}\hline 1&2\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\uparrow\\\downarrow\end{smallmatrix}+\sqrt{\tfrac{1}{2}}\begin{array}{|c|c|}\hline 1&1\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\uparrow\\\downarrow\end{smallmatrix}\right]\right\rangle=\sqrt{\tfrac{1}{5}}\begin{array}{|c|c|}\hline 1&1\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\downarrow\\\downarrow\end{smallmatrix}+\sqrt{\tfrac{2}{5}}\begin{array}{|c|c|}\hline 1&2\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\uparrow\\\downarrow\end{smallmatrix}+\sqrt{\tfrac{2}{5}}\begin{array}{|c|c|}\hline 1&1\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\uparrow\\\downarrow\end{smallmatrix}$$

ℓ=1 p=shell LS states combined to states of definite J =3/2 at L=2

M$_J$=3/2

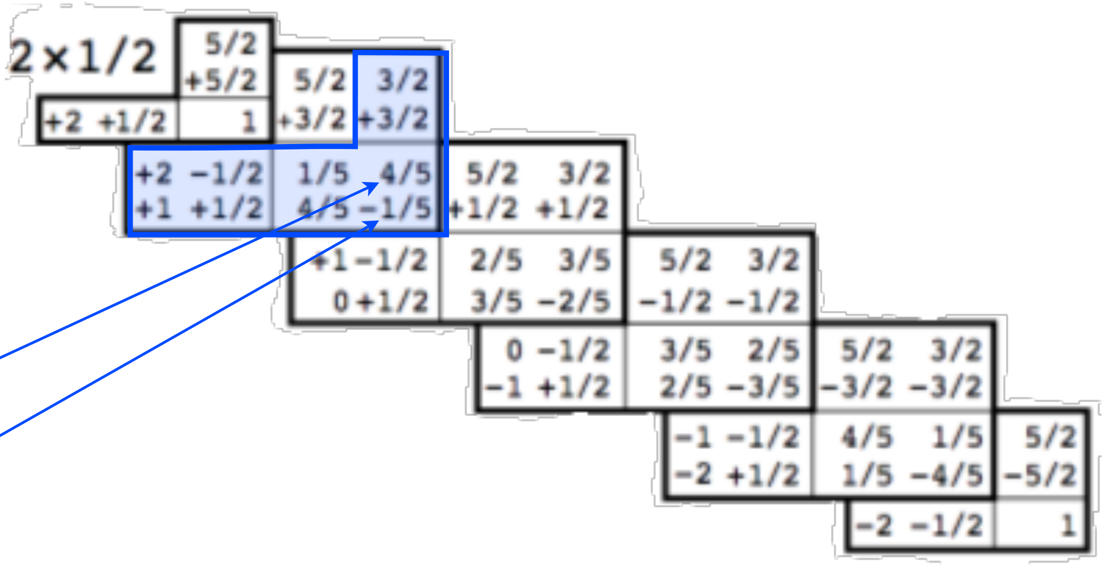
$$\left|{}^4S_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle=\left|\begin{array}{c}\boxed{1}\\\boxed{2}\\\boxed{3}\end{array}\ \uparrow\uparrow\uparrow\right\rangle,\left|{}^4S_{J=\frac{3}{2}}\ \tfrac{1}{2}\right\rangle=\left|\begin{array}{c}\boxed{1}\\\boxed{2}\\\boxed{3}\end{array}\ \uparrow\uparrow\downarrow\right\rangle,\left|{}^4S_{J=\frac{3}{2}}\ \tfrac{-1}{2}\right\rangle=\left|\begin{array}{c}\boxed{1}\\\boxed{2}\\\boxed{3}\end{array}\ \uparrow\downarrow\downarrow\right\rangle,\left|{}^4S_{J=\frac{3}{2}}\ \tfrac{-3}{2}\right\rangle=\left|\begin{array}{c}\boxed{1}\\\boxed{2}\\\boxed{3}\end{array}\ \downarrow\downarrow\downarrow\right\rangle$$

quartet ${}^4S$ J=$\frac{3}{2}$,
M$_J$ = $\frac{+3}{2},\frac{+1}{2},\frac{-1}{2},\frac{-3}{2}$.

$$\left|{}^2D_{J=\frac{5}{2}}\ \tfrac{5}{2}\right\rangle=\left|d^{L=2}_{M=2}\chi^{1/2}_{1/2}\right\rangle$$

Doublet ${}^2D$, J=$\frac{5}{2}$ M$_J$=$\frac{5}{2}$,

$$=\left|\begin{array}{cc}\boxed{1}&\boxed{1}\\\boxed{2}&\end{array}\ \begin{array}{c}\uparrow\uparrow\\\downarrow\end{array}\right\rangle$$



2×1/2	5/2 +5/2
+2 +1/2	1

	5/2 +3/2	3/2 +3/2
+2 −1/2	1/5	4/5
+1 +1/2	4/5	−1/5

	5/2 +1/2	3/2 +1/2
+1 −1/2	2/5	3/5
0 +1/2	3/5	−2/5

	5/2 −1/2	3/2 −1/2
0 −1/2	3/5	2/5
−1 +1/2	2/5	−3/5

	5/2 −3/2	3/2 −3/2
−1 −1/2	4/5	1/5
−2 +1/2	1/5	−4/5

	5/2 −5/2
−2 −1/2	1

$$\left|{}^2D_{J=\frac{5}{2}}\ \tfrac{3}{2}\right\rangle=\sqrt{\tfrac{1}{5}}\left|d^{L=2}_{M=2}\chi^{1/2}_{-1/2}\right\rangle+\sqrt{\tfrac{4}{5}}\left|d^{L=2}_{M=1}\chi^{1/2}_{1/2}\right\rangle$$

Doublet ${}^2D$, J=$\frac{5}{2}$ M$_J$=$\frac{3}{2}$

$$=\sqrt{\tfrac{1}{5}}\left|\begin{array}{cc}\boxed{1}&\boxed{1}\\\boxed{2}&\end{array}\ \begin{array}{c}\uparrow\downarrow\\\downarrow\end{array}\right\rangle+\sqrt{\tfrac{4}{5}}\left|\left[\sqrt{\tfrac{1}{2}}\begin{array}{cc}\boxed{1}&\boxed{2}\\\boxed{2}&\end{array}\ \begin{array}{c}\uparrow\uparrow\\\downarrow\end{array}+\sqrt{\tfrac{1}{2}}\begin{array}{cc}\boxed{1}&\boxed{1}\\\boxed{3}&\end{array}\ \begin{array}{c}\uparrow\uparrow\\\downarrow\end{array}\right]\right\rangle=\sqrt{\tfrac{1}{5}}\begin{array}{cc}\boxed{1}&\boxed{1}\\\boxed{2}&\end{array}\ \begin{array}{c}\uparrow\downarrow\\\downarrow\end{array}+\sqrt{\tfrac{2}{5}}\begin{array}{cc}\boxed{1}&\boxed{2}\\\boxed{2}&\end{array}\ \begin{array}{c}\uparrow\uparrow\\\downarrow\end{array}+\sqrt{\tfrac{2}{5}}\begin{array}{cc}\boxed{1}&\boxed{1}\\\boxed{3}&\end{array}\ \begin{array}{c}\uparrow\uparrow\\\downarrow\end{array}$$

$$\left|{}^2D_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle=\sqrt{\tfrac{4}{5}}\left|d^{L=2}_{M=2}\chi^{1/2}_{-1/2}\right\rangle-\sqrt{\tfrac{1}{5}}\left|d^{L=2}_{M=1}\chi^{1/2}_{+1/2}\right\rangle$$

Doublet ${}^2D$, J=$\frac{3}{2}$ M$_J$=$\frac{3}{2}$

ℓ=1 p=shell LS states combined to states of definite J =3/2 at L=2

M_J=3/2

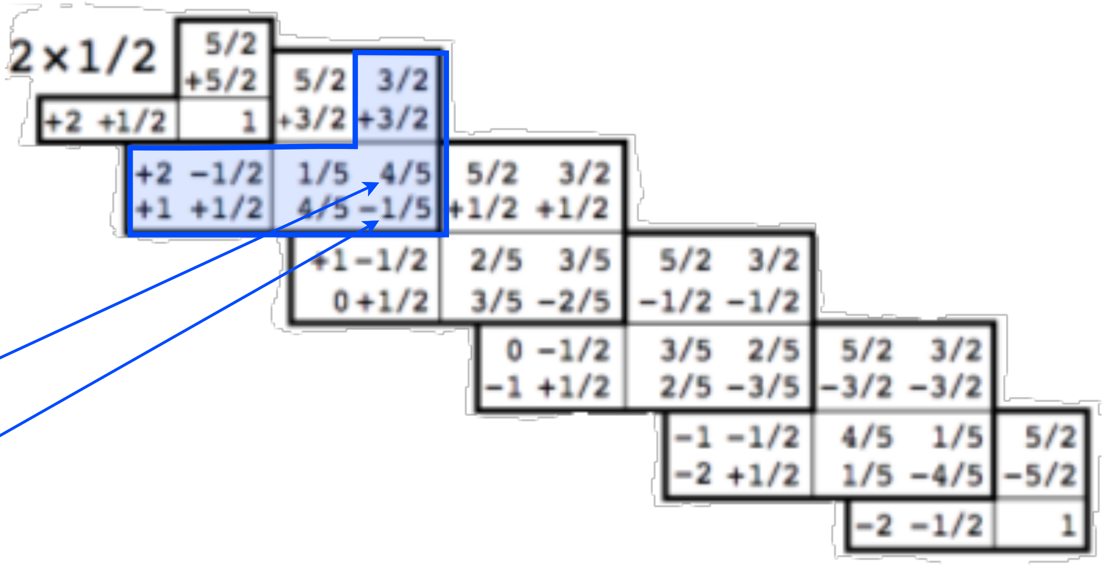
$$\left|{}^4S_{J=\frac{3}{2}}^{\;\frac{3}{2}}\right\rangle=\left|\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline\end{array}\;\uparrow\uparrow\uparrow\right\rangle,\left|{}^4S_{J=\frac{3}{2}}^{\;\frac{1}{2}}\right\rangle=\left|\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline\end{array}\;\uparrow\uparrow\downarrow\right\rangle,\left|{}^4S_{J=\frac{3}{2}}^{\;\frac{-1}{2}}\right\rangle=\left|\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline\end{array}\;\uparrow\downarrow\downarrow\right\rangle,\left|{}^4S_{J=\frac{3}{2}}^{\;\frac{-3}{2}}\right\rangle=\left|\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline\end{array}\;\downarrow\downarrow\downarrow\right\rangle$$

quartet ${}^4S$ J=$\frac{3}{2}$,
$M_J=\frac{+3}{2},\frac{+1}{2},\frac{-1}{2},\frac{-3}{2}$.

$$\left|{}^2D_{J=\frac{5}{2}}^{\;\frac{5}{2}}\right\rangle=\left|d_{M=2}^{L=2}\chi_{1/2}^{1/2}\right\rangle$$

Doublet ${}^2D$, J=$\frac{5}{2}$ $M_J=\frac{5}{2}$,

$$=\left|\begin{array}{|c|c|}\hline 1&1\\ \hline 2\\ \cline{1-1}\end{array}\;\begin{matrix}\uparrow\uparrow\\ \downarrow\end{matrix}\right\rangle$$



2×1/2

		5/2										
		+5/2	5/2	3/2								
+2	+1/2	1	+3/2	+3/2								
+2	−1/2		1/5	4/5	5/2	3/2						
+1	+1/2		4/5	−1/5	+1/2	+1/2						
+1	−1/2				2/5	3/5	5/2	3/2				
0	+1/2				3/5	−2/5	−1/2	−1/2				
0	−1/2						3/5	2/5	5/2	3/2		
−1	+1/2						2/5	−3/5	−3/2	−3/2		
−1	−1/2								4/5	1/5	5/2	
−2	+1/2								1/5	−4/5	−5/2	
−2	−1/2										1	

$$\left|{}^2D_{J=\frac{5}{2}}^{\;\frac{3}{2}}\right\rangle=\sqrt{\tfrac{1}{5}}\left|d_{M=2}^{L=2}\chi_{-1/2}^{1/2}\right\rangle+\sqrt{\tfrac{4}{5}}\left|d_{M=1}^{L=2}\chi_{1/2}^{1/2}\right\rangle$$

Doublet ${}^2D$, J=$\frac{5}{2}$ $M_J=\frac{3}{2}$

$$=\sqrt{\tfrac{1}{5}}\left|\begin{array}{|c|c|}\hline 1&1\\ \hline 2\\ \cline{1-1}\end{array}\;\begin{matrix}\uparrow\downarrow\\ \downarrow\end{matrix}\right\rangle+\sqrt{\tfrac{4}{5}}\left|\left[\sqrt{\tfrac{1}{2}}\begin{array}{|c|c|}\hline 1&2\\ \hline 2\\ \cline{1-1}\end{array}\;\begin{matrix}\uparrow\uparrow\\ \downarrow\end{matrix}+\sqrt{\tfrac{1}{2}}\begin{array}{|c|c|}\hline 1&1\\ \hline 3\\ \cline{1-1}\end{array}\;\begin{matrix}\uparrow\uparrow\\ \downarrow\end{matrix}\right]\right\rangle=\sqrt{\tfrac{1}{5}}\begin{array}{|c|c|}\hline 1&1\\ \hline 2\\ \cline{1-1}\end{array}\;\begin{matrix}\uparrow\downarrow\\ \downarrow\end{matrix}+\sqrt{\tfrac{2}{5}}\begin{array}{|c|c|}\hline 1&2\\ \hline 2\\ \cline{1-1}\end{array}\;\begin{matrix}\uparrow\uparrow\\ \downarrow\end{matrix}+\sqrt{\tfrac{2}{5}}\begin{array}{|c|c|}\hline 1&1\\ \hline 3\\ \cline{1-1}\end{array}\;\begin{matrix}\uparrow\uparrow\\ \downarrow\end{matrix}$$

$$\left|{}^2D_{J=\frac{3}{2}}^{\;\frac{3}{2}}\right\rangle=\sqrt{\tfrac{4}{5}}\left|d_{M=2}^{L=2}\chi_{-1/2}^{1/2}\right\rangle-\sqrt{\tfrac{1}{5}}\left|d_{M=1}^{L=2}\chi_{+1/2}^{1/2}\right\rangle$$

Doublet ${}^2D$, J=$\frac{3}{2}$ $M_J=\frac{3}{2}$

$$=\sqrt{\tfrac{4}{5}}\left|\begin{array}{|c|c|}\hline 1&1\\ \hline 2\\ \cline{1-1}\end{array}\;\begin{matrix}\uparrow\downarrow\\ \downarrow\end{matrix}\right\rangle-\sqrt{\tfrac{1}{5}}\left|\left[\sqrt{\tfrac{1}{2}}\begin{array}{|c|c|}\hline 1&2\\ \hline 2\\ \cline{1-1}\end{array}\;\begin{matrix}\uparrow\uparrow\\ \downarrow\end{matrix}+\sqrt{\tfrac{1}{2}}\begin{array}{|c|c|}\hline 1&1\\ \hline 3\\ \cline{1-1}\end{array}\;\begin{matrix}\uparrow\uparrow\\ \downarrow\end{matrix}\right]\right\rangle=\sqrt{\tfrac{4}{5}}\begin{array}{|c|c|}\hline 1&1\\ \hline 2\\ \cline{1-1}\end{array}\;\begin{matrix}\uparrow\downarrow\\ \downarrow\end{matrix}-\sqrt{\tfrac{1}{10}}\begin{array}{|c|c|}\hline 1&2\\ \hline 2\\ \cline{1-1}\end{array}\;\begin{matrix}\uparrow\uparrow\\ \downarrow\end{matrix}-\sqrt{\tfrac{1}{10}}\begin{array}{|c|c|}\hline 1&1\\ \hline 3\\ \cline{1-1}\end{array}\;\begin{matrix}\uparrow\uparrow\\ \downarrow\end{matrix}$$

AMOP reference links on pages 2-4

4.25.18 class 26: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

$(S_3)^*(U(3))\subset U(6)$ models of p^3 electronic spin-orbit states and couplings

[2,1] tableau states lowered by $\mathbf{L}_-=\sqrt{2}(E_{21}+E_{32})$

Top-(J,M) states   to mid-level states

$\ell=1$ p=shell LS states combined to states of definite J

J=3/2 at L=0 (^{4}S),   J=5/2 at L=2 (^{2}D)

C-G coupling; J=3/2 at L=2 (^{2}D), ➡ J=3/2 at L=1 (^{2}P),   J=1/2 at L=1 (^{2}P)

Spin-orbit state assembly formula and Slater determinants

$\ell=1$ p=shell LSJ states transformed to Slater determinants from J=3/2 (^{4}S)

Slater functions for J=5/2,   J=3/2 (^{2}D)

Slater functions for J=3/2 (^{2}P),   J=1/2 (^{2}P)

Summary of states and level connection paths

Symmetry dimension accounting

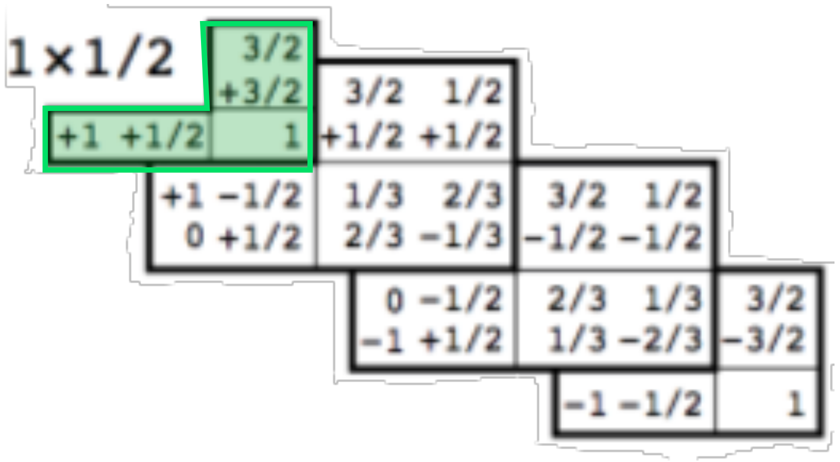
Spin-orbit Hamiltonian matrix calculation

Application to spin-orbit and entanglement break-up scattering

ℓ=1 *p*=shell LS states combined to states of definite J =3/2 at L=1 M$_J$=3/2

$$\left|{}^2P_{J=\frac{3}{2}}^{\frac{3}{2}}\right\rangle = \underline{\left|p_{M=1}^{L=1}\chi_{+1/2}^{1/2}\right\rangle}$$

$\underline{\text{Doublet }{}^2P,\ \text{J}=\tfrac{3}{2}\ \ \text{M}_\text{J}=\tfrac{3}{2}}$



1×1/2	3/2					
	+3/2	3/2	1/2			
+1 +1/2	1	+1/2	+1/2			
+1 −1/2		1/3	2/3	3/2	1/2	
0 +1/2		2/3	−1/3	−1/2	−1/2	
0 −1/2				2/3	1/3	3/2
−1 +1/2				1/3	−2/3	−3/2
−1 −1/2						1

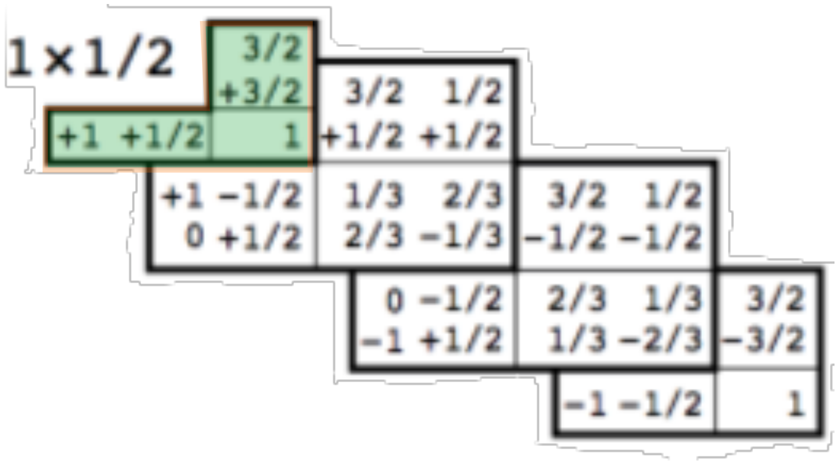
ℓ=1 *p*=shell LS states combined to states of definite J =3/2 at L=1

M_J=3/2

$$\left|{}^2P_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle=\left|p^{L=1}_{M=1}\chi^{1/2}_{+1/2}\right\rangle$$

Doublet ${}^2P$, J$=\frac{3}{2}$ M$_J=\frac{3}{2}$

$$=\left|\left[\sqrt{\tfrac{1}{2}}\ \begin{array}{|c|c|}\hline 1 & 2\\ \hline 2 & \multicolumn{1}{c}{}\\ \cline{1-1}\end{array}\ \begin{array}{c}\uparrow\uparrow\\ \downarrow\end{array}-\sqrt{\tfrac{1}{2}}\ \begin{array}{|c|c|}\hline 1 & 1\\ \hline 3 & \multicolumn{1}{c}{}\\ \cline{1-1}\end{array}\ \begin{array}{c}\uparrow\uparrow\\ \downarrow\end{array}\right]\right\rangle$$



1×1/2	3/2 +3/2	3/2 +1/2	1/2 +1/2	3/2 −1/2	1/2 −1/2	3/2 −3/2
+1 +1/2	1					
+1 −1/2		1/3	2/3			
0 +1/2		2/3	−1/3			
0 −1/2				2/3	1/3	
−1 +1/2				1/3	−2/3	
−1 −1/2						1

ℓ=1 p=shell LS states combined to states of definite J =3/2 at L=1

M$_J$=1/2

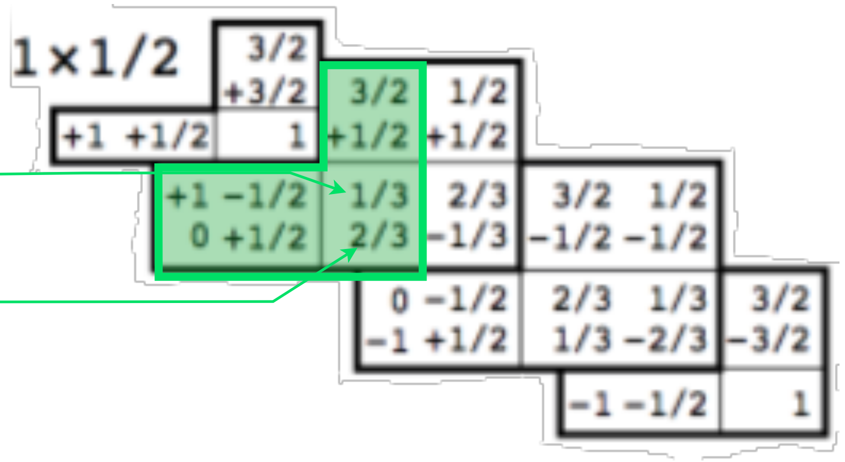
$$\left|{}^2P_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle = \left|p^{L=1}_{M=1}\chi^{1/2}_{+1/2}\right\rangle$$

Doublet ${}^2P$, J=$\frac{3}{2}$ M$_J$=$\frac{3}{2}$

$$= \left|\left[\sqrt{\tfrac{1}{2}}\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\ \begin{array}{cc}\uparrow & \uparrow \\ \downarrow & \end{array} - \sqrt{\tfrac{1}{2}}\ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\ \begin{array}{cc}\uparrow & \uparrow \\ \downarrow & \end{array}\right]\right\rangle$$

$$\left|{}^2P_{J=\frac{3}{2}}\ \tfrac{1}{2}\right\rangle = \sqrt{\tfrac{1}{3}}\left|p^{L=1}_{M=1}\chi^{1/2}_{-1/2}\right\rangle + \sqrt{\tfrac{2}{3}}\left|p^{L=1}_{M=0}\chi^{1/2}_{+1/2}\right\rangle$$

Doublet ${}^2P$, J=$\frac{3}{2}$ M$_J$=$\frac{1}{2}$



1×1/2	3/2 +3/2	3/2 +1/2	1/2 +1/2	3/2 −1/2	1/2 −1/2	3/2 −3/2
+1 +1/2	1					
+1 −1/2		1/3	2/3			
0 +1/2		2/3	−1/3			
0 −1/2				2/3	1/3	
−1 +1/2				1/3	−2/3	
−1 −1/2						1

ℓ=1 p=shell LS states combined to states of definite J =3/2 at L=1

M$_J$=1/2

$$\left|{}^2P_{J=\frac{3}{2}}\ {}^{\frac{3}{2}}_{\ }\right\rangle = \underline{\left|p^{L=1}_{M=1}\chi^{1/2}_{+1/2}\right\rangle}$$

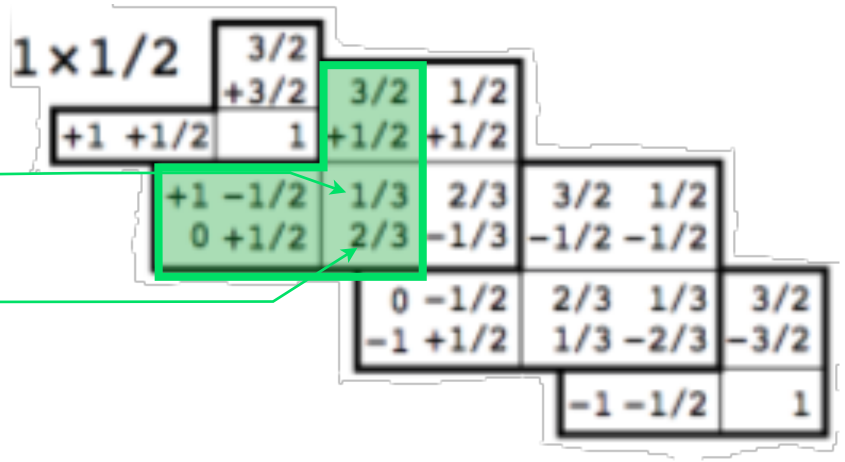
Doublet ^2P, J=$\frac{3}{2}$ M$_J$=$\frac{3}{2}$

$$= \left|\left[\sqrt{\tfrac{1}{2}}\ \begin{array}{|c|c|}\hline 1 & 2\\ \hline 2 \\ \cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\uparrow\\ \downarrow\ \ \end{smallmatrix} - \sqrt{\tfrac{1}{2}}\ \begin{array}{|c|c|}\hline 1 & 1\\ \hline 3 \\ \cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\uparrow\\ \downarrow\ \ \end{smallmatrix}\right]\right\rangle$$

$$\left|{}^2P_{J=\frac{3}{2}}\ {}^{\frac{1}{2}}_{\ }\right\rangle = \sqrt{\tfrac{1}{3}}\left|p^{L=1}_{M=1}\chi^{1/2}_{-1/2}\right\rangle + \sqrt{\tfrac{2}{3}}\left|p^{L=1}_{M=0}\chi^{1/2}_{+1/2}\right\rangle$$

Doublet ^2P, J=$\frac{3}{2}$ M$_J$=$\frac{1}{2}$

$$= \sqrt{\tfrac{1}{3}}\left|\left[\sqrt{\tfrac{1}{2}}\ \begin{array}{|c|c|}\hline 1 & 2\\ \hline 2 \\ \cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\downarrow\\ \downarrow\ \ \end{smallmatrix} - \sqrt{\tfrac{1}{2}}\ \begin{array}{|c|c|}\hline 1 & 1\\ \hline 3 \\ \cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\downarrow\\ \downarrow\ \ \end{smallmatrix}\right]\right\rangle + \sqrt{\tfrac{2}{3}}\left|\left[\tfrac{-1}{2}\ \begin{array}{|c|c|}\hline 1 & 2\\ \hline 3 \\ \cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\uparrow\\ \downarrow\ \ \end{smallmatrix} + \tfrac{\sqrt{3}}{2}\ \begin{array}{|c|c|}\hline 1 & 3\\ \hline 2 \\ \cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\uparrow\\ \downarrow\ \ \end{smallmatrix}\right]\right\rangle$$



1×1/2	3/2 +3/2				
+1 +1/2	1	3/2 +1/2	1/2 +1/2		
+1 −1/2 0 +1/2		1/3 2/3	2/3 −1/3	3/2 1/2 −1/2 −1/2	
0 −1/2 −1 +1/2				2/3 1/3 1/3 −2/3	3/2 −3/2
−1 −1/2					1

ℓ=1 p=shell LS states combined to states of definite J =3/2 at L=1

MJ=1/2

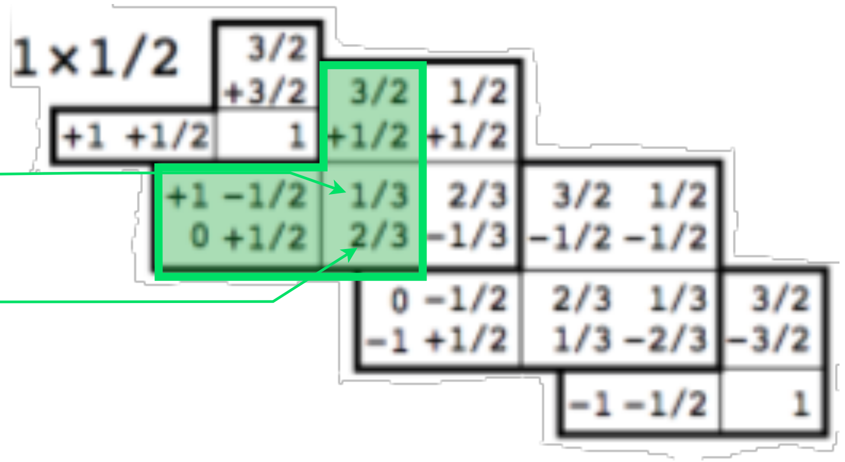
$$\left|{}^2P^{\frac{3}{2}}_{J=\frac{3}{2}}\right\rangle = \left|p^{L=1}_{M=1}\chi^{1/2}_{+1/2}\right\rangle \qquad \text{Doublet } {}^2P,\ \mathrm{J}=\tfrac{3}{2}\ \mathrm{M_J}=\tfrac{3}{2}$$

$$= \left|\left[\sqrt{\tfrac{1}{2}}\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\ \begin{array}{l}\uparrow\uparrow\\ \downarrow\end{array} - \sqrt{\tfrac{1}{2}}\ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\ \begin{array}{l}\uparrow\uparrow\\ \downarrow\end{array}\right]\right\rangle$$

$$\left|{}^2P^{\frac{1}{2}}_{J=\frac{3}{2}}\right\rangle = \sqrt{\tfrac{1}{3}}\left|p^{L=1}_{M=1}\chi^{1/2}_{-1/2}\right\rangle + \sqrt{\tfrac{2}{3}}\left|p^{L=1}_{M=0}\chi^{1/2}_{+1/2}\right\rangle \qquad \text{Doublet } {}^2P,\ \mathrm{J}=\tfrac{3}{2}\ \mathrm{M_J}=\tfrac{1}{2}$$

$$= \sqrt{\tfrac{1}{3}}\left|\left[\sqrt{\tfrac{1}{2}}\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\ \begin{array}{l}\uparrow\downarrow\\ \downarrow\end{array} - \sqrt{\tfrac{1}{2}}\ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\ \begin{array}{l}\uparrow\downarrow\\ \downarrow\end{array}\right]\right\rangle + \sqrt{\tfrac{2}{3}}\left|\left[\tfrac{-1}{2}\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\ \begin{array}{l}\uparrow\uparrow\\ \downarrow\end{array} + \tfrac{\sqrt{3}}{2}\ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\ \begin{array}{l}\uparrow\uparrow\\ \downarrow\end{array}\right]\right\rangle$$

$$= \sqrt{\tfrac{1}{6}}\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\ \begin{array}{l}\uparrow\downarrow\\ \downarrow\end{array} - \sqrt{\tfrac{1}{6}}\ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\ \begin{array}{l}\uparrow\downarrow\\ \downarrow\end{array} \qquad - \sqrt{\tfrac{1}{6}}\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\ \begin{array}{l}\uparrow\uparrow\\ \downarrow\end{array} + \tfrac{1}{\sqrt{2}}\ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\ \begin{array}{l}\uparrow\uparrow\\ \downarrow\end{array}$$



1×1/2

	3/2 +3/2
+1 +1/2	1

	3/2 +1/2	1/2 +1/2
+1 −1/2	1/3	2/3
0 +1/2	2/3	−1/3

	3/2 −1/2	1/2 −1/2
0 −1/2	2/3	1/3
−1 +1/2	1/3	−2/3

	3/2 −3/2
−1 −1/2	1

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ℓ=1 *p*=shell LS states combined to states of definite J =1/2 at L=1
MJ=1/2

$$\left|{}^2P_{J=\frac{3}{2}}^{\frac{3}{2}}\right\rangle = \left|p^{L=1}_{M=1}\chi^{1/2}_{+1/2}\right\rangle$$

Doublet ${}^2P$, J=$\frac{3}{2}$ M$_J$=$\frac{3}{2}$

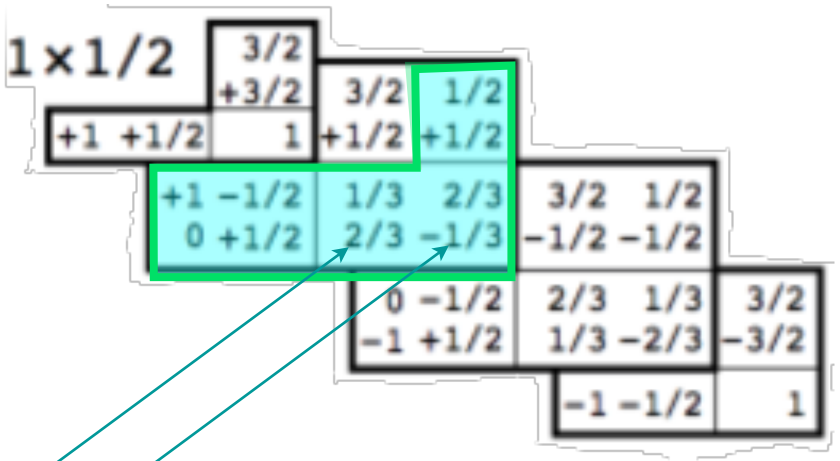
$$=\left|\left[\sqrt{\tfrac{1}{2}}\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\phantom{\boxed{2}}\end{smallmatrix}\ \begin{smallmatrix}\uparrow\uparrow\\\downarrow\end{smallmatrix} - \sqrt{\tfrac{1}{2}}\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\phantom{\boxed{1}}\end{smallmatrix}\ \begin{smallmatrix}\uparrow\uparrow\\\downarrow\end{smallmatrix}\right]\right\rangle$$

$$\left|{}^2P_{J=\frac{3}{2}}^{\frac{1}{2}}\right\rangle = \sqrt{\tfrac{1}{3}}\left|p^{L=1}_{M=1}\chi^{1/2}_{-1/2}\right\rangle + \sqrt{\tfrac{2}{3}}\left|p^{L=1}_{M=0}\chi^{1/2}_{+1/2}\right\rangle$$

Doublet ${}^2P$, J=$\frac{3}{2}$ M$_J$=$\frac{1}{2}$

$$=\sqrt{\tfrac{1}{3}}\left|\left[\sqrt{\tfrac{1}{2}}\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\phantom{\boxed{2}}\end{smallmatrix}\ \begin{smallmatrix}\uparrow\downarrow\\\downarrow\end{smallmatrix} - \sqrt{\tfrac{1}{2}}\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\phantom{\boxed{1}}\end{smallmatrix}\ \begin{smallmatrix}\uparrow\downarrow\\\downarrow\end{smallmatrix}\right]\right\rangle + \sqrt{\tfrac{2}{3}}\left|\left[\tfrac{-1}{2}\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\phantom{\boxed{2}}\end{smallmatrix}\ \begin{smallmatrix}\uparrow\uparrow\\\downarrow\end{smallmatrix} + \tfrac{\sqrt{3}}{2}\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{2}\phantom{\boxed{3}}\end{smallmatrix}\ \begin{smallmatrix}\uparrow\uparrow\\\downarrow\end{smallmatrix}\right]\right\rangle$$

$$=\sqrt{\tfrac{1}{6}}\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\phantom{\boxed{2}}\end{smallmatrix}\ \begin{smallmatrix}\uparrow\downarrow\\\downarrow\end{smallmatrix} - \sqrt{\tfrac{1}{6}}\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\phantom{\boxed{1}}\end{smallmatrix}\ \begin{smallmatrix}\uparrow\downarrow\\\downarrow\end{smallmatrix} \qquad - \sqrt{\tfrac{1}{6}}\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\phantom{\boxed{2}}\end{smallmatrix}\ \begin{smallmatrix}\uparrow\uparrow\\\downarrow\end{smallmatrix} + \tfrac{1}{\sqrt{2}}\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{2}\phantom{\boxed{3}}\end{smallmatrix}\ \begin{smallmatrix}\uparrow\uparrow\\\downarrow\end{smallmatrix}$$



$$\left|{}^2P_{J=\frac{1}{2}}^{\frac{1}{2}}\right\rangle = \sqrt{\tfrac{2}{3}}\underline{\left|p^{L=1}_{M=1}\chi^{1/2}_{-1/2}\right\rangle - \sqrt{\tfrac{1}{3}}\left|p^{L=1}_{M=0}\chi^{1/2}_{+1/2}\right\rangle}$$

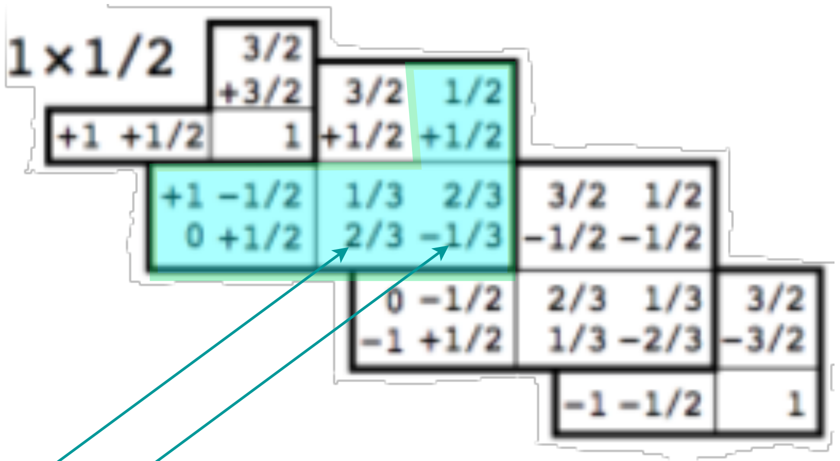
Doublet $\underline{{}^2P\text{, J}=\frac{1}{2}}$ M$_J$=$\frac{1}{2}$

ℓ=1 p=shell LS states combined to states of definite J =1/2 at L=1 M$_J$=1/2

$$\left|{}^2P_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle=\left|p^{L=1}_{M=1}\chi^{1/2}_{+1/2}\right\rangle$$

Doublet ${}^2P$, J=$\frac{3}{2}$ M$_J$=$\frac{3}{2}$

$$=\left|\left[\sqrt{\tfrac{1}{2}}\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\uparrow\\\downarrow\end{smallmatrix}-\sqrt{\tfrac{1}{2}}\begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\uparrow\\\downarrow\end{smallmatrix}\right]\right\rangle$$



1×1/2	3/2 +3/2	3/2 +1/2	1/2 +1/2	3/2 −1/2	1/2 −1/2	3/2 −3/2
+1 +1/2	1					
+1 −1/2		1/3	2/3			
0 +1/2		2/3	−1/3			
0 −1/2				2/3	1/3	
−1 +1/2				1/3	−2/3	
−1 −1/2						1

$$\left|{}^2P_{J=\frac{3}{2}}\ \tfrac{1}{2}\right\rangle=\sqrt{\tfrac{1}{3}}\left|p^{L=1}_{M=1}\chi^{1/2}_{-1/2}\right\rangle+\sqrt{\tfrac{2}{3}}\left|p^{L=1}_{M=0}\chi^{1/2}_{+1/2}\right\rangle$$

Doublet ${}^2P$, J=$\frac{3}{2}$ M$_J$=$\frac{1}{2}$

$$=\sqrt{\tfrac{1}{3}}\left|\left[\sqrt{\tfrac{1}{2}}\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\downarrow\\\downarrow\end{smallmatrix}-\sqrt{\tfrac{1}{2}}\begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\downarrow\\\downarrow\end{smallmatrix}\right]\right\rangle+\sqrt{\tfrac{2}{3}}\left|\left[\tfrac{-1}{2}\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\uparrow\\\downarrow\end{smallmatrix}+\tfrac{\sqrt{3}}{2}\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\uparrow\\\downarrow\end{smallmatrix}\right]\right\rangle$$

$$=\sqrt{\tfrac{1}{6}}\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\downarrow\\\downarrow\end{smallmatrix}-\sqrt{\tfrac{1}{6}}\begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\downarrow\\\downarrow\end{smallmatrix}\qquad-\sqrt{\tfrac{1}{6}}\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\uparrow\\\downarrow\end{smallmatrix}+\tfrac{1}{\sqrt{2}}\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\uparrow\\\downarrow\end{smallmatrix}$$

$$\left|{}^2P_{J=\frac{1}{2}}\ \tfrac{1}{2}\right\rangle=\underline{\sqrt{\tfrac{2}{3}}\left|p^{L=1}_{M=1}\chi^{1/2}_{-1/2}\right\rangle-\sqrt{\tfrac{1}{3}}\left|p^{L=1}_{M=0}\chi^{1/2}_{+1/2}\right\rangle}$$

Doublet ${}^2P$, J=$\frac{1}{2}$ M$_J$=$\frac{1}{2}$

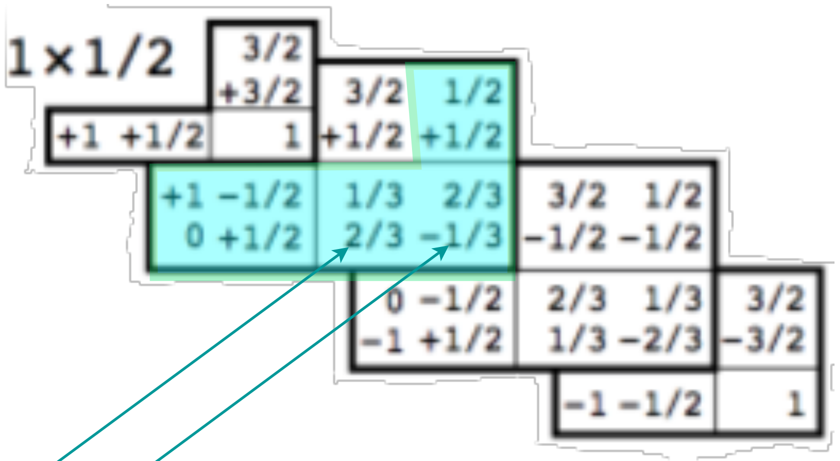
$$=\sqrt{\tfrac{2}{3}}\left|\left[\sqrt{\tfrac{1}{2}}\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\downarrow\\\downarrow\end{smallmatrix}-\sqrt{\tfrac{1}{2}}\begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\downarrow\\\downarrow\end{smallmatrix}\right]\right\rangle-\sqrt{\tfrac{1}{3}}\left|\left[\tfrac{-1}{2}\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\uparrow\\\downarrow\end{smallmatrix}+\tfrac{\sqrt{3}}{2}\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\uparrow\\\downarrow\end{smallmatrix}\right]\right\rangle$$

ℓ=1 p=shell LS states combined to states of definite J =1/2 at L=1 M

$$\left|{}^2P_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle = \left|p^{L=1}_{M=1}\chi^{1/2}_{+1/2}\right\rangle$$

Doublet ${}^2P$, J=$\frac{3}{2}$ M$_J$=$\frac{3}{2}$

$$= \left|\left[\sqrt{\tfrac{1}{2}}\begin{array}{|c|c|}\hline 1&2\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\uparrow\\ \downarrow\end{smallmatrix} - \sqrt{\tfrac{1}{2}}\begin{array}{|c|c|}\hline 1&1\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\uparrow\\ \downarrow\end{smallmatrix}\right]\right\rangle$$



1×1/2		3/2					
		+3/2	3/2	1/2			
+1	+1/2	1	+1/2	+1/2			
+1	−1/2		1/3	2/3	3/2	1/2	
0	+1/2		2/3	−1/3	−1/2	−1/2	
0	−1/2				2/3	1/3	3/2
−1	+1/2				1/3	−2/3	−3/2
−1	−1/2						1

$$\left|{}^2P_{J=\frac{3}{2}}\ \tfrac{1}{2}\right\rangle = \sqrt{\tfrac{1}{3}}\left|p^{L=1}_{M=1}\chi^{1/2}_{-1/2}\right\rangle + \sqrt{\tfrac{2}{3}}\left|p^{L=1}_{M=0}\chi^{1/2}_{+1/2}\right\rangle$$

Doublet ${}^2P$, J=$\frac{3}{2}$ M$_J$=$\frac{1}{2}$

$$= \sqrt{\tfrac{1}{3}}\left|\left[\sqrt{\tfrac{1}{2}}\begin{array}{|c|c|}\hline 1&2\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\downarrow\\ \downarrow\end{smallmatrix} - \sqrt{\tfrac{1}{2}}\begin{array}{|c|c|}\hline 1&1\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\downarrow\\ \downarrow\end{smallmatrix}\right]\right\rangle + \sqrt{\tfrac{2}{3}}\left|\left[\tfrac{-1}{2}\begin{array}{|c|c|}\hline 1&2\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\uparrow\\ \downarrow\end{smallmatrix} + \tfrac{\sqrt{3}}{2}\begin{array}{|c|c|}\hline 1&3\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\uparrow\\ \downarrow\end{smallmatrix}\right]\right\rangle$$

$$= \sqrt{\tfrac{1}{6}}\begin{array}{|c|c|}\hline 1&2\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\downarrow\\ \downarrow\end{smallmatrix} - \sqrt{\tfrac{1}{6}}\begin{array}{|c|c|}\hline 1&1\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\downarrow\\ \downarrow\end{smallmatrix} \qquad - \sqrt{\tfrac{1}{6}}\begin{array}{|c|c|}\hline 1&2\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\uparrow\\ \downarrow\end{smallmatrix} + \tfrac{1}{\sqrt{2}}\begin{array}{|c|c|}\hline 1&3\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\uparrow\\ \downarrow\end{smallmatrix}$$

$$\left|{}^2P_{J=\frac{1}{2}}\ \tfrac{1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\left|p^{L=1}_{M=1}\chi^{1/2}_{-1/2}\right\rangle - \sqrt{\tfrac{1}{3}}\left|p^{L=1}_{M=0}\chi^{1/2}_{+1/2}\right\rangle$$

Doublet ${}^2P$, J=$\frac{1}{2}$ M$_J$=$\frac{1}{2}$

$$= \sqrt{\tfrac{2}{3}}\left|\left[\sqrt{\tfrac{1}{2}}\begin{array}{|c|c|}\hline 1&2\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\downarrow\\ \downarrow\end{smallmatrix} - \sqrt{\tfrac{1}{2}}\begin{array}{|c|c|}\hline 1&1\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\downarrow\\ \downarrow\end{smallmatrix}\right]\right\rangle - \sqrt{\tfrac{1}{3}}\left|\left[\tfrac{-1}{2}\begin{array}{|c|c|}\hline 1&2\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\uparrow\\ \downarrow\end{smallmatrix} + \tfrac{\sqrt{3}}{2}\begin{array}{|c|c|}\hline 1&3\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\uparrow\\ \downarrow\end{smallmatrix}\right]\right\rangle$$

$$= \sqrt{\tfrac{1}{3}}\begin{array}{|c|c|}\hline 1&2\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\downarrow\\ \downarrow\end{smallmatrix} - \sqrt{\tfrac{1}{3}}\begin{array}{|c|c|}\hline 1&1\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\downarrow\\ \downarrow\end{smallmatrix} \qquad + \sqrt{\tfrac{1}{12}}\begin{array}{|c|c|}\hline 1&2\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\uparrow\\ \downarrow\end{smallmatrix} - \tfrac{1}{2}\begin{array}{|c|c|}\hline 1&3\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\uparrow\\ \downarrow\end{smallmatrix}$$

AMOP
reference links
on pages 2-4

4.25.18 class 26: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

$(S_3)^*(U(3)) \subset U(6)$ models of p^3 electronic spin-orbit states and couplings

[2,1] tableau states lowered by $\mathbf{L}_- = \sqrt{2}(E_{21}+E_{32})$

Top-(J,M) states to mid-level states

$\ell=1$ p=shell LS states combined to states of definite J

J=3/2 at L=0 (^4S), J=5/2 at L=2 (^2D)

C-G coupling; J=3/2 at L=2 (^2D), J=3/2 at L=1 (^2P), J=1/2 at L=1 (^2P)

➡ Spin-orbit state assembly formula and Slater determinants

$\ell=1$ p=shell LSJ states transformed to Slater determinants from J=3/2 (^4S)

Slater functions for J=5/2, J=3/2 (^2D)

Slater functions for J=3/2 (^2P), J=1/2 (^2P)

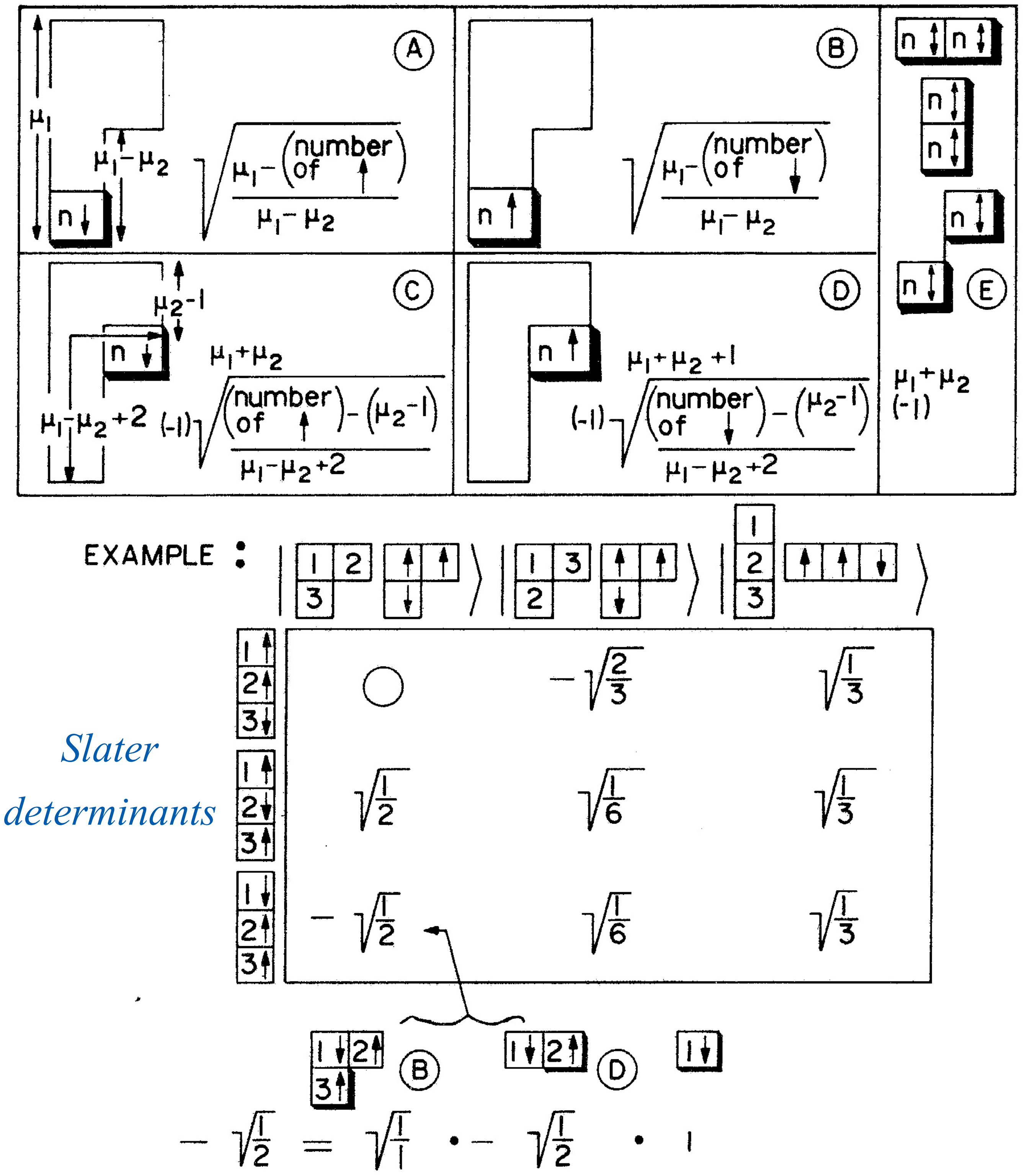
Summary of states and level connection paths

Symmetry dimension accounting

Spin-orbit Hamiltonian matrix calculation

Application to spin-orbit and entanglement break-up scattering

Introducing atomic spin-orbit state assembly formula and Slater determinants



Slater determinants

FIG. 5. Assembly formula for combining orbital and spin states. Each column state (Slater determinant) on the left-hand side of the sample table has a definite spin (arrow)on each orbital state (number). The formulas will give the overlap of this Slater state with a given orbital tableau state if we first write the spins within this orbital tableau in exactly the same way. Then we proceed to remove boxes with numbered spins starting with the highest number(s). Each "removal" gives a factor depending on what is being removed and where (cases A–E). All of the numbers in the formulas refer to the condition of the tableau just before the box outlined in the figure is removed.

Introducing atomic spin-orbit state assembly formula and Slater determinants

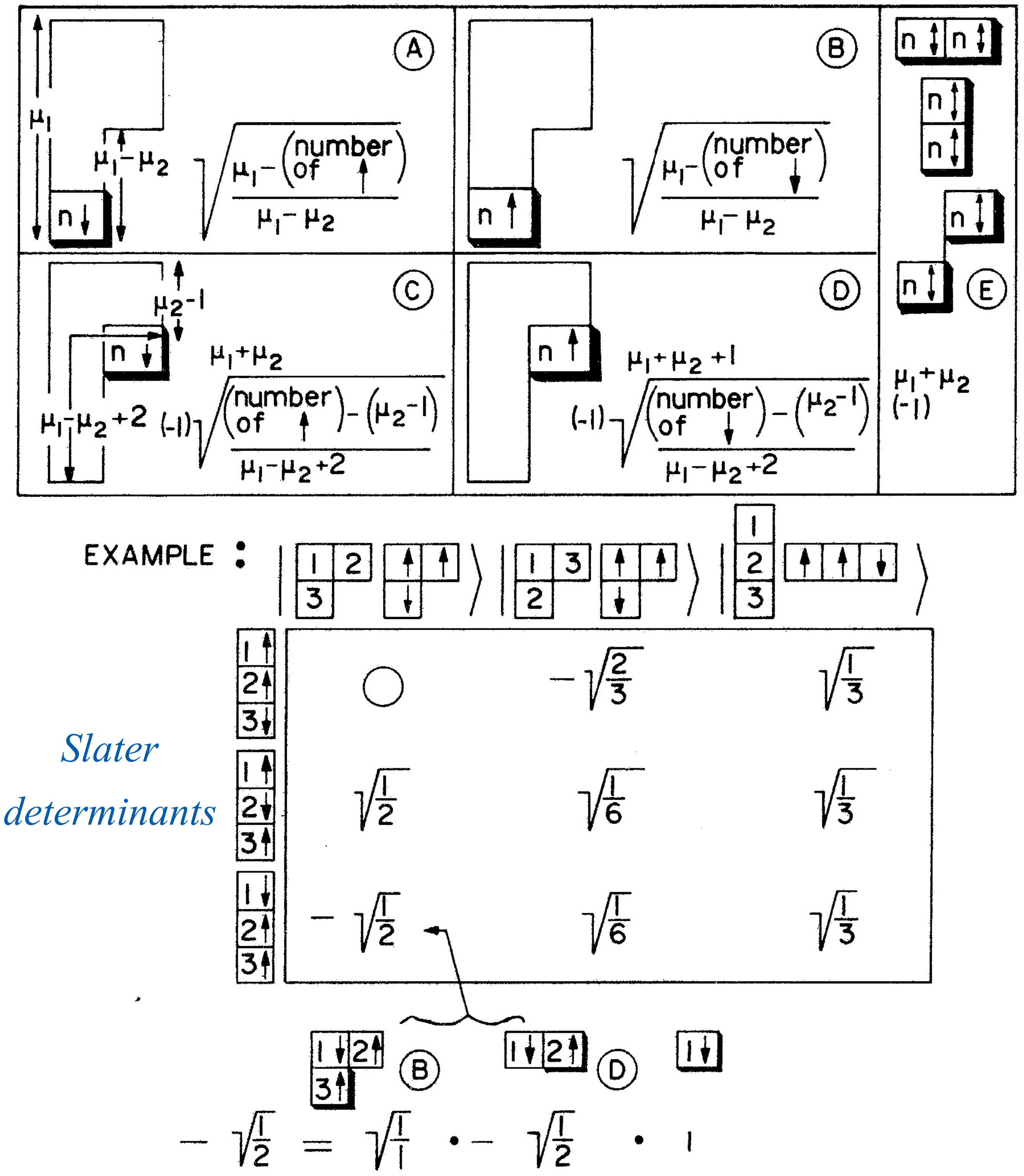


FIG. 5. Assembly formula for combining orbital and spin states. Each column state (Slater determinant) on the left-hand side of the sample table has a definite spin (arrow) on each orbital state (number). The formulas will give the overlap of this Slater state with a given orbital tableau state if we first write the spins within this orbital tableau in exactly the same way. Then we proceed to remove boxes with numbered spins starting with the highest number(s). Each "removal" gives a factor depending on what is being removed and where (cases A–E). All of the numbers in the formulas refer to the condition of the tableau just before the box outlined in the figure is removed.

The simplest assembly:

Slater determinants

	1 2 ↑ ↓	1 ↑ ↓ 2
1↑ 2↓	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$
1↑ 2↓	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$

AMOP
reference links
on pages 2-4

4.23.18 class 25: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

$(S_3)^*(U(3))\subset U(6)$ models of p^3 electronic spin-orbit states and couplings

[2,1] tableau states lowered by $\mathbf{L}_- = \sqrt{2}(E_{21}+E_{32})$

Top-(J,M) states   thru mid-level states

ℓ=1 *p*=shell LS states combined to states of definite J

J=3/2 at L=0 (^{4}S).   J=5/2 at L=2 (^{2}D)
Clebsch-Gordon coupling; J=3/2 at L=2 (^{2}D)
J=3/2 at L=1 (^{2}P)
J=1/2 at L=1 (^{2}P)

Spin-orbit state assembly formula and Slater determinants

➡ The simplest assembly (Detailed)

ℓ=1 *p*=shell LSJ states transformed to Slater determinants from J=3/2 (^{4}S)

Slater functions for J=5/2 (^{2}D)
Slater functions for J=3/2 (^{2}D)
Slater functions for J=3/2 (^{2}P)

Application to spin-orbit and entanglement break-up scattering

Introducing atomic spin-orbit state assembly formula and Slater determinants

Slater determinant state key:

$$a=1\uparrow, b=1\downarrow, c=2\uparrow, d=2\downarrow$$

The simplest assembly:

	$\boxed{1}\boxed{2}\;\begin{matrix}\boxed{\uparrow}\\ \boxed{\downarrow}\end{matrix}$	$\begin{matrix}\boxed{1}\\ \boxed{2}\end{matrix}\;\boxed{\uparrow}\boxed{\downarrow}$
$\begin{matrix}\boxed{1\uparrow}\\ \boxed{2\downarrow}\end{matrix}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$
$\begin{matrix}\boxed{1\downarrow}\\ \boxed{2\uparrow}\end{matrix}$	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$

$$\boxed{1}\boxed{2}\qquad\begin{matrix}\boxed{\uparrow}\\ \boxed{\downarrow}\end{matrix}$$

$$\left(\frac{\left|1,2\right\rangle+\left|2,1\right\rangle}{\sqrt{2}}\right)\left(\frac{\left|\uparrow,\downarrow\right\rangle-\left|\downarrow,\uparrow\right\rangle}{\sqrt{2}}\right)$$

$$\begin{matrix}\boxed{1}\\ \boxed{2}\end{matrix}\qquad\boxed{\uparrow}\boxed{\downarrow}$$

$$\left(\frac{\left|1,2\right\rangle-\left|2,1\right\rangle}{\sqrt{2}}\right)\left(\frac{\left|\uparrow,\downarrow\right\rangle+\left|\downarrow,\uparrow\right\rangle}{\sqrt{2}}\right)$$

	$\boxed{1}\boxed{2}\;\begin{matrix}\boxed{\uparrow}\\ \boxed{\downarrow}\end{matrix}$
$\begin{matrix}\boxed{1\uparrow}\\ \boxed{2\downarrow}\end{matrix}=\begin{matrix}a\\ d\end{matrix}=\frac{ad-da}{\sqrt{2}}$	
$\begin{matrix}\boxed{1\downarrow}\\ \boxed{2\uparrow}\end{matrix}=\begin{matrix}b\\ c\end{matrix}=\frac{bc-cb}{\sqrt{2}}$	

Introducing atomic spin-orbit state assembly formula and Slater determinants

Slater determinant state key:

$a=1\uparrow, b=1\downarrow, c=2\uparrow, d=2\downarrow$

The simplest assembly:

	$\boxed{1}\boxed{2}\ \begin{matrix}\boxed{\uparrow}\\\boxed{\downarrow}\end{matrix}$	$\begin{matrix}\boxed{1}\\\boxed{2}\end{matrix}\ \boxed{\uparrow}\boxed{\downarrow}$
$\begin{matrix}\boxed{1\uparrow}\\\boxed{2\downarrow}\end{matrix}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$
$\begin{matrix}\boxed{1\downarrow}\\\boxed{2\uparrow}\end{matrix}$	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$

$\boxed{1}\boxed{2}\qquad\begin{matrix}\boxed{\uparrow}\\\boxed{\downarrow}\end{matrix}$

$$\left(\frac{|1,2\rangle+|2,1\rangle}{\sqrt{2}}\right)\left(\frac{|\uparrow,\downarrow\rangle-|\downarrow,\uparrow\rangle}{\sqrt{2}}\right)$$

$$\frac{1}{2}\left(|1\uparrow,2\downarrow\rangle-|2\downarrow,1\uparrow\rangle+|2\uparrow,1\downarrow\rangle-|1\downarrow,2\uparrow\rangle\right)$$

$\begin{matrix}\boxed{1}\\\boxed{2}\end{matrix}\qquad\boxed{\uparrow}\boxed{\downarrow}$

$$\left(\frac{|1,2\rangle-|2,1\rangle}{\sqrt{2}}\right)\left(\frac{|\uparrow,\downarrow\rangle+|\downarrow,\uparrow\rangle}{\sqrt{2}}\right)$$

	$\boxed{1}\boxed{2}\ \begin{matrix}\boxed{\uparrow}\\\boxed{\downarrow}\end{matrix}$
$\begin{matrix}\boxed{1\uparrow}\\\boxed{2\downarrow}\end{matrix} = \begin{matrix}a\\d\end{matrix} = \frac{ad-da}{\sqrt{2}}$	
$\begin{matrix}\boxed{1\downarrow}\\\boxed{2\uparrow}\end{matrix} = \begin{matrix}b\\c\end{matrix} = \frac{bc-cb}{\sqrt{2}}$	

Introducing atomic spin-orbit state assembly formula and Slater determinants

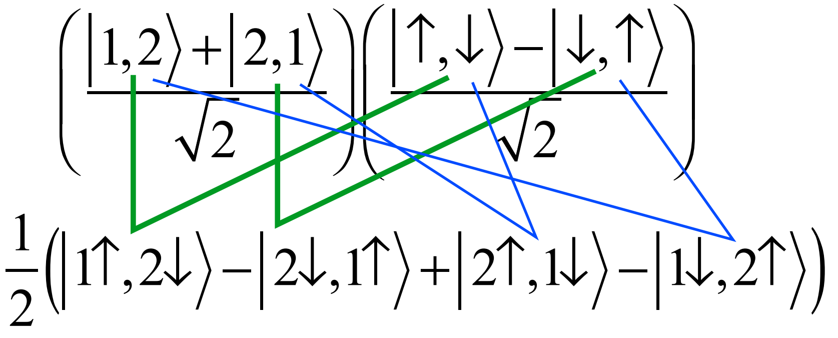
Slater determinant state key:

$a=1\uparrow, b=1\downarrow, c=2\uparrow, d=2\downarrow$

The simplest assembly:

	$\begin{array}{\|c\|c\|}\hline 1&2\\\hline\end{array}\ \begin{array}{\|c\|}\hline\uparrow\\\hline\downarrow\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 1\\\hline 2\\\hline\end{array}\ \begin{array}{\|c\|c\|}\hline\uparrow&\downarrow\\\hline\end{array}$
$\begin{array}{\|c\|}\hline 1\uparrow\\\hline 2\downarrow\\\hline\end{array}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$
$\begin{array}{\|c\|}\hline 1\downarrow\\\hline 2\uparrow\\\hline\end{array}$	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$

$\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\qquad\begin{array}{|c|}\hline\uparrow\\\hline\downarrow\\\hline\end{array}$



$$\left(\frac{|1,2\rangle+|2,1\rangle}{\sqrt{2}}\right)\left(\frac{|\uparrow,\downarrow\rangle-|\downarrow,\uparrow\rangle}{\sqrt{2}}\right)$$

$$\frac{1}{2}\left(|1\uparrow,2\downarrow\rangle-|2\downarrow,1\uparrow\rangle+|2\uparrow,1\downarrow\rangle-|1\downarrow,2\uparrow\rangle\right)$$

$$\frac{1}{2}\left(\quad ad \quad - \quad da \quad + \quad cb \quad - \quad bc \quad\right)$$

$\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\qquad\begin{array}{|c|c|}\hline\uparrow&\downarrow\\\hline\end{array}$

$$\left(\frac{|1,2\rangle-|2,1\rangle}{\sqrt{2}}\right)\left(\frac{|\uparrow,\downarrow\rangle+|\downarrow,\uparrow\rangle}{\sqrt{2}}\right)$$

	$\begin{array}{\|c\|c\|}\hline 1&2\\\hline\end{array}\ \begin{array}{\|c\|}\hline\uparrow\\\hline\downarrow\\\hline\end{array}$ $\frac{1}{2}\left(ad-da+cb-bc\right)$
$\begin{array}{\|c\|}\hline 1\uparrow\\\hline 2\downarrow\\\hline\end{array}=\begin{matrix}a\\d\end{matrix}=\frac{ad-da}{\sqrt{2}}$	
$\begin{array}{\|c\|}\hline 1\downarrow\\\hline 2\uparrow\\\hline\end{array}=\begin{matrix}b\\c\end{matrix}=\frac{bc-cb}{\sqrt{2}}$	

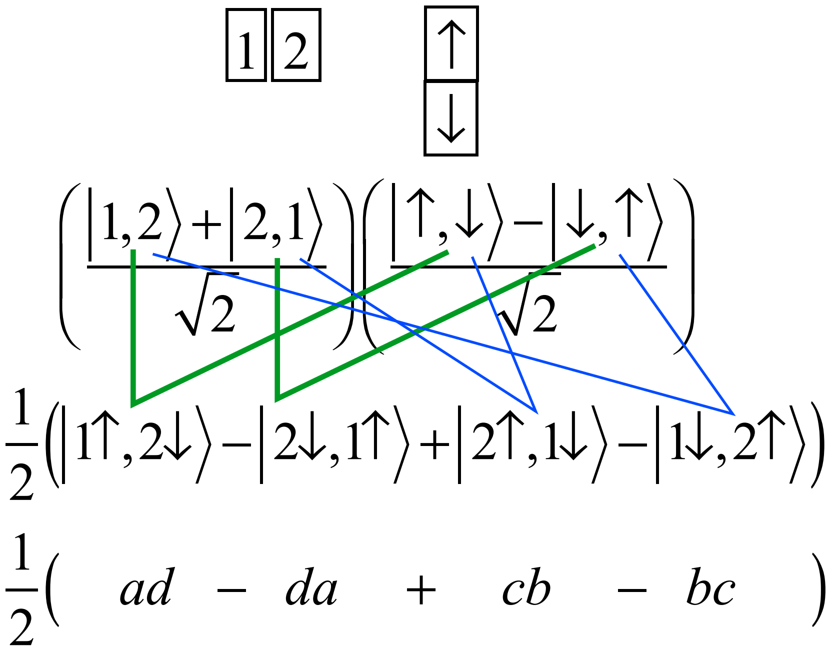
Introducing atomic spin-orbit state assembly formula and Slater determinants

Slater determinant state key:

$$a=1\uparrow, b=1\downarrow, c=2\uparrow, d=2\downarrow$$

The simplest assembly:

	$\boxed{1}\boxed{2}$ $\begin{array}{\|c\|}\hline \uparrow\\ \hline \downarrow\\ \hline\end{array}$	$\begin{array}{\|c\|}\hline 1\\ \hline 2\\ \hline\end{array}$ $\boxed{\uparrow}\boxed{\downarrow}$
$\begin{array}{\|c\|}\hline 1\uparrow\\ \hline 2\downarrow\\ \hline\end{array}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$
$\begin{array}{\|c\|}\hline 1\downarrow\\ \hline 2\uparrow\\ \hline\end{array}$	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$

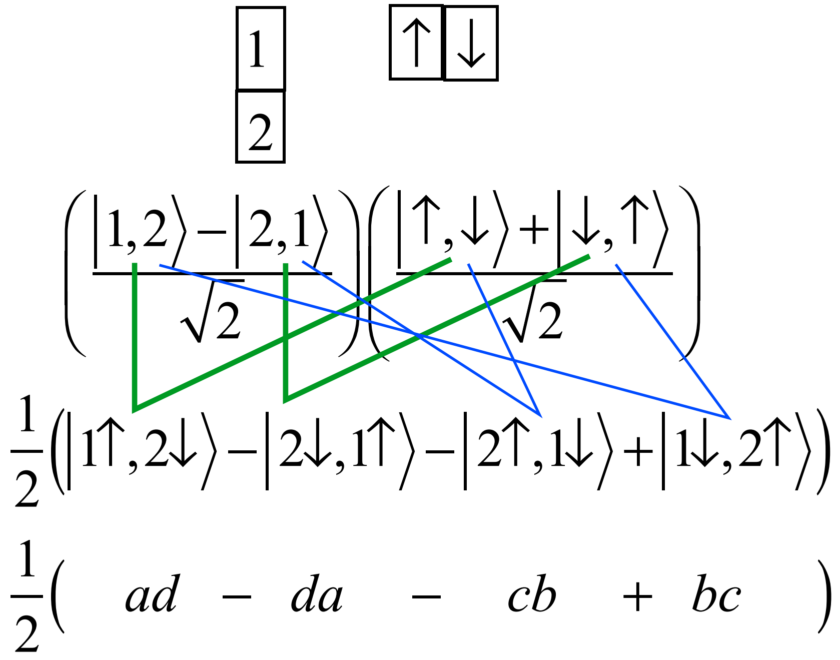


$\boxed{1}\boxed{2}$ $\begin{array}{|c|}\hline \uparrow\\ \hline \downarrow\\ \hline\end{array}$

$$\left(\frac{|1,2\rangle+|2,1\rangle}{\sqrt{2}}\right)\left(\frac{|\uparrow,\downarrow\rangle-|\downarrow,\uparrow\rangle}{\sqrt{2}}\right)$$

$$\frac{1}{2}\left(|1\uparrow,2\downarrow\rangle-|2\downarrow,1\uparrow\rangle+|2\uparrow,1\downarrow\rangle-|1\downarrow,2\uparrow\rangle\right)$$

$$\frac{1}{2}\left(\quad ad \quad - \quad da \quad + \quad cb \quad - \quad bc \quad\right)$$



$\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array}$ $\boxed{\uparrow}\boxed{\downarrow}$

$$\left(\frac{|1,2\rangle-|2,1\rangle}{\sqrt{2}}\right)\left(\frac{|\uparrow,\downarrow\rangle+|\downarrow,\uparrow\rangle}{\sqrt{2}}\right)$$

$$\frac{1}{2}\left(|1\uparrow,2\downarrow\rangle-|2\downarrow,1\uparrow\rangle-|2\uparrow,1\downarrow\rangle+|1\downarrow,2\uparrow\rangle\right)$$

$$\frac{1}{2}\left(\quad ad \quad - \quad da \quad - \quad cb \quad + \quad bc \quad\right)$$

	$\boxed{1}\boxed{2}$ $\begin{array}{\|c\|}\hline \uparrow\\ \hline \downarrow\\ \hline\end{array}$ $\frac{1}{2}(ad-da+cb-bc)$	$\begin{array}{\|c\|}\hline 1\\ \hline 2\\ \hline\end{array}$ $\boxed{\uparrow}\boxed{\downarrow}$ $\frac{1}{2}(ad-da-cb+bc)$
$\begin{array}{\|c\|}\hline 1\uparrow\\ \hline 2\downarrow\\ \hline\end{array} = \begin{matrix}a\\ d\end{matrix} = \frac{ad-da}{\sqrt{2}}$		
$\begin{array}{\|c\|}\hline 1\downarrow\\ \hline 2\uparrow\\ \hline\end{array} = \begin{matrix}b\\ c\end{matrix} = \frac{bc-cb}{\sqrt{2}}$		

Introducing atomic spin-orbit state assembly formula and Slater determinants

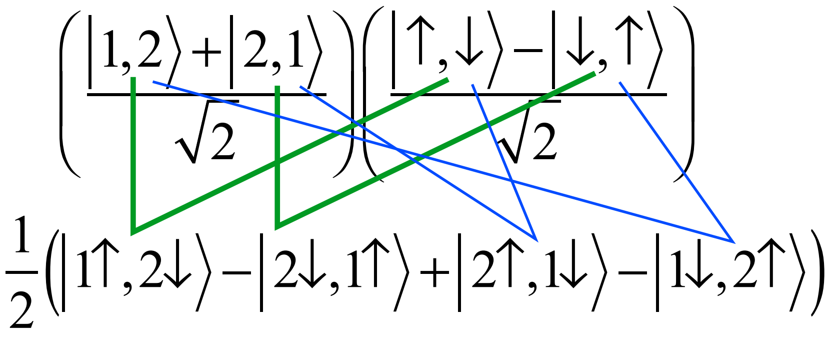
Slater determinant state key:

$a=1\uparrow, b=1\downarrow, c=2\uparrow, d=2\downarrow$

The simplest assembly:

	$\boxed{1}\boxed{2}\ \begin{array}{\|c\|}\hline \uparrow\\\hline \downarrow\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 1\\\hline 2\\\hline\end{array}\ \boxed{\uparrow}\boxed{\downarrow}$
$\begin{array}{\|c\|}\hline 1\uparrow\\\hline 2\downarrow\\\hline\end{array}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$
$\begin{array}{\|c\|}\hline 1\downarrow\\\hline 2\uparrow\\\hline\end{array}$	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$

$\boxed{1}\boxed{2}\quad \begin{array}{|c|}\hline \uparrow\\\hline \downarrow\\\hline\end{array}$

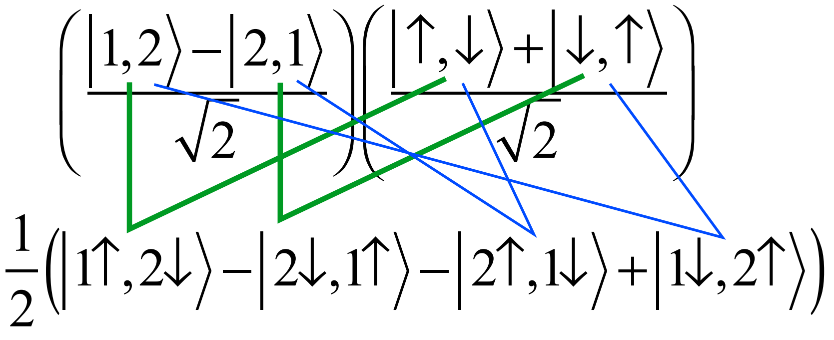


$$\left(\frac{|1,2\rangle+|2,1\rangle}{\sqrt{2}}\right)\left(\frac{|\uparrow,\downarrow\rangle-|\downarrow,\uparrow\rangle}{\sqrt{2}}\right)$$

$$\frac{1}{2}\left(|1\uparrow,2\downarrow\rangle-|2\downarrow,1\uparrow\rangle+|2\uparrow,1\downarrow\rangle-|1\downarrow,2\uparrow\rangle\right)$$

$$\frac{1}{2}\left(\ ad\ -\ da\ +\ cb\ -\ bc\ \right)$$

$\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\quad \boxed{\uparrow}\boxed{\downarrow}$



$$\left(\frac{|1,2\rangle-|2,1\rangle}{\sqrt{2}}\right)\left(\frac{|\uparrow,\downarrow\rangle+|\downarrow,\uparrow\rangle}{\sqrt{2}}\right)$$

$$\frac{1}{2}\left(|1\uparrow,2\downarrow\rangle-|2\downarrow,1\uparrow\rangle-|2\uparrow,1\downarrow\rangle+|1\downarrow,2\uparrow\rangle\right)$$

$$\frac{1}{2}\left(\ ad\ -\ da\ -\ cb\ +\ bc\ \right)$$

	$\boxed{1}\boxed{2}\ \begin{array}{\|c\|}\hline \uparrow\\\hline \downarrow\\\hline\end{array}$ $\frac{1}{2}(ad-da+cb-bc)$	$\begin{array}{\|c\|}\hline 1\\\hline 2\\\hline\end{array}\ \boxed{\uparrow}\boxed{\downarrow}$ $\frac{1}{2}(ad-da-cb+bc)$
$\begin{array}{\|c\|}\hline 1\uparrow\\\hline 2\downarrow\\\hline\end{array} = \begin{array}{c} a\\ d\end{array} = \frac{ad-da}{\sqrt{2}}$	$\frac{1}{2}\frac{1}{\sqrt{2}}+\frac{1}{2}\frac{1}{\sqrt{2}}=\sqrt{\frac{1}{2}}$	
$\begin{array}{\|c\|}\hline 1\downarrow\\\hline 2\uparrow\\\hline\end{array} = \begin{array}{c} b\\ c\end{array} = \frac{bc-cb}{\sqrt{2}}$		

Introducing atomic spin-orbit state assembly formula and Slater determinants

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$a=1\uparrow, b=1\downarrow, c=2\uparrow, d=2\downarrow$

The simplest assembly:

	$\boxed{1}\boxed{2}\ \begin{matrix}\boxed{\uparrow}\\\boxed{\downarrow}\end{matrix}$	$\begin{matrix}\boxed{1}\\\boxed{2}\end{matrix}\ \boxed{\uparrow}\boxed{\downarrow}$
$\begin{matrix}\boxed{1\uparrow}\\\boxed{2\downarrow}\end{matrix}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$
$\begin{matrix}\boxed{1\downarrow}\\\boxed{2\uparrow}\end{matrix}$	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$

$$\boxed{1}\boxed{2}\qquad\begin{matrix}\boxed{\uparrow}\\\boxed{\downarrow}\end{matrix}$$

$$\left(\frac{\lvert 1,2\rangle+\lvert 2,1\rangle}{\sqrt{2}}\right)\left(\frac{\lvert\uparrow,\downarrow\rangle-\lvert\downarrow,\uparrow\rangle}{\sqrt{2}}\right)$$

$$\frac{1}{2}\left(\lvert 1\uparrow,2\downarrow\rangle-\lvert 2\downarrow,1\uparrow\rangle+\lvert 2\uparrow,1\downarrow\rangle-\lvert 1\downarrow,2\uparrow\rangle\right)$$

$$\frac{1}{2}\left(\ ad\ -\ da\ +\ cb\ -\ bc\ \right)$$

$$\begin{matrix}\boxed{1}\\\boxed{2}\end{matrix}\qquad\boxed{\uparrow}\boxed{\downarrow}$$

$$\left(\frac{\lvert 1,2\rangle-\lvert 2,1\rangle}{\sqrt{2}}\right)\left(\frac{\lvert\uparrow,\downarrow\rangle+\lvert\downarrow,\uparrow\rangle}{\sqrt{2}}\right)$$

$$\frac{1}{2}\left(\lvert 1\uparrow,2\downarrow\rangle-\lvert 2\downarrow,1\uparrow\rangle-\lvert 2\uparrow,1\downarrow\rangle+\lvert 1\downarrow,2\uparrow\rangle\right)$$

$$\frac{1}{2}\left(\ ad\ -\ da\ -\ cb\ +\ bc\ \right)$$

	$\boxed{1}\boxed{2}\ \begin{matrix}\boxed{\uparrow}\\\boxed{\downarrow}\end{matrix}$ $\frac{1}{2}\left(ad-da+cb-bc\right)$	$\begin{matrix}\boxed{1}\\\boxed{2}\end{matrix}\ \boxed{\uparrow}\boxed{\downarrow}$ $\frac{1}{2}\left(ad-da-cb+bc\right)$
$\begin{matrix}\boxed{1\uparrow}\\\boxed{2\downarrow}\end{matrix} = \begin{matrix}a\\d\end{matrix} = \frac{ad-da}{\sqrt{2}}$	$\frac{1}{2}\frac{1}{\sqrt{2}}+\frac{1}{2}\frac{1}{\sqrt{2}}=\sqrt{\frac{1}{2}}$	$\frac{1}{2}\frac{1}{\sqrt{2}}+\frac{1}{2}\frac{1}{\sqrt{2}}=\sqrt{\frac{1}{2}}$
$\begin{matrix}\boxed{1\downarrow}\\\boxed{2\uparrow}\end{matrix} = \begin{matrix}b\\c\end{matrix} = \frac{bc-cb}{\sqrt{2}}$		

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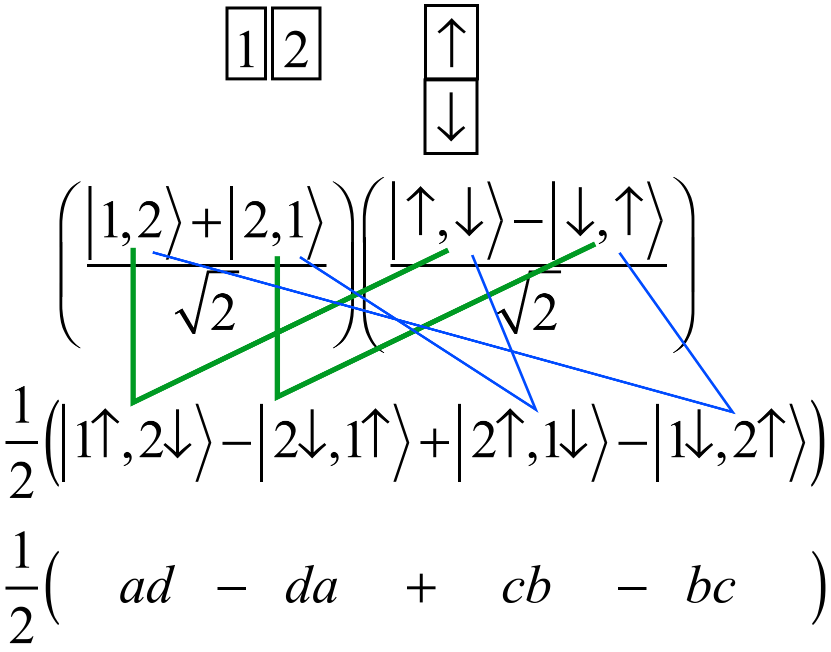
$$a=1\uparrow,b=1\downarrow,c=2\uparrow,d=2\downarrow$$

The simplest assembly:

	$\boxed{1}\boxed{2}\ \begin{array}{\|c\|}\hline \uparrow\\\hline \downarrow\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 1\\\hline 2\\\hline\end{array}\ \boxed{\uparrow}\boxed{\downarrow}$
$\begin{array}{\|c\|}\hline 1\uparrow\\\hline 2\downarrow\\\hline\end{array}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$
$\begin{array}{\|c\|}\hline 1\downarrow\\\hline 2\uparrow\\\hline\end{array}$	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$

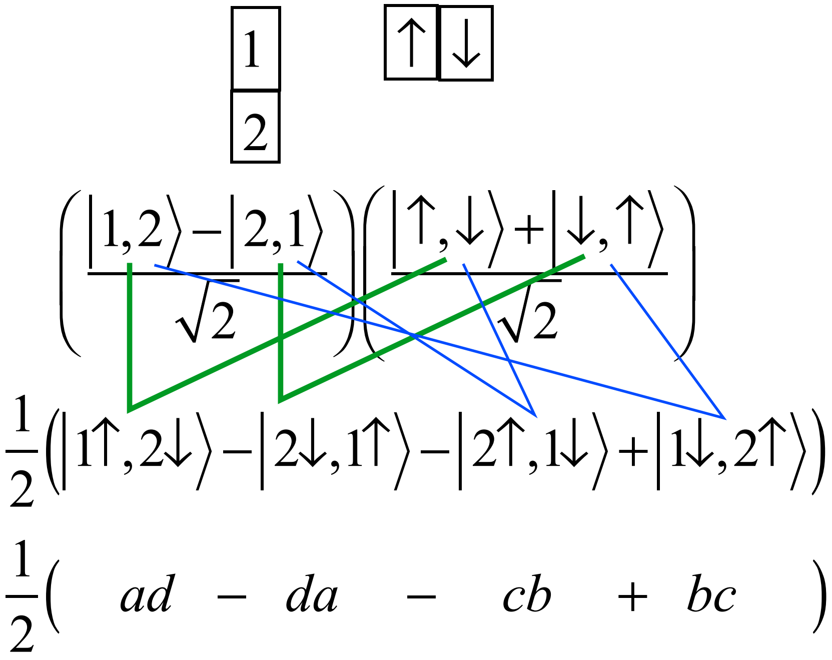


$$\boxed{1}\boxed{2}\qquad\begin{array}{|c|}\hline \uparrow\\\hline \downarrow\\\hline\end{array}$$

$$\left(\frac{|1,2\rangle+|2,1\rangle}{\sqrt{2}}\right)\left(\frac{|\uparrow,\downarrow\rangle-|\downarrow,\uparrow\rangle}{\sqrt{2}}\right)$$

$$\frac{1}{2}\left(|1\uparrow,2\downarrow\rangle-|2\downarrow,1\uparrow\rangle+|2\uparrow,1\downarrow\rangle-|1\downarrow,2\uparrow\rangle\right)$$

$$\frac{1}{2}\left(\ ad\ -\ da\ +\ cb\ -\ bc\ \right)$$



$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\qquad\boxed{\uparrow}\boxed{\downarrow}$$

$$\left(\frac{|1,2\rangle-|2,1\rangle}{\sqrt{2}}\right)\left(\frac{|\uparrow,\downarrow\rangle+|\downarrow,\uparrow\rangle}{\sqrt{2}}\right)$$

$$\frac{1}{2}\left(|1\uparrow,2\downarrow\rangle-|2\downarrow,1\uparrow\rangle-|2\uparrow,1\downarrow\rangle+|1\downarrow,2\uparrow\rangle\right)$$

$$\frac{1}{2}\left(\ ad\ -\ da\ -\ cb\ +\ bc\ \right)$$

	$\boxed{1}\boxed{2}\ \begin{array}{\|c\|}\hline \uparrow\\\hline \downarrow\\\hline\end{array}$ $\frac{1}{2}(ad-da+cb-bc)$	$\begin{array}{\|c\|}\hline 1\\\hline 2\\\hline\end{array}\ \boxed{\uparrow}\boxed{\downarrow}$ $\frac{1}{2}(ad-da-cb+bc)$
$\begin{array}{\|c\|}\hline 1\uparrow\\\hline 2\downarrow\\\hline\end{array}=\begin{array}{c}a\\d\end{array}=\frac{ad-da}{\sqrt{2}}$	$\frac{1}{2}\frac{1}{\sqrt{2}}+\frac{1}{2}\frac{1}{\sqrt{2}}=\sqrt{\frac{1}{2}}$	$\frac{1}{2}\frac{1}{\sqrt{2}}+\frac{1}{2}\frac{1}{\sqrt{2}}=\sqrt{\frac{1}{2}}$
$\begin{array}{\|c\|}\hline 1\downarrow\\\hline 2\uparrow\\\hline\end{array}=\begin{array}{c}b\\c\end{array}=\frac{bc-cb}{\sqrt{2}}$	$-\frac{1}{2}\frac{1}{\sqrt{2}}-\frac{1}{2}\frac{1}{\sqrt{2}}=-\sqrt{\frac{1}{2}}$	

Introducing atomic spin-orbit state assembly formula and Slater determinants

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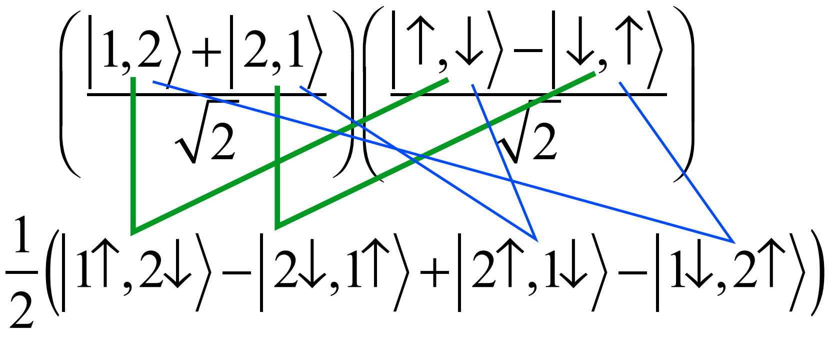
The simplest assembly:

	$\boxed{1}\boxed{2}\ \begin{matrix}\boxed{\uparrow}\\\boxed{\downarrow}\end{matrix}$	$\begin{matrix}\boxed{1}\\\boxed{2}\end{matrix}\ \boxed{\uparrow}\boxed{\downarrow}$
$\begin{matrix}\boxed{1\uparrow}\\\boxed{2\downarrow}\end{matrix}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$
$\begin{matrix}\boxed{1\downarrow}\\\boxed{2\uparrow}\end{matrix}$	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$

$\boxed{1}\boxed{2}\qquad\begin{matrix}\boxed{\uparrow}\\\boxed{\downarrow}\end{matrix}$

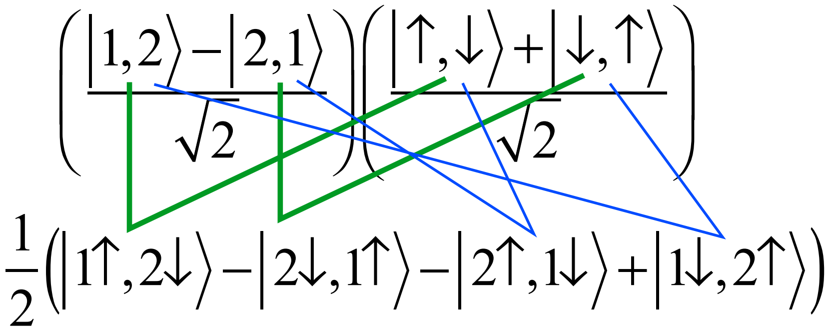


$$\left(\frac{|1,2\rangle+|2,1\rangle}{\sqrt{2}}\right)\left(\frac{|\uparrow,\downarrow\rangle-|\downarrow,\uparrow\rangle}{\sqrt{2}}\right)$$

$$\frac{1}{2}\left(|1\uparrow,2\downarrow\rangle-|2\downarrow,1\uparrow\rangle+|2\uparrow,1\downarrow\rangle-|1\downarrow,2\uparrow\rangle\right)$$

$$\frac{1}{2}(\quad ad \quad - \quad da \quad + \quad cb \quad - \quad bc \quad)$$

$\begin{matrix}\boxed{1}\\\boxed{2}\end{matrix}\qquad\boxed{\uparrow}\boxed{\downarrow}$



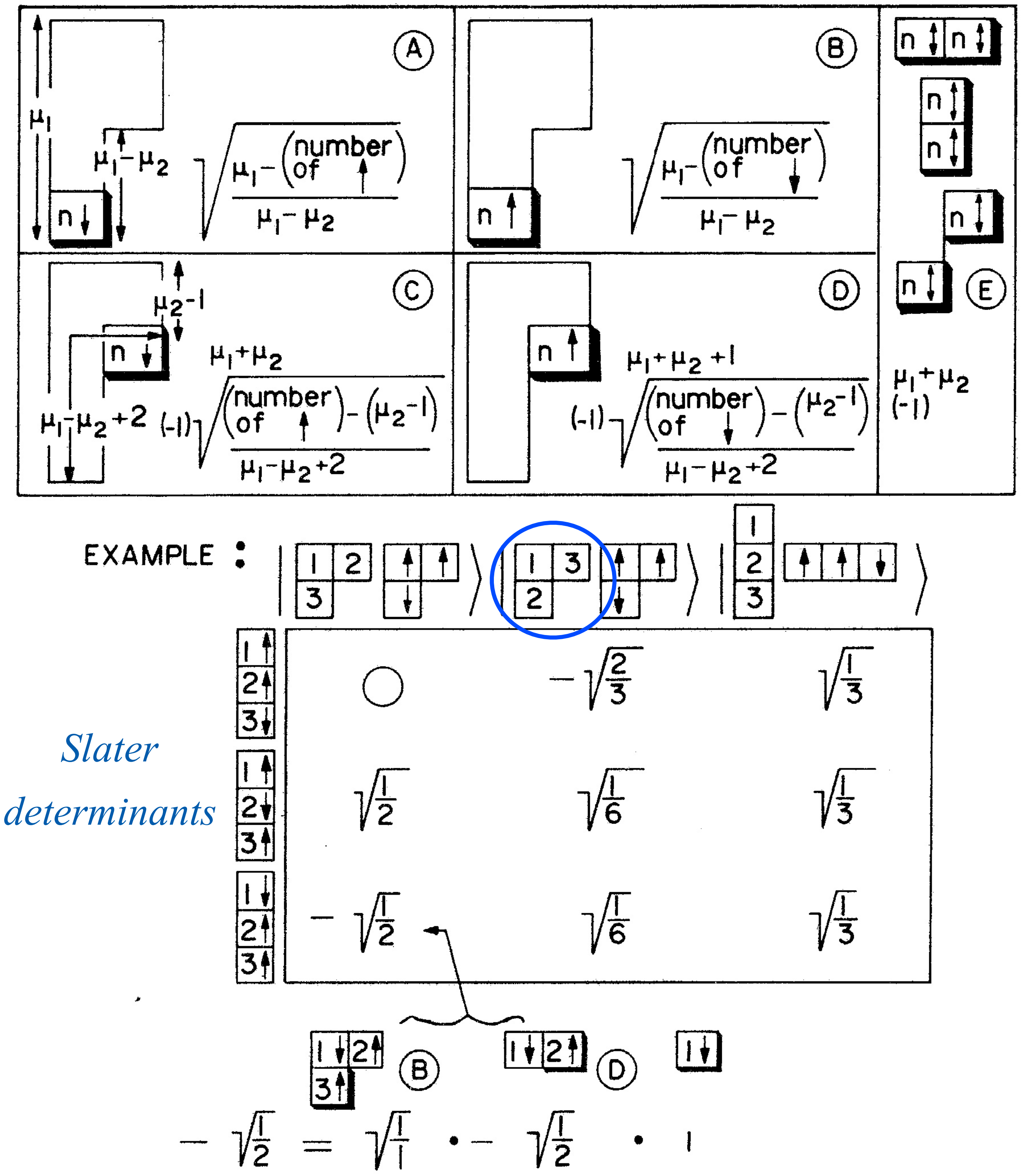
$$\left(\frac{|1,2\rangle-|2,1\rangle}{\sqrt{2}}\right)\left(\frac{|\uparrow,\downarrow\rangle+|\downarrow,\uparrow\rangle}{\sqrt{2}}\right)$$

$$\frac{1}{2}\left(|1\uparrow,2\downarrow\rangle-|2\downarrow,1\uparrow\rangle-|2\uparrow,1\downarrow\rangle+|1\downarrow,2\uparrow\rangle\right)$$

$$\frac{1}{2}(\quad ad \quad - \quad da \quad - \quad cb \quad + \quad bc \quad)$$

	$\boxed{1}\boxed{2}\ \begin{matrix}\boxed{\uparrow}\\\boxed{\downarrow}\end{matrix}$ $\frac{1}{2}(ad-da+cb-bc)$	$\begin{matrix}\boxed{1}\\\boxed{2}\end{matrix}\ \boxed{\uparrow}\boxed{\downarrow}$ $\frac{1}{2}(ad-da-cb+bc)$
$\begin{matrix}\boxed{1\uparrow}\\\boxed{2\downarrow}\end{matrix} = \begin{matrix}a\\d\end{matrix} = \frac{ad-da}{\sqrt{2}}$	$\frac{1}{2}\frac{1}{\sqrt{2}}+\frac{1}{2}\frac{1}{\sqrt{2}}=\sqrt{\frac{1}{2}}$	$\frac{1}{2}\frac{1}{\sqrt{2}}+\frac{1}{2}\frac{1}{\sqrt{2}}=\sqrt{\frac{1}{2}}$
$\begin{matrix}\boxed{1\downarrow}\\\boxed{2\uparrow}\end{matrix} = \begin{matrix}b\\c\end{matrix} = \frac{bc-cb}{\sqrt{2}}$	$-\frac{1}{2}\frac{1}{\sqrt{2}}-\frac{1}{2}\frac{1}{\sqrt{2}}=-\sqrt{\frac{1}{2}}$	$\frac{1}{2}\frac{1}{\sqrt{2}}+\frac{1}{2}\frac{1}{\sqrt{2}}=\sqrt{\frac{1}{2}}$

Introducing atomic spin-orbit state assembly formula and Slater determinants

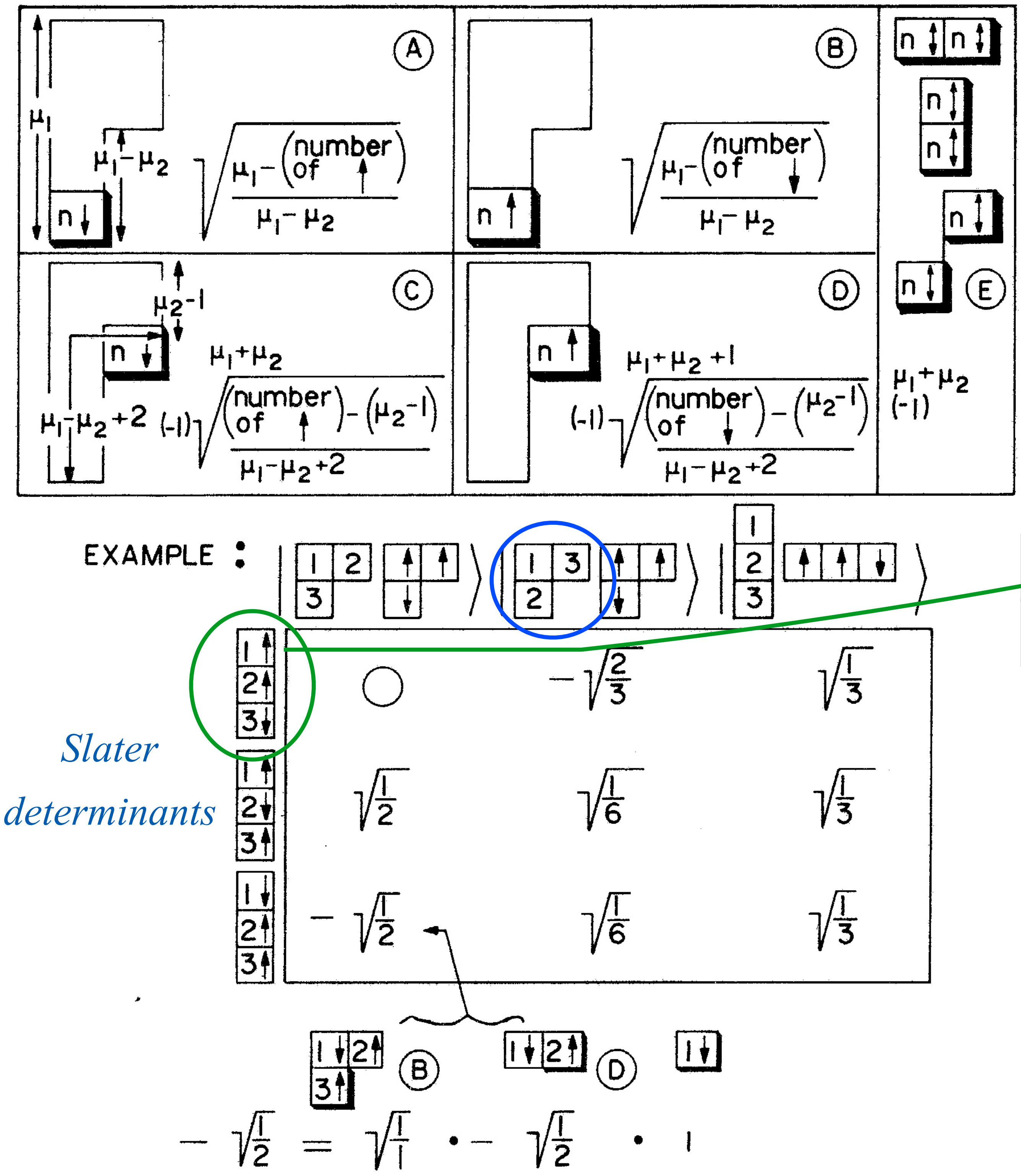


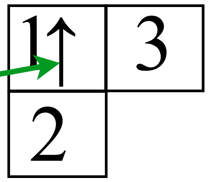
Slater determinants

FIG. 5. Assembly formula for combining orbital and spin states. Each column state (Slater determinant) on the left-hand side of the sample table has a definite spin (arrow)on each orbital state (number). The formulas will give the overlap of this Slater state with a given orbital tableau state if we first write the spins within this orbital tableau in exactly the same way. Then we proceed to remove boxes with numbered spins starting with the highest number(s). Each "removal" gives a factor depending on what is being removed and where (cases A–E). All of the numbers in the formulas refer to the condition of the tableau just before the box outlined in the figure is removed.

1	3
2	

Introducing atomic spin-orbit state assembly formula and Slater determinants





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Introducing atomic spin-orbit state assembly formula and Slater determinants

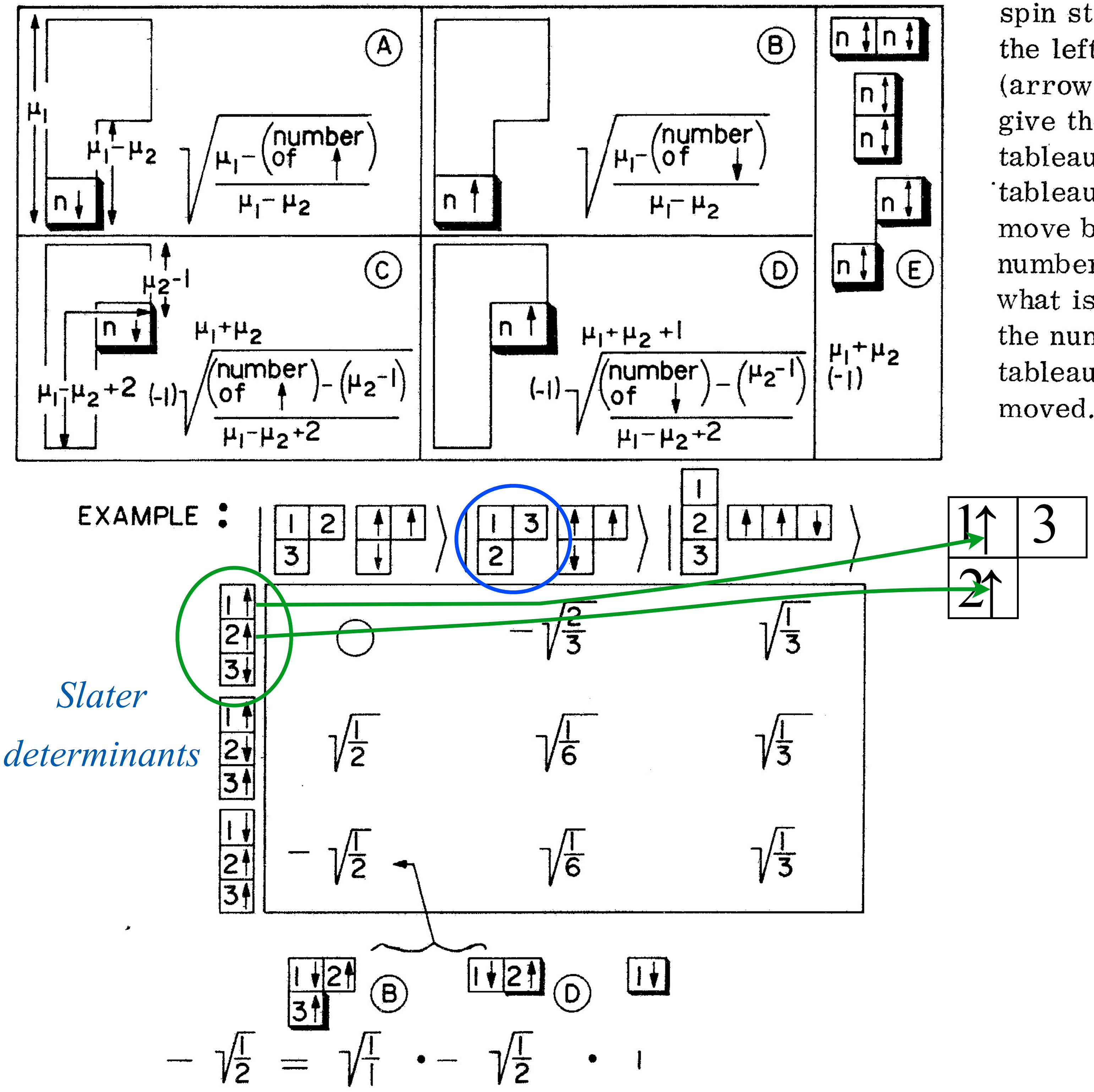
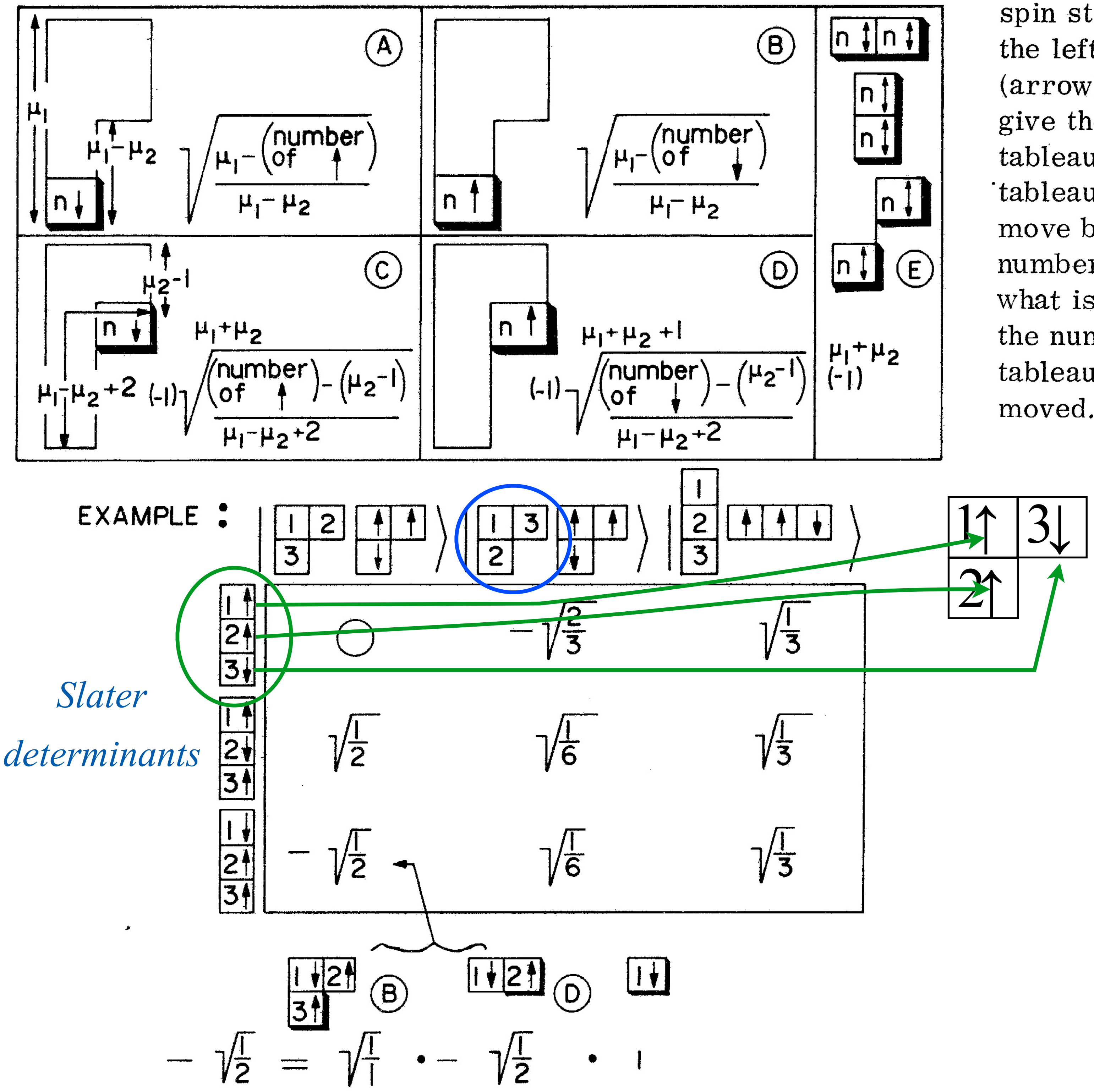


FIG. 5. Assembly formula for combining orbital and spin states. Each column state (Slater determinant) on the left-hand side of the sample table has a definite spin (arrow) on each orbital state (number). The formulas will give the overlap of this Slater state with a given orbital tableau state if we first write the spins within this orbital tableau in exactly the same way. Then we proceed to remove boxes with numbered spins starting with the highest number(s). Each "removal" gives a factor depending on what is being removed and where (cases A–E). All of the numbers in the formulas refer to the condition of the tableau just before the box outlined in the figure is removed.

Introducing atomic spin-orbit state assembly formula and Slater determinants



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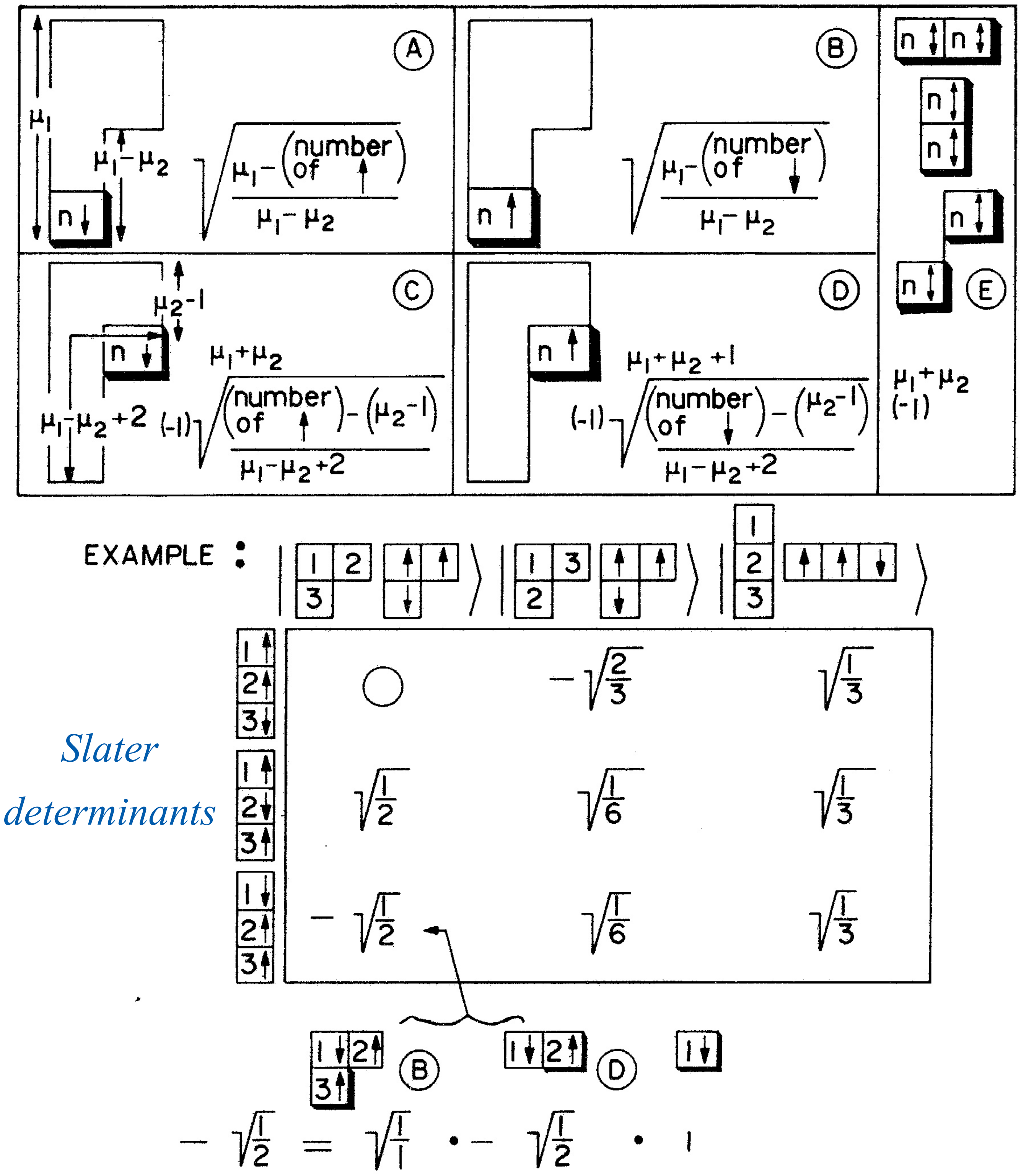
Introducing atomic spin-orbit state assembly formula and Slater determinants



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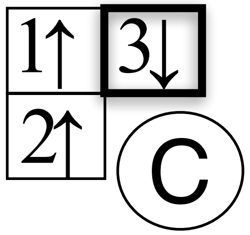
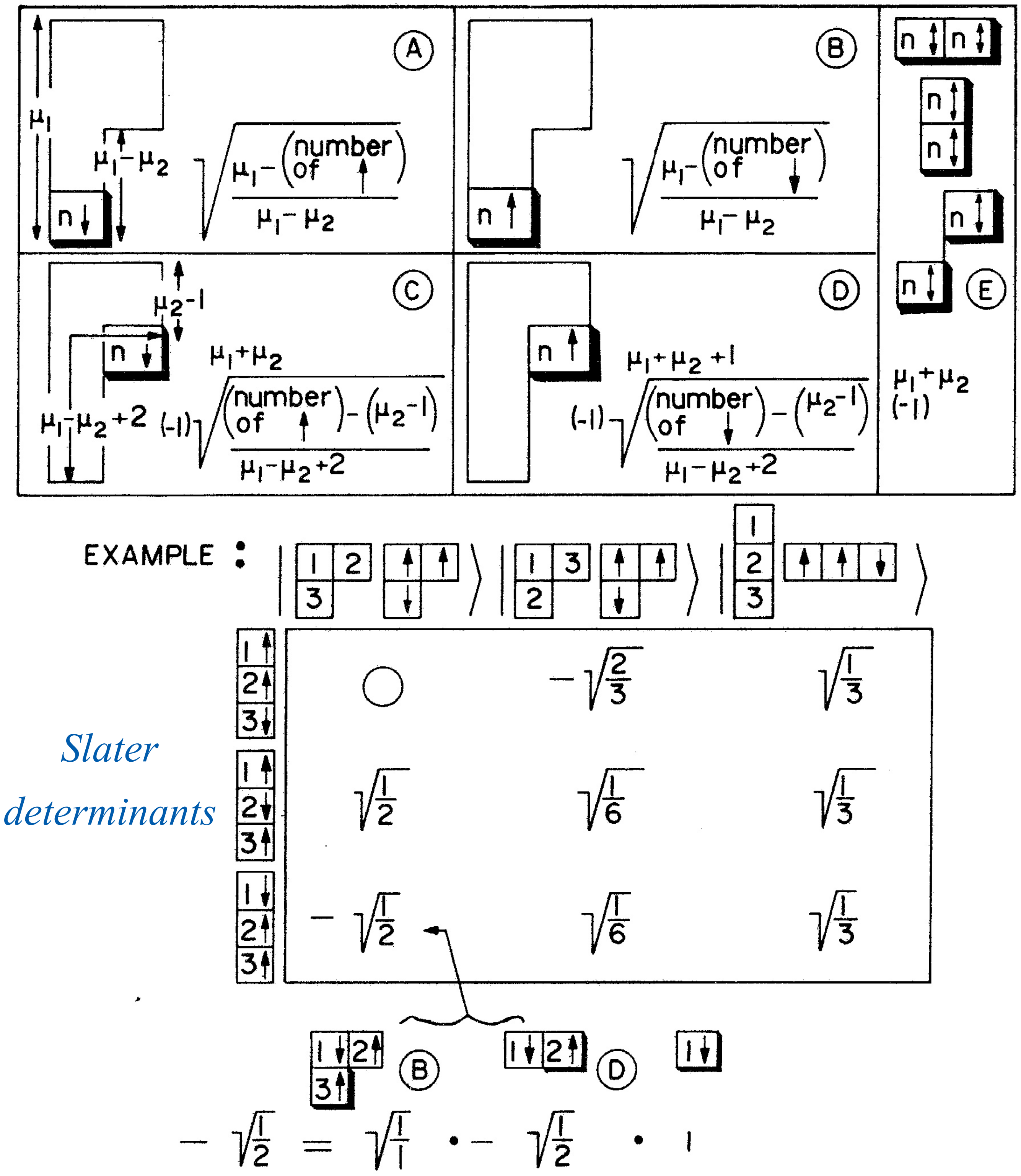


Introducing atomic spin-orbit state assembly formula and Slater determinants



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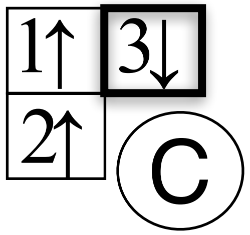
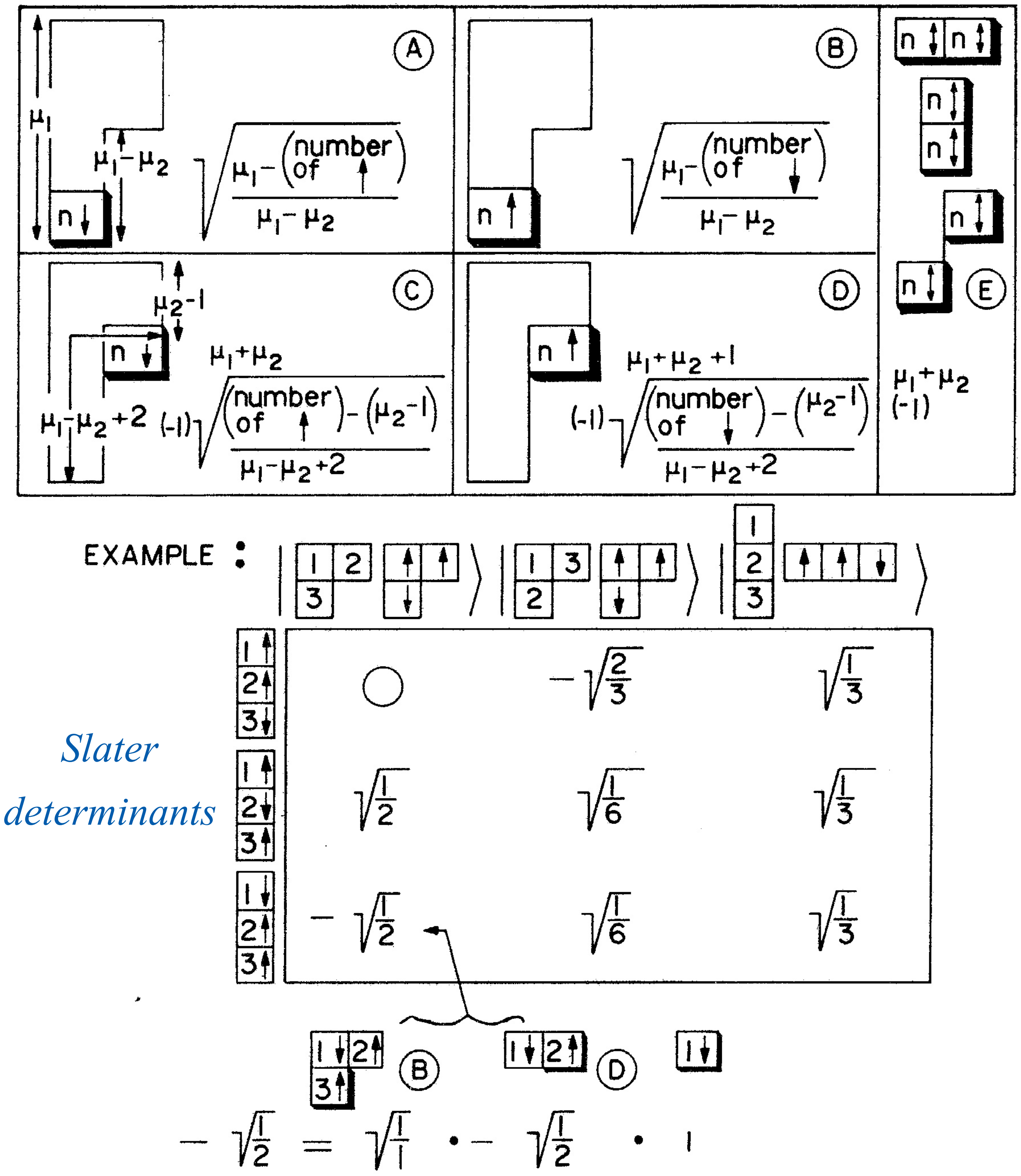
Slater determinants



$$(-)\sqrt{\frac{2-0}{3}}$$

$\mu_1=2,\mu_2=1$

Introducing atomic spin-orbit state assembly formula and Slater determinants



Slater determinants

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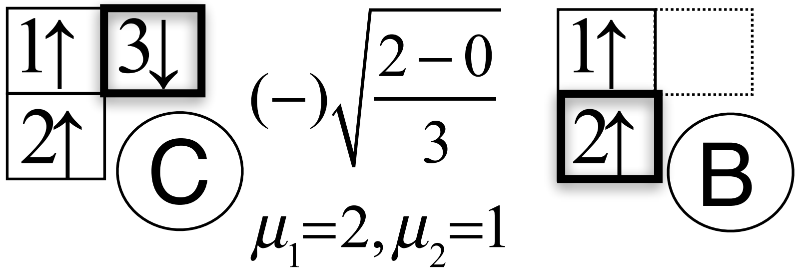
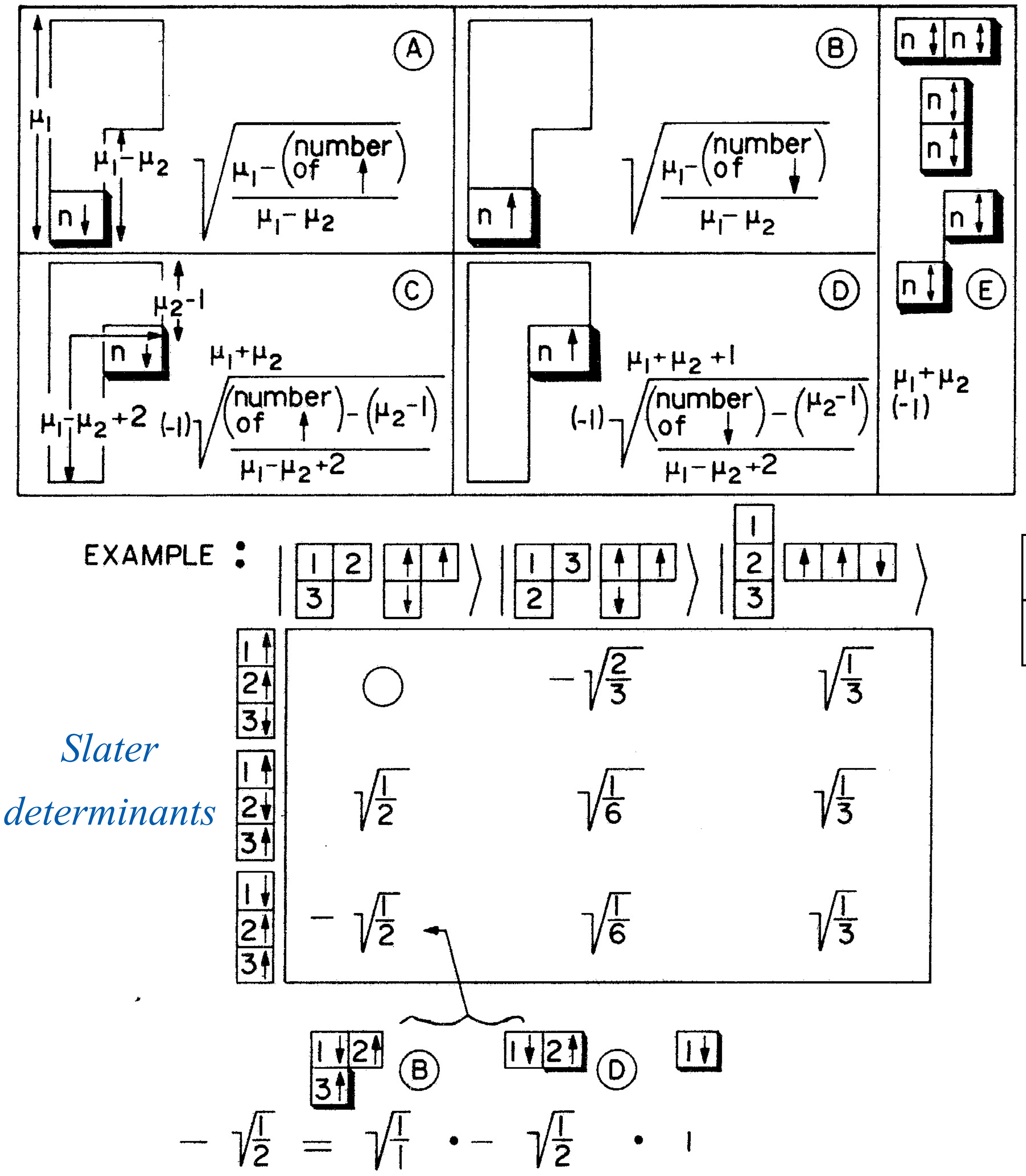


$(-)\sqrt{\frac{2-0}{3}}$

$\mu_1=2,\mu_2=1$

Introducing atomic spin-orbit state assembly formula and Slater determinants



Slater determinants

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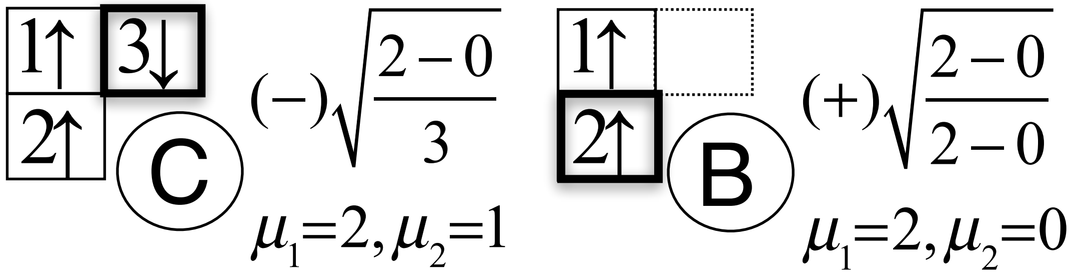
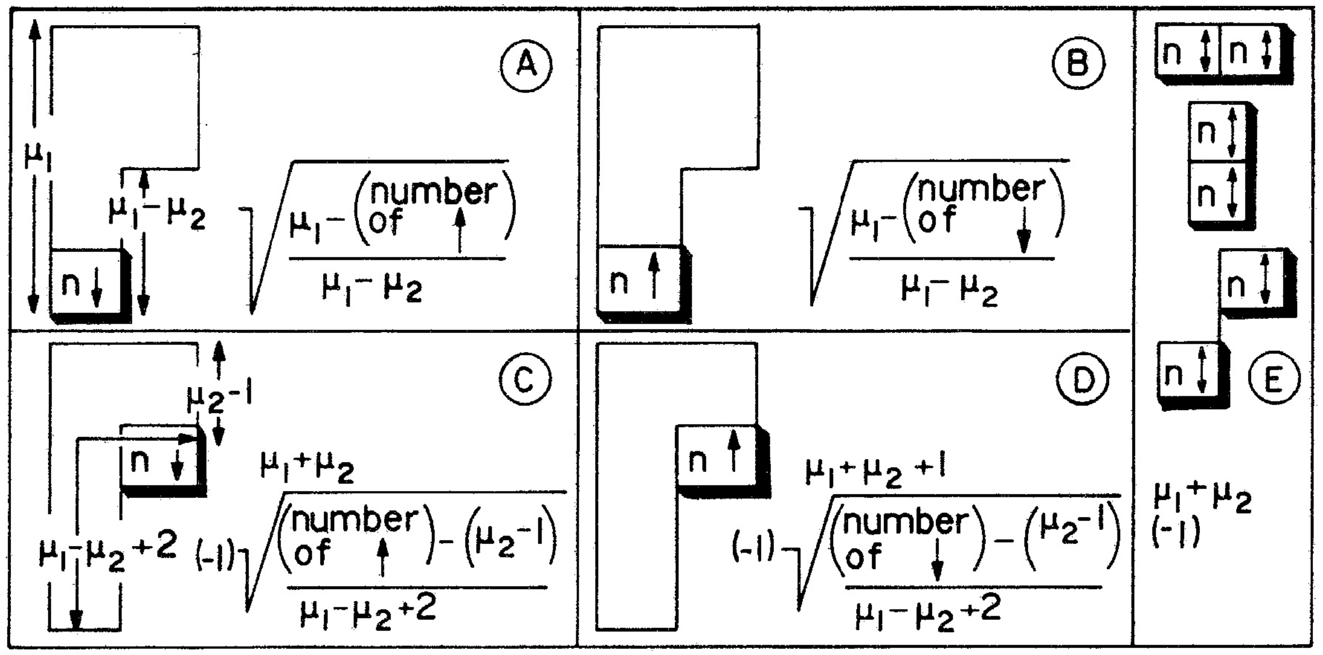


$(-)\sqrt{\frac{2-0}{3}}$ $\mu_1=2, \mu_2=1$

$(+)\sqrt{\frac{2-0}{2-0}}$ $\mu_1=2, \mu_2=0$

Introducing atomic spin-orbit state assembly formula and Slater determinants



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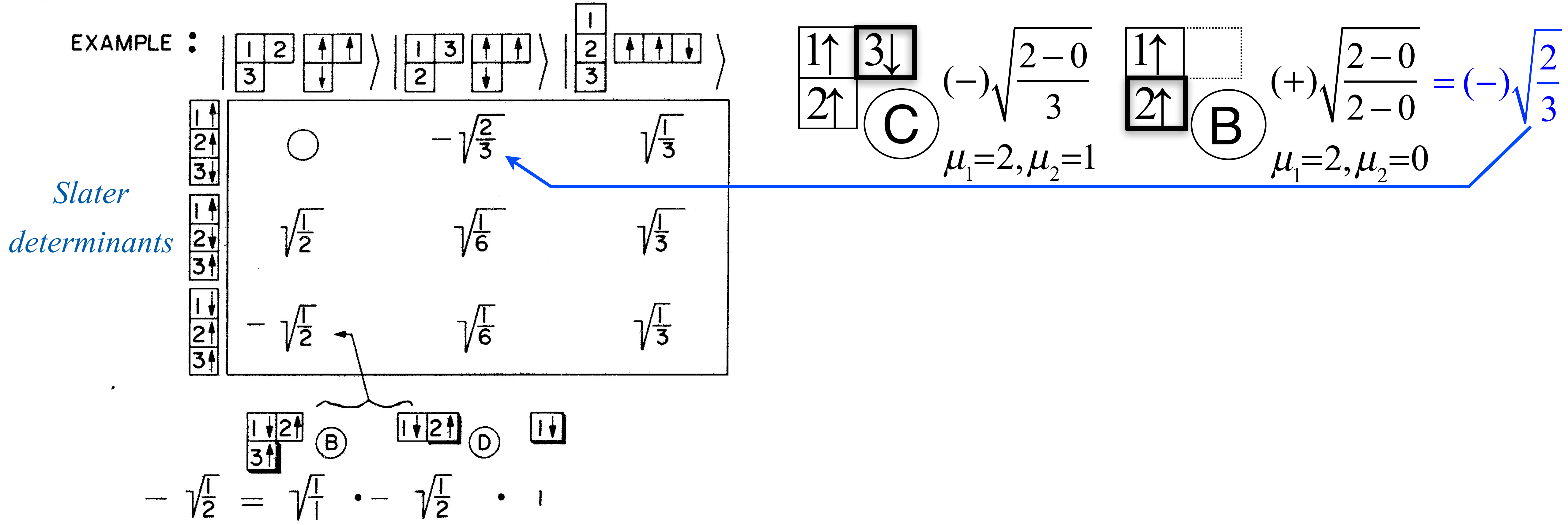


Slater determinants

$(-)\sqrt{\frac{2-0}{3}}$ $\mu_1=2,\mu_2=1$

$(+)\sqrt{\frac{2-0}{2-0}} = (-)\sqrt{\frac{2}{3}}$ $\mu_1=2,\mu_2=0$

Introducing atomic spin-orbit state assembly formula and Slater determinants

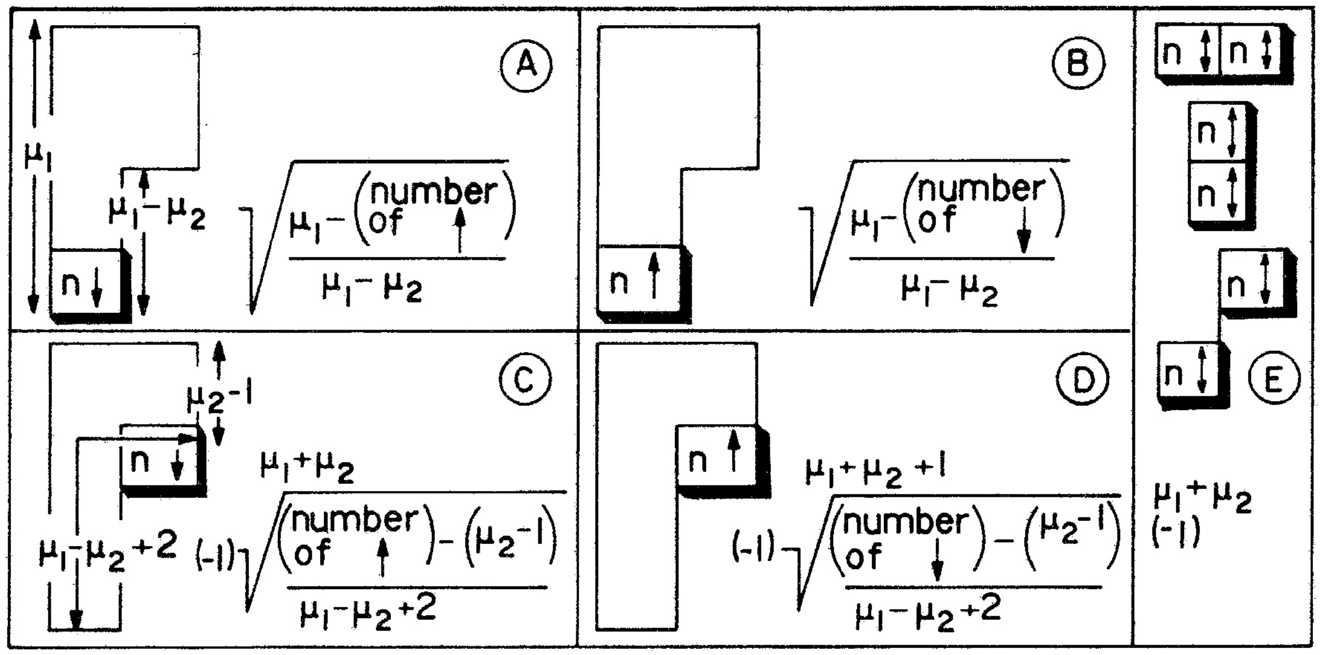
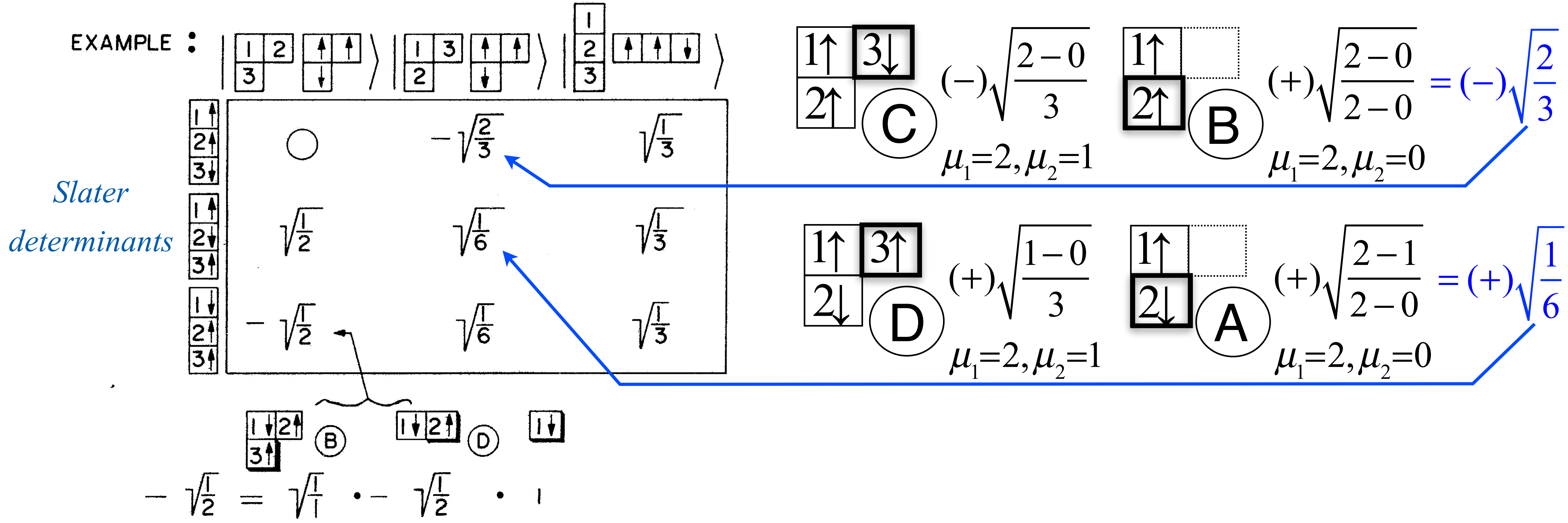


FIG. 5. Assembly formula for combining orbital and spin states. Each column state (Slater determinant) on the left-hand side of the sample table has a definite spin (arrow)on each orbital state (number). The formulas will give the overlap of this Slater state with a given orbital tableau state if we first write the spins within this orbital tableau in exactly the same way. Then we proceed to remove boxes with numbered spins starting with the highest number(s). Each "removal" gives a factor depending on what is being removed and where (cases A–E). All of the numbers in the formulas refer to the condition of the tableau just before the box outlined in the figure is removed.



	$\vert$ 1 2 / 3 , ↑ ↑ / ↓ $\rangle$	$\vert$ 1 3 / 2 , ↑ ↑ / ↓ $\rangle$	$\vert$ 1 / 2 / 3 , ↑ ↑ ↓ $\rangle$
1↑ 2↑ 3↓	0	$-\sqrt{\frac{2}{3}}$	$\sqrt{\frac{1}{3}}$
1↑ 2↓ 3↑	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{3}}$
1↓ 2↑ 3↑	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{3}}$

1↓ 2↑ / 3↑ (B)   1↓ 2↑ (D)   1↓

$$-\sqrt{\tfrac{1}{2}}=\sqrt{\tfrac{1}{1}}\cdot-\sqrt{\tfrac{1}{2}}\cdot 1$$

1↑ 3↓ / 2↑ (C) $(-)\sqrt{\frac{2-0}{3}}$ $\mu_1=2,\mu_2=1$ 1↑ / 2↑ (B) $(+)\sqrt{\frac{2-0}{2-0}}$ $\mu_1=2,\mu_2=0$ $=(-)\sqrt{\frac{2}{3}}$

1↑ 3↑ / 2↓ (D) $(+)\sqrt{\frac{1-0}{3}}$ $\mu_1=2,\mu_2=1$ 1↑ / 2↓ (A) $(+)\sqrt{\frac{2-1}{2-0}}$ $\mu_1=2,\mu_2=0$ $=(+)\sqrt{\frac{1}{6}}$

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Spin-orbit state assembly formula and Slater determinants

➡ Extra assembly table

$\ell=1$ p=shell LSJ states transformed to Slater determinants from J=3/2 (^{4}S)

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Note change in assembly matrix for two spin down…

	$\begin{array}{ll}\boxed{1}&\boxed{2}\\\boxed{3}&\end{array}\quad\begin{array}{ll}\boxed{\uparrow}&\boxed{\downarrow}\\\boxed{\downarrow}&\end{array}$	$\begin{array}{ll}\boxed{1}&\boxed{3}\\\boxed{2}&\end{array}\quad\begin{array}{ll}\boxed{\uparrow}&\boxed{\downarrow}\\\boxed{\downarrow}&\end{array}$	$\begin{array}{l}\boxed{1}\\\boxed{2}\\\boxed{3}\end{array}\quad\boxed{\uparrow}\boxed{\downarrow}\boxed{\downarrow}$
$\begin{array}{l}\boxed{1\uparrow}\\\boxed{2\downarrow}\\\boxed{3\downarrow}\end{array}$	$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{6}}$	$\frac{1}{\sqrt{3}}$
$\begin{array}{l}\boxed{1\downarrow}\\\boxed{2\uparrow}\\\boxed{3\downarrow}\end{array}$	$\frac{-1}{\sqrt{2}}$	$\frac{-1}{\sqrt{6}}$	$\frac{1}{\sqrt{3}}$
$\begin{array}{l}\boxed{1\downarrow}\\\boxed{2\downarrow}\\\boxed{3\uparrow}\end{array}$	0	$\frac{+2}{\sqrt{6}}$	$\frac{1}{\sqrt{3}}$

Note change in assembly matrix for two spin down…

	$\begin{array}{ll} 1 & 2 \\ 3 & \end{array}$ $\begin{array}{ll} \uparrow & \downarrow \\ \downarrow & \end{array}$	$\begin{array}{ll} 1 & 3 \\ 2 & \end{array}$ $\begin{array}{ll} \uparrow & \downarrow \\ \downarrow & \end{array}$	$\begin{array}{l} 1 \\ 2 \\ 3 \end{array}$ $\begin{array}{lll} \uparrow & \downarrow & \downarrow \end{array}$
$\begin{array}{l} 1\uparrow \\ 2\downarrow \\ 3\downarrow \end{array}$	$\begin{array}{ll} 1\uparrow & 2\downarrow \\ \mathbf{3}\downarrow & \end{array}$ $\sqrt{\frac{2-1}{2-1}}$ Ⓐ $+\sqrt{\frac{1-0}{0+2}}$ Ⓒ $\boxed{\frac{1}{\sqrt{2}}}$	$\begin{array}{ll} 1\uparrow & \mathbf{3}\downarrow \\ 2\downarrow & \end{array}$ $-\sqrt{\frac{1-0}{1+2}}$ Ⓒ $\sqrt{\frac{2-1}{2-0}}$ Ⓐ $\boxed{\frac{-1}{\sqrt{6}}}$	$\begin{array}{l} 1\uparrow \\ 2\downarrow \\ \mathbf{3}\downarrow \end{array}$ $\sqrt{\frac{3-1}{3-0}}$ Ⓐ $\sqrt{\frac{2-1}{2-0}}$ Ⓐ $\boxed{\frac{1}{\sqrt{3}}}$
$\begin{array}{l} 1\downarrow \\ 2\uparrow \\ 3\downarrow \end{array}$	$\begin{array}{ll} 1\downarrow & 2\uparrow \\ \mathbf{3}\downarrow & \end{array}$ $\sqrt{\frac{2-1}{2-1}}$ Ⓐ $-\sqrt{\frac{1-0}{0+2}}$ Ⓓ $\boxed{\frac{-1}{\sqrt{2}}}$	$\begin{array}{ll} 1\downarrow & \mathbf{3}\downarrow \\ 2\uparrow & \end{array}$ $-\sqrt{\frac{1-0}{1+2}}$ Ⓒ $\sqrt{\frac{2-1}{2-0}}$ Ⓑ $\boxed{\frac{-1}{\sqrt{6}}}$	$\begin{array}{l} 1\downarrow \\ 2\uparrow \\ \mathbf{3}\downarrow \end{array}$ $\sqrt{\frac{3-1}{3-0}}$ Ⓐ $\sqrt{\frac{2-1}{2-0}}$ Ⓑ $\boxed{\frac{1}{\sqrt{3}}}$
$\begin{array}{l} 1\downarrow \\ 2\downarrow \\ 3\uparrow \end{array}$	$\begin{array}{ll} 1\downarrow & 2\downarrow \\ \mathbf{3}\uparrow & \end{array}$ $\sqrt{\frac{2-2}{2-1}}$ Ⓑ $\sqrt{\frac{0-0}{0+2}}$ Ⓒ $\boxed{0}$	$\begin{array}{ll} 1\downarrow & \mathbf{3}\uparrow \\ 2\downarrow & \end{array}$ $+\sqrt{\frac{2-0}{1+2}}$ Ⓓ $\sqrt{\frac{2-0}{2-0}}$ Ⓐ $\boxed{\frac{+2}{\sqrt{6}}}$	$\begin{array}{l} 1\downarrow \\ 2\downarrow \\ \mathbf{3}\uparrow \end{array}$ $\sqrt{\frac{3-2}{3-0}}$ Ⓑ $\sqrt{\frac{2-0}{2-0}}$ Ⓐ $\boxed{\frac{1}{\sqrt{3}}}$

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M_J=3/2,..

$$\left|{}^4S_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle=\left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\ \uparrow\uparrow\uparrow\right\rangle,\left|{}^4S_{J=\frac{3}{2}}\ \tfrac{1}{2}\right\rangle=\left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\ \uparrow\uparrow\downarrow\right\rangle,\left|{}^4S_{J=\frac{3}{2}}\ \tfrac{-1}{2}\right\rangle=\left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\ \uparrow\downarrow\downarrow\right\rangle,\left|{}^4S_{J=\frac{3}{2}}\ \tfrac{-3}{2}\right\rangle=\left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\ \downarrow\downarrow\downarrow\right\rangle$$

quartet ${}^4S$ J=$\frac{3}{2}$,

$\mathrm{M}_J = \frac{+3}{2}, \frac{+1}{2}, \frac{-1}{2}, \frac{-3}{2}$.

Slater determinant state key:

a=1↑,b=1↓,c=2↑,d=2↓,e=3↑,f=3↓

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ℓ=1 p=shell LSJ states transformed to Slater determinants fromJ= 3/2 at L=0

M$_J$=3/2,..

$$\left|{}^4S_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle=\left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\ \uparrow\uparrow\uparrow\right\rangle,\left|{}^4S_{J=\frac{3}{2}}\ \tfrac{1}{2}\right\rangle=\left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\ \uparrow\uparrow\downarrow\right\rangle,\left|{}^4S_{J=\frac{3}{2}}\ \tfrac{-1}{2}\right\rangle=\left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\ \uparrow\downarrow\downarrow\right\rangle,\left|{}^4S_{J=\frac{3}{2}}\ \tfrac{-3}{2}\right\rangle=\left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\ \downarrow\downarrow\downarrow\right\rangle$$

quartet ${}^4S$ J=$\frac{3}{2}$,

M$_J$ = $\frac{+3}{2},\frac{+1}{2},\frac{-1}{2},\frac{-3}{2}$.

$$\left|{}^4S_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle=\left|\begin{array}{|c|}\hline 1\uparrow\\\hline 2\uparrow\\\hline 3\uparrow\\\hline\end{array}\ \begin{array}{c}a\\c\\e\end{array}\right\rangle,\left|{}^4S_{J=\frac{3}{2}}\ \tfrac{1}{2}\right\rangle=\left|\begin{array}{|c|}\hline 1\uparrow\\\hline 2\uparrow\\\hline 3\downarrow\\\hline\end{array}\ \begin{array}{c}a\\c\\f\end{array}\right\rangle,\left|{}^4S_{J=\frac{3}{2}}\ \tfrac{-1}{2}\right\rangle=\left|\begin{array}{|c|}\hline 1\uparrow\\\hline 2\downarrow\\\hline 3\downarrow\\\hline\end{array}\ \begin{array}{c}a\\d\\f\end{array}\right\rangle,\left|{}^4S_{J=\frac{3}{2}}\ \tfrac{-3}{2}\right\rangle=\left|\begin{array}{|c|}\hline 1\downarrow\\\hline 2\downarrow\\\hline 3\downarrow\\\hline\end{array}\ \begin{array}{c}b\\d\\f\end{array}\right\rangle$$

Slater determinant state key:

$a=1\uparrow, b=1\downarrow, c=2\uparrow, d=2\downarrow, e=3\uparrow, f=3\downarrow$

ℓ=1 *p*=shell LSJ states transformed to Slater determinants fromJ= 3/2 at L=0

M$_J$=3/2,..

$$\left|{}^4S_{J=\frac{3}{2}}^{\ \frac{3}{2}}\right\rangle=\left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\ \uparrow\uparrow\uparrow\right\rangle,\left|{}^4S_{J=\frac{3}{2}}^{\ \frac{1}{2}}\right\rangle=\left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\ \uparrow\uparrow\downarrow\right\rangle,\left|{}^4S_{J=\frac{3}{2}}^{\ \frac{-1}{2}}\right\rangle=\left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\ \uparrow\downarrow\downarrow\right\rangle,\left|{}^4S_{J=\frac{3}{2}}^{\ \frac{-3}{2}}\right\rangle=\left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\ \downarrow\downarrow\downarrow\right\rangle$$

quartet ${}^4S$ J=$\frac{3}{2}$,

$$M_J=\frac{+3}{2},\frac{+1}{2},\frac{-1}{2},\frac{-3}{2}.$$

$$\left|{}^4S_{J=\frac{3}{2}}^{\ \frac{3}{2}}\right\rangle=\left|\begin{array}{|c|}\hline 1\uparrow\\\hline 2\uparrow\\\hline 3\uparrow\\\hline\end{array}\ \begin{matrix}a\\c\\e\end{matrix}\right\rangle,\left|{}^4S_{J=\frac{3}{2}}^{\ \frac{1}{2}}\right\rangle=\left|\begin{array}{|c|}\hline 1\uparrow\\\hline 2\uparrow\\\hline 3\downarrow\\\hline\end{array}\ \begin{matrix}a\\c\\f\end{matrix}\right\rangle,\left|{}^4S_{J=\frac{3}{2}}^{\ \frac{-1}{2}}\right\rangle=\left|\begin{array}{|c|}\hline 1\uparrow\\\hline 2\downarrow\\\hline 3\downarrow\\\hline\end{array}\ \begin{matrix}a\\d\\f\end{matrix}\right\rangle,\left|{}^4S_{J=\frac{3}{2}}^{\ \frac{-3}{2}}\right\rangle=\left|\begin{array}{|c|}\hline 1\downarrow\\\hline 2\downarrow\\\hline 3\downarrow\\\hline\end{array}\ \begin{matrix}b\\d\\f\end{matrix}\right\rangle$$

$$\left|{}^4S_{J=\frac{3}{2}}^{\ \frac{3}{2}}\right\rangle=\left|\begin{matrix}a\\c\\e\end{matrix}\right\rangle,\left|{}^4S_{J=\frac{3}{2}}^{\ \frac{1}{2}}\right\rangle=\left|\begin{matrix}a\\c\\f\end{matrix}\right\rangle,\left|{}^4S_{J=\frac{3}{2}}^{\ \frac{-1}{2}}\right\rangle=\left|\begin{matrix}a\\d\\f\end{matrix}\right\rangle,\left|{}^4S_{J=\frac{3}{2}}^{\ \frac{-3}{2}}\right\rangle=\left|\begin{matrix}b\\d\\f\end{matrix}\right\rangle$$

Slater determinant state key:

a=1↑,b=1↓,c=2↑,d=2↓,e=3↑,f=3↓

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ℓ=1 p=shell LSJ states transformed to Slater determinants from J=5/2 at L=2

M$_J$=5/2,..

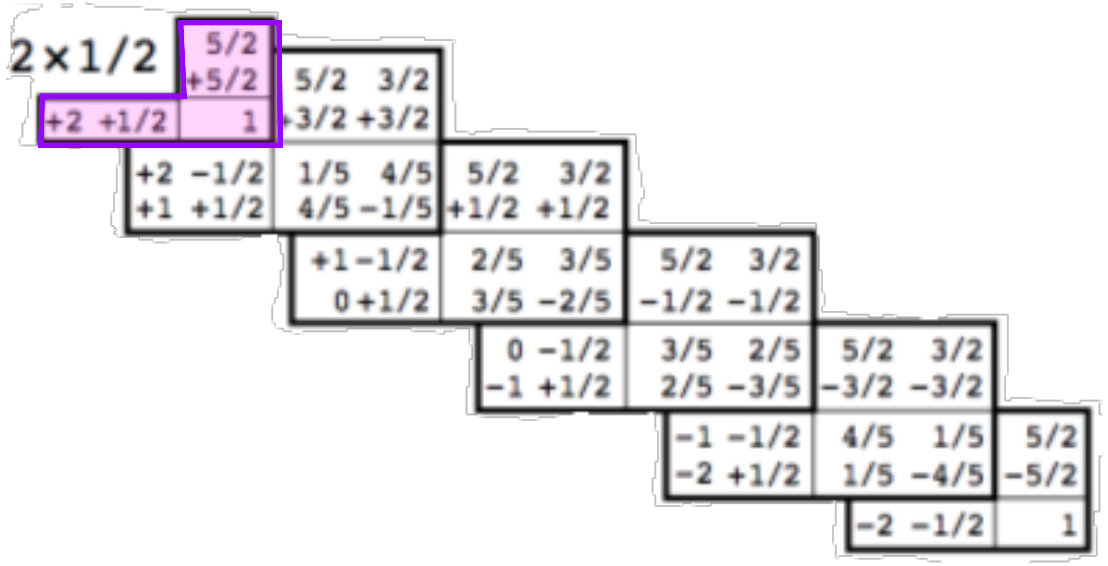
$$\left|{}^4S_{J=\frac{3}{2}}^{\frac{3}{2}}\right\rangle=\left|\begin{matrix}a\\c\\e\end{matrix}\right\rangle,\left|{}^4S_{J=\frac{3}{2}}^{\frac{1}{2}}\right\rangle=\left|\begin{matrix}a\\c\\f\end{matrix}\right\rangle,\left|{}^4S_{J=\frac{3}{2}}^{\frac{-1}{2}}\right\rangle=\left|\begin{matrix}a\\d\\f\end{matrix}\right\rangle,\left|{}^4S_{J=\frac{3}{2}}^{\frac{-3}{2}}\right\rangle=\left|\begin{matrix}b\\d\\f\end{matrix}\right\rangle$$

quartet ${}^4S$ J=$\frac{3}{2}$,

$M_J=\frac{+3}{2},\frac{+1}{2},\frac{-1}{2},\frac{-3}{2}$.

$$\left|{}^2D_{J=\frac{5}{2}}^{\frac{5}{2}}\right\rangle=\underline{\left|d_{M=2}^{L=2}\chi_{1/2}^{1/2}\right\rangle}$$

Doublet ${}^2D$, $\underline{\text{J}=\frac{5}{2}\ \text{M}_\text{J}=\frac{5}{2}}$,



2×1/2

5/2 +5/2 | +2 +1/2 | 1

5/2 3/2 +3/2 +3/2 | +2 −1/2 | 1/5 4/5 | +1 +1/2 | 4/5 −1/5

5/2 3/2 +1/2 +1/2 | +1 −1/2 | 2/5 3/5 | 0 +1/2 | 3/5 −2/5

5/2 3/2 −1/2 −1/2 | 0 −1/2 | 3/5 2/5 | −1 +1/2 | 2/5 −3/5

5/2 3/2 −3/2 −3/2 | −1 −1/2 | 4/5 1/5 | −2 +1/2 | 1/5 −4/5

5/2 −5/2 | −2 −1/2 | 1

Slater determinant state key:

a=1↑,b=1↓,c=2↑,d=2↓,e=3↑,f=3↓

ℓ=1 p=shell LSJ states transformed to Slater determinants from J=5/2 at L=2

M_J=5/2

$$\left|{}^4S^{\frac{3}{2}}_{J=\frac{3}{2}}\right\rangle=\left|\begin{matrix}a\\c\\e\end{matrix}\right\rangle,\left|{}^4S^{\frac{1}{2}}_{J=\frac{3}{2}}\right\rangle=\left|\begin{matrix}a\\c\\f\end{matrix}\right\rangle,\left|{}^4S^{\frac{-1}{2}}_{J=\frac{3}{2}}\right\rangle=\left|\begin{matrix}a\\d\\f\end{matrix}\right\rangle,\left|{}^4S^{\frac{-3}{2}}_{J=\frac{3}{2}}\right\rangle=\left|\begin{matrix}b\\d\\f\end{matrix}\right\rangle$$

quartet ${}^4S$ J=$\frac{3}{2}$,
$M_J=\frac{+3}{2},\frac{+1}{2},\frac{-1}{2},\frac{-3}{2}$.

$$\left|{}^2D^{\frac{5}{2}}_{J=\frac{5}{2}}\right\rangle=\left|\underline{d^{L=2}_{M=2}\chi^{1/2}_{1/2}}\right\rangle$$

Doublet ${}^2D$, J=$\frac{5}{2}$ $\underline{M_J=\frac{5}{2}}$,

$$=\left|\begin{array}{|c|c|}\hline 1 & 1\\ \hline 2 & \\ \cline{1-1}\end{array}\quad\begin{matrix}\uparrow\uparrow\\ \downarrow\end{matrix}\right\rangle$$

$$\begin{array}{|c|c|}\hline 1\uparrow & \boxed{1\uparrow}\\ \hline 2\downarrow & \\ \cline{1-1}\end{array}\ (+)\sqrt{\frac{2-2}{2-1}}=0$$

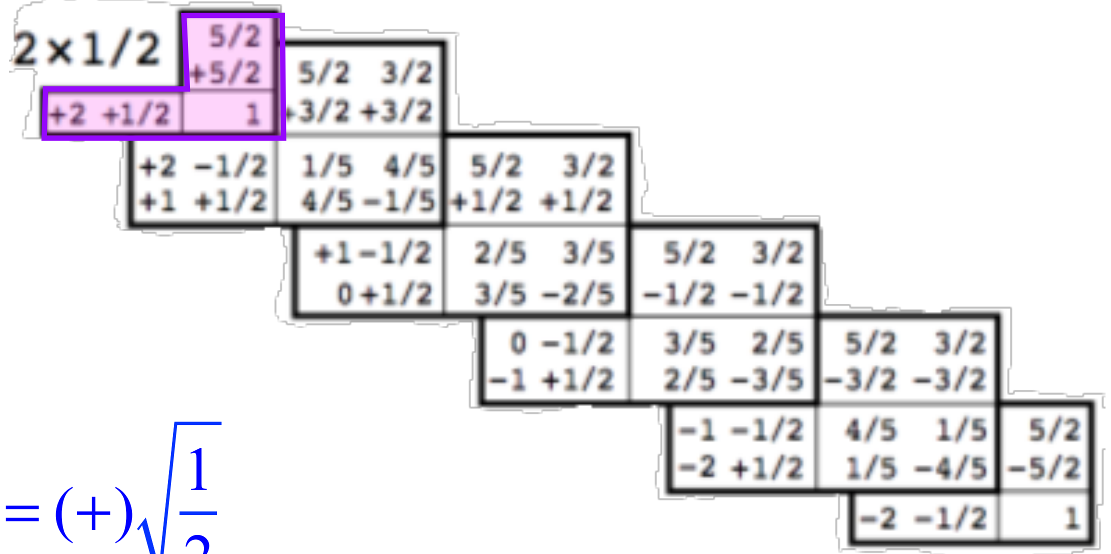
Ⓐ μ_1=2, μ_2=1

$$\begin{array}{|c|c|}\hline 1\uparrow & 1\downarrow\\ \hline \boxed{2\uparrow} & \\ \cline{1-1}\end{array}\ (+)\sqrt{\frac{2-1}{2-1}}$$

Ⓑ μ_1=2, μ_2=1

$$\begin{array}{|c|c|}\hline 1\uparrow & \boxed{1\downarrow}\\ \hline \ & \\ \cline{1-1}\end{array}\ (+)\sqrt{\frac{1-0}{0+2}}=(+)\sqrt{\frac{1}{2}}$$

Ⓒ μ_1=1, μ_2=1



2×1/2

		5/2 +5/2
+2	+1/2	1

		5/2	3/2
		+3/2	+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

		5/2	3/2
		+1/2	+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

		5/2	3/2
		−1/2	−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

		5/2	3/2
		−3/2	−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

		5/2 −5/2
−2	−1/2	1

Slater determinant state key:

a=1↑,b=1↓,c=2↑,d=2↓,e=3↑,f=3↓

ℓ=1 p=shell LSJ states transformed to Slater determinants from J=5/2 at L=2

M$_J$=5/2

$$\left|{}^4S_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle = \left|\begin{matrix}a\\c\\e\end{matrix}\right\rangle, \left|{}^4S_{J=\frac{3}{2}}\ \tfrac{1}{2}\right\rangle = \left|\begin{matrix}a\\c\\f\end{matrix}\right\rangle, \left|{}^4S_{J=\frac{3}{2}}\ \tfrac{-1}{2}\right\rangle = \left|\begin{matrix}a\\d\\f\end{matrix}\right\rangle, \left|{}^4S_{J=\frac{3}{2}}\ \tfrac{-3}{2}\right\rangle = \left|\begin{matrix}b\\d\\f\end{matrix}\right\rangle$$

quartet ${}^4S$ J=$\frac{3}{2}$,
$\mathrm{M}_J = \frac{+3}{2}, \frac{+1}{2}, \frac{-1}{2}, \frac{-3}{2}$.

$$\left|{}^2D_{J=\frac{5}{2}}\ \tfrac{5}{2}\right\rangle = \left|\underline{d^{L=2}_{M=2}\chi^{1/2}_{1/2}}\right\rangle = \left|\begin{array}{|c|c|}\hline 1 & 1\\ \hline 2 & \\ \cline{1-1}\end{array}\ \begin{matrix}\uparrow\uparrow\\ \downarrow\end{matrix}\right\rangle$$

Doublet ${}^2D$, J=$\frac{5}{2}$ $\underline{\mathrm{M_J}=\frac{5}{2}}$,

$$\begin{array}{|c|c|}\hline 1\uparrow & \boxed{1\uparrow}\\ \hline 2\downarrow & \\ \cline{1-1}\end{array}\ \text{Ⓐ}\ (+)\sqrt{\frac{2-2}{2-1}} = 0$$

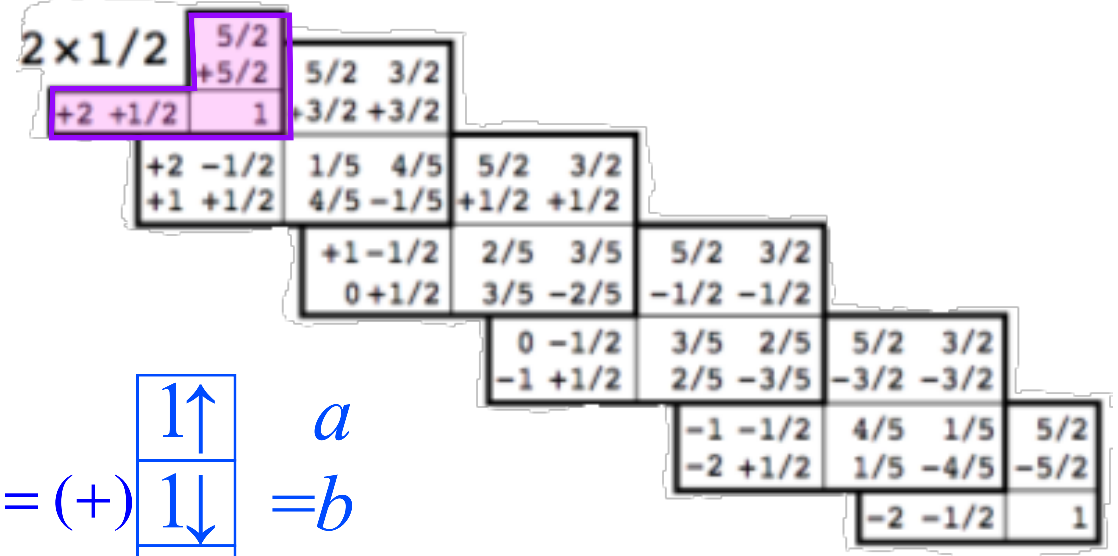
$\mu_1=2, \mu_2=1$

$$\begin{array}{|c|c|}\hline 1\uparrow & 1\downarrow\\ \hline \boxed{2\uparrow} & \\ \cline{1-1}\end{array}\ \text{Ⓑ}\ (+)\sqrt{\frac{2-1}{2-1}}$$

$\mu_1=2, \mu_2=1$

$$\begin{array}{|c|c|}\hline 1\uparrow & \boxed{1\downarrow}\\ \hline \ & \\ \cline{1-1}\end{array}\ \text{Ⓔ}\ (+)\sqrt{1}\qquad \mu_1=1, \mu_2=1$$

$$= (+)\begin{array}{|c|}\hline 1\uparrow\\ \hline 1\downarrow\\ \hline 2\uparrow\\ \hline\end{array}\ \begin{matrix}a\\ =b\\ c\end{matrix}$$



2×1/2	5/2 +5/2
+2 +1/2	1

	5/2 +3/2	3/2 +3/2
+2 −1/2	1/5	4/5
+1 +1/2	4/5	−1/5

	5/2 +1/2	3/2 +1/2
+1 −1/2	2/5	3/5
0 +1/2	3/5	−2/5

	5/2 −1/2	3/2 −1/2
0 −1/2	3/5	2/5
−1 +1/2	2/5	−3/5

	5/2 −3/2	3/2 −3/2
−1 −1/2	4/5	1/5
−2 +1/2	1/5	−4/5

	5/2 −5/2
−2 −1/2	1

Slater determinant state key:

a=1↑,b=1↓,c=2↑,d=2↓,e=3↑,f=3↓

AMOP reference links on pages 2-4

4.25.18 class 26: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

$(S_3)^*(U(3)) \subset U(6)$ models of p^3 electronic spin-orbit states and couplings

[2,1] tableau states lowered by $\mathbf{L}_- = \sqrt{2}(E_{21} + E_{32})$

Top-(J,M) states to mid-level states

$\ell=1$ p=shell LS states combined to states of definite J

J=3/2 at L=0 (^4S), J=5/2 at L=2 (^2D)

C-G coupling; J=3/2 at L=2 (^2D), J=3/2 at L=1 (^2P), J=1/2 at L=1 (^2P)

Spin-orbit state assembly formula and Slater determinants

Extra assembly table

$\ell=1$ p=shell LSJ states transformed to Slater determinants from J=3/2 (^4S)

Slater functions for J=5/2, J=3/2 (^2D)

Slater functions for J=3/2 (^2P), J=1/2 (^2P)

Summary of states and level connection paths

Symmetry dimension accounting

Spin-orbit Hamiltonian matrix calculation

Application to spin-orbit and entanglement break-up scattering

ℓ=1 p=shell LSJ states transformed to Slater determinants from J=3/2 at L=2

M$_J$=3/2

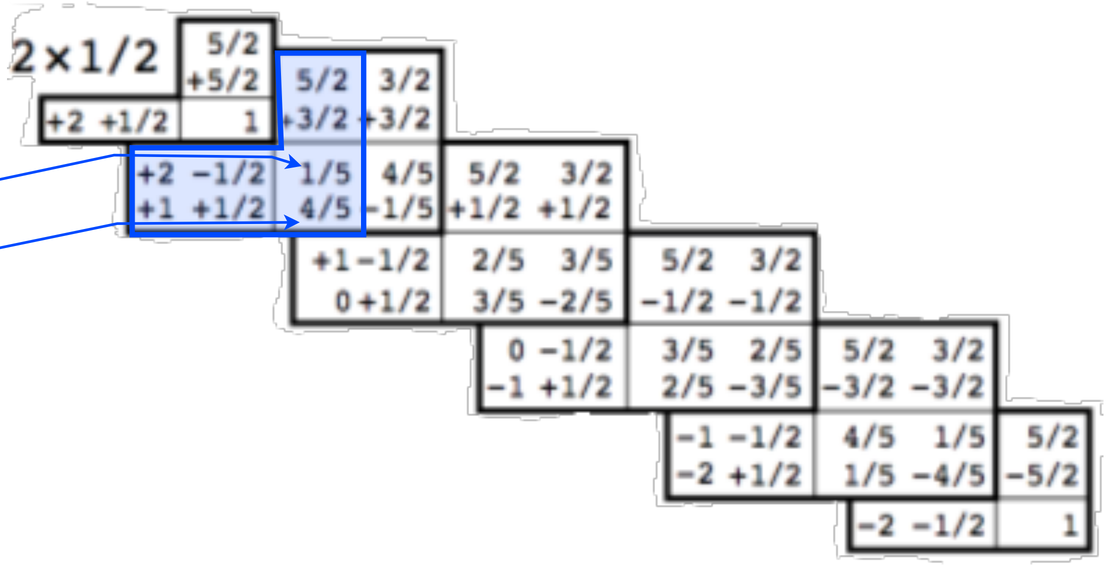
$$\left|{}^4S_{J=\frac{3}{2}}^{\;\frac{3}{2}}\right\rangle=\left|\begin{matrix}a\\c\\e\end{matrix}\right\rangle,\left|{}^4S_{J=\frac{3}{2}}^{\;\frac{1}{2}}\right\rangle=\left|\begin{matrix}a\\c\\f\end{matrix}\right\rangle,\left|{}^4S_{J=\frac{3}{2}}^{\;\frac{-1}{2}}\right\rangle=\left|\begin{matrix}a\\d\\f\end{matrix}\right\rangle,\left|{}^4S_{J=\frac{3}{2}}^{\;\frac{-3}{2}}\right\rangle=\left|\begin{matrix}b\\d\\f\end{matrix}\right\rangle$$

quartet ${}^4S$ J=$\frac{3}{2}$,
$M_J=\frac{+3}{2},\frac{+1}{2},\frac{-1}{2},\frac{-3}{2}$.

$$\left|{}^2D_{J=\frac{5}{2}}^{\;\frac{5}{2}}\right\rangle=\left|d_{M=2}^{L=2}\chi_{1/2}^{1/2}\right\rangle$$

Doublet ${}^2D$, J=$\frac{5}{2}$ M$_J$=$\frac{5}{2}$,

$$=\left|\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\;\begin{matrix}\uparrow\uparrow\\\downarrow\end{matrix}\right\rangle=\begin{matrix}a\\b\\c\end{matrix}$$



2×1/2	5/2 +5/2	5/2 +3/2	3/2 +3/2	5/2 +1/2	3/2 +1/2	5/2 −1/2	3/2 −1/2	5/2 −3/2	3/2 −3/2	5/2 −5/2
+2 +1/2	1									
+2 −1/2		1/5	4/5							
+1 +1/2		4/5	−1/5							
+1 −1/2				2/5	3/5					
0 +1/2				3/5	−2/5					
0 −1/2						3/5	2/5			
−1 +1/2						2/5	−3/5			
−1 −1/2								4/5	1/5	
−2 +1/2								1/5	−4/5	
−2 −1/2										1

$$\left|{}^2D_{J=\frac{5}{2}}^{\;\frac{3}{2}}\right\rangle=\underline{\sqrt{\tfrac{1}{5}}\left|d_{M=2}^{L=2}\chi_{-1/2}^{1/2}\right\rangle+\sqrt{\tfrac{4}{5}}\left|d_{M=1}^{L=2}\chi_{1/2}^{1/2}\right\rangle}$$

Doublet ${}^2D$, $\underline{\text{J}=\frac{5}{2}\ \text{M}_\text{J}=\frac{3}{2}}$

ℓ=1 p=shell LSJ states transformed to Slater determinants from J=5/2 at L=2 M$_J$=3/2

$$\left|{}^4S^{\frac32}_{J=\frac32}\right\rangle=\left|\begin{matrix}a\\c\\e\end{matrix}\right\rangle,\left|{}^4S^{\frac12}_{J=\frac32}\right\rangle=\left|\begin{matrix}a\\c\\f\end{matrix}\right\rangle,\left|{}^4S^{\frac{-1}{2}}_{J=\frac32}\right\rangle=\left|\begin{matrix}a\\d\\f\end{matrix}\right\rangle,\left|{}^4S^{\frac{-3}{2}}_{J=\frac32}\right\rangle=\left|\begin{matrix}b\\d\\f\end{matrix}\right\rangle$$

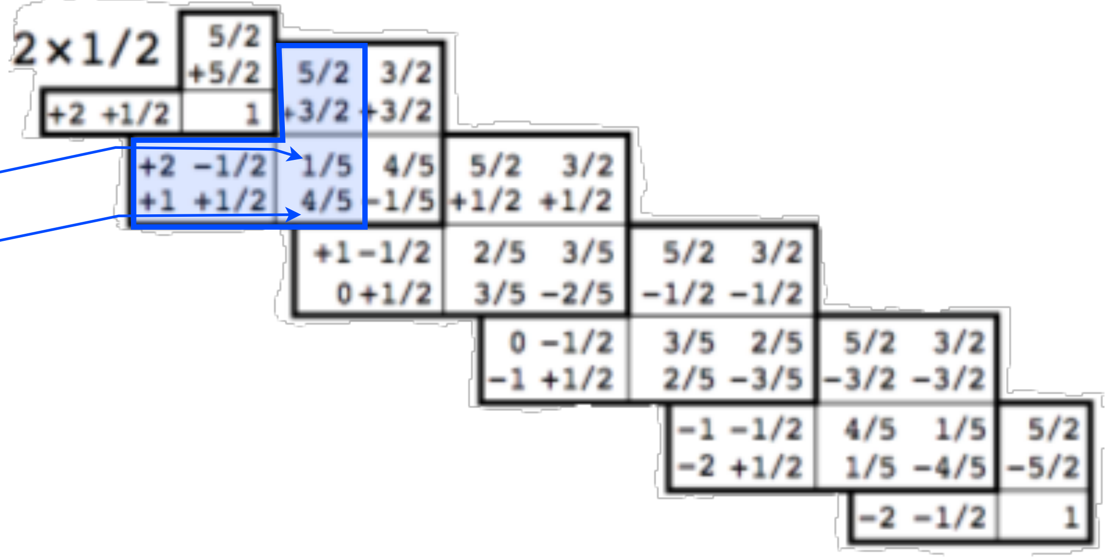
quartet ${}^4S$ J=$\frac32$,

M$_J$ = $\frac{+3}{2},\frac{+1}{2},\frac{-1}{2},\frac{-3}{2}$.

$$\left|{}^2D^{\frac52}_{J=\frac52}\right\rangle=\left|d^{L=2}_{M=2}\chi^{1/2}_{1/2}\right\rangle$$

Doublet ${}^2D$, J=$\frac52$ M$_J$=$\frac52$,

$$=\left|\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\;\begin{matrix}\uparrow\uparrow\\\downarrow\end{matrix}\right\rangle\;\begin{matrix}a\\=b\\c\end{matrix}$$



2×1/2

	5/2 +5/2					
+2 +1/2	1					

	5/2 +3/2	3/2 +3/2
+2 −1/2	1/5	4/5
+1 +1/2	4/5	−1/5

	5/2 +1/2	3/2 +1/2
+1 −1/2	2/5	3/5
0 +1/2	3/5	−2/5

	5/2 −1/2	3/2 −1/2
0 −1/2	3/5	2/5
−1 +1/2	2/5	−3/5

	5/2 −3/2	3/2 −3/2
−1 −1/2	4/5	1/5
−2 +1/2	1/5	−4/5

	5/2 −5/2
−2 −1/2	1

$$\left|{}^2D^{\frac32}_{J=\frac52}\right\rangle=\sqrt{\tfrac15}\left|d^{L=2}_{M=2}\chi^{1/2}_{-1/2}\right\rangle+\sqrt{\tfrac45}\left|d^{L=2}_{M=1}\chi^{1/2}_{1/2}\right\rangle$$

Doublet ${}^2D$, J=$\frac52$ M$_J$=$\frac32$

$$=\sqrt{\tfrac15}\left|\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\;\begin{matrix}\uparrow\downarrow\\\downarrow\end{matrix}\right\rangle+\sqrt{\tfrac45}\left|\left[\sqrt{\tfrac12}\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\;\begin{matrix}\uparrow\uparrow\\\downarrow\end{matrix}+\sqrt{\tfrac12}\begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\;\begin{matrix}\uparrow\uparrow\\\downarrow\end{matrix}\right]\right\rangle=\sqrt{\tfrac15}\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\;\begin{matrix}\uparrow\downarrow\\\downarrow\end{matrix}+\sqrt{\tfrac25}\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\;\begin{matrix}\uparrow\uparrow\\\downarrow\end{matrix}+\sqrt{\tfrac25}\begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\;\begin{matrix}\uparrow\uparrow\\\downarrow\end{matrix}$$

$$\begin{array}{|c|c|}\hline 1\uparrow&1\downarrow\\\hline 2\downarrow\\\cline{1-1}\end{array}\;Ⓐ\;(+)\sqrt{\frac{2-1}{2-1}}\quad \mu_1=2,\mu_2=1$$

$$\begin{array}{|c|c|}\hline 1\uparrow&1\downarrow\\\hline\end{array}\;Ⓔ\;(+)\sqrt{1}\quad \mu_1=1,\mu_2=1$$

$$=(+)\begin{array}{|c|}\hline 1\uparrow\\\hline 1\downarrow\\\hline 2\downarrow\\\hline\end{array}\;\begin{matrix}a\\=b\\d\end{matrix}$$

ℓ=1 p=shell LSJ states transformed to Slater determinants from J=5/2 at L=2

M$_J$=3/2

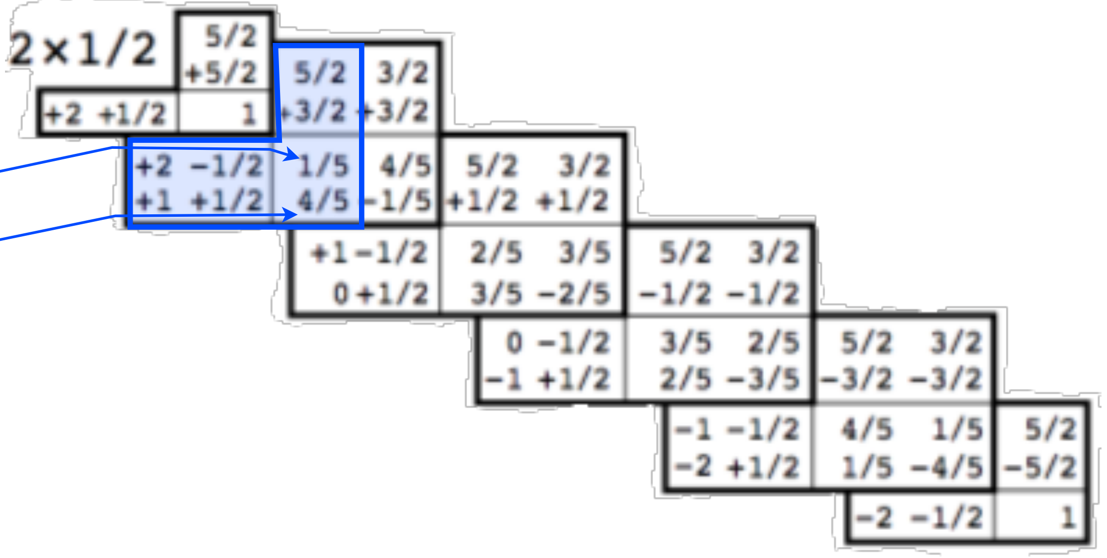
$$\left|{}^4S_{J=\frac{3}{2}}\ {}^{\frac{3}{2}}\right\rangle = \left|\begin{matrix} a\\ c\\ e\end{matrix}\right\rangle, \left|{}^4S_{J=\frac{3}{2}}\ {}^{\frac{1}{2}}\right\rangle = \left|\begin{matrix} a\\ c\\ f\end{matrix}\right\rangle, \left|{}^4S_{J=\frac{3}{2}}\ {}^{\frac{-1}{2}}\right\rangle = \left|\begin{matrix} a\\ d\\ f\end{matrix}\right\rangle, \left|{}^4S_{J=\frac{3}{2}}\ {}^{\frac{-3}{2}}\right\rangle = \left|\begin{matrix} b\\ d\\ f\end{matrix}\right\rangle$$

quartet ${}^4S$ J=$\frac{3}{2}$,

M$_J$ = $\frac{+3}{2},\frac{+1}{2},\frac{-1}{2},\frac{-3}{2}$.

$$\left|{}^2D_{J=\frac{5}{2}}\ {}^{\frac{5}{2}}\right\rangle = \left|d^{L=2}_{M=2}\chi^{1/2}_{1/2}\right\rangle$$ Doublet ${}^2D$, J=$\frac{5}{2}$ M$_J$=$\frac{5}{2}$,

$$= \left|\begin{array}{l}\boxed{1}\boxed{1}\\ \boxed{2}\end{array}\quad \begin{array}{l}\uparrow\uparrow\\ \downarrow\end{array}\right\rangle \quad = \begin{matrix} a\\ b\\ c\end{matrix}$$



2×1/2	5/2
	+5/2
+2 +1/2	1

	5/2	3/2
	+3/2	+3/2
+2 −1/2	1/5	4/5
+1 +1/2	4/5	−1/5

	5/2	3/2
	+1/2	+1/2
+1 −1/2	2/5	3/5
0 +1/2	3/5	−2/5

	5/2	3/2
	−1/2	−1/2
0 −1/2	3/5	2/5
−1 +1/2	2/5	−3/5

	5/2	3/2
	−3/2	−3/2
−1 −1/2	4/5	1/5
−2 +1/2	1/5	−4/5

	5/2
	−5/2
−2 −1/2	1

$$\left|{}^2D_{J=\frac{5}{2}}\ {}^{\frac{3}{2}}\right\rangle = \underline{\sqrt{\tfrac{1}{5}}\left|d^{L=2}_{M=2}\chi^{1/2}_{-1/2}\right\rangle + \sqrt{\tfrac{4}{5}}\left|d^{L=2}_{M=1}\chi^{1/2}_{1/2}\right\rangle}$$ Doublet ${}^2D$, $\underline{\text{J}=\frac{5}{2}\ \text{M}_J=\frac{3}{2}}$

$$=\sqrt{\tfrac{1}{5}}\left|\begin{array}{l}\boxed{1}\boxed{1}\\ \boxed{2}\end{array}\ \begin{array}{l}\uparrow\downarrow\\ \downarrow\end{array}\right\rangle + \sqrt{\tfrac{4}{5}}\left|\left[\sqrt{\tfrac{1}{2}}\begin{array}{l}\boxed{1}\boxed{2}\\ \boxed{2}\end{array}\ \begin{array}{l}\uparrow\uparrow\\ \downarrow\end{array} + \sqrt{\tfrac{1}{2}}\begin{array}{l}\boxed{1}\boxed{1}\\ \boxed{3}\end{array}\ \begin{array}{l}\uparrow\uparrow\\ \downarrow\end{array}\right]\right\rangle = \sqrt{\tfrac{1}{5}}\begin{array}{l}\boxed{1}\boxed{1}\\ \boxed{2}\end{array}\ \begin{array}{l}\uparrow\downarrow\\ \downarrow\end{array} + \sqrt{\tfrac{2}{5}}\begin{array}{l}\boxed{1}\boxed{2}\\ \boxed{2}\end{array}\ \begin{array}{l}\uparrow\uparrow\\ \downarrow\end{array} + \sqrt{\tfrac{2}{5}}\begin{array}{l}\boxed{1}\boxed{1}\\ \boxed{3}\end{array}\ \begin{array}{l}\uparrow\uparrow\\ \downarrow\end{array}$$

$$\begin{array}{l}\boxed{1\uparrow}\boxed{1\downarrow}\\ \boxed{2\downarrow}\end{array}\ Ⓐ\ (+)\sqrt{\frac{2-1}{2-1}}\quad \mu_1=2,\mu_2=1$$

$$\boxed{1\uparrow}\boxed{1\downarrow}\ Ⓔ\ (+)\sqrt{1}\quad \mu_1=1,\mu_2=1$$

$$\begin{array}{l}\boxed{1\uparrow}\boxed{2\uparrow}\\ \boxed{2\downarrow}\end{array}\ Ⓔ\ (-)\sqrt{\frac{1}{1}}\quad \mu_1=2,\mu_2=1$$

$$=(+)\begin{array}{l}\boxed{1\uparrow}\\ \boxed{1\downarrow}\\ \boxed{2\downarrow}\end{array}\ \begin{matrix} a\\ =b\\ d\end{matrix} \qquad =(-)\begin{array}{l}\boxed{1\uparrow}\\ \boxed{2\uparrow}\\ \boxed{2\downarrow}\end{array}\ \begin{matrix} a\\ =\text{-}c\\ d\end{matrix}$$

ℓ=1 p=shell LSJ states transformed to Slater determinants from J=5/2 at L=2

M$_J$=3/2

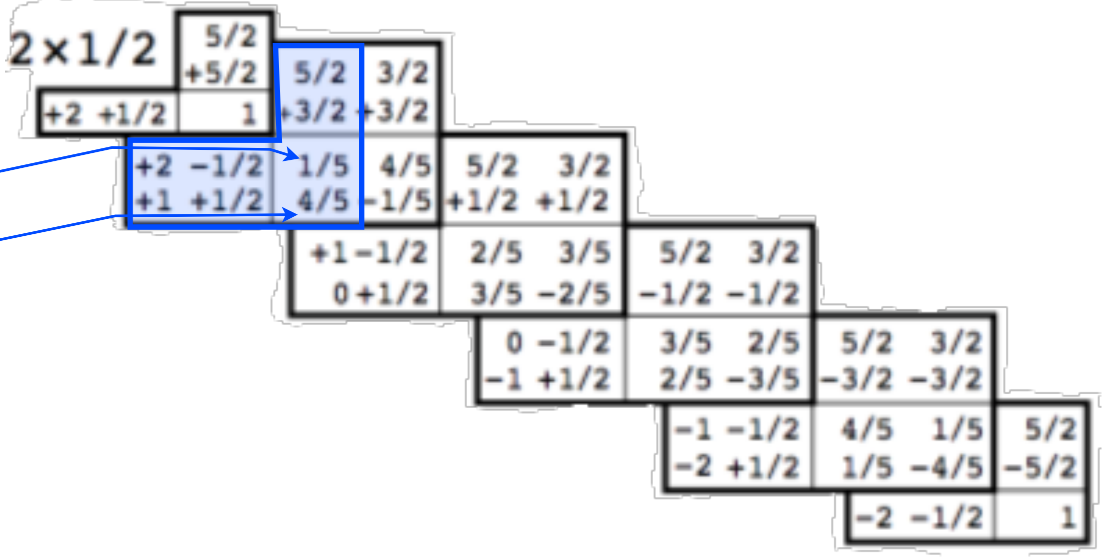
$$\left|{}^4S_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle=\left|\begin{matrix}a\\c\\e\end{matrix}\right\rangle,\ \left|{}^4S_{J=\frac{3}{2}}\ \tfrac{1}{2}\right\rangle=\left|\begin{matrix}a\\c\\f\end{matrix}\right\rangle,\ \left|{}^4S_{J=\frac{3}{2}}\ \tfrac{-1}{2}\right\rangle=\left|\begin{matrix}a\\d\\f\end{matrix}\right\rangle,\ \left|{}^4S_{J=\frac{3}{2}}\ \tfrac{-3}{2}\right\rangle=\left|\begin{matrix}b\\d\\f\end{matrix}\right\rangle$$

quartet ${}^4S$ J=$\frac{3}{2}$,
M$_J$ = $\frac{+3}{2},\frac{+1}{2},\frac{-1}{2},\frac{-3}{2}$.

$$\left|{}^2D_{J=\frac{5}{2}}\ \tfrac{5}{2}\right\rangle=\left|d^{L=2}_{M=2}\chi^{1/2}_{1/2}\right\rangle$$

Doublet ${}^2D$, J=$\frac{5}{2}$ M$_J$=$\frac{5}{2}$,

$$=\left|\begin{array}{|c|c|}\hline 1&1\\\hline 2&\\\cline{1-1}\end{array}\ \begin{matrix}\uparrow\uparrow\\\downarrow\end{matrix}\right\rangle \ \begin{matrix}a\\=b\\c\end{matrix}$$



2×1/2	5/2 +5/2
+2 +1/2	1

	5/2 +3/2	3/2 +3/2
+2 −1/2	1/5	4/5
+1 +1/2	4/5	−1/5

	5/2 +1/2	3/2 +1/2
+1 −1/2	2/5	3/5
0 +1/2	3/5	−2/5

	5/2 −1/2	3/2 −1/2
0 −1/2	3/5	2/5
−1 +1/2	2/5	−3/5

	5/2 −3/2	3/2 −3/2
−1 −1/2	4/5	1/5
−2 +1/2	1/5	−4/5

	5/2 −5/2
−2 −1/2	1

$$\left|{}^2D_{J=\frac{5}{2}}\ \tfrac{3}{2}\right\rangle=\underline{\sqrt{\tfrac{1}{5}}\left|d^{L=2}_{M=2}\chi^{1/2}_{-1/2}\right\rangle+\sqrt{\tfrac{4}{5}}\left|d^{L=2}_{M=1}\chi^{1/2}_{1/2}\right\rangle}$$

Doublet ${}^2D$, $\underline{\text{J}=\frac{5}{2}\ \text{M}_\text{J}=\frac{3}{2}}$

$$=\sqrt{\tfrac{1}{5}}\left|\begin{array}{|c|c|}\hline 1&1\\\hline 2&\\\cline{1-1}\end{array}\ \begin{matrix}\uparrow\downarrow\\\downarrow\end{matrix}\right\rangle+\sqrt{\tfrac{4}{5}}\left|\left[\sqrt{\tfrac{1}{2}}\begin{array}{|c|c|}\hline 1&2\\\hline 2&\\\cline{1-1}\end{array}\ \begin{matrix}\uparrow\uparrow\\\downarrow\end{matrix}+\sqrt{\tfrac{1}{2}}\begin{array}{|c|c|}\hline 1&1\\\hline 3&\\\cline{1-1}\end{array}\ \begin{matrix}\uparrow\uparrow\\\downarrow\end{matrix}\right]\right\rangle=\sqrt{\tfrac{1}{5}}\begin{array}{|c|c|}\hline 1&1\\\hline 2&\\\cline{1-1}\end{array}\ \begin{matrix}\uparrow\downarrow\\\downarrow\end{matrix}+\sqrt{\tfrac{2}{5}}\begin{array}{|c|c|}\hline 1&2\\\hline 2&\\\cline{1-1}\end{array}\ \begin{matrix}\uparrow\uparrow\\\downarrow\end{matrix}+\sqrt{\tfrac{2}{5}}\begin{array}{|c|c|}\hline 1&1\\\hline 3&\\\cline{1-1}\end{array}\ \begin{matrix}\uparrow\uparrow\\\downarrow\end{matrix}$$

$$\begin{array}{|c|c|}\hline 1\uparrow&1\downarrow\\\hline 2\downarrow&\\\cline{1-1}\end{array}\ (+)\sqrt{\frac{2-1}{2-1}}\qquad \begin{array}{|c|c|}\hline 1\uparrow&1\downarrow\\\hline\end{array}\ (+)\sqrt{1}$$

ⓐ μ_1=2, μ_2=1 (A)

(E) μ_1=1, μ_2=1

$$=(+)\begin{array}{|c|}\hline 1\uparrow\\\hline 1\downarrow\\\hline 2\downarrow\\\hline\end{array}\ \begin{matrix}a\\=b\\d\end{matrix}$$

$$\begin{array}{|c|c|}\hline 1\uparrow&2\uparrow\\\hline 2\downarrow&\\\cline{1-1}\end{array}\ (-)\sqrt{\frac{1}{1}}$$

(E) μ_1=2, μ_2=1

$$=(-)\begin{array}{|c|}\hline 1\uparrow\\\hline 2\uparrow\\\hline 2\downarrow\\\hline\end{array}\ \begin{matrix}a\\=\text{-}c\\d\end{matrix}$$

$$\begin{array}{|c|c|}\hline 1\uparrow&1\downarrow\\\hline 3\uparrow&\\\cline{1-1}\end{array}\ (+)\sqrt{\frac{2-1}{2-1}}\qquad \begin{array}{|c|c|}\hline 1\uparrow&1\downarrow\\\hline\end{array}\ (+)\sqrt{1}$$

(D) μ_1=2, μ_2=1

(E) μ_1=1, μ_2=1

$$=(+)\begin{array}{|c|}\hline 1\uparrow\\\hline 1\downarrow\\\hline 3\uparrow\\\hline\end{array}\ \begin{matrix}a\\=b\\e\end{matrix}$$

ℓ=1 p=shell LSJ states transformed to Slater determinants from J=3/2 at L=2

M$_J$=3/2

$$\left|{}^4S^{\frac{3}{2}}_{J=\frac{3}{2}}\right\rangle=\left|\begin{matrix}a\\c\\e\end{matrix}\right\rangle,\left|{}^4S^{\frac{1}{2}}_{J=\frac{3}{2}}\right\rangle=\left|\begin{matrix}a\\c\\f\end{matrix}\right\rangle,\left|{}^4S^{\frac{-1}{2}}_{J=\frac{3}{2}}\right\rangle=\left|\begin{matrix}a\\d\\f\end{matrix}\right\rangle,\left|{}^4S^{\frac{-3}{2}}_{J=\frac{3}{2}}\right\rangle=\left|\begin{matrix}b\\d\\f\end{matrix}\right\rangle$$

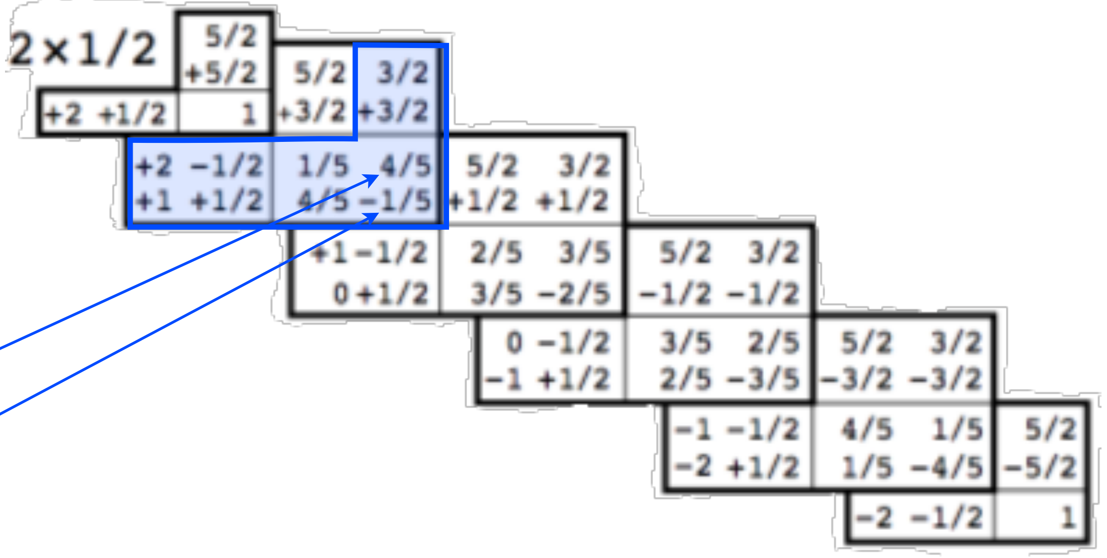
quartet ${}^4S$ J=$\frac{3}{2}$,

$M_J=\frac{+3}{2},\frac{+1}{2},\frac{-1}{2},\frac{-3}{2}.$

$$\left|{}^2D^{\frac{5}{2}}_{J=\frac{5}{2}}\right\rangle=\left|d^{L=2}_{M=2}\chi^{1/2}_{1/2}\right\rangle$$

Doublet ${}^2D$, J=$\frac{5}{2}$ M$_J$=$\frac{5}{2}$,

$$=\left|\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\quad\begin{matrix}\uparrow\uparrow\\\downarrow\end{matrix}\right\rangle\quad\begin{matrix}a\\=b\\c\end{matrix}$$



2×1/2	5/2 +5/2	
+2 +1/2	1	
	5/2 +3/2	3/2 +3/2
+2 −1/2	1/5	4/5
+1 +1/2	4/5	−1/5
	5/2 +1/2	3/2 +1/2
+1 −1/2	2/5	3/5
0 +1/2	3/5	−2/5
	5/2 −1/2	3/2 −1/2
0 −1/2	3/5	2/5
−1 +1/2	2/5	−3/5
	5/2 −3/2	3/2 −3/2
−1 −1/2	4/5	1/5
−2 +1/2	1/5	−4/5
	5/2 −5/2	
−2 −1/2	1	

$$\left|{}^2D^{\frac{3}{2}}_{J=\frac{5}{2}}\right\rangle=\sqrt{\tfrac{1}{5}}\left|d^{L=2}_{M=2}\chi^{1/2}_{-1/2}\right\rangle+\sqrt{\tfrac{4}{5}}\left|d^{L=2}_{M=1}\chi^{1/2}_{1/2}\right\rangle$$

Doublet ${}^2D$, J=$\frac{5}{2}$ M$_J$=$\frac{3}{2}$

$$=\sqrt{\tfrac{1}{5}}\left|\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\quad\begin{matrix}\uparrow\downarrow\\\downarrow\end{matrix}\right\rangle+\sqrt{\tfrac{4}{5}}\left|\left[\sqrt{\tfrac{1}{2}}\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\quad\begin{matrix}\uparrow\uparrow\\\downarrow\end{matrix}+\sqrt{\tfrac{1}{2}}\begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\quad\begin{matrix}\uparrow\uparrow\\\downarrow\end{matrix}\right]\right\rangle=\sqrt{\tfrac{1}{5}}\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\quad\begin{matrix}\uparrow\downarrow\\\downarrow\end{matrix}+\sqrt{\tfrac{2}{5}}\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\quad\begin{matrix}\uparrow\uparrow\\\downarrow\end{matrix}+\sqrt{\tfrac{2}{5}}\begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\quad\begin{matrix}\uparrow\uparrow\\\downarrow\end{matrix}$$

$$\begin{matrix}a\\=b\\d\end{matrix}\qquad\qquad\begin{matrix}a\\=\text{-}c\\d\end{matrix}\qquad\qquad\begin{matrix}a\\=b\\e\end{matrix}$$

$$\left|{}^2D^{\frac{3}{2}}_{J=\frac{3}{2}}\right\rangle=\sqrt{\tfrac{4}{5}}\left|d^{L=2}_{M=2}\chi^{1/2}_{-1/2}\right\rangle-\sqrt{\tfrac{1}{5}}\left|d^{L=2}_{M=1}\chi^{1/2}_{+1/2}\right\rangle$$

Doublet ${}^2D$, J=$\frac{3}{2}$ M$_J$=$\frac{3}{2}$

ℓ=1 p=shell LSJ states transformed to Slater determinants from J=3/2 at L=2

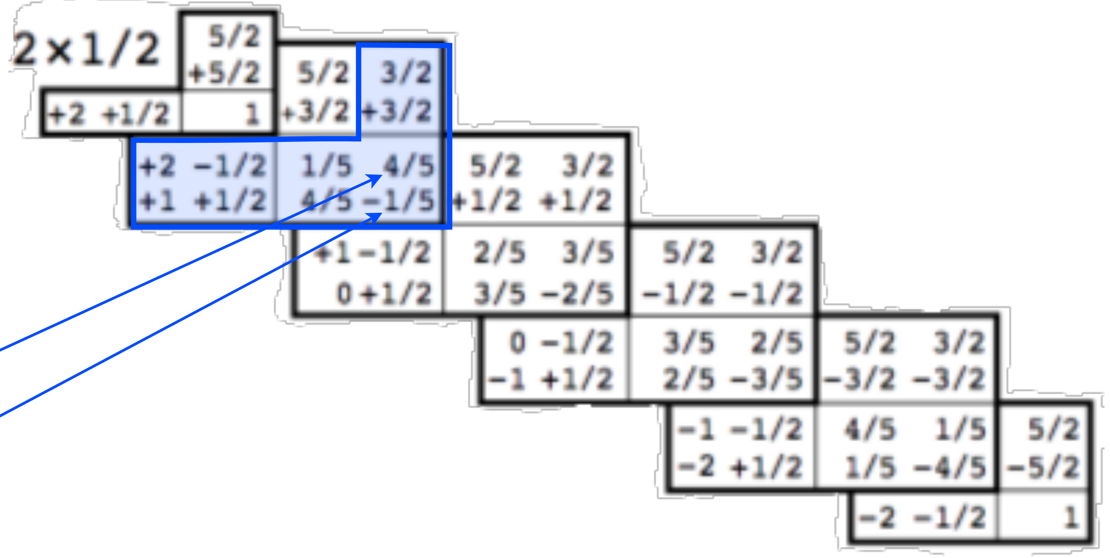
M$_J$=3/2

$$\left|{}^4S^{\frac32}_{J=\frac32}\right\rangle=\left|\begin{matrix}a\\c\\e\end{matrix}\right\rangle,\left|{}^4S^{\frac12}_{J=\frac32}\right\rangle=\left|\begin{matrix}a\\c\\f\end{matrix}\right\rangle,\left|{}^4S^{\frac{-1}{2}}_{J=\frac32}\right\rangle=\left|\begin{matrix}a\\d\\f\end{matrix}\right\rangle,\left|{}^4S^{\frac{-3}{2}}_{J=\frac32}\right\rangle=\left|\begin{matrix}b\\d\\f\end{matrix}\right\rangle$$

quartet ${}^4S$ J=$\frac32$, M$_J$ = $\frac{+3}{2},\frac{+1}{2},\frac{-1}{2},\frac{-3}{2}$.

$$\left|{}^2D^{\frac52}_{J=\frac52}\right\rangle=\left|d^{L=2}_{M=2}\chi^{1/2}_{1/2}\right\rangle$$ Doublet ${}^2D$, J=$\frac52$ M$_J$=$\frac52$,

$$=\left|\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\;\begin{matrix}\uparrow&\uparrow\\\downarrow&\end{matrix}\right\rangle=\begin{matrix}a\\b\\c\end{matrix}$$



2×1/2

+2 +1/2	5/2 +5/2
	1

	5/2 +3/2	3/2 +3/2
+2 −1/2	1/5	4/5
+1 +1/2	4/5	−1/5

	5/2 +1/2	3/2 +1/2
+1 −1/2	2/5	3/5
0 +1/2	3/5	−2/5

	5/2 −1/2	3/2 −1/2
0 −1/2	3/5	2/5
−1 +1/2	2/5	−3/5

	5/2 −3/2	3/2 −3/2
−1 −1/2	4/5	1/5
−2 +1/2	1/5	−4/5

	5/2 −5/2
−2 −1/2	1

$$\left|{}^2D^{\frac32}_{J=\frac52}\right\rangle=\sqrt{\tfrac15}\left|d^{L=2}_{M=2}\chi^{1/2}_{-1/2}\right\rangle+\sqrt{\tfrac45}\left|d^{L=2}_{M=1}\chi^{1/2}_{1/2}\right\rangle$$ Doublet ${}^2D$, J=$\frac52$ M$_J$=$\frac32$

$$=\sqrt{\tfrac15}\left|\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\;\begin{matrix}\uparrow&\downarrow\\\downarrow&\end{matrix}\right\rangle+\sqrt{\tfrac45}\left|\left[\sqrt{\tfrac12}\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\;\begin{matrix}\uparrow&\uparrow\\\downarrow&\end{matrix}+\sqrt{\tfrac12}\begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\;\begin{matrix}\uparrow&\uparrow\\\downarrow&\end{matrix}\right]\right\rangle=\sqrt{\tfrac15}\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\;\begin{matrix}\uparrow&\downarrow\\\downarrow&\end{matrix}+\sqrt{\tfrac25}\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\;\begin{matrix}\uparrow&\uparrow\\\downarrow&\end{matrix}+\sqrt{\tfrac25}\begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\;\begin{matrix}\uparrow&\uparrow\\\downarrow&\end{matrix}$$

$$=\sqrt{\tfrac15}\begin{matrix}a\\b\\d\end{matrix}-\sqrt{\tfrac25}\begin{matrix}a\\c\\d\end{matrix}+\sqrt{\tfrac25}\begin{matrix}a\\b\\e\end{matrix}$$

$$\left|{}^2D^{\frac32}_{J=\frac32}\right\rangle=\sqrt{\tfrac45}\left|d^{L=2}_{M=2}\chi^{1/2}_{-1/2}\right\rangle-\sqrt{\tfrac15}\left|d^{L=2}_{M=1}\chi^{1/2}_{+1/2}\right\rangle$$ Doublet ${}^2D$, J=$\frac32$ M$_J$=$\frac32$

$$=\sqrt{\tfrac45}\left|\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\;\begin{matrix}\uparrow&\downarrow\\\downarrow&\end{matrix}\right\rangle-\sqrt{\tfrac15}\left|\left[\sqrt{\tfrac12}\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\;\begin{matrix}\uparrow&\uparrow\\\downarrow&\end{matrix}+\sqrt{\tfrac12}\begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\;\begin{matrix}\uparrow&\uparrow\\\downarrow&\end{matrix}\right]\right\rangle=\sqrt{\tfrac45}\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\;\begin{matrix}\uparrow&\downarrow\\\downarrow&\end{matrix}-\sqrt{\tfrac{1}{10}}\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\;\begin{matrix}\uparrow&\uparrow\\\downarrow&\end{matrix}-\sqrt{\tfrac{1}{10}}\begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\;\begin{matrix}\uparrow&\uparrow\\\downarrow&\end{matrix}$$

$$=\sqrt{\tfrac45}\begin{matrix}a\\b\\d\end{matrix}+\sqrt{\tfrac{1}{10}}\begin{matrix}a\\c\\d\end{matrix}-\sqrt{\tfrac{1}{10}}\begin{matrix}a\\b\\e\end{matrix}$$

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\;\begin{matrix}\uparrow&\downarrow\\\downarrow&\end{matrix}=\begin{matrix}a\\b\\d\end{matrix}\qquad\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\;\begin{matrix}\uparrow&\uparrow\\\downarrow&\end{matrix}=-\begin{matrix}a\\c\\d\end{matrix}\qquad\begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\;\begin{matrix}\uparrow&\uparrow\\\downarrow&\end{matrix}=\begin{matrix}a\\b\\e\end{matrix}$$

AMOP
reference links
on pages 2-4

4.25.18 class 26: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

$(S_3)^*(U(3))\subset U(6)$ models of p^3 electronic spin-orbit states and couplings

[2,1] tableau states lowered by $\mathbf{L}_-=\sqrt{2}(E_{21}+E_{32})$

Top-(J,M) states to mid-level states

$\ell=1$ p=shell LS states combined to states of definite J

J=3/2 at L=0 (^{4}S), J=5/2 at L=2 (^{2}D)

C-G coupling; J=3/2 at L=2 (^{2}D), J=3/2 at L=1 (^{2}P), J=1/2 at L=1 (^{2}P)

Spin-orbit state assembly formula and Slater determinants

Extra assembly table

$\ell=1$ p=shell LSJ states transformed to Slater determinants from J=3/2 (^{4}S)

Slater functions for J=5/2, J=3/2 (^{2}D)

Slater functions for J=3/2 (^{2}P), J=1/2 (^{2}P)

Summary of states and level connection paths

Symmetry dimension accounting

Spin-orbit Hamiltonian matrix calculation

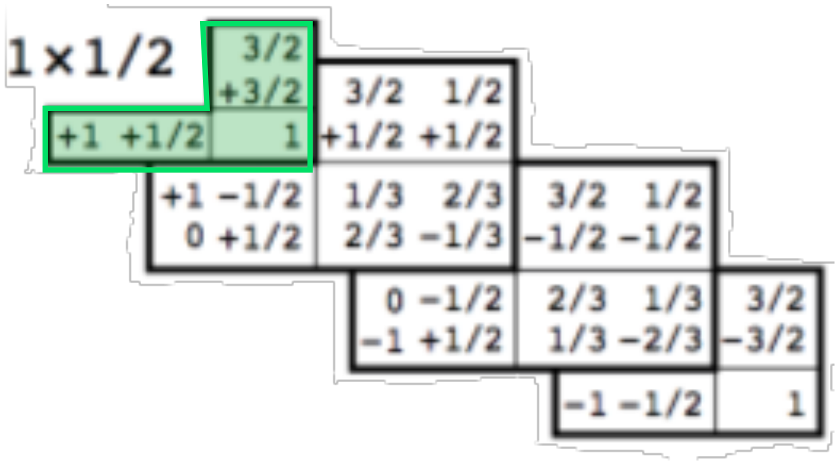
Application to spin-orbit and entanglement break-up scattering

ℓ=1 *p*=shell LSJ states transformed to Slater determinants from J=3/2 at L=1

M$_J$=3/2

$$\left|{}^2P_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle = \underline{\left|p^{L=1}_{M=1}\chi^{1/2}_{+1/2}\right\rangle}$$

Doublet ${}^2P$, $\underline{\text{J}=\frac{3}{2}\ \text{M}_J=\frac{3}{2}}$



1×1/2	3/2 +3/2			
+1 +1/2	1	3/2 1/2 +1/2 +1/2		
	+1 −1/2 0 +1/2	1/3 2/3 2/3 −1/3	3/2 1/2 −1/2 −1/2	
		0 −1/2 −1 +1/2	2/3 1/3 1/3 −2/3	3/2 −3/2
			−1 −1/2	1

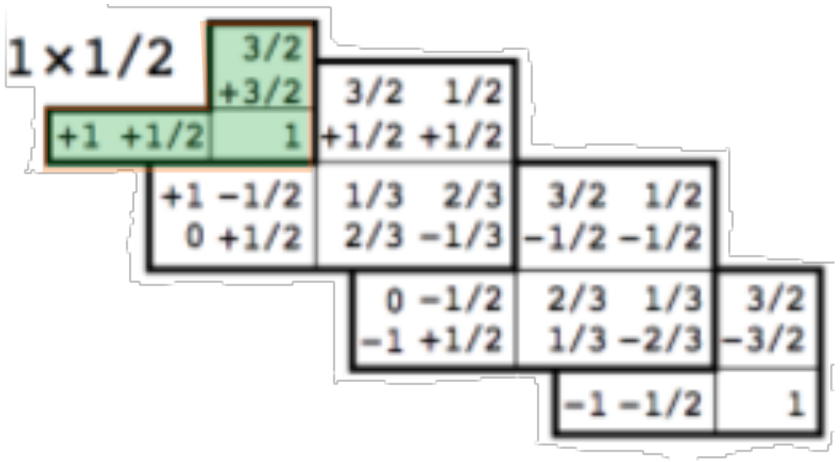
ℓ=1 p=shell LSJ states transformed to Slater determinants from J=3/2 at L=1

M$_J$=3/2

$$\left|{}^2P_{J=\frac{3}{2}}\ {}_{\frac{3}{2}}\right\rangle = \underline{\left|p^{L=1}_{M=1}\chi^{1/2}_{+1/2}\right\rangle} \qquad \text{Doublet } {}^2P,\ \underline{\mathrm{J}=\tfrac{3}{2}\ \ \mathrm{M_J}=\tfrac{3}{2}}$$

$$= \sqrt{\tfrac{1}{2}}\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\ \begin{matrix}\uparrow\uparrow \\ \downarrow\end{matrix} \qquad -\sqrt{\tfrac{1}{2}}\ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\ \begin{matrix}\uparrow\uparrow \\ \downarrow\end{matrix}$$

$$= -\sqrt{\tfrac{1}{2}}\ \begin{matrix} a \\ c \\ d \end{matrix} \qquad -\sqrt{\tfrac{1}{2}}\ \begin{matrix} a \\ b \\ e \end{matrix}$$



1×1/2	3/2 +3/2		
+1 +1/2	1	3/2 1/2 +1/2 +1/2	
+1 −1/2 0 +1/2		1/3 2/3 2/3 −1/3	3/2 1/2 −1/2 −1/2
0 −1/2 −1 +1/2			2/3 1/3 1/3 −2/3	3/2 −3/2
−1 −1/2			1

ℓ=1 p=shell LSJ states transformed to Slater determinants from J=3/2 at L=1

M$_J$=1/2

$$\left|{}^{2}P_{J=\frac{3}{2}}{}^{\frac{3}{2}}\right\rangle = \left|p^{L=1}_{M=1}\chi^{1/2}_{+1/2}\right\rangle$$

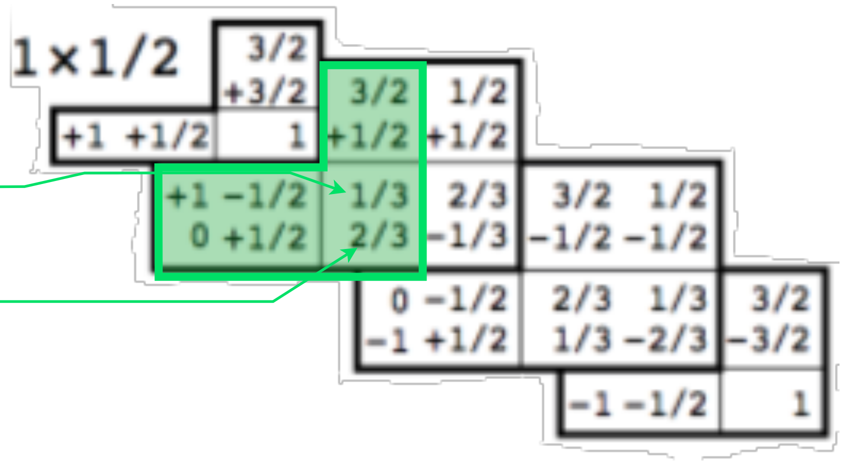
Doublet ^2P, J=$\frac{3}{2}$ M$_J$=$\frac{3}{2}$

$$= \sqrt{\tfrac{1}{2}}\;\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \\ \cline{1-1}\end{array}\;\begin{array}{c}\uparrow\uparrow\\ \downarrow\end{array} \qquad -\sqrt{\tfrac{1}{2}}\;\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \\ \cline{1-1}\end{array}\;\begin{array}{c}\uparrow\uparrow\\ \downarrow\end{array}$$

$$= -\sqrt{\tfrac{1}{2}}\;\begin{array}{c} a \\ c \\ d \end{array} \qquad -\sqrt{\tfrac{1}{2}}\;\begin{array}{c} a \\ b \\ e \end{array}$$

$$\left|{}^{2}P_{J=\frac{3}{2}}{}^{\frac{1}{2}}\right\rangle = \sqrt{\tfrac{1}{3}}\left|p^{L=1}_{M=1}\chi^{1/2}_{-1/2}\right\rangle + \sqrt{\tfrac{2}{3}}\left|p^{L=1}_{M=0}\chi^{1/2}_{+1/2}\right\rangle$$

Doublet ^2P, J=$\frac{3}{2}$ M$_J$=$\frac{1}{2}$



1×1/2	3/2 +3/2	
+1 +1/2	1	
	3/2 +1/2	1/2 +1/2
+1 −1/2	1/3	2/3
0 +1/2	2/3	−1/3
	3/2 −1/2	1/2 −1/2
0 −1/2	2/3	1/3
−1 +1/2	1/3	−2/3
	3/2 −3/2	
−1 −1/2	1	

ℓ=1 p=shell LSJ states transformed to Slater determinants from J=3/2 at L=1 MJ=1/2

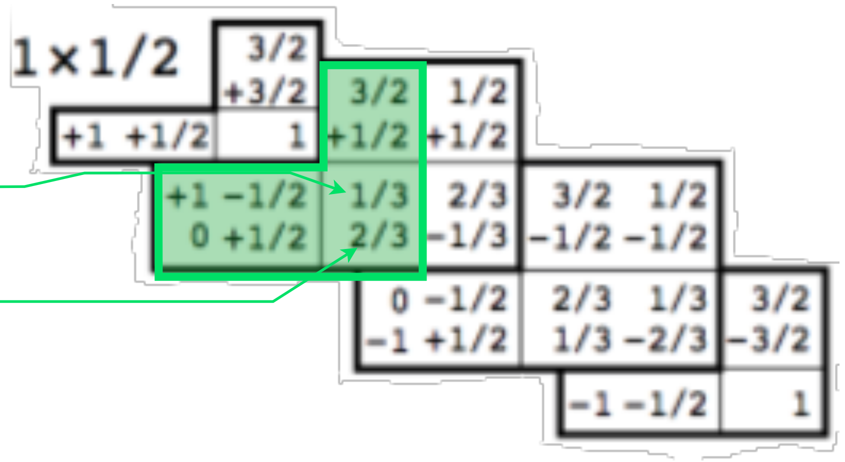
$$\left|{}^2P_{J=\frac{3}{2}}\ {}_{\frac{3}{2}}\right\rangle = \left|p^{L=1}_{M=1}\chi^{1/2}_{+1/2}\right\rangle \qquad \text{Doublet } {}^2P,\ \mathrm{J}=\tfrac{3}{2}\ \mathrm{M_J}=\tfrac{3}{2}$$

$$= \sqrt{\tfrac{1}{2}}\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\ \begin{matrix}\uparrow\uparrow\\ \downarrow\end{matrix} \quad - \sqrt{\tfrac{1}{2}}\ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\ \begin{matrix}\uparrow\uparrow\\ \downarrow\end{matrix}$$

$$= \ -\sqrt{\tfrac{1}{2}}\ \begin{matrix}a\\ c\\ d\end{matrix} \quad - \sqrt{\tfrac{1}{2}}\ \begin{matrix}a\\ b\\ e\end{matrix}$$

$$\left|{}^2P_{J=\frac{3}{2}}\ {}_{\frac{1}{2}}\right\rangle = \sqrt{\tfrac{1}{3}}\left|p^{L=1}_{M=1}\chi^{1/2}_{-1/2}\right\rangle + \sqrt{\tfrac{2}{3}}\left|p^{L=1}_{M=0}\chi^{1/2}_{+1/2}\right\rangle \qquad \text{Doublet } {}^2P,\ \mathrm{J}=\tfrac{3}{2}\ \mathrm{M_J}=\tfrac{1}{2}$$

$$= \sqrt{\tfrac{1}{3}}\left|\left[\sqrt{\tfrac{1}{2}}\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\ \begin{matrix}\uparrow\downarrow\\ \downarrow\end{matrix} - \sqrt{\tfrac{1}{2}}\ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\ \begin{matrix}\uparrow\downarrow\\ \downarrow\end{matrix}\ \right]\right\rangle + \sqrt{\tfrac{2}{3}}\left|\left[\tfrac{-1}{2}\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\ \begin{matrix}\uparrow\uparrow\\ \downarrow\end{matrix} + \tfrac{\sqrt{3}}{2}\ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\ \begin{matrix}\uparrow\uparrow\\ \downarrow\end{matrix}\ \right]\right\rangle$$



1×1/2

		3/2 +3/2
+1	+1/2	1

		3/2 +1/2	1/2 +1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

		3/2 −1/2	1/2 −1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

		3/2 −3/2
−1	−1/2	1

ℓ=1 p=shell LSJ states transformed to Slater determinants from J=3/2 at L=1 M$_J$=1/2

$$\left|{}^2P^{\frac{3}{2}}_{J=\frac{3}{2}}\right\rangle=\left|p^{L=1}_{M=1}\chi^{1/2}_{+1/2}\right\rangle$$

Doublet ${}^2P$, J=$\frac{3}{2}$ M$_J$=$\frac{3}{2}$

$$=\sqrt{\tfrac{1}{2}}\begin{array}{|c|c|}\hline 1&2\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\;\begin{array}{l}\uparrow\uparrow\\\downarrow\end{array}\quad-\sqrt{\tfrac{1}{2}}\begin{array}{|c|c|}\hline 1&1\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\;\begin{array}{l}\uparrow\uparrow\\\downarrow\end{array}$$

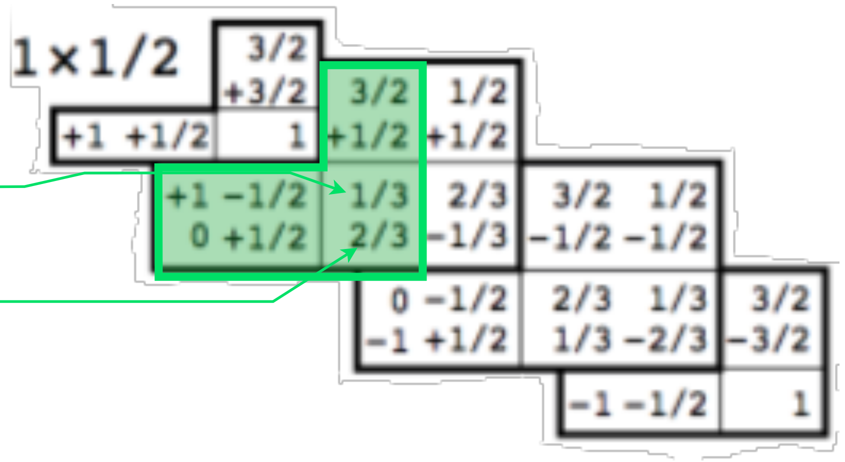
$$=\;-\sqrt{\tfrac{1}{2}}\begin{matrix}a\\c\\d\end{matrix}\qquad-\sqrt{\tfrac{1}{2}}\begin{matrix}a\\b\\e\end{matrix}$$

$$\left|{}^2P^{\frac{1}{2}}_{J=\frac{3}{2}}\right\rangle=\sqrt{\tfrac{1}{3}}\left|p^{L=1}_{M=1}\chi^{1/2}_{-1/2}\right\rangle+\sqrt{\tfrac{2}{3}}\left|p^{L=1}_{M=0}\chi^{1/2}_{+1/2}\right\rangle$$

Doublet ${}^2P$, J=$\frac{3}{2}$ M$_J$=$\frac{1}{2}$

$$=\sqrt{\tfrac{1}{3}}\left|\left[\sqrt{\tfrac{1}{2}}\begin{array}{|c|c|}\hline 1&2\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\;\begin{array}{l}\uparrow\downarrow\\\downarrow\end{array}-\sqrt{\tfrac{1}{2}}\begin{array}{|c|c|}\hline 1&1\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\;\begin{array}{l}\uparrow\downarrow\\\downarrow\end{array}\right]\right\rangle+\sqrt{\tfrac{2}{3}}\left|\left[\tfrac{-1}{2}\begin{array}{|c|c|}\hline 1&2\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\;\begin{array}{l}\uparrow\uparrow\\\downarrow\end{array}+\tfrac{\sqrt{3}}{2}\begin{array}{|c|c|}\hline 1&3\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\;\begin{array}{l}\uparrow\uparrow\\\downarrow\end{array}\right]\right\rangle$$

$$=\sqrt{\tfrac{1}{6}}\begin{array}{|c|c|}\hline 1&2\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\;\begin{array}{l}\uparrow\downarrow\\\downarrow\end{array}-\sqrt{\tfrac{1}{6}}\begin{array}{|c|c|}\hline 1&1\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\;\begin{array}{l}\uparrow\downarrow\\\downarrow\end{array}\qquad-\sqrt{\tfrac{1}{6}}\begin{array}{|c|c|}\hline 1&2\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\;\begin{array}{l}\uparrow\uparrow\\\downarrow\end{array}+\tfrac{1}{\sqrt{2}}\begin{array}{|c|c|}\hline 1&3\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\;\begin{array}{l}\uparrow\uparrow\\\downarrow\end{array}$$



1×1/2	3/2 +3/2	3/2 +1/2	1/2 +1/2	3/2 −1/2	1/2 −1/2	3/2 −3/2
+1 +1/2	1					
+1 −1/2		1/3	2/3			
0 +1/2		2/3	−1/3			
0 −1/2				2/3	1/3	
−1 +1/2				1/3	−2/3	
−1 −1/2						1

ℓ=1 p=shell LSJ states transformed to Slater determinants from J=3/2 at L=1

M$_J$=1/2

$$\left|{}^2P_{J=\frac{3}{2}\,\frac{3}{2}}\right\rangle=\left|p^{L=1}_{M=1}\chi^{1/2}_{+1/2}\right\rangle$$

Doublet ^2P, J=$\frac{3}{2}$ M$_J$=$\frac{3}{2}$

$$=\sqrt{\tfrac{1}{2}}\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\begin{smallmatrix}\uparrow\uparrow\\\downarrow\end{smallmatrix}-\sqrt{\tfrac{1}{2}}\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\begin{smallmatrix}\uparrow\uparrow\\\downarrow\end{smallmatrix}$$

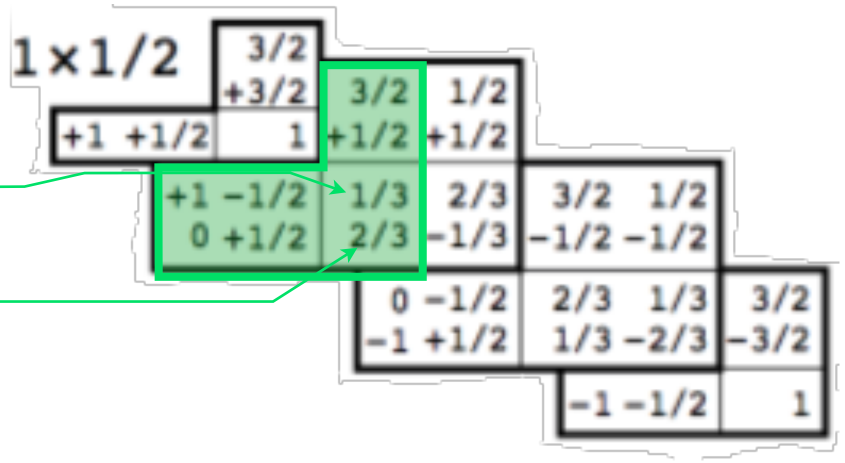
$$=-\sqrt{\tfrac{1}{2}}\begin{matrix}a\\c\\d\end{matrix}\qquad-\sqrt{\tfrac{1}{2}}\begin{matrix}a\\b\\e\end{matrix}$$

$$\left|{}^2P_{J=\frac{3}{2}\,\frac{1}{2}}\right\rangle=\sqrt{\tfrac{1}{3}}\left|p^{L=1}_{M=1}\chi^{1/2}_{-1/2}\right\rangle+\sqrt{\tfrac{2}{3}}\left|p^{L=1}_{M=0}\chi^{1/2}_{+1/2}\right\rangle$$

Doublet ^2P, J=$\frac{3}{2}$ M$_J$=$\frac{1}{2}$

$$=\sqrt{\tfrac{1}{3}}\left|\left[\sqrt{\tfrac{1}{2}}\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\begin{smallmatrix}\uparrow\downarrow\\\downarrow\end{smallmatrix}-\sqrt{\tfrac{1}{2}}\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\begin{smallmatrix}\uparrow\downarrow\\\downarrow\end{smallmatrix}\right]\right\rangle+\sqrt{\tfrac{2}{3}}\left|\left[\tfrac{-1}{2}\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\begin{smallmatrix}\uparrow\uparrow\\\downarrow\end{smallmatrix}+\tfrac{\sqrt{3}}{2}\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{2}\end{smallmatrix}\begin{smallmatrix}\uparrow\uparrow\\\downarrow\end{smallmatrix}\right]\right\rangle$$

$$=\sqrt{\tfrac{1}{6}}\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{2}\end{smallmatrix}\begin{smallmatrix}\uparrow\downarrow\\\downarrow\end{smallmatrix}-\sqrt{\tfrac{1}{6}}\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}\begin{smallmatrix}\uparrow\downarrow\\\downarrow\end{smallmatrix}\qquad-\sqrt{\tfrac{1}{6}}\begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}\begin{smallmatrix}\uparrow\uparrow\\\downarrow\end{smallmatrix}+\tfrac{1}{\sqrt{2}}\begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{2}\end{smallmatrix}\begin{smallmatrix}\uparrow\uparrow\\\downarrow\end{smallmatrix}$$



		3/2	3/2	1/2	3/2	1/2	3/2
1×1/2		+3/2	+1/2	+1/2	−1/2	−1/2	−3/2
+1	+1/2	1					
+1	−1/2		1/3	2/3			
0	+1/2		2/3	−1/3			
0	−1/2				2/3	1/3	
−1	+1/2				1/3	−2/3	
−1	−1/2						1

$$\begin{smallmatrix}\boxed{1\uparrow}\boxed{1\downarrow}\\\boxed{3\downarrow}\end{smallmatrix}\ \text{A}\ (+)\sqrt{\frac{2-1}{2-1}}\quad \mu_1=2,\mu_2=1$$

$$\boxed{1\uparrow}\boxed{1\downarrow}\ \text{E}\ (+)\sqrt{1}\quad \mu_1=1,\mu_2=1$$

$$\begin{smallmatrix}\boxed{1\downarrow}\boxed{2\uparrow}\\\boxed{2\downarrow}\end{smallmatrix}\ \text{E}\ (-)\sqrt{\frac{1}{1}}\quad \mu_1=2,\mu_2=1$$

$$=(+)\begin{matrix}\boxed{1\uparrow}\\\boxed{1\downarrow}\\\boxed{3\downarrow}\end{matrix}\ \begin{matrix}a\\=b\\f\end{matrix}$$

$$=(-)\begin{matrix}\boxed{1\downarrow}\\\boxed{2\uparrow}\\\boxed{2\downarrow}\end{matrix}\ \begin{matrix}b\\=\text{-}c\\d\end{matrix}$$

ℓ=1 p=shell LSJ states transformed to Slater determinants from J=3/2 at L=1

M$_J$=1/2

$$\left|{}^2P_{J=\frac{3}{2}}{}_{\frac{3}{2}}\right\rangle = \left|p^{L=1}_{M=1}\chi^{1/2}_{+1/2}\right\rangle$$

Doublet ${}^2P$, J=$\frac{3}{2}$ M$_J$=$\frac{3}{2}$

$$= \sqrt{\tfrac{1}{2}}\begin{array}{l}\boxed{1}\boxed{2}\\ \boxed{2}\end{array}\begin{array}{l}\uparrow\uparrow\\ \downarrow\end{array} \quad -\sqrt{\tfrac{1}{2}}\begin{array}{l}\boxed{1}\boxed{1}\\ \boxed{3}\end{array}\begin{array}{l}\uparrow\uparrow\\ \downarrow\end{array}$$

$$= -\sqrt{\tfrac{1}{2}}\begin{array}{c}a\\ c\\ d\end{array} \quad -\sqrt{\tfrac{1}{2}}\begin{array}{c}a\\ b\\ e\end{array}$$

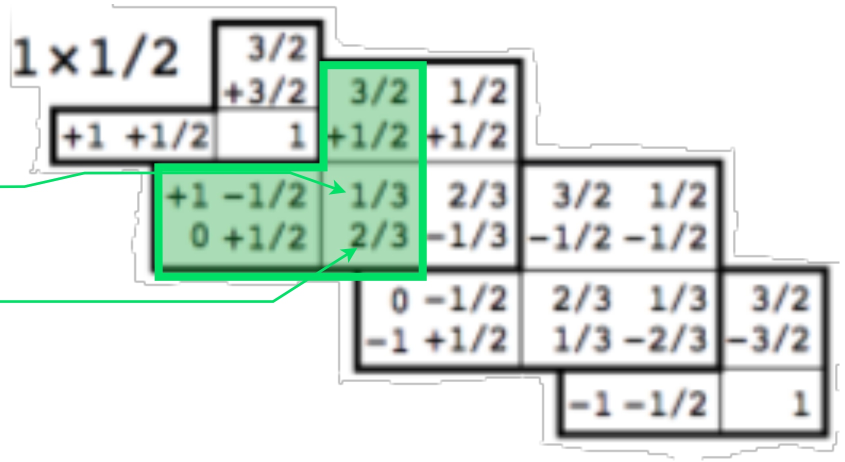
$$\left|{}^2P_{J=\frac{3}{2}}{}_{\frac{1}{2}}\right\rangle = \sqrt{\tfrac{1}{3}}\left|p^{L=1}_{M=1}\chi^{1/2}_{-1/2}\right\rangle + \sqrt{\tfrac{2}{3}}\left|p^{L=1}_{M=0}\chi^{1/2}_{+1/2}\right\rangle$$

Doublet ${}^2P$, J=$\frac{3}{2}$ M$_J$=$\frac{1}{2}$

$$= \sqrt{\tfrac{1}{3}}\left|\left[\sqrt{\tfrac{1}{2}}\begin{array}{l}\boxed{1}\boxed{2}\\ \boxed{2}\end{array}\begin{array}{l}\uparrow\downarrow\\ \downarrow\end{array} -\sqrt{\tfrac{1}{2}}\begin{array}{l}\boxed{1}\boxed{1}\\ \boxed{3}\end{array}\begin{array}{l}\uparrow\downarrow\\ \downarrow\end{array}\right]\right\rangle + \sqrt{\tfrac{2}{3}}\left|\left[\tfrac{-1}{2}\begin{array}{l}\boxed{1}\boxed{2}\\ \boxed{3}\end{array}\begin{array}{l}\uparrow\uparrow\\ \downarrow\end{array} + \tfrac{\sqrt{3}}{2}\begin{array}{l}\boxed{1}\boxed{3}\\ \boxed{2}\end{array}\begin{array}{l}\uparrow\uparrow\\ \downarrow\end{array}\right]\right\rangle$$

$$= \sqrt{\tfrac{1}{6}}\begin{array}{l}\boxed{1}\boxed{2}\\ \boxed{2}\end{array}\begin{array}{l}\uparrow\downarrow\\ \downarrow\end{array} -\sqrt{\tfrac{1}{6}}\begin{array}{l}\boxed{1}\boxed{1}\\ \boxed{3}\end{array}\begin{array}{l}\uparrow\downarrow\\ \downarrow\end{array} \quad -\sqrt{\tfrac{1}{6}}\begin{array}{l}\boxed{1}\boxed{2}\\ \boxed{3}\end{array}\begin{array}{l}\uparrow\uparrow\\ \downarrow\end{array} + \tfrac{1}{\sqrt{2}}\begin{array}{l}\boxed{1}\boxed{3}\\ \boxed{2}\end{array}\begin{array}{l}\uparrow\uparrow\\ \downarrow\end{array}$$

$$= -\sqrt{\tfrac{1}{6}}\begin{array}{c}b\\ c\\ d\end{array} \quad -\sqrt{\tfrac{1}{6}}\begin{array}{c}a\\ b\\ f\end{array}$$



1×1/2

	3/2 +3/2					
+1 +1/2	1	3/2 +1/2	1/2 +1/2			
+1 −1/2 0 +1/2		1/3 2/3	2/3 −1/3	3/2 1/2 −1/2 −1/2		
		0 −1/2 −1 +1/2		2/3 1/3 1/3 −2/3	3/2 −3/2	
				−1 −1/2	1	

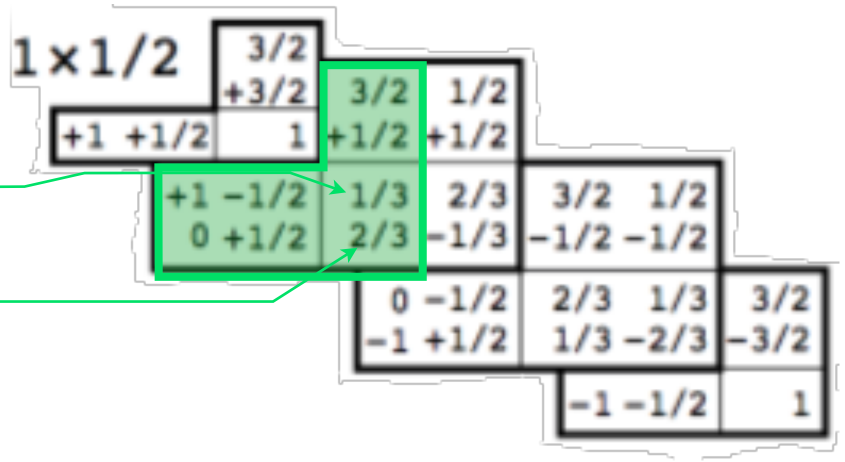
ℓ=1 p=shell LSJ states transformed to Slater determinants from J=3/2 at L=1

M$_J$=1/2

$$\left|{}^2P_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle = \left|p^{L=1}_{M=1}\chi^{1/2}_{+1/2}\right\rangle \qquad \text{Doublet } {}^2P,\ \mathrm{J}=\tfrac{3}{2}\ \ \mathrm{M_J}=\tfrac{3}{2}$$

$$= \sqrt{\tfrac{1}{2}}\ \begin{array}{|c|c|}\hline 1&2\\ \hline 2\\ \cline{1-1}\end{array}\ \begin{array}{cc}\uparrow&\uparrow\\ \downarrow&\end{array} \ -\sqrt{\tfrac{1}{2}}\ \begin{array}{|c|c|}\hline 1&1\\ \hline 3\\ \cline{1-1}\end{array}\ \begin{array}{cc}\uparrow&\uparrow\\ \downarrow&\end{array}$$

$$= -\sqrt{\tfrac{1}{2}}\ \begin{matrix}a\\ c\\ d\end{matrix} \qquad -\sqrt{\tfrac{1}{2}}\ \begin{matrix}a\\ b\\ e\end{matrix}$$



1×1/2	3/2	3/2	1/2	3/2	1/2	3/2
	+3/2	+1/2	+1/2	−1/2	−1/2	−3/2
+1 +1/2	1					
+1 −1/2		1/3	2/3			
0 +1/2		2/3	−1/3			
0 −1/2				2/3	1/3	
−1 +1/2				1/3	−2/3	
−1 −1/2						1

$$\left|{}^2P_{J=\frac{3}{2}}\ \tfrac{1}{2}\right\rangle = \sqrt{\tfrac{1}{3}}\left|p^{L=1}_{M=1}\chi^{1/2}_{-1/2}\right\rangle + \sqrt{\tfrac{2}{3}}\left|p^{L=1}_{M=0}\chi^{1/2}_{+1/2}\right\rangle \qquad \text{Doublet } {}^2P,\ \mathrm{J}=\tfrac{3}{2}\ \ \mathrm{M_J}=\tfrac{1}{2}$$

$$= \sqrt{\tfrac{1}{3}}\left|\left[\sqrt{\tfrac{1}{2}}\ \begin{array}{|c|c|}\hline 1&2\\ \hline 2\\ \cline{1-1}\end{array}\ \begin{array}{cc}\uparrow&\downarrow\\ \downarrow&\end{array} -\sqrt{\tfrac{1}{2}}\ \begin{array}{|c|c|}\hline 1&1\\ \hline 3\\ \cline{1-1}\end{array}\ \begin{array}{cc}\uparrow&\downarrow\\ \downarrow&\end{array}\ \right]\right\rangle + \sqrt{\tfrac{2}{3}}\left|\left[\tfrac{-1}{2}\ \begin{array}{|c|c|}\hline 1&2\\ \hline 3\\ \cline{1-1}\end{array}\ \begin{array}{cc}\uparrow&\uparrow\\ \downarrow&\end{array} + \tfrac{\sqrt{3}}{2}\ \begin{array}{|c|c|}\hline 1&3\\ \hline 2\\ \cline{1-1}\end{array}\ \begin{array}{cc}\uparrow&\uparrow\\ \downarrow&\end{array}\ \right]\right\rangle$$

$$= \sqrt{\tfrac{1}{6}}\ \begin{array}{|c|c|}\hline 1&2\\ \hline 2\\ \cline{1-1}\end{array}\ \begin{array}{cc}\uparrow&\downarrow\\ \downarrow&\end{array} -\sqrt{\tfrac{1}{6}}\ \begin{array}{|c|c|}\hline 1&1\\ \hline 3\\ \cline{1-1}\end{array}\ \begin{array}{cc}\uparrow&\downarrow\\ \downarrow&\end{array} \qquad -\sqrt{\tfrac{1}{6}}\ \begin{array}{|c|c|}\hline 1&2\\ \hline 3\\ \cline{1-1}\end{array}\ \begin{array}{cc}\uparrow&\uparrow\\ \downarrow&\end{array} + \tfrac{1}{\sqrt{2}}\ \begin{array}{|c|c|}\hline 1&3\\ \hline 2\\ \cline{1-1}\end{array}\ \begin{array}{cc}\uparrow&\uparrow\\ \downarrow&\end{array}$$

$$= -\sqrt{\tfrac{1}{6}}\ \begin{matrix}b\\ c\\ d\end{matrix} \qquad -\sqrt{\tfrac{1}{6}}\ \begin{matrix}a\\ b\\ f\end{matrix} \qquad -\sqrt{\tfrac{1}{6}}\left(\tfrac{1}{\sqrt{2}}\ \begin{matrix}a\\ d\\ e\end{matrix}\ \ \text{-}\tfrac{1}{\sqrt{2}}\ \begin{matrix}b\\ c\\ e\end{matrix}\right) + \tfrac{1}{\sqrt{2}}\left(\tfrac{-2}{\sqrt{6}}\ \begin{matrix}a\\ c\\ f\end{matrix}\ +\tfrac{1}{\sqrt{6}}\ \begin{matrix}a\\ d\\ e\end{matrix}\ +\tfrac{1}{\sqrt{6}}\ \begin{matrix}b\\ c\\ e\end{matrix}\right)$$

	$\begin{array}{\|c\|c\|}\hline 1&2\\ \hline 3\\ \cline{1-1}\end{array}\ \begin{array}{cc}\uparrow&\uparrow\\ \downarrow&\end{array}$	$\begin{array}{\|c\|c\|}\hline 1&3\\ \hline 2\\ \cline{1-1}\end{array}\ \begin{array}{cc}\uparrow&\uparrow\\ \downarrow&\end{array}$	$\begin{array}{\|c\|}\hline 1\\ \hline 2\\ \hline 3\\ \hline\end{array}\ \uparrow\uparrow\downarrow$
$\begin{matrix}a\\ c\\ f\end{matrix}$ $\begin{array}{\|c\|}\hline 1\uparrow\\ \hline 2\uparrow\\ \hline 3\downarrow\\ \hline\end{array}$	0	$\frac{-2}{\sqrt{6}}$	$\frac{1}{\sqrt{3}}$
$\begin{matrix}a\\ d\\ e\end{matrix}$ $\begin{array}{\|c\|}\hline 1\uparrow\\ \hline 2\downarrow\\ \hline 3\uparrow\\ \hline\end{array}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{3}}$
$\begin{matrix}b\\ c\\ e\end{matrix}$ $\begin{array}{\|c\|}\hline 1\downarrow\\ \hline 2\uparrow\\ \hline 3\uparrow\\ \hline\end{array}$	$\frac{-1}{\sqrt{2}}$	$\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{3}}$

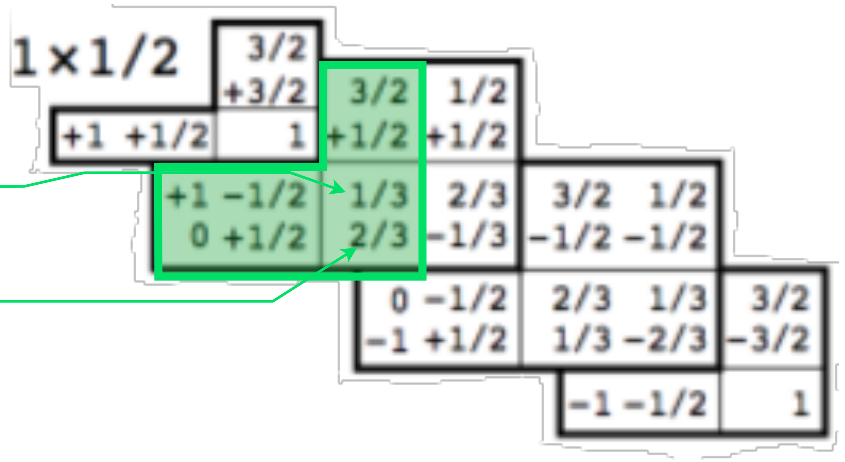
ℓ=1 p=shell LSJ states transformed to Slater determinants from J=3/2 at L=1

MJ=1/2

$$\left|{}^2P_{J=\frac{3}{2}}^{\;\frac{3}{2}}\right\rangle = \left|p^{L=1}_{M=1}\chi^{1/2}_{+1/2}\right\rangle \qquad \text{Doublet } {}^2P,\ \mathrm{J}=\tfrac{3}{2}\ \mathrm{M_J}=\tfrac{3}{2}$$

$$= \sqrt{\tfrac{1}{2}}\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\ \begin{array}{l}\uparrow\uparrow\\ \downarrow\end{array} \quad -\sqrt{\tfrac{1}{2}}\ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\ \begin{array}{l}\uparrow\uparrow\\ \downarrow\end{array}$$

$$= -\sqrt{\tfrac{1}{2}}\ \begin{array}{c}a\\ c\\ d\end{array} \quad -\sqrt{\tfrac{1}{2}}\ \begin{array}{c}a\\ b\\ e\end{array}$$



1×1/2

		3/2				
		+3/2	3/2	1/2		
+1	+1/2	1	+1/2	+1/2		
+1	−1/2		1/3	2/3	3/2	1/2
0	+1/2		2/3	−1/3	−1/2	−1/2
0	−1/2				2/3	1/3	3/2
−1	+1/2				1/3	−2/3	−3/2
−1	−1/2						1

$$\left|{}^2P_{J=\frac{3}{2}}^{\;\frac{1}{2}}\right\rangle = \sqrt{\tfrac{1}{3}}\left|p^{L=1}_{M=1}\chi^{1/2}_{-1/2}\right\rangle + \sqrt{\tfrac{2}{3}}\left|p^{L=1}_{M=0}\chi^{1/2}_{+1/2}\right\rangle \qquad \text{Doublet } {}^2P,\ \mathrm{J}=\tfrac{3}{2}\ \mathrm{M_J}=\tfrac{1}{2}$$

$$= \sqrt{\tfrac{1}{3}}\left|\left[\sqrt{\tfrac{1}{2}}\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\ \begin{array}{l}\uparrow\downarrow\\ \downarrow\end{array} -\sqrt{\tfrac{1}{2}}\ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\ \begin{array}{l}\uparrow\downarrow\\ \downarrow\end{array}\right]\right\rangle + \sqrt{\tfrac{2}{3}}\left|\left[\tfrac{-1}{2}\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\ \begin{array}{l}\uparrow\uparrow\\ \downarrow\end{array} + \tfrac{\sqrt{3}}{2}\ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\ \begin{array}{l}\uparrow\uparrow\\ \downarrow\end{array}\right]\right\rangle$$

$$= \sqrt{\tfrac{1}{6}}\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\ \begin{array}{l}\uparrow\downarrow\\ \downarrow\end{array} -\sqrt{\tfrac{1}{6}}\ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\ \begin{array}{l}\uparrow\downarrow\\ \downarrow\end{array} \quad -\sqrt{\tfrac{1}{6}}\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\ \begin{array}{l}\uparrow\uparrow\\ \downarrow\end{array} + \tfrac{1}{\sqrt{2}}\ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\ \begin{array}{l}\uparrow\uparrow\\ \downarrow\end{array}$$

$$= -\sqrt{\tfrac{1}{6}}\ \begin{array}{c}b\\ c\\ d\end{array} \quad -\sqrt{\tfrac{1}{6}}\ \begin{array}{c}a\\ b\\ f\end{array} \quad -\sqrt{\tfrac{1}{6}}\left(\tfrac{1}{\sqrt{2}}\ \begin{array}{c}a\\ d\\ e\end{array} \ -\tfrac{1}{\sqrt{2}}\ \begin{array}{c}b\\ c\\ e\end{array}\right) + \tfrac{1}{\sqrt{2}}\left(\tfrac{-2}{\sqrt{6}}\ \begin{array}{c}a\\ c\\ f\end{array} + \tfrac{1}{\sqrt{6}}\ \begin{array}{c}a\\ d\\ e\end{array} + \tfrac{1}{\sqrt{6}}\ \begin{array}{c}b\\ c\\ e\end{array}\right)$$

$$= -\sqrt{\tfrac{1}{6}}\ \begin{array}{c}b\\ c\\ d\end{array} \quad -\sqrt{\tfrac{1}{6}}\ \begin{array}{c}a\\ b\\ f\end{array} \qquad \tfrac{1}{\sqrt{3}}\ \begin{array}{c}b\\ c\\ e\end{array} \quad -\tfrac{1}{\sqrt{3}}\ \begin{array}{c}a\\ c\\ f\end{array}$$

	$\begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\ \begin{array}{l}\uparrow\uparrow\\ \downarrow\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\ \begin{array}{l}\uparrow\uparrow\\ \downarrow\end{array}$	$\begin{array}{\|c\|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}\ \uparrow\uparrow\downarrow$
a c f $\begin{array}{\|c\|}\hline 1\uparrow \\ \hline 2\uparrow \\ \hline 3\downarrow \\ \hline\end{array}$	0	$\frac{-2}{\sqrt{6}}$	$\frac{1}{\sqrt{3}}$
a d e $\begin{array}{\|c\|}\hline 1\uparrow \\ \hline 2\downarrow \\ \hline 3\uparrow \\ \hline\end{array}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{3}}$
b c e $\begin{array}{\|c\|}\hline 1\downarrow \\ \hline 2\uparrow \\ \hline 3\uparrow \\ \hline\end{array}$	$\frac{-1}{\sqrt{2}}$	$\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{3}}$

ℓ=1 p=shell LSJ states transformed to Slater determinants from J=3/2 at L=1 $\mathrm{M_J}$=1/2

$$\left|{}^2P_{J=\frac{3}{2}}{}^{\frac{3}{2}}\right\rangle = \left|p^{L=1}_{M=1}\chi^{1/2}_{+1/2}\right\rangle \qquad \text{Doublet } {}^2P,\ \mathrm{J}{=}\tfrac{3}{2}\ \mathrm{M_J}{=}\tfrac{3}{2}$$

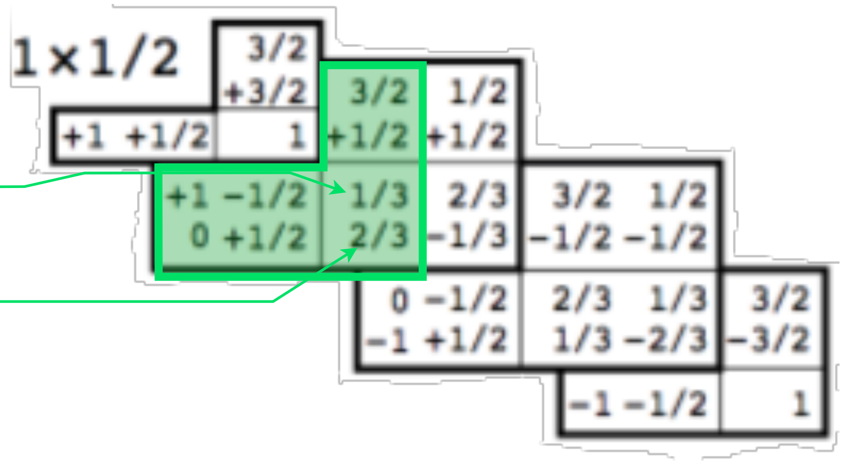
$$= \sqrt{\tfrac{1}{2}}\ \begin{array}{l}\boxed{1}\boxed{2}\\ \boxed{2}\end{array}\ \begin{array}{l}\uparrow\uparrow\\ \downarrow\end{array} \quad -\sqrt{\tfrac{1}{2}}\ \begin{array}{l}\boxed{1}\boxed{1}\\ \boxed{3}\end{array}\ \begin{array}{l}\uparrow\uparrow\\ \downarrow\end{array}$$

$$= -\sqrt{\tfrac{1}{2}}\ \begin{array}{c}a\\ c\\ d\end{array} \quad -\sqrt{\tfrac{1}{2}}\ \begin{array}{c}a\\ b\\ e\end{array}$$

$$\left|{}^2P_{J=\frac{3}{2}}{}^{\frac{1}{2}}\right\rangle = \sqrt{\tfrac{1}{3}}\left|p^{L=1}_{M=1}\chi^{1/2}_{-1/2}\right\rangle + \sqrt{\tfrac{2}{3}}\left|p^{L=1}_{M=0}\chi^{1/2}_{+1/2}\right\rangle \qquad \text{Doublet } {}^2P,\ \mathrm{J}{=}\tfrac{3}{2}\ \mathrm{M_J}{=}\tfrac{1}{2}$$

$$= \sqrt{\tfrac{1}{6}}\ \begin{array}{l}\boxed{1}\boxed{2}\\ \boxed{2}\end{array}\ \begin{array}{l}\uparrow\downarrow\\ \downarrow\end{array} -\sqrt{\tfrac{1}{6}}\ \begin{array}{l}\boxed{1}\boxed{1}\\ \boxed{3}\end{array}\ \begin{array}{l}\uparrow\downarrow\\ \downarrow\end{array} \quad -\sqrt{\tfrac{1}{6}}\ \begin{array}{l}\boxed{1}\boxed{2}\\ \boxed{3}\end{array}\ \begin{array}{l}\uparrow\uparrow\\ \downarrow\end{array} + \tfrac{1}{\sqrt{2}}\ \begin{array}{l}\boxed{1}\boxed{3}\\ \boxed{2}\end{array}\ \begin{array}{l}\uparrow\uparrow\\ \downarrow\end{array}$$

$$= -\sqrt{\tfrac{1}{6}}\ \begin{array}{c}b\\ c\\ d\end{array} \quad -\sqrt{\tfrac{1}{6}}\ \begin{array}{c}a\\ b\\ f\end{array} \quad \tfrac{1}{\sqrt{3}}\ \begin{array}{c}b\\ c\\ e\end{array} \quad -\tfrac{1}{\sqrt{3}}\ \begin{array}{c}a\\ c\\ f\end{array}$$



1×1/2	3/2 +3/2	3/2 +1/2	1/2 +1/2	3/2 −1/2	1/2 −1/2	3/2 −3/2
+1 +1/2	1					
+1 −1/2		1/3	2/3			
0 +1/2		2/3	−1/3			
0 −1/2				2/3	1/3	
−1 +1/2				1/3	−2/3	
−1 −1/2						1

	$\begin{array}{l}\boxed{1}\boxed{2}\\ \boxed{3}\end{array}\ \begin{array}{l}\uparrow\uparrow\\ \downarrow\end{array}$	$\begin{array}{l}\boxed{1}\boxed{3}\\ \boxed{2}\end{array}\ \begin{array}{l}\uparrow\uparrow\\ \downarrow\end{array}$	$\begin{array}{l}\boxed{1}\\ \boxed{2}\\ \boxed{3}\end{array}\ \uparrow\uparrow\downarrow$
a $\boxed{1\uparrow}$ c $\boxed{2\uparrow}$ f $\boxed{3\downarrow}$	0	$\frac{-2}{\sqrt{6}}$	$\frac{1}{\sqrt{3}}$
a $\boxed{1\uparrow}$ d $\boxed{2\downarrow}$ e $\boxed{3\uparrow}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{3}}$
b $\boxed{1\downarrow}$ c $\boxed{2\uparrow}$ e $\boxed{3\uparrow}$	$\frac{-1}{\sqrt{2}}$	$\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{3}}$

AMOP reference links on pages 2-4

4.25.18 class 26: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

$(S_3)^*(U(3)) \subset U(6)$ models of p^3 electronic spin-orbit states and couplings

[2,1] tableau states lowered by $\mathbf{L}_- = \sqrt{2}(E_{21}+E_{32})$

Top-(J,M) states to mid-level states

$\ell=1$ p=shell LS states combined to states of definite J

J=3/2 at L=0 (^4S), J=5/2 at L=2 (^2D)

C-G coupling; J=3/2 at L=2 (^2D), J=3/2 at L=1 (^2P), J=1/2 at L=1 (^2P)

Spin-orbit state assembly formula and Slater determinants

Extra assembly table

$\ell=1$ p=shell LSJ states transformed to Slater determinants from J=3/2 (^4S)

Slater functions for J=5/2, J=3/2 (^2D)

Slater functions for J=3/2 (^2P), J=1/2 (^2P)

Summary of states and level connection paths

Symmetry dimension accounting

Spin-orbit Hamiltonian matrix calculation

Application to spin-orbit and entanglement break-up scattering

ℓ=1 p=shell LSJ states transformed to Slater determinants from J=1/2 at L=1 M$_J$=1/2

$$\left|{}^2P^{\frac32}_{J=\frac32}\right\rangle = \left|p^{L=1}_{M=1}\chi^{1/2}_{+1/2}\right\rangle$$

Doublet ${}^2P$, J=$\frac32$ M$_J$=$\frac32$

$$= \sqrt{\tfrac12}\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\begin{array}{c}\uparrow\uparrow\\ \downarrow\end{array} \quad -\sqrt{\tfrac12}\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\begin{array}{c}\uparrow\uparrow\\ \downarrow\end{array}$$

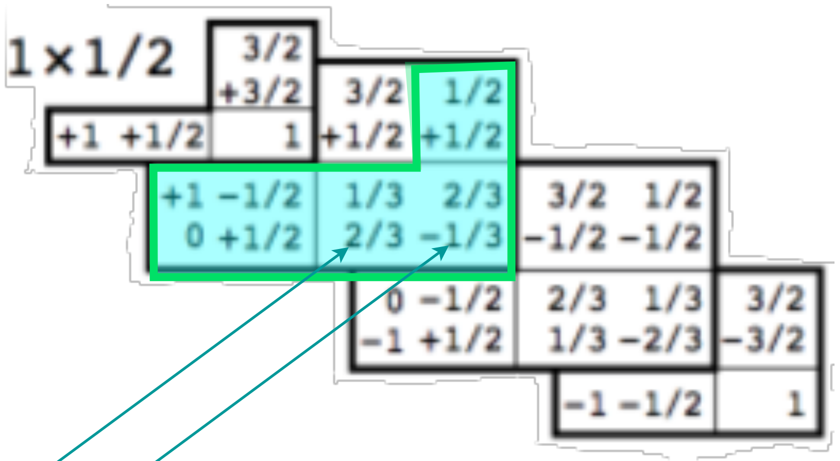
$$= -\sqrt{\tfrac12}\begin{array}{c}a\\ c\\ d\end{array} \quad -\sqrt{\tfrac12}\begin{array}{c}a\\ b\\ e\end{array}$$

$$\left|{}^2P^{\frac12}_{J=\frac32}\right\rangle = \sqrt{\tfrac13}\left|p^{L=1}_{M=1}\chi^{1/2}_{-1/2}\right\rangle + \sqrt{\tfrac23}\left|p^{L=1}_{M=0}\chi^{1/2}_{+1/2}\right\rangle$$

Doublet ${}^2P$, J=$\frac32$ M$_J$=$\frac12$

$$= \sqrt{\tfrac16}\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\begin{array}{c}\uparrow\downarrow\\ \downarrow\end{array} - \sqrt{\tfrac16}\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\begin{array}{c}\uparrow\downarrow\\ \downarrow\end{array} \quad - \sqrt{\tfrac16}\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\begin{array}{c}\uparrow\uparrow\\ \downarrow\end{array} + \tfrac{1}{\sqrt2}\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\begin{array}{c}\uparrow\uparrow\\ \downarrow\end{array}$$

$$= -\sqrt{\tfrac16}\begin{array}{c}b\\ c\\ d\end{array} \quad -\sqrt{\tfrac16}\begin{array}{c}a\\ b\\ f\end{array} \quad \tfrac{1}{\sqrt3}\begin{array}{c}b\\ c\\ e\end{array} \quad -\tfrac{1}{\sqrt3}\begin{array}{c}a\\ c\\ f\end{array}$$



1×1/2	3/2 +3/2	3/2 +1/2	1/2 +1/2	3/2 −1/2	1/2 −1/2	3/2 −3/2
+1 +1/2	1					
+1 −1/2		1/3	2/3			
0 +1/2		2/3	−1/3			
0 −1/2				2/3	1/3	
−1 +1/2				1/3	−2/3	
−1 −1/2						1

$$\left|{}^2P^{\frac12}_{J=\frac12}\right\rangle = \underline{\sqrt{\tfrac23}\left|p^{L=1}_{M=1}\chi^{1/2}_{-1/2}\right\rangle - \sqrt{\tfrac13}\left|p^{L=1}_{M=0}\chi^{1/2}_{+1/2}\right\rangle}$$

Doublet ${}^2P$, $\underline{\text{J}=\frac12\ \text{M}_J=\frac12}$

ℓ=1 p=shell LSJ states transformed to Slater determinants from J=1/2 at L=1

MJ=1/2

$$\left|{}^2P^{\frac{3}{2}}_{J=\frac{3}{2}}\right\rangle=\left|p^{L=1}_{M=1}\chi^{1/2}_{+1/2}\right\rangle$$

Doublet ${}^2P$, J=$\frac{3}{2}$ M$_J$=$\frac{3}{2}$

$$=\sqrt{\tfrac{1}{2}}\begin{array}{ll}\boxed{1}&\boxed{2}\\\boxed{2}&\end{array}\begin{array}{l}\uparrow\uparrow\\\downarrow\end{array}\quad-\sqrt{\tfrac{1}{2}}\begin{array}{ll}\boxed{1}&\boxed{1}\\\boxed{3}&\end{array}\begin{array}{l}\uparrow\uparrow\\\downarrow\end{array}$$

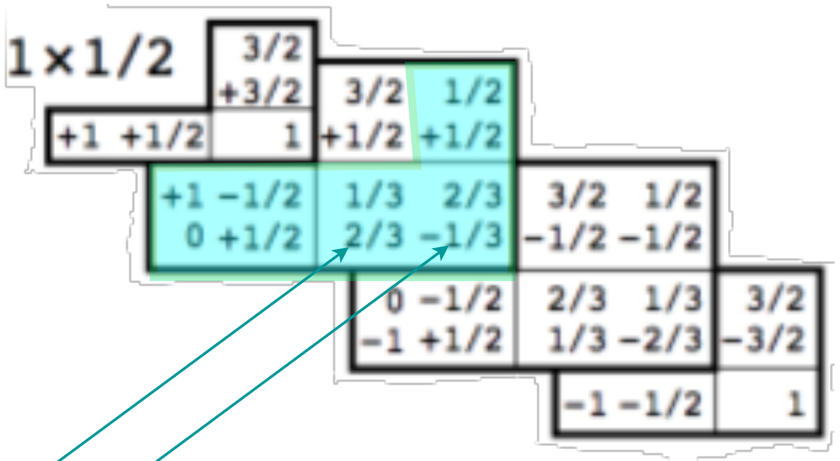
$$=-\sqrt{\tfrac{1}{2}}\begin{array}{c}a\\c\\d\end{array}\quad-\sqrt{\tfrac{1}{2}}\begin{array}{c}a\\b\\e\end{array}$$

$$\left|{}^2P^{\frac{1}{2}}_{J=\frac{3}{2}}\right\rangle=\sqrt{\tfrac{1}{3}}\left|p^{L=1}_{M=1}\chi^{1/2}_{-1/2}\right\rangle+\sqrt{\tfrac{2}{3}}\left|p^{L=1}_{M=0}\chi^{1/2}_{+1/2}\right\rangle$$

Doublet ${}^2P$, J=$\frac{3}{2}$ M$_J$=$\frac{1}{2}$

$$=\sqrt{\tfrac{1}{6}}\begin{array}{ll}\boxed{1}&\boxed{2}\\\boxed{2}&\end{array}\begin{array}{l}\uparrow\downarrow\\\downarrow\end{array}-\sqrt{\tfrac{1}{6}}\begin{array}{ll}\boxed{1}&\boxed{1}\\\boxed{3}&\end{array}\begin{array}{l}\uparrow\downarrow\\\downarrow\end{array}\quad-\sqrt{\tfrac{1}{6}}\begin{array}{ll}\boxed{1}&\boxed{2}\\\boxed{3}&\end{array}\begin{array}{l}\uparrow\uparrow\\\downarrow\end{array}+\tfrac{1}{\sqrt{2}}\begin{array}{ll}\boxed{1}&\boxed{3}\\\boxed{2}&\end{array}\begin{array}{l}\uparrow\uparrow\\\downarrow\end{array}$$

$$=-\sqrt{\tfrac{1}{6}}\begin{array}{c}b\\c\\d\end{array}\quad-\sqrt{\tfrac{1}{6}}\begin{array}{c}a\\b\\f\end{array}\quad\tfrac{1}{\sqrt{3}}\begin{array}{c}b\\c\\e\end{array}\quad-\tfrac{1}{\sqrt{3}}\begin{array}{c}a\\c\\f\end{array}$$



1×1/2

		3/2
		+3/2
+1	+1/2	1

		3/2	1/2
		+1/2	+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

		3/2	1/2
		−1/2	−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

		3/2
		−3/2
−1	−1/2	1

$$\left|{}^2P^{\frac{1}{2}}_{J=\frac{1}{2}}\right\rangle=\underline{\sqrt{\tfrac{2}{3}}\left|p^{L=1}_{M=1}\chi^{1/2}_{-1/2}\right\rangle-\sqrt{\tfrac{1}{3}}\left|p^{L=1}_{M=0}\chi^{1/2}_{+1/2}\right\rangle}$$

Doublet ${}^2P$, $\underline{\text{J}=\frac{1}{2}\ \text{M}_J=\frac{1}{2}}$

$$=\sqrt{\tfrac{2}{3}}\left|\left[\sqrt{\tfrac{1}{2}}\begin{array}{ll}\boxed{1}&\boxed{2}\\\boxed{2}&\end{array}\begin{array}{l}\uparrow\downarrow\\\downarrow\end{array}-\sqrt{\tfrac{1}{2}}\begin{array}{ll}\boxed{1}&\boxed{1}\\\boxed{3}&\end{array}\begin{array}{l}\uparrow\downarrow\\\downarrow\end{array}\right]\right\rangle-\sqrt{\tfrac{1}{3}}\left|\left[\tfrac{-1}{2}\begin{array}{ll}\boxed{1}&\boxed{2}\\\boxed{3}&\end{array}\begin{array}{l}\uparrow\uparrow\\\downarrow\end{array}+\tfrac{\sqrt{3}}{2}\begin{array}{ll}\boxed{1}&\boxed{3}\\\boxed{2}&\end{array}\begin{array}{l}\uparrow\uparrow\\\downarrow\end{array}\right]\right\rangle$$

ℓ=1 p=shell LSJ states transformed to Slater determinants from J=1/2 at L=1 M$_J$=1/2

$$\left|{}^2P_{J=\frac{3}{2}}\ {}_{\frac{3}{2}}\right\rangle=\left|p^{L=1}_{M=1}\chi^{1/2}_{+1/2}\right\rangle$$

Doublet ${}^2P$, J=$\frac{3}{2}$ M$_J$=$\frac{3}{2}$

$$=\sqrt{\tfrac{1}{2}}\ \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\uparrow\\\downarrow\end{smallmatrix}\qquad-\sqrt{\tfrac{1}{2}}\ \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\uparrow\\\downarrow\end{smallmatrix}$$

$$=\ -\sqrt{\tfrac{1}{2}}\ \begin{matrix}a\\c\\d\end{matrix}\qquad-\ \sqrt{\tfrac{1}{2}}\ \begin{matrix}a\\b\\e\end{matrix}$$

$$\left|{}^2P_{J=\frac{3}{2}}\ {}_{\frac{1}{2}}\right\rangle=\sqrt{\tfrac{1}{3}}\left|p^{L=1}_{M=1}\chi^{1/2}_{-1/2}\right\rangle+\sqrt{\tfrac{2}{3}}\left|p^{L=1}_{M=0}\chi^{1/2}_{+1/2}\right\rangle$$

Doublet ${}^2P$, J=$\frac{3}{2}$ M$_J$=$\frac{1}{2}$

$$=\sqrt{\tfrac{1}{6}}\ \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\downarrow\\\downarrow\end{smallmatrix}\ -\sqrt{\tfrac{1}{6}}\ \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\downarrow\\\downarrow\end{smallmatrix}\qquad-\sqrt{\tfrac{1}{6}}\ \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\uparrow\\\downarrow\end{smallmatrix}\ +\tfrac{1}{\sqrt{2}}\ \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\uparrow\\\downarrow\end{smallmatrix}$$

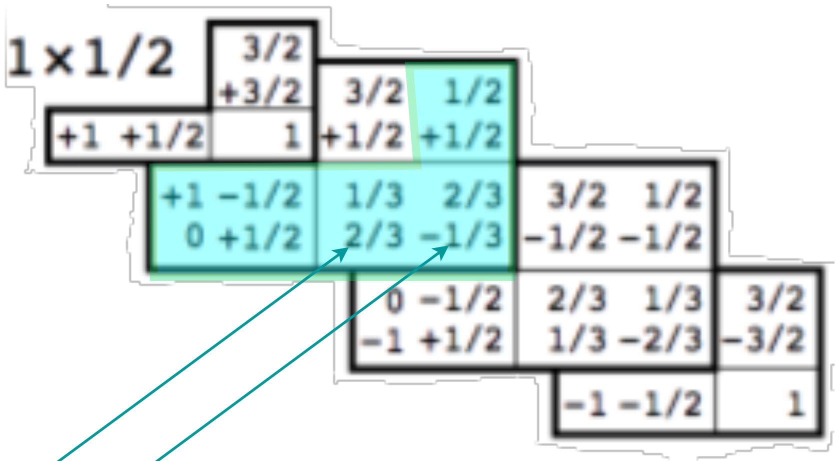
$$=\ -\sqrt{\tfrac{1}{6}}\ \begin{matrix}b\\c\\d\end{matrix}\qquad-\sqrt{\tfrac{1}{6}}\ \begin{matrix}a\\b\\f\end{matrix}\qquad\tfrac{1}{\sqrt{3}}\ \begin{matrix}b\\c\\e\end{matrix}\qquad-\tfrac{1}{\sqrt{3}}\ \begin{matrix}a\\c\\f\end{matrix}$$

$$\left|{}^2P_{J=\frac{1}{2}}\ {}_{\frac{1}{2}}\right\rangle=\underline{\sqrt{\tfrac{2}{3}}\left|p^{L=1}_{M=1}\chi^{1/2}_{-1/2}\right\rangle-\sqrt{\tfrac{1}{3}}\left|p^{L=1}_{M=0}\chi^{1/2}_{+1/2}\right\rangle}$$

Doublet ${}^2P$, J=$\frac{1}{2}$ M$_J$=$\frac{1}{2}$

$$=\sqrt{\tfrac{2}{3}}\left|\left[\sqrt{\tfrac{1}{2}}\ \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\downarrow\\\downarrow\end{smallmatrix}\ -\sqrt{\tfrac{1}{2}}\ \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\downarrow\\\downarrow\end{smallmatrix}\ \right]\right\rangle-\sqrt{\tfrac{1}{3}}\left|\left[\tfrac{-1}{2}\ \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\uparrow\\\downarrow\end{smallmatrix}\ +\tfrac{\sqrt{3}}{2}\ \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\uparrow\\\downarrow\end{smallmatrix}\ \right]\right\rangle$$

$$=\sqrt{\tfrac{1}{3}}\ \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\downarrow\\\downarrow\end{smallmatrix}\ -\sqrt{\tfrac{1}{3}}\ \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\downarrow\\\downarrow\end{smallmatrix}\qquad+\sqrt{\tfrac{1}{12}}\ \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\uparrow\\\downarrow\end{smallmatrix}\ -\tfrac{1}{2}\ \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\ \begin{smallmatrix}\uparrow\uparrow\\\downarrow\end{smallmatrix}$$



1×1/2	3/2 +3/2					
+1 +1/2	1	3/2 +1/2	1/2 +1/2			
+1 −1/2 0 +1/2		1/3 2/3	2/3 −1/3	3/2 −1/2	1/2 −1/2	
0 −1/2 −1 +1/2				2/3 1/3	1/3 −2/3	3/2 −3/2
−1 −1/2						1

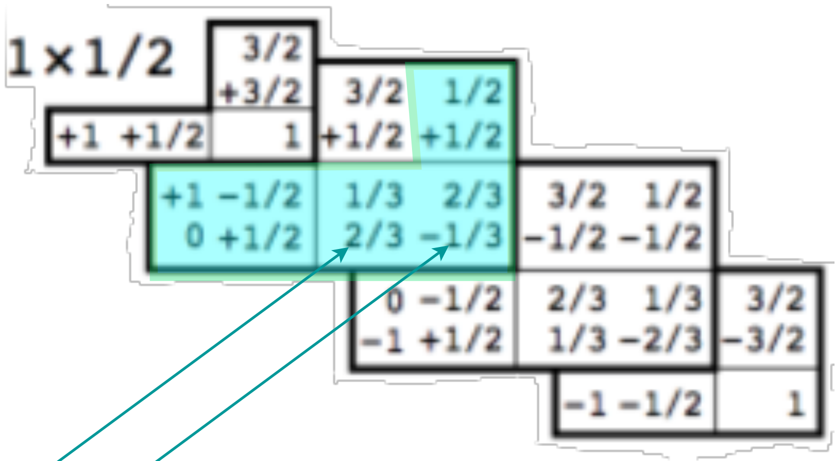
ℓ=1 p=shell LSJ states transformed to Slater determinants from J=1/2 at L=1 M$_J$=1/2

$$\left|{}^2P^{\,\frac{3}{2}}_{J=\frac{3}{2}}\right\rangle = \left|p^{L=1}_{M=1}\chi^{1/2}_{+1/2}\right\rangle$$

Doublet ${}^2P$, J=$\frac{3}{2}$ M$_J$=$\frac{3}{2}$

$$= \sqrt{\tfrac{1}{2}}\begin{array}{ll}\boxed{1}&\boxed{2}\\\boxed{2}&\end{array}\begin{array}{c}\uparrow\uparrow\\\downarrow\end{array} \quad -\sqrt{\tfrac{1}{2}}\begin{array}{ll}\boxed{1}&\boxed{1}\\\boxed{3}&\end{array}\begin{array}{c}\uparrow\uparrow\\\downarrow\end{array}$$

$$= -\sqrt{\tfrac{1}{2}}\begin{array}{c}a\\c\\d\end{array} \quad -\sqrt{\tfrac{1}{2}}\begin{array}{c}a\\b\\e\end{array}$$


1×1/2

		3/2 +3/2	
+1 +1/2		1	
		3/2 +1/2	1/2 +1/2
+1 −1/2 0 +1/2		1/3 2/3	2/3 −1/3
		3/2 −1/2	1/2 −1/2
0 −1/2 −1 +1/2		2/3 1/3	1/3 −2/3
		3/2 −3/2	
−1 −1/2		1	

$$\left|{}^2P^{\,\frac{1}{2}}_{J=\frac{3}{2}}\right\rangle = \sqrt{\tfrac{1}{3}}\left|p^{L=1}_{M=1}\chi^{1/2}_{-1/2}\right\rangle + \sqrt{\tfrac{2}{3}}\left|p^{L=1}_{M=0}\chi^{1/2}_{+1/2}\right\rangle$$

Doublet ${}^2P$, J=$\frac{3}{2}$ M$_J$=$\frac{1}{2}$

$$= \sqrt{\tfrac{1}{6}}\begin{array}{ll}\boxed{1}&\boxed{2}\\\boxed{2}&\end{array}\begin{array}{c}\uparrow\downarrow\\\downarrow\end{array} -\sqrt{\tfrac{1}{6}}\begin{array}{ll}\boxed{1}&\boxed{1}\\\boxed{3}&\end{array}\begin{array}{c}\uparrow\downarrow\\\downarrow\end{array} \quad -\sqrt{\tfrac{1}{6}}\begin{array}{ll}\boxed{1}&\boxed{2}\\\boxed{3}&\end{array}\begin{array}{c}\uparrow\uparrow\\\downarrow\end{array} + \tfrac{1}{\sqrt{2}}\begin{array}{ll}\boxed{1}&\boxed{3}\\\boxed{2}&\end{array}\begin{array}{c}\uparrow\uparrow\\\downarrow\end{array}$$

$$= -\sqrt{\tfrac{1}{6}}\begin{array}{c}b\\c\\d\end{array} \quad -\sqrt{\tfrac{1}{6}}\begin{array}{c}a\\b\\f\end{array} \quad \tfrac{1}{\sqrt{3}}\begin{array}{c}b\\c\\e\end{array} \quad -\tfrac{1}{\sqrt{3}}\begin{array}{c}a\\c\\f\end{array}$$

$$\left|{}^2P^{\,\frac{1}{2}}_{J=\frac{1}{2}}\right\rangle = \underline{\sqrt{\tfrac{2}{3}}\left|p^{L=1}_{M=1}\chi^{1/2}_{-1/2}\right\rangle - \sqrt{\tfrac{1}{3}}\left|p^{L=1}_{M=0}\chi^{1/2}_{+1/2}\right\rangle}$$

Doublet ${}^2P$, J=$\frac{1}{2}$ $\underline{\text{M}_J=\frac{1}{2}}$

$$= \sqrt{\tfrac{1}{3}}\begin{array}{ll}\boxed{1}&\boxed{2}\\\boxed{2}&\end{array}\begin{array}{c}\uparrow\downarrow\\\downarrow\end{array} -\sqrt{\tfrac{1}{3}}\begin{array}{ll}\boxed{1}&\boxed{1}\\\boxed{3}&\end{array}\begin{array}{c}\uparrow\downarrow\\\downarrow\end{array} \quad +\sqrt{\tfrac{1}{12}}\begin{array}{ll}\boxed{1}&\boxed{2}\\\boxed{3}&\end{array}\begin{array}{c}\uparrow\uparrow\\\downarrow\end{array} - \tfrac{1}{2}\begin{array}{ll}\boxed{1}&\boxed{3}\\\boxed{2}&\end{array}\begin{array}{c}\uparrow\uparrow\\\downarrow\end{array}$$

$$= -\sqrt{\tfrac{1}{3}}\begin{array}{c}b\\c\\d\end{array} \quad -\sqrt{\tfrac{1}{3}}\begin{array}{c}a\\b\\f\end{array} \quad -\tfrac{1}{\sqrt{6}}\begin{array}{c}b\\c\\e\end{array} \quad \tfrac{1}{\sqrt{6}}\begin{array}{c}a\\c\\f\end{array}$$

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4.25.18 class 26: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

$(S_3)^*(U(3))\subset U(6)$ models of p^3 electronic spin-orbit states and couplings

[2,1] tableau states lowered by $\mathbf{L}_- = \sqrt{2}(E_{21}+E_{32})$

Top-(J,M) states to mid-level states

ℓ=1 *p*=shell LS states combined to states of definite J

J=3/2 at L=0 (^{4}S), J=5/2 at L=2 (^{2}D)

C-G coupling; J=3/2 at L=2 (^{2}D), J=3/2 at L=1 (^{2}P), J=1/2 at L=1 (^{2}P)

Spin-orbit state assembly formula and Slater determinants

Extra assembly table

ℓ=1 *p*=shell LSJ states transformed to Slater determinants from J=3/2 (^{4}S)

Slater functions for J=5/2, J=3/2 (^{2}D)

Slater functions for J=3/2 (^{2}P), J=1/2 (^{2}P)

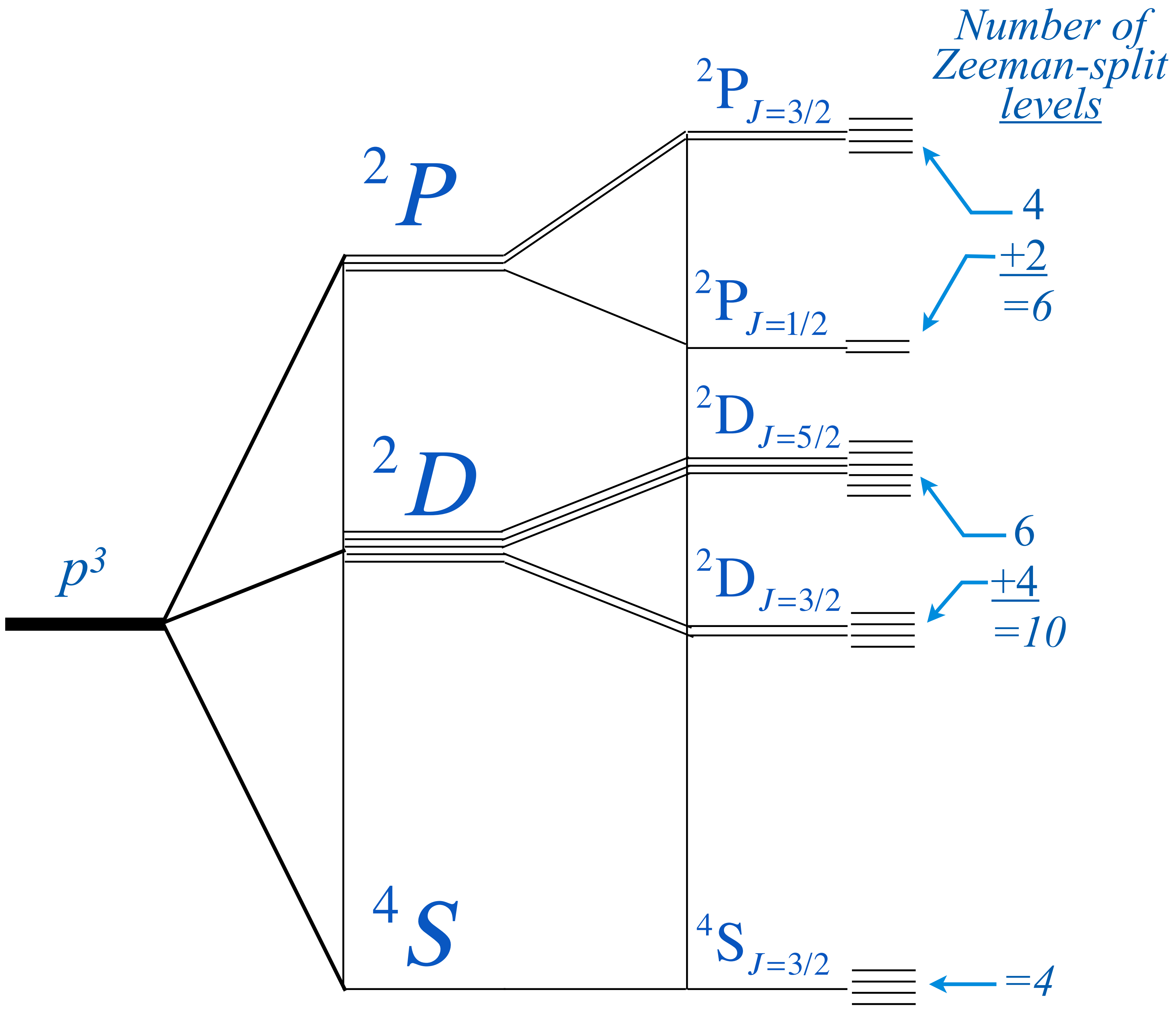
➡ Summary of states and level connection paths

Symmetry dimension accounting

Spin-orbit Hamiltonian matrix calculation

Application to spin-orbit and entanglement break-up scattering

$\ell=1$ $p^3=$ spin-orbit levels and Slater states



= 6+10+4

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$(S_3)^*(U(3)) \subset U(6)$ models of p^3 electronic spin-orbit states and couplings

[2,1] tableau states lowered by $\mathbf{L}_- = \sqrt{2}(E_{21}+E_{32})$
  Top-(J,M) states   to mid-level states
$\ell=1$ p=shell LS states combined to states of definite J
J=3/2 at L=0 (^4S),   J=5/2 at L=2 (^2D)
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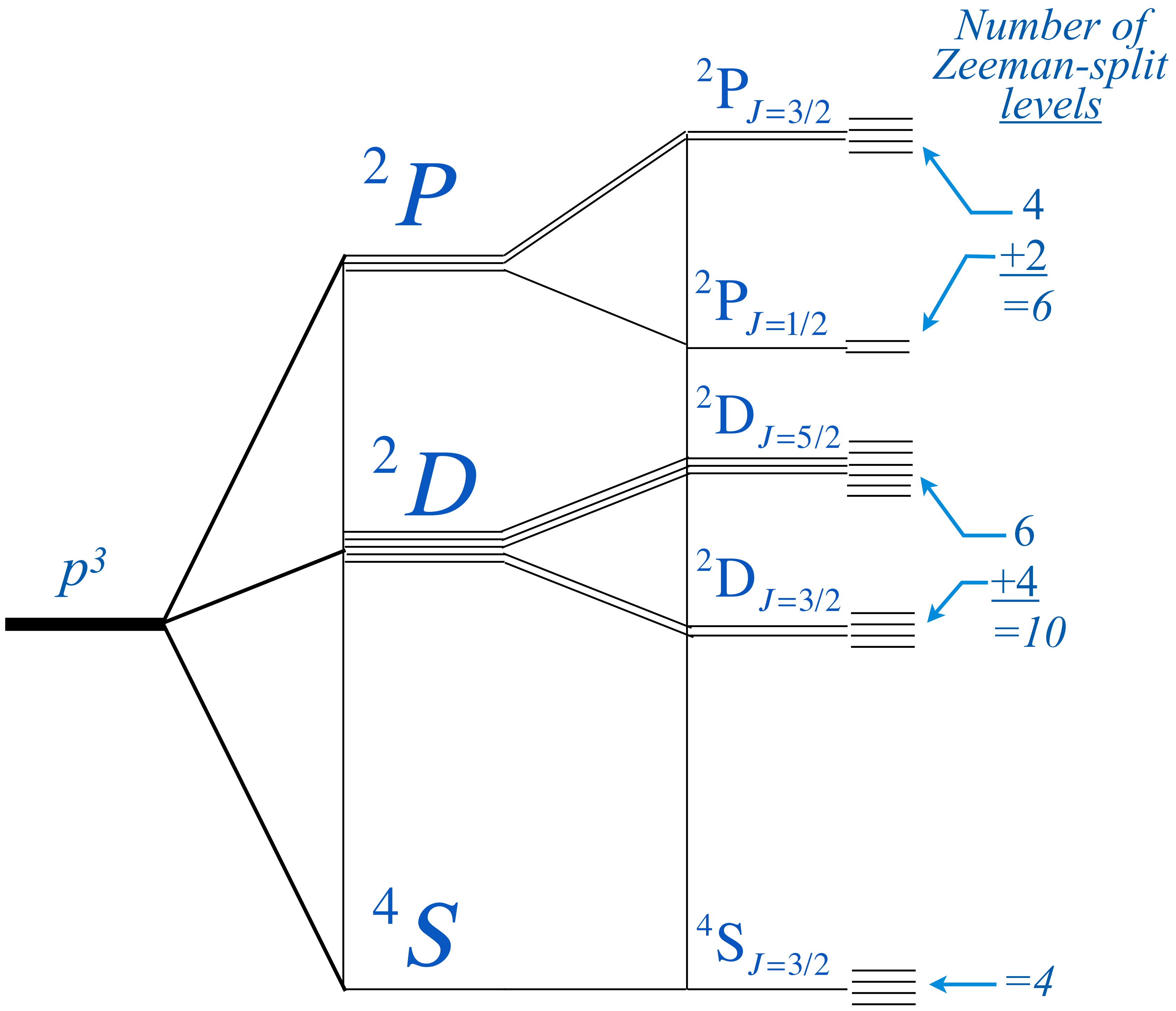
ℓ=1 p^3= spin-orbit levels and Slater states

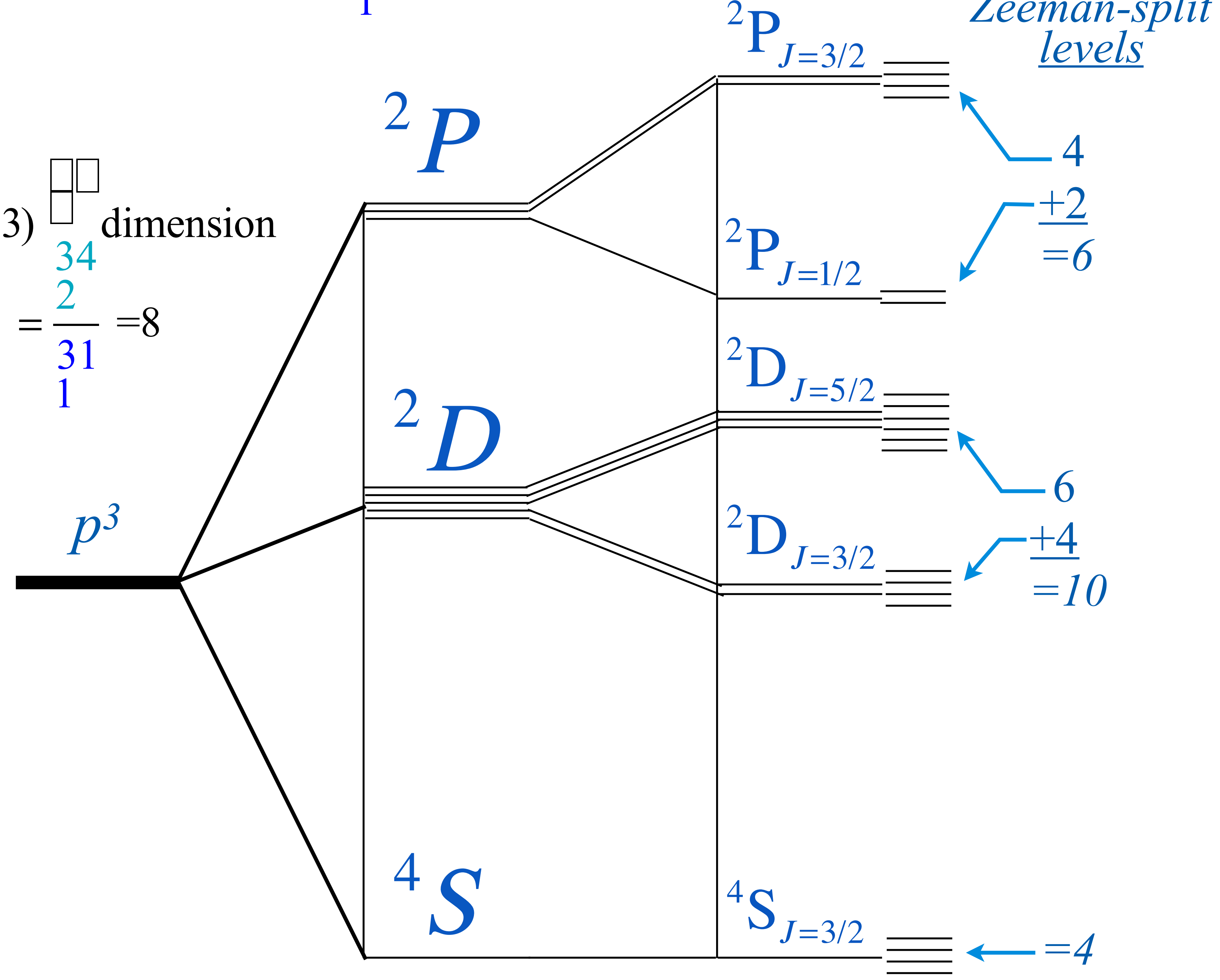


$$U(6)\text{ dimension grand total: } N = \frac{\begin{matrix}6\\5\\4\end{matrix}}{\begin{matrix}3\\2\\1\end{matrix}} = \frac{120}{6} = 20 = 6+10+4$$

$\ell=1$ $p^3=$ spin-orbit levels and Slater states

$U(2)\ \begin{smallmatrix}\square\square\\ \square\end{smallmatrix}$ dimension $= \dfrac{\begin{smallmatrix}2\,3\\ 1\end{smallmatrix}}{\begin{smallmatrix}3\,1\\ 1\end{smallmatrix}} = 2$

$U(3)\ \begin{smallmatrix}\square\square\\ \square\end{smallmatrix}$ dimension $= \dfrac{\begin{smallmatrix}3\,4\\ 2\end{smallmatrix}}{\begin{smallmatrix}3\,1\\ 1\end{smallmatrix}} = 8$



Number of Zeeman-split levels

p^3

${}^2P$

${}^2D$

${}^4S$

${}^2P_{J=3/2}$ — 4

${}^2P_{J=1/2}$ — +2 =6

${}^2D_{J=5/2}$ — 6

${}^2D_{J=3/2}$ — +4 =10

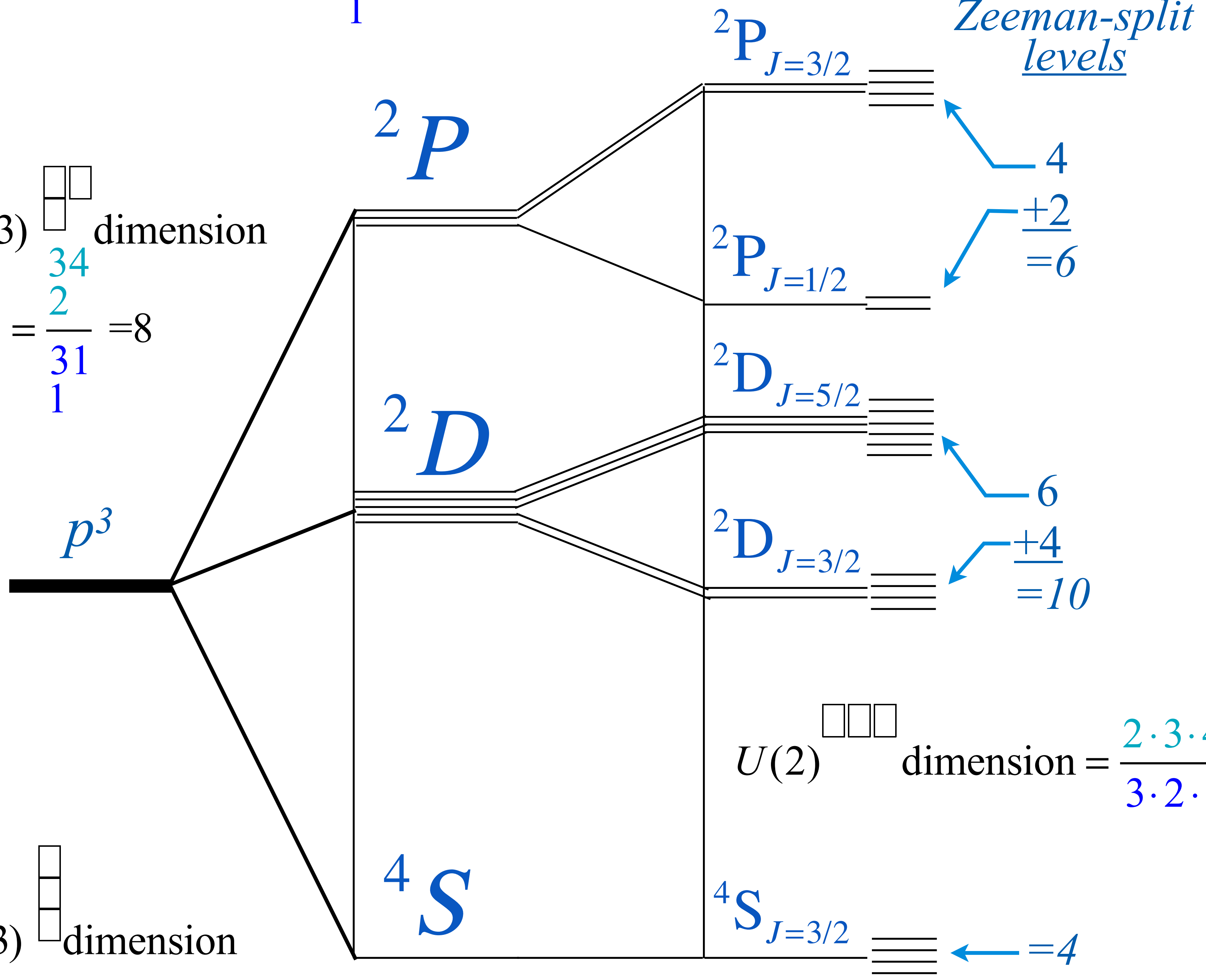
${}^4S_{J=3/2}$ — =4

$U(6)$ dimension grand total: $N = \dfrac{\begin{smallmatrix}6\\ 5\\ 4\end{smallmatrix}}{\begin{smallmatrix}3\\ 2\\ 1\end{smallmatrix}} = \dfrac{120}{6} = 20 = 6+10+4$

ℓ=1 p^3= spin-orbit levels and Slater states

$$U(2)\ \begin{matrix}\square\square\\ \square\ \end{matrix}\ \text{dimension} = \frac{\begin{matrix}23\\1\end{matrix}}{\begin{matrix}31\\1\end{matrix}} = 2$$

$$U(3)\ \begin{matrix}\square\square\\ \square\ \end{matrix}\ \text{dimension} = \frac{\begin{matrix}34\\2\end{matrix}}{\begin{matrix}31\\1\end{matrix}} = 8$$



$$U(2)\ \square\square\square\ \text{dimension} = \frac{2\cdot3\cdot4}{3\cdot2\cdot1} = 4$$

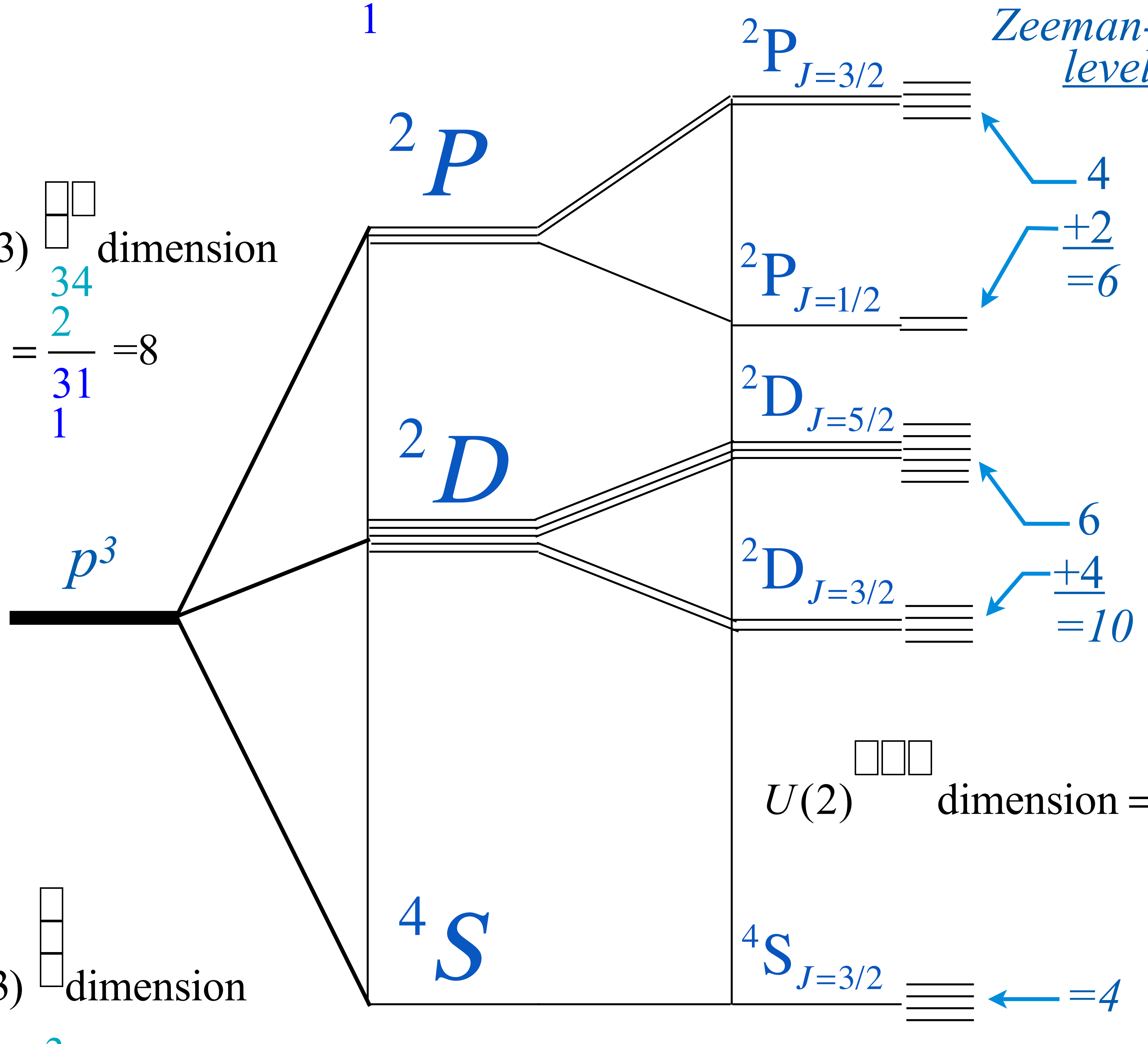
$$\left|{}^4S_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle = \left|\begin{matrix}a\\c\\e\end{matrix}\right\rangle$$

$$U(3)\ \begin{matrix}\square\\ \square\\ \square\end{matrix}\ \text{dimension} = \frac{\begin{matrix}3\\2\\1\end{matrix}}{\begin{matrix}3\\2\\1\end{matrix}} = 1$$

$$U(6)\ \text{dimension grand total: } N = \frac{\begin{matrix}6\\5\\4\end{matrix}}{\begin{matrix}3\\2\\1\end{matrix}} = \frac{120}{6} = 20 = 6+10+4$$

ℓ=1 p^3= spin-orbit levels and Slater states

$$U(2)\ \begin{array}{|c|c|}\hline &\\\hline &\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\ \text{dimension} = \frac{\begin{matrix}2&3\\1&\end{matrix}}{\begin{matrix}3&1\\1&\end{matrix}} = 2$$



$$U(3)\ \begin{array}{|c|c|}\hline &\\\hline &\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\ \text{dimension} = \frac{\begin{matrix}3&4\\2&\end{matrix}}{\begin{matrix}3&1\\1&\end{matrix}} = 8$$

$$\left|{}^2D_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle = \sqrt{\tfrac{4}{5}}\begin{matrix}a\\b\\d\end{matrix} + \sqrt{\tfrac{1}{10}}\begin{matrix}a\\c\\d\end{matrix} - \sqrt{\tfrac{1}{10}}\begin{matrix}a\\b\\e\end{matrix}$$

$$U(2)\ \begin{array}{|c|c|c|}\hline &&\\\hline\end{array}\ \text{dimension} = \frac{2\cdot 3\cdot 4}{3\cdot 2\cdot 1} = 4$$

$$U(3)\ \begin{array}{|c|}\hline \\\hline \\\hline \\\hline\end{array}\ \text{dimension} = \frac{\begin{matrix}3\\2\\1\end{matrix}}{\begin{matrix}3\\2\\1\end{matrix}} = 1$$

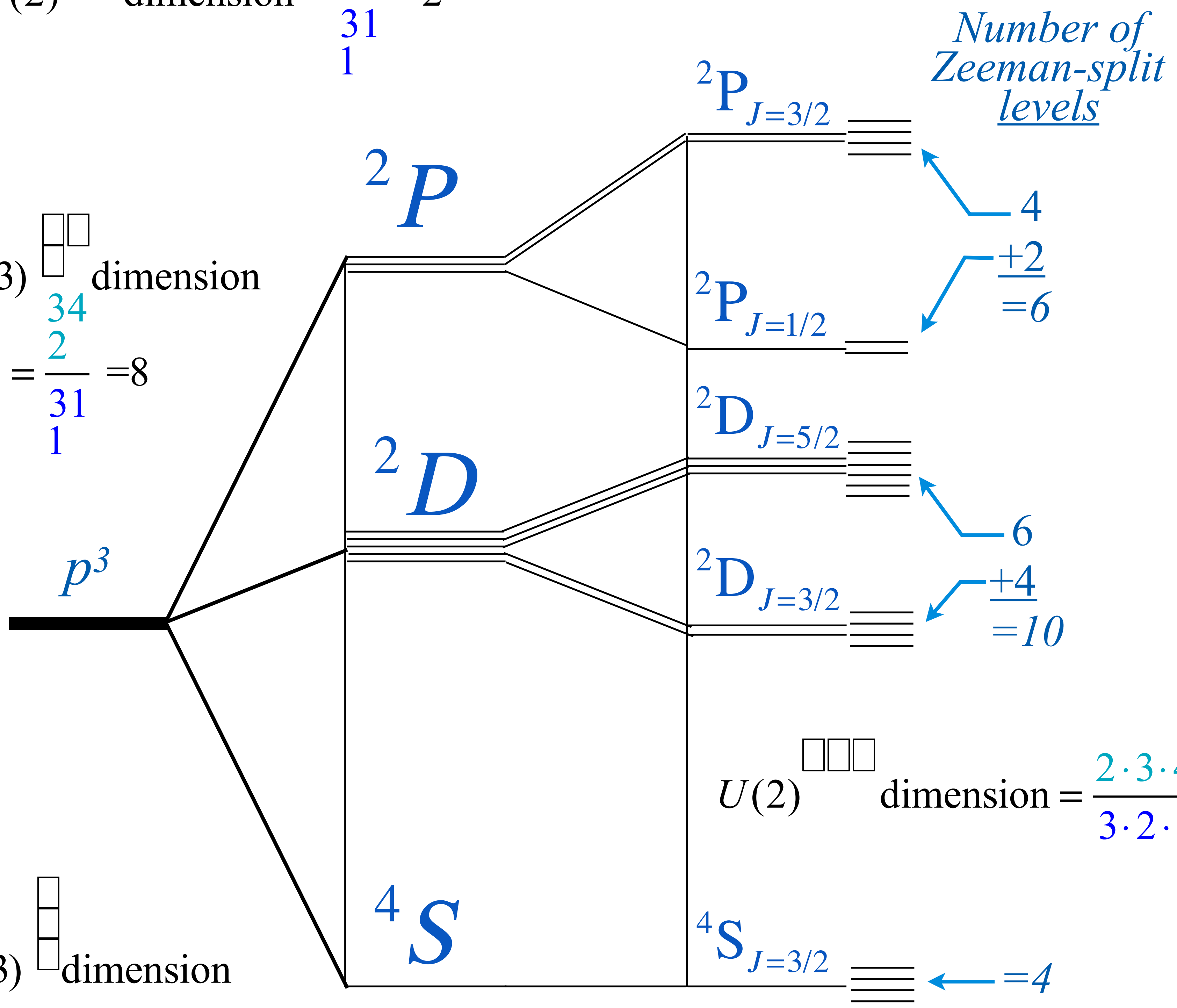
$$\left|{}^4S_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle = \left|\ \begin{matrix}a\\c\\e\end{matrix}\ \right\rangle$$

$$U(6)\ \text{dimension grand total: } N = \frac{\begin{matrix}6\\5\\4\end{matrix}}{\begin{matrix}3\\2\\1\end{matrix}} = \frac{120}{6} = 20 = 6+10+4$$

ℓ=1 p^3= spin-orbit levels and Slater states

$$U(2)^{\begin{smallmatrix}\square&\square\\\square\end{smallmatrix}}\ \text{dimension}=\frac{\begin{matrix}2&3\\1\end{matrix}}{\begin{matrix}3&1\\1\end{matrix}}=2$$

$$U(3)^{\begin{smallmatrix}\square&\square\\\square\end{smallmatrix}}\ \text{dimension}=\frac{\begin{matrix}3&4\\2\end{matrix}}{\begin{matrix}3&1\\1\end{matrix}}=8$$



$$U(2)^{\square\square\square}\ \text{dimension}=\frac{2\cdot3\cdot4}{3\cdot2\cdot1}=4$$

$$\left|{}^2D_{J=\frac{5}{2}}\ {}^{\frac{5}{2}}\right\rangle=\begin{matrix}a\\b\\c\end{matrix}$$

$$\left|{}^2D_{J=\frac{3}{2}}\ {}^{\frac{3}{2}}\right\rangle=\sqrt{\tfrac{4}{5}}\ \begin{matrix}a\\b\\d\end{matrix}\ +\sqrt{\tfrac{1}{10}}\ \begin{matrix}a\\c\\d\end{matrix}\ -\ \sqrt{\tfrac{1}{10}}\ \begin{matrix}a\\b\\e\end{matrix}$$

$$\left|{}^4S_{J=\frac{3}{2}}\ {}^{\frac{3}{2}}\right\rangle=\left|\ \begin{matrix}a\\c\\e\end{matrix}\right\rangle$$

$$U(3)^{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}\ \text{dimension}=\frac{\begin{matrix}3\\2\\1\end{matrix}}{\begin{matrix}3\\2\\1\end{matrix}}=1$$

$$U(6)\ \text{dimension grand total: }N=\frac{\begin{matrix}6\\5\\4\end{matrix}}{\begin{matrix}3\\2\\1\end{matrix}}=\frac{120}{6}=20=6+10+4$$

ℓ=1 p^3= spin-orbit levels and Slater states

$$U(2)\ \begin{array}{ll}\square&\square\\\square&\end{array}\ \text{dimension}=\frac{\begin{smallmatrix}2&3\\1&\end{smallmatrix}}{\begin{smallmatrix}3&1\\1&\end{smallmatrix}}=2$$

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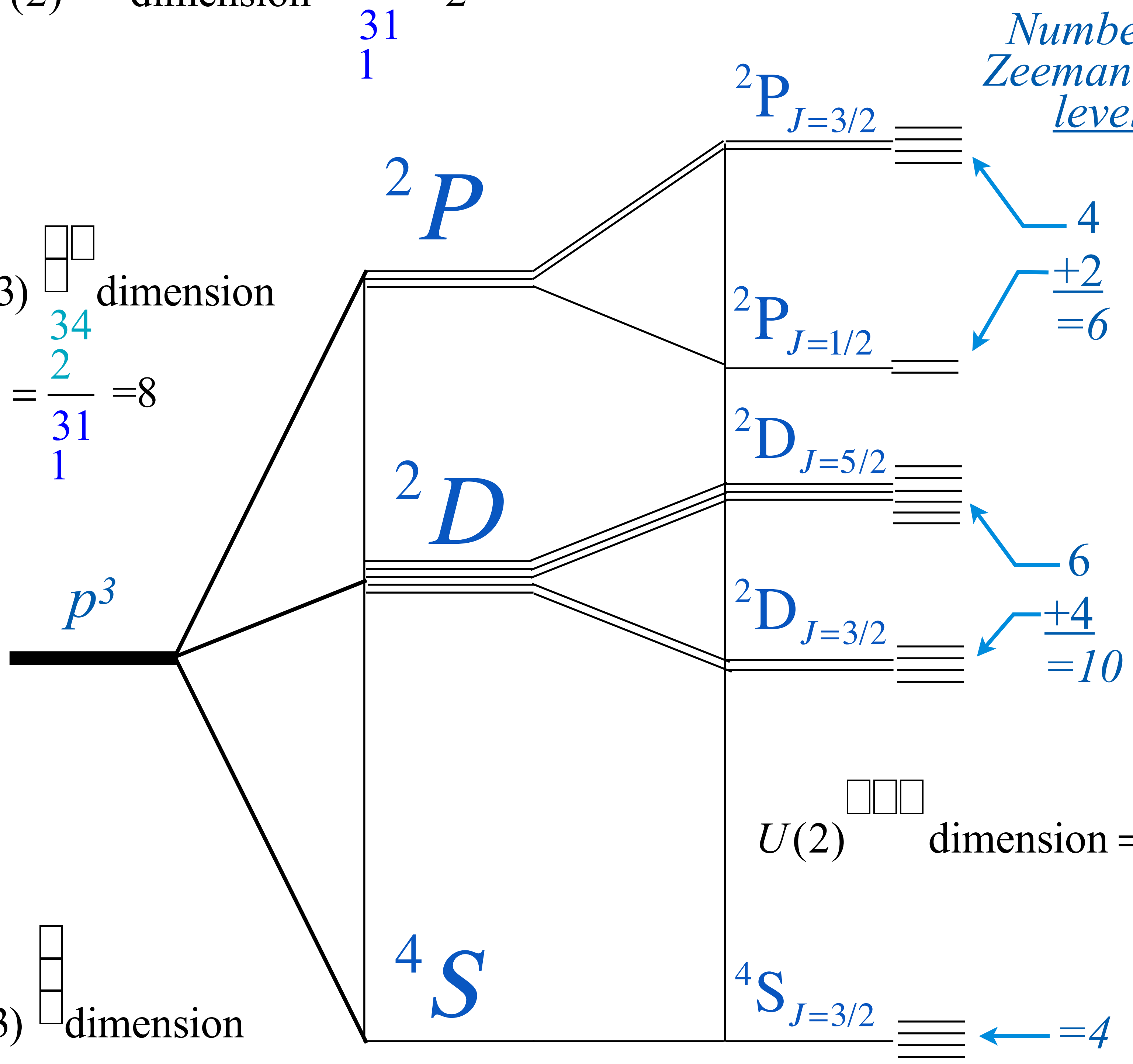


$$\left|{}^2P_{J=\frac{1}{2}}\ \tfrac{1}{2}\right\rangle=-\sqrt{\tfrac{1}{3}}\begin{smallmatrix}b\\c\\d\end{smallmatrix}-\sqrt{\tfrac{1}{3}}\begin{smallmatrix}a\\b\\f\end{smallmatrix}-\tfrac{1}{\sqrt{6}}\begin{smallmatrix}b\\c\\e\end{smallmatrix}+\tfrac{1}{\sqrt{6}}\begin{smallmatrix}a\\c\\f\end{smallmatrix}$$

$$\left|{}^2D_{J=\frac{5}{2}}\ \tfrac{5}{2}\right\rangle=\begin{smallmatrix}a\\b\\c\end{smallmatrix}$$

$$\left|{}^2D_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle=\sqrt{\tfrac{4}{5}}\begin{smallmatrix}a\\b\\d\end{smallmatrix}+\sqrt{\tfrac{1}{10}}\begin{smallmatrix}a\\c\\d\end{smallmatrix}-\sqrt{\tfrac{1}{10}}\begin{smallmatrix}a\\b\\e\end{smallmatrix}$$

$$U(2)\ \begin{array}{lll}\square&\square&\square\end{array}\ \text{dimension}=\frac{2\cdot3\cdot4}{3\cdot2\cdot1}=4$$

$$\left|{}^4S_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle=\left|\begin{smallmatrix}a\\c\\e\end{smallmatrix}\right\rangle$$

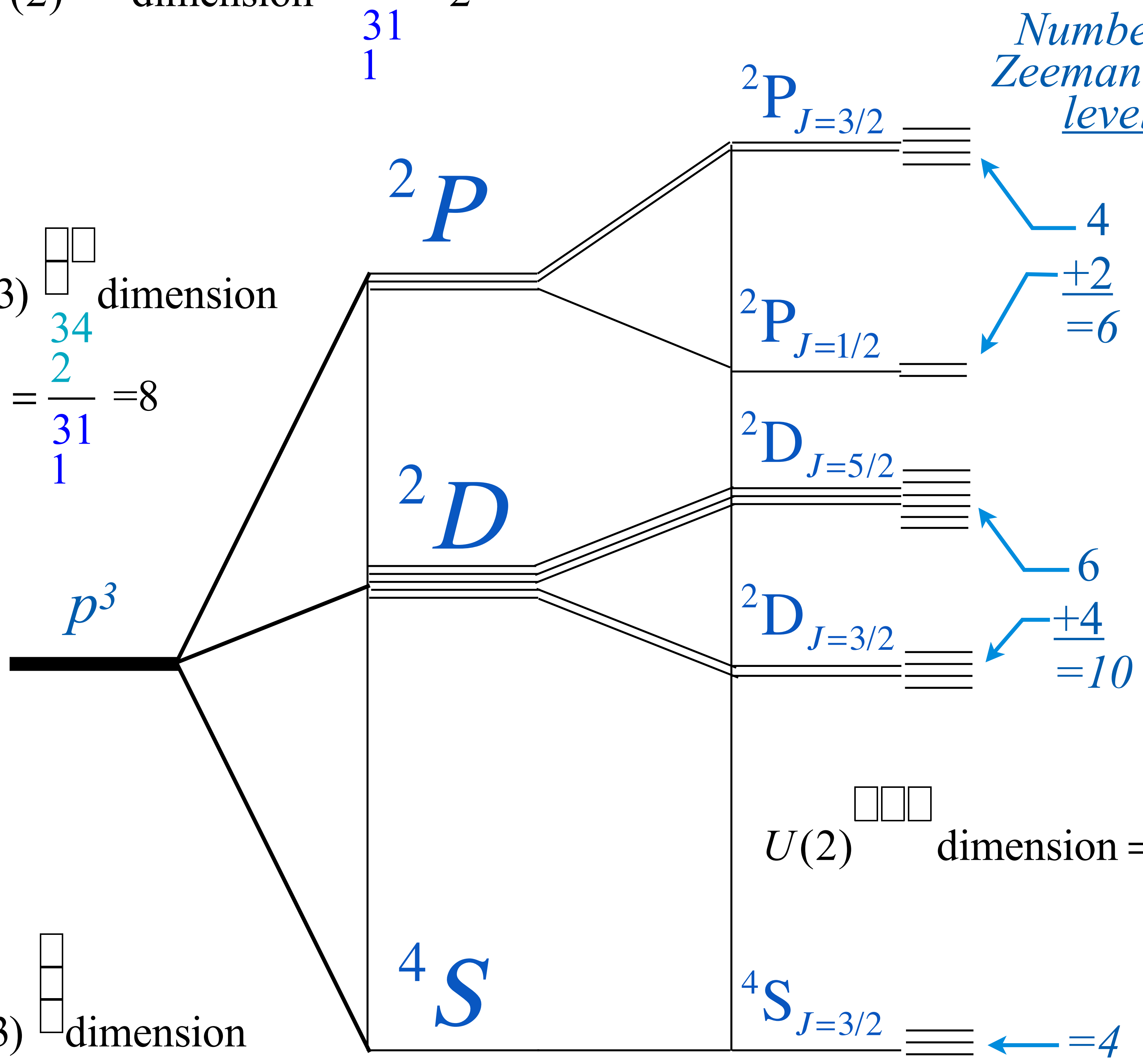
$$U(3)\ \begin{array}{l}\square\\\square\\\square\end{array}\ \text{dimension}=\frac{\begin{smallmatrix}3\\2\\1\end{smallmatrix}}{\begin{smallmatrix}3\\2\\1\end{smallmatrix}}=1$$

$$U(6)\ \text{dimension grand total: }N=\frac{\begin{smallmatrix}6\\5\\4\end{smallmatrix}}{\begin{smallmatrix}3\\2\\1\end{smallmatrix}}=\frac{120}{6}=20=6+10+4$$

$\ell=1$ p^3= spin-orbit levels and Slater states

$$U(2)\ \begin{smallmatrix}\square\square\\\square\end{smallmatrix}\ \text{dimension}=\frac{\begin{matrix}23\\1\end{matrix}}{\begin{matrix}31\\1\end{matrix}}=2$$

$$U(3)\ \begin{smallmatrix}\square\square\\\square\end{smallmatrix}\ \text{dimension}=\frac{\begin{matrix}34\\2\end{matrix}}{\begin{matrix}31\\1\end{matrix}}=8$$



$$\left|{}^2P_{J=\frac{3}{2}}\ \tfrac{1}{2}\right\rangle=-\sqrt{\tfrac{1}{6}}\begin{matrix}b\\c\\d\end{matrix}-\sqrt{\tfrac{1}{6}}\begin{matrix}a\\b\\f\end{matrix}+\tfrac{1}{\sqrt{3}}\begin{matrix}b\\c\\e\end{matrix}-\tfrac{1}{\sqrt{3}}\begin{matrix}a\\c\\f\end{matrix}$$

$$\left|{}^2P_{J=\frac{1}{2}}\ \tfrac{1}{2}\right\rangle=-\sqrt{\tfrac{1}{3}}\begin{matrix}b\\c\\d\end{matrix}-\sqrt{\tfrac{1}{3}}\begin{matrix}a\\b\\f\end{matrix}-\tfrac{1}{\sqrt{6}}\begin{matrix}b\\c\\e\end{matrix}+\tfrac{1}{\sqrt{6}}\begin{matrix}a\\c\\f\end{matrix}$$

$$\left|{}^2D_{J=\frac{5}{2}}\ \tfrac{5}{2}\right\rangle=\begin{matrix}a\\b\\c\end{matrix}$$

$$\left|{}^2D_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle=\sqrt{\tfrac{4}{5}}\begin{matrix}a\\b\\d\end{matrix}+\sqrt{\tfrac{1}{10}}\begin{matrix}a\\c\\d\end{matrix}-\sqrt{\tfrac{1}{10}}\begin{matrix}a\\b\\e\end{matrix}$$

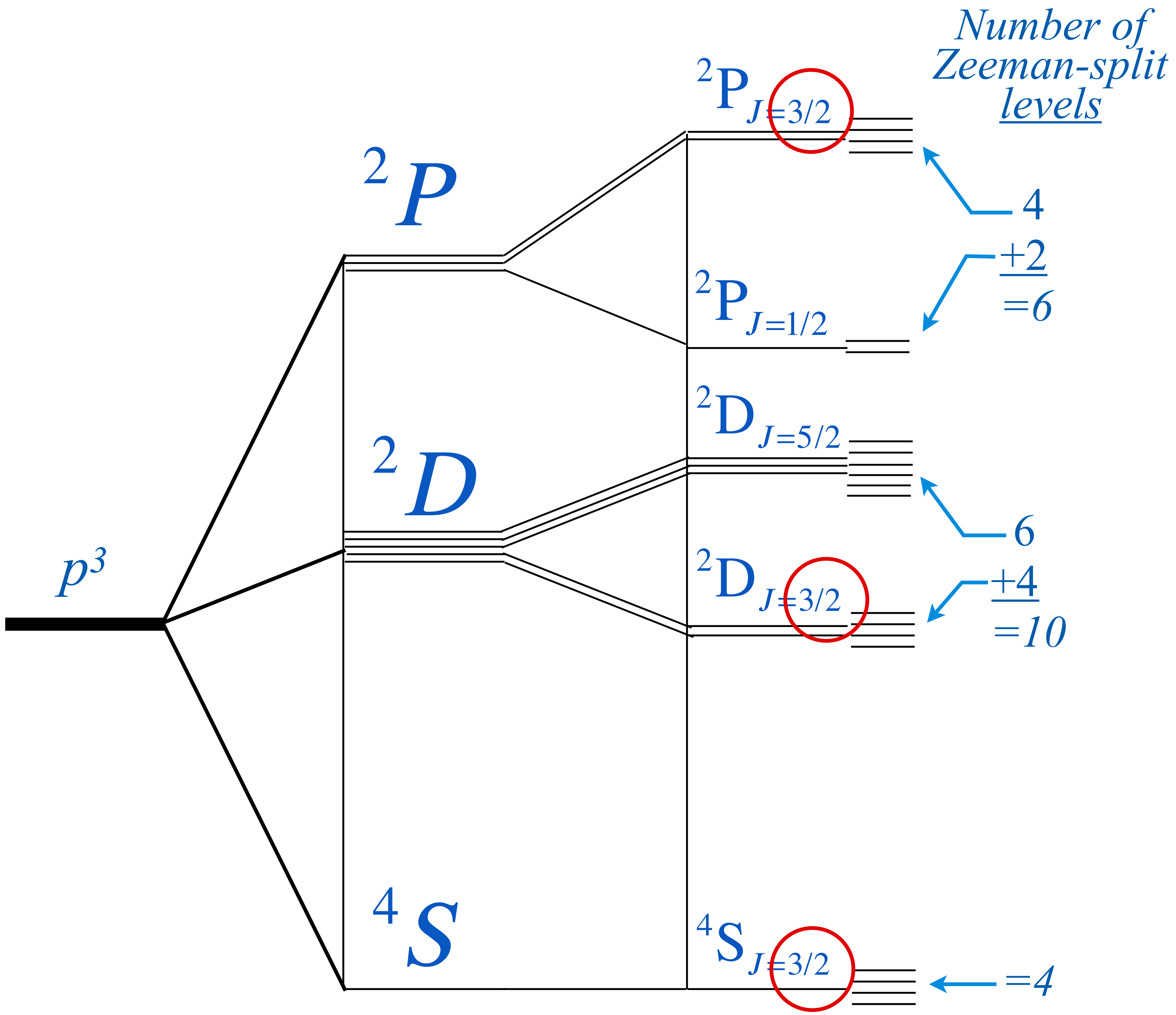
$$U(2)\ \square\square\square\ \text{dimension}=\frac{2\cdot3\cdot4}{3\cdot2\cdot1}=4$$

$$U(3)\ \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}\ \text{dimension}=\frac{\begin{matrix}3\\2\\1\end{matrix}}{\begin{matrix}3\\2\\1\end{matrix}}=1$$

$$\left|{}^4S_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle=\left|\begin{matrix}a\\c\\e\end{matrix}\right\rangle$$

$$U(6)\ \text{dimension grand total: } N=\frac{\begin{matrix}6\\5\\4\end{matrix}}{\begin{matrix}3\\2\\1\end{matrix}}=\frac{120}{6}=20=6+10+4$$

$\ell=1$ p^3=configuration spin-orbit Hamiltonian in Slater determinant basis



$$\left|{}^2P_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle = -\sqrt{\tfrac{1}{2}}\begin{matrix}a\\c\\d\end{matrix} - \sqrt{\tfrac{1}{2}}\begin{matrix}a\\b\\e\end{matrix}$$

$$\left|{}^2P_{J=\frac{1}{2}}\ \tfrac{1}{2}\right\rangle = -\sqrt{\tfrac{1}{3}}\begin{matrix}b\\c\\d\end{matrix} - \sqrt{\tfrac{1}{3}}\begin{matrix}a\\b\\f\end{matrix} - \tfrac{1}{\sqrt{6}}\begin{matrix}b\\c\\e\end{matrix} + \tfrac{1}{\sqrt{6}}\begin{matrix}a\\c\\f\end{matrix}$$

$$\left|{}^2D_{J=\frac{5}{2}}\ \tfrac{5}{2}\right\rangle = \begin{matrix}a\\b\\c\end{matrix}$$

$$\left|{}^2D_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle = \sqrt{\tfrac{4}{5}}\begin{matrix}a\\b\\d\end{matrix} + \sqrt{\tfrac{1}{10}}\begin{matrix}a\\c\\d\end{matrix} - \sqrt{\tfrac{1}{10}}\begin{matrix}a\\b\\e\end{matrix}$$

$$\left|{}^4S_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle = \left|\begin{matrix}a\\c\\e\end{matrix}\right\rangle$$

$$\left|{}^4S_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle = \left|\begin{matrix}a\\c\\e\end{matrix}\right\rangle, \left|{}^4S_{J=\frac{3}{2}}\ \tfrac{1}{2}\right\rangle = \left|\begin{matrix}a\\c\\f\end{matrix}\right\rangle, \left|{}^4S_{J=\frac{3}{2}}\ \tfrac{-1}{2}\right\rangle = \left|\begin{matrix}a\\d\\f\end{matrix}\right\rangle, \left|{}^4S_{J=\frac{3}{2}}\ \tfrac{-3}{2}\right\rangle = \left|\begin{matrix}b\\d\\f\end{matrix}\right\rangle$$

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4.25.18 class 26: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

$(S_3)^*(U(3)) \subset U(6)$ models of p^3 electronic spin-orbit states and couplings

[2,1] tableau states lowered by $\mathbf{L}_- = \sqrt{2}(E_{21}+E_{32})$

Top-(J,M) states to mid-level states

$\ell=1$ p=shell LS states combined to states of definite J

J=3/2 at L=0 (^4S), J=5/2 at L=2 (^2D)

C-G coupling; J=3/2 at L=2 (^2D), J=3/2 at L=1 (^2P), J=1/2 at L=1 (^2P)

Spin-orbit state assembly formula and Slater determinants

Extra assembly table

$\ell=1$ p=shell LSJ states transformed to Slater determinants from J=3/2 (^4S)

Slater functions for J=5/2, J=3/2 (^2D)

Slater functions for J=3/2 (^2P), J=1/2 (^2P)

Summary of states and level connection paths

Symmetry dimension accounting

Spin-orbit Hamiltonian matrix calculation

Application to spin-orbit and entanglement break-up scattering

ℓ=1 p^3=configuration spin-orbit Hamiltonian in Slater determinant basis

$$U(6) \text{ bases: } \left\{ \left|a\right\rangle \equiv \left|1\uparrow\right\rangle \ ,\ \left|b\right\rangle \equiv \left|1\downarrow\right\rangle \ ,\ \left|c\right\rangle \equiv \left|2\uparrow\right\rangle \ ,\ \left|d\right\rangle \equiv \left|2\downarrow\right\rangle \ ,\ \left|e\right\rangle \equiv \left|3\uparrow\right\rangle \ ,\ \left|f\right\rangle \equiv \left|3\downarrow\right\rangle \right\}$$

U(6) tensors of rank-1 (Axial orbit momentum ℓ-vector and spin momentum s-vector) Lect.24 *p.16*

$$\left\langle \mathbf{v}^{1\,1}_{\bar{1}\,1} \right\rangle = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \bar{1} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \bar{1} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \frac{1}{\sqrt{2}} \qquad \left\langle \mathbf{v}^{11}_{00} \right\rangle = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \bar{1} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \bar{1} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix} \frac{1}{2} \qquad \left\langle \mathbf{v}^{11}_{1\bar{1}} \right\rangle = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \bar{1} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \bar{1} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \frac{1}{\sqrt{2}} = \mathbf{v}^1_1 \otimes \mathbf{v}^1_{\bar{1}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \cdot & \bar{1} & \cdot \\ \cdot & \cdot & \bar{1} \\ \cdot & \cdot & \cdot \end{pmatrix} \otimes \begin{pmatrix} \cdot & \cdot \\ 1 & \cdot \end{pmatrix}$$

$$= -\sqrt{\tfrac{1}{2}}(E_{cb} + E_{ed}) \qquad = \tfrac{1}{2}(E_{aa} - E_{bb} - E_{ee} + E_{ff}) \qquad = -\sqrt{\tfrac{1}{2}}(E_{bc} + E_{de})$$

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$$= -\sqrt{\tfrac{1}{2}}(E_{cb} + E_{ed}) \qquad = \tfrac{1}{2}(E_{aa} - E_{bb} - E_{ee} + E_{ff}) \qquad = -\sqrt{\tfrac{1}{2}}(E_{bc} + E_{de})$$

Spin-Orbit Hamiltonian:

$$H_{spin-orbit} = \xi \sum_{\alpha=1}^{n} \vec{\ell}(electron\,\alpha) \bullet \vec{\mathrm{s}}(electron\,\alpha)$$

$$= \xi(V^{11}_{00} - V^{11}_{1\bar{1}} - V^{11}_{\bar{1}1}) = \xi\left[\tfrac{1}{2}(E_{aa} - E_{bb} - E_{ee} + E_{ff}) + \sqrt{\tfrac{1}{2}}(E_{bc} + E_{de} + E_{bc} + E_{de}) \right]$$

AMOP
reference links
on pages 2-4

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➡ Individual matrix components

Application to spin-orbit and entanglement break-up scattering

Calculating p^3 spin-orbit Hamiltonian matrix for J=3/2

$$\left\langle {}^{2}P_{J=\frac{3}{2}}^{\frac{3}{2}} \middle| H_{s\text{-}o} \middle| {}^{4}S_{J=\frac{3}{2}}^{\frac{3}{2}} \right\rangle$$

$$H_{s\text{-}o} = \xi\left[\frac{1}{2}(E_{aa} - E_{bb} - E_{ee} + E_{ff}) + \sqrt{\frac{1}{2}}(E_{bc} + E_{de} + E_{cb} + E_{ed}) \right]$$

$$-\sqrt{\tfrac{1}{2}}\left\langle \begin{matrix} a \\ c \\ d \end{matrix} \right| - \sqrt{\tfrac{1}{2}}\left\langle \begin{matrix} a \\ b \\ e \end{matrix} \right| = \left\langle {}^{2}P_{J=\frac{3}{2}}^{\frac{3}{2}} \middle| \frac{H_{s\text{-}o}}{\xi} \middle| {}^{4}S_{J=\frac{3}{2}}^{\frac{3}{2}} \right\rangle = \left| \begin{matrix} a \\ c \\ e \end{matrix} \right\rangle$$

Calculating p^3 spin-orbit Hamiltonian matrix for J=3/2

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has diagonal-E_{nn} part:

…that gives zero

$$-\sqrt{\tfrac{1}{2}}\left\langle \begin{matrix} a \\ c \\ d \end{matrix} \right| - \sqrt{\tfrac{1}{2}}\left\langle \begin{matrix} a \\ b \\ e \end{matrix} \right| \frac{1}{2}(E_{aa}-E_{bb}-E_{ee}+E_{ff}) \left| \begin{matrix} a \\ c \\ e \end{matrix} \right\rangle$$

Calculating p^3 spin-orbit Hamiltonian matrix for J=3/2

$$\left\langle {}^2P^{\frac{3}{2}}_{J=\frac{3}{2}} \middle| H_{s\text{-}o} \middle| {}^4S^{\frac{3}{2}}_{J=\frac{3}{2}} \right\rangle$$

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…that gives zero

…and off-diagonal-E_{ab} part:

$$-\sqrt{\tfrac{1}{2}}\left\langle \begin{matrix} a \\ c \\ d \end{matrix} \right| - \sqrt{\tfrac{1}{2}}\left\langle \begin{matrix} a \\ b \\ e \end{matrix} \right| \sqrt{\frac{1}{2}}(E_{bc}+E_{de}+E_{cb}+E_{ed}) \left| \begin{matrix} a \\ c \\ e \end{matrix} \right\rangle$$

Calculating p^3 spin-orbit Hamiltonian matrix for J=3/2

$$\left\langle {}^2P_{J=\frac{3}{2}}\ \tfrac{3}{2} \right| H_{s\text{-}o} \left| {}^4S_{J=\frac{3}{2}}\ \tfrac{3}{2} \right\rangle$$

$$H_{s\text{-}o} = \xi\left[\frac{1}{2}(E_{aa} - E_{bb} - E_{ee} + E_{ff}) + \sqrt{\frac{1}{2}}(E_{bc}+E_{de}+E_{cb}+E_{ed})\right]$$

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$$-\sqrt{\tfrac{1}{2}}\left\langle \begin{matrix} a\\ c\\ d\end{matrix} \right| - \sqrt{\tfrac{1}{2}}\left\langle \begin{matrix} a\\ b\\ e\end{matrix} \right| \sqrt{\frac{1}{2}}(E_{bc}+E_{de}+E_{cb}+E_{ed}) \left| \begin{matrix} a\\ c\\ e\end{matrix} \right\rangle$$

$$-\tfrac{1}{2}\left(\left\langle \begin{matrix} a\\ c\\ d\end{matrix} \right| + \left\langle \begin{matrix} a\\ b\\ e\end{matrix} \right| \right) \left(\left| \begin{matrix} a\\ b\\ e\end{matrix} \right\rangle + \left| \begin{matrix} a\\ c\\ d\end{matrix} \right\rangle + 0 + 0 \right)$$

Calculating p^3 spin-orbit Hamiltonian matrix for J=3/2

$$\left\langle {}^{2}P^{\frac{3}{2}}_{J=\frac{3}{2}} \middle| H_{s\text{-}o} \middle| {}^{4}S^{\frac{3}{2}}_{J=\frac{3}{2}} \right\rangle$$

$$H_{s\text{-}o} = \xi\left[\frac{1}{2}(E_{aa}-E_{bb}-E_{ee}+E_{ff})+\sqrt{\frac{1}{2}}(E_{bc}+E_{de}+E_{cb}+E_{ed})\right]$$

$$-\sqrt{\tfrac{1}{2}}\left\langle \begin{matrix}a\\c\\d\end{matrix}\right| -\sqrt{\tfrac{1}{2}}\left\langle \begin{matrix}a\\b\\e\end{matrix}\right| = \left\langle {}^{2}P^{\frac{3}{2}}_{J=\frac{3}{2}}\right| \frac{H_{s\text{-}o}}{\xi} \left| {}^{4}S^{\frac{3}{2}}_{J=\frac{3}{2}}\right\rangle = \left|\begin{matrix}a\\c\\e\end{matrix}\right\rangle$$

has diagonal-E_{nn} part:

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$$-\sqrt{\tfrac{1}{2}}\left\langle \begin{matrix}a\\c\\d\end{matrix}\right| -\sqrt{\tfrac{1}{2}}\left\langle \begin{matrix}a\\b\\e\end{matrix}\right| \frac{1}{2}(E_{aa}-E_{bb}-E_{ee}+E_{ff}) \left|\begin{matrix}a\\c\\e\end{matrix}\right\rangle$$

…and off-diagonal-E_{ab} part:

$$-\sqrt{\tfrac{1}{2}}\left\langle \begin{matrix}a\\c\\d\end{matrix}\right| -\sqrt{\tfrac{1}{2}}\left\langle \begin{matrix}a\\b\\e\end{matrix}\right| \sqrt{\frac{1}{2}}(E_{bc}+E_{de}+E_{cb}+E_{ed}) \left|\begin{matrix}a\\c\\e\end{matrix}\right\rangle$$

$$-\tfrac{1}{2}\left(\left\langle \begin{matrix}a\\c\\d\end{matrix}\right| + \left\langle \begin{matrix}a\\b\\e\end{matrix}\right|\right) \left(\left|\begin{matrix}a\\b\\e\end{matrix}\right\rangle + \left|\begin{matrix}a\\c\\d\end{matrix}\right\rangle + 0 + 0\right)$$

$$\frac{-1}{2}\begin{array}{|c|c|c}\hline 0 & 1 & \left|\begin{matrix}a\\b\\e\end{matrix}\right\rangle \\ \hline 1 & 0 & +\left|\begin{matrix}a\\c\\d\end{matrix}\right\rangle \\ \hline\end{array}$$

Calculating p^3 spin-orbit Hamiltonian matrix for J=3/2

$$\left\langle {}^2P_{J=\frac{3}{2}}^{\frac{3}{2}} \middle| H_{s\text{-}o} \middle| {}^4S_{J=\frac{3}{2}}^{\frac{3}{2}} \right\rangle$$

$$H_{s\text{-}o} = \xi\left[\tfrac{1}{2}(E_{aa} - E_{bb} - E_{ee} + E_{ff}) + \sqrt{\tfrac{1}{2}}(E_{bc}+E_{de}+E_{cb}+E_{ed})\right]$$

$$-\sqrt{\tfrac{1}{2}}\left\langle \begin{matrix}a\\c\\d\end{matrix} \right| - \sqrt{\tfrac{1}{2}}\left\langle \begin{matrix}a\\b\\e\end{matrix} \right| = \left\langle {}^2P_{J=\frac{3}{2}}^{\frac{3}{2}} \middle| \frac{H_{s\text{-}o}}{\xi} \middle| {}^4S_{J=\frac{3}{2}}^{\frac{3}{2}} \right\rangle = \left| \begin{matrix}a\\c\\e\end{matrix} \right\rangle$$

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$$-\sqrt{\tfrac{1}{2}}\left\langle \begin{matrix}a\\c\\d\end{matrix} \right| - \sqrt{\tfrac{1}{2}}\left\langle \begin{matrix}a\\b\\e\end{matrix} \right| \tfrac{1}{2}(E_{aa}-E_{bb}-E_{ee}+E_{ff}) \left| \begin{matrix}a\\c\\e\end{matrix} \right\rangle$$

…and off-diagonal-E_{ab} part:

$$-\sqrt{\tfrac{1}{2}}\left\langle \begin{matrix}a\\c\\d\end{matrix} \right| - \sqrt{\tfrac{1}{2}}\left\langle \begin{matrix}a\\b\\e\end{matrix} \right| \sqrt{\tfrac{1}{2}}(E_{bc}+E_{de}+E_{cb}+E_{ed}) \left| \begin{matrix}a\\c\\e\end{matrix} \right\rangle$$

$$-\tfrac{1}{2}\left(\left\langle \begin{matrix}a\\c\\d\end{matrix} \right| + \left\langle \begin{matrix}a\\b\\e\end{matrix} \right| \right) \qquad \left(\left| \begin{matrix}a\\b\\e\end{matrix} \right\rangle + \left| \begin{matrix}a\\c\\d\end{matrix} \right\rangle + 0 + 0 \right)$$

$$\frac{-1}{2}\begin{array}{|c|c|l} 0 & 1 & \left| \begin{matrix}a\\b\\e\end{matrix} \right\rangle \\ \hline 1 & 0 & +\left| \begin{matrix}a\\c\\d\end{matrix} \right\rangle \end{array}$$

Result: -½ -½ =-1

Calculating *p*³ spin-orbit Hamiltonian matrix for J=3/2

$$\left\langle {}^2P_{J=\frac{3}{2}}\ \tfrac{3}{2}\right| H_{s\text{-}o} \left| {}^4S_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle$$

$$H_{s\text{-}o} = \xi\left[\tfrac{1}{2}(E_{aa} - E_{bb} - E_{ee} + E_{ff}) + \sqrt{\tfrac{1}{2}}(E_{bc}+E_{de}+E_{cb}+E_{ed})\right]$$

$$\left\langle {}^2D_{J=\frac{3}{2}}\ \tfrac{3}{2}\right| H_{s\text{-}o} \left| {}^4S_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle$$

$$-\sqrt{\tfrac{1}{2}}\left\langle \begin{matrix} a \\ c \\ d \end{matrix}\right| - \sqrt{\tfrac{1}{2}}\left\langle \begin{matrix} a \\ b \\ e \end{matrix}\right| = \left\langle {}^2P_{J=\frac{3}{2}}\ \tfrac{3}{2}\right| \frac{H_{s\text{-}o}}{\xi} \left| {}^4S_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle = \left| \begin{matrix} a \\ c \\ e \end{matrix}\right\rangle$$

has diagonal-E_{nn} part:

…that gives zero

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…and off-diagonal-E_{ab} part:

$$\sqrt{\tfrac{4}{5}}\left\langle \begin{matrix} a \\ b \\ d \end{matrix}\right| + \sqrt{\tfrac{1}{10}}\left\langle \begin{matrix} a \\ c \\ d \end{matrix}\right| - \sqrt{\tfrac{1}{10}}\left\langle \begin{matrix} a \\ b \\ e \end{matrix}\right| = \left\langle {}^2D_{J=\frac{3}{2}}\ \tfrac{3}{2}\right| \frac{H_{s\text{-}o}}{\xi} \left| {}^4S_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle = \left| \begin{matrix} a \\ c \\ e \end{matrix}\right\rangle$$

$$-\sqrt{\tfrac{1}{2}}\left\langle \begin{matrix} a \\ c \\ d \end{matrix}\right| - \sqrt{\tfrac{1}{2}}\left\langle \begin{matrix} a \\ b \\ e \end{matrix}\right| \sqrt{\tfrac{1}{2}}(E_{bc}+E_{de}+E_{cb}+E_{ed}) \left| \begin{matrix} a \\ c \\ e \end{matrix}\right\rangle$$

$$-\tfrac{1}{2}\left(\left\langle \begin{matrix} a \\ c \\ d \end{matrix}\right| + \left\langle \begin{matrix} a \\ b \\ e \end{matrix}\right|\right) \left(\left| \begin{matrix} a \\ b \\ e \end{matrix}\right\rangle + \left| \begin{matrix} a \\ c \\ d \end{matrix}\right\rangle + 0 + 0\right)$$

$$\frac{-1}{2}\ \begin{array}{|c|c|c}\hline 0 & 1 & \left|\begin{matrix} a \\ b \\ e \end{matrix}\right\rangle \\ \hline 1 & 0 & +\left|\begin{matrix} a \\ c \\ d \end{matrix}\right\rangle \\ \hline \end{array}$$

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Calculating p^3 spin-orbit Hamiltonian matrix for J=3/2

$$H_{s\text{-}o} = \xi\left[\tfrac{1}{2}(E_{aa} - E_{bb} - E_{ee} + E_{ff}) + \sqrt{\tfrac{1}{2}}(E_{bc}+E_{de}+E_{cb}+E_{ed})\right]$$

$$\left\langle {}^2P_{J=\frac{3}{2}}\ \tfrac{3}{2}\right| H_{s\text{-}o} \left| {}^4S_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle$$

$$\left\langle {}^2D_{J=\frac{3}{2}}\ \tfrac{3}{2}\right| H_{s\text{-}o} \left| {}^4S_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle$$

$$-\sqrt{\tfrac{1}{2}}\left\langle a\,c\,d\right| - \sqrt{\tfrac{1}{2}}\left\langle a\,b\,e\right| = \left\langle {}^2P_{J=\frac{3}{2}}\ \tfrac{3}{2}\right| \frac{H_{s\text{-}o}}{\xi} \left| {}^4S_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle = \left| a\,c\,e\right\rangle$$

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$$-\sqrt{\tfrac{1}{2}}\left\langle a\,c\,d\right| - \sqrt{\tfrac{1}{2}}\left\langle a\,b\,e\right| \ \tfrac{1}{2}(E_{aa}-E_{bb}-E_{ee}+E_{ff}) \left| a\,c\,e\right\rangle$$

…and off-diagonal-E_{ab} part:

$$\sqrt{\tfrac{4}{5}}\left\langle a\,b\,d\right| + \sqrt{\tfrac{1}{10}}\left\langle a\,c\,d\right| - \sqrt{\tfrac{1}{10}}\left\langle a\,b\,e\right| = \left\langle {}^2D_{J=\frac{3}{2}}\ \tfrac{3}{2}\right| \frac{H_{s\text{-}o}}{\xi} \left| {}^4S_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle = \left| a\,c\,e\right\rangle$$

$$-\sqrt{\tfrac{1}{2}}\left\langle a\,c\,d\right| - \sqrt{\tfrac{1}{2}}\left\langle a\,b\,e\right| \ \sqrt{\tfrac{1}{2}}(E_{bc}+E_{de}+E_{cb}+E_{ed}) \left| a\,c\,e\right\rangle$$

$$\left(\sqrt{8}\left\langle a\,b\,d\right| + \left\langle a\,c\,d\right| - \left\langle a\,b\,e\right|\right)$$

$\sqrt{\tfrac{1}{10}}\sqrt{\tfrac{1}{2}}$

0	0	-1	$\left\| a\,b\,e\right\rangle$
0	1	0	$+\left\| a\,c\,d\right\rangle$

$$-\tfrac{1}{2}\left(\left\langle a\,c\,d\right| + \left\langle a\,b\,e\right|\right) \left(\left| a\,b\,e\right\rangle + \left| a\,c\,d\right\rangle + 0 + 0\right)$$

$\frac{-1}{2}$

0	1	$\left\| a\,b\,e\right\rangle$
1	0	$+\left\| a\,c\,d\right\rangle$

Result: -½ -½ =-1

Calculating p^3 spin-orbit Hamiltonian matrix for J=3/2

$$H_{s\text{-}o}=\xi\left[\tfrac{1}{2}(E_{aa}-E_{bb}-E_{ee}+E_{ff})+\sqrt{\tfrac{1}{2}}(E_{bc}+E_{de}+E_{cb}+E_{ed})\right]$$

$$\left\langle {}^{2}P_{J=\frac{3}{2}}\ \tfrac{3}{2}\right|H_{s\text{-}o}\left|{}^{4}S_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle$$

$$\left\langle {}^{2}D_{J=\frac{3}{2}}\ \tfrac{3}{2}\right|H_{s\text{-}o}\left|{}^{4}S_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle$$

$$-\sqrt{\tfrac{1}{2}}\left\langle \begin{matrix}a\\c\\d\end{matrix}\right|-\sqrt{\tfrac{1}{2}}\left\langle \begin{matrix}a\\b\\e\end{matrix}\right|=\left\langle {}^{2}P_{J=\frac{3}{2}}\ \tfrac{3}{2}\right|\frac{H_{s\text{-}o}}{\xi}\left|{}^{4}S_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle=\left|\begin{matrix}a\\c\\e\end{matrix}\right\rangle$$

has diagonal-E_{nn} part: …that gives zero

$$-\sqrt{\tfrac{1}{2}}\left\langle \begin{matrix}a\\c\\d\end{matrix}\right|-\sqrt{\tfrac{1}{2}}\left\langle \begin{matrix}a\\b\\e\end{matrix}\right|\tfrac{1}{2}(E_{aa}-E_{bb}-E_{ee}+E_{ff})\left|\begin{matrix}a\\c\\e\end{matrix}\right\rangle$$

…and off-diagonal-E_{ab} part:

$$\sqrt{\tfrac{4}{5}}\left\langle \begin{matrix}a\\b\\d\end{matrix}\right|+\sqrt{\tfrac{1}{10}}\left\langle \begin{matrix}a\\c\\d\end{matrix}\right|-\sqrt{\tfrac{1}{10}}\left\langle \begin{matrix}a\\b\\e\end{matrix}\right|=\left\langle {}^{2}D_{J=\frac{3}{2}}\ \tfrac{3}{2}\right|\frac{H_{s\text{-}o}}{\xi}\left|{}^{4}S_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle=\left|\begin{matrix}a\\c\\e\end{matrix}\right\rangle$$

$$-\sqrt{\tfrac{1}{2}}\left\langle \begin{matrix}a\\c\\d\end{matrix}\right|-\sqrt{\tfrac{1}{2}}\left\langle \begin{matrix}a\\b\\e\end{matrix}\right|\sqrt{\tfrac{1}{2}}(E_{bc}+E_{de}+E_{cb}+E_{ed})\left|\begin{matrix}a\\c\\e\end{matrix}\right\rangle$$

	$\sqrt{8}\langle abd\vert$	$+\langle acd\vert$	$-\langle abe\vert$	
$\sqrt{\frac{1}{10}}\sqrt{\frac{1}{2}}$	0	0	-1	$\vert abe\rangle$
	0	1	0	$+\vert acd\rangle$

Result: (1/√20)(-1 +1 =0

$$-\tfrac{1}{2}\left(\left\langle \begin{matrix}a\\c\\d\end{matrix}\right|+\left\langle \begin{matrix}a\\b\\e\end{matrix}\right|\right)\left(\left|\begin{matrix}a\\b\\e\end{matrix}\right\rangle+\left|\begin{matrix}a\\c\\d\end{matrix}\right\rangle+0+0\right)$$

	$\langle acd\vert$	$\langle abe\vert$	
$\frac{-1}{2}$	0	1	$\vert abe\rangle$
	1	0	$+\vert acd\rangle$

Result: -½ -½ =-1

Calculating p^3 spin-orbit Hamiltonian matrix for J=3/2

$$\left\langle {}^2D_{J=\frac{3}{2}}\ \tfrac{3}{2} \right| H_{s\text{-}o} \left| {}^2P_{J=\frac{3}{2}}\ \tfrac{3}{2} \right\rangle$$

$$\sqrt{\tfrac{4}{5}}\left\langle \begin{matrix}a\\b\\d\end{matrix}\right| + \sqrt{\tfrac{1}{10}}\left\langle \begin{matrix}a\\c\\d\end{matrix}\right| - \sqrt{\tfrac{1}{10}}\left\langle \begin{matrix}a\\b\\e\end{matrix}\right| = \left\langle {}^2D_{J=\frac{3}{2}}\ \tfrac{3}{2} \right| \frac{H_{s\text{-}o}}{\xi} \left| {}^2P_{J=\frac{3}{2}}\ \tfrac{3}{2} \right\rangle = -\sqrt{\tfrac{1}{2}}\left| \begin{matrix}a\\c\\d\end{matrix}\right\rangle - \sqrt{\tfrac{1}{2}}\left| \begin{matrix}a\\b\\e\end{matrix}\right\rangle$$

The diagonal-E_{nn} part is not identically zero:

$$\left(\sqrt{\tfrac{4}{5}}\left\langle \begin{matrix}a\\b\\d\end{matrix}\right| + \sqrt{\tfrac{1}{10}}\left\langle \begin{matrix}a\\c\\d\end{matrix}\right| - \sqrt{\tfrac{1}{10}}\left\langle \begin{matrix}a\\b\\e\end{matrix}\right|\right)\frac{1}{2}(E_{aa}-E_{bb}-E_{ee}+E_{ff})\ -\left(\sqrt{\tfrac{1}{2}}\left| \begin{matrix}a\\c\\d\end{matrix}\right\rangle + \sqrt{\tfrac{1}{2}}\left| \begin{matrix}a\\b\\e\end{matrix}\right\rangle\right)$$

Calculating p^3 spin-orbit Hamiltonian matrix for J=3/2

$$\left\langle {}^2D^{\frac{3}{2}}_{J=\frac{3}{2}} \middle| H_{s\text{-}o} \middle| {}^2P^{\frac{3}{2}}_{J=\frac{3}{2}} \right\rangle$$

$$\sqrt{\tfrac{4}{5}}\left\langle\begin{matrix}a\\b\\d\end{matrix}\right| + \sqrt{\tfrac{1}{10}}\left\langle\begin{matrix}a\\c\\d\end{matrix}\right| - \sqrt{\tfrac{1}{10}}\left\langle\begin{matrix}a\\b\\e\end{matrix}\right| = \left\langle {}^2D^{\frac{3}{2}}_{J=\frac{3}{2}} \middle| \frac{H_{s\text{-}o}}{\xi} \middle| {}^2P^{\frac{3}{2}}_{J=\frac{3}{2}} \right\rangle = -\sqrt{\tfrac{1}{2}}\left|\begin{matrix}a\\c\\d\end{matrix}\right\rangle - \sqrt{\tfrac{1}{2}}\left|\begin{matrix}a\\b\\e\end{matrix}\right\rangle$$

Here diagonal-E_{nn} part is not identically zero:

$$\left(\sqrt{\tfrac{4}{5}}\left\langle\begin{matrix}a\\b\\d\end{matrix}\right| + \sqrt{\tfrac{1}{10}}\left\langle\begin{matrix}a\\c\\d\end{matrix}\right| - \sqrt{\tfrac{1}{10}}\left\langle\begin{matrix}a\\b\\e\end{matrix}\right|\right)\frac{1}{2}(E_{aa}-E_{bb}-E_{ee}+E_{ff})\ -\left(\sqrt{\tfrac{1}{2}}\left|\begin{matrix}a\\c\\d\end{matrix}\right\rangle + \sqrt{\tfrac{1}{2}}\left|\begin{matrix}a\\b\\e\end{matrix}\right\rangle\right)$$

$$\left(\sqrt{\tfrac{1}{10}}\left\langle\begin{matrix}a\\c\\d\end{matrix}\right| - \sqrt{\tfrac{1}{10}}\left\langle\begin{matrix}a\\b\\e\end{matrix}\right|\right)\ \frac{-1}{2\sqrt{2}}(E_{aa}-E_{bb}-E_{ee}+0)\left(\left|\begin{matrix}a\\c\\d\end{matrix}\right\rangle + \left|\begin{matrix}a\\b\\e\end{matrix}\right\rangle\right)$$

$$(E_{aa}-E_{bb}-E_{ee}+0)\left(\left|\begin{matrix}a\\c\\d\end{matrix}\right\rangle + \left|\begin{matrix}a\\b\\e\end{matrix}\right\rangle\right)$$

diagonal-E_{nn} part
changes righthand ket

Calculating p^3 spin-orbit Hamiltonian matrix for J=3/2

$$\left\langle {}^2D^{\frac{3}{2}}_{J=\frac{3}{2}}\right| H_{s\text{-}o}\left| {}^2P^{\frac{3}{2}}_{J=\frac{3}{2}}\right\rangle$$

$$\sqrt{\tfrac{4}{5}}\left\langle \begin{matrix}a\\b\\d\end{matrix}\right| + \sqrt{\tfrac{1}{10}}\left\langle \begin{matrix}a\\c\\d\end{matrix}\right| - \sqrt{\tfrac{1}{10}}\left\langle \begin{matrix}a\\b\\e\end{matrix}\right| = \left\langle {}^2D^{\frac{3}{2}}_{J=\frac{3}{2}}\right| \frac{H_{s\text{-}o}}{\xi}\left| {}^2P^{\frac{3}{2}}_{J=\frac{3}{2}}\right\rangle = -\sqrt{\tfrac{1}{2}}\left| \begin{matrix}a\\c\\d\end{matrix}\right\rangle - \sqrt{\tfrac{1}{2}}\left| \begin{matrix}a\\b\\e\end{matrix}\right\rangle$$

Here diagonal-E_{nn} part is not identically zero:

$$\left(\sqrt{\tfrac{4}{5}}\left\langle \begin{matrix}a\\b\\d\end{matrix}\right| + \sqrt{\tfrac{1}{10}}\left\langle \begin{matrix}a\\c\\d\end{matrix}\right| - \sqrt{\tfrac{1}{10}}\left\langle \begin{matrix}a\\b\\e\end{matrix}\right|\right)\frac{1}{2}(E_{aa}-E_{bb}-E_{ee}+E_{ff})\ -\left(\sqrt{\tfrac{1}{2}}\left| \begin{matrix}a\\c\\d\end{matrix}\right\rangle + \sqrt{\tfrac{1}{2}}\left| \begin{matrix}a\\b\\e\end{matrix}\right\rangle\right)$$

$$\left(\sqrt{\tfrac{1}{10}}\left\langle \begin{matrix}a\\c\\d\end{matrix}\right| - \sqrt{\tfrac{1}{10}}\left\langle \begin{matrix}a\\b\\e\end{matrix}\right|\right)\ \frac{-1}{2\sqrt{2}}(E_{aa}-E_{bb}-E_{ee}+0)\left(\left| \begin{matrix}a\\c\\d\end{matrix}\right\rangle + \left| \begin{matrix}a\\b\\e\end{matrix}\right\rangle\right)$$

$$(E_{aa}-E_{bb}-E_{ee}+0)\left(\left| \begin{matrix}a\\c\\d\end{matrix}\right\rangle + \left| \begin{matrix}a\\b\\e\end{matrix}\right\rangle\right)$$

diagonal-E_{nn} part changes righthand ket

$$=\left(E_{aa}\left| \begin{matrix}a\\c\\d\end{matrix}\right\rangle + E_{aa}\left| \begin{matrix}a\\b\\e\end{matrix}\right\rangle - E_{bb}\left| \begin{matrix}a\\b\\e\end{matrix}\right\rangle - E_{ee}\left| \begin{matrix}a\\b\\e\end{matrix}\right\rangle\right)$$

Calculating p^3 spin-orbit Hamiltonian matrix for J=3/2

$$\left\langle {}^2D^{\frac{3}{2}}_{J=\frac{3}{2}}\right| H_{s\text{-}o} \left| {}^2P^{\frac{3}{2}}_{J=\frac{3}{2}}\right\rangle$$

$$\sqrt{\tfrac{4}{5}}\left\langle \begin{matrix}a\\b\\d\end{matrix}\right| + \sqrt{\tfrac{1}{10}}\left\langle \begin{matrix}a\\c\\d\end{matrix}\right| - \sqrt{\tfrac{1}{10}}\left\langle \begin{matrix}a\\b\\e\end{matrix}\right| = \left\langle {}^2D^{\frac{3}{2}}_{J=\frac{3}{2}}\right| \frac{H_{s\text{-}o}}{\xi} \left| {}^2P^{\frac{3}{2}}_{J=\frac{3}{2}}\right\rangle = -\sqrt{\tfrac{1}{2}}\left| \begin{matrix}a\\c\\d\end{matrix}\right\rangle - \sqrt{\tfrac{1}{2}}\left| \begin{matrix}a\\b\\e\end{matrix}\right\rangle$$

Here diagonal-E_{nn} part is not identically zero:

$$\left(\sqrt{\tfrac{4}{5}}\left\langle \begin{matrix}a\\b\\d\end{matrix}\right| + \sqrt{\tfrac{1}{10}}\left\langle \begin{matrix}a\\c\\d\end{matrix}\right| - \sqrt{\tfrac{1}{10}}\left\langle \begin{matrix}a\\b\\e\end{matrix}\right|\right) \frac{1}{2}(E_{aa}-E_{bb}-E_{ee}+E_{ff}) - \left(\sqrt{\tfrac{1}{2}}\left| \begin{matrix}a\\c\\d\end{matrix}\right\rangle + \sqrt{\tfrac{1}{2}}\left| \begin{matrix}a\\b\\e\end{matrix}\right\rangle\right)$$

$$\left(\sqrt{\tfrac{1}{10}}\left\langle \begin{matrix}a\\c\\d\end{matrix}\right| - \sqrt{\tfrac{1}{10}}\left\langle \begin{matrix}a\\b\\e\end{matrix}\right|\right) \frac{-1}{2\sqrt{2}}(E_{aa}-E_{bb}-E_{ee}+0)\left(\left| \begin{matrix}a\\c\\d\end{matrix}\right\rangle + \left| \begin{matrix}a\\b\\e\end{matrix}\right\rangle\right)$$

$$(E_{aa}-E_{bb}-E_{ee}+0)\left(\left| \begin{matrix}a\\c\\d\end{matrix}\right\rangle + \left| \begin{matrix}a\\b\\e\end{matrix}\right\rangle\right)$$

diagonal-E_{nn} part
changes righthand ket

$$= \left(E_{aa}\left| \begin{matrix}a\\c\\d\end{matrix}\right\rangle + E_{aa}\left| \begin{matrix}a\\b\\e\end{matrix}\right\rangle - E_{bb}\left| \begin{matrix}a\\b\\e\end{matrix}\right\rangle - E_{ee}\left| \begin{matrix}a\\b\\e\end{matrix}\right\rangle\right)$$

$$= \left(\left| \begin{matrix}a\\c\\d\end{matrix}\right\rangle + \left| \begin{matrix}a\\b\\e\end{matrix}\right\rangle - \left| \begin{matrix}a\\b\\e\end{matrix}\right\rangle - \left| \begin{matrix}a\\b\\e\end{matrix}\right\rangle\right) = \left(\left| \begin{matrix}a\\c\\d\end{matrix}\right\rangle - \left| \begin{matrix}a\\b\\e\end{matrix}\right\rangle\right)$$

Calculating p^3 spin-orbit Hamiltonian matrix for J=3/2

$$\left\langle {}^2D_{J=\frac{3}{2}}^{\frac{3}{2}}\right| H_{s\text{-}o} \left| {}^2P_{J=\frac{3}{2}}^{\frac{3}{2}}\right\rangle$$

$$\sqrt{\tfrac{4}{5}}\left\langle\begin{matrix}a\\b\\d\end{matrix}\right| + \sqrt{\tfrac{1}{10}}\left\langle\begin{matrix}a\\c\\d\end{matrix}\right| - \sqrt{\tfrac{1}{10}}\left\langle\begin{matrix}a\\b\\e\end{matrix}\right| = \left\langle {}^2D_{J=\frac{3}{2}}^{\frac{3}{2}}\right| \frac{H_{s\text{-}o}}{\xi} \left| {}^2P_{J=\frac{3}{2}}^{\frac{3}{2}}\right\rangle = -\sqrt{\tfrac{1}{2}}\left|\begin{matrix}a\\c\\d\end{matrix}\right\rangle - \sqrt{\tfrac{1}{2}}\left|\begin{matrix}a\\b\\e\end{matrix}\right\rangle$$

Non-zero diagonal-E_{nn} contribution:

Here diagonal-E_{nn} part is not identically zero:

$$\left(\sqrt{\tfrac{4}{5}}\left\langle\begin{matrix}a\\b\\d\end{matrix}\right| + \sqrt{\tfrac{1}{10}}\left\langle\begin{matrix}a\\c\\d\end{matrix}\right| - \sqrt{\tfrac{1}{10}}\left\langle\begin{matrix}a\\b\\e\end{matrix}\right|\right) \frac{1}{2}(E_{aa}-E_{bb}-E_{ee}+E_{ff}) \; -\left(\sqrt{\tfrac{1}{2}}\left|\begin{matrix}a\\c\\d\end{matrix}\right\rangle + \sqrt{\tfrac{1}{2}}\left|\begin{matrix}a\\b\\e\end{matrix}\right\rangle\right)$$

$$\frac{-1}{2\sqrt{2}}\sqrt{\tfrac{1}{10}}\left(\left\langle\begin{matrix}a\\c\\d\end{matrix}\right| - \left\langle\begin{matrix}a\\b\\e\end{matrix}\right|\right)\left(\left|\begin{matrix}a\\c\\d\end{matrix}\right\rangle - \left|\begin{matrix}a\\b\\e\end{matrix}\right\rangle\right) = \left(\sqrt{\tfrac{1}{10}}\left\langle\begin{matrix}a\\c\\d\end{matrix}\right| - \sqrt{\tfrac{1}{10}}\left\langle\begin{matrix}a\\b\\e\end{matrix}\right|\right) \frac{-1}{2\sqrt{2}}(E_{aa}-E_{bb}-E_{ee}+0)\left(\left|\begin{matrix}a\\c\\d\end{matrix}\right\rangle + \left|\begin{matrix}a\\b\\e\end{matrix}\right\rangle\right)$$

$$(E_{aa}-E_{bb}-E_{ee}+0)\left(\left|\begin{matrix}a\\c\\d\end{matrix}\right\rangle + \left|\begin{matrix}a\\b\\e\end{matrix}\right\rangle\right)$$

diagonal-E_{nn} part
changes righthand ket

$$= \left(E_{aa}\left|\begin{matrix}a\\c\\d\end{matrix}\right\rangle + E_{aa}\left|\begin{matrix}a\\b\\e\end{matrix}\right\rangle - E_{bb}\left|\begin{matrix}a\\b\\e\end{matrix}\right\rangle - E_{ee}\left|\begin{matrix}a\\b\\e\end{matrix}\right\rangle\right)$$

$$= \left(\left|\begin{matrix}a\\c\\d\end{matrix}\right\rangle + \left|\begin{matrix}a\\b\\e\end{matrix}\right\rangle - \left|\begin{matrix}a\\b\\e\end{matrix}\right\rangle - \left|\begin{matrix}a\\b\\e\end{matrix}\right\rangle\right) = \left(\left|\begin{matrix}a\\c\\d\end{matrix}\right\rangle - \left|\begin{matrix}a\\b\\e\end{matrix}\right\rangle\right)$$

Calculating p^3 spin-orbit Hamiltonian matrix for J=3/2

$$\left\langle {}^2D_{J=\frac{3}{2}}\ \tfrac{3}{2}\right| H_{s\text{-}o} \left| {}^2P_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle$$

$$\sqrt{\tfrac{4}{5}}\left\langle \begin{matrix}a\\b\\d\end{matrix}\right| + \sqrt{\tfrac{1}{10}}\left\langle \begin{matrix}a\\c\\d\end{matrix}\right| - \sqrt{\tfrac{1}{10}}\left\langle \begin{matrix}a\\b\\e\end{matrix}\right| = \left\langle {}^2D_{J=\frac{3}{2}}\ \tfrac{3}{2}\right| \frac{H_{s\text{-}o}}{\xi} \left| {}^2P_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle = -\sqrt{\tfrac{1}{2}}\left| \begin{matrix}a\\c\\d\end{matrix}\right\rangle - \sqrt{\tfrac{1}{2}}\left| \begin{matrix}a\\b\\e\end{matrix}\right\rangle$$

Non-zero diagonal-E_{nn} contribution: Here diagonal-E_{nn} part is not identically zero:

$$\boxed{\frac{-1}{2\sqrt{2}}\sqrt{\tfrac{1}{10}}\left(\left\langle \begin{matrix}a\\c\\d\end{matrix}\right\|\left. \begin{matrix}a\\c\\d\end{matrix}\right\rangle + \left\langle \begin{matrix}a\\b\\e\end{matrix}\right\|\left. \begin{matrix}a\\b\\e\end{matrix}\right\rangle\right) = \overset{\downarrow}{-\sqrt{\tfrac{1}{20}}} = \Bigg(}\ \sqrt{\tfrac{4}{5}}\left\langle \begin{matrix}a\\b\\d\end{matrix}\right| + \sqrt{\tfrac{1}{10}}\left\langle \begin{matrix}a\\c\\d\end{matrix}\right| - \sqrt{\tfrac{1}{10}}\left\langle \begin{matrix}a\\b\\e\end{matrix}\right|\Bigg) \frac{1}{2}(E_{aa}-E_{bb}-E_{ee}+E_{ff})\ -\left(\sqrt{\tfrac{1}{2}}\left| \begin{matrix}a\\c\\d\end{matrix}\right\rangle + \sqrt{\tfrac{1}{2}}\left| \begin{matrix}a\\b\\e\end{matrix}\right\rangle\right)$$

Calculating p^3 spin-orbit Hamiltonian matrix for J=3/2

$$\left\langle {}^2D_{J=\frac{3}{2}}^{\frac{3}{2}}\right| H_{s\text{-}o} \left| {}^2P_{J=\frac{3}{2}}^{\frac{3}{2}}\right\rangle$$

$$\sqrt{\tfrac{4}{5}}\left\langle\begin{array}{c}a\\b\\d\end{array}\right| + \sqrt{\tfrac{1}{10}}\left\langle\begin{array}{c}a\\c\\d\end{array}\right| - \sqrt{\tfrac{1}{10}}\left\langle\begin{array}{c}a\\b\\e\end{array}\right| = \left\langle {}^2D_{J=\frac{3}{2}}^{\frac{3}{2}}\right| \frac{H_{s\text{-}o}}{\xi} \left| {}^2P_{J=\frac{3}{2}}^{\frac{3}{2}}\right\rangle = -\sqrt{\tfrac{1}{2}}\left|\begin{array}{c}a\\c\\d\end{array}\right\rangle - \sqrt{\tfrac{1}{2}}\left|\begin{array}{c}a\\b\\e\end{array}\right\rangle$$

Non-zero diagonal-E_{nn} contribution:

Here diagonal-E_{nn} part is not identically zero:

$$\boxed{\frac{-1}{2\sqrt{2}}\sqrt{\tfrac{1}{10}}\left(\left\langle\begin{array}{c}a\\c\\d\end{array}\right\|\left.\begin{array}{c}a\\c\\d\end{array}\right\rangle + \left\langle\begin{array}{c}a\\b\\e\end{array}\right\|\left.\begin{array}{c}a\\b\\e\end{array}\right\rangle\right) = \overset{\downarrow}{-\sqrt{\tfrac{1}{20}}}} = \left(\sqrt{\tfrac{4}{5}}\left\langle\begin{array}{c}a\\b\\d\end{array}\right| + \sqrt{\tfrac{1}{10}}\left\langle\begin{array}{c}a\\c\\d\end{array}\right| - \sqrt{\tfrac{1}{10}}\left\langle\begin{array}{c}a\\b\\e\end{array}\right|\right)\frac{1}{2}(E_{aa}-E_{bb}-E_{ee}+E_{ff}) - \left(\sqrt{\tfrac{1}{2}}\left|\begin{array}{c}a\\c\\d\end{array}\right\rangle + \sqrt{\tfrac{1}{2}}\left|\begin{array}{c}a\\b\\e\end{array}\right\rangle\right)$$

Off-diagonal-E_{nn} contributions:

$$\left(\sqrt{\tfrac{4}{5}}\left\langle\begin{array}{c}a\\b\\d\end{array}\right| + \sqrt{\tfrac{1}{10}}\left\langle\begin{array}{c}a\\c\\d\end{array}\right| - \sqrt{\tfrac{1}{10}}\left\langle\begin{array}{c}a\\b\\e\end{array}\right|\right)\sqrt{\frac{1}{2}}(E_{bc}+E_{de}+E_{cb}+E_{ed}) - \left(\sqrt{\tfrac{1}{2}}\left|\begin{array}{c}a\\c\\d\end{array}\right\rangle + \sqrt{\tfrac{1}{2}}\left|\begin{array}{c}a\\b\\e\end{array}\right\rangle\right)$$

Calculating p^3 spin-orbit Hamiltonian matrix for J=3/2

$$\left\langle {}^2D^{\frac32}_{J=\frac32}\right| H_{s\text{-}o} \left| {}^2P^{\frac32}_{J=\frac32}\right\rangle$$

$$\sqrt{\tfrac45}\left\langle\begin{matrix}a\\b\\d\end{matrix}\right| + \sqrt{\tfrac1{10}}\left\langle\begin{matrix}a\\c\\d\end{matrix}\right| - \sqrt{\tfrac1{10}}\left\langle\begin{matrix}a\\b\\e\end{matrix}\right| = \left\langle {}^2D^{\frac32}_{J=\frac32}\right| \frac{H_{s\text{-}o}}{\xi} \left| {}^2P^{\frac32}_{J=\frac32}\right\rangle = -\sqrt{\tfrac12}\left|\begin{matrix}a\\c\\d\end{matrix}\right\rangle - \sqrt{\tfrac12}\left|\begin{matrix}a\\b\\e\end{matrix}\right\rangle$$

Non-zero diagonal-E_{nn} contribution:

Here diagonal-E_{nn} part is not identically zero:

$$\boxed{\frac{-1}{2\sqrt2}\sqrt{\tfrac1{10}}\left(\left\langle\begin{matrix}a\\c\\d\end{matrix}\right\|\left.\begin{matrix}a\\c\\d\end{matrix}\right\rangle + \left\langle\begin{matrix}a\\b\\e\end{matrix}\right\|\left.\begin{matrix}a\\b\\e\end{matrix}\right\rangle\right) = \overset{\downarrow}{-\sqrt{\tfrac1{20}}}} = \left(\sqrt{\tfrac45}\left\langle\begin{matrix}a\\b\\d\end{matrix}\right| + \sqrt{\tfrac1{10}}\left\langle\begin{matrix}a\\c\\d\end{matrix}\right| - \sqrt{\tfrac1{10}}\left\langle\begin{matrix}a\\b\\e\end{matrix}\right|\right)\frac12(E_{aa}-E_{bb}-E_{ee}+E_{ff}) - \left(\sqrt{\tfrac12}\left|\begin{matrix}a\\c\\d\end{matrix}\right\rangle + \sqrt{\tfrac12}\left|\begin{matrix}a\\b\\e\end{matrix}\right\rangle\right)$$

Off-diagonal-E_{nn} contributions:

$$\left(\sqrt{\tfrac45}\left\langle\begin{matrix}a\\b\\d\end{matrix}\right| + \sqrt{\tfrac1{10}}\left\langle\begin{matrix}a\\c\\d\end{matrix}\right| - \sqrt{\tfrac1{10}}\left\langle\begin{matrix}a\\b\\e\end{matrix}\right|\right)\sqrt{\frac12}(E_{bc}+E_{de}+E_{cb}+E_{ed}) - \left(\sqrt{\tfrac12}\left|\begin{matrix}a\\c\\d\end{matrix}\right\rangle + \sqrt{\tfrac12}\left|\begin{matrix}a\\b\\e\end{matrix}\right\rangle\right)$$

$$-\tfrac12\left(E_{bc}\left|\begin{matrix}a\\c\\d\end{matrix}\right\rangle + E_{ed}\left|\begin{matrix}a\\c\\d\end{matrix}\right\rangle + E_{cb}\left|\begin{matrix}a\\b\\e\end{matrix}\right\rangle + E_{de}\left|\begin{matrix}a\\b\\e\end{matrix}\right\rangle\right)$$

$$-\tfrac12\left(\left|\begin{matrix}a\\b\\d\end{matrix}\right\rangle + \left|\begin{matrix}a\\c\\e\end{matrix}\right\rangle + \left|\begin{matrix}a\\c\\e\end{matrix}\right\rangle + \left|\begin{matrix}a\\b\\d\end{matrix}\right\rangle\right) = -\left(\left|\begin{matrix}a\\b\\d\end{matrix}\right\rangle + \left|\begin{matrix}a\\c\\e\end{matrix}\right\rangle\right)$$

Calculating p^3 spin-orbit Hamiltonian matrix for J=3/2

$$\left\langle {}^2D^{\frac32}_{J=\frac32}\right| H_{s\text{-}o} \left| {}^2P^{\frac32}_{J=\frac32}\right\rangle$$

$$\sqrt{\tfrac45}\left\langle\begin{matrix}a\\b\\d\end{matrix}\right| + \sqrt{\tfrac{1}{10}}\left\langle\begin{matrix}a\\c\\d\end{matrix}\right| - \sqrt{\tfrac{1}{10}}\left\langle\begin{matrix}a\\b\\e\end{matrix}\right| = \left\langle {}^2D^{\frac32}_{J=\frac32}\right| \frac{H_{s\text{-}o}}{\xi} \left| {}^2P^{\frac32}_{J=\frac32}\right\rangle = -\sqrt{\tfrac12}\left|\begin{matrix}a\\c\\d\end{matrix}\right\rangle - \sqrt{\tfrac12}\left|\begin{matrix}a\\b\\e\end{matrix}\right\rangle$$

Non-zero diagonal-E_{nn} contribution:

Here diagonal-E_{nn} part is not identically zero:

$$\boxed{\frac{-1}{2\sqrt2}\sqrt{\tfrac{1}{10}}\left(\left\langle\begin{matrix}a\\c\\d\end{matrix}\right\|\left.\begin{matrix}a\\c\\d\end{matrix}\right\rangle + \left\langle\begin{matrix}a\\b\\e\end{matrix}\right\|\left.\begin{matrix}a\\b\\e\end{matrix}\right\rangle\right) = \overset{\downarrow}{-\sqrt{\tfrac{1}{20}}}} = \left(\sqrt{\tfrac45}\left\langle\begin{matrix}a\\b\\d\end{matrix}\right| + \sqrt{\tfrac{1}{10}}\left\langle\begin{matrix}a\\c\\d\end{matrix}\right| - \sqrt{\tfrac{1}{10}}\left\langle\begin{matrix}a\\b\\e\end{matrix}\right|\right)\frac12(E_{aa}-E_{bb}-E_{ee}+E_{ff}) - \left(\sqrt{\tfrac12}\left|\begin{matrix}a\\c\\d\end{matrix}\right\rangle + \sqrt{\tfrac12}\left|\begin{matrix}a\\b\\e\end{matrix}\right\rangle\right)$$

Off-diagonal-E_{nn} contributions:

$$\left(\sqrt{\tfrac45}\left\langle\begin{matrix}a\\b\\d\end{matrix}\right| + \sqrt{\tfrac{1}{10}}\left\langle\begin{matrix}a\\c\\d\end{matrix}\right| - \sqrt{\tfrac{1}{10}}\left\langle\begin{matrix}a\\b\\e\end{matrix}\right|\right)\sqrt{\frac12}(E_{bc}+E_{de}+E_{cb}+E_{ed}) - \left(\sqrt{\tfrac12}\left|\begin{matrix}a\\c\\d\end{matrix}\right\rangle + \sqrt{\tfrac12}\left|\begin{matrix}a\\b\\e\end{matrix}\right\rangle\right)$$

$$-\tfrac12\left(E_{bc}\left|\begin{matrix}a\\c\\d\end{matrix}\right\rangle + E_{ed}\left|\begin{matrix}a\\c\\d\end{matrix}\right\rangle + E_{cb}\left|\begin{matrix}a\\b\\e\end{matrix}\right\rangle + E_{de}\left|\begin{matrix}a\\b\\e\end{matrix}\right\rangle\right)$$

$$-\tfrac12\left(\left|\begin{matrix}a\\b\\d\end{matrix}\right\rangle + \left|\begin{matrix}a\\c\\e\end{matrix}\right\rangle + \left|\begin{matrix}a\\c\\e\end{matrix}\right\rangle + \left|\begin{matrix}a\\b\\d\end{matrix}\right\rangle\right) = -\left(\left|\begin{matrix}a\\b\\d\end{matrix}\right\rangle + \left|\begin{matrix}a\\c\\e\end{matrix}\right\rangle\right)$$

$$\left(\sqrt8\left\langle\begin{matrix}a\\b\\d\end{matrix}\right| + \left\langle\begin{matrix}a\\c\\d\end{matrix}\right| - \left\langle\begin{matrix}a\\b\\e\end{matrix}\right|\right)$$

$-\sqrt{\frac{1}{10}}$	$\sqrt8$	0	-0	$\left\lvert\begin{matrix}a\\b\\d\end{matrix}\right\rangle$
	0	0	-0	$+\left\lvert\begin{matrix}a\\c\\e\end{matrix}\right\rangle$

$$= -\sqrt{\tfrac{8}{10}} = -\sqrt{\tfrac45}$$

Calculating p^3 spin-orbit Hamiltonian matrix for J=3/2

$$\left\langle {}^2D_{J=\frac32}^{\frac32}\right| H_{s\text{-}o} \left| {}^2P_{J=\frac32}^{\frac32}\right\rangle$$

$$\sqrt{\tfrac45}\left\langle\begin{matrix}a\\b\\d\end{matrix}\right| + \sqrt{\tfrac1{10}}\left\langle\begin{matrix}a\\c\\d\end{matrix}\right| - \sqrt{\tfrac1{10}}\left\langle\begin{matrix}a\\b\\e\end{matrix}\right| = \left\langle {}^2D_{J=\frac32}^{\frac32}\right| \frac{H_{s\text{-}o}}{\xi} \left| {}^2P_{J=\frac32}^{\frac32}\right\rangle = -\sqrt{\tfrac12}\left|\begin{matrix}a\\c\\d\end{matrix}\right\rangle - \sqrt{\tfrac12}\left|\begin{matrix}a\\b\\e\end{matrix}\right\rangle$$

Non-zero diagonal-E_{nn} contribution:

Here diagonal-E_{nn} part is not identically zero:

$$\boxed{\frac{-1}{2\sqrt2}\sqrt{\frac1{10}}\left(\left\langle\begin{matrix}a\\c\\d\end{matrix}\middle\|\begin{matrix}a\\c\\d\end{matrix}\right\rangle + \left\langle\begin{matrix}a\\b\\e\end{matrix}\middle\|\begin{matrix}a\\b\\e\end{matrix}\right\rangle\right) = \overset{\downarrow}{-\sqrt{\frac1{20}}} =} \left(\sqrt{\tfrac45}\left\langle\begin{matrix}a\\b\\d\end{matrix}\right| + \sqrt{\tfrac1{10}}\left\langle\begin{matrix}a\\c\\d\end{matrix}\right| - \sqrt{\tfrac1{10}}\left\langle\begin{matrix}a\\b\\e\end{matrix}\right|\right) \frac12\left(E_{aa}-E_{bb}-E_{ee}+E_{ff}\right) - \left(\sqrt{\tfrac12}\left|\begin{matrix}a\\c\\d\end{matrix}\right\rangle + \sqrt{\tfrac12}\left|\begin{matrix}a\\b\\e\end{matrix}\right\rangle\right)$$

Off-diagonal-E_{nn} contributions:

$$\left(\sqrt{\tfrac45}\left\langle\begin{matrix}a\\b\\d\end{matrix}\right| + \sqrt{\tfrac1{10}}\left\langle\begin{matrix}a\\c\\d\end{matrix}\right| - \sqrt{\tfrac1{10}}\left\langle\begin{matrix}a\\b\\e\end{matrix}\right|\right) \sqrt{\frac12\left(E_{bc}+E_{de}+E_{cb}+E_{ed}\right)} - \left(\sqrt{\tfrac12}\left|\begin{matrix}a\\c\\d\end{matrix}\right\rangle + \sqrt{\tfrac12}\left|\begin{matrix}a\\b\\e\end{matrix}\right\rangle\right)$$

$$-\tfrac12\left(E_{bc}\left|\begin{matrix}a\\c\\d\end{matrix}\right\rangle + E_{ed}\left|\begin{matrix}a\\c\\d\end{matrix}\right\rangle + E_{cb}\left|\begin{matrix}a\\b\\e\end{matrix}\right\rangle + E_{de}\left|\begin{matrix}a\\b\\e\end{matrix}\right\rangle\right)$$

$$-\tfrac12\left(\left|\begin{matrix}a\\b\\d\end{matrix}\right\rangle + \left|\begin{matrix}a\\c\\e\end{matrix}\right\rangle + \left|\begin{matrix}a\\c\\e\end{matrix}\right\rangle + \left|\begin{matrix}a\\b\\d\end{matrix}\right\rangle\right) = -\left(\left|\begin{matrix}a\\b\\d\end{matrix}\right\rangle + \left|\begin{matrix}a\\c\\e\end{matrix}\right\rangle\right)$$

$$\left(\sqrt8\left\langle\begin{matrix}a\\b\\d\end{matrix}\right| + \left\langle\begin{matrix}a\\c\\d\end{matrix}\right| - \left\langle\begin{matrix}a\\b\\e\end{matrix}\right|\right)$$

$-\sqrt{\frac1{10}}$

$\sqrt8$	0	-0	$\left\|\begin{matrix}a\\b\\d\end{matrix}\right\rangle$
0	0	-0	$+\left\|\begin{matrix}a\\c\\e\end{matrix}\right\rangle$

$$= -\sqrt{\frac8{10}} = -\sqrt{\frac45}$$

Total Result: $-\sqrt{\frac45} - \sqrt{\frac1{20}} = -\frac{4}{\sqrt{20}} - \frac{1}{\sqrt{20}} = -\frac{5}{\sqrt{20}} = -\sqrt{\frac54}$

Calculating *p*³ spin-orbit Hamiltonian matrix for J=3/2

$$\left\langle {}^{2}D_{J=\frac{3}{2}}\ \tfrac{3}{2}\right| H_{s\text{-}o} \left| {}^{2}P_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle \quad \left\langle {}^{2}D_{J=\frac{3}{2}}\ \tfrac{3}{2}\right| H_{s\text{-}o} \left| {}^{4}S_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle$$

$$\left\langle {}^{2}P_{J=\frac{3}{2}}\ \tfrac{3}{2}\right| H_{s\text{-}o} \left| {}^{4}S_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle$$

$$\begin{array}{c|ccc|}
 & \left| {}^{2}D_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle & \left| {}^{2}P_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle & \left| {}^{4}S_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle \\
\hline
\left\langle {}^{2}D_{J=\frac{3}{2}}\ \tfrac{3}{2}\right| & 0 & -\sqrt{\frac{5}{4}} & 0 \\
\left\langle {}^{2}P_{J=\frac{3}{2}}\ \tfrac{3}{2}\right| & -\sqrt{\frac{5}{4}} & 0 & -1 \\
\left\langle {}^{4}S_{J=\frac{3}{2}}\ \tfrac{3}{2}\right| & 0 & -1 & 0
\end{array}$$

Calculating p^3 spin-orbit Hamiltonian matrix for J=3/2

Secular equation:

$$\det\begin{vmatrix} \lambda & -\frac{\sqrt{5}}{2} & 0 \\ -\frac{\sqrt{5}}{2} & \lambda & -1 \\ 0 & -1 & \lambda \end{vmatrix}$$

$$\left\langle {}^2D_{J=\frac{3}{2}}\ \tfrac{3}{2} \right| H_{s\text{-}o} \left| {}^2P_{J=\frac{3}{2}}\ \tfrac{3}{2} \right\rangle \quad \left\langle {}^2D_{J=\frac{3}{2}}\ \tfrac{3}{2} \right| H_{s\text{-}o} \left| {}^4S_{J=\frac{3}{2}}\ \tfrac{3}{2} \right\rangle$$

$$\left\langle {}^2P_{J=\frac{3}{2}}\ \tfrac{3}{2} \right| H_{s\text{-}o} \left| {}^4S_{J=\frac{3}{2}}\ \tfrac{3}{2} \right\rangle$$

$$\begin{array}{c|ccc|} & \left| {}^2D_{J=\frac{3}{2}}\ \tfrac{3}{2} \right\rangle & \left| {}^2P_{J=\frac{3}{2}}\ \tfrac{3}{2} \right\rangle & \left| {}^4S_{J=\frac{3}{2}}\ \tfrac{3}{2} \right\rangle \\ \hline \left\langle {}^2D_{J=\frac{3}{2}}\ \tfrac{3}{2} \right| & 0 & -\sqrt{\frac{5}{4}} & 0 \\ \left\langle {}^2P_{J=\frac{3}{2}}\ \tfrac{3}{2} \right| & -\sqrt{\frac{5}{4}} & 0 & -1 \\ \left\langle {}^4S_{J=\frac{3}{2}}\ \tfrac{3}{2} \right| & 0 & -1 & 0 \end{array}$$

Calculating p^3 spin-orbit Hamiltonian matrix for J=3/2

$$\left\langle {}^2D_{J=\frac{3}{2}}\ \tfrac{3}{2}\right| H_{s\text{-}o} \left| {}^2P_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle \quad \left\langle {}^2D_{J=\frac{3}{2}}\ \tfrac{3}{2}\right| H_{s\text{-}o} \left| {}^4S_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle$$

$$\left\langle {}^2P_{J=\frac{3}{2}}\ \tfrac{3}{2}\right| H_{s\text{-}o} \left| {}^4S_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle$$

Secular equation:

$$\det\begin{vmatrix} \lambda & -\frac{\sqrt{5}}{2} & 0 \\ -\frac{\sqrt{5}}{2} & \lambda & -1 \\ 0 & -1 & \lambda \end{vmatrix} = \lambda\begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} + \frac{\sqrt{5}}{2}\begin{vmatrix} \frac{-\sqrt{5}}{2} & -1 \\ 0 & \lambda \end{vmatrix} = 0$$

$$\lambda(\lambda^2-1)+\tfrac{\sqrt{5}}{2}(\tfrac{-\sqrt{5}}{2})\lambda = 0$$

$$\begin{array}{c|ccc|} & \left| {}^2D_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle & \left| {}^2P_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle & \left| {}^4S_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle \\ \hline \left\langle {}^2D_{J=\frac{3}{2}}\ \tfrac{3}{2}\right| & 0 & -\sqrt{\frac{5}{4}} & 0 \\ \left\langle {}^2P_{J=\frac{3}{2}}\ \tfrac{3}{2}\right| & -\sqrt{\frac{5}{4}} & 0 & -1 \\ \left\langle {}^4S_{J=\frac{3}{2}}\ \tfrac{3}{2}\right| & 0 & -1 & 0 \end{array}$$

Calculating p^3 spin-orbit Hamiltonian matrix for J=3/2

$$\left\langle {}^2D_{J=\frac{3}{2}}\ \tfrac{3}{2}\right| H_{s\text{-}o} \left| {}^2P_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle \quad \left\langle {}^2D_{J=\frac{3}{2}}\ \tfrac{3}{2}\right| H_{s\text{-}o} \left| {}^4S_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle$$

$$\left\langle {}^2P_{J=\frac{3}{2}}\ \tfrac{3}{2}\right| H_{s\text{-}o} \left| {}^4S_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle$$

Secular equation:

$$\det\begin{vmatrix} \lambda & -\frac{\sqrt{5}}{2} & 0 \\ -\frac{\sqrt{5}}{2} & \lambda & -1 \\ 0 & -1 & \lambda \end{vmatrix} = \lambda\begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} + \frac{\sqrt{5}}{2}\begin{vmatrix} \frac{-\sqrt{5}}{2} & -1 \\ 0 & \lambda \end{vmatrix} = 0$$

$$\lambda(\lambda^2-1)+\tfrac{\sqrt{5}}{2}(\tfrac{-\sqrt{5}}{2})\lambda = 0$$

$$\lambda^3-\lambda-\tfrac{5}{4}\lambda = 0 = \lambda(\lambda^2-\tfrac{9}{4}) = \lambda(\lambda-\tfrac{3}{2})(\lambda+\tfrac{3}{2})$$

$$\begin{array}{c|ccc|} & \left| {}^2D_{J=\frac{3}{2}}\ \frac{3}{2}\right\rangle & \left| {}^2P_{J=\frac{3}{2}}\ \frac{3}{2}\right\rangle & \left| {}^4S_{J=\frac{3}{2}}\ \frac{3}{2}\right\rangle \\ \hline \left\langle {}^2D_{J=\frac{3}{2}}\ \frac{3}{2}\right| & 0 & -\sqrt{\frac{5}{4}} & 0 \\ \left\langle {}^2P_{J=\frac{3}{2}}\ \frac{3}{2}\right| & -\sqrt{\frac{5}{4}} & 0 & -1 \\ \left\langle {}^4S_{J=\frac{3}{2}}\ \frac{3}{2}\right| & 0 & -1 & 0 \end{array}$$

Calculating p^3 spin-orbit Hamiltonian matrix for J=3/2

$$\left\langle {}^2D_{J=\frac{3}{2}}\ \tfrac{3}{2}\right| H_{s\text{-}o} \left| {}^2P_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle \quad \left\langle {}^2D_{J=\frac{3}{2}}\ \tfrac{3}{2}\right| H_{s\text{-}o} \left| {}^4S_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle$$

$$\left\langle {}^2P_{J=\frac{3}{2}}\ \tfrac{3}{2}\right| H_{s\text{-}o} \left| {}^4S_{J=\frac{3}{2}}\ \tfrac{3}{2}\right\rangle$$

Secular equation:

$$\det\begin{vmatrix} \lambda & -\frac{\sqrt{5}}{2} & 0 \\ -\frac{\sqrt{5}}{2} & \lambda & -1 \\ 0 & -1 & \lambda \end{vmatrix} = \lambda\begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} + \frac{\sqrt{5}}{2}\begin{vmatrix} \frac{-\sqrt{5}}{2} & -1 \\ 0 & \lambda \end{vmatrix} = 0$$

$$\lambda(\lambda^2-1)+\tfrac{\sqrt{5}}{2}(\tfrac{-\sqrt{5}}{2})\lambda = 0$$

$$\lambda^3-\lambda-\tfrac{5}{4}\lambda = 0 = \lambda(\lambda^2-\tfrac{9}{4}) = \lambda(\lambda-\tfrac{3}{2})(\lambda+\tfrac{3}{2})$$

	$\left\| {}^2D_{J=\frac{3}{2}}\ \frac{3}{2}\right\rangle$	$\left\| {}^2P_{J=\frac{3}{2}}\ \frac{3}{2}\right\rangle$	$\left\| {}^4S_{J=\frac{3}{2}}\ \frac{3}{2}\right\rangle$
$\left\langle {}^2D_{J=\frac{3}{2}}\ \frac{3}{2}\right\|$	0	$-\sqrt{\frac{5}{4}}$	0
$\left\langle {}^2P_{J=\frac{3}{2}}\ \frac{3}{2}\right\|$	$-\sqrt{\frac{5}{4}}$	0	-1
$\left\langle {}^4S_{J=\frac{3}{2}}\ \frac{3}{2}\right\|$	0	-1	0

Projectors: Eigenvalues:

$$P_0 = \begin{pmatrix} \frac{4}{9} & 0 & \frac{-2\sqrt{5}}{9} \\ 0 & 0 & 0 \\ \frac{-2\sqrt{5}}{9} & 0 & \frac{5}{9} \end{pmatrix}$$

λ= 0

$$P_{+3/2} = \begin{pmatrix} \frac{5}{18} & \frac{-\sqrt{5}}{6} & \frac{\sqrt{5}}{9} \\ \frac{-\sqrt{5}}{6} & \frac{1}{2} & \frac{-1}{3} \\ \frac{\sqrt{5}}{9} & \frac{-1}{3} & \frac{2}{9} \end{pmatrix}$$

λ= +3/2

$$P_{-3/2} = \begin{pmatrix} \frac{5}{18} & \frac{\sqrt{5}}{6} & \frac{\sqrt{5}}{9} \\ \frac{\sqrt{5}}{6} & \frac{1}{2} & \frac{1}{3} \\ \frac{\sqrt{5}}{9} & \frac{1}{3} & \frac{2}{9} \end{pmatrix}$$

λ= -3/2

Calculating p^3 spin-orbit Hamiltonian matrix for J=3/2

$$\left\langle {}^2D_{J=\frac{3}{2}}^{\frac{3}{2}} \middle| H_{s\text{-}o} \middle| {}^2P_{J=\frac{3}{2}}^{\frac{3}{2}} \right\rangle \quad \left\langle {}^2D_{J=\frac{3}{2}}^{\frac{3}{2}} \middle| H_{s\text{-}o} \middle| {}^4S_{J=\frac{3}{2}}^{\frac{3}{2}} \right\rangle$$

$$\left\langle {}^2P_{J=\frac{3}{2}}^{\frac{3}{2}} \middle| H_{s\text{-}o} \middle| {}^4S_{J=\frac{3}{2}}^{\frac{3}{2}} \right\rangle$$

Secular equation:

$$\det\begin{vmatrix} \lambda & -\frac{\sqrt{5}}{2} & 0 \\ -\frac{\sqrt{5}}{2} & \lambda & -1 \\ 0 & -1 & \lambda \end{vmatrix} = \lambda\begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} + \frac{\sqrt{5}}{2}\begin{vmatrix} \frac{-\sqrt{5}}{2} & -1 \\ 0 & \lambda \end{vmatrix} = 0$$

$$\lambda(\lambda^2 - 1) + \tfrac{\sqrt{5}}{2}(\tfrac{-\sqrt{5}}{2})\lambda = 0$$

$$\lambda^3 - \lambda - \tfrac{5}{4}\lambda = 0 = \lambda(\lambda^2 - \tfrac{9}{4}) = \lambda(\lambda - \tfrac{3}{2})(\lambda + \tfrac{3}{2})$$

$$\begin{array}{c|ccc|} & \left| {}^2D_{J=\frac{3}{2}}^{\frac{3}{2}} \right\rangle & \left| {}^2P_{J=\frac{3}{2}}^{\frac{3}{2}} \right\rangle & \left| {}^4S_{J=\frac{3}{2}}^{\frac{3}{2}} \right\rangle \\ \hline \left\langle {}^2D_{J=\frac{3}{2}}^{\frac{3}{2}} \right| & 0 & -\sqrt{\frac{5}{4}} & 0 \\ \left\langle {}^2P_{J=\frac{3}{2}}^{\frac{3}{2}} \right| & -\sqrt{\frac{5}{4}} & 0 & -1 \\ \left\langle {}^4S_{J=\frac{3}{2}}^{\frac{3}{2}} \right| & 0 & -1 & 0 \end{array}$$

Projectors: Eigenvalues:

$$P_0 = \begin{pmatrix} \frac{4}{9} & 0 & \frac{-2\sqrt{5}}{9} \\ 0 & 0 & 0 \\ \frac{-2\sqrt{5}}{9} & 0 & \frac{5}{9} \end{pmatrix}$$

λ= 0

$$P_{+3/2} = \begin{pmatrix} \frac{5}{18} & \frac{-\sqrt{5}}{6} & \frac{\sqrt{5}}{9} \\ \frac{-\sqrt{5}}{6} & \frac{1}{2} & \frac{-1}{3} \\ \frac{\sqrt{5}}{9} & \frac{-1}{3} & \frac{2}{9} \end{pmatrix}$$

λ= +3/2

$$P_{-3/2} = \begin{pmatrix} \frac{5}{18} & \frac{\sqrt{5}}{6} & \frac{\sqrt{5}}{9} \\ \frac{\sqrt{5}}{6} & \frac{1}{2} & \frac{1}{3} \\ \frac{\sqrt{5}}{9} & \frac{1}{3} & \frac{2}{9} \end{pmatrix}$$

λ= -3/2

Eigenvectors:

$$|0\rangle = \frac{1}{3}\begin{pmatrix} -2 \\ 0 \\ \sqrt{5} \end{pmatrix}, \quad \left|\tfrac{+3}{2}\right\rangle = \frac{1}{3\sqrt{2}}\begin{pmatrix} -\sqrt{5} \\ 3 \\ -2 \end{pmatrix}, \quad \left|\tfrac{-3}{2}\right\rangle = \frac{1}{3\sqrt{2}}\begin{pmatrix} -\sqrt{5} \\ 3 \\ -2 \end{pmatrix}$$

The ℓ=1 p=shell in a nutshell

quartet ${}^4S$:

$$\begin{array}{cc}L=0 & S=\frac{3}{2}\\ M=0 & \mu=\frac{3}{2}\end{array}\left|\begin{array}{c}\boxed{1}\\\boxed{2}\\\boxed{3}\end{array}\ \uparrow\uparrow\uparrow\right\rangle,\quad \begin{array}{cc}L=0 & S=\frac{3}{2}\\ M=0 & \mu=\frac{1}{2}\end{array}\left|\begin{array}{c}\boxed{1}\\\boxed{2}\\\boxed{3}\end{array}\ \uparrow\uparrow\downarrow\right\rangle\quad \begin{array}{cc}L=0 & S=\frac{3}{2}\\ M=0 & \mu=\frac{-1}{2}\end{array}\left|\begin{array}{c}\boxed{1}\\\boxed{2}\\\boxed{3}\end{array}\ \uparrow\downarrow\downarrow\right\rangle,\quad \begin{array}{cc}L=0 & S=\frac{3}{2}\\ M=0 & \mu=\frac{-3}{2}\end{array}\left|\begin{array}{c}\boxed{1}\\\boxed{2}\\\boxed{3}\end{array}\ \downarrow\downarrow\downarrow\right\rangle.$$

Doublet ${}^2D$, M=2:

$$\begin{array}{c}L=2,\ S=\frac{1}{2}\\ M=2,\ \mu=\frac{1}{2}\end{array}\left|\begin{array}{cc}\boxed{1}&\boxed{1}\\\boxed{2}&\end{array}\ \begin{array}{c}\uparrow\uparrow\\\downarrow\end{array}\right\rangle,\quad \begin{array}{c}L=2,\ S=-\frac{1}{2}\\ M=2,\ \mu=\frac{1}{2}\end{array}\left|\begin{array}{cc}\boxed{1}&\boxed{1}\\\boxed{2}&\end{array}\ \begin{array}{c}\uparrow\downarrow\\\downarrow\end{array}\right\rangle.$$

Doublet ${}^2D$, M=1:

$$\begin{array}{c}L=2,\ S=\frac{1}{2}\\ M=1,\ \mu=\frac{1}{2}\end{array}\ \frac{1}{\sqrt{2}}\left|\begin{array}{cc}\boxed{1}&\boxed{2}\\\boxed{2}&\end{array}\ \begin{array}{c}\uparrow\uparrow\\\downarrow\end{array}\right\rangle+\frac{1}{\sqrt{2}}\left|\begin{array}{cc}\boxed{1}&\boxed{1}\\\boxed{3}&\end{array}\ \begin{array}{c}\uparrow\uparrow\\\downarrow\end{array}\right\rangle,$$

$$\begin{array}{c}L=2,\ S=\frac{1}{2}\\ M=1,\ \mu=\frac{-1}{2}\end{array}\ +\frac{1}{\sqrt{2}}\left|\begin{array}{cc}\boxed{1}&\boxed{2}\\\boxed{2}&\end{array}\ \begin{array}{c}\uparrow\downarrow\\\downarrow\end{array}\right\rangle+\frac{1}{\sqrt{2}}\left|\begin{array}{cc}\boxed{1}&\boxed{1}\\\boxed{3}&\end{array}\ \begin{array}{c}\uparrow\downarrow\\\downarrow\end{array}\right\rangle,$$

Doublet ${}^2P$, M=1:

$$\begin{array}{c}L=1,\ S=\frac{1}{2}\\ M=1,\ \mu=\frac{1}{2}\end{array}\ \frac{1}{\sqrt{2}}\left|\begin{array}{cc}\boxed{1}&\boxed{2}\\\boxed{2}&\end{array}\ \begin{array}{c}\uparrow\uparrow\\\downarrow\end{array}\right\rangle-\frac{1}{\sqrt{2}}\left|\begin{array}{cc}\boxed{1}&\boxed{1}\\\boxed{3}&\end{array}\ \begin{array}{c}\uparrow\uparrow\\\downarrow\end{array}\right\rangle,$$

$$\begin{array}{c}L=1,\ S=\frac{1}{2}\\ M=1,\ \mu=\frac{-1}{2}\end{array}\ +\frac{1}{\sqrt{2}}\left|\begin{array}{cc}\boxed{1}&\boxed{2}\\\boxed{2}&\end{array}\ \begin{array}{c}\uparrow\downarrow\\\downarrow\end{array}\right\rangle-\frac{1}{\sqrt{2}}\left|\begin{array}{cc}\boxed{1}&\boxed{1}\\\boxed{3}&\end{array}\ \begin{array}{c}\uparrow\downarrow\\\downarrow\end{array}\right\rangle,$$

Doublet ${}^2D$, M=0:

$$\begin{array}{c}L=2,\ S=\frac{1}{2}\\ M=0,\ \mu=\frac{1}{2}\end{array}\ \frac{\sqrt{3}}{2}\left|\begin{array}{cc}\boxed{1}&\boxed{2}\\\boxed{3}&\end{array}\ \begin{array}{c}\uparrow\uparrow\\\downarrow\end{array}\right\rangle+\frac{1}{2}\left|\begin{array}{cc}\boxed{1}&\boxed{3}\\\boxed{2}&\end{array}\ \begin{array}{c}\uparrow\uparrow\\\downarrow\end{array}\right\rangle,$$

$$\begin{array}{c}L=2,\ S=\frac{1}{2}\\ M=0,\ \mu=\frac{-1}{2}\end{array}\ +\frac{\sqrt{3}}{2}\left|\begin{array}{cc}\boxed{1}&\boxed{2}\\\boxed{3}&\end{array}\ \begin{array}{c}\uparrow\downarrow\\\downarrow\end{array}\right\rangle+\frac{1}{2}\left|\begin{array}{cc}\boxed{1}&\boxed{3}\\\boxed{2}&\end{array}\ \begin{array}{c}\uparrow\downarrow\\\downarrow\end{array}\right\rangle,$$

Doublet ${}^2P$, M=0:

$$\begin{array}{c}L=1,\ S=\frac{1}{2}\\ M=0,\ \mu=\frac{1}{2}\end{array}\ \frac{-1}{2}\left|\begin{array}{cc}\boxed{1}&\boxed{2}\\\boxed{3}&\end{array}\ \begin{array}{c}\uparrow\uparrow\\\downarrow\end{array}\right\rangle+\frac{\sqrt{3}}{2}\left|\begin{array}{cc}\boxed{1}&\boxed{1}\\\boxed{3}&\end{array}\ \begin{array}{c}\uparrow\uparrow\\\downarrow\end{array}\right\rangle,$$

$$\begin{array}{c}L=1,\ S=\frac{1}{2}\\ M=0,\ \mu=\frac{-1}{2}\end{array}\ -\frac{1}{2}\left|\begin{array}{cc}\boxed{1}&\boxed{2}\\\boxed{3}&\end{array}\ \begin{array}{c}\uparrow\downarrow\\\downarrow\end{array}\right\rangle+\frac{\sqrt{3}}{2}\left|\begin{array}{cc}\boxed{1}&\boxed{3}\\\boxed{2}&\end{array}\ \begin{array}{c}\uparrow\downarrow\\\downarrow\end{array}\right\rangle,$$

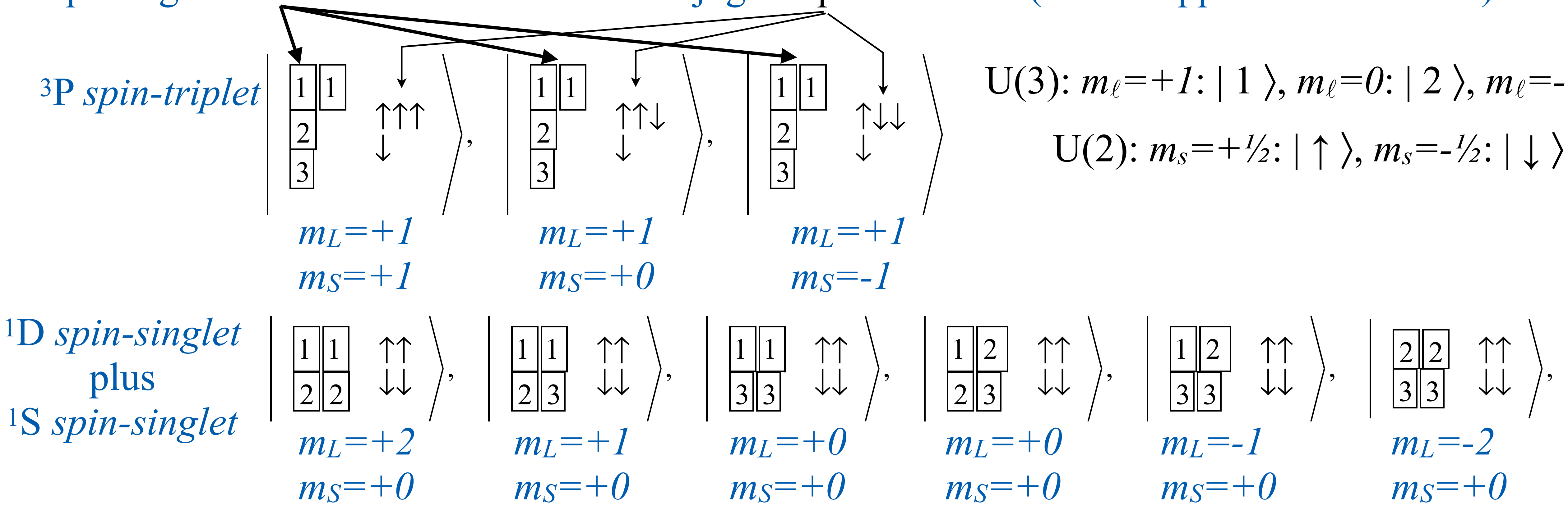
⋮ ⋮ (*M*=-*1* row) ⋮ ⋮

Doublet ${}^2D$, $M=-2$:

$$\begin{array}{c}L=2,\ S=\frac{1}{2}\\ M=-2,\ \mu=\frac{1}{2}\end{array}\left|\begin{array}{cc}\boxed{2}&\boxed{3}\\\boxed{3}&\end{array}\ \begin{array}{c}\uparrow\uparrow\\\downarrow\end{array}\right\rangle,\quad \begin{array}{c}L=2,\ S=\frac{1}{2}\\ M=-2,\ \mu=\frac{-1}{2}\end{array}\left|\begin{array}{cc}\boxed{2}&\boxed{3}\\\boxed{3}&\end{array}\ \begin{array}{c}\uparrow\downarrow\\\downarrow\end{array}\right\rangle.$$

U(3)×U(2) approach: Coupling total orbit-L tableaus to total spin *S* tableaus

A state satisfying Pauli-antisymmetry (Exclusion principle) can be simply represented by putting an orbital tableaus next to a conjugate spin tableaus. (Rows flipped with columns)



U(3): $m_\ell=+1$: $|\,1\,\rangle$, $m_\ell=0$: $|\,2\,\rangle$, $m_\ell=-1$: $|\,3\,\rangle$

U(2): $m_s=+½$: $|\uparrow\rangle$, $m_s=-½$: $|\downarrow\rangle$

These involve fairly complicated S_n-coupled U(3)×U(2) combinations that will be developed later.

An elementary development using U(6) combinations of so called *Slater determinants* is done first.

AMOP reference links on pages 2-4

4.16.18 class 23: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

$(S_n)^*(U(m))$ shell model of electrostatic quadrupole-quadrupole-e interactions

- Marrying spin $s=\frac{1}{2}$ and orbital $\ell=1$ together: U(3)×U(2)
    - The $\ell=1$ p=shell in a nutshell
- U(6)⊃U(3)×U(2) approach: Coupling spin-orbit ($s=\frac{1}{2}$, $\ell=1$) tableaus
- Introducing atomic spin-orbit state assembly formula
    - Slater determinants
    - p-shell Spin-orbit calculations (not finished)
- Clebsch Gordan coefficients. (Rev. Mod. Phys. annual gift)
- S_n projection for atomic spin and orbit states
    - Review of Mach-Mock (particle-state) principle
    - Tableau P-operators on orbits
    - Tableau P-operators on spin
- Fermi-Dirac-Pauli anti-symmetric p^3-states
- Boson operators and symmetric p^2-states
    - Connecting to angular momentum
    - Projecting to angular momentum

U(6)⊃U(3)×U(2) approach: Coupling spin-orbit (s=½, ℓ=1) tableaus

Six states of a single (s=½) electron in (ℓ=1) p-shell labeled by *a* to *f*.

$U(6)$ bases: $\left\{ \left|a\right\rangle \equiv \left|1\uparrow\right\rangle \;,\; \left|b\right\rangle \equiv \left|1\downarrow\right\rangle \;,\; \left|c\right\rangle \equiv \left|2\uparrow\right\rangle \;,\; \left|d\right\rangle \equiv \left|2\downarrow\right\rangle \;,\; \left|e\right\rangle \equiv \left|3\uparrow\right\rangle \;,\; \left|f\right\rangle \equiv \left|3\downarrow\right\rangle \right\}$

U(6)⊃U(3)×U(2) approach: Coupling spin-orbit (s=½, ℓ=1) tableaus

Six states of a single (s=½) electron in (ℓ=1) p-shell labeled by a to f.

$$U(6) \text{ bases: } \left\{ \left|a\right\rangle \equiv \left|1\uparrow\right\rangle , \left|b\right\rangle \equiv \left|1\downarrow\right\rangle , \left|c\right\rangle \equiv \left|2\uparrow\right\rangle , \left|d\right\rangle \equiv \left|2\downarrow\right\rangle , \left|e\right\rangle \equiv \left|3\uparrow\right\rangle , \left|f\right\rangle \equiv \left|3\downarrow\right\rangle \right\}$$

U(6) tensor operators are outer products of U(3) $\mathbf{v}^k_q$($orbit$) with U(2) $\mathbf{v}^\lambda_\sigma$($spin$) operators

$$\left\langle \begin{matrix} \ell & \frac{1}{2} \\ m' & \mu' \end{matrix} \middle| v^{k\,\lambda}_{q\,\sigma} \middle| \begin{matrix} \ell & \frac{1}{2} \\ m & \mu \end{matrix} \right\rangle = \left\langle \begin{matrix} \ell \\ m' \end{matrix} \middle| v^{k}_{q} \middle| \begin{matrix} \ell \\ m \end{matrix} \right\rangle \left\langle \begin{matrix} \frac{1}{2} \\ \mu' \end{matrix} \middle| v^{\lambda}_{\sigma} \middle| \begin{matrix} \frac{1}{2} \\ \mu \end{matrix} \right\rangle$$

U(6)⊃U(3)×U(2) approach: Coupling spin-orbit ($s=½$, $\ell=1$) tableaus

Six states of a single ($s=½$) electron in ($\ell=1$) p-shell labeled by a to f.

$$U(6)\text{ bases: }\left\{\left|a\right\rangle\equiv\left|1\uparrow\right\rangle\;,\;\left|b\right\rangle\equiv\left|1\downarrow\right\rangle\;,\;\left|c\right\rangle\equiv\left|2\uparrow\right\rangle\;,\;\left|d\right\rangle\equiv\left|2\downarrow\right\rangle\;,\;\left|e\right\rangle\equiv\left|3\uparrow\right\rangle\;,\;\left|f\right\rangle\equiv\left|3\downarrow\right\rangle\right\}$$

U(6) tensor operators are outer products of U(3) $\mathbf{v}^k{}_q(orbit)$ with U(2) $\mathbf{v}^\lambda{}_\sigma(spin)$ operators

$$\left\langle \begin{matrix}\ell & \frac{1}{2}\\ m' & \mu'\end{matrix}\right| v^{k\,\lambda}_{q\,\sigma} \left|\begin{matrix}\ell & \frac{1}{2}\\ m & \mu\end{matrix}\right\rangle = \left\langle \begin{matrix}\ell\\ m'\end{matrix}\right| v^{k}_{q} \left|\begin{matrix}\ell\\ m\end{matrix}\right\rangle \left\langle \begin{matrix}\frac{1}{2}\\ \mu'\end{matrix}\right| v^{\lambda}_{\sigma} \left|\begin{matrix}\frac{1}{2}\\ \mu\end{matrix}\right\rangle$$

$$\left\langle\mathbf{v}^2_{\bar{2}}\right\rangle=\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\\1&\cdot&\cdot\end{pmatrix}\quad \left\langle\mathbf{v}^2_{\bar{1}}\right\rangle=\begin{pmatrix}\cdot&\cdot&\cdot\\1&\cdot&\cdot\\\cdot&\bar{1}&\cdot\end{pmatrix}\tfrac{1}{\sqrt{2}}\quad \left\langle\mathbf{v}^2_{0}\right\rangle=\begin{pmatrix}1&\cdot&\cdot\\\cdot&\bar{2}&\cdot\\\cdot&\cdot&1\end{pmatrix}\tfrac{1}{\sqrt{6}}\quad \left\langle\mathbf{v}^2_{1}\right\rangle=\begin{pmatrix}\cdot&\bar{1}&\cdot\\\cdot&\cdot&1\\\cdot&\cdot&\cdot\end{pmatrix}\tfrac{1}{\sqrt{2}}\quad \left\langle\mathbf{v}^2_{2}\right\rangle=\begin{pmatrix}\cdot&\cdot&1\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$$

$$\left\langle\mathbf{v}^1_{\bar{1}}\right\rangle=\begin{pmatrix}\cdot&\cdot\\1&\cdot\end{pmatrix}\quad \left\langle\mathbf{v}^1_{0}\right\rangle=\begin{pmatrix}1&\cdot\\\cdot&\bar{1}\end{pmatrix}\tfrac{1}{\sqrt{2}}\quad \left\langle\mathbf{v}^1_{1}\right\rangle=\begin{pmatrix}\cdot&\bar{1}\\\cdot&\cdot\end{pmatrix}$$

$$\left\langle\mathbf{v}^0_{0}\right\rangle=\begin{pmatrix}1&\cdot\\\cdot&1\end{pmatrix}\tfrac{1}{\sqrt{3}}$$

$$\left\langle\mathbf{v}^1_{\bar{1}}\right\rangle=\begin{pmatrix}\cdot&\cdot&\cdot\\1&\cdot&\cdot\\\cdot&1&\cdot\end{pmatrix}\tfrac{1}{\sqrt{2}}\quad \left\langle\mathbf{v}^1_{0}\right\rangle=\begin{pmatrix}1&\cdot&\cdot\\\cdot&0&\cdot\\\cdot&\cdot&\bar{1}\end{pmatrix}\tfrac{1}{\sqrt{2}}\quad \left\langle\mathbf{v}^1_{1}\right\rangle=\begin{pmatrix}\cdot&\bar{1}&\cdot\\\cdot&\cdot&\bar{1}\\\cdot&\cdot&\cdot\end{pmatrix}\tfrac{1}{\sqrt{2}}$$

$$\left\langle\mathbf{v}^0_{0}\right\rangle=\begin{pmatrix}1&\cdot&\cdot\\\cdot&1&\cdot\\\cdot&\cdot&1\end{pmatrix}\tfrac{1}{\sqrt{3}}$$

Notational compaction:

$\bar{1}\equiv-1,\ \bar{2}\equiv-2,\quad etc.$

U(6)⊃U(3)×U(2) approach: Coupling spin-orbit (s=½, ℓ=1) tableaus

Six states of a single (s=½) electron in (ℓ=1) p-shell labeled by a to f.

$U(6)$ bases: $\left\{\left|a\right\rangle \equiv \left|1\uparrow\right\rangle , \left|b\right\rangle \equiv \left|1\downarrow\right\rangle , \left|c\right\rangle \equiv \left|2\uparrow\right\rangle , \left|d\right\rangle \equiv \left|2\downarrow\right\rangle , \left|e\right\rangle \equiv \left|3\uparrow\right\rangle , \left|f\right\rangle \equiv \left|3\downarrow\right\rangle\right\}$

U(6) tensor operators are outer products of U(3) $\mathbf{v}^k{}_q$(*orbit*) with U(2) $\mathbf{v}^\lambda{}_\sigma$(*spin*) operators

$$\left\langle \begin{matrix} \ell & \frac{1}{2} \\ m' & \mu' \end{matrix} \middle| v^{k\,\lambda}_{q\,\sigma} \middle| \begin{matrix} \ell & \frac{1}{2} \\ m & \mu \end{matrix} \right\rangle = \left\langle \begin{matrix} \ell \\ m' \end{matrix} \middle| v^{k}_{q} \middle| \begin{matrix} \ell \\ m \end{matrix} \right\rangle \left\langle \begin{matrix} \frac{1}{2} \\ \mu' \end{matrix} \middle| v^{\lambda}_{\sigma} \middle| \begin{matrix} \frac{1}{2} \\ \mu \end{matrix} \right\rangle$$

$$\left\langle \mathbf{v}^2_{\bar{2}} \right\rangle = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \end{pmatrix} \quad \left\langle \mathbf{v}^2_{\bar{1}} \right\rangle = \begin{pmatrix} \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \\ \cdot & \bar{1} & \cdot \end{pmatrix}\tfrac{1}{\sqrt{2}} \quad \left\langle \mathbf{v}^2_{0} \right\rangle = \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & \bar{2} & \cdot \\ \cdot & \cdot & 1 \end{pmatrix}\tfrac{1}{\sqrt{6}} \quad \left\langle \mathbf{v}^2_{1} \right\rangle = \begin{pmatrix} \cdot & \bar{1} & \cdot \\ \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \end{pmatrix}\tfrac{1}{\sqrt{2}} \quad \left\langle \mathbf{v}^2_{2} \right\rangle = \begin{pmatrix} \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

$$\left\langle \mathbf{v}^1_{\bar{1}} \right\rangle = \begin{pmatrix} \cdot & \cdot \\ 1 & \cdot \end{pmatrix} \quad \left\langle \mathbf{v}^1_{0} \right\rangle = \begin{pmatrix} 1 & \cdot \\ \cdot & \bar{1} \end{pmatrix}\tfrac{1}{\sqrt{2}} \quad \left\langle \mathbf{v}^1_{1} \right\rangle = \begin{pmatrix} \cdot & \bar{1} \\ \cdot & \cdot \end{pmatrix}$$

$$\left\langle \mathbf{v}^0_{0} \right\rangle = \begin{pmatrix} 1 & \cdot \\ \cdot & 1 \end{pmatrix}\tfrac{1}{\sqrt{3}}$$

$$\left\langle \mathbf{v}^1_{\bar{1}} \right\rangle = \begin{pmatrix} \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \end{pmatrix}\tfrac{1}{\sqrt{2}} \quad \left\langle \mathbf{v}^1_{0} \right\rangle = \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & \bar{1} \end{pmatrix}\tfrac{1}{\sqrt{2}} \quad \left\langle \mathbf{v}^1_{1} \right\rangle = \begin{pmatrix} \cdot & \bar{1} & \cdot \\ \cdot & \cdot & \bar{1} \\ \cdot & \cdot & \cdot \end{pmatrix}\tfrac{1}{\sqrt{2}}$$

Notational compaction:

$\bar{1} \equiv -1, \;\; \bar{2} \equiv -2, \quad$ *etc.*

$$\left\langle \mathbf{v}^0_{0} \right\rangle = \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix}\tfrac{1}{\sqrt{3}}$$

$\frac{1}{\sqrt{2}}(-E_{cb}-E_{ed}) =$

$$\left\langle \mathbf{v}^{11}_{\bar{1}1} \right\rangle = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \bar{1} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \bar{1} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}\tfrac{1}{\sqrt{2}} \quad \left\langle \mathbf{v}^{11}_{00} \right\rangle = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \bar{1} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \bar{1} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}\tfrac{1}{2} \quad \left\langle \mathbf{v}^{11}_{1\bar{1}} \right\rangle = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \bar{1} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \bar{1} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}\tfrac{1}{\sqrt{2}} = \mathbf{v}^1_1 \otimes \mathbf{v}^1_{\bar{1}} = \tfrac{1}{\sqrt{2}}\begin{pmatrix} \cdot & \bar{1} & \cdot \\ \cdot & \cdot & \bar{1} \\ \cdot & \cdot & \cdot \end{pmatrix} \otimes \begin{pmatrix} \cdot & \cdot \\ 1 & \cdot \end{pmatrix}$$

$= \frac{1}{\sqrt{2}}(-E_{bc}-E_{de})$

$$\left\langle \mathbf{v}^{11}_{\bar{1}\bar{1}} \right\rangle = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix}\tfrac{1}{\sqrt{2}} \quad \left\langle \mathbf{v}^{00}_{00} \right\rangle = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}\tfrac{1}{\sqrt{6}} \quad \left\langle \mathbf{v}^{11}_{11} \right\rangle = \begin{pmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}\tfrac{1}{\sqrt{2}}$$

p-shell Spin-orbit calculation