AMOP reference links on following page

1.24.18 class 4.0: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

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...and some other stuff for next class.

AMOP reference links *(Updated list given on 2nd page of each class presentation)*

Web Resources - front page

UAF Physics UTube channel

2014 AMOP

2017 Group Theory for QM

2018 AMOP

Frame Transformation Relations And Multipole Transitions In Symmetric Polyatomic Molecules - RMP-1978 (Alt Scanned version)
Rotational energy surfaces and high- J eigenvalue structure of polyatomic molecules - Harter - Patterson - 1984
Galloping waves and their relativistic properties - ajp-1985-Harter
Asymptotic eigensolutions of fourth and sixth rank octahedral tensor operators - Harter-Patterson-JMP-1979
Nuclear spin weights and gas phase spectral structure of 12C60 and 13C60 buckminsterfullerene -Harter-Reimer-Cpl-1992 - (Alt1, Alt2 Erratum)

Theory of hyperfine and superfine levels in symmetric polyatomic molecules.

I) Trigonal and tetrahedral molecules: Elementary spin-1/2 cases in vibronic ground states - PRA-1979-Harter-Patterson (Alt scan)
II) Elementary cases in octahedral hexafluoride molecules - Harter-PRA-1981 (Alt scan)

Rotation-vibration scalar coupling zeta coefficients and spectroscopic band shapes of buckminsterfullerene - Weeks-Harter-CPL-1991 (Alt scan)
Fullerene symmetry reduction and rotational level fine structure/ the Buckyball isotopomer 12C 13C59 - jcp-Reimer-Harter-1997 (HiRez)
Molecular Eigensolution Symmetry Analysis and Fine Structure - IJMS-harter-mitchell-2013

Rotation–vibration spectra of icosahedral molecules.

I) Icosahedral symmetry analysis and fine structure - harter-weeks-jcp-1989
II) Icosahedral symmetry, vibrational eigenfrequencies, and normal modes of buckminsterfullerene - weeks-harter-jcp-1989
III) Half-integral angular momentum - harter-reimer-jcp-1991

QTCA Unit 10 Ch 30 - 2013
AMOP Ch 32 Molecular Symmetry and Dynamics - 2019
AMOP Ch 0 Space-Time Symmetry - 2019

RESONANCE AND REVIVALS

I) QUANTUM ROTOR AND INFINITE-WELL DYNAMICS - ISMSLi2012 (Talk) https://kb.osu.edu/dspace/handle/1811/52324
II) Comparing Half-integer Spin and Integer Spin - Alva-ISMS-Ohio2013-R777 (Talks)
III) Quantum Resonant Beats and Revivals in the Morse Oscillators and Rotors - (2013-Li-Diss)

Rovibrational Spectral Fine Structure Of Icosahedral Molecules - Cpl 1986 (Alt Scan)
Gas Phase Level Structure of C60 Buckyball and Derivatives Exhibiting Broken Icosahedral Symmetry - reimer-diss-1996
Resonance and Revivals in Quantum Rotors - Comparing Half-integer Spin and Integer Spin - Alva-ISMS-Ohio2013-R777 (Talk)
Quantum Revivals of Morse Oscillators and Farey-Ford Geometry - Li-Harter-cpl-2013
Wave Node Dynamics and Revival Symmetry in Quantum Rotors - harter - jms - 2001

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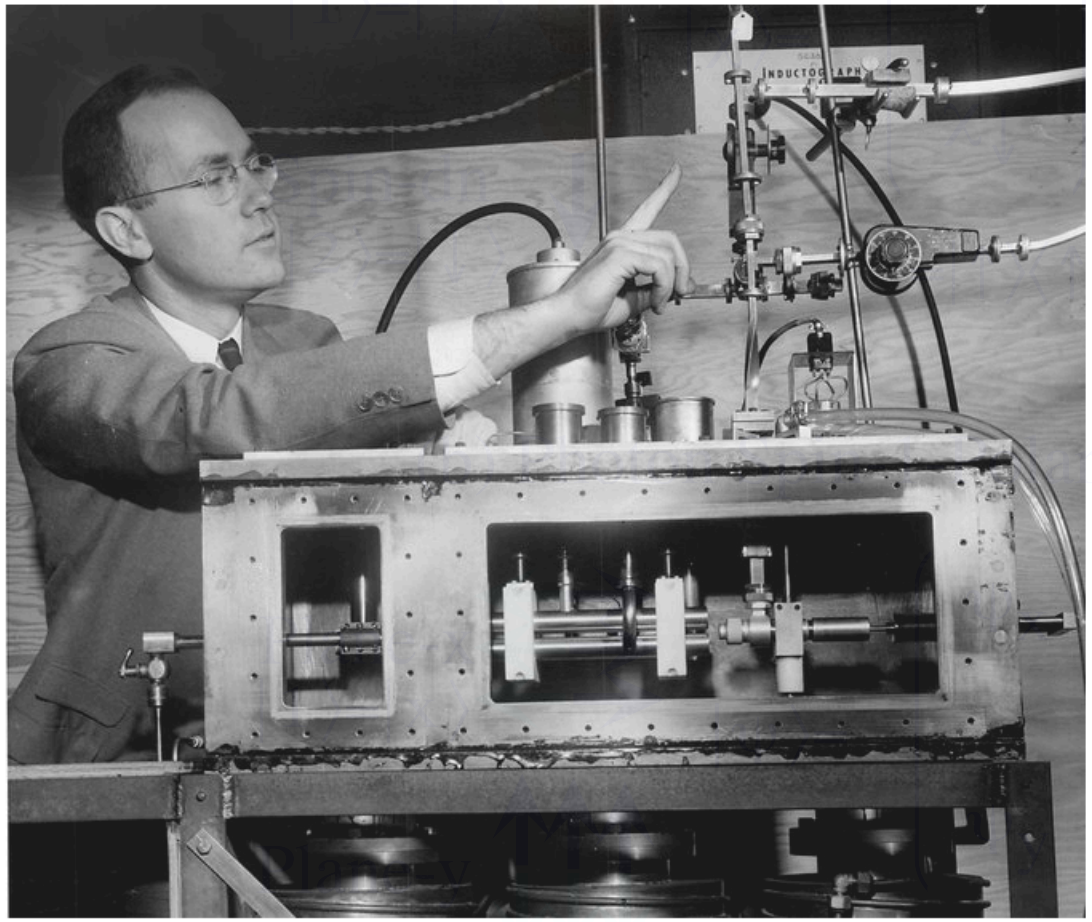
(a) Electron Spin-1/2-Polarization

Charles H. Townes, Who Paved Way for the Laser in Daily Life, Dies at 99

By ROBERT D. McFADDEN JAN. 28, 2015



Charles Townes in 1955. Eddie Hausner/The New York Times

The New York Times — Most Popular | Video | My Account

Today's Headlines

Thursday, January 29, 2015

He had an "a-ha!" moment. Sitting on a park bench in Washington one April morning in 1951, pondering how to stimulate molecular energy to create shorter wavelengths, he conceived of a device he called a maser, for microwave amplification by stimulated emission of radiation. It would use molecules to nudge other molecules, and amplify their thrust by getting them to resonate like tuning forks and line up in a powerful beam.

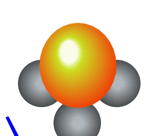
He and two graduate students, James P. Gordon and H. J. Zeigler, built his maser in 1953 and patented their creation. It was the first device operating on the principles of the laser, although it amplified microwave radiation rather than infrared or visible light radiation.

Five years later, Dr. Townes and Dr. Schawlow, who was his brother-in-law and would win the 1981 Nobel Prize in Physics for work on laser spectroscopy, drew a blueprint for a laser. They called it an optical maser, a term that never caught on, and through Bell Laboratories they secured the first laser patent in 1959, a year before Dr. Maiman's first working model.

(c) Ammonia (NH_3) Inversion States

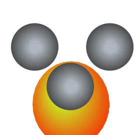
N-UP

$|1\rangle = |UP\rangle$

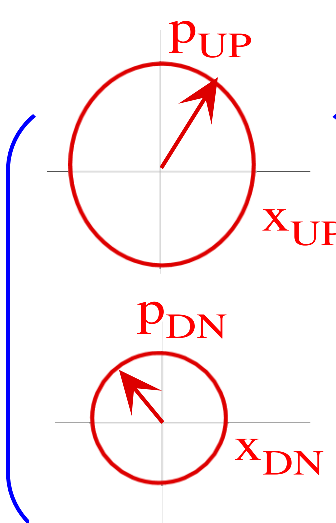


N-DN

$|2\rangle = |DN\rangle$



$$|\nu\rangle = \begin{pmatrix} \nu_{UP} \\ \nu_{DN} \end{pmatrix} = \begin{pmatrix} \langle UP|\nu\rangle \\ \langle DN|\nu\rangle \end{pmatrix} = $$



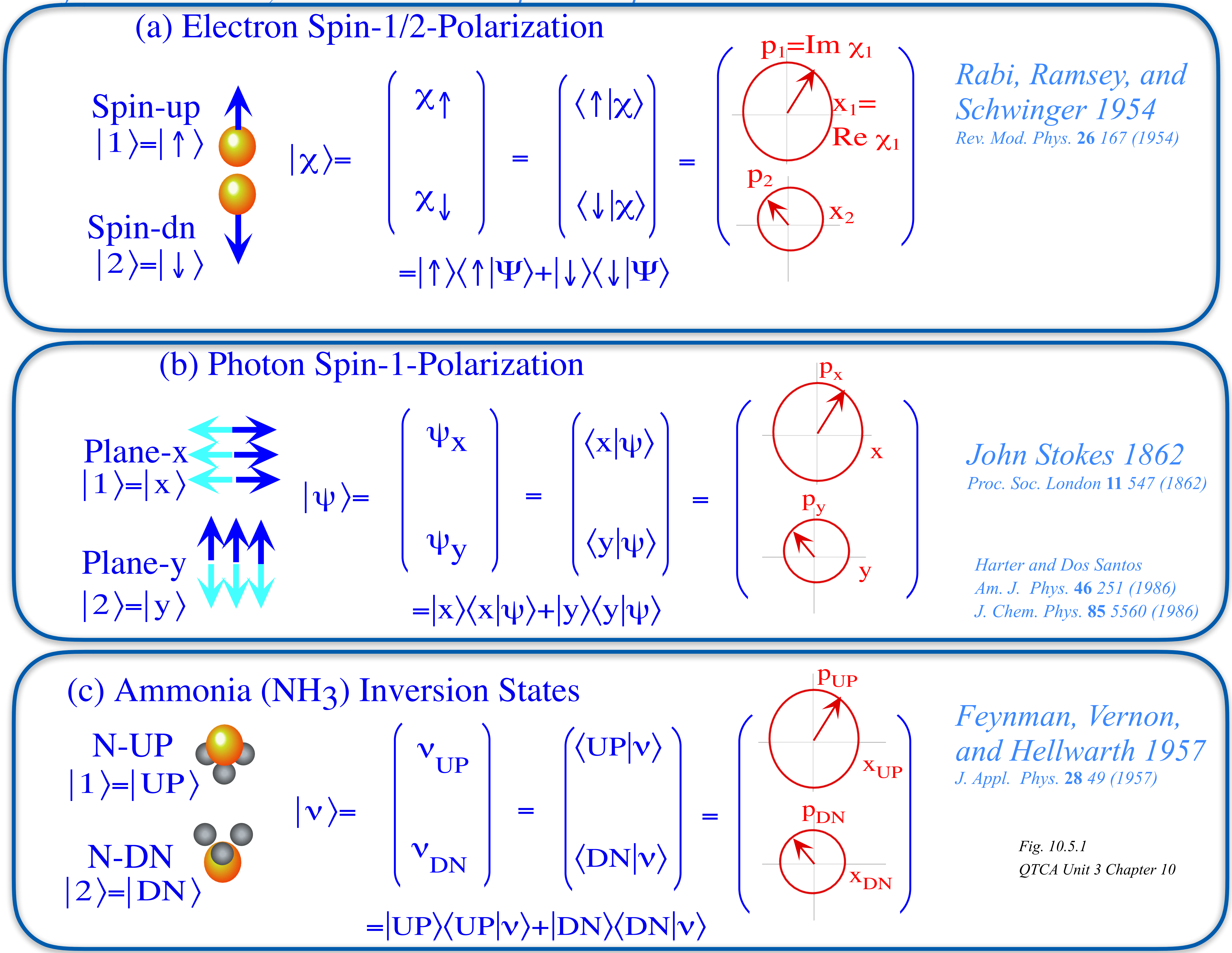
$$= |UP\rangle\langle UP|\nu\rangle + |DN\rangle\langle DN|\nu\rangle$$

Feynman, Vernon, and Hellwarth 1957

J. Appl. Phys. **28** 49 (1957)

Fig. 10.5.1
QTCA Unit 3 Chapter 10

Three famous 2-state systems and two-complex-component coordinates



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Newton-Hooke equation

First start with 2-by-2 Hermitian (self-conjugate) matrix

$$\mathbf{H} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H}^\dagger$$

H_{jk} matrix must obey: $(H_{jk})^* = H_{kj}$

that operates on 2-D complex Dirac ket vector $|\Psi\rangle$.

Both have 4 parameters *($2^2 = 2+2$)*

$$|\Psi\rangle = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

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$$\begin{pmatrix} i\dot{x}_1 - \dot{p}_1 \\ i\dot{x}_2 - \dot{p}_2 \end{pmatrix} = \begin{pmatrix} Ax_1 + Bx_2 + Cp_2 + iAp_1 + iBp_2 - iCx_2 \\ Bx_1 + Dx_2 - Cp_1 + iBp_1 + iDp_2 + iCx_1 \end{pmatrix}$$

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Finally a 2nd time derivative (Assume *constant* A, B, D, and *let* $C=0$) gives 2nd-order classical Newton-Hooke-like equation: $|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$

$$\begin{aligned}\ddot{x}_1 &= A\dot{p}_1 + B\dot{p}_2 - C\dot{x}_2 \\ &= -A\left(Ax_1 + Bx_2 + Cp_2\right) - B\left(Bx_1 + Dx_2 - Cp_1\right) - C\left(Bp_1 + Dp_2 + Cx_1\right) \\ &= -\left(A^2 + B^2 + C^2\right)x_1 - \left(AB + BD\right)x_2 - C\left(A + D\right)p_2\end{aligned}$$

$$\begin{aligned}\ddot{x}_2 &= B\dot{p}_1 + D\dot{p}_2 + C\dot{x}_1 \\ &= -B\left(Ax_1 + Bx_2 + Cp_2\right) - D\left(Bx_1 + Dx_2 - Cp_1\right) + C\left(Ap_1 + Bp_2 - Cx_2\right) \\ &= -\left(AB + BD\right)x_1 - \left(B^2 + D^2 + C^2\right)x_2 + C\left(A + D\right)p_1\end{aligned}$$

For constant A, B, C, *and* D

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$$= -\left(A^2+B^2+C^2\right)x_1 - \left(AB+BD\right)x_2 - C\left(A+D\right)p_2$$

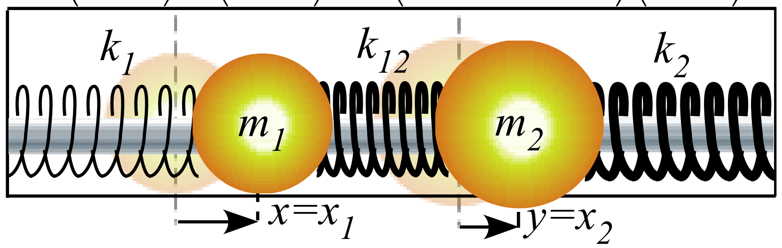
$$\ddot{x}_2 = B\dot{p}_1 + D\dot{p}_2 + C\dot{x}_1$$
$$= -B\left(Ax_1+Bx_2+Cp_2\right) - D\left(Bx_1+Dx_2-Cp_1\right) + C\left(Ap_1+Bp_2-Cx_2\right)$$
$$= -\left(AB+BD\right)x_1 - \left(B^2+D^2+C^2\right)x_2 + C\left(A+D\right)p_1$$

For constant $A, B, C,$ *and* D

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} A^2+B^2 & AB+BD \\ AB+BD & B^2+D^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

For C=0 Is form of 2D Hooke harmonic oscillator

$$\frac{\partial^2}{\partial t^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \equiv \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$



ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

$$i\hbar|\dot{\Psi}(t)\rangle=\mathbf{H}|\Psi(t)\rangle$$

$$|\ddot{\mathbf{x}}\rangle=-\mathbf{K}\bullet|\mathbf{x}\rangle$$

Newton-Hooke equation

First start with 2-by-2 Hermitian (self-conjugate) matrix

$$\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D\end{pmatrix}=\mathbf{H}^\dagger$$

that operates on 2-D complex Dirac ket vector $|\Psi\rangle$.

$$|\Psi\rangle=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}=\begin{pmatrix}a_1\\ a_2\end{pmatrix}$$

Separate real x_k and imaginary p_k parts of Ψ_k amplitudes to convert the complex 1st-order equation $i\partial_t\Psi=\mathbf{H}\Psi$ into pairs of ***real*** 1st-order differential equations.

$$\dot{x}_1=Ap_1+Bp_2-Cx_2 \qquad \dot{p}_1=-Ax_1-Bx_2-Cp_2$$
$$\dot{x}_2=Bp_1+Dp_2+Cx_1 \qquad \dot{p}_2=-Bx_1-Dx_2+Cp_1$$

Then start with classical Hamiltonian. (Designed to give same result.)

$$H_c=\frac{A}{2}\left(p_1^2+x_1^2\right)+B\left(x_1x_2+p_1p_2\right)+C\left(x_1p_2-x_2p_1\right)+\frac{D}{2}\left(p_2^2+x_2^2\right)$$

Then Hamilton's equations of motion are the following.

$$\dot{x}_1=\frac{\partial H_c}{\partial p_1}=Ap_1+Bp_2-Cx_2 \qquad \dot{p}_1=-\frac{\partial H_c}{\partial x_1}=-\left(Ax_1+Bx_2+Cp_2\right)$$
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QM vs. Classical Equations are identical

Finally a 2nd time derivative (Assume *constant* A, B, D, and *let* $C=0$) gives 2nd-order classical Newton-Hooke-like equation: $|\ddot{\mathbf{x}}\rangle=-\mathbf{K}\bullet|\mathbf{x}\rangle$

$$\ddot{x}_1=A\dot{p}_1+B\dot{p}_2-C\dot{x}_2$$
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$$=-\left(A^2+B^2+C^2\right)x_1-\left(AB+BD\right)x_2-C\left(A+D\right)p_2$$

$$\ddot{x}_2=B\dot{p}_1+D\dot{p}_2+C\dot{x}_1$$
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For constant A*,*B*,*C*, and* D

$$\begin{pmatrix}\ddot{x}_1\\ \ddot{x}_2\end{pmatrix}=-\begin{pmatrix}A^2+B^2 & AB+BD\\ AB+BD & B^2+D^2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

For $C=0$ *Is form of 2D Hooke harmonic oscillator*

$$\frac{\partial^2}{\partial t^2}\begin{pmatrix}x_1\\ x_2\end{pmatrix}\equiv\begin{pmatrix}\ddot{x}_1\\ \ddot{x}_2\end{pmatrix}=-\begin{pmatrix}K_{11} & K_{12}\\ K_{21} & K_{22}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

ABD-to- K_{ij} *or* k_a *connection formulae* $(m_1=1=m_2)$

$$K_{11}=A^2+B^2=k_1+k_{12}, \qquad K_{12}=AB+BD=-k_{12},$$
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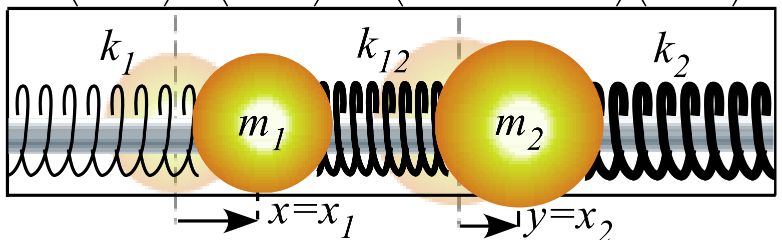

Here is an operator view of the QM-Classical connection: Take Schrodinger operator $i\partial_t=\mathbf{H}$ (with $C=0$) and square it!

…for "natural" units ($\hbar=1$)

$$i\frac{\partial}{\partial t}=\begin{pmatrix}A & B\\ B & D\end{pmatrix}\Rightarrow\left(i\frac{\partial}{\partial t}\right)^2=\begin{pmatrix}A & B\\ B & D\end{pmatrix}^2\Rightarrow-\frac{\partial^2}{\partial t^2}=\begin{pmatrix}A^2+B^2 & AB+BD\\ AB+BD & B^2+D^2\end{pmatrix}$$

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$$\begin{aligned}\ddot{x}_1&=A\dot{p}_1+B\dot{p}_2-C\dot{x}_2\\&=-A\left(Ax_1+Bx_2+Cp_2\right)-B\left(Bx_1+Dx_2-Cp_1\right)-C\left(Bp_1+Dp_2+Cx_1\right)\\&=-\left(A^2+B^2+C^2\right)x_1-\left(AB+BD\right)x_2-C\left(A+D\right)p_2\end{aligned}$$

$$\begin{aligned}\ddot{x}_2&=B\dot{p}_1+D\dot{p}_2+C\dot{x}_1\\&=-B\left(Ax_1+Bx_2+Cp_2\right)-D\left(Bx_1+Dx_2-Cp_1\right)+C\left(Ap_1+Bp_2-Cx_2\right)\\&=-\left(AB+BD\right)x_1-\left(B^2+D^2+C^2\right)x_2+C\left(A+D\right)p_1\end{aligned}$$

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$$\begin{pmatrix}\ddot{x}_1\\ \ddot{x}_2\end{pmatrix}=-\begin{pmatrix}A^2+B^2 & AB+BD\\ AB+BD & B^2+D^2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

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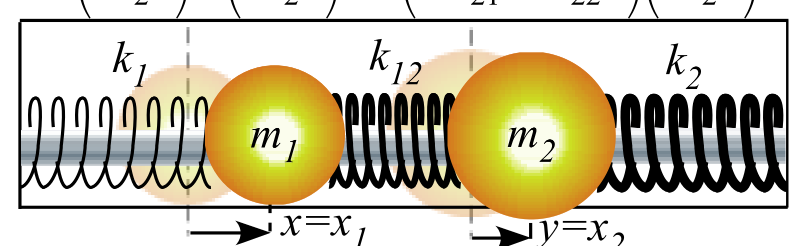

Here is an operator view of the QM-Classical connection: Take Schrodinger operator $i\partial_t=\mathbf{H}$ (with $C\neq 0$) and square it!

Now it's *Complex*

…for "natural" units ($\hbar=1$)

$$i\frac{\partial}{\partial t}=\begin{pmatrix}A & B-iC\\ B+iC & D\end{pmatrix}\Rightarrow\left(i\frac{\partial}{\partial t}\right)^2=\begin{pmatrix}A & B-iC\\ B+iC & D\end{pmatrix}^2\Rightarrow-\frac{\partial^2}{\partial t^2}=\begin{pmatrix}A^2+B^2+C^2 & AB+BD-i(AC+CD)\\ AB+BD+i(AC+CD) & B^2+D^2+C^2\end{pmatrix}$$

Conclusion: 2-state Schro-equation $i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ is like "square-root" of Newton-Hooke. $\sqrt{|\ddot{\mathbf{x}}\rangle=-\mathbf{K}\bullet|\mathbf{x}\rangle}$

Symmetry group $\mathscr{G}$ representations=>*AMOP* Hamiltonian **H** (or K) matrices, irreps $\mathscr{D}^{(\alpha)}$
=>*AMOP* wave functions $\Psi^{(\alpha)}$, eigensolutions

$\mathscr{G}$ = U(2) = Unitary group of dimension 2

Memoriam: Charles H. Townes 1916-2015 and his famous 2-state system: NH_3 maser in 1955

Earlier 2-state systems: 1863 John Stokes optical polarization , 1954 Rabi, Ramsey, and Schwinger NMR (MRI)

ANALOGY: (1) Classical 2-state motion $(\partial/\partial t)^2\mathbf{x}=-\mathbf{K}\cdot\mathbf{x}$ *vs (2) Quantum 2-state motion* $ih(\partial/\partial t)\mathbf{\Psi}=\mathbf{H}\cdot\mathbf{\Psi}$

Hamilton-Pauli spinor symmetry and **σ***-expansion of* $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu=\omega_A\boldsymbol{\sigma}_A+\omega_B\boldsymbol{\sigma}_B+\omega_C\boldsymbol{\sigma}_C+\omega_o\boldsymbol{\sigma}_o$

ABCD Time evolution operator $\mathbf{U}(t)=e^{-i\mathbf{H}t}$ *; its evaluation and visualization*

ABCD symmetry operator $\{\boldsymbol{\sigma}_A,\boldsymbol{\sigma}_B,\boldsymbol{\sigma}_C\}$ *product algebra for spinor-vector operators* $\sigma_a=\boldsymbol{\sigma}\cdot\mathbf{a}$

Spinor-vector operator products $(\boldsymbol{\sigma}\cdot\mathbf{a})(\boldsymbol{\sigma}\cdot\mathbf{a})$

Crazy-Thing Theorem: $e^{-i\sigma_a\Theta}=\cos\Theta-i\sigma_a\sin\Theta$

U(2) transformation matrices and related R(3) rotations in ABC-space

Mysterious factors of 2 or ½ on 2D spinors or 3D vectors

2D {↑,↓} spinor space ½ as fast as 3D {ABC} spin-vectors

Hamiltonian for NMR: 3D Spin Moment Vector $\mathbf{m}=(m_x, m_y, m_z)$ *in field* $\mathbf{B}=(B_x, B_y, B_z)$

State coordinates using Euler-angle rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case

Spin-1/2 (2D-complex spinor) case

The ABC's of U(2) dynamics-Archetypes

Asymmetric-Diagonal A-Type motion

Bilateral-Balanced B-Type motion

Circular-Coriolis… C-Type motion

$ABCD$ Symmetry operator analysis and $U(2)$ spinors

Decompose the Hamiltonian operator $\mathbf{H}$ into four *ABCD symmetry operators*
(Labeled to provide dynamic mnemonics as well as colorful analogies)

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = A\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + D\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = A\mathbf{e}_{11} + B\boldsymbol{\sigma}_B + C\boldsymbol{\sigma}_C + D\mathbf{e}_{22}$$

$$= \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{H} = \frac{A-D}{2}\boldsymbol{\sigma}_A + B\boldsymbol{\sigma}_B + C\boldsymbol{\sigma}_C + \frac{A+D}{2}\boldsymbol{\sigma}_0$$

Symmetry archetypes: *A (Asymmetric-diagonal)*| *B (Bilateral-balanced)*| *C (complex, circular, chiral, cyclotron, ...*

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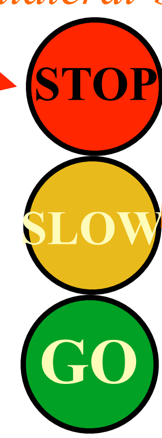
$$\mathbf{H} = \frac{A-D}{2}\boldsymbol{\sigma}_A + B\boldsymbol{\sigma}_B + C\boldsymbol{\sigma}_C + \frac{A+D}{2}\boldsymbol{\sigma}_0$$

Are there three C_2 subgroups?
$C_2^{(A)}$, $C_2^{(B)}$, $C_2^{(C)}$,

Symmetry archetypes: *A (Asymmetric-diagonal)*| *B (Bilateral-balanced)*| *C (complex, circular, chiral, cyclotron, Coriolis, centrifugal, curly, and circulating-current-carrying...)*

Standing waves

Motivation for coloring scheme:
The Traffic Signal



Moving waves

ABCD Symmetry operator analysis and U(2) spinors

Decompose the Hamiltonian operator **H** into four *ABCD symmetry operators*

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$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = A\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + D\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = A\mathbf{e}_{11} + B\boldsymbol{\sigma}_B + C\boldsymbol{\sigma}_C + D\mathbf{e}_{22}$$

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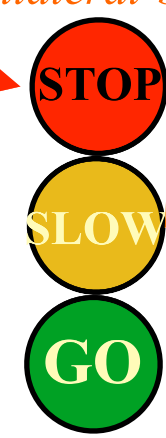

Moving waves ($C\neq 0$)

Standing waves ($C=0$)

Yes, and a LOT more!

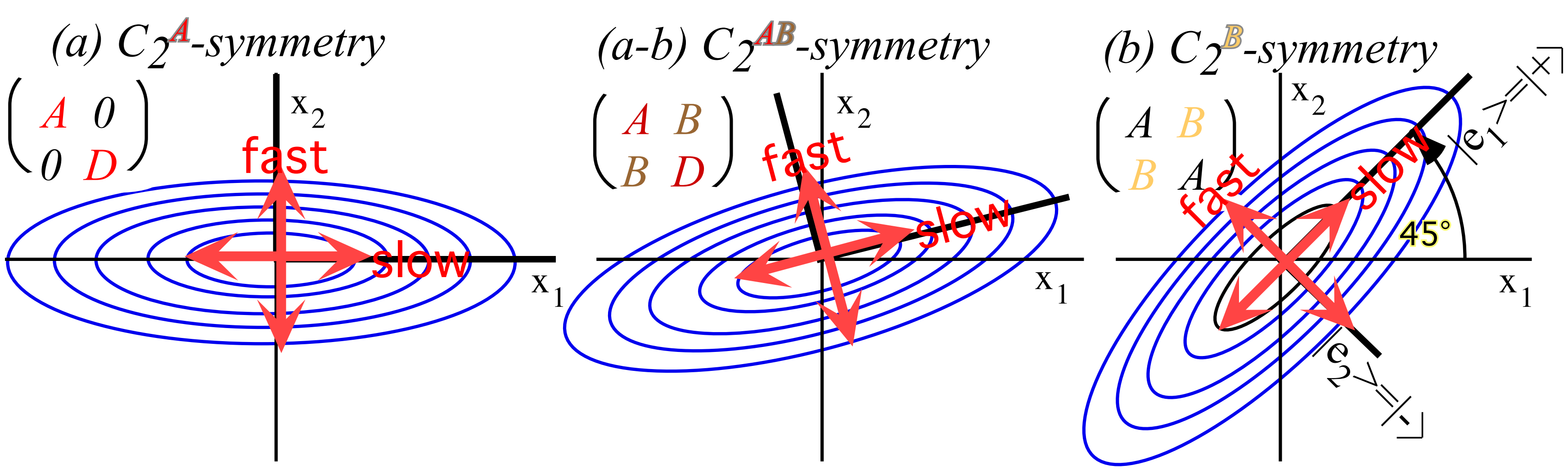

Fig. 10.1.2 Potentials for (a) $C_2{}^A$-asymmetric-diagonal, (ab) $C_2{}^{AB}$-mixed , (b) $C_2{}^B$-bilateral U(2)system.

OBJECTIVE: Evaluate and (*most* important!) *visualize* matrix-exponent solutions.

$$\left|\Psi(t)\right\rangle = e^{-i\mathbf{H}\cdot t}\left|\Psi(0)\right\rangle \quad \text{is solution to: } i\hbar\partial_t\left|\Psi(t)\right\rangle = \mathbf{H}\left|\Psi(t)\right\rangle \quad (let:\ \hbar=1)$$

Hamilton generalized Euler's expansion $e^{-i\omega t} = \cos\omega t - i\sin\omega t$ so matrix exponential becomes powerful.

$$e^{-i\mathbf{H}\cdot t} = e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}\cdot t} = e^{-i\frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot t - iB\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cdot t - iC\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cdot t - i\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cdot t}$$

$\sigma_A = \sigma_Z$ $\quad$ $\sigma_B = \sigma_X$ $\quad$ $\sigma_C = \sigma_Y$

where:
$$\vec{\omega}\cdot t = \begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\cdot t = \begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix}\cdot t \text{ and: } \omega_0 = \frac{A+D}{2}$$

Key pieces of mathematical bookkeeping

ABCD Time evolution operator

$$\mathbf{U}(t) = e^{-i\mathbf{H}\cdot t}$$

Need shorthand notations for:
$$e^{-i\mathbf{H}\cdot t} = e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}\cdot t}$$
$$= e^{-i\omega_A\sigma_A\cdot t - i\omega_B\sigma_B\cdot t - i\omega_C\sigma_C\cdot t - i\omega_0\cdot t}$$
$$= e^{-i[\omega_A\sigma_A + \omega_B\sigma_B + \omega_C\sigma_C]\cdot t}e^{-i\omega_0\cdot t}$$
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$\sigma_A = \sigma_Z$ $\quad\sigma_B = \sigma_X$ $\quad\sigma_C = \sigma_Y$

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$\vec{\sigma}\bullet\vec{\omega}\cdot t$

where: $\vec{\omega}\cdot t = \begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\cdot t = \begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix}\cdot t$ and: $\omega_0 = \frac{A+D}{2}$

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Need shorthand notations for:

$$e^{-i\mathbf{H}\cdot t} = e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}\cdot t} = e^{-i\omega_A\sigma_A\cdot t - i\omega_B\sigma_B\cdot t - i\omega_C\sigma_C\cdot t - i\omega_0\cdot t}$$

$$= e^{-i\overbrace{[\omega_A\sigma_A + \omega_B\sigma_B + \omega_C\sigma_C]}^{\vec{\omega}\bullet\vec{\sigma}=\vec{\sigma}\bullet\vec{\omega}}\cdot t}e^{-i\omega_0\cdot t}$$

$$= e^{-i[\vec{\sigma}\bullet\vec{\omega}]\cdot t}e^{-i\omega_0\cdot t}$$

Weird dot-product: $\vec{\omega}\bullet\vec{\sigma} = \vec{\sigma}\bullet\vec{\omega}$

ordinary vector ($\vec{\omega}$) — *matrix-operator vector* ($\vec{\sigma}$)

OBJECTIVE: Evaluate and (*most* important!) *visualize* matrix-exponent solutions.

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$$\sigma_A = \sigma_Z \qquad \sigma_B = \sigma_X \qquad \sigma_C = \sigma_Y$$

$$= e^{-i\sigma_\varphi\varphi}e^{-i\omega_0\cdot t} = e^{-i\vec{\sigma}\bullet\vec{\omega}\cdot t}e^{-i\omega_0\cdot t}$$

$$\sigma_\varphi\varphi = \vec{\sigma}\bullet\vec{\varphi} = \vec{\sigma}\bullet\vec{\omega}\cdot t$$

where: $\vec{\varphi} = \vec{\omega}\cdot t = \begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\cdot t = \begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix}\cdot t$ and: $\omega_0 = \dfrac{A+D}{2}$

Key pieces of mathematical bookkeeping

Need shorthand notations for: $e^{-i\mathbf{H}\cdot t} = e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}\cdot t} = e^{-i\omega_A\sigma_A\cdot t - i\omega_B\sigma_B\cdot t - i\omega_C\sigma_C\cdot t - i\omega_0\cdot t}$

Weird dot-product: $\vec{\omega}\bullet\vec{\sigma} = \vec{\sigma}\bullet\vec{\omega} = \sigma_\omega$

ordinary vector

matrix-operator vector

ω-component of operator vector **σ**

$$= e^{-i\overbrace{[\omega_A\sigma_A + \omega_B\sigma_B + \omega_C\sigma_C]}^{\vec{\omega}\bullet\vec{\sigma} = \vec{\sigma}\bullet\vec{\omega} = \sigma_\omega}\cdot t}e^{-i\omega_0\cdot t}$$

$$= e^{-i\vec{\sigma}\bullet\vec{\varphi}}e^{-i\omega_0\cdot t} = e^{-i\sigma_\varphi\varphi}e^{-i\omega_0\cdot t}$$

Symmetry group $\mathscr{G}$ representations=>*AMOP* Hamiltonian **H** (or K) matrices, irreps $\mathscr{D}^{(\alpha)}$
=>*AMOP* wave functions $\Psi^{(\alpha)}$, eigensolutions

$\mathscr{G}$ = U(2) = Unitary group of dimension 2

Memoriam: Charles H. Townes 1916-2015 and his famous 2-state system: NH_3 maser in 1955

Earlier 2-state systems: 1863 John Stokes optical polarization , 1954 Rabi, Ramsey, and Schwinger NMR (MRI)

ANALOGY: (1) Classical 2-state motion $(\partial/\partial t)^2\mathbf{x}=-\mathbf{K}\cdot\mathbf{x}$ *vs (2) Quantum 2-state motion* $ih(\partial/\partial t)\mathbf{\Psi}=\mathbf{H}\cdot\mathbf{\Psi}$

Hamilton-Pauli spinor symmetry and $\boldsymbol{\sigma}$*-expansion of* $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu=\omega_A\boldsymbol{\sigma}_A+\omega_B\boldsymbol{\sigma}_B+\omega_C\boldsymbol{\sigma}_C+\omega_0\boldsymbol{\sigma}_0$

ABCD Time evolution operator $\mathbf{U}(t)=e^{-i\mathbf{H}t}$ *; its evaluation and visualization*

ABCD symmetry operator $\{\boldsymbol{\sigma}_A,\boldsymbol{\sigma}_B,\boldsymbol{\sigma}_C\}$ *product algebra for spinor-vector operators* $\sigma_a=\boldsymbol{\sigma}\cdot\mathbf{a}$

Spinor-vector operator products $(\boldsymbol{\sigma}\cdot\mathbf{a})(\boldsymbol{\sigma}\cdot\mathbf{a})$

Crazy-Thing Theorem: $e^{-i\sigma_a\Theta}=\cos\Theta - i\sigma_a\sin\Theta$

U(2) transformation matrices and related R(3) rotations in ABC-space

Mysterious factors of 2 or ½ on 2D spinors or 3D vectors

2D $\{\uparrow,\downarrow\}$ *spinor space ½ as fast as 3D {ABC} spin-vectors*

Hamiltonian for NMR: 3D Spin Moment Vector $\mathbf{m}=(m_x, m_y, m_z,)$ *in field* $\mathbf{B}=(B_x, B_y, B_z)$

State coordinates using Euler-angle rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case

Spin-1/2 (2D-complex spinor) case

The ABC's of U(2) dynamics-Archetypes

Asymmetric-Diagonal A-Type motion

Bilateral-Balanced B-Type motion

Circular-Coriolis… C-Type motion

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	σ_X	σ_Y	σ_Z
σ_X	**1**		
σ_Y		**1**	
σ_Z			**1**

U(2) generator product table

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Symmetry group $\mathscr{G}$ representations=>*AMOP* Hamiltonian **H** (or K) matrices, irreps $\mathscr{D}^{(\alpha)}$
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$\mathscr{G}$ = U(2) = Unitary group of dimension 2

Memoriam: Charles H. Townes 1916-2015 and his famous 2-state system: NH_3 maser in 1955

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ANALOGY: (1) Classical 2-state motion $(\partial/\partial t)^2\mathbf{x}=-\mathbf{K}\cdot\mathbf{x}$ vs (2) Quantum 2-state motion $ih(\partial/\partial t)\mathbf{\Psi}=\mathbf{H}\cdot\mathbf{\Psi}$

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So there are an ∞ number of $C_2{}^{(a)}$ sub-groups

So-called *anti-commutation(* $\sigma_X\sigma_Y = -\sigma_Y\sigma_X$, $\sigma_X\sigma_Z = -\sigma_Z\sigma_X$ etc.) kills off-diagonal terms:

So: $\sigma_a{}^2 = \mathbf{1}$

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$\sigma_a\,\sigma_b$-products form a dot ($\bullet$) *and* cross ($\times$) *U(2)-algebra* that generalizes products $\sigma_X\sigma_Y = i\sigma_Z$, $\sigma_Z\sigma_X = i\,\sigma_Y$, $\sigma_Y\sigma_Z = i\,\sigma_X$, etc. ...

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$$\begin{array}{lll} a_Xb_X\sigma_X\sigma_X & +a_Xb_Y\sigma_X\sigma_Y & +a_Xb_Z\sigma_X\sigma_Z \\ = +a_Yb_X\sigma_Y\sigma_X & +a_Yb_Y\sigma_Y\sigma_Y & +a_Yb_Z\sigma_Y\sigma_Z \\ +a_Zb_X\sigma_Z\sigma_X & +a_Zb_Y\sigma_Z & +a_Zb_Z\sigma_Z\sigma_Z \end{array} = \begin{array}{lll} a_Xb_X\mathbf{1} & +a_Xb_Y\sigma_X\sigma_Y & -a_Xb_Z\sigma_Z\sigma_X \\ -a_Yb_X\sigma_X\sigma_Y & +a_Yb_Y\mathbf{1} & +a_Yb_Z\sigma_Y\sigma_Z \\ +a_Zb_X\sigma_Z\sigma_X & -a_Zb_Y\sigma_Y\sigma_Z & +a_Zb_Z\mathbf{1} \end{array}$$

$$= (a_Xb_X + a_Yb_Y + a_Zb_Z)\mathbf{1} + i(a_Yb_Z - a_Zb_Y)\sigma_X + i(a_Zb_X - a_Xb_Z)\sigma_Y + i(a_Xb_Y - a_Yb_X)\sigma_Z$$

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$$\begin{aligned} = &\ a_Xb_X\mathbf{1} + a_Xb_Y\sigma_X\sigma_Y - a_Xb_Z\sigma_Z\sigma_X \\ &- a_Yb_X\sigma_X\sigma_Y + a_Yb_Y\mathbf{1} + a_Yb_Z\sigma_Y\sigma_Z \\ &+ a_Zb_X\sigma_Z\sigma_X - a_Zb_Y\sigma_Y\sigma_Z + a_Zb_Z\mathbf{1} \end{aligned}$$

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$$\begin{aligned}
&\quad a_Xb_X\sigma_X\sigma_X + a_Xb_Y\sigma_X\sigma_Y + a_Xb_Z\sigma_X\sigma_Z \qquad a_Xb_X\mathbf{1} + a_Xb_Y\sigma_X\sigma_Y - a_Xb_Z\sigma_Z\sigma_X \\
&= +a_Yb_X\sigma_Y\sigma_X + a_Yb_Y\sigma_Y\sigma_Y + a_Yb_Z\sigma_Y\sigma_Z = -a_Yb_X\sigma_X\sigma_Y + a_Yb_Y\mathbf{1} + a_Yb_Z\sigma_Y\sigma_Z \\
&\quad +a_Zb_X\sigma_Z\sigma_X + a_Zb_Y\sigma_Z + a_Zb_Z\sigma_Z\sigma_Z \qquad +a_Zb_X\sigma_Z\sigma_X - a_Zb_Y\sigma_Y\sigma_Z + a_Zb_Z\mathbf{1}
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(Recall complex variable result.)

$$\begin{aligned}
A^*B &= (A_X + iA_Y)^*(B_X + iB_Y) = (A_X - iA_Y)(B_X + iB_Y) \\
&= (A_XB_X + A_YB_Y) + i(A_XB_Y - A_YB_X) \\
&= (\mathbf{A}\bullet\mathbf{B}) + i(\mathbf{A}\times\mathbf{B})_Z
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U(2) generator product table

Symmetry group $\mathscr{G}$ representations=>*AMOP* Hamiltonian **H** (or K) matrices, irreps $\mathscr{D}^{(\alpha)}$ =>*AMOP* wave functions $\Psi^{(\alpha)}$, eigensolutions

$\mathscr{G}$ = U(2) = Unitary group of dimension 2

Memoriam: Charles H. Townes 1916-2015 and his famous 2-state system: NH_3 maser in 1955

Earlier 2-state systems: 1863 John Stokes optical polarization , 1954 Rabi, Ramsey, and Schwinger NMR (MRI)

ANALOGY: (1) Classical 2-state motion $(\partial/\partial t)^2\mathbf{x}=-\mathbf{K}\cdot\mathbf{x}$ *vs (2) Quantum 2-state motion* $ih(\partial/\partial t)\mathbf{\Psi}=\mathbf{H}\cdot\mathbf{\Psi}$

Hamilton-Pauli spinor symmetry and $\boldsymbol{\sigma}$*-expansion of* $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu=\omega_A\boldsymbol{\sigma}_A+\omega_B\boldsymbol{\sigma}_B+\omega_C\boldsymbol{\sigma}_C+\omega_o\boldsymbol{\sigma}_o$

ABCD Time evolution operator $\mathbf{U}(t)=e^{-i\mathbf{H}t}$ *; its evaluation and visualization*

ABCD symmetry operator $\{\boldsymbol{\sigma}_A,\boldsymbol{\sigma}_B,\boldsymbol{\sigma}_C\}$ *product algebra for spinor-vector operators* $\sigma_a=\boldsymbol{\sigma}\cdot\mathbf{a}$

Spinor-vector operator products $(\boldsymbol{\sigma}\cdot\mathbf{a})(\boldsymbol{\sigma}\cdot\mathbf{a})$

Crazy-Thing Theorem: $e^{-i\sigma_a\Theta}=\cos\Theta-i\sigma_a\sin\Theta$

U(2) transformation matrices and related R(3) rotations in ABC-space

Mysterious factors of 2 or ½ on 2D spinors or 3D vectors

2D {↑,↓} spinor space ½ as fast as 3D {ABC} spin-vectors

Hamiltonian for NMR: 3D Spin Moment Vector $\mathbf{m}=(m_x, m_y, m_z,)$ *in field* $\mathbf{B}=(B_x, B_y, B_z)$

State coordinates using Euler-angle rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case

Spin-1/2 (2D-complex spinor) case

The ABC's of U(2) dynamics-Archetypes

Asymmetric-Diagonal A-Type motion

Bilateral-Balanced B-Type motion

Circular-Coriolis… C-Type motion

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Hamilton replaces *(-i)* with $-i\sigma_\varphi$ in the $e^{-i\varphi}$ power series above to get a sequence of terms just like it.

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$$= \frac{\varphi_A\sigma_A + \varphi_B\sigma_B + \varphi_C\sigma_C}{\sqrt{\varphi_A^2 + \varphi_B^2 + \varphi_C^2}}$$

Hamilton replaces *(-i)* with $-i\sigma_\varphi$ in the $e^{-i\varphi}$ power series above to get a sequence of terms just like it.

$$(-i\sigma_\varphi)^0 = +\mathbf{1},\ (-i\sigma_\varphi)^1 = -i\sigma_\varphi,\ (-i\sigma_\varphi)^2 = -\mathbf{1},\ (-i\sigma_\varphi)^3 = +i\sigma_\varphi,\ (-i\sigma_\varphi)^4 = +\mathbf{1},\ (-i\sigma_\varphi)^5 = -i\sigma_\varphi,\ etc.$$

This allows Hamilton to generalize Euler's rotation $e^{-i\varphi}$ to $e^{-i\sigma_\varphi\varphi}$ for any $\sigma_\varphi\varphi = (\vec{\sigma}\bullet\vec{\varphi}) = \varphi_A\sigma_A + \varphi_B\sigma_B + \varphi_C\sigma_C = (\vec{\sigma}\bullet\hat{\varphi})\varphi$

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σ_Y	$-i\sigma_Z$	1	$i\sigma_X$
σ_Z	$i\sigma_Y$	$-i\sigma_X$	1

U(2) generator product table

OBJECTIVE: Evaluate and (*most* important!) *visualize* matrix-exponent solutions.

$$\left|\Psi(t)\right\rangle = e^{-i\mathbf{H}\cdot t}\left|\Psi(0)\right\rangle = (\mathbf{1}\cos\varphi - i\sigma_\varphi \sin\varphi)\left|\Psi(0)\right\rangle e^{-i\omega_0\cdot t}$$

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$\sigma_A = \sigma_Z$ $\sigma_B = \sigma_X$ $\sigma_C = \sigma_Y$

$$= e^{-i\sigma_\varphi \varphi} e^{-i\omega_0\cdot t} = e^{-i\vec{\sigma}\bullet\vec{\omega}\cdot t} e^{-i\omega_0\cdot t}$$

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Key pieces of mathematical bookkeeping

Symmetry relations make spinors $\{\sigma_X = \sigma_B, \sigma_Y = \sigma_C, \sigma_Z = \sigma_A\}$ or quaternions $\{\mathbf{i} = -i\sigma_X, \mathbf{j} = -i\sigma_Y, \mathbf{k} = -i\sigma_Z\}$ into a powerful *U(2)-algebra*.

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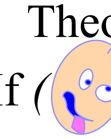
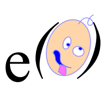
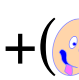
Note even powers of *(-i)* are ± 1 and odd powers of *(-i)* are $\pm i$.:

$$(-i)^0 = +1,\ (-i)^1 = -i,\ (-i)^2 = -1,\ (-i)^3 = +i,\ (-i)^4 = +1,\ (-i)^5 = -i,\ etc.$$

The Crazy Thing Theorem:

If ()^2 = -**1**

Then:

e()φ = $\mathbf{1}\cos\varphi$ + ()$\sin\varphi$

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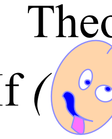
If $)^2 = -\mathbf{1}$

Then:

$e^{(\text{crazy thing})\varphi} = \mathbf{1}\cos\varphi + (\text{crazy thing})\sin\varphi$

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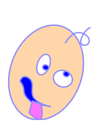
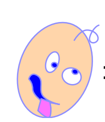
Here:  = *-i*

Crazy thing is just -√-1

Here:  $= -i\sigma_\varphi = -i(\vec{\sigma}\bullet\hat{\varphi}) = -i\frac{(\vec{\sigma}\bullet\vec{\varphi})}{\varphi}$

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U(2) generator product table

Symmetry group $\mathscr{G}$ representations=>*AMOP* Hamiltonian **H** (or K) matrices, irreps $\mathscr{D}^{(\alpha)}$
=>*AMOP* wave functions $\Psi^{(\alpha)}$, eigensolutions

$\mathscr{G}$ = U(2) = Unitary group of dimension 2

Memoriam: Charles H. Townes 1916-2015 and his famous 2-state system: NH_3 maser in 1955

Earlier 2-state systems: 1863 John Stokes optical polarization , 1954 Rabi, Ramsey, and Schwinger NMR (MRI)

ANALOGY: (1) Classical 2-state motion $(\partial/\partial t)^2\mathbf{x}=-\mathbf{K}\cdot\mathbf{x}$ vs (2) Quantum 2-state motion $ih(\partial/\partial t)\mathbf{\Psi}=\mathbf{H}\cdot\mathbf{\Psi}$

Hamilton-Pauli spinor symmetry and **σ***-expansion of* $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu=\omega_A\boldsymbol{\sigma}_A+\omega_B\boldsymbol{\sigma}_B+\omega_C\boldsymbol{\sigma}_C+\omega_o\boldsymbol{\sigma}_o$

ABCD Time evolution operator $\mathbf{U}(t)=e^{-i\mathbf{H}t}$ *; its evaluation and visualization*

ABCD symmetry operator $\{\boldsymbol{\sigma}_A,\boldsymbol{\sigma}_B,\boldsymbol{\sigma}_C\}$ *product algebra for spinor-vector operators* $\sigma_a=\boldsymbol{\sigma}\cdot\mathbf{a}$

Spinor-vector operator products $(\boldsymbol{\sigma}\cdot\mathbf{a})(\boldsymbol{\sigma}\cdot\mathbf{a})$

Crazy-Thing Theorem: $e^{-i\sigma_a\Theta}=\cos\Theta-i\sigma_a\sin\Theta$

U(2) transformation matrices and related R(3) rotations in ABC-space

Mysterious factors of 2 or ½ on 2D spinors or 3D vectors

2D {↑,↓} spinor space ½ as fast as 3D {ABC} spin-vectors

Hamiltonian for NMR: 3D Spin Moment Vector $\mathbf{m}=(m_x, m_y, m_z)$ *in field* $\mathbf{B}=(B_x, B_y, B_z)$

State coordinates using Euler-angle rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case

Spin-1/2 (2D-complex spinor) case

The ABC's of U(2) dynamics-Archetypes

Asymmetric-Diagonal A-Type motion

Bilateral-Balanced B-Type motion

Circular-Coriolis… C-Type motion

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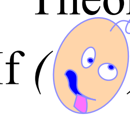
If ( $)^2 = -\mathbf{1}$

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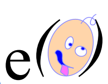
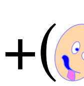
e()θ $= \mathbf{1}\cos\theta +$ ()$\sin\theta$

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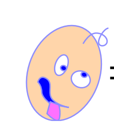
Here:  $= -i\sigma_{\varphi} = -i(\vec{\sigma}\bullet\hat{\varphi}) = -i\frac{(\vec{\sigma}\bullet\vec{\varphi})}{\varphi}$

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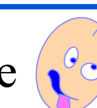
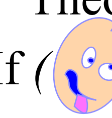
Example 1: A or Z rotation

The  Crazy Thing Theorem:

If ()^2 = -**1**

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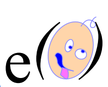
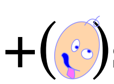
e()θ = **1**cosθ+()sinθ

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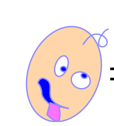
Here:  $= -i\sigma_{\varphi} = -i(\vec{\sigma}\bullet\hat{\varphi}) = -i\frac{(\vec{\sigma}\bullet\vec{\varphi})}{\varphi}$

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Example 1:
A or Z rotation

$$e^{-i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\varphi_C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\varphi_C - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\sin\varphi_C$$

$$= \begin{pmatrix} \cos\varphi_C & -\sin\varphi_C \\ \sin\varphi_C & \cos\varphi_C \end{pmatrix}$$

Example 2:
C or Y rotation

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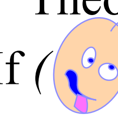
If ()^2= **-1**

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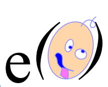
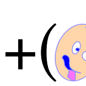
e()θ = **1**cosθ+()sinθ

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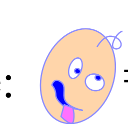
Here:  $= -i\sigma_\varphi = -i(\vec{\sigma}\bullet\hat{\varphi}) = -i\frac{(\vec{\sigma}\bullet\vec{\varphi})}{\varphi}$

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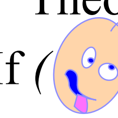
If ()^2 = -**1**

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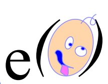
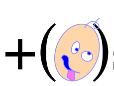
e()θ = **1**cosθ + ()sinθ

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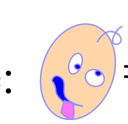
Here:  $= -i\sigma_\varphi = -i(\vec{\sigma}\bullet\hat{\varphi}) = -i\frac{(\vec{\sigma}\bullet\vec{\varphi})}{\varphi}$

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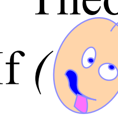
If ()^2 = -**1**

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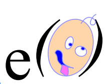
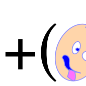
e^()θ = **1**cosθ + ()sinθ

Polar coordinates for unit axis vector $\hat{\omega}$ *or* $\hat{\varphi}$

$$\hat{\omega}_X = \hat{\omega}_B = \cos\varphi\ \sin\vartheta = \hat{\varphi}_B$$

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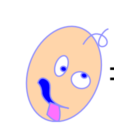
Any $\varphi = \omega t$*-axial rotation*

Here:  $= -i\sigma_\varphi = -i(\vec{\sigma}\bullet\hat{\varphi}) = -i\frac{(\vec{\sigma}\bullet\vec{\varphi})}{\varphi}$

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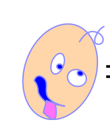
Example 2: C or Y rotation

We test these operators by making them rotate each other....

Here:  $= -i\sigma_{\varphi} = -i(\vec{\sigma}\bullet\hat{\varphi}) = -i\frac{(\vec{\sigma}\bullet\vec{\varphi})}{\varphi}$

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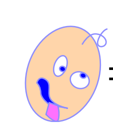
Here:  $= -i\sigma_{\varphi} = -i(\vec{\boldsymbol{\sigma}}\bullet\hat{\varphi}) = -i\frac{(\vec{\boldsymbol{\sigma}}\bullet\vec{\varphi})}{\varphi}$

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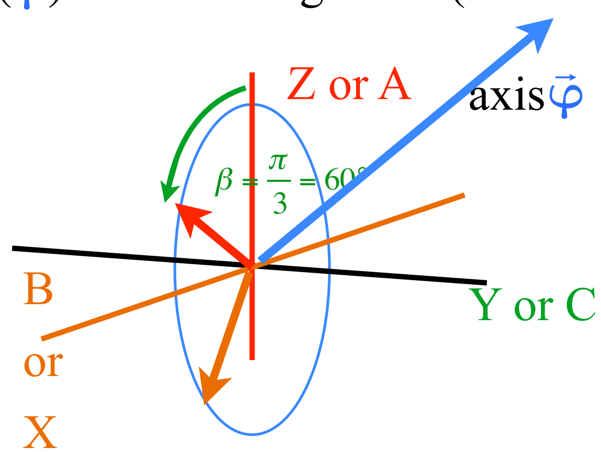
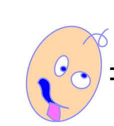


Here:  $= -i\sigma_{\varphi} = -i(\vec{\sigma}\bullet\hat{\varphi}) = -i\frac{(\vec{\sigma}\bullet\vec{\varphi})}{\varphi}$

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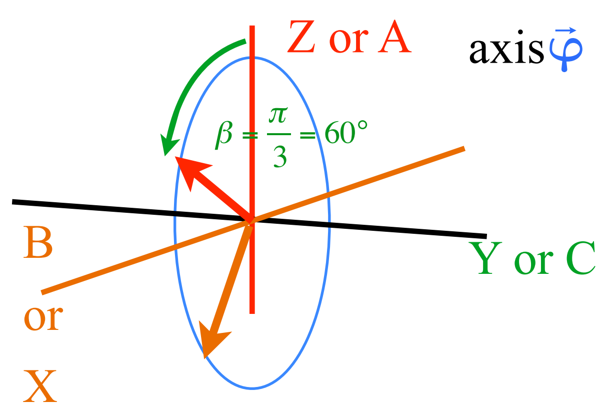
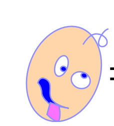


Here:  $= -i\sigma_{\varphi} = -i(\vec{\sigma}\bullet\hat{\varphi}) = -i\frac{(\vec{\sigma}\bullet\vec{\varphi})}{\varphi}$

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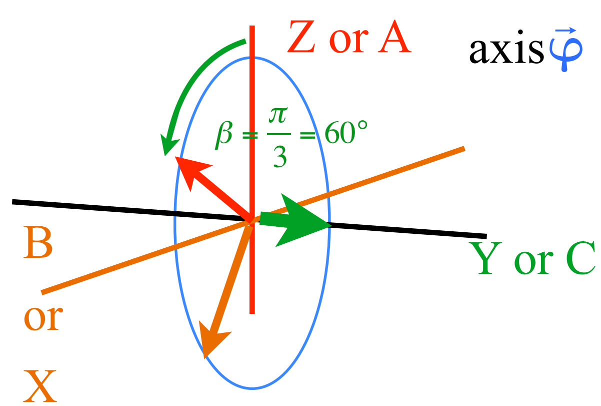
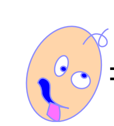


Here:  $= -i\sigma_{\varphi} = -i(\vec{\sigma}\bullet\hat{\varphi}) = -i\frac{(\vec{\sigma}\bullet\vec{\varphi})}{\varphi}$

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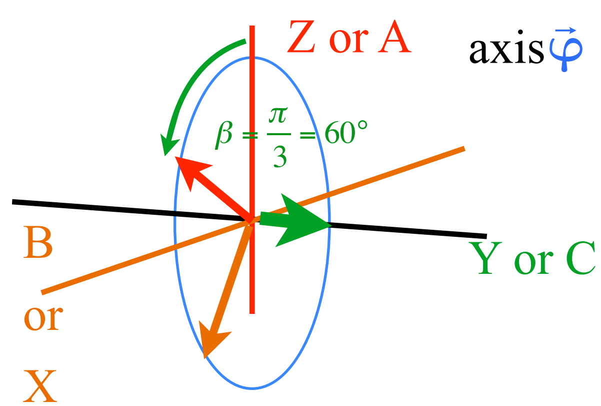
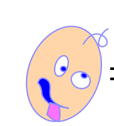


Here:  $= -i\sigma_{\varphi} = -i(\vec{\sigma}\bullet\hat{\varphi}) = -i\frac{(\vec{\sigma}\bullet\vec{\varphi})}{\varphi}$

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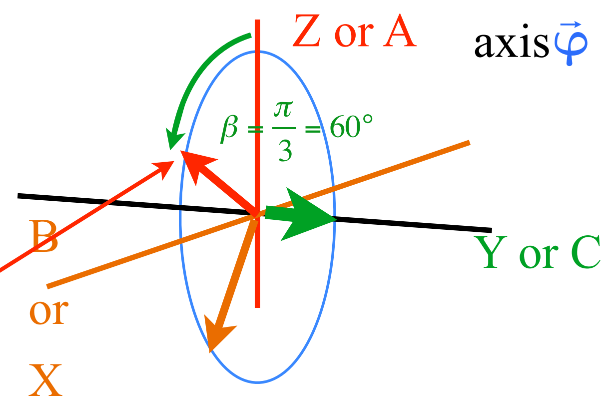
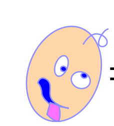


Here:  $= -i\sigma_{\varphi} = -i(\vec{\sigma}\bullet\hat{\varphi}) = -i\frac{(\vec{\sigma}\bullet\vec{\varphi})}{\varphi}$

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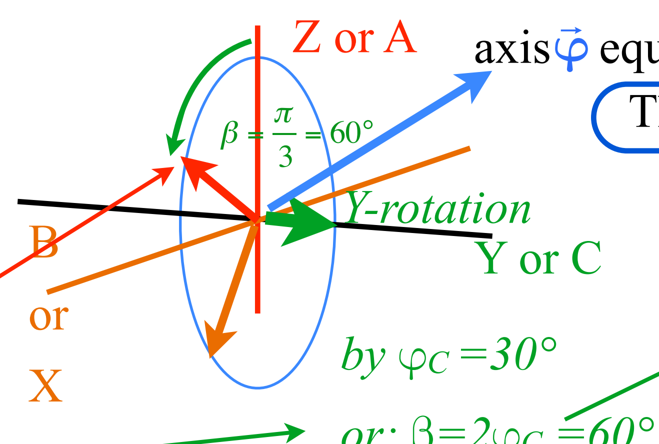
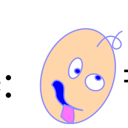


axis $\vec{\varphi}$ equals $\vec{\mathbf{e}}_C\varphi_C$ here

The *3D*-rotation is by 2φ, *twice* the *2D* angle φ.

Here:  $= -i\sigma_{\varphi} = -i(\vec{\boldsymbol{\sigma}}\bullet\hat{\varphi}) = -i\frac{(\vec{\boldsymbol{\sigma}}\bullet\vec{\varphi})}{\varphi}$

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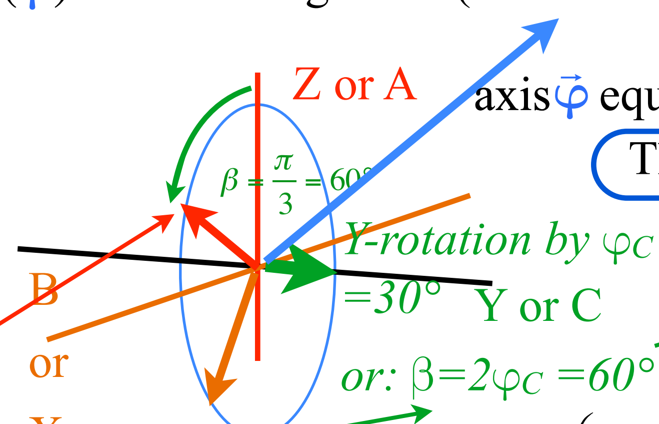
$$= \begin{pmatrix} \cos\varphi_C & -\sin\varphi_C \\ \sin\varphi_C & \cos\varphi_C\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix} \cos\varphi_C & \sin\varphi_C \\ -\sin\varphi_C & \cos\varphi_C\end{pmatrix}$$
$$= \begin{pmatrix} \cos^2\varphi_C - \sin^2\varphi_C & 2\sin\varphi_C\cos\varphi_C \\ 2\sin\varphi_C\cos\varphi_C & \sin^2\varphi_C - \cos^2\varphi_C\end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\cos 2\varphi_C + \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\sin 2\varphi_C$$
$$= \boldsymbol{\sigma}_A \cos 2\varphi_C + \boldsymbol{\sigma}_B \sin 2\varphi_C$$



axis $\vec{\varphi}$ equals $\vec{\mathbf{e}}_C\varphi_C$ here

The *3D*-rotation is by 2φ, *twice* the *2D* angle φ.

$\vec{\varphi}=\vec{\omega}\cdot t$ equal to $\vec{\omega}$ only at $t=1$ but $\hat{\varphi}=\hat{\omega}$ always.

$$\hat{\varphi} = \begin{pmatrix} \hat{\varphi}_A \\ \hat{\varphi}_B \\ \hat{\varphi}_C \end{pmatrix} = \begin{pmatrix} \varphi_A \\ \varphi_B \\ \varphi_C \end{pmatrix}\frac{1}{\sqrt{\varphi_A^2 + \varphi_B^2 + \varphi_C^2}} = \begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\frac{1}{\sqrt{\omega_A^2 + \omega_B^2 + \omega_C^2}}$$

OBJECTIVE: Evaluate and (*most* important!) *visualize* matrix-exponent solutions.

ABCD Time evolution operator $\mathbf{U}(t)=e^{-i\mathbf{H}\cdot t}$

$$|\Psi(t)\rangle = e^{-i\mathbf{H}\cdot t}|\Psi(0)\rangle \quad = (\mathbf{1}\cos\varphi - i\sigma_\varphi \sin\varphi)e^{-i\omega_0\cdot t}$$

Hamilton generalized Euler's expansion $e^{-i\Omega t} = \cos\Omega t - i\sin\Omega t$ so matrix exponential becomes powerful.

$$e^{-i\mathbf{H}\cdot t} = e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D\end{pmatrix}t} = e^{-i\frac{A-D}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}t - iB\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}t - iC\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}t - i\frac{A+D}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}t} = e^{-i(\omega_0\sigma_0 + \vec{\omega}\bullet\vec{\sigma})\cdot t} = e^{-i\omega_0\cdot t}(\mathbf{1}\cos\omega t - i\sigma_\varphi \sin\omega\cdot t)$$

$\sigma_A = \sigma_Z$ $\quad \sigma_B = \sigma_X$ $\quad \sigma_C = \sigma_Y$

where: $\vec{\varphi} = \begin{pmatrix}\varphi_A \\ \varphi_B \\ \varphi_C\end{pmatrix} = \vec{\omega}\cdot t = \begin{pmatrix}\omega_A \\ \omega_B \\ \omega_C\end{pmatrix}\cdot t = \begin{pmatrix}\frac{A-D}{2} \\ B \\ C\end{pmatrix}\cdot t$ and: $\omega_0 = \dfrac{A+D}{2}$

$$e^{-i\varphi} = 1\ \cos\varphi \ - \ i\ \sin\varphi$$

generalizes to:

$$e^{-i\sigma_\varphi\varphi} = \mathbf{1}\cos\varphi \ - \ i\ \sigma_\varphi \sin\varphi$$

$$e^{-i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\varphi_A} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\cos\varphi_A - i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\sin\varphi_A$$

$$= \begin{pmatrix}\cos\varphi_A - i\sin\varphi_A & 0 \\ 0 & \cos\varphi_A + i\sin\varphi_A\end{pmatrix} = \begin{pmatrix} e^{-i\varphi_A} & 0 \\ 0 & e^{i\varphi_A}\end{pmatrix}$$

Example 1: A or Z rotation

$$e^{-i\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\varphi_C} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\cos\varphi_C - i\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\sin\varphi_C$$

$$= \begin{pmatrix}\cos\varphi_C & -\sin\varphi_C \\ \sin\varphi_C & \cos\varphi_C\end{pmatrix}$$

Example 2: C or Y rotation

3D axis vector $\vec{\varphi}=\vec{\omega}\cdot t$ corresponds to generator $\sigma_\varphi = \sigma_A\hat{\varphi}_A + \sigma_B\hat{\varphi}_B + \sigma_C\hat{\varphi}_C$ of *rotation* $e^{-i\sigma_\varphi\varphi} = R(\vec{\varphi}) = \mathbf{1}\cos\varphi - i\sigma_\varphi \sin\varphi$ about axis $\vec{\varphi}$.

Any 2-by-2 σ_μ-matrix may be rotated by any $R(\vec{\varphi})$ matrix acting *twice* (fore-and-aft^{-1}) to give: $\sigma_\mu^{(\vec{\varphi}\text{-rotated})} = R(\vec{\varphi})\sigma_\mu R^{-1}(\vec{\varphi}) = R(\vec{\varphi})\sigma_\mu R^\dagger(\vec{\varphi})$

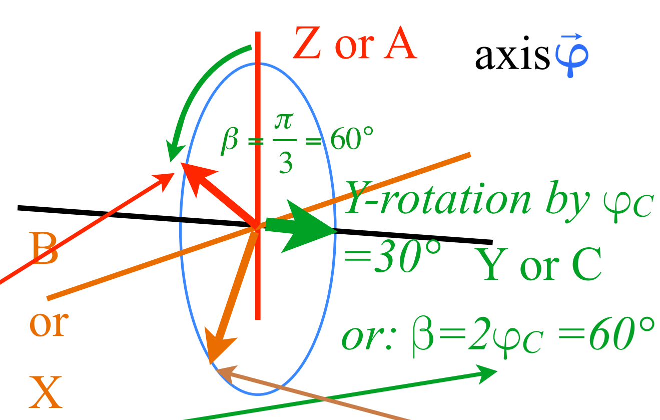
$$\mathbf{R}(\varphi_C)\cdot\boldsymbol{\sigma}_A\cdot\mathbf{R}^{-1}(\varphi_C)$$

$$= \begin{pmatrix}\cos\varphi_C & -\sin\varphi_C \\ \sin\varphi_C & \cos\varphi_C\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\cos\varphi_C & \sin\varphi_C \\ -\sin\varphi_C & \cos\varphi_C\end{pmatrix}$$

$$= \begin{pmatrix}\cos^2\varphi_C - \sin^2\varphi_C & 2\sin\varphi_C\cos\varphi_C \\ 2\sin\varphi_C\cos\varphi_C & \sin^2\varphi_C - \cos^2\varphi_C\end{pmatrix}$$

$$= \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\cos 2\varphi_C + \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\sin 2\varphi_C$$

$$= \boldsymbol{\sigma}_A \cos 2\varphi_C + \boldsymbol{\sigma}_B \sin 2\varphi_C$$



$$\mathbf{R}(\varphi_C)\cdot\boldsymbol{\sigma}_B\cdot\mathbf{R}^{-1}(\varphi_C)$$

$$= \begin{pmatrix}\cos\varphi_C & -\sin\varphi_C \\ \sin\varphi_C & \cos\varphi_C\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\cos\varphi_C & \sin\varphi_C \\ -\sin\varphi_C & \cos\varphi_C\end{pmatrix}$$

$$= \begin{pmatrix}-2\sin\varphi_C\cos\varphi_C & \cos^2\varphi_C - \sin^2\varphi_C \\ \cos^2\varphi_C - \sin^2\varphi_C & 2\sin\varphi_C\cos\varphi_C\end{pmatrix}$$

$$= \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\sin 2\varphi_C + \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\cos 2\varphi_C$$

$$= -\boldsymbol{\sigma}_A \sin 2\varphi_C + \boldsymbol{\sigma}_B \cos 2\varphi_C$$

The *3D*-rotation is by 2φ, *twice* the *2D* angle φ.

Symmetry group $\mathscr{G}$ representations=>*AMOP* Hamiltonian **H** (or K) matrices, irreps $\mathscr{D}^{(\alpha)}$
=>*AMOP* wave functions $\Psi^{(\alpha)}$, eigensolutions

$\mathscr{G}$ = U(2) = Unitary group of dimension 2

Memoriam: Charles H. Townes 1916-2015 and his famous 2-state system: NH_3 maser in 1955

Earlier 2-state systems: 1863 John Stokes optical polarization , 1954 Rabi, Ramsey, and Schwinger NMR (MRI)

ANALOGY: (1) Classical 2-state motion $(\partial/\partial t)^2\mathbf{x}=-\mathbf{K}\cdot\mathbf{x}$ *vs (2) Quantum 2-state motion* $ih(\partial/\partial t)\boldsymbol{\Psi}=\mathbf{H}\cdot\boldsymbol{\Psi}$

Hamilton-Pauli spinor symmetry and **σ***-expansion of* $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu=\omega_A\boldsymbol{\sigma}_A+\omega_B\boldsymbol{\sigma}_B+\omega_C\boldsymbol{\sigma}_C+\omega_o\boldsymbol{\sigma}_o$

ABCD Time evolution operator $\mathbf{U}(t)=e^{-i\mathbf{H}t}$ *; its evaluation and visualization*

ABCD symmetry operator $\{\boldsymbol{\sigma}_A,\boldsymbol{\sigma}_B,\boldsymbol{\sigma}_C\}$ *product algebra for spinor-vector operators* $\sigma_a=\boldsymbol{\sigma}\cdot\mathbf{a}$

Spinor-vector operator products $(\boldsymbol{\sigma}\cdot\mathbf{a})(\boldsymbol{\sigma}\cdot\mathbf{a})$

Crazy-Thing Theorem: $e^{-i\sigma_a\Theta}=\cos\Theta-i\sigma_a\sin\Theta$

U(2) transformation matrices and related R(3) rotations in ABC-space

Mysterious factors of 2 or ½ on 2D spinors or 3D vectors

2D $\{\uparrow,\downarrow\}$ *spinor space ½ as fast as 3D {ABC} spin-vectors*

Hamiltonian for NMR: 3D Spin Moment Vector $\mathbf{m}=(m_x, m_y, m_z,)$ *in field* $\mathbf{B}=(B_x, B_y, B_z)$

State coordinates using Euler-angle rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case | *Spin-1/2 (2D-complex spinor) case*

The ABC's of U(2) dynamics-Archetypes

Asymmetric-Diagonal A-Type motion

Bilateral-Balanced B-Type motion

Circular-Coriolis… C-Type motion

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

Notation for 2D Spinor space

$$\mathbf{H} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \omega_0 \sigma_0 + \omega_A \sigma_A + \omega_B \sigma_B + \omega_C \sigma_C = \omega_0\sigma_0 + \vec{\omega} \bullet \vec{\sigma} = \omega_0 \mathbf{1} + \omega \boldsymbol{\sigma}_{\omega}$$

Symmetry archetypes: *A (Asymmetric-diagonal)*| *B (Bilateral-balanced)*| *C (Chiral-circular-complex...)*

The { σ_1, σ_A , σ_B , σ_C } are the well known *Pauli-spin operators* { $\sigma_1 = \sigma_0$, $\sigma_B = \sigma_X$, $\sigma_C = \sigma_Y$, $\sigma_A = \sigma_Z$ }

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$$= \omega_0 \sigma_0 + \omega_A \sigma_A + \omega_B \sigma_B + \omega_C \sigma_C = \omega_0\sigma_0 + \vec{\omega}\bullet\vec{\sigma} = \omega_0 \mathbf{1} + \omega \boldsymbol{\sigma}_\omega$$

$$= \Omega_0 \mathbf{1} + \Omega_A \mathbf{S}_A + \Omega_B \mathbf{S}_B + \Omega_C \mathbf{S}_C = \Omega_0 \mathbf{1} + \vec{\Omega}\bullet\vec{\mathbf{S}}$$

$$= \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} + 2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + 2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}$$

Notation for 3D Vector space

0^{th} component unchanged

components A, B, C switch 1/2-factor from ω-velocity to S-momentum

Symmetry archetypes: *A (Asymmetric-diagonal)* | *B (Bilateral-balanced)* | *C (Chiral-circular-complex...)*

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$$= \omega_0 \sigma_0 + \omega_A \sigma_A + \omega_B \sigma_B + \omega_C \sigma_C = \omega_0\sigma_0 + \vec{\omega}\bullet\vec{\sigma} = \omega_0 \mathbf{1} + \omega \boldsymbol{\sigma}_\omega$$

Notation for 3D Vector space

$$= \Omega_0 \mathbf{1} + \Omega_A \mathbf{S}_A + \Omega_B \mathbf{S}_B + \Omega_C \mathbf{S}_C = \Omega_0 \mathbf{1} + \vec{\Omega}\bullet\vec{\mathbf{S}}$$

$$= \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} + 2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + 2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}$$

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The { $\mathbf{1}$, S_A, S_B, S_C } are the *Jordan-Angular-Momentum operators* { $\mathbf{1} = \sigma_0$, $S_B = S_X$, $S_C = S_Y$, $S_A = S_Z$ }
(Often labeled { J_X, J_Y, J_Z })

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$$\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}=\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+\frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}+B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}+C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$=\omega_0\,\sigma_0+\omega_A\,\sigma_A+\omega_B\,\sigma_B+\omega_C\,\sigma_C=\omega_0\sigma_0+\vec{\omega}\bullet\vec{\sigma}=\omega_0\mathbf{1}+\omega\boldsymbol{\sigma}_\omega$$

Notation for 2D Spinor space

$$=\Omega_0\,\mathbf{1}+\Omega_A\,\mathbf{S}_A+\Omega_B\,\mathbf{S}_B+\Omega_C\,\mathbf{S}_C=\Omega_0\mathbf{1}+\vec{\Omega}\bullet\vec{\mathbf{S}}$$

$$=\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+(A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}+2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}+2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}$$

Notation for 3D Vector space

0th component unchanged

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(Often labeled { J_X, J_Y, J_Z })

Notation for 2D Spinor space

where: $\vec{\varphi}=\vec{\omega}\cdot t=\begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\cdot t=\begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix}\cdot t$ and: $\omega_0=\frac{A+D}{2}$

$$e^{-i\mathbf{H}\cdot t}=e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}\cdot t}=e^{-i(\omega_0\sigma_0+\vec{\omega}\bullet\vec{\sigma})\cdot t}=e^{-i\omega_0\cdot t}e^{-i\,\vec{\omega}\bullet\vec{\sigma}\cdot t}=e^{-i\omega_0\cdot t}e^{-i\boldsymbol{\sigma}_\omega\omega\cdot t}=e^{-i\omega_0\cdot t}\left(\mathbf{1}\cos\omega\cdot t-i\boldsymbol{\sigma}_\omega\sin\omega\cdot t\right)$$

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$$\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}=\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+\frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}+B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}+C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$=\omega_0\sigma_0+\omega_A\sigma_A+\omega_B\sigma_B+\omega_C\sigma_C=\omega_0\sigma_0+\vec{\omega}\bullet\vec{\sigma}=\omega_0\mathbf{1}+\omega\boldsymbol{\sigma}_\omega$$

Notation for 3D Vector space

$$=\Omega_0\mathbf{1}+\Omega_A\mathbf{S}_A+\Omega_B\mathbf{S}_B+\Omega_C\mathbf{S}_C=\Omega_0\mathbf{1}+\vec{\Omega}\bullet\vec{\mathbf{S}}$$

$$=\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+(A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}+2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}+2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}$$

0th component unchanged

components A, B, C switch 1/2-factor from ω-velocity to S-momentum

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(Often labeled { J_X, J_Y , J_Z })

Notation for 2D Spinor space

where: $\vec{\boldsymbol{\varphi}}=\vec{\boldsymbol{\omega}}\cdot t=\begin{pmatrix}\omega_A\\ \omega_B\\ \omega_C\end{pmatrix}\cdot t=\begin{pmatrix}\frac{A-D}{2}\\ B\\ C\end{pmatrix}\cdot t$ and: $\omega_0=\frac{A+D}{2}$

$$e^{-i\mathbf{H}\cdot t}=e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}\cdot t}=e^{-i(\omega_0\sigma_0+\vec{\omega}\bullet\vec{\sigma})\cdot t}=e^{-i\omega_0\cdot t}e^{-i\,\vec{\omega}\bullet\vec{\sigma}\cdot t}=e^{-i\omega_0\cdot t}e^{-i\boldsymbol{\sigma}_\omega\omega\cdot t}=e^{-i\omega_0\cdot t}\left(\mathbf{1}\cos\omega\cdot t-i\boldsymbol{\sigma}_\omega\sin\omega\cdot t\right)$$

$$=e^{-i(\Omega_0\mathbf{1}+\vec{\Omega}\bullet\vec{\mathbf{S}})\cdot t}=e^{-i\Omega_0\cdot t}e^{-i\,\vec{\Omega}\cdot t\bullet\mathbf{S}}=e^{-i\Omega_0\cdot t}\left(\mathbf{1}\cos\frac{\Omega\cdot t}{2}-i\boldsymbol{\sigma}_\omega\sin\frac{\Omega\cdot t}{2}\right)$$

Notation for 3D Vector space

where: $\vec{\boldsymbol{\Theta}}=\vec{\boldsymbol{\Omega}}\cdot t=\begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix}\cdot t=\begin{pmatrix}A-D\\ 2B\\ 2C\end{pmatrix}\cdot t$ and: $\Omega_0=\frac{A+D}{2}$

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$$\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}=\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+\frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}+B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}+C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Notation for 2D Spinor space

$$= \omega_0 \sigma_0 + \omega_A \sigma_A + \omega_B \sigma_B + \omega_C \sigma_C = \omega_0\sigma_0 + \vec{\boldsymbol{\omega}}\bullet\vec{\boldsymbol{\sigma}} = \omega_0 \mathbf{1} + \omega\boldsymbol{\sigma}_\omega$$

$$= \Omega_0 \mathbf{1} + \Omega_A \mathbf{S}_A + \Omega_B \mathbf{S}_B + \Omega_C \mathbf{S}_C = \Omega_0\mathbf{1} + \vec{\boldsymbol{\Omega}}\bullet\vec{\mathbf{S}}$$

$$=\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+(A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}+2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}+2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}$$

Notation for 3D Vector space

0^{th} component unchanged

components *A*, *B*, *C* switch 1/2-factor from ω-velocity to S-momentum

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(Often labeled { J_X, J_Y, J_Z })

"Crank" vector (2D-Spinor)

$$\vec{\boldsymbol{\varphi}}=\begin{pmatrix} \varphi_A \\ \varphi_B \\ \varphi_C \end{pmatrix}=\vec{\boldsymbol{\omega}}\cdot t=\begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\cdot t=\begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix}\cdot t$$

Notation for 2D Spinor space

where: $\vec{\boldsymbol{\varphi}}=\vec{\boldsymbol{\omega}}\cdot t=\begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\cdot t=\begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix}\cdot t$ and: $\omega_0=\frac{A+D}{2}$

$$e^{-i\mathbf{H}\cdot t}=e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}t}=e^{-i(\omega_0\sigma_0+\vec{\boldsymbol{\omega}}\bullet\vec{\boldsymbol{\sigma}})\cdot t}=e^{-i\omega_0\cdot t}e^{-i\,\vec{\boldsymbol{\omega}}\bullet\vec{\boldsymbol{\sigma}}\cdot t}=e^{-i\omega_0\cdot t}e^{-i\boldsymbol{\sigma}_\omega\omega\cdot t}=e^{-i\omega_0\cdot t}\left(\mathbf{1}\cos\omega\cdot t-i\boldsymbol{\sigma}_\omega\sin\omega\cdot t\right)$$

$$=e^{-i(\Omega_0\mathbf{1}+\vec{\boldsymbol{\Omega}}\bullet\vec{\mathbf{S}})\cdot t}=e^{-i\Omega_0\cdot t}e^{-i\,\vec{\boldsymbol{\Omega}}\cdot t\bullet\mathbf{S}} \qquad =e^{-i\Omega_0\cdot t}\left(\mathbf{1}\cos\frac{\Omega\cdot t}{2}-i\boldsymbol{\sigma}_\omega\sin\frac{\Omega\cdot t}{2}\right)$$

"Crank" vector (3D-Vector)

$$\vec{\boldsymbol{\Theta}}=\begin{pmatrix} \Theta_A \\ \Theta_B \\ \Theta_C \end{pmatrix}=\vec{\boldsymbol{\Omega}}\cdot t=\begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix}\cdot t=\begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}\cdot t$$

Notation for 3D Vector space

where: $\vec{\boldsymbol{\Theta}}=\vec{\boldsymbol{\Omega}}\cdot t=\begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix}\cdot t=\begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}\cdot t$ and: $\Omega_0=\frac{A+D}{2}$

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U(2) transformation matrices and related R(3) rotations in ABC-space

Mysterious factors of 2 or ½ on 2D spinors or 3D vectors

2D $\{\uparrow,\downarrow\}$ *spinor space ½ as fast as 3D {ABC} spin-vectors*

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Spin-1 (3D-real vector) case

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The ABC's of U(2) dynamics-Archetypes

Asymmetric-Diagonal A-Type motion

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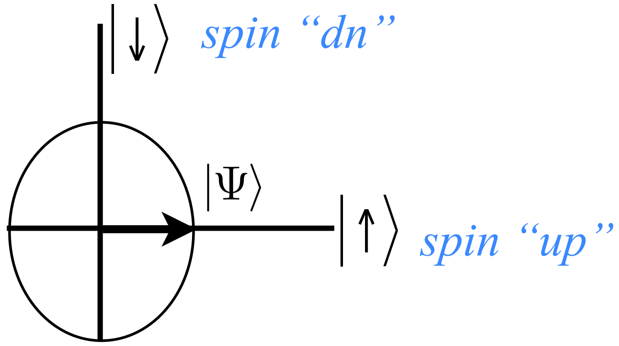
The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

U(2):2D Spinor {|↑⟩,|↓⟩}-space (complex)

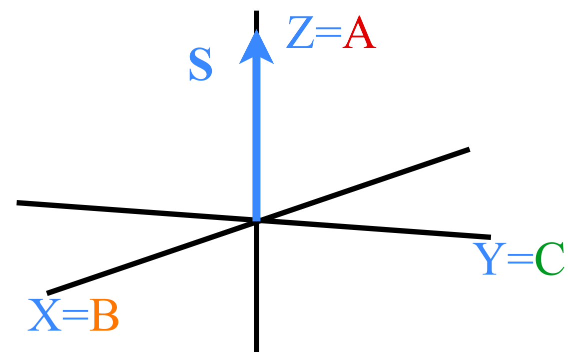
R(3):3D Spin Vector {S_X, S_Y, S_Z}-space (real)

State vector $|\Psi\rangle = |\uparrow\rangle\langle\uparrow|\Psi\rangle + |\downarrow\rangle\langle\downarrow|\Psi\rangle$

Spin vector $\mathbf{S} = |X\rangle\langle X|\mathbf{S}\rangle + |Y\rangle\langle Y|\mathbf{S}\rangle + |Z\rangle\langle Z|\mathbf{S}\rangle$



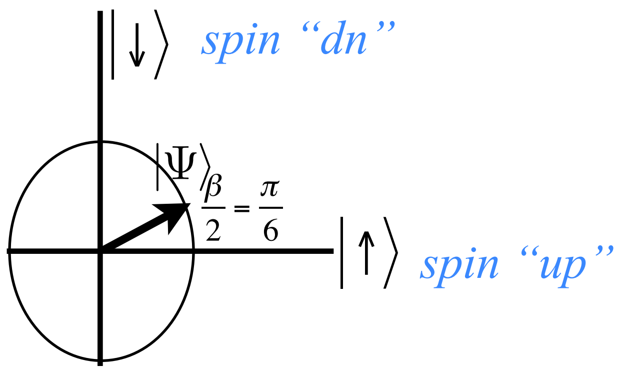
$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}$$



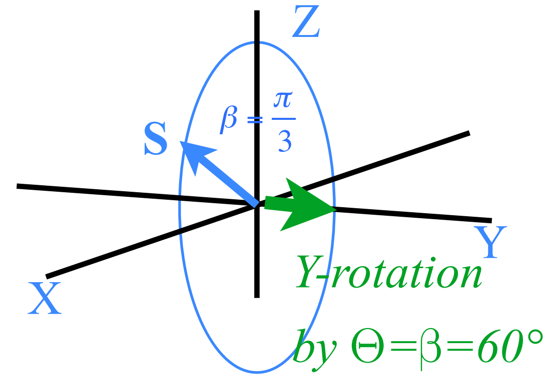
$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$$

$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix} = \begin{pmatrix}\cos\frac{\beta}{2} & -\sin\frac{\beta}{2}\\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}$$

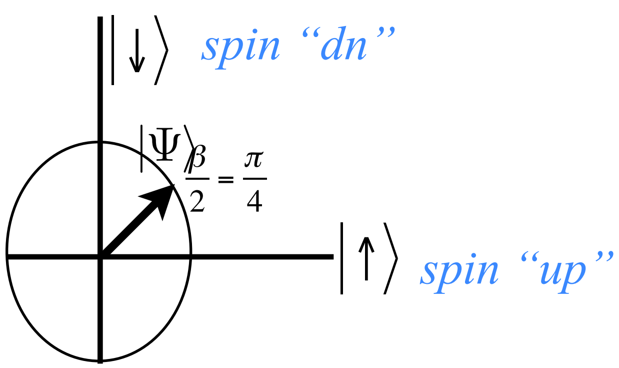
$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix} = \begin{pmatrix}\cos\beta & 0 & \sin\beta\\ 0 & 1 & 0\\ -\sin\beta & 0 & \cos\beta\end{pmatrix}\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$$



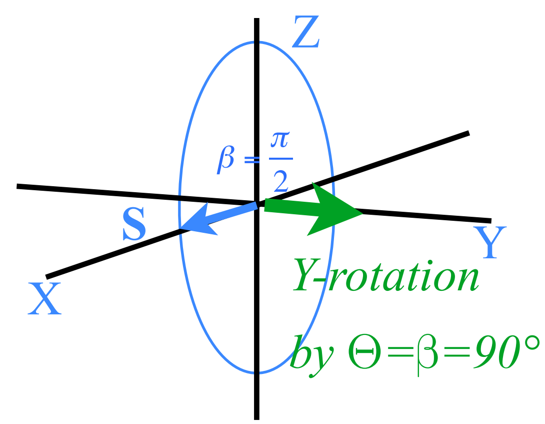
$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix} = \begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}$$



$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix} = \begin{pmatrix}\sin\beta\\ 0\\ \cos\beta\end{pmatrix} = \begin{pmatrix}\sqrt{3}/2\\ 0\\ 1/2\end{pmatrix}$$



$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix} = \begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$$



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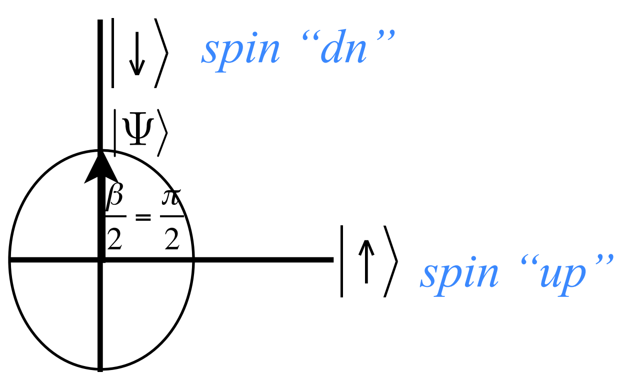

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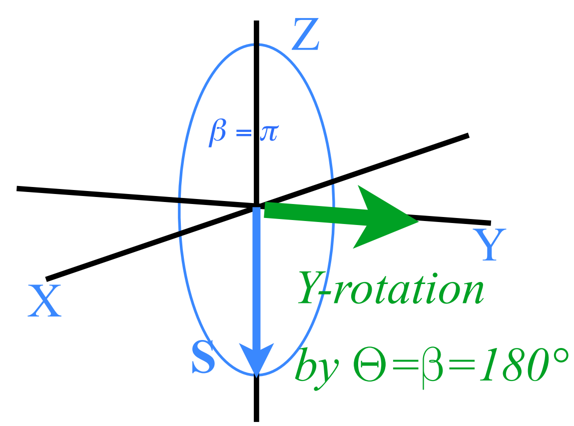

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Life in 2D Spinor space is "Half-Fast"

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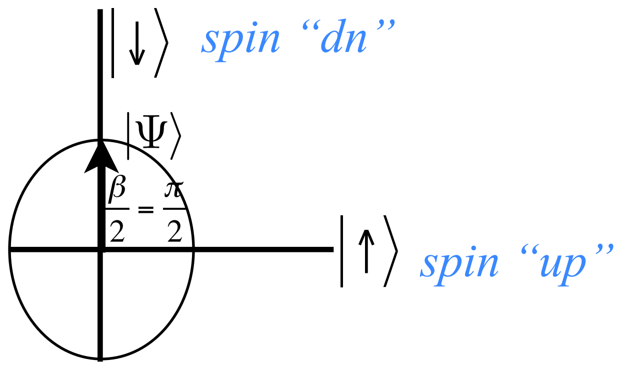

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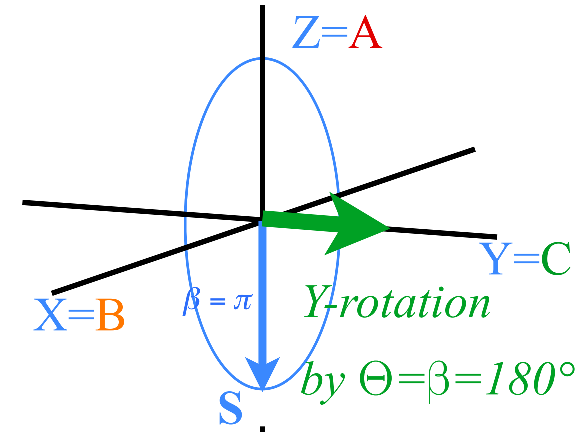

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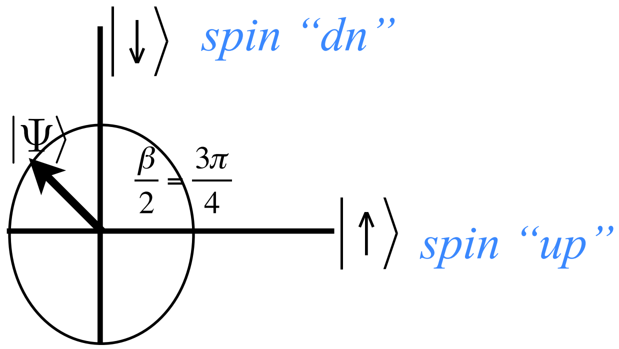
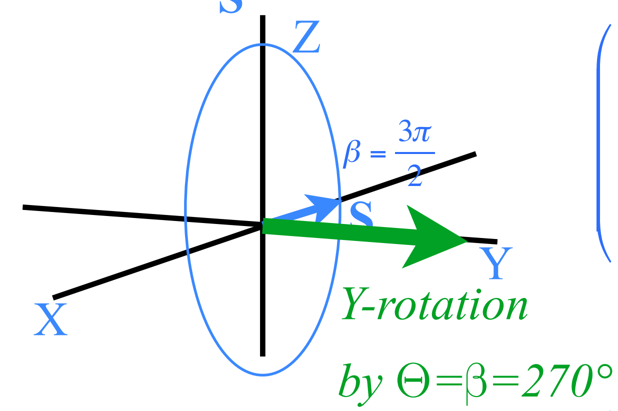


$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix}=\begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix}=\begin{pmatrix}\frac{-1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$$



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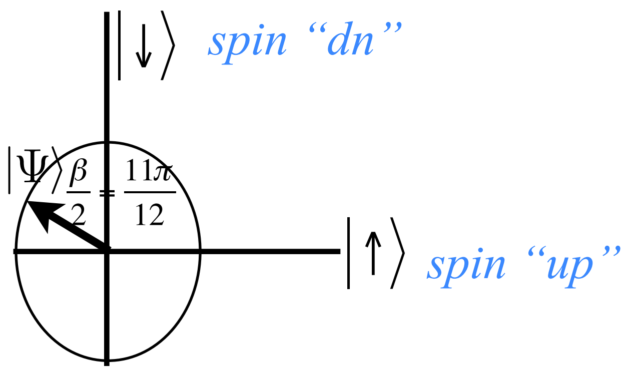
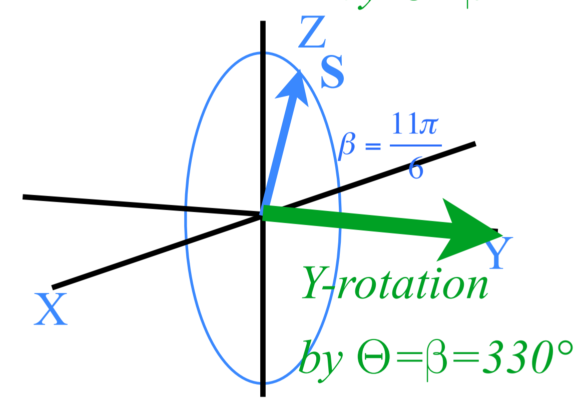


$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix}=\begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix}=\begin{pmatrix}\frac{-\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}$$



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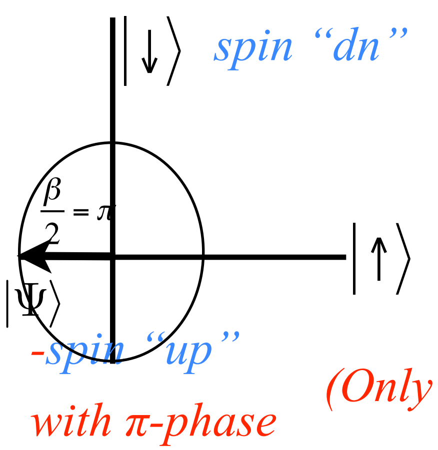

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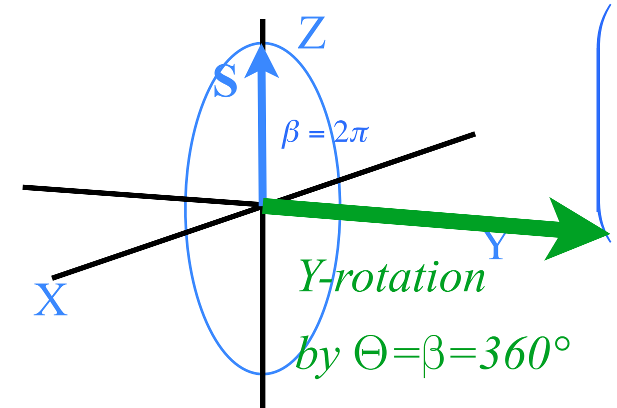
(Only “half-way” home after 2π =360° rotation)



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$$= gB_Z \quad \sigma_A \quad + \quad gB_X \quad \sigma_X \quad + gB_Y \quad \sigma_Y \quad = \vec{\omega}\bullet\vec{\sigma} = \omega\sigma_\omega$$

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Two views of Hamilton crank vector $\Omega(\varphi,\vartheta)$
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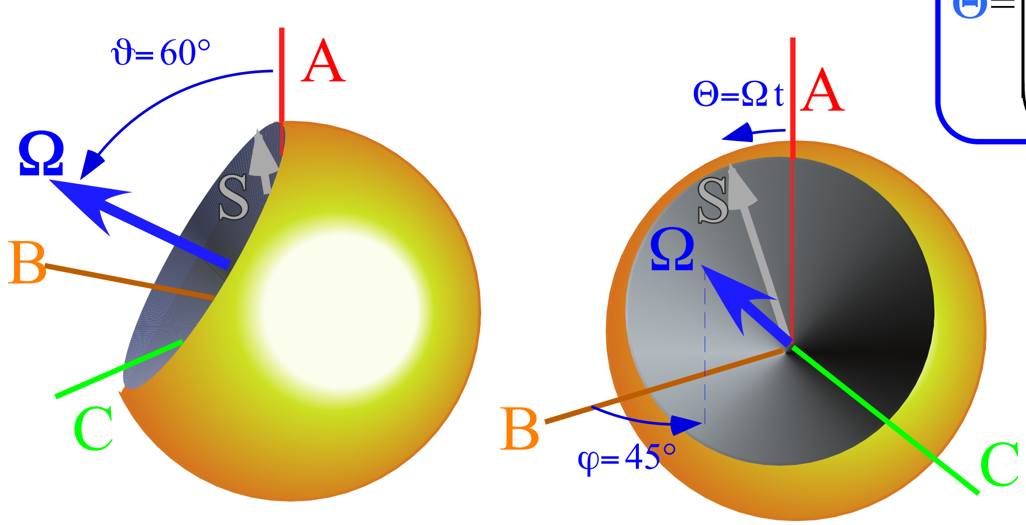

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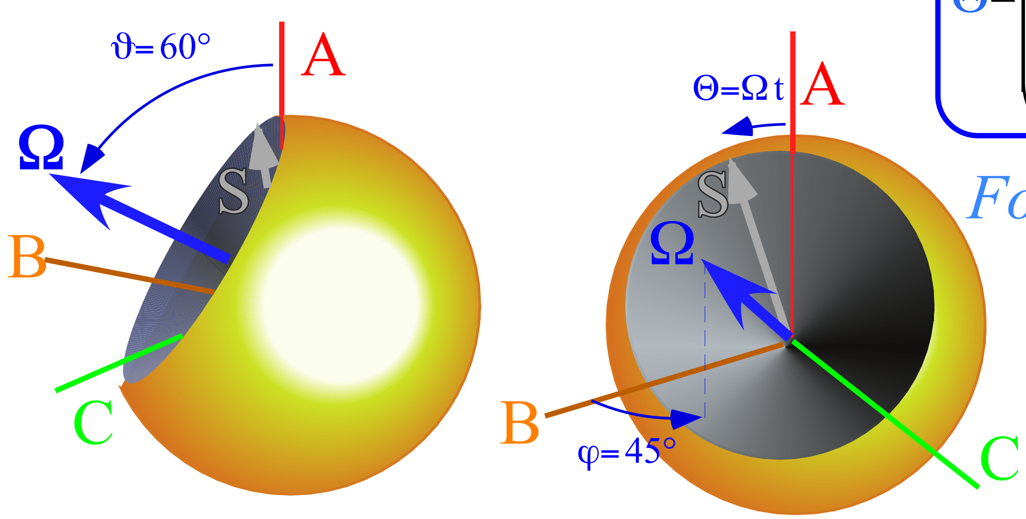

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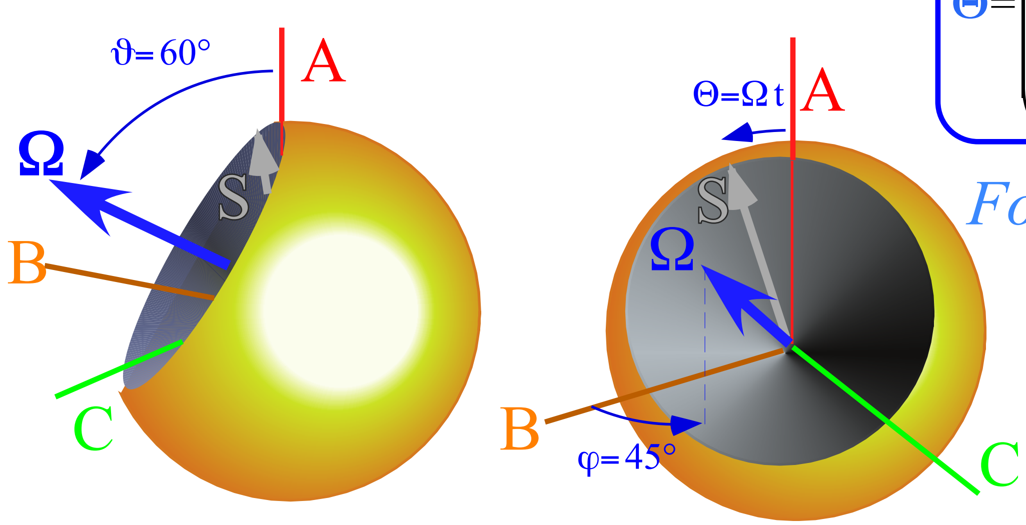

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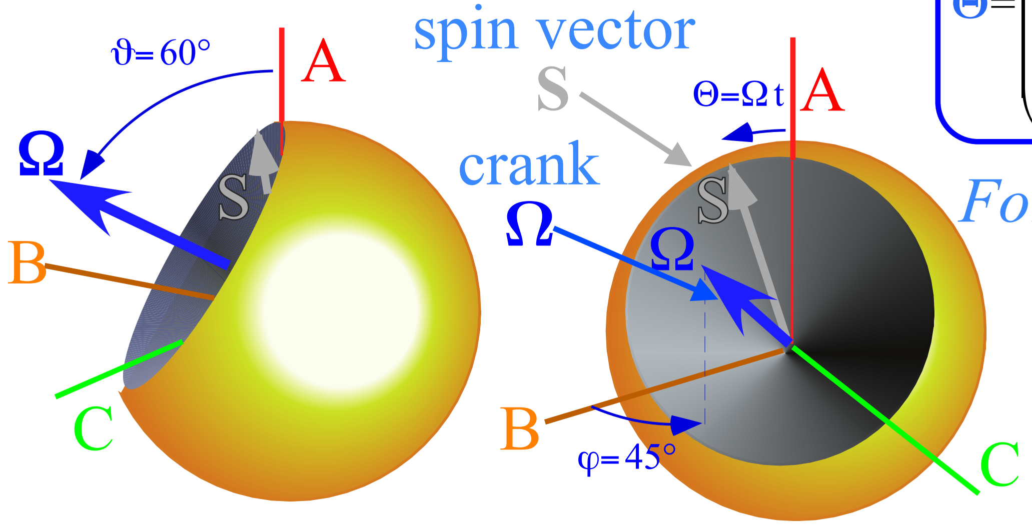

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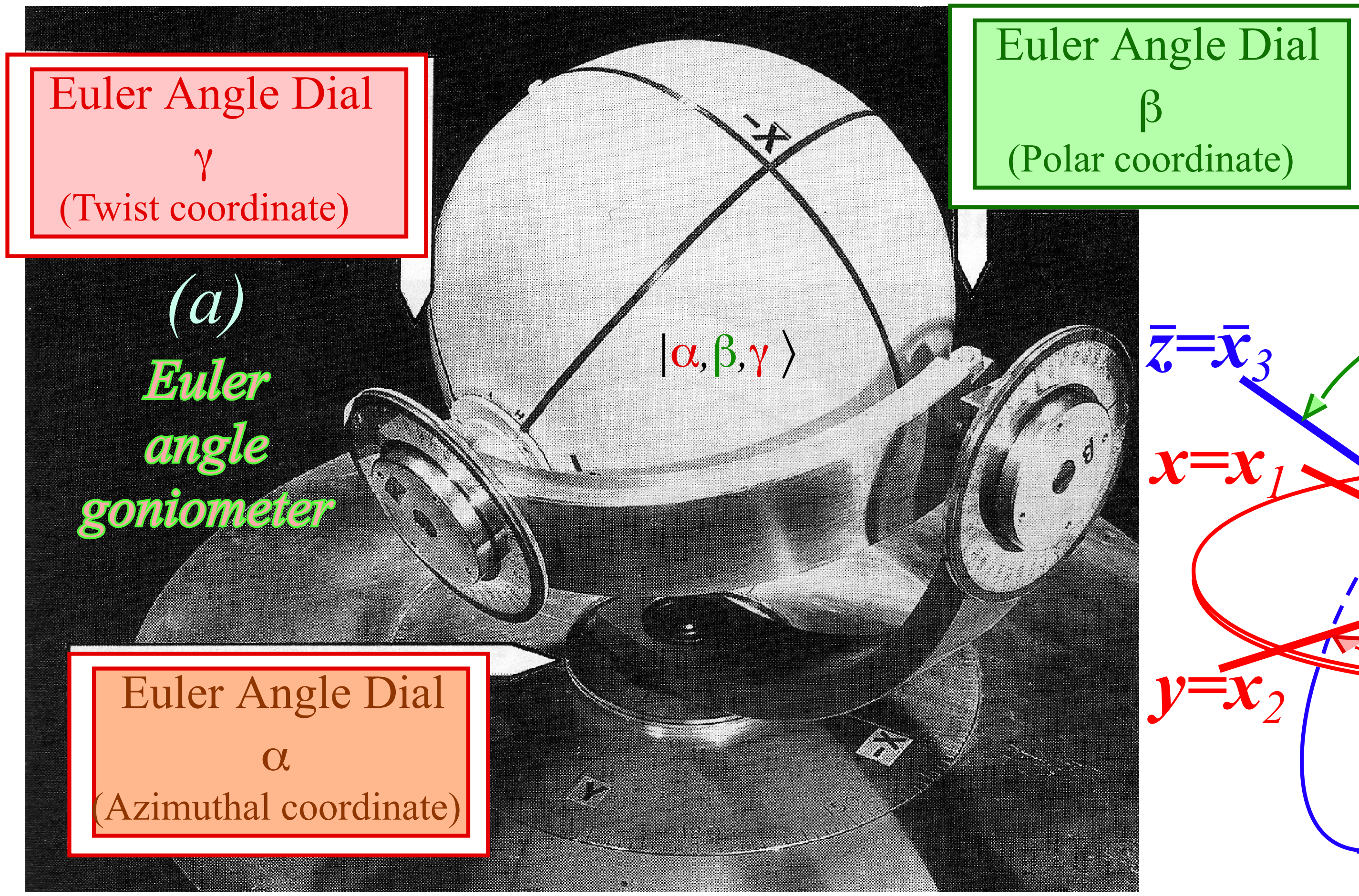
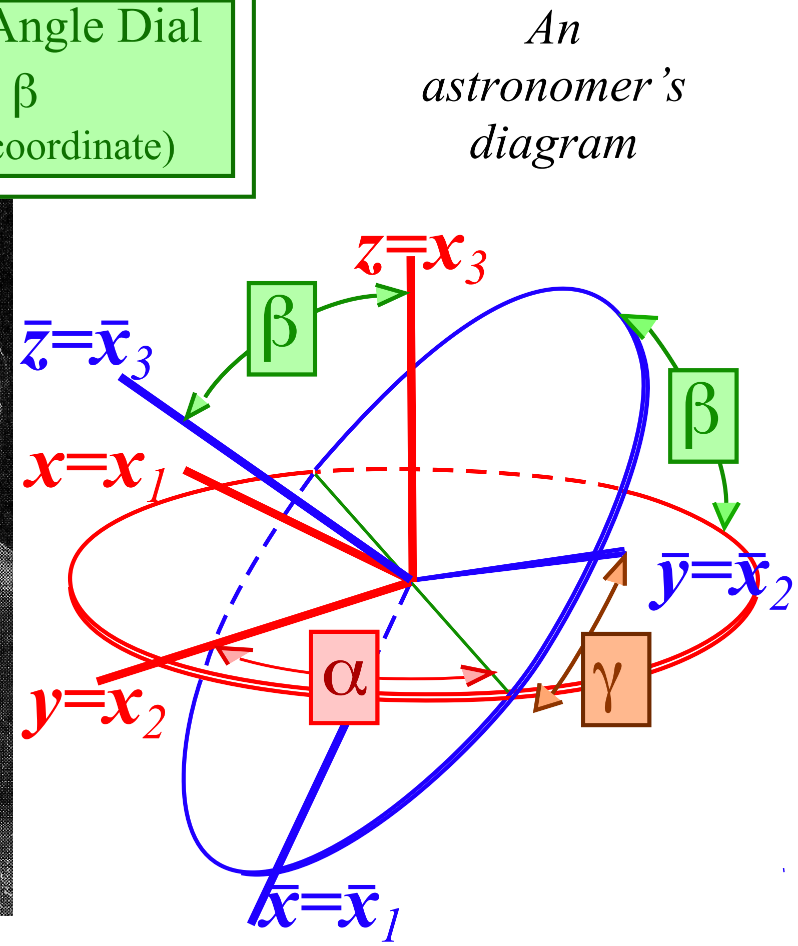
Euler's state definition using rotations **R**$(\alpha,0,0)$, **R**$(0,\beta,0)$, *and* **R**$(0,0,\gamma)$

Spin-1 (3D-real vector) case

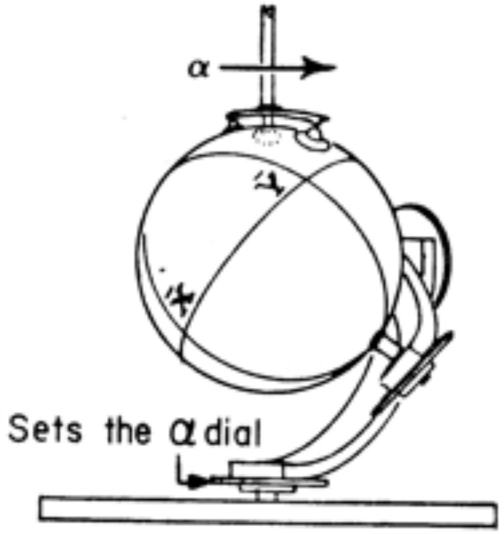




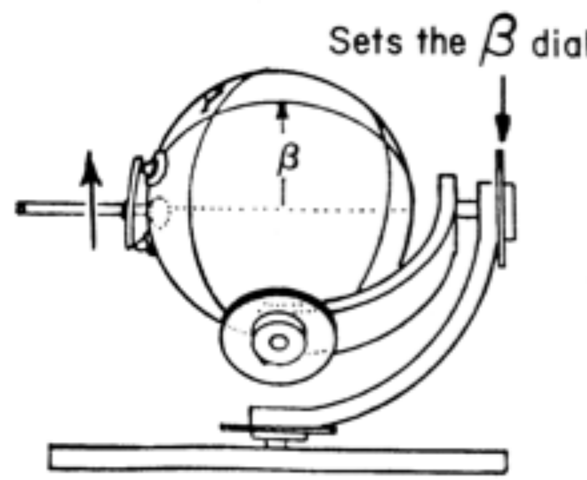
Euler's rotation state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case

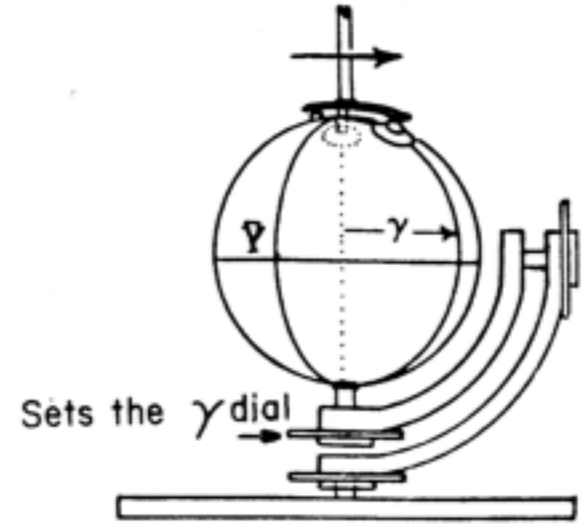
Third rotation $\mathbf{R}(\alpha 0 0)$



Second rotation $\mathbf{R}(0 \beta 0)$



First rotation $\mathbf{R}(0 0 \gamma)$



$$\left\langle R\left(\alpha\beta\gamma\right)\right\rangle = \left\langle R\left(\alpha 0 0\right)\right\rangle \qquad \left\langle R\left(0\beta 0\right)\right\rangle \qquad \left\langle R\left(0 0\gamma\right)\right\rangle$$

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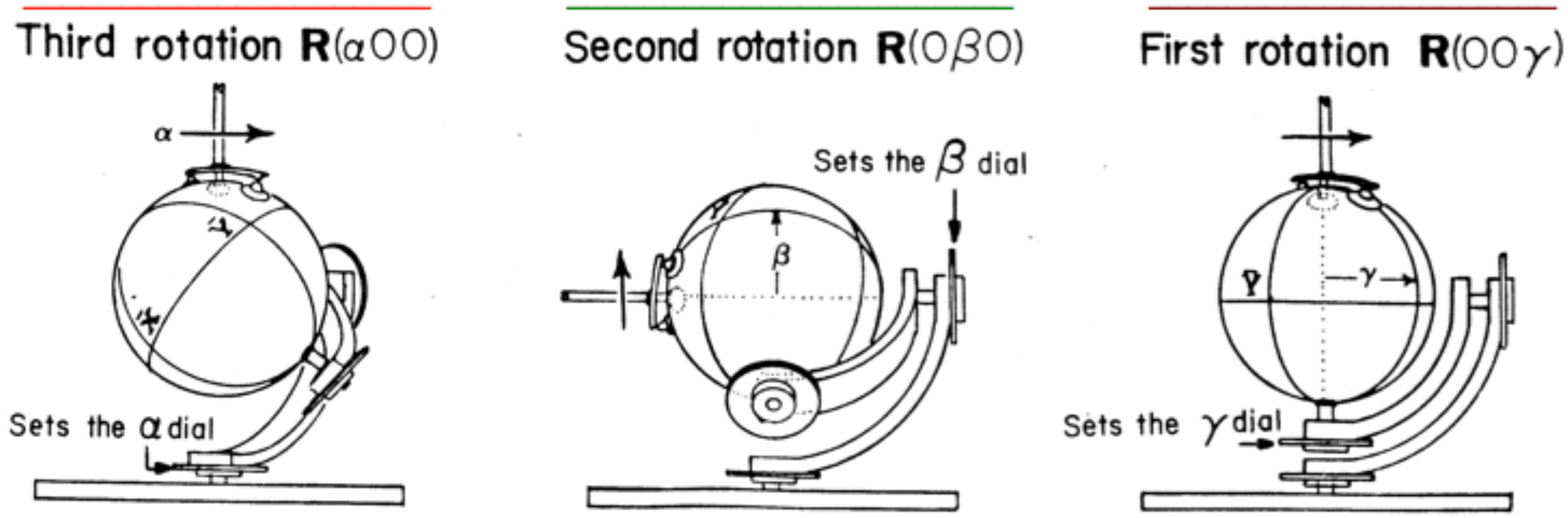

$$\langle R(\alpha\beta\gamma)\rangle = \langle R(\alpha 00)\rangle \quad \langle R(0\beta 0)\rangle \quad \langle R(00\gamma)\rangle$$

$$= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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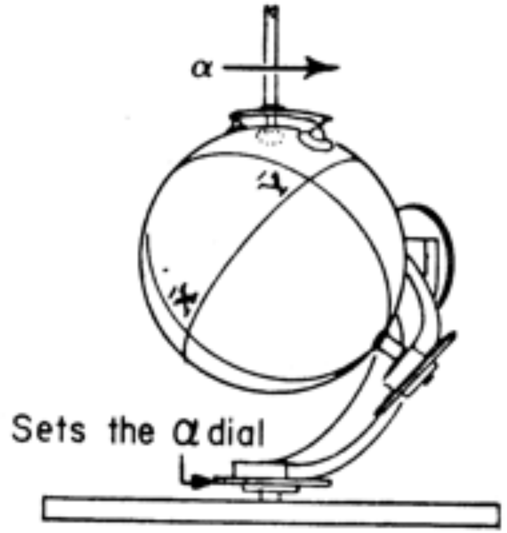
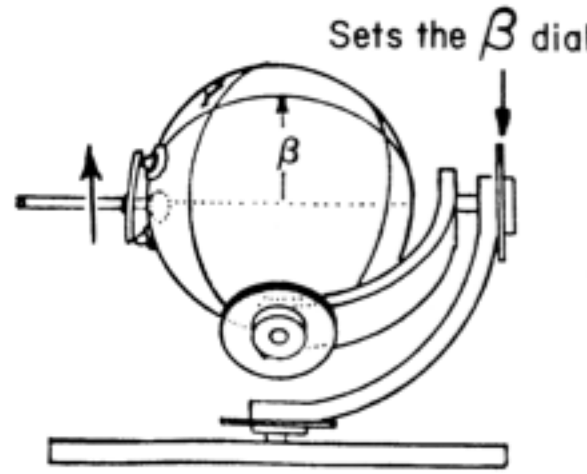
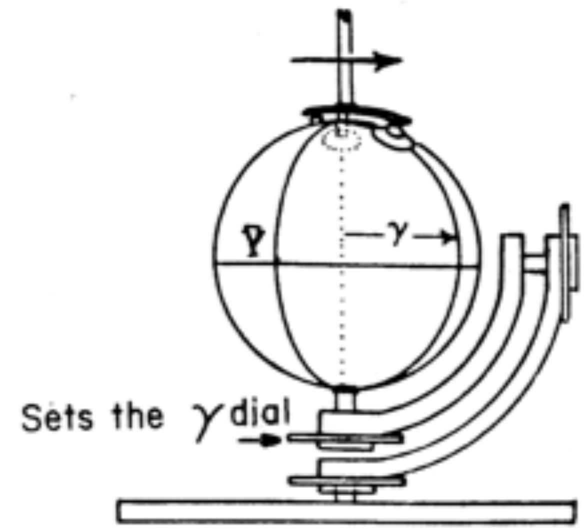


Second rotation $\mathbf{R}(0\beta 0)$



First rotation $\mathbf{R}(00\gamma)$



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$$|\mathbf{e}_{\bar{X}}\rangle = R(\alpha\beta\gamma)|\mathbf{e}_X\rangle \qquad |\mathbf{e}_{\bar{Y}}\rangle = R(\alpha\beta\gamma)|\mathbf{e}_Y\rangle \qquad |\mathbf{e}_{\bar{Z}}\rangle = R(\alpha\beta\gamma)|\mathbf{e}_Z\rangle$$

$$\left(\langle \mathbf{e}_A | R(\alpha\beta\gamma) | \mathbf{e}_B\rangle\right) = \begin{matrix} \langle \mathbf{e}_X | \\ \langle \mathbf{e}_Y | \\ \langle \mathbf{e}_Z | \end{matrix} \begin{pmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta \\ \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta \\ -\cos\gamma\sin\beta & \sin\gamma\sin\beta & \cos\beta \end{pmatrix}$$

Note lab-frame polar coordinates of Z(body)

Euler's rotation state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

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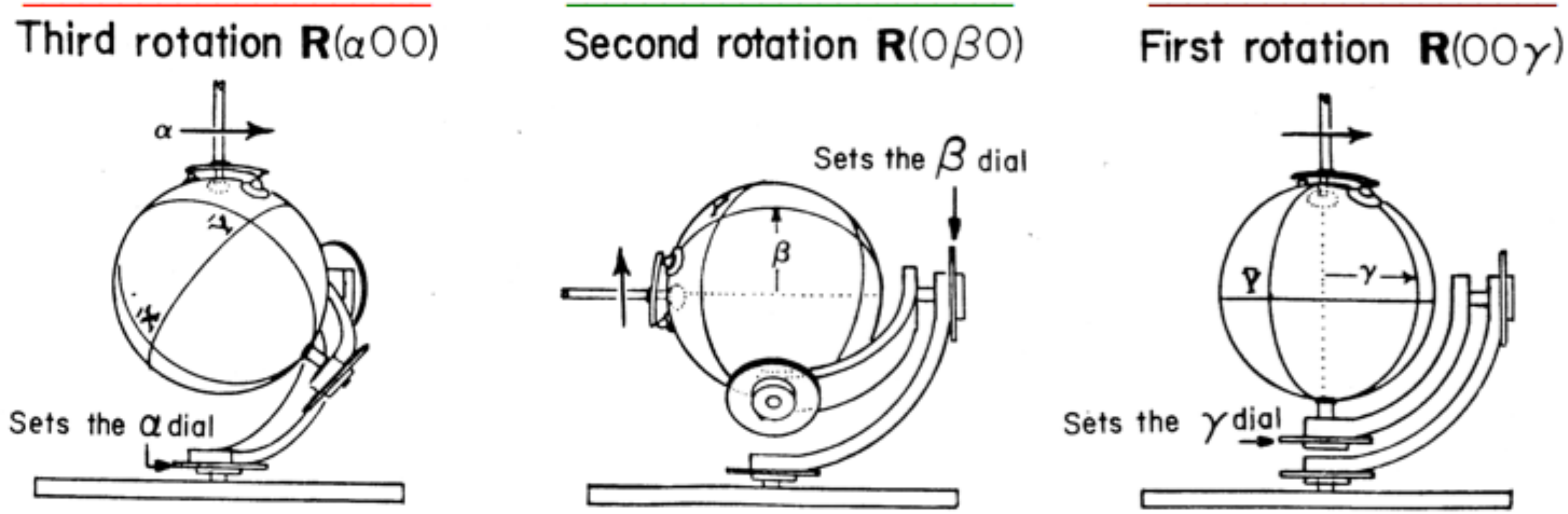

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Note lab-frame polar coordinates of Z(body)

...and body-frame polar coordinates of Z(lab)

Euler's rotation state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, and $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case

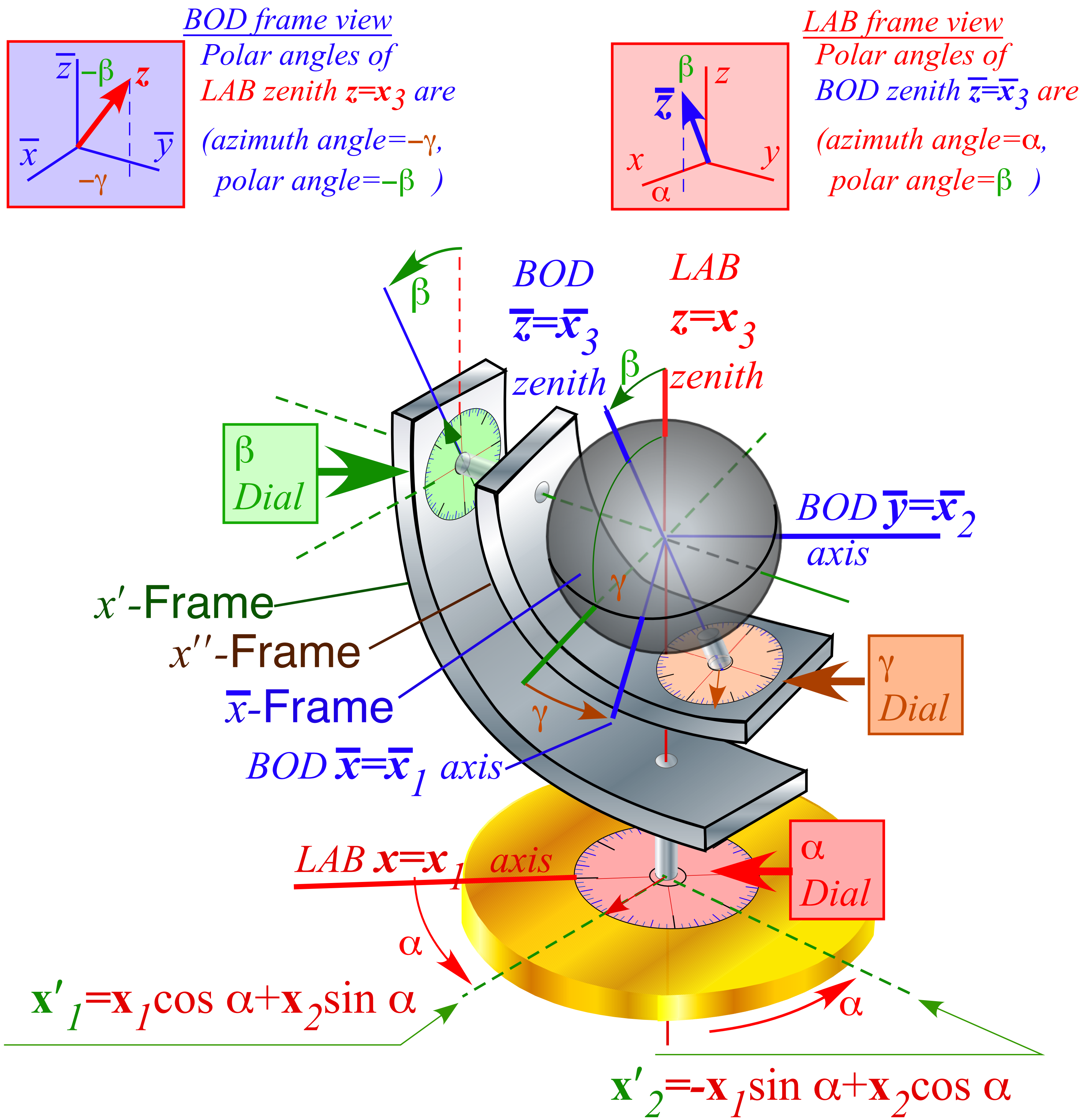

Fig. 10.A.3-4 Mechanical device demonstrating Euler angles (α,β,γ)

Euler's rotation state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, and $\mathbf{R}(0,0,\gamma)$

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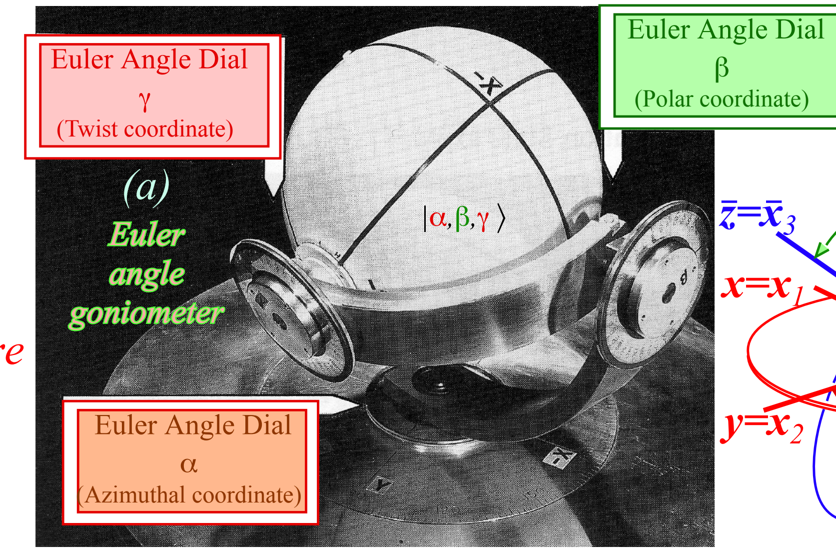
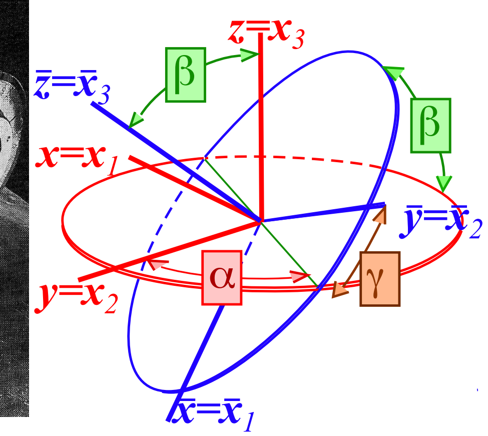
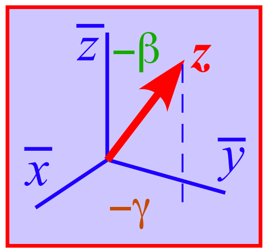
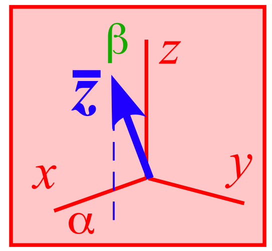


An astronomer's diagram

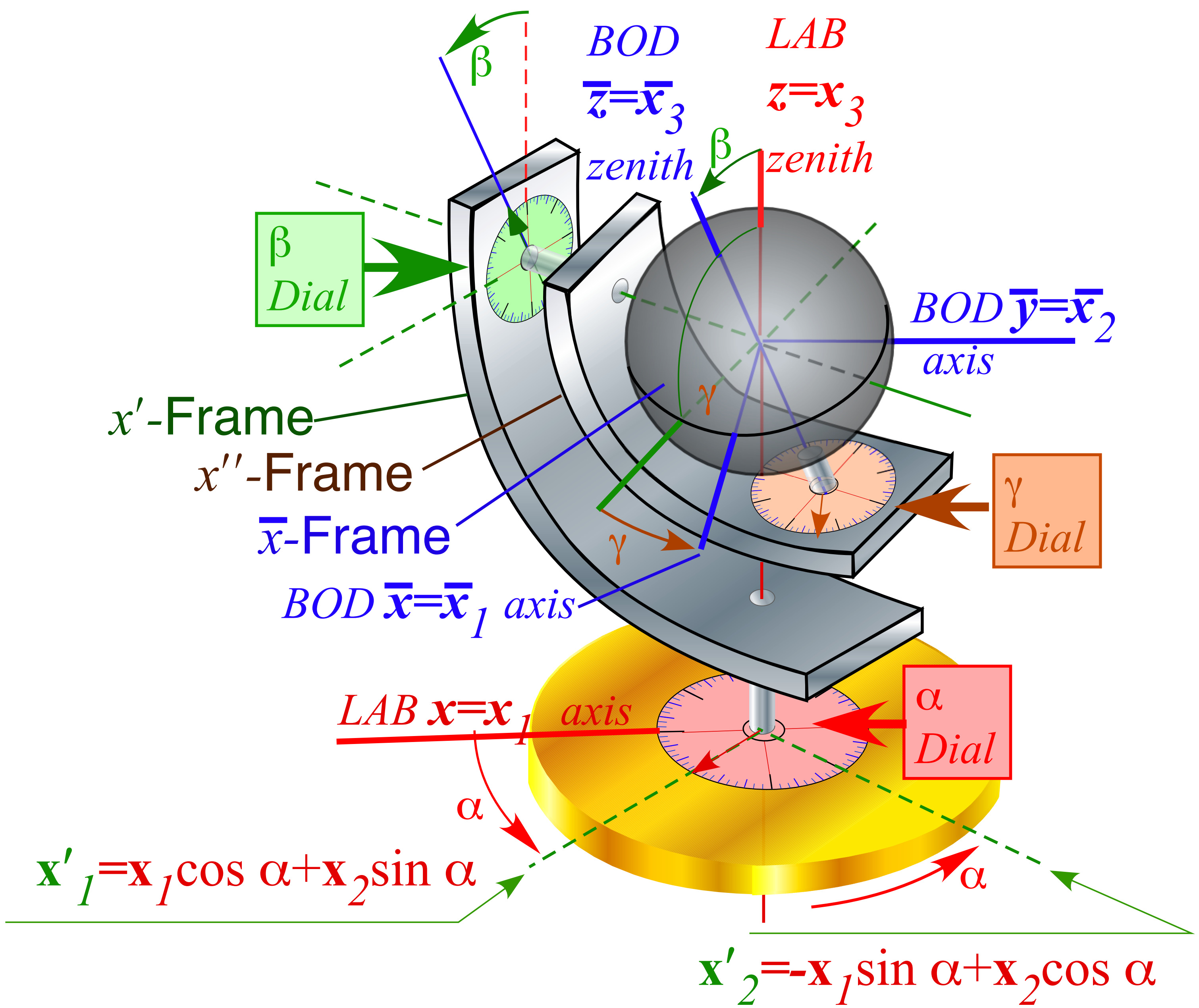




BOD frame view
Polar angles of LAB zenith $z=x_3$ are
(azimuth angle $=-\gamma$,
polar angle $=-\beta$)



LAB frame view
Polar angles of BOD zenith $\bar{z}=\bar{x}_3$ are
(azimuth angle $=\alpha$,
polar angle $=\beta$)



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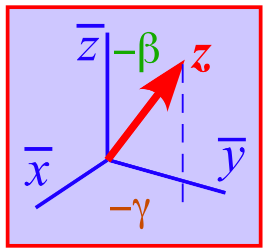
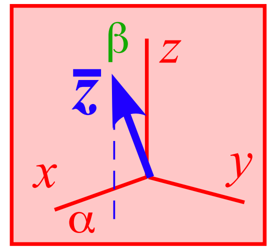
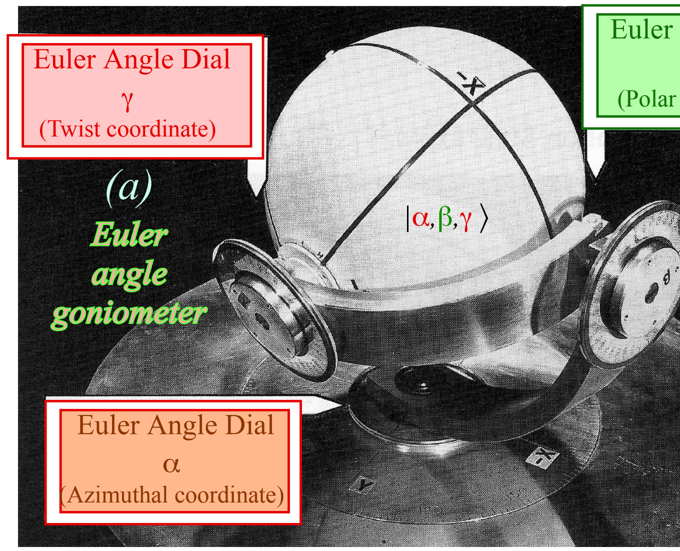
Spin-1 (3D-real vector) case



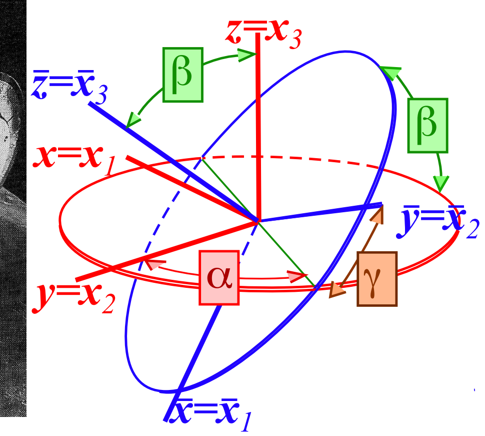
BOD frame view
Polar angles of LAB zenith $z=x_3$ *are (azimuth angle=*$-\gamma$*, polar angle=*$-\beta$ *)*

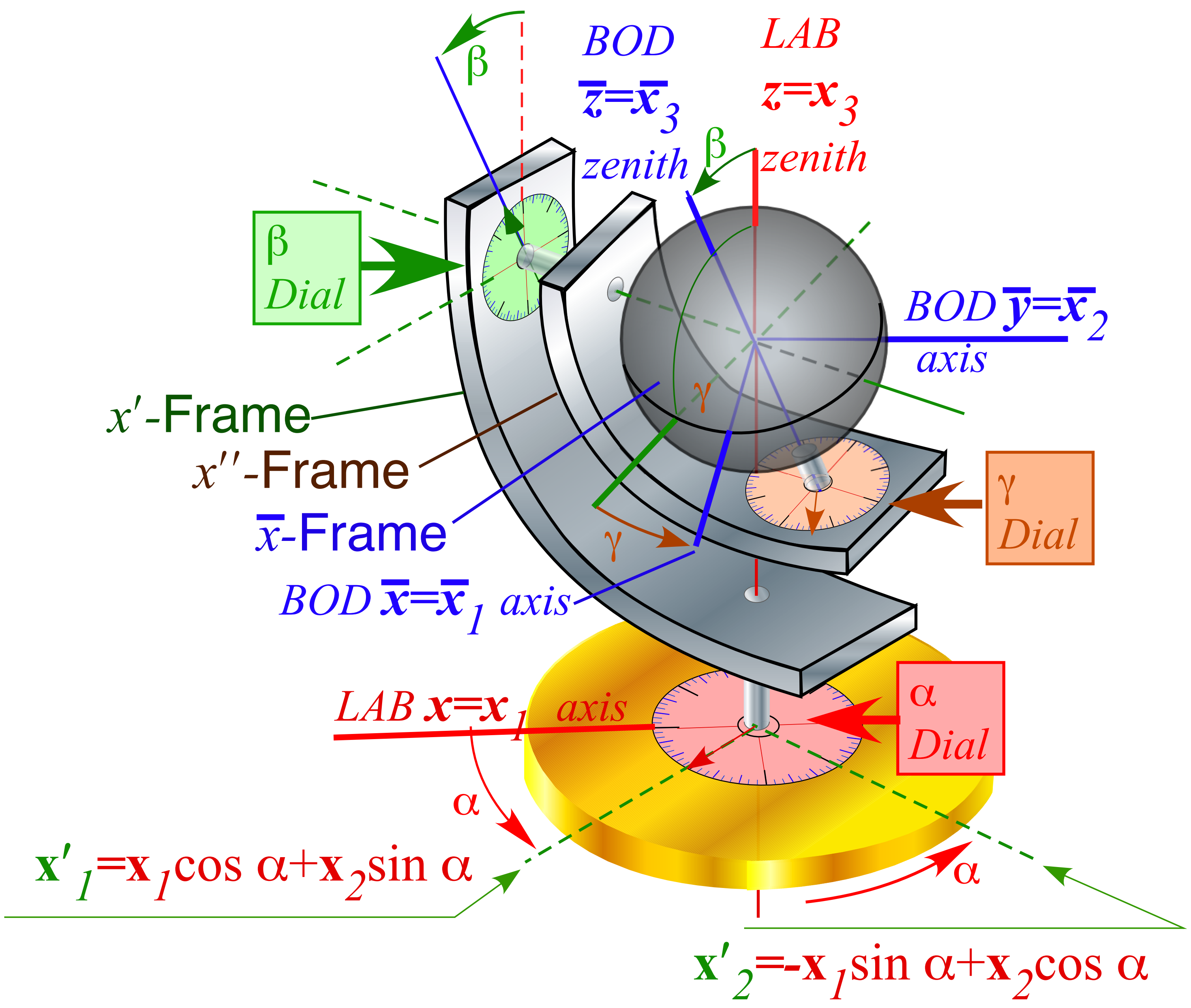


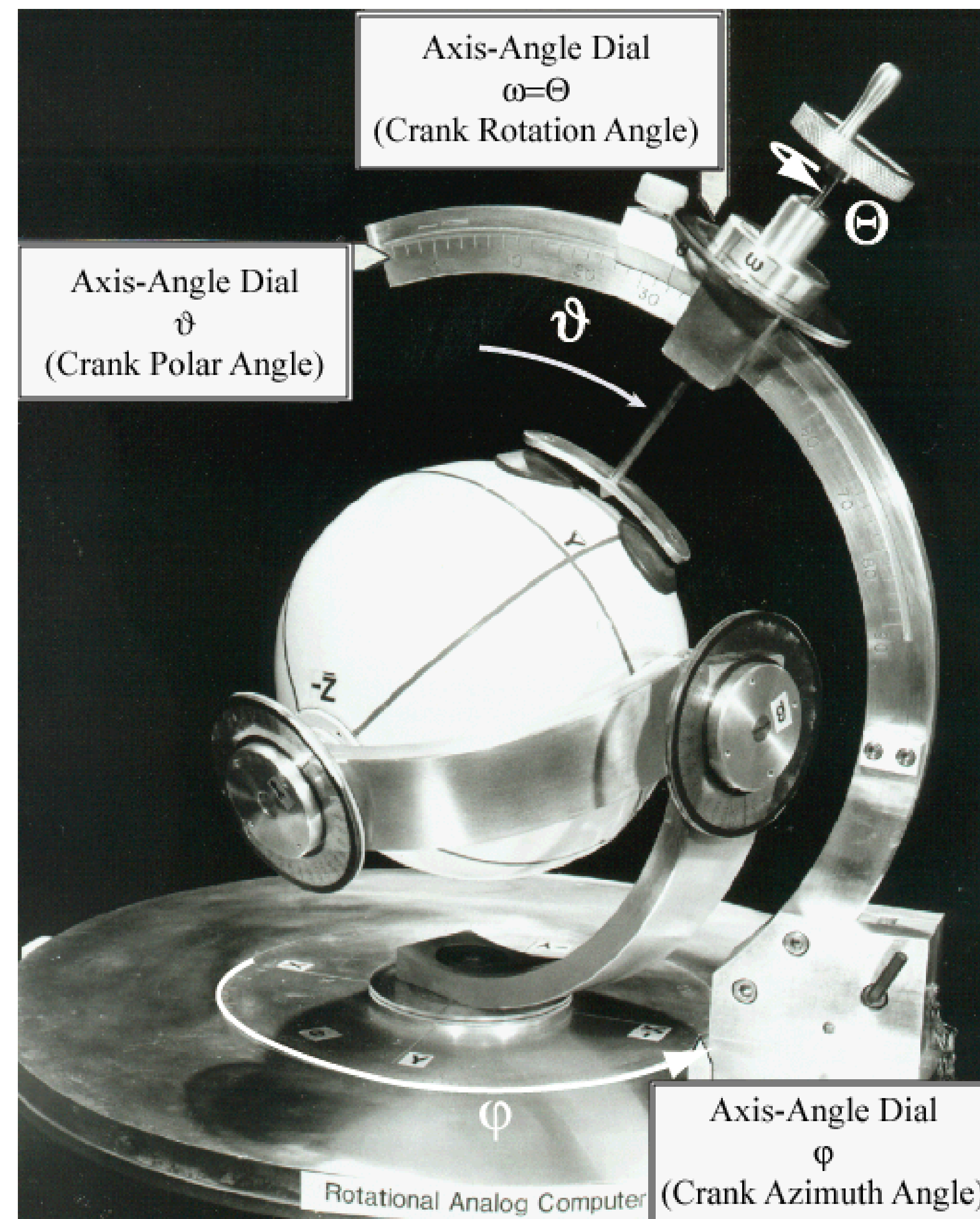
LAB frame view
Polar angles of BOD zenith $\bar{z}=\bar{x}_3$ *are (azimuth angle=*α*, polar angle=*β *)*



An astronomer's diagram







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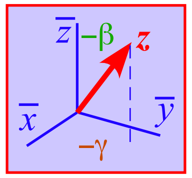
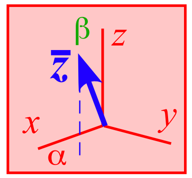
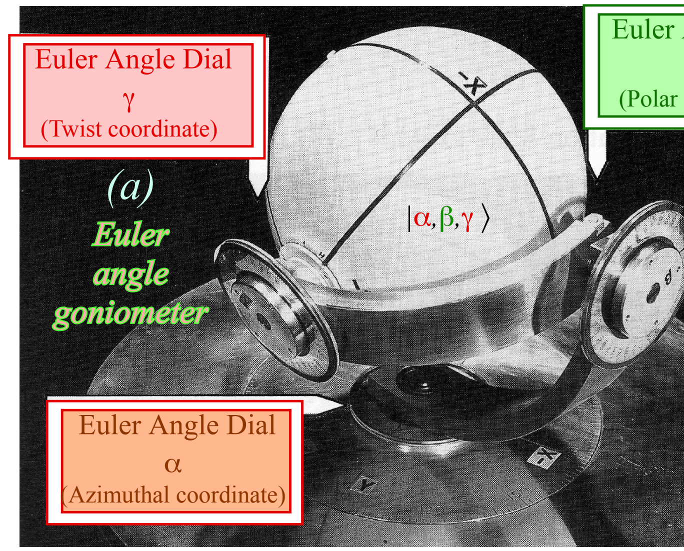
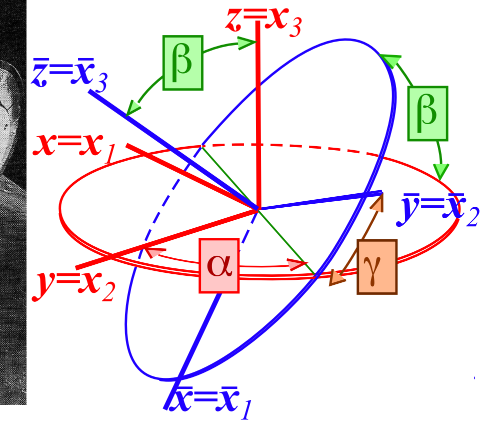


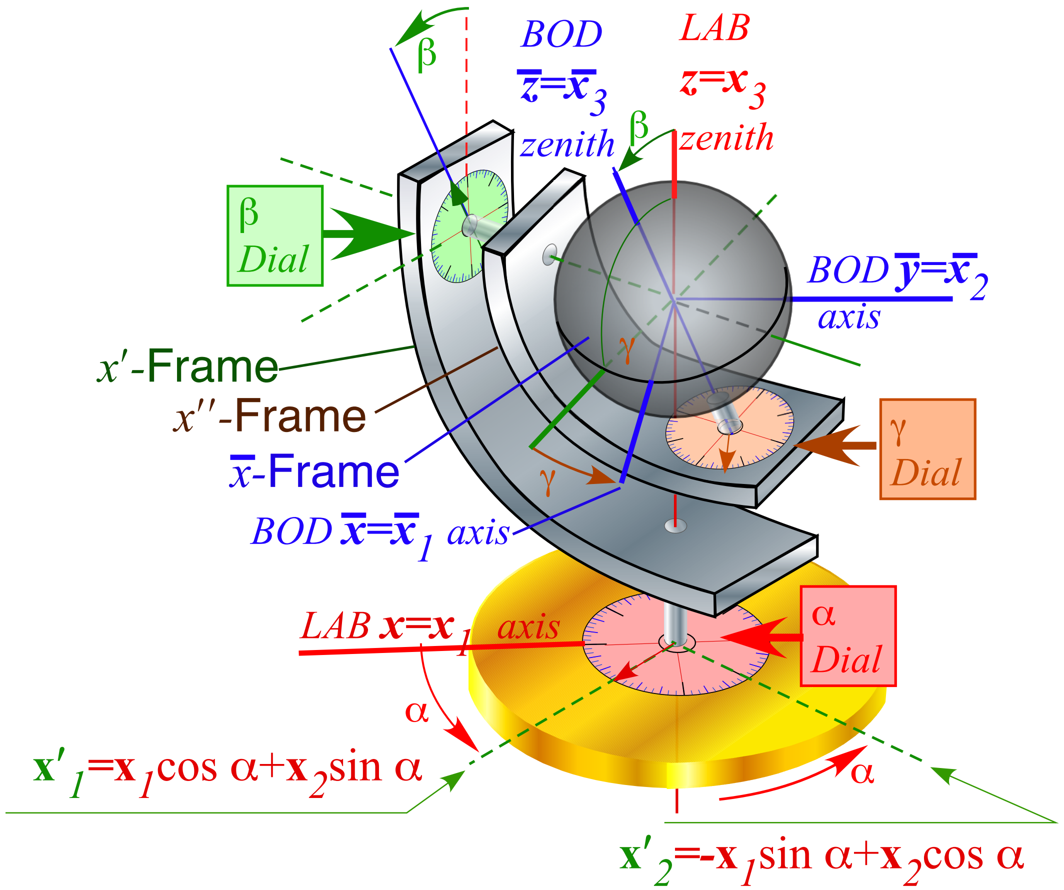
BOD frame view
Polar angles of LAB zenith $z=x_3$ are (azimuth angle $=-\gamma$, polar angle $=-\beta$)

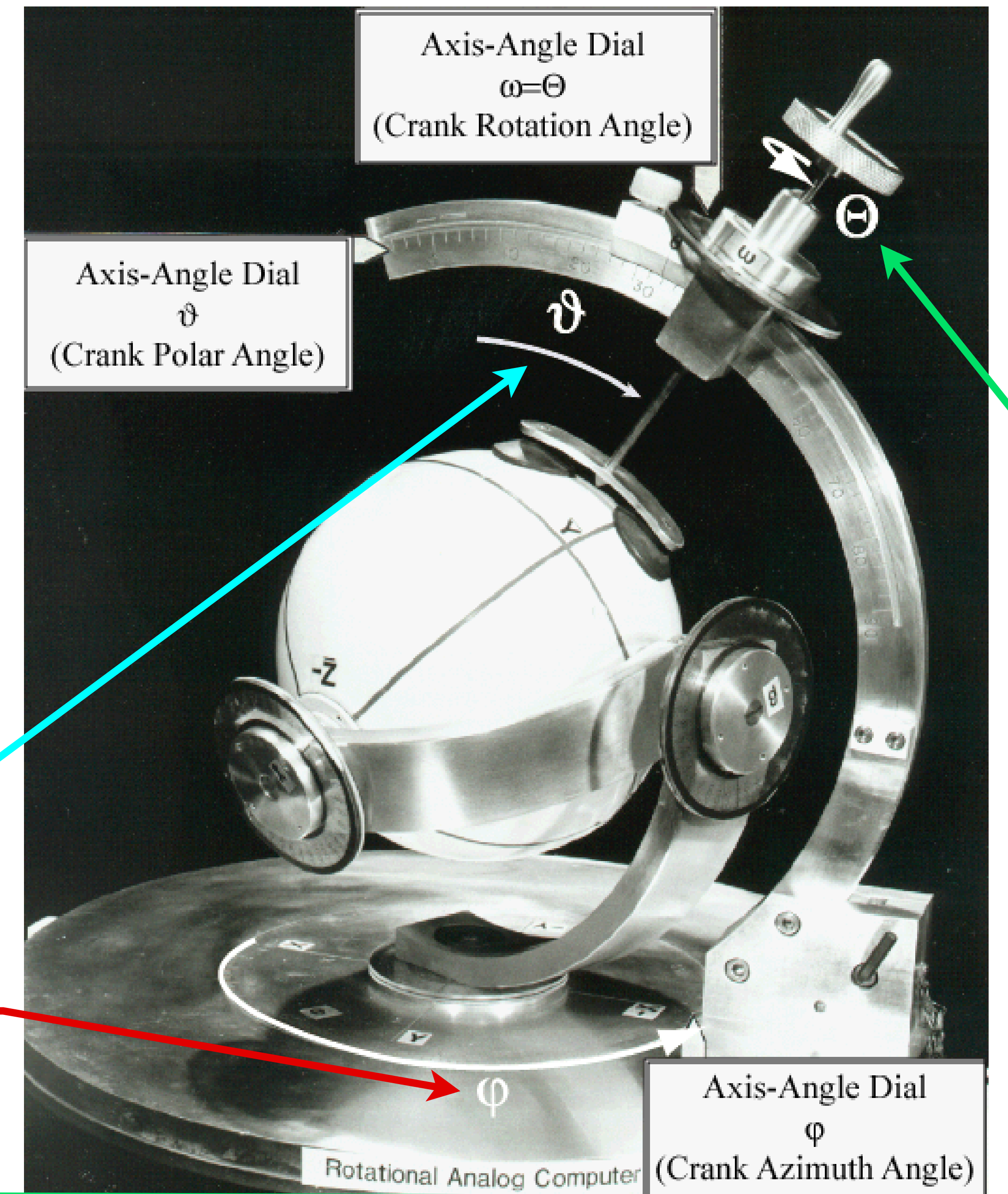


LAB frame view
Polar angles of BOD zenith $\bar{z}=\bar{x}_3$ are (azimuth angle $=\alpha$, polar angle $=\beta$)









Polar crank coordinates for unit axis vector $\hat{\boldsymbol{\omega}}$ or $\hat{\boldsymbol{\varphi}}$

$$\hat{\omega}_X = \hat{\omega}_B = \cos\varphi\,\sin\vartheta = \hat{\varphi}_B$$

$$\hat{\omega}_Y = \hat{\omega}_C = \sin\varphi\,\sin\vartheta = \hat{\varphi}_C$$

$$\hat{\omega}_Z = \hat{\omega}_A = \cos\vartheta = \hat{\varphi}_A$$

Crank turn angle: $\Theta=\Omega t=2\omega t$

Fig. 10.A.3-4 Mechanical device demonstrating Euler angles (α,β,γ)

Symmetry group $\mathscr{G}$ representations=>*AMOP* Hamiltonian **H** (or K) matrices, irreps $\mathscr{D}^{(\alpha)}$
=>*AMOP* wave functions $\Psi^{(\alpha)}$, eigensolutions

$\mathscr{G}$ = U(2) = Unitary group of dimension 2

Memoriam: Charles H. Townes 1916-2015 and his famous 2-state system: NH_3 maser in 1955

Earlier 2-state systems: 1863 John Stokes optical polarization , 1954 Rabi, Ramsey, and Schwinger NMR (MRI)

ANALOGY: (1) Classical 2-state motion $(\partial/\partial t)^2\mathbf{x}=-\mathbf{K}\cdot\mathbf{x}$ *vs (2) Quantum 2-state motion* $ih(\partial/\partial t)\mathbf{\Psi}=\mathbf{H}\cdot\mathbf{\Psi}$

Hamilton-Pauli spinor symmetry and $\boldsymbol{\sigma}$*-expansion of* $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu=\omega_A\boldsymbol{\sigma}_A+\omega_B\boldsymbol{\sigma}_B+\omega_C\boldsymbol{\sigma}_C+\omega_o\boldsymbol{\sigma}_o$

ABCD Time evolution operator $\mathbf{U}(t)=e^{-i\mathbf{H}t}$ *; its evaluation and visualization*

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Spinor-vector operator products $(\boldsymbol{\sigma}\cdot\mathbf{a})(\boldsymbol{\sigma}\cdot\mathbf{a})$

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U(2) transformation matrices and related R(3) rotations in ABC-space

Mysterious factors of 2 or ½ on 2D spinors or 3D vectors

2D {↑,↓} spinor space ½ as fast as 3D {ABC} spin-vectors

Hamiltonian for NMR: 3D Spin Moment Vector $\mathbf{m}=(m_x, m_y, m_z,)$ *in field* $\mathbf{B}=(B_x, B_y, B_z)$

➡ *State coordinates using Euler-angle rotations* $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case

➡ *Spin-1/2 (2D-complex spinor) case*

The ABC's of U(2) dynamics-Archetypes

Asymmetric-Diagonal A-Type motion

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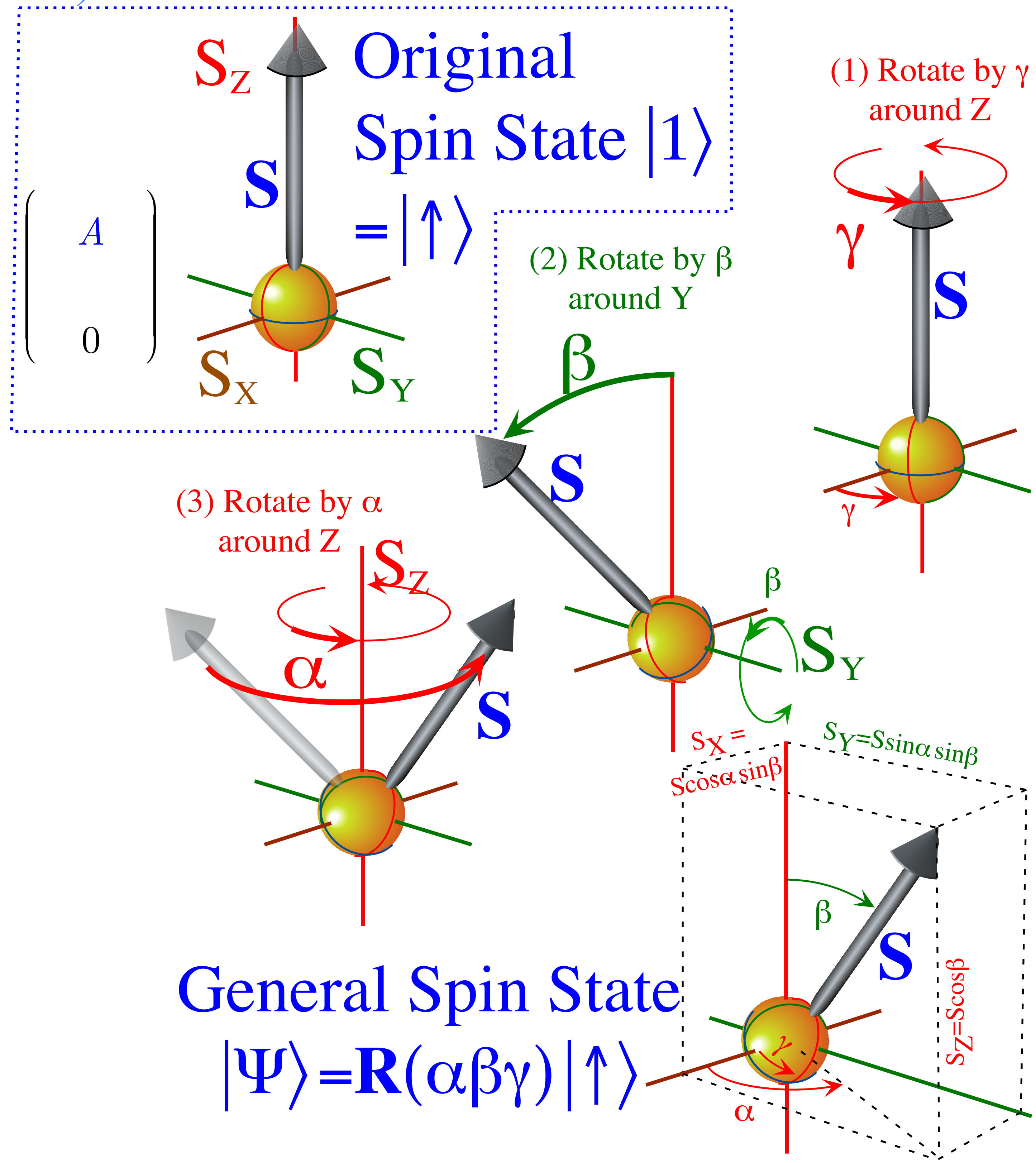
Euler's rotation state definition using rotations **R**$(\alpha,0,0)$, **R**$(0,\beta,0)$, *and* **R**$(0,0,\gamma)$

Spin-1/2 (2D-complex spinor) case

$$\left|a\right\rangle = \mathbf{R}\left(\alpha\beta\gamma\right)\left|\uparrow\right\rangle$$

$$= \mathbf{R}[\alpha \text{ about } Z]\cdot\mathbf{R}[\beta \text{ about } Y]\cdot\mathbf{R}[\gamma \text{ about } Z]\left|\uparrow\right\rangle$$

$$= \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix}\begin{pmatrix} A \\ 0 \end{pmatrix}$$



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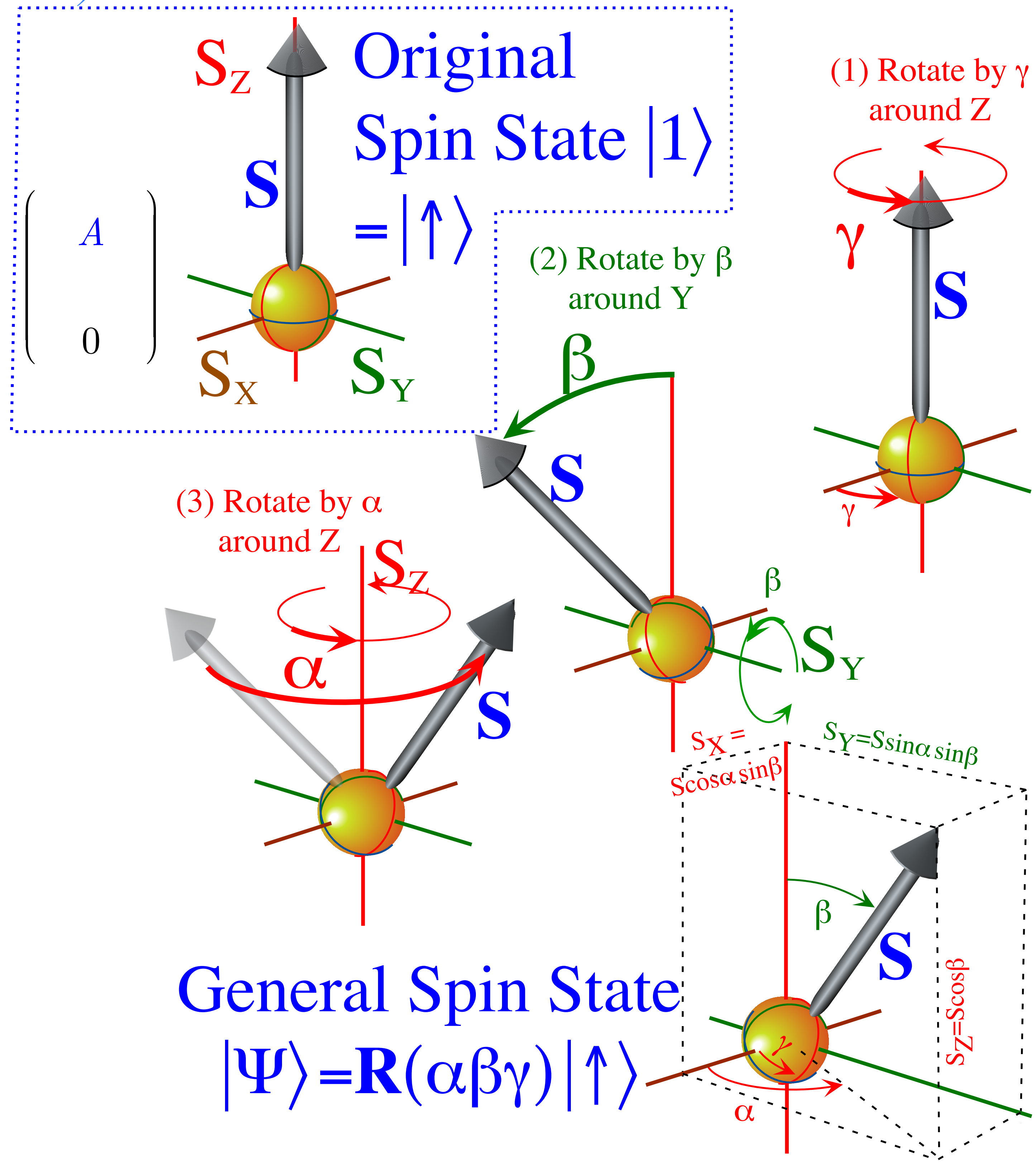
$$= \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} A \\ 0 \end{pmatrix}$$

$$= A\begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$



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Spin-1 (3D-real vector) case ◀ (related) ▶ *Spin-1/2 (2D-complex spinor) case*

The ABC's of U(2) dynamics-Archetypes

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Bilateral-Balanced B-Type motion

Circular-Coriolis… C-Type motion

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

Asymmetry $S_A = S_Z$, Balance $S_B = S_X$, and Chirality $S_C = S_Y$

Each point $\{E_1, E_2\}$ defines *2D-HO phase space* or analogous Ψ-space given by 2D amplitude array:
This defines real *3D spin vector* (S_A, S_B, S_C) "pointing" to a polarization ellipse or state.

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

$$\text{Asymmetry } S_A = \frac{1}{2}\left(a\middle|\sigma_A\middle|a\right) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_1 - a_2^* a_2\right] = \frac{1}{2}\left[x_1^2 + p_1^2 - x_2^2 - p_2^2\right]$$

$$\text{Balance } S_B = \frac{1}{2}\left(a\middle|\sigma_B\middle|a\right) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_2 + a_2^* a_1\right] = \left[p_1 p_2 + x_1 x_2\right]$$

$$\text{Chirality } S_C = \frac{1}{2}\left(a\middle|\sigma_C\middle|a\right) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{-i}{2}\left[a_1^* a_2 - a_2^* a_1\right] = \left[x_1 p_2 - x_2 p_1\right]$$

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

Asymmetry $S_A = S_Z$, Balance $S_B = S_X$, and Chirality $S_C = S_Y$

Each point $\{E_1, E_2\}$ defines *2D-HO phase space* or analogous Ψ-space given by 2D amplitude array: $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = A \begin{pmatrix} e^{-i\frac{\alpha}{2}} \cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}} \sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}}$

This defines real *3D spin vector* (S_A, S_B, S_C) "pointing" to a polarization ellipse or state.

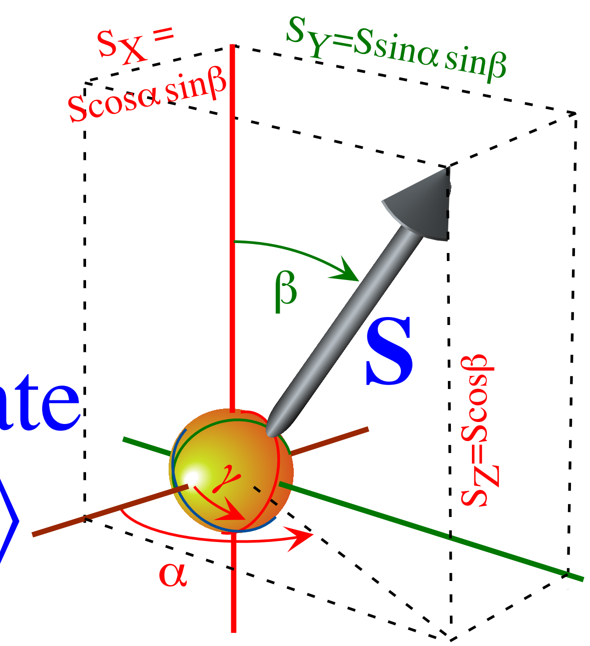
Asymmetry $S_A = \frac{1}{2}(a|\sigma_A|a) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_1 - a_2^* a_2\right] = \frac{1}{2}\left[x_1^2 + p_1^2 - x_2^2 - p_2^2\right] = \frac{I}{2}[\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}] = \frac{I}{2}\cos\beta$

Balance $S_B = \frac{1}{2}(a|\sigma_B|a) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_2 + a_2^* a_1\right] = \left[p_1 p_2 + x_1 x_2\right] = I\left[-\sin\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2} + \cos\frac{\alpha+\gamma}{2}\cos\frac{\alpha-\gamma}{2}\right]\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{I}{2}\cos\alpha\sin\beta$

Chirality $S_C = \frac{1}{2}(a|\sigma_C|a) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{-i}{2}\left[a_1^* a_2 - a_2^* a_1\right] = \left[x_1 p_2 - x_2 p_1\right] = I\left[\cos\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2} - \cos\frac{\alpha-\gamma}{2}\cdot -\sin\frac{\alpha+\gamma}{2}\right]\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{I}{2}\sin\alpha\sin\beta$

General Spin State

$|\Psi\rangle = \mathbf{R}(\alpha\beta\gamma)|\uparrow\rangle$



3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

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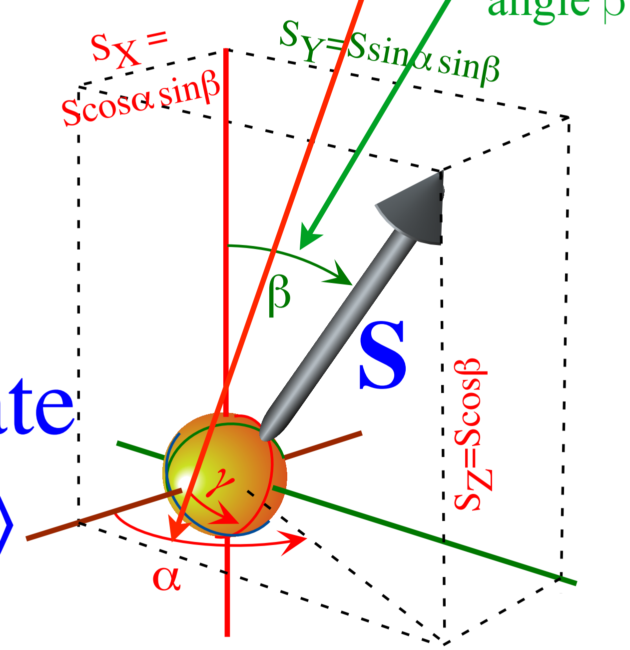
$$\text{Balance } S_B = \frac{1}{2}\left(a\middle|\sigma_B\middle|a\right) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_2 + a_2^* a_1\right] = \left[p_1 p_2 + x_1 x_2\right] = I\left[-\sin\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2} + \cos\frac{\alpha+\gamma}{2}\cos\frac{\alpha-\gamma}{2}\right]\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{I}{2}\cos\alpha\sin\beta$$

$$\text{Chirality } S_C = \frac{1}{2}\left(a\middle|\sigma_C\middle|a\right) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{-i}{2}\left[a_1^* a_2 - a_2^* a_1\right] = \left[x_1 p_2 - x_2 p_1\right] = I\left[\cos\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2} - \cos\frac{\alpha-\gamma}{2}\cdot -\sin\frac{\alpha+\gamma}{2}\right]\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{I}{2}\sin\alpha\sin\beta$$

azimuth angle α

polar angle β



General Spin State

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3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

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$$\text{Asymmetry } S_A=\frac{1}{2}\langle a|\sigma_A|a\rangle=\frac{1}{2}\begin{pmatrix}a_1^* & a_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix}=\frac{1}{2}\left[a_1^*a_1-a_2^*a_2\right]=\frac{1}{2}\left[x_1^2+p_1^2-x_2^2-p_2^2\right]=\frac{I}{2}\left[\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right]=\frac{I}{2}\cos\beta$$

$$\text{Balance } S_B=\frac{1}{2}\langle a|\sigma_B|a\rangle=\frac{1}{2}\begin{pmatrix}a_1^* & a_2^*\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix}=\frac{1}{2}\left[a_1^*a_2+a_2^*a_1\right]=\left[p_1p_2+x_1x_2\right]=I\left[-\sin\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2}+\cos\frac{\alpha+\gamma}{2}\cos\frac{\alpha-\gamma}{2}\right]\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{I}{2}\cos\alpha\sin\beta$$

$$\text{Chirality } S_C=\frac{1}{2}\langle a|\sigma_C|a\rangle=\frac{1}{2}\begin{pmatrix}a_1^* & a_2^*\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix}=\frac{-i}{2}\left[a_1^*a_2-a_2^*a_1\right]=\left[x_1p_2-x_2p_1\right]=I\left[\cos\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2}-\cos\frac{\alpha-\gamma}{2}\cdot-\sin\frac{\alpha+\gamma}{2}\right]\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{I}{2}\sin\alpha\sin\beta$$

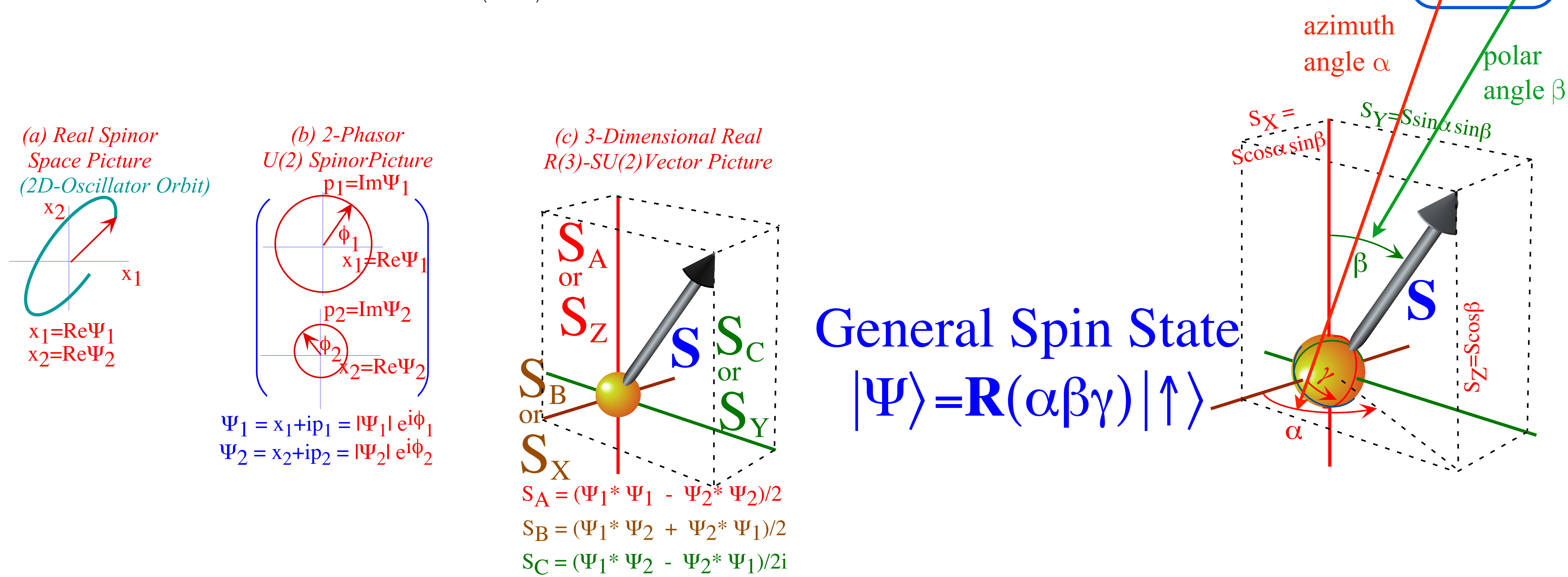

Fig. 10.5.2 Spinor, phasor, and vector descriptions of 2-state systems .

Symmetry group $\mathscr{G}$ representations=>*AMOP* Hamiltonian **H** (or K) matrices, irreps $\mathscr{D}^{(\alpha)}$
=>*AMOP* wave functions $\Psi^{(\alpha)}$, eigensolutions

$\mathscr{G}$ = U(2) = Unitary group of dimension 2

Memoriam: Charles H. Townes 1916-2015 and his famous 2-state system: NH_3 maser in 1955

Earlier 2-state systems: 1863 John Stokes optical polarization , 1954 Rabi, Ramsey, and Schwinger NMR (MRI)

ANALOGY: (1) Classical 2-state motion $(\partial/\partial t)^2\mathbf{x}=-\mathbf{K}\cdot\mathbf{x}$ *vs (2) Quantum 2-state motion* $ih(\partial/\partial t)\mathbf{\Psi}=\mathbf{H}\cdot\mathbf{\Psi}$

Hamilton-Pauli spinor symmetry and **σ***-expansion of* $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu=\omega_A\boldsymbol{\sigma}_A+\omega_B\boldsymbol{\sigma}_B+\omega_C\boldsymbol{\sigma}_C+\omega_o\boldsymbol{\sigma}_o$

ABCD Time evolution operator $\mathbf{U}(t)=e^{-i\mathbf{H}t}$ *; its evaluation and visualization*

ABCD symmetry operator $\{\boldsymbol{\sigma}_A,\boldsymbol{\sigma}_B,\boldsymbol{\sigma}_C\}$ *product algebra for spinor-vector operators* $\sigma_a=\boldsymbol{\sigma}\cdot\mathbf{a}$

Spinor-vector operator products $(\boldsymbol{\sigma}\cdot\mathbf{a})(\boldsymbol{\sigma}\cdot\mathbf{a})$

Crazy-Thing Theorem: $e^{-i\sigma_a\Theta}=\cos\Theta-i\sigma_a\sin\Theta$

U(2) transformation matrices and related R(3) rotations in ABC-space

Mysterious factors of 2 or ½ on 2D spinors or 3D vectors

2D {↑,↓} spinor space ½ as fast as 3D {ABC} spin-vectors

Hamiltonian for NMR: 3D Spin Moment Vector $\mathbf{m}=(m_x, m_y, m_z,)$ *in field* $\mathbf{B}=(B_x, B_y, B_z)$

State coordinates using Euler-angle rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case ← (related) → *Spin-1/2 (2D-complex spinor) case*

The ABC's of U(2) dynamics-Archetypes

→ *Asymmetric-Diagonal A-Type motion*

Bilateral-Balanced B-Type motion

Circular-Coriolis… C-Type motion

The ABC's of U(2) dynamics

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

$$= \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

Asymmetric Diagonal A-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^A|1\rangle & \langle 1|\mathbf{H}^A|2\rangle \\ \langle 2|\mathbf{H}^A|1\rangle & \langle 2|\mathbf{H}^A|2\rangle \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_A}{2}\sigma_A$$

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$$= \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

$$= \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

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Crank : $\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 0 \\ 0 \end{pmatrix}$ *Eigen − Spin* : $\vec{\mathbf{S}} = \begin{pmatrix} S_A \\ S_B \\ S_C \end{pmatrix} = \begin{pmatrix} \pm S \\ 0 \\ 0 \end{pmatrix}$

The ABC's of U(2) dynamics

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$
$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\boldsymbol{\Omega}}}{2}\bullet\boldsymbol{\sigma}$$

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$$= \frac{A+D}{2}\,\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\,\sigma_A$$

$$= \frac{A+D}{2}\,\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\,\sigma_B + \frac{\Omega_C}{2}\,\sigma_C + \frac{\Omega_A}{2}\,\sigma_A$$

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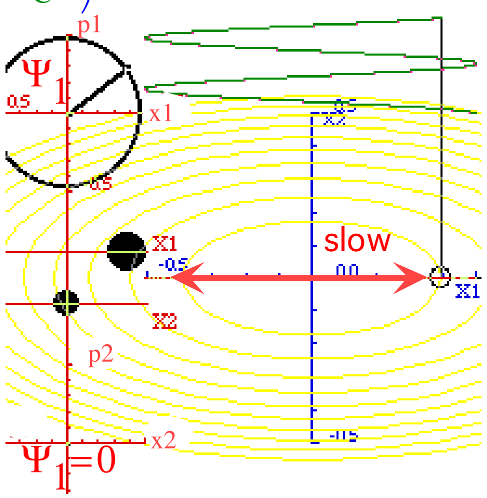
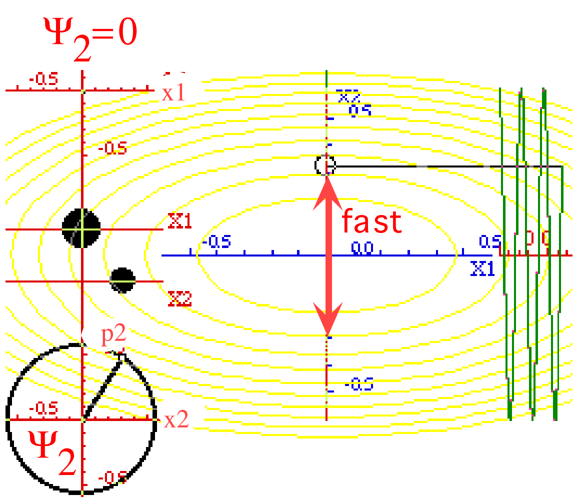
Crank : $\vec{\boldsymbol{\Omega}} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 0 \\ 0 \end{pmatrix}$ *Eigen − Spin* : $\vec{\mathbf{S}} = \begin{pmatrix} S_A \\ S_B \\ S_C \end{pmatrix} = \begin{pmatrix} \pm S \\ 0 \\ 0 \end{pmatrix}$





The *ABC's of U(2) dynamics*

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$$= \frac{A+D}{2}\,\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\,\sigma_A$$

$$= \frac{A+D}{2}\,\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\,\sigma_B + \frac{\Omega_C}{2}\,\sigma_C + \frac{\Omega_A}{2}\,\sigma_A$$

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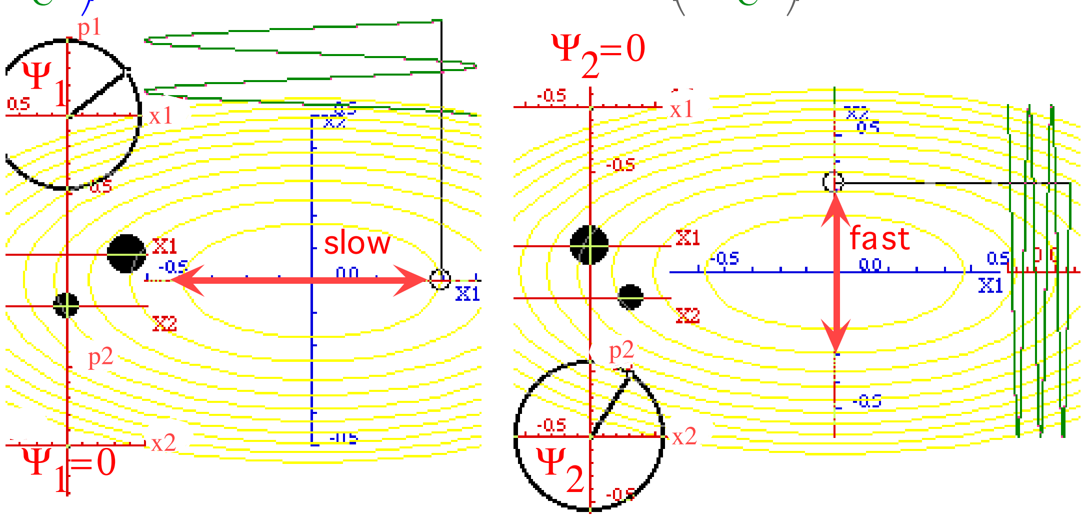

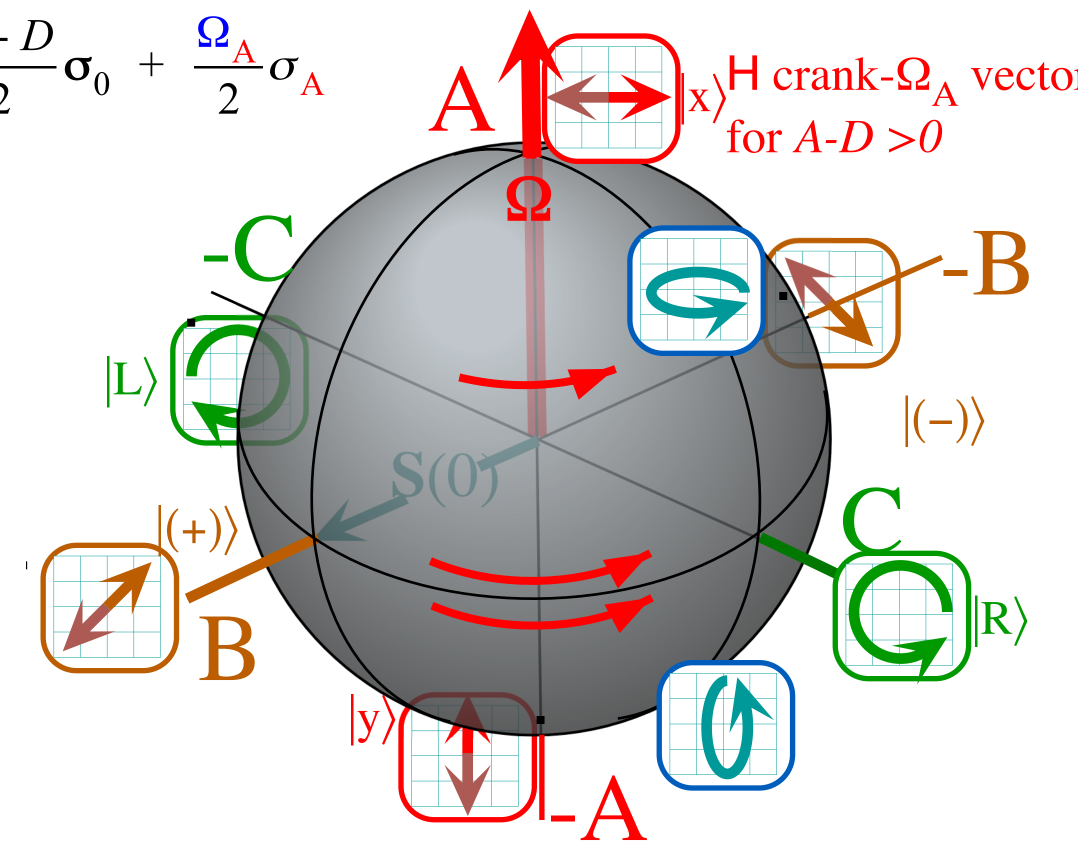

The ABC's of U(2) dynamics

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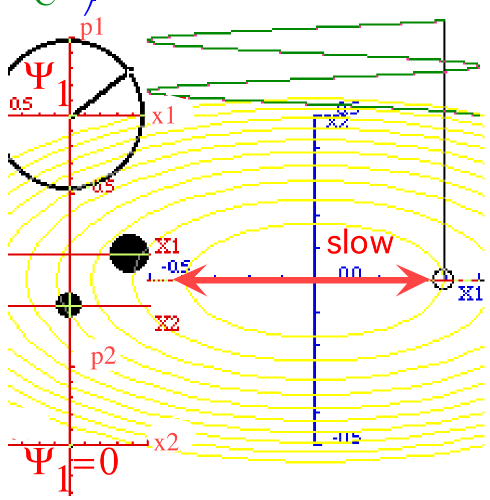
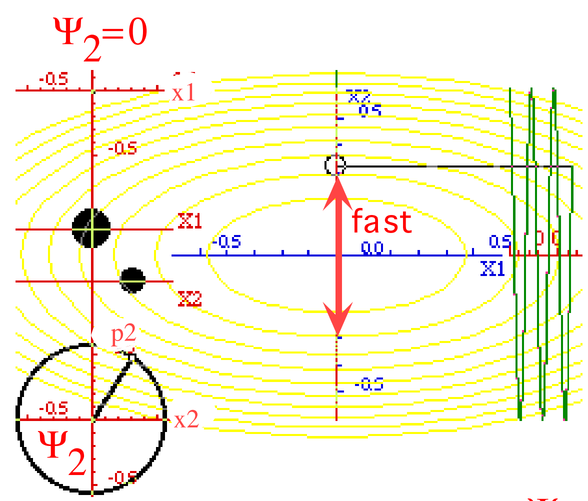
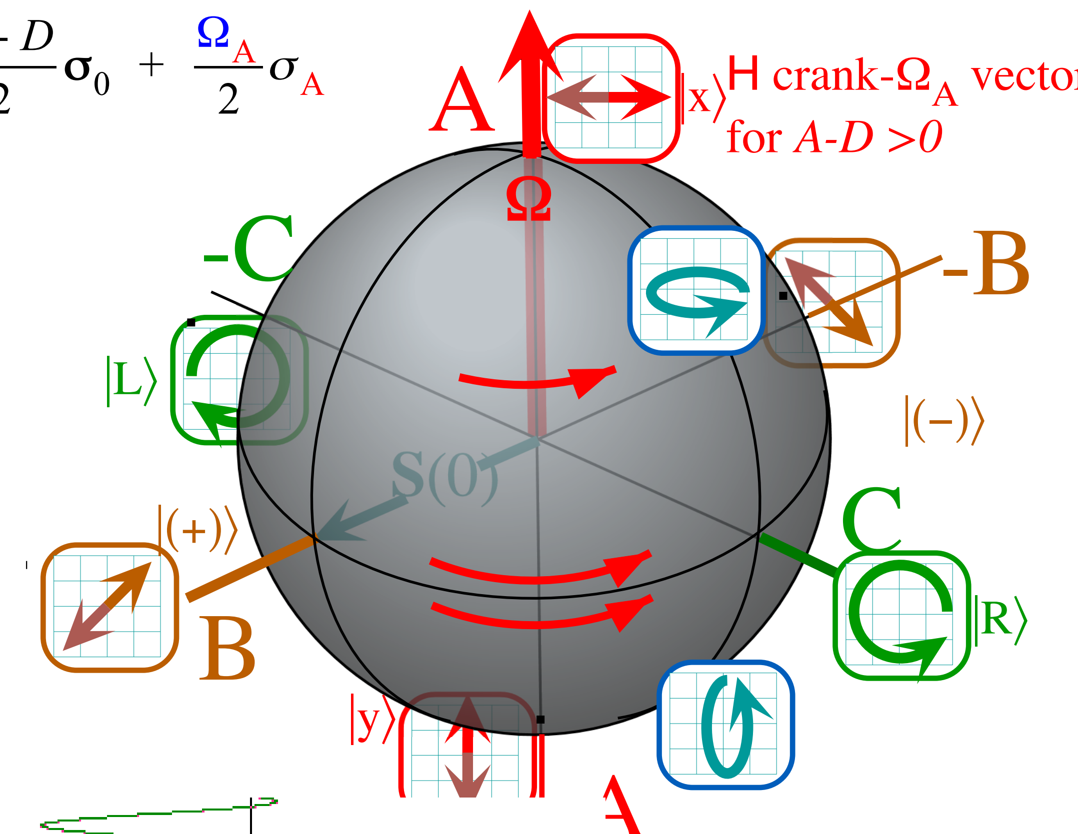
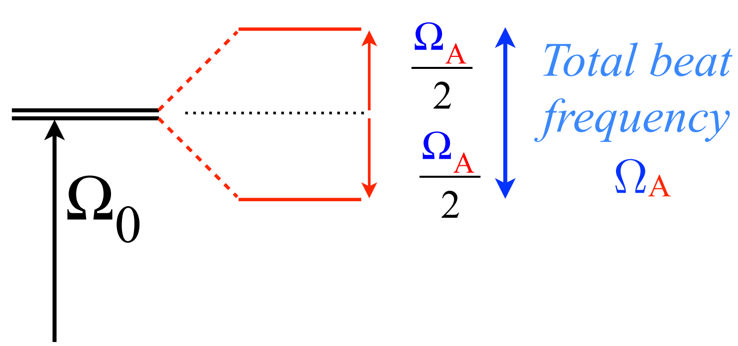
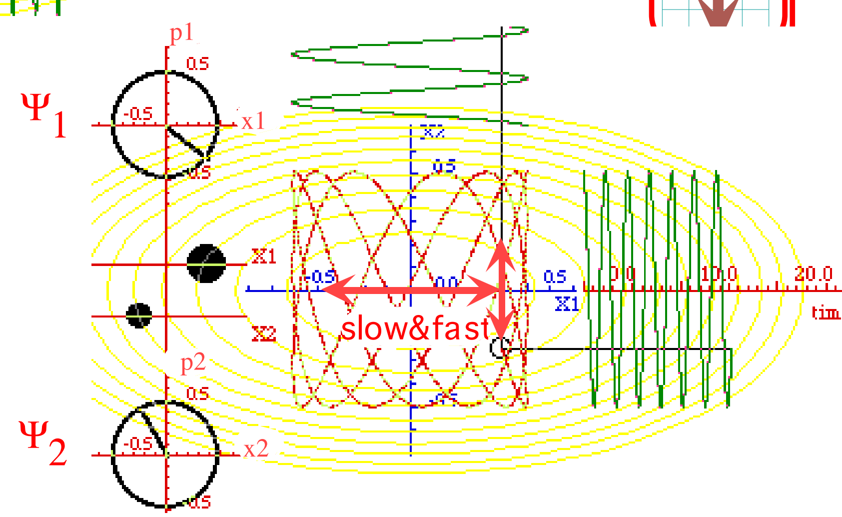






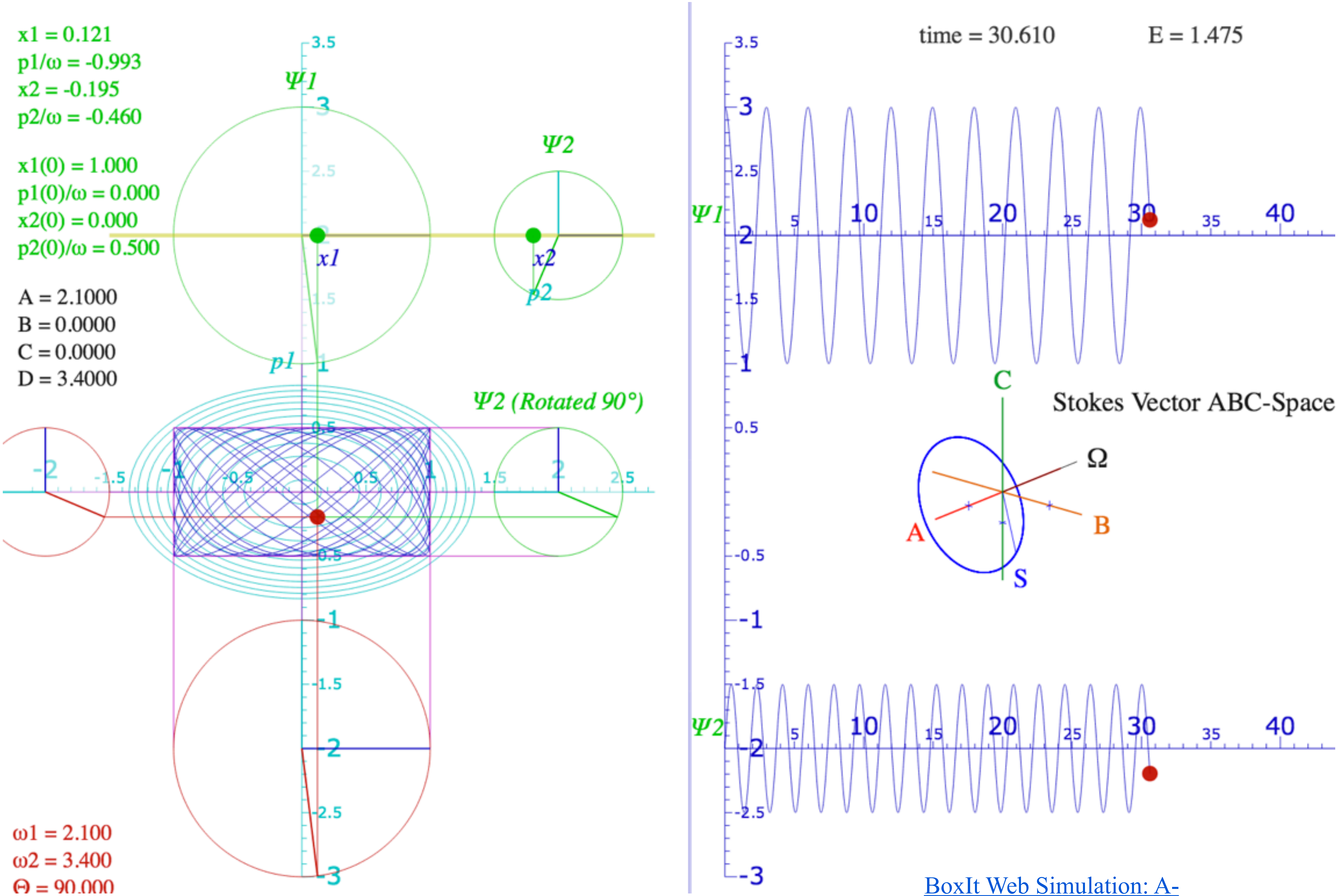
Beat dynamics:





[BoxIt (A-Type) Web Simulation]

A-Type elliptical polarized motion



BoxIt Web Simulation: A-Type with A=2.1, D=3.4

Symmetry group $\mathscr{G}$ representations=>*AMOP* Hamiltonian **H** (or K) matrices, irreps $\mathscr{D}^{(\alpha)}$
=>*AMOP* wave functions $\Psi^{(\alpha)}$, eigensolutions

$\mathscr{G}$ = U(2) = Unitary group of dimension 2

Memoriam: Charles H. Townes 1916-2015 and his famous 2-state system: NH_3 maser in 1955

Earlier 2-state systems: 1863 John Stokes optical polarization , 1954 Rabi, Ramsey, and Schwinger NMR (MRI)

ANALOGY: (1) Classical 2-state motion $(\partial/\partial t)^2\mathbf{x}=-\mathbf{K}\cdot\mathbf{x}$ *vs (2) Quantum 2-state motion* $ih(\partial/\partial t)\mathbf{\Psi}=\mathbf{H}\cdot\mathbf{\Psi}$

Hamilton-Pauli spinor symmetry and **σ***-expansion of* $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu=\omega_A\boldsymbol{\sigma}_A+\omega_B\boldsymbol{\sigma}_B+\omega_C\boldsymbol{\sigma}_C+\omega_o\boldsymbol{\sigma}_o$

ABCD Time evolution operator $\mathbf{U}(t)=e^{-i\mathbf{H}t}$ *; its evaluation and visualization*

ABCD symmetry operator $\{\boldsymbol{\sigma}_A,\boldsymbol{\sigma}_B,\boldsymbol{\sigma}_C\}$ *product algebra for spinor-vector operators* $\sigma_a=\boldsymbol{\sigma}\cdot\mathbf{a}$

Spinor-vector operator products $(\boldsymbol{\sigma}\cdot\mathbf{a})(\boldsymbol{\sigma}\cdot\mathbf{a})$

Crazy-Thing Theorem: $e^{-i\sigma_a\Theta}=\cos\Theta - i\sigma_a\sin\Theta$

U(2) transformation matrices and related R(3) rotations in ABC-space

Mysterious factors of 2 or ½ on 2D spinors or 3D vectors

2D {↑,↓} spinor space ½ as fast as 3D {ABC} spin-vectors

Hamiltonian for NMR: 3D Spin Moment Vector $\mathbf{m}=(m_x, m_y, m_z,)$ *in field* $\mathbf{B}=(B_x, B_y, B_z)$

State coordinates using Euler-angle rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case ◀ (related) ▶ *Spin-1/2 (2D-complex spinor) case*

The ABC's of U(2) dynamics-Archetypes

Asymmetric-Diagonal A-Type motion

➡ *Bilateral-Balanced B-Type motion*

Circular-Coriolis… C-Type motion

The ABC's of U(2) dynamics

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{A+D}{2}\,\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\,\sigma_A$$

$$= \frac{A+D}{2}\,\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\,\sigma_B + \frac{\Omega_C}{2}\,\sigma_C + \frac{\Omega_A}{2}\,\sigma_A$$

$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

Bilateral-Balanced B-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^B|1\rangle & \langle 1|\mathbf{H}^B|2\rangle \\ \langle 2|\mathbf{H}^B|1\rangle & \langle 2|\mathbf{H}^B|2\rangle \end{pmatrix} = \begin{pmatrix} \Omega_0 & B \\ B & \Omega_0 \end{pmatrix} = \Omega_0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \Omega_0\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B$$

The *ABC*'s of *U*(2) dynamics

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

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$$= \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

$$= \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

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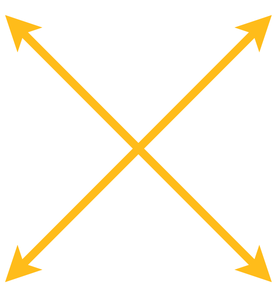
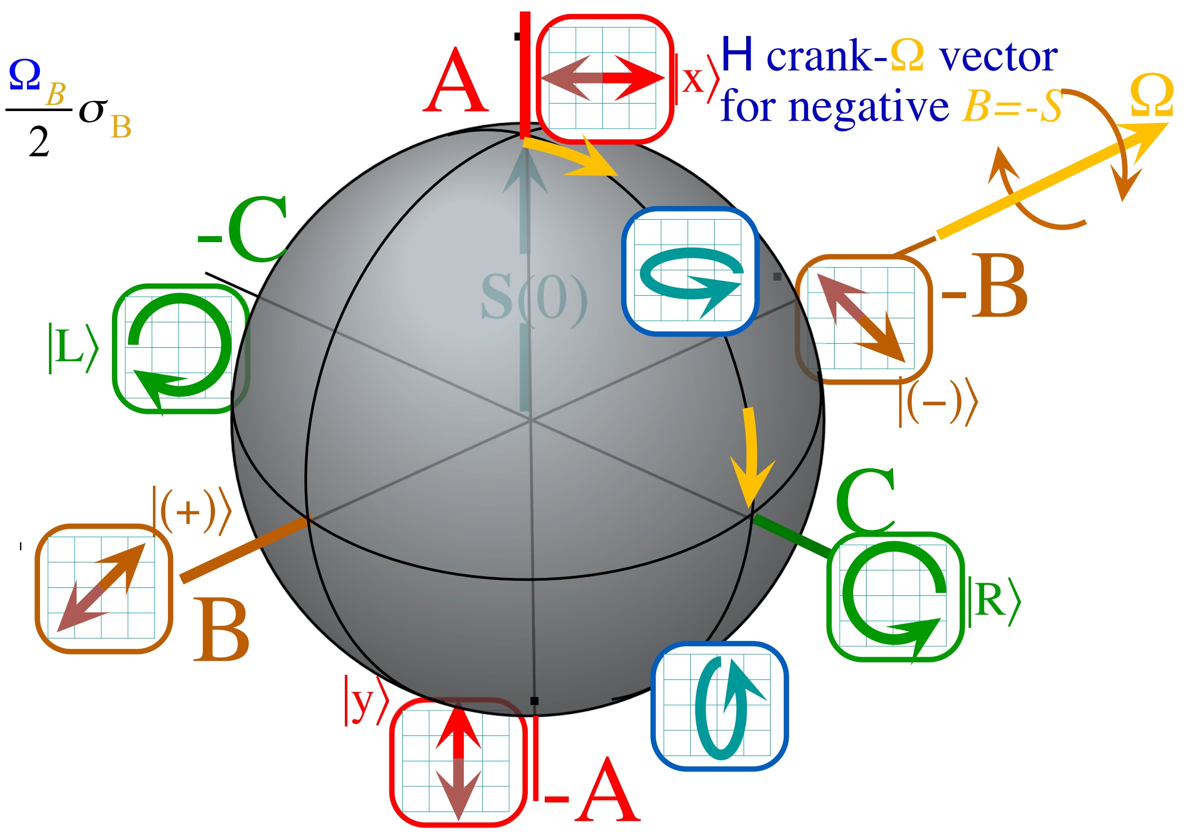
$$\begin{pmatrix}\langle 1|\mathbf{H}^B|1\rangle & \langle 1|\mathbf{H}^B|2\rangle\\ \langle 2|\mathbf{H}^B|1\rangle & \langle 2|\mathbf{H}^B|2\rangle\end{pmatrix} = \begin{pmatrix}\Omega_0 & B\\ B & \Omega_0\end{pmatrix} = \Omega_0\begin{pmatrix}1&0\\0&1\end{pmatrix} + B\begin{pmatrix}0&1\\1&0\end{pmatrix} = \Omega_0\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B$$

$$\textit{Crank}:\ \vec{\Omega} = \begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix} = \begin{pmatrix}0\\ 2B\\ 0\end{pmatrix} \qquad \textit{Eigen-Spin}:\vec{\mathbf{S}} = \begin{pmatrix}S_A\\ S_B\\ S_C\end{pmatrix} = \begin{pmatrix}0\\ \pm S\\ 0\end{pmatrix}$$





The ABC's of U(2) dynamics

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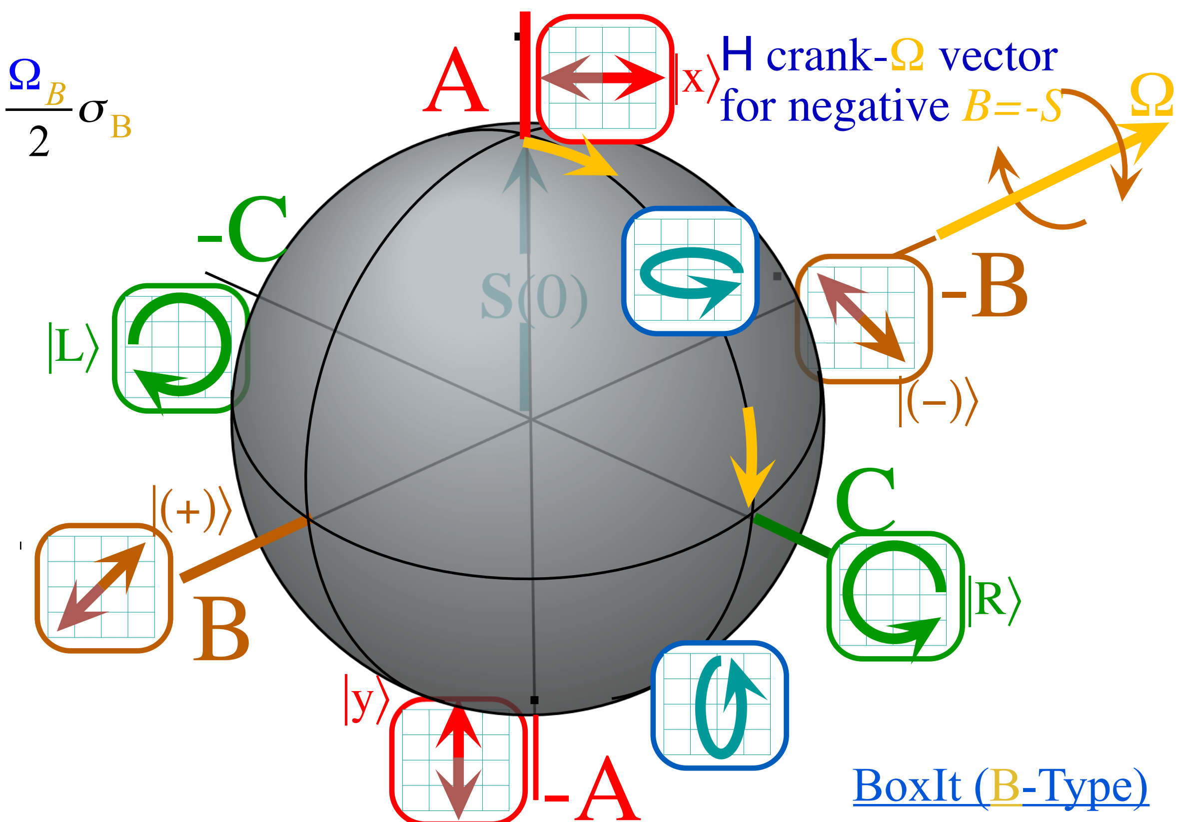
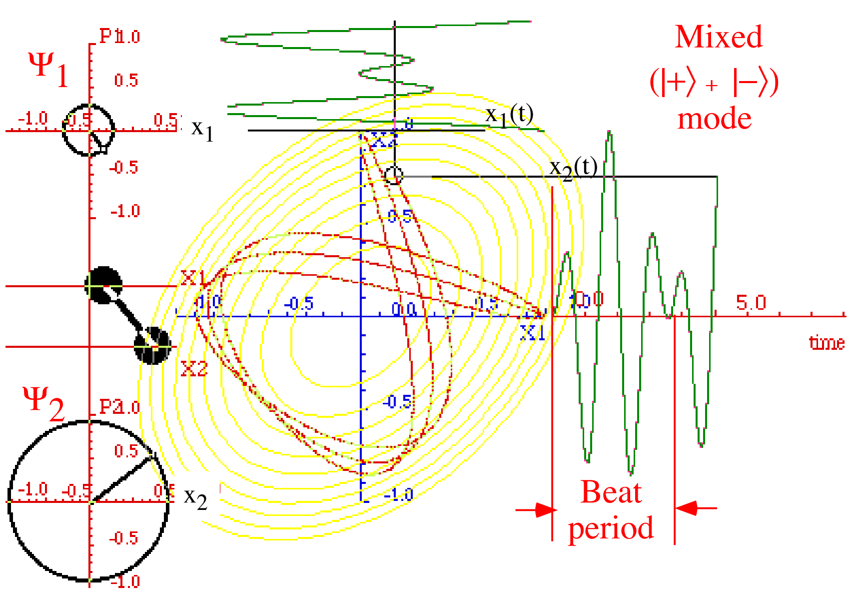


BoxIt (B-Type) Web Simulation

Beat dynamics:



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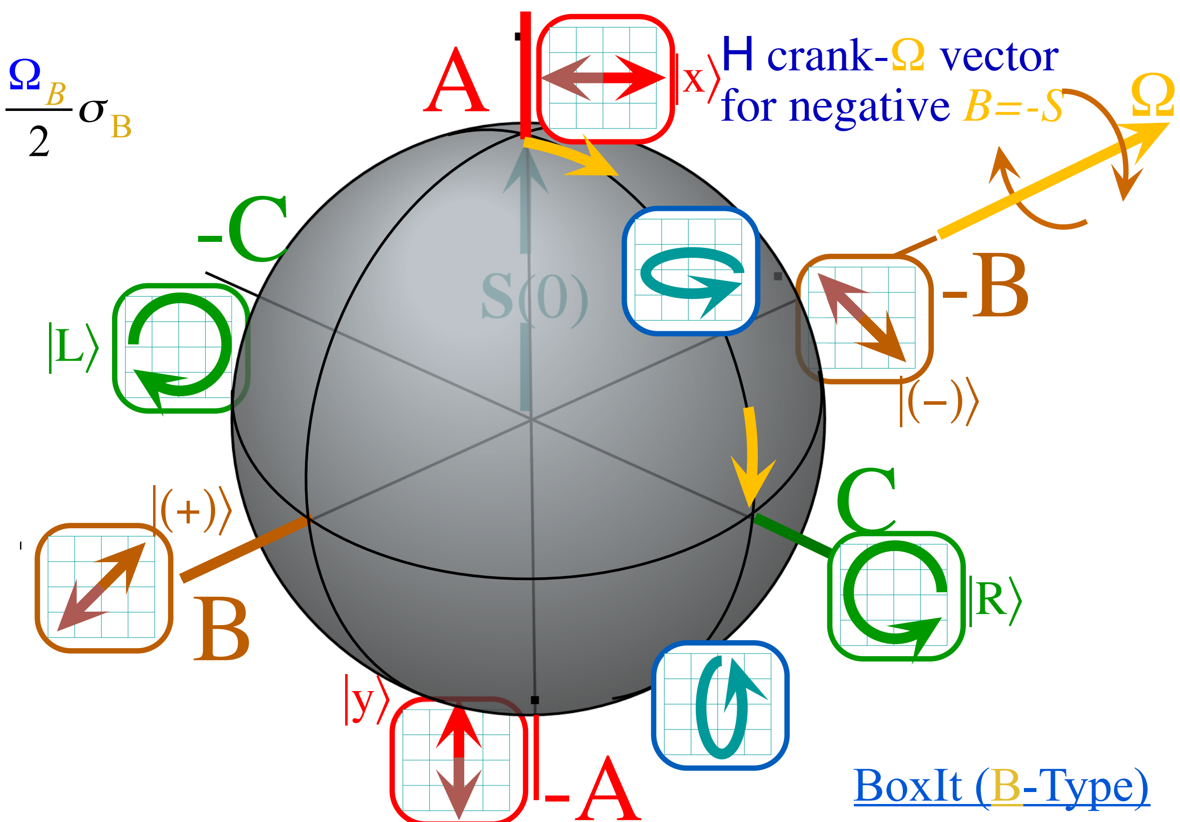
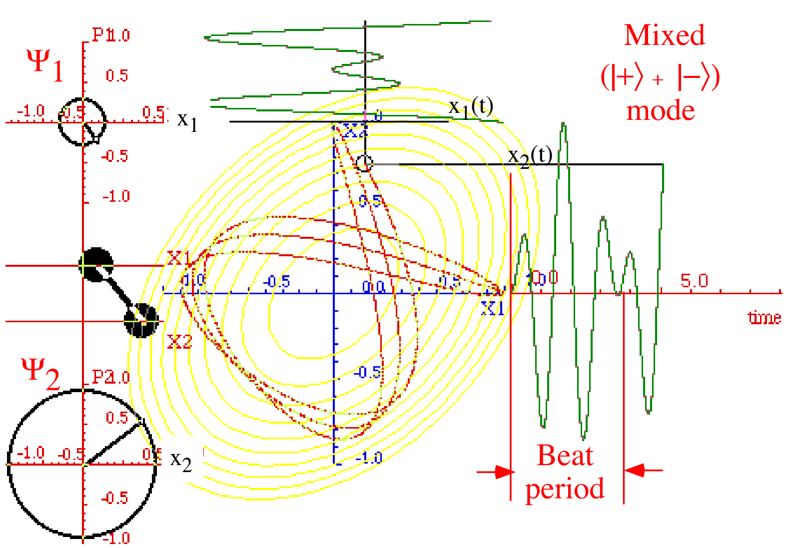
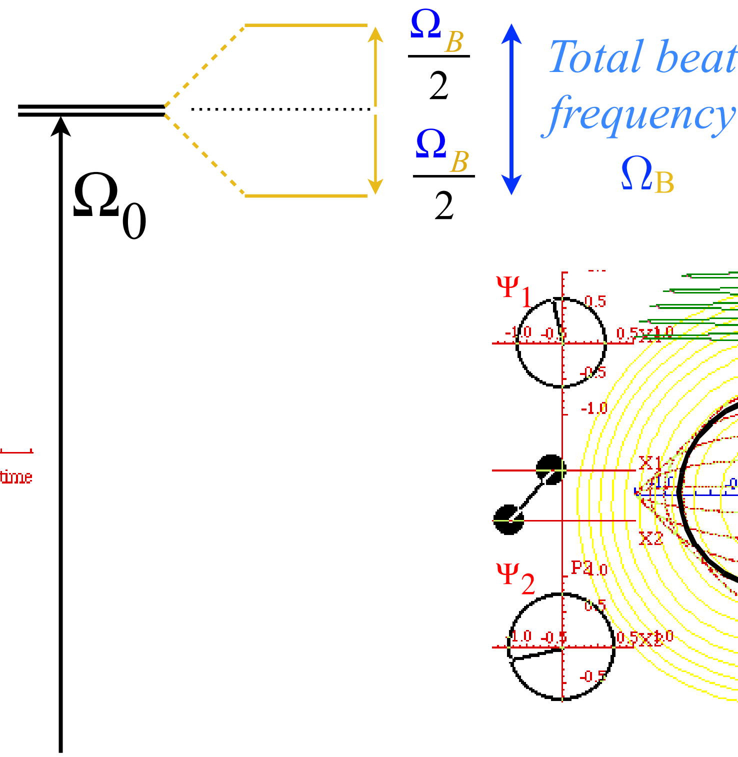
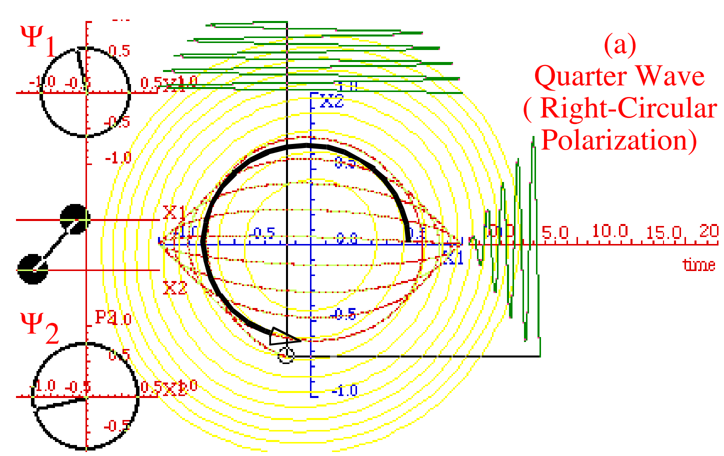


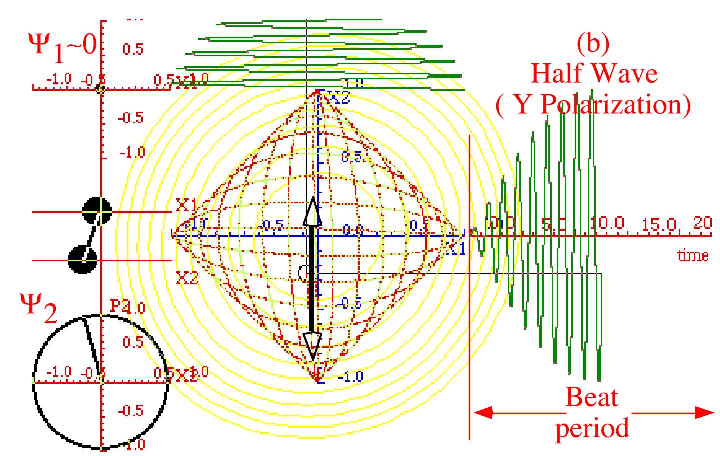
BoxIt (B-Type) Web Simulation

Beat dynamics:

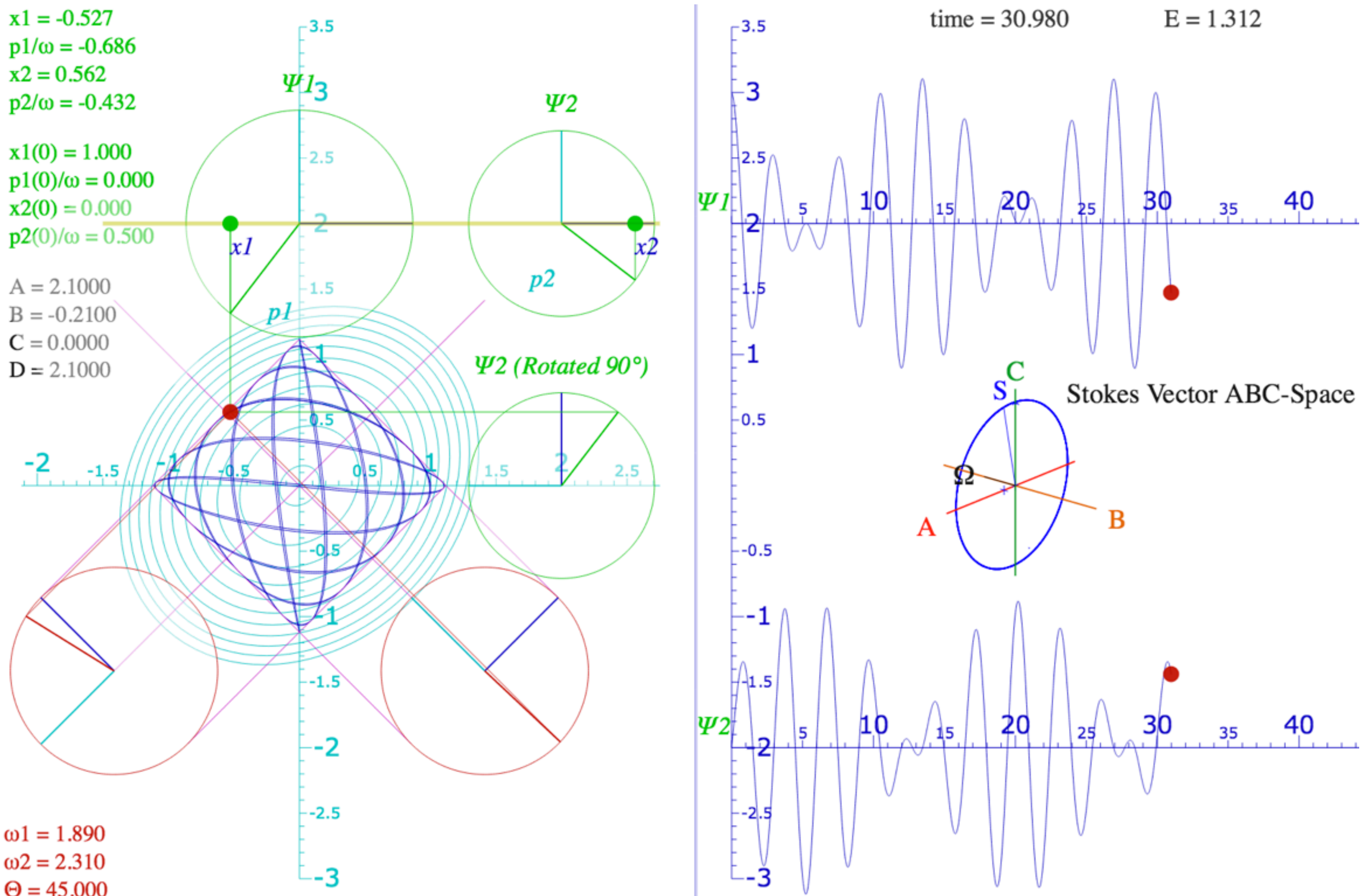








B-Type elliptical polarized motion



[BoxIt Web Simulation:](#)
[B-Type with A, D=2.1; B=-0.21](#)

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U(2) transformation matrices and related R(3) rotations in ABC-space

Mysterious factors of 2 or ½ on 2D spinors or 3D vectors

2D {↑,↓} spinor space ½ as fast as 3D {ABC} spin-vectors

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$$= \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

$$= \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

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Circular-Coriolis... C-Type motion

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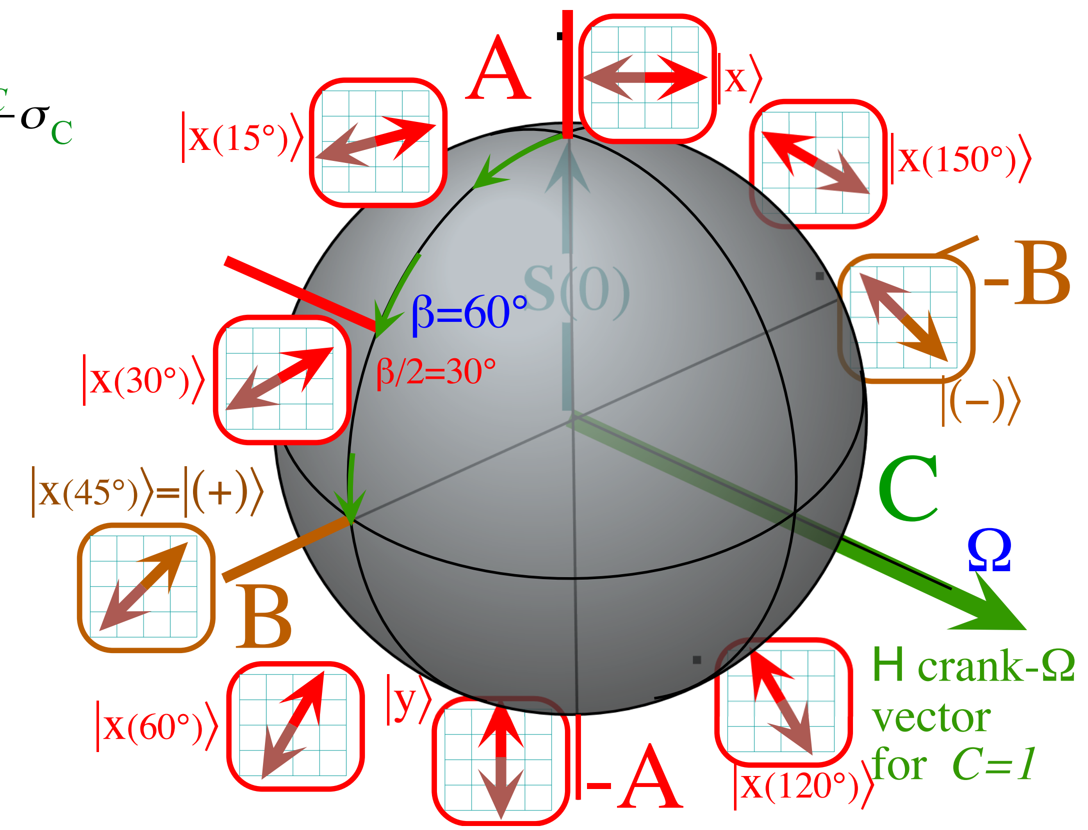
$$Crank: \vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2C \end{pmatrix} \qquad Eigen-Spin: \vec{\mathbf{S}} = \begin{pmatrix} S_A \\ S_B \\ S_C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \pm S \end{pmatrix}$$



BoxIt (C-Type)
Web Simulation

The ABC's of U(2) dynamics

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$
$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

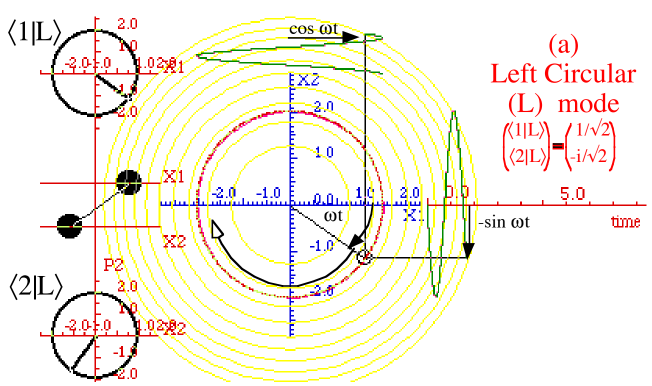
$$= \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

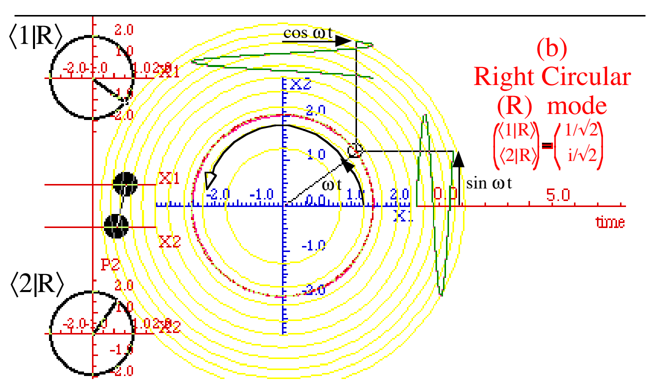
$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

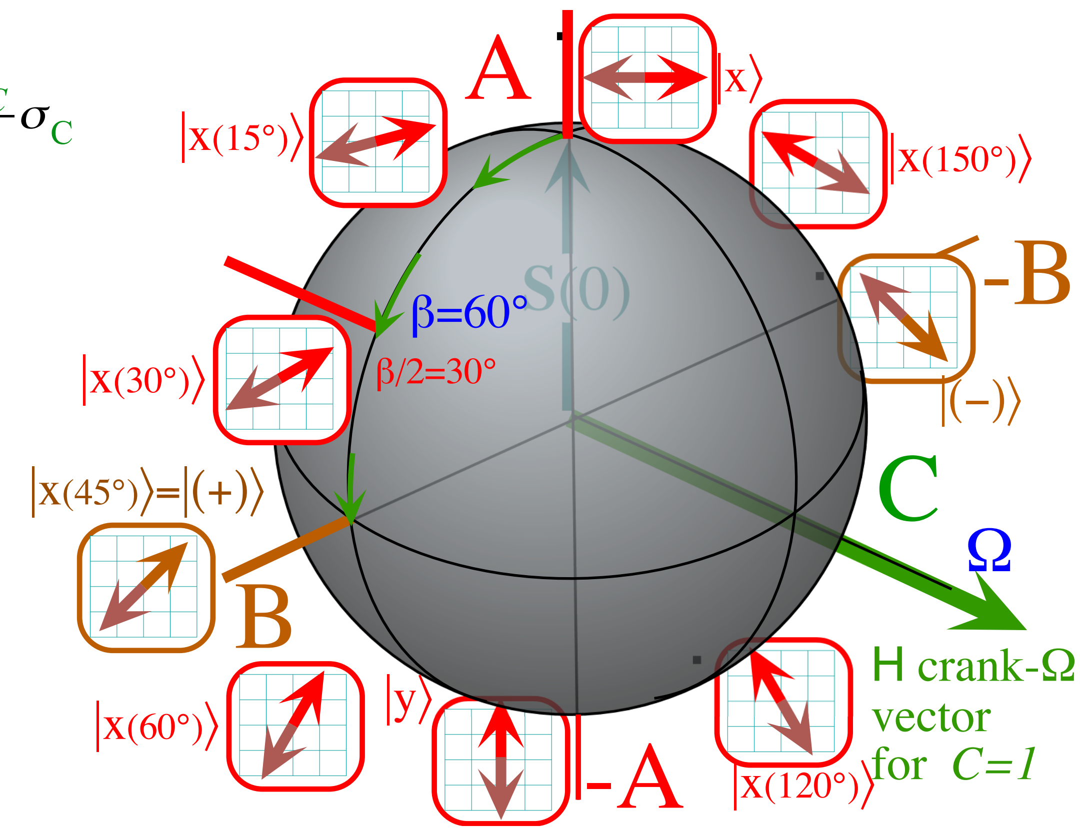
Circular-Coriolis... C-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^C|1\rangle & \langle 1|\mathbf{H}^C|2\rangle \\ \langle 2|\mathbf{H}^C|1\rangle & \langle 2|\mathbf{H}^C|2\rangle \end{pmatrix} = \begin{pmatrix} \Omega_0 & -iC \\ iC & \Omega_0 \end{pmatrix} = \Omega_0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \Omega_0\boldsymbol{\sigma}_0 + \frac{\Omega_C}{2}\sigma_C$$

$$\textit{Crank}: \vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2C \end{pmatrix} \qquad \textit{Eigen-Spin}: \vec{\mathbf{S}} = \begin{pmatrix} S_A \\ S_B \\ S_C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \pm S \end{pmatrix}$$







BoxIt (C-Type)
Web Simulation

The ABC's of U(2) dynamics

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

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$$= \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

$$= \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

Circular-Coriolis... C-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^C|1\rangle & \langle 1|\mathbf{H}^C|2\rangle \\ \langle 2|\mathbf{H}^C|1\rangle & \langle 2|\mathbf{H}^C|2\rangle \end{pmatrix} = \begin{pmatrix} \Omega_0 & -iC \\ iC & \Omega_0 \end{pmatrix} = \Omega_0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \Omega_0\boldsymbol{\sigma}_0 + \frac{\Omega_C}{2}\sigma_C$$

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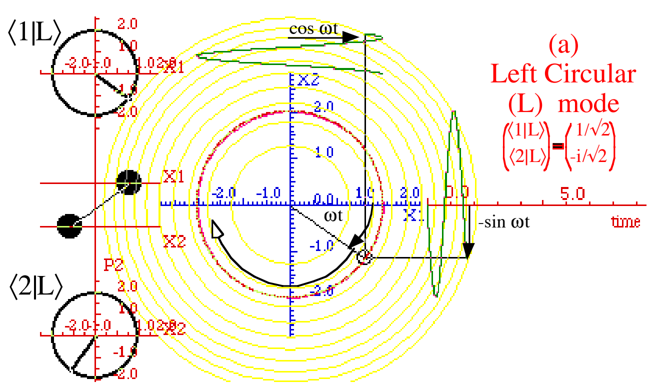
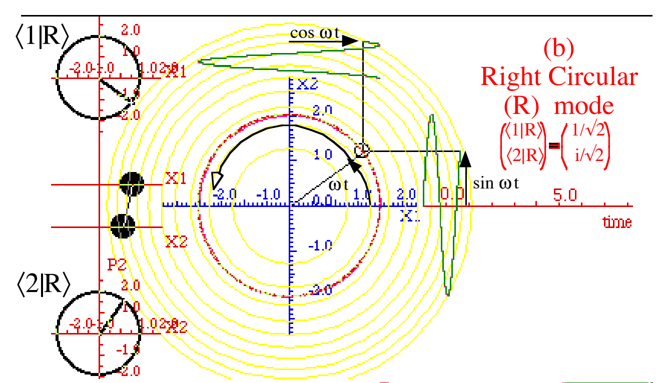
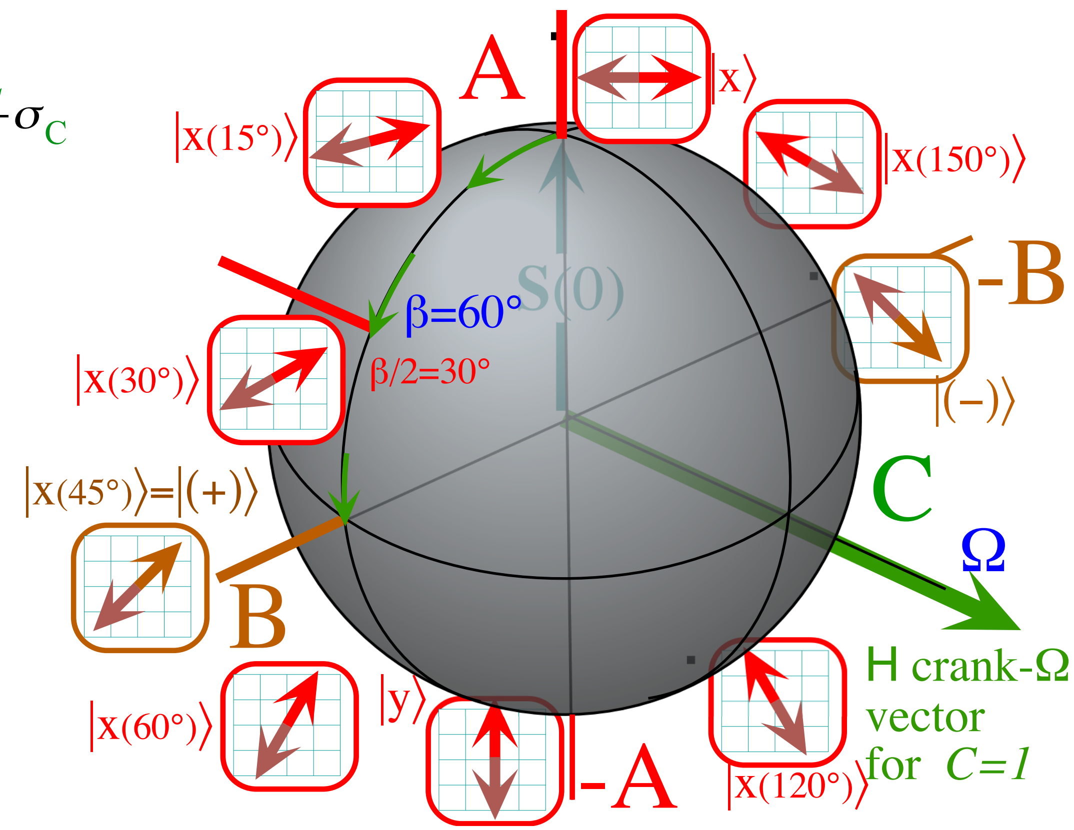
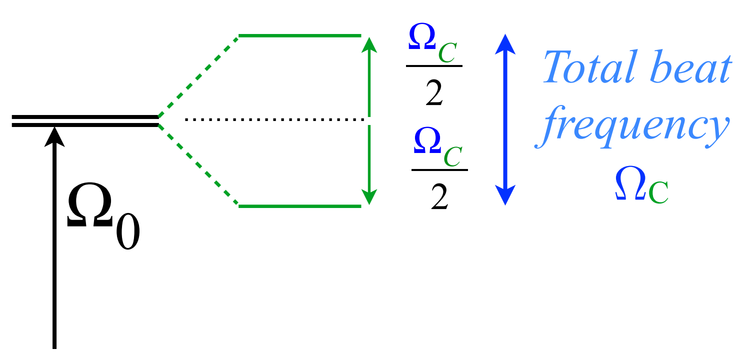
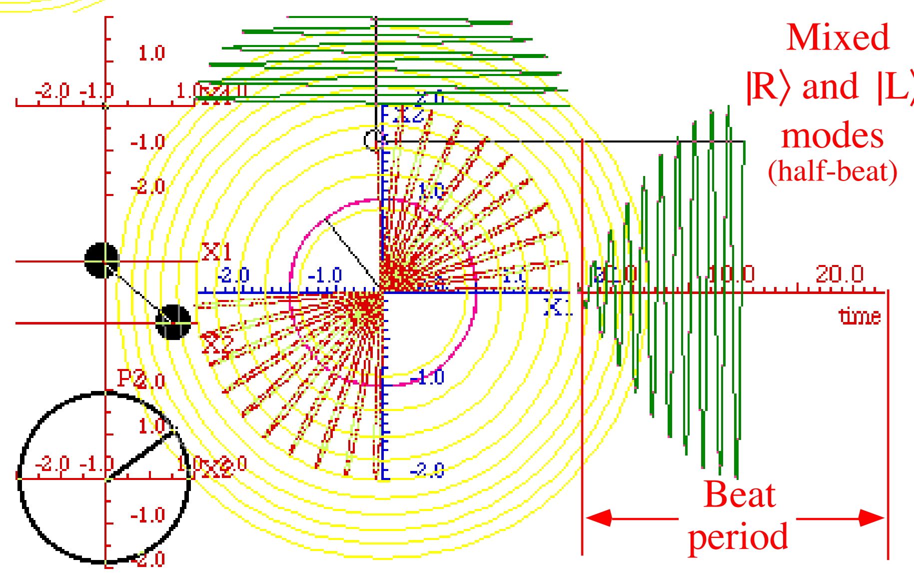






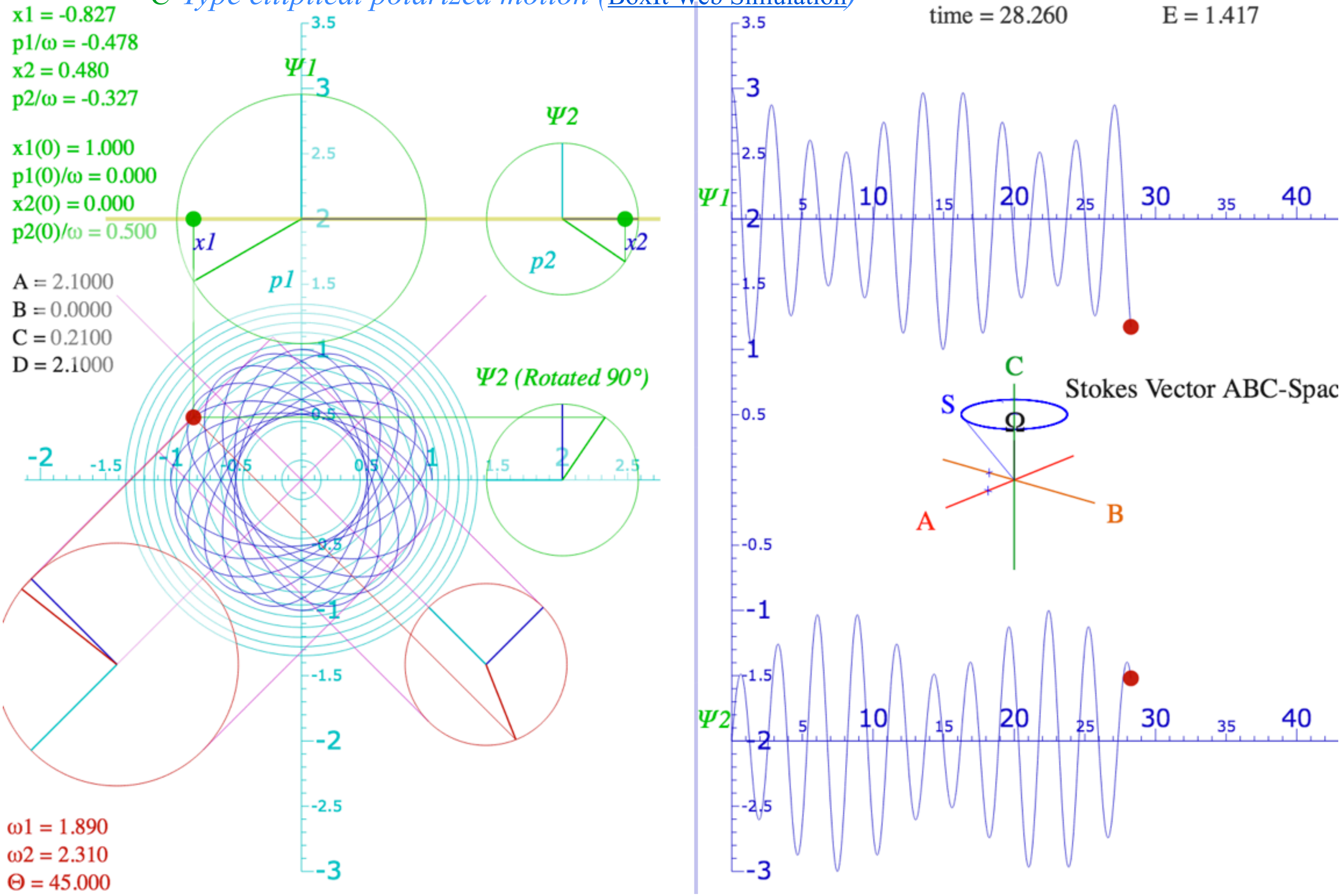
Beat dynamics:





BoxIt (C-Type)
Web Simulation

C-Type elliptical polarized motion (BoxIt Web Simulation)



BoxIt Web Simulation:
C-Type with A, D=2.1; C=-0.21

Review: Fundamental Euler **R***(αβγ) and Darboux* **R**[φϑΘ] *representations of U(2) and R(3)*

Euler **R***(αβγ) derived from Darboux* **R**[φϑΘ] *and vice versa*

Euler **R***(αβγ) rotation* Θ =0−4π*-sequence* [φϑ] *fixed*

R(3)-U(2) slide rule for converting **R***(αβγ)* ↔ **R**[φϑΘ] *and Sundial*

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

The ABC's of U(2) dynamics-Archetypes

Asymmetric-Diagonal A-Type motion

Bilateral-Balanced B-Type motion

Circular-Coriolis… C-Type motion

The ABC's of U(2) dynamics-Mixed modes

→ *AB-Type motion and Wigner's Avoided-Symmetry-Crossings*

ABC-Type elliptical polarized motion

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates

Euler Angle (αβγ) ellipse coordinates

The ABC's of U(2) dynamics-Mixed modes

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$
$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

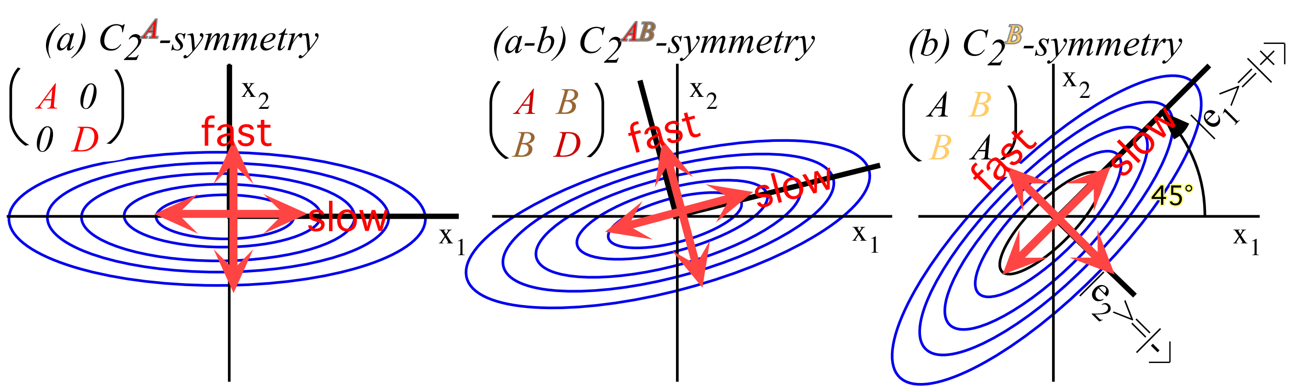
$$= \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

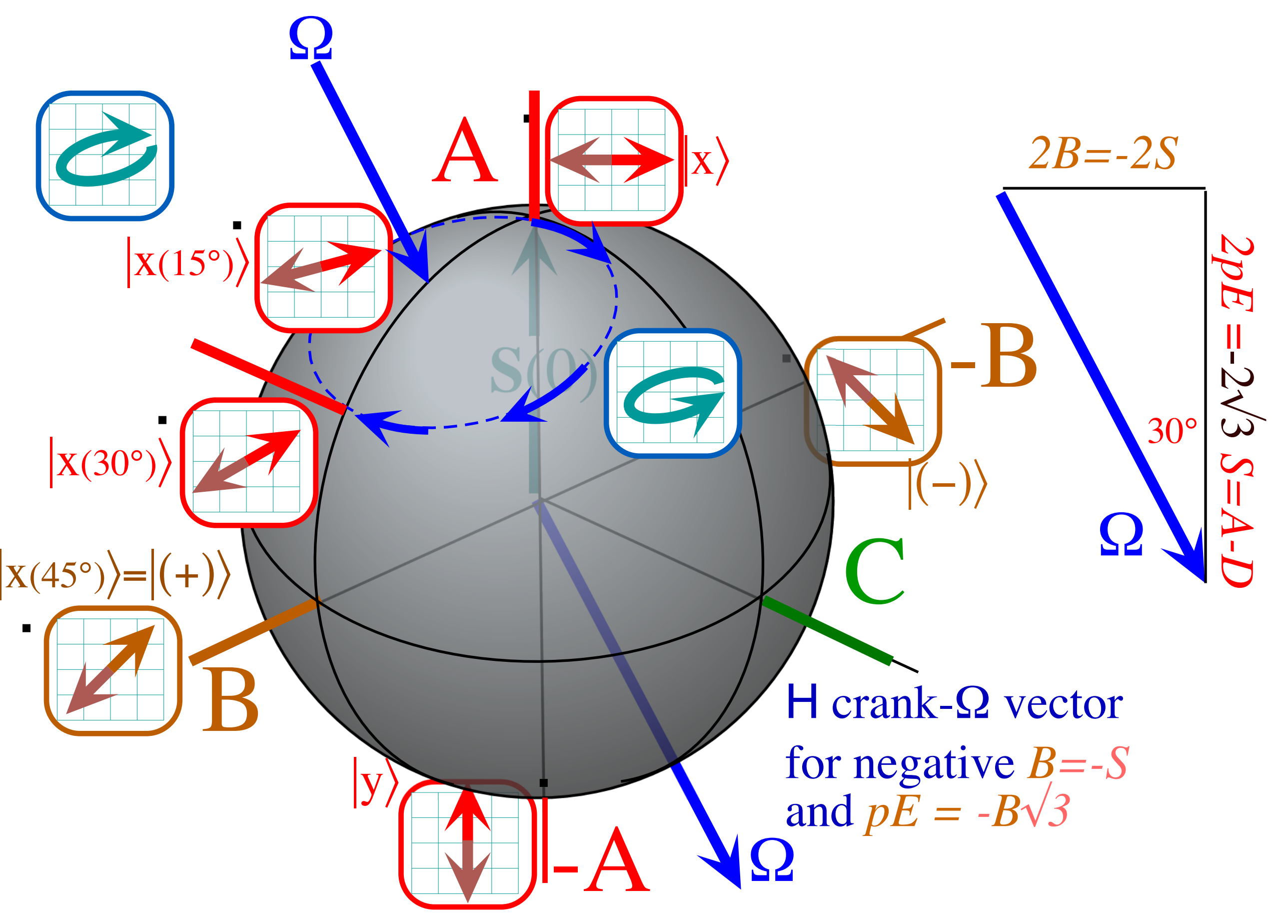
$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

Tilted-plane polarization AB-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^{AB}|1\rangle & \langle 1|\mathbf{H}^{AB}|2\rangle \\ \langle 2|\mathbf{H}^{AB}|1\rangle & \langle 2|\mathbf{H}^{AB}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B \\ B & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \Omega_0\boldsymbol{\sigma}_0 + \frac{\Omega_A}{2}\sigma_A + \frac{\Omega_B}{2}\sigma_B$$

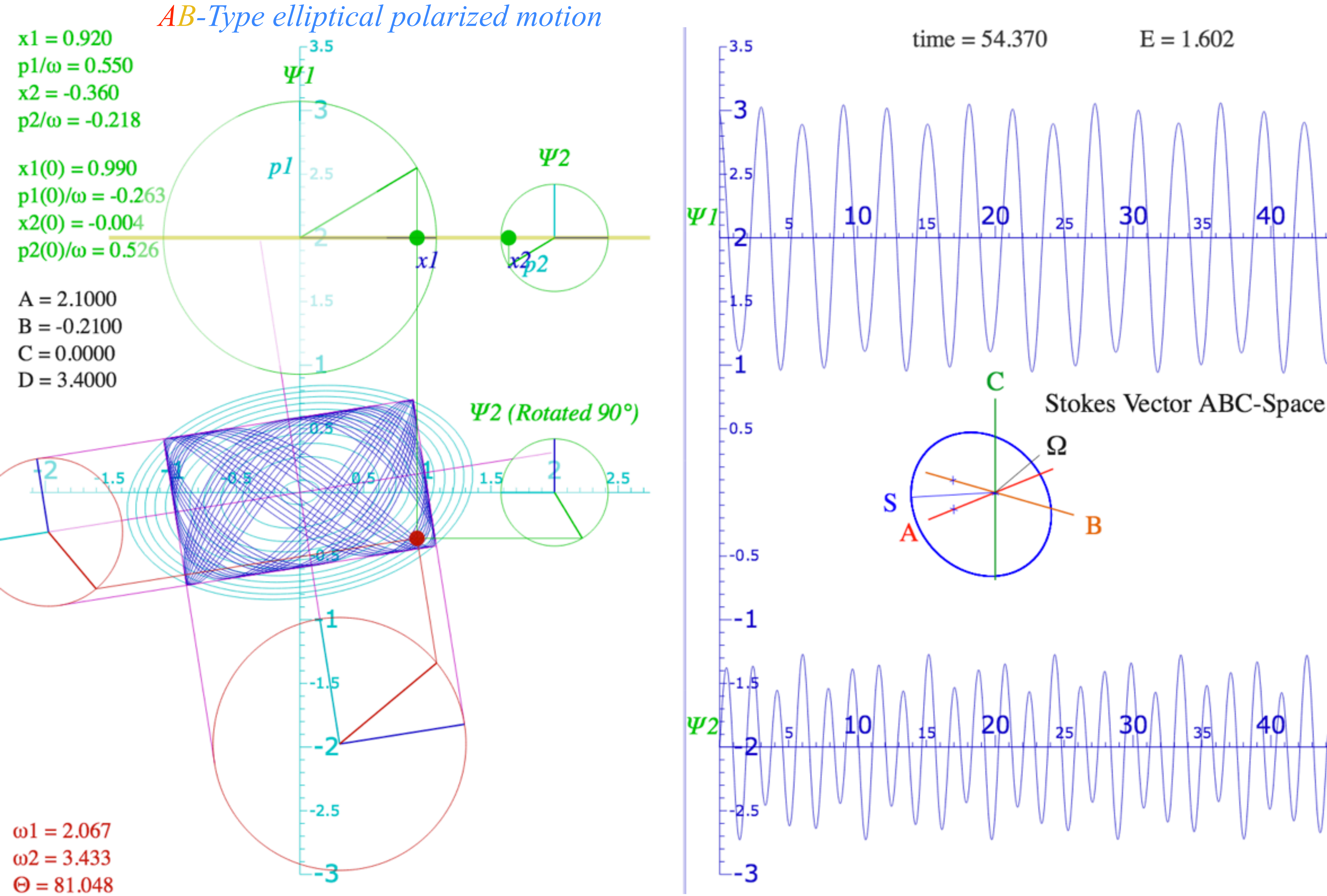
$$\text{Crank}:\ \vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 0 \end{pmatrix} \qquad \text{Eigen-Spin}: \vec{\mathbf{S}} = \pm S\cdot\hat{\Omega}$$





Beat dynamics:

BoxIt (AB-Type Motion)
Web Simulation



[BoxIt Web Simulation:
AB-Type with A=2.1; B=-0.21; D=3.4]

A to B to A Symmetry breaking described by hyperbolic eigenvalues of $A\boldsymbol{\sigma}_A + B\boldsymbol{\sigma}_B = \mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$

$\mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$ Secular equation: $\varepsilon^2 - 0 \cdot \varepsilon - (A^2 + B^2)$ gives *hyperbolic* energy levels: $\varepsilon = \pm\sqrt{A^2 + B^2}$

Here we display eigenvalues and eigenvectors while holding B constant and varying A. Obviously it can be done vice-versa and with varying C, too.

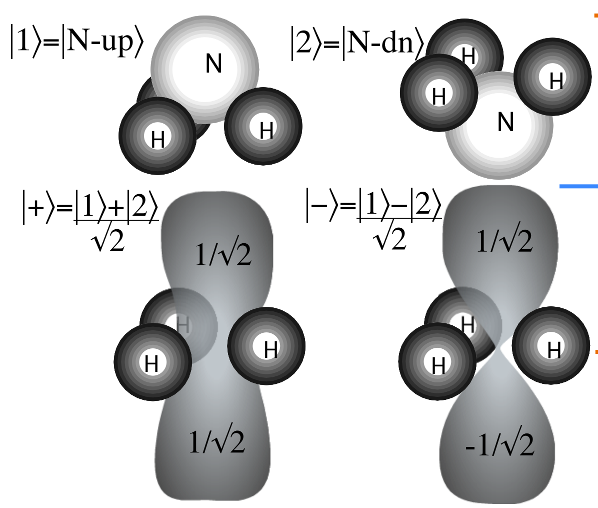


Fig. 10.3.2 Ammonia (NH_3) inversion states (a) Base states (b) C_2-Eigenstates

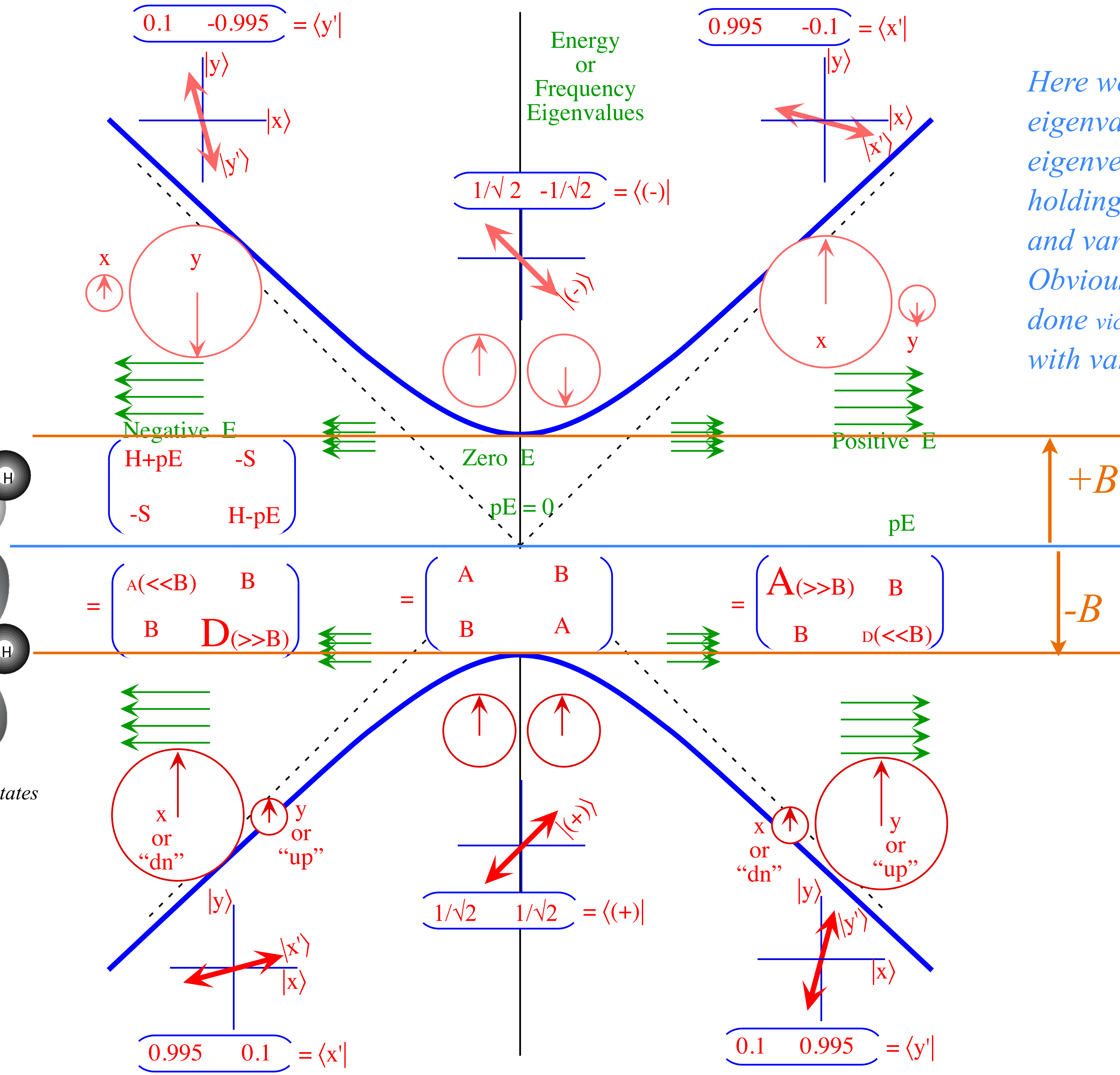


Fig. 10.3.1 (b) Wigner avoided level crossing. (Fixed tunneling B=-S and variable A-D=pE field.)

A to B to A Symmetry breaking described by hyperbolic eigenvalues of $A\boldsymbol{\sigma}_A + B\boldsymbol{\sigma}_B = \mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$

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$\mathbf{H}(B-basis)$ $\qquad$ $\mathbf{H}(A-basis)$

$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}} \begin{pmatrix} +A & B \\ B & -A \end{pmatrix} \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}}$$

Here we display eigenvalues and eigenvectors while holding B constant and varying A. Obviously it can be done vice-versa and with varying C, too.

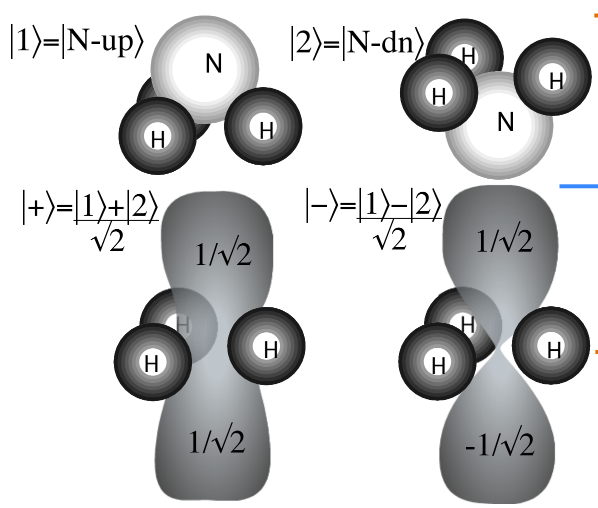


Fig. 10.3.2 Ammonia (NH_3) inversion states
(a) Base states (b) C_2-Eigenstates

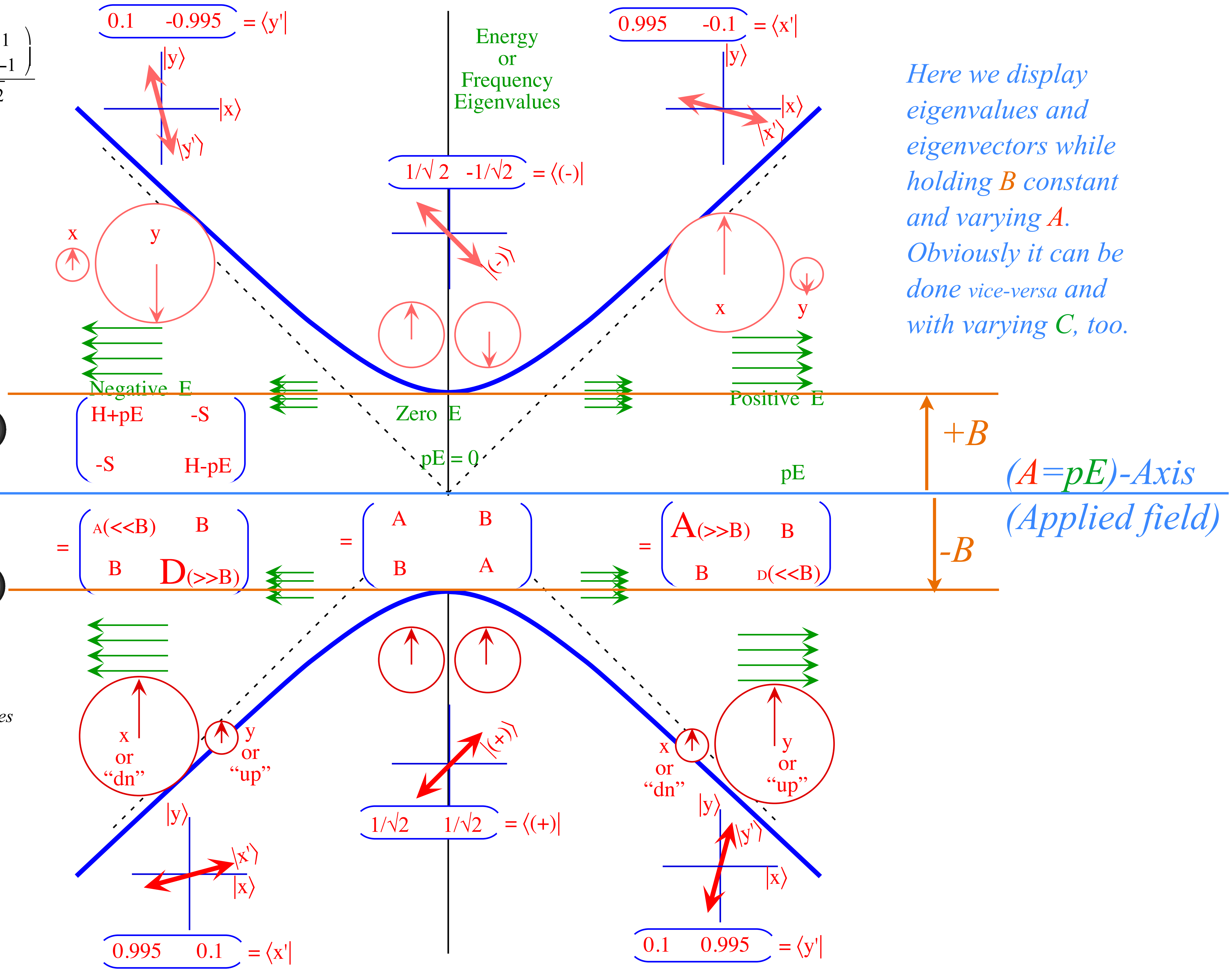


Fig. 10.3.1 (b) Wigner avoided level crossing. (Fixed tunneling B=-S and variable A-D=pE field.)

Review of Lecture 6: C_2 symmetry is 2D oscillators and three famous 2-state systems

Review of Lecture 6: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ *vs. Classical 2D-HO:* $\partial^2_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Review of Lecture 6: Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$-expansion in ABCD-Types) $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$

Deriving $\boldsymbol{\sigma}$-exponential time evolution (or revolution) operator $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

Spinor arithmetic like complex arithmetic

Spinor vector algebra like complex vector algebra

Spinor exponentials like complex exponentials (“Crazy-Thing”-Theorem)

Geometry of U(2) evolution (or R(3) revolution) operator $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

The “mysterious” factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

2D Spinor vs 3D vector rotation

NMR Hamiltonian: 3D Spin Moment $\mathbf{m}$ *in* $\mathbf{B}$ *field*

Euler’s state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case

Spin-1/2 (2D-complex spinor) case

➡ *3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states*

Asymmetry $S_A=S_Z$, *Balance* $S_B=S_X$, *and Chirality* $S_C=S_Y$

➡ *Polarization ellipse and spinor state dynamics*

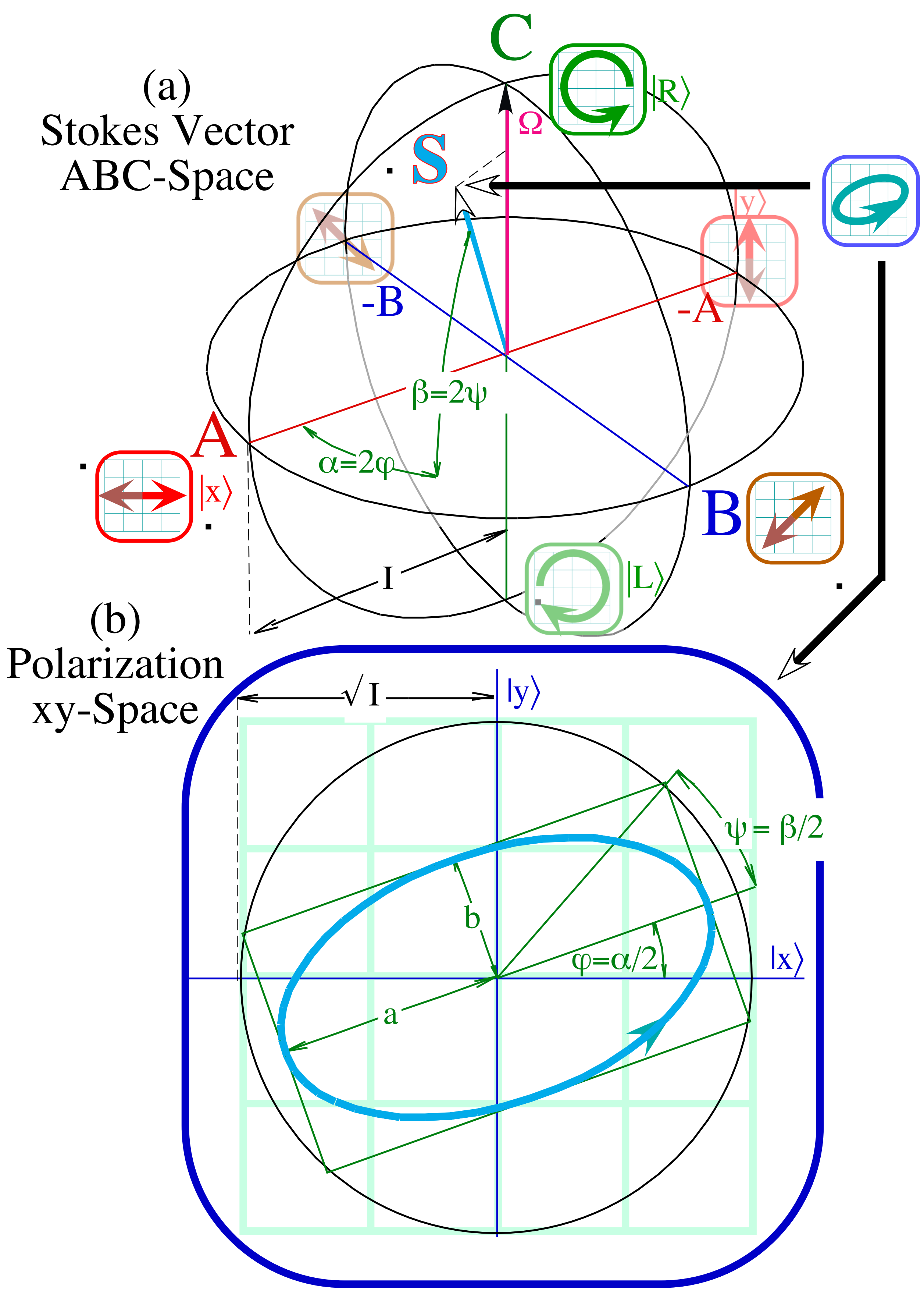


Fig. 10.B.3 Polarization variables (a) Stokes real-vector space (ABC) (b) Complex xy-spinor-space (x_1,x_2).

Polarization ellipse and spinor state dynamics

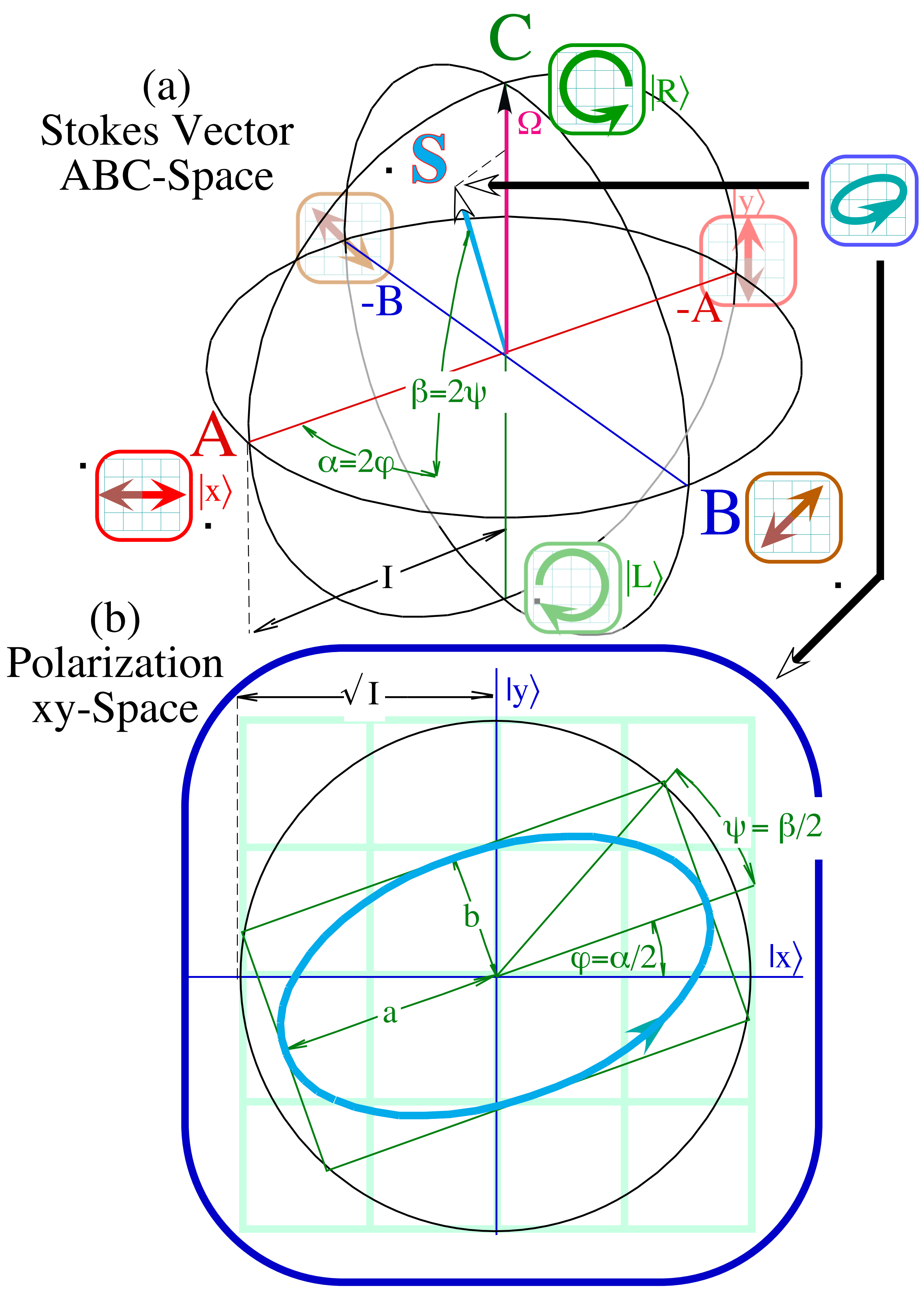


Fig. 10.B.3 Polarization variables (a) Stokes real-vector space (ABC) (b) Complex xy-spinor-space (x_1,x_2).

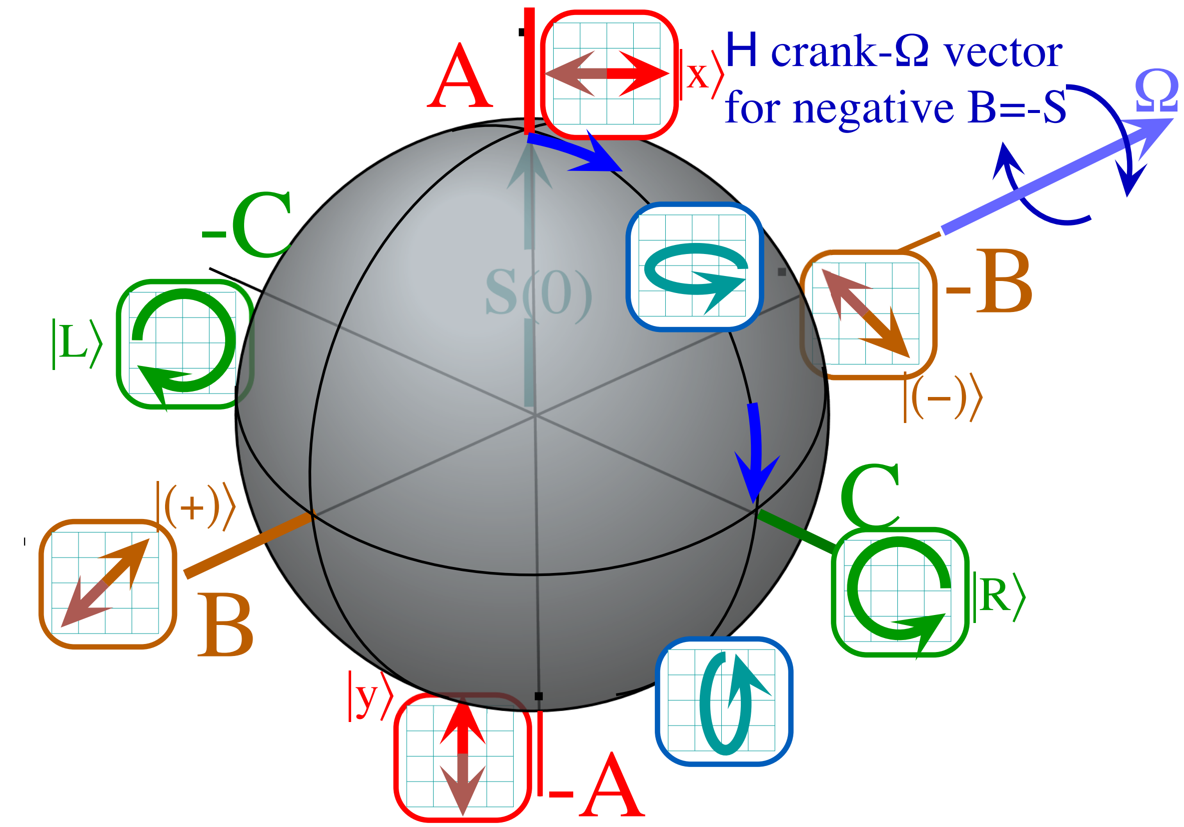
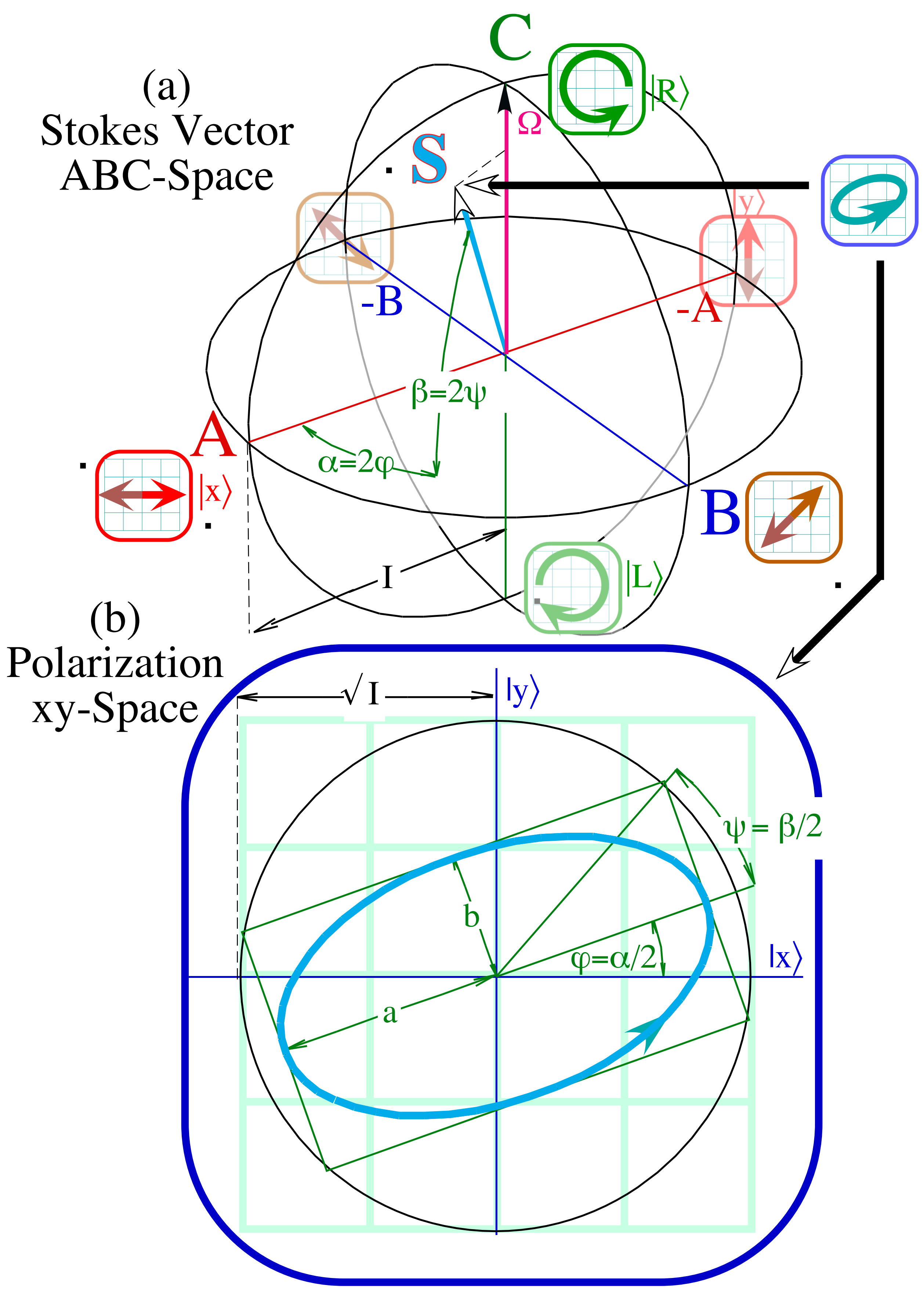


Fig. 10.5.5 Time evolution of a B-type beat. **S***-vector rotates from A to C to -A to -C and back to A.*

Polarization ellipse and spinor state dynamics



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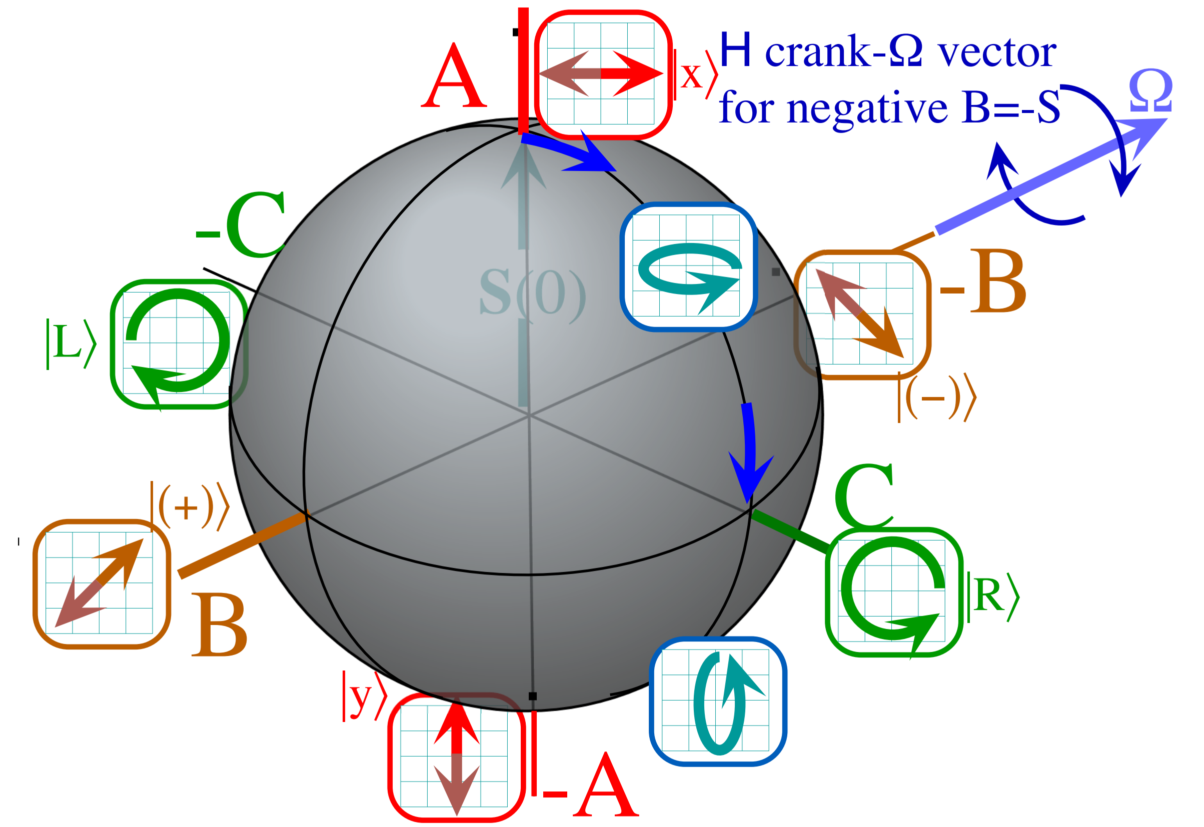
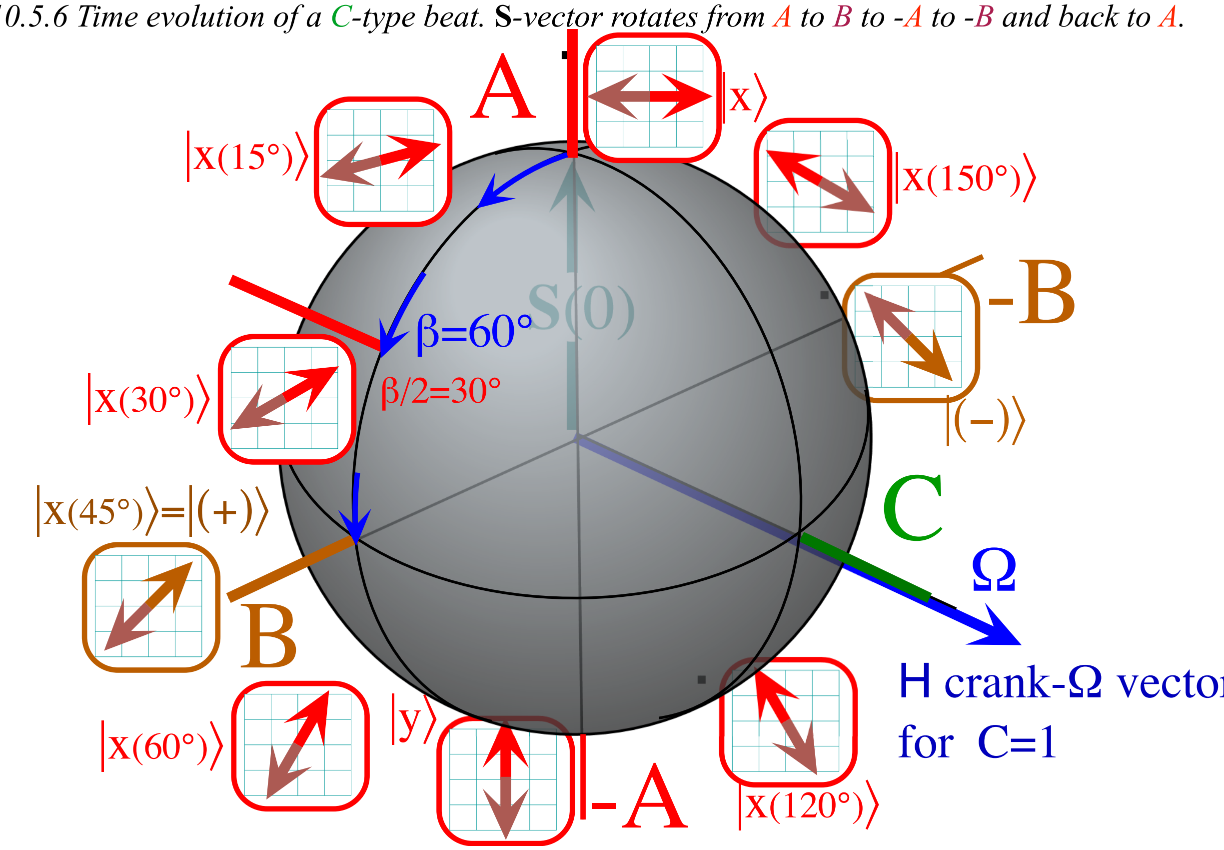


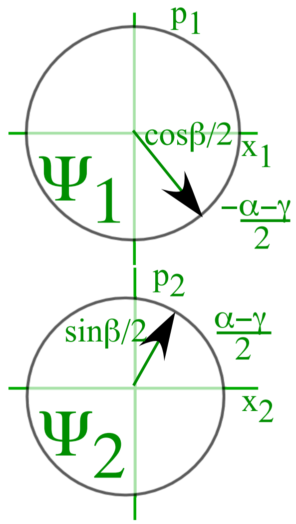
Fig. 10.5.5 Time evolution of a B-type beat. **S**-vector rotates from A to C to -A to -C and back to A.

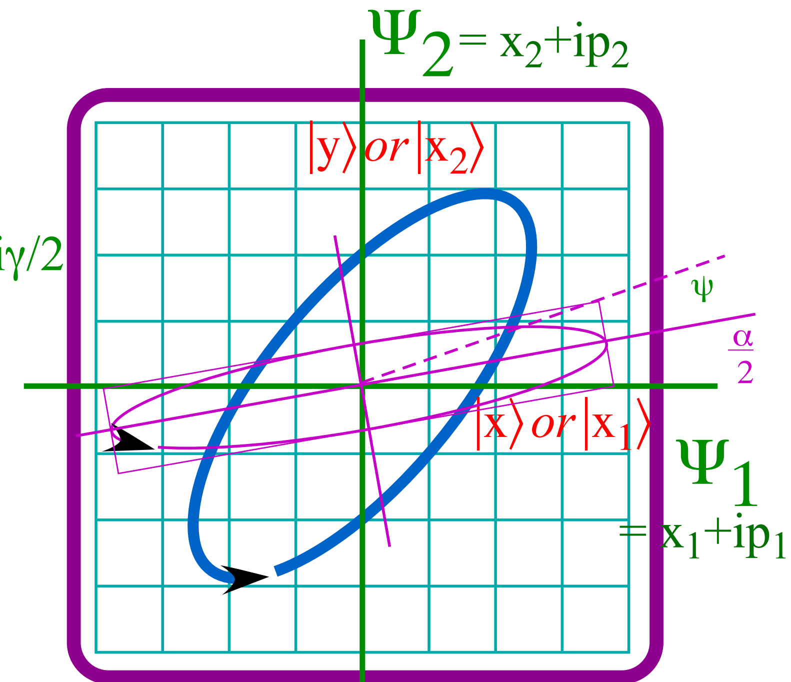
Fig. 10.5.6 Time evolution of a C-type beat. **S**-vector rotates from A to B to -A to -B and back to A.



U(2) World : Complex 2D Spinors

2-State ket $|\Psi\rangle = \begin{pmatrix}\Psi_1 \\ \Psi_2\end{pmatrix} = \begin{pmatrix}\sqrt{N}e^{-i\alpha/2}\cos\beta/2 \\ \sqrt{N}e^{i\alpha/2}\sin\beta/2\end{pmatrix} e^{-i\gamma/2}$





R(3) World : Real 3D Vectors

$|\Psi\rangle$ State

Spin Vector S $= \begin{pmatrix} S_B \\ S_C \\ S_A \end{pmatrix} = \begin{pmatrix} N\sin\beta\cos\alpha \\ N\sin\beta\sin\alpha \\ N\cos\beta \end{pmatrix}\frac{1}{2}$

(for α=15° β=45°)

H-Operator $\begin{pmatrix} A & B\text{-}iC \\ B+iC & D \end{pmatrix}$

Angular velocity

$\boldsymbol{\Omega} = \begin{pmatrix} \Omega_B \\ \Omega_C \\ \Omega_A \end{pmatrix} = \begin{pmatrix} 2B \\ 2C \\ A\text{-}D \end{pmatrix} = \begin{pmatrix} \Omega\sin\vartheta\cos\varphi \\ \Omega\sin\vartheta\sin\varphi \\ \Omega\cos\vartheta \end{pmatrix}$