Complex powers of second order non-homogeneous elliptic differential operators with degenerating symbols in the spaces $L_p(\mathbb{R}^n)$.[*]

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Abstract

We study complex powers of the second order elliptic differential operators with complex coefficients in the lower term. The negative powers are realized as potential type operators, and the positive powers as inverse approximate operators. We give also the description of the domailn of the positive powers.

2000 MSC: 31B10, 47A58, 47G10.
Keywords: Potential type operators, Fractional powers of differential operators.

1 Introduction

At present there are a great deal of investigations on complex powers of second order differential operators within the framework of the spaces $L_p \equiv L_p(\mathbb{R}^n)$ (for a comprehensive list of references and detailed discussions we refer to the books [12], [13] and survey papers [14], [9], [10]). Some first results in this area are known due to the papers [15], [16] by S. G. Samko, who constructed the inversion of the Riesz potentials I^α, which are known to be negative powers of the Laplace operator $-\Delta$, and described the range $I^\alpha(L_p)$ in terms of hypersingular integrals. These results have been extended to other classical operators of mathematical physics, such as the heat operator, the wave operator, the Klein-Gordon-Fock and Shrödinger operators, and others (see references mentioned).

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[*] Supported by Russian grant RFFI 00-01-00046a.

[†] First author also acknowledges CONACyT project no. 35521-E (Mexico, D.F.).

Here we consider an arbitrary second order differential operator with complex coefficients in the lower order terms:

$$-P(D,D)+\sum_{j=1}^{n}\gamma_j\frac{\partial}{\partial x_j},\quad \gamma_j\in\mathbb{C}, \tag{1.1}$$

where $P(x,x)$ is an elliptic real quadratic form. Negative powers of the operator (1.1)

$$J_\gamma^\alpha\varphi=\Big(-P(D,D)+\sum_{j=1}^{n}\gamma_j\frac{\partial}{\partial x_j}\Big)^{-\frac{\alpha}{2}}\varphi,\quad \mathrm{Re}\,\alpha>0, \tag{1.2}$$

are initially defined on "nice" functions φ via the Fourier transform as follows,

$$FJ_\gamma^\alpha\varphi(\xi)=j_{\alpha,\gamma}(\xi)F\varphi(\xi),$$

where

$$j_{\alpha,\gamma}=\begin{cases}(P(\xi,\xi)-i\gamma\cdot\xi)^{-\frac{\alpha}{2}}, & \mathrm{Re}\,\gamma=(\mathrm{Re}\,\gamma_1,\mathrm{Re}\,\gamma_2\ldots,\mathrm{Re}\,\gamma_n)\neq(0,0,\ldots,0),\\ (P(\xi,\xi)-i\gamma\cdot\xi-i0)^{-\frac{\alpha}{2}}, & \mathrm{Re}\,\gamma=(0,0,\ldots,0).\end{cases} \tag{1.3}$$

They are realized as potential type operators with explicitly written kernels for $0<\mathrm{Re}\,\alpha<n+1,\ 1\le p<\frac{n+1}{\mathrm{Re}\,\alpha}$ and treated in the distributional sense for the other values of the parameters α and p ($\mathrm{Re}\,\alpha>0,\ 1\le p\le\infty$). Here the following sufficient condition arises for the potential J_γ^α to be defined on all of L_p:

$$\frac{\mathrm{Re}\,\gamma_1}{\mathrm{Im}\,\gamma_1}=\frac{\mathrm{Re}\,\gamma_2}{\mathrm{Im}\,\gamma_2}=\cdots=\frac{\mathrm{Re}\,\gamma_n}{\mathrm{Im}\,\gamma_n}. \tag{1.4}$$

If this condition is violated, then the kernel of corresponding potential has, in fact, an exponential-like growth at infinity. That is why we assume condition (1.4) to be fulfilled in all that follows.

We observe that natural difficulties arising in the inversion problem for the potential (1.2) are caused by the fact that its symbol

$$m_{\alpha,\gamma}(\xi)=P(\xi,\xi)-i\gamma\cdot\xi,\quad \gamma=(\gamma_1,\gamma_2,\ldots,\gamma_n),$$

degenerates on an ellipsoid if $\mathrm{Re}\,\gamma=0$, and on the intersection of the ellipsoid and a hyperplane if $\mathrm{Re}\,\gamma\neq0$ (in contrast to the case $\gamma\in\mathbb{R}^n$, considered in [1], [2], which is known to be elliptic, the symbol (1.3) has singularities "spread" over these sets). To overcome these difficulties we apply the method of Approximative Inverse Operators (AIO), which has been proved to be an effective tool for inverting potential-type operators in the non-elliptic case. Some results in this area are covered by the book [12] and the surveys [14], [9], [10].

Within the framework of this method we construct the inverse operator to the operator J_γ^α as the limit (in the L_p norm or almost everywhere) of the sequence of convolutions with summable kernels. We also describe the range $J_\gamma^\alpha(L_p),\ 0 < \operatorname{Re}\alpha < n+1,\ 1 \le p < \frac{n+1}{\operatorname{Re}\alpha}$, in terms of the inverse operator. In fact, we obtain an explicit expression for the positive powers

$$\Big(- P(D,D) + \sum_{j=1}^{n} \gamma_j \frac{\partial}{\partial x_j} \Big)^{\frac{\alpha}{2}},\ \operatorname{Re}\alpha > 0$$

and describe their domains. We do not restrict ourselves to the case $\operatorname{Re}\alpha < n+1$, $1 \le p < \frac{n+1}{\operatorname{Re}\alpha}$ when the operator J_γ^α is well-defined on the whole L_p and consider the case $\operatorname{Re}\alpha > 0,\ 1 \le p < \infty$ as well. In the case when $\operatorname{Re}\alpha$ is "large" we describe the classes $J_\gamma^\alpha(L_p) \cap (L_r + L_s)$.

Let us justify our interest to the consideration of such classes. One of the main problems in potential theory regards to the description of functions represented as potentials with densities in L_p, that is, the description of the range $J^\alpha(L_p)$ of the potential J^α. The kernel of the potential may grow at infinity (as in the case under the consideration when $\operatorname{Re}\alpha$ is large); it may also have other "bad" properties (as, for example, singularities, "spread" over different sets in $\mathbb{R}^n$, which may be locally unsummable). In this case the integral $J^\alpha\varphi$ does not converge, generally speaking, and the potential J^α is treated as an operator acting from L_p into a space of distributions on a suitable space of test functions where the kernel of potential must be a convolute. In this case the description of the range $J^\alpha(L_p)$ seems to be a problem. But sometimes it is possible to describe the class $J^\alpha(L_p) \cap L_r$ (or $J^\alpha(L_p) \cap (L_r + L_s)$).

The paper is organized as follows. First, we dwell on the model case when $P(D,D) = \Delta$ in (1.1) and obtain our results for the potential

$$I_\gamma^\alpha \varphi = \Big(- \Delta + \sum_{j=1}^{n} \gamma_j \frac{\partial}{\partial x_j} \Big)^{-\frac{\alpha}{2}} \varphi, \quad \operatorname{Re}\alpha > 0. \tag{1.5}$$

Then on the basis of equality

$$J_\gamma^\alpha = \mathcal{L}^{-1} I_{\widetilde{\gamma}}^\alpha \mathcal{L},\ \widetilde{\gamma} = \mathcal{B}^\tau \gamma,$$

where $(\mathcal{L}\varphi)(x) = \varphi((B^\tau)^{-1}x)$, B is the matrix such that $P(Bx, Bx) = |x|^2$, we extend these results to the general case of the operator (1.1).

We also note that the case $\operatorname{Re}\gamma = (0, 0, \ldots, 0)$ is, in fact, the case of acoustic potentials, because of the relation

$$(1.6) \qquad I_\gamma^\alpha = |b|^{-1} U^{-1} H^\alpha U, \quad U\varphi(x) = \exp\Big\{ -i\frac{b \cdot x}{|b|}\Big\}\varphi\Big(\frac{x}{|b|}\Big), \quad \gamma = 2bi, \ b \in \mathbb{R}^n.$$

Here H^α is the acoustic potential realizing negative powers of the Helmholtz operator. In this case we mainly use the results on inversion of the potentials $H^\alpha\varphi$, $0 < \operatorname{Re}\alpha < n+1$, $\varphi \in L_p$ $1 \le p < \frac{2n}{n+\operatorname{Re}\alpha-1}$, obtained in [20], with slight changes for our needs.

Some results presented here, in the case $0 < \operatorname{Re}\alpha < n+1$, $1 \le p < \frac{n+1}{\operatorname{Re}\alpha}$, were announced in [5].

2 Preliminaries

We begin with the following notations: $\langle f, g\rangle = \int_{\mathbb{R}^n} \overline{f(x)} g(x) dx$; $Ff(\xi) \equiv \widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{ix\cdot\xi} dx$ is the Fourier transform and $F^{-1}f(x) = (2\pi)^{-n} Ff(-x)$ is the inverse Fourier transform; $[a]$ denotes the integral part of a real number a; $L_p \equiv L_p(\mathbb{R}^n)$; $L_{p,\mu}$ is the space of measurable functions f such that $(1+|x|)^{-\frac{\mu}{p}} f(x) \in L_p$; $W(x, \varepsilon) = (4\pi\varepsilon)^{-\frac{n}{2}} \exp\{-\frac{|x|^2}{4\varepsilon}\}$, $\varepsilon > 0$, is the Gauss-Weierstrass kernel, $W_\varepsilon\varphi(x) = \int_{\mathbb{R}^n} W(x-y, \varepsilon)\varphi(y) dy$; $S \equiv S(\mathbb{R}^n)$ is the Schwartz class of rapidly decreasing smooth functions and S' is the space of tempered distributions. Let V be an arbitrary closed set in $\mathbb{R}^n$. The Samko-Lizorkin space ([6], [7]) $\Phi_V \equiv \Phi_V(\mathbb{R}^n)$ is defined to consist of Schwartz functions φ such that $\widehat{\varphi}$ and all its derivatives vanish on V; the dual space $\Psi_V \equiv \Psi_V(\mathbb{R}^n) = F\Phi_V$ is a topological space equipped with the countable set of pairwise coordinated comparable norms

$$\|\psi\|_N = \sup_{\substack{x \in \mathbb{R}^n \setminus 0 \\ |k| \le N}} \Big[\max\Big\{\sqrt{1+|x|^2}, \frac{1}{\rho(x, V)}\Big\}\Big]^N |\mathcal{D}^k \psi(x)|, \quad N = 0, 1, 2, \ldots,$$

where

$$\mathcal{D}^k\psi(x) = \frac{\partial^{|k|}}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}} \psi(x), \ \ k = (k_1, \ldots, k_n), \ \ |k| = k_1 + \ldots + k_n, \ \ \rho(x, V) = \min_{y \in V} |x-y|.$$

The symbols Φ'_V, Ψ'_V denote the spaces of distributions on Φ_V, Ψ_V respectively.

Theorem 2.1 *([12], [17]) A function $g(x)$ which agrees with the distribution $g \in \Psi'_V$ is a multiplier in Ψ_V if and only if $g(x) \in C^\infty(\mathbb{R}^n \setminus V)$ and for each multiindex k there exist a natural $\nu(k)$ and a positive $c(k)$, such that*

$$|\mathcal{D}^k g(x)| \le c(k) \Big[\max\Big\{\sqrt{1+|x|^2}, \frac{1}{\rho(x, V)}\Big\}\Big]^{\nu(k)}, \quad x \in \mathbb{R}^n \setminus V.$$

In what follows we use only two cases, when i) $V = \{\xi \in \mathbb{R}^n : \lambda \cdot \xi = 0,\ \lambda \in \mathbb{R}^n\}$ and ii) $V = \{\xi \in \mathbb{R}^n : |\xi + b|^2 - |b|^2 = 0,\ b \in \mathbb{R}^n\}$ and the following information about the corresponding spaces Φ_V (see [12], [13], [18]). In the case i) the space Φ_V is dense in L_p, $1 < p < \infty$, and a function $\omega \in S$ belongs to Φ_V if and only if $\int_{\mathbb{R}} t^k \omega(x - \lambda t)dt = 0$ for any $k = 0, 1, \dots$ and all $x \in \mathbb{R}^n$. In the case ii) the denseness of Φ_V in L_p is known for $2 \le p < \infty$ only (see Remark 4.8 below).

Theorem 2.2 *([12]) Let $f \in L_1$ and suppose $\frac{\partial^m}{\partial x_1 \dots \partial x_m} f(x) \in L_p,\ m = 1, \dots, n$ for almost all $x \in \mathbb{R}^n$ and some $p \in (1, 2]$, where $k_1, \dots, k_m$ are pairwise different. Then $Ff \in L_1$.*

We also use the following result.

Theorem 2.3 *([11]) Let the function f(x,z) be analytic in z in some domain $D \subset \mathbb{C}$ for almost all $x \in \Omega \subset \mathbb{R}^n$ and admits an integrable dominant, that is, $|f(x,z)| \le F(x) \in L_1$. Then the integral $\int_\Omega f(x,z)dx$ is an analytic function in D.*

Consider the function

$$g_\gamma^\alpha(x) = \frac{\left(\sum_{j=1}^n \gamma_j^2\right)^{\frac{n-\alpha}{2}}}{2^{n-1}\pi^{n/2}\Gamma(\alpha/2)} |x|^{\frac{\alpha-n}{2}} K_{\frac{n-\alpha}{2}}\left(\frac{\sqrt{\sum_{j=1}^n \gamma_j^2}}{2}|x|\right) \exp\left\{\frac{\gamma \cdot x}{2}\right\}.$$

Here $K_\nu(z)$ is the McDonald function of order ν, and z^α is the main branch of the multivalued function $z^\alpha = \exp\{\alpha(\ln|z| + i \arg z + 2\pi k i\}$, being analytic in $\mathbb{C} \setminus (-\infty, 0]$. The "size" of McDonald's function near the origin ($|z| < 1$) is estimated by

$$|K_\nu(z)| \le c \begin{cases} |z|^{-\mathrm{Re}\,\nu}, & |\mathrm{Re}\,\nu| \ge \delta,\quad \delta \in (0, M), \\ |z|^{-\mathrm{Re}\,\nu}(c(M) + \ln 1/|z|), & |\mathrm{Re}\,\nu| \ge 0. \end{cases} \tag{2.1}$$

We also recall that $K_\nu(z)$ has the following asymptotic behavior at infinity:

$$K_\nu(z) \sim \sqrt{\pi/2z}\, e^{-z}, \quad |z| \longrightarrow \infty. \tag{2.2}$$

Now we define negative powers (1.5) for $\varphi \in L_p$ as convolutions with the kernel $g_\gamma^\alpha(x)$. They are treated in the regular sense (as integrals over $\mathbb{R}^n$) for $0 < \mathrm{Re}\,\alpha < n+1,\ 1 \le p < \frac{n+1}{\mathrm{Re}\,\alpha}$ and in the distributional sense in the remaining cases. As was mentioned above, the condition (1.4) appears here as sufficient for the mentioned realization and as necessary at least in the case $0 < \mathrm{Re}\,\alpha < n,\ 1 \le p < \frac{n+1}{\mathrm{Re}\,\alpha}$.

Indeed, by (2.2) we have

$$|g^\alpha_\gamma(x)| \sim c|x|^{\frac{\operatorname{Re}\alpha-n-1}{2}} \exp\Big\{\operatorname{Re}\frac{\gamma\cdot x}{2} - \frac{|x|}{2}\operatorname{Re}\sqrt{\sum_{j=1}^n \gamma_j^2}\Big\}, \quad |x| \longrightarrow \infty.$$

Let us assume that $\delta \equiv \operatorname{Re}\sqrt{\sum_{j=1}^n \gamma_j^2} < |\operatorname{Re}\gamma|$. Then for any x in the cone

$$\frac{x}{|x|}\cdot\frac{\operatorname{Re}\gamma}{|\operatorname{Re}\gamma|} > \frac{\delta+\varepsilon_0}{|\operatorname{Re}\gamma|}, \quad 0<\varepsilon_0<|\operatorname{Re}\gamma|-\delta$$

we have

$$\exp\Big\{\operatorname{Re}\frac{\gamma\cdot x}{2} - \frac{|x|}{2}\operatorname{Re}\sqrt{\sum_{j=1}^n \gamma_j^2}\Big\} = \exp\Big\{\frac{|x||\operatorname{Re}\gamma|}{2}\Big(\frac{x}{|x|}\cdot\frac{\operatorname{Re}\gamma}{|\operatorname{Re}\gamma|} - \frac{\delta}{|\operatorname{Re}\gamma|}\Big)\Big\} \geq \exp\Big\{\frac{|x|}{4}\varepsilon_0\Big\}$$

Thus it is sufficient, and in the case $0 < \operatorname{Re}\alpha < n,\ 1 \leq p < \frac{n+1}{\operatorname{Re}\alpha}$ is also necessary, to assume that

$$|\operatorname{Re}\gamma| - \operatorname{Re}\sqrt{\sum_{j=1}^n \gamma_j^2} \leq 0,$$

which is equivalent to the following:

$$|\operatorname{Re}\gamma||\operatorname{Im}\gamma| \leq |\operatorname{Re}\gamma\cdot\operatorname{Im}\gamma| \tag{2.3}$$

On the other hand, as is seen from the identity

$$\Big(\sum_{j=1}^n a_j b_j\Big)^2 = \sum_{j=1}^n a_j^2 \sum_{j=1}^n b_j^2 - \frac{1}{2}\sum_{j=1}^n\sum_{i=1}^n (a_i b_j - a_j b_i)^2,$$

only the inequality in (2.3) is possible, which is exactly the condition (1.4).

Remark 2.4 *We assume the condition (1.4) to be fulfilled everywhere below and write* $\gamma = \lambda a$, *where* $\lambda\in\mathbb{R}^n$, $a\in\mathbb{C}$. *Thus the function* $g^\alpha_\gamma(x)$ *takes the form*

$$g^\alpha_\gamma(x) \equiv g^\alpha_{\lambda a}(x) = \frac{(|\lambda||a|)^{\frac{n-\alpha}{2}}}{2^{n-1}\pi^{n/2}\Gamma(\alpha/2)}|x|^{\frac{\alpha-n}{2}} K_{\frac{n-\alpha}{2}}\left(\frac{|\lambda|\sqrt{a}}{2}|x|\right) e^{a\frac{\lambda\cdot x}{2}}. \tag{2.4}$$

Without loss of generality we also assume that $\operatorname{Re} a \geq 0$.

As was mentioned in the case $\operatorname{Re} a = 0$ (we usually write $\lambda a = 2bi,\ b \in \mathbb{R}^n$) the operator I^α_{2bi} is connected by (1.6) with the acoustic potential H^α defined in [20] for $\varphi \in L_p$, $1 \le p < \frac{2n}{n+\operatorname{Re}\alpha-1}$ as

$$H^\alpha\varphi(x) = \int_{\mathbb{R}^n} h_\alpha(y)\varphi(x-y)dy, \quad h_\alpha(x) = i\frac{2^{-\frac{n+\alpha}{2}}\pi^{1-\frac{n}{2}}}{\Gamma(\frac{\alpha}{2})}|x|^{\frac{\alpha-n}{2}}H^1_{\frac{n-\alpha}{2}}(|x|),$$

$H^1_\nu(z)$ being the first Hankel function.

The following statement follows from (2.4), (2.1), (2.2).

Lemma 2.5 *The function $g^\alpha_{\lambda a}(x)$ admits the estimates*

$$\text{(2.5)} \qquad |g^\alpha_{\lambda a}(x)| \le c \begin{cases} |x|^{\operatorname{Re}\alpha-n}, \quad |x|<1, \quad 0<\operatorname{Re}\alpha<n, \\ |x|^{\operatorname{Re}\alpha-n}\left(1+\ln\frac{1}{|x|}\right), \quad |x|<1, \quad 0<\operatorname{Re}\alpha\le n, \\ 1, \quad |x|<1, \quad n<\operatorname{Re}\alpha<n+1, \\ |x|^{\frac{\operatorname{Re}\alpha-n-1}{2}}, \quad |x|>1, \quad 0<\operatorname{Re}\alpha<n+1. \end{cases}$$

Moreover,

$$\text{(2.6)} \qquad \int_{\delta<|x|<u} |g^\alpha_{\lambda a}(x)|^q dx \le c\int_\delta^\mu t^{q\frac{\operatorname{Re}\alpha-n-1}{2}+\frac{n-1}{2}}dt,$$

where $0 < \operatorname{Re}\alpha < \infty$, $0 < \delta < \mu \le \infty$ *and* c *is independent on* μ. *In particular,* $g^\alpha_{\lambda a}(x) \in L_q$ *when* $0 < \operatorname{Re}\alpha < n+1$, $1 - \operatorname{Re}\alpha/n < 1/q < 1 - \operatorname{Re}\alpha/(n+1)$, $1 \le q \le \infty$.

3 Integral representation and $L_p \to L_q$ estimates for potential $I^\alpha_{\lambda a}$

For $0 < \operatorname{Re}\alpha < n+1$, $1 \le p < \frac{n+1}{\operatorname{Re}\alpha}$ we define the potential (1.5) for $\varphi \in L_p$ as follows,

$$\text{(3.1)} \qquad I^\alpha_{\lambda a}\varphi(x) = \int_{\mathbb{R}^n} g^\alpha_{\lambda a}(y)\varphi(x-y)dy, \qquad 0<\operatorname{Re}\alpha<n+1$$

(see (3.3), (3.5) below for the justification of this definition). For the remaining values of the parameters p, $\operatorname{Re}\alpha$, we also define the potential $I^\alpha_{\lambda a}$ on the whole L_p, treating it in the sense of Φ'_V distributions, where

$$V = \begin{cases} V_1 \equiv \{\xi \in \mathbb{R}^n : \lambda\cdot\xi = 0\}, \quad \operatorname{Re} a \ne 0, \\ V_2 \equiv \{\xi \in \mathbb{R}^n : |\xi+b|^2 - |b|^2 = 0\}, \quad \operatorname{Re} a = 0, (\lambda a = 2bi, b \in \mathbb{R}^n) \end{cases}$$

Namelly, we set

$$\langle I^{\alpha}_{\lambda a}\varphi, \omega\rangle = \langle \varphi, (I^{\alpha}_{\lambda a})^{*}\omega\rangle, \quad \omega \in \Phi_V,$$

where $(I^{\alpha}_{\lambda a})^{*} = I^{\overline{\alpha}}_{\lambda \overline{a}}$. The correctness of this definition is justified by the Gelfand-Shilov theorem ([3]).

The following theorem provides $L_p \to L_q$ estimates for the operator $I^{\alpha}_{\lambda a}$. We make no distinction here between the cases $\operatorname{Re} a = 0$ and $\operatorname{Re} a \neq 0$, which makes it convenient to use the notation $I^{\alpha}_{\lambda a}$ for both cases.

Theorem 3.1 *Let* $0 < \operatorname{Re}\alpha < n+1, \quad 1 \le p < \frac{n+1}{\operatorname{Re}\alpha}, \quad \varphi \in L_p$.

1. *The integral (3.1) converges for almost all* $x \in \mathbb{R}^n$. *The operator* $I^{\alpha}_{\lambda a}$ *is bounded from* L_p *into* L_q, $1 \le p \le q \le \infty$, *where*

$$\begin{aligned} &(3.2) \qquad \frac{1}{p} - \frac{\operatorname{Re}\alpha}{n} \le \frac{1}{q} \le \frac{1}{p} - \frac{\operatorname{Re}\alpha}{n+1}, \quad p \neq 1, \\ &\qquad\qquad \frac{1}{p} - \frac{\operatorname{Re}\alpha}{n} < \frac{1}{q} < \frac{1}{p} - \frac{\operatorname{Re}\alpha}{n+1}, \quad p = 1, \end{aligned}$$

except when both $p = n/\operatorname{Re}\alpha$ *and* $q = \infty$. *It is also of the weak* $(1, \frac{n+1}{n+1-\operatorname{Re}\alpha})$ *and* $(1, \frac{n}{n-\operatorname{Re}\alpha})$*-types, that is,*

$$\operatorname{mes}\{x \in \mathbb{R}^n : \ |I^{\alpha}_{\lambda a}\varphi(x)| > \delta\} \le \left(c\frac{\|\varphi\|_1}{\delta}\right)^{r}, \quad \varphi \in L_1, \quad \delta > 0,$$

where $r = \frac{n}{n-\operatorname{Re}\alpha}$ *for* $0 < \operatorname{Re}\alpha < n$ *or* $r = \frac{n+1}{n+1-\operatorname{Re}\alpha}$, *for* $0 < \operatorname{Re}\alpha < n+1$ *and the constant* c *does not depend either on* φ *or on* δ.

2. *If* $p \le r \le q$, *where* r *satisfies (3.2) (including the case* $p = 1$*), then* $I^{\alpha}_{\lambda a}$ *is bounded from* L_p *into* $L_{r,\gamma}$, $\gamma > n(1/r - 1/q)$. *In particular, one may take* $\gamma > np\operatorname{Re}\alpha/(n+1)$ *when* $p = r$.

3. *If* $1 \le p < \infty$, $\operatorname{Re}\alpha \ge n/p$, *except the case* $p = 1$ *and* $\operatorname{Re}\alpha = n$, *then the function* $I^{\alpha}_{\lambda a}\varphi(x)$ *is continuous on* $\mathbb{R}^n$ *for* $\varphi \in L_p$ *such that* $\lim_{|x|\to\infty} I^{\alpha}_{\lambda a}\varphi(x) = 0$.

PROOF. Let χ be the characteristic function of the unit ball. Represent the integral (3.1) as follows:

$$I^{\alpha}_{\lambda a}\varphi(x) = \int_{\mathbb{R}^n} g^{\alpha}_{\lambda a}(y)\chi(y)\varphi(x-y)dy + \int_{\mathbb{R}^n} g^{\alpha}_{\lambda a}(y)[1-\chi(y)]\varphi(x-y)dy \equiv I^{\alpha}_{\lambda a,0}\varphi(x) + I^{\alpha}_{\lambda a,\infty}\varphi(x).$$

By the Hardy-Littlewood-Sobolev theorem ([4], [19]) the operator $I^{\alpha}_{\lambda a,0}$, is bounded from L_p into L_q, $\frac{1}{q}=\frac{1}{p}-\frac{\operatorname{Re}\alpha}{n}$ and is of the weak $(1,\frac{n}{n-\operatorname{Re}\alpha})$-type when $0<\operatorname{Re}\alpha<n$. It is also bounded from L_p into L_q, $\frac{1}{q}>1-\frac{\operatorname{Re}\alpha}{n}$ for $0<\operatorname{Re}\alpha\leq n$, by Young's theorem. For $n<\operatorname{Re}\alpha<n+1$ the operator $I^{\alpha}_{\lambda a,0}$ is obviously bounded from L_p into L_q for any $1\leq p\leq q\leq\infty$ (since $g^{\alpha}_{\lambda a}\chi\in L_1\cap L_\infty$).

By standard Martsinkievich arguments and due to (2.6) one can show that the operator $I^{\alpha}_{\lambda a,\infty}$ is bounded from L_p into L_q, $\frac{1}{q}=\frac{1}{p}-\frac{\operatorname{Re}\alpha}{n+1}$ and is of the weak $(1,\frac{n+1}{n+1-\operatorname{Re}\alpha})$ type for $0<\operatorname{Re}\alpha<n+1$. It is also bounded from L_1 into L_∞ since $g^{\alpha}_{\lambda a}[1-\chi]\in L_\infty$.

The above implies *1*. Statement *2* is obtained by usual arguments. Let us prove statement *3*. We have

$$|I^{\alpha}_{\lambda a}\varphi(x+y)-I^{\alpha}_{\lambda a}\varphi(x)|\leq\begin{cases}\|g^{\alpha}_{\lambda a}(x+y)-g^{\alpha}_{\lambda a}(x)\|_{p'}\|\varphi\|_p, & \text{if} \quad p\neq 1,\\ \|g^{\alpha}_{\lambda a}\|_{p'}\|\varphi(x+y)-\varphi(x)\|_p, & \text{if} \quad p=1,\end{cases}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $\frac{n}{p}\leq\operatorname{Re}\alpha<\frac{n+1}{2}$, except in the case $\operatorname{Re}\alpha=n$, $p=1$. Thus, *3* follows by virtue of continuity of a function in L_p in the L_p norm $(1\leq p<\infty)$. □

This result slightly improves the corresponding result from [20] in the case $\operatorname{Re} a=0$.

We point out an interesting problem: to give a characterization of all pairs $(1/p,1/q)$ such that the operator H^{α} is bounded from L_p into L_q, that is, to construct the $\mathcal{L}$ - characteristic of the potential H^{α}. This problem looks very nontrivial and remains still to be open. In [8] the authors made an attempt to construct the $\mathcal{L}$ - characteristic of the operator H^{α}, but their construction has some gaps.

Lemma 3.2 *Let* $\operatorname{Re}\alpha>0, \operatorname{Re} a\neq 0$, $\varphi\in\Phi_{V_1}$. *Then*

$$FI^{\alpha}_{\lambda a}\varphi(\xi)=(|\xi|^2-ia\lambda\cdot\xi)^{-\alpha/2}\widehat{\varphi}(\xi),\quad \xi\in\mathbb{R}^n. \tag{3.3}$$

Proof. Let $\operatorname{Im} a=0$ first. From the results of [1] we obtain

$$\int_{\mathbb{R}^n}g^{\alpha}_{\lambda a}(y)\varphi(x-y)\ dy=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{-ix\cdot\xi}\frac{\widehat{\varphi}(\xi)}{(|\xi|^2-ia\lambda\cdot\xi)^{\alpha/2}}\ d\xi,\quad \varphi\in\Phi_{V_1}.$$

Let $\xi=\omega_\lambda(\eta)$, where ω_λ is a fixed rotation of $\mathbb{R}^n$ such that $\omega_\lambda(1,0,\ldots,0)=\frac{\lambda}{|\lambda|}$. We have

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{-ix\cdot\xi}\frac{\widehat{\varphi}(\xi)}{(|\xi|^2-ia\lambda\cdot\xi)^{\alpha/2}}\,d\xi=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{-i\omega^{-1}_\lambda(x)\cdot\eta}\frac{\widehat{\varphi}(\omega_\lambda(\eta))}{(|\eta|^2-ia|\lambda|\eta_1)^{\alpha/2}}\,d\eta.$$

Hence,

$$\int_{\mathbb{R}^n}g^{\alpha}_{\lambda a}(y)\varphi(x-y)\,dy=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{-i\omega^{-1}_\lambda(x)\cdot\eta}\frac{\widehat{\varphi}(\omega_\lambda(\eta))}{(|\eta|^2-ia|\lambda|\eta_1)^{\alpha/2}}\,d\eta. \tag{3.4}$$

The last equality is extended by analyticity to the case $a \in \mathbb{C}$, $\operatorname{Re} a > 0$. From the analyticity of the McDonald function, the estimate (2.5), and due to Theorem 2.3 we obtain analyticity of the left-hand side of (3.4) in a small neighborhood of each point a, $\operatorname{Re} a > 0$.

Let $\psi(\eta) = \widehat{\varphi}(\omega_\lambda(\eta))$, $\widetilde{\eta} = (\eta_2, \ldots, \eta_n) \in \mathbb{R}^{n-1}$, $\widetilde{V}_1 = \omega_\lambda^{-1}(V_1) = \{\eta \in \mathbb{R}^n : \eta_1 = 0\}$ and let Ω be a ball in $\mathbb{R}^n$ containing all zeroes of the function $|\eta|^2 - ia|\lambda|\eta_1$. We have

$$\sup_{\eta\in\Omega} \left| \frac{\psi(\eta)}{(|\eta|^2 - ia|\lambda|\eta_1)^{\alpha/2}} \right| \le c \sup_{\eta\in\Omega} \left| \frac{\psi(\eta) - \sum_{k=0}^{\left[\frac{\operatorname{Re}\alpha}{2}\right]} \frac{1}{k!} \frac{\partial^k}{\partial \eta_1^k} \psi(0, \widetilde{\eta}) \eta_1^k}{|\eta_1|^{\operatorname{Re}\alpha/2}} \right| < \infty$$

(here we use the relation $\widehat{\varphi}(\omega_\lambda(\eta)) \in \Psi_{\widetilde{V}_1}$). The function under the integral sign in the right-hand side of (3.4) is estimated by $c|\widehat{\varphi}(\omega_\lambda(\eta))|$ outside of the ball Ω. As above, due to Theorem 2.3 we obtain the analyticity of the right side of (3.4) in $\mathbb{C} \setminus (-\infty, 0]$. Thus, (3.4) is proved for $a \in \mathbb{C}$, $\operatorname{Re} a > 0$. Now we have

$$\widehat{I^\alpha_{\lambda a}\ \varphi}(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, dx \int_{\mathbb{R}^n} e^{-i\omega_\lambda^{-1}(x)\cdot\eta} \frac{\widehat{\varphi}(\omega_\lambda(\eta))}{(|\eta|^2 - ia|\lambda|\eta_1)^{\alpha/2}}\, d\eta =$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{iy\cdot\omega_\lambda^{-1}(\xi)}\, dy \int_{\mathbb{R}^n} e^{-iy\cdot\eta} \frac{\widehat{\varphi}(\omega_\lambda(\eta))}{(|\eta|^2 - ia|\lambda|\eta_1)^{\alpha/2}}\, d\eta = \frac{\widehat{\varphi}(\xi)}{(|\xi|^2 - ia\lambda\cdot\xi)^{\alpha/2}}.$$

□

Corollary 3.3 *If* $\operatorname{Re}\alpha > 0$, $\operatorname{Re} a \neq 0$ *then* $I^\alpha_{\lambda a}(\Phi_{V_1}) = \Phi_{V_1}$.

PROOF. Since $\rho(x, V_1) = \frac{1}{|\lambda|}|\lambda \cdot x|$, we only have to note that $(|\xi|^2 - ia\lambda\cdot\xi)^\delta$, $\delta \in \mathbb{C}$ is a multiplier in Ψ_{V_1} by Theorem 2.1 and then apply (3.3). □

Lemma 3.4 *Let* $\operatorname{Re}\alpha > 0$, $\operatorname{Re} a = 0$, $\lambda a = 2bi, b \in \mathbb{R}^n$, $\varphi \in \Phi_{V_2}$. *Then*

$$FI^\alpha_{2bi}\varphi(\xi) = (|\xi + b|^2 - |b|^2 - i0)^{-\alpha/2}\widehat{\varphi}(\xi), \quad \xi \in \mathbb{R}^n. \tag{3.5}$$

PROOF. We have

$$(F(|\xi|^2 - |b|^2 - i0)^{-\alpha/2})(x) \overset{(S')}{=} (2\pi)^n e^{-ib\cdot x} g_{\alpha,2bi}(x),$$

whence

$$\widehat{g_{\alpha,2bi}}(\xi) \overset{(S')}{=} (|\xi + b|^2 - |b|^2 - i0)^{-\alpha/2}.$$

For $0 < \operatorname{Re}\alpha < 2$, $\varphi \in \Phi_{V_2}$ we have

$$\int_{\mathbb{R}^n} \overline{g_{\alpha,2bi}(y)}\varphi(y)\,dy = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} (|\xi+b|^2 - |b|^2 - i0)^{-\overline{\alpha}/2}\widehat{\varphi}(\xi)\,d\xi.$$

The last equality is valid for all $\operatorname{Re}\alpha > 0$ by Theorem 2.3. Setting $\omega(y) = \overline{\varphi(x-y)}$, we arrive at the equality

$$\int_{\mathbb{R}^n} g_{\alpha,2bi}(y)\omega(x-y)\,dy = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} (|\xi+b|^2 - |b|^2 - i0)^{-\alpha/2}e^{-ix\cdot\xi}\widehat{\omega}(\xi)\,d\xi$$

which yields (3.5), since the function $g(x) = (|\xi+b|^2 - |b|^2)^{\gamma}$, $\gamma \in \mathbb{C}$, is a Ψ_{V_2} multiplier (by Theorem 2.1). □

Corollary 3.5 *If* $\operatorname{Re}\alpha > 0,\ b \in \mathbb{R}^n$ *then* $I^{\alpha}_{2bi}(\Phi_{V_2}) = \Phi_{V_2}$.

4 Inversion and description of potential $I^{\alpha}_{\lambda a}$ with L_p-densities

4.1 The case $\operatorname{Re} a > 0$.

Within the framework of the AIO method we construct the inverse to the $I^{\alpha}_{\lambda a}$ operator in the form

$$K^{\alpha}_{\lambda a}f(x) = \lim_{\varepsilon\to 0} K^{\alpha}_{\lambda a,\varepsilon}f(x) = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} K_{\alpha,\lambda a,\varepsilon}(x-y)f(y)dy, \tag{4.1}$$

the limit in (4.1) being taken in the L_p norm or almost everywhere. Here

$$K_{\alpha,\lambda a,\varepsilon}(x) = (F^{-1}k_{\alpha,\lambda a,\varepsilon})(x),\quad k_{\alpha,\lambda a,\varepsilon}(\xi) = (|\xi|^2 - ia\lambda\cdot\xi)^{\frac{\alpha}{2}}e^{-\varepsilon|\xi|^2}\Omega_s(\alpha,\varepsilon,\lambda\cdot\xi),$$

$$\Omega_s(\alpha,\varepsilon,\lambda\cdot\xi) = \begin{cases} \frac{(\lambda\cdot\xi)^{2s}}{[(\lambda\cdot\xi)^2+\varepsilon^2]^s}, & \operatorname{Re}\alpha < 2n,\\ 1\quad, & \operatorname{Re}\alpha \ge 2n,\end{cases}$$

and s is a natural number, $s > \max\{\frac{1}{2}(n - [\operatorname{Re}\alpha/2]), 0\}$.

Lemma 4.1 *The relation* $K_{\alpha,\lambda a,\varepsilon} \in L_1 \cap L_\infty$ *is valid.*

PROOF. Since $k_{\alpha,\lambda a,\varepsilon} \in L_1$, it suffices to prove that $Fk_{\alpha,\lambda a,\varepsilon}(\omega_\lambda(\cdot)) \in L_1$, where $y = \omega_\lambda(\xi)$ is the rotation defined in the proof of Lemma 3.2. We have

$$\frac{\partial^m}{\partial x_{k_1} \ldots \partial x_{k_m}} k_{\alpha,\lambda a,\varepsilon}(\omega_\lambda(\xi)) = e^{-\varepsilon|\xi|^2}(|\xi|^2 - ia|\lambda|\xi_1)^{\frac{\alpha}{2}-m}\Omega_s(\alpha,\varepsilon,\lambda\cdot\xi) \times$$
$$\begin{cases} P_m^1(|\xi|^2 - ia|\lambda|\xi_1)\xi_{k_1}\cdots\xi_{k_m}, \quad \text{if } \ k_j \neq 1, \quad j = 1,\ldots,m, \\ P_m^1(|\xi|^2 - ia|\lambda|\xi_1)(2 - \frac{i}{\xi_1}a|\lambda|)) - \frac{\delta_\alpha}{\xi_1[(|\lambda|\xi_1)^2+\varepsilon^2]}P_m^2(|\xi|^2 - ia|\lambda|\xi_1)]\xi_{k_1}\cdots\xi_{k_m}, \\ \text{if exists } \ j_0 \ \text{ such that } k_{j_0} = 1, \end{cases}$$

where $P_m^j(x)$ are polinomials of order m and $\delta_\alpha = 0$ for $\operatorname{Re}\alpha \geq 2n$. Hence $Fk_{\alpha,\lambda a,\varepsilon}(\omega_\lambda(\cdot)) \in L_1$ by Theorem 2.2. □

Consider the following auxiliary operator

$$A_\varepsilon\varphi(x) = \frac{1}{2}\int_{\mathbb{R}} e^{-\varepsilon|t|}\varphi(x - \lambda t)dt, \qquad x \in \mathbb{R}^n.$$

It is easily seen that

$$\widehat{A_\varepsilon\varphi}(\xi) = \frac{2\varepsilon\widehat{\varphi}(\xi)}{(\lambda\cdot\xi)^2 + \varepsilon^2}, \qquad \varphi \in S.$$

Set further

$$M_\varepsilon\varphi(x) = \sum_{k=1}^{s} C_s^k(-\varepsilon)^k A_\varepsilon^k\varphi(x)$$

where

$$A_\varepsilon^k\varphi(x) = \frac{1}{2^k}\int_{\mathbb{R}}\ldots\int_{\mathbb{R}} e^{-\varepsilon(|t_1|+\ldots+|t_k|)}\varphi(x - \lambda(t_1+\ldots+t_k))dt_1\ldots dt_k.$$

The following theorem provides the inversion formula for the potential $I_{\lambda a}^\alpha\varphi, \ \varphi \in L_p$.

Theorem 4.2 *Let* $\operatorname{Re}\alpha > 0$, $1 \leq p \leq \infty$, $\varphi \in L_p$, $f \in L_r + L_s$, $1 \leq r, \ s \leq \infty$. *If* $f(x) = I_{\lambda a}^\alpha\varphi(x)$ *a.e. for* $0 < \operatorname{Re}\alpha < n+1$, *or* $f = I_{\lambda a}^\alpha\varphi$ *in the sense of* Φ'_{V_1}*-distributions, then*

$$K_{\lambda a}^\alpha f(x) = \varphi(x), \tag{4.2}$$

where $K_{\lambda a}^\alpha$ *is the operator (4.1), the limit in (4.1) being understood as follows:*

1. *In the sense of a.e. convergence for* $1 \leq p < \infty$ *if* $0 < \operatorname{Re}\alpha < 2n$, *and for* $1 \leq p \leq \infty$ *if* $\operatorname{Re}\alpha \geq 2n$;

2. *In the* L_p*-norm for* $1 < p < \infty$ *if* $0 < \operatorname{Re}\alpha < 2n$, *and for* $1 \leq p < \infty$ *if* $\operatorname{Re}\alpha \geq 2n$.

In the case $p = 1$, $0 < \operatorname{Re}\alpha < 2n$, *if* $K^{\alpha}_{\lambda a}$ *is understood as the limit in the* L_1 *norm in (4.1), then (4.2) holds if and only if*

$$\int_{\mathbb{R}} \varphi(x - \lambda \cdot t)dt = 0, \qquad for\ almost\ all\ \ x \in \mathbb{R}^n. \tag{4.3}$$

Proof. We base ourselves on the equality

$$K^{\alpha}_{\lambda a,\varepsilon} f(x) = \delta_\alpha M_\varepsilon W_\varepsilon \varphi(x) + W_\varepsilon \varphi(x), \tag{4.4}$$

where $\delta_\alpha = 1$ if $0 < \operatorname{Re}\alpha < 2n$, and $\delta_\alpha = 0$ if $\operatorname{Re}\alpha \geq 2n$. Indeed, since $k_{\alpha,\lambda a,\varepsilon}$ is a Ψ_{V_1} multiplier (by Theorem 2.1), for $\omega \in \Phi_{V_1}$ we have

$$\begin{aligned} \langle K^{\alpha}_{\lambda a,\varepsilon} f, \omega \rangle &= \langle f, \overline{K^{\alpha}_{\lambda a,\varepsilon}\overline{\omega}} \rangle = \langle I^{\alpha}_{\lambda a}\varphi, \overline{K^{\alpha}_{\lambda a,\varepsilon}\overline{\omega}} \rangle = \langle \varphi, \overline{I^{\alpha}_{\lambda a} K^{\alpha}_{\lambda a,\varepsilon}\overline{\omega}} \rangle \\ &= \langle \varphi, (\delta_\alpha M_\varepsilon W_\varepsilon + W_\varepsilon)\omega \rangle = \langle (\delta_\alpha M_\varepsilon W_\varepsilon + W_\varepsilon)\varphi, \omega \rangle. \end{aligned} \tag{4.5}$$

Here the equality

$$\overline{I^{\alpha}_{\lambda a} K^{\alpha}_{\lambda a,\varepsilon}\overline{\omega}} = (\delta_\alpha M_\varepsilon W_\varepsilon + W_\varepsilon)\omega, \quad \omega \in \Phi_{V_1}$$

is easily verified via Fourier transform. The last equality of the chain (4.5) is justified by application of Fubini's theorem. One can find r_1, s_1, p_1 such that $1 < r_1,\ s_1,\ p_1 < \infty$, $K^{\alpha}_{\lambda a,\varepsilon} f \in L_{r_1} + L_{s_1}$, and $(\delta_\alpha M_\varepsilon W_\varepsilon + W_\varepsilon)\varphi \in L_{p_1}$. Let $\omega \in S$. We choose a sequence $\omega_k \in \Phi_{V_1}$, approximating the function ω in the $L_{p_1'}$, $L_{r_1'}$, and $L_{s_1'}$ norms simultaneously (the possibility of such approximation is proved in [18]). Passing to the limit as $k \longrightarrow \infty$ in the formula

$$\langle K^{\alpha}_{\lambda a,\varepsilon} f, \omega_k \rangle = \langle (\delta_\alpha M_\varepsilon W_\varepsilon + W_\varepsilon)\varphi, \omega_k \rangle, \quad \omega_k \in \Phi_{V_1},$$

we arrive at the equality

$$\langle K^{\alpha}_{\lambda a,\varepsilon} f, \omega \rangle = \langle (\delta_\alpha M_\varepsilon W_\varepsilon + W_\varepsilon)\varphi, \omega \rangle,\ \omega \in S,$$

which yields (4.4).

Now the theorem is proved in the case $\operatorname{Re}\alpha \geq 2n$, since the Gauss-Weierstrass integral $W_\varepsilon\varphi$ in the right side of (4.4) converges to $\varphi (\in L_p)$ in the L_p norm for $1 \leq p < \infty$ and a.e. for $1 \leq p \leq \infty$.

Let $0 < \operatorname{Re}\alpha < 2n$. It suffices to prove (except the necessity of condition (4.3)) that

$$\lim_{\varepsilon \longrightarrow 0} \varepsilon^k A^k_\varepsilon \varphi(x) = 0 \tag{4.6}$$

in the L_p-norm or a.e.

Let us choose $q>1$ such that p lies between q and 2. For $\varphi\in\Phi_{V_1}$ by the estimate

$$(4.7)\ \|\varepsilon^k A_\varepsilon^k\varphi\|_p\le c\varepsilon^k\|\varphi\|_p\int_{\mathbb{R}^k}e^{-\varepsilon(|t_1|+\ldots+|t_k|)}dt_1\ldots dt_k\le c\|\varphi\|_p,\ \varphi\in L_p,\ 1\le p\le\infty,$$

and the interpolation inequality

$$\|u\|_p\le\|u\|_r^\theta\|u\|_q^{1-\theta},\quad 1\le r<p<q\le\infty,\quad \frac{1}{p}=\frac{\theta}{r}+\frac{1-\theta}{q},\quad 0\le\theta\le 1,$$

we have

$$\|\varepsilon^k A_\varepsilon^k\varphi\|_p\le\|\varepsilon^k A_\varepsilon^k\varphi\|_q^{1-\theta}\|\varepsilon^k A_\varepsilon^k\varphi\|_2^\theta\le c\|\varphi\|_q^{1-\theta}\|\varepsilon^k(\lambda\cdot\xi+i\varepsilon)^{-k}\widehat{\varphi}(\xi)\|_2^\theta\longrightarrow 0,\ \varepsilon\longrightarrow 0.$$

Due to denseness of Φ_{V_1} in L_p, $1<p<\infty$ and the uniform estimate (4.7) we obtain

$$\overset{(L_p)}{\lim_{\varepsilon\longrightarrow 0}}\varepsilon^k A_\varepsilon^k\varphi=0,\quad\varphi\in L_p,\quad 1<p<\infty.$$

Let now $\varphi\in L_1$ and $\int_{\mathbb{R}}\varphi(x-\lambda t)dt=0$, $x\in\mathbb{R}^n$. Set $\psi(x)=\varphi(\omega_\lambda(x))$, $\widetilde{x}=(x_2,\ldots,x_n)\in\mathbb{R}^{n-1}$. We note that $\int_{\mathbb{R}}\psi(x_1-|\lambda|t,\widetilde{x})dt=0$. Consider

$$\begin{aligned}2\varepsilon A_\varepsilon\varphi(x) &= \varepsilon\int_{\mathbb{R}}e^{-\varepsilon|t|}\psi(x_1-|\lambda|t,\widetilde{x})dt=\varepsilon\int_{\mathbb{R}}e^{-\varepsilon|t|}\psi(x_1-|\lambda|t,\widetilde{x})dt-\\ &\varepsilon e^{-\varepsilon\frac{x_1}{|\lambda|}}\int_{\mathbb{R}}\psi(x_1-|\lambda|t,\widetilde{x})dt=\varepsilon\int_{\mathbb{R}}\Big[\exp\{-\varepsilon\frac{|x_1-\eta|}{|\lambda|}\}-\exp\{-\varepsilon\frac{|x_1|}{|\lambda|}\}\Big]\psi(\eta,\widetilde{x})d\eta\end{aligned}$$

By the Minkovskii inequality we obtain

$$\begin{aligned}&\|\varepsilon A_\varepsilon\varphi\|_1\le\frac{\varepsilon}{2}\int_{\mathbb{R}}\|\psi(\eta,\cdot)\|_{L_1(\mathbb{R}^{n-1}_{\widetilde{x}})}d\eta\int_{\mathbb{R}}\Big|\exp\{-\varepsilon\frac{|x_1-\eta|}{|\lambda|}\}-\exp\{-\varepsilon\frac{|x_1|}{|\lambda|}\}\Big|dx_1\\ &=\frac{1}{2}\int_{\mathbb{R}}\|\psi(\eta,\cdot)\|_{L_1(\mathbb{R}^{n-1}_{\widetilde{x}})}d\eta\int_{\mathbb{R}}\Big|\exp\{-\frac{|x_1-\varepsilon\eta|}{|\lambda|}\}-\exp\{-\frac{|x_1|}{|\lambda|}\}\Big|dx_1\equiv\int_{\mathbb{R}}F_\varepsilon(\eta)d\eta.\end{aligned}$$

We note that

$$|F_\varepsilon(\eta)|\le c\|\psi(\eta,\cdot)\|_{L_1(\mathbb{R}^{n-1}_{\widetilde{x}})}\in L_1(\mathbb{R})\quad\text{and}\quad F_\varepsilon(\eta)\longrightarrow 0,\ \ \varepsilon\to 0$$

for almost all $\eta\in\mathbb{R}$, whence

$$\|\varepsilon A_\varepsilon\varphi\|_1\longrightarrow 0,\quad\varepsilon\to 0$$

by the Lebesgue dominated convergence theorem. It can be easily shown that

$$\varepsilon^k A_\varepsilon^k \varphi(x) \le c\frac{\varepsilon}{2^{k-1}} A_{2^{1-k}\varepsilon}\varphi(x), \quad \text{and} \quad \varepsilon A_\varepsilon \varphi \xrightarrow{(L_1)} 0, \quad \varepsilon \to 0, \tag{4.8}$$

for $\varphi \in L_1$ and $\varphi(x) \ge 0$. Hence, for any $\varphi \in L_1$ satisfying (4.3) we get

$$\overset{(L_1)}{\lim_{\varepsilon\to 0}} \varepsilon^k A_\varepsilon^k \varphi = 0.$$

Conversely, let $\varphi \in L_1$ and $\|M_\varepsilon W_\varepsilon \varphi\|_1 \longrightarrow 0$, as $\varepsilon \to 0$. We have

$$\||M_\varepsilon W_\varepsilon \varphi\|_1 \ge (2\pi)^n \| \exp\{-\varepsilon|\xi|^2\}[(\lambda\cdot\xi)^2/((\lambda\cdot\xi)^2+\varepsilon^2)]^s \widehat{\varphi}(\xi)\|_\infty \longrightarrow 0, \;\; \varepsilon \to 0,$$

which implies $\widehat{\varphi}(\eta) = 0, \;\; \eta \in V_1$. From [17] we have the formula

$$\int_{\mathbb{R}} \varphi(x - \lambda\cdot\xi)d\xi = \frac{(2\pi)^{1-n}}{|\lambda|}\langle \delta_{V_1}, e^{-iy\cdot x}\widehat{\varphi}(y)\rangle,$$

where δ_{V_1} is the Dirac δ - function supported on V_1. Now, to prove the necessity of the condition (4.4) we only have to note that the right-hand side of the previous formula is equal to 0.

Let us prove (4.6) for $\varphi \in L_p, \; 1 \le p < \infty$, where the limit is interpreted in the "a.e." sense. In the case $p = 1$ we obviously have $\varepsilon A_\varepsilon \varphi(x) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$ a.e. in $\mathbb{R}^n$, which together with the estimate (4.8) yields (4.6).

Let now $1 < p < \infty$. By virtue of (4.9) it remains to prove that the limit $\overset{a.e.}{\lim_{\varepsilon\longrightarrow 0}} \varepsilon^k A_\varepsilon^k \varphi(x)$ exists (since the limit in the L_p norm coincides with the a.e. limit). Denote

$$\Lambda\psi(x) = \overline{\lim_{\varepsilon\longrightarrow 0}} \varepsilon^k A_\varepsilon^k \psi(x) - \underline{\lim_{\varepsilon\longrightarrow 0}} \varepsilon^k A_\varepsilon^k \psi(x), \quad x \in \mathbb{R}^n,$$

and represent the function φ as $\varphi = g + h$, where $h \in S$ and $\|g\|_p < \delta$, δ being an arbitrarily small positive number. Setting $\psi(x) = g(\omega_\lambda(x)$, we have

$$\begin{aligned} |\varepsilon^k A_\varepsilon^k g(x)| &= 2^{-k}\varepsilon^k \Big| \int_{\mathbb{R}} \dots \int_{\mathbb{R}} e^{-\varepsilon(t_1+\dots+t_k)} \psi(x_1 - |\lambda|(|t_1|+\dots+|t_k|), \widetilde{x}) dt_1 \dots dt_k \Big| \\ &\le c(M^k \psi(\cdot, \widetilde{x}))(x), \end{aligned}$$

where

$$(M\psi(\cdot,\widetilde{x}))(x_1) = \sup_{\varepsilon>0} \frac{1}{2\varepsilon}\int_{x_1-\varepsilon}^{x_1+\varepsilon} |\psi(x_1,\widetilde{x})|\, dx_1$$

is the maximal operator applied in the variable x_1, $\widetilde{x} = (x_2, x_3, \ldots, x_n) \in \mathbb{R}^{n-1}$. Due to weak boundedness of the maximal operator M in L_p, $1 \le p \le \infty$ we have

$$\text{mes}\,\{x \in \mathbb{R}^n : \ \Lambda\varphi(x) > \delta\} \le \text{mes}\,\{x : \ (M\psi(\cdot, \widetilde{x_1}))(x_1) > \delta/2\} \le \left(\frac{c}{\delta}\right)^p \|g\|_p^p, \ \delta > 0.$$

Since $\|g\|_p$ is arbitrarily small, we have $\text{mes}\,\{x \in \mathbb{R}^n : \ \Lambda\varphi(x) > \delta\} = 0, \ \delta > 0$, hence the mentioned limit exists. □

The following theorem provides the representability of a function $f \in L_r + L_s$ by the potential $I^\alpha_{\lambda a}\varphi, \ \varphi \in L_p$.

Theorem 4.3 *Let* $\text{Re}\,\alpha > 0$, $1 < p < \infty$, $f \in L_r + L_s$, $1 \le r, s \le \infty$.

1. *If* $K^\alpha_{\lambda a,\varepsilon} f$ *converges in the* L_p *norm as* $\varepsilon \longrightarrow 0$, $1 \le p < \infty$, *then*

$$f \overset{(\Phi'_{V_1})}{=} I^\alpha_{\lambda a}\varphi, \quad \textit{where} \quad \varphi = \overset{(L_p)}{\lim_{\varepsilon \to 0}} K^\alpha_{\lambda a,\varepsilon} f.$$

2. *If* $\sup_{\varepsilon > o} \|K^\alpha_{\lambda a,\varepsilon} f\|_p < \infty, \ 1 < p < \infty$ *then there exists* $\varphi \in L_p$ *such that*

$$f \overset{(\Phi'_{V_1})}{=} I^\alpha_{\lambda a}\varphi.$$

3. *For* $0 \le \text{Re}\,\alpha < n+1, \ 1 \le p < \frac{n+1}{\text{Re}\,\alpha}$ *the equality* $f(x) = I^\alpha_{\lambda a}\varphi(x)$ *in 1, 2 also holds a.e. in* $\mathbb{R}^n$.

Proof. Let $f \in L_r + L_s$ and $K^\alpha_{\lambda a,\varepsilon} f = \varphi \in L_p, \ 1 \le p < \infty$. Then for any $\omega \in \Phi_{V_1}$ we have:

$$\begin{aligned} \langle I^\alpha_{\lambda a}\varphi, \omega\rangle &= \langle \varphi, \overline{I^\alpha_{\lambda a}\overline{\omega}}\rangle = \langle \overset{(L_p)}{\lim_{\varepsilon \to 0}} K^\alpha_{\lambda a,\varepsilon} f, \overline{I^\alpha_{\lambda a}\overline{\omega}}\rangle = \lim_{\varepsilon \to 0} \langle K^\alpha_{\lambda a,\varepsilon} f, \overline{I^\alpha_{\lambda a}\overline{\omega}}\rangle \\ &= \lim_{\varepsilon \to 0} \langle f, \overline{K^\alpha_{\lambda a,\varepsilon} I^\alpha_{\lambda a}\overline{\omega}}\rangle = \lim_{\varepsilon \to 0} \langle f, (\delta_\alpha M_\varepsilon W_\varepsilon + W_\varepsilon)\omega\rangle. \end{aligned}$$

The third of these equalities follows from the fact that convergence in the L_p norm implies convergence in Φ'_{V_1}. Let us show that

$$\langle f, (\delta_\alpha M_\varepsilon W_\varepsilon + W_\varepsilon)\omega\rangle \longrightarrow \langle f, \omega\rangle, \quad \omega \in \Phi_{V_1}, \quad \varepsilon \to 0.$$

Indeed, for $f \in L_p$, $1 < p \le \infty$ by Theorem 4.2 we have

$$|\langle f, (\delta_\alpha M_\varepsilon W_\varepsilon + W_\varepsilon)\omega\rangle - \langle f, \omega\rangle| \le \|f\|_p \|(\delta_\alpha M_\varepsilon W_\varepsilon + W_\varepsilon)\omega - \omega\|_{p'} \longrightarrow 0, \quad \varepsilon \to 0.$$

For $f \in L_1$ this relation is justified by the Lebesgue dominated convergence theorem. Now the case $f \in L_r + L_s$, $1 \le r,\ s \le \infty$ turns to be a simple consequence of the previous one. Thus

$$\langle I^\alpha_{\lambda a}\varphi, \omega\rangle = \langle f, \omega\rangle, \quad \omega \in \Phi_{V_1}.$$

Let $f(x) \in L_r + L_s$ and $\sup_{\varepsilon>0} \|K^\alpha_{\lambda a,\varepsilon} f(x)\|_p < \infty$. We have

$$|\langle K^\alpha_{\lambda a,\varepsilon} f, \omega\rangle| \le c\|\omega\|_{p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

where c does not depend on ε. Thus by weak compactness of $L_p,\ 1 < p < \infty$, there exist a subsequence ε_k and a function $\varphi \in L_p$ such that

$$\langle K^\alpha_{\lambda a,\varepsilon_k} f, \omega\rangle \longrightarrow \langle \varphi, \omega\rangle, \quad \varepsilon \longrightarrow 0, \quad \omega \in L_{p'}.$$

Now, for φ and $\omega \in \Phi_{V_1}$ we have as above:

$$\langle I^\alpha_{\lambda a}\varphi, \omega\rangle = \langle \varphi, \overline{I^\alpha_{\lambda a}\overline{\omega}}\rangle = \lim_{\varepsilon_k \to 0} \langle K^\alpha_{\lambda a,\varepsilon_k} f, \overline{I^\alpha_{\lambda a}\overline{\omega}}\rangle = \lim_{\varepsilon_k \to 0} \langle f, K^\alpha_{\lambda a,\varepsilon_k} I^\alpha_{\lambda a}\omega\rangle = \langle f, \omega\rangle.$$

For $0 < \operatorname{Re}\alpha < n+1,\ \ 1 \le p < \frac{n+1}{\operatorname{Re}\alpha},\ \varphi \in L_p$, by Theorem 3.1 we have $I^\alpha_{\lambda a}\varphi \in L_q$, where q is described in the statement of the mentioned theorem. This relation and the equality $I^\alpha_{\lambda a}\varphi \overset{(\Phi'_{V_1})}{=} f$ imply that $I^\alpha_{\lambda a}\varphi(x) = f(x)$ for almost all $x \in \mathbb{R}^n$. $\square$

Let $I^\alpha_{\lambda a}(L_p)$ $(\subset \Phi'_{V_1})$ denote the range of the operator $I^\alpha_{\lambda a}$. One of the main results of this paper is the following.

Theorem 4.4 *Let* $0 < \operatorname{Re}\alpha < n+1$, $1 < p < \frac{n+1}{\operatorname{Re}\alpha}$. *Then*

$$I^\alpha_{\lambda a}(L_p) = \{f \in L_q : K^\alpha_{\lambda a} f \in L_p\} = \{f \in L_q : \sup_{\varepsilon>0} \|K^\alpha_{\lambda a,\varepsilon} f\|_p < \infty\},$$

where

$$\frac{1}{p} - \frac{\operatorname{Re}\alpha}{n} \le \frac{1}{q} \le \frac{1}{p} - \frac{\operatorname{Re}\alpha}{n+1}, \quad 1 \le q \le \infty,$$

except when both $p = n/\operatorname{Re}\alpha$ *and* $q = \infty$. *Here*

$$K^\alpha_{\lambda a} f = \overset{(L_p)}{\lim_{\varepsilon \to 0}} K^\alpha_{\lambda a,\varepsilon} f.$$

The statement of this theorem follows from Theorems 4.2 and 4.3.

4.2 The case $\operatorname{Re} a = 0$.

Here as above we write $\lambda a = 2bi,\ b \in \mathbb{R}^n$. Set

$$N_{2bi}^{\alpha} f(x) = \lim_{\varepsilon\to 0} N_{2bi,\varepsilon}^{\alpha} f(x) = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} N_{\alpha,2bi,\varepsilon}(y) f(x-y)dy. \tag{4.9}$$

Here $N_{\alpha,2bi,\varepsilon}(x) = |b|^{n+\alpha} e^{ib\cdot x} S_{\varepsilon,\alpha}(|b|x)$,

$$S_{\varepsilon,\alpha}(x) = \frac{(2\pi)^{-\frac{n}{2}}}{|x|^{\frac{n-2}{2}}} \int_0^{\infty} t^{n/2} e^{-\varepsilon t^2} \omega(\alpha,\varepsilon,t) J_{\frac{n-2}{2}}(t|x|)dt$$

$$\omega(\alpha,\varepsilon,t) = \begin{cases} \frac{(t^2-1)^{n+\alpha/2}}{(t^2+(\varepsilon+i)^2)^n}\ , & \operatorname{Re}\alpha < 2n, \\ (t^2-1-i0)^{\frac{\alpha}{2}} \quad , & \operatorname{Re}\alpha \ge 2n, \end{cases}$$

We note that

$$FN_{\alpha,2bi,\varepsilon}(\xi) = e^{-\varepsilon|\xi|^2}(|\xi|^2 + 2b\cdot\xi - i0)^{\frac{\alpha}{2}}, \quad \operatorname{Re}\alpha \ge 2n, \quad a.e.\ \ \xi \in \mathbb{R}^n.$$

Theorem 4.5 *([20]) Let* $0 < \operatorname{Re}\alpha < n+1,\ f = H^{\alpha}\varphi,\ \ \varphi \in L_p,\ \ 1 \le p < \frac{2n}{n+\operatorname{Re}\alpha-1}$. *Then*

$$S^{\alpha} f(x) = \varphi(x),$$

where

$$S^{\alpha} f(x) = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} S_{\varepsilon,\alpha}(y) f(x-y)dy,$$

the limit being taken in the $L_{p,\mu}$ *- norm,* $\mu > \frac{n+\operatorname{Re}\alpha-1}{2}p$ *or a.e.*

Theorem 4.6 *Let* $\operatorname{Re}\alpha > 0,\ 1 \le p \le \infty,\ \varphi \in L_p,\ f \in L_r + L_s,\ 1 \le r,s \le \infty$.

1. *If* $f(x) = I_{2bi}^{\alpha}\varphi(x),\ x \in \mathbb{R}^n,\ 0 < \operatorname{Re}\alpha < 2n, 1 \le p < \frac{2n}{n+\operatorname{Re}\alpha-1}$, *then*

   $$N_{2bi}^{\alpha} f(x) = \varphi(x), \tag{4.10}$$

   where N_{2bi}^{α} *is the operator (4.9), the limit in (4.9) being understood as the a.e. limit or in the norm of the weighted space* $L_{p,\mu}$, $\mu > \frac{n+\operatorname{Re}\alpha-1}{2}p$.

2. *If* $f = I_{2bi}^{\alpha}\varphi$ *in the sense of* Φ'_{V_2} *distributions and* $\operatorname{Re}\alpha \ge 2n$, *then (4.10) holds, where* N_{2bi}^{α} *is the operator (4.9), the limit in (4.9) being taken in the* L_p *norm for* $1 \le p < \infty$, *or a.e. for* $1 \le p \le \infty$.

Proof. The first statement of this theorem can be derived from the corresponding result of [20] due to relation (1.6) and Theorem 4.5. The second statement is obvious in view of the relation $N^{\alpha}_{2bi,\varepsilon}f = W_{\varepsilon}\varphi$, which can be proved in just the same way as in the previous subsection. $\square$

Theorem 4.7 *Let* $\operatorname{Re}\alpha > 0$, $1 < p < \infty$, $f \in L_r + L_s$, $1 \le r, s \le \infty$.

1. *If* $N^{\alpha}_{2bi,\varepsilon}f$ *converges in the* L_p *norm as* $\varepsilon \longrightarrow 0$, $1 \le p < \infty$, *then*

$$f \overset{(\Phi'_{V_2})}{=} I^{\alpha}_{2bi}\varphi, \quad \textit{where} \quad \varphi = \overset{(L_p)}{\lim_{\varepsilon\to 0}} N^{\alpha}_{2bi,\varepsilon}f.$$

2. *If* $\sup_{\varepsilon>o} \|N^{\alpha}_{2bi,\varepsilon}f\|_p < \infty,\ 1 < p < \infty$, *then there exists* $\varphi \in L_p$ *such that*

$$f \overset{(\Phi'_{V_2})}{=} I^{\alpha}_{2bi}\varphi.$$

3. *For* $0 \le \operatorname{Re}\alpha < n+1,\ 1 \le p < \frac{n+1}{\operatorname{Re}\alpha}$, $p \le 2$, *the equality* $f(x) = I^{\alpha}_{2bi}\varphi(x)$ *in 1, 2 also holds a.e.*

The proof is quite similar to that of Theorem 4.3.

Remark 4.8 *The unnatural restrictions in the theorems of this section are tightly connected with the denseness in* L_p *of the Samko-Lizorkin spaces* Φ_V. *The denseness of* Φ_V, *mes* $V = 0$ *in* L_p *is known (see [12], [18]) for* $2 \le p < \infty$. *In the case* $1 < p < 2$ *it has been proved in [12] and [18] for special types of sets only (the so-called quasibroken sets). In particular, an important case when* V *is a sphere (*$V = V_2$ *in our considerations) remains still to be open.*

5 The case of the operator $-P(D,D) + \sum_{j=1}^{n} \gamma_j \frac{\partial}{\partial x_j}$.

Here we apply the results of the previous sections to the operator (1.2). As above, we assume the condition (1.4) to be fulfilled and write $\gamma = \mu a$, $\mu \in \mathbb{R}^n$, $a \in \mathbb{C}$, $\operatorname{Re} a \ge 0$. Let us restrict ourselves to the case $\operatorname{Re} a \ne 0$. First we note that the statement of Theorem 3.1 is valid for the operator $J^{\alpha}_{\mu a}$. This means that the potential $J^{\alpha}_{\mu a}\varphi$, $\varphi \in L_p$, $1 \le p < \frac{n+1}{\operatorname{Re}\alpha}$, realized as the convolution

$$J^{\alpha}_{\mu a}\varphi(x) = |detB| \int_{\mathbb{R}^n} g_{\alpha,\mu B^{\tau}a}(B^{\tau}y)\varphi(x-y)dy, \quad 0 < \operatorname{Re}\alpha < n+1,$$

satisfies the same mapping properties as the potential $I^{\alpha}_{\lambda a}$ $(\lambda = B^{\tau}\mu)$ under the same conditions on the parameters $(1 \le p < \frac{n+1}{\operatorname{Re}\alpha},\ 0 < \operatorname{Re}\alpha < n+1)$.

For the remaining values of parameters we also define $J^{\alpha}_{\mu a}$ on the whole L_p as in Section 3, setting

$$\langle J^{\alpha}_{\mu a}\varphi, \omega\rangle = \langle \varphi, J^{\overline{\alpha}}_{\mu \overline{a}}\omega\rangle, \quad \omega \in \Phi_V,$$

where $V = \{\xi \in \mathbb{R}^n : P(\xi,\xi) - ia\mu\cdot\xi = 0\}$.

Further we have

$$FJ^{\alpha}_{\mu a}\varphi(\xi) = (P(\xi,\xi) - ia\mu\cdot\xi)^{-\frac{\alpha}{2}}\widehat{\varphi}(\xi), \quad \varphi \in \Phi_V, \quad \xi \in \mathbb{R}^n,$$

and, moreover, $J^{\alpha}_{\mu a}(\Phi_V) = \Phi_V$, $\operatorname{Re}\alpha > 0,\ \operatorname{Re} a \ne 0$.

Set

$$\mathcal{K}^{\alpha}_{\mu a} f(x) = \lim_{\varepsilon\to 0} \mathcal{K}^{\alpha}_{\mu a,\varepsilon} f(x), \quad \mathcal{K}^{\alpha}_{\mu a,\varepsilon} = \mathcal{L}^{-1} K^{\alpha}_{\mu B^{\tau} a,\varepsilon}\mathcal{L}, \tag{5.1}$$

where $K^{\alpha}_{\lambda a,\varepsilon}$ is the operator from (4.1).

Theorem 5.1 *Let* $\operatorname{Re}\alpha > 0$, $1 \le p \le \infty$, $\varphi \in L_p$, $f \in L_r + L_s$, $1 \le r,\ s \le \infty$. *If* $f(x) = J^{\alpha}_{\lambda a}\varphi(x)$ *a.e. for* $0 < \operatorname{Re}\alpha < n+1$, *or* $f = J^{\alpha}_{\mu a}\varphi$ *in the sense of* Φ'_V *distributions, then*

$$\mathcal{K}^{\alpha}_{\mu a} f(x) = \varphi(x), \tag{5.2}$$

where $\mathcal{K}^{\alpha}_{\mu a}$ *is given by (5.1), the limit in (5.1) being understood as follows:*

1. *In the "a.e." sense for* $1 \le p < \infty$ *if* $0 < \operatorname{Re}\alpha < 2n$, *and for* $1 \le p \le \infty$ *if* $\operatorname{Re}\alpha \ge 2n$;
2. *In the* L_p*-norm for* $1 < p < \infty$ *if* $0 < \operatorname{Re}\alpha < 2n$, *and for* $1 \le p < \infty$ *if* $\operatorname{Re}\alpha \ge 2n$.

If $p = 1$, $0 < \operatorname{Re}\alpha < 2n$, *and the limit in (5.1) is taken in the* L_1 *norm, then (5.2) holds if and only if*

$$\int_{\mathbb{R}} \varphi(x - \mu\cdot t)dt = 0 \qquad \textit{a.e. in } \mathbb{R}^n.$$

Theorem 5.2 *Let* $\operatorname{Re}\alpha > 0$, $1 < p < \infty$, $f \in L_r + L_s$, $1 \le r, s \le \infty$.

1. *If* $\mathcal{K}^{\alpha}_{\mu a,\varepsilon} f$ *converges in the* L_p *norm as* $\varepsilon \longrightarrow 0$, $1 \le p < \infty$, *then*

$$f \overset{(\Phi'_V)}{=} J^{\alpha}_{\mu a}\varphi, \quad \textit{where} \quad \varphi = \overset{(L_p)}{\lim_{\varepsilon\to 0}} \mathcal{K}^{\alpha}_{\mu a,\varepsilon} f.$$

2. *If* $\sup_{\varepsilon>o}\|\mathcal{K}^{\alpha}_{\mu a,\varepsilon}f\|_p<\infty,\ 1<p<\infty$*, then there exists* $\varphi\in L_p$ *such that*

$$f\overset{(\Phi'_V)}{=}J^{\alpha}_{\lambda a}\varphi.$$

3. *For* $0\le \operatorname{Re}\alpha<n+1,\ 1\le p<\frac{n+1}{\operatorname{Re}\alpha}$ *the equality* $f(x)=J^{\alpha}_{\lambda a}\varphi(x)$ *in 1, 2 also holds a.e.*

Theorem 5.3 *Let* $0<\operatorname{Re}\alpha<n+1$*,* $1<p<\frac{n+1}{\operatorname{Re}\alpha}$ *and* $J^{\alpha}_{\mu a}(L_p)$ $(\subset\Phi'_V)$ *denotes the range of the operator* $J^{\alpha}_{\mu a}$*. Then*

$$J^{\alpha}_{\mu a}(L_p)=I^{\alpha}_{\mu B^{\tau}a}(L_p)=I^{\alpha}_{\mu a}(L_p)$$

where the description of the range $I^{\alpha}_{\mu a}(L_p)$ *is covered by Theorem 4.4.*

Acknowledgements. The authors express their gratitude to Professors S. G. Samko and N. K. Karapetyants for useful discussions of the results.

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