AMOP reference links on following page

2.07.18 class 8.0: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

Symmetry group $\mathscr{G}$=U(2) representations, 2D HO Hamiltonian $\mathbf{H}=\hbar\omega_{ab}\mathbf{a}_a^\dagger\mathbf{a}_b$ operators, 2D HO wave eigenfunctions $\Psi_{n,m}$, and coherent $[\alpha]$ states

Factoring 2D-HO Hamiltonian $\mathbf{H}=\frac{A}{2}\left(\mathbf{p}_1^2+\mathbf{x}_1^2\right)+B\left(\mathbf{x}_1\mathbf{x}_2+\mathbf{p}_1\mathbf{p}_2\right)+C\left(\mathbf{x}_1\mathbf{p}_2-\mathbf{x}_2\mathbf{p}_1\right)+\frac{D}{2}\left(\mathbf{p}_2^2+\mathbf{x}_2^2\right)$

2D-Oscillator basic states and operations

Commutation relations

Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry

Anti-commutation relations

Two-dimensional (or 2-particle) base states: ket-kets and bra-bras

Outer product arrays

Entangled 2-particle states

Two-particle (or 2-dimensional) matrix operators

U(2) Hamiltonian and irreducible representations

2D-Oscillator states and related 3D angular momentum multiplets

R(3) Angular momentum generators by U(2) analysis

Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$

SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Mostly Notation and Bookkeeping :

AMOP reference links *(Updated list given on 2nd page of each class presentation)*

Web Resources - front page

UAF Physics UTube channel

2014 AMOP

2017 Group Theory for QM

2018 AMOP

Frame Transformation Relations And Multipole Transitions In Symmetric Polyatomic Molecules - RMP-1978 (Alt Scanned version)
Rotational energy surfaces and high- J eigenvalue structure of polyatomic molecules - Harter - Patterson - 1984
Galloping waves and their relativistic properties - ajp-1985-Harter
Asymptotic eigensolutions of fourth and sixth rank octahedral tensor operators - Harter-Patterson-JMP-1979
Nuclear spin weights and gas phase spectral structure of 12C60 and 13C60 buckminsterfullerene -Harter-Reimer-Cpl-1992 - (Alt1, Alt2 Erratum)

Theory of hyperfine and superfine levels in symmetric polyatomic molecules.

I) Trigonal and tetrahedral molecules: Elementary spin-1/2 cases in vibronic ground states - PRA-1979-Harter-Patterson (Alt scan)
II) Elementary cases in octahedral hexafluoride molecules - Harter-PRA-1981 (Alt scan)

Rotation-vibration scalar coupling zeta coefficients and spectroscopic band shapes of buckminsterfullerene - Weeks-Harter-CPL-1991 (Alt scan)
Fullerene symmetry reduction and rotational level fine structure/ the Buckyball isotopomer 12C 13C59 - jcp-Reimer-Harter-1997 (HiRez)
Molecular Eigensolution Symmetry Analysis and Fine Structure - IJMS-harter-mitchell-2013

Rotation–vibration spectra of icosahedral molecules.

I) Icosahedral symmetry analysis and fine structure - harter-weeks-jcp-1989
II) Icosahedral symmetry, vibrational eigenfrequencies, and normal modes of buckminsterfullerene - weeks-harter-jcp-1989
III) Half-integral angular momentum - harter-reimer-jcp-1991

QTCA Unit 10 Ch 30 - 2013
AMOP Ch 32 Molecular Symmetry and Dynamics - 2019
AMOP Ch 0 Space-Time Symmetry - 2019

RESONANCE AND REVIVALS

I) QUANTUM ROTOR AND INFINITE-WELL DYNAMICS - ISMSLi2012 (Talk) OSU knowledge Bank
II) Comparing Half-integer Spin and Integer Spin - Alva-ISMS-Ohio2013-R777 (Talks)
III) Quantum Resonant Beats and Revivals in the Morse Oscillators and Rotors - (2013-Li-Diss)

Rovibrational Spectral Fine Structure Of Icosahedral Molecules - Cpl 1986 (Alt Scan)
Gas Phase Level Structure of C60 Buckyball and Derivatives Exhibiting Broken Icosahedral Symmetry - reimer-diss-1996
Resonance and Revivals in Quantum Rotors - Comparing Half-integer Spin and Integer Spin - Alva-ISMS-Ohio2013-R777 (Talk)
Quantum Revivals of Morse Oscillators and Farey-Ford Geometry - Li-Harter-cpl-2013
Wave Node Dynamics and Revival Symmetry in Quantum Rotors - harter - jms - 2001

**In development - a Web based AMOP Reference page, with more options/control over display*

Symmetry group $\mathscr{G}$=U(2) representations, 2D HO Hamiltonian $\mathbf{H}=\hbar\omega_{ab}\mathbf{a}_a^\dagger\mathbf{a}_b$ operators, 2D HO wave eigenfunctions $\Psi_{n,m}$, and coherent $[\alpha]$ states

Factoring 2D-HO Hamiltonian $\mathbf{H}=\frac{A}{2}\left(\mathbf{p}_1^2+\mathbf{x}_1^2\right)+B\left(\mathbf{x}_1\mathbf{x}_2+\mathbf{p}_1\mathbf{p}_2\right)+C\left(\mathbf{x}_1\mathbf{p}_2-\mathbf{x}_2\mathbf{p}_1\right)+\frac{D}{2}\left(\mathbf{p}_2^2+\mathbf{x}_2^2\right)$

- *2D-Oscillator basic states and operations*
  - *Commutation relations*
  - *Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry*
    - *Anti-commutation relations*
  - *Two-dimensional (or 2-particle) base states: ket-kets and bra-bras*
    - *Outer product arrays*
    - *Entangled 2-particle states*
- *Two-particle (or 2-dimensional) matrix operators*
  - *U(2) Hamiltonian and irreducible representations*
  - *2D-Oscillator states and related 3D angular momentum multiplets*

R(3) Angular momentum generators by U(2) analysis

Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$

SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Mostly Notation and Bookkeeping :

2D-Oscillator basic states and operations

First rewrite a classical 2-D Hamiltonian (Class-4 p16) with a thick-tip pen! (They're **operators** now!)

$$\mathbf{H} = \frac{A}{2}\left(\mathbf{p}_1^2 + \mathbf{x}_1^2\right) + B\left(\mathbf{x}_1\mathbf{x}_2 + \mathbf{p}_1\mathbf{p}_2\right) + C\left(\mathbf{x}_1\mathbf{p}_2 - \mathbf{x}_2\mathbf{p}_1\right) + \frac{D}{2}\left(\mathbf{p}_2^2 + \mathbf{x}_2^2\right)$$

The *ABCD* matrix from Class 4

$$\mathbf{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$$

(Mass factors $\sqrt{M}$, spring constants K_{ij}, and Planck $\hbar$ constants are absorbed into *A, B, C,* and *D* constants used in Class 4 to 6.)

2D-Oscillator basic states and operations

First rewrite a classical 2-D Hamiltonian (Class-4 p16) with a thick-tip pen! (They're **operators** now!)

$$\mathbf{H} = \frac{A}{2}\left(\mathbf{p}_1^2 + \mathbf{x}_1^2\right) + B\left(\mathbf{x}_1\mathbf{x}_2 + \mathbf{p}_1\mathbf{p}_2\right) + C\left(\mathbf{x}_1\mathbf{p}_2 - \mathbf{x}_2\mathbf{p}_1\right) + \frac{D}{2}\left(\mathbf{p}_2^2 + \mathbf{x}_2^2\right)$$

The *ABCD* matrix from Class 4

$$\mathbf{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$$

(Mass factors $\sqrt{M}$, spring constants K_{ij}, and Planck $\hbar$ constants are absorbed into *A, B, C,* and *D* constants used in Class 4 to 6.)

Define **a** and **a**† **operators**

$\mathbf{a}_1 = (\mathbf{x}_1 + i\,\mathbf{p}_1)/\sqrt{2}$ $\quad \mathbf{a}^\dagger{}_1 = (\mathbf{x}_1 - i\,\mathbf{p}_1)/\sqrt{2}$ $\quad \mathbf{a}_2 = (\mathbf{x}_2 + i\,\mathbf{p}_2)/\sqrt{2}$ $\quad \mathbf{a}^\dagger{}_2 = (\mathbf{x}_2 - i\,\mathbf{p}_2)/\sqrt{2}$

Each system dimension $\mathbf{x}_1$ and $\mathbf{x}_2$ is assumed orthogonal, neither being constrained by the other.

2D-Oscillator basic states and operations

First rewrite a classical 2-D Hamiltonian (Class-4 p16) with a thick-tip pen! (They're **operators** now!)

$$\mathbf{H} = \frac{A}{2}\left(\mathbf{p}_1^2 + \mathbf{x}_1^2\right) + B\left(\mathbf{x}_1\mathbf{x}_2 + \mathbf{p}_1\mathbf{p}_2\right) + C\left(\mathbf{x}_1\mathbf{p}_2 - \mathbf{x}_2\mathbf{p}_1\right) + \frac{D}{2}\left(\mathbf{p}_2^2 + \mathbf{x}_2^2\right)$$

The *ABCD* matrix from Class 4

$$\mathbf{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$$

Define **a** and **a**† **operators**

$\mathbf{a}_1 = (\mathbf{x}_1 + i\,\mathbf{p}_1)/\sqrt{2}$ $\quad$ $\mathbf{a}^\dagger{}_1 = (\mathbf{x}_1 - i\,\mathbf{p}_1)/\sqrt{2}$ $\quad$ $\mathbf{a}_2 = (\mathbf{x}_2 + i\,\mathbf{p}_2)/\sqrt{2}$ $\quad$ $\mathbf{a}^\dagger{}_2 = (\mathbf{x}_2 - i\,\mathbf{p}_2)/\sqrt{2}$

Solve for $\mathbf{x}_k$ and $\mathbf{p}_k$ **operators**

$\mathbf{x}_1 = (\mathbf{a}^\dagger{}_1 + \mathbf{a}_1)/\sqrt{2}$ $\quad$ $\mathbf{p}_1 = i\,(\mathbf{a}^\dagger{}_1 - \mathbf{a}_1)/\sqrt{2}$ $\quad$ $\mathbf{x}_2 = (\mathbf{a}^\dagger{}_2 + \mathbf{a}_2)/\sqrt{2}$ $\quad$ $\mathbf{p}_2 = i\,(\mathbf{a}^\dagger{}_2 - \mathbf{a}_2)/\sqrt{2}$

Each system dimension $\mathbf{x}_1$ and $\mathbf{x}_2$ is assumed orthogonal, neither being constrained by the other.

Symmetry group $\mathscr{G}$=U(2) representations, 2D HO Hamiltonian $\mathbf{H}=\hbar\omega_{ab}\mathbf{a}_a^\dagger\mathbf{a}_b$ operators, 2D HO wave eigenfunctions $\Psi_{n,m}$, and coherent [α] states

Factoring 2D-HO Hamiltonian $\mathbf{H}=\frac{A}{2}\left(\mathbf{p}_1^2+\mathbf{x}_1^2\right)+B\left(\mathbf{x}_1\mathbf{x}_2+\mathbf{p}_1\mathbf{p}_2\right)+C\left(\mathbf{x}_1\mathbf{p}_2-\mathbf{x}_2\mathbf{p}_1\right)+\frac{D}{2}\left(\mathbf{p}_2^2+\mathbf{x}_2^2\right)$

2D-Oscillator basic states and operations

Commutation relations

Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry

Anti-commutation relations

Two-dimensional (or 2-particle) base states: ket-kets and bra-bras

Outer product arrays

Entangled 2-particle states

Two-particle (or 2-dimensional) matrix operators

U(2) Hamiltonian and irreducible representations

2D-Oscillator states and related 3D angular momentum multiplets

R(3) Angular momentum generators by U(2) analysis

Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$

SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Mostly Notation and Bookkeeping :

2D-Oscillator basic states and operations

First rewrite a classical 2-D Hamiltonian (Class-4 p16) with a thick-tip pen! (They're **operators** now!)

$$\mathbf{H} = \frac{A}{2}\left(\mathbf{p}_1^2 + \mathbf{x}_1^2\right) + B\left(\mathbf{x}_1\mathbf{x}_2 + \mathbf{p}_1\mathbf{p}_2\right) + C\left(\mathbf{x}_1\mathbf{p}_2 - \mathbf{x}_2\mathbf{p}_1\right) + \frac{D}{2}\left(\mathbf{p}_2^2 + \mathbf{x}_2^2\right)$$

The *ABCD* matrix from Class 4

$$\mathbf{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix}$$

Define **a** and **a**† **operators**

$\mathbf{a}_1 = (\mathbf{x}_1 + i\,\mathbf{p}_1)/\sqrt{2}$ $\quad$ $\mathbf{a}^\dagger{}_1 = (\mathbf{x}_1 - i\,\mathbf{p}_1)/\sqrt{2}$ $\quad$ $\mathbf{a}_2 = (\mathbf{x}_2 + i\,\mathbf{p}_2)/\sqrt{2}$ $\quad$ $\mathbf{a}^\dagger{}_2 = (\mathbf{x}_2 - i\,\mathbf{p}_2)/\sqrt{2}$

Define $\mathbf{x}_k$ and $\mathbf{p}_k$ **operators**

$\mathbf{x}_1 = (\mathbf{a}^\dagger{}_1 + \mathbf{a}_1)/\sqrt{2}$ $\quad$ $\mathbf{p}_1 = i\,(\mathbf{a}^\dagger{}_1 - \mathbf{a}_1)/\sqrt{2}$ $\quad$ $\mathbf{x}_2 = (\mathbf{a}^\dagger{}_2 + \mathbf{a}_2)/\sqrt{2}$ $\quad$ $\mathbf{p}_2 = i\,(\mathbf{a}^\dagger{}_2 - \mathbf{a}_2)/\sqrt{2}$

Each system dimension $\mathbf{x}_1$ and $\mathbf{x}_2$ is assumed orthogonal, neither being constrained by the other.
This includes an axiom of *inter-dimensional commutivity*.

$$[\,\mathbf{x}_1\,,\,\mathbf{p}_2] = \mathbf{0} = [\,\mathbf{x}_2\,,\,\mathbf{p}_1]\,,\quad [\,\mathbf{a}_1\,,\,\mathbf{a}^\dagger{}_2] = \mathbf{0} = [\,\mathbf{a}_2\,,\,\mathbf{a}^\dagger{}_1]$$

2D-Oscillator basic states and operations

First rewrite a classical 2-D Hamiltonian (Class-4 p16) with a thick-tip pen! (They're **operators** now!)

$$\mathbf{H} = \frac{A}{2}\left(\mathbf{p}_1^2 + \mathbf{x}_1^2\right) + B\left(\mathbf{x}_1\mathbf{x}_2 + \mathbf{p}_1\mathbf{p}_2\right) + C\left(\mathbf{x}_1\mathbf{p}_2 - \mathbf{x}_2\mathbf{p}_1\right) + \frac{D}{2}\left(\mathbf{p}_2^2 + \mathbf{x}_2^2\right)$$

The $ABCD$ matrix from Class 4

$$\mathbf{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$$

Define **a** and **a**† **operators**

$\mathbf{a}_1 = (\mathbf{x}_1 + i\,\mathbf{p}_1)/\sqrt{2}$ $\quad \mathbf{a}^\dagger{}_1 = (\mathbf{x}_1 - i\,\mathbf{p}_1)/\sqrt{2}$ $\quad \mathbf{a}_2 = (\mathbf{x}_2 + i\,\mathbf{p}_2)/\sqrt{2}$ $\quad \mathbf{a}^\dagger{}_2 = (\mathbf{x}_2 - i\,\mathbf{p}_2)/\sqrt{2}$

Solve for $\mathbf{x}_k$ and $\mathbf{p}_k$ **operators**

$\mathbf{x}_1 = (\mathbf{a}^\dagger{}_1 + \mathbf{a}_1)/\sqrt{2}$ $\quad \mathbf{p}_1 = i\,(\mathbf{a}^\dagger{}_1 - \mathbf{a}_1)/\sqrt{2}$ $\quad \mathbf{x}_2 = (\mathbf{a}^\dagger{}_2 + \mathbf{a}_2)/\sqrt{2}$ $\quad \mathbf{p}_2 = i\,(\mathbf{a}^\dagger{}_2 - \mathbf{a}_2)/\sqrt{2}$

Each system dimension $\mathbf{x}_1$ and $\mathbf{x}_2$ is assumed orthogonal, neither being constrained by the other. This includes an axiom of *inter-dimensional commutivity*.

$$[\,\mathbf{x}_1, \mathbf{p}_2] = \mathbf{0} = [\,\mathbf{x}_2, \mathbf{p}_1],\quad [\,\mathbf{a}_1, \mathbf{a}^\dagger{}_2] = \mathbf{0} = [\,\mathbf{a}_2, \mathbf{a}^\dagger{}_1]$$

Commutation relations within space-1 (x_1) or space-2 (x_2) space are those of a 1D-oscillator. Class-7

$$[\,\mathbf{a}_1, \mathbf{a}^\dagger{}_1] = \mathbf{1},\quad [\,\mathbf{a}_2, \mathbf{a}^\dagger{}_2] = \mathbf{1}$$

2D-Oscillator basic states and operations

First rewrite a classical 2-D Hamiltonian (Class-4 p16) with a thick-tip pen! (They're **operators** now!)

$$\mathbf{H} = \frac{A}{2}\left(\mathbf{p}_1^2 + \mathbf{x}_1^2\right) + B\left(\mathbf{x}_1\mathbf{x}_2 + \mathbf{p}_1\mathbf{p}_2\right) + C\left(\mathbf{x}_1\mathbf{p}_2 - \mathbf{x}_2\mathbf{p}_1\right) + \frac{D}{2}\left(\mathbf{p}_2^2 + \mathbf{x}_2^2\right)$$

The *ABCD* matrix from Class 4

$$\mathbf{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$$

Define **a** and $\mathbf{a}^\dagger$ **operators**

$\mathbf{a}_1 = (\mathbf{x}_1 + i\,\mathbf{p}_1)/\sqrt{2}$ $\quad$ $\mathbf{a}^\dagger{}_1 = (\mathbf{x}_1 - i\,\mathbf{p}_1)/\sqrt{2}$ $\quad$ $\mathbf{a}_2 = (\mathbf{x}_2 + i\,\mathbf{p}_2)/\sqrt{2}$ $\quad$ $\mathbf{a}^\dagger{}_2 = (\mathbf{x}_2 - i\,\mathbf{p}_2)/\sqrt{2}$

Solve for $\mathbf{x}_k$ and $\mathbf{p}_k$ **operators**

$\mathbf{x}_1 = (\mathbf{a}^\dagger{}_1 + \mathbf{a}_1)/\sqrt{2}$ $\quad$ $\mathbf{p}_1 = i\,(\mathbf{a}^\dagger{}_1 - \mathbf{a}_1)/\sqrt{2}$ $\quad$ $\mathbf{x}_2 = (\mathbf{a}^\dagger{}_2 + \mathbf{a}_2)/\sqrt{2}$ $\quad$ $\mathbf{p}_2 = i\,(\mathbf{a}^\dagger{}_2 - \mathbf{a}_2)/\sqrt{2}$

Each system dimension $\mathbf{x}_1$ and $\mathbf{x}_2$ is assumed orthogonal, neither being constrained by the other.
This includes an axiom of *inter-dimensional commutivity*.

$$[\,\mathbf{x}_1, \mathbf{p}_2] = \mathbf{0} = [\,\mathbf{x}_2, \mathbf{p}_1], \quad [\,\mathbf{a}_1, \mathbf{a}^\dagger{}_2] = \mathbf{0} = [\,\mathbf{a}_2, \mathbf{a}^\dagger{}_1]$$

Commutation relations within space-1 (x_1) or space-2 (x_2) space are those of a 1D-oscillator.

$$[\,\mathbf{a}_1, \mathbf{a}^\dagger{}_1] = \mathbf{1}, \quad [\,\mathbf{a}_2, \mathbf{a}^\dagger{}_2] = \mathbf{1}$$

This applies in general to *N*-dimensional oscillator problems.

$$[\,\mathbf{a}_m, \mathbf{a}_n] = \mathbf{a}_m\mathbf{a}_n - \mathbf{a}_n\mathbf{a}_m = \mathbf{0}$$

$$[\,\mathbf{a}_m, \mathbf{a}^\dagger{}_n] = \mathbf{a}_m\mathbf{a}^\dagger{}_n - \mathbf{a}^\dagger{}_n\mathbf{a}_m = \delta_{mn}\mathbf{1}$$

$$[\,\mathbf{a}^\dagger{}_m, \mathbf{a}^\dagger{}_n] = \mathbf{a}^\dagger{}_m\mathbf{a}^\dagger{}_n - \mathbf{a}^\dagger{}_n\mathbf{a}^\dagger{}_m = \mathbf{0}$$

2D-Oscillator basic states and operations

First rewrite a classical 2-D Hamiltonian (Class-4 p16) with a thick-tip pen! (They're **operators** now!)

$$\mathbf{H} = \frac{A}{2}\left(\mathbf{p}_1^2 + \mathbf{x}_1^2\right) + B\left(\mathbf{x}_1\mathbf{x}_2 + \mathbf{p}_1\mathbf{p}_2\right) + C\left(\mathbf{x}_1\mathbf{p}_2 - \mathbf{x}_2\mathbf{p}_1\right) + \frac{D}{2}\left(\mathbf{p}_2^2 + \mathbf{x}_2^2\right)$$

The *ABCD* matrix from Class 4

$$\mathbf{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix}$$

Define **a** and **a**† **operators**

$\mathbf{a}_1 = (\mathbf{x}_1 + i\,\mathbf{p}_1)/\sqrt{2}$ $\quad$ $\mathbf{a}^\dagger_1 = (\mathbf{x}_1 - i\,\mathbf{p}_1)/\sqrt{2}$ $\quad$ $\mathbf{a}_2 = (\mathbf{x}_2 + i\,\mathbf{p}_2)/\sqrt{2}$ $\quad$ $\mathbf{a}^\dagger_2 = (\mathbf{x}_2 - i\,\mathbf{p}_2)/\sqrt{2}$

Solve for $\mathbf{x}_k$ and $\mathbf{p}_k$ **operators**

$\mathbf{x}_1 = (\mathbf{a}^\dagger_1 + \mathbf{a}_1)/\sqrt{2}$ $\quad$ $\mathbf{p}_1 = i\,(\mathbf{a}^\dagger_1 - \mathbf{a}_1)/\sqrt{2}$ $\quad$ $\mathbf{x}_2 = (\mathbf{a}^\dagger_2 + \mathbf{a}_2)/\sqrt{2}$ $\quad$ $\mathbf{p}_2 = i\,(\mathbf{a}^\dagger_2 - \mathbf{a}_2)/\sqrt{2}$

Each system dimension $\mathbf{x}_1$ and $\mathbf{x}_2$ is assumed orthogonal, neither being constrained by the other. This includes an axiom of *inter-dimensional commutivity*.

$$[\,\mathbf{x}_1, \mathbf{p}_2] = \mathbf{0} = [\,\mathbf{x}_2, \mathbf{p}_1], \quad [\,\mathbf{a}_1, \mathbf{a}^\dagger_2] = \mathbf{0} = [\,\mathbf{a}_2, \mathbf{a}^\dagger_1]$$

Commutation relations within space-1 (x_1) or space-2 (x_2) space are those of a 1D-oscillator.

$$[\,\mathbf{a}_1, \mathbf{a}^\dagger_1] = \mathbf{1}, \quad [\,\mathbf{a}_2, \mathbf{a}^\dagger_2] = \mathbf{1}$$

This applies in general to *N*-dimensional oscillator problems.

$$[\,\mathbf{a}_m, \mathbf{a}_n] = \mathbf{a}_m\mathbf{a}_n - \mathbf{a}_n\mathbf{a}_m = \mathbf{0}$$

$$[\,\mathbf{a}_m, \mathbf{a}^\dagger_n] = \mathbf{a}_m\mathbf{a}^\dagger_n - \mathbf{a}^\dagger_n\mathbf{a}_m = \delta_{mn}\mathbf{1}$$

$$[\,\mathbf{a}^\dagger_m, \mathbf{a}^\dagger_n] = \mathbf{a}^\dagger_m\mathbf{a}^\dagger_n - \mathbf{a}^\dagger_n\mathbf{a}^\dagger_m = \mathbf{0}$$

New symmetrized $\mathbf{a}^\dagger_m\mathbf{a}_n$ operators replace the old ket-bras $|m\rangle\langle n|$ that define semi-classical **H** matrix.

$$\mathbf{H} = H_{11}\left(\mathbf{a}_1^\dagger\mathbf{a}_1 + \mathbf{1}/2\right) + H_{12}\mathbf{a}_1^\dagger\mathbf{a}_2 + H_{21}\mathbf{a}_2^\dagger\mathbf{a}_1 + H_{22}\left(\mathbf{a}_2^\dagger\mathbf{a}_2 + \mathbf{1}/2\right)$$

$$= A\left(\mathbf{a}_1^\dagger\mathbf{a}_1 + \mathbf{1}/2\right) + (B - iC)\mathbf{a}_1^\dagger\mathbf{a}_2 + (B + iC)\mathbf{a}_2^\dagger\mathbf{a}_1 + D\left(\mathbf{a}_2^\dagger\mathbf{a}_2 + \mathbf{1}/2\right)$$

$$\mathbf{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix}$$

2D-Oscillator basic states and operations

First rewrite a classical 2-D Hamiltonian (Class-4 p16) with a thick-tip pen! (They're **operators** now!)

$$\mathbf{H}=\frac{A}{2}\left(\mathbf{p}_1^2+\mathbf{x}_1^2\right)+B\left(\mathbf{x}_1\mathbf{x}_2+\mathbf{p}_1\mathbf{p}_2\right)+C\left(\mathbf{x}_1\mathbf{p}_2-\mathbf{x}_2\mathbf{p}_1\right)+\frac{D}{2}\left(\mathbf{p}_2^2+\mathbf{x}_2^2\right)$$

The $ABCD$ matrix from Class 4

$$\mathbf{H}=\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22}\end{pmatrix}=\begin{pmatrix} A & B-iC \\ B+iC & D\end{pmatrix}$$

Define **a** and **a**† **operators**

$\mathbf{a}_1=(\mathbf{x}_1+i\,\mathbf{p}_1)/\sqrt{2}$ $\quad\mathbf{a}^\dagger{}_1=(\mathbf{x}_1-i\,\mathbf{p}_1)/\sqrt{2}$ $\quad\mathbf{a}_2=(\mathbf{x}_2+i\,\mathbf{p}_2)/\sqrt{2}$ $\quad\mathbf{a}^\dagger{}_2=(\mathbf{x}_2-i\,\mathbf{p}_2)/\sqrt{2}$

Solve for $\mathbf{x}_k$ and $\mathbf{p}_k$ **operators**

$\mathbf{x}_1=(\mathbf{a}^\dagger{}_1+\mathbf{a}_1)/\sqrt{2}$ $\quad\mathbf{p}_1=i\,(\mathbf{a}^\dagger{}_1-\mathbf{a}_1)/\sqrt{2}$ $\quad\mathbf{x}_2=(\mathbf{a}^\dagger{}_2+\mathbf{a}_2)/\sqrt{2}$ $\quad\mathbf{p}_2=i\,(\mathbf{a}^\dagger{}_2-\mathbf{a}_2)/\sqrt{2}$

Each system dimension $\mathbf{x}_1$ and $\mathbf{x}_2$ is assumed orthogonal, neither being constrained by the other. This includes an axiom of *inter-dimensional commutivity*.

$$[\,\mathbf{x}_1,\mathbf{p}_2]=\mathbf{0}=[\,\mathbf{x}_2,\mathbf{p}_1],\quad[\,\mathbf{a}_1,\mathbf{a}^\dagger{}_2]=\mathbf{0}=[\,\mathbf{a}_2,\mathbf{a}^\dagger{}_1]$$

Commutation relations within space-1 (x_1) or space-2 (x_2) space are those of a 1D-oscillator.

$$[\,\mathbf{a}_1,\mathbf{a}^\dagger{}_1]=\mathbf{1},\quad[\,\mathbf{a}_2,\mathbf{a}^\dagger{}_2]=\mathbf{1}$$

This applies in general to N-dimensional oscillator problems.

$[\,\mathbf{a}_m,\mathbf{a}_n]=\mathbf{a}_m\mathbf{a}_n-\mathbf{a}_n\mathbf{a}_m=\mathbf{0}$ $\quad[\,\mathbf{a}_m,\mathbf{a}^\dagger{}_n]=\mathbf{a}_m\mathbf{a}^\dagger{}_n-\mathbf{a}^\dagger{}_n\mathbf{a}_m=\delta_{mn}\mathbf{1}$ $\quad[\,\mathbf{a}^\dagger{}_m,\mathbf{a}^\dagger{}_n]=\mathbf{a}^\dagger{}_m\mathbf{a}^\dagger{}_n-\mathbf{a}^\dagger{}_n\mathbf{a}^\dagger{}_m=\mathbf{0}$

New symmetrized $\mathbf{a}^\dagger{}_m\mathbf{a}_n$ operators replace the old ket-bras $|m\rangle\langle n|$ that define semi-classical **H** matrix.

$$\mathbf{H}=H_{11}\left(\mathbf{a}_1^\dagger\mathbf{a}_1+\mathbf{1}/2\right)+H_{12}\mathbf{a}_1^\dagger\mathbf{a}_2+H_{21}\mathbf{a}_2^\dagger\mathbf{a}_1+H_{22}\left(\mathbf{a}_2^\dagger\mathbf{a}_2+\mathbf{1}/2\right)$$

$$=A\left(\mathbf{a}_1^\dagger\mathbf{a}_1+\mathbf{1}/2\right)+(B-iC)\mathbf{a}_1^\dagger\mathbf{a}_2+(B+iC)\mathbf{a}_2^\dagger\mathbf{a}_1+D\left(\mathbf{a}_2^\dagger\mathbf{a}_2+\mathbf{1}/2\right)$$

$$\mathbf{H}=\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22}\end{pmatrix}=\begin{pmatrix} A & B-iC \\ B+iC & D\end{pmatrix}$$

Both are elementary "place-holders" for parameters H_{mn} or A, $B\pm iC$, and D.

$$|m\rangle\langle n|\rightarrow\left(\mathbf{a}_m^\dagger\mathbf{a}_n+\mathbf{a}_n\mathbf{a}_m^\dagger\right)/2=\mathbf{a}_m^\dagger\mathbf{a}_n+\delta_{m,n}\mathbf{1}/2$$

Operator arithmetic detailed:

$$\mathbf{a}_1\mathbf{a}_1^\dagger = \tfrac{1}{\sqrt{2}}\left(\mathbf{x}_1 + i\mathbf{p}_1\right)\tfrac{1}{\sqrt{2}}\left(\mathbf{x}_1 - i\mathbf{p}_1\right) = \tfrac{1}{2}\left(\mathbf{x}_1^2 + \mathbf{p}_1^2 - i(\mathbf{x}_1\mathbf{p}_1 - \mathbf{p}_1\mathbf{x}_1)\right) = \tfrac{1}{2}\left(\mathbf{x}_1^2 + \mathbf{p}_1^2 + \tfrac{\hbar}{2}\mathbf{1}\right)$$

$$\mathbf{a}_2\mathbf{a}_2^\dagger = \tfrac{1}{\sqrt{2}}\left(\mathbf{x}_2 + i\mathbf{p}_2\right)\tfrac{1}{\sqrt{2}}\left(\mathbf{x}_2 - i\mathbf{p}_2\right) = \tfrac{1}{2}\left(\mathbf{x}_2^2 + \mathbf{p}_2^2 - i(\mathbf{x}_2\mathbf{p}_2 - \mathbf{p}_2\mathbf{x}_2)\right) = \tfrac{1}{2}\left(\mathbf{x}_2^2 + \mathbf{p}_2^2 + \tfrac{\hbar}{2}\mathbf{1}\right)$$

$$\mathbf{x}_1\mathbf{x}_2 = \tfrac{1}{\sqrt{2}}\left(\mathbf{a}_1^\dagger + \mathbf{a}_1\right)\tfrac{1}{\sqrt{2}}\left(\mathbf{a}_2^\dagger + \mathbf{a}_2\right) = \tfrac{1}{2}\left(\mathbf{a}_1^\dagger\mathbf{a}_2^\dagger + \mathbf{a}_1^\dagger\mathbf{a}_2 + \mathbf{a}_1\mathbf{a}_2^\dagger + \mathbf{a}_1\mathbf{a}_2\right)$$

$$\mathbf{p}_1\mathbf{p}_2 = \tfrac{i}{\sqrt{2}}\left(\mathbf{a}_1^\dagger - \mathbf{a}_1\right)\tfrac{i}{\sqrt{2}}\left(\mathbf{a}_2^\dagger - \mathbf{a}_2\right) = \tfrac{-1}{2}\left(\mathbf{a}_1^\dagger\mathbf{a}_2^\dagger - \mathbf{a}_1^\dagger\mathbf{a}_2 - \mathbf{a}_1\mathbf{a}_2^\dagger + \mathbf{a}_1\mathbf{a}_2\right)$$

$$\mathbf{x}_1\mathbf{x}_2 + \mathbf{p}_1\mathbf{p}_2 = (\mathbf{a}_1^\dagger\mathbf{a}_2 + \mathbf{a}_1\mathbf{a}_2^\dagger) = (\mathbf{a}_1^\dagger\mathbf{a}_2 + \mathbf{a}_2^\dagger\mathbf{a}_1)$$

$$\mathbf{x}_1\mathbf{p}_2 = \tfrac{1}{\sqrt{2}}\left(\mathbf{a}_1^\dagger + \mathbf{a}_1\right)\tfrac{i}{\sqrt{2}}\left(\mathbf{a}_2^\dagger - \mathbf{a}_2\right) = \tfrac{i}{2}\left(\mathbf{a}_1^\dagger\mathbf{a}_2^\dagger - \mathbf{a}_1^\dagger\mathbf{a}_2 + \mathbf{a}_1\mathbf{a}_2^\dagger - \mathbf{a}_1\mathbf{a}_2\right)$$

$$-\mathbf{x}_2\mathbf{p}_1 = \tfrac{-1}{\sqrt{2}}\left(\mathbf{a}_2^\dagger + \mathbf{a}_2\right)\tfrac{i}{\sqrt{2}}\left(\mathbf{a}_1^\dagger - \mathbf{a}_1\right) = \tfrac{-i}{2}\left(\mathbf{a}_2^\dagger\mathbf{a}_1^\dagger + \mathbf{a}_2\mathbf{a}_1^\dagger - \mathbf{a}_2^\dagger\mathbf{a}_1 - \mathbf{a}_2\mathbf{a}_1\right)$$

$$\mathbf{x}_1\mathbf{p}_2 - \mathbf{x}_2\mathbf{p}_1 = -i\mathbf{a}_1^\dagger\mathbf{a}_2 + i\mathbf{a}_1\mathbf{a}_2^\dagger) = -i\mathbf{a}_1^\dagger\mathbf{a}_2 + i\mathbf{a}_2^\dagger\mathbf{a}_1$$

Weird 2D HO Hamiltonian cooked up to match U(2) quantum **H**-equation with classical **K**-equation

$$\mathbf{H} = \frac{A}{2}\left(\mathbf{p}_1^2 + \mathbf{x}_1^2\right) + B\left(\mathbf{x}_1\mathbf{x}_2 + \mathbf{p}_1\mathbf{p}_2\right) + C\left(\mathbf{x}_1\mathbf{p}_2 - \mathbf{x}_2\mathbf{p}_1\right) + \frac{D}{2}\left(\mathbf{p}_2^2 + \mathbf{x}_2^2\right)$$

New symmetrized $\mathbf{a}^\dagger{}_m\mathbf{a}_n$ operators replace the old ket-bras $|m\rangle\langle n|$ that define semi-classical **H** matrix.

$$\mathbf{H} = H_{11}\left(\mathbf{a}_1^\dagger\mathbf{a}_1 + \mathbf{1}/2\right) + H_{12}\mathbf{a}_1^\dagger\mathbf{a}_2 + H_{21}\mathbf{a}_2^\dagger\mathbf{a}_1 + H_{22}\left(\mathbf{a}_2^\dagger\mathbf{a}_2 + \mathbf{1}/2\right)$$

$$= A\left(\mathbf{a}_1^\dagger\mathbf{a}_1 + \mathbf{1}/2\right) + (B - iC)\mathbf{a}_1^\dagger\mathbf{a}_2 + (B + iC)\mathbf{a}_2^\dagger\mathbf{a}_1 + D\left(\mathbf{a}_2^\dagger\mathbf{a}_2 + \mathbf{1}/2\right)$$

$$\mathbf{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$$

Symmetry group $\mathscr{G}$=U(2) representations, 2D HO Hamiltonian $\mathbf{H}=\hbar\omega_{ab}\mathbf{a}_a{}^\dagger\mathbf{a}_b$ operators, 2D HO wave eigenfunctions $\Psi_{n,m}$, and coherent [α] states

Factoring 2D-HO Hamiltonian $\mathbf{H}=\frac{A}{2}\left(\mathbf{p}_1^2+\mathbf{x}_1^2\right)+B\left(\mathbf{x}_1\mathbf{x}_2+\mathbf{p}_1\mathbf{p}_2\right)+C\left(\mathbf{x}_1\mathbf{p}_2-\mathbf{x}_2\mathbf{p}_1\right)+\frac{D}{2}\left(\mathbf{p}_2^2+\mathbf{x}_2^2\right)$

- *2D-Oscillator basic states and operations*
  - *Commutation relations*
  - *Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry*
    - *Anti-commutation relations*
  - *Two-dimensional (or 2-particle) base states: ket-kets and bra-bras*
    - *Outer product arrays*
    - *Entangled 2-particle states*
- *Two-particle (or 2-dimensional) matrix operators*
  - *U(2) Hamiltonian and irreducible representations*
  - *2D-Oscillator states and related 3D angular momentum multiplets*

R(3) Angular momentum generators by U(2) analysis

Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$

SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Mostly Notation and Bookkeeping

Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry

Commutivity is known as *Bose symmetry.* Bose and Einstein discovered this symmetry of light quanta. ($\mathbf{a}_m$, $\mathbf{a}^\dagger{}_n$) operators called *Boson operators* create or destroy *quanta* or "particles" known as *Bosons*.

Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry

Commutivity is known as *Bose symmetry.* Bose and Einstein discovered this symmetry of light quanta. $(\mathbf{a}_m, \mathbf{a}^\dagger{}_n)$ operators called *Boson operators* create or destroy *quanta* or "particles" known as *Bosons*.

If $\mathbf{a}^\dagger{}_m$ raises electromagnetic mode quantum number m to $m+1$ it is said to create a *photon*.

Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry

Commutivity is known as *Bose symmetry.* Bose and Einstein discovered this symmetry of light quanta. ($\mathbf{a}_m$, $\mathbf{a}^\dagger{}_n$) operators called *Boson operators* create or destroy *quanta* or "particles" known as *Bosons*.

If $\mathbf{a}^\dagger{}_m$ raises electromagnetic mode quantum number m to $m+1$ it is said to create a *photon*.

If $\mathbf{a}^\dagger{}_m$ raises crystal vibration mode quantum number m to $m+1$ it is said to create a *phonon*.

Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry

Commutivity is known as *Bose symmetry.* Bose and Einstein discovered this symmetry of light quanta. ($\mathbf{a}_m$, $\mathbf{a}^\dagger_n$) operators called *Boson operators* create or destroy *quanta* or "particles" known as *Bosons*.

If $\mathbf{a}^\dagger_m$ raises electromagnetic mode quantum number m to $m+1$ it is said to create a *photon*.

If $\mathbf{a}^\dagger_m$ raises crystal vibration mode quantum number m to $m+1$ it is said to create a *phonon*.

If $\mathbf{a}^\dagger_m$ raises liquid ^{4}He rotational quantum number m to $m+1$ it is said to create a *roton*.

Symmetry group $\mathscr{G}$=U(2) representations, 2D HO Hamiltonian $\mathbf{H}=\hbar\omega_{ab}\mathbf{a}_a^\dagger\mathbf{a}_b$ operators,
2D HO wave eigenfunctions $\Psi_{n,m}$, and coherent [α] states

Factoring 2D-HO Hamiltonian $\mathbf{H}=\frac{A}{2}\left(\mathbf{p}_1^2+\mathbf{x}_1^2\right)+B\left(\mathbf{x}_1\mathbf{x}_2+\mathbf{p}_1\mathbf{p}_2\right)+C\left(\mathbf{x}_1\mathbf{p}_2-\mathbf{x}_2\mathbf{p}_1\right)+\frac{D}{2}\left(\mathbf{p}_2^2+\mathbf{x}_2^2\right)$

2D-Oscillator basic states and operations
- *Commutation relations*
- *Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry*
  - *Anti-commutation relations*
- *Two-dimensional (or 2-particle) base states: ket-kets and bra-bras*
  - *Outer product arrays*
  - *Entangled 2-particle states*

Two-particle (or 2-dimensional) matrix operators
- *U(2) Hamiltonian and irreducible representations*
- *2D-Oscillator states and related 3D angular momentum multiplets*

R(3) Angular momentum generators by U(2) analysis

Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$

SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Mostly Notation and Bookkeeping

Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry

Commutivity is known as *Bose symmetry.* Bose and Einstein discovered this symmetry of light quanta. $(\mathbf{a}_m, \mathbf{a}^\dagger{}_n)$ operators called *Boson operators* create or destroy *quanta* or "particles" known as *Bosons*.

If $\mathbf{a}^\dagger{}_m$ raises electromagnetic mode quantum number m to $m+1$ it is said to create a *photon*.

If $\mathbf{a}^\dagger{}_m$ raises crystal vibration mode quantum number m to $m+1$ it is said to create a *phonon*.

If $\mathbf{a}^\dagger{}_m$ raises liquid ^{4}He rotational quantum number m to $m+1$ it is said to create a *roton*.

Anti-commutivity is named *Fermi-Dirac symmetry* or *anti-symmetry*. It is found in electron waves.

Fermi operators $(\mathbf{c}_m,\mathbf{c}_n)$ are defined to create *Fermions* and use *anti-commutators* $\{\mathbf{A},\mathbf{B}\} = \mathbf{AB}+\mathbf{BA}$.

$$\{\mathbf{c}_m,\mathbf{c}_n\}=\mathbf{c}_m\mathbf{c}_n+\mathbf{c}_n\mathbf{c}_m=\mathbf{0} \qquad \{\mathbf{c}_m,\mathbf{c}^\dagger{}_n\}=\mathbf{c}_m\mathbf{c}^\dagger{}_n+\mathbf{c}^\dagger{}_n\mathbf{c}_m=\delta_{mn}\mathbf{1} \qquad \{\mathbf{c}^\dagger{}_m,\mathbf{c}^\dagger{}_n\}=\mathbf{c}^\dagger{}_m\mathbf{c}^\dagger{}_n+\mathbf{c}^\dagger{}_n\mathbf{c}^\dagger{}_m=\mathbf{0}$$

Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry

Commutivity is known as *Bose symmetry.* Bose and Einstein discovered this symmetry of light quanta. $(\mathbf{a}_m, \mathbf{a}^\dagger_n)$ operators called *Boson operators* create or destroy *quanta* or "particles" known as *Bosons*.

If $\mathbf{a}^\dagger_m$ raises electromagnetic mode quantum number m to $m+1$ it is said to create a *photon*.

If $\mathbf{a}^\dagger_m$ raises crystal vibration mode quantum number m to $m+1$ it is said to create a *phonon*.

If $\mathbf{a}^\dagger_m$ raises liquid ^{4}He rotational quantum number m to $m+1$ it is said to create a *roton*.

Anti-commutivity is named *Fermi-Dirac symmetry* or *anti-symmetry*. It is found in electron waves.

Fermi operators $(\mathbf{c}_m,\mathbf{c}_n)$ are defined to create *Fermions* and use *anti-commutators* $\{\mathbf{A},\mathbf{B}\} = \mathbf{AB}+\mathbf{BA}$.

$$\{\mathbf{c}_m,\mathbf{c}_n\}=\mathbf{c}_m\mathbf{c}_n+\mathbf{c}_n\mathbf{c}_m=\mathbf{0} \qquad \{\mathbf{c}_m,\mathbf{c}^\dagger_n\}=\mathbf{c}_m\mathbf{c}^\dagger_n+\mathbf{c}^\dagger_n\mathbf{c}_m=\delta_{mn}\mathbf{1} \qquad \{\mathbf{c}^\dagger_m,\mathbf{c}^\dagger_n\}=\mathbf{c}^\dagger_m\mathbf{c}^\dagger_n+\mathbf{c}^\dagger_n\mathbf{c}^\dagger_m=\mathbf{0}$$

Fermi $\mathbf{c}^\dagger_n$ has a rigid birth-control policy; they are allowed just one Fermion or else, none at all.

Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry

Commutivity is known as *Bose symmetry.* Bose and Einstein discovered this symmetry of light quanta. ($\mathbf{a}_m$, $\mathbf{a}^\dagger{}_n$) operators called *Boson operators* create or destroy *quanta* or "particles" known as *Bosons*.

If $\mathbf{a}^\dagger{}_m$ raises electromagnetic mode quantum number m to $m+1$ it is said to create a *photon*.

If $\mathbf{a}^\dagger{}_m$ raises crystal vibration mode quantum number m to $m+1$ it is said to create a *phonon*.

If $\mathbf{a}^\dagger{}_m$ raises liquid ^{4}He rotational quantum number m to $m+1$ it is said to create a *roton*.

Anti-commutivity is named *Fermi-Dirac symmetry* or *anti-symmetry*. It is found in electron waves.

Fermi operators ($\mathbf{c}_m$,$\mathbf{c}_n$) are defined to create *Fermions* and use *anti-commutators* $\{\mathbf{A},\mathbf{B}\} = \mathbf{AB}+\mathbf{BA}$.

$$\{\mathbf{c}_m,\mathbf{c}_n\}=\mathbf{c}_m\mathbf{c}_n+\mathbf{c}_n\mathbf{c}_m=\mathbf{0} \qquad \{\mathbf{c}_m,\mathbf{c}^\dagger{}_n\}=\mathbf{c}_m\mathbf{c}^\dagger{}_n+\mathbf{c}^\dagger{}_n\mathbf{c}_m=\delta_{mn}\mathbf{1} \qquad \{\mathbf{c}^\dagger{}_m,\mathbf{c}^\dagger{}_n\}=\mathbf{c}^\dagger{}_m\mathbf{c}^\dagger{}_n+\mathbf{c}^\dagger{}_n\mathbf{c}^\dagger{}_m=\mathbf{0}$$

Fermi $\mathbf{c}^\dagger{}_n$ has a rigid birth-control policy; they are allowed just one Fermion or else, none at all.

Creating two Fermions of the same type is punished by death. This is because x=-x implies x=0.

$$\mathbf{c}^\dagger{}_m\mathbf{c}^\dagger{}_m\,|0\rangle = -\,\mathbf{c}^\dagger{}_m\mathbf{c}^\dagger{}_m\,|0\rangle = \mathbf{0}$$

Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry

Commutivity is known as *Bose symmetry.* Bose and Einstein discovered this symmetry of light quanta. ($\mathbf{a}_m$, $\mathbf{a}^\dagger{}_n$) operators called *Boson operators* create or destroy *quanta* or "particles" known as *Bosons*.

If $\mathbf{a}^\dagger{}_m$ raises electromagnetic mode quantum number m to $m+1$ it is said to create a *photon*.

If $\mathbf{a}^\dagger{}_m$ raises crystal vibration mode quantum number m to $m+1$ it is said to create a *phonon*.

If $\mathbf{a}^\dagger{}_m$ raises liquid ^{4}He rotational quantum number m to $m+1$ it is said to create a *roton*.

Anti-commutivity is named *Fermi-Dirac symmetry* or *anti-symmetry*. It is found in electron waves.

Fermi operators ($\mathbf{c}_m$,$\mathbf{c}_n$) are defined to create *Fermions* and use *anti-commutators* $\{\mathbf{A},\mathbf{B}\} = \mathbf{AB}+\mathbf{BA}$.

$$\{\mathbf{c}_m,\mathbf{c}_n\}=\mathbf{c}_m\mathbf{c}_n+\mathbf{c}_n\mathbf{c}_m=\mathbf{0} \qquad \{\mathbf{c}_m,\mathbf{c}^\dagger{}_n\}=\mathbf{c}_m\mathbf{c}^\dagger{}_n+\mathbf{c}^\dagger{}_n\mathbf{c}_m=\delta_{mn}\mathbf{1} \qquad \{\mathbf{c}^\dagger{}_m,\mathbf{c}^\dagger{}_n\}=\mathbf{c}^\dagger{}_m\mathbf{c}^\dagger{}_n+\mathbf{c}^\dagger{}_n\mathbf{c}^\dagger{}_m=\mathbf{0}$$

Fermi $\mathbf{c}^\dagger{}_n$ has a rigid birth-control policy; they are allowed just one Fermion or else, none at all.

Creating two Fermions of the same type is punished by death. This is because $x=-x$ implies $x=0$.

$$\mathbf{c}^\dagger{}_m\mathbf{c}^\dagger{}_m\,|0\rangle = -\,\mathbf{c}^\dagger{}_m\mathbf{c}^\dagger{}_m\,|0\rangle = \mathbf{0}$$

That no two indistinguishable Fermions can be in the same state, is called the *Pauli exclusion principle*.

Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry

Commutivity is known as *Bose symmetry.* Bose and Einstein discovered this symmetry of light quanta. ($\mathbf{a}_m$, $\mathbf{a}^\dagger{}_n$) operators called *Boson operators* create or destroy *quanta* or "particles" known as *Bosons*.

If $\mathbf{a}^\dagger{}_m$ raises electromagnetic mode quantum number m to $m+1$ it is said to create a *photon*.

If $\mathbf{a}^\dagger{}_m$ raises crystal vibration mode quantum number m to $m+1$ it is said to create a *phonon*.

If $\mathbf{a}^\dagger{}_m$ raises liquid ^{4}He rotational quantum number m to $m+1$ it is said to create a *roton*.

Anti-commutivity is named *Fermi-Dirac symmetry* or *anti-symmetry*. It is found in electron waves.

Fermi operators ($\mathbf{c}_m$,$\mathbf{c}_n$) are defined to create *Fermions* and use *anti-commutators* $\{\mathbf{A},\mathbf{B}\} = \mathbf{AB}+\mathbf{BA}$.

$$\{\mathbf{c}_m,\mathbf{c}_n\}=\mathbf{c}_m\mathbf{c}_n+\mathbf{c}_n\mathbf{c}_m=\mathbf{0} \qquad \{\mathbf{c}_m,\mathbf{c}^\dagger{}_n\}=\mathbf{c}_m\mathbf{c}^\dagger{}_n+\mathbf{c}^\dagger{}_n\mathbf{c}_m=\delta_{mn}\mathbf{1} \qquad \{\mathbf{c}^\dagger{}_m,\mathbf{c}^\dagger{}_n\}=\mathbf{c}^\dagger{}_m\mathbf{c}^\dagger{}_n+\mathbf{c}^\dagger{}_n\mathbf{c}^\dagger{}_m=\mathbf{0}$$

Fermi $\mathbf{c}^\dagger{}_n$ has a rigid birth-control policy; they are allowed just one Fermion or else, none at all.

Creating two Fermions of the same type is punished by death. This is because $x=-x$ implies $x=0$.

$$\mathbf{c}^\dagger{}_m\mathbf{c}^\dagger{}_m\,|0\rangle = -\,\mathbf{c}^\dagger{}_m\mathbf{c}^\dagger{}_m\,|0\rangle = \mathbf{0}$$

That no two indistinguishable Fermions can be in the same state, is called the *Pauli exclusion principle*.

Quantum numbers of $n=0$ and $n=1$ are the only allowed eigenvalues of the number operator $\mathbf{c}^\dagger{}_m\mathbf{c}_m$.

$$\mathbf{c}^\dagger{}_m\mathbf{c}_m\,|0\rangle = \mathbf{0}\ ,\ \ \mathbf{c}^\dagger{}_m\mathbf{c}_m\,|1\rangle = |1\rangle\ ,\ \ \mathbf{c}^\dagger{}_m\mathbf{c}_m\,|n\rangle = \mathbf{0}\ \ \text{for: } n>1$$

Symmetry group $\mathscr{G}$=U(2) representations, 2D HO Hamiltonian $\mathbf{H}=\hbar\omega_{ab}\mathbf{a}_a^\dagger\mathbf{a}_b$ operators, 2D HO wave eigenfunctions $\Psi_{n,m}$, and coherent [α] states

Factoring 2D-HO Hamiltonian $\mathbf{H}=\frac{A}{2}\left(\mathbf{p}_1^2+\mathbf{x}_1^2\right)+B\left(\mathbf{x}_1\mathbf{x}_2+\mathbf{p}_1\mathbf{p}_2\right)+C\left(\mathbf{x}_1\mathbf{p}_2-\mathbf{x}_2\mathbf{p}_1\right)+\frac{D}{2}\left(\mathbf{p}_2^2+\mathbf{x}_2^2\right)$

2D-Oscillator basic states and operations

Commutation relations

Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry

Anti-commutation relations

Two-dimensional (or 2-particle) base states: ket-kets and bra-bras

Outer product arrays

Entangled 2-particle states

Two-particle (or 2-dimensional) matrix operators

U(2) Hamiltonian and irreducible representations

2D-Oscillator states and related 3D angular momentum multiplets

R(3) Angular momentum generators by U(2) analysis

Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$

SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Mostly Notation and Bookkeeping :

Two-dimensional (or 2-particle) base states: ket-kets and bra-bras

A state for a particle in two-dimensions (or two one-dimensional particles) is a *"ket-ket"* $|n_1\rangle|n_2\rangle$

It is outer product of the kets for each single dimension or particle.

The dual description is done similarly using *"bra-bras"* $\langle n_2|\langle n_1| = (|n_1\rangle|n_2\rangle)^\dagger$

Two-dimensional (or 2-particle) base states: ket-kets and bra-bras

A state for a particle in two-dimensions (or two one-dimensional particles) is a *"ket-ket"* $|n_1\rangle|n_2\rangle$
It is outer product of the kets for each single dimension or particle.
The dual description is done similarly using *"bra-bras"* $\langle n_2|\langle n_1| = (|n_1\rangle|n_2\rangle)^\dagger$

This applies to all types of states $|\Psi_1\rangle|\Psi_2\rangle$: eigenstates $|n_1\rangle|n_2\rangle$, or $\langle n_2|\langle n_1|$,
position states $|x_1\rangle|x_2\rangle$ and $\langle x_2|\langle x_1|$, coherent states $|\alpha_1\rangle|\alpha_2\rangle$ and $\langle\alpha_2|\langle\alpha_1|$, or whatever.

Two-dimensional (or 2-particle) base states: ket-kets and bra-bras

A state for a particle in two-dimensions (or two one-dimensional particles) is a *"ket-ket"* $|n_1\rangle|n_2\rangle$
It is outer product of the kets for each single dimension or particle.
The dual description is done similarly using *"bra-bras"* $\langle n_2|\langle n_1| = (|n_1\rangle|n_2\rangle)^\dagger$

This applies to all types of states $|\Psi_1\rangle|\Psi_2\rangle$: eigenstates $|n_1\rangle|n_2\rangle$, or $\langle n_2|\langle n_1|$,
position states $|x_1\rangle|x_2\rangle$ and $\langle x_2|\langle x_1|$, coherent states $|\alpha_1\rangle|\alpha_2\rangle$ and $\langle\alpha_2|\langle\alpha_1|$, or whatever.

Scalar product is defined so that each kind of particle or dimension
will "find" each other and ignore the presence of other kind(s).

$$\langle x_2|\langle x_1||\Psi_1\rangle|\Psi_2\rangle = \langle x_1|\Psi_1\rangle\langle x_2|\Psi_2\rangle$$

Two-dimensional (or 2-particle) base states: ket-kets and bra-bras

A state for a particle in two-dimensions (or two one-dimensional particles) is a *"ket-ket"* $|n_1\rangle|n_2\rangle$
It is outer product of the kets for each single dimension or particle.
The dual description is done similarly using *"bra-bras"* $\langle n_2|\langle n_1| = (|n_1\rangle|n_2\rangle)^\dagger$

This applies to all types of states $|\Psi_1\rangle|\Psi_2\rangle$: eigenstates $|n_1\rangle|n_2\rangle$, or $\langle n_2|\langle n_1|$,
position states $|x_1\rangle|x_2\rangle$ and $\langle x_2|\langle x_1|$, coherent states $|\alpha_1\rangle|\alpha_2\rangle$ and $\langle \alpha_2|\langle \alpha_1|$, or whatever.

Scalar product is defined so that each kind of particle or dimension will "find" each other and ignore the presence of other kind(s).

$$\langle x_2 |\langle x_1 ||\Psi_1\rangle|\Psi_2\rangle = \langle x_1 |\Psi_1\rangle\langle x_2 |\Psi_2\rangle$$

Probability axiom-1 gives correct probability for finding particle-1 at x_1 *and* particle-2 at x_2,
if state $|\Psi_1\rangle|\Psi_2\rangle$ must choose between *all* (x_1 , x_2).

$$\left|\langle x_1,x_2|\Psi_1,\Psi_2\rangle\right|^2=\left|\langle x_2|\langle x_1||\Psi_1\rangle|\Psi_2\rangle\right|^2$$

$$=\left|\langle x_1|\Psi_1\rangle\right|^2\left|\langle x_2|\Psi_2\rangle\right|^2$$

Two-dimensional (or 2-particle) base states: ket-kets and bra-bras

A state for a particle in two-dimensions (or two one-dimensional particles) is a *"ket-ket"* $|n_1\rangle|n_2\rangle$
It is outer product of the kets for each single dimension or particle.
The dual description is done similarly using *"bra-bras"* $\langle n_2|\langle n_1| = (|n_1\rangle|n_2\rangle)^\dagger$

This applies to all types of states $|\Psi_1\rangle|\Psi_2\rangle$: eigenstates $|n_1\rangle|n_2\rangle$, or $\langle n_2|\langle n_1|$,
position states $|x_1\rangle|x_2\rangle$ and $\langle x_2|\langle x_1|$, coherent states $|\alpha_1\rangle|\alpha_2\rangle$ and $\langle \alpha_2|\langle \alpha_1|$, or whatever.

Scalar product is defined so that each kind of particle or dimension will "find" each other and ignore the presence of other kind(s).

$$\langle x_2 |\langle x_1 ||\Psi_1\rangle|\Psi_2\rangle = \langle x_1 |\Psi_1\rangle\langle x_2 |\Psi_2\rangle$$

Probability axiom-1 gives correct probability for finding particle-1 at x_1 *and* particle-2 at x_2,
if state $|\Psi_1\rangle|\Psi_2\rangle$ must choose between *all* (x_1 , x_2).

$$\left|\langle x_1,x_2|\Psi_1,\Psi_2\rangle\right|^2=\left|\langle x_2|\langle x_1||\Psi_1\rangle|\Psi_2\rangle\right|^2$$

$$=\left|\langle x_1|\Psi_1\rangle\right|^2\left|\langle x_2|\Psi_2\rangle\right|^2$$

Product of individual probabilities $\left|\langle x_1|\Psi_1\rangle\right|^2$ and $\left|\langle x_2|\Psi_2\rangle\right|^2$ respects standard Bayesian probability theory.

Two-dimensional (or 2-particle) base states: ket-kets and bra-bras

A state for a particle in two-dimensions (or two one-dimensional particles) is a *"ket-ket"* $|n_1\rangle|n_2\rangle$
It is outer product of the kets for each single dimension or particle.
The dual description is done similarly using *"bra-bras"* $\langle n_2|\langle n_1| = (|n_1\rangle|n_2\rangle)^\dagger$

This applies to all types of states $|\Psi_1\rangle|\Psi_2\rangle$: eigenstates $|n_1\rangle|n_2\rangle$, or $\langle n_2|\langle n_1|$,
position states $|x_1\rangle|x_2\rangle$ and $\langle x_2|\langle x_1|$, coherent states $|\alpha_1\rangle|\alpha_2\rangle$ and $\langle\alpha_2|\langle\alpha_1|$, or whatever.

Scalar product is defined so that each kind of particle or dimension will "find" each other and ignore the presence of other kind(s).

$$\langle x_2 |\langle x_1 ||\Psi_1\rangle|\Psi_2\rangle = \langle x_1 |\Psi_1\rangle\langle x_2 |\Psi_2\rangle$$

Probability axiom-1 gives correct probability for finding particle-1 at x_1 *and* particle-2 at x_2,
if state $|\Psi_1\rangle|\Psi_2\rangle$ must choose between *all* (x_1 , x_2).

$$\left|\langle x_1, x_2|\Psi_1, \Psi_2\rangle\right|^2 = \left|\langle x_2|\langle x_1||\Psi_1\rangle|\Psi_2\rangle\right|^2$$
$$= \left|\langle x_1|\Psi_1\rangle\right|^2 \left|\langle x_2|\Psi_2\rangle\right|^2$$

Product of individual probabilities $\left|\langle x_1|\Psi_1\rangle\right|^2$ and $\left|\langle x_2|\Psi_2\rangle\right|^2$ respects standard Bayesian probability theory.

Note common shorthand *big-bra-big-ket* notation $\langle x_1, x_2|\Psi_1, \Psi_2\rangle = \langle x_2|\langle x_1||\Psi_1\rangle|\Psi_2\rangle$

Two-dimensional (or 2-particle) base states: ket-kets and bra-bras

A state for a particle in two-dimensions (or two one-dimensional particles) is a *"ket-ket"* $|n_1\rangle|n_2\rangle$
It is outer product of the kets for each single dimension or particle.
The dual description is done similarly using *"bra-bras"* $\langle n_2|\langle n_1| = (|n_1\rangle|n_2\rangle)^\dagger$

This applies to all types of states $|\Psi_1\rangle|\Psi_2\rangle$: eigenstates $|n_1\rangle|n_2\rangle$, or $\langle n_2|\langle n_1|$,
position states $|x_1\rangle|x_2\rangle$ and $\langle x_2|\langle x_1|$, coherent states $|\alpha_1\rangle|\alpha_2\rangle$ and $\langle\alpha_2|\langle\alpha_1|$, or whatever.

Scalar product is defined so that each kind of particle or dimension will "find" each other and ignore the presence of other kind(s).

$$\langle x_2|\langle x_1||\Psi_1\rangle|\Psi_2\rangle = \langle x_1|\Psi_1\rangle\langle x_2|\Psi_2\rangle$$

Probability axiom-1 gives correct probability for finding particle-1 at x_1 <u>*and*</u> particle-2 at x_2,
if state $|\Psi_1\rangle|\Psi_2\rangle$ must choose between <u>*all*</u> (x_1, x_2).

$$\left|\langle x_1,x_2|\Psi_1,\Psi_2\rangle\right|^2=\left|\langle x_2|\langle x_1||\Psi_1\rangle|\Psi_2\rangle\right|^2$$
$$=\left|\langle x_1|\Psi_1\rangle\right|^2\left|\langle x_2|\Psi_2\rangle\right|^2$$

Product of individual probabilities $\left|\langle x_1|\Psi_1\rangle\right|^2$ and $\left|\langle x_2|\Psi_2\rangle\right|^2$ respects standard Bayesian probability theory.

Note common shorthand *big-bra-big-ket* notation $\langle x_1,x_2|\Psi_1,\Psi_2\rangle = \langle x_2|\langle x_1||\Psi_1\rangle|\Psi_2\rangle$

Must ask a perennial modern question: *"How are these structures stored in a computer program?"*
The usual answer is in *outer product* or *tensor arrays*. Next pages show sketches of these objects.

Symmetry group $\mathscr{G}$=U(2) representations, 2D HO Hamiltonian $\mathbf{H}=\hbar\omega_{ab}\mathbf{a}_a^\dagger\mathbf{a}_b$ operators, 2D HO wave eigenfunctions $\Psi_{n,m}$, and coherent [α] states

Factoring 2D-HO Hamiltonian $\mathbf{H}=\frac{A}{2}\left(\mathbf{p}_1^2+\mathbf{x}_1^2\right)+B\left(\mathbf{x}_1\mathbf{x}_2+\mathbf{p}_1\mathbf{p}_2\right)+C\left(\mathbf{x}_1\mathbf{p}_2-\mathbf{x}_2\mathbf{p}_1\right)+\frac{D}{2}\left(\mathbf{p}_2^2+\mathbf{x}_2^2\right)$

2D-Oscillator basic states and operations
- *Commutation relations*
- *Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry*
  - *Anti-commutation relations*
- *Two-dimensional (or 2-particle) base states: ket-kets and bra-bras*
  - *Outer product arrays*
  - *Entangled 2-particle states*

Two-particle (or 2-dimensional) matrix operators
- *U(2) Hamiltonian and irreducible representations*
- *2D-Oscillator states and related 3D angular momentum multiplets*

R(3) Angular momentum generators by U(2) analysis

Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$

SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Mostly Notation and Bookkeeping :

Outer product arrays

Start with an elementary ket basis for each dimension or particle type-1 and type-2.

$Type-1$ $\qquad$ $Type-2$ $\qquad$ $\cdots$

$$|0_1\rangle=\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix},|1_1\rangle=\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix},|2_1\rangle=\begin{pmatrix}0\\0\\1\\\vdots\end{pmatrix},\cdots \qquad |0_2\rangle=\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix},|1_2\rangle=\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix},|2_2\rangle=\begin{pmatrix}0\\0\\1\\\vdots\end{pmatrix},\cdots$$

Outer product arrays

Start with an elementary ket basis for each dimension or particle type-1 and type-2.

$$\begin{array}{lll} Type-1 & Type-2 & \cdots \end{array}$$

$$|0_1\rangle=\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix},\ |1_1\rangle=\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix},\ |2_1\rangle=\begin{pmatrix}0\\0\\1\\\vdots\end{pmatrix},\ \cdots \qquad |0_2\rangle=\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix},\ |1_2\rangle=\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix},\ |2_2\rangle=\begin{pmatrix}0\\0\\1\\\vdots\end{pmatrix},\ \cdots$$

Outer products are constructed for the states that might have non-negligible amplitudes.

$$|0_1\rangle|0_2\rangle=\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}=\left(\begin{array}{c}1\\0\\0\\\vdots\\\hline 0\\0\\0\\\vdots\\\hline 0\\0\\0\\\vdots\end{array}\right),\ |0_1\rangle|1_2\rangle=\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix}=\left(\begin{array}{c}0\\1\\0\\\vdots\\\hline 0\\0\\0\\\vdots\\\hline 0\\0\\0\\\vdots\end{array}\right),\ \cdots|1_1\rangle|0_2\rangle=\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix}\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}=\left(\begin{array}{c}0\\0\\0\\\vdots\\\hline 1\\0\\0\\\vdots\\\hline 0\\0\\0\\\vdots\end{array}\right),\ \cdots|1_1\rangle|2_2\rangle=\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix}\begin{pmatrix}0\\0\\1\\\vdots\end{pmatrix}=\left(\begin{array}{c}0\\0\\0\\\vdots\\\hline 0\\0\\1\\\vdots\\\hline 0\\0\\0\\\vdots\end{array}\right)$$

$$=\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix}\otimes\begin{pmatrix}0\\0\\1\\\vdots\end{pmatrix},$$

Kronecker outer ($\otimes$) product notation

Outer product arrays

Start with an elementary ket basis for each dimension or particle type-1 and type-2.

$$Type-1 \qquad\qquad Type-2 \qquad\qquad \cdots$$

$$\left|0_1\right\rangle=\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix},\left|1_1\right\rangle=\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix},\left|2_1\right\rangle=\begin{pmatrix}0\\0\\1\\\vdots\end{pmatrix},\cdots \qquad \left|0_2\right\rangle=\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix},\left|1_2\right\rangle=\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix},\left|2_2\right\rangle=\begin{pmatrix}0\\0\\1\\\vdots\end{pmatrix},\cdots$$

Outer products are constructed for the states that might have non-negligible amplitudes.

$$\left|0_1\right\rangle\left|0_2\right\rangle=\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}=\begin{pmatrix}1\\0\\0\\\vdots\\\hline 0\\0\\0\\\vdots\\\hline 0\\0\\0\\\vdots\end{pmatrix},\left|0_1\right\rangle\left|1_2\right\rangle=\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix}=\begin{pmatrix}0\\1\\0\\\vdots\\\hline 0\\0\\0\\\vdots\\\hline 0\\0\\0\\\vdots\end{pmatrix},\cdots\left|1_1\right\rangle\left|0_2\right\rangle=\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix}\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}=\begin{pmatrix}0\\0\\0\\\vdots\\\hline 1\\0\\0\\\vdots\\\hline 0\\0\\0\\\vdots\end{pmatrix},\cdots\left|1_1\right\rangle\left|2_2\right\rangle=\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix}\begin{pmatrix}0\\0\\1\\\vdots\end{pmatrix}=\begin{pmatrix}0\\0\\0\\\vdots\\\hline 0\\0\\1\\\vdots\\\hline 0\\0\\0\\\vdots\end{pmatrix},$$

Herein lies conflict between standard ∞-D analysis and finite computers

Outer product arrays

Start with an elementary ket basis for each dimension or particle type-1 and type-2.

$$\textit{Type}-1 \qquad\qquad\qquad\qquad \textit{Type}-2 \qquad\qquad\qquad\qquad \cdots$$

$$|0_1\rangle = \begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix},\ |1_1\rangle = \begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix},\ |2_1\rangle = \begin{pmatrix}0\\0\\1\\\vdots\end{pmatrix},\ \cdots \qquad |0_2\rangle = \begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix},\ |1_2\rangle = \begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix},\ |2_2\rangle = \begin{pmatrix}0\\0\\1\\\vdots\end{pmatrix},\ \cdots$$

Outer products are constructed for the states that might have non-negligible amplitudes.

$$|0_1\rangle|0_2\rangle = \begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix} = \begin{pmatrix}1\\0\\0\\\vdots\\\overline{0}\\0\\0\\\vdots\\\overline{0}\\0\\0\\\vdots\end{pmatrix},\ |0_1\rangle|1_2\rangle = \begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix} = \begin{pmatrix}0\\1\\0\\\vdots\\\overline{0}\\0\\0\\\vdots\\\overline{0}\\0\\0\\\vdots\end{pmatrix},\ \cdots|1_1\rangle|0_2\rangle = \begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix}\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix} = \begin{pmatrix}0\\0\\0\\\vdots\\\overline{1}\\0\\0\\\vdots\\\overline{0}\\0\\0\\\vdots\end{pmatrix},\ \cdots|1_1\rangle|2_2\rangle = \begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix}\begin{pmatrix}0\\0\\1\\\vdots\end{pmatrix} = \begin{pmatrix}0\\0\\0\\\vdots\\\overline{0}\\0\\1\\\vdots\\\overline{0}\\0\\0\\\vdots\end{pmatrix},$$

Herein lies conflict between standard ∞-D analysis and finite computers

Make adjustable-size finite phasor arrays for each particle/dimension.

Outer product arrays

Start with an elementary ket basis for each dimension or particle type-1 and type-2.

$$\begin{array}{ll} Type-1 & Type-2 \quad \cdots \\ \left|0_1\right\rangle=\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix},\left|1_1\right\rangle=\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix},\left|2_1\right\rangle=\begin{pmatrix}0\\0\\1\\\vdots\end{pmatrix},\cdots & \left|0_2\right\rangle=\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix},\left|1_2\right\rangle=\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix},\left|2_2\right\rangle=\begin{pmatrix}0\\0\\1\\\vdots\end{pmatrix},\cdots \end{array}$$

Outer products are constructed for the states that might have non-negligible amplitudes.

$$\left|0_1\right\rangle\left|0_2\right\rangle=\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}=\begin{pmatrix}1\\0\\0\\\vdots\\\overline{0}\\0\\0\\\vdots\\\overline{0}\\0\\0\\\vdots\end{pmatrix},\left|0_1\right\rangle\left|1_2\right\rangle=\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix}=\begin{pmatrix}0\\1\\0\\\vdots\\\overline{0}\\0\\0\\\vdots\\\overline{0}\\0\\0\\\vdots\end{pmatrix},\cdots\left|1_1\right\rangle\left|0_2\right\rangle=\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix}\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}=\begin{pmatrix}0\\0\\0\\\vdots\\\overline{1}\\0\\0\\\vdots\\\overline{0}\\0\\0\\\vdots\end{pmatrix},\cdots\left|1_1\right\rangle\left|2_2\right\rangle=\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix}\begin{pmatrix}0\\0\\1\\\vdots\end{pmatrix}=\begin{pmatrix}0\\0\\0\\\vdots\\\overline{0}\\0\\1\\\vdots\\\overline{0}\\0\\0\\\vdots\end{pmatrix},$$

Herein lies conflict between standard ∞-D analysis and finite computers

Make adjustable-size finite phasor arrays for each particle/dimension.

Convergence is achieved by orderly upgrades in the number of phasors to a point where results do not change.

A 2-wave state product has a lexicographic *(00, 01, 02, ...10, 11, 12,..., 20, 21, 22, ..)* array indexing.

Outer product arrays

Start with an elementary ket basis for each dimension or particle type-1 and type-2.

$$\textit{Type}-1 \qquad\qquad \textit{Type}-2 \qquad\qquad \cdots$$

$$|0_1\rangle = \begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}, |1_1\rangle = \begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix}, |2_1\rangle = \begin{pmatrix}0\\0\\1\\\vdots\end{pmatrix}, \cdots \qquad |0_2\rangle = \begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}, |1_2\rangle = \begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix}, |2_2\rangle = \begin{pmatrix}0\\0\\1\\\vdots\end{pmatrix}, \cdots$$

Outer products are constructed for the states that might have non-negligible amplitudes.

$$|0_1\rangle|0_2\rangle = \begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix} = \begin{pmatrix}1\\0\\0\\\vdots\\\hline 0\\0\\0\\\vdots\\\hline 0\\0\\0\\\vdots\end{pmatrix}, |0_1\rangle|1_2\rangle = \begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix} = \begin{pmatrix}0\\1\\0\\\vdots\\\hline 0\\0\\0\\\vdots\\\hline 0\\0\\0\\\vdots\end{pmatrix}, \cdots |1_1\rangle|0_2\rangle = \begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix}\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix} = \begin{pmatrix}0\\0\\0\\\vdots\\\hline 1\\0\\0\\\vdots\\\hline 0\\0\\0\\\vdots\end{pmatrix}, \cdots |1_1\rangle|2_2\rangle = \begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix}\begin{pmatrix}0\\0\\1\\\vdots\end{pmatrix} = \begin{pmatrix}0\\0\\0\\\vdots\\\hline 0\\0\\1\\\vdots\\\hline 0\\0\\0\\\vdots\end{pmatrix},$$

Herein lies conflict between standard ∞-D analysis and finite computers

Make adjustable-size finite phasor arrays for each particle/dimension.

Convergence is achieved by orderly upgrades in the number of phasors to a point where results do not change.

A 2-wave state product has a lexicographic *(00, 01, 02, ...10, 11, 12,..., 20, 21, 22, ..)* array indexing.

"Little-Endian" indexing *(...01,02,03..10,11,12,13 ... 20,21,22,23,...)*

Least significant digit at (right) END

$$|\Psi_1\rangle|\Psi_2\rangle = \begin{pmatrix}\langle 0|\Psi_1\rangle\\\langle 1|\Psi_1\rangle\\\langle 2|\Psi_1\rangle\\\vdots\end{pmatrix} \otimes \begin{pmatrix}\langle 0|\Psi_2\rangle\\\langle 1|\Psi_2\rangle\\\langle 2|\Psi_2\rangle\\\vdots\end{pmatrix} = \begin{pmatrix}\langle 0|\Psi_1\rangle\langle 0|\Psi_2\rangle\\\langle 0|\Psi_1\rangle\langle 1|\Psi_2\rangle\\\langle 0|\Psi_1\rangle\langle 2|\Psi_2\rangle\\\vdots\\\hline \langle 1|\Psi_1\rangle\langle 0|\Psi_2\rangle\\\langle 1|\Psi_1\rangle\langle 1|\Psi_2\rangle\\\langle 1|\Psi_1\rangle\langle 2|\Psi_2\rangle\\\vdots\\\hline \langle 2|\Psi_1\rangle\langle 0|\Psi_2\rangle\\\langle 2|\Psi_1\rangle\langle 1|\Psi_2\rangle\\\langle 2|\Psi_1\rangle\langle 2|\Psi_2\rangle\\\vdots\end{pmatrix} = \begin{pmatrix}\langle 0_1 0_2|\Psi_1\Psi_2\rangle\\\langle 0_1 1_2|\Psi_1\Psi_2\rangle\\\langle 0_1 2_2|\Psi_1\Psi_2\rangle\\\vdots\\\hline \langle 1_1 0_2|\Psi_1\Psi_2\rangle\\\langle 1_1 1_2|\Psi_1\Psi_2\rangle\\\langle 1_1 2_2|\Psi_1\Psi_2\rangle\\\vdots\\\hline \langle 2_1 0_2|\Psi_1\Psi_2\rangle\\\langle 2_1 1_2|\Psi_1\Psi_2\rangle\\\langle 2_1 2_2|\Psi_1\Psi_2\rangle\\\vdots\end{pmatrix}$$

Outer product arrays

Start with an elementary ket basis for each dimension or particle type-1 and type-2.

$$Type-1 \qquad\qquad Type-2 \qquad \cdots$$

$$\left|0_1\right\rangle=\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix},\left|1_1\right\rangle=\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix},\left|2_1\right\rangle=\begin{pmatrix}0\\0\\1\\\vdots\end{pmatrix},\cdots \quad \left|0_2\right\rangle=\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix},\left|1_2\right\rangle=\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix},\left|2_2\right\rangle=\begin{pmatrix}0\\0\\1\\\vdots\end{pmatrix},\cdots$$

Outer products are constructed for the states that might have non-negligible amplitudes.

$$\left|0_1\right\rangle\left|0_2\right\rangle=\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}=\begin{pmatrix}1\\0\\0\\\vdots\\\hline 0\\0\\0\\\vdots\\\hline 0\\0\\0\\\vdots\end{pmatrix},\left|0_1\right\rangle\left|1_2\right\rangle=\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix}=\begin{pmatrix}0\\1\\0\\\vdots\\\hline 0\\0\\0\\\vdots\\\hline 0\\0\\0\\\vdots\end{pmatrix},\cdots\left|1_1\right\rangle\left|0_2\right\rangle=\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix}\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}=\begin{pmatrix}0\\0\\0\\\vdots\\\hline 1\\0\\0\\\vdots\\\hline 0\\0\\0\\\vdots\end{pmatrix},\cdots\left|1_1\right\rangle\left|2_2\right\rangle=\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix}\begin{pmatrix}0\\0\\1\\\vdots\end{pmatrix}=\begin{pmatrix}0\\0\\0\\\vdots\\\hline 0\\0\\1\\\vdots\\\hline 0\\0\\0\\\vdots\end{pmatrix},$$

Herein lies conflict between standard ∞-D analysis and finite computers

Make adjustable-size finite phasor arrays for each particle/dimension.

Convergence is achieved by orderly upgrades in the number of phasors to a point where results do not change.

A 2-wave state product has a lexicographic *(00, 01, 02, ...10, 11, 12,..., 20, 21, 22, ..)* array indexing.

"Little-Endian" indexing *(...01,02,03..10,11,12,13 ... 20,21,22,23,...)*

Least significant digit at (right) END

$$\left|\Psi_1\right\rangle\left|\Psi_2\right\rangle=\begin{pmatrix}\langle 0|\Psi_1\rangle\\\langle 1|\Psi_1\rangle\\\langle 2|\Psi_1\rangle\\\vdots\end{pmatrix}\otimes\begin{pmatrix}\langle 0|\Psi_2\rangle\\\langle 1|\Psi_2\rangle\\\langle 2|\Psi_2\rangle\\\vdots\end{pmatrix}=\begin{pmatrix}\langle 0|\Psi_1\rangle\langle 0|\Psi_2\rangle\\\langle 0|\Psi_1\rangle\langle 1|\Psi_2\rangle\\\langle 0|\Psi_1\rangle\langle 2|\Psi_2\rangle\\\vdots\\\hline\langle 1|\Psi_1\rangle\langle 0|\Psi_2\rangle\\\langle 1|\Psi_1\rangle\langle 1|\Psi_2\rangle\\\langle 1|\Psi_1\rangle\langle 2|\Psi_2\rangle\\\vdots\\\hline\langle 2|\Psi_1\rangle\langle 0|\Psi_2\rangle\\\langle 2|\Psi_1\rangle\langle 1|\Psi_2\rangle\\\langle 2|\Psi_1\rangle\langle 2|\Psi_2\rangle\\\vdots\end{pmatrix}=\begin{pmatrix}\langle 0_10_2|\Psi_1\Psi_2\rangle\\\langle 0_11_2|\Psi_1\Psi_2\rangle\\\langle 0_12_2|\Psi_1\Psi_2\rangle\\\vdots\\\hline\langle 1_10_2|\Psi_1\Psi_2\rangle\\\langle 1_11_2|\Psi_1\Psi_2\rangle\\\langle 1_12_2|\Psi_1\Psi_2\rangle\\\vdots\\\hline\langle 2_10_2|\Psi_1\Psi_2\rangle\\\langle 2_11_2|\Psi_1\Psi_2\rangle\\\langle 2_12_2|\Psi_1\Psi_2\rangle\\\vdots\end{pmatrix}$$

or anti-lexicographic *(00, 10, 20, ...01, 11, 21,..., 02, 12, 22, ..)* array indexing

"Big-Endian" indexing *(...00,10,20..01,11,21,31 ... 02,12,22,32...)*

Most significant digit at (right) END

Outer product arrays

Start with an elementary ket basis for each dimension or particle type-1 and type-2.

$$\begin{array}{ll} Type-1 & Type-2 \quad \cdots \\ \left|0_1\right\rangle=\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}, \left|1_1\right\rangle=\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix}, \left|2_1\right\rangle=\begin{pmatrix}0\\0\\1\\\vdots\end{pmatrix}, \cdots & \left|0_2\right\rangle=\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}, \left|1_2\right\rangle=\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix}, \left|2_2\right\rangle=\begin{pmatrix}0\\0\\1\\\vdots\end{pmatrix}, \cdots \end{array}$$

Outer products are constructed for the states that might have non-negligible amplitudes.

$$\left|0_1\right\rangle\left|0_2\right\rangle=\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}=\begin{pmatrix}1\\0\\0\\\vdots\\\hline 0\\0\\0\\\vdots\\\hline 0\\0\\0\\\vdots\end{pmatrix}, \left|0_1\right\rangle\left|1_2\right\rangle=\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix}=\begin{pmatrix}0\\1\\0\\\vdots\\\hline 0\\0\\0\\\vdots\\\hline 0\\0\\0\\\vdots\end{pmatrix}, \cdots\left|1_1\right\rangle\left|0_2\right\rangle=\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix}\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}=\begin{pmatrix}0\\0\\0\\\vdots\\\hline 1\\0\\0\\\vdots\\\hline 0\\0\\0\\\vdots\end{pmatrix}, \cdots\left|1_1\right\rangle\left|2_2\right\rangle=\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix}\begin{pmatrix}0\\0\\1\\\vdots\end{pmatrix}=\begin{pmatrix}0\\0\\0\\\vdots\\\hline 0\\0\\1\\\vdots\\\hline 0\\0\\0\\\vdots\end{pmatrix},$$

Herein lies conflict between standard ∞-D analysis and finite computers

Make adjustable-size finite phasor arrays for each particle/dimension.

Convergence is achieved by orderly upgrades in the number of phasors to a point where results do not change.

A 2-wave state product has a lexicographic *(00, 01, 02, ...10, 11, 12,..., 20, 21, 22, ..)* array indexing.

$$\left|\Psi_1\right\rangle\left|\Psi_2\right\rangle=\begin{pmatrix}\langle 0|\Psi_1\rangle\\\langle 1|\Psi_1\rangle\\\langle 2|\Psi_1\rangle\\\vdots\end{pmatrix}\otimes\begin{pmatrix}\langle 0|\Psi_2\rangle\\\langle 1|\Psi_2\rangle\\\langle 2|\Psi_2\rangle\\\vdots\end{pmatrix}=\begin{pmatrix}\langle 0|\Psi_1\rangle\langle 0|\Psi_2\rangle\\\langle 0|\Psi_1\rangle\langle 1|\Psi_2\rangle\\\langle 0|\Psi_1\rangle\langle 2|\Psi_2\rangle\\\vdots\\\hline\langle 1|\Psi_1\rangle\langle 0|\Psi_2\rangle\\\langle 1|\Psi_1\rangle\langle 1|\Psi_2\rangle\\\langle 1|\Psi_1\rangle\langle 2|\Psi_2\rangle\\\vdots\\\hline\langle 2|\Psi_1\rangle\langle 0|\Psi_2\rangle\\\langle 2|\Psi_1\rangle\langle 1|\Psi_2\rangle\\\langle 2|\Psi_1\rangle\langle 2|\Psi_2\rangle\\\vdots\end{pmatrix}=\begin{pmatrix}\langle 0_10_2|\Psi_1\Psi_2\rangle\\\langle 0_11_2|\Psi_1\Psi_2\rangle\\\langle 0_12_2|\Psi_1\Psi_2\rangle\\\vdots\\\hline\langle 1_10_2|\Psi_1\Psi_2\rangle\\\langle 1_11_2|\Psi_1\Psi_2\rangle\\\langle 1_12_2|\Psi_1\Psi_2\rangle\\\vdots\\\hline\langle 2_10_2|\Psi_1\Psi_2\rangle\\\langle 2_11_2|\Psi_1\Psi_2\rangle\\\langle 2_12_2|\Psi_1\Psi_2\rangle\\\vdots\end{pmatrix}$$

shorthand *big-bra-big-ket* notation

$$|\Psi\rangle=\begin{pmatrix}\langle 0_10_2|\Psi\rangle\\\langle 0_11_2|\Psi\rangle\\\langle 0_12_2|\Psi\rangle\\\vdots\\\hline\langle 1_10_2|\Psi\rangle\\\langle 1_11_2|\Psi\rangle\\\langle 1_12_2|\Psi\rangle\\\vdots\\\hline\langle 2_10_2|\Psi\rangle\\\langle 2_11_2|\Psi\rangle\\\langle 2_12_2|\Psi\rangle\\\vdots\end{pmatrix}=\begin{pmatrix}\Psi_{00}\\\Psi_{01}\\\Psi_{02}\\\vdots\\\hline\Psi_{10}\\\Psi_{11}\\\Psi_{12}\\\vdots\\\hline\Psi_{20}\\\Psi_{21}\\\Psi_{22}\\\vdots\end{pmatrix}$$

"Little-Endian" indexing *(...01,02,03..10,11,12,13 ... 20,21,22,23,...)*

Least significant digit at (right) END

Symmetry group $\mathscr{G}$=U(2) representations, 2D HO Hamiltonian $\mathbf{H}=\hbar\omega_{ab}\mathbf{a}_a{}^\dagger\mathbf{a}_b$ operators, 2D HO wave eigenfunctions $\Psi_{n,m}$, and coherent [α] states

Factoring 2D-HO Hamiltonian $\mathbf{H}=\frac{A}{2}\left(\mathbf{p}_1^2+\mathbf{x}_1^2\right)+B\left(\mathbf{x}_1\mathbf{x}_2+\mathbf{p}_1\mathbf{p}_2\right)+C\left(\mathbf{x}_1\mathbf{p}_2-\mathbf{x}_2\mathbf{p}_1\right)+\frac{D}{2}\left(\mathbf{p}_2^2+\mathbf{x}_2^2\right)$

2D-Oscillator basic states and operations
- *Commutation relations*
- *Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry*
  - *Anti-commutation relations*
- *Two-dimensional (or 2-particle) base states: ket-kets and bra-bras*
  - *Outer product arrays*
  - *Entangled 2-particle states*

Two-particle (or 2-dimensional) matrix operators
- *U(2) Hamiltonian and irreducible representations*
- *2D-Oscillator states and related 3D angular momentum multiplets*

R(3) Angular momentum generators by U(2) analysis

Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$

SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Mostly Notation and Bookkeeping :

Entangled 2-particle states (Analogy with matrix array)

A matrix operator **M** is rarely a single nilpotent operator $|1\rangle\langle 2|$ or idempotent $|1\rangle\langle 1|$.

Entangled 2-particle states (Analogy with matrix array)

A matrix operator **M** is rarely a single nilpotent operator $|1\rangle\langle 2|$ or idempotent $|1\rangle\langle 1|$.

A two-particle state $|\Psi\rangle$ is rarely a single outer product $|\Psi_1\rangle|\Psi_2\rangle$ of 1-particle states $|\Psi_1\rangle$ and $|\Psi_2\rangle$.
(Even rarer is $|\Psi_1\rangle|\Psi_1\rangle$.)

Entangled 2-particle states (Analogy with matrix array)

A matrix operator **M** is rarely a single nilpotent operator $|1\rangle\langle 2|$ or idempotent $|1\rangle\langle 1|$.

A two-particle state $|\Psi\rangle$ is rarely a single outer product $|\Psi_1\rangle|\Psi_2\rangle$ of 1-particle states $|\Psi_1\rangle$ and $|\Psi_2\rangle$.
(Even rarer is $|\Psi_1\rangle|\Psi_1\rangle$.)

ANALOGY:

A general *n-by-n* matrix **M** operator is a combination of n^2 terms: $\mathbf{M} = \sum_{j=1}^{n} \sum_{k=1}^{n} M_{j,k} |j\rangle\langle k|$

...that *might* be diagonalized to a combination of n projectors: $\mathbf{M} = \sum_{e=1}^{n} \mu_e |e\rangle\langle e|$

Entangled 2-particle states (Analogy with matrix array)

A matrix operator **M** is rarely a single nilpotent operator $|1\rangle\langle 2|$ or idempotent $|1\rangle\langle 1|$.

A two-particle state $|\Psi\rangle$ is rarely a single outer product $|\Psi_1\rangle|\Psi_2\rangle$ of 1-particle states $|\Psi_1\rangle$ and $|\Psi_2\rangle$.

(Even rarer is $|\Psi_1\rangle|\Psi_1\rangle$.)

ANALOGY:

A general *n-by-n* matrix **M** operator is a combination of n^2 terms: $\mathbf{M} = \sum_{j=1}^{n} \sum_{k=1}^{n} M_{j,k} |j\rangle\langle k|$

...that *might* be diagonalized to a combination of n projectors: $\mathbf{M} = \sum_{e=1}^{n} \mu_e |e\rangle\langle e|$

So a general two-particle state $|\Psi\rangle$ is a combination of *entangled* products: $|\Psi\rangle = \sum_j \sum_k \psi_{j,k} |\Psi_j\rangle |\Psi_k\rangle$

Entangled 2-particle states (Analogy with matrix array)

A matrix operator **M** is rarely a single nilpotent operator $|1\rangle\langle 2|$ or idempotent $|1\rangle\langle 1|$.

A two-particle state $|\Psi\rangle$ is rarely a single outer product $|\Psi_1\rangle|\Psi_2\rangle$ of 1-particle states $|\Psi_1\rangle$ and $|\Psi_2\rangle$.
(Even rarer is $|\Psi_1\rangle|\Psi_1\rangle$.)

ANALOGY:

A general *n-by-n* matrix **M** operator is a combination of n^2 terms: $\mathbf{M} = \sum_{j=1}^{n}\sum_{k=1}^{n} M_{j,k}|j\rangle\langle k|$

...that *might* be diagonalized to a combination of n projectors: $\mathbf{M} = \sum_{e=1}^{n} \mu_e |e\rangle\langle e|$

So a general two-particle state $|\Psi\rangle$ is a combination of *entangled* products: $|\Psi\rangle = \sum_j \sum_k \psi_{j,k} |\Psi_j\rangle|\Psi_k\rangle$

...that *might* be *de-entangled* to a combination of n terms: $|\Psi\rangle = \sum_e \phi_e |\varphi_e\rangle|\varphi_e\rangle$

Symmetry group $\mathscr{G}$=U(2) representations, 2D HO Hamiltonian $\mathbf{H}=\hbar\omega_{ab}\mathbf{a}_a{}^\dagger\mathbf{a}_b$ operators,
2D HO wave eigenfunctions $\Psi_{n,m}$, and coherent [α] states

Factoring 2D-HO Hamiltonian $\mathbf{H}=\frac{A}{2}\left(\mathbf{p}_1^2+\mathbf{x}_1^2\right)+B\left(\mathbf{x}_1\mathbf{x}_2+\mathbf{p}_1\mathbf{p}_2\right)+C\left(\mathbf{x}_1\mathbf{p}_2-\mathbf{x}_2\mathbf{p}_1\right)+\frac{D}{2}\left(\mathbf{p}_2^2+\mathbf{x}_2^2\right)$

2D-Oscillator basic states and operations
Commutation relations
Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry
Anti-commutation relations
Two-dimensional (or 2-particle) base states: ket-kets and bra-bras
Outer product arrays
Entangled 2-particle states
Two-particle (or 2-dimensional) matrix operators
U(2) Hamiltonian and irreducible representations
2D-Oscillator states and related 3D angular momentum multiplets
R(3) Angular momentum generators by U(2) analysis
Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$
SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Mostly Notation and Bookkeeping :

Two-particle (or 2-dimensional) matrix operators

When 2-particle operator $\mathbf{a}_k$ acts on a 2-particle state, $\mathbf{a}_k$ "finds" its type-k state but ignores the others.

$$\mathbf{a}_1^\dagger|n_1 n_2\rangle = \mathbf{a}_1^\dagger|n_1\rangle|n_2\rangle = \sqrt{n_1+1}|n_1+1\ n_2\rangle \qquad \mathbf{a}_2^\dagger|n_1 n_2\rangle = |n_1\rangle\mathbf{a}_2^\dagger|n_2\rangle = \sqrt{n_2+1}|n_1\ n_2+1\rangle$$

$$\mathbf{a}_1|n_1 n_2\rangle = \mathbf{a}_1|n_1\rangle|n_2\rangle = \sqrt{n_1}|n_1-1\ n_2\rangle \qquad \mathbf{a}_2|n_1 n_2\rangle = |n_1\rangle\mathbf{a}_2|n_2\rangle = \sqrt{n_2}|n_1\ n_2-1\rangle$$

$\mathbf{a}_1$"finds" its type-1 $\qquad$ $\mathbf{a}_2$"finds" its type-2

Two-particle (or 2-dimensional) matrix operators

When 2-particle operator $\mathbf{a}_k$ acts on a 2-particle state, $\mathbf{a}_k$ "finds" its type-k state but ignores the others.

$$\mathbf{a}_1^\dagger |n_1 n_2\rangle = \mathbf{a}_1^\dagger |n_1\rangle |n_2\rangle = \sqrt{n_1+1}\,|n_1+1\; n_2\rangle \qquad \mathbf{a}_2^\dagger |n_1 n_2\rangle = |n_1\rangle \mathbf{a}_2^\dagger |n_2\rangle = \sqrt{n_2+1}\,|n_1\; n_2+1\rangle$$

$$\mathbf{a}_1 |n_1 n_2\rangle = \mathbf{a}_1 |n_1\rangle |n_2\rangle = \sqrt{n_1}\,|n_1-1\; n_2\rangle \qquad \mathbf{a}_2 |n_1 n_2\rangle = |n_1\rangle \mathbf{a}_2 |n_2\rangle = \sqrt{n_2}\,|n_1\; n_2-1\rangle$$

$\mathbf{a}_1$"finds" its type-1 $\qquad$ $\mathbf{a}_2$"finds" its type-2

General definition of the 2D oscillator base state.

$$|n_1 n_2\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{n_1}\left(\mathbf{a}_2^\dagger\right)^{n_2}}{\sqrt{n_1!\,n_2!}}|0\,0\rangle$$

Two-particle (or 2-dimensional) matrix operators

When 2-particle operator $\mathbf{a}_k$ acts on a 2-particle state, $\mathbf{a}_k$ "finds" its type-k state but ignores the others.

$$\mathbf{a}_1^\dagger|n_1 n_2\rangle = \mathbf{a}_1^\dagger|n_1\rangle|n_2\rangle = \sqrt{n_1+1}\,|n_1+1\ n_2\rangle \qquad \mathbf{a}_2^\dagger|n_1 n_2\rangle = |n_1\rangle\mathbf{a}_2^\dagger|n_2\rangle = \sqrt{n_2+1}\,|n_1\ n_2+1\rangle$$

$$\mathbf{a}_1|n_1 n_2\rangle = \mathbf{a}_1|n_1\rangle|n_2\rangle = \sqrt{n_1}\,|n_1-1\ n_2\rangle \qquad \mathbf{a}_2|n_1 n_2\rangle = |n_1\rangle\mathbf{a}_2|n_2\rangle = \sqrt{n_2}\,|n_1\ n_2-1\rangle$$

$\mathbf{a}_1$"finds" its type-1 $\qquad$ $\mathbf{a}_2$"finds" its type-2

The *ABCD* matrix from Class 4

$$\mathbf{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$$

General definition of the 2D oscillator base state.

$$|n_1 n_2\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{n_1}\left(\mathbf{a}_2^\dagger\right)^{n_2}}{\sqrt{n_1!\,n_2!}}|0\,0\rangle$$

$$\mathbf{H} = H_{11}\left(\mathbf{a}_1^\dagger\mathbf{a}_1 + \mathbf{1}/2\right) + H_{12}\mathbf{a}_1^\dagger\mathbf{a}_2 + H_{21}\mathbf{a}_2^\dagger\mathbf{a}_1 + H_{22}\left(\mathbf{a}_2^\dagger\mathbf{a}_2 + \mathbf{1}/2\right)$$

The $\mathbf{a}_m{}^\dagger\mathbf{a}_n$ combinations in the *ABCD* Hamiltonian **H** have fairly simple matrix elements.

$$\mathbf{H} = A\left(\mathbf{a}_1^\dagger\mathbf{a}_1 + \mathbf{1}/2\right) + (B - iC)\mathbf{a}_1^\dagger\mathbf{a}_2 + (B + iC)\mathbf{a}_2^\dagger\mathbf{a}_1 + D\left(\mathbf{a}_2^\dagger\mathbf{a}_2 + \mathbf{1}/2\right)$$

Two-particle (or 2-dimensional) matrix operators

When 2-particle operator $\mathbf{a}_k$ acts on a 2-particle state, $\mathbf{a}_k$ "finds" its type-k state but ignores the others.

$$\mathbf{a}_1^\dagger|n_1 n_2\rangle = \mathbf{a}_1^\dagger|n_1\rangle|n_2\rangle = \sqrt{n_1+1}|n_1+1\, n_2\rangle$$

$$\mathbf{a}_1|n_1 n_2\rangle = \mathbf{a}_1|n_1\rangle|n_2\rangle = \sqrt{n_1}|n_1-1\, n_2\rangle$$

$\mathbf{a}_1$"finds" its type-1

$$\mathbf{a}_2^\dagger|n_1 n_2\rangle = |n_1\rangle\mathbf{a}_2^\dagger|n_2\rangle = \sqrt{n_2+1}|n_1\, n_2+1\rangle$$

$$\mathbf{a}_2|n_1 n_2\rangle = |n_1\rangle\mathbf{a}_2|n_2\rangle = \sqrt{n_2}|n_1\, n_2-1\rangle$$

$\mathbf{a}_2$"finds" its type-2

The *ABCD* matrix from Class 4

$$\mathbf{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$$

General definition of the 2D oscillator base state.

$$|n_1 n_2\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{n_1}\left(\mathbf{a}_2^\dagger\right)^{n_2}}{\sqrt{n_1!\, n_2!}}|0\,0\rangle$$

$$\mathbf{H} = H_{11}\left(\mathbf{a}_1^\dagger\mathbf{a}_1 + \mathbf{1}/2\right) + H_{12}\mathbf{a}_1^\dagger\mathbf{a}_2 + H_{21}\mathbf{a}_2^\dagger\mathbf{a}_1 + H_{22}\left(\mathbf{a}_2^\dagger\mathbf{a}_2 + \mathbf{1}/2\right)$$

The $\mathbf{a}_m{}^\dagger\mathbf{a}_n$ combinations in the *ABCD* Hamiltonian $\mathbf{H}$ have fairly simple matrix elements.

$$\mathbf{a}_1^\dagger\mathbf{a}_1|n_1 n_2\rangle = n_1|n_1\, n_2\rangle$$

$$\mathbf{a}_1^\dagger\mathbf{a}_2|n_1 n_2\rangle = \sqrt{n_1+1}\sqrt{n_2}|n_1+1\, n_2-1\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_1|n_1 n_2\rangle = \sqrt{n_1}\sqrt{n_2+1}|n_1-1\, n_2+1\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_2|n_1 n_2\rangle = n_2|n_1\, n_2\rangle$$

$$\mathbf{H} = A\left(\mathbf{a}_1^\dagger\mathbf{a}_1 + \mathbf{1}/2\right) + (B-iC)\mathbf{a}_1^\dagger\mathbf{a}_2 + (B+iC)\mathbf{a}_2^\dagger\mathbf{a}_1 + D\left(\mathbf{a}_2^\dagger\mathbf{a}_2 + \mathbf{1}/2\right)$$

Two-particle (or 2-dimensional) matrix operators

When 2-particle operator $\mathbf{a}_k$ acts on a 2-particle state, $\mathbf{a}_k$ "finds" its type-k state but ignores the others.

$$\mathbf{a}_1^\dagger|n_1 n_2\rangle = \mathbf{a}_1^\dagger|n_1\rangle|n_2\rangle = \sqrt{n_1+1}\,|n_1+1\;n_2\rangle \qquad \mathbf{a}_2^\dagger|n_1 n_2\rangle = |n_1\rangle\mathbf{a}_2^\dagger|n_2\rangle = \sqrt{n_2+1}\,|n_1\;n_2+1\rangle$$

$$\mathbf{a}_1|n_1 n_2\rangle = \mathbf{a}_1|n_1\rangle|n_2\rangle = \sqrt{n_1}\,|n_1-1\;n_2\rangle \qquad \mathbf{a}_2|n_1 n_2\rangle = |n_1\rangle\mathbf{a}_2|n_2\rangle = \sqrt{n_2}\,|n_1\;n_2-1\rangle$$

$\mathbf{a}_1$"finds" its type-1 $\qquad$ $\mathbf{a}_2$"finds" its type-2

The *ABCD* matrix from Class 4

$$\mathbf{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$$

General definition of the 2D oscillator base state.

$$|n_1 n_2\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{n_1}\left(\mathbf{a}_2^\dagger\right)^{n_2}}{\sqrt{n_1!\,n_2!}}|0\,0\rangle$$

$$\mathbf{H} = H_{11}\left(\mathbf{a}_1^\dagger\mathbf{a}_1 + \mathbf{1}/2\right) + H_{12}\mathbf{a}_1^\dagger\mathbf{a}_2 + H_{21}\mathbf{a}_2^\dagger\mathbf{a}_1 + H_{22}\left(\mathbf{a}_2^\dagger\mathbf{a}_2 + \mathbf{1}/2\right)$$

The $\mathbf{a}_m{}^\dagger\mathbf{a}_n$ combinations in the *ABCD* Hamiltonian $\mathbf{H}$ have fairly simple matrix elements.

$$\mathbf{a}_1^\dagger\mathbf{a}_1|n_1 n_2\rangle = n_1|n_1\;n_2\rangle \qquad \mathbf{a}_1^\dagger\mathbf{a}_2|n_1 n_2\rangle = \sqrt{n_1+1}\sqrt{n_2}\,|n_1+1\;n_2-1\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_1|n_1 n_2\rangle = \sqrt{n_1}\sqrt{n_2+1}\,|n_1-1\;n_2+1\rangle \qquad \mathbf{a}_2^\dagger\mathbf{a}_2|n_1 n_2\rangle = n_2|n_1\;n_2\rangle$$

$$\mathbf{H} = A\left(\mathbf{a}_1^\dagger\mathbf{a}_1 + \mathbf{1}/2\right) + (B-iC)\mathbf{a}_1^\dagger\mathbf{a}_2 + (B+iC)\mathbf{a}_2^\dagger\mathbf{a}_1 + D\left(\mathbf{a}_2^\dagger\mathbf{a}_2 + \mathbf{1}/2\right)$$

$\langle\mathbf{H}\rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$

	$\|00\rangle$	$\|01\rangle$	$\|02\rangle$	$\cdots$	$\|10\rangle$	$\|11\rangle$	$\|12\rangle$	$\cdots$	$\|20\rangle$	$\|21\rangle$	$\|22\rangle$	$\cdots$
$\langle 00\|$	0			$\cdots$	$\cdot$			$\cdots$				$\cdots$
$\langle 01\|$		D		$\cdots$	$B+iC$	$\cdot$		$\cdots$				
$\langle 02\|$			$2D$	$\cdots$		$\sqrt{2}(B+iC)$	$\cdot$	$\cdots$				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$				$\cdots$
$\langle 10\|$	$\cdot$	$B-iC$		$\cdots$	A			$\cdots$	$\cdot$			$\cdots$
$\langle 11\|$		$\cdot$	$\sqrt{2}(B-iC)$	$\cdots$		$A+D$		$\cdots$	$\sqrt{2}(B+iC)$	$\cdot$		$\cdots$
$\langle 12\|$			$\cdot$	$\cdots$			$A+2D$	$\cdots$		$\sqrt{4}(B+iC)$	$\cdot$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$\langle 20\|$					$\cdot$	$\sqrt{2}(B-iC)$		$\cdots$	$2A$			$\cdots$
$\langle 21\|$						$\cdot$	$\sqrt{4}(B-iC)$	$\cdots$		$2A+D$		$\cdots$
$\langle 22\|$							$\cdot$	$\cdots$			$2A+2D$	$\cdots$
$\vdots$					$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

"Little-Endian" indexing
(...01,02,03..10,11,12,13 ... 20,21,22,23,...)

Two-particle (or 2-dimensional) matrix operators

When 2-particle operator $\mathbf{a}_k$ acts on a 2-particle state, $\mathbf{a}_k$ "finds" its type-k state but ignores the others.

$$\mathbf{a}_1^\dagger|n_1 n_2\rangle = \mathbf{a}_1^\dagger|n_1\rangle|n_2\rangle = \sqrt{n_1+1}\,|n_1+1\; n_2\rangle \qquad \mathbf{a}_2^\dagger|n_1 n_2\rangle = |n_1\rangle\mathbf{a}_2^\dagger|n_2\rangle = \sqrt{n_2+1}\,|n_1\; n_2+1\rangle$$

$$\mathbf{a}_1|n_1 n_2\rangle = \mathbf{a}_1|n_1\rangle|n_2\rangle = \sqrt{n_1}\,|n_1-1\; n_2\rangle \qquad \mathbf{a}_2|n_1 n_2\rangle = |n_1\rangle\mathbf{a}_2|n_2\rangle = \sqrt{n_2}\,|n_1\; n_2-1\rangle$$

$\mathbf{a}_1$"finds" its type-1 $\qquad$ $\mathbf{a}_2$"finds" its type-2

The *ABCD* matrix from Class 4

$$\mathbf{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$$

General definition of the 2D oscillator base state.

$$|n_1 n_2\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{n_1}\left(\mathbf{a}_2^\dagger\right)^{n_2}}{\sqrt{n_1!\,n_2!}}|0\,0\rangle$$

$$\mathbf{H} = H_{11}\left(\mathbf{a}_1^\dagger\mathbf{a}_1 + \mathbf{1}/2\right) + H_{12}\mathbf{a}_1^\dagger\mathbf{a}_2 + H_{21}\mathbf{a}_2^\dagger\mathbf{a}_1 + H_{22}\left(\mathbf{a}_2^\dagger\mathbf{a}_2 + \mathbf{1}/2\right)$$

The $\mathbf{a}_m{}^\dagger\mathbf{a}_n$ combinations in the *ABCD* Hamiltonian $\mathbf{H}$ have fairly simple matrix elements.

$$\mathbf{a}_1^\dagger\mathbf{a}_1|n_1 n_2\rangle = n_1|n_1\; n_2\rangle \qquad \mathbf{a}_1^\dagger\mathbf{a}_2|n_1 n_2\rangle = \sqrt{n_1+1}\sqrt{n_2}\,|n_1+1\; n_2-1\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_1|n_1 n_2\rangle = \sqrt{n_1}\sqrt{n_2+1}\,|n_1-1\; n_2+1\rangle \qquad \mathbf{a}_2^\dagger\mathbf{a}_2|n_1 n_2\rangle = n_2|n_1\; n_2\rangle$$

$$\mathbf{H} = A\left(\mathbf{a}_1^\dagger\mathbf{a}_1 + \mathbf{1}/2\right) + (B-iC)\mathbf{a}_1^\dagger\mathbf{a}_2 + (B+iC)\mathbf{a}_2^\dagger\mathbf{a}_1 + D\left(\mathbf{a}_2^\dagger\mathbf{a}_2 + \mathbf{1}/2\right)$$

$\langle\mathbf{H}\rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$

	$\lvert 00\rangle$	$\lvert 01\rangle$	$\lvert 02\rangle$	$\cdots$	$\lvert 10\rangle$	$\lvert 11\rangle$	$\lvert 12\rangle$	$\cdots$	$\lvert 20\rangle$	$\lvert 21\rangle$	$\lvert 22\rangle$	$\cdots$
$\langle 00\rvert$	0			$\cdots$	$\cdot$			$\cdots$				
$\langle 01\rvert$		D		$\cdots$	$B+iC$	$\cdot$		$\cdots$				
$\langle 02\rvert$			$2D$	$\cdots$		$\sqrt{2}(B+iC)$	$\cdot$	$\cdots$				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$				$\cdots$
$\langle 10\rvert$	$\cdot$	$B-iC$		$\cdots$	A			$\cdots$	$\cdot$			$\cdots$
$\langle 11\rvert$		$\cdot$	$\sqrt{2}(B-iC)$	$\cdots$		$A+D$		$\cdots$	$\sqrt{2}(B+iC)$	$\cdot$		$\cdots$
$\langle 12\rvert$			$\cdot$	$\cdots$			$A+2D$	$\cdots$		$\sqrt{4}(B+iC)$	$\cdot$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$\langle 20\rvert$					$\cdot$	$\sqrt{2}(B-iC)$		$\cdots$	$2A$			$\cdots$
$\langle 21\rvert$						$\cdot$	$\sqrt{4}(B-iC)$	$\cdots$		$2A+D$		$\cdots$
$\langle 22\rvert$							$\cdot$	$\cdots$			$2A+2D$	$\cdots$
$\vdots$					$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

"Little-Endian" indexing
(...01,02,03..10,11,12,13 ... 20,21,22,23,...)

Two-particle (or 2-dimensional) matrix operators

When 2-particle operator $\mathbf{a}_k$ acts on a 2-particle state, $\mathbf{a}_k$ "finds" its type-k state but ignores the others.

$$\mathbf{a}_1^\dagger|n_1 n_2\rangle = \mathbf{a}_1^\dagger|n_1\rangle|n_2\rangle = \sqrt{n_1+1}\,|n_1+1\; n_2\rangle$$
$$\mathbf{a}_1|n_1 n_2\rangle = \mathbf{a}_1|n_1\rangle|n_2\rangle = \sqrt{n_1}\,|n_1-1\; n_2\rangle$$

$\mathbf{a}_1$"finds" its type-1

$$\mathbf{a}_2^\dagger|n_1 n_2\rangle = |n_1\rangle\mathbf{a}_2^\dagger|n_2\rangle = \sqrt{n_2+1}\,|n_1\; n_2+1\rangle$$
$$\mathbf{a}_2|n_1 n_2\rangle = |n_1\rangle\mathbf{a}_2|n_2\rangle = \sqrt{n_2}\,|n_1\; n_2-1\rangle$$

$\mathbf{a}_2$"finds" its type-2

The *ABCD* matrix from Class 4

$$\mathbf{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$$

General definition of the 2D oscillator base state.

$$|n_1 n_2\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{n_1}\left(\mathbf{a}_2^\dagger\right)^{n_2}}{\sqrt{n_1!\,n_2!}}|0\,0\rangle$$

$$\mathbf{H} = H_{11}\left(\mathbf{a}_1^\dagger\mathbf{a}_1 + \mathbf{1}/2\right) + H_{12}\mathbf{a}_1^\dagger\mathbf{a}_2 + H_{21}\mathbf{a}_2^\dagger\mathbf{a}_1 + H_{22}\left(\mathbf{a}_2^\dagger\mathbf{a}_2 + \mathbf{1}/2\right)$$

The $\mathbf{a}_m{}^\dagger\mathbf{a}_n$ combinations in the *ABCD* Hamiltonian $\mathbf{H}$ have fairly simple matrix elements.

$$\mathbf{a}_1^\dagger\mathbf{a}_1|n_1 n_2\rangle = n_1|n_1\; n_2\rangle \qquad \mathbf{a}_1^\dagger\mathbf{a}_2|n_1 n_2\rangle = \sqrt{n_1+1}\sqrt{n_2}\,|n_1+1\; n_2-1\rangle$$
$$\mathbf{a}_2^\dagger\mathbf{a}_1|n_1 n_2\rangle = \sqrt{n_1}\sqrt{n_2+1}\,|n_1-1\; n_2+1\rangle \qquad \mathbf{a}_2^\dagger\mathbf{a}_2|n_1 n_2\rangle = n_2|n_1\; n_2\rangle$$

$$\mathbf{H} = A\left(\mathbf{a}_1^\dagger\mathbf{a}_1 + \mathbf{1}/2\right) + (B-iC)\mathbf{a}_1^\dagger\mathbf{a}_2 + (B+iC)\mathbf{a}_2^\dagger\mathbf{a}_1 + D\left(\mathbf{a}_2^\dagger\mathbf{a}_2 + \mathbf{1}/2\right)$$

$\langle\mathbf{H}\rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$

	$\|00\rangle$	$\|01\rangle$	$\|02\rangle$	$\cdots$	$\|10\rangle$	$\|11\rangle$	$\|12\rangle$	$\cdots$	$\|20\rangle$	$\|21\rangle$	$\|22\rangle$	$\cdots$
$\langle 00\|$	0			$\cdots$	$\cdot$			$\cdots$				$\cdots$
$\langle 01\|$		D		$\cdots$	$B+iC$	$\cdot$		$\cdots$				$\cdots$
$\langle 02\|$			$2D$	$\cdots$		$\sqrt{2}(B+iC)$	$\cdot$	$\cdots$				$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$				$\cdots$
$\langle 10\|$	$\cdot$	$B-iC$		$\cdots$	A			$\cdots$	$\cdot$			$\cdots$
$\langle 11\|$		$\cdot$	$\sqrt{2}(B-iC)$	$\cdots$		$A+D$		$\cdots$	$\sqrt{2}(B+iC)$	$\cdot$		$\cdots$
$\langle 12\|$			$\cdot$	$\cdots$			$A+2D$	$\cdots$		$\sqrt{4}(B+iC)$	$\cdot$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$\langle 20\|$					$\cdot$	$\sqrt{2}(B-iC)$		$\cdots$	$2A$			$\cdots$
$\langle 21\|$						$\cdot$	$\sqrt{4}(B-iC)$	$\cdots$		$2A+D$		$\cdots$
$\langle 22\|$							$\cdot$	$\cdots$			$2A+2D$	$\cdots$
$\vdots$					$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

"Little-Endian" indexing
(...01,02,03..10,11,12,13 ... 20,21,22,23,...)

Symmetry group $\mathscr{G}$=U(2) representations, 2D HO Hamiltonian $\mathbf{H}=\hbar\omega_{ab}\mathbf{a}_a^\dagger\mathbf{a}_b$ operators, 2D HO wave eigenfunctions $\Psi_{n,m}$, and coherent [α] states

Factoring 2D-HO Hamiltonian $\mathbf{H}=\frac{A}{2}\left(\mathbf{p}_1^2+\mathbf{x}_1^2\right)+B\left(\mathbf{x}_1\mathbf{x}_2+\mathbf{p}_1\mathbf{p}_2\right)+C\left(\mathbf{x}_1\mathbf{p}_2-\mathbf{x}_2\mathbf{p}_1\right)+\frac{D}{2}\left(\mathbf{p}_2^2+\mathbf{x}_2^2\right)$

2D-Oscillator basic states and operations
- *Commutation relations*
- *Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry*
  - *Anti-commutation relations*
- *Two-dimensional (or 2-particle) base states: ket-kets and bra-bras*
  - *Outer product arrays*
  - *Entangled 2-particle states*

Two-particle (or 2-dimensional) matrix operators
- *U(2) Hamiltonian and irreducible representations*
- *2D-Oscillator states and related 3D angular momentum multiplets*

R(3) Angular momentum generators by U(2) analysis

Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$

SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Mostly Notation and Bookkeeping :

U(2)-2D-HO Hamiltonian and irreducible representations

"Little-Endian" indexing
(...01,02,03..10,11,12,13 ... 20,21,22,23,...)

$\mathbf{H} =$

$$A\left(\mathbf{a}_1^\dagger\mathbf{a}_1+\mathbf{1}/2\right)+\left(B-iC\right)\mathbf{a}_1^\dagger\mathbf{a}_2$$
$$+\left(B+iC\right)\mathbf{a}_2^\dagger\mathbf{a}_1+D\left(\mathbf{a}_2^\dagger\mathbf{a}_2+\mathbf{1}/2\right)$$

$\langle\mathbf{H}\rangle = A(\mathbf{1}/2)+D(\mathbf{1}/2)+$

	$\vert 00\rangle$	$\vert 01\rangle$	$\vert 02\rangle$	$\cdots$	$\vert 10\rangle$	$\vert 11\rangle$	$\vert 12\rangle$	$\cdots$	$\vert 20\rangle$	$\vert 21\rangle$	$\vert 22\rangle$	$\cdots$
$\langle 00\vert$	0			$\cdots$	$\cdot$			$\cdots$				$\cdots$
$\langle 01\vert$		D		$\cdots$	$B+iC$	$\cdot$		$\cdots$				$\cdots$
$\langle 02\vert$			$2D$	$\cdots$		$\sqrt{2}(B+iC)$	$\cdot$	$\cdots$				$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$				$\cdots$
$\langle 10\vert$	$\cdot$	$B-iC$		$\cdots$	A			$\cdots$	$\cdot$			$\cdots$
$\langle 11\vert$		$\cdot$	$\sqrt{2}(B-iC)$	$\cdots$		$A+D$		$\cdots$	$\sqrt{2}(B+iC)$	$\cdot$		$\cdots$
$\langle 12\vert$			$\cdot$	$\cdots$			$A+2D$	$\cdots$		$\sqrt{4}(B+iC)$	$\cdot$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$\langle 20\vert$					$\cdot$	$\sqrt{2}(B-iC)$		$\cdots$	$2A$			$\cdots$
$\langle 21\vert$						$\cdot$	$\sqrt{4}(B-iC)$	$\cdots$		$2A+D$		$\cdots$
$\langle 22\vert$							$\cdot$	$\cdots$			$2A+2D$	$\cdots$
$\vdots$					$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Example:

$\mathbf{a}_1^\dagger\mathbf{a}_2\vert 02\rangle=\sqrt{0+1}\sqrt{2}\vert 0+1\;2-1\rangle=\sqrt{2}\vert 1\,1\rangle$

$\mathbf{a}_1^\dagger\mathbf{a}_2\vert n_1n_2\rangle=\sqrt{n_1+1}\sqrt{n_2}\vert n_1+1\;n_2-1\rangle$

$$\mathbf{a}_1^\dagger\mathbf{a}_1\vert n_1n_2\rangle=n_1\vert n_1n_2\rangle$$
$$\mathbf{a}_2^\dagger\mathbf{a}_1\vert n_1n_2\rangle=\sqrt{n_1}\sqrt{n_2+1}\vert n_1-1\;n_2+1\rangle$$
$$\mathbf{a}_1^\dagger\mathbf{a}_2\vert n_1n_2\rangle=\sqrt{n_1+1}\sqrt{n_2}\vert n_1+1\;n_2-1\rangle$$
$$\mathbf{a}_2^\dagger\mathbf{a}_2\vert n_1n_2\rangle=n_2\vert n_1n_2\rangle$$

Rearrangement of rows and columns brings the matrix to a block-diagonal form.

Base states $\vert n_1\rangle\vert n_2\rangle$ with the same *total quantum number* $\nu= n_1 + n_2$ define each block.

U(2)-2D-HO Hamiltonian and irreducible representations

"Little-Endian" indexing (*...01,02,03..10,11,12,13 ... 20,21,22,23,...*)

$$\mathbf{H} = A\left(\mathbf{a}_1^\dagger\mathbf{a}_1 + 1/2\right) + (B - iC)\mathbf{a}_1^\dagger\mathbf{a}_2 + (B + iC)\mathbf{a}_2^\dagger\mathbf{a}_1 + D\left(\mathbf{a}_2^\dagger\mathbf{a}_2 + 1/2\right)$$

$$\langle\mathbf{H}\rangle = A(1/2) + D(1/2) +$$

	$\|00\rangle$	$\|01\rangle$	$\|02\rangle$	$\cdots$	$\|10\rangle$	$\|11\rangle$	$\|12\rangle$	$\cdots$	$\|20\rangle$	$\|21\rangle$	$\|22\rangle$	$\cdots$
$\langle 00\|$	0			$\cdots$	$\cdot$			$\cdots$				$\cdots$
$\langle 01\|$		D		$\cdots$	$B+iC$	$\cdot$		$\cdots$				$\cdots$
$\langle 02\|$			$2D$	$\cdots$		$\sqrt{2}(B+iC)$	$\cdot$	$\cdots$				$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$				$\cdots$
$\langle 10\|$	$\cdot$	$B-iC$		$\cdots$	A			$\cdots$	$\cdot$			$\cdots$
$\langle 11\|$		$\cdot$	$\sqrt{2}(B-iC)$	$\cdots$		$A+D$		$\cdots$	$\sqrt{2}(B+iC)$	$\cdot$		$\cdots$
$\langle 12\|$		*Example:*	$\cdot$	$\cdots$			$A+2D$	$\cdots$		$\sqrt{4}(B+iC)$	$\cdot$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$\langle 20\|$					$\cdot$	$\sqrt{2}(B-iC)$		$\cdots$	$2A$			$\cdots$
$\langle 21\|$						$\cdot$	$\sqrt{4}(B-iC)$	$\cdots$		$2A+D$		$\cdots$
$\langle 22\|$							$\cdot$	$\cdots$			$2A+2D$	$\cdots$
$\vdots$					$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

$$\mathbf{a}_1^\dagger\mathbf{a}_2|02\rangle = \sqrt{0+1}\sqrt{2}|0+1\ 2-1\rangle = \sqrt{2}|1\,1\rangle$$

$$\mathbf{a}_1^\dagger\mathbf{a}_2|n_1 n_2\rangle = \sqrt{n_1+1}\sqrt{n_2}|n_1+1\ n_2-1\rangle$$

$$\mathbf{a}_1^\dagger\mathbf{a}_1|n_1 n_2\rangle = n_1|n_1 n_2\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_1|n_1 n_2\rangle = \sqrt{n_1}\sqrt{n_2+1}|n_1-1\ n_2+1\rangle$$

$$\mathbf{a}_1^\dagger\mathbf{a}_2|n_1 n_2\rangle = \sqrt{n_1+1}\sqrt{n_2}|n_1+1\ n_2-1\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_2|n_1 n_2\rangle = n_2|n_1 n_2\rangle$$

Rearrangement of rows and columns brings the matrix to a block-diagonal form.

Base states $|n_1\rangle|n_2\rangle$ with the same *total quantum number* $\nu = n_1 + n_2$ define each block.

Group reorganized "Little-Endian" indexing (*...01,02,03..10,11,12,13 ... 20,21,22,23,...*)

$$\langle\mathbf{H}\rangle = A(1/2) + D(1/2) +$$

	$\|00\rangle$	$\|01\rangle$	$\|10\rangle$	$\|02\rangle$	$\|11\rangle$	$\|20\rangle$	$\|03\rangle$	$\|12\rangle$	$\|21\rangle$	$\|30\rangle$	$\cdots$
$\langle 00\|$	0										
$\langle 01\|$		D	$B+iC$								
$\langle 10\|$		$B-iC$	A								
$\langle 02\|$				$2D$	$\sqrt{2}(B+iC)$						
$\langle 11\|$				$\sqrt{2}(B-iC)$	$A+D$	$\sqrt{2}(B+iC)$					
$\langle 20\|$					$\sqrt{2}(B-iC)$	$2A$					
$\langle 03\|$							$3D$	$\sqrt{3}(B+iC)$			
$\langle 12\|$							$\sqrt{3}(B-iC)$	$A+2D$	$\sqrt{4}(B+iC)$		
$\langle 21\|$								$\sqrt{4}(B-iC)$	$2A+D$	$\sqrt{3}(B+iC)$	
$\langle 30\|$									$\sqrt{3}(B-iC)$	$3A$	
$\vdots$											

Vacuum ($\nu=0$)

Fundamental ($\nu=1$) vibrational sub-space

Overtone ($\nu=2$) vibrational sub-space

Overtone ($\nu=3$) vibrational sub-space

$$\mathbf{H}^A = A\left(\mathbf{a}_1^\dagger\mathbf{a}_1 + 1/2\right) + D\left(\mathbf{a}_2^\dagger\mathbf{a}_2 + 1/2\right)$$

$$\varepsilon^A_{n_1 n_2} = A\left(n_1 + \frac{1}{2}\right) + D\left(n_2 + \frac{1}{2}\right) = \frac{A+D}{2}(n_1 + n_2 + 1) + \frac{A-D}{2}(n_1 - n_2)$$

Symmetry group $\mathscr{G}$=U(2) representations, 2D HO Hamiltonian $\mathbf{H}=\hbar\omega_{ab}\mathbf{a}_a{}^\dagger\mathbf{a}_b$ operators, 2D HO wave eigenfunctions $\Psi_{n,m}$, and coherent [α] states

Factoring 2D-HO Hamiltonian $\mathbf{H}=\frac{A}{2}\left(\mathbf{p}_1^2+\mathbf{x}_1^2\right)+B\left(\mathbf{x}_1\mathbf{x}_2+\mathbf{p}_1\mathbf{p}_2\right)+C\left(\mathbf{x}_1\mathbf{p}_2-\mathbf{x}_2\mathbf{p}_1\right)+\frac{D}{2}\left(\mathbf{p}_2^2+\mathbf{x}_2^2\right)$

2D-Oscillator basic states and operations

Commutation relations

Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry

Anti-commutation relations

Two-dimensional (or 2-particle) base states: ket-kets and bra-bras

Outer product arrays

Entangled 2-particle states

Two-particle (or 2-dimensional) matrix operators

U(2) Hamiltonian and irreducible representations

2D-Oscillator states ← *related 3D angular momentum multiplets*

R(3) Angular momentum generators by U(2) analysis

Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$

SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Mostly Notation and Bookkeeping :

2D-Oscillator states *and related 3D angular momentum multiplets*

Fundamental eigenstates

The first step is to diagonalize the fundamental 2-by-2 matrix .

$$\langle \mathbf{H} \rangle_{\upsilon=1}^{Fundamental} = \begin{array}{c|cc} n_1, n_2 & |1,0\rangle & |0,1\rangle \\ \hline \langle 1,0| & A & B - iC \\ \langle 0,1| & B + iC & D \end{array} + \frac{A+D}{2}\mathbf{1}$$

Group reorganized "Big-Endian" indexing

(...*00*,*10*,*20*..*01*,*11*,*21*,*31* ...*02*,*12*,*22*,*32*...)

2D-Oscillator states *and related 3D angular momentum multiplets*

Fundamental eigenstates

The first step is to diagonalize the fundamental 2-by-2 matrix .

$\langle \mathbf{H} \rangle_{\upsilon=1}^{Fundamental} =$

n_1, n_2	$\vert 1,0\rangle$	$\vert 0,1\rangle$
$\langle 1,0\vert$	A	$B-iC$
$\langle 0,1\vert$	$B+iC$	D

$+\frac{A+D}{2}\mathbf{1}$

Group reorganized "Big-Endian" indexing

(...*00*,*10*,*20*..*01*,*11*,*21*,*31* ...*02*,*12*,*22*,*32*...)

Recall decomposition of **H** (Class-4 p16)

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} + \frac{A+D}{2}\mathbf{1} = (A+D)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{2} + 2C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{1}{2} + (A-D)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{2}$$

Fundamental eigenstates

The first step is to diagonalize the fundamental 2-by-2 matrix .

$$\langle \mathbf{H} \rangle_{\upsilon=1}^{Fundamental} = \begin{array}{c|cc} n_1,n_2 & |1,0\rangle & |0,1\rangle \\ \hline \langle 1,0| & A & B-iC \\ \langle 0,1| & B+iC & D \end{array} + \frac{A+D}{2}\mathbf{1}$$

Group reorganized "Big-Endian" indexing

(...*00*,*10*,*20*..*01*,*11*,*21*,*31* ...*02*,*12*,*22*,*32*...)

Recall decomposition of **H** (Class-4 p16)

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} + \frac{A+D}{2}\mathbf{1} = (A+D)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{2} + 2C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{1}{2} + (A-D)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{2}$$

in terms of Jordan-Pauli spin operators.

$$\mathbf{H} = \Omega_0\mathbf{1} + \mathbf{\Omega}\bullet\vec{\mathbf{S}} = \Omega_0\mathbf{1} + \Omega_B\mathbf{S}_B + \Omega_C\mathbf{S}_C + \Omega_A\mathbf{S}_A \quad \textit{(ABC Optical vector notation)}$$

$$= \Omega_0\mathbf{1} + \Omega_X\mathbf{S}_X + \Omega_Y\mathbf{S}_Y + \Omega_Z\mathbf{S}_Z \quad \textit{(XYZ Electron spin notation)}$$

2D-Oscillator states and related 3D angular momentum multiplets

Fundamental eigenstates

The first step is to diagonalize the fundamental 2-by-2 matrix .

$$\langle \mathbf{H} \rangle^{Fundamental}_{\upsilon=1} = \begin{array}{c|cc} n_1,n_2 & |1,0\rangle & |0,1\rangle \\ \hline \langle 1,0| & A & B-iC \\ \langle 0,1| & B+iC & D \end{array} + \frac{A+D}{2}\mathbf{1}$$

Group reorganized "Big-Endian" indexing
(...00,10,20..01,11,21,31 ...02,12,22,32...)

Recall decomposition of **H** (Class-4 p16)

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} + \frac{A+D}{2}\mathbf{1} = (A+D)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{2} + 2C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{1}{2} + (A-D)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{2}$$

in terms of Jordan-Pauli spin operators.

$$\mathbf{H} = \Omega_0\mathbf{1} + \mathbf{\Omega}\bullet\vec{\mathbf{S}} = \Omega_0\mathbf{1} + \Omega_B\mathbf{S}_B + \Omega_C\mathbf{S}_C + \Omega_A\mathbf{S}_A \quad (ABC \text{ Optical vector notation})$$

$$= \Omega_0\mathbf{1} + \Omega_X\mathbf{S}_X + \Omega_Y\mathbf{S}_Y + \Omega_Z\mathbf{S}_Z \quad (XYZ \text{ Electron spin notation})$$

Frequency eigenvalues $\omega_\pm$ of $\mathbf{H}\text{-}\Omega_0\mathbf{1}/2$ and *fundamental transition frequency* $\Omega = \omega_+ - \omega_-$:

$$\omega_\pm = \frac{\Omega_0 \pm \Omega}{2} = \frac{A+D \pm \sqrt{(2B)^2 + (2C)^2 + (A-D)^2}}{2} = \frac{A+D}{2} \pm \sqrt{\left(\frac{A-D}{2}\right)^2 + B^2 + C^2}$$

Polar angles (φ,ϑ) of $+\mathbf{\Omega}$-vector (or polar angles $(\varphi,\vartheta\pm\pi)$ of $-\mathbf{\Omega}$-vector) gives **H** eigenvectors.

$$|\omega_+\rangle = \begin{pmatrix} e^{-i\varphi/2}\cos\frac{\vartheta}{2} \\ e^{i\varphi/2}\sin\frac{\vartheta}{2} \end{pmatrix}, \quad |\omega_-\rangle = \begin{pmatrix} -e^{-i\varphi/2}\sin\frac{\vartheta}{2} \\ e^{i\varphi/2}\cos\frac{\vartheta}{2} \end{pmatrix} \quad \text{where: } \begin{cases} \cos\vartheta = \frac{A\text{-}D}{\Omega} \\ \tan\varphi = \frac{C}{B} \end{cases}$$

Finding eigenvectors Class 5 p72

2D-Oscillator states and related 3D angular momentum multiplets

Fundamental eigenstates

The first step is to diagonalize the fundamental 2-by-2 matrix .

$$\langle \mathbf{H}\rangle_{\upsilon=1}^{Fundamental} = \begin{array}{c|cc} n_1,n_2 & |1,0\rangle & |0,1\rangle \\ \hline \langle 1,0| & A & B-iC \\ \langle 0,1| & B+iC & D \end{array} + \frac{A+D}{2}\mathbf{1}$$

Group reorganized "Big-Endian" indexing
(...00,10,20..01,11,21,31 ...02,12,22,32...)

Recall decomposition of **H** (Class-4 p16)

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} + \frac{A+D}{2}\mathbf{1} = (A+D)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{2} + 2C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{1}{2} + (A-D)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{2}$$

in terms of Jordan-Pauli spin operators.

$$\mathbf{H} = \Omega_0\mathbf{1} + \mathbf{\Omega}\bullet\vec{\mathbf{S}} = \Omega_0\mathbf{1} + \Omega_B\mathbf{S}_B + \Omega_C\mathbf{S}_C + \Omega_A\mathbf{S}_A \quad (ABC \textit{ Optical vector notation})$$
$$= \Omega_0\mathbf{1} + \Omega_X\mathbf{S}_X + \Omega_Y\mathbf{S}_Y + \Omega_Z\mathbf{S}_Z \quad (XYZ \textit{ Electron spin notation})$$

Frequency eigenvalues $\omega_\pm$ of $\mathbf{H}$-$\Omega_0\mathbf{1}/2$ and *fundamental transition frequency* $\Omega = \omega_+ - \omega_-$:

$$\omega_\pm = \frac{\Omega_0 \pm \Omega}{2} = \frac{A+D \pm \sqrt{(2B)^2 + (2C)^2 + (A-D)^2}}{2} = \frac{A+D}{2} \pm \sqrt{\left(\frac{A-D}{2}\right)^2 + B^2 + C^2}$$

Polar angles (φ,ϑ) of $+\mathbf{\Omega}$-vector (or polar angles $(\varphi,\vartheta\pm\pi)$ of $-\mathbf{\Omega}$-vector) gives **H** eigenvectors.

$$|\omega_+\rangle = \begin{pmatrix} e^{-i\varphi/2}\cos\frac{\vartheta}{2} \\ e^{i\varphi/2}\sin\frac{\vartheta}{2} \end{pmatrix}, \quad |\omega_-\rangle = \begin{pmatrix} -e^{-i\varphi/2}\sin\frac{\vartheta}{2} \\ e^{i\varphi/2}\cos\frac{\vartheta}{2} \end{pmatrix} \quad \text{where: } \begin{cases} \cos\vartheta = \frac{A-D}{\Omega} \\ \tan\varphi = \frac{C}{B} \end{cases}$$

Finding eigenvectors Class 5 p72

More important for the general solution, are the *eigen-creation operators* $\mathbf{a}^\dagger_+$ *and* $\mathbf{a}^\dagger_-$ defined by

$$\mathbf{a}_+^\dagger = e^{-i\varphi/2}\left(\cos\frac{\vartheta}{2}\mathbf{a}_1^\dagger + e^{i\varphi}\sin\frac{\vartheta}{2}\mathbf{a}_2^\dagger\right), \quad \mathbf{a}_-^\dagger = e^{-i\varphi/2}\left(-\sin\frac{\vartheta}{2}\mathbf{a}_1^\dagger + e^{i\varphi}\cos\frac{\vartheta}{2}\mathbf{a}_2^\dagger\right)$$

2D-Oscillator states and related 3D angular momentum multiplets

Fundamental eigenstates

The first step is to diagonalize the fundamental 2-by-2 matrix .

$$\langle \mathbf{H} \rangle_{\upsilon=1}^{Fundamental} = \begin{array}{c|cc} n_1,n_2 & |1,0\rangle & |0,1\rangle \\ \hline \langle 1,0| & A & B-iC \\ \langle 0,1| & B+iC & D \end{array} + \frac{A+D}{2}\mathbf{1}$$

Group reorganized "Big-Endian" indexing
(...00,10,20..01,11,21,31 ...02,12,22,32...)

Recall decomposition of **H** (Class-4 p16)

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} + \frac{A+D}{2}\mathbf{1} = (A+D)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{2} + 2C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{1}{2} + (A-D)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{2}$$

in terms of Jordan-Pauli spin operators.

$$\mathbf{H} = \Omega_0\mathbf{1} + \boldsymbol{\Omega}\bullet\vec{\mathbf{S}} = \Omega_0\mathbf{1} + \Omega_B\mathbf{S}_B + \Omega_C\mathbf{S}_C + \Omega_A\mathbf{S}_A \quad \text{(ABC Optical vector notation)}$$

$$= \Omega_0\mathbf{1} + \Omega_X\mathbf{S}_X + \Omega_Y\mathbf{S}_Y + \Omega_Z\mathbf{S}_Z \quad \text{(XYZ Electron spin notation)}$$

Frequency eigenvalues $\omega_{\pm}$ of $\mathbf{H}$-$\Omega_0\mathbf{1}/2$ and *fundamental transition frequency* $\Omega = \omega_+ - \omega_-$:

$$\omega_{\pm} = \frac{\Omega_0 \pm \Omega}{2} = \frac{A+D \pm \sqrt{(2B)^2 + (2C)^2 + (A-D)^2}}{2} = \frac{A+D}{2} \pm \sqrt{\left(\frac{A-D}{2}\right)^2 + B^2 + C^2}$$

Polar angles (φ,ϑ) of $+\boldsymbol{\Omega}$-vector (or polar angles $(\varphi,\vartheta\pm\pi)$ of $-\boldsymbol{\Omega}$-vector) gives **H** eigenvectors.

$$|\omega_+\rangle = \begin{pmatrix} e^{-i\varphi/2}\cos\frac{\vartheta}{2} \\ e^{i\varphi/2}\sin\frac{\vartheta}{2} \end{pmatrix}, \quad |\omega_-\rangle = \begin{pmatrix} -e^{-i\varphi/2}\sin\frac{\vartheta}{2} \\ e^{i\varphi/2}\cos\frac{\vartheta}{2} \end{pmatrix} \quad \text{where: } \begin{cases} \cos\vartheta = \frac{A\text{-}D}{\Omega} \\ \tan\varphi = \frac{C}{B} \end{cases}$$

Finding eigenvectors Class 5 p72

More important for the general solution, are the *eigen-creation operators* $\mathbf{a}^\dagger_+$ *and* $\mathbf{a}^\dagger_-$ defined by

$$\mathbf{a}_+^\dagger = e^{-i\varphi/2}\left(\cos\frac{\vartheta}{2}\mathbf{a}_1^\dagger + e^{i\varphi}\sin\frac{\vartheta}{2}\mathbf{a}_2^\dagger\right), \quad \mathbf{a}_-^\dagger = e^{-i\varphi/2}\left(-\sin\frac{\vartheta}{2}\mathbf{a}_1^\dagger + e^{i\varphi}\cos\frac{\vartheta}{2}\mathbf{a}_2^\dagger\right)$$

create **H** eigenstates directly from the ground state.

$$\mathbf{a}_+^\dagger|0\rangle = |\omega_+\rangle, \quad \mathbf{a}_-^\dagger|0\rangle = |\omega_-\rangle$$

2D-Oscillator states *and related 3D angular momentum multiplets*

Setting *(B=0=C)* and *(A=ω₊)* and *(D=ω₋)* gives diagonal block matrices.

Group reorganized
"Little-Endian" indexing
(...01,02,03..10,11,12,13 ...
20,21,22,23,...)

$\langle \mathbf{H} \rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$

	$\|00\rangle$	$\|01\rangle$	$\|10\rangle$	$\|02\rangle$	$\|11\rangle$	$\|20\rangle$	$\|03\rangle$	$\|12\rangle$	$\|21\rangle$	$\|30\rangle$	$\cdots$
$\langle 00\|$	0										
$\langle 01\|$		ω_-									
$\langle 10\|$			ω_+								
$\langle 02\|$				$2\omega_-$							
$\langle 11\|$					$\omega_+ + \omega_-$						
$\langle 20\|$						$2\omega_+$					
$\langle 03\|$							$3\omega_-$				
$\langle 12\|$								$\omega_+ + 2\omega_-$			
$\langle 21\|$									$2\omega_+ + \omega_-$		
$\langle 30\|$										$3\omega_+$	
$\vdots$											

$$\omega_+ - \omega_- = \Omega$$
$$= \sqrt{(2B)^2 + (2C)^2 + (A-D)^2}$$
$$= A - D$$

$$\mathbf{H}^A = A\left(\mathbf{a}_1^\dagger \mathbf{a}_1 + \mathbf{1}/2\right) + D\left(\mathbf{a}_2^\dagger \mathbf{a}_2 + \mathbf{1}/2\right)$$

2D-Oscillator states and related 3D angular momentum multiplets

Setting *(B=0=C)* and *(A=ω₊)* and *(D=ω₋)* gives diagonal block matrices.

Group reorganized "Little-Endian" indexing *(...01,02,03..10,11,12,13 ... 20,21,22,23,...)*

$\langle \mathbf{H} \rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$

	$\vert 00\rangle$	$\vert 01\rangle$	$\vert 10\rangle$	$\vert 02\rangle$	$\vert 11\rangle$	$\vert 20\rangle$	$\vert 03\rangle$	$\vert 12\rangle$	$\vert 21\rangle$	$\vert 30\rangle$	$\cdots$
$\langle 00\vert$	0										
$\langle 01\vert$		ω_-									
$\langle 10\vert$			ω_+								
$\langle 02\vert$				$2\omega_-$							
$\langle 11\vert$					$\omega_+ + \omega_-$						
$\langle 20\vert$						$2\omega_+$					
$\langle 03\vert$							$3\omega_-$				
$\langle 12\vert$								$\omega_+ + 2\omega_-$			
$\langle 21\vert$									$2\omega_+ + \omega_-$		
$\langle 30\vert$										$3\omega_+$	
$\vdots$											

$$\omega_+ - \omega_- = \Omega$$
$$= \sqrt{(2B)^2 + (2C)^2 + (A-D)^2}$$
$$= A - D$$

$$\mathbf{H}^A = A\left(\mathbf{a}_1^\dagger \mathbf{a}_1 + \mathbf{1}/2\right) + D\left(\mathbf{a}_2^\dagger \mathbf{a}_2 + \mathbf{1}/2\right)$$

$$\varepsilon^A_{n_1 n_2} = A\left(n_1 + \frac{1}{2}\right) + D\left(n_2 + \frac{1}{2}\right) = \frac{A+D}{2}(n_1 + n_2 + 1) + \frac{A-D}{2}(n_1 - n_2)$$

2D-Oscillator states and related 3D angular momentum multiplets

Group reorganized "Little-Endian" indexing (...01,02,03..10,11,12,13 ... 20,21,22,23,...)

Setting $(B=0=C)$ and $(A=\omega_+)$ and $(D=\omega_-)$ gives diagonal block matrices.

$\langle \mathbf{H} \rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$

	$\lvert 00\rangle$	$\lvert 01\rangle$	$\lvert 10\rangle$	$\lvert 02\rangle$	$\lvert 11\rangle$	$\lvert 20\rangle$	$\lvert 03\rangle$	$\lvert 12\rangle$	$\lvert 21\rangle$	$\lvert 30\rangle$	$\cdots$
$\langle 00\rvert$	0										
$\langle 01\rvert$		ω_-									
$\langle 10\rvert$			ω_+								
$\langle 02\rvert$				$2\omega_-$							
$\langle 11\rvert$					$\omega_+ + \omega_-$						
$\langle 20\rvert$						$2\omega_+$					
$\langle 03\rvert$							$3\omega_-$				
$\langle 12\rvert$								$\omega_+ + 2\omega_-$			
$\langle 21\rvert$									$2\omega_+ + \omega_-$		
$\langle 30\rvert$										$3\omega_+$	
$\vdots$											

$$\omega_+ - \omega_- = \Omega$$
$$= \sqrt{(2B)^2 + (2C)^2 + (A-D)^2}$$
$$= A - D$$

$$\mathbf{H}^A = A\left(\mathbf{a}_1^\dagger \mathbf{a}_1 + \mathbf{1}/2\right) + D\left(\mathbf{a}_2^\dagger \mathbf{a}_2 + \mathbf{1}/2\right)$$

$$\varepsilon^A_{n_1 n_2} = A\left(n_1 + \frac{1}{2}\right) + D\left(n_2 + \frac{1}{2}\right) = \frac{A+D}{2}\left(n_1 + n_2 + 1\right) + \frac{A-D}{2}\left(n_1 - n_2\right)$$
$$= \Omega_0\left(n_1 + n_2 + 1\right) + \frac{\Omega}{2}\left(n_1 - n_2\right) = \Omega_0\left(\upsilon + 1\right) + \Omega\, m$$

2D-Oscillator states and related 3D angular momentum multiplets

Setting (B=0=C) and (A=ω_+) and (D=ω_-) gives diagonal block matrices.

Group reorganized
"Little-Endian" indexing
(...01,02,03..10,11,12,13 ...
20,21,22,23,...)

	$\lvert 00\rangle$	$\lvert 01\rangle$	$\lvert 10\rangle$	$\lvert 02\rangle$	$\lvert 11\rangle$	$\lvert 20\rangle$	$\lvert 03\rangle$	$\lvert 12\rangle$	$\lvert 21\rangle$	$\lvert 30\rangle$	$\cdots$
$\langle 00\rvert$	0										
$\langle 01\rvert$		ω_-									
$\langle 10\rvert$			ω_+								
$\langle 02\rvert$				$2\omega_-$							
$\langle 11\rvert$					$\omega_+ + \omega_-$						
$\langle 20\rvert$						$2\omega_+$					
$\langle 03\rvert$							$3\omega_-$				
$\langle 12\rvert$								$\omega_+ + 2\omega_-$			
$\langle 21\rvert$									$2\omega_+ + \omega_-$		
$\langle 30\rvert$										$3\omega_+$	
$\vdots$											

$\upsilon = n_1 + n_2 = 0\ block$

$\upsilon = n_1 + n_2 = 1\ block$

$\upsilon = n_1 + n_2 = 2\ block$

$\upsilon = n_1 + n_2 = 3\ block$

$\langle \mathbf{H} \rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$

$$\omega_+ - \omega_- = \Omega$$

$$= \sqrt{(2B)^2 + (2C)^2 + (A-D)^2}$$

$$= A - D$$

$$\mathbf{H}^A = A\left(\mathbf{a}_1^\dagger \mathbf{a}_1 + \mathbf{1}/2\right) + D\left(\mathbf{a}_2^\dagger \mathbf{a}_2 + \mathbf{1}/2\right)$$

$$\varepsilon^A_{n_1 n_2} = A\left(n_1 + \frac{1}{2}\right) + D\left(n_2 + \frac{1}{2}\right) = \frac{A+D}{2}(n_1 + n_2 + 1) + \frac{A-D}{2}(n_1 - n_2)$$

$$= \Omega_0 (n_1 + n_2 + 1) + \frac{\Omega}{2}(n_1 - n_2) = \Omega_0(\upsilon + 1) + \Omega\, m$$

Define *total quantum number* υ=*2j* and half-difference or *asymmetry quantum number m*

$$\upsilon = n_1 + n_2 = 2j \qquad j = \frac{n_1 + n_2}{2} = \frac{\upsilon}{2} \qquad m = \frac{n_1 - n_2}{2}$$

2D-Oscillator states and related 3D angular momentum multiplets

Group reorganized "Little-Endian" indexing (...01,02,03..10,11,12,13 ... 20,21,22,23,...)

Setting (B=0=C) and ($A=\omega_+$) and ($D=\omega_-$) gives diagonal block matrices.

$j=0$: $\upsilon = n_1 + n_2 = 0$ *block*

$j=\frac{1}{2}$: $\upsilon = n_1 + n_2 = 1$ *block*

$j=1$: $\upsilon = n_1 + n_2 = 2$ *block*

$j=\frac{3}{2}$: $\upsilon = n_1 + n_2 = 3$ *block*

$\langle \mathbf{H} \rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$

	$\vert 00\rangle$	$\vert 01\rangle$	$\vert 10\rangle$	$\vert 02\rangle$	$\vert 11\rangle$	$\vert 20\rangle$	$\vert 03\rangle$	$\vert 12\rangle$	$\vert 21\rangle$	$\vert 30\rangle$	$\cdots$
$\langle 00\vert$	0										
$\langle 01\vert$		ω_-									
$\langle 10\vert$			ω_+								
$\langle 02\vert$				$2\omega_-$							
$\langle 11\vert$					$\omega_+ + \omega_-$						
$\langle 20\vert$						$2\omega_+$					
$\langle 03\vert$							$3\omega_-$				
$\langle 12\vert$								$\omega_+ + 2\omega_-$			
$\langle 21\vert$									$2\omega_+ + \omega_-$		
$\langle 30\vert$										$3\omega_+$	
$\vdots$											

$$\omega_+ - \omega_- = \Omega = \sqrt{(2B)^2 + (2C)^2 + (A-D)^2} = A - D$$

$$\mathbf{H}^A = A\left(\mathbf{a}_1^\dagger \mathbf{a}_1 + \mathbf{1}/2\right) + D\left(\mathbf{a}_2^\dagger \mathbf{a}_2 + \mathbf{1}/2\right)$$

$$\varepsilon^A_{n_1 n_2} = A\left(n_1 + \frac{1}{2}\right) + D\left(n_2 + \frac{1}{2}\right) = \frac{A+D}{2}(n_1+n_2+1) + \frac{A-D}{2}(n_1 - n_2)$$

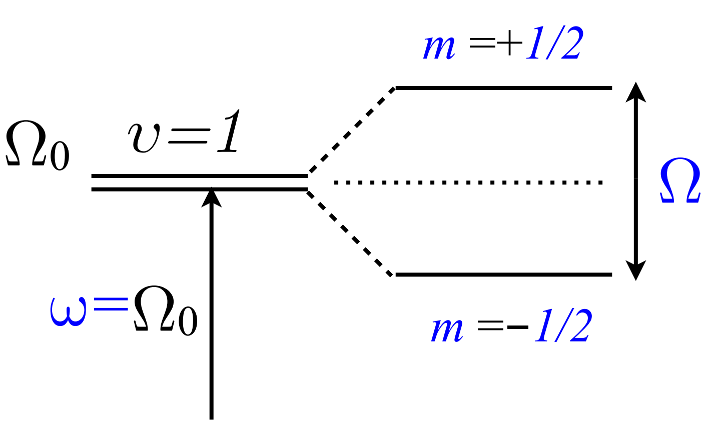
$$= \Omega_0 (n_1 + n_2 + 1) + \frac{\Omega}{2}(n_1 - n_2) = \Omega_0(\upsilon + 1) + \Omega\, m$$

Define *total quantum number* $\upsilon=2j$ and half-difference or *asymmetry quantum number m*

$$\upsilon = n_1 + n_2 = 2j \qquad j = \frac{n_1 + n_2}{2} = \frac{\upsilon}{2} \qquad m = \frac{n_1 - n_2}{2}$$

$\upsilon+1=2j+1$ multiplies *base frequency* $\omega=\Omega_0$

m multiplies *beat frequency* Ω



$$\omega_+ = \Omega_0 + \Omega\left(+\tfrac{1}{2}\right)$$

$$\omega_- = \Omega_0 + \Omega\left(-\tfrac{1}{2}\right)$$

2D-Oscillator states and related 3D angular momentum multiplets

Fundamental eigenstates

The first step is to diagonalize the fundamental 2-by-2 matrix .

$$\langle \mathbf{H} \rangle^{Fundamental}_{\upsilon=1} = \begin{array}{c|cc} n_1,n_2 & |1,0\rangle & |0,1\rangle \\ \hline \langle 1,0| & A & B-iC \\ \langle 0,1| & B+iC & D \end{array} + \frac{A+D}{2}\mathbf{1}$$

Group reorganized "Big-Endian" indexing

(...00,10,20..01,11,21,31 ...02,12,22,32...)

Recall decomposition of **H** (Class-4 p16)

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} + \frac{A+D}{2}\mathbf{1} = (A+D)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{2} + 2C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{1}{2} + (A-D)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{2}$$

in terms of Jordan-Pauli spin operators.

$$\mathbf{H} = \Omega_0 \mathbf{1} + \mathbf{\Omega} \bullet \vec{\mathbf{S}} = \Omega_0 \mathbf{1} + \Omega_B \mathbf{S}_B + \Omega_C \mathbf{S}_C + \Omega_A \mathbf{S}_A \quad (ABC \text{ Optical vector notation})$$

$$= \Omega_0 \mathbf{1} + \Omega_X \mathbf{S}_X + \Omega_Y \mathbf{S}_Y + \Omega_Z \mathbf{S}_Z \quad (XYZ \text{ Electron spin notation})$$

Frequency eigenvalues $\omega_\pm$ of $\mathbf{H}$-$\Omega_0\mathbf{1}/2$ and *fundamental transition frequency* $\Omega = \omega_+ - \omega_-$:

$$\omega_\pm = \frac{\Omega_0 \pm \Omega}{2} = \frac{A+D \pm \sqrt{(2B)^2 + (2C)^2 + (A-D)^2}}{2} = \frac{A+D}{2} \pm \sqrt{\left(\frac{A-D}{2}\right)^2 + B^2 + C^2}$$

Polar angles (φ,ϑ) of $+\mathbf{\Omega}$-vector (or polar angles $(\varphi,\vartheta\pm\pi)$ of $-\mathbf{\Omega}$-vector) gives **H** eigenvectors.

$$|\omega_+\rangle = \begin{pmatrix} e^{-i\varphi/2}\cos\frac{\vartheta}{2} \\ e^{i\varphi/2}\sin\frac{\vartheta}{2} \end{pmatrix}, \quad |\omega_-\rangle = \begin{pmatrix} -e^{-i\varphi/2}\sin\frac{\vartheta}{2} \\ e^{i\varphi/2}\cos\frac{\vartheta}{2} \end{pmatrix} \quad \text{where:} \begin{cases} \cos\vartheta = \frac{A\text{-}D}{\Omega} \\ \tan\varphi = \frac{C}{B} \end{cases}$$

More important for the general solution, are the *eigen-creation operators* $\mathbf{a}^\dagger_+$ *and* $\mathbf{a}^\dagger_-$ defined by

$$\mathbf{a}^\dagger_+ = e^{-i\varphi/2}\left(\cos\frac{\vartheta}{2}\mathbf{a}^\dagger_1 + e^{i\varphi}\sin\frac{\vartheta}{2}\mathbf{a}^\dagger_2\right), \quad \mathbf{a}^\dagger_- = e^{-i\varphi/2}\left(-\sin\frac{\vartheta}{2}\mathbf{a}^\dagger_1 + e^{i\varphi}\cos\frac{\vartheta}{2}\mathbf{a}^\dagger_2\right)$$

create **H** eigenstates directly from the ground state.

$$\mathbf{a}^\dagger_+|0\rangle = |\omega_+\rangle, \quad \mathbf{a}^\dagger_-|0\rangle = |\omega_-\rangle$$

Symmetry group $\mathscr{G}$=U(2) representations, 2D HO Hamiltonian $\mathbf{H}=\hbar\omega_{ab}\mathbf{a}_a{}^\dagger\mathbf{a}_b$ operators,
2D HO wave eigenfunctions $\Psi_{n,m}$, and coherent [α] states

Factoring 2D-HO Hamiltonian $\mathbf{H}=\frac{A}{2}\left(\mathbf{p}_1^2+\mathbf{x}_1^2\right)+B\left(\mathbf{x}_1\mathbf{x}_2+\mathbf{p}_1\mathbf{p}_2\right)+C\left(\mathbf{x}_1\mathbf{p}_2-\mathbf{x}_2\mathbf{p}_1\right)+\frac{D}{2}\left(\mathbf{p}_2^2+\mathbf{x}_2^2\right)$

2D-Oscillator basic states and operations
- *Commutation relations*
- *Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry*
  - *Anti-commutation relations*
- *Two-dimensional (or 2-particle) base states: ket-kets and bra-bras*
  - *Outer product arrays*
  - *Entangled 2-particle states*

Two-particle (or 2-dimensional) matrix operators
- *U(2) Hamiltonian and irreducible representations*
- *2D-Oscillator states related 3D angular momentum multiplets*

R(3) Angular momentum generators by U(2) analysis
Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$
SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Mostly Notation and Bookkeeping :

2D-Oscillator states and related 3D angular momentum multiplets

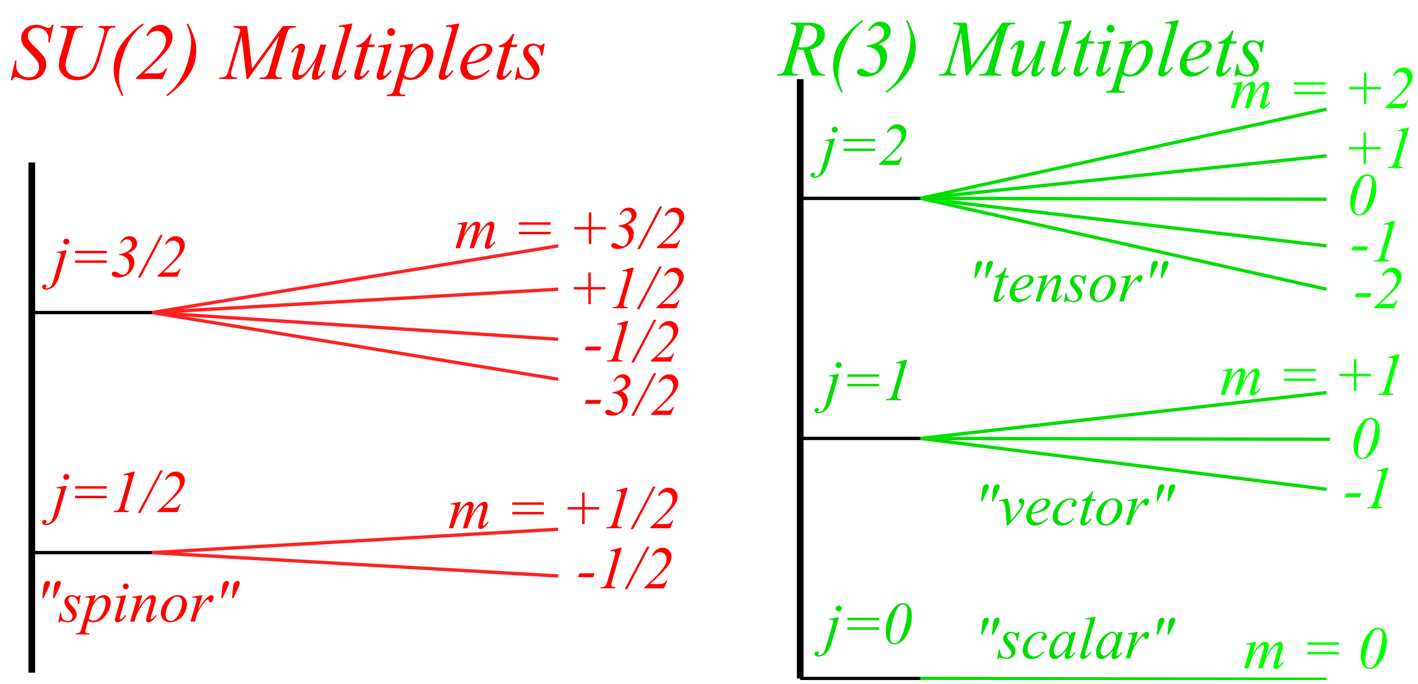
Setting $(B=0=C)$ and $(A=\omega_+)$ and $(D=\omega_-)$ gives diagonal block matrices.

Group reorganized
"Little-Endian" indexing
(...01,02,03..10,11,12,13 ...
20,21,22,23,...)

$\langle \mathbf{H} \rangle = A(1/2) + D(1/2) +$

	$\vert 00\rangle$	$\vert 01\rangle$	$\vert 10\rangle$	$\vert 02\rangle$	$\vert 11\rangle$	$\vert 20\rangle$	$\vert 03\rangle$	$\vert 12\rangle$	$\vert 21\rangle$	$\vert 30\rangle$	$\cdots$
$\langle 00\vert$	0										
$\langle 01\vert$		ω_-									
$\langle 10\vert$			ω_+								
$\langle 02\vert$				$2\omega_-$							
$\langle 11\vert$					$\omega_+ + \omega_-$						
$\langle 20\vert$						$2\omega_+$					
$\langle 03\vert$							$3\omega_-$				
$\langle 12\vert$								$\omega_+ + 2\omega_-$			
$\langle 21\vert$									$2\omega_+ + \omega_-$		
$\langle 30\vert$										$3\omega_+$	
$\vdots$											

$$\omega_+ - \omega_- = \Omega$$
$$= \sqrt{(2B)^2 + (2C)^2 + (A-D)^2}$$
$$= A - D$$



2D-Oscillator states and related 3D angular momentum multiplets

Group reorganized
"Little-Endian" indexing
(...01,02,03..10,11,12,13 ...
20,21,22,23,...)

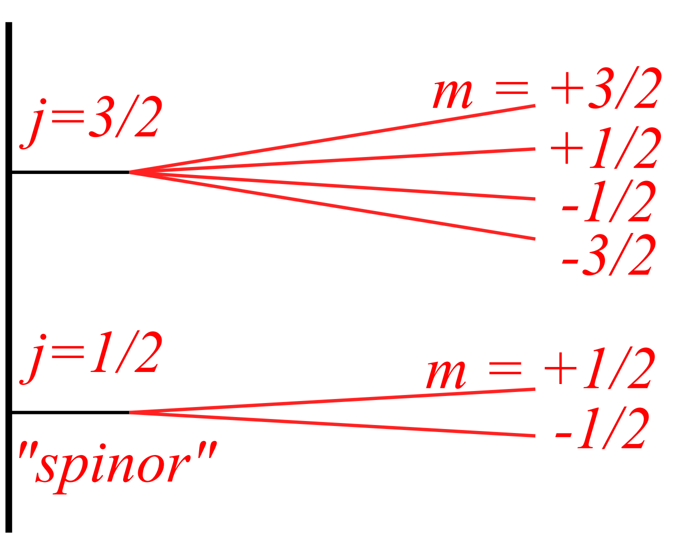
Setting (B=0=C) and ($A=\omega_+$) and ($D=\omega_-$) gives diagonal block matrices.

$\langle \mathbf{H} \rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$

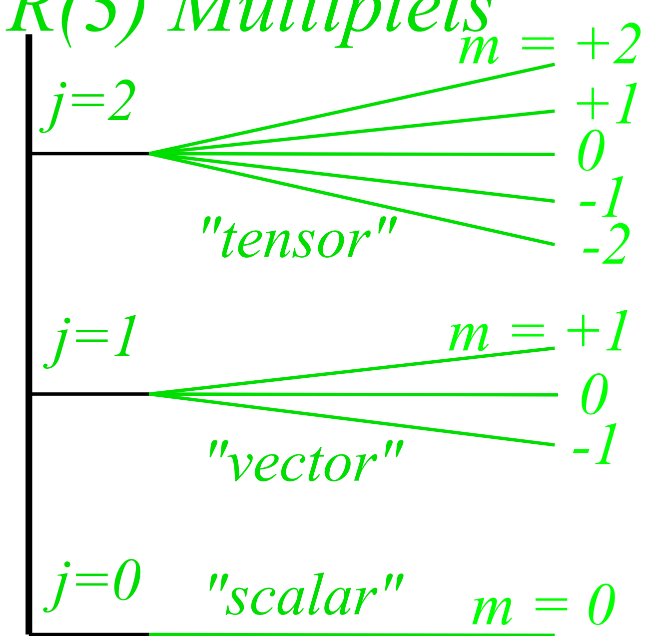
	$\lvert 00\rangle$	$\lvert 01\rangle$	$\lvert 10\rangle$	$\lvert 02\rangle$	$\lvert 11\rangle$	$\lvert 20\rangle$	$\lvert 03\rangle$	$\lvert 12\rangle$	$\lvert 21\rangle$	$\lvert 30\rangle$	$\cdots$
$\langle 00\rvert$	0										
$\langle 01\rvert$		ω_-									
$\langle 10\rvert$			ω_+								
$\langle 02\rvert$				$2\omega_-$							
$\langle 11\rvert$					$\omega_+ + \omega_-$						
$\langle 20\rvert$						$2\omega_+$					
$\langle 03\rvert$							$3\omega_-$				
$\langle 12\rvert$								$\omega_+ + 2\omega_-$			
$\langle 21\rvert$									$2\omega_+ + \omega_-$		
$\langle 30\rvert$										$3\omega_+$	
$\vdots$											

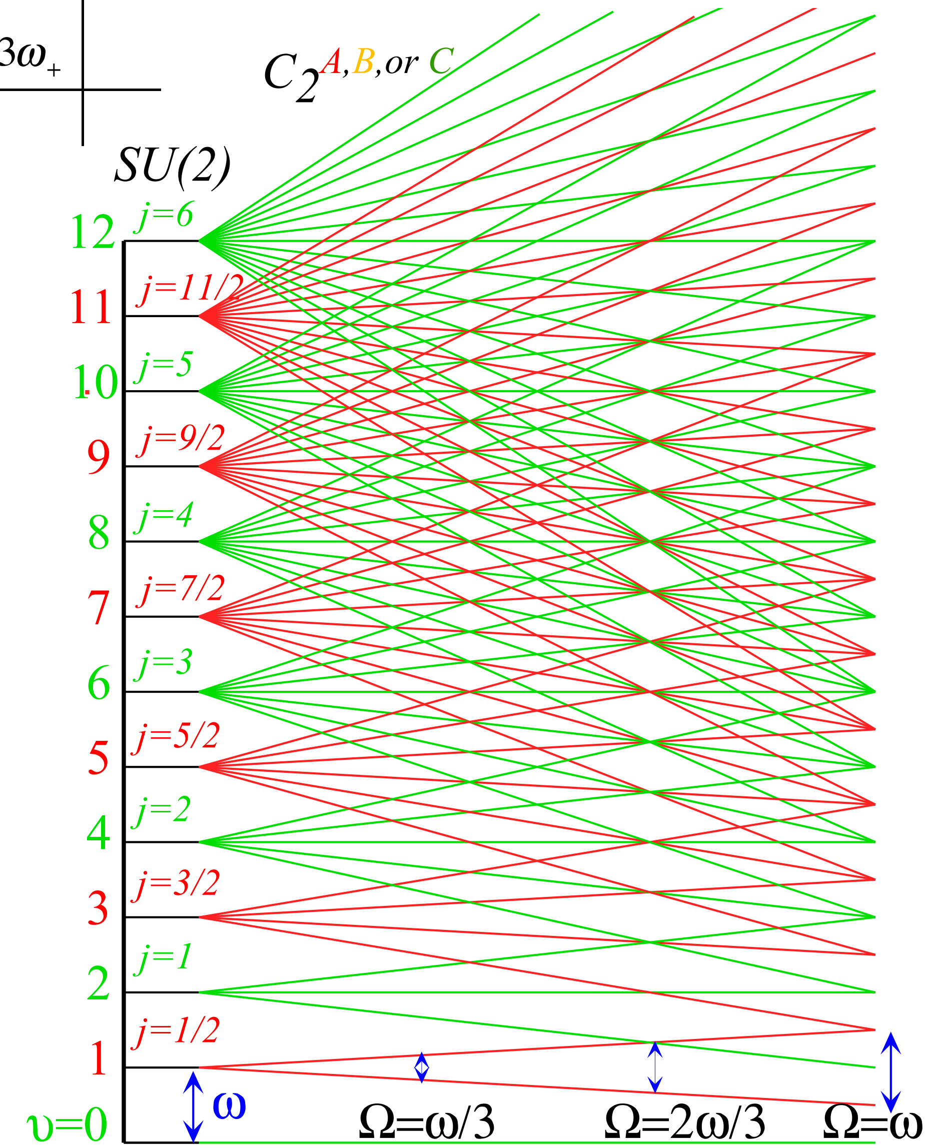
$$\omega_+ - \omega_- = \Omega$$
$$= \sqrt{(2B)^2 + (2C)^2 + (A-D)^2}$$
$$= A - D$$

SU(2) Multiplets



R(3) Multiplets





2D-Oscillator states and related 3D angular momentum multiplets

Structure of U(2)

$$\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle = |n_1 n_2\rangle$$

$j=0$: $\left|\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right\rangle = |00\rangle$ "scalar"

$j=\frac{1}{2}$: $\left|\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right\rangle = |10\rangle = |\uparrow\rangle$, $\left|\begin{smallmatrix} 1/2 \\ -1/2 \end{smallmatrix}\right\rangle = |01\rangle = |\downarrow\rangle$ "spinor"

$j=1$: $\left|\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right\rangle = |20\rangle$, $\left|\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right\rangle = |11\rangle$, $\left|\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right\rangle = |02\rangle$ "3-vector"

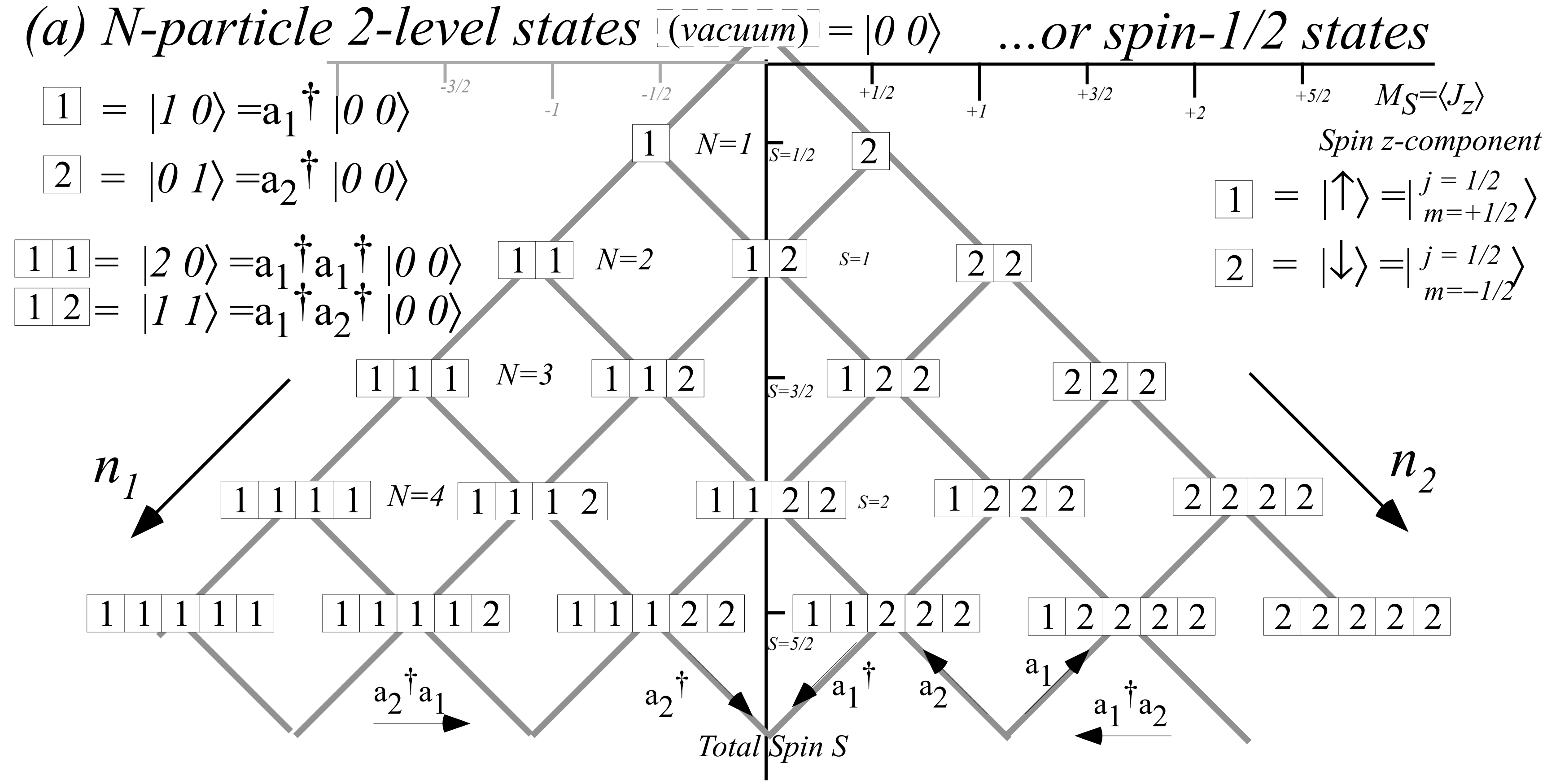
$j=\frac{3}{2}$: $\left|\begin{smallmatrix} 3/2 \\ 1/2 \end{smallmatrix}\right\rangle = |30\rangle$, $\left|\begin{smallmatrix} 3/2 \\ 1/2 \end{smallmatrix}\right\rangle = |21\rangle$, $\left|\begin{smallmatrix} 3/2 \\ -1/2 \end{smallmatrix}\right\rangle = |12\rangle$, $\left|\begin{smallmatrix} 3/2 \\ -3/2 \end{smallmatrix}\right\rangle = |03\rangle$ "4-spinor"

$j=2$: $\left|\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right\rangle = |40\rangle$, $\left|\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right\rangle = |31\rangle$, $\left|\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right\rangle = |22\rangle$, $\left|\begin{smallmatrix} 2 \\ -1 \end{smallmatrix}\right\rangle = |13\rangle$, $\left|\begin{smallmatrix} 2 \\ -2 \end{smallmatrix}\right\rangle = |04\rangle$ "tensor"

$\vdots$

$$j = \frac{\upsilon}{2} = \frac{n_1 + n_2}{2} \qquad n_1 = j + m = 2\upsilon + m$$

$$m = \frac{n_1 - n_2}{2} \qquad n_2 = j - m = 2\upsilon - m$$



Symmetry group $\mathscr{G}$=U(2) representations, 2D HO Hamiltonian $\mathbf{H}=\hbar\omega_{ab}\mathbf{a}_a^\dagger\mathbf{a}_b$ operators,
2D HO wave eigenfunctions $\Psi_{n,m}$, and coherent $[\alpha]$ states

Factoring 2D-HO Hamiltonian $\mathbf{H}=\frac{A}{2}\left(\mathbf{p}_1^2+\mathbf{x}_1^2\right)+B\left(\mathbf{x}_1\mathbf{x}_2+\mathbf{p}_1\mathbf{p}_2\right)+C\left(\mathbf{x}_1\mathbf{p}_2-\mathbf{x}_2\mathbf{p}_1\right)+\frac{D}{2}\left(\mathbf{p}_2^2+\mathbf{x}_2^2\right)$

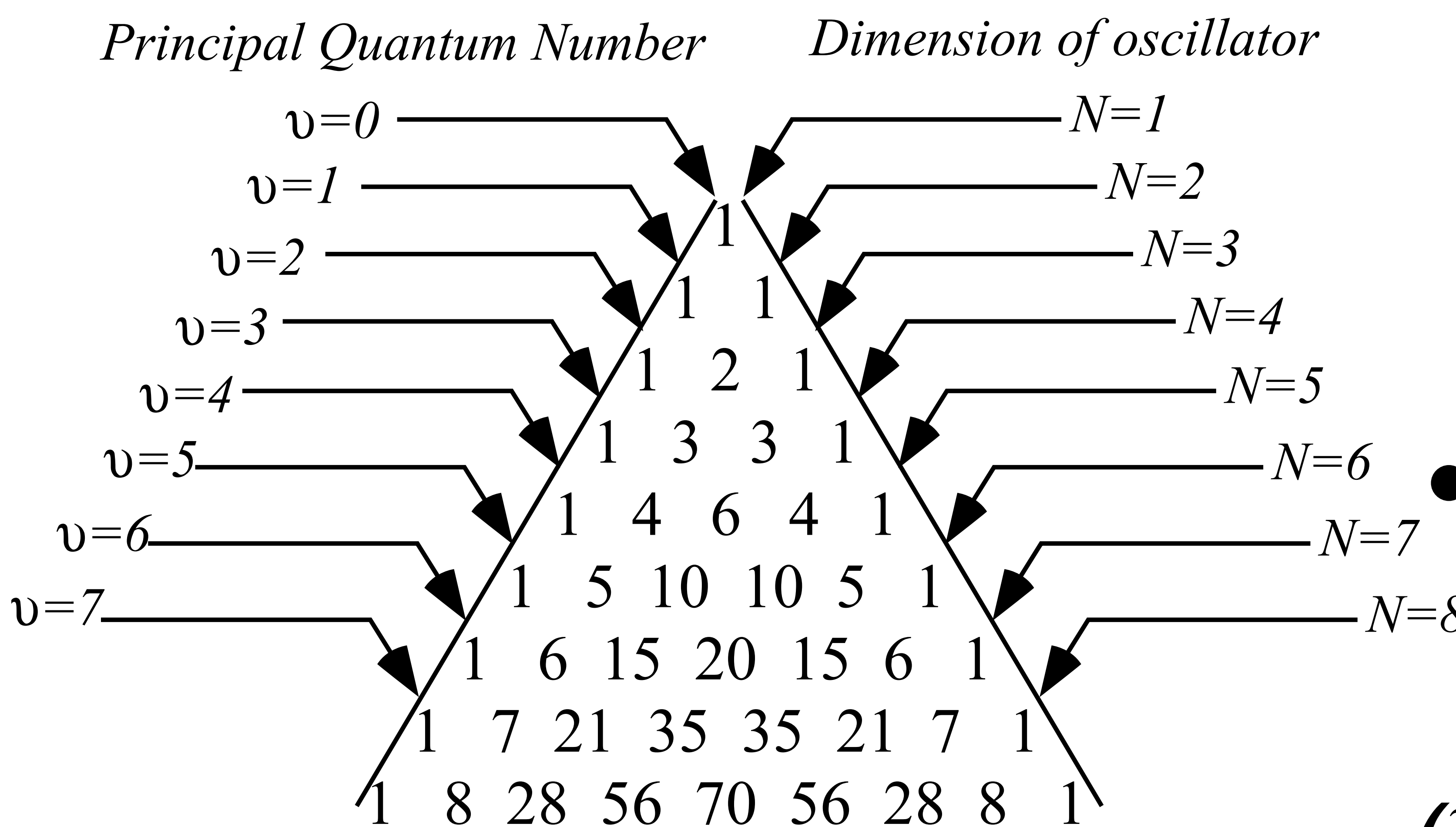
2D-Oscillator basic states and operations
Commutation relations
Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry
Anti-commutation relations
Two-dimensional (or 2-particle) base states: ket-kets and bra-bras
Outer product arrays
Entangled 2-particle states
Two-particle (or 2-dimensional) matrix operators
U(2) Hamiltonian and irreducible representations
2D-Oscillator states related 3D angular momentum multiplets
R(3) Angular momentum generators by U(2) analysis
Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$
SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Mostly Notation and Bookkeeping :

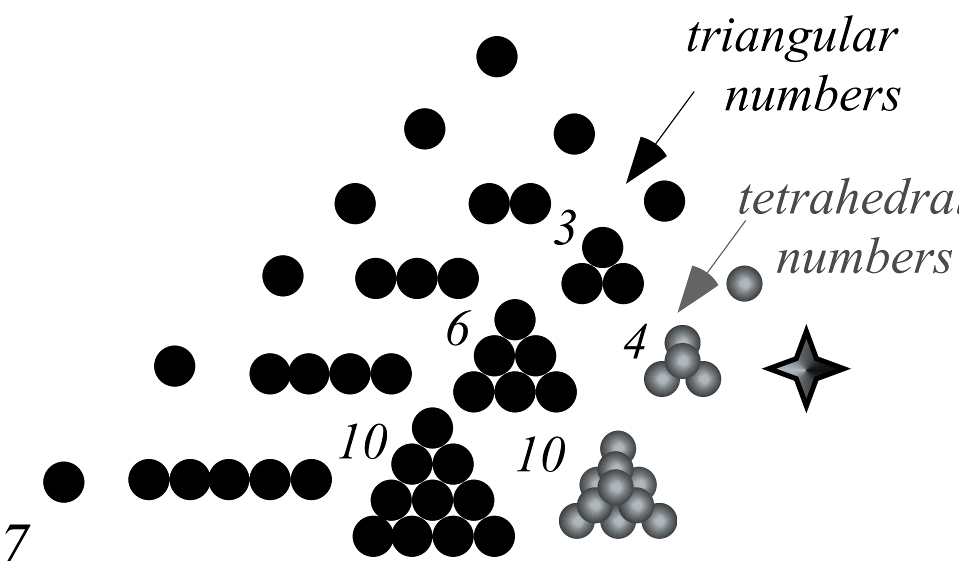
ND-Oscillator eigensolutions

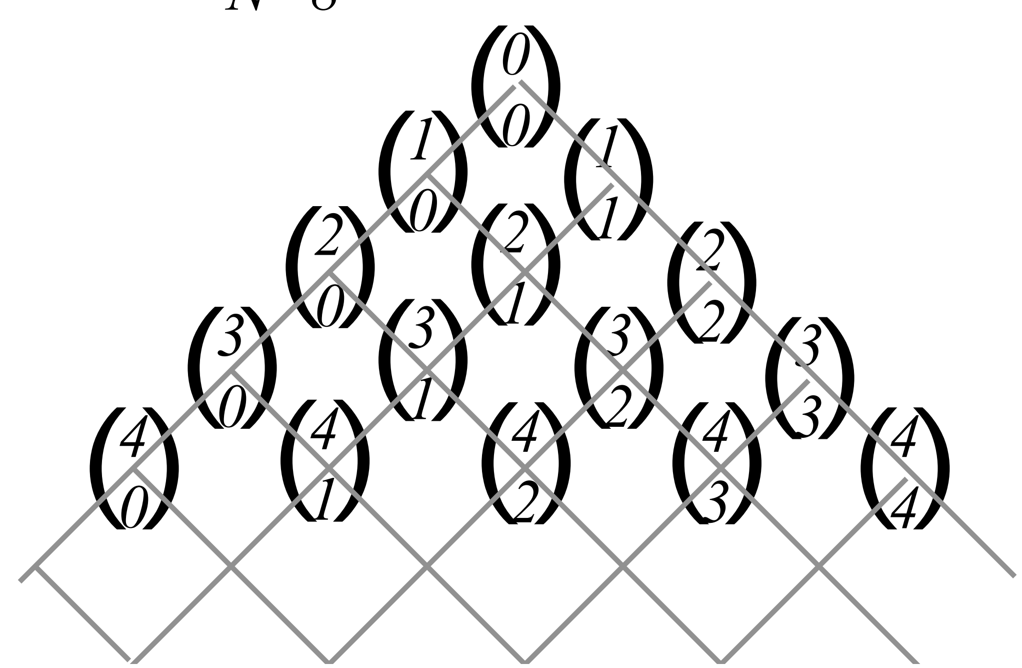
Introducing U(N)

(a) N-D Oscillator Degeneracy ℓ of quamtum level υ



(b) Stacking numbers





(c) Binomial coefficients

$$\frac{(N-1+\upsilon)!}{(N-1)!\,\upsilon!} = \binom{N-1+\upsilon}{\upsilon} = \binom{N-1+\upsilon}{N-1}$$

Introducing U(3)

(b) N-particle 3-level states ...or spin-1 states

[1] $= |1\ 0\ 0\rangle = a_1^\dagger\ |0\ 0\ 0\rangle$

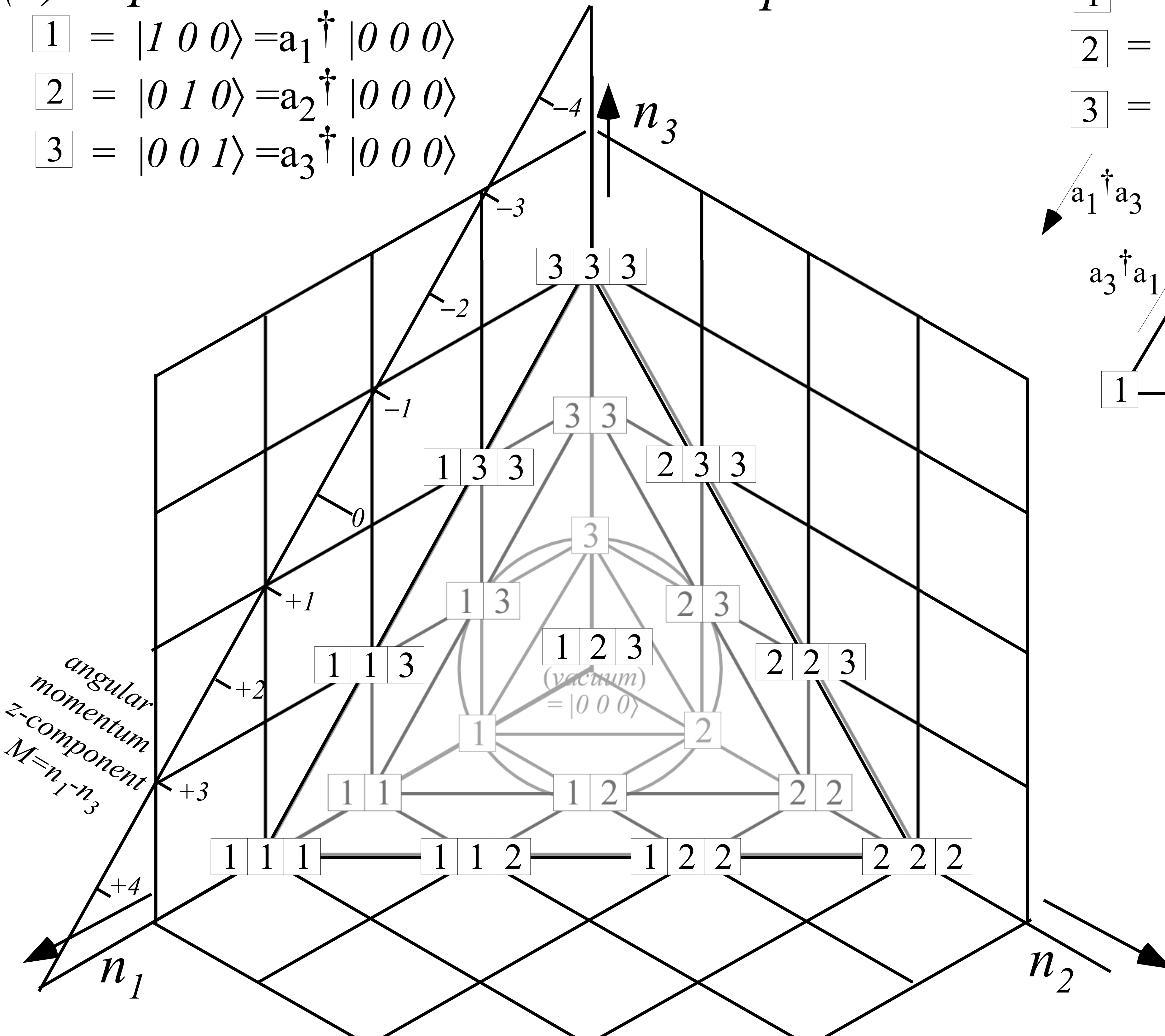
[2] $= |0\ 1\ 0\rangle = a_2^\dagger\ |0\ 0\ 0\rangle$

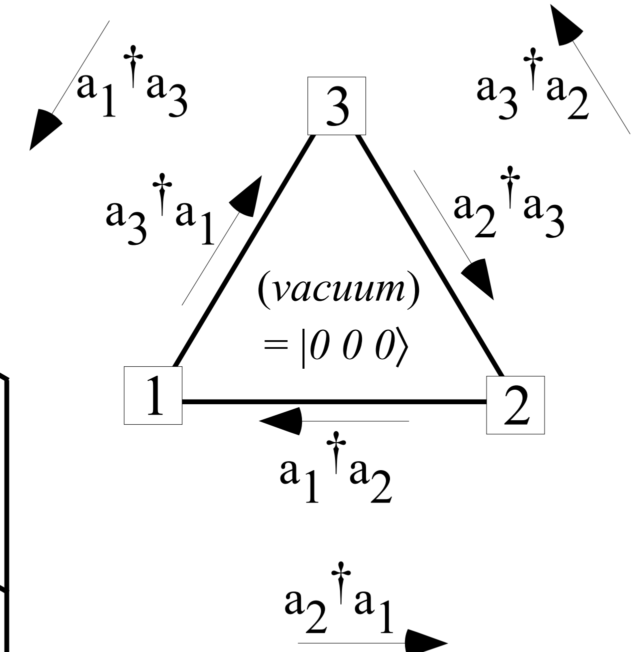
[3] $= |0\ 0\ 1\rangle = a_3^\dagger\ |0\ 0\ 0\rangle$

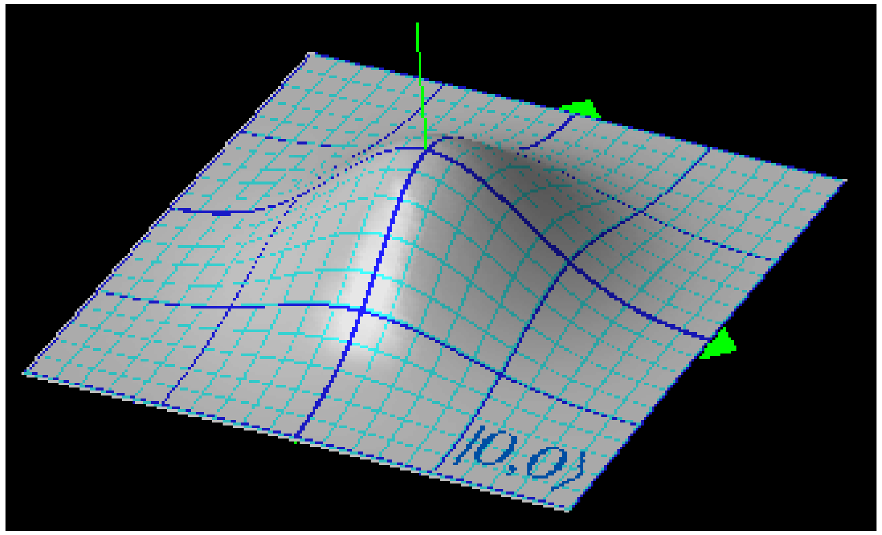
[1] $= |\uparrow\rangle = \left|{j=1 \atop m=+1}\right\rangle$

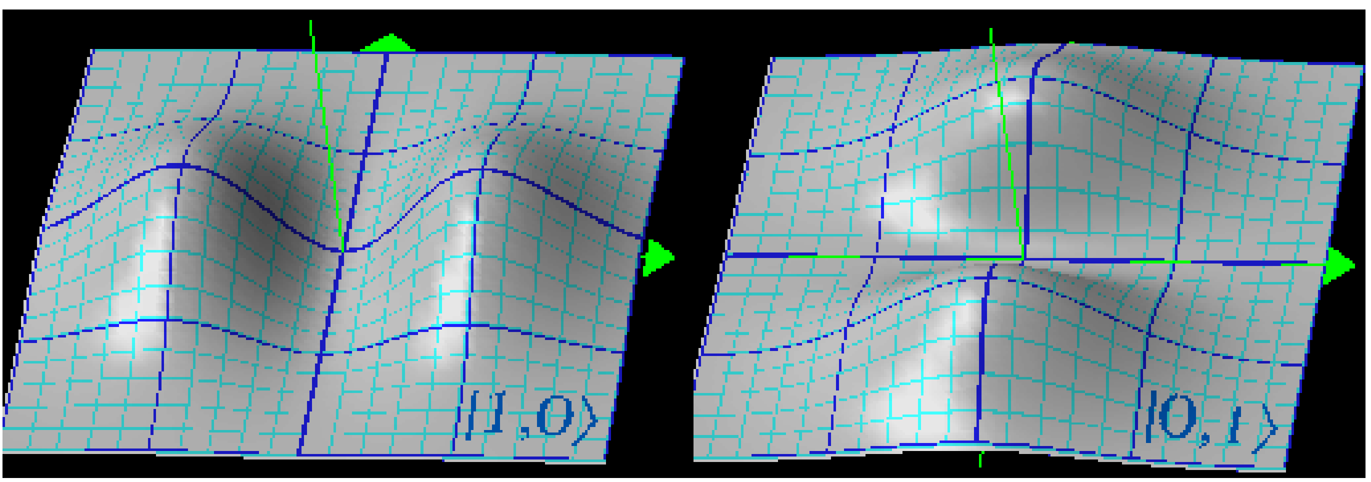
[2] $= |\leftrightarrow\rangle = \left|{j=1 \atop m=0}\right\rangle$

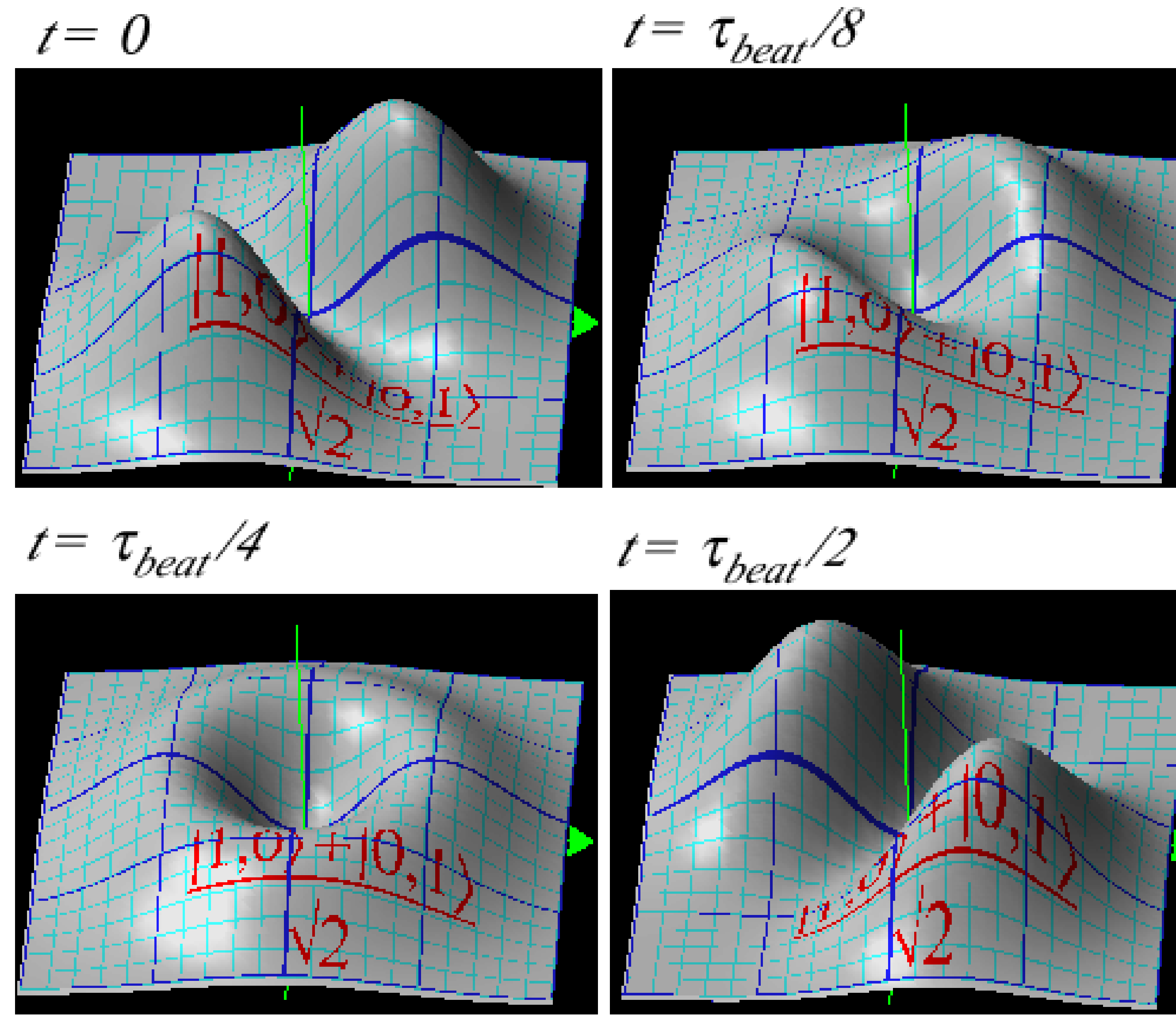
[3] $= |\downarrow\rangle = \left|{j=1 \atop m=-1}\right\rangle$











$$) = \frac{1}{2}\left|\psi_{10}(x_1,x_2)e^{-i\omega_{10}t} + \psi_{01}(x_1,x_2)e^{-i\omega_{01}t}\right|^2 e^{-\left(x_1^2+x_2^2\right)} = \frac{e^{-\left(x_1^2+x_2^2\right)}}{2\pi}\left|\sqrt{2}x_1e^{-i\omega_{10}t} + \sqrt{2}x_1e^{-i\omega_{(}}\right.$$

$$= \frac{e^{-\left(x_1^2+x_2^2\right)}}{\pi}\left(x_1^2 + x_2^2 + 2x_1x_2\cos(\omega_{10} - \omega_{01})t\right) = \frac{e^{-\left(x_1^2+x_2^2\right)}}{\pi}\begin{cases} \left|x_1 + x_2\right|^2 & for:\ t=0 \\ x_1^2 + x_2^2 & for:\ t=\tau_{\text{beat}}/4 \\ \left|x_1 - x_2\right|^2 & for:\ t=\tau_{\text{beat}}/2 \end{cases}$$

Symmetry group $\mathscr{G}$=U(2) representations, 2D HO Hamiltonian $\mathbf{H}=\hbar\omega_{ab}\mathbf{a}_a^\dagger\mathbf{a}_b$ operators, 2D HO wave eigenfunctions $\Psi_{n,m}$, and coherent [α] states

Factoring 2D-HO Hamiltonian $\mathbf{H}=\frac{A}{2}\left(\mathbf{p}_1^2+\mathbf{x}_1^2\right)+B\left(\mathbf{x}_1\mathbf{x}_2+\mathbf{p}_1\mathbf{p}_2\right)+C\left(\mathbf{x}_1\mathbf{p}_2-\mathbf{x}_2\mathbf{p}_1\right)+\frac{D}{2}\left(\mathbf{p}_2^2+\mathbf{x}_2^2\right)$

- *2D-Oscillator basic states and operations*
  - *Commutation relations*
  - *Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry*
    - *Anti-commutation relations*
  - *Two-dimensional (or 2-particle) base states: ket-kets and bra-bras*
    - *Outer product arrays*
    - *Entangled 2-particle states*
- *Two-particle (or 2-dimensional) matrix operators*
  - *U(2) Hamiltonian and irreducible representations*
  - *2D-Oscillator states related 3D angular momentum multiplets*

R(3) Angular momentum generators by U(2) analysis

Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$

SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Mostly Notation and Bookkeeping :

R(3) Angular momentum generators by U(2) analysis

Class.4 p71-75

(υ=*1*) or (j=*1/2*) block **H** matrices of U(2) oscillator

Use irreps of unit operator $\mathbf{S}_0 = \mathbf{1}$ and spin operators $\{ \mathbf{S}_X, \mathbf{S}_Y, \mathbf{S}_Z \}$. (also known as: $\{ \mathbf{S}_B, \mathbf{S}_C, \mathbf{S}_A \}$)

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + 2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} + (A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

R(3) Angular momentum generators by U(2) analysis

Class.4 p71-75

($\upsilon=1$) or ($j=1/2$) block **H** matrices of U(2) oscillator

Use irreps of unit operator $\mathbf{S}_0=\mathbf{1}$ and spin operators $\{\mathbf{S}_X,\mathbf{S}_Y,\mathbf{S}_Z\}$. (also known as: $\{\mathbf{S}_B,\mathbf{S}_C,\mathbf{S}_A\}$)

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + 2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} + (A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

($\upsilon=2$) or ($j=1$) 3-by-3 block uses their vector irreps.

$$\begin{pmatrix} 2A & \sqrt{2}(B-iC) & \cdot \\ \sqrt{2}(B+iC) & A+D & \sqrt{2}(B-iC) \\ \cdot & \sqrt{2}(B+iC) & 2D \end{pmatrix} = (A+D)\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} + 2B\begin{pmatrix} \cdot & \frac{\sqrt{2}}{2} & \cdot \\ \frac{\sqrt{2}}{2} & \cdot & \frac{\sqrt{2}}{2} \\ \cdot & \frac{\sqrt{2}}{2} & \cdot \end{pmatrix} + 2C\begin{pmatrix} \cdot & -i\frac{\sqrt{2}}{2} & \cdot \\ i\frac{\sqrt{2}}{2} & \cdot & -i\frac{\sqrt{2}}{2} \\ \cdot & i\frac{\sqrt{2}}{2} & \cdot \end{pmatrix} + (A-D)\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

R(3) Angular momentum generators by U(2) analysis

Class.4 p71-75

($\upsilon=1$) or ($j=1/2$) block **H** matrices of U(2) oscillator

Use irreps of unit operator $\mathbf{S}_0 = \mathbf{1}$ and spin operators $\{\mathbf{S}_X, \mathbf{S}_Y, \mathbf{S}_Z\}$. (also known as: $\{\mathbf{S}_B, \mathbf{S}_C, \mathbf{S}_A\}$)

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + 2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} + (A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

($\upsilon=2$) or ($j=1$) 3-by-3 block uses their vector irreps.

$$\begin{pmatrix} 2A & \sqrt{2}(B-iC) & \cdot \\ \sqrt{2}(B+iC) & A+D & \sqrt{2}(B-iC) \\ \cdot & \sqrt{2}(B+iC) & 2D \end{pmatrix} = (A+D)\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} + 2B\begin{pmatrix} \cdot & \frac{\sqrt{2}}{2} & \cdot \\ \frac{\sqrt{2}}{2} & \cdot & \frac{\sqrt{2}}{2} \\ \cdot & \frac{\sqrt{2}}{2} & \cdot \end{pmatrix} + 2C\begin{pmatrix} \cdot & -i\frac{\sqrt{2}}{2} & \cdot \\ i\frac{\sqrt{2}}{2} & \cdot & -i\frac{\sqrt{2}}{2} \\ \cdot & i\frac{\sqrt{2}}{2} & \cdot \end{pmatrix} + (A-D)\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

($\upsilon=3$) or ($j=3/2$) 4-by-4 block uses Dirac spinor irreps.

$$\begin{pmatrix} 3A & \sqrt{3}(B-iC) & & \\ \sqrt{3}(B+iC) & 2A+D & \sqrt{4}(B-iC) & \\ & \sqrt{4}(B+iC) & A+2D & \sqrt{3}(B-iC) \\ & & \sqrt{3}(B+iC) & 3D \end{pmatrix} = \frac{3(A+D)}{2}\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} + 2B\begin{pmatrix} \cdot & \frac{\sqrt{3}}{2} & \cdot & \cdot \\ \frac{\sqrt{3}}{2} & \cdot & \frac{\sqrt{4}}{2} & \cdot \\ \cdot & \frac{\sqrt{4}}{2} & \cdot & \frac{\sqrt{3}}{2} \\ \cdot & \cdot & \frac{\sqrt{3}}{2} & \cdot \end{pmatrix} + 2C\begin{pmatrix} \cdot & -i\frac{\sqrt{3}}{2} & \cdot & \cdot \\ i\frac{\sqrt{3}}{2} & \cdot & -i\frac{\sqrt{4}}{2} & \cdot \\ \cdot & i\frac{\sqrt{4}}{2} & \cdot & -i\frac{\sqrt{3}}{2} \\ \cdot & \cdot & i\frac{\sqrt{3}}{2} & \cdot \end{pmatrix} + (A-D)\begin{pmatrix} \frac{3}{2} & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{2} & \cdot & \cdot \\ \cdot & \cdot & -\frac{1}{2} & \cdot \\ \cdot & \cdot & \cdot & -\frac{3}{2} \end{pmatrix}$$

R(3) Angular momentum generators by U(2) analysis

Class.4 p71-75

($\upsilon=1$) or ($j=1/2$) block **H** matrices of U(2) oscillator

Use irreps of unit operator $\mathbf{S}_0 = \mathbf{1}$ and spin operators $\{ \mathbf{S}_X, \mathbf{S}_Y, \mathbf{S}_Z \}$. (also known as: $\{ \mathbf{S}_B, \mathbf{S}_C, \mathbf{S}_A \}$)

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + 2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} + (A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

($\upsilon=2$) or ($j=1$) 3-by-3 block uses their vector irreps.

$$\begin{pmatrix} 2A & \sqrt{2}(B-iC) & \cdot \\ \sqrt{2}(B+iC) & A+D & \sqrt{2}(B-iC) \\ \cdot & \sqrt{2}(B+iC) & 2D \end{pmatrix} = (A+D)\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} + 2B\begin{pmatrix} \cdot & \frac{\sqrt{2}}{2} & \cdot \\ \frac{\sqrt{2}}{2} & \cdot & \frac{\sqrt{2}}{2} \\ \cdot & \frac{\sqrt{2}}{2} & \cdot \end{pmatrix} + 2C\begin{pmatrix} \cdot & -i\frac{\sqrt{2}}{2} & \cdot \\ i\frac{\sqrt{2}}{2} & \cdot & -i\frac{\sqrt{2}}{2} \\ \cdot & i\frac{\sqrt{2}}{2} & \cdot \end{pmatrix} + (A-D)\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

($\upsilon=3$) or ($j=3/2$) 4-by-4 block uses Dirac spinor irreps.

$$\begin{pmatrix} 3A & \sqrt{3}(B-iC) & & \\ \sqrt{3}(B+iC) & 2A+D & \sqrt{4}(B-iC) & \\ & \sqrt{4}(B+iC) & A+2D & \sqrt{3}(B-iC) \\ & & \sqrt{3}(B+iC) & 3D \end{pmatrix} = \frac{3(A+D)}{2}\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} + 2B\begin{pmatrix} \cdot & \frac{\sqrt{3}}{2} & \cdot & \cdot \\ \frac{\sqrt{3}}{2} & \cdot & \frac{\sqrt{4}}{2} & \cdot \\ \cdot & \frac{\sqrt{4}}{2} & \cdot & \frac{\sqrt{3}}{2} \\ \cdot & \cdot & \frac{\sqrt{3}}{2} & \cdot \end{pmatrix} + 2C\begin{pmatrix} \cdot & -i\frac{\sqrt{3}}{2} & \cdot & \cdot \\ i\frac{\sqrt{3}}{2} & \cdot & -i\frac{\sqrt{4}}{2} & \cdot \\ \cdot & i\frac{\sqrt{4}}{2} & \cdot & -i\frac{\sqrt{3}}{2} \\ \cdot & \cdot & i\frac{\sqrt{3}}{2} & \cdot \end{pmatrix} + (A-D)\begin{pmatrix} \frac{3}{2} & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{2} & \cdot & \cdot \\ \cdot & \cdot & -\frac{1}{2} & \cdot \\ \cdot & \cdot & \cdot & -\frac{3}{2} \end{pmatrix}$$

($\upsilon=2j$) or ($2j+1$)-by-($2j+1$) block uses $D^{(j)}(\mathbf{s}_\mu)$ irreps of U(2) or R(3).

$$\langle \mathbf{H} \rangle^{j-block} = 2j\Omega_0 \langle \mathbf{1} \rangle^j + \Omega_X \langle \mathbf{s}_X \rangle^j + \Omega_Y \langle \mathbf{s}_Y \rangle^j + \Omega_Z \langle \mathbf{s}_Z \rangle^j$$

R(3) Angular momentum generators by U(2) analysis

($\upsilon=1$) or ($j=1/2$) block **H** matrices of U(2) oscillator

Use irreps of unit operator $\mathbf{S}_0 = \mathbf{1}$ and spin operators $\{ \mathbf{S}_X, \mathbf{S}_Y, \mathbf{S}_Z \}$. (also known as: $\{ \mathbf{S}_B, \mathbf{S}_C, \mathbf{S}_A \}$)

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + 2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} + (A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

($\upsilon=2$) or ($j=1$) 3-by-3 block uses their vector irreps.

$$\begin{pmatrix} 2A & \sqrt{2}(B-iC) & \cdot \\ \sqrt{2}(B+iC) & A+D & \sqrt{2}(B-iC) \\ \cdot & \sqrt{2}(B+iC) & 2D \end{pmatrix} = (A+D)\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} + 2B\begin{pmatrix} \cdot & \frac{\sqrt{2}}{2} & \cdot \\ \frac{\sqrt{2}}{2} & \cdot & \frac{\sqrt{2}}{2} \\ \cdot & \frac{\sqrt{2}}{2} & \cdot \end{pmatrix} + 2C\begin{pmatrix} \cdot & -i\frac{\sqrt{2}}{2} & \cdot \\ i\frac{\sqrt{2}}{2} & \cdot & -i\frac{\sqrt{2}}{2} \\ \cdot & i\frac{\sqrt{2}}{2} & \cdot \end{pmatrix} + (A-D)\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

($\upsilon=3$) or ($j=3/2$) 4-by-4 block uses Dirac spinor irreps.

$$\begin{pmatrix} 3A & \sqrt{3}(B-iC) & & \\ \sqrt{3}(B+iC) & 2A+D & \sqrt{4}(B-iC) & \\ & \sqrt{4}(B+iC) & A+2D & \sqrt{3}(B-iC) \\ & & \sqrt{3}(B+iC) & 3D \end{pmatrix} = \frac{3(A+D)}{2}\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} + 2B\begin{pmatrix} \cdot & \frac{\sqrt{3}}{2} & \cdot & \cdot \\ \frac{\sqrt{3}}{2} & \cdot & \frac{\sqrt{4}}{2} & \cdot \\ \cdot & \frac{\sqrt{4}}{2} & \cdot & \frac{\sqrt{3}}{2} \\ \cdot & \cdot & \frac{\sqrt{3}}{2} & \cdot \end{pmatrix} + 2C\begin{pmatrix} \cdot & -i\frac{\sqrt{3}}{2} & \cdot & \cdot \\ i\frac{\sqrt{3}}{2} & \cdot & -i\frac{\sqrt{4}}{2} & \cdot \\ \cdot & i\frac{\sqrt{4}}{2} & \cdot & -i\frac{\sqrt{3}}{2} \\ \cdot & \cdot & i\frac{\sqrt{3}}{2} & \cdot \end{pmatrix} + (A-D)\begin{pmatrix} \frac{3}{2} & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{2} & \cdot & \cdot \\ \cdot & \cdot & -\frac{1}{2} & \cdot \\ \cdot & \cdot & \cdot & -\frac{3}{2} \end{pmatrix}$$

($\upsilon=2j$) or ($2j+1$)-by-($2j+1$) block uses $D^{(j)}(\mathbf{s}_\mu)$ irreps of U(2) or R(3).

$$\langle \mathbf{H} \rangle^{j-block} = 2j\Omega_0 \langle \mathbf{1} \rangle^j + \Omega_X \langle \mathbf{s}_X \rangle^j + \Omega_Y \langle \mathbf{s}_Y \rangle^j + \Omega_Z \langle \mathbf{s}_Z \rangle^j$$

All j-block matrix operators factor into *raise-n-lower* operators $\mathbf{s}_\pm = \mathbf{s}_X \pm i\mathbf{s}_Y$ plus the diagonal $\mathbf{s}_Z$

$$\langle \mathbf{H} \rangle^{j-block} = 2j\Omega_0 \langle \mathbf{1} \rangle^j + \left[(\Omega_X - i\Omega_Y)\langle \mathbf{s}_X + i\mathbf{s}_Y \rangle^j + (\Omega_X + i\Omega_Y)\langle \mathbf{s}_X - i\mathbf{s}_Y \rangle^j \right]/2 + \Omega_Z \langle \mathbf{s}_Z \rangle^j$$

Symmetry group $\mathscr{G}$=U(2) representations, 2D HO Hamiltonian $\mathbf{H}=\hbar\omega_{ab}\mathbf{a}_a^\dagger\mathbf{a}_b$ operators, 2D HO wave eigenfunctions $\Psi_{n,m}$, and coherent $[\alpha]$ states

Factoring 2D-HO Hamiltonian $\mathbf{H}=\frac{A}{2}\left(\mathbf{p}_1^2+\mathbf{x}_1^2\right)+B\left(\mathbf{x}_1\mathbf{x}_2+\mathbf{p}_1\mathbf{p}_2\right)+C\left(\mathbf{x}_1\mathbf{p}_2-\mathbf{x}_2\mathbf{p}_1\right)+\frac{D}{2}\left(\mathbf{p}_2^2+\mathbf{x}_2^2\right)$

2D-Oscillator basic states and operations
- *Commutation relations*
- *Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry*
  - *Anti-commutation relations*
- *Two-dimensional (or 2-particle) base states: ket-kets and bra-bras*
  - *Outer product arrays*
  - *Entangled 2-particle states*

Two-particle (or 2-dimensional) matrix operators
- *U(2) Hamiltonian and irreducible representations*
- *2D-Oscillator states related 3D angular momentum multiplets*

R(3) Angular momentum generators by U(2) analysis

Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$

SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Mostly Notation and Bookkeeping :

Angular momentum raise-n-lower operators $\mathbf{S}_+$ and $\mathbf{S}_-$

Class.8 p82-85 (this class)

$\mathbf{s}_+ = \mathbf{s}_X + i\mathbf{s}_Y$ and $\mathbf{s}_- = \mathbf{s}_X - i\mathbf{s}_Y = \mathbf{s}_+^\dagger$

Starting with *j=1/2* we see that $\mathbf{S}_+$ is an *elementary projection operator* $\mathbf{e}_{12} = |1\rangle\langle 2| = \mathbf{P}_{12}$

$$\langle \mathbf{s}_+ \rangle^{\frac{1}{2}} = D^{\frac{1}{2}}(\mathbf{s}_+) = D^{\frac{1}{2}}(\mathbf{s}_X + i\mathbf{s}_Y) = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + i \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathbf{P}_{12}$$

Such operators can be upgraded to *creation-destruction operator* combinations $\mathbf{a}^\dagger\mathbf{a}$

$$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2 = \mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow \quad , \qquad \mathbf{s}_- = \left(\mathbf{a}_1^\dagger \mathbf{a}_2\right)^\dagger = \mathbf{a}_2^\dagger \mathbf{a}_1 = \mathbf{a}_\downarrow^\dagger \mathbf{a}_\uparrow$$

Angular momentum raise-n-lower operators $\mathbf{S}_+$ and $\mathbf{S}_-$

Class.8 p82-85 (this class)

$\mathbf{s}_+ = \mathbf{s}_X + i\mathbf{s}_Y$ and $\mathbf{s}_- = \mathbf{s}_X - i\mathbf{s}_Y = \mathbf{s}_+^\dagger$

Starting with *j=1/2* we see that $\mathbf{S}+$ is an *elementary projection operator* $\mathbf{e}_{12} = |1\rangle\langle 2| = \mathbf{P}_{12}$

$$\langle \mathbf{s}_+ \rangle^{\frac{1}{2}} = D^{\frac{1}{2}}(\mathbf{s}_+) = D^{\frac{1}{2}}(\mathbf{s}_X + i\mathbf{s}_Y) = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + i \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathbf{P}_{12}$$

Such operators can be upgraded to *creation-destruction operator* combinations $\mathbf{a}^\dagger\mathbf{a}$

$$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2 = \mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow \quad , \qquad \mathbf{s}_- = \left(\mathbf{a}_1^\dagger \mathbf{a}_2\right)^\dagger = \mathbf{a}_2^\dagger \mathbf{a}_1 = \mathbf{a}_\downarrow^\dagger \mathbf{a}_\uparrow$$

Hamilton-Pauli-Jordan representation of $\mathbf{s}_Z$ is: $\langle \mathbf{s}_Z \rangle^{\left(\frac{1}{2}\right)} = D^{\left(\frac{1}{2}\right)}(\mathbf{s}_Z) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

This suggests an $\mathbf{a}^\dagger\mathbf{a}$ form for $\mathbf{s}_Z$.

$$\mathbf{s}_Z = \tfrac{1}{2}\left(\mathbf{a}_1^\dagger \mathbf{a}_1 - \mathbf{a}_2^\dagger \mathbf{a}_2\right) = \tfrac{1}{2}\left(\mathbf{a}_\uparrow^\dagger \mathbf{a}_\uparrow - \mathbf{a}_\downarrow^\dagger \mathbf{a}_\downarrow\right)$$

Angular momentum raise-n-lower operators $\mathbf{S}_+$ and $\mathbf{S}_-$

$\mathbf{s}_+ = \mathbf{s}_X + i\mathbf{s}_Y$ and $\mathbf{s}_- = \mathbf{s}_X - i\mathbf{s}_Y = \mathbf{s}_+^\dagger$

Starting with *j=1/2* we see that $\mathbf{S}+$ is an *elementary projection operator* $\mathbf{e}_{12} = |1\rangle\langle 2| = \mathbf{P}_{12}$

$$\langle \mathbf{s}_+ \rangle^{\frac{1}{2}} = D^{\frac{1}{2}}(\mathbf{s}_+) = D^{\frac{1}{2}}(\mathbf{s}_X + i\mathbf{s}_Y) = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + i \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathbf{P}_{12}$$

Such operators can be upgraded to *creation-destruction operator* combinations $\mathbf{a}^\dagger\mathbf{a}$

$$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2 = \mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow \quad , \qquad \mathbf{s}_- = (\mathbf{a}_1^\dagger \mathbf{a}_2)^\dagger = \mathbf{a}_2^\dagger \mathbf{a}_1 = \mathbf{a}_\downarrow^\dagger \mathbf{a}_\uparrow$$

Hamilton-Pauli-Jordan representation of $\mathbf{s}_Z$ is: $\langle \mathbf{s}_Z \rangle^{\left(\frac{1}{2}\right)} = D^{\left(\frac{1}{2}\right)}(\mathbf{s}_Z) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

This suggests an $\mathbf{a}^\dagger\mathbf{a}$ form for $\mathbf{s}_Z$.

$$\mathbf{s}_Z = \tfrac{1}{2}(\mathbf{a}_1^\dagger \mathbf{a}_1 - \mathbf{a}_2^\dagger \mathbf{a}_2) = \tfrac{1}{2}(\mathbf{a}_\uparrow^\dagger \mathbf{a}_\uparrow - \mathbf{a}_\downarrow^\dagger \mathbf{a}_\downarrow)$$

Let $\mathbf{a}_1^\dagger = \mathbf{a}_\uparrow^\dagger$ create up-spin $\uparrow$

$$|1\rangle = |\uparrow\rangle = \left| \begin{matrix} 1/2 \\ +1/2 \end{matrix} \right\rangle = \mathbf{a}_1^\dagger |0\rangle = \mathbf{a}_\uparrow^\dagger |0\rangle$$

Angular momentum raise-n-lower operators $\mathbf{S}_+$ and $\mathbf{S}_-$

$$\mathbf{s}_+ = \mathbf{s}_X + i\mathbf{s}_Y \quad \text{and} \quad \mathbf{s}_- = \mathbf{s}_X - i\mathbf{s}_Y = \mathbf{s}_+^\dagger$$

Starting with *j=1/2* we see that $\mathbf{S}+$ is an *elementary projection operator* $\mathbf{e}_{12} = |1\rangle\langle 2| = \mathbf{P}_{12}$

$$\langle \mathbf{s}_+ \rangle^{\frac{1}{2}} = D^{\frac{1}{2}}(\mathbf{s}_+) = D^{\frac{1}{2}}(\mathbf{s}_X + i\mathbf{s}_Y) = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + i \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathbf{P}_{12}$$

Such operators can be upgraded to *creation-destruction operator* combinations $\mathbf{a}^\dagger\mathbf{a}$

$$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2 = \mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow \quad , \qquad \mathbf{s}_- = \left(\mathbf{a}_1^\dagger \mathbf{a}_2\right)^\dagger = \mathbf{a}_2^\dagger \mathbf{a}_1 = \mathbf{a}_\downarrow^\dagger \mathbf{a}_\uparrow$$

Hamilton-Pauli-Jordan representation of $\mathbf{s}_Z$ is: $\langle \mathbf{s}_Z \rangle^{\left(\frac{1}{2}\right)} = D^{\left(\frac{1}{2}\right)}(\mathbf{s}_Z) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

This suggests an $\mathbf{a}^\dagger\mathbf{a}$ form for $\mathbf{s}_Z$.

$$\mathbf{s}_Z = \tfrac{1}{2}\left(\mathbf{a}_1^\dagger \mathbf{a}_1 - \mathbf{a}_2^\dagger \mathbf{a}_2\right) = \tfrac{1}{2}\left(\mathbf{a}_\uparrow^\dagger \mathbf{a}_\uparrow - \mathbf{a}_\downarrow^\dagger \mathbf{a}_\downarrow\right)$$

Let $\mathbf{a}_1^\dagger = \mathbf{a}_\uparrow^\dagger$ create up-spin $\uparrow$

$$|1\rangle = |\uparrow\rangle = \left| \begin{matrix} 1/2 \\ +1/2 \end{matrix} \right\rangle = \mathbf{a}_1^\dagger|0\rangle = \mathbf{a}_\uparrow^\dagger|0\rangle$$

Let $\mathbf{a}_2^\dagger = \mathbf{a}_\downarrow^\dagger$ create dn-spin $\downarrow$

$$|2\rangle = |\downarrow\rangle = \left| \begin{matrix} 1/2 \\ -1/2 \end{matrix} \right\rangle = \mathbf{a}_2^\dagger|0\rangle = \mathbf{a}_\downarrow^\dagger|0\rangle$$

Angular momentum raise-n-lower operators $\mathbf{S}_+$ and $\mathbf{S}_-$

$\mathbf{s}_+ = \mathbf{s}_X + i\mathbf{s}_Y$ and $\mathbf{s}_- = \mathbf{s}_X - i\mathbf{s}_Y = \mathbf{s}_+^\dagger$

Starting with *j=1/2* we see that $\mathbf{S}+$ is an *elementary projection operator* $\mathbf{e}_{12} = |1\rangle\langle 2| = \mathbf{P}_{12}$

$$\langle \mathbf{s}_+ \rangle^{\frac{1}{2}} = D^{\frac{1}{2}}(\mathbf{s}_+) = D^{\frac{1}{2}}(\mathbf{s}_X + i\mathbf{s}_Y) = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + i \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathbf{P}_{12}$$

Such operators can be upgraded to *creation-destruction operator* combinations $\mathbf{a}^\dagger\mathbf{a}$

$$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2 = \mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow \;, \qquad \mathbf{s}_- = (\mathbf{a}_1^\dagger \mathbf{a}_2)^\dagger = \mathbf{a}_2^\dagger \mathbf{a}_1 = \mathbf{a}_\downarrow^\dagger \mathbf{a}_\uparrow$$

Hamilton-Pauli-Jordan representation of $\mathbf{s}_Z$ is: $\langle \mathbf{s}_Z \rangle^{\left(\frac{1}{2}\right)} = D^{\left(\frac{1}{2}\right)}(\mathbf{s}_Z) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

This suggests an $\mathbf{a}^\dagger\mathbf{a}$ form for $\mathbf{s}_Z$.

$$\mathbf{s}_Z = \tfrac{1}{2}(\mathbf{a}_1^\dagger \mathbf{a}_1 - \mathbf{a}_2^\dagger \mathbf{a}_2) = \tfrac{1}{2}(\mathbf{a}_\uparrow^\dagger \mathbf{a}_\uparrow - \mathbf{a}_\downarrow^\dagger \mathbf{a}_\downarrow)$$

Let $\mathbf{a}_1^\dagger = \mathbf{a}_\uparrow^\dagger$ create up-spin ↑

$$|1\rangle = |\uparrow\rangle = \left| \begin{matrix} 1/2 \\ +1/2 \end{matrix} \right\rangle = \mathbf{a}_1^\dagger |0\rangle = \mathbf{a}_\uparrow^\dagger |0\rangle$$

Let $\mathbf{a}_2^\dagger = \mathbf{a}_\downarrow^\dagger$ create dn-spin ↓

$$|2\rangle = |\downarrow\rangle = \left| \begin{matrix} 1/2 \\ -1/2 \end{matrix} \right\rangle = \mathbf{a}_2^\dagger |0\rangle = \mathbf{a}_\downarrow^\dagger |0\rangle$$

destroys dn-spin ↓

creates up-spin ↑

to raise angular momentum by one $\hbar$ unit

$$\mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow |\downarrow\rangle = |\uparrow\rangle \quad \text{or:} \quad \mathbf{a}_1^\dagger \mathbf{a}_2 |2\rangle = |1\rangle$$

Angular momentum raise-n-lower operators $\mathbf{S}_+$ and $\mathbf{S}_-$

$\mathbf{s}_+ = \mathbf{s}_X + i\mathbf{s}_Y$ and $\mathbf{s}_- = \mathbf{s}_X - i\mathbf{s}_Y = \mathbf{s}_+^\dagger$

Starting with *j=1/2* we see that $\mathbf{S}+$ is an *elementary projection operator* $\mathbf{e}_{12} = |1\rangle\langle 2| = \mathbf{P}_{12}$

$$\langle \mathbf{s}_+ \rangle^{\frac{1}{2}} = D^{\frac{1}{2}}(\mathbf{s}_+) = D^{\frac{1}{2}}(\mathbf{s}_X + i\mathbf{s}_Y) = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + i \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathbf{P}_{12}$$

Such operators can be upgraded to *creation-destruction operator* combinations $\mathbf{a}^\dagger\mathbf{a}$

$$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2 = \mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow \quad , \qquad \mathbf{s}_- = (\mathbf{a}_1^\dagger \mathbf{a}_2)^\dagger = \mathbf{a}_2^\dagger \mathbf{a}_1 = \mathbf{a}_\downarrow^\dagger \mathbf{a}_\uparrow$$

Hamilton-Pauli-Jordan representation of $\mathbf{s}_Z$ is: $\langle \mathbf{s}_Z \rangle^{\left(\frac{1}{2}\right)} = D^{\left(\frac{1}{2}\right)}(\mathbf{s}_Z) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

This suggests an $\mathbf{a}^\dagger\mathbf{a}$ form for $\mathbf{s}_Z$.

$$\mathbf{s}_Z = \tfrac{1}{2}(\mathbf{a}_1^\dagger \mathbf{a}_1 - \mathbf{a}_2^\dagger \mathbf{a}_2) = \tfrac{1}{2}(\mathbf{a}_\uparrow^\dagger \mathbf{a}_\uparrow - \mathbf{a}_\downarrow^\dagger \mathbf{a}_\downarrow)$$

Let $\mathbf{a}_1^\dagger = \mathbf{a}_\uparrow^\dagger$ create up-spin ↑

$$|1\rangle = |\uparrow\rangle = \left| \begin{matrix} 1/2 \\ +1/2 \end{matrix} \right\rangle = \mathbf{a}_1^\dagger |0\rangle = \mathbf{a}_\uparrow^\dagger |0\rangle$$

Let $\mathbf{a}_2^\dagger = \mathbf{a}_\downarrow^\dagger$ create dn-spin ↓

$$|2\rangle = |\downarrow\rangle = \left| \begin{matrix} 1/2 \\ -1/2 \end{matrix} \right\rangle = \mathbf{a}_2^\dagger |0\rangle = \mathbf{a}_\downarrow^\dagger |0\rangle$$

destroys dn-spin ↓

creates up-spin ↑

to raise angular momentum by one $\hbar$ unit

$$\mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow |\downarrow\rangle = |\uparrow\rangle \quad \text{or:} \quad \mathbf{a}_1^\dagger \mathbf{a}_2 |2\rangle = |1\rangle$$

$\mathbf{s}_- = \mathbf{a}_2^\dagger \mathbf{a}_1 = \mathbf{a}_\downarrow^\dagger \mathbf{a}_\uparrow$ destroys up-spin ↑

creates dn-spin ↓

to lower angular momentum by one $\hbar$ unit

$$\mathbf{a}_\downarrow^\dagger \mathbf{a}_\uparrow |\uparrow\rangle = |\downarrow\rangle \quad \text{or:} \quad \mathbf{a}_2^\dagger \mathbf{a}_1 |1\rangle = |2\rangle$$

Symmetry group $\mathscr{G}$=U(2) representations, 2D HO Hamiltonian $\mathbf{H}=\hbar\omega_{ab}\mathbf{a}_a{}^\dagger\mathbf{a}_b$ operators,
2D HO wave eigenfunctions $\Psi_{n,m}$, and coherent [α] states

Factoring 2D-HO Hamiltonian $\mathbf{H}=\frac{A}{2}\left(\mathbf{p}_1^2+\mathbf{x}_1^2\right)+B\left(\mathbf{x}_1\mathbf{x}_2+\mathbf{p}_1\mathbf{p}_2\right)+C\left(\mathbf{x}_1\mathbf{p}_2-\mathbf{x}_2\mathbf{p}_1\right)+\frac{D}{2}\left(\mathbf{p}_2^2+\mathbf{x}_2^2\right)$

2D-Oscillator basic states and operations
- *Commutation relations*
- *Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry*
  - *Anti-commutation relations*
- *Two-dimensional (or 2-particle) base states: ket-kets and bra-bras*
  - *Outer product arrays*
  - *Entangled 2-particle states*

Two-particle (or 2-dimensional) matrix operators
- *U(2) Hamiltonian and irreducible representations*
- *2D-Oscillator states related 3D angular momentum multiplets*

Mostly Notation and Bookkeeping :

R(3) Angular momentum generators by U(2) analysis

Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$

SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Class 8 p49-54 (this class)

U(2) boson oscillator states $|n_1,n_2\rangle$

Oscillator total quanta: $\upsilon=(n_1+n_2)$

$$|n_1 n_2\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{n_1}\left(\mathbf{a}_2^\dagger\right)^{n_2}}{\sqrt{n_1!\,n_2!}}|0\,0\rangle$$

SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Class 8 p49-54 (this class)

U(2) boson oscillator states $|n_1,n_2\rangle$ = R(3) spin or rotor states $\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$

Oscillator total quanta: $\upsilon=(n_1+n_2)$ Rotor total momenta: $j=\upsilon/2$

$$\left|n_1 n_2\right\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{n_1}\left(\mathbf{a}_2^\dagger\right)^{n_2}}{\sqrt{n_1!\,n_2!}}\left|0\;0\right\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{j+m}\left(\mathbf{a}_2^\dagger\right)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}\left|0\;0\right\rangle = \left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$$

$SU(2)\subset U(2)$ oscillators vs. $R(3)\subset O(3)$ rotors

Class 8 p49-54 (this class)

$U(2)$ boson oscillator states $|n_1,n_2\rangle$ = R(3) spin or rotor states $\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$

Oscillator total quanta: $\upsilon=(n_1+n_2)$ Rotor total momenta: $j=\upsilon/2$ and z-momenta: $m=(n_1-n_2)/2$

$$|n_1 n_2\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{n_1}\left(\mathbf{a}_2^\dagger\right)^{n_2}}{\sqrt{n_1!\,n_2!}}|0\,0\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{j+m}\left(\mathbf{a}_2^\dagger\right)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}|0\,0\rangle = \left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$$

$$j=\upsilon/2=(n_1+n_2)/2$$
$$m=(n_1-n_2)/2$$

$$n_1=j+m$$
$$n_2=j-m$$

$SU(2)\subset U(2)$ oscillators vs. $R(3)\subset O(3)$ rotors

Class 8 p49-54 (this class)

$U(2)$ boson oscillator states $|n_1,n_2\rangle$ = R(3) spin or rotor states $\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$

Oscillator total quanta: $\upsilon=(n_1+n_2)$ Rotor total momenta: $j=\upsilon/2$ and z-momenta: $m=(n_1-n_2)/2$

$$\left|n_1 n_2\right\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{n_1}\left(\mathbf{a}_2^\dagger\right)^{n_2}}{\sqrt{n_1!\,n_2!}}\left|0\,0\right\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{j+m}\left(\mathbf{a}_2^\dagger\right)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}\left|0\,0\right\rangle = \left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$$

$j=\upsilon/2=(n_1+n_2)/2$

$m=(n_1-n_2)/2$

$n_1=j+m$

$n_2=j-m$

$U(2)$ boson oscillator states = $U(2)$ spinor states

$$\left|n_\uparrow n_\downarrow\right\rangle = \frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{n_1}\left(\mathbf{a}_\downarrow^\dagger\right)^{n_2}}{\sqrt{n_\uparrow!\,n_\downarrow!}}\left|0\,0\right\rangle = \frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{j+m}\left(\mathbf{a}_\downarrow^\dagger\right)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}\left|0\,0\right\rangle = \left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$$

$SU(2)\subset U(2)$ oscillators vs. $R(3)\subset O(3)$ rotors

Class 8 p49-54 (this class)

$U(2)$ boson oscillator states $|n_1,n_2\rangle$ = R(3) spin or rotor states $\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$

Oscillator total quanta: $\upsilon=(n_1+n_2)$ Rotor total momenta: $j=\upsilon/2$ and z-momenta: $m=(n_1-n_2)/2$

$$\left|n_1 n_2\right\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{n_1}\left(\mathbf{a}_2^\dagger\right)^{n_2}}{\sqrt{n_1!\,n_2!}}\left|0\,0\right\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{j+m}\left(\mathbf{a}_2^\dagger\right)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}\left|0\,0\right\rangle = \left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$$

$j=\upsilon/2=(n_1+n_2)/2$

$m=(n_1-n_2)/2$

$n_1=j+m$

$n_2=j-m$

$U(2)$ boson oscillator states = $U(2)$ spinor states

$$\left|n_\uparrow n_\downarrow\right\rangle = \frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{n_1}\left(\mathbf{a}_\downarrow^\dagger\right)^{n_2}}{\sqrt{n_\uparrow!\,n_\downarrow!}}\left|0\,0\right\rangle = \frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{j+m}\left(\mathbf{a}_\downarrow^\dagger\right)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}\left|0\,0\right\rangle = \left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$$

Oscillator $\mathbf{a}^\dagger\mathbf{a}$...

$$\mathbf{a}_1^\dagger\mathbf{a}_2\left|n_1 n_2\right\rangle = \sqrt{n_1+1}\sqrt{n_2}\left|n_1+1\; n_2-1\right\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_1\left|n_1 n_2\right\rangle = \sqrt{n_1}\sqrt{n_2+1}\left|n_1-1\; n_2+1\right\rangle$$

SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Class 8 p49-54 (this class)

U(2) boson oscillator states $|n_1,n_2\rangle$ = R(3) spin or rotor states $\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$

Oscillator total quanta: $\upsilon=(n_1+n_2)$ Rotor total momenta: $j=\upsilon/2$ and z-momenta: $m=(n_1-n_2)/2$

$$\left|n_1 n_2\right\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{n_1}\left(\mathbf{a}_2^\dagger\right)^{n_2}}{\sqrt{n_1!\,n_2!}}\left|0\,0\right\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{j+m}\left(\mathbf{a}_2^\dagger\right)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}\left|0\,0\right\rangle = \left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$$

$j=\upsilon/2=(n_1+n_2)/2$

$m=(n_1-n_2)/2$

$n_1=j+m$

$n_2=j-m$

U(2) boson oscillator states = *U(2)* spinor states

$$\left|n_\uparrow n_\downarrow\right\rangle = \frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{n_1}\left(\mathbf{a}_\downarrow^\dagger\right)^{n_2}}{\sqrt{n_\uparrow!\,n_\downarrow!}}\left|0\,0\right\rangle = \frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{j+m}\left(\mathbf{a}_\downarrow^\dagger\right)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}\left|0\,0\right\rangle = \left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$$

Oscillator $\mathbf{a}^\dagger\mathbf{a}$ give $\mathbf{s}_+$ matrices.

$$\mathbf{a}_1^\dagger\mathbf{a}_2\left|n_1 n_2\right\rangle=\sqrt{n_1+1}\sqrt{n_2}\left|n_1+1\; n_2-1\right\rangle \Rightarrow \mathbf{s}_+\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle=\sqrt{j+m+1}\sqrt{j-m}\left|\begin{smallmatrix} j \\ m+1 \end{smallmatrix}\right\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_1\left|n_1 n_2\right\rangle=\sqrt{n_1}\sqrt{n_2+1}\left|n_1-1\; n_2+1\right\rangle$$

$SU(2)\subset U(2)$ oscillators vs. $R(3)\subset O(3)$ rotors

Class 8 p49-54 (this class)

$U(2)$ boson oscillator states $|n_1,n_2\rangle$ = R(3) spin or rotor states $\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$

Oscillator total quanta: $\upsilon=(n_1+n_2)$ Rotor total momenta: $j=\upsilon/2$ and z-momenta: $m=(n_1-n_2)/2$

$$\left|n_1 n_2\right\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{n_1}\left(\mathbf{a}_2^\dagger\right)^{n_2}}{\sqrt{n_1!\,n_2!}}\left|0\;0\right\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{j+m}\left(\mathbf{a}_2^\dagger\right)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}\left|0\;0\right\rangle = \left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$$

$j=\upsilon/2=(n_1+n_2)/2$

$m=(n_1-n_2)/2$

$n_1=j+m$

$n_2=j-m$

$U(2)$ boson oscillator states = $U(2)$ spinor states

$$\left|n_\uparrow n_\downarrow\right\rangle = \frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{n_1}\left(\mathbf{a}_\downarrow^\dagger\right)^{n_2}}{\sqrt{n_\uparrow!\,n_\downarrow!}}\left|0\;0\right\rangle = \frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{j+m}\left(\mathbf{a}_\downarrow^\dagger\right)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}\left|0\;0\right\rangle = \left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$$

Oscillator $\mathbf{a}^\dagger\mathbf{a}$ give $\mathbf{s}_\pm$ matrices.

$$\mathbf{a}_1^\dagger\mathbf{a}_2\left|n_1 n_2\right\rangle=\sqrt{n_1+1}\sqrt{n_2}\left|n_1+1\;n_2-1\right\rangle \Rightarrow \mathbf{s}_+\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle=\sqrt{j+m+1}\sqrt{j-m}\left|\begin{smallmatrix} j \\ m+1 \end{smallmatrix}\right\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_1\left|n_1 n_2\right\rangle=\sqrt{n_1}\sqrt{n_2+1}\left|n_1-1\;n_2+1\right\rangle \Rightarrow \mathbf{s}_-\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle=\sqrt{j+m}\sqrt{j-m+1}\left|\begin{smallmatrix} j \\ m-1 \end{smallmatrix}\right\rangle$$

SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Class 8 p49-54 (this class)

U(2) boson oscillator states $|n_1,n_2\rangle$ = R(3) spin or rotor states $\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$

Oscillator total quanta: $\upsilon=(n_1+n_2)$ Rotor total momenta: $j=\upsilon/2$ and z-momenta: $m=(n_1-n_2)/2$

$$\left|n_1 n_2\right\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{n_1}\left(\mathbf{a}_2^\dagger\right)^{n_2}}{\sqrt{n_1!\,n_2!}}\left|0\,0\right\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{j+m}\left(\mathbf{a}_2^\dagger\right)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}\left|0\,0\right\rangle = \left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$$

$$j=\upsilon/2=(n_1+n_2)/2 \qquad m=(n_1-n_2)/2$$

$$n_1=j+m \qquad n_2=j-m$$

U(2) boson oscillator states = *U(2)* spinor states

$$\left|n_\uparrow n_\downarrow\right\rangle = \frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{n_1}\left(\mathbf{a}_\downarrow^\dagger\right)^{n_2}}{\sqrt{n_\uparrow!\,n_\downarrow!}}\left|0\,0\right\rangle = \frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{j+m}\left(\mathbf{a}_\downarrow^\dagger\right)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}\left|0\,0\right\rangle = \left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$$

Oscillator $\mathbf{a}^\dagger\mathbf{a}$ give $\mathbf{s}_\pm$ matrices.

$$\mathbf{a}_1^\dagger\mathbf{a}_2\left|n_1 n_2\right\rangle=\sqrt{n_1+1}\sqrt{n_2}\left|n_1+1\;n_2-1\right\rangle \Rightarrow \mathbf{s}_+\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle=\sqrt{j+m+1}\sqrt{j-m}\left|\begin{smallmatrix} j \\ m+1 \end{smallmatrix}\right\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_1\left|n_1 n_2\right\rangle=\sqrt{n_1}\sqrt{n_2+1}\left|n_1-1\;n_2+1\right\rangle \Rightarrow \mathbf{s}_-\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle=\sqrt{j+m}\sqrt{j-m+1}\left|\begin{smallmatrix} j \\ m-1 \end{smallmatrix}\right\rangle$$

1/2-difference of number-ops is $\mathbf{s}_Z$ eigenvalue.

$$\left.\begin{aligned}\mathbf{a}_1^\dagger\mathbf{a}_1\left|n_1 n_2\right\rangle&=n_1\left|n_1 n_2\right\rangle\\ \mathbf{a}_2^\dagger\mathbf{a}_2\left|n_1 n_2\right\rangle&=n_2\left|n_1 n_2\right\rangle\end{aligned}\right\} \mathbf{s}_Z\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle=\tfrac{1}{2}\left(\mathbf{a}_1^\dagger\mathbf{a}_1-\mathbf{a}_2^\dagger\mathbf{a}_2\right)\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle=\frac{n_1-n_2}{2}\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle=m\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$$

SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Class 8 p49-54 (this class)

U(2) boson oscillator states $|n_1,n_2\rangle$ = R(3) spin or rotor states $\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$

Oscillator total quanta: $\upsilon=(n_1+n_2)$ Rotor total momenta: $j=\upsilon/2$ and z-momenta: $m=(n_1-n_2)/2$

$$|n_1 n_2\rangle = \frac{(\mathbf{a}_1^\dagger)^{n_1}(\mathbf{a}_2^\dagger)^{n_2}}{\sqrt{n_1!\,n_2!}}|0\,0\rangle = \frac{(\mathbf{a}_1^\dagger)^{j+m}(\mathbf{a}_2^\dagger)^{j-m}}{\sqrt{(j+m)!(j-m)!}}|0\,0\rangle = \left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$$

$j=\upsilon/2=(n_1+n_2)/2$

$m=(n_1-n_2)/2$

$n_1=j+m$

$n_2=j-m$

U(2) boson oscillator states = *U(2)* spinor states

$$|n_\uparrow n_\downarrow\rangle = \frac{(\mathbf{a}_\uparrow^\dagger)^{n_1}(\mathbf{a}_\downarrow^\dagger)^{n_2}}{\sqrt{n_\uparrow!\,n_\downarrow!}}|0\,0\rangle = \frac{(\mathbf{a}_\uparrow^\dagger)^{j+m}(\mathbf{a}_\downarrow^\dagger)^{j-m}}{\sqrt{(j+m)!(j-m)!}}|0\,0\rangle = \left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$$

Oscillator $\mathbf{a}^\dagger\mathbf{a}$ give $\mathbf{s}_\pm$ matrices.

$$\mathbf{a}_1^\dagger\mathbf{a}_2|n_1 n_2\rangle=\sqrt{n_1+1}\sqrt{n_2}|n_1+1\; n_2-1\rangle \Rightarrow \mathbf{s}_+\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle=\sqrt{j+m+1}\sqrt{j-m}\left|\begin{smallmatrix} j \\ m+1 \end{smallmatrix}\right\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_1|n_1 n_2\rangle=\sqrt{n_1}\sqrt{n_2+1}|n_1-1\; n_2+1\rangle \Rightarrow \mathbf{s}_-\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle=\sqrt{j+m}\sqrt{j-m+1}\left|\begin{smallmatrix} j \\ m-1 \end{smallmatrix}\right\rangle$$

1/2-difference of number-ops is $\mathbf{s}_Z$ eigenvalue.

$$\left.\begin{matrix}\mathbf{a}_1^\dagger\mathbf{a}_1|n_1 n_2\rangle=n_1|n_1 n_2\rangle \\ \mathbf{a}_2^\dagger\mathbf{a}_2|n_1 n_2\rangle=n_2|n_1 n_2\rangle\end{matrix}\right\}\mathbf{s}_Z\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle=\tfrac{1}{2}(\mathbf{a}_1^\dagger\mathbf{a}_1-\mathbf{a}_2^\dagger\mathbf{a}_2)\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle=\frac{n_1-n_2}{2}\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle=m\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$$

j=1 vector $\mathbf{s}_+$

$$D^1(\mathbf{s}_+)=D^1(\mathbf{s}_X+i\mathbf{s}_Y)=\begin{pmatrix} \cdot & \frac{\sqrt{2}}{2} & \cdot \\ \frac{\sqrt{2}}{2} & \cdot & \frac{\sqrt{2}}{2} \\ \cdot & \frac{\sqrt{2}}{2} & \cdot \end{pmatrix}+i\begin{pmatrix} \cdot & -i\frac{\sqrt{2}}{2} & \cdot \\ i\frac{\sqrt{2}}{2} & \cdot & -i\frac{\sqrt{2}}{2} \\ \cdot & i\frac{\sqrt{2}}{2} & \cdot \end{pmatrix}=\begin{pmatrix} \cdot & \sqrt{2} & \cdot \\ 0 & \cdot & \sqrt{2} \\ \cdot & 0 & \cdot \end{pmatrix},$$

...and $\mathbf{s}_Z$

$$D^1(\mathbf{s}_Z)=\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

$SU(2)\subset U(2)$ oscillators vs. $R(3)\subset O(3)$ rotors

Class 8 p49-54 (this class)

U(2) boson oscillator states $|n_1,n_2\rangle$ = R(3) spin or rotor states $\left|\begin{smallmatrix} j \\ m\end{smallmatrix}\right\rangle$

Oscillator total quanta: $\upsilon=(n_1+n_2)$ Rotor total momenta: $j=\upsilon/2$ and z-momenta: $m=(n_1-n_2)/2$

$$\left|n_1 n_2\right\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{n_1}\left(\mathbf{a}_2^\dagger\right)^{n_2}}{\sqrt{n_1!\,n_2!}}\left|0\,0\right\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{j+m}\left(\mathbf{a}_2^\dagger\right)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}\left|0\,0\right\rangle = \left|\begin{smallmatrix} j \\ m\end{smallmatrix}\right\rangle$$

$j=\upsilon/2=(n_1+n_2)/2$

$m=(n_1-n_2)/2$

$n_1=j+m$

$n_2=j-m$

U(2) boson oscillator states = *U(2)* spinor states

$$\left|n_\uparrow n_\downarrow\right\rangle = \frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{n_1}\left(\mathbf{a}_\downarrow^\dagger\right)^{n_2}}{\sqrt{n_\uparrow!\,n_\downarrow!}}\left|0\,0\right\rangle = \frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{j+m}\left(\mathbf{a}_\downarrow^\dagger\right)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}\left|0\,0\right\rangle = \left|\begin{smallmatrix} j \\ m\end{smallmatrix}\right\rangle$$

Oscillator $\mathbf{a}^\dagger\mathbf{a}$ give $\mathbf{s}_\pm$ matrices.

$$\mathbf{a}_1^\dagger\mathbf{a}_2\left|n_1 n_2\right\rangle=\sqrt{n_1+1}\sqrt{n_2}\left|n_1+1\; n_2-1\right\rangle \Rightarrow \mathbf{s}_+\left|\begin{smallmatrix} j \\ m\end{smallmatrix}\right\rangle=\sqrt{j+m+1}\sqrt{j-m}\left|\begin{smallmatrix} j \\ m+1\end{smallmatrix}\right\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_1\left|n_1 n_2\right\rangle=\sqrt{n_1}\sqrt{n_2+1}\left|n_1-1\; n_2+1\right\rangle \Rightarrow \mathbf{s}_-\left|\begin{smallmatrix} j \\ m\end{smallmatrix}\right\rangle=\sqrt{j+m}\sqrt{j-m+1}\left|\begin{smallmatrix} j \\ m-1\end{smallmatrix}\right\rangle$$

1/2-difference of number-ops is $\mathbf{s}_Z$ eigenvalue.

$$\mathbf{a}_1^\dagger\mathbf{a}_1\left|n_1 n_2\right\rangle=n_1\left|n_1 n_2\right\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_2\left|n_1 n_2\right\rangle=n_2\left|n_1 n_2\right\rangle$$

$$\mathbf{s}_Z\left|\begin{smallmatrix} j \\ m\end{smallmatrix}\right\rangle=\tfrac{1}{2}\left(\mathbf{a}_1^\dagger\mathbf{a}_1-\mathbf{a}_2^\dagger\mathbf{a}_2\right)\left|\begin{smallmatrix} j \\ m\end{smallmatrix}\right\rangle=\frac{n_1-n_2}{2}\left|\begin{smallmatrix} j \\ m\end{smallmatrix}\right\rangle=m\left|\begin{smallmatrix} j \\ m\end{smallmatrix}\right\rangle$$

j=1 vector $\mathbf{s}_+$

$$D^1(\mathbf{s}_+)=D^1(\mathbf{s}_X+i\mathbf{s}_Y)=\begin{pmatrix} \cdot & \frac{\sqrt{2}}{2} & \cdot \\ \frac{\sqrt{2}}{2} & \cdot & \frac{\sqrt{2}}{2} \\ \cdot & \frac{\sqrt{2}}{2} & \cdot \end{pmatrix}+i\begin{pmatrix} \cdot & -i\frac{\sqrt{2}}{2} & \cdot \\ i\frac{\sqrt{2}}{2} & \cdot & -i\frac{\sqrt{2}}{2} \\ \cdot & i\frac{\sqrt{2}}{2} & \cdot \end{pmatrix}=\begin{pmatrix} \cdot & \sqrt{2} & \cdot \\ 0 & \cdot & \sqrt{2} \\ \cdot & 0 & \cdot \end{pmatrix},$$

...and $\mathbf{s}_Z$

$$D^1(\mathbf{s}_Z)=\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

j=3/2 spinor $\mathbf{s}_+$

$$D^{\frac{3}{2}}(\mathbf{s}_+)=\begin{pmatrix} \cdot & \sqrt{3} & \cdot & \cdot \\ 0 & \cdot & \sqrt{4} & \cdot \\ \cdot & 0 & \cdot & \sqrt{3} \\ \cdot & \cdot & 0 & \cdot \end{pmatrix}=\left(D^{\frac{3}{2}}(\mathbf{s}_-)\right)^\dagger,$$

...and $\mathbf{s}_Z$

$$D^{\frac{3}{2}}(\mathbf{s}_Z)=\begin{pmatrix} \frac{3}{2} & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{2} & \cdot & \cdot \\ \cdot & \cdot & -\frac{1}{2} & \cdot \\ \cdot & \cdot & \cdot & -\frac{3}{2} \end{pmatrix}$$

$SU(2)\subset U(2)$ oscillators vs. $R(3)\subset O(3)$ rotors

Class 8 p49-54 (this class)

$U(2)$ boson oscillator states $|n_1,n_2\rangle$ = R(3) spin or rotor states $\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$

Oscillator total quanta: $\upsilon=(n_1+n_2)$ Rotor total momenta: $j=\upsilon/2$ and z-momenta: $m=(n_1-n_2)/2$

$$\left|n_1 n_2\right\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{n_1}\left(\mathbf{a}_2^\dagger\right)^{n_2}}{\sqrt{n_1!\,n_2!}}\left|0\,0\right\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{j+m}\left(\mathbf{a}_2^\dagger\right)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}\left|0\,0\right\rangle = \left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$$

$j=\upsilon/2=(n_1+n_2)/2$

$m=(n_1-n_2)/2$

$n_1=j+m$

$n_2=j-m$

$U(2)$ boson oscillator states = $U(2)$ spinor states

$$\left|n_\uparrow n_\downarrow\right\rangle = \frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{n_1}\left(\mathbf{a}_\downarrow^\dagger\right)^{n_2}}{\sqrt{n_\uparrow!\,n_\downarrow!}}\left|0\,0\right\rangle = \frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{j+m}\left(\mathbf{a}_\downarrow^\dagger\right)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}\left|0\,0\right\rangle = \left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$$

Oscillator $\mathbf{a}^\dagger\mathbf{a}$ give $\mathbf{s}_\pm$ matrices.

$$\mathbf{a}_1^\dagger\mathbf{a}_2\left|n_1 n_2\right\rangle=\sqrt{n_1+1}\sqrt{n_2}\left|n_1+1\; n_2-1\right\rangle \Rightarrow \mathbf{s}_+\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle=\sqrt{j+m+1}\sqrt{j-m}\left|\begin{smallmatrix} j \\ m+1 \end{smallmatrix}\right\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_1\left|n_1 n_2\right\rangle=\sqrt{n_1}\sqrt{n_2+1}\left|n_1-1\; n_2+1\right\rangle \Rightarrow \mathbf{s}_-\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle=\sqrt{j+m}\sqrt{j-m+1}\left|\begin{smallmatrix} j \\ m-1 \end{smallmatrix}\right\rangle$$

1/2-difference of number-ops is $\mathbf{s}_Z$ eigenvalue.

$$\left.\begin{aligned}\mathbf{a}_1^\dagger\mathbf{a}_1\left|n_1 n_2\right\rangle&=n_1\left|n_1 n_2\right\rangle\\ \mathbf{a}_2^\dagger\mathbf{a}_2\left|n_1 n_2\right\rangle&=n_2\left|n_1 n_2\right\rangle\end{aligned}\right\}\; \mathbf{s}_Z\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle=\tfrac{1}{2}\left(\mathbf{a}_1^\dagger\mathbf{a}_1-\mathbf{a}_2^\dagger\mathbf{a}_2\right)\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle=\frac{n_1-n_2}{2}\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle=m\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$$

j=1 vector $\mathbf{s}_+$ ...and $\mathbf{s}_Z$

$$D^1(\mathbf{s}_+)=D^1(\mathbf{s}_X+i\mathbf{s}_Y)=\begin{pmatrix} \cdot & \frac{\sqrt{2}}{2} & \cdot \\ \frac{\sqrt{2}}{2} & \cdot & \frac{\sqrt{2}}{2} \\ \cdot & \frac{\sqrt{2}}{2} & \cdot \end{pmatrix}+i\begin{pmatrix} \cdot & -i\frac{\sqrt{2}}{2} & \cdot \\ i\frac{\sqrt{2}}{2} & \cdot & -i\frac{\sqrt{2}}{2} \\ \cdot & i\frac{\sqrt{2}}{2} & \cdot \end{pmatrix}=\begin{pmatrix} \cdot & \sqrt{2} & \cdot \\ 0 & \cdot & \sqrt{2} \\ \cdot & 0 & \cdot \end{pmatrix},\qquad D^1(\mathbf{s}_Z)=\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

j=3/2 spinor $\mathbf{s}_+$ ...and $\mathbf{s}_Z$

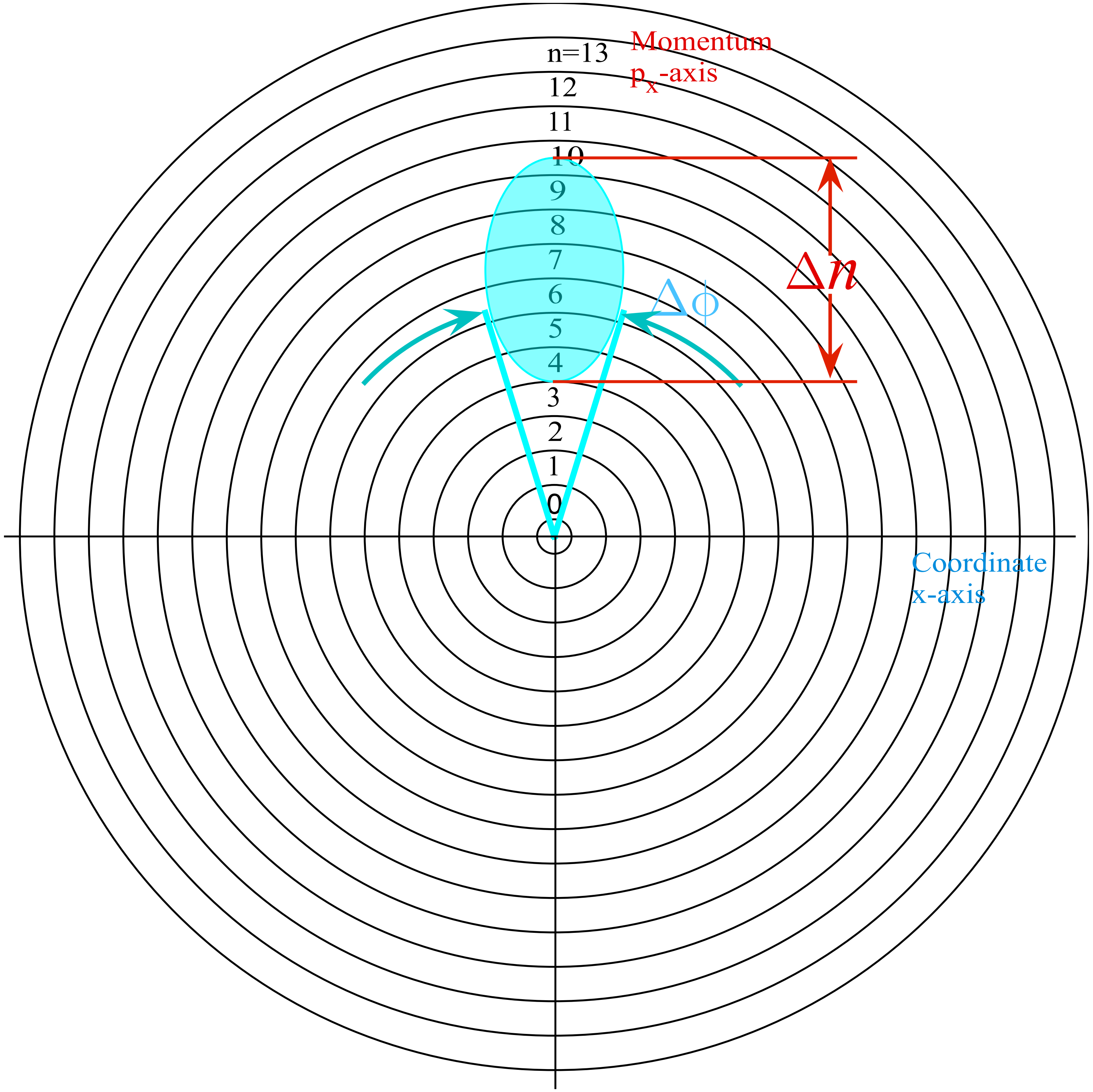
$$D^{\frac{3}{2}}(\mathbf{s}_+)=\begin{pmatrix} \cdot & \sqrt{3} & \cdot & \cdot \\ 0 & \cdot & \sqrt{4} & \cdot \\ \cdot & 0 & \cdot & \sqrt{3} \\ \cdot & \cdot & 0 & \cdot \end{pmatrix}=\left(D^{\frac{3}{2}}(\mathbf{s}_-)\right)^\dagger,\qquad D^{\frac{3}{2}}(\mathbf{s}_Z)=\begin{pmatrix} \frac{3}{2} & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{2} & \cdot & \cdot \\ \cdot & \cdot & -\frac{1}{2} & \cdot \\ \cdot & \cdot & \cdot & -\frac{3}{2} \end{pmatrix}$$

j=2 tensor $\mathbf{s}_+$ ...and $\mathbf{s}_Z$

$$D^2(\mathbf{s}_+)=\begin{pmatrix} \cdot & \sqrt{4} & \cdot & \cdot & \cdot \\ 0 & \cdot & \sqrt{3} & \cdot & \cdot \\ \cdot & 0 & \cdot & \sqrt{3} & \cdot \\ \cdot & \cdot & 0 & \cdot & \sqrt{4} \\ \cdot & \cdot & \cdot & 0 & \cdot \end{pmatrix}=\left(D^2(\mathbf{s}_-)\right)^\dagger,\qquad D^2(\mathbf{s}_Z)=\begin{pmatrix} 2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & -2 \end{pmatrix}$$

Properties of 1D-HO coherent state

Coherent wave packet uncertainty relation: $\Delta n \cdot \Delta\phi > \pi / n$



???? Some uncertainty remains about this uncertainty ????