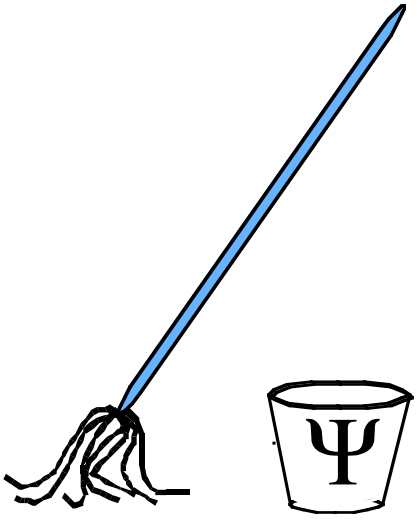

QM for AMOP

Chapter 8
Fourier Symmetry Analysis

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CHAPTER 8. FOURIER SYMMETRY ANALYSIS 3

Fourier analysis is most useful when there is a symmetry G in which all the coordinate points are indistinguishable. For an unbounded x-continuum, G is an infinite translational symmetry group labeled T. For a bounded x_p-ring of "quantum dots" the symmetry G is an N-cyclic rotation group labeled C_N. In Chapter 8 a fictitious hexagonal beam analyzer with C_6 symmetry is considered. The transfer matrix eigensolutions of such a device are found using a modern form of Fourier analysis known as group representation theory or symmetry analysis, one of the most powerful tools in quantum theory. The symmetry of the bounded Bohr x-ring continuum is also discussed.

Chapter 8. Fourier Symmetry Analysis

From where do the wavefunctions like $\Psi = e^{i(kx - \omega t)}$ come? One answer to this involves the concept of *symmetry analysis* and *group representation theory*. These sound like big names for what is still regarded as a pretty scary mathematical subject. However, the basic ideas of this powerful tool are actually quite simple as we hope to show now. Most of the needed algebraic work has been done in Ch. 3 regarding spectral decomposition. The physical ideas of Fourier analysis and Bohr ring waves are in Ch. 7. Symmetry group representation theory is really just a beautiful generalization of *Fourier analysis* that gives eigensolutions of "difficult" operators using simple properties of commuting symmetry operators.

8.1. Introducing Cyclic Symmetry: A C_6 example

A ring of quantum dots was introduced in Section 7.3 as a model for finite Fourier analysis. The Fourier tranformation matrix was discussed with examples for *N=1, 2, 3, 4, 5,* and *6*. The idea of cyclic symmetry C_N was broached as a property of the matrices in Fig. 7.3.3 and Fig. 7.3.5. Here that idea is put on a more solid footing.

(a) Cyclic symmetry C_N: A 6-quantum-dot analyzer

Suppose someone invents some beam analyzer that takes an *N*-state beam and sorts it into *N* beams arrayed around a circular device as imagined in Fig. 8.1.1 for *N=6*. Let each beam path entering the device contain particles in one of *N* states $\{|0\rangle, |1\rangle, |2\rangle, \ldots, |N\text{-}1\rangle\}$ after which the device does things which causes the beams to interfere or be otherwise modified before recombining and counting.

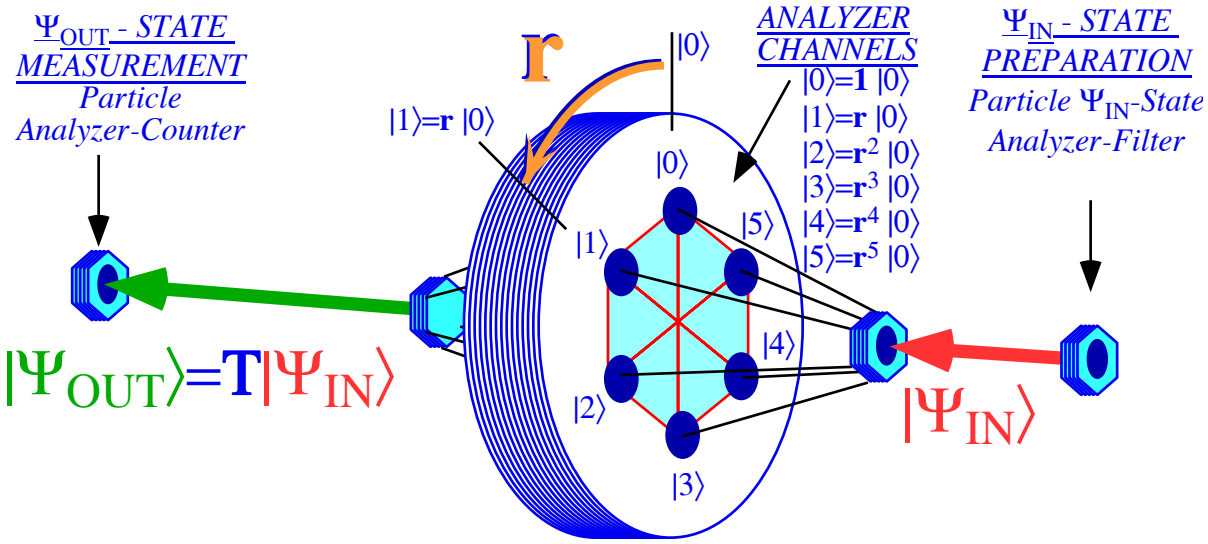


Fig. 8.1.1 Generic N-state (C_N) beam analyzer experiment with (N = 6) channels

We are intentionally being vague about the nature of the states. (After all, this device hasn't even been invented yet!) Let us just say they are some kind of hyper-polarization states. (Put a prefix like 'hyper' on something ordinary and people stop asking questions.) The point is that by just knowing the symmetry of a device it is possible to work out a lot of the quantum mechanics without knowing so much of the underlying

details. It is a lot like the photon polarization and electron spin problems discussed in Chapter 1. Electron and photon "spin" are physically quite different but use much of the same mathematical theory.

By *symmetry*, we mean any operators **r**, $\mathbf{r}^2$,.. that do not alter the analyzer experiment no matter how many times you apply them. In particular, suppose a *60°* rotational operator **r** indicated in Fig. 8.1.1. could be done some night by the lab janitor, so when the physicists show up the next morning all their experiments work the same as the day before.

However, it is important to state what we mean the janitor's **r**-operation to do. He could just rotate the whole lab building by *60°*. That, indeed, is a symmetry, but not one we will discuss until later. Besides, a rotation like that happens every four hours as the Earth turns; no janitor needed! This is called the *symmetry of isotropy of space*. It is a *continuous* or *Lie symmetry* for which *60°* has no special significance.

Instead, what we have in mind for the janitor to do is rotate just the analyzer in the center of Fig. 8.1.1 by *60°* as indicated in the figure. Well, that analyzer looks pretty heavy, so, instead we'll ask that the janitor just rotate the little input source and the little output counter both by minus *60°*, which is operation $\mathbf{r}^{-1}=\mathbf{r}^5$. This does the same as a whole-Earth/lab rotation by *-60°* (which no one detects) followed by a positive *60°* rotation of the big analyzer to "upright" leaving input and output devices behind at *-60°*.

It is important to understand that all transformations are *relative transformations*; something gets moved or mapped relative to something else. You've probably heard it quoted, "Everything's relative!" Well, that's often garbage, but here it isn't. Rotations, Lorentz transformations, and our analyzer operators **T** (Recall Fig. 1.6.1), and **r** in Fig. 8.1.1 are all mappings of one vector or thing relative to another.

By the way, our helpful suggestion to the janitor won't help much if the input and output devices are big analyzers, too. It was noted in Chapter 1 that filters and counters are analyzers set in certain ways. But, the analyzer in Fig. 8.1.1 is a more powerful one than heretofore discussed. (And, isn't better always bigger?) So let's assume that the janitor can easily do $\mathbf{r}^{-1} = \mathbf{r}^5$ to the smaller input and output devices whose in and out states are written as follows in Dirac notation,

$$|\Psi_{OUT\,(r^{-1})}\rangle = \mathbf{r}^{-1}|\Psi_{OUT}\rangle \ , \qquad |\Psi_{IN\,(r^{-1})}\rangle = \mathbf{r}^{-1}\,|\Psi_{IN}\rangle \ . \qquad (8.1.1)$$

Symmetry of the transformation operator **T** means it does exactly the same relative thing to any state $|\Psi_{IN}\rangle$ as it does to the janitor-rotated state $|\Psi_{IN\,(r^{-1})}\rangle$, that is

$$|\Psi_{OUT}\rangle = \mathbf{T}\,|\Psi_{IN}\rangle \qquad \text{implies:} \qquad |\Psi_{OUT\,(r^{-1})}\rangle = \mathbf{T}\,|\Psi_{IN\,(r^{-1})}\rangle \qquad (8.1.2a)$$

or

$$\mathbf{r}^{-1}|\Psi_{OUT}\rangle = \mathbf{T}\,\mathbf{r}^{-1}|\Psi_{IN}\rangle \qquad (8.1.2b)$$

$$|\Psi_{OUT}\rangle = \mathbf{r}\,\mathbf{T}\,\mathbf{r}^{-1}|\Psi_{IN}\rangle \qquad (8.1.2c)$$

If this is true for all input states $|\Psi_{IN}\rangle$ then it follows that effect of analyzer operator **T** in (8.1.2a) and in (8.1.2c) are indistinguishable, or **T** is *invariant* to **r**

$$\mathbf{T} = \mathbf{r}\,\mathbf{T}\,\mathbf{r}^{-1} \quad \text{or:} \quad \mathbf{r}^{-1}\mathbf{T}\,\mathbf{r} = \mathbf{T} \qquad (8.1.2d)$$

or, that **r** *commutes* with **T**; the latter being the most common way to say that **T** *has* **r**-*symmetry*.

$$\mathbf{T}\,\mathbf{r} = \mathbf{r}\,\mathbf{T} \qquad (8.1.2e)$$

All the above parts of equation (8.1.2) are really the same requirement for **r**-*symmetry of* **T**.

Note: This is not the same as just multiplying both sides of $|\Psi_{OUT}\rangle = \mathbf{T}\,|\Psi_{IN}\rangle$ by $\mathbf{r}$ or $\mathbf{r}^{-1}$ which just gives a whole-Earth/lab rotation, that is, operate with $\mathbf{r}^{-1}$ and insert the identity ($\mathbf{r}\,\mathbf{r}^{-1}=\mathbf{1}$) to get

$$\mathbf{r}^{-1}\,|\Psi_{OUT}\rangle = \mathbf{r}^{-1}\,\mathbf{T}\,|\Psi_{IN}\rangle = \mathbf{r}^{-1}\,\mathbf{T}\,\mathbf{r}\,\mathbf{r}^{-1}\,|\Psi_{IN}\rangle\,. \qquad (8.1.3a)$$

This reduces to an expression *similar* to the original $|\Psi_{OUT}\rangle = \mathbf{T}\,|\Psi_{IN}\rangle$

$$|\Psi_{OUT\,(r^{-1})}\rangle = \mathbf{r}^{-1}\,\mathbf{T}\,|\Psi_{IN}\rangle = \mathbf{r}^{-1}\,\mathbf{T}\,\mathbf{r}\,|\Psi_{IN\,(r^{-1})}\rangle = \mathbf{T}_{(r^{-1})}\,|\Psi_{IN\,(r^{-1})}\rangle \qquad (8.1.3b)$$

where $\mathbf{T}_{(r^{-1})}$ is a *similarity transformation* $\mathbf{r}^{-1}\mathbf{T}\,\mathbf{r}$ of $\mathbf{T}$. (This is an active transformation; devices move.)

$$\mathbf{T}_{(r^{-1})} = \mathbf{r}^{-1}\,\mathbf{T}\,\mathbf{r} \qquad (8.1.3c)$$

These relations hold true for any analyzer operator $\mathbf{T}$ whether it has symmetry or not.

For $\mathbf{T}$ to have $\mathbf{r}$-symmetry it is necessary that the similarity transformation leaves $\mathbf{T}$ unchanged or invariant ($\mathbf{T}_{(r^{-1})} = \mathbf{T}$), as in (8.1.2d).To recap

An analyzer has $\mathbf{r}$*-symmetry if and only if its operator* $\mathbf{T}$ *commutes with* $\mathbf{r}$, *that is* ($\mathbf{T}\,\mathbf{r} = \mathbf{r}\,\mathbf{T}$).

(b) C_N Symmetry groups and representations

Now, the janitor, having fooled the physicists once, does it again the next night, by rotating by $\mathbf{r}$ one more time giving the same positions as if $\mathbf{r}^2$ had been done the first night. Then a combination of $\mathbf{r}^2$ and $\mathbf{r}^3$ is tried. (This just gives $\mathbf{r}^{-1} = \mathbf{r}^5$ the inverse of which was tried on the first night.) All of these products are symmetries if the factors are. (So the physicists end up getting fooled night after night for almost a week of different positions! Saturday, they have to take off since they read right-to-left.)

If operators $\mathbf{a}$ and $\mathbf{b}$ commute with an analyzer $\mathbf{T}$-matrix then so do all their products

$$\text{If: } \mathbf{aT}=\mathbf{Ta} \quad \text{and} \quad \mathbf{bT}=\mathbf{T\,b} \quad \text{then} \quad \mathbf{abT}=\mathbf{Tab} \quad \text{and} \quad \mathbf{baT}=\mathbf{T\,ba} \qquad (8.1.4a)$$

$$\text{and inverses. If: } \mathbf{aT}=\mathbf{Ta} \quad \text{then} \quad \mathbf{a}^{-1}\mathbf{T}=\mathbf{Ta}^{-1} \qquad (8.1.4b)$$

This shows that the set of unitary operators that commute with a particular $\mathbf{T}$-operator must satisfy the group axioms (1-4) stated in Sec. 2.2. This set is called a *symmetry group* $G=\{\,\mathbf{a}\,,\,\mathbf{b}\,,\,\mathbf{c}\,,...,\,\mathbf{g}\,,..\}$ of the operator $\mathbf{T}$. We are supposing that the analyzer matrix $\mathbf{T}$ associated with the experiment in Fig. 8.1.1 has an N-cyclic symmetry group $C_6 = \{\,\mathbf{1}\,,\,\mathbf{r}\,,\,\mathbf{r}^2\,,\,\mathbf{r}^3,\,\mathbf{r}^4,\,\mathbf{r}^5\}$ of six ($N=6$) operators that have the following *group multiplication table*. We put the inverses of the first column in the top row so $\mathbf{1}$ is on the diagonal.

C_6	$\mathbf{1}$	$\mathbf{r}^5$	$\mathbf{r}^4$	$\mathbf{r}^3$	$\mathbf{r}^2$	$\mathbf{r}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^5$	$\mathbf{r}^4$	$\mathbf{r}^3$	$\mathbf{r}^2$	$\mathbf{r}$
$\mathbf{r}$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^5$	$\mathbf{r}^4$	$\mathbf{r}^3$	$\mathbf{r}^2$
$\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^5$	$\mathbf{r}^4$	$\mathbf{r}^3$
$\mathbf{r}^3$	$\mathbf{r}^3$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^5$	$\mathbf{r}^4$
$\mathbf{r}^4$	$\mathbf{r}^4$	$\mathbf{r}^3$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^5$
$\mathbf{r}^5$	$\mathbf{r}^5$	$\mathbf{r}^4$	$\mathbf{r}^3$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$

(8.1.5a)

$$\mathbf{1}=\begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix} \qquad (8.1.5b)$$

Think of the table as a matrix in a basis $\{|0\rangle|1\rangle|2\rangle|3\rangle|4\rangle|5\rangle\}$ defined by operators $\{\mathbf{1},\mathbf{r},\mathbf{r}^2,\mathbf{r}^3,\mathbf{r}^4,\mathbf{r}^5\}$.

This makes a matrix representation for each operator using the channel states as a basis by simply replacing each operator's table entry by a "1" in that position of its matrix and "0" or "dot" (·) elsewhere.

$$\mathbf{r}=\begin{array}{c}\begin{matrix}|0\rangle&|1\rangle&|2\rangle&|3\rangle&|4\rangle&|5\rangle\end{matrix}\\ \begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&1\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix}\end{array},\ \mathbf{r}^2=\begin{array}{c}\begin{matrix}|0\rangle&|1\rangle&|2\rangle&|3\rangle&|4\rangle&|5\rangle\end{matrix}\\ \begin{pmatrix}\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix}\end{array},\ \mathbf{r}^3=\begin{array}{c}\begin{matrix}|0\rangle&|1\rangle&|2\rangle&|3\rangle&|4\rangle&|5\rangle\end{matrix}\\ \begin{pmatrix}\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\end{pmatrix}\end{array},\ \mathbf{r}^4=\begin{array}{c}\begin{matrix}|0\rangle&|1\rangle&|2\rangle&|3\rangle&|4\rangle&|5\rangle\end{matrix}\\ \begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix}\end{array},\ \mathbf{r}^5=\begin{array}{c}\begin{matrix}|0\rangle&|1\rangle&|2\rangle&|3\rangle&|4\rangle&|5\rangle\end{matrix}\\ \begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}\end{array}$$

(8.1.5c)

These are sometimes called the *regular permutation representations* because they permute each of the *p*-position states. The first column of matrix $\mathbf{r}^{p-1}$ represents the basic ket definition $|p\rangle = \mathbf{r}^{p-1}|1\rangle$ as follows.

$$|0\rangle = \mathbf{1}|0\rangle\ ,\ |1\rangle = \mathbf{r}|0\rangle\ ,\ |2\rangle = \mathbf{r}^2|0\rangle\ ,\ |3\rangle = \mathbf{r}^3|0\rangle\ ,\ |4\rangle = \mathbf{r}^4|0\rangle\ ,\ |5\rangle = \mathbf{r}^5|0\rangle \qquad (8.1.6a)$$

The **r**-transform is unitary $\mathbf{r}^\dagger=\mathbf{r}^{-1}$. The Hermitian conjugate of these relations is the basic bra definition.

$$\langle 0| = \langle 0|\mathbf{1}\ ,\ \langle 1| = \langle 0|\mathbf{r}^{-1}\ ,\ \langle 2| = \langle 0|\mathbf{r}^{-2}\ ,\ \langle 3| = \langle 0|\mathbf{r}^{-3}\ ,\ \langle 4| = \langle 0|\mathbf{r}^{-4}\ ,\ \langle 5| = \langle 0|\mathbf{r}^{-5} \qquad (8.1.6b)$$

These defintions may be summed up by defining a *representation matrix* $\mathsf{R}(\mathbf{g})$ with components $\mathsf{R}_{pq}(\mathbf{g})$.

$$\mathsf{R}_{pq}(\mathbf{g})= \langle p|\mathbf{g}|q\rangle \qquad (8.1.6c)$$

(c) So what's a group representation?

To use a more "kosher" mathematical language we should say that the representation matrices in (8.1.5b-c) are functions $\mathsf{R}(\mathbf{g})$ of the group $\mathsf{G}=\{\mathbf{1},\mathbf{g}_1,\mathbf{g}_2,\ldots\}=C_6=\{\mathbf{1},\mathbf{r},\mathbf{r}^2,\mathbf{r}^3,\mathbf{r}^4,\mathbf{r}^5\}$. That is, every group operator gets mapped onto a matrix so that the matrix $\mathsf{R}(\mathbf{g}_1\mathbf{g}_2)$ of a group product $\mathbf{g}_1\mathbf{g}_2$ is the matrix product $\mathsf{R}(\mathbf{g}_1)\bullet\ \mathsf{R}(\mathbf{g}_2)$ of the factors.

$$\mathsf{R}(\mathbf{g}_1)\bullet\ \mathsf{R}(\mathbf{g}_2) = \mathsf{R}(\mathbf{g}_1\bullet\mathbf{g}_2) \qquad (8.1.7a)$$

Stated simply, "*The product of representations must equal the representation of the product.*" The matrices in (8.1.5b-c) must obey the group multiplication table (8.1.5a)! It is easy to see that the first matrix (8.1.5b) satisfies this requirement trivially.

$$\mathsf{R}(\mathbf{1})\bullet\ \mathsf{R}(\mathbf{1}) = \mathsf{R}(\mathbf{1}\bullet\mathbf{1}) = \mathsf{R}(\mathbf{1}) \qquad (8.1.7b)$$

The remainder have to satisfy it because of definition (8.1.6) involve bras and kets which obey Axioms 1-4, that is, $\mathsf{R}(\mathbf{g})$ is a *unitary representation*. The conjugation axiom $(\langle p|q\rangle^*=\langle q|p\rangle)$ implies that the †-conugate $(\mathsf{R}^\dagger_{pq}=\mathsf{R}^*_{qp})$ of a representation must be the representation of the group inverse $\mathbf{r}^\dagger=\mathbf{r}^{-1}$.

$$\mathsf{R}_{pq}(\mathbf{g}^\dagger)= \langle p|\mathbf{g}^\dagger|q\rangle = (\langle q|\mathbf{g}|p\rangle)^*= (\mathsf{R}_{qp}(\mathbf{g}))^* \qquad (8.1.8a)$$

Stated more simply this is simply demanding operator unitarity from its representations.

$$\mathsf{R}^\dagger(\mathbf{g}) = \mathsf{R}(\mathbf{g}^\dagger) = \mathsf{R}(\mathbf{g}^{-1}) = \mathsf{R}^{-1}(\mathbf{g}) \qquad (8.1.8b)$$

All of the above are properties that are invariant to a change-of-basis transformation $\mathsf{U}^\dagger\mathsf{U}=\mathsf{1}$. Given $\mathsf{R}^U(\mathbf{g}) = \mathsf{U}\,\mathsf{R}(\mathbf{g})\mathsf{U}^\dagger$, it follows that the new R^U matrices also satisfy (8.1.7) thru (8.1.8). For example,

$$\mathsf{R}^U(\mathbf{g}_1)\mathsf{R}^U(\mathbf{g}_1) = \mathsf{U}\,\mathsf{R}(\mathbf{g}_1)\mathsf{U}^\dagger\mathsf{U}\,\mathsf{R}(\mathbf{g}_2)\mathsf{U}^\dagger= \mathsf{U}\,\mathsf{R}(\mathbf{g}_1)\mathsf{R}(\mathbf{g}_2)\mathsf{U}^\dagger = \mathsf{R}^U(\mathbf{g}_1\mathbf{g}_2) \qquad (8.1.9)$$

Now we discuss finding and applying the diagonalizong transformation or *d-tran* of $\mathsf{R}(\mathbf{g})$.

8.2 C_N Spectral Decomposition: Solving a C_6 transfer matrix

The main analyzer of Fig. 8.1.1 is supposed to have C_6 symmetry. However, it is also supposed to do some things that we haven't let single analyzers do to an incoming base state $|\Psi_{IN}\rangle=|p\rangle$, and that is, *mix it up*! No longer will a base state $|1\rangle$ or $|2\rangle$ just fly on through with nothing more than an extra phase attached, so it just comes out $e^{i\Omega_1}|1\rangle$ or $e^{i\Omega_2}|2\rangle$. From now on, each base state $|p\rangle$ is going to get treated to a full-blown transformation matrix T that is not necessarily diagonal. A general base state $|\Psi_{IN}\rangle$ will be output as $|\Psi_{OUT}\rangle$, as follows,

$$\begin{pmatrix} \langle 0|\Psi_{\rm OUT}\rangle \\ \langle 1|\Psi_{\rm OUT}\rangle \\ \langle 2|\Psi_{\rm OUT}\rangle \\ \langle 3|\Psi_{\rm OUT}\rangle \\ \langle 4|\Psi_{\rm OUT}\rangle \\ \langle 5|\Psi_{\rm OUT}\rangle \end{pmatrix} = \begin{pmatrix} T_{00} & T_{01} & T_{02} & T_{03} & T_{04} & T_{05} \\ T_{10} & T_{11} & T_{12} & T_{13} & T_{14} & T_{15} \\ T_{20} & T_{21} & T_{22} & T_{23} & T_{24} & T_{25} \\ T_{30} & T_{31} & T_{32} & T_{33} & T_{34} & T_{35} \\ T_{40} & T_{41} & T_{42} & T_{43} & T_{44} & T_{45} \\ T_{50} & T_{51} & T_{52} & T_{53} & T_{54} & T_{55} \end{pmatrix} \bullet \begin{pmatrix} \langle 0|\Psi_{\rm IN}\rangle \\ \langle 1|\Psi_{\rm IN}\rangle \\ \langle 2|\Psi_{\rm IN}\rangle \\ \langle 3|\Psi_{\rm IN}\rangle \\ \langle 4|\Psi_{\rm IN}\rangle \\ \langle 5|\Psi_{\rm IN}\rangle \end{pmatrix} \qquad (8.2.1a)$$

where off-diagonal *(p≠q)* matrix elements

$$T_{pq} = \langle p|\,\mathrm{T}\,|q\rangle \qquad (8.2.1b)$$

of **T** are not all zero if $|p\rangle$ and $|q\rangle$ do not belong to **T**'s "own" eigenbasis. (Bilingual redundancy, again.)

So, are we ready to diagonalize a general six-by-six matrix? No way, Jose'! But, here is where symmetry analysis rides to the rescue. If we can diagonalize the **r**-matrix in (8.1.5) then, barring appearance of nilpotents or other obnoxious gremlins, we may be able to also diagonalize the **T**-matrix (8.2.1). This is because (8.2.1) isn't just any old six-by-six matrix; it has C_6 symmetry and must therefore commute with each of its symmetry operators like **r**. Recall **T r = r T** in (8.1.2). This means that **T** and **r** share projectors $\mathbf{P}_k$ as shown in (3.1.37). Diagonalize **r** and you may have diagonalized **T** as well!

(a) Spectral decomposition of symmetry operators $\mathbf{r}^p$

The problem of analyzing (8.2.1) is then reduced to diagonalizing **r** in (8.1.5a), another six-by-six matrix, albeit a simpler one. But wait! No matrix need bother us. The minimal equation for **r** is simply

$$\mathbf{r}^N = \mathbf{1} \ (N=6, \text{ here.}) \qquad (8.2.2)$$

and all its eigenvalues are the roots of unity given before by (7.3.5) and displayed in Fig. 7.3.3.

$$\chi_m = \left(r_N\right)^m = \left(e^{-i2\pi/N}\right)^m = e^{-i2\pi\, m/N} \quad \text{where: } m = 0,1,2,\ldots,N-1 \qquad (8.2.3)$$

(Again, *N=6*). The spectral projectors of **r** follow easily. To help understand this recall that a spectral decomposition of any matrix **M** come with beautiful and powerful consequential relations. First, **M**'s *eigen-projector* $\mathbf{P}_k$ satisfies: $\mathbf{M}\mathbf{P}_k = \varepsilon_k \mathbf{P}_k$ and *orthonormality* $\mathbf{P}_j\mathbf{P}_k = \delta_{jk}\mathbf{P}_k$. Then there is *completeness*

$$\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_n \,. \qquad (3.1.15d)_{\text{repeated}}$$

and *spectral decomposition of operator* **M**, and *functional spectral decomposition of an operator* **M** .

$$\mathbf{M} = \varepsilon_1 \mathbf{P}_1 + \varepsilon_2 \mathbf{P}_2 + \ldots + \varepsilon_n \mathbf{P}_n \qquad (3.1.15e)_{\text{repeated}}$$

$$\mathrm{f}(\mathbf{M}) = \mathrm{f}(\varepsilon_1)\mathbf{P}_1 + \mathrm{f}(\varepsilon_2)\mathbf{P}_2 + \ldots + \mathrm{f}(\varepsilon_n)\mathbf{P}_n \qquad (3.1.17)_{\text{repeated}}$$

Applying the spectral relations using the eigenvalues (roots) in (8.2.3) gives a functional (power) spectral decomposition *(3.1.17)repeated* of all powers $\mathbf{r}^p$ of rotation operator $\mathbf{r}$ by putting $(\chi_m)^p$ before each $\mathbf{P}^m$.

$$\begin{aligned}
\mathbf{1} &= \mathbf{P}^0 + \mathbf{P}^1 + \mathbf{P}^2 + \mathbf{P}^3 + \mathbf{P}^4 + \mathbf{P}^5 \\
\mathbf{r} &= \mathbf{P}^0 + \chi_1\mathbf{P}^1 + \chi_2\mathbf{P}^2 + \chi_3\mathbf{P}^3 + \chi_4\mathbf{P}^4 + \chi_5\mathbf{P}^5 \\
\mathbf{r}^2 &= \mathbf{P}^0 + \chi_1^2\mathbf{P}^1 + \chi_2^2\mathbf{P}^2 + \chi_3^2\mathbf{P}^3 + \chi_4^2\mathbf{P}^4 + \chi_5^2\mathbf{P}^5 \\
\mathbf{r}^3 &= \mathbf{P}^0 + \chi_1^3\mathbf{P}^1 + \chi_2^3\mathbf{P}^2 + \chi_3^3\mathbf{P}^3 + \chi_4^3\mathbf{P}^4 + \chi_5^3\mathbf{P}^5 \\
\mathbf{r}^4 &= \mathbf{P}^0 + \chi_1^4\mathbf{P}^1 + \chi_2^4\mathbf{P}^2 + \chi_3^4\mathbf{P}^3 + \chi_4^4\mathbf{P}^4 + \chi_5^4\mathbf{P}^5 \\
\mathbf{r}^5 &= \mathbf{P}^0 + \chi_1^5\mathbf{P}^1 + \chi_2^5\mathbf{P}^2 + \chi_3^5\mathbf{P}^3 + \chi_4^5\mathbf{P}^4 + \chi_5^5\mathbf{P}^5
\end{aligned} \quad \text{where: } \chi_m^p = \left(\chi_m\right)^p = e^{-i2\pi(mp)/N} \qquad (8.2.4a)$$

Apart from the normalization, the $\mathbf{P}^m$-to-$\mathbf{r}^p$ relation above is a unitary linear combination having the same Fourier transformation coefficients $\langle k_m|x_p\rangle$ as (7.3.10a). The inverse $\mathbf{r}^p$-to-$\mathbf{P}^m$ relation is obtained by transpose conjugating the coefficients $\chi_m{}^p$ above to give coefficients just like $\langle x_p|k_m\rangle$ in (7.3.10b).

$$(\chi_m{}^p)^* = \sqrt{N}\langle k_m|x_p\rangle^* = \sqrt{N}\langle x_p|k_m\rangle = e^{i2\pi(mp)/N} = \rho_p{}^m \qquad (8.2.4b)$$

Then divide all by the norm $N=6$ to make the following idempotent projectors.

$$\begin{aligned}
\mathbf{P}^0 &= \left(\mathbf{1} + \mathbf{r} + \mathbf{r}^2 + \mathbf{r}^3 + \mathbf{r}^4 + \mathbf{r}^5\right)/6 \\
\mathbf{P}^1 &= \left(\mathbf{1} + \rho_1\mathbf{r} + \rho_2\mathbf{r}^2 + \rho_3\mathbf{r}^3 + \rho_4\mathbf{r}^4 + \rho_5\mathbf{r}^5\right)/6 \\
\mathbf{P}^2 &= \left(\mathbf{1} + \rho_1^2\mathbf{r} + \rho_2^2\mathbf{r}^2 + \rho_3^2\mathbf{r}^3 + \rho_4^2\mathbf{r}^4 + \rho_5^2\mathbf{r}^5\right)/6 \\
\mathbf{P}^3 &= \left(\mathbf{1} + \rho_1^3\mathbf{r} + \rho_2^3\mathbf{r}^2 + \rho_3^3\mathbf{r}^3 + \rho_4^3\mathbf{r}^4 + \rho_5^3\mathbf{r}^5\right)/6 \\
\mathbf{P}^4 &= \left(\mathbf{1} + \rho_1^4\mathbf{r} + \rho_2^4\mathbf{r}^2 + \rho_3^4\mathbf{r}^3 + \rho_4^4\mathbf{r}^4 + \rho_5^4\mathbf{r}^5\right)/6 \\
\mathbf{P}^5 &= \left(\mathbf{1} + \rho_1^5\mathbf{r} + \rho_2^5\mathbf{r}^2 + \rho_3^5\mathbf{r}^3 + \rho_4^5\mathbf{r}^4 + \rho_5^5\mathbf{r}^5\right)/6
\end{aligned} \quad \text{where: } \rho_p^m = \chi_m^{*p} = e^{i2\pi(pm)/N} \qquad (8.2.4c)$$

Operating on the first position state with these projectors gives the desired eigenstates of the **T**-matrix. The norm is $\langle 1|\mathbf{P}^m|1\rangle = 1/N$. (Recall $(3.1.13)_{\text{example}}$) Its root $1/\sqrt{N}$ results to give normalized eigenkets.

$$|k_m\rangle = \mathbf{P}^m|0\rangle\sqrt{N} = \sum_{p=0}^{N-1}\rho_p^m\mathbf{r}^p|0\rangle\sqrt{N}\,/\,N = \sum_{p=0}^{N-1}e^{i2\pi(pm)/N}|p\rangle/\sqrt{N} \qquad (8.2.5a)$$

The inverse ket relations give position states $|x_p\rangle=|p\rangle$ in terms of wave $|k_m\rangle$ eigenkets.

$$|p\rangle = \mathbf{r}^p|0\rangle = \sum_{p=0}^{N-1}\chi_p^p\mathbf{P}^m|0\rangle\sqrt{N} = \sum_{p=0}^{N-1}e^{-i2\pi(mp)/N}|k_m\rangle/\sqrt{N} \qquad (8.2.5b)$$

The preceding ket relations (8.2.5) and their operator equivalents (8.2.4) are the discrete-N Fourier transformations whose N-*by*-N transformation matrices are pictured for $N=1, 2, 3, 4, 5,$ and 6 in Fig. 7.3.3 and for $N=24$ in Fig. 7.3.5. The physical transformation is between N "quantum-dot" position point $|p\rangle$-states *(*$|x_p\rangle=|p\rangle$*)* and their N quantum momentum Fourier-wave $|k_m\rangle$-states. Much of the above is mathematical "legalese" which gets short-circuited in the calculations that are described next.

(b) Writing transfer operator T in terms of symmetry operators $\mathbf{r}^p$

In order for main analyzer **T**-matrix (8.2.1) to have C_N symmetry, it must commute with all the rotation operator **r**-matrices in (2.7.5). **T** does this by being a linear combination of $\mathbf{r}^p$ as follows.

$$\mathbf{T} = A\,\mathbf{1} + B\,\mathbf{r} + C\,\mathbf{r}^2 + D\,\mathbf{r}^3 + C'\,\mathbf{r}^4 + B'\,\mathbf{r}^5, \qquad (8.2.6)$$

The $\mathbf{r}^p$-matrices in (2.7.5) are thus combined to give the general C_6 -symmetric **T**-matrix relation (8.2.1).

$$\begin{pmatrix} \langle 0|\Psi_{OUT}\rangle \\ \langle 1|\Psi_{OUT}\rangle \\ \langle 2|\Psi_{OUT}\rangle \\ \langle 3|\Psi_{OUT}\rangle \\ \langle 4|\Psi_{OUT}\rangle \\ \langle 5|\Psi_{OUT}\rangle \end{pmatrix} = \begin{pmatrix} A & B' & C' & D & C & B \\ B & A & B' & C' & D & C \\ C & B & A & B' & C' & D \\ D & C & B & A & B' & C' \\ C' & D & C & B & A & B' \\ B' & C' & D & C & B & A \end{pmatrix} \bullet \begin{pmatrix} \langle 0|\Psi_{IN}\rangle \\ \langle 1|\Psi_{IN}\rangle \\ \langle 2|\Psi_{IN}\rangle \\ \langle 3|\Psi_{IN}\rangle \\ \langle 4|\Psi_{IN}\rangle \\ \langle 5|\Psi_{IN}\rangle \end{pmatrix} \qquad (8.2.7)$$

The undetermined coefficients *A, B, C, D, C′*, and *B′* correspond to all the *transition amplitudes* that state $|0\rangle$ could possibly have to other states $|0\rangle$, $|1\rangle$, $|2\rangle$, $|3\rangle$, $|4\rangle$, and $|5\rangle$ as indicated by arrows in Fig. 8.2.1a.

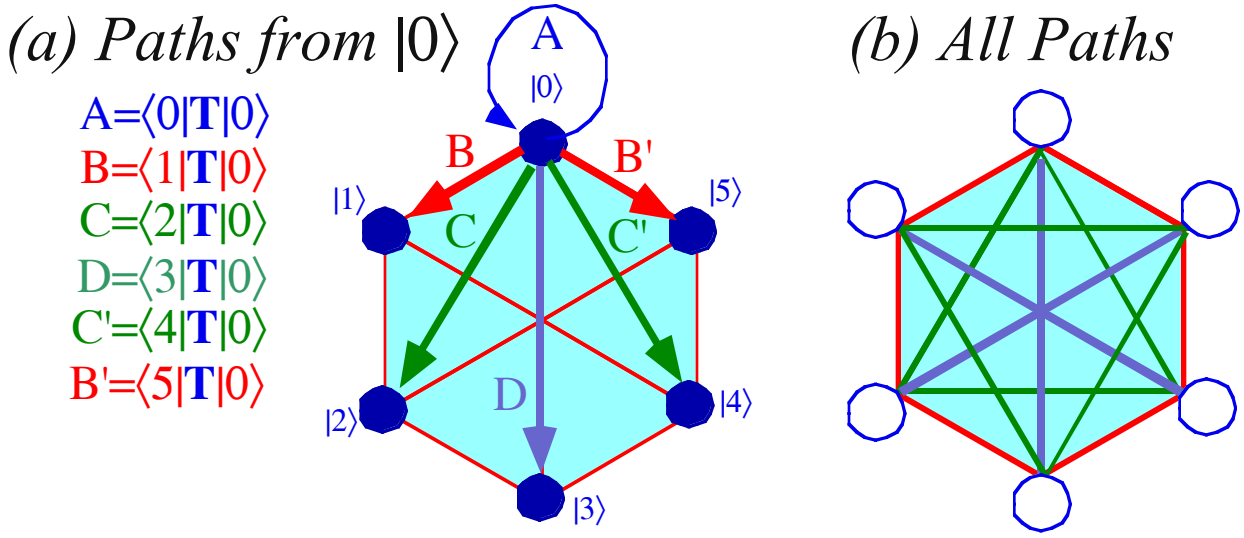


Fig. 8.2.1 Generic 6-channel (C_6) beam transitions (a) Amplitudes (b) Paths

In order that the system really have C_6 symmetry, the next state $|1\rangle$ must make the same amplitudes to the states $|1\rangle$, $|2\rangle$, $|3\rangle$, $|4\rangle$, $|5\rangle$, and $|6\rangle$, respectively, and so on for $|2\rangle$, $|3\rangle$, $|4\rangle$, and $|5\rangle$. All the equivalent paths are indicated in Fig. 8.2.1b.

The expression of a quantum operator, such as the analyzer transfer matrix **T**, in terms of its symmetry operators, such as the $\mathbf{r}^p$, is a deep and important idea which will be used a lot in the rest of this text. It is useful if, as the case is here, the $\mathbf{r}^p$ and **T** have the same set of eigenstates or projectors so that a (presumably!) easy spectral decomposition of the former also solves the latter. Also, it is useful to label by symmetry operators both the system coordinate base states, as in (8.1.6), and the transfer or transition amplitudes or *paths* between the base states, as in Fig. 8.2.1.

(c) Spectral decomposition of transfer operator T

Now a C_6-symmetric **T** operator equation with these *A, B, C,.. amplitudes* must be diagonalized if represented in the symmetry projected $|k_m\rangle$ basis (8.2.5).

$$\begin{pmatrix} \langle k_0|\Psi_{OUT}\rangle \\ \langle k_1|\Psi_{OUT}\rangle \\ \langle k_2|\Psi_{OUT}\rangle \\ \langle k_3|\Psi_{OUT}\rangle \\ \langle k_4|\Psi_{OUT}\rangle \\ \langle k_5|\Psi_{OUT}\rangle \end{pmatrix} = \begin{pmatrix} \varepsilon(k_0) & 0 & 0 & 0 & 0 & 0 \\ 0 & \varepsilon(k_1) & 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon(k_2) & B & 0 & 0 \\ 0 & 0 & 0 & \varepsilon(k_3) & 0 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon(k_4) & 0 \\ 0 & 0 & 0 & 0 & 0 & \varepsilon(k_5) \end{pmatrix} \bullet \begin{pmatrix} \langle k_0|\Psi_{IN}\rangle \\ \langle k_1|\Psi_{IN}\rangle \\ \langle k_2|\Psi_{IN}\rangle \\ \langle k_3|\Psi_{IN}\rangle \\ \langle k_4|\Psi_{IN}\rangle \\ \langle k_5|\Psi_{IN}\rangle \end{pmatrix} \quad (8.2.8)$$

This is because **T** in (8.2.6) is a combination of symmetry operators (2.7.5) and all the symmetry operators have $|k_m\rangle$ as eigenvectors with eigenvalues (8.2.3).

$$\mathbf{r}^p |k_m\rangle = \mathbf{r}^p \mathbf{P}^m|1\rangle = e^{-i2\pi\, mp/N} \mathbf{P}^m|1\rangle = e^{-i2\pi\, mp/6}|k_m\rangle \quad (8.2.9)$$

Eigensolutions for **r**-operators are examples of elementary *Bloch symmetry conditions*.

$$\mathbf{r}|k_m\rangle = e^{-ik_m a}|k_m\rangle = e^{-i2\pi\, m/6}|k_m\rangle \text{ where: } k_m = \frac{2\pi}{Na}m \quad (8.2.10)$$

It says that a translation by distance a ($60°$ rotation **r** along analyzer circumference) sees each phase timer advance forward by $k_m a$ consistent with pictures Fig. 7.3.3 of Bloch $(m)_N$ waves. (Remember: phasor clocks turn clockwise with time, a negative angle.) Bloch symmetry is based upon the **r** -eigenoperator relation $\mathbf{r}\,\mathbf{P}^m = \chi_m \mathbf{P}^m$ with *(m)-th*-root-of-unity eigenvalues $\chi_m = e^{-i2\pi\, m/N}$ of **r** from (8.2.3).

An eigenvalue formula for all possible C_6 symmetric T-matrices

To compute the **T**-eigenvalues we just have to substitute the **r**-values of (8.2.9) into (8.2.6)!

$$\langle k_m|\mathbf{T}|k_m\rangle = A\langle k_m|\mathbf{1}|k_m\rangle + B\langle k_m|\mathbf{r}|k_m\rangle + C\langle k_m|\mathbf{r}^2|k_m\rangle + D\langle k_m|\mathbf{r}^3|k_m\rangle + C'\langle k_m|\mathbf{r}^4|k_m\rangle + B'\langle k_m|\mathbf{r}^5|k_m\rangle$$
$$= A + B\,e^{-ik_m a} + C\,e^{-i2k_m a} + D\,e^{-i3k_m a} + C'\,e^{i2k_m a} + B'\,e^{ik_m a} \quad (8.2.11a)$$

(Note: $e^{-i4k_m a} = e^{i2k_m a}$ since *-4 mod 6 = 2 mod 6*. Also, $e^{-i5k_m a} = e^{ik_m a}$ since *-5 mod 6 = 1 mod 6*) Another way to derive eigenvalues is to put $|k_m\rangle$ into a matrix eigenequation (8.2.7) for **T**.

$$\begin{pmatrix} A & B' & C' & D & C & B \\ B & A & B' & C' & D & C \\ C & B & A & B' & C' & D \\ D & C & B & A & B' & C' \\ C' & D & C & B & A & B' \\ B' & C' & D & C & B & A \end{pmatrix} \bullet \begin{pmatrix} 1 \\ e^{ik_m a} \\ e^{i2k_m a} \\ e^{i3k_m a} \\ e^{-i2k_m a} \\ e^{-ik_m a} \end{pmatrix} = \varepsilon(k_m) \begin{pmatrix} 1 \\ e^{ik_m a} \\ e^{i2k_m a} \\ e^{i3k_m a} \\ e^{-i2k_m a} \\ e^{-ik_m a} \end{pmatrix} \quad (8.2.11b)$$

The first row multiplication shows gives the same eigenvalue.

$$\varepsilon(k_m) = A + B\,e^{-ik_m a} + C\,e^{-i2k_m a} + D\,e^{-i3k_m a} + C'\,e^{i2k_m a} + B'\,e^{ik_m a} \quad (8.2.11c)$$

It is important to understand what has been accomplished. A general eigenvalue and eigenvector formula has been derived *for all possible matrices* **T** *that have the symmetry* C_6 of this particular "thought-experimental" problem. That is pretty neat, and it is just the beginning of a powerful set of symmetry tools!

What do the k_m- eigensolutions mean?

The physical interpretation of C_N eigensolutions is well known to electrical engineers. The ket in (8.2.11b) is a 6-phase generalization of the voltage in 3-phase wires commonly used to transport 220V power. A C_3 example shown in Fig. 8.2.2 resembles the 2_3=-1_3-row of the C_3 table in Fig.7.3.3 with a time-phase of $t=5\pi/6$. (The 2_3=-1_3-bra (row) is the †-conjugate of a 1_3=-2_3-ket (column) eigenvector.) The result is a $(k=1)$-wave moving left to right in Fig. 8.2.2a or clockwise in Fig. 8.2.2b. (Recall: phasor-ahead feeds into phasor-behind. Imaginary ImΨ precedes the real ReΨ in time since phasors turn like clocks.)

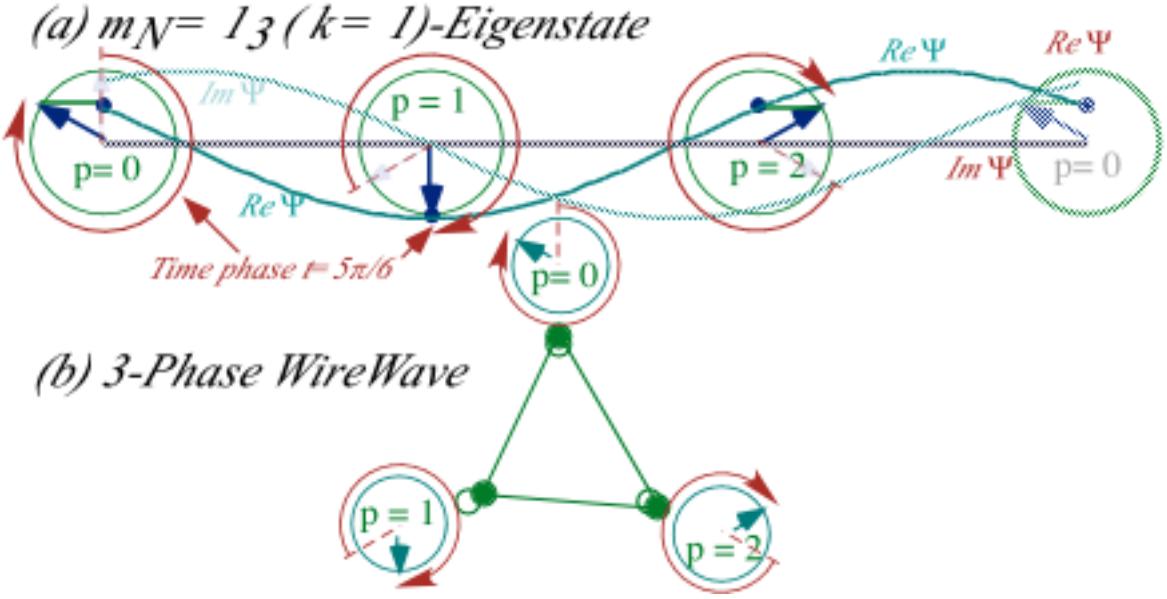


Fig. 8.2.2 (k=1) 3-channel (C_3) wave eigenstate (a) Real and imaginary waves (b) Phasors

A beam with all amplitudes equally dephased from their next neighbor is a $|k_m\rangle$-state that is not changed by a cyclically wired device that has C_N symmetry such as the C_6 analyzer sketched in Fig. 8.2.1. Also, if the **T**-matrix is *unitary* ($\mathbf{T}^\dagger=\mathbf{T}^{-1}$), $|k_m\rangle$-state eigenvalues $\varepsilon(k_m)$ must be unitary, too.

$$\varepsilon(k_m)^* = 1/\,\varepsilon(k_m) \qquad \text{or:}\ \ \varepsilon(k_m) = e^{i\phi m} \tag{8.2.12}$$

So the effect of the analyzer on an *eigenchannel* $|k_m\rangle$-state can only be to add an overall phase ϕ_m to it.

$$\mathbf{T}\,|k_m\rangle = e^{i\phi m}\,|k_m\rangle \tag{8.2.13}$$

The phase ϕ_m is sometimes called an *eigenchannel phase-shift* or *eigenphase* ϕ_m . Below we write the *eigenchannel basis representation* of the $\mathbf{T}\,|k_m\rangle$-equation for a general input state $|\Psi_{IN}\rangle$ with arbitrary values for its *N-eigenchannel-amplitudes* $\langle k_m|\Psi_{IN}\rangle$ of (8.2.7). (This means the *N-channel-amplitudes* $\langle p|\Psi_{IN}\rangle$ in the original representation (8.2.6) are arbitrary, too.) Below is for general $|\Psi_{IN}\rangle$.

$$\begin{pmatrix} \langle k_0|\Psi_{OUT}\rangle \\ \langle k_1|\Psi_{OUT}\rangle \\ \langle k_2|\Psi_{OUT}\rangle \\ \langle k_3|\Psi_{OUT}\rangle \\ \langle k_4|\Psi_{OUT}\rangle \\ \langle k_5|\Psi_{OUT}\rangle \end{pmatrix} = \begin{pmatrix} e^{i\phi_0} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{i\phi_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{i\phi_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i\phi_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{i\phi_4} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{i\phi_5} \end{pmatrix} \bullet \begin{pmatrix} \langle k_0|\Psi_{IN}\rangle \\ \langle k_1|\Psi_{IN}\rangle \\ \langle k_2|\Psi_{IN}\rangle \\ \langle k_3|\Psi_{IN}\rangle \\ \langle k_4|\Psi_{IN}\rangle \\ \langle k_5|\Psi_{IN}\rangle \end{pmatrix} \tag{8.2.14}$$

(d) OK, where did those e^{ikx} wavefunctions come from?

Every student of differential equations is told early on to try the exponential solutions e^{At} or e^{iat} in independent variable t with little reason given except, "It works!...sometimes." Now we can see why and when such solutions work. The key to our exponential eigenfunctions $\psi_{k_m}(x_p)= e^{ik_m x_p}/\sqrt{N}$ was C_N symmetry which demanded in (2.7.5) that we use roots of unity, that is, the roots of the minimal equation $\mathbf{r}^N=\mathbf{1}$ for symmetry operator $\mathbf{r}$.

If we let N approach infinity ($N\to\infty$) the symmetry approaches continuous translation symmetry C_∞, and the eigenfunctions $\psi_{k_m}(x_p)$ approach plane waves $\psi_k(x)= e^{ikx}/\sqrt{2\pi}$ such as given by (2.6.20b) in Sec. 2.6b. Symmetry demands independence or invariance to translation of the independent variable x. In other words, you should get the same differential equation no matter whether you let the origin be at $x=0$ or at $x=2{,}517$ in Timbuktu. For example, the differential equation

$$\frac{d^2\psi}{dx^2}+2\gamma\frac{d\psi}{dx}+k^2\psi=0 \qquad (8.2.15)$$

does have C_∞ symmetry so e^{ikx} will work, but an equation like

$$\frac{d^2\psi}{dx^2}+2\gamma x\frac{d\psi}{dx}+k^2x^2\psi=0 \qquad (8.2.16)$$

does not have C_∞ symmetry because of the x-dependence; it's not the same equation in Timbuktu. An example of a C_N -symmetric differential equation is Matieu's equation for waves in a periodic solid.

$$\frac{d^2\psi}{dx^2}+k^2\cos(Nx)\psi=0$$

All that we have said applies as well when the independent variable is time t. For example, the differential equation

$$\frac{d^2\psi}{dt^2}+2\Gamma\frac{d\psi}{dt}+\omega^2\psi=0$$

does have C_∞ symmetry so $e^{i\omega t}$ will work. An example of a C_N -symmetric time differential equation is Mathieu's equation for a periodic force. Later we use C_N -symmetry to help solve this type of equation.

$$\frac{d^2\psi}{dt^2}+k^2\cos(Nt)\psi=0$$

8.3 Related Symmetry Analysis Examples

The homo-cyclic two-dot C_2 and three-dot C_3 sytems are sketched below in the way the C_6 system was sketched in Fig. 8.2.1. The transfer matrix equations (8.3.1) have eigenket tables (8.3.2).

$$\begin{pmatrix} \langle 0|\Psi_{OUT}\rangle \\ \langle 1|\Psi_{OUT}\rangle \end{pmatrix} = \begin{pmatrix} A & B \\ B & A \end{pmatrix} \begin{pmatrix} \langle 0|\Psi_{IN}\rangle \\ \langle 1|\Psi_{IN}\rangle \end{pmatrix} \quad (8.3.1a)$$

$$\begin{pmatrix} \langle 0|\Psi_{OUT}\rangle \\ \langle 1|\Psi_{OUT}\rangle \\ \langle 2|\Psi_{OUT}\rangle \end{pmatrix} = \begin{pmatrix} A & B' & B \\ B & A & B' \\ B' & B & A \end{pmatrix} \begin{pmatrix} \langle 0|\Psi_{IN}\rangle \\ \langle 1|\Psi_{IN}\rangle \\ \langle 2|\Psi_{IN}\rangle \end{pmatrix} \quad (8.3.1b)$$

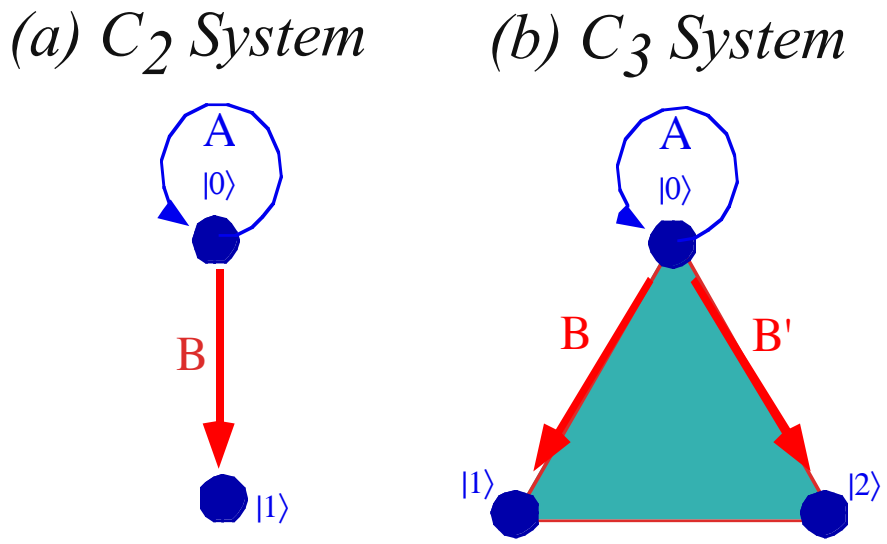


Fig. 8.3.1 Generic N-channel (C_N) quantum dot systems. (a)N=2 (b) N=3

(8.3.2a)

C_2	$\|x_0\rangle = \mathbf{R}^0\|0\rangle$	$\|x_1\rangle = \mathbf{R}^1\|0\rangle$	
$\|(0)_2\rangle$	1	1	$/\sqrt{2}$
$\|(1)_2\rangle$	1	−1	$/\sqrt{2}$

(8.3.2b)

C_3	$\|x_0\rangle = \mathbf{r}^0\|0\rangle$	$\|x_1\rangle = \mathbf{r}^1\|0\rangle$	$\|x_2\rangle = \mathbf{r}^2\|0\rangle$	
$\|(0)_3\rangle$	1	1	1	$/\sqrt{3}$
$\|(1)_3\rangle$	1	$e^{2\pi i/3}$	$e^{-2\pi i/3}$	$/\sqrt{3}$
$\|(2)_3\rangle$	1	$e^{-2\pi i/3}$	$e^{2\pi i/3}$	$/\sqrt{3}$

The eigenket tables are from Fig. 7.3.3. Each phasor in the ⟨bra| table for C_3 in Fig. 7.3.3 is replaced by its complex conjugate to make kets. A preceding Fig. 8.2.2 shows a $|(1)_3\rangle$ wave with eigen-phase shift of $-5\pi/6$. The corresponding transfer matrix eigenvalues $\langle m_N|\mathbf{T}|m_N\rangle$ in terms of parameters $A,B,..$ are left as exercises.

Besides such cyclic C_N systems there are an enormous number of ways to connect N-dots in ways that have more or less symmetry. A few of these are considered below and in problems. Most of the interesting (Also, read "doable!") quantum problems have an underlying symmetry.

(a) Dihedral symmetry D_2

Two 4-dot symmetries are shown in Fig. 8.3.2 below with transfer matrix relations.

$$\begin{pmatrix} \langle 0|\Psi_{OUT}\rangle \\ \langle 1|\Psi_{OUT}\rangle \\ \langle 2|\Psi_{OUT}\rangle \\ \langle 3|\Psi_{OUT}\rangle \end{pmatrix} = \begin{pmatrix} A & B & C & C \\ B & A & C & C \\ C & C & A' & B' \\ C & C & B' & A' \end{pmatrix} \begin{pmatrix} \langle 0|\Psi_{IN}\rangle \\ \langle 1|\Psi_{IN}\rangle \\ \langle 2|\Psi_{IN}\rangle \\ \langle 3|\Psi_{IN}\rangle \end{pmatrix} \qquad (8.3.3a)$$

$$\begin{pmatrix} \langle 0|\Psi_{OUT}\rangle \\ \langle 1|\Psi_{OUT}\rangle \\ \langle 2|\Psi_{OUT}\rangle \\ \langle 3|\Psi_{OUT}\rangle \end{pmatrix} = \begin{pmatrix} A & B & B' & C \\ B & A & C & B' \\ B' & C & A & B \\ C & B' & B & A \end{pmatrix} \begin{pmatrix} \langle 0|\Psi_{IN}\rangle \\ \langle 1|\Psi_{IN}\rangle \\ \langle 2|\Psi_{IN}\rangle \\ \langle 3|\Psi_{IN}\rangle \end{pmatrix} \qquad (8.3.3b)$$

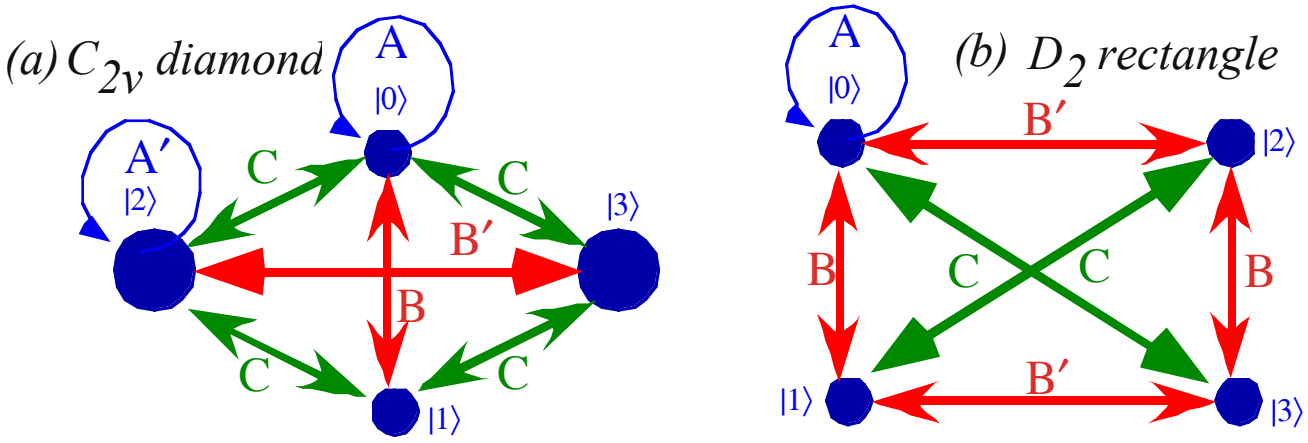


Fig. 8.3.2 Generic 4-channel (D_2) quantum dot systems. (a)Diamond C_{2v} (b) Rectangular D_2.

Consider the rectangular D_2 system. Its transfer matrix may be written in terms of four operators.

$$\mathbf{T} = A\,\mathbf{1} + B\,\mathbf{R}_x + B'\,\mathbf{R}_y + C\,\mathbf{R}_z$$

$$\begin{pmatrix} A & B & B' & C \\ B & A & C & B' \\ B' & C & A & B \\ C & B' & B & A \end{pmatrix} = A\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} + B'\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad (8.3.4)$$

Each of the operators $\mathbf{R_x}$, $\mathbf{R_y}$, or $\mathbf{R_z}$, corresponds to *180°*-rotations around *x*, *y*, or *z* axes, respectively, the effect of which is indicated in Fig. 8.3.1b by transfer path arrows labeled *B*, *B′*, and *C*, respectively. A transfer path *B′* along the *x*-direction is done by a *y*-rotation $\mathbf{R_y}$, while *B* along *y* is done by $\mathbf{R_x}$.

D_2 group structure

The multiplication table for the *Verrgrupe* (4-group) is quite famous and relevant to quantum theory.

1	R_x	R_y	R_z
R_x	1	R_z	R_y
R_y	R_z	1	R_x
R_z	R_y	R_x	1

(8.3.5a)

Its structure reduces to a few simple products. The first is *(xyz)*-cyclic: It holds for *(zxy)* and *(yzx)*, too.

$$\mathbf{R_x}\,\mathbf{R_y} = \mathbf{R_y}\,\mathbf{R_x} = \mathbf{R_z}, \qquad (8.3.5b)$$

$$\mathbf{R_x}^2 = \mathbf{R_y}^2 = \mathbf{R_z}^2 = \mathbf{1}. \qquad (8.3.5c)$$

D_2 spectral decomposition: The old "1=1•1 trick" again

The latter (8.3.5c) are of immediate interest to a quantum algebraist because they give minimal equations.

$$\mathbf{R_x}^2 - \mathbf{1} = \mathbf{0}, \quad (8.3.5d) \qquad \mathbf{R_y}^2 - \mathbf{1} = \mathbf{0}. \quad (8.3.5e)$$

From the roots (±1) of each minimal equation is constructed a spectral decomposition of $\mathbf{R_x}$ and $\mathbf{R_y}$. This is the simplest application of the Chapter 3 projector formula (3.1.15a) you will probably ever see.

$$\begin{aligned} \mathbf{P}_x^+ &= \frac{\mathbf{1}+\mathbf{R}_x}{2} \\ \mathbf{P}_x^- &= \frac{\mathbf{1}-\mathbf{R}_x}{2} \end{aligned} \quad (8.3.6a) \qquad \begin{aligned} \mathbf{P}_y^+ &= \frac{\mathbf{1}+\mathbf{R}_y}{2} \\ \mathbf{P}_y^- &= \frac{\mathbf{1}-\mathbf{R}_y}{2} \end{aligned} \quad (8.3.6b)$$

This spectrally decomposes $\mathbf{R_x}$ and $\mathbf{R_y}$ separately. We can do $\mathbf{R_z}$, too, but all three must be done *together*.

$$\begin{aligned} \mathbf{1} &= \mathbf{P}_x^+ + \mathbf{P}_x^- \\ \mathbf{R}_x &= \mathbf{P}_x^+ - \mathbf{P}_x^- \end{aligned} \quad (8.3.7a) \qquad \begin{aligned} \mathbf{1} &= \mathbf{P}_y^+ + \mathbf{P}_y^- \\ \mathbf{R}_y &= \mathbf{P}_y^+ - \mathbf{P}_y^- \end{aligned} \quad (8.3.7b)$$

To make projectors for the whole D_2 symmetry together we use the old "**1=1•1** trick" from (3.1.36).

$$\mathbf{1} = \mathbf{1}\cdot\mathbf{1} = \left(\mathbf{P}_x^+ + \mathbf{P}_x^-\right)\cdot\left(\mathbf{P}_y^+ + \mathbf{P}_y^-\right) = \mathbf{P}_x^+\cdot\mathbf{P}_y^+ + \mathbf{P}_x^-\cdot\mathbf{P}_y^+ + \mathbf{P}_x^+\cdot\mathbf{P}_y^- + \mathbf{P}_x^-\cdot\mathbf{P}_y^- \quad (8.3.8)$$

The result are *irreducible* projectors $\mathbf{P}^{(i)}$ for the whole D_2 symmetry. Irreducible means $\mathrm{Trace}\mathbf{R}(\mathbf{P}^{(i)})=1$.

$$\begin{aligned} \mathbf{P}^{++} &\equiv \mathbf{P}_x^+\cdot\mathbf{P}_y^+ = \frac{\left(\mathbf{1}+\mathbf{R}_x\right)\cdot\left(\mathbf{1}+\mathbf{R}_y\right)}{2\cdot 2} = \frac{1}{4}\left(\mathbf{1}+\mathbf{R}_x+\mathbf{R}_y+\mathbf{R}_z\right) \\ \mathbf{P}^{-+} &\equiv \mathbf{P}_x^-\cdot\mathbf{P}_y^+ = \frac{\left(\mathbf{1}-\mathbf{R}_x\right)\cdot\left(\mathbf{1}+\mathbf{R}_y\right)}{2\cdot 2} = \frac{1}{4}\left(\mathbf{1}-\mathbf{R}_x+\mathbf{R}_y-\mathbf{R}_z\right) \\ \mathbf{P}^{+-} &\equiv \mathbf{P}_x^+\cdot\mathbf{P}_y^- = \frac{\left(\mathbf{1}+\mathbf{R}_x\right)\cdot\left(\mathbf{1}-\mathbf{R}_y\right)}{2\cdot 2} = \frac{1}{4}\left(\mathbf{1}+\mathbf{R}_x-\mathbf{R}_y-\mathbf{R}_z\right) \\ \mathbf{P}^{--} &\equiv \mathbf{P}_x^-\cdot\mathbf{P}_y^- = \frac{\left(\mathbf{1}-\mathbf{R}_x\right)\cdot\left(\mathbf{1}-\mathbf{R}_y\right)}{2\cdot 2} = \frac{1}{4}\left(\mathbf{1}-\mathbf{R}_x-\mathbf{R}_y+\mathbf{R}_z\right) \end{aligned} \quad (8.3.9a)$$

Each $\mathbf{P}$ is multiplied by its own eigenvalue (±1) of $\mathbf{1}$, $\mathbf{R_x}$, $\mathbf{R_y}$, and $\mathbf{R_z}$ in the D_2 spectral decomposition.

$$\begin{aligned} \mathbf{1} &= (+1)\mathbf{P}^{++} + (+1)\mathbf{P}^{-+} + (+1)\mathbf{P}^{+-} + (+1)\mathbf{P}^{--} \quad (\textit{completeness}) \\ \mathbf{R}_x &= (+1)\mathbf{P}^{++} + (-1)\mathbf{P}^{-+} + (+1)\mathbf{P}^{+-} + (-1)\mathbf{P}^{--} \\ \mathbf{R}_y &= (+1)\mathbf{P}^{++} + (+1)\mathbf{P}^{-+} + (-1)\mathbf{P}^{+-} + (-1)\mathbf{P}^{--} \\ \mathbf{R}_z &= (+1)\mathbf{P}^{++} + (-1)\mathbf{P}^{-+} + (-1)\mathbf{P}^{+-} + (+1)\mathbf{P}^{--} \end{aligned} \quad (8.3.9b)$$

Spectral decomposition of D_2 transfer matrices

Spectral decomposition applies to transfer matrix (8.3.4) and yields its eigenvalue spectrum.

$$\begin{aligned} \langle{+}{+}|\mathbf{T}|{+}{+}\rangle &= \varepsilon^{++} = A\langle\mathbf{1}\rangle + B\langle\mathbf{R}_x\rangle + B'\langle\mathbf{R}_y\rangle + C\langle\mathbf{R}_z\rangle = A + B + B' + C \\ \langle{-}{+}|\mathbf{T}|{-}{+}\rangle &= \varepsilon^{-+} = A\langle\mathbf{1}\rangle + B\langle\mathbf{R}_x\rangle + B'\langle\mathbf{R}_y\rangle + C\langle\mathbf{R}_z\rangle = A - B + B' - C \\ \langle{+}{-}|\mathbf{T}|{+}{-}\rangle &= \varepsilon^{+-} = A\langle\mathbf{1}\rangle + B\langle\mathbf{R}_x\rangle + B'\langle\mathbf{R}_y\rangle + C\langle\mathbf{R}_z\rangle = A + B - B' - C \\ \langle{-}{-}|\mathbf{T}|{-}{-}\rangle &= \varepsilon^{--} = A\langle\mathbf{1}\rangle + B\langle\mathbf{R}_x\rangle + B'\langle\mathbf{R}_y\rangle + C\langle\mathbf{R}_z\rangle = A - B - B' + C \end{aligned} \quad (8.3.10)$$

Again, this is a formula for *all possible* D_2-symmetric operators in this device space of Fig. 8.3.2b. Higher symmetry, such as "square" or *tetragonal* D_4–symmetry is obtained if parameters B and B' are equal. Then the

eigenvalues ε^{+-} and ε^{-+} become equal or *degenerate*. Such a symmetry is non-commutative or *non-Abelian* and requires further theory which will be taken up in a later chapter.

(b) Outer product structure: Double qubit registers

One of the things that makes group algebra powerful is the concept of an *outer (×) product* of two groups. You may have noticed that the D_2 group multiplication table was divided up so that the C_2 subgroup $\{\mathbf{1}, \mathbf{R_x}\}$ was isolated from the rest. The outer product is appropriate when two isolated "factors" correspond to orthogonal or independent systems such as two separate particles or two dimensions or two qubits.

D_2 is product $C_2 \times C_2$

An outer product of the eigenvalue tables in (8.3.2a) yields the D_2 eigenvalue table. This is basically what was happening in the algebraic maneuver of (8.3.8) based upon the old "**1=1•1**" trick.

$$\begin{array}{c|cc} C_2^x & \mathbf{1} & \mathbf{R}_x \\ \hline + & 1 & 1 \\ - & 1 & -1 \end{array} \times \begin{array}{c|cc} C_2^y & \mathbf{1} & \mathbf{R}_y \\ \hline + & 1 & 1 \\ - & 1 & -1 \end{array} = \begin{array}{c|cccc} C_2^x \times C_2^y & \mathbf{1}\cdot\mathbf{1} & \mathbf{R}_x\cdot\mathbf{1} & \mathbf{1}\cdot\mathbf{R}_y & \mathbf{R}_x\cdot\mathbf{R}_y \\ \hline +\cdot+ & 1\cdot1 & 1\cdot1 & 1\cdot1 & 1\cdot1 \\ -\cdot+ & 1\cdot1 & -1\cdot1 & 1\cdot1 & -1\cdot1 \\ \hline +\cdot- & 1\cdot1 & 1\cdot1 & 1\cdot(-1) & 1\cdot(-1) \\ -\cdot- & 1\cdot1 & -1\cdot1 & 1\cdot(-1) & -1\cdot(-1) \end{array} \qquad (8.3.11a)$$

$$= \begin{array}{c|cccc} D_2 & \mathbf{1} & \mathbf{R}_x & \mathbf{R}_y & \mathbf{R}_z \\ \hline ++ & 1 & 1 & 1 & 1 \\ -+ & 1 & -1 & 1 & -1 \\ \hline +- & 1 & 1 & -1 & -1 \\ -- & 1 & -1 & -1 & 1 \end{array} \qquad (8.3.11b)$$

Note that the numbers in (8.3.11b) are exactly the coefficients of *A, B, B′*, and *C* in the eigenvalue formulas for ε^{++}, ε^{-+}, ε^{+-}, and ε^{--} in (8.3.10). So the×-product makes this calculation very easy indeed.

The outer product requires every operator in D_2 to be *uniquely* a product of one element in C_2^x and one element in C_2^y. The elements in C_2^x must commute with all those in C_2^y so each product is unique.

$$C_2^x \times C_2^y = \{\mathbf{1}, \mathbf{R}_x\} \times \{\mathbf{1}, \mathbf{R}_y\} = \begin{array}{c|cc} C_2^x \times C_2^y & \mathbf{1} & \mathbf{R}_y \\ \hline \mathbf{1} & \mathbf{1}\cdot\mathbf{1} & \mathbf{1}\cdot\mathbf{R}_y \\ \mathbf{R}_x & \mathbf{R}_x\cdot\mathbf{1} & \mathbf{R}_x\cdot\mathbf{R}_y \end{array} \qquad (8.3.11c)$$

$$= \{\mathbf{1}, \mathbf{R}_x, \mathbf{R}_y, \mathbf{R}_z\} = D_2$$

If a group G has g operators and a group H has h members, then $G \times H$ must have exactly gh members. It can be a great help to find a symmetry group is an outer product of its parts.

Multiple outer products are possible. The $D_2 = C_2 \times C_2$ system is like a double-binary or 4-bit register. A $C_2 \times C_2 \times C_2$ system is a triple-binary or 8-bit register known as *1-byte*. A double-binary D_2 register differs from a quadrary (C_4) register as a 1-byte binary systemis not a single octal (C_8) system.

Big-endian versus Little-endian

Computer scientists differ on whether the right ending bit should be the most significant bit (and least rapidly changing) or least significant bit and most often changing. (The former is called the Big-Endian

convention while the latter is called the Little-Endian convention after a perjorative folk-song.) The sequence *(00, 01, 10, 11)* is Little-Endian and more like our decimal numbering system. The sequence *(00, 10, 01, 11)* or in (8.3.11) (++, -+, +-, --) is Big-Endian and what we are using here.

C_6 is product $C_3\times C_2$ *(but C_4 is NOT* $C_2\times C_2$*)*

Our first example, the cyclic group C_6, is a composite $C_3\times C_2$ of two of its subgroups C_2 and C_3 as shown below. Here the eigenvalue table (8.3.2a) of C_2 is crossed with the C_3 table (8.3.2b).

C_3	**1**	**r**	$\mathbf{r}^2$
$(0)_3$	1	1	1
$(1)_3$	1	$e^{2\pi i/3}$	$e^{-2\pi i/3}$
$(2)_3$	1	$e^{-2\pi i/3}$	$e^{2\pi i/3}$

$\times$

C_2	**1**	**R**
$(0)_2$	1	1
$(1)_2$	1	−1

$=$

$C_3\times C_2$	**1**	**r**	$\mathbf{r}^2$	$\mathbf{1}\cdot\mathbf{R}$	$\mathbf{r}\cdot\mathbf{R}$	$\mathbf{r}^2\cdot\mathbf{R}$
$(0)_3\cdot(0)_2$	$1\cdot1$	$1\cdot1$	$1\cdot1$	$1\cdot1$	$1\cdot1$	$1\cdot1$
$(1)_3\cdot(0)_2$	$1\cdot1$	$e^{2\pi i/3}\cdot1$	$e^{-2\pi i/3}\cdot1$	$1\cdot1$	$e^{2\pi i/3}\cdot1$	$e^{-2\pi i/3}\cdot1$
$(2)_3\cdot(0)_2$	$1\cdot1$	$e^{-2\pi i/3}\cdot1$	$e^{2\pi i/3}\cdot1$	$1\cdot1$	$e^{-2\pi i/3}\cdot1$	$e^{2\pi i/3}\cdot1$
$(0)_3\cdot(1)_2$	$1\cdot1$	$1\cdot1$	$1\cdot1$	$1\cdot(-1)$	$1\cdot(-1)$	$1\cdot(-1)$
$(1)_3\cdot(1)_2$	$1\cdot1$	$1\cdot1$	$e^{-2\pi i/3}\cdot1$	$1\cdot(-1)$	$e^{2\pi i/3}\cdot(-1)$	$e^{-2\pi i/3}\cdot(-1)$
$(2)_3\cdot(1)_2$	$1\cdot1$	$e^{-2\pi i/3}\cdot1$	$1\cdot1$	$1\cdot(-1)$	$e^{-2\pi i/3}\cdot(-1)$	$e^{2\pi i/3}\cdot(-1)$

$=$

$C_3\times C_2=C_6$	**1**	$\mathbf{r}=h^2$	$\mathbf{r}^2=h^4$	$\mathbf{R}=\mathbf{h}^3$	$\mathbf{r}\cdot\mathbf{R}=h$	$\mathbf{r}^2\cdot\mathbf{R}=h^5$
$(0)_3\cdot(0)_2=(0)_6$	1	1	1	1	1	1
$(1)_3\cdot(0)_2=(2)_6$	1	$e^{2\pi i/3}$	$e^{-2\pi i/3}$	1	$e^{2\pi i/3}$	$e^{-2\pi i/3}$
$(2)_3\cdot(0)_2=(4)_6$	1	$e^{-2\pi i/3}$	$e^{2\pi i/3}$	1	$e^{-2\pi i/3}$	$e^{2\pi i/3}$
$(0)_3\cdot(1)_2=(3)_6$	1	1	1	-1	-1	-1
$(1)_3\cdot(1)_2=(5)_6$	1	$e^{2\pi i/3}$	$e^{-2\pi i/3}$	-1	$-e^{2\pi i/3}$	$-e^{-2\pi i/3}$
$(2)_3\cdot(1)_2=(1)_6$	1	$e^{-2\pi i/3}$	$e^{2\pi i/3}$	−1	$-e^{-2\pi i/3}$	$-e^{2\pi i/3}$

(8.3.12)

The tricky part is to identify the C_6 waves $(k)_6$ that belong to a each product $(m)_3.(n)_2$. That is,

$$e^{i(k)_6 x}=e^{i(m)_3 x}e^{i(n)_2 x}=e^{i\left(m\frac{2\pi}{3}+n\frac{2\pi}{2}\right)x}=e^{i(2m+3n)\frac{2\pi}{6}x}. \qquad (8.3.13a)$$

The desired k-value is: $k = (2m + 3n) \bmod 6$ (8.3.13b)

For, example, the last row of (8.3.12) belongs to C_6 wave $k=(2.2+3.1) \bmod 6 = 7 \bmod 6 = 1$ or $(1)_6$. The result is a reordered C_6 table, but otherwise it is the same as the one first drawn in Fig. 7.3.3. Verify!

Symmetry Catalog

Cataloging the number of symmetry groups of a given order N is a difficult problem with a long history. But, for commutative or Abelian groups considered so far, it reduces to finding all the distinct outer products $C_p\times C_q\times C_r\times C_s\times C_t$ … of cyclic groups such that $N=pqrst$…. is a product of primes. Product $C_p\times C_q$ is the same as C_{pq} if p and q share no factor in common so we don't include C_{pq} in the catalog if p and q are prime since then $C_{pq} = C_p\times C_q$ as in the case of $C_6 = C_2\times C_3$ above. But we do include both $C_p\times C_p$ and C_{pp} which <u>are</u> distinct as were $C_2\times C_2$ and C_4 above. If $N=p^P$ is a power of a prime such as $N=8=2^3$, then a distinct group exists for each *partition* of the power P. For example, $P=3 =1+2 = 1+1+1$ has three distinct prime base-$(p=2)$ groups: C_8 and $C_4\times C_2$ and $C_2\times C_2\times C_2$ are all distinct symmetries.

Problems for Chapter 8.

Subgroup soup

8.1.1 (a) The C_6 symmetry group has subgroups. List all of them except C_6 itself.

(b) Do the same for the symmetry groups C_3 , C_4 , and C_5 . What is special about groups C_N of prime order N?

Ttrace'o **g**

8.1.2 (a) By group axioms (Sec. 2.2) show each row and column of a group table has an operator **g** only once.

(b) Use (a) to show that the regular representation trace *TraceR(***g***)* is zero for all but "do-nothing" unit operator **g=1**.

Turn-about's fair play

8.2.1 Suppose we are given the eigenvalues $\{\tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5\}$ of a unitary C_6 transfer matrix **T** in (8.2.1).

(a) Can the $\{\tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5\}$ be any old complex numbers? What restrictions, if any, apply?

(b) Can one give a formula for all 36 components T_{pq} of **T** in terms of $\{\tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5\}$? If so do it. If not expalin why not and under what conditions you may be able to do it.

A Hex on pairing

8.2.2 Suppose the C_6 transfer matrix **T** is the form of the *Pairing operato*r, that is all components equal $T_{pq} = T$.

(a) Derive the resulting eigenvalue spectrum.

(b) What, if any, limitations need to be placed on parameter T?

(c) Discuss which waves belong to which eigenvalues

Phase o'Hex

8.2.3 (a) Could the hexagonal C_6 analyzer be wired so input $|even\ sites\rangle$=(1,0,1,0,1,0) comes out $e^{i\phi}\ |even\rangle$?
What k_m-eigenstates make up $|even\ sites\rangle$? Does your "rewiring" maintain C_6 symmetry?

(b) Could the C_6 analyzer be wired so input $|even\ sites\rangle$ comes out $e^{i\phi}\ |odd\ sites\rangle$=(0,1,0,1,0,1)?
What k_m-eigenstates make up $|odd\ sites\rangle$? Does your "rewiring" maintain C_6 symmetry?

(c) Could the C_6 analyzer be wired so input $|odd\ symm\rangle$=(1,-1,1,-1,1,-1) comes out $e^{i\phi}\ |odd\ symm\rangle$?
What k_m-eigenstates make up $|odd\ symm\rangle$? Does your "rewiring" maintain C_6 symmetry?

(d) Could the C_6 analyzer be wired so input $|\ odd\ symm\rangle$ comes out $e^{i\phi}\ |even\ symm\rangle$=(1,1,1,1,1,1)?
What k_m-eigenstates make up $|\ even\ symm\rangle$? Does your "rewiring" maintain C_6 symmetry?

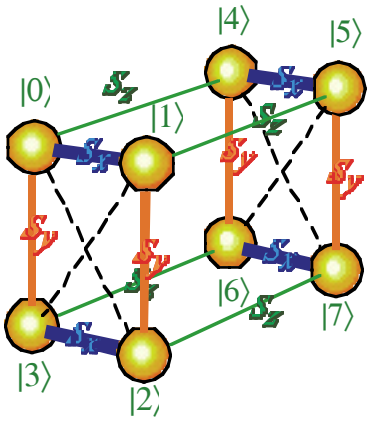
Little diamond

8.3.1. The symmetry eigensolution analysis of the C_{2v} diamond quantum dot device in Fig. 8.3.2(a) is a little different than its D_2 cousin in Fig. 8.3.2(b). Symmerty multiplication table and spectral decomposition is essentially the same but the transfer **T**-operator is not such a simple linear combination of symmetry operators. Represent the symmetry and give a decomposition of symmetry and **T**-matrix. (Note that x and y-plane mirror reflections are symmetry operators, too. There was no distinction between rotations and reflections in the D_2 problem.)

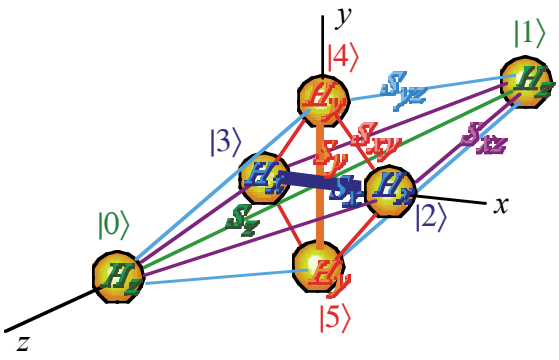
Double Crossed

8.3.2. Complete a symmetry catalog of commutative (Abelian) groups in terms of distinct $C_p \times C_q \times ...$ cross products.
(a) for order N=8. (b) N=9. (c) N=10. (d) N=11. (e) N=12. (f) N=16.

Problem 8.3.3 "Big box"



Problem 8.3.4 "Big diamond"



Big box

8.3.3. Give a complete symmetry eigensolution analysis of the D_{2h} device pictured here. First show that the full symmetry with horizontal reflection group C_h ={**1**, σ_{xy}(thru z-axis)} is $C_2 \times C_2 \times C_h = C_2 \times C_2 \times C_2$ which is called D_{2h}.
Derive character table of D_{2h} using the cross product trick of (8.3.11).

Big diamond

8.3.4. Give a complete symmetry eigensolution analysis of the D_{2h} device pictured above.

Ttrace'o **P**

8.3.5. Before (8.3.9a) it is noted that *TraceR(***P***)=1* means projector **P** is irreducible, that is, not a sum $\mathbf{P}=\mathbf{P}_1+\mathbf{P}_2$ of other "smaller" projectors. Explain this and verify by constructing the representation of the $\mathbf{P}^{++}$,… in (8.3.9).