QM
for
AMOP

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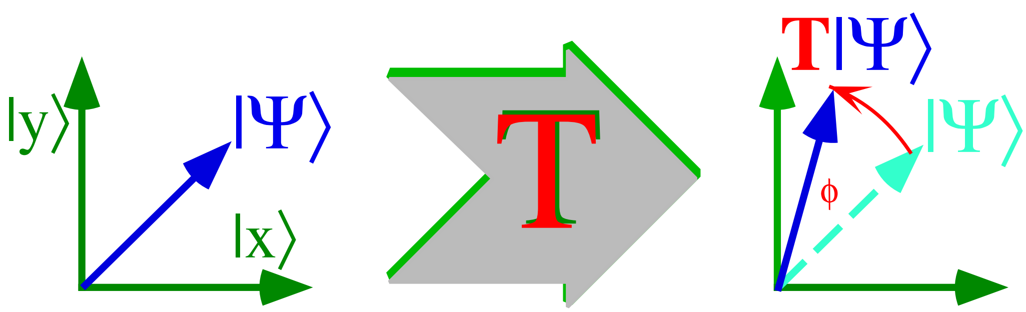
Chapter 2
Transformation and Transfer Operators and Matrices



Four axioms of quantum theory are based on the physics introduced in Chapter 1 and related to mathematical operations such as scalar products, matrix multiplication, change of basis, and projection. The four quantum axioms are related to the four axioms of a mathematical symmetry group as well as the all-important requirement of *unitarity* or "probability conservation." The "mother of all groups," the unitary group U(N), is introduced for later use.

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2.1 Transformation Amplitude Matrices: Quantum Axioms

We have seen how quantum amplitudes like $\langle x|x'\rangle$ are entries in arrays or *transformation matrices* like (1.2.5), for example. It is time to state mathematical and physical axioms which these quantities obey. This will help in reviewing the preceding sections and in establishing the mathematical basis of quantum theory.

The quantum axioms will be stated as clearly and as physically as possible. Like the establishment of a constitution for democratic country, it is imperative to base them on previous experience and above all make them reasonable, that is, *self-consistent*. In mathematics, their can be no proof of any part of a set of axioms. If such proof is found, (this happens rarely), then the axiom is upgraded to a theorem.

The same applies to physical theory, the main difference is that physical axioms in a given theory may eventually be "proved" by incorporation within a more fundamental theory. An example of this ocurred when classical mechanics became superseded by quantum theory. Before quantum theory, the Newton's Laws such as conservation of momentum were physical axioms. Quantum theory (as we will see later on) "proves" that momentum is conserved, but only on the average and after many trials and counts. (Sound familiar? We have already seen that quantum theory, like a blow-dried weatherman, only predicts probabilities; the "hard-edged" classical world is gone forever.)

Let us now state a set of quantum axioms that seem to underlie the marvelous theory that has replaced the classical Newtonian theory which had reigned for nearly two centuries before 1913.

(a) Fundamental quantum axioms

Our statement of axioms will be based on an n-state system whose sorters always sort quantum beams into no more than n sub-beams. (Recall Fig. 1.1.1) Our motivation for the axioms will be based upon observed behavior for the 2-state systems of photon polarization. The axioms are concerned with the properties of n^2 complex quantum amplitudes $\langle j|k'\rangle$ arrayed in the following n-by-n *transformation matrix*.

$$T(b \leftarrow b') = \begin{pmatrix} \langle 1|1'\rangle & \langle 1|2'\rangle & \cdots & \langle 1|n'\rangle \\ \langle 2|1'\rangle & \langle 2|2'\rangle & \cdots & \langle 2|n'\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle n|1'\rangle & \langle n|2'\rangle & \cdots & \langle n|n'\rangle \end{pmatrix} \qquad (2.1.1)$$

(1) The probability axiom

Our first axiom concerns the physical interpretation of amplitudes.

Axiom 1: The absolute square $|\langle j|k'\rangle|^2 = \langle j|k'\rangle^* \langle j|k'\rangle$ *gives the probability for occurrence in state-j of a system that started in state-k'=1',2',..,or n' from one sorter and then was forced to choose between states j=1,2,...,n by another sorter.*

This idea of probability was introduced first in Sect. 1.2b (See eq. (1.2.12).) The "forced-to-choose" clause and the word "state" are the kickers here. These concepts arise from the properties of "sorters" which have been described in the preceding sections. Some mysterious things are going on inside these sorters and analyzers so that a particle or system is forced to choose one and only one of n-states. That, in turn, makes the idea of a state mysterious, too. Which comes first, the chicken or the egg?

The next axiom is also a debatable one. Most quantum theories demand it, as will ours.

(2) The conjugation or inversion axiom

The second axiom concerns going backwards through a sorter or the reversal of amplitudes.

Axiom 2: The complex conjugate of an amplitude gives its reverse: $\langle j|k'\rangle^* = \langle k'|j\rangle$ (2.1.2)

We appeal to the idea of time-reversal and Planck's phase factor (1.2.7) when justifying this one. Conjugation of $e^{-i\omega t}$ yields $(e^{-i\omega t})^*=e^{i\omega t}$ which can be interpreted as changing the sign of time ($t\rightarrow -t$), that is, going backwards. It also could have been a sign change of frequency ($\omega\rightarrow -\omega$), but the sign of ω is fixed by convention to be positive in elementary non-relativistic quantum theory. Interestingly, when it comes to doing a conjugation of the deBroglie phase factor e^{ikx} to $(e^{ikx})^*=e^{-ikx}$, we always interpret this as a change of sign of the wavevector ($k\rightarrow -k$) of momentum, that is, conjugation *really* makes things go backwards. Clearly, we haven't heard the last discussion of Axiom 2. Now an axiom that looks air-tight (but isn't).

(3) The orthonormality or identity axiom

The third axiom concerns the amplitude for "re measurement" by the same analyzer.

Axiom 3: If identical analyzers are used twice or more the amplitude for a passed state-k is one, and for all others it is forever zero: $\langle j|k\rangle = \delta_{jk} = \left\{ \begin{array}{c} 1 \text{ if: } j=k \\ 0 \text{ if: } j\neq k \end{array} \right\} = \langle j'|k'\rangle$ (2.1.3)

You might think that a "caveman" could prove this axiom. We showed in Fig. 1.1.4 an easy experiment of repeated analysis of x-polarization. Apart from losses due to crystalline imperfections and absorption, this seems to "prove" unit probability for x-to-x. Probability, yes, but amplitude? Not necessarily! By axiom-1, probability is an absolute square which kills any phase factors an amplitude might have. The caveman could only prove Axiom-3 up to an arbitrary phase factor.

$$\langle j|k\rangle = e^{i\phi}\delta_{jk}$$

Unit phase is a mathematical convention we can live with. Now comes the "axiom of axioms.".

(4) The completeness or closure axiom

The fourth axiom concerns the "Do-nothing" property of an ideal analyzer, that is, a sorter followed by an "unsorter" or "put-back-togetherer" as introduced in Sec. 1.3.

Axiom 4. Ideal sorting followed by ideal recombination of amplitudes has no effect:

$$\langle j''|m'\rangle = \sum_{k=1}^{n} \langle j''|k\rangle\langle k|m'\rangle \qquad (2.1.4)$$

This axiom contains much if not all of the physical mystery of quantum phenomena. The idea is that a system initially in a prime state-m' is first sorted by an analyzer into states $k=1, 2, ..., n$ each with amplitude

$\langle k | m' \rangle$ and then each of those is recombined (summed over *k*) and sorted by another analyzer into one of the doubly-prime states $j''= 1'', 2'', \ldots , n''$, each with an additional amplitude $\langle j'' | k \rangle$. The claim is that the amplitude for each of the doubly-prime j''-states is none other than $\langle j'' | m' \rangle$, exactly what it would have been if *k*-analyzer had never intervened!

The upper halves of previous Figs. 1.3.8 and 1.3.9 show examples of this. How it can go wrong due to "peeking" or "dephasing" is shown by the lower halves of the same figures. To be precise, the figures show a case where the doubly-prime analyzer is the same kind of analyzer as the singly-prime sorter that would produce the initial state-m', that is, $j''=j'$. For this special case (2.1.4) becomes

$$\langle j' | m' \rangle = \sum_{k=1}^{n} \langle j' | k \rangle \langle k | m' \rangle \tag{2.1.5}$$

You can "verify" this case of Axiom-4. Using orthonormality Axiom-3 gives

$$\delta_{jm} = \sum_{k=1}^{n} \langle j' | k \rangle \langle k | m' \rangle = \sum_{k=1}^{n} \langle k | j' \rangle^* \langle k | m' \rangle \tag{2.1.6}$$

where conjugation axiom-2 is used, too. Now, for $m' = j'$ this becomes

$$1 = \sum_{k=1}^{n} \langle k | j' \rangle^* \langle k | j' \rangle = \sum_{k=1}^{n} \left| \langle k | j' \rangle \right|^2 = \sum_{k=1}^{n} P\left(j' \, to \, k \right) \tag{2.1.7}$$

According to the probability axiom-1 this states that the sum of probabilities for all k-channels equals one which is consistent with the definition of probability in (1.1.2b). It means we have completely accounted for all possible states that a *k*-analyzer can sort. The *k*-states are then called a *complete set of states*.

A unit amplitude final x'-beam was shown in Fig. 1.3.9 along with a zero amplitude final y'-beam emerging from an ideal "do-nothing" analyzer with absolutely no "peeking" allowed. If the final x'y'-sorter-counter had been replaced by an *xy*-analyzer or even a general Ψ-analyzer, it should still be impossible to see any effect of the intervening "do-nothing" *xy*-analyzer. This is the general physical consequence of Axiom-4. Again, precise "proof" of this or other axioms is currently impossible. They are just some "laws" that we must live with for awhile...probably, quite awhile.

(b) Matrix bra-kets: bra-and-ket vectors and representations

The quantum axioms are strongly connected to mathematical axioms of linear algebra and unitary matrix group theory. As mentioned before in Secs. 1.2a and c, *Dirac notation* for vectors is a result of "dissecting" a transformation matrix. Dirac invented a notation for entire columns and entire rows of a *T*-matrix; they are called *kets* $|k\rangle$ and *bras* $\langle j|$ respectively. The general *T*-matrix below shows its bras and kets.

$$\begin{array}{cccc} |1'\rangle & |2'\rangle & \cdots & |n'\rangle \\ \downarrow & \downarrow & \cdots & \downarrow \end{array} \tag{2.1.8}$$

$$T(b \leftarrow b') = \begin{pmatrix} \langle 1|1'\rangle & \langle 1|2'\rangle & \cdots & \langle 1|n'\rangle \\ \langle 2|1'\rangle & \langle 2|2'\rangle & \cdots & \langle 2|n'\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle n|1'\rangle & \langle n|2'\rangle & \cdots & \langle n|n'\rangle \end{pmatrix} \begin{array}{c} \leftarrow \langle 1| \\ \leftarrow \langle 2| \\ \vdots \\ \leftarrow \langle n| \end{array}$$

These bras and kets are examples of what we will call *abstract* mathematical quantities. They stand for a definite physical object, in this case, definite states occupied by particles or systems. In order to view these objects or store them in a computer, you need a *representation* of them, that is, a list of numbers such as the *column vectors* representing kets which stand for primed states 1', 2', as pointed out in (2.1.8) or below,

$$\left|1'\right\rangle \to \begin{pmatrix} \langle 1|1'\rangle \\ \langle 2|1'\rangle \\ \vdots \\ \langle n|1'\rangle \end{pmatrix}, \quad \left|2'\right\rangle \to \begin{pmatrix} \langle 1|2'\rangle \\ \langle 2|2'\rangle \\ \vdots \\ \langle n|2'\rangle \end{pmatrix}, \tag{2.1.9a}$$

or the *row vectors* representing bras standing for unprimed states-1, 2,..as pointed out below.

$$\langle 1| \to \begin{pmatrix} \langle 1|1'\rangle & \langle 1|2'\rangle & \cdots & \langle 1|n'\rangle \end{pmatrix}, \qquad \langle 2| \to \begin{pmatrix} \langle 2|1'\rangle & \langle 2|2'\rangle & \cdots & \langle 2|n'\rangle \end{pmatrix} \tag{2.1.9b}$$

This Dirac abstraction is achieved by literally "abstracting" bras or kets from Axiom-4. Starting with an Axiom-4 equation for any basis $\{|1\rangle,|2\rangle,\cdots,|n\rangle\}$ or $\{|\bar{1}\rangle,|\bar{2}\rangle,\cdots,|\bar{n}\rangle\}$ or $\{|1'\rangle,|2'\rangle,\cdots,|n'\rangle\}$, *etc*.

$$\langle m''|j'\rangle = \sum_{k=1}^{n} \langle m''|k\rangle\langle k|j'\rangle = \sum_{\bar{k}=1}^{n} \langle m''|\bar{k}\rangle\langle \bar{k}|j'\rangle = \sum_{k'=1}^{n} \langle m''|k'\rangle\langle k'|j'\rangle = etc.\,, \tag{2.1.10a}$$

one can "rip-off" its bra $\langle m''|$ to give a ket $|j'\rangle$

$$|j'\rangle = \sum_{k=1}^{n} |k\rangle\langle k|j'\rangle = \sum_{\bar{k}=1}^{n} |\bar{k}\rangle\langle \bar{k}|j'\rangle = \sum_{k'=1}^{n} |k'\rangle\langle k'|j'\rangle = etc.\,, \tag{2.1.10b}$$

or the j'-ket can be "ripped-off" to expose a bra. (Recall the finale of 1999 US World Soccer Cup victory.)

$$\langle m''| = \sum_{k=1}^{n} \langle m''|k\rangle\langle k| = \sum_{\bar{k}=1}^{n} \langle m''|\bar{k}\rangle\langle \bar{k}| = \sum_{k'=1}^{n} \langle m''|k'\rangle\langle k'| = etc. \tag{2.1.10c}$$

A representation depends upon the basis you are using. Each case of the (2.1.10b) uses different transformation matrix coefficients and base kets to describe a single abstract ket. The following are all different representations of the same ket $|1'\rangle$. The third one is the j'-representation of $|1'\rangle$ which is composed of all zero components but one according to Axiom-3. (It's being represented in its own basis.)

$$|1'\rangle \to \begin{pmatrix} \langle 1|1'\rangle \\ \langle 2|1'\rangle \\ \vdots \\ \langle n|1'\rangle \end{pmatrix}, \text{ or: } |1'\rangle \to \begin{pmatrix} \langle \bar{1}|1'\rangle \\ \langle \bar{2}|1'\rangle \\ \vdots \\ \langle \bar{n}|1'\rangle \end{pmatrix}, \text{ or: } |1'\rangle \to \begin{pmatrix} \langle 1'|1'\rangle \\ \langle 2'|1'\rangle \\ \vdots \\ \langle n'|1'\rangle \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, etc.$$

The same applies to each of the bra representations (2.1.10c). Here are some representations of bra $\langle 2'|$.

$$, \ \langle 2'| \to \begin{pmatrix} \langle 2'|1\rangle & \langle 2'|2\rangle & \cdots & \langle 2'|n\rangle \end{pmatrix}, \text{ or: } \langle 2'| \to \begin{pmatrix} \langle 2'|\bar{1}\rangle & \langle 2'|\bar{2}\rangle & \cdots & \langle 2'|\bar{n}\rangle \end{pmatrix},$$
$$\text{or: } \langle 2'| \to \begin{pmatrix} \langle 2'|1'\rangle & \langle 2'|2'\rangle & \cdots & \langle 2'|n'\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 & \cdots & 0 \end{pmatrix}, etc.$$

The last is the j'-representation of $\langle 2'|$ using its own (primed) basis.

Note: It is very tempting to replace the "mapping arrows" ($\to$) by equal signs (=). This is often done in many a sloppy modern physics text (including this one, later on!), but don't do it until you are sure of which basis you are using and are settled into using that basis exclusively. Once one is "married" to a given basis it is permissible to dispense with some formalities!

Abstract forms (2.1.10b-c) can be reassembled with a general state-ket $|\Psi\rangle$ or bra $\langle\Psi|$ to give the following two Dirac forms of Axiom-4 for an arbitrary state-Ψ.

$$\langle m''|\Psi\rangle = \sum_{k=1}^{n} \langle m''|k\rangle\langle k|\Psi\rangle = \sum_{\bar{k}=1}^{n} \langle m''|\bar{k}\rangle\langle \bar{k}|\Psi\rangle = \sum_{k'=1}^{n} \langle m''|k'\rangle\langle k'|\Psi\rangle = etc.\,, \qquad (2.1.11a)$$

$$\langle \Psi|j'\rangle = \sum_{k=1}^{n} \langle \Psi|k\rangle\langle k|j'\rangle = \sum_{\bar{k}=1}^{n} \langle \Psi|\bar{k}\rangle\langle \bar{k}|j'\rangle = \sum_{k'=1}^{n} \langle \Psi|k'\rangle\langle k'|j'\rangle = etc. \qquad (2.1.11b)$$

(1) Transformation matrix operation: Change of basis

Axiom-4 has an important mathematical interpretation. You may view it as a *matrix-vector transformation* which effects a *change of basis*. Compare the Dirac algebraic form (2.1.11a) to the following matrix-acting-on-column-vector multiplication.

$$\begin{pmatrix} \langle 1''|\Psi\rangle \\ \langle 2''|\Psi\rangle \\ \vdots \\ \langle n''|\Psi\rangle \end{pmatrix} = \begin{pmatrix} \langle 1''|1'\rangle & \langle 1''|2'\rangle & \cdots & \langle 1''|n'\rangle \\ \langle 2''|1'\rangle & \langle 2''|2'\rangle & \cdots & \langle 2''|n'\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle n''|1'\rangle & \langle n''|2'\rangle & \cdots & \langle n''|n'\rangle \end{pmatrix} \begin{pmatrix} \langle 1'|\Psi\rangle \\ \langle 2'|\Psi\rangle \\ \vdots \\ \langle n'|\Psi\rangle \end{pmatrix} = \begin{pmatrix} \langle 1''|\bar{1}\rangle & \langle 1''|\bar{2}\rangle & \cdots & \langle 1''|\bar{n}\rangle \\ \langle 2''|\bar{1}\rangle & \langle 2''|\bar{2}\rangle & \cdots & \langle 2''|\bar{n}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle n''|\bar{1}\rangle & \langle n''|\bar{2}\rangle & \cdots & \langle n''|\bar{n}\rangle \end{pmatrix} \begin{pmatrix} \langle \bar{1}|\Psi\rangle \\ \langle \bar{2}|\Psi\rangle \\ \vdots \\ \langle \bar{n}|\Psi\rangle \end{pmatrix} = etc.$$

$(2.1.11a)_{\text{representation}}$

A 2-by-2 example of this was first given by (1.2.6b). The Dirac algebraic form for the bra expression (2.1.11b) has the following row-vector-on-matrix representation.

$$\begin{pmatrix} \langle \Psi|1'\rangle & \langle \Psi|2'\rangle & \cdots & \langle \Psi|n'\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \langle \Psi|1''\rangle & \langle \Psi|2''\rangle & \cdots & \langle \Psi|n''\rangle \end{pmatrix} \begin{pmatrix} \langle 1''|1'\rangle & \langle 1''|2'\rangle & \cdots & \langle 1''|n'\rangle \\ \langle 2''|1'\rangle & \langle 2''|2'\rangle & \cdots & \langle 2''|n'\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle n''|1'\rangle & \langle n''|2'\rangle & \cdots & \langle n''|n'\rangle \end{pmatrix} = etc. \qquad (2.1.11b)_{\text{representation}}$$

These are the bra and ket forms of general quantum coordinate transformations.

(2) Transformation matrix products: Unitary groups

Axiom-4 has another even more important mathematical interpretation. You may view it as a *transformation matrix product* which forms a *unitary transformation group U(n)*. Axiom-4 is basically a matrix product as seen by comparing the following two representations. First, the original Dirac form

$$\langle j''|m'\rangle = \sum_{k=1}^{n} \langle j''|k\rangle\langle k|m'\rangle \qquad (2.1.12a)$$

and then here is the same thing in matrix form.

$$\begin{pmatrix} \langle 1''|1'\rangle & \langle 1''|2'\rangle & \cdots & \langle 1''|n'\rangle \\ \langle 2''|1'\rangle & \langle 2''|2'\rangle & \cdots & \langle 2''|n'\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle n''|1'\rangle & \langle n''|2'\rangle & \cdots & \langle n''|n'\rangle \end{pmatrix} = \begin{pmatrix} \langle 1''|1\rangle & \langle 1''|2\rangle & \cdots & \langle 1''|n\rangle \\ \langle 2''|1\rangle & \langle 2''|2\rangle & \cdots & \langle 2''|n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle n''|1\rangle & \langle n''|2\rangle & \cdots & \langle n''|n\rangle \end{pmatrix} \bullet \begin{pmatrix} \langle 1|1'\rangle & \langle 1|2'\rangle & \cdots & \langle 1|n'\rangle \\ \langle 2|1'\rangle & \langle 2|2'\rangle & \cdots & \langle 2|n'\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle n|1'\rangle & \langle n|2'\rangle & \cdots & \langle n|n'\rangle \end{pmatrix} \qquad (2.1.12b)$$

You could also write this is as the following *matrix product*

$$T_{j''m'}\begin{pmatrix} prime \\ to \\ double-prime \end{pmatrix} = \sum_{k=1}^{n} T_{j''k}\begin{pmatrix} unprimed \\ to \\ double-prime \end{pmatrix} T_{k\,m'}\begin{pmatrix} prime \\ to \\ unprimed \end{pmatrix} \qquad (2.1.12c)$$

or abstractly as follows. $\mathbf{T}(b'' \leftarrow b') = \mathbf{T}(b'' \leftarrow b) \bullet \mathbf{T}(b \leftarrow b')$ (2.1.12d)

The latter **T**'s are our first example of *abstract group operators*. They are quantum coordinate transformation operators that deliver us from one basis to another through transformations such as (2.1.11). Eq. (2.1.12) is their composition rule: If **T**(B ← A) goes from A to B and **T**(C ← B) goes from B to C then the product **T**(C ← B)•**T**(B ← A) goes directly from A to C, that is, it equals **T**(C ← A) .

These operators stand for all the possible transformations that are allowed in quantum space. Their matrix representations represent all the means for "getting around" in the state space. As we will see later, they also represent (practically) all the possible quantum analyzers that can be built in laboratory experiments involving the n-state system being studied. Remember that each abstract mathematical quantity corresponds to some "thing" out there. Bras and kets correspond to systems of particles and operators like the **T**'s correspond to devices which "do things" to the systems, that is, transport or "taxi" the system from one state to another.

(3) Scalar products: Invariance and Hermitian conjugation

Finally, Axiom-4 has a very simple mathematical interpretation. You may view it as a *scalar product* between two arbitrary state vectors, a state-ket $\left|\Psi\right\rangle$ and bra $\left\langle\Phi\right|$. First, the general Axiom-4

$$\left\langle\Phi\middle|\Psi\right\rangle = \sum_{k=1}^{n} \left\langle\Phi\middle|k\right\rangle\left\langle k\middle|\Psi\right\rangle = \sum_{\bar{k}=1}^{n} \left\langle\Phi\middle|\bar{k}\right\rangle\left\langle\bar{k}\middle|\Psi\right\rangle = etc. \qquad (2.1.13a)$$

and then the representations of Axiom-4 as a "dot" product of row and column vectors.

$$\left\langle\Phi\middle|\Psi\right\rangle = \begin{pmatrix} \left\langle\Phi\middle|1\right\rangle & \left\langle\Phi\middle|2\right\rangle & \cdots & \left\langle\Phi\middle|n\right\rangle \end{pmatrix}\begin{pmatrix} \left\langle 1\middle|\Psi\right\rangle \\ \left\langle 2\middle|\Psi\right\rangle \\ \vdots \\ \left\langle n\middle|\Psi\right\rangle \end{pmatrix} = \begin{pmatrix} \left\langle\Phi\middle|\bar{1}\right\rangle & \left\langle\Phi\middle|\bar{2}\right\rangle & \cdots & \left\langle\Phi\middle|\bar{n}\right\rangle \end{pmatrix}\begin{pmatrix} \left\langle \bar{1}\middle|\Psi\right\rangle \\ \left\langle \bar{2}\middle|\Psi\right\rangle \\ \vdots \\ \left\langle \bar{n}\middle|\Psi\right\rangle \end{pmatrix} = etc. \qquad (2.1.13b)$$

In Sec. b(1-2) we saw matrix multiplication operations (2.1.11) and (2.1.12). Note that these are just combinations of the elementary scalar product shown in (2.1.13), that is, a row "dot-multiplied" with a column. Matrix operation on a vector in (2.1.11) involves n scalar products. Matrix multiplication (2.1.12) involves n^2 scalar products. It helps to see how Axiom-4 relates all these operations.

The scalar product $\left\langle\Phi\middle|\Psi\right\rangle$ is one of few abstract quantum quantities that has the same numerical value in all possible representations. All the possible sums in (2.1.13) give the same number $\left\langle\Phi\middle|\Psi\right\rangle$ even though the various representations of individual abstract bra $\left\langle\Phi\right|$ and ket $\left|\Psi\right\rangle$ will have wildly different numbers in them. The scalar products are said to be *invariant to change of bases*. This invariance is called *unitary invariance* for reasons that will be discussed shortly.

The scalar product can be expressed entirely in terms of ket components by using the conjugation axiom-2 to convert bra components $\left\langle\Phi\middle|k\right\rangle$ to conjugated ket components $\left\langle k\middle|\Phi\right\rangle^* = \left\langle\Phi\middle|k\right\rangle$.

$$\langle\Phi|\Psi\rangle = \sum_{k=1}^{n} \langle k|\Phi\rangle^* \langle k|\Psi\rangle = \sum_{\bar{k}=1}^{n} \langle \bar{k}|\Phi\rangle^* \langle \bar{k}|\Psi\rangle = etc. \qquad (2.1.14)$$

$$\langle\Phi|\Psi\rangle = \begin{pmatrix} \langle 1|\Phi\rangle^* & \langle 2|\Phi\rangle^* & \cdots & \langle n|\Phi\rangle^* \end{pmatrix} \begin{pmatrix} \langle 1|\Psi\rangle \\ \langle 2|\Psi\rangle \\ \vdots \\ \langle n|\Psi\rangle \end{pmatrix} = \begin{pmatrix} \langle \bar{1}|\Phi\rangle^* & \langle \bar{2}|\Phi\rangle^* & \cdots & \langle \bar{n}|\Phi\rangle^* \end{pmatrix} \begin{pmatrix} \langle \bar{1}|\Psi\rangle \\ \langle \bar{2}|\Psi\rangle \\ \vdots \\ \langle \bar{n}|\Psi\rangle \end{pmatrix} = etc.$$

The mathematical "sex change" operation of converting a ket to a bra is given the technical name of *transpose conjugation †* . (Perhaps, the dagger symbol † indicates the use of a knife or scapel.) It is also called *Hermitian conjugation* after the mathematician Hermite. (No, Hermite did not undergo any such operation, so far as we know.) As you can see from (2.1.14), this operation on column vectors is relatively painless and, indeed, receives far less publicity than its human equivalent. It simply involves the flipping of a column to become a row or vice-versa (called *transposing* ^{T}) followed or preceded by complex conjugation (*) of every amplitude in the row or column.

The notation for Hermitian conjugation is used as follows. First the abstract form is as follows.

$$\langle\Phi|^\dagger = |\Phi\rangle \text{ , or: } \quad |\Phi\rangle^\dagger = \langle\Phi| \qquad (2.1.15)$$

A matrix representation of this *conjugation †* is a row-to-column conversion with complex conjugation.

$$\begin{pmatrix} \langle\Phi|1\rangle & \langle\Phi|2\rangle & \cdots & \langle\Phi|n\rangle \end{pmatrix}^\dagger = \begin{pmatrix} \langle\Phi|1\rangle^* \\ \langle\Phi|2\rangle^* \\ \vdots \\ \langle\Phi|n\rangle^* \end{pmatrix} = \begin{pmatrix} \langle 1|\Phi\rangle \\ \langle 2|\Phi\rangle \\ \vdots \\ \langle n|\Phi\rangle \end{pmatrix} \text{, or: } \begin{pmatrix} \langle 1|\Phi\rangle \\ \langle 2|\Phi\rangle \\ \vdots \\ \langle n|\Phi\rangle \end{pmatrix}^\dagger = \begin{pmatrix} \langle 1|\Phi\rangle^* & \langle 2|\Phi\rangle^* & \cdots & \langle n|\Phi\rangle^* \end{pmatrix}$$

$$= \begin{pmatrix} \langle\Phi|1\rangle & \langle\Phi|2\rangle & \cdots & \langle\Phi|n\rangle \end{pmatrix}$$

Axiom-2 was used, again. Note: two Hermitian conjugations cancel: $\mathbf{A}^{\dagger\dagger}=\mathbf{A}$. (Don't try this at home!)

(4) Particle expectation number: Norms and normalization

A most important physical quantity is the scalar product of a state vector $|\Psi\rangle$ with itself, conjugated, of course. (You've probably noticed that "same-sex" marriages are prohibited in this quantum state theory. Actually, it can be done, but they're not real. Or invariant. In fact, they're usually very complex. For these you must wait until we discuss such taboo topics as *n-body mechanics* and *tensor operator theory*. Can't wait, can you?) The self product of $|\Psi\rangle$ with itself (which is always real) is quite analogous to scalar product.

$$\langle\Psi|\Psi\rangle = \sum_{k=1}^{n} \langle\Psi|k\rangle\langle k|\Psi\rangle = \sum_{k=1}^{n} \langle k|\Psi\rangle^*\langle k|\Psi\rangle = \sum_{k=1}^{n} \left|\langle k|\Psi\rangle\right|^2 = \begin{pmatrix} \langle 1|\Psi\rangle^* & \langle 2|\Psi\rangle^* & \cdots & \langle n|\Psi\rangle^* \end{pmatrix} \begin{pmatrix} \langle 1|\Psi\rangle \\ \langle 2|\Psi\rangle \\ \vdots \\ \langle n|\Psi\rangle \end{pmatrix} \qquad (2.1.16)$$

$\langle\Psi|\Psi\rangle$ is called the *particle number expectation* or *total probability* for a general quantum state $|\Psi\rangle$.

According to Axiom-1, the probability axiom, $\langle\Psi|\Psi\rangle$ is just the sum of the channel probabilities $|\langle k|\Psi\rangle|^2$ for each k-channel. Initially, we expect the probability sum to be unity as in (1.12b) or (2.1.7). However, after a particle or system is dragged through several analyzers having counters and other such dissipative devices, the total probability may be reduced significantly. An example was the *x*-beam after "peeking" which had a state vector $|\Psi\rangle = \begin{pmatrix} \sqrt{0.75} & 0 \end{pmatrix}$ for which the total probability is only 75%.

$\langle\Psi|\Psi\rangle$ is also called a *state norm*. For "permissive" states, the norm can be any real number. However, states that are being used as *base states* are expected to set a better example. Base states are supposed to be a complete set of kets $\{|1\rangle, |2\rangle, \cdots, |n\rangle\}$ and bras $\{\langle 1|, \langle 2|, \cdots, \langle n|\}$ which satisfy all Axioms 1-4. In particular, they must be normal and satisfy the *ortho-normalization conditions* of Axiom-3.

$$\langle i|j\rangle = \delta_{ij} = \begin{cases} 1 & \text{if: } i=j \\ 0 & \text{if: } i \neq j \end{cases} \tag{2.1.17}$$

The *Kronecker delta function* δ_{ij} is intended to guarantee unit norm $\langle j|j\rangle = 1$ (*for all j*) and, hence, unit probability for all base states. But, state vectors become "sub-normal" or abnormal in "dissipative" places.(5)

Axiom-4 totally abstracted: Projectors

Axiom-4 has one more level of abstraction: a complete "rip-off" of its bras and kets as follows.

$$\langle j''|m'\rangle = \sum_{k=1}^{n} \langle j''|k\rangle\langle k|m'\rangle \quad \Rightarrow \quad \mathbf{1} = \sum_{k=1}^{n} |k\rangle\langle k| \tag{2.1.18}$$

The result is a sum of "ket-bras" $|k\rangle\langle k|$ which are quite the opposite of the "bra-kets" like $\langle k|k\rangle$ which are scalars. (By axiom-3 $\langle k|k\rangle=1$.) The ket-bras $|k\rangle\langle k|$ are elementary examples of *tensor operators* and are called *projection operators* or *projectors* $\mathbf{P}_k$.

$$\mathbf{P}_k = |k\rangle\langle k| \tag{2.1.19}$$

The axiom-4 sum to the identity operator **1** is called a *completeness relation*.

$$\mathbf{1} = \sum_{k=1}^{n} \mathbf{P}_k = \sum_{k=1}^{n} |k\rangle\langle k| \tag{2.1.20}$$

The action of a projection operator $\mathbf{P}_k$ on a general state vector $|\Psi\rangle$ yields a projection or "shadow" $|k\rangle\langle k|\Psi\rangle$ of the original vector in the direction of the *k-th* base ket, as shown in the Fig. 2.1.1.

$$\mathbf{P}_k|\Psi\rangle = |k\rangle\langle k|\Psi\rangle \tag{2.1.21}$$

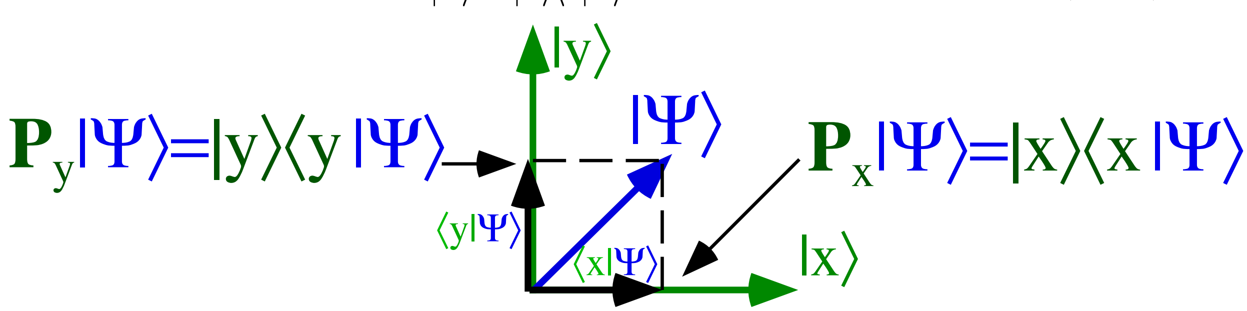

Fig. 2.1.1 Projection operators $\mathbf{P}_k$ *map state ket onto base axes*

To construct a projection matrix representation of a $\mathbf{P}_k$ simply reattach a $\mathbf{P}_k$'s "ripped-off" bras and kets for the desired basis, say the φ-tilted polarization bases $\{|x'\rangle, |y'\rangle\}$.

$$\begin{pmatrix} \langle x'|\mathbf{P}_x|x'\rangle & \langle x'|\mathbf{P}_x|y'\rangle \\ \langle y'|\mathbf{P}_x|x'\rangle & \langle y'|\mathbf{P}_x|y'\rangle \end{pmatrix} = \begin{pmatrix} \langle x'|x\rangle\langle x|x'\rangle & \langle x'|x\rangle\langle x|y'\rangle \\ \langle y'|x\rangle\langle x|x'\rangle & \langle y'|x\rangle\langle x|y'\rangle \end{pmatrix}, \text{ where: } \begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

Projector $\mathbf{P}_x = |x\rangle\langle x|$ is what is called an *outer* or *Kronecker tensor (⊗) product* of ket and bra .

$$\begin{pmatrix} \langle x'|\mathbf{P}_x|x'\rangle & \langle x'|\mathbf{P}_x|y'\rangle \\ \langle y'|\mathbf{P}_x|x'\rangle & \langle y'|\mathbf{P}_x|y'\rangle \end{pmatrix} = \begin{pmatrix} \langle x'|x\rangle\langle x|x'\rangle & \langle x'|x\rangle\langle x|y'\rangle \\ \langle y'|x\rangle\langle x|x'\rangle & \langle y'|x\rangle\langle x|y'\rangle \end{pmatrix} = \begin{pmatrix} \langle x'|x\rangle \\ \langle y'|x\rangle \end{pmatrix} \otimes \begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \end{pmatrix} \quad (2.1.22)$$

The x'y'-representations for both $\mathbf{P}_x$ and $\mathbf{P}_y$ are worked out below.

$$\mathbf{P}_x = |x\rangle\langle x| \rightarrow \begin{pmatrix} \cos\phi \\ -\sin\phi \end{pmatrix} \otimes \begin{pmatrix} \cos\phi & -\sin\phi \end{pmatrix} \qquad \mathbf{P}_y = |y\rangle\langle y| \rightarrow \begin{pmatrix} \sin\phi \\ \cos\phi \end{pmatrix} \otimes \begin{pmatrix} \sin\phi & \cos\phi \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\phi & -\sin\phi\cos\phi \\ -\sin\phi\cos\phi & \sin^2\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{(\text{for } \phi=0)}, \qquad = \begin{pmatrix} \sin^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \cos^2\phi \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_{(\text{for } \phi=0)}$$

The case φ=0 gives *xy*-representations of $\mathbf{P}_x$ and $\mathbf{P}_y$ which each contain all "0" except for a single "1". Such matrices are called *elementary unit tensor representations* or *unit dyads*.

2.2 Transformation Operators: Unitarity and Group Axioms

If there is one thing you should always remember about quantum operators or any sort of mathematical operator, it is this. *Always begin with a base-state definition of an operator*. This simple rule will save you endless headache and hair-pulling.

(a) Base ket and bra transformations

As an example, consider a rotational transformation **T=R**(ϕ) similar to the one used for the rotated polarizer in Sec. 1.2. You want a transformation operator that takes each Cartesian ket $\{|x\rangle, |y\rangle\}$ and maps it into a new rotated basis $\{|\bar{x}\rangle = \mathbf{T}|x\rangle, |\bar{y}\rangle = \mathbf{T}|y\rangle\}$ given by the following and shown in Fig. 2.2.1.

$$\left\{|\bar{x}\rangle = \mathbf{T}|x\rangle = \cos\phi|x\rangle + \sin\phi|y\rangle, \qquad |\bar{y}\rangle = \mathbf{T}|y\rangle = -\sin\phi|x\rangle + \cos\phi|y\rangle\right\} \tag{2.2.1}$$

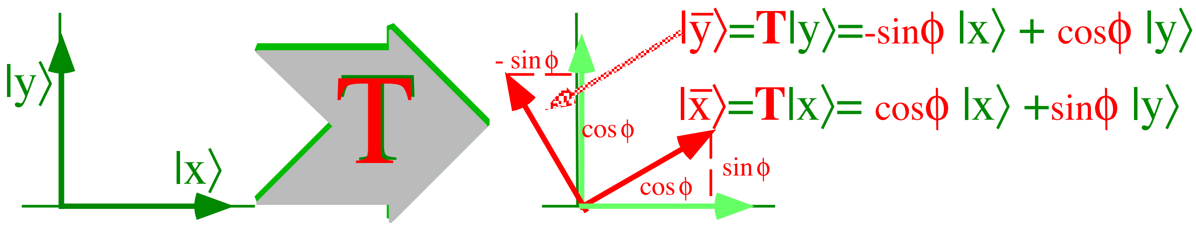

*Fig. 2.2.1 Transformation **T** maps unit kets into rotated ones*

Once this basic ket-vector definition **T** is given, all else follows. Scalar products of (2.2.1) with bra $\langle x|$ and then by bra $\langle y|$ gives the four transformation matrix components. Because of axiom-2 orthonormality $\langle j|k\rangle = \delta_{jk}$ or $\langle x|x\rangle = 1,\ \langle x|y\rangle = 0,\ etc.$, each product $\langle j|\bar{k}\rangle$ gives just one term.

$$\begin{pmatrix} \langle x|\bar{x}\rangle & \langle x|\bar{y}\rangle \\ \langle y|\bar{x}\rangle & \langle y|\bar{y}\rangle \end{pmatrix} = \begin{pmatrix} \langle x|\mathbf{T}|x\rangle & \langle x|\mathbf{T}|y\rangle \\ \langle y|\mathbf{T}|x\rangle & \langle y|\mathbf{T}|y\rangle \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \tag{2.2.2}$$

The general n-state form of transformation (2.2.1) is

$$|\bar{k}\rangle = \mathbf{T}|k\rangle = \sum_{j=1}^{n} |j\rangle\langle j|\bar{k}\rangle = \sum_{j=1}^{n} |j\rangle\langle j|\mathbf{T}|k\rangle \ , \ \text{where: } \langle j|\bar{k}\rangle = \langle j|\mathbf{T}|k\rangle \tag{2.2.3}$$

Axiom-2 requires that all base states satisfy orthonormality , and so must the new $\{|\bar{x}\rangle, |\bar{y}\rangle\}$ bases.

$$\langle j|k\rangle = \delta_{jk} = \langle \bar{j}|\bar{k}\rangle$$

In order for the transformation **T** to preserve orthonormality, the operator which transforms old bras $\{\langle x|, \langle y|\}$ into the new bras $\{\langle \bar{x}|, \langle \bar{y}|\}$ must be an inverse $\mathbf{T}^{-1}$ to the transformation **T** that transformed the kets in the basic definition (2.2.1). Only then does a scalar product stay the same. That is,

$$\left\{\langle \bar{x}| = \langle x|\mathbf{T}^{-1},\ \langle \bar{y}| = \langle y|\mathbf{T}^{-1}\right\} \quad \text{so that: } \langle \bar{j}|\bar{k}\rangle = \langle j|\mathbf{T}^{-1}\mathbf{T}|k\rangle = \langle j|\mathbf{1}|k\rangle = \langle j|k\rangle \ . \tag{2.2.4}$$

However, bras are obtained by Hermitian conjugation according to (2.1.15)

$$\langle \bar{k}| = |\bar{k}\rangle^\dagger = \left(\mathbf{T}|k\rangle\right)^\dagger = \left(\sum_{j=1}^{n} |j\rangle\langle j|\bar{k}\rangle\right)^\dagger$$

How do we dagger (†) a whole bunch of terms? The answer, is to †-stab (Hermite-transpose) each part.

$$\langle \bar{k}| = |\bar{k}\rangle^\dagger = \left(\mathbf{T}|k\rangle\right)^\dagger = \left(\sum_{j=1}^{n} \langle \bar{k}|j\rangle\langle j|\right) = \langle k|\mathbf{T}^\dagger \text{ , where: } \langle \bar{k}|j\rangle = \langle k|\mathbf{T}^\dagger|j\rangle = \langle j|\bar{k}\rangle^* \qquad (2.2.5a)$$

Here Hermitian conjugate $T^\dagger$ of an operator-matrix T is defined. True to form, its matrix representation is the transpose-conjugate of the original T-matrix. Also, from (2.2.4) it follows that the dagger-transform operator $\mathbf{T}^\dagger$ that transforms bras is just the inverse of ket transformer $\mathbf{T}$.

$$\langle k|\mathbf{T}^\dagger|j\rangle = \langle j|\bar{k}\rangle^* = \langle \bar{k}|j\rangle = \langle j|\mathbf{T}|k\rangle^* = \langle k|\mathbf{T}^{-1}|j\rangle, \quad \text{or:} \quad \mathbf{T}^\dagger = \mathbf{T}^{-1} \qquad (2.2.5b)$$

The ability to invert a matrix by simply transposing and conjugating is a <u>great</u> computational luxury, especially if the matrices are large. An operator $\mathbf{U}$ that satisfies $\mathbf{U}^\dagger = \mathbf{U}^{-1}$ is called a *unitary operator*. Such an operator preserves the unit norm as well as all bra-ket scalar products.

For our simple example of a ϕ-rotation the bras transform as follows.

$$\left\{\langle \bar{x}| = \langle x|\mathbf{T}^{-1} = \langle x|\cos\phi + \langle y|\sin\phi, \qquad \langle \bar{y}| = \langle y|\mathbf{T}^{-1} = -\langle x|\sin\phi + \langle y|\cos\phi\right\} \qquad (2.2.6a)$$

In this case the inverse is simply the transpose ($\mathbf{T}^T$); no conjugation is needed since the matrix is real.

$$\begin{pmatrix} \langle x|\bar{x}\rangle & \langle x|\bar{y}\rangle \\ \langle y|\bar{x}\rangle & \langle y|\bar{y}\rangle \end{pmatrix}^\dagger = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}^\dagger = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} = \begin{pmatrix} \langle x|\mathbf{T}^\dagger|x\rangle & \langle x|\mathbf{T}^\dagger|y\rangle \\ \langle y|\mathbf{T}^\dagger|x\rangle & \langle y|\mathbf{T}^\dagger|y\rangle \end{pmatrix} = \begin{pmatrix} \langle \bar{x}|x\rangle & \langle \bar{x}|y\rangle \\ \langle \bar{y}|x\rangle & \langle \bar{y}|y\rangle \end{pmatrix} \qquad (2.2.6b)$$

(In this case, a ±sign flip of angle ϕ would do the trick, too.) An operator $\mathbf{O}$ that satisfies $\mathbf{O}^T = \mathbf{O}^{-1}$ is called an *orthogonal operator*. It keeps real unit vectors orthogonal as it transforms them.

(b) Bra-ket vector component transformations

Once the T-matrix is known, it is a simple matter to derive the transformation rules for components of any ket vector $|\Psi\rangle$ that lives in this ket space. It is important to remember that the abstract $|\Psi\rangle$ is not the thing that changes in a change-of-basis transformation. $|\Psi\rangle$ is just being expressed in two equivalent ways. (In other words, it has two "aliases" as shown in the equations (2.2.7) and Fig. 2.2.2 below.)

$$|\Psi\rangle = |x\rangle\langle x|\Psi\rangle + |y\rangle\langle y|\Psi\rangle = |\bar{x}\rangle\langle \bar{x}|\Psi\rangle + |\bar{y}\rangle\langle \bar{y}|\Psi\rangle \qquad (2.2.7)$$

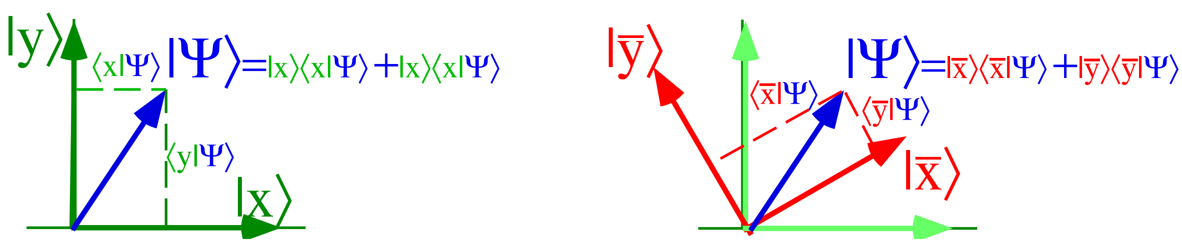

Fig. 2.2.2 Same vector $|\Psi\rangle$ *with two sets of coordinate bases. ("Passive" or "Alias" transformation)*

A change-of-basis transformation gives one "alias" , say $\left(\langle \bar{x}|\Psi\rangle, \langle \bar{y}|\Psi\rangle\right)$, in terms of another, say $\left(\langle x|\Psi\rangle, \langle y|\Psi\rangle\right)$. Here, the transformation is obtained either by multiplying $\langle \bar{x}|$ and $\langle \bar{y}|$ in (2.2.6a) on the right by $|\Psi\rangle$ or multiplying $|\Psi\rangle$ in (2.2.7) on the left by $\langle \bar{x}|$ or $\langle \bar{y}|$. Below are the results.

$$\begin{aligned}\langle\bar{x}|\Psi\rangle &= \langle\bar{x}|x\rangle\langle x|\Psi\rangle + \langle\bar{x}|y\rangle\langle y|\Psi\rangle = \cos\phi\langle x|\Psi\rangle + \sin\phi\langle y|\Psi\rangle \\ \langle\bar{y}|\Psi\rangle &= \langle\bar{y}|x\rangle\langle x|\Psi\rangle + \langle\bar{y}|y\rangle\langle y|\Psi\rangle = -\sin\phi\langle x|\Psi\rangle + \cos\phi\langle y|\Psi\rangle\end{aligned} \quad (2.2.8a)$$

Matrix form for this is the following which uses the inverse (2.2.6b) of transformation matrix (2.2.2).

$$\begin{pmatrix}\langle\bar{x}|\Psi\rangle \\ \langle\bar{y}|\Psi\rangle\end{pmatrix} = \begin{pmatrix}\langle\bar{x}|x\rangle & \langle\bar{x}|y\rangle \\ \langle\bar{y}|x\rangle & \langle\bar{y}|y\rangle\end{pmatrix}\begin{pmatrix}\langle x|\Psi\rangle \\ \langle y|\Psi\rangle\end{pmatrix} = \begin{pmatrix}\cos\phi & \sin\phi \\ -\sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}\langle x|\Psi\rangle \\ \langle y|\Psi\rangle\end{pmatrix} \quad (2.2.8a)$$

It is also inverse to our first change-of-basis example (1.2.6b). See (2.1.11) for general *n*-state formulas.

Even for a simple example like our ϕ-rotation it is difficult to derive the relations (2.2.8) just by looking at the components of a general vector in Fig. 2.2.2. (Try it!) This is one of the reasons for making the rule stated at the beginning of this Section 2.2. *You should work with bases first*.

It is important not to confuse a change-of-basis or "alias" transformation with an "active" or "alibi" transformation in which the operator $\mathbf{T}$ is used to move a state vector $|\Psi\rangle$ into a new vector $\mathbf{T}|\Psi\rangle = |\Psi(T)\rangle$ as shown in Fig. 2.2.3. Rotation $\mathbf{T}$ acted on the bases in Fig. 2.2.1-2 as $|\Psi\rangle$ stood still.

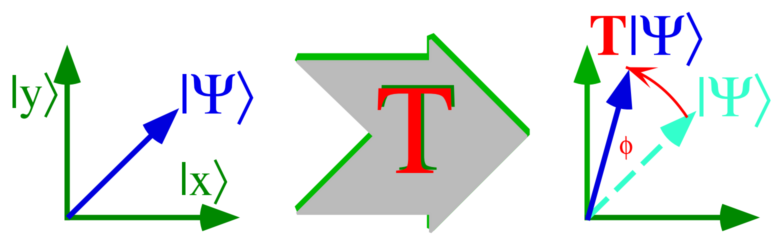

Fig. 2.2.3 Same basis but vector $|\Psi\rangle$ moves elsewhere. ("Active" or "Alibi" transformation)

Active transformation operations generally stand for analyzers or other parts of the physical space-time environment. Representations of an active transformation $\mathbf{T}|\Psi\rangle$ are made by attaching to it the bras and kets for whichever basis you want to use, say in this case, the original $\{|x\rangle,|y\rangle\}$.

$$\begin{aligned}\langle x|\mathbf{T}|\Psi\rangle &= \langle x|\mathbf{T}|x\rangle\langle x|\Psi\rangle + \langle x|\mathbf{T}|y\rangle\langle y|\Psi\rangle \\ \langle y|\mathbf{T}|\Psi\rangle &= \langle y|\mathbf{T}|x\rangle\langle x|\Psi\rangle + \langle y|\mathbf{T}|y\rangle\langle y|\Psi\rangle\end{aligned} \quad (2.2.9a)$$

The matrix form for this active transformation is

$$\begin{pmatrix}\langle x|\mathbf{T}|\Psi\rangle \\ \langle y|\mathbf{T}|\Psi\rangle\end{pmatrix} = \begin{pmatrix}\langle x|\Psi(T)\rangle \\ \langle y|\Psi(T)\rangle\end{pmatrix} = \begin{pmatrix}\langle x|\mathbf{T}|x\rangle & \langle x|\mathbf{T}|y\rangle \\ \langle y|\mathbf{T}|x\rangle & \langle y|\mathbf{T}|y\rangle\end{pmatrix}\begin{pmatrix}\langle x|\Psi\rangle \\ \langle y|\Psi\rangle\end{pmatrix} = \begin{pmatrix}\cos\phi & -\sin\phi \\ \sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}\langle x|\Psi\rangle \\ \langle y|\Psi\rangle\end{pmatrix} \quad (2.2.9b)$$

Polarization devices that do this operation and many more will be discussed later.

(c) Group axioms

The axioms of *group theory* are mathematical axioms set down by Everiste Galois shortly before he was killed in a duel at the age of 21. They are closely related to the axioms of quantum theory which we have stated so far. Groups play a key role in the development of quantum theory, particularly at its advanced levels.

A *group* G is a set of operations $G=\{a, b, c, ...\}$ or *elements* that can be combined in group products to give other operations in the same set. Examples are rotations or permutations, the latter of which occupied Galois' attention. Below are listed the axioms which all groups must satisfy. (The first we've already stated.)

Each is discussed as to its relevance to the set U={**A**, **B**, **C**, ...} of all unitary transformation matrices which satisfy the quantum axioms 1-4 for an n-state system. As will be seen, U is a group which is labeled the *n-dimensional unitary group U(n)*.

(1) The closure axiom

Products ab = c are defined between any two group elements a and b, and the result c is contained in the group.

Products **A B=C** of transformation operators are defined by their matrix representations as are the operators themselves according to quantum axiom-4 (The closure or completeness axiom.) as explained in Sec. 2.1b(2). In Sec. 2.2(a) it was shown that all transformation matrices are unitary. Given $\mathbf{A}^\dagger\mathbf{A} = \mathbf{1}$ and $\mathbf{B}^\dagger\mathbf{B} = \mathbf{1}$ we must prove that the product **A B=C** is also unitary. Inserting $\mathbf{A}^\dagger\mathbf{A} = \mathbf{1}$ between $\mathbf{B}^\dagger$ and **B** gives

$$\mathbf{B}^\dagger\mathbf{A}^\dagger\mathbf{A}\,\mathbf{B} = \mathbf{B}^\dagger\mathbf{B} = \mathbf{1} \tag{2.2.10a}$$

or

$$\mathbf{C}^\dagger\mathbf{C} = \mathbf{1} \tag{2.2.10b}$$

where showing

$$\mathbf{C}^\dagger = (\mathbf{A}\,\mathbf{B})^\dagger = \mathbf{B}^\dagger\mathbf{A}^\dagger \tag{2.2.10c}$$

is left as an exercise.

(2) The associativity axiom

Products (ab)c and a(bc) are equal for all elements a, b, and c in the group .

Associativity is automatically guaranteed for matrix products on which the algebraic significance of quantum axiom-4 is based.(You should be able to prove this even if it's not a familiar axiom.)

(3) The identity axiom

There is a unique element 1 (the identity) such that 1·a = a = a·1 for all elements a in the group ..

According to the quantum axiom-2 (the orthonormality or identity axiom) there is a unique unit transformation matrix $\langle j|k\rangle = \delta_{jk}$ that does nothing. It represents the perfect "do-nothing" analyzer. We indicate the corresponding abstract operator by **1** in the abstract completeness relation (2.1.20).

(4) The inverse axiom

For all elements a in the group there is an inverse element a^{-1} *such that* $a^{-1}a = 1 = a \cdot a^{-1}$.

This seems to follow since we already know that the Hermitian conjugate $\mathbf{A}^\dagger$ of any operator **A** is also its inverse $\mathbf{A}^\dagger=\mathbf{A}^{-1}$. (Recall (2.2.5b).) There is, however, a catch. For infinite state systems one cannot guarantee that $\mathbf{A}^\dagger\mathbf{A} = \mathbf{1}$ also implies $\mathbf{A}\mathbf{A}^\dagger = \mathbf{1}$, or vice-versa. However, it does follow for finite systems. (See exercises.) In any case, for a transformation matrix satisfying quantum axioms 1-4, even with n=∞, it can be proved that all operators commute with their conjugates and inverses.

This is true in spite of the fact that unitary group operators do not generally commute with each other.

$$\mathbf{AB} \neq \mathbf{BA} \textit{ (for some } \mathbf{A},\mathbf{B} \textit{ in } U(n) \textit{ for } n\geq 1)$$

Groups that satisfy a fifth postulate of commutivity are called *Abelian* groups.

(5) The commutative axiom (Abelian groups only)

All elements a in an Abelian group are mutually commuting: a·b = b·a.

Only the one-dimensional unitary group

$$U(1) = \{1,..., e^{i\alpha}, ...\} \text{ where: } (-\pi < \alpha \leq \pi) \qquad (2.2.11)$$

is Abelian. It consists of all possible phase factors for a single-state (1D) system, and these numbers obviously commute and form a group labeled by a single real parameter which is the phase angle α.

(d) U(n) group dimension

The group $U(n)$ is the set of all *n-by-n* transformation matrices having components $U_{ij} = T_{ij}$ each of which satisfy n^2 unitarity equations of the form

$$\sum_{j=1}^{n} U^\dagger_{ij} U_{jk} = \left(U^\dagger U\right)_{ik} = (\mathbf{1})_{ik} = \delta_{ik} = \sum_{j=1}^{n} U^*_{ji} U_{jk} \qquad (2.2.12)$$

The n^2 components U_{ij} are generally complex numbers amounting to $2n^2$ real parameters. So the number of independent real parameters to label $U(n)$ operators is $2n^2 - n^2 = n^2$ and is called the unitary *group dimension*, the number of *quantum coordinates*. A 2-state system's U(2) dimension is $2^2=4$.

(e) SU(n) group dimension

For most of quantum theory the over-all phase of a system is unmeasureable and ignorable. To avoid dealing with such a phase one generally restricts attention to matrices of $U(n)$ of unit determinant.

$$\det\ |U| = 1 \qquad (2.2.13)$$

Such operators form a subgroup of $U(n)$ called the *special unimodular group SU(n)*. Because, of equation (2.2.13) the $SU(n)$ dimension is n^2 -1, one less than that of $U(n)$. For $SU(2)$, which we study first, this dimension is $2^2 - 1 = 3$, so the number of 2-state quantum coordinates is just three. ($2^2 - 1 = 3$)

Problems for Ch. 2

Daggers

2.1.1. Prove identities for the following. Use representations to help with abstract cases.

(a) $(\mathbf{AB})^\dagger =$? (In terms of $(\mathbf{A})^\dagger$ and $(\mathbf{B})^\dagger$.)

(b) $(\mathbf{A}|\psi\rangle)^\dagger =$? (In terms of $(\mathbf{A})^\dagger$ and $(|\psi\rangle)^\dagger$ =?.)

(c) $(\langle\phi|\psi\rangle)^\dagger =$? (In terms of $\langle\phi|\psi\rangle^*$=?.)

(d) $(\langle\phi|\mathbf{A}|\psi\rangle)^\dagger =$? (In terms of $(\mathbf{A})^\dagger$ and $(|\psi\rangle)^\dagger$ =?, etc..)

(e) $(\mathbf{ABC})^\dagger =$? $(\mathbf{ABC})^{-1} =$?

(f) $(\langle\phi|\mathbf{ABC}|\psi\rangle)^\dagger =$?

(g) $(|\phi\rangle\langle\psi|)^\dagger =$?

Transforming Backwards and Forwards

2.2.1. Suppose the following basic definition of a transformation $\mathbf{T}$ from a basis $\{|1\rangle,|2\rangle,|3\rangle\}$ to another basis $\{|1'\rangle,|2'\rangle,|3'\rangle\}$:

$$|1'\rangle = \mathbf{T}|1\rangle = (\,|1\rangle - |2\rangle\,)/\sqrt{2},\ \ |2'\rangle = \mathbf{T}|2\rangle = (\,|1\rangle + |2\rangle\,)/\sqrt{2},\ \ |3'\rangle = \mathbf{T}|3\rangle = |3\rangle/i,\ \ (i=e^{\pi i/2})$$

(a) Construct the $3x3$ matrix representation of $\mathbf{T}$ and of $\mathbf{T}^\dagger$ in the basis $\{|1\rangle,|2\rangle,|3\rangle\}$.

(b) Construct the $3x3$ matrix representation of $\mathbf{T}$ and of $\mathbf{T}^\dagger$ in the basis $\{|1'\rangle,|2'\rangle,|3'\rangle\}$.

(c) Write in <u>matrix form</u> a change-of-basis transformation for prime representation of a ket $|\psi\rangle$, that is, $\{\langle 1'|\psi\rangle,\langle 2'|\psi\rangle,\langle 3'|\psi\rangle\}$, in terms of its original representation $\{\langle 1|\psi\rangle,\langle 2|\psi\rangle,\langle 3|\psi\rangle\}$.

(c)† Write in <u>matrix form</u> a change-of-basis transformation for prime representation of a bra $\langle\psi|$, that is, $\{\langle\psi|1'\rangle,\langle\psi|2'\rangle,\langle\psi|3'\rangle\}$, in terms of its original representation $\{\langle\psi|1\rangle,\langle\psi|2\rangle,\langle\psi|3\rangle\}$.

(d) Write in <u>matrix form</u> a change-of-basis transformation for prime representation of operator $\mathbf{U}$, that is, $\{\langle 1'|\mathbf{U}|1'\rangle,\langle 1'|\mathbf{U}|2'\rangle,..\}$, in terms of its original representation $\{\langle 1|\mathbf{U}|1\rangle,\langle 1|\mathbf{U}|2\rangle,..\}$.

(e) Are any of the (a-b) results..Unitary matrices? .. Hermitean matrices? ..Orthogonal matrices?

(f) Are any of the matrices from (a) and equal to those from (b)? Which, if any, of the (a)-(b) equalities are a general result? Why or why not? (Prove or give a counter example.)

Mirror-Mirror

2.2.2. A clothing store lets you examine your new suit in a device that consists of two vertical planar mirrors. Mirror X extends along the x-axis. Mirror Φ extends along an axis that is rotated counter clockwise by angle ϕ around the vertical hinge that forms the intersection of the two mirrors at (x,y)=(0,0). You stand somewhere between the two mirrors and try various ϕ while looking at any of several images of your necktie (or necklace) which is located where you're standing at $\mathbf{n}$=(x,y). (Neglect vertical z-axis.) *Start from basic (basis-vector) definitions <u>only</u> to work the following questions. Deriving amplitudes directly will be marked down*.

(a) Represent the transformation $\mathbf{T}$(X) that describes reflections by mirror X in xy-basis. Compute its effect on necktie point **n**. Sketch top view of this mapping.

(b) Represent the transformation $\mathbf{T}(\Phi)$ that describes reflections by mirror Φ in xy-basis. Compute its effect on necktie point **n**. Sketch top view of this mapping.

(c) Represent the transformation $\mathbf{T}(\Phi X)$ that describes reflections by mirror X followed by mirror Φ in the xy-basis. Compute its effect on necktie point **n**. Sketch top view of this mapping. What familiar operation is this? Express as group product.

(d) Represent the transformation $\mathbf{T}(X\Phi)$ that describes reflections by mirror Φ followed by mirror X in the xy-basis. Compute its effect on necktie point **n**. Sketch top view of this mapping. What familiar operation is this? Express as group product. Do $\mathbf{T}$(X) and $\mathbf{T}(\Phi)$ ever commute?

(e) Calculate the determinants and trace of each of the resulting operations (a) thru (d). Comment on the physical or geometric significance, if any, of these numbers.