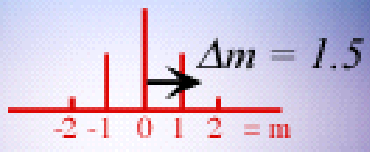
Quantum Theory for the Computer Age

W. G. Harter-University of Arkansas



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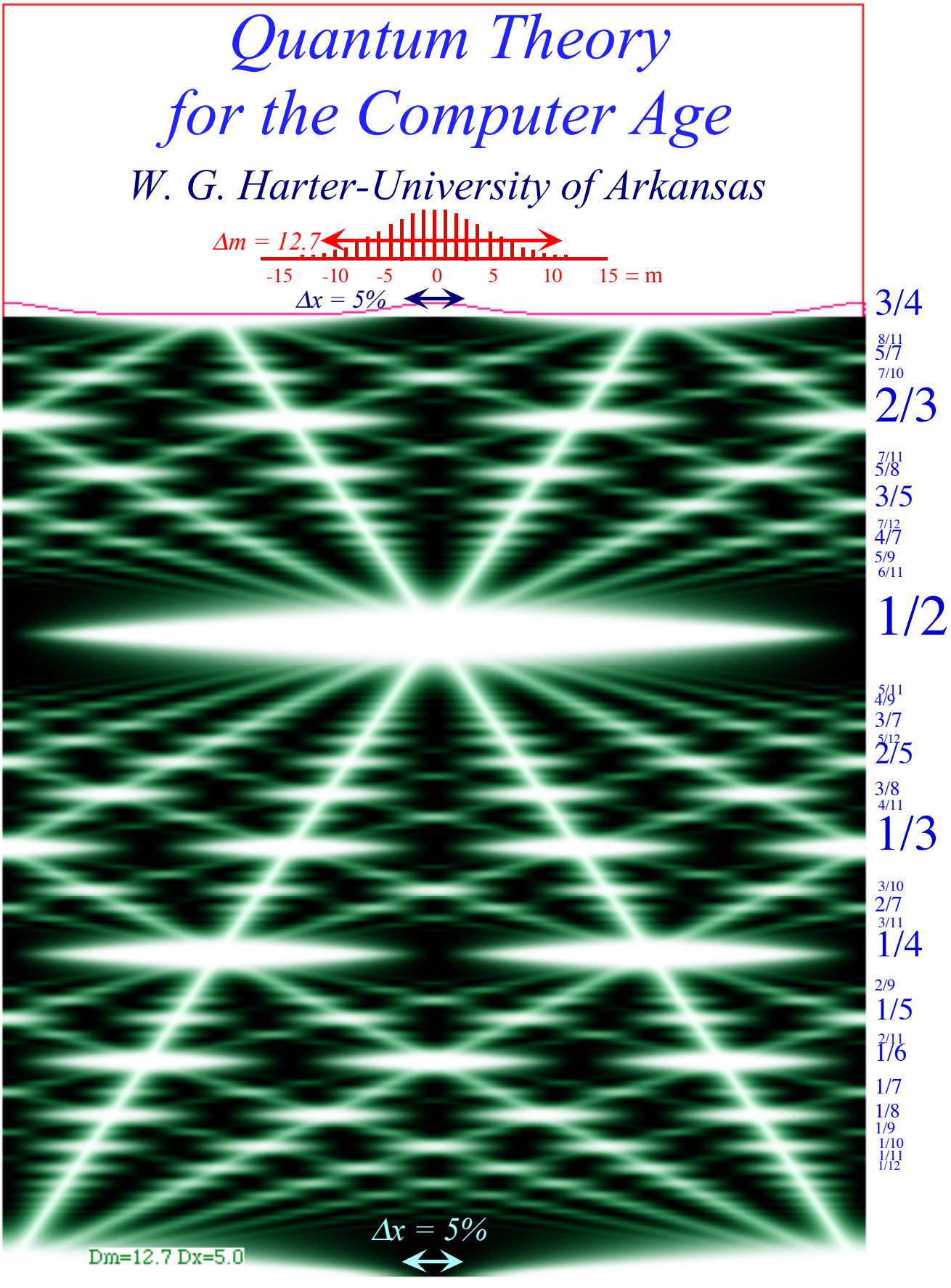
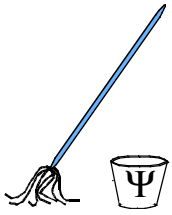


Quantum Theory for the Computer Age

An Introduction to Analysis
for
Atomic, Molecular, and Optical Physics



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Quantum Theory for the Computer Age (QM for AMOP) W. G. Harter

Preface

Research in quantum theory and its applications to atomic, molecular, and optical physics has grown enormously in the past half century as have related fields involving condensed matter. A new industry known as nanotechnology is just one of the results of a renaissance based on quantum mechanics. Such industry and research is largely built around an information economy, that is, computers and telecommunication.

Unfortunately, the teaching of quantum theory has not advanced as quickly. Most quantum texts follow an approach developed when slide-rules were the principle means of doing numerical computation. The notes and texts of several of the early masters, including Born, Fermi, Oppenheimer, Landau and Schwinger, form the basis of much of what current textbooks contain. This has set an orthodoxy from which few deviate.

Notable exceptions to conventional texts are ones based on lectures by Richard Feynman. The third volume of *The Feynman Lectures on Physics* by Feynman, Leighton, and Sands *(Addison Wesley 1964)* is a fresh approach to quantum theory. His unorthodox approach has survived to the present, indeed, his set of lectures are found in popular bookstores as well as in virtually every technical outlet or library in the world.

It is the Feynman approach which motivates the present work. Mentoring by Feynman and Bill Wagner (coauthor with Ferando Morinigo of Gravitation Feynman's book) during my introductory graduate career at Cal Tech influences this work immeasurably. The Feynman approach is characterized by an abundant use of physical analogies and pictures. While he never undersold solid mathematics, he did comment once that the disappearance of formal mathematics would "only set physics back about a week!"

One interpretation of Feynman's comment is that mathematics ought to be designed to fit the physics, not the other way around. In other words, physical insight ought to be prime mover and the main goal. Appropriate mathematics is found (or invented, if necessary) to solidify details. The result of this approach, as I hope this book shows, is better mathematics *and* physics with elegant theory, powerful computation, and most important, a set of insightful tools that uncover new directions and inventions.

One new feature of this book is something that, early on, Feynman warned against, but later he adopted fairly enthusiastically. That is the use of computer thought experiments involving both the classical commercial machines and (as yet mythical) "quantum computers." In 1964 Feynman warned me against then new (classical) computers, "Watch out! I know guys that got sucked into those things. They're so seductive; you think you can solve anything with them!" But, by 1981 Feynman was giving lectures on computation, something he did off and on until his untimely death in 1988.

Computers play a key role in this book and one whose time has come. For over two decades I have been developing computer animations, graphics and simulations to help visualize classical and quantum phenomena. The most important outcomes of this effort have been improved physical analogies of the type that Feynman was so good at creating and did so (mostly) unaided by computers.

Times have changed, and it is difficult to say how Feynman would react to having several giga-pixel "eyes" staring back from each room in old Bridge Lab. But, we are the ones who must decide how these things are used. The approach of this book has been to make computers useful, not just for number crunching, but for the conceptual and theoretical development as well, particularly with regard to visualization of physics in space and time. Harvard educator, Howard Gardner, has noted that the human visual system has geometric pattern processing that, while less precise, handle data more quickly than its verbal or math logic processors.

An approach to quantum physics that attempts to harness a largely untapped visual human intelligence is still an academically unorthodox one. Indeed, a group of mathematicians known as Bourbakians essentially rejected figures in publications. Such images were considered childish and misleading. This small group of intellectual fundamentalists rebelled against Henri Poincare who helped create relativity and quantum theories of great and lasting value. Now after many decades, could Bourbakian claim results of greater value?

Still, it is striking that many seminal theoretical physicists including Einstein and Schwinger wrote reams of formulae without figures. In this, Feynman is a notable exception, but a prevailing attitude seems to be that analysis and calculation come first and then, perhaps, a few diagrams might be allowed.

However, if you believe as I do that physics is primarily an experimental science, then why not experiment? Whether useful ideas come from "real" lab experiments or computer lab simulations, it only matters how useful they are. One is cautious not to be misled by either one. Modern personal computers lend a graphical approach to theoretical physics that is becoming a powerful research tool. Many new basic ideas and fundamental blind spots are being exposed that might have otherwise not been found out.

Moreover, a geometric approach, such as developed in this book, should appeal to modern cyber-savvy students and enhance learning of quantum theory at all levels. We can only hope that Feynman, a pioneer in graphical physics visualization, would support this development.

About the Programs: LearnIt and CodeIt

The first tier of computer programs in this book is the *LearnIt* series consisting of *OscillIt*, *QuantIt*, *WaveIt*, etc. listed in tables below. These are (hopefully) user-friendly applications that produced many of the figures in this book. They also provide animated visualizations of physical phenomena or analogies thereof. Indeed, they are like analog computers that make text figures come alive for experimentation. Such programs were an essential aid to my ability to discover new ideas. Clearly, this needs to be made more available

The suffix *"It"* attached to many of these programs is derived from the *FaceIt* interface invented by Dan Kampemier, founder of *FaceWare* in Urbana, IL. It was one of the first worldwide programming projects to enhance the new Apple MacIntosh graphical user/programmer interface (GUI or GPI) and allow menus, controls, text editors, spreadsheets, movie or graphics windows to be conveniently created and used. I participated as a developer and user in *FaceWare* from 1985 until around 1993.

One advantage of *FaceWare* was that it allowed one to learn and teach *useful* root-level programming simultaneously with physics course material. The disadvantage was that it worked only on an Apple CPU and then (after Kampemier gave up) only on classic operating systems OS 7-9 or cloned OS 10.1-4. It will always be risky for research and teaching projects to develop software that relies on one type of app, GPI, or CPU. It also penalizes students who may be unable or unwilling to buy particular tool or platform.

Fortunately, there is a solution that involves high speed web browsers that are free and universally capable of running applications developed using *Java* and *HTML* programming/debugging interfaces that now exist on most of them. Dr. T. C. Reimer has pioneered in converting *LearnIt* apps from original object *Pascal*, *FORTRAN*, and *C++* code to the modern hypertext format. Eventually, web-based text figures and formulae can become control panels for their underlying *LearnIt* applications that run seamlessly on any device.

Also, it will be possible to build a tree of programming projects for a given course that we call a *CodeIt* system. Students saw-off one or more branches of *CodeIt* trees to build their own applications as homework or lab projects. Eventually, they can build applications of sufficient complexity to aid in their thesis or dissertation research projects. Also, select *CodeIt* applications may be added to either *LearnIt* or *CodeIt* collections.

Tables below correlate the first few text chapters with some *LearnIt* programs.

Unit 1 Wave Amplitudes and Analyzers

	QuantIt	*OscillIt*	*ColorU2*	*WaveIt*	*RelativIt*	*BohrIt*	*GuideIt*	*BandIt*	*AvoidIt*	*CoulIt*	*AnalIt*
1.1	X										
1.2	X	x									
1.3	X	x	x								
1.A		x									x
1.B		x									x
2.1		x									
2.2											
3.1											
3.2											

Unit 2 Wave Dyanmics (Spacetime and per-Spacetime)

	QuantIt	OscillIt	ColorU2	WaveIt	RelativIt	BohrIt	GuideIt	BandIt	AvoidIt	CoulIt	AnalIt
4.1			x	x							
4.2				X							
4.3				X	x						
4.4	.	.	.	X	.	x					
5.1				x	X						x
5.2				x	X	x					
5.3	.	.	.	.	X	.					
5.4	.	.	.	.	X	.	.				
6.1					x		x				
6.2					x		x				
6.3							X				

Unit 3 Fourier Analysis and Symmetry

	QuantIt	OscillIt	ColorU2	WaveIt	RelativIt	BohrIt	GuideIt	BandIt	AvoidIt	CoulIt	AnalIt
7.1						x					
7.2								x			
7.3	.	.	.	X	.	.	.	.			
8.1				X		x					
8.2				X		x					
8.3	.	.	.	x	.	x	.	.			
9.1			x								
9.2											
9.3				X		X					
9.4	.	.	.	X	.	X					
10.1			X	x		x					
10.2			X	x		x		.			
10.3	.	.	X	.x	.	x	.	.	X		
10.4			X								

Unit 4 Wave Equations

	QuantIt	OscillIt	ColorU2	WaveIt	RelativIt	BohrIt	GuideIt	BandIt	AvoidIt	CoulIt	AnalIt
11.1-2				.x		X					
11.3				.		X		X			
11.4	.	.	.	.x	.	X	.	X			
11.5		x		x	x	X		X			
12.1			.			X		X			
12.2				x		X		X			
12.3			x	X		X		x			
13.1	.	.	.	.	.	.		X			
13.2	.	.	.	.	.	.		X			
13.3	.	.	.x	.	.	.		X			

Unit 5 Periodic Potentials

	SwingIt	*OscillIt*	*ColorU2*	*WaveIt*	*RelativIt*	*BohrIt*	*GuideIt*	*BandIt*	*AvoidIt*	*CoulIt*	*AnalIt*
14.1				x		x		X			
14.2				x		x		X			
15.1	.	.		.x	.	x	.	X			
15.2				.		.		X			
15.3				.		.		X			
15.4	.	.	.	.x	.	.	.	X			
15.5			.	x		X		X			
16.1	X	x	x					X			
16.2	X	x	x	.		X		X			
16.3	.X	.	.x	.x	.	X		X			

Unit 6 Time Dependent Perturbation

	QuantIt	*OscillIt*	*ColorU2*	*WaveIt*	*RelativIt*	*BohrIt*	*GuideIt*	*BandIt*	*AvoidIt*	*CoulIt*	*AnalIt*
17.1		x									
17.2		x		.		.					
18.1				.		.					
18.2	.	.	.	.	.	.	.				
18.3			.								
19.1			X						X		

Unit 7 Quantum Harmonic Oscillators

	QuantIt	*OscillIt*	*ColorU2*	*WaveIt*	*RelativIt*	*BohrIt*	*GuideIt*	*BandIt*	*AvoidIt*	*CoulIt*	*AnalIt*
20.1		x				X					
20.2		x		.		.X					
20.3		x		.		.X					
21.1	.	.	.x	.	.	.	.				
21.2			.x								
21.3			x								
22.1				.							
22.2	.	.	.	.	.						

Unit 8 Oscillation, Spin, and Rotation (Under development)

	QuantIt	*OscillIt*	*ColorU2*	*WaveIt*	*RelativIt*	*BohrIt*	*GuideIt*	*BandIt*	*AvoidIt*	*CoulIt*	*AnalIt*
23.1	X										
23.2	X			.		.					
23.3	X			.		.					
23.4	.	.	.	.	.	.	.				
24.1			.								
24.2											
25.1				.							
25.2	.	.	.	.	.						

About the Subject Matter: A Brief Guide

This book is a *spectral* approach to quantum theory. Oscillatory phenomena including wave polarization, wave dynamics, resonance, and interference are emphasized. A student of wave optics should feel quite at home. The quantum psi-wavefunction is related in Chapter 1 to an electromagnetic *E*-wave field, and most waves treated in Chapters 1 through 4 relate to plane electromagnetic waves.

However, the plane waves of Chapter 4 through 6 are viewed in a new light that shows that quantum theory and relativity are quite the same subject with far simpler logic than exists in previous treatments of either one. Chapter 4 derives relativistic Doppler and Lorentz transformations by wave interference, and Chapter 5 develops relativistic matter-wave dispersion in a few simple steps. Light and matter make their own space-time coordinate manifolds by elementary spectral interference. This is a new result and one of several in this book that have only recently been published. Detailed study of elementary spectral components and their beat frequencies have shown to be a useful research tools as well as good pedagogy.

The word *spectral* has many connotations that need to be related as well as distinguished. Frequency spectra from prisms and gratings are well known *physical* phenomena since Newton, and modern (quantum) spectroscopy has increased accuracy to one part in 10^{16} or better. *Mathematical* spectra or eigenvalues of matrices often relate to laboratory spectra or quanta, but one must be careful to distinguish these two uses of the word. Chapters 1 through 3 carefully relate and distinguish physical phenomena from their mathematical descriptions, that is, distinguish physical mysteries from mathematical ones.

A key quantum concept, the *transformation matrix* $T_{ab}=\langle a'|b\rangle$, is introduced in Chapter 1 and the first example is a 2-by-2 polarization rotation matrix. For a physicist, *T* gives outcomes of polarization experiments. Wavefunctions $\langle x|\psi\rangle=\psi(x)$ form another example of *T* as do wave-based Lorentz transformations in Chapter 4. But, simple 2-by-2 examples in Chapters 1-3 are quite sufficient to introduce Dirac bra-ket notation for transfer-operators describing polarization analyzers and projection-operators describing filters. “Own-states” or eigenstates of analyzer-filters are introduced as states which analyzers make or which filters pass.

Chapter 2 develops four quantum *axioms* which *T*–matrices obey as *physical* objects. As *mathematical* objects *T*-matrices relate eigenvectors of one operator to those of another. Chapter 3 connects the physical axioms to mathematical *theorems* in matrix algebra and to axioms for group algebra based on the spectral decomposition of matrices. Algebraic spectral theorems then begin to show why group algebra is powerful and fundamental to analyzing quantum spectra.

Efficient use of group algebra motivated by physics is one of the most powerful features of this book and it is introduced and explained as it is used throughout. This begins again in Chapters 7, 8, and 9 with the treatment of a “quantum-dot” system consisting of a square *(N=4)* or hexagonal *(N=6)* nano-corral. By introducing discrete versions of Bohr’s earliest problem, an electron-on-a-ring, it is easier to introduce Fourier theory and its symmetry. Also, it corresponds to nano-devices currently being developed.

A discrete *N-by-N* Fourier transformation matrix made of *Nth* roots of unity $\langle x_p|k_m\rangle=e^{imp2\pi/N}$ diagonalizes symmetry operators that satisfy $\mathbf{r}^N=\mathbf{1}$. (Such a *T*-matrix is known as a C_N-group character table.) At the same time the Fourier *T*-matrix diagonalizes all matrices that have C_N–symmetry since all such matrices are linear combinations of $\mathbf{r}, \mathbf{r}^2, \mathbf{r}^3,\ldots, \mathbf{r}^N$. This provides, in Chapter 8, all possible eigensolutions of all possible *N*-dot transfer matrices. The same is done for *N*-dot evolution operators or Hamiltonians $\mathbf{H}=H\mathbf{1}+S\mathbf{r}+T\mathbf{r}^2+\ldots$

in Chapter 9 and makes an elementary introduction to band theory. The approach also provides a way to introduce Schrodinger time dynamics while showing effective ways to build and solve non-trivial examples. Finally, it clarifies Bohr-matter-wave revivals and their space-time "fractal coordinates" in Chapter 9.

A general 2-by-2 Hamiltonian $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$ is analyzed in Chapter 10 using analogy with coupled pendulums. **H** is expressed as a linear combination $\mathbf{H}=(A+D)/2\,\sigma_0+(A-D)/2\,\sigma_A+B\,\sigma_B+C\,\sigma_C$ of reflection-symmetry operators $\sigma_0=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_A=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\sigma_B=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_C=\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ to give **H** archetypes *Type-A (Asymmetric-diagonal)* $\mathbf{H}_A=\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$, *Type-B (Bilateral-balanced)* $\mathbf{H}_B=\begin{pmatrix} A & B \\ B & A \end{pmatrix}=A\mathbf{1}+B\sigma_B$, and *Type-C (Circular-complex-chiral-coriolis)* $\mathbf{H}_C=\begin{pmatrix} A & -iC \\ iC & A \end{pmatrix}=A\mathbf{1}+C\sigma_C$. Mixed types *AB*, *AC*, *BC*, and *ABC*, lead to discussion of avoided-level-crossing.

Reflection operators (σ_A,σ_B,σ_C), apart from our pedagogical *ABC*-labels, are well-known Pauli spinors (σ_Z,σ_X,σ_Y), but, it is not so well known that (apart from an *i*-factor) they belong to Hamilton's *quaternion* or *hyper-complex (***1,i,j,k***)*-numbers found in 1841. Modern quantum theory owes a lot to this *U(2)* algebra.

Furthermore, Hamilton's observation of mirror-reflection properties ($\sigma^2=\mathbf{1}$) greatly increases their utility so they generate both quantum rotations (with the *i*-factor: $e^{i\sigma_a R}=\mathbf{1}\cos R+i\sigma_a\sin R$) and Lorentz transformations (without the *i*-factor: $e^{\sigma_a L}=\mathbf{1}\cosh L+\sigma_a\sinh L$). This motivates a logical development of quantum theory of spin, rotation, and relativistic wave mechanics. The *ABC* coupled-oscillator analogy helps make spin and quasi-spin-analogies less mysterious. In addition, there are some quite deep reasons for pursuing the coupled oscillator analogy.

An isotropic 2D-oscillator *(A=D, B=0=C)* has full *U(2)* symmetry and so *U(2)* leads to a much simpler theory of both quantum angular momentum and relativistic quantum field theory. Both electron spins and orbits and photon spins and orbits are simplified and unified by this in later chapters. The first *U(2)* examples treated in Chapter 10 are photon-polarization and electron spin (as introduced in Chapter 1), and the NH_3 maser doublet. These set the stage for more advanced 2-state and *N*-state symmetry analysis later on.

The oldest and most prevalent (yet least studied) 2-state or *U(2)* system is a pair of plane waves. A pair of counter-propagating plane waves are used in aforementioned Chapters 4 to 6 to derive Lorentz-Einstein relativity and quantum matter wave dispersion, two pillars of quantum theory. The *U(2)*-wave pair system returns in Chapters 11 to 14 as a basis for analyzing eigenstates in potential barriers and wells. Using a 2-by-2 crossing or C-matrix and the scattering or S-matrix does this. The *C*-matrix is unimodular (as is a Lorentz matrix) while the *S*-matrix is unitary (as is a rotation matrix) with *eigenphase* eigenvalues $e^{i\delta}$.

The concept of *eigenchannels*, which are *S*-matrix eigensolutions, is developed in Chapter 13. The properties of eigenchannels and eigenphases are analyzed by *ABC-U(2)* symmetry particularly in resonance situations where they are sensitive functions of energy and of interest for electronic-photonic-devices.

One result in Chapter 14 is an alternative to band theory in Chapter 9 that is more appropriate to treat modern super-lattice nano structures and photon band-gap devices. An important distinction is shown between *resonant* and *non-resonant* eigenchannels. The former have their largest wave amplitude *inside* a nano-structure and resemble bound state waves, while the latter pile up *outside* and resemble scattering waves. Generalization of this applies to related ebb-and-flow of molecular, atomic, nuclear, and sub-nuclear waves.

Wave symmetry analysis involving general non-Abelian (non-commutative) group theory is described in Chapter 15 using a novel approach. Again, the physical props are quantum-well or quantum-dot structures introduced before in Chapter 13 and 14. Concepts of *symmetry-relativity-duality* are introduced. These require that all transformations be defined as one wave relative to another, essentially a clarification of earlier Axioms 1-4 in Chapter 2. The result is two mutually commuting or *intertwining* groups: "outside" *global* or lab-defined symmetry operators $\{\mathbf{g},\mathbf{g}',...\}$ and "inside" *local* or body-defined symmetry operators $\{\bar{\mathbf{g}},\bar{\mathbf{g}}',...\}$.

The result of this extra care is an increase in computational and analytic capability with a lot simpler logic. A general Hamiltonian-matrix or *S*-matrix is constructed and classified in Chapter 15, as in C_N analysis of Chapter 9 or *ABC-U(2)* analysis of Chapter 10, by its combination of symmetry operators. However, unlike Chapter 9, this symmetry is *non*-commutative, and so the matrix must be built from "inside" local operators in order to commute with all "outside" global operators. A spectral decomposition of either group leads to a related decomposition of the intertwining dual and a desired reduction of the *H* or *S*-matrix. The final result tells how much "insider" wave (like a resonant eigenchannel) and "outsider" wave (like a non-resonant eigenchannel) is present in each spectral component. The physical insight provided is considerable.

Chapter 16 rounds out the discussion of band symmetry and wave mechanics using the Fourier analysis introduced in Chapters 7 and 8. Also reintroduced are coupled pendulum models of Chapters 10 and 11 that relate Schrodinger waves in a variable potential *V(x)* to waves along a "shower curtain" (coupled pendulums) of variable height $\ell(x)$. Momentum or *k*-basis representations $\langle k'|H|k\rangle$ of Hamiltonian are compared to the standard position or *x*-basis representation $\langle x'|H|x\rangle$. Resulting computational advantages (as well as *dis*advantages) are shown using an analogy between a space-periodic potential *V(x)* and a time-periodic force *F(t)* on a single pendulum. Linear resonance response is compared to multiplicative resonance or *parametric resonance*, the latter being relevant since a potential *V(x)* acts by multiplying $\psi(x)$.

This sets up the discussion of time dependent perturbations in Chapters 17 through 19. Classical electromagnetic perturbations are described using full vector-scalar potentials $\left(\mathbf{A}(x,t),\Phi(x,t)\right)$ needed to build a relativistic quantum field theory. However, the non-relativistic Schrodinger approach is developed first to satisfy prevailing electronic-photonic customs. Time is a parameter rather than a part of space-time and perturbing fields and operators are explicit functions of time governed by outside input.

Chapter 18 derives first-order perturbation theory of elementary E•r dipole resonance and Fermi-Golden-Rule constant-transition-rate theory and compares it to linear resonance of classical Lorentz theory. Chapter 19 goes beyond perturbation theory for a two-state system where the *U(2)*-parameters *A(t)*, *B(t)*, *C(t)*,and *D(t)* are explicit functions of time and discusses parametric resonance and Rabi NMR oscillation.

Chapters 20 to 22 develop the quantum theory of harmonic oscillation and quantum electromagnetic fields. Two-dimensional oscillator theory of Chapter 18 exploits the *U(2)-ABC*-parameterization of Chapter 10 to begin relating *U(2)* spin-*up*-spin-*dn* to three-dimensional *ABC*-rotation and *R(3)* quantum angular momentum. It also leads to *super-symmetry* since it applies to a single particle oscillating in 2D or to two particles (coupled pendulums) each oscillating in 1D. Odd oscillator quanta *n=1, 3, 5,...* correspond to half-integer spin *j=1/2, 3/2, 5/2,..* with odd-particle-permutation parity. Even oscillator quanta *n=0, 2, 4,...* correspond to integer angular quanta *l=0, 1, 2,..* with even-permutation parity. One is Bose-like the other is Fermi-like.

Chapters 23 to 25 develop the quantum theory of real *R(3)* rotation symmetry and angular momentum using the *U(2)* oscillator basis and Hamilton reflection symmetry. The development also uses Schwinger a-$a^\dagger$ operator algebra and Casimir invariants. The physical props are molecular or quantum rotors that carry an intrinsic Cartesian reference frame. The full symmetry is an intertwining dual $R_{LAB}(3)*\overline{R_{BODY}(3)}$ group with global-local properties introduced in Chapter 15. As before, it leads easily to eigensolutions which here are the Wigner transformation matrices $D^{J*}_{M_{Lab}N_{Body}}$ of both half-integer-J (spinor) and integer-J (vector, tensor,…). Orbital harmonics Y^ℓ are special cases of D-functions for integer J=ℓ: $D^{\ell*}_{M,0}=Y^\ell_M$ where the intrinsic momentum N_{Body} is set to zero and ignored. Group algebra reduces difficult issues of phase and normalization.

Group algebra also simplifies problems of coupled rotors. The most famous of these are spin-orbit (fine-structure) and spin-spin (hyperfine-structure) problems introduced in Chapter 25. Visualizing and deriving coupling transformation matrices (Clebsch-Gordan coefficients) is aided considerably by a dual-symmetry approach. This is particularly helpful for building and analyzing molecular states whose respect for various local symmetries may ebb-and-flow enormously with excitation energy.

Chapter 26 to 28 introduces atomic orbital and shell structure beginning with Coulomb orbitals that have the angular Y^ℓ–wave (derived in Chapter 23) and a radial $R_{n\ell}$-wave. The coulomb field has an important symmetry $R(4)=R(3)\times R(3)$ that is related to the rotor symmetry of Chapter 23 and aids in calculations of eigenvalues and energy matrices. Rydberg orbitals discussed in Chapter 28 represent a large area of research in atomic spectroscopy. They are also relevant for understanding excitons in condensed matter. They should be featured as important general phenomena.

The final chapters are devoted to multiparticle systems, an enormous and ever-increasing field. Topics chosen are a tiny sampling but ones that exhibit symmetry and correlation (entanglement) effects and tools for dealing with them. The underlying symmetry of N-identical particles (molecules, nucleons, electrons, photons,..) that may occupy M quantum states is generally taken to be $U(M)\times S_N$ where S_N is the permutation symmetry of N particles. Nuclear, atomic, and molecular orbital shell theory are historically the first areas to develop this analysis. Chapter 30 and 31 introduce unitary analysis of atomic and molecular shell structure.

$U(M)\times S_N$ is part of a larger dual intertwining symmetry $U(M)\times\overline{U(N)}$ which is a most important example of the “inside*outside” quantum duality treated in Chapters 16 and 24. The $U(M)$ redefines the M-states of whichever particles they may occupy while $\overline{U(N)}$ redefines the N-particles between whatever states they may be in. The ideas of particles and states are put onto more equal and general “quasi-particle” footing. Examples are given of nuclear spins having resonantly enhanced effects on whole polyatomic molecular wavefunctions. Similar correlative effects in solids and BEC ensembles are possible.

The insight and computational power provided by these types of symmetry analyses is enormous and still largely unexplored. As quantum theory advances into the computer age, and particularly if there is to be a *quantum-computer age*, this kind of analysis is likely to advance from relative obscurity to serve its time as a methodology of quite some utility.

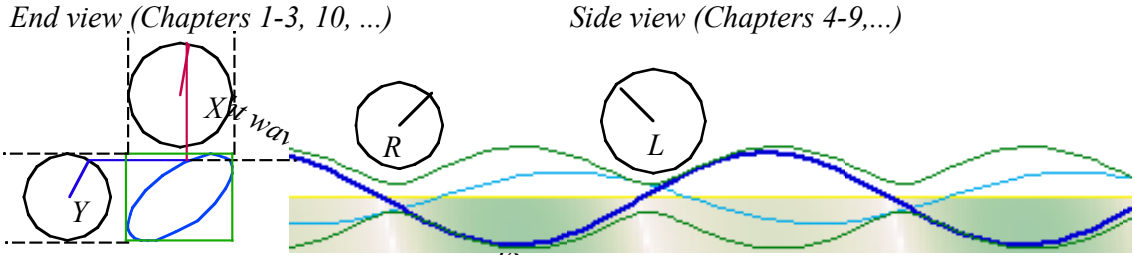
Optical Views of Quantum Mechanics

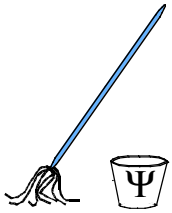
The origins of quantum theory and relativity are deeply connected with light and wave optics. Planck's axiom E=hν was, at first, a shot in the dark, so to speak, that clarified the statistical properties of low temperature electromagnetic radiation. The history of this incredible result is found at the beginning of most texts on quantum mechanics and modern physics.

This text also uses light to develop quantum theory, but in a simpler and more direct way that avoids at first the complexity of quantum statistical mechanics. The first two units focus instead on the oscillatory wave and resonance properties of light but treat the quantum counter as a black box.

The elementary objects of thought will, for the first two units, be coherent and mostly spectrally pure laser light beams. Unit 1 concerns optical polarization, that is, light beams veiwed head-on. Unit 2 concerns wave propagation, that is, light beams viewed (as best we can) from the side. In either view, (See figure below) much can be learned by modeling it as a two-state or coupled-oscillator system.

From such simple elements we develop the concept and properties of quantum matter waves by appealing to spacetime symmetry required for optical waves. It is a minimalist approach based upon analogies. It seeks to develop as much physics as possible with the simplest and least number of axioms. William Occam (*1285-1349*) put forth ideas known as Occam's razors to cut axioms to a minimum in order to explain the most phenomena. We hope we can use his ideas effectively in this introduction to quantum phenomena.



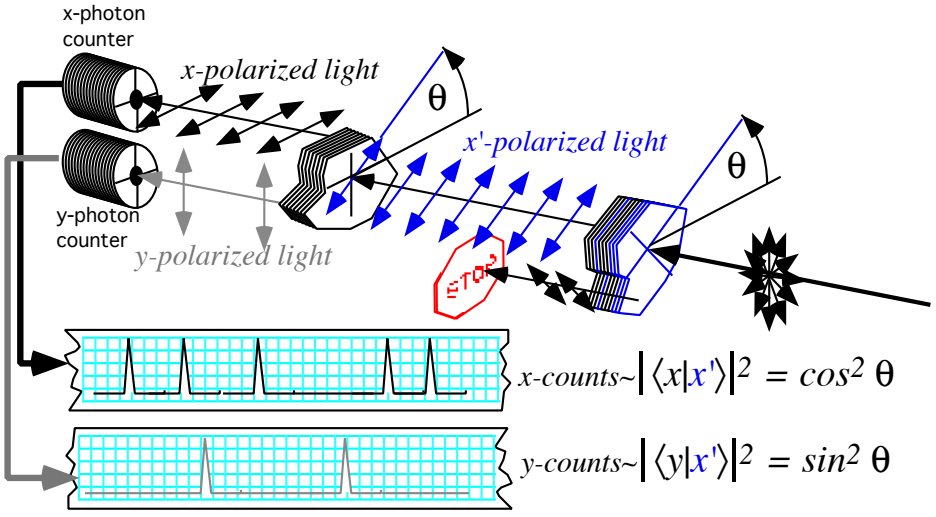


QM for AMOP

W. G. Harter

Unit 1 Quantum Amplitudes

Basic quantum amplitudes, analysis, and Dirac notation is introduced by thought experiments involving optical beams with polarization devices. Concepts such as state vectors, matrix operators, and eigensolutions are introduced via physics of beam splitters, analyzers, and counters. Operator spectral decomposition is related to state filtering and projection operators. Symmetry group operators, matrix spectral decomposition, and perturbation theory are introduced.



Chapter 1
Amplitudes, Analyzers and Matrices

The Dirac bra-ket transformation matrix ⟨a|b⟩ or amplitude array is introduced as the main object of study in quantum theory and related to experiments with beam sorters and analyzers. Quantum counting with and without “peeking” or dephasing is simulated and analyzed from several points of view.

Unit 1 Quantum Amplitudes

Chapter 1. Amplitudes, Analyzers and Matrices

We begin our description of quantum theory using Feynman's ideas of *particle beams* and *quantum analyzers*. A "beam-analyzer" approach lets us discuss modern atomic, molecular, and quantum optical experiments more easily than a more conventional "wavefunction-potential" approach which will be described later. Many of the newer experiments involve beams of atoms or photons which take turns undergoing "analysis." The same is true for early seminal experiments in the beginning of quantum mechanics such as those of Stern-Gerlach, Davisson-Germer, or Brown-Twiss. Our "beam-analyzer" approach will involve "thought experiments" and computer simulations based on such classic experiments.

There are theoretical reasons for using a "beam-analyzer" approach. It is more fundamental; the "wavefunction-potential" approach is a special case of the former. Also, philosophical discussion of beam-analyzer mechanics is less of a pain in the neck because many of the mysterious aspects of quantum theory are stated up-front. (In science, as in politics, a "cover-up" is usually worse than the crime.) Finally, powerful mathematical and numerical techniques are more easily motivated and understood via a "beam-analyzer" approach. This helps to demystify mathematical concepts such as operators and state vectors which might otherwise become confused with the real mystery which lies in the physics.

1.1 Beam Sorters

The fundamental idea of beam analysis is fairly simple. The basic unit is an *elementary beam sorter* which is sketched in Fig. 1.1.1. A beam sorter splits a beam of particles coming from the right into some number n of *channels*. In each channel one finds particles in some physical condition or *state* that is distinguishable from those found in neighboring channels. (The words "find" and, particularly the word "state" need to be clarified, as we will see.)

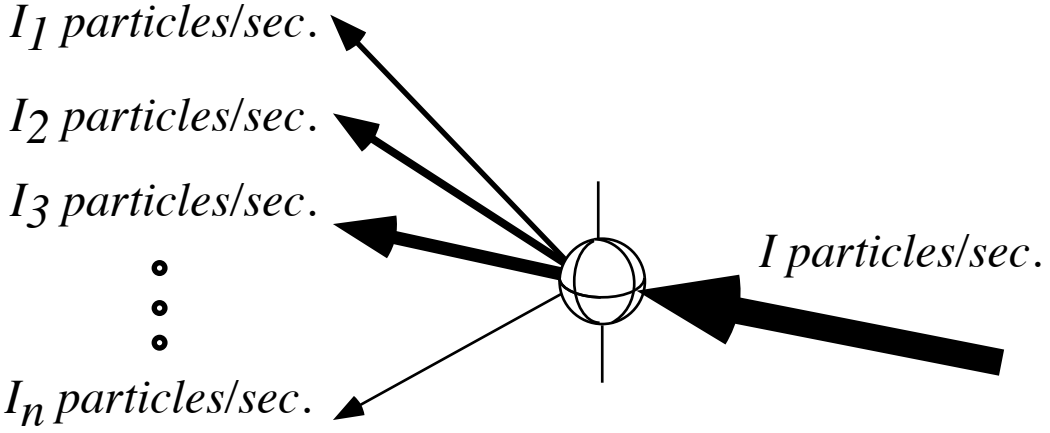


Fig. 1.1.1 Elementary beam sorter for n-state beam

Every particle that enters an elementary sorter winds up in one of the n channels ; no particles are lost or exempt. (Particles which can decay or otherwise mutate will be discussed later, but the analysis is the same; it just involves additional channels which are called *decay* or *inelastic scattering* channels.)

The initial beam (Right hand side of Fig. 1.1.1) has an *intensity* or *beam current I* . This is the number of particles per second passing a given point in that channel. This is distributed among the *n* channels which have currents $I_1, I_2, I_3 \ldots I_n$, respectively. Particle conservation requires that these channel intensities sum up to the total *I*.

$$I = I_1 + I_2 + I_3 + \ldots + I_n \qquad (1.1.1)$$

One job of quantum mechanics is to compute *relative intensities* or *probabilities* P_k defined by

$$P_k = I_k / I \qquad (1.1.2a)$$

where

$$1 = P_1 + P_2 + P_3 + \ldots + P_n \qquad (1.1.2b)$$

follows from (1.1.1). Later, this gets "puffed up" into an operator equation called a completeness relation.

The "quantum" nature of a beam-analyzer is tacitly being assumed here. In other words, we have already begun sneaking in some pretty mysterious concepts. First, the idea of a *particle* is a quintessential quantum concept that has been (and probably will continue to be) a real mystery. It is one of those concepts that humans have taken for granted (or granite) since before the Greeks coined the word *Atmos* while observing that great stones are made of bits of sand. Perhaps, what we really mean is an *elementary* particle like an electron or a photon as opposed to a *composite* particle like Buckyball (C_{60}) or a flake of dandruff. However, that is neither a necessary nor sufficient description. For awhile, the phrase "elementary particle" wass disappearing from the modern physics lexicon as it becomes increasingly clear nothing in nature is limited by our preconceived classical notion of a grain of sand. All "stuffs", meaning all forms of energy, have fundamental quantum behavior which can only mimic our preconceived notions of particles.

Second, the fact that an atomic beam can only be sorted into a finite (quantized) number *n* of split beams was a very big surprise when it first was observed, particularly by Stern and Gerlach whose Ag^+ beam split into exactly two parts! (See Fig. 1.1.2) The curious finite splitting of beams is, perhaps, most responsible for our concept of a *quantum state*. Indeed, Goudschmitt and Uhlenbeck proposed the idea of spin-up and spin-down states of electron spin polarization to help explain a number of atomic phenomena including the Stern-Gerlach experiment.

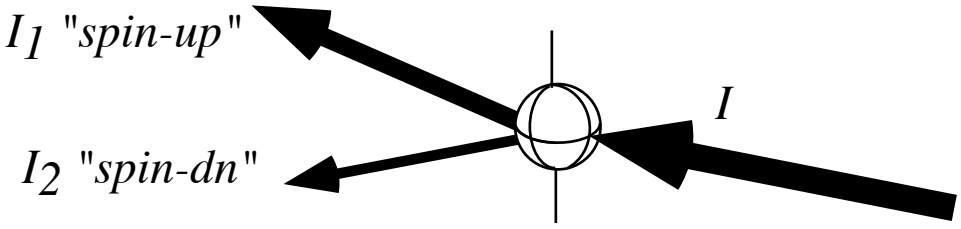


Fig. 1.1.2 Stern-Gerlach beam sorter for 2-state electron spin beam

Idealized versions of the Stern-Gerlach experiment and other two-state systems will be used to develop quantum theory in our beginning chapters. Feynman starts his description with three-state systems since their three-dimensional state-space is simple and much like the one we live in. Our choice of two-state systems is similarly motivated by the desire for simplicity and familiarity, however it uses an ultimately simpler and more fundamental analogy that goes back to 1860-1870's optical polarization theory of Poincare and Stokes.

We introduce quantum theory *vis-a-vis* photon-spin polarization, electron-spin polarization and nuclear (proton) spin-polarization; they all use similar mathematics. It also applies to NH_3 inversion-doublet states that gave us the first coherent radiation source or *maser* and marked a beginning of the laser revolution. A great deal of physics can be learned from the 2-state systems, and it also shows how to begin dealing with general *n*-state systems and much of quantum physics. Let's begin with some examples.

(a) Photon-beam polarization sorters

Consider some beam sorting experiments that a caveman could do by peering through calcite crystals. Each crystal magically gives two beams and two images, one with light polarized along the crystal's optical *x*-axis and a split-off beam having only *y*-polarized light as shown in Fig. 1.1.3 below.

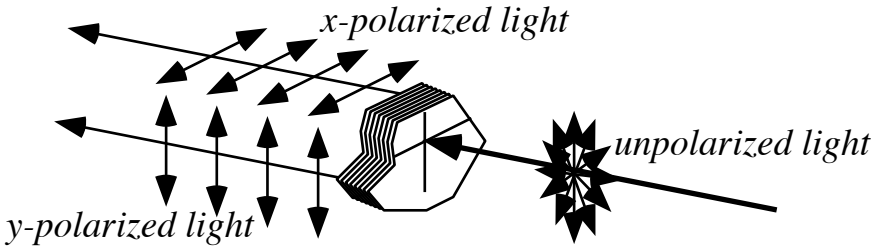


Fig. 1.1.3 Primitive photon beam sorter for 2-state polarization

If a second crystal catches the *x*-beam of the first crystal while blocking its *y*-beam, then the *y*-beam from the second crystal will disappear when the crystals' optical axes are aligned as shown in Fig. 1.1.4.

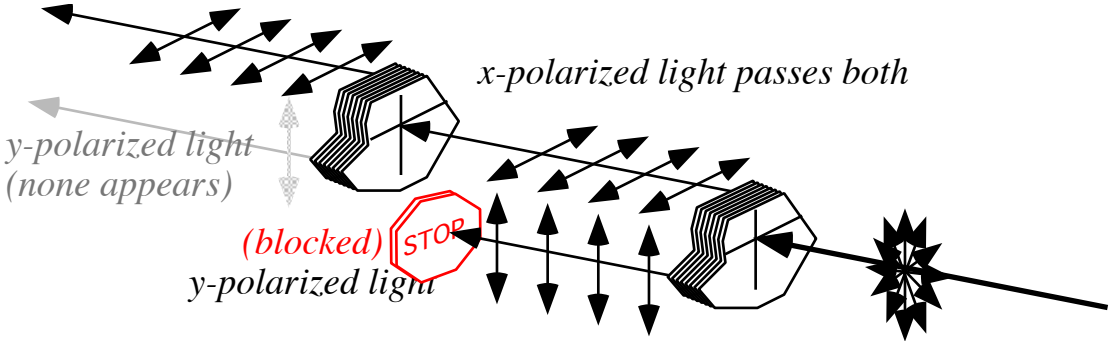


Fig. 1.1.4 Photon beam sorters in series. Second one examines x-beam of the first.

Modern optics labs have more sophisticated (and expensive) polarization sorters such as the Brewster prism sketched in Fig. 1.1.5. This takes advantage of fact that light reflected from a dielectric interface is nearly 100% polarized parallel to the reflection plane for a certain (Brewster's) angle of reflection.

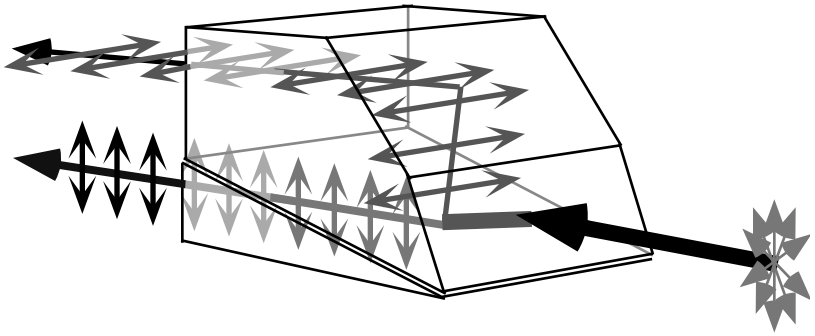


Fig. 1.1.5 Example of modern optical polarization sorter: The Brewster prism

(b) Electron-beam spin polarization sorters

Electron polarizers seem more mysterious than photon polarizers. The first ones used expensive vacuum and electron optics technology. As electronics evolved from vacuum tubes to semiconductors to *micro-m* optical fibers to *nano-m* wires, tiny *spintronic* polarizers have been developed. Here we will start with the old stuff.

A rough sketch of a Stern-Gerlach spin polarizer is shown in Fig. 1.1.6. It consists of asymmetric magnetic poles that produce a **B**-field with a large *z*-component and a field gradient tensor $\nabla\mathbf{B}$ with a large *zz*-component. The hapless electron is injected at right angles to the **B**-or *z*-axis, say, along the *y*-or beam axis. We presume electron spin angular momentum **S** and the magnetic dipole moment **m** are related by a constant of proportionality known as the *gyro magnetic ratio* γ.

$$\mathbf{S} = \gamma\,\mathbf{m} \qquad (1.1.3)$$

A classical scenario for what happens next goes something like the following. On entry **S** and **m** are pointing more or less up-z-axis and moving with the electron right-to-left along the *y*-or beam axis in Fig. 1.1.6 below.

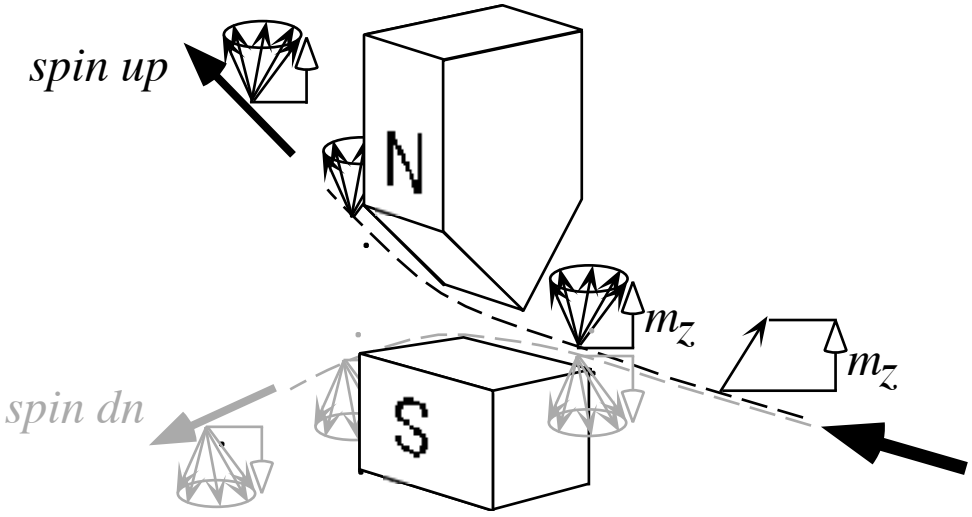


*Fig. 1.1.6 Electron beam sorting by non-uniform **B**-field (Stern-Gerlach polarizer)*

First, the **B**-field starts the electron spin and magnetic moment precessing like a conical helicopter blade around the **B** or *z*-axis thereby essentially freezing the *z*-component S_z or m_z of the spin moment and averaging the *x*- and *y*-components to zero. (See Fig. 1.1.6) Then the *zz*-gradient grabs the *z*-component m_z of electronic magnetic moment **m** with a force vector **F** in the direction $\mathbf{e_z}$ of the **B**-field gradient.

$$\mathbf{F} = \mathbf{m}\bullet\nabla\mathbf{B} = \mathbf{e}_z\left(\langle m_z\rangle\frac{\partial B_z}{\partial z}\right) \quad . \qquad (1.1.4)$$

This accelerates the "helicopter" in the z-direction at a rate proportional to the z-moment component m_z that the electron had when it first encountered the **B**-field. According to this, the final beam z-deflection is proportional to the initial z-component m_z or $S_z = \gamma m_z$ for each electron. So you might expect a randomly polarized beam to become smeared with a secant distribution up and down the left wall of the laboratory.

NOT! To practically everyone's surprise just two spots show up. The upper spot corresponds to a spin component of $S_z=+\hbar/2$ (called spin-up) and the lower spot to a spin component of $S_z=-\hbar/2$ (called spin-dn) where Planck's constant is $\hbar = h/2\pi = 1.05\ E\text{-}34\ Js$. No in-between values of S_z such as zero or $\pm 0.1\hbar/2$ or $\pm 0.25\hbar/2$ are ever seen no matter how much the original beam is randomized. Each electron spin vector **S** seems to behave like a political extremist; it chooses either to be completely up or completely down with respect to the **B**-field. Nothing in between is ever seen. Furthermore, each electron seems to have exactly one-half quanta ($\hbar/2$) of angular momentum permanently buried in its belly. This came as a surprise to those who were just getting used to the early ideas of Bohr quantum theory which said that the smallest quantum of angular momentum or action was the Planck $\hbar$ unit.

The Stern-Gerlach experiment also is remarkable since the electron in question is dragging along an entire silver atom which out-weighs it by a factor of about 300,000. (The experiment used a beam of Ag^+ cations.) One could imagine dragging a 500 pound hog around by its ear!

This experiment appears to be a good deal more complicated than the cave-man polarization experiments. We shall put off discussion of its details until later, but even then, the deep-down details of electronic spin and structure remain mysterious to this day. Quantum electrodynamics (QED) has come a long way but many mysteries remain. If you can give a cogent sub-electronic theory of electron structure which explains the detailed origin of its spin 1/2 you might have a Nobel prize Fedexed to your doorstep by Friday.

For now, we can only treat spin phenomena as part of a given set of physical axioms and construct a mathematical analog for the behavior. Just such a mathematical structure is called *spinor analysis* by Jordan and Pauli who (re)discovered it around 1920. It is similar to *quaternion algebra* which was discovered by W. R. Hamilton around 1843, more than half a century before the first Stern-Gerlach experiments. The modern name for this mathematics is *U(2) group algebra* and that is one of the many mathematical ideas we will be developing. It is a credit to the efficiency of *U(2)* mathematics that it applies to both electronic and optical spin or polarization. However it does so by ignoring some obvious physical differences between these two quite diverse phenomena! We will try to address some of these as we go along.

1.2 Beam Sorters in Series: Transformation Matrices

The fun begins with investigation of beam sorters in series such as Fig. 1.2.1 below. This is the same as Fig. 1.1.4 except that the optical axis of the first crystal is tilted relative to the second by an angle θ. Without tilt (θ=0) the *x*-polarized beam from the first crystal is 100% sorted to the *x*-beam exit of the second crystal and nothing shows up at the *y*-polarized exit as we saw in Fig. 1.1.4. But, for even a slight tilt, as in Fig. 1.2.1, there will appear a weak beam exiting the *y*-polarized exit of the second sorter and the *x*-polarized beam will be reduced in intensity by just the amount that gets diverted into the *y*-beam. We indicate the rotated optical axes by primed (x', y') labels of the polarization.

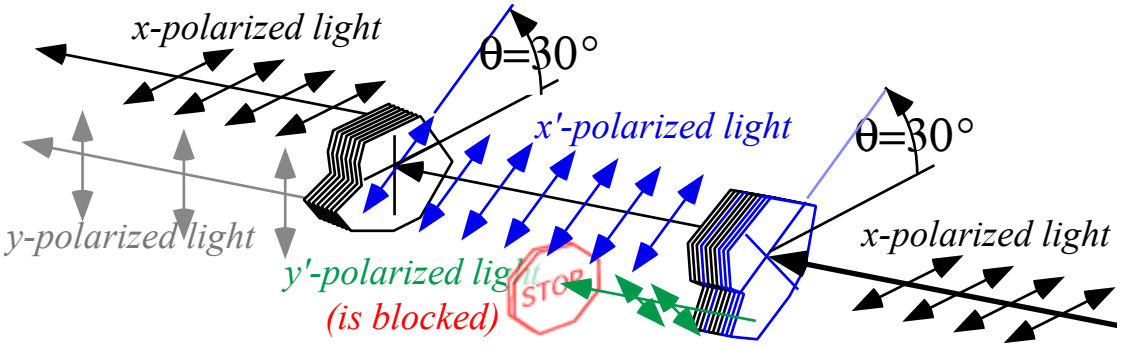


Fig. 1.2.1 Photon beam sorters in series with the first one y-blocked and tilted by angle θ.

We will use this example to introduce what is probably the single most important mathematical object in quantum theory: the *transformation matrix*. It is possible, with a little classical hand-waving, to visualize and understand the quantum transformation matrix for optical polarization. For electron polarization, which we consider subsequently, the transformation matrix and its interpretation will, at first, seem quite mysterious. Later, we will see that they are both representations of the same thing. (And, they are both mysterious, but in a nicer sort of way.)

(a) Transformation matrices for optical polarization

How does the tilted x'-polarization in Fig. 1.2.1 get transformed into *y*-polarized light and how much gets transformed? Consider the following model for an optically active crystal. Let it have two kinds of charged masses held by very strong springs. First there are the X-masses that can only slide and oscillate along the optical *x*-axis, and then there are the Y-masses which can only oscillate along the *y*-axis perpendicular to x. In other words, an *x*-polarized E-field can only wiggle the X-masses which then pass on an *x*-polarized polarization wave that comes out somewhere on the other side of the crystal, and similarly for *y*-polarized waves which come out somewhere else. In calcite the *x*-waves go at different speeds than the *y*-waves in order to produce optical beam splitting but ideally either transmits the same intensity.

Now when an electric wave with x'-polarization tilted by angle θ hits the crystal, both the X-masses and the Y-masses get stimulated in proportion to the projections *cos* θ and *sin* θ of the x'-direction on their respective oscillation tracks X and Y. This is sketched in Fig. 1.2.2.

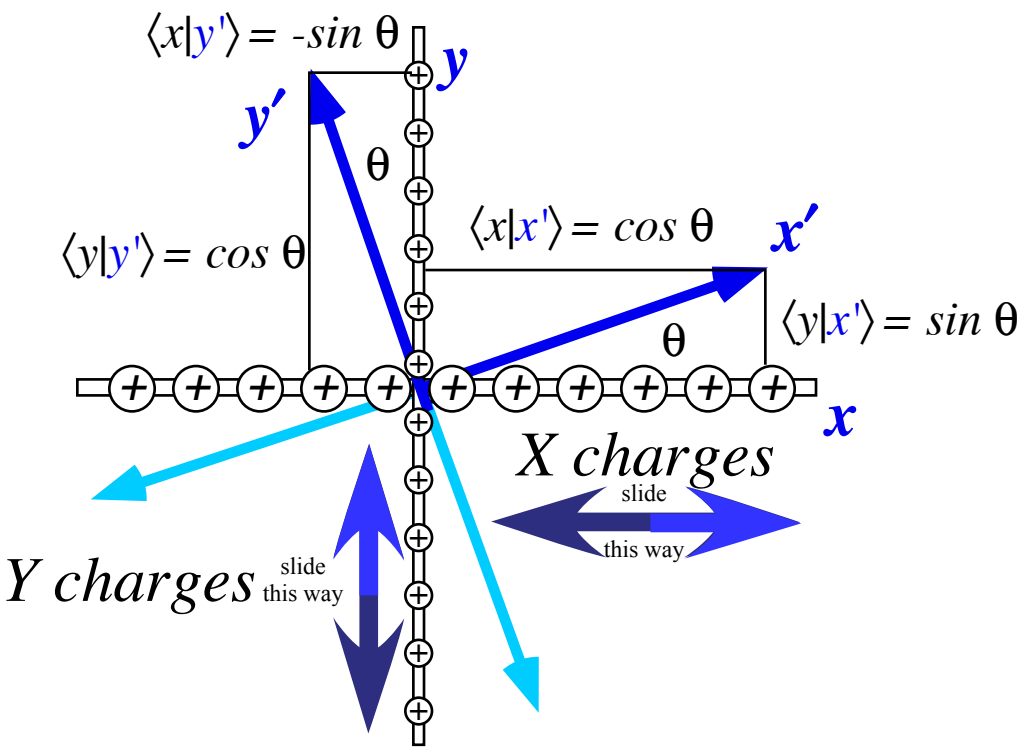


Fig. 1.2.2 Geometry of photon beam sorter for input polarizations (x',y') tilted by angle θ.

The resulting X and Y output amplitudes due to incoming x'-polarization are given by the following *Dirac bra-ket notation*.

$$\begin{aligned} X \text{ output amplitude due to } x' \text{ input} &= \langle x|x'\rangle = \cos\theta, \\ Y \text{ output amplitude due to } x' \text{ input} &= \langle y|x'\rangle = \sin\theta \end{aligned} \qquad (1.2.1a)$$

If we had instead focused the y' beam (and blocked x'), then the X and Y outputs would have been the following according to Fig. 1.2.2. Note in particular that a +y' field drives X-charges negatively (-sinθ).

$$\begin{aligned} X \text{ output amplitude due to } y' \text{ input} &= \langle x|y'\rangle = -\sin\theta, \\ Y \text{ output amplitude due to } y' \text{ input} &= \langle y|y'\rangle = \cos\theta \end{aligned} \qquad (1.2.1b)$$

An array of these amplitudes is called the *transformation matrix* for the ideal polarization experiments of the type sketched in Fig. 1.2.1. The first column (1.2.1a) represents the x'-beam going in Fig. 1.2.1.

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad (1.2.1c)$$

The second column (1.2.1b) does the same for a y'-beam experiment.

The array (1.2.1c) is also a standard *coordinate transformation matrix* for rotation of coordinate axes. All quantum transformation matrices are some kind of mathematical coordinate transformation, though few are as obvious as this one. Transformation group theory is very useful in quantum mechanics.

How do you visualize and understand a transformation matrix? Dirac has given us a neat way to do so with his clever bra-ket notation. Let's take transformation (1.2.1c) apart again into its separate columns (1.2.1a) and 1.1.5b). Such columns are called *ket-vectors* or *kets* by Dirac.

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$
$$\Downarrow \qquad \Downarrow \qquad\qquad \Downarrow \qquad \Downarrow$$
$$|x'\rangle = \begin{pmatrix} \langle x|x'\rangle \\ \langle y|x'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad |y'\rangle = \begin{pmatrix} \langle x|y'\rangle \\ \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \tag{1.2.2}$$

The kets $|x'\rangle$ and $|y'\rangle$ are just a funny notation for the unit vectors **x'** and **y'** indicated by the darker arrows in Fig. 1.2.2. But, their quantum mechanical significance is a bit deeper; the kets are each examples of a *polarization state vector* $|\Psi\rangle$ of a photon. The amplitudes $\langle x|\Psi\rangle$ and $\langle y|\Psi\rangle$ relate any state $|\Psi\rangle$ to the original (untilted θ=0) x and *y*-polarization states that come out of the θ=0 sorter, that is, to the basic unit vector basis $|x\rangle$ and $|y\rangle$ or **x** and **y** in Fig. 1.2.2 which are represented as follows.

$$|x\rangle = \begin{pmatrix} \langle x|x\rangle \\ \langle y|x\rangle \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |y\rangle = \begin{pmatrix} \langle x|y\rangle \\ \langle y|y\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.2.3}$$

(1.2.3) is just (1.2.2) with θ=0 . This relation is expressed using vector sums. In Dirac notation we write

$$\begin{aligned} |x'\rangle &= |x\rangle\langle x|x'\rangle + |y\rangle\langle y|x'\rangle, & |y'\rangle &= |x\rangle\langle x|y'\rangle + |y\rangle\langle y|y'\rangle, \\ &= |x\rangle(\cos\theta) + |y\rangle(\sin\theta), & &= |x\rangle(-\sin\theta) + |y\rangle(\cos\theta). \end{aligned} \tag{1.2.4a}$$

The same thing in Gibbs vector notation would be

$$\begin{aligned} \mathbf{x}' &= \mathbf{x}(\mathbf{x}\bullet\mathbf{x}') + \mathbf{y}(\mathbf{y}\bullet\mathbf{x}'), & \mathbf{y}' &= \mathbf{x}(\mathbf{x}\bullet\mathbf{y}') + \mathbf{y}(\mathbf{y}\bullet\mathbf{y}'), \\ &= \mathbf{x}\,(\cos\theta) + \mathbf{y}\,(\sin\theta), & &= \mathbf{x}\,(-\sin\theta) + \mathbf{y}\,(\cos\theta)\,. \end{aligned} \tag{1.2.4b}$$

By comparing these two notations it's clear that the transformation matrix of bra-kets corresponds to an array of dot or *scalar products*. The dot products of unit vectors are often called *direction cosines*.

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} (\mathbf{x}\bullet\mathbf{x}') & (\mathbf{x}\bullet\mathbf{y}') \\ (\mathbf{y}\bullet\mathbf{x}') & (\mathbf{y}\bullet\mathbf{y}') \end{pmatrix} \tag{1.2.5}$$

Equations (1.2.4) apply to any state $|\Psi\rangle$, not just $|x'\rangle$ or $|y'\rangle$, and to any valid quantum basis kets, not just $|x\rangle$ and $|y\rangle$. Any state can be expanded in any basis

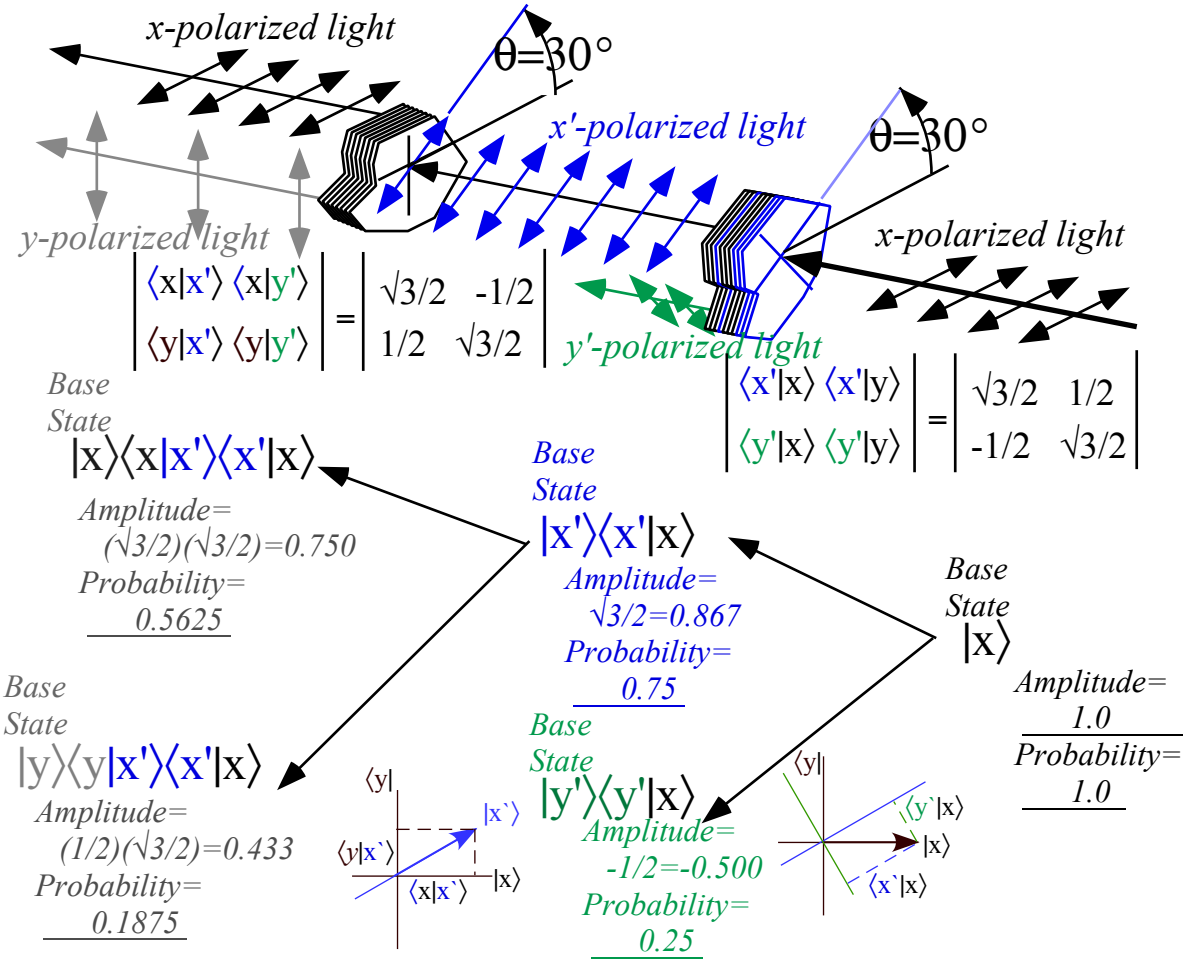
$$|\Psi\rangle = |x\rangle\langle x|\Psi\rangle + |y\rangle\langle y|\Psi\rangle = |x'\rangle\langle x'|\Psi\rangle + |y'\rangle\langle y'|\Psi\rangle \tag{1.2.6a}$$

Transformation matrices relate the amplitudes ($\langle x|\Psi\rangle$,$\langle y|\Psi\rangle$) of one basis to ($\langle x'|\Psi\rangle$,$\langle y'|\Psi\rangle$) of another.

$$\begin{pmatrix} \langle x|\Psi\rangle \\ \langle y|\Psi\rangle \end{pmatrix} = \begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} \begin{pmatrix} \langle x'|\Psi\rangle \\ \langle y'|\Psi\rangle \end{pmatrix}, \text{ or } \begin{pmatrix} \Psi_x \\ \Psi_y \end{pmatrix} = \begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} \begin{pmatrix} \Psi_{x'} \\ \Psi_{y'} \end{pmatrix} \tag{1.2.6b}$$

What's the state I'm in? The ideas behind projection

To get a feeling for doing quantum calculations we trace through a chain of polarization sorting experiments using the chain from Fig. 1.2.1 with a particular pure state $|x\rangle$ entering as shown below. (This is in contrast to the random mess going into Fig. 1.2.1.) We want to calculate what comes out in each channel or branch-b, namely (a) a *base state* $|b\rangle$, (b) its *amplitude* $\langle b|\Psi\rangle$, and (c) its *probability* $|\langle b|\Psi\rangle|^2$.



First is the transformation matrix $\langle b|c\rangle$ for each sorter that outputs branch b given input channel-c. The transformation matrices are given below each sorter in the figure above following (1.2.2-5).

Then the state in branch-b is $|b\rangle\langle b|\Psi\rangle$ where $|\Psi\rangle$ is whatever state came in the sorter input channel. In the x'-polarized light channel the state is simply $|x'\rangle\langle x'|x\rangle = |x'\rangle\sqrt{3}/2$. The number $\langle x'|x\rangle = \sqrt{3}/2$ is the *amplitude* of the base state of the branch or channel *base state* $|x'\rangle$, while $|\langle x'|x\rangle|^2 = 3/4$ is the *probability* or *branching ratio* for counting a particle there. This is true because any photon that makes it to branch or channel-x' must be an x'-polarized particle that is in state $|x'\rangle$ if its probability to be there is 100% or else an *attenuated* state $|x'\rangle\langle x'|x\rangle$ if its probability $|\langle x'|x\rangle|^2$ to arrive is less than one.

This process of writing $|b\rangle\langle b|\Psi\rangle$ is repeated for the next sorter in the chain only now $|\Psi\rangle$ is the previously attenuated state $|x'\rangle\langle x'|x\rangle$. So, the y-polarized branch ends up with state $|y\rangle\langle y|x'\rangle\langle x'|x\rangle$ with an even smaller amplitude $\langle y|x'\rangle\langle x'|x\rangle$ as shown in the lower left hand corner of the figure. Each use of $|b\rangle\langle b|$ is called a *projection operation* and is discussed in Sec. 2.1.(b) 5.

(b) WHOA! That analogy is TOO simple! Planck's energy and quantum counts

If simple 2D-rotation was all there was to quantum theory we probably wouldn't have courses for it! In fact, the transformation matrices, even for optical polarization, are a little more complicated than the preceding analogies might first indicate.

We mentioned that we were dealing with electromagnetic waves and charge oscillations in our simple model. We need to say a little more about this. Static (DC) polarization is fairly simple by comparison. Optical polarization involves high frequency (AC) dynamics and *resonance* phenomena. This is true for most of nature's processes, particularly those in the quantum domain where all the amplitudes wiggle incessantly like so many fidgety children. Gibb's vector notation, such as (1.2.4b) was designed for DC vectors. Dirac notation is designed for AC vectors, and AC theory uses *complex variables*.

There is more. The simple truth is this: *all quantum amplitudes are complex numbers*. At the very least they have a (sometimes hidden) time-dependent factor $e^{-i\omega t}$ given by

$$e^{-i\omega t} = \cos \omega t - i \sin \omega t \qquad (1.2.7a)$$

where the angular frequency $\omega=2\pi\nu$ or frequency ν is related by *Planck's constant* $h=2\pi\hbar=6.63E\text{-}34Js$

$$\varepsilon = h\nu = \hbar\omega \qquad (1.2.7b)$$

to the energy ε of a quantum state. This will be one of our most important axioms of quantum mechanics, when we get around to formal axiomization. Energy is Mother Nature's heart rate and heart beat.

In the case of light, (1.2.7b) is the equation for the energy of a single quantum of light, or *photon,* the smallest piece of energy you can extract from a light beam of a given frequency or color. Eq. (1.2.7b) is, perhaps, the first equation of quantum theory, historically and fundamentally, the basis for at least two Nobel prizes and still, many decades later, just as mysterious as it was when first stated in 1905.

However, for decades (1863-1905) classical polarization theory would ignore (1.2.7b) because the huge number of photons in a typical light beam makes it appear to be a continuous wave. The angular frequency $\omega=2\pi\nu$ of light in (1.2.7a) is presumed to be known (by color if visible) and classical resonance theory of Lorentz usually predicts polarization response due to a light beam very accurately. Generally, the beam itself was described by a complex electric field amplitude vector $\left(E_x, E_y\right)$ which is proportional, by some constant factor f, to our unit $\left(\Psi_x, \Psi_y\right)$ vector in (1.2.6b).

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} E_x(0)e^{-i\omega t} \\ E_y(0)e^{-i\omega t} \end{pmatrix} = f \begin{pmatrix} \Psi_x \\ \Psi_y \end{pmatrix} \qquad (1.2.8)$$

Classical theory did not consider the energy $\varepsilon = \hbar\omega$ of one photon, only the *Poynting energy flux S* or *energy density U* of a whole light beam. According to Maxwell's electromagnetic wave theory the density or flux is proportional to the sum of absolute squares of the complex amplitudes.

$$S = cU, \text{ where: } U = \varepsilon_0\left(\left|E_x\right|^2 + \left|E_y\right|^2\right) = \varepsilon_0\left(E_x^* E_x + E_y^* E_y\right) = \varepsilon_0\left(E_x(0)^2 + E_y(0)^2\right) \quad (1.2.9a)$$

Constant speed of light $c = 2.997E^8\ ms^{-1}$ and electrostatic constant $\varepsilon_0 = 8.842E\text{-}12\ C^2N^{-1}m^2$ are given.

For a beam of *n*-photons, the energy density is *U* (*Joules per cubic meter*) and energy flux *S=cU* (*Joules per square meter per second*. Photons are assumed to have velocity c.) To relate this to that of a single photon we must equate *U* to *n*-times Planck's energy $\hbar\omega$ in eq. (1.2.7b) divided by beam or cavity volume *V*.)

$$\frac{n\hbar\omega}{V} = U = \varepsilon_0\left(\left|E_x\right|^2 + \left|E_y\right|^2\right) = \varepsilon_0 f^2\left(\left|\Psi_x\right|^2 + \left|\Psi_y\right|^2\right) \tag{1.2.10}$$

Particle number *n* or beam intensity *I* is found by dividing $U{\cdot}V$ by the quantum energy $\hbar\omega$.

$$n = I = \frac{V{\cdot}U}{\hbar\omega} = \frac{V\varepsilon_0}{\hbar\omega}\left(\left|E_x\right|^2 + \left|E_y\right|^2\right) = \frac{V\varepsilon_0}{\hbar\omega} f^2\left(\left|\Psi_x\right|^2 + \left|\Psi_y\right|^2\right) = \frac{V\varepsilon_0}{\hbar\omega} f^2 n \tag{1.2.11a}$$

For a single photon $n=I=1$ the E-field amplitude and Ψ-amplitude scale factor is given

$$f = \sqrt{\frac{\hbar\omega}{V\varepsilon_0}} = \sqrt{\left|E_x\right|^2 + \left|E_y\right|^2}\ , \quad \text{for one-photon: } n = I = 1 \tag{1.2.11b}$$

This is the *quantum field constant f* used much later on. Note that the Ψ-amplitude squares sum to the particle number per unit time.

$$n = I = \left|\Psi_x\right|^2 + \left|\Psi_y\right|^2 = \langle x|\Psi\rangle^* \langle x|\Psi\rangle + \langle y|\Psi\rangle^* \langle y|\Psi\rangle \tag{1.2.11c}$$

This is called a *normalization condition*. For one particle ($n=1$) it is called *unit normalization*.

At first, we will set $n=I=1$, and deal with only one quantum (photon) at a time. Then each term $\Psi_x^*\Psi_x$ or $\Psi_y^*\Psi_y$ of (1.2.11c) gives the *probability* that the photon with Ψ-polarization will be found in the *x*-polarization state or the *y*-polarization state, respectively. With this statement we first confront the awful truth of quantum theory. To really do the "caveman" polarization experiment accurately we need to wait a million years or so, until the 20-th century when *photon counters* are invented. Then we buy two of these gadgets and stick them at the ends of the x and *y*-beams as shown in Fig. 1.2.3 below.

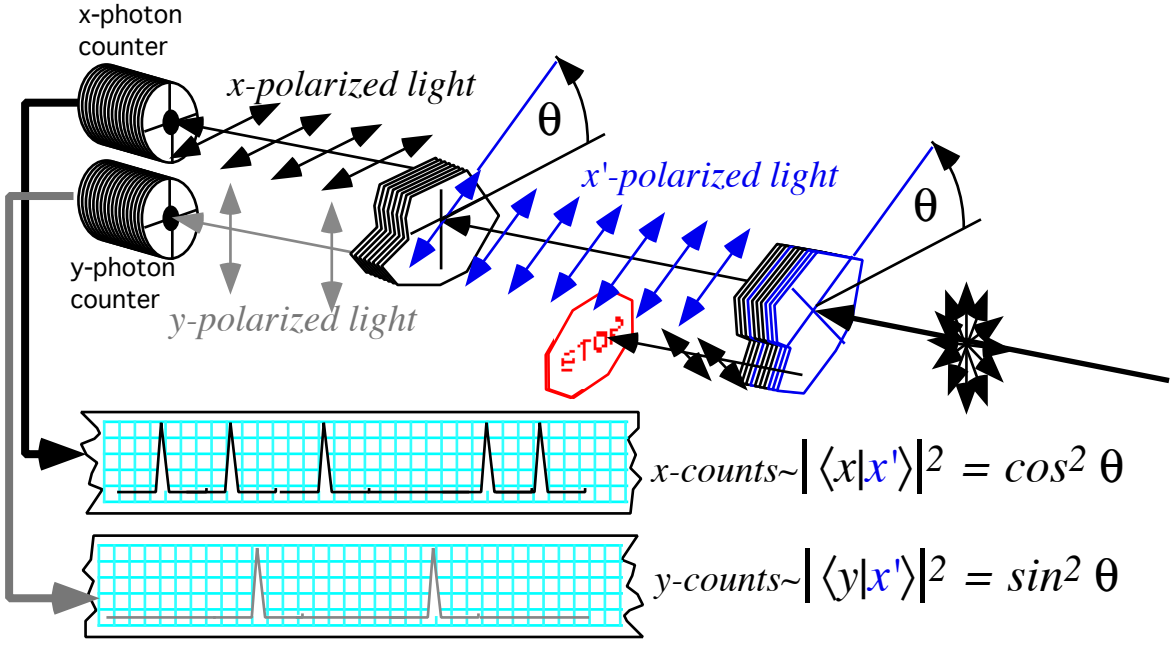


Fig. 1.2.3 Photon x-y beam sorting and quantum (photon) counting of an x'-state

After waiting for hundreds, thousands, or millions of counts the relative numbers of *x*-photon counts to *y*-photon counts will gradually approach the predicted ratios listed above in Fig. 1.2.3 of

$$\left|\Psi_x\right|^2 = \left|\langle x|\Psi\rangle\right|^2 = \left|\langle x|x'\rangle\right|^2 = \cos^2\theta \qquad \text{and} \qquad \left|\Psi_y\right|^2 = \left|\langle y|\Psi\rangle\right|^2 = \left|\langle y|x'\rangle\right|^2 = \sin^2\theta \qquad (1.2.12)$$

In other words, the quantum experiment for large numbers of photons will correspond to the classical predictions of 1863. This is an example of *quantum-classical correspondence*, quantum physics usually yields classical physics in the limit of large quantum numbers or large numbers of observations.

Otherwise, quantum amplitudes yield information in the form of probabilities and statistical distributions. The absolute square $\left|\langle x|x'\rangle\right|^2$ of amplitude $\langle x|x'\rangle$ is the probability that one photon in the x'-beam will register a count in the *x*-counter of Fig. 1.2.3. The (complex in general) amplitude $\langle x|x'\rangle$ is called the *probability amplitude* for a x' to x transformation. We read amplitudes right to left (x' goes into x) like Hebrew because, perhaps, many of the originators of quantum theory were Jewish. Also, always remember that we square the amplitude to get the probability.

It is instructive to see some of the limitations of quantum theory early on. You might wonder, "Can quantum theory tell if a particular x'-photon will go to the *x*-counter or to the *y*-counter?" The answer appears to be a resounding NO! Not even Mother Nature, as crafty as she is, seems to know. Or you might ask, "Can we tell exactly *when* a photon will make its decision to be x or y?" Again, NO! As we will see later, monochromatic light beams (meaning single frequency or color) are particularly reluctant to say when (or where) their individual photons are going to show up.

However, quantum theory can predict correlation statistics about the time distribution of counts, but this depends on the properties of the wizard behind the curtain on the right of Fig. 1.2.3 who is cooking up the photon beam as well as the nature of the photon counters themselves. For the time being, we will pay no attention to the wizard behind the curtain. Also, counters are assumed 100% efficient.

(c) Transformation matrices for electron spin polarization

As we said in Section 1.1.(b) electron polarization is not as easily visualized as the photon polarization. The same goes for the transformation matrices and corresponding amplitudes even though (as we will eventually see) their mathematics is virtually identical.

Indeed, the idea that an electron could and should be described as a wave was even more mysterious than the idea that light waves could be viewed as particles. Electrodynamics of the late 1800's had electrons labeled as a particles and light labeled as waves. Relativity and quantum mechanics have gone a long way toward showing the similarity of these two types of quantum energy-momentum, while also emphasizing their differences. Modern "super-unified" field theories continue attempts to unite them and all particles while modern experiments continue, more often than not, to distinguish them.

With this in mind we introduce an ideal electron polarization transformation experiment analogous to the photon polarization experiment in Fig. 1.2.3. This is shown below in Fig. 1.2.4.

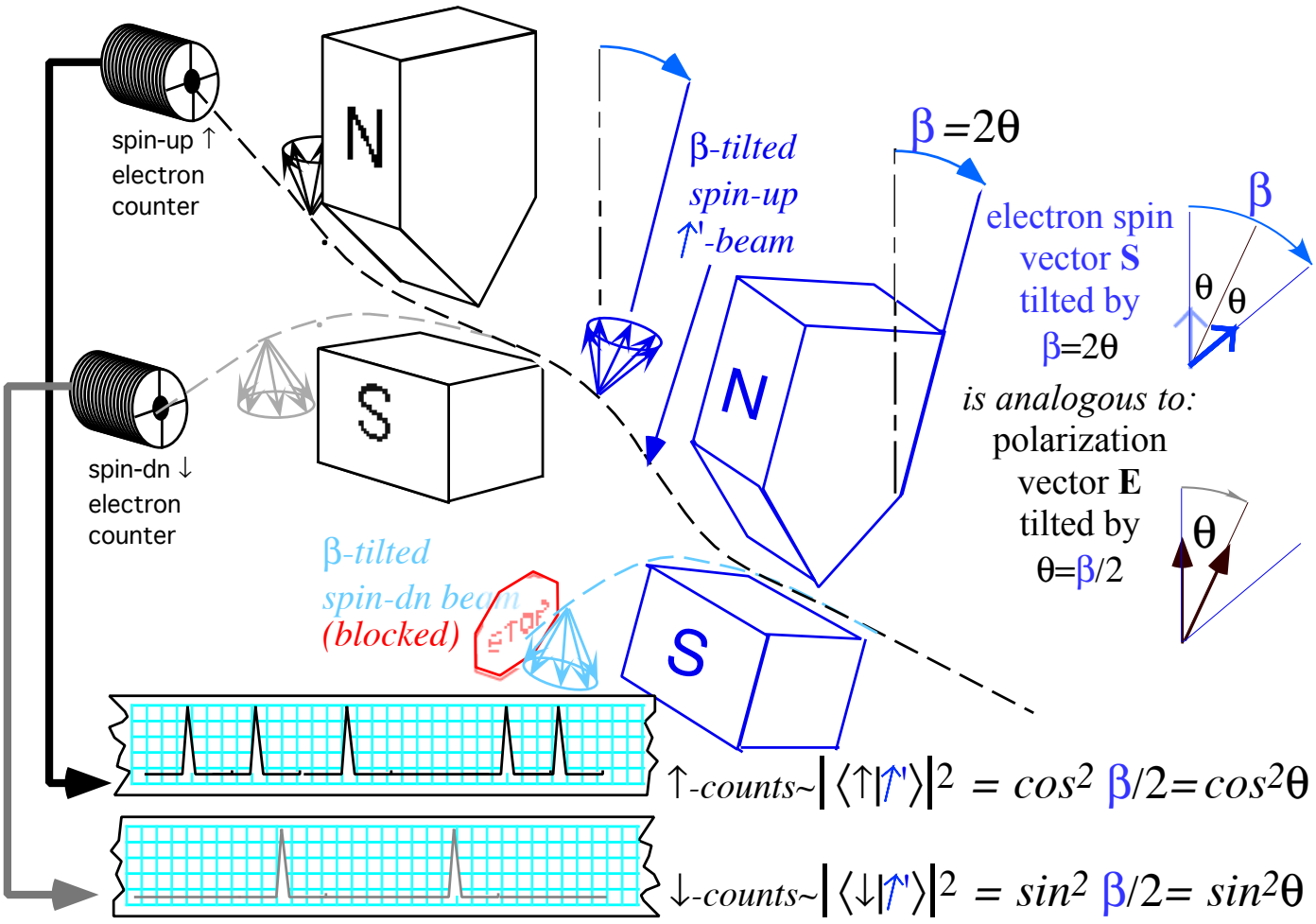


Fig. 1.2.4 Electron up-dn-spin counting of a tilted spin-up (↑′)-state

This is analogous to the polarization experiment first discussed after Fig. 1.2.3. Only now it is a tilted spin-up (↑′) electrons that have to decide whether to choose spin up (↑) or spin dn (↓). The up (↑) and dn (↓) output amplitudes due to incoming β-tilted spin-up (↑′) are as follows.

(↑) output amplitude due to (↑′) input = $\langle\uparrow|\uparrow'\rangle = \cos\beta/2 = \cos\theta$ (1.2.12a)

(↓) output amplitude due to (↑′) input = $\langle\downarrow|\uparrow'\rangle = \sin\beta/2 = \sin\theta$ (1.2.12b)

Comparison with (1.2.1a) shows that we use half the tilt angle (β/2) in the sine and cosine, while the photon formulas used the whole angle (θ). This is because spin-up is *180°* from spin-down while *x*-polarization is only *90°* from *y*-polarization. Tilting *x*-polarization by θ=*90°* makes it *y*-polarization according to (1.2.1a) where cos *90°*=0 and sin *90°*=1, but tilting spin-up (↑) into spin-dn (↓) requires twice the angle or β=2θ=*180°* according to (1.2.12a-b) where cos *180°/2* =0 and sin *180°/2* =1. This half-angle geometry is one of many mysterious features of the strange half-quantum spin ħ/2 of an electron to be treated in later Units.

NOTE: From now on a prime means "tilted" so ↑′ and ↑′ mean the same tilted spin-up. Similarly for spin-dn.

An array of spin-1/2 amplitudes is called a *spinor transformation matrix*, and it describes the ideal electron spin experiments of the type sketched in Fig. 1.2.4. It is the same as the analogous photon polarization transformation matrix except the polarization angle (θ) is replaced by a half angle (β/2).

$$\begin{pmatrix} \langle\uparrow|\uparrow'\rangle & \langle\uparrow|\downarrow'\rangle \\ \langle\downarrow|\uparrow'\rangle & \langle\downarrow|\downarrow'\rangle \end{pmatrix} = \begin{pmatrix} \cos\beta/2 & -\sin\beta/2 \\ \sin\beta/2 & \cos\beta/2 \end{pmatrix} \tag{1.2.13}$$

Once again we extract columns which are called *ket-vectors* or *kets* by Dirac in analogy to (1.2.2).

$$\begin{pmatrix} \langle\uparrow|\uparrow'\rangle & \langle\uparrow|\downarrow'\rangle \\ \langle\downarrow|\uparrow'\rangle & \langle\downarrow|\downarrow'\rangle \end{pmatrix} = \begin{pmatrix} \cos\beta/2 & -\sin\beta/2 \\ \sin\beta/2 & \cos\beta/2 \end{pmatrix}$$
$$\Downarrow \qquad \Downarrow$$
$$\Downarrow \qquad\qquad \Downarrow \tag{1.2.14}$$
$$|\uparrow'\rangle = \begin{pmatrix} \langle\uparrow|\uparrow'\rangle \\ \langle\downarrow|\uparrow'\rangle \end{pmatrix} = \begin{pmatrix} \cos\beta/2 \\ \sin\beta/2 \end{pmatrix}, \quad |\downarrow'\rangle = \begin{pmatrix} \langle\uparrow|\downarrow'\rangle \\ \langle\downarrow|\downarrow'\rangle \end{pmatrix} = \begin{pmatrix} -\sin\beta/2 \\ \cos\beta/2 \end{pmatrix}$$

The first column above represents the tilted up-($\uparrow'$)-beam going in to be split Fig. 1.2.1. The second column does the same for a tilted down-($\downarrow'$)-beam experiment. Prime (′) means "β-tilted" here.

The kets are each an example of an *electron spin-state vector* $|\chi\rangle$. The amplitudes $\langle\uparrow|\chi\rangle$ and $\langle\downarrow|\chi\rangle$ relate <u>any</u> state $|\chi\rangle$ to the original (untilted β=0) spin-up and spin-dn states that come out of a β=0 sorter, that is, to the basic <u>unit</u> vector basis $|\uparrow\rangle$ and $|\downarrow\rangle$ which are represented as follows.

$$|\uparrow\rangle = \begin{pmatrix} \langle\uparrow|\uparrow\rangle \\ \langle\downarrow|\uparrow\rangle \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle = \begin{pmatrix} \langle\uparrow|\downarrow\rangle \\ \langle\downarrow|\downarrow\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.2.15}$$

(1.2.15) is just (1.2.14) with β=0 . This relation is expressed using vector sums in Dirac notation .

$$\begin{aligned} |\uparrow'\rangle &= |\uparrow\rangle\langle\uparrow|\uparrow'\rangle + |\downarrow\rangle\langle\downarrow|\uparrow'\rangle, \quad |\downarrow'\rangle = |\uparrow\rangle\langle\uparrow|\downarrow'\rangle + |\downarrow\rangle\langle\downarrow|\downarrow'\rangle, \\ &= |\uparrow\rangle\left(\cos\frac{\beta}{2}\right) + |\downarrow\rangle\left(\sin\frac{\beta}{2}\right), \qquad = |\uparrow\rangle\left(-\sin\frac{\beta}{2}\right) + |\downarrow\rangle\left(\cos\frac{\beta}{2}\right). \end{aligned} \tag{1.2.16}$$

Equations (1.2.16) apply to <u>any</u> spin state $|\chi\rangle$, not just $|\uparrow'\rangle$ or $|\downarrow'\rangle$, and to any valid quantum basis kets, not just $|\uparrow\rangle$ and $|\downarrow\rangle$. Any spin state can be expanded in any spin basis

$$|\chi\rangle = |\uparrow\rangle\langle\uparrow|\chi\rangle + |\downarrow\rangle\langle\downarrow|\chi\rangle = |\uparrow'\rangle\langle\uparrow'|\chi\rangle + |\downarrow'\rangle\langle\downarrow'|\chi\rangle \tag{1.2.17a}$$

Transformation matrices relate the amplitudes ($\langle\uparrow|\chi\rangle$,$\langle\downarrow|\chi\rangle$) of one basis to ($\langle\uparrow'|\chi\rangle$,$\langle\downarrow'|\chi\rangle$) of another.

$$\begin{pmatrix} \langle\uparrow|\chi\rangle \\ \langle\downarrow|\chi\rangle \end{pmatrix} = \begin{pmatrix} \langle\uparrow|\uparrow'\rangle & \langle\uparrow|\downarrow'\rangle \\ \langle\downarrow|\uparrow'\rangle & \langle\downarrow|\downarrow'\rangle \end{pmatrix} \begin{pmatrix} \langle\uparrow'|\chi\rangle \\ \langle\downarrow'|\chi\rangle \end{pmatrix}, \text{ or } \begin{pmatrix} \chi_\uparrow \\ \chi_\downarrow \end{pmatrix} = \begin{pmatrix} \langle\uparrow|\uparrow'\rangle & \langle\uparrow|\downarrow'\rangle \\ \langle\downarrow|\uparrow'\rangle & \langle\downarrow|\downarrow'\rangle \end{pmatrix} \begin{pmatrix} \chi_{\uparrow'} \\ \chi_{\downarrow'} \end{pmatrix} \tag{1.2.17b}$$

If this seems like *deja vu* (again!) from eqs. (1.2.2-6), it should. The mathematics is very similar, and we are setting the stage for general quantum theory. Electron physics, on the other hand, is quite different from photon physics when it comes to interpreting the amplitudes.

(d) Amplitudes of What? Fermi vs. Bose

As noted in Fig. 1.2.4 the electron spin-up or down count probabilities per unit time were proportional to the absolute squares of the respective amplitudes. (Actually, they're <u>equal</u> if you choose the right time units.)

The same was true for photon polarization *x*- or y counts in Fig. 1.2.3, but that was no surprise because classical field intensities have always been absolute squares of the polarization field amplitudes or wave functions.

Furthermore, the classical polarization amplitudes have had real physical meaning since about 1863 or a little before. Appropriately scaled (Recall eq. (1.2.11b).) they stood for E-fields, so many Volts per meter, or some such. At least, we thought they did!

Now along comes these post-bellum electron spin amplitudes which seem to have absolutely no physical significance whatsoever. Only their absolute squares are observable, and those are probabilities which are dimensionless apart from the (per time unit) that accompanies all particle counting experiments. There is no real physical polarization wave field analogous to the photon E-field that can be associated with an electron or even a big cloud of electrons. Why is this?

It turns out that electrons and all objects called *Fermions* which have half-quantum spins are incurable "loners." They avoid being near or like their own kind. Why this should be is still quite a mystery. Perhaps, they are embarrassed by only having half the quantum spin of the other guys and don't want anyone to find out. Whatever! So quantum physics chalks this up as one more axiom called by various names like the *Pauli principle* or *Fermi symmetry* after Wolfgang Pauli and Enrico Fermi. We will discuss these principles and their consequences in later chapters. One consequence is that electrons seem unable to get together in the same state in order to make a classical field. Fermions are tireless individualists that hold up atoms and solids but seem incapable of "unionizing" into a coherent beam.

On the other hand, photons and all objects called *Bosons* which have integral quantum spins, are real "party animals" and "copy-cat frat-rats" of the particle world and ever influenced by peer pressure. Their behavior is attributed to *Bose-Einstein symmetry* (yet another axiom) which is named after Albert Einstein and Nahari Bose. One consequence is that bosons love to swarm together into a single state and copy each other's behavior. This results in observable classical fields made of enormous numbers of bosons such those in a laser beam or atomic Bose condensate beams or liquid ^{4}He.

One remarkable exception is the super conductive states of electrons in various substances such as certain metals, rare-earth cuprates, and alkaline-doped solid C_{60} (buckminsterfullerite). It is thought that the electrons double up into pairs and become composite bosons which can then participate in a coherent quantum current.

To summarize: many quantum amplitudes like $\langle\uparrow|\uparrow'\rangle$ or $\langle x|\Psi\rangle$ are *probability amplitudes* only with no simple classical wave amplitude interpretation. Some see them as ghostly waves of "potential existence" or call them waves of "nothing in nothing." Only their square (probability) is real.

What are Photon Counters? Schrodinger's Cat (and Mouse)

A photon counter is like an explosive device set up to "go off" when it is disturbed by just the right stimulus. A classical analogy might be a mousetrap rigged to light a match which sets off a firecracker in a nitro factory. From miles away you can hear a big "BOOOM!!" and be pretty sure that just then a mouse nibbled its (very last) piece of cheese. Note it was the nitro factory that went “BOOOM.” Mice don’t go “BOOOM.” By analogy, photons don’t go “click” ... that’s a counting device you’re hearing.

Some photon counters use the *photoelectric effect* to select only photons that have an energy or frequency greater than a certain *threshold* value $E_{threshold} = h\,\nu_{threshold} = \hbar\,\omega_{threshold}$. As discussed later, an electron in an atom or molecule may be ejected from the atom in the presence of photons with energy above the *ionization threshold* for that system, but not if the photon energy is below that value. Ejected electrons are then free to be accelerated by a voltage set in the counter.This in turn ejects more electrons. After several stages of this kind of ejection and acceleration, an exponentially growing avalanche of electrons is recorded as a current "boom" or "click" that is counted. Nevertheless, photons don’t go “click.”

Counters have two important properties; they are amplifiers that result in *macroscopic* and, more importantly, *irreversible* effects. After the "Boom!" or the electronic avalanche, there is essentially zero probability for ever seeing an "un-Boom!" or "un-avalanche" in the lifetime of the experiment. This is quite in contrast to coherent quantum processes which allow an analyzer to sort or split up a photon and then coherently "un-sort" or reassemble the same photon at the analyzer exit as will be seen next in Fig. 1.3.1.

It is useful to describe the photon state inside the analyzer as a coherent linear combination of the form $|\Psi\rangle=\alpha|\Psi_{up}\rangle+\beta|\Psi_{dn}\rangle$. In contrast, the presence of a counter makes such a description less useful because the energy added by the amplification process will, at the very least, randomly speed up the phase of one or both of the amplitudes α or β. Moreover, it may involve coupling with huge sets of quantum states outside of the $|\Psi_{up}\rangle$ and $|\Psi_{dn}\rangle$ being studied with each outside channel siphoning its share of energy and accumulated phase and intensity. Such a complex state combination is likely to make reversibility quite impossible.

Schrodinger described a whimsical counting experiment known as *Schrodinger's Cat*. It involved a cat sleeping next to a device that poisons the poor animal only if a certain state, say $|\Psi_{up}\rangle$, is counted in the "quantum part" of the device. Schrodinger may have speculated that the state of the system could be written as a combination $\alpha|\Psi_{up}Cat_{dead}\rangle+\beta|\Psi_{dn}Cat_{alive}\rangle$, that is a cat $|\alpha|$-percent-dead and $|\beta|$-percent-alive.

However, this is nonsense since the experiment proposed is just a particularly complicated (and cruel) counting amplifier. The amplitudes α or β have no predictive value beyond the statistical probabilities $|\alpha|^2$ or else $|\beta|^2$ which might, under the most ideal conditions, just give approximate actuary for, say, a billion cat experiments. Such a Frankenstein laboratory cat experiment is unlikely to have any more coherence than a "Schrodinger's mouse" experiment mentioned above.

Nevertheless, the term *Schrodinger's Cat* is used to describe macroscopic and coherently reversible coupled-atom-cavity experiments that have been studied. There is no readily apparent theoretical limit to the size and complexity of a state in which quantum amplitudes like α and β may be coherently phased, and now there is evidence that quantum coherence in necessary to explain some biochemical pathways. Could actual cats (or mice) participate in coherent quantum experiments anytime in the foreseeable future? Meow!

1.3 Beam Analyzers: Fundamental Quantum Processes

To describe the weirdness and beauty of quantum physics we will discuss experiments involving elementary *quantum analyzers*. The quantum analyzer we will use here consists of two beam sorters of the type discussed in Section 1.2 placed back-to-back. More correctly we will pair up a beam sorter with a *beam un-sorter* or "put-back-together-er" that exactly undoes the splitting caused by its sorting companion.

Is this even possible? Yes, we know it is, provided you choose your apparatus judiciously and construct it carefully. One of the deepest axioms for much of atomic and molecular physics is the idea of perfect *time-reversal symmetry* at a sub-microscopic level. Roughly speaking, it says, "Anything you can do, I can do backwards." or "If you do, I can undo!" Let's see some examples of analyzer experiments.

(a) Optical polarization analyzers

Optical polarization analyzers are routinely constructed in a modern laser laboratory. Two Brewster prisms of the type sketched in Fig. 1.1.5 have only to be carefully mounted back-to-back as shown in Fig. 1.3.1. By "carefully" we mean that one must adjust split beam paths so that the output beam has exactly the same polarization and intensity as the input beam, that is, $|\Psi_{OUT}\rangle = |\Psi_{IN}\rangle$. Ideally, this would mean matching the x and y optical path lengths to within a fraction of the 0.5 micron wavelength of light. An accuracy of ±one ten-millionths of a meter might be enough. It's definitely not a "caveman" device!

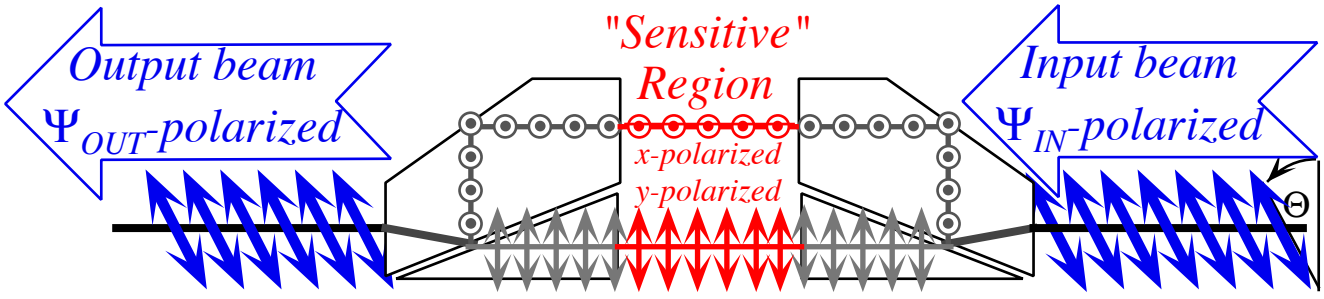


Fig. 1.3.1 Anatomy of ideal optical polarization analyzer

The purpose of these analyzers is let us to "tickle" or "perturb" each photon as it goes through a "sensitive" region where it has been temporarily sorted into to two different polarization states. Then you see what happens to its polarization as it emerges supposedly "reassembled." It is important to understand how incredibly sensitive the photon state is to what happens between the points where it is sorted on the right hand side of Fig. 1.3.1 and "un-sorted" on the left hand side. In between sorting and un-sorting, the physicists play the role of the Marquis de Sade with the poor photons in such devices. Soon the "Photon-Rights" groups may be picketing their darkened laboratories!

A theorist avoids "photon sadism" by simulating analyzer experiments on a computer. The following discussions use the *Quantit* program which draws visual representations of the amplitudes in split beams. A *QuantIt* simulation of a "do-nothing" analyzer set up like Fig. 1.3.1 is shown in the Fig. 1.3.2 . The analyzer is shown receiving a beam of $\theta=\beta/2=30°$ polarized photons from the right and sending out the same polarization toward the left. (It "does nothing" to polarization.) In between there is a "high road" beam that is *x*-polarized ($\theta=\beta/2=0°$) and a "low road" beam that is *y*-polarized ($\theta=\beta/2=90°$).

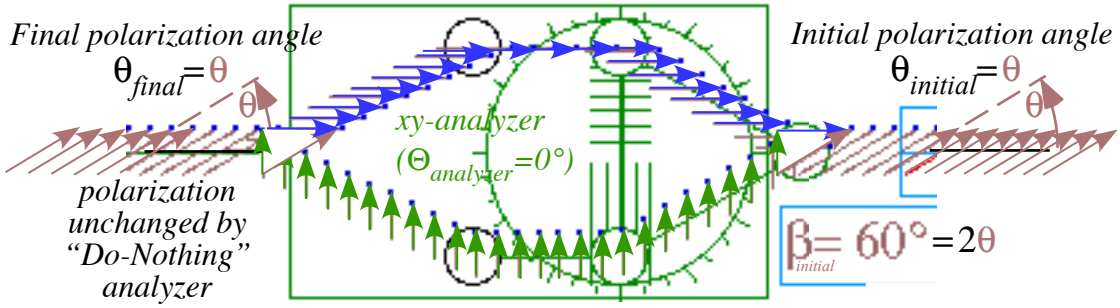
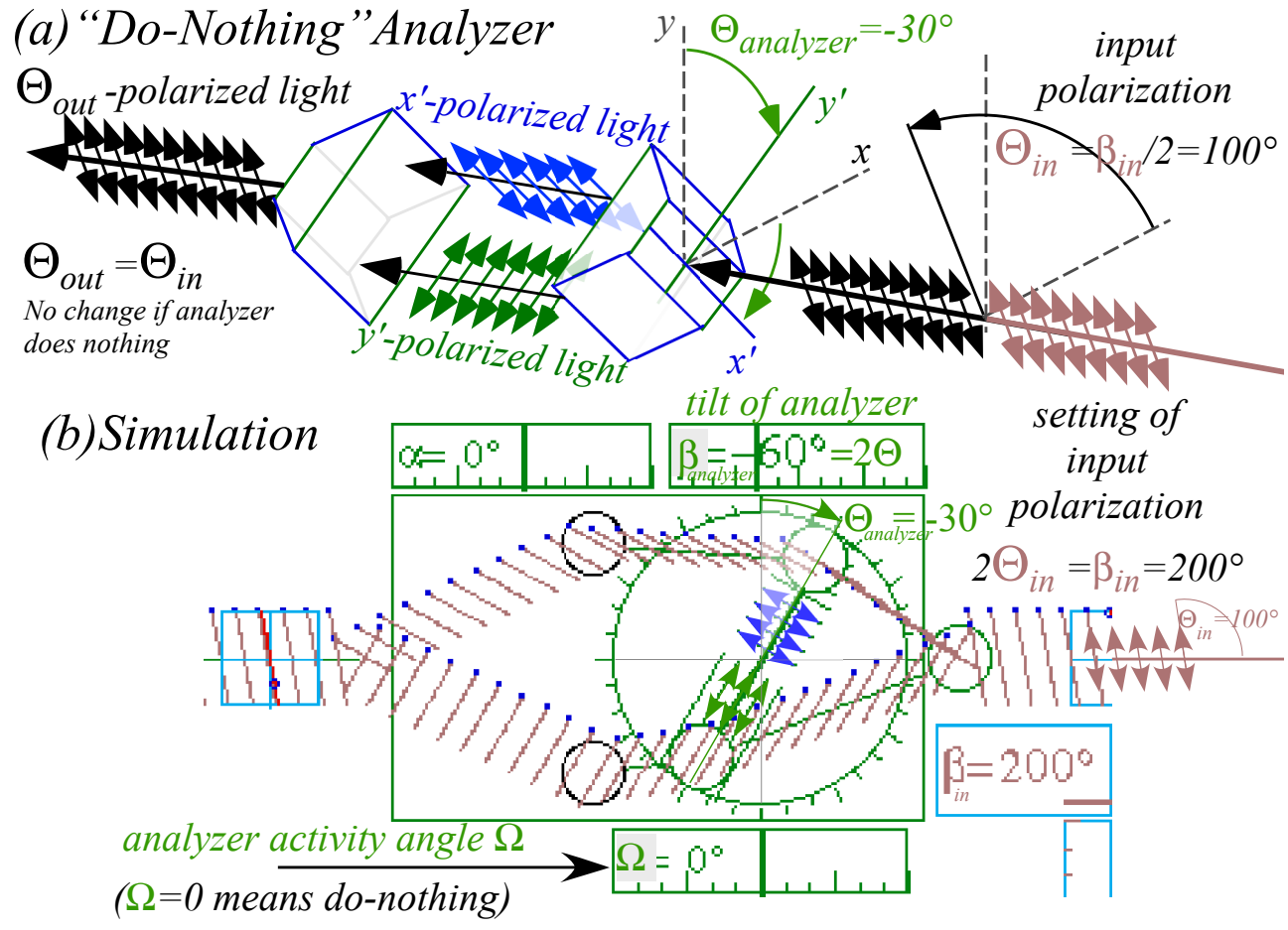


Fig. 1.3.2 Computer sketch of simulated polarization analyzer in "do-nothing" mode

Note that polarization **E**-vectors are indicated by little lines drawn in the plane of the beam paths since it is impractical in this kind of figure to draw them as they really are (transverse to beam) while showing beam paths, too. Another "do-nothing" example is shown in the side-bar below. Following that are some examples of analyzer experiments and configurations that "do something."

Simulation of "Do-Nothing" Analyzers by QuantIt

Various analyzer configurations are simulated by *QuantIt* as shown by a do-nothing ($\Omega=0°$) analyzer in the figure below. Initially polarized state enters from the right, perhaps from another analyzer that is set initially to input angle β_{in} . (Here it is set initially to $\Theta_{in}= 100°$ or $\beta_{in}= 200°$.) *QuantIt* uses electron spin-tilt-β-angles that double photon-*x*-polarization tilt Θ-angles: $\beta=2\Theta$. (Recall electron vs. photon analogy in Fig. 1.2.4.)



To calculate Θ-analyzer output imagine another analyzer, say, a basic *xy*-analyzer, parked off to the left to analyze the output beam into *x* and *y*-components. The amplitudes in its *x* and *y*-channels will be a sum over *x′* and *y′*-paths whose amplitudes (1.2.2) are $\langle x'|\Theta_{in}\rangle=\cos(\Theta_{in}-\Theta)$ and $\langle y'|\Theta_{in}\rangle=\sin(\Theta_{in}-\Theta)$ where relative angle is $\Theta_{in}-\Theta$ of Θ_{in} to Θ-analyzer axes-(*x′*,*y′*) related to lab axes by $\langle x|x'\rangle=\cos\Theta=\langle y|y'\rangle$ and $\langle y|x'\rangle=\sin\Theta=-\langle x|y'\rangle$.
Output *x-component* is: $\langle x|\Theta_{out}\rangle=\langle x|x'\rangle\langle x'|\Theta_{in}\rangle+\langle x|y'\rangle\langle y'|\Theta_{in}\rangle=cos\Theta cos(\Theta_{in}-\Theta)-sin\Theta sin(\Theta_{in}-\Theta)=cos\Theta_{in}$
Output *y-component* is: $\langle y|\Theta_{out}\rangle=\langle y|x'\rangle\langle x'|\Theta_{in}\rangle+\langle y|y'\rangle\langle y'|\Theta_{in}\rangle=sin\Theta cos(\Theta_{in}-\Theta)-cos\Theta sin(\Theta_{in}-\Theta)=sin\Theta_{in}$.
(Recall $cos(a+b)=cos\,a\,cos\,b-sina\,sin\,b$ and $sin(a+b)=sin\,a\,cos\,b+cos\,a\,sin\,b$)
If the Θ-analyzer vanished the *xy*-analyzer sees output $\langle x|\Theta_{in}\rangle=cos\Theta_{in}$ and $\langle y|\Theta_{in}\rangle=sin\Theta_{in}$ straightaway.
So $\Theta_{out}=\Theta_{in}$ is unchanged by a "Do-nothing" and having it is the same as having no analyzer.

(1) Optical analyzers in sorter-counter configuration

An analyzer can easily be reduced to a simple sorter-counter of the type discussed in Section 1.2. You just block the ends of the *x*-high road and the *y*-low road with counters as shown in Fig. 1.3.3.

$x\text{-counts}\sim|\langle x|x'\rangle|^2$
$=\cos^2\theta=0.75$

$y\text{-counts}\sim|\langle y|x'\rangle|^2$
$=\sin^2\theta=0.25$

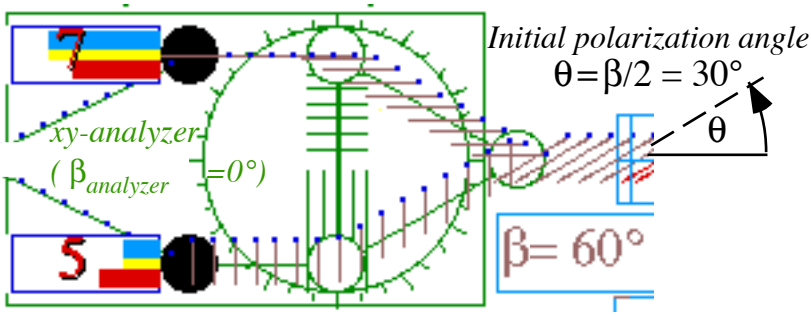


Fig. 1.3.3 Computer sketch of simulated polarization analyzer set up as a sorter-counter

This is the most extreme photon "perturbation" that analyzers can do; it kills any and all photons that venture into this experiment. Not nice! But, one gets an accurate "body-count" to compare with the quantum predictions from (1.2.12) which are shown on the left hand side of Fig. 1.3.3.

The predictions for (θ=β/2=*30°*) are 75% probability for *x*-polarized photons and 25% probability for *y*-polarized photons. This particular experiment involving 12 photons came out with 7 counts of *x*-photons and 5 counts of *y*-photons. Was that pretty good? Sorry! *It doesn't mean a damn thing!* It could have just as well come out with twelve *x*-counts and zero *y*-counts or even *vice-versa*. Or worse, it could have come out 9 to 3 exactly as predicted, and some people who were really stupid would brag that they were doing fabulously great physics. (Then you could ask them what they predicted for 13 photons!)

Quantum predictions of this sort only begin to become meaningful for large numbers of counts. Statistics with small samples is generally not very useful for confirming or disproving a particular theory. Fortunately, there is an endless supply of photons just dying to be sacrificed for your research project.

(2) Optical analyzers in a filter configuration (Polaroid© sunglasses)

Another setting for an analyzer closes one of the paths and leaves the other open. The closed path may have a photon counter as shown below in Fig. 1.3.4. That option does not affect the experiment.

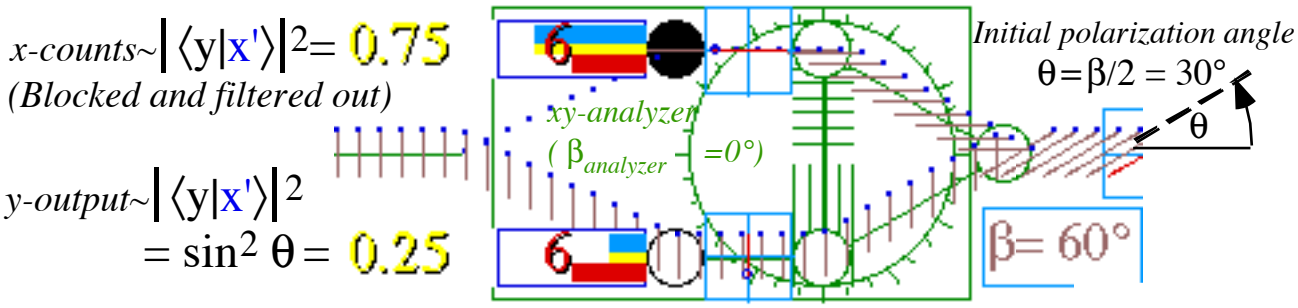


Fig. 1.3.4 Simulated polarization analyzer set up to filter out the x-polarized photons

This experiment simply kills (and counts) all the photons that choose the *x*-polarized path but lets the photons that choose the *y*-polarized path go on through. (The latter are counted by default since there are only two choices here.) Our theory says that only 25% should take the *y*-path on the average just as it did when both paths were blocked in Fig. 1.3.3. This makes sense according to classical arguments since the *y*-component of the initial *30°* polarization E-vector is half of |E| (sin *30°* = 1/2) and that corresponds to the one-quarter intensity ($|1/2|^2 = 1/4$) getting through this *y*-pass filter.

However, any given quantum experiment will probably deviate from the prediction. After twelve "throws" this particular experiment paid off 6 *y*-photons. That is 50% which is twice the house odds. (Wanna' bet even odds on the next twelve, you old riverboat gambler, you!?)

A lot of money has been "won" by an "experiment" like this, but not by betting on single photon events. Edwin Land sold many Polaroid© sunglasses which pass mostly *y*-polarized light and block the *x*-polarized light that accounts for most of the glare reflected from roads by the Brewster effect. You should take heed. Any experiment, like this or ones we will study later, might become a multi-million $ invention if you use your imagination creatively.

(3) Optical analyzers in the "control" configuration: Half or Quarter wave plates

Now let's see an example of an analyzer configuration that does what analyzers are really intended to do. Recall that a "do-nothing" analyzer was ever so carefully adjusted so the two paths were the same to within a tiny fraction of an optical wave oscillation. This was necessary to assure that the polarization of the input beam is reproduced as perfectly as possible in the output beam.

A "do-something" analyzer requires the same care and precision in adjusting the two paths, but the two paths are given a non-zero difference in optical path length or *relative phase* Ω. If this phase difference can be accurately set and controlled over a range of 2π ($-\pi<\Omega<\pi$) then the analyzer is capable of completely controlling the output polarization. If the *x*-path gains a half-wave or phase of $\Omega=\pi$ relative to the *y*-path, then input *30°* polarization becomes tilted by an angle of θ=150° as shown in Fig. 1.3.5a. Such a device is called a

half-wave plate. If the *x*-path gains a quarter-wave or phase of $\Omega=\pi/2$ relative to the *y*-path, then input *30°* polarization becomes *elliptical* polarization as shown in Fig. 1.3.5b. This device is called a *quarter-wave plate*.

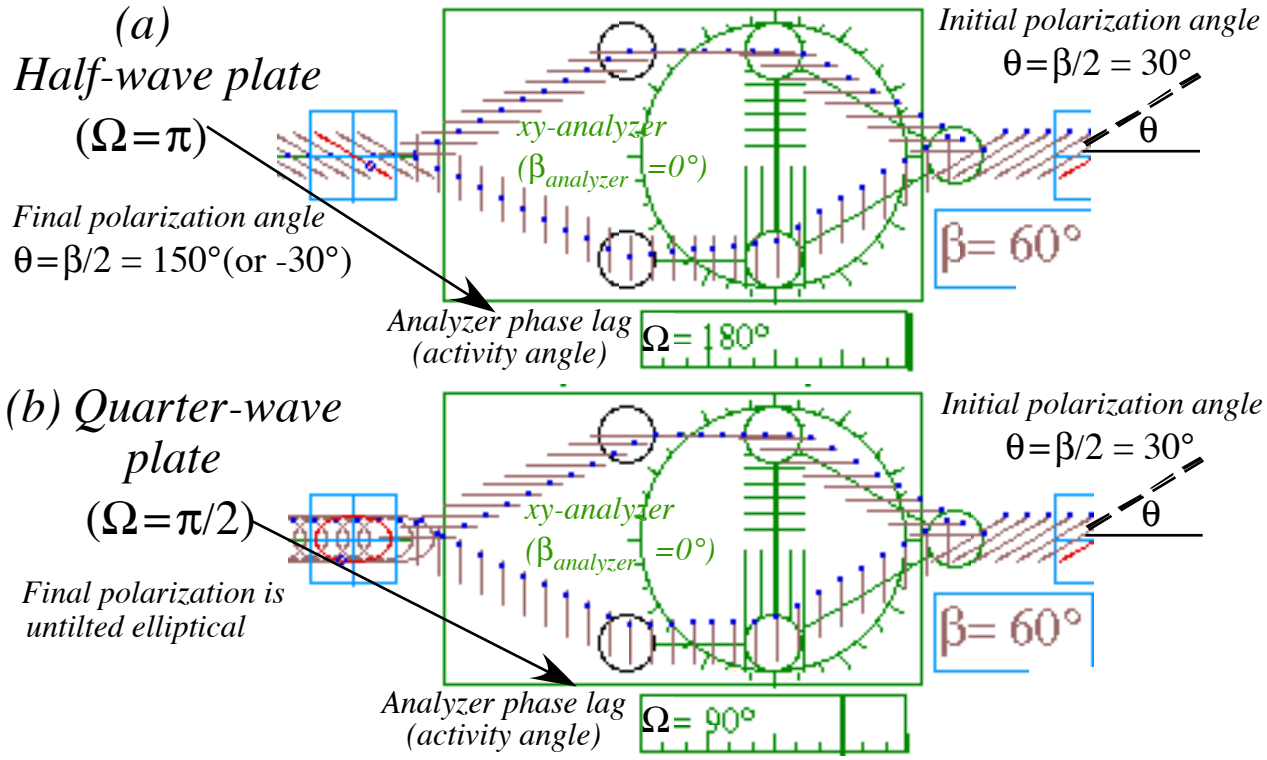


Fig. 1.3.5 Polarization control set to shift phase by (a) Half-wave ($\Omega=\pi$) , (b) Quarter wave ($\Omega=\pi/2$)

To understand what happened in Fig. 1.3.5 consider the input $\theta=30°$ polarization state vector given by the $|x'\rangle$ ket in (1.2.2) or (1.2.4a). Then we shift its *x*-phase by an angle $\Omega=\pi$ or $\pi/2$, respectively, relative to the *y*-phase, by multiplying Ψ_x by phase factor $e^{-i\Omega}$.

$$\begin{pmatrix} e^{-i\Omega} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}=\begin{pmatrix} e^{-i\pi} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} -\sqrt{3}/2 \\ 1/2 \end{pmatrix} \text{ for } \Omega=\pi, \quad (1.3.1a)$$

$$\begin{pmatrix} e^{-i\Omega} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}=\begin{pmatrix} e^{-i\pi/2} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} -i\sqrt{3}/2 \\ 1/2 \end{pmatrix} \text{ for } \Omega=\pi/2. \quad (1.3.1b)$$

Phase shifting is done using 2x2 complex *matrix operators* such as *T* or the *phase-balanced form R* below.

$$T=\begin{pmatrix} e^{-i\Omega} & 0 \\ 0 & 1 \end{pmatrix}, \text{ or: } \quad R=\begin{pmatrix} e^{-i\Omega/2} & 0 \\ 0 & e^{+i\Omega/2} \end{pmatrix}=e^{+i\Omega/2}T. \quad (1.3.1c)$$

Whenever possible, one tries to represent analyzers by complex operators **T** which act on input state kets $|\Psi_{IN}\rangle$ to give the resulting output state kets $|\Psi_{OUT}\rangle$. Phase-balanced forms are designed to preserve overall phase.

$$|\Psi_{OUT}\rangle=\mathbf{T}|\Psi_{IN}\rangle \quad (1.3.2)$$

A classical picture of the resulting time behavior is shown in Fig. 1.3.6. This is obtained by plotting the real parts of the vectors in (1.3.1a-b) after they have been multiplied by the Planck time-frequency factor $e^{-i\omega t}$ from (1.2.7) as follows. The simulations in Fig. 1.3.5 apply the phase-balanced forms *R* in each case.

$$\begin{pmatrix}\Psi_x(t)\\ \Psi_y(t)\end{pmatrix}=\begin{pmatrix}\Psi_x(0)\\ \Psi_y(0)\end{pmatrix}e^{-i\omega t}=\begin{pmatrix}-\sqrt{3}/2\\ 1/2\end{pmatrix}e^{-i\omega t},\qquad \begin{pmatrix}\Psi_x(t)\\ \Psi_y(t)\end{pmatrix}=\begin{pmatrix}\Psi_x(0)\\ \Psi_y(0)\end{pmatrix}e^{-i\omega t}=\begin{pmatrix}-i\sqrt{3}/2\\ 1/2\end{pmatrix}e^{-i\omega t}$$

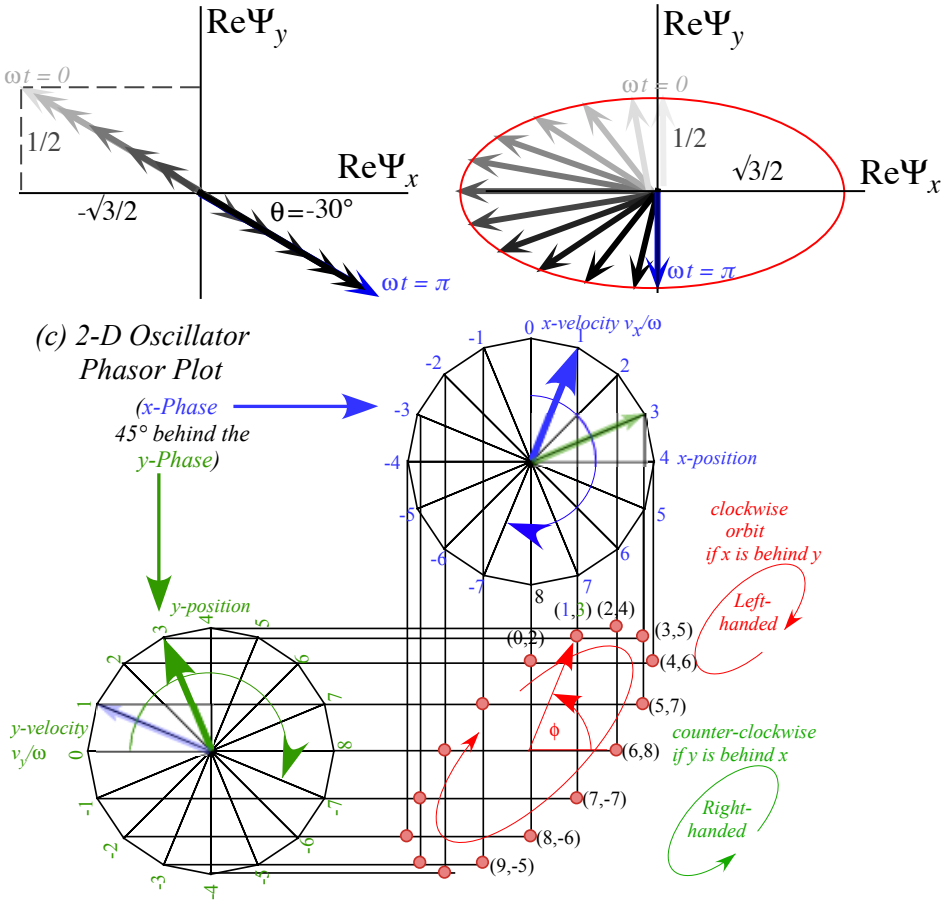
$$\begin{pmatrix}\mathrm{Re}\,\Psi_x(t)\\ \mathrm{Re}\,\Psi_y(t)\end{pmatrix}=\begin{pmatrix}-(\sqrt{3}/2)\cos\omega t\\ (1/2)\cos\omega t\end{pmatrix}\ \text{for}\ \Omega=\pi,\qquad \begin{pmatrix}\mathrm{Re}\,\Psi_x(t)\\ \mathrm{Re}\,\Psi_y(t)\end{pmatrix}=\begin{pmatrix}(-\sqrt{3}/2)\sin\omega t\\ (1/2)\cos\omega t\end{pmatrix}\ \text{for}\ \Omega=\pi/2.$$

(1.3.3a) (1.3.3b)

The phase factor of (-1) in the *x*-component of eq. (1.3.3a) simply reflects the polarization oscillation plane through the *x*-axis as shown in Fig. 1.3.6a and on the left of Fig. 1.3.5a. The phase factor of (*i*) in the *x*-component of eq. (1.3.3b) causes the real polarization vector to trace an elliptical path in the *xy*-plane as shown in Fig. 1.3.6b and on the left (output) of Fig. 1.3.5b.

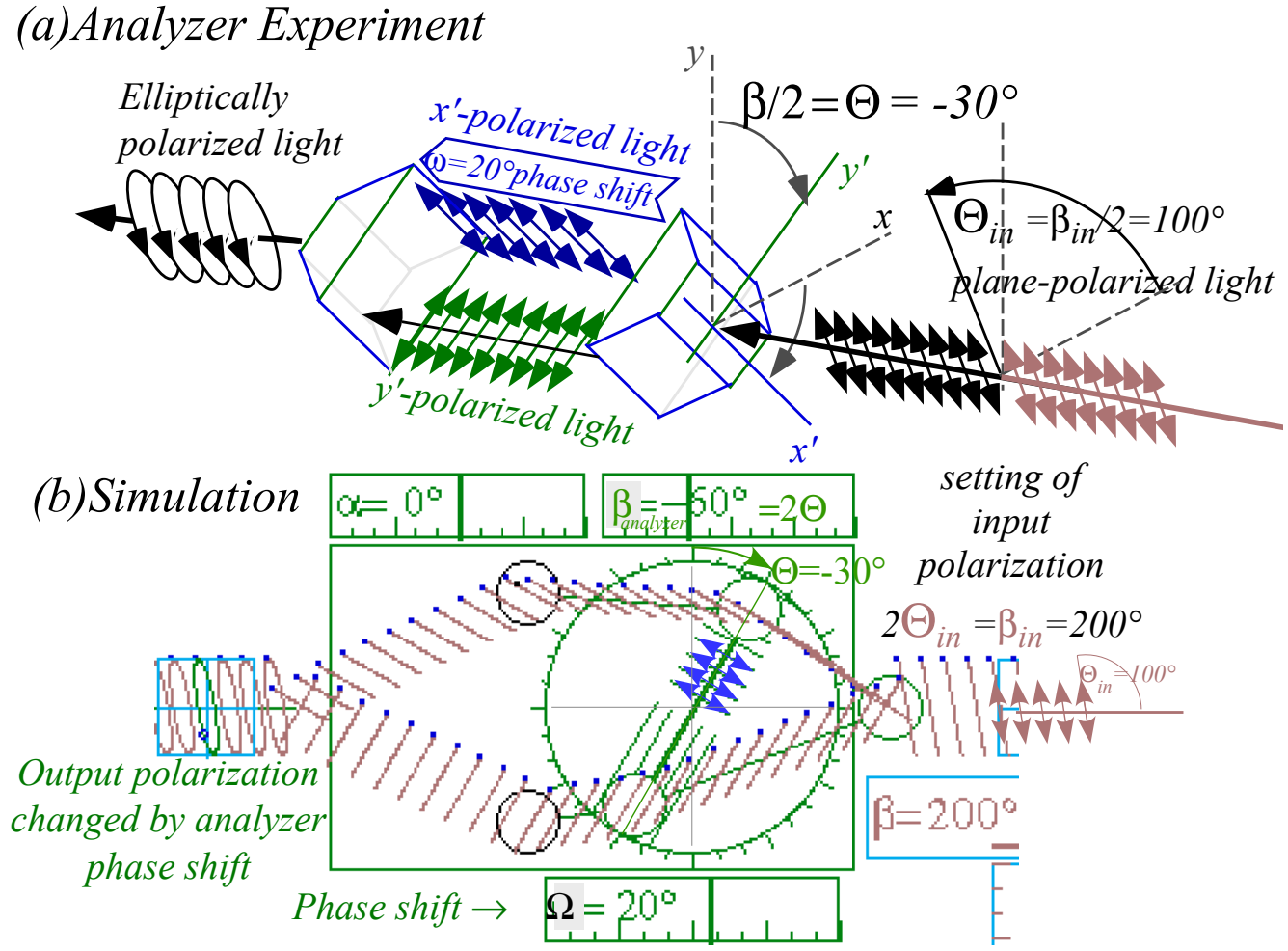
Note the polar form of *i* is $e^{i\pi/2}$ with polar angle $\pi/2$ or $90°$. Therefore, we say that the oscillation with the *i*-factor or $\pi/2$ phase is $90°$ counter-clockwise to an oscillation that has no extra phase factor. An *i*-factor *x*-oscillation is $90°$ behind *y* but (*-i*) makes *x* go $90°$ ahead (giving anti-clockwise rotation as in Fig. 1.3.6b) since the $e^{-i\omega t}$ phase of Planck advances clockwise with time. The factor $e^{-i\Omega}$ with $\Omega=-45°$ gives a tilted clockwise rotating elliptical polarization as in Fig. 1.3.6(c).

Fig. 1.3.6 Polarization states for (a) Half-wave (Ω=π) , (b) Quarter wave (Ω=π/2) (c) (Ω=–π/4)



Simulation of Active or "Do-Something" Analyzers

Suppose an active analyzer, that is a "do-something" analyzer, shifts its high beam by a phase angle (Ω=*20°*) as indicated above the analyzer in the figure below. As a result the initial plane-*100°*-polarization coming in gets transformed to an elliptical polarization going out. How do we calculate the Ψ_{out} state now? The key is to put in the Ω-phase shift in the right place.



The output calculation is almost the same as it is for the "do-nothing" analyzer except now an extra phase factor $e^{-i\Omega} = 0.94\text{-}i\ 0.34$ is tacked onto factors for the *x'*-path as was discussed after (1.3.1).

$$x\text{-output: } \langle x|\Psi_{out}\rangle = \langle x|x'\rangle e^{-i\Omega}\langle x'|\Psi_{in}\rangle + \langle x|y'\rangle\langle y'|\Psi_{in}\rangle = e^{-i\Omega}\cos\Theta\cos(\Theta_{in}-\Theta) - \sin\Theta\sin(\Theta_{in}-\Theta)$$

$$y\text{-output: } \langle y|\Psi_{out}\rangle = \langle y|x'\rangle e^{-i\Omega}\langle x'|\Psi_{in}\rangle + \langle y|y'\rangle\langle y'|\Psi_{in}\rangle = e^{-i\Omega}\sin\Theta\cos(\Theta_{in}-\Theta) + \cos\Theta\sin(\Theta_{in}-\Theta)$$

The numerical results for these amplitudes are as follows. Both of these need to be given in polar form.

$$x\text{-output: } \langle x|\Psi_{out}\rangle = (0.94\text{-i}0.34)0.87(-0.64) - (-0.5)(0.77) \quad = -0.140 + i0.189 = 0.235e^{i2.2}$$

$$y\text{-output: } \langle y|\Psi_{out}\rangle = (0.94\text{-i}0.34)(-0.5)(-0.64) + (0.87)(0.77) = 0.966 - i0.109 = 0.972e^{-i0.11}$$

The *x* and *y*-probabilities are $(0.235)^2$=*0.055 (5.5%)* and$(0.972)^2$=*0.945 (94.5%)*, respectively. Multiplying the amplitudes by the Planck time phasor $e^{-iEt/\hbar}$ gives their time dependence. (See (1.3.3)) The *x*-amplitude is running *2.2-(-0.1)=2.3* radians or *131°* behind the *y*-amplitude. This makes a slightly tilted ellipse. The detailed geometry of polarization ellipsometry is discussed in Chapter 10.

(4) Optical analyzers in a "peeking" configuration

The following is a preliminary discussion about one of the most peculiar aspects of quantum theory. It falls under a large and fairly nebulous topic of *quantum measurement* or *quantum observation*. The main idea, which we will discuss later on, is that the information about a given system comes in quantum chunks just like its energy and momenta ; indeed, information is energy and momenta since information requires frequency bandwidth. So information about a quantum system comes at a price, and the price is that the energy and state of the system is affected at the quantum level.

Suppose a new analyzer is constructed with both ports open just like the original "do-nothing" analyzer first shown in Fig. 1.3.1 and 1.3.2. Only the new analyzer has a "spy" or an "*x*-eye" which "peeks" and looks out for a passing *x*-photon as shown in Fig. 1.3.5. It may also have another "*y*-eye" which "peeks" at the lower *y*-path to check if a *y*-photon went by. However, for a two-state system, one good eye is assumed to be enough to tell which way the photon went.

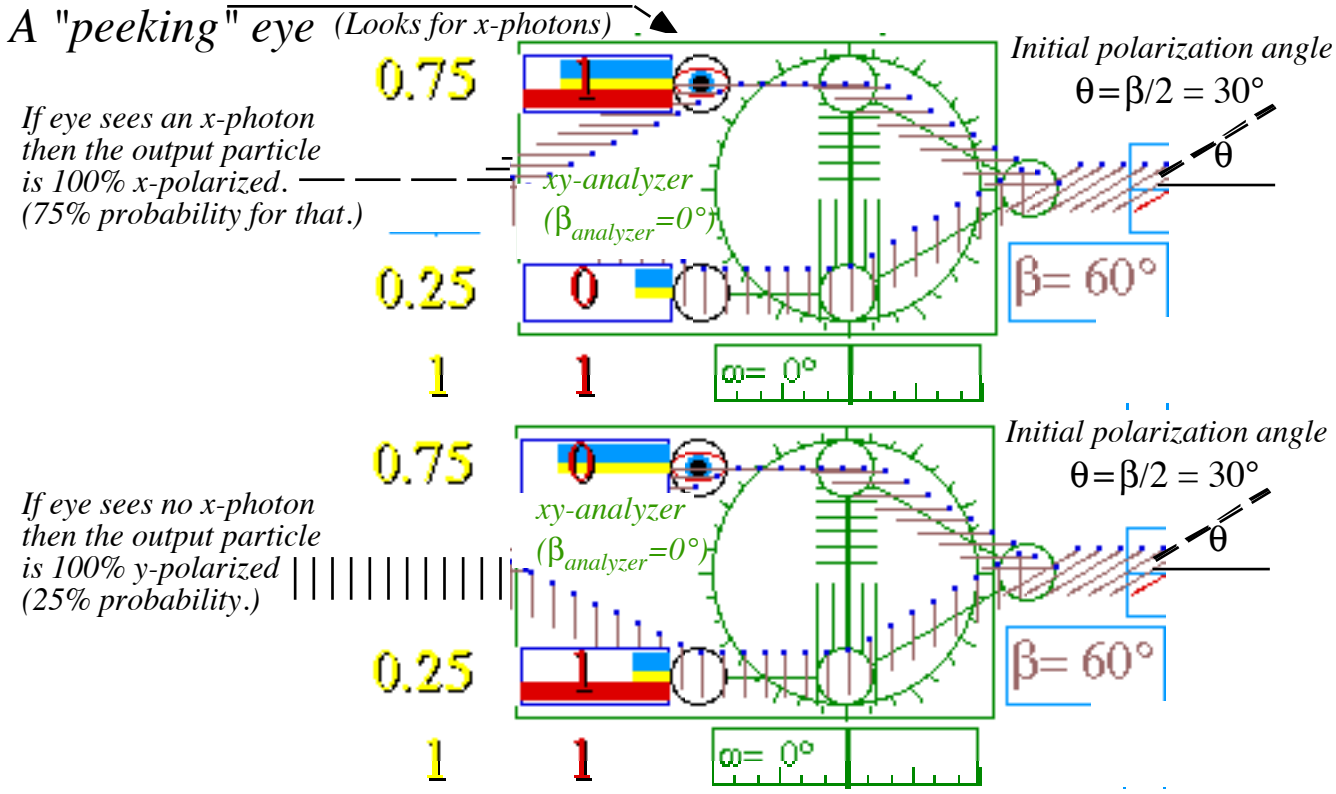


Fig. 1.3.7 Simulated polarization analyzer set up to "peek" if the photon is x-or y-polarized

It should be emphasized that the "peeking" is as delicate as possible, a sort of "ideal peeking." It does not change the *x*-polarization of any of the *x*-photons it "sees." Nor does it change the *y*-polarization of any of the *y*-photons. Neither does it alter the predictions that our *30°* polarized input beam would, if so analyzed, yield 75% of the photons to be *x*-polarized and the remaining 25% to be *y*-polarized.

However, it cannot avoid altering the state of the incoming beam. If a *x*-photon is "seen" then that is exactly what comes out; the output photon is 100% *x*-polarized. If no *x*-photon is seen then the output is a 100% *y*-polarized photon. This analyzer can only output *x*-polarized or *y*-polarized photons, and nothing in

between. Furthermore, after counting many photons the *x*-polarized photons will, on the average, be 75% of the population and the *y*-photons will account for the remaining 25%.

Such "perfect-peeking" is like a sorter counter in Fig. 1.3.3 which, after each counter records a count, recreates a (randomly phased) photon with the same polarization as the one it absorbed.

This output is very different from that of the "do-nothing" analyzer whose output was entirely made up of *30°* polarized photons, that is each photon had 75% intensity in the *x*-direction ($\Psi_x = \sqrt{3}/2, |\Psi_x|^2 = 0.75$) and 25% intensity in the *y*-direction ($\Psi_y = 1/2, |\Psi_y|^2 = 0.25$). In contrast, the "peeked" photons are either all x and no y ($|\Psi_x|^2 = 0.75$, and $|\Psi_y|^2 = 0.00$) as are 75% of the photons, or else all y and no x ($|\Psi_x|^2 = 0.00$, and $|\Psi_y|^2 = 0.25$) as are the remaining 25%. It's as though peeking produces two separate beams within the final beam neither of which have 100% probability individually, but together their probabilities add to 1. (This is discussed in greater detail after Fig. 1.3.9b just ahead.)

One of the most bizarre aspects of quantum theory is that by looking to see if a particle chooses the *x*-path or the *y*-path we seem to cause it become exactly one of the possible results that we are looking for; in this case, either x or else y but not a combination of both. This is sometimes called *wave amplitude collapse* since the amplitude for the "other" path (or paths for more than two states) seems to instantly be shut off. However, this little bit of jargon is highly suspicious and perhaps it should be deleted from our vocabulary.

It may seem paradoxical that by turning on the "*x*-eye" and seeing an *x*-photon on the *x*-path instantly shuts off any chance that the "*y*-eye" will see a photon down on the *y*-path at that moment or until another photon comes along. Could the *x*-path observer use amplitude collapse to communicate instantaneously (or faster than light) with the distant *y*-path observer?

No! Remember the *x*-photon that the "*x*-eye" recorded was not a sure thing until it actually happened, rather it had a probability of only 75%. The *y*-path observer is not going to notice any "shut-off" if that particular instant doesn't give him a *y*-photon count. The poor *y*-observer is only getting 25% of the photons, anyway. Furthermore, there is a 25% probability that after the *x*-observer turns on his "*x*-eye" the first photon goes instead to the *y*-observer. What kind of message is that? It is a garbled one with exactly zero information. Amplitude collapse is an example of what has been called "spooky action at a distance." So far "spooky action" has not been made to work for us in spite of the fact that modern quantum theory has many examples of this sort of thing for over a century.

(b) Effects of "peeking" : Coherent versus incoherent beams

Understanding the effects of "peeking" is one of the most difficult parts of fundamental quantum mechanics. To help in this we now compare the output of an *xy*-"*Do nothing*" analyzer like the one in Fig. 1.3.2 with that of an *xy*-"*Peeker*" analyzer like the one in Fig. 1.3.7. A second "counting" analyzer tilted by an angle of $\beta/2 = 30°$ is set to count the output of the "Do-nothing" in the upper left half of Fig. 1.3.8 and the same $\beta/2 = 30°$ counter counts output of the *xy*-"Peeker" in the lower left half. For each case the initial beam (Extreme right of Fig. 1.3.8) is the same $\beta/2 = 30°$ input that has been used in all the previous cases, but the resulting counts on the extreme left are very different. It is important to see why.

The $\beta/2 = 30°$ tilted counter will sort any beam into $\beta/2 = 30°$ tilted *xy*-polarized beams with a $|x'\rangle$ -beam going to the upper counter and a $|y'\rangle$ -beam going to the lower counter. The initial input beam on the extreme right of Fig. 1.3.8 is a pure $|x'\rangle$-beam, and the "Do-nothing" analyzer, true to its name, does nothing to change $|x'\rangle$-polarization. So all photons go straight to the upper path and into the $|x'\rangle$ counter 100% of the time. The $|y'\rangle$-counter gets exactly zero counts.

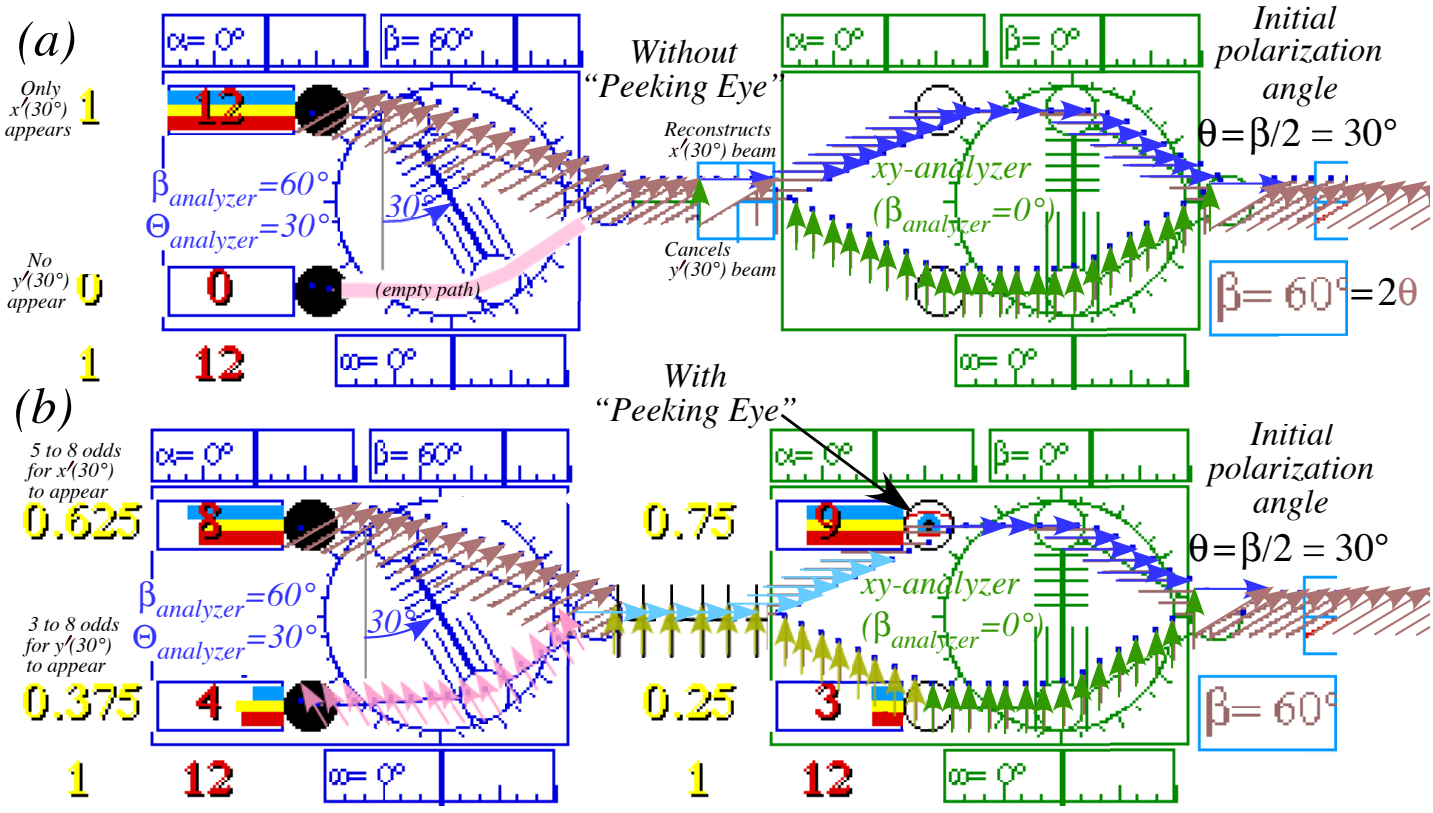


Fig. 1.3.8 Output with β/2=30° input to: (a) Coherent xy-"Do nothing" or (b) Incoherent xy-"Peeking" devices

However, the 'Peeking" analyzer in the lower right half of Fig. 1.3.8 will put out two kinds of polarized beams; an *x*-polarized beam with 75% of the intensity and a *y*-polarized beam with 25% of the intensity, as we discussed earlier. (Sec. 1.3 a(5)). These each get sorted and counted separately.

A sketch in Fig. 1.3.9 below of the polarization vector components or amplitudes can explain this. The upper right hand portion shows the initial $|x'\rangle$-beam polarization being resolved into its *x*- and *y*-components

so they split into the $|x\rangle$-beam and the $|y\rangle$-beam which are then recombined to make a final $|x'\rangle$-beam output on the upper left hand side. The orthogonal $|y'\rangle$ component has zero amplitude.

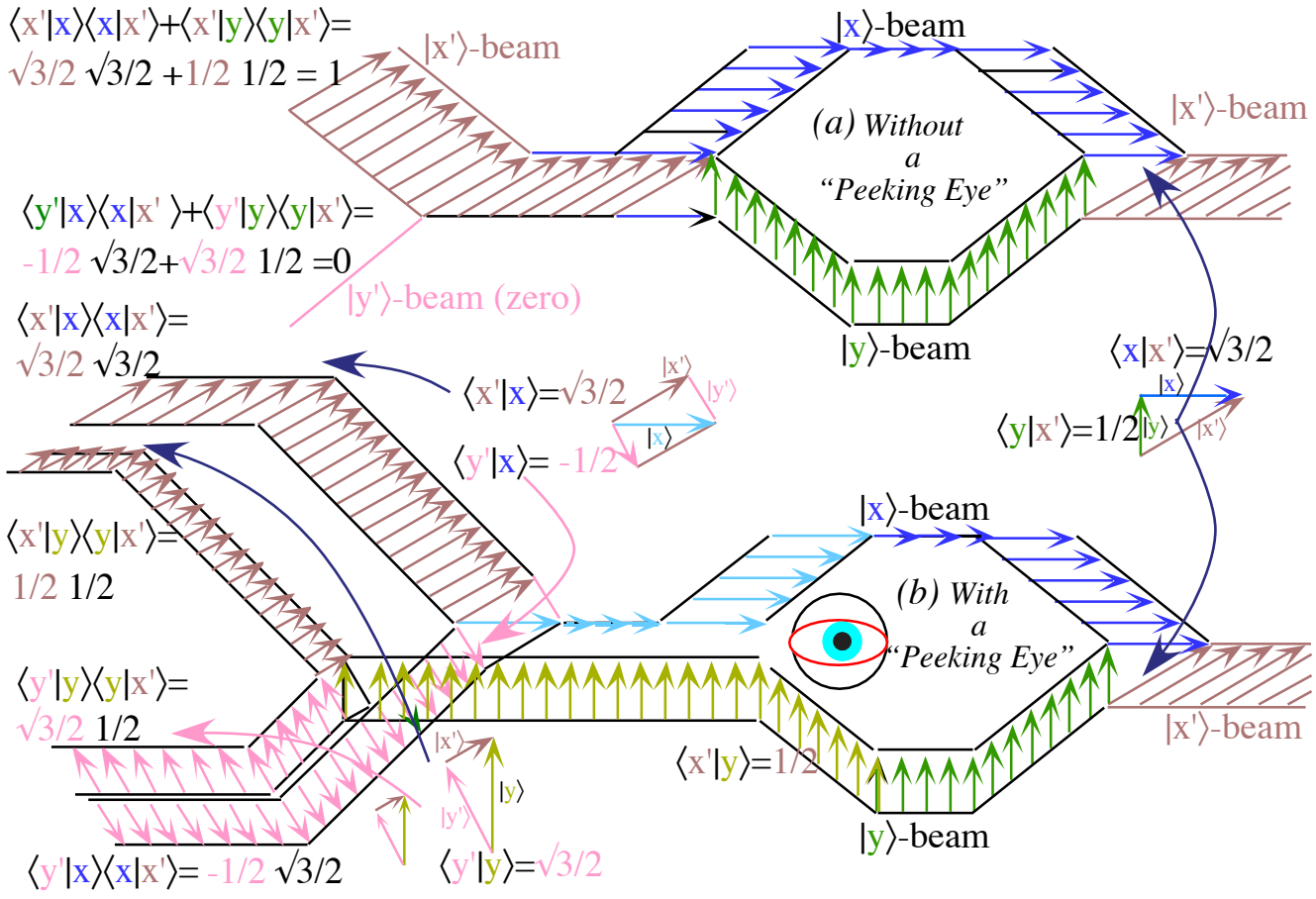


Fig. 1.3.9 Beams-amplitudes of (a) xy-"Do nothing" and (b) xy-"Peeking" analyzer each with $|x'\rangle$ input

A more detailed view of components and amplitudes is shown in the lower portion where we imagine that the "peeking eye" has caused the two beams to be distinguished. In other words, something has happened to make the $|x\rangle$-beam or $|y\rangle$-beams "dirty" so that they cannot recombine to make a pure $|x'\rangle$-beam going up to the $|x'\rangle$ counter and a zero beam going down to the $|y'\rangle$ counter. This "dirt" is simply a *random phase* which an "eye" must add to every photon it "sees." Without this "dirt" the lower two amplitudes cancel to 0 through perfect *destructive interference* while the upper beams add up to 1 through perfect *constructive interference*. ("Constructive interference" is an oxymoron in the"dirty" classical world we're used to, but it is absolutely essential for understanding wave behavior in the quantum world.)

(1) Amplitude products

We now consider the effect of successive analyzers such as the example in Fig. 1.3.8 and 9 where an *xy*-analyzer acts on an initial state $|x'\rangle$ and is followed by an x'y'-counter. This example provides the opportunity to introduce more of the fundamentals of quantum analysis and mathematics.

According to our discussion in Section 1.2 b, when a beam in state $|\Psi\rangle$ enters a sorter it gets sorted into a series of beams $\{|b_1\rangle,|b_2\rangle,...\}$ of amplitude $\langle b_1|\Psi\rangle$, $\langle b_2|\Psi\rangle$, ..., respectively. (So far only two-state systems have been shown.) If one of these beams (say, $|b_m\rangle$) encounters another analyzer or sorter it gets sorted into another series of beams $\{|c_1\rangle,|c_2\rangle,...\}$ of amplitude $\langle c_1|b_m\rangle\langle b_m|\Psi\rangle$, $\langle c_2|b_m\rangle\langle b_m|\Psi\rangle$, ..., respectively. Note: each sorter amplitude $\langle c_1|b_m\rangle$, $\langle c_2|b_m\rangle$, ..., gets multiplied by the input amplitude $\langle b_m|\Psi\rangle$. This is because polarization response is *linear* in the input amplitude, as indeed, are all analogous quantum processes. Output is proportional to input.

This process happens again each time a beam runs into another sorter and makes another set of "baby beams." For example, an amplitude of the form shown in Fig. 1.3.10

$$\langle e_p|D_oC_nB_m|\Psi\rangle=\langle e_p|d_o\rangle\langle d_o|c_n\rangle\langle c_n|b_m\rangle\langle b_m|\Psi\rangle \qquad (1.3.4)$$

is the contribution to the $|e_p\rangle$ beam of the e-sorter by a particle that went through $|d_o\rangle$ beam of a d-sorter after passing the $|c_n\rangle$ beam of the c-sorter, after passing the $|b_m\rangle$ beam of the b-sorter from the input $|\Psi\rangle$ beam. We just read this series backwards from finish to start. Recall that you read amplitudes like Hebrew, right to left, in going forward from start to finish or (correct terminology) initial to final.

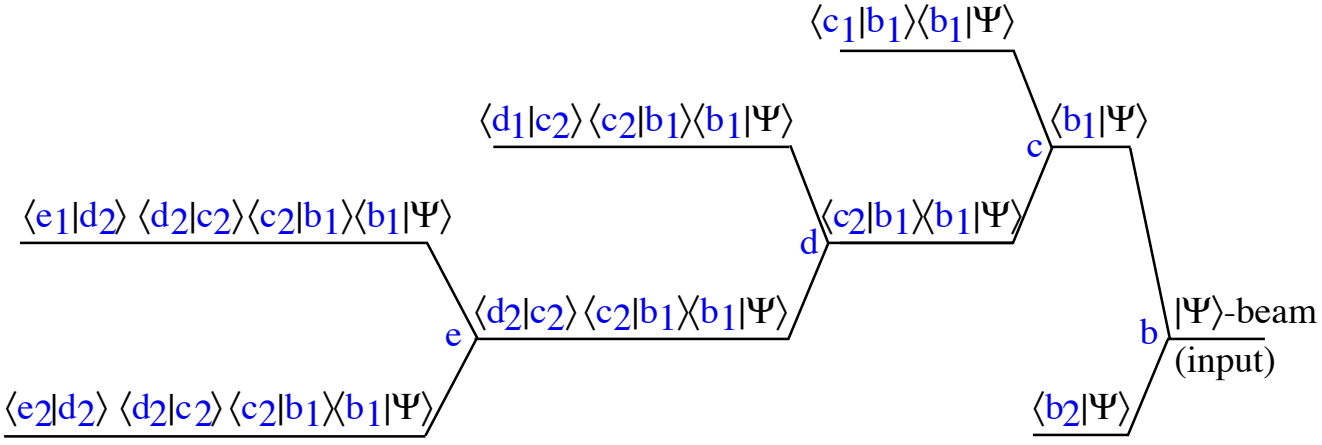


Fig. 1.3.10 Beams-amplitude products for successive beam sorting

(2) Amplitude sums

Analyzers recombine their sorted beams and the resulting analyzer output amplitude is a sum of the amplitudes of its beams. For example, the final amplitude of the x' or y'-counters in the upper left hand part of Fig. 1.3.9 is given by the following sum. (Recall that the initial state is $|\Psi\rangle=|x'\rangle$.)

$$\begin{aligned}\text{Amp. at x'-counter} &= \langle x'|x\rangle\langle x|\Psi\rangle + \langle x'|y\rangle\langle y|\Psi\rangle = \langle x'|x\rangle\langle x|x'\rangle + \langle x'|y\rangle\langle y|x'\rangle \\ \text{Amp. at y'-counter} &= \langle y'|x\rangle\langle x|\Psi\rangle + \langle y'|y\rangle\langle y|\Psi\rangle = \langle y'|x\rangle\langle x|x'\rangle + \langle y'|y\rangle\langle y|x'\rangle\end{aligned} \qquad (1.3.5)$$

Amplitudes are diagrammed as *30°-60°* right triangle segments in Fig. 1.3.9 according to Sec. 1.2a

$$\begin{pmatrix} \langle x'|x\rangle & \langle x'|y\rangle \\ \langle y'|x\rangle & \langle y'|y\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad \begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$
$$= \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix}, \text{ for } \theta=30°, \qquad = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}, \text{ for } \theta=30° \qquad (1.3.6)$$

(Notice that the two transformation matrices are inverses of each other.) Substituting in (1.3.5) gives the correct amplitudes for the "do nothing" analyzer.

$$\text{Amp. at x'-counter} = \langle x'|x\rangle\langle x|x'\rangle + \langle x'|y\rangle\langle y|x'\rangle = \frac{3}{4} + \frac{1}{4} = 1 \qquad (1.3.7a)$$

$$\text{Amp. at y'-counter} = \langle y'|x\rangle\langle x|x'\rangle + \langle y'|y\rangle\langle y|x'\rangle = -\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} = 0 \qquad (1.3.7b)$$

The two pairs of terms of (1.3.7) are sketched by two pairs of parallel beams in the lower left of Fig. 1.3.9. For an ideal "do nothing" analyzer, the first two in (1.3.7a) sum to 1 to give the same x'-beam that came as input, but the second two in (1.3.7b) cancel causing the y'-beam to vanish.

(3) Random phase effects ("Dirty" beams)

However, if any of the four beams is "dirty", that is, has a random phase then the counter output is very different. Suppose the "*x*-eye" tags each *x*-photon with a phase $e^{i\phi}$ where ϕ is a different random number for each photon so ϕ ranges over the unit circle ($-\pi<\phi<\pi$). Then (1.3.7) becomes

$$\text{Amp. at x'-counter} = \langle x'|x\rangle\left(e^{i\phi}\right)\langle x|x'\rangle + \langle x'|y\rangle\langle y|x'\rangle = \frac{3}{4}\left(e^{i\phi}\right) + \frac{1}{4} \qquad (1.3.8a)$$

$$\text{Amp. at y'-counter} = \langle y'|x\rangle\left(e^{i\phi}\right)\langle x|x'\rangle + \langle y'|y\rangle\langle y|x'\rangle = -\frac{\sqrt{3}}{4}\left(e^{i\phi}\right) + \frac{\sqrt{3}}{4} \qquad (1.3.8b)$$

Recall that statistical probabilities and count rates are determined by the absolute square of amplitudes according to (1.2.12). The count probabilities from (1.3.8) are as follows.

$$\begin{aligned}\text{x'-count probability} &= \left|\frac{3}{4}\left(e^{i\phi}\right) + \frac{1}{4}\right|^2 = \left(\frac{3}{4}\left(e^{i\phi}\right) + \frac{1}{4}\right)^*\left(\frac{3}{4}\left(e^{i\phi}\right) + \frac{1}{4}\right) \\ &= \frac{3^2}{4^2} + \frac{1^2}{4^2} + \frac{3}{4^2}\left(e^{-i\phi} + e^{i\phi}\right) = \frac{5}{8} + \frac{3}{16}\left(e^{-i\phi} + e^{i\phi}\right)\end{aligned} \qquad (1.3.9a)$$

$$\begin{aligned}\text{y'-count probability} &= \left|\frac{-\sqrt{3}}{4}\left(e^{i\phi}\right) + \frac{\sqrt{3}}{4}\right|^2 = \left(\frac{-\sqrt{3}}{4}\left(e^{i\phi}\right) + \frac{\sqrt{3}}{4}\right)^*\left(\frac{-\sqrt{3}}{4}\left(e^{i\phi}\right) + \frac{\sqrt{3}}{4}\right) \\ &= \frac{3}{4^2} + \frac{3}{4^2} - \frac{3}{4^2}\left(e^{-i\phi} + e^{i\phi}\right) = \frac{3}{8} - \frac{3}{16}\left(e^{-i\phi} + e^{i\phi}\right)\end{aligned} \qquad (1.3.9b)$$

Recall also, that probabilities are useful predictors only for large numbers of trials, that is, after many photons, so the average of $e^{i\phi}$ or $e^{-i\phi}$ approaches zero if ϕ is random. Then (1.3.8) reduces to

x'-count probability = 5/8 = 0.625, y'-count probability = 3/8 = 0.375, (1.3.9c)

which agrees with the simulation in the lower half of Fig. 1.3.8. Main idea: "Dirty" beams don't interfere.

(4) Summing amplitudes or probabilities?

This example exposes one of the important differences between quantum theory and classical physics. Let's assume that the act of "peeking" or "measurement" will, at the very least, jiggle the phase of the object of the measurement. Let's see how "Jiggling" causes count rates to exactly match what classical probability analysis would give. According to classical reasoning the probability that a x'-photon shows up in the x'-counter via an *xy*-analyzer equals a sum of probability products *P(x' to x)* times *P(x to x')* for the *x*-path and *P (x' to y)* times *P(y to x')* for the *y*-path. Read products below from right to left. (QM uses Hebrew style.)

$$\left(\begin{array}{c}\text{Probability that}\\ \text{photon in x'-input}\\ \textit{becomes}\\ \text{photon in x'-counter}\end{array}\right)_{classical} =$$

$$\left(\begin{array}{c}\text{probability that}\\ \text{photon in x-beam}\\ \textit{becomes}\\ \text{photon in x'-counter}\end{array}\right) * \left(\begin{array}{c}\text{probability that}\\ \text{photon in x'-input}\\ \textit{becomes}\\ \text{photon in x-beam}\end{array}\right) + \left(\begin{array}{c}\text{probability that}\\ \text{photon in y-beam}\\ \textit{becomes}\\ \text{photon in x'-counter}\end{array}\right) * \left(\begin{array}{c}\text{probability that}\\ \text{photon in x'-input}\\ \textit{becomes}\\ \text{photon in y-beam}\end{array}\right)$$

Now replace the probability factors by their squared-amplitude values *P(x to x')* $= \left|\langle x'|x\rangle\right|^2$, *etc.*.

$$\left(\begin{array}{c}\text{Probability that}\\ \text{photon in x'-input}\\ \textit{becomes}\\ \text{photon in x'-counter}\end{array}\right)_{classical} =$$

$$\left(\left|\langle x'|x\rangle\right|^2\right) * \left(\left|\langle x|x'\rangle\right|^2\right) + \left(\left|\langle x'|y\rangle\right|^2\right) * \left(\left|\langle y|x'\rangle\right|^2\right) = \left(\left|\frac{\sqrt{3}}{2}\right|^2\right) * \left(\left|\frac{\sqrt{3}}{2}\right|^2\right) + \left(\left|\frac{-1}{2}\right|^2\right) * \left(\left|\frac{1}{2}\right|^2\right) = \frac{5}{8}$$

This what we got in (1.3.9) by ignoring the phase terms. Indeed, the square of (1.3.8a) is as follows.

$$\left(\begin{array}{c}\text{Quantum probability}\\ \text{at x'-counter}\end{array}\right) = \left|\langle x'|x\rangle\left(e^{i\phi}\right)\langle x|x'\rangle + \langle x'|y\rangle\langle y|x'\rangle\right|^2$$

$$= \left|\langle x'|x\rangle\langle x|x'\rangle\right|^2 + \left|\langle x'|y\rangle\langle y|x'\rangle\right|^2 + e^{-i\phi}\langle x'|x\rangle^*\langle x|x'\rangle^*\langle x'|y\rangle\langle y|x'\rangle + e^{i\phi}\langle x'|x\rangle\langle x|x'\rangle\langle x'|y\rangle^*\langle y|x'\rangle^*$$

$$= (\quad \text{classical probability} \quad) + (\quad \text{Phase-sensitive or } \textit{quantum interference} \text{ terms} \quad)$$

(1.3.10)

The difference between quantum and classical probability predictions comes down to what we call phase-sensitive or *quantum interference* terms. Classical and quantum predictions differ after many counts only if phase factors $e^{i\phi}$ do not average to zero. Otherwise the quantum results reduce to the classical ones.

To summarize: Quantum probability is a square of the sum of probability amplitudes for all indistinguishable paths as in the following example.

$$\left(\begin{array}{c}\text{Quantum probability}\\ \text{at x'-counter}\end{array}\right) = \left|\langle x'|x\rangle\langle x|x'\rangle + \langle x'|y\rangle\langle y|x'\rangle\right|^2 \qquad (1.3.11a)$$

Classical probability has a smaller number of terms from this that are just a sum of the squares of the probability amplitudes for all paths, distinguishable or not, as in the following example.

$$\left(\begin{array}{c}\text{Classical probability}\\ \text{at x'-counter}\end{array}\right) = \left|\langle x'|x\rangle\langle x|x'\rangle\right|^2 + \left|\langle x'|y\rangle\langle y|x'\rangle\right|^2 \qquad (1.3.11b)$$

Here is a general rule : If you know what path a system took, (or better, if Mother Nature knows) don't bother with the full quantum square-of-a-sum. Just take the sum-of-the-squares because it is quite certain that the interference terms will average to zero.

Further comments about complex amplitudes

The polarization amplitudes used so far are idealized in a number of ways. The examples given in Sec. 1.2a are real sines and cosines that depend on geometry (tilt angles) alone. As noted in Sec 1.2b time evolution adds complex factors. Also amplitudes for anomalous polarization response (such as occurs near resonance) are complex because the output is not always in phase with the input. Many interesting quantum processes involve relative phases that are extremely sensitive to input parameters.

(c) Electron polarization analyzers

Electron or ion beam polarization analyzers are definitely high-tech experiments. High vacuums are essential and sophisticated magnetic steering fields are needed to level off and recombine the sorted split beams. A rough sketch of an ideal ion beam analyzer is shown in Fig. 1.3.11 below. It is analogous to the optical polarization analyzer shown before in Fig. 1.3.1.

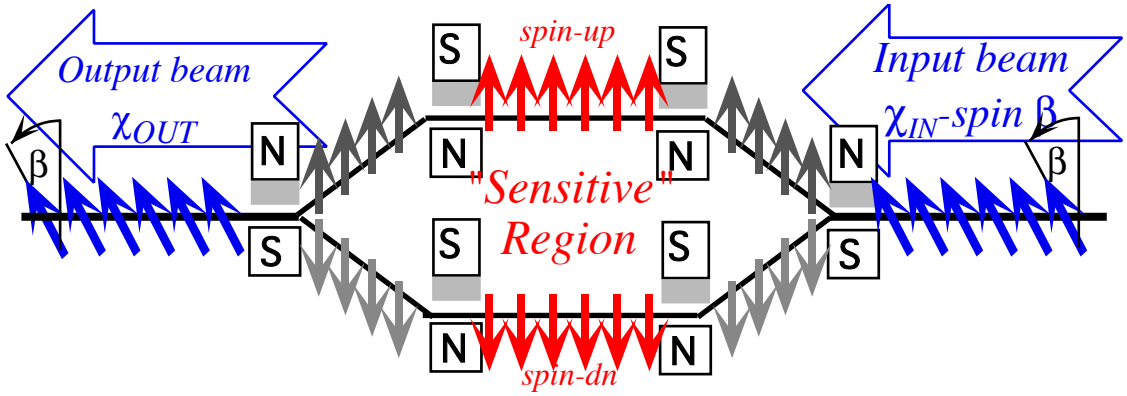


Fig. 1.3.11 Anatomy of ideal electron or ion spin polarization analyzer

In analogy with the optical case, one tests it in a "do-nothing" configuration. It should be possible to adjust it so all input electron spin states come out the same in the output beam, that is, $|\chi_{OUT}\rangle = |\chi_{IN}\rangle$ for all possible $|\chi_{IN}\rangle$. Once this is achieved, then changes can be made in the "sensitive" region where we can "do things" to the separated spin-up and spin-dn beams before they try to recombine.

Appendix 1.A. Review of Complex Algebra

Complex algebra depends on $(i^2=-1)$ and the DeMoivre identities (1) or their inverses (2).

$$e^{i\omega t} = \cos\omega t + i\sin\omega t, \quad (1.A.1a) \qquad \cos\omega t = (e^{i\omega t} + e^{-i\omega t})/2 \quad (1.A.2a)$$

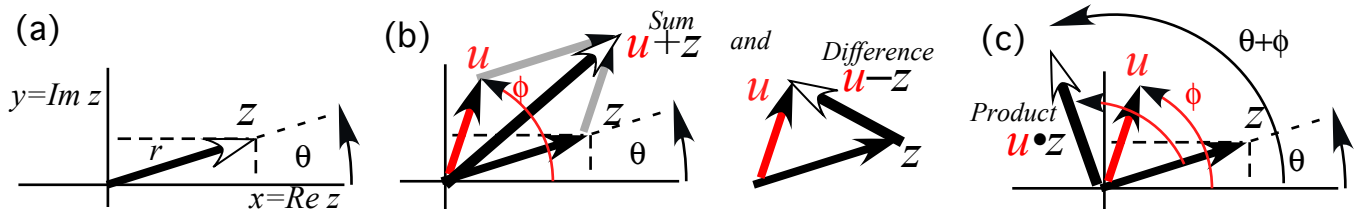
$$e^{-i\omega t} = \cos\omega t - i\sin\omega t, \quad (1.A.1b) \qquad \sin\omega t = (e^{i\omega t} - e^{-i\omega t})/2i \quad (1.A.2b)$$

This allows any complex number z to be written in *Cartesian* form $(z=x+iy)$ or else *polar* form $(z=re^{i\theta})$ where:

$$x = \cos\theta \quad (1.A.3a) \qquad r = \sqrt{(x^2+y^2)} \quad (1.A.4a)$$

$$y = \sin\theta \quad (1.A.3b) \qquad \theta = ATAN2(y,x) = \tan^{-1}(y/x) \quad (1.A.4b)$$

are polar-Cartesian coordinate transformations. Caution! Use *ATAN2*, not $\tan^{-1}$ to assure correct angle.



Linear operations (such as addition or subtraction) favor use of Cartesian forms.

$$z+z' = (x+x') + i(y+y') \quad (1.A.5a) \qquad z-z' = (x-x') + i(y-y') \quad (1.A.5b)$$

Non-linear operations (such as multiplication or division) favor the use of polar forms.

$$z\cdot z' = (re^{i\theta})\cdot(r'e^{i\theta'}) = r\cdot r' e^{i(\theta+\theta')} \quad (1.A.6a) \qquad z/z' = (re^{i\theta})/(r'e^{i\theta'}) = r/r' e^{i(\theta-\theta')} \quad (1.A.6b)$$

Multiplication of $z=re^{i\theta}$ by unitary $u=e^{i\phi}$ (for which $r_u^2=|u^*u|^2=1$) gives a rotation of z as seen here.

$$z' = u\cdot z = (e^{i\phi})\cdot(re^{i\theta}) = re^{i(\theta+\phi)} \quad (1.A.6c)$$

In Cartesian form this equation is a matrix rotation operation.

$$z' = x' + iy' = (\cos\phi + i\sin\phi)\cdot(x+iy) = re^{i(\theta+\phi)} \quad (1.A.7a)$$

$$z' = x' + iy' = (x\cos\phi - y\sin\phi) + i(x\sin\phi + y\cos\phi) = r(\cos(\theta+\phi) + i\sin(\theta+\phi)) \quad (1.A.7b)$$

Equating real parts gives x-rotation and equating imaginary parts gives y-rotation.

$$x' = x\cos\phi - y\sin\phi = r\cos\theta\cos\phi - r\sin\theta\sin\phi = r\cos(\theta+\phi) \quad (1.A.8a)$$

$$y' = x\sin\phi + y\cos\phi = r\cos\theta\sin\phi + r\sin\theta\cos\phi = r\sin(\theta+\phi) \quad (1.A.8b)$$

Angle-sum trig identities for $\cos(\theta+\phi)$ and $\sin(\theta+\phi)$ are re-derived in an "automatic" trigonometry.

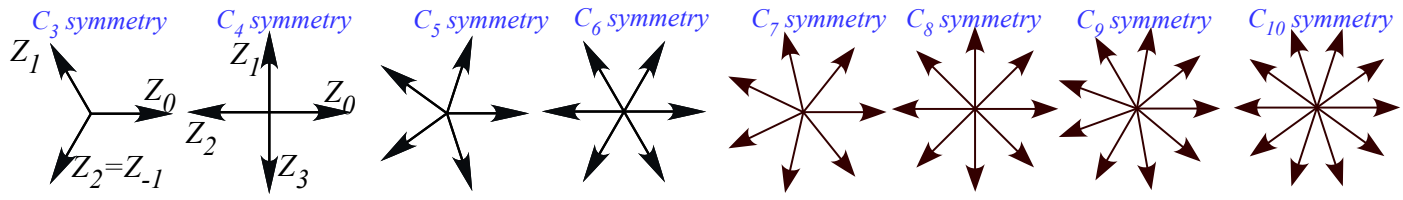
Roots of complex numbers are no problem. For example, solving $z^n = 1$ for $z = (1)^{1/n}$ is simple if we write a polar form of unity: $1 = 1\cdot e^{2\pi i}$. The zeroth z_0 root is 1, and the first root z_1 is

$$z_1 = (1)^{1/n} = (e^{2\pi i})^{1/n} = e^{2\pi i/n} \quad (1.A.9)$$

There are n roots $z_k=z_1^{\,k}$ because if $(z_1^{\,n})$ equals one, so does $(z_1^{\,n})^2=(z_1^{\,2})^n$ and $(z_1^{\,3})^n$... and so on.

$$z_1 = e^{2\pi i/n}, z_2 = (z_1)^2 = e^{2(2\pi i/n)}, z_3 = (z_1)^3 = e^{3(2\pi i/n)}, \ldots, z_n = (z_1)^n = 1 = z_0 \quad (1.A.10)$$

If these are plotted they form an n-sided regular polygon. Such diagrams help to do z_k arithmetic.



Quantum theory is intimately connected to symmetry, and this arithmetic is the foundation of symmetry analysis and Fourier theory.

Appendix 1.B. Complex Response of Oscillators

To learn quantum theory it helps to know oscillators. A classical stimulated harmonic oscillator equation

$$\frac{d^2z}{dt^2}+2\Gamma\frac{dz}{dt}+\omega_0^2 z=a \qquad (1.B.1)$$

is solved by complex functions $e^{-i\omega t}$ describing a *monochromatic* (single frequency ω_s) stimulus

$$a(t)=a(0)e^{-i\omega_s t}\ . \qquad (1.B.2)$$

as well as a response at the same frequency if its amplitude is proportional to that of the stimulus.

$$z_{\text{response}}(t)=G_{\omega_0}(\omega_s)a(t) \qquad (1.B.3)$$

The complex proportionality factor G depends upon the stimulus frequency ω_s , the natural frequency ω_0 , and damping constant Γ , only. Because the equation is linear and time independent the G factor should not depend on the amplitude A_s of the stimulus. It may help to think of the oscillator as a 'black box' that responds linearly to input as shown below in Fig. 1.B.1.

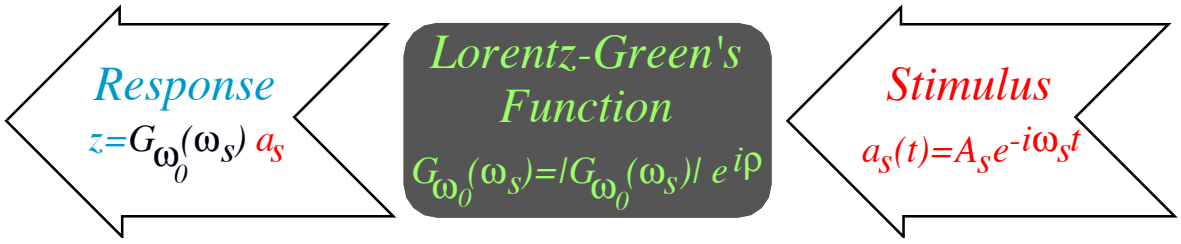


Fig.1.B.1 Black-box diagram of oscillator response to monochromatic stimulus

Now we substitute z_{response} into the classical oscillator equation of motion (1.B.1) and solve for *Lorentzian response function* or *classical Green's function* $G_{\omega_0}(\omega_s)$ of frequency ω_s .

$$G_{\omega_0}(\omega_s)=\frac{1}{\omega_0^2-\omega_s^2-i2\Gamma\omega_s}=\text{Re}\,G_{\omega_0}(\omega_s)+i\,\text{Im}\,G_{\omega_0}(\omega_s)=\left|G_{\omega_0}(\omega_s)\right|e^{i\rho} \qquad (1.B.4)$$

The real and imaginary parts of the Green's function are as follows:

$$\text{Re}\,G_{\omega_0}(\omega_s)=\frac{\omega_0^2-\omega_s^2}{\left(\omega_0^2-\omega_s^2\right)^2+\left(2\Gamma\omega_s\right)^2}\ , \qquad (1.B.5a)$$

$$\text{Im}\,G_{\omega_0}(\omega_s)=\frac{2\Gamma\omega_s}{\left(\omega_0^2-\omega_s^2\right)^2+\left(2\Gamma\omega_s\right)^2}\ , \qquad (1.B.5b)$$

while its magnitude $\left|G(\omega_s)\right|$ and polar angle ρ are the following:

$$\left|G_{\omega_0}(\omega_s)\right|=\frac{1}{\sqrt{\left(\omega_0^2-\omega_s^2\right)^2+\left(2\Gamma\omega_s\right)^2}}\ , \qquad (1.B.5c)$$

$$\rho=\tan^{-1}\left(\frac{(2\Gamma\omega_s)}{\omega_0^2-\omega_s^2}\right) \qquad (1.B.5d)$$

The angle ρ is the response *phase lag*, that is, the phase angle by which the response oscillation lags behind the phase $(-\omega_s t)$ of the stimulating oscillation.

$$z_{\text{response}}(t)=\left|G_{\omega_0}(\omega_s)\right|a(0)e^{-i(\omega_s t-\rho)} \qquad (1.B.5)$$

It may help to visualize stimulus and response phasors as a pair rigidly rotating at rate ω_s . The response phasor lags ρ radians behind the stimulus as shown below in Fig. 1.B.2.

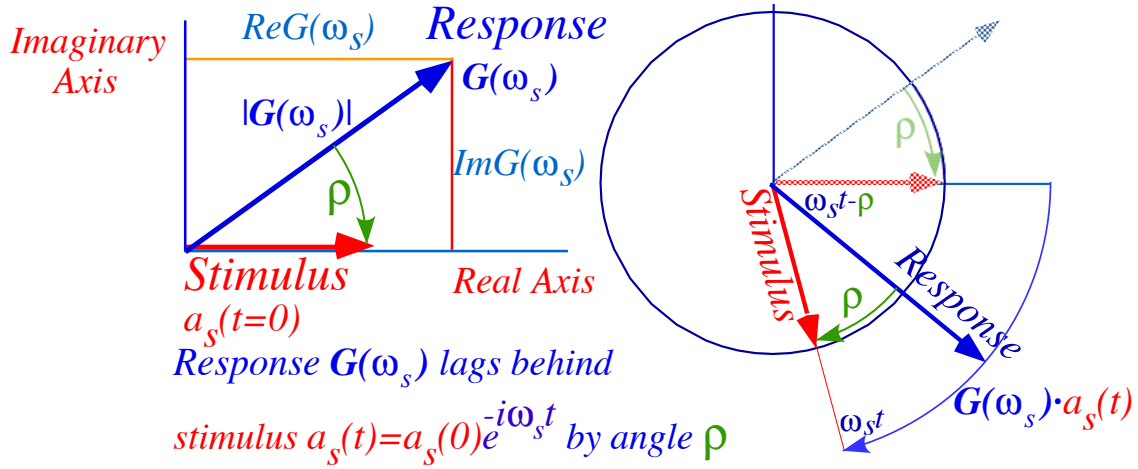


Fig.1.B.2 Oscillator response and stimulus phasors rotate rigidly at angular rate ω_s .

Complex algebra such as (1.B.5) is indispensible for analyzing oscillatory phenomena and devices. The most prevalent applications to electrical engineering (And, this is why *ABS, (xy)->(polar)*, etc. keys are on your calculator!) began when Nikoli Tesla showed to Edison and Westinghouse (This took some time!) that their DC wiring was impractical and AC was the way to go in building a large power grid.

In optics as well as quantum mechanics there are many relations of the form (1.B.3) for which the *phasor* diagrams of Fig. 1.B.2 are helpful. Our description of quantum waves would be impossibly cumbersome without complex phasors and algebra.

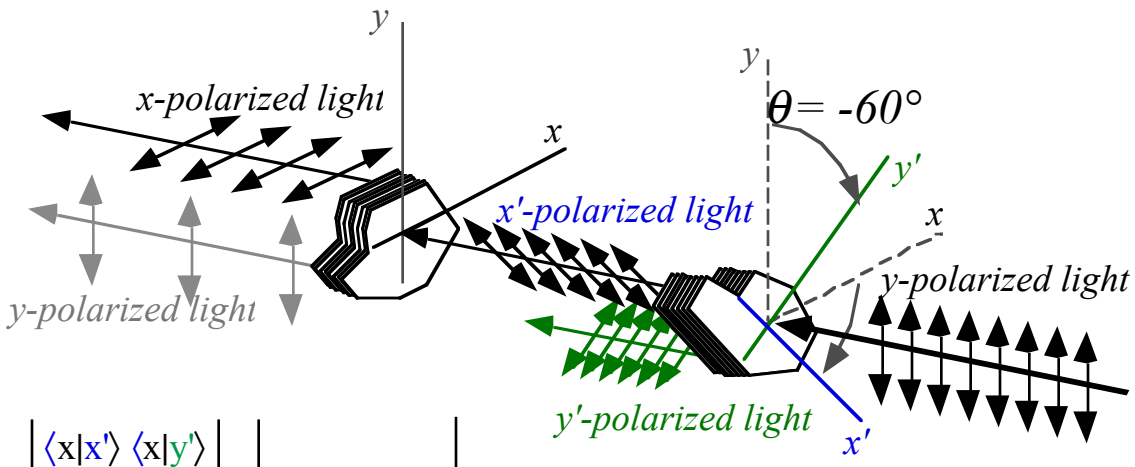
The idea of a phasor is based on that of *phase space*, certainly one of the oldest ideas of physics. Indeed, it goes back to the ancient astronomers tracking the phase of the moon. The real axis of a phasor is what we call the "is" of an oscillator, that is, where it is now. The imaginary component in Fig. 1.B.1 is what we will call the "gonna'be" component, that is, where the oscillator is going to be in *1/4*-cycle if the phasor continues rotating clockwise at the same frequency. (The imaginary component is also a velocity or momentum component in ω units.) A helpful mnemonic is used in the billionaire business world:

Imagination precedes reality by one quarter!

In Unit 2 and throughout the rest of the text, phasors will be important tools.

Problems for Ch.1

1.2.1. A y-polarized light beam of unit amplitude (1 photon/sec.) enters the analyzer system as shown below. Fill in the blanks with numbers or symbols that tell as much as possible about what is present at each channel or branch.



$$\begin{vmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{vmatrix} = \begin{vmatrix} \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ \end{vmatrix}$$

$$\begin{vmatrix} \langle x'|x\rangle & \langle x'|y\rangle \\ \langle y'|x\rangle & \langle y'|y\rangle \end{vmatrix} = \begin{vmatrix} \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ \end{vmatrix}$$

State of output x channel

|___⟩ _____

Amplitude= _____

Probability= _____

State of x' channel

|___⟩ _____

Amplitude= _____

Probability= _____

State of input channel

|y⟩ 1.0

Amplitude= 1.0

Probability= 1.0

State of output y channel

|___⟩ _____

Amplitude= _____

Probability= _____

State of y' channel

|___⟩ _____

Amplitude= _____

Probability= _____

A Dim View

1.2.2 (a) How far away from KUAF (10^5 Watts at 91.3 MHz) do you only get 1 photon/m^{2}s?

(b) How far away from a 10^5 Watt green light source do you only get 1 photon/m^{2}s? Can you guesstimate the threshold of visibility (in green photon/m^{2}s) for your eye?

Give E-field amplitude in each case. Assume (incorrectly) scalar isotropic coherent wave sources.

Photonic Zeno

1.2.3 Imagine a series of N polarization beam sorters like the ones in Fig. 1.2.1 or 1.2.3 are placed so the top x-output beam of each goes into the next sorter in line which is rotated clockwise by an angle ϕ relative to the one before. Suppose unit amplitude x-polarization ($\Psi_x = 1$, $\Psi_y = 0$) comes into the first sorter in the series.

(a) What angle ϕ makes the amplitude $1/2^N$ coming out of this series ? (Zeno attenuation)

(b) What angle ϕ makes the intensity $1/2^N$ coming out of this series ? (Zeno depletion)

(c) Suppose the objective is to have as much y-polarization as is practical come out of this series. How does the output amplitude and intensity vary with the number N?

How many (N) sorters are needed to give 99% photon conversion efficiency ?

Electronic Zeno

1.2.4. Imagine a series of N electron beam sorters like the ones in Fig. 1.1.6 or 1.2.4 are placed so the top ↑- (up) output beam of each goes into the next sorter in line which is rotated clockwise by an angle ϕ relative to the one before. Suppose unit amplitude ↑- spin ($\Psi\uparrow = 1$, $\Psi\downarrow = 0$) comes into the first sorter in the series.

(a) What angle ϕ makes the amplitude $1/2^N$ coming out of this series ? (Zeno attenuation)

(b) What angle ϕ makes the intensity $1/2^N$ coming out of this series ? (Zeno depletion)

(c) Suppose the objective is to maximize ↓-spin (down) output from this series. How does the output amplitude and intensity vary with the number N?

How many (N) sorters are needed to give 99% electron conversion efficiency ?

(This is called adiabatic reversal.)

Fashion Plates

1.3.1. The effects of a 1/4-wave and a 1/2-wave plate were described in (1.3.1) to (1.3.3) and sketched in Fig. 1.3.6.

(a)Do the same for a "whole-wave" plate. (Give Ψ and sketch Re Ψ trajectory.)

(b)Do the same for a 1/3-wave plate. (Give Ψ and sketch Re Ψ trajectory.)

Good vibrations

1.B.1 An atomic oscillator is stimulated by a force that is sinusoidally oscillating at a frequency ω_S that is EQUAL to the natural resonance frequency ω_O of the atom. Then the atom's oscillation will be...(a) in phase with (b) 45° ahead of (c) 45° behind (d) 90° ahead of (e) 90° behind (f) 135° ahead of (f) 135° behind (f) 180° out of phase with .... the phase of the force.

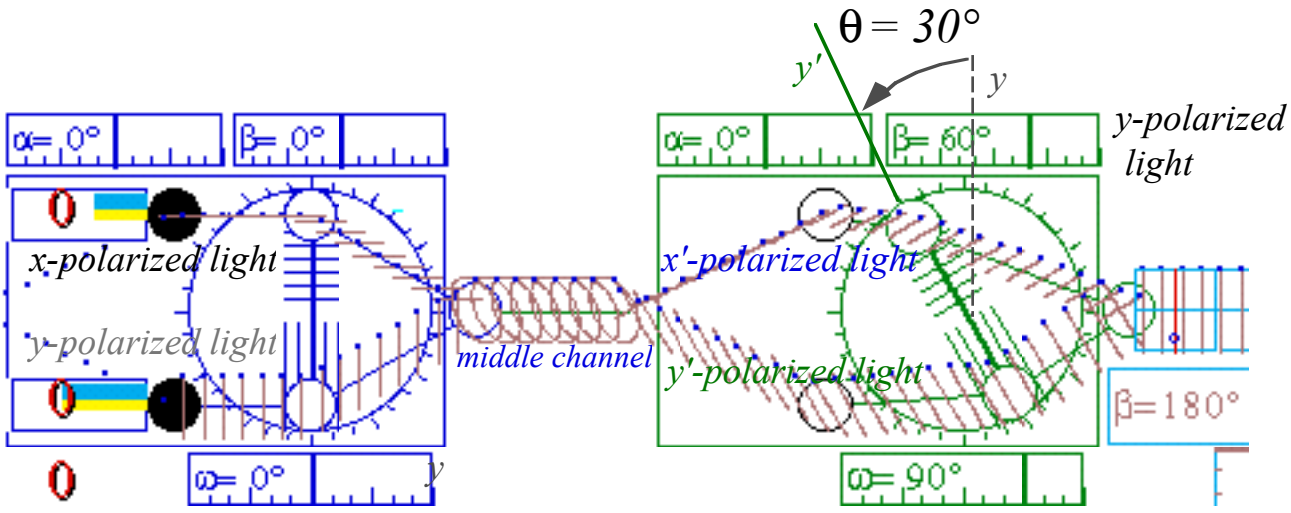
Keep the phase baby

1.B.1 An atomic oscillator is stimulated by a force sinusoidally oscillating at a frequency ω_S that is much LESS than the natural resonance frequency ω_O of the atom. Then the atom's oscillation will be more or less...(a) in phase with (b) 45° ahead of (c) 45° behind (d) 90° ahead of (e) 90° behind (f) 135° ahead of (f) 135° behind (f) 180° out of phase with .... the phase of the force.

Ellipsis in Middle

1.3.1 A y-polarized light beam of unit amplitude (1 photon/sec.) enters an active analyzer that is tilted by 30° as shown below. The active analyzer puts a ω = 90° phase factor e-iω in the x' beam.

Fill in the blanks with numbers or symbols that tell as much as possible about what is present at each channel or branch.



$$\begin{vmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{vmatrix} = \begin{vmatrix} \_\_\_\_ & \_\_\_\_ \\ \_\_\_\_ & \_\_\_\_ \end{vmatrix} \qquad \begin{vmatrix} \langle x'|x\rangle & \langle x'|y\rangle \\ \langle y'|x\rangle & \langle y'|y\rangle \end{vmatrix} = \begin{vmatrix} \_\_\_\_ & \_\_\_\_ \\ \_\_\_\_ & \_\_\_\_ \end{vmatrix}$$

State of output x channel

|__⟩_____

Amplitude=

_____.

Probability=

_____.

State of x' channel

|__⟩_____

Amplitude=

_____.

Probability=

_____

State of input channel

|y⟩1.0

Amplitude=

1.0

Probability=

1.0

State of middle channel

_____

Probability=

_____.

State of output y channel

|__⟩_____

Amplitude=

_____.

Probability=

_____.

State of y' channel

|__⟩_____

Amplitude=

_____.

Probability=

_____

Note: The x' and y' quantities will not be graded. Use for work.

A Quantum Internet

1.3.2. The sorter network shown in the third figure consists of three kinds of polarization sorters a, b, and c each with their main x-axis (a_1-state) rotated by θ_a, θ_b, θ_c, respectively. The shaded triangles (pointing to the right toward the input side) split their input beams while the white triangles (pointing to the left) recombine the two beams of opposite polarization into one beam. (They are sorters reversed and flipped over.)

Let : $\theta_a = \theta$, $\theta_b = 2\theta$, $\theta_c = 3\theta$. Let input be x-polarized state $(\psi_x,\psi_y) = (1,0)$.

(a) How many distinct "Feynman paths" are there from input to each output channel?

(b) If the recombination is done with "peeking" or dephasing of some kind, compute the probability at each channel for the angle θ=45°.

(c) If the recombination is done without "peeking" or dephasing of some kind, compute the probability at each channel for the angle θ=45°.

(d) For each of the two cases of "peeking" and not "peeking" give a formula for probability in each channel as function of angle θ. (Check: Do your probabilities add up to one?)

4. (Extra credit) A "peeking" analyzer gives the same probabilities in Fig. 1.3.8-9 whether it totally "collapses" the wave amplitudes or just jiggles one of their phases. Is there an experiment that could tell the difference? Tell why not, or describe one that would.