Boundedness for some convolution and twisted convolution operators

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We consider $L_p - L_q$ boundedness problems for some special integral operators with oscillation both in the symbols and in the kernels. Such kernels have singularities on the unit sphere S^{n-1} and at infinity, and are smooth on $\mathbb{R}^n \setminus S^{n-1}$. The main purpose is to show how these features affect the boundedness picture. Some further problems are discussed.

1. Introduction

Consider the well known potential type operator

$$I_\theta^\alpha f(x) = \int_{\mathbb{R}^n} \frac{\theta(y)}{|y|^{n-\alpha}} f(x-y) d\mu(y), \quad 0 < \operatorname{Re} \alpha < n,$$

with smooth bounded characteristic function θ. Here $d\mu(y)$ denotes the usual Lebesgue volume measure in $\mathbb{R}^n$. The kernels of such operators have singularities only at the origin and at infinity. For $\theta(x) = 1$ the symbol of I_1^α is $\gamma_\alpha |\xi|^{-\alpha}$, where γ_α is some normalizing constant. It is well known that this operator is bounded from L_p to L_q if and only if p, q, α satisfy the Sobolev relation, i.e., $1/q = 1/p - \operatorname{Re} \alpha / n$, where $0 < \operatorname{Re} \alpha < n$. This condition is also sufficient for the boundedness of I_θ^α.

The natural continuation of this study is the consideration of potential type operators with kernels having singularities on sets in $\mathbb{R}^n$. As a matter of fact, there has been a great deal of research on boundedness properties of convolutions with such kernels, in particular, having singularities on the unit sphere. Such operators arise naturally in concrete problems of mathematical physics (without claiming completeness we refer to [10] – [13], [17] [21]). In particular, Miyachi [10], [11] investigated operators with symbols $a(|\xi|)|\xi|^{-\alpha} e^{i|\xi|}$, $0 \leq \alpha < n$ in connection with the Cauchy problem for the wave equation. Here $a(|\xi|)$ is a smooth cut-off function, $a(|\xi|) = 1$ for $|\xi| > 2$ and $a(|\xi|) = 0$, $|\xi| \leq 1$. See also a paper by Strichartz [21] on boundedness results for the convolution operator with kernel $k(x) = (1 - |x|^2)^{-\alpha/2}$, $|x| \leq 1$ and $k(x) = 0$, if $|x| > 1$, $0 < \alpha < (n+1)/2$. Note further, that the kernels of such operators have no singularities at infinity.

We start by considering the convolution operators $A^\alpha f = \Omega_\alpha * f$, where Ω_α is the distributional (inverse) Fourier transform of the symbol $m_\alpha(\xi) = |\xi|^{-\alpha} e^{i|\xi|}$. They are represented, at least for $(n+1)/2 < \operatorname{Re}\alpha < n$, as

$$A^\alpha f(x) = \int_{\mathbb{R}^n} \Omega_\alpha(y)\ f(x-y)\ d\mu(y), \quad f \in S. \tag{1.1}$$

Introducing oscillation in the multiplier (i.e., substituting $|\xi|^{-\alpha}$ with $|\xi|^{-\alpha} e^{i|\xi|}$) we change the boundedness properties essentially. It appears that the kernels Ω_α have singularities on the sphere S^{n-1} and at infinity, and are smooth on $\mathbb{R}^n \setminus S^{n-1}$. As a consequence, the set of $(1/p, 1/q)$ for which the corresponding operator is $L_p - L_q$ bounded has positive Lebesgue plane measure ($\operatorname{Re}\alpha \neq 0$). This is what Theorem 3.1 states, and the conditions in this theorem are sufficient and necessary.

The second step is to introduce, in addition, some kind of oscillation in the kernel, and to investigate how much it affects on the boundedness properties. To this end we consider a special type of operators, namely, twisted convolution operators. Thus, we consider operators $B^\alpha f = \Omega_\alpha \times f$, which, for $(n+1)/2 < \operatorname{Re}\alpha < n$ are given by

$$B^\alpha f(x) = \int_{\mathbb{R}^n} \Omega_\alpha(y)\ f(x-y)\ e^{i \operatorname{Im}\langle x,y\rangle}\ d\mu(y), \quad f \in S. \tag{1.2}$$

Speaking about twisted convolutions we always naturally assume the dimension n is even, and identify $\mathbb{R}^n$ with $\mathbb{C}^{n/2}$. Thus, $\langle x,y\rangle$ denotes the inner product in $\mathbb{C}^{n/2}$ of vectors x, y regarded as elements of $\mathbb{C}^{n/2}$. We obtain Theorem 4.2, where we give sufficient conditions for boundedness of B^α. It naturally appears that the set of points $(1/p, 1/q)$ of "boundedness" for B^α is considerably large, containing that for A^α. Needless to say, investigation of the necessity of our conditions seems to be a very difficult problem, because in this case we can use neither the theory of multipliers nor many other methods of harmonic analysis, which work effectively in the usual convolution case. For example, twisted convolution, in general, no longer has the symmetry property, a very important feature for the usual ones (see [2], [8] for examples of such operators). Certainly, these affect also in obtaining of the sufficient conditions.

The results can be obtained for more general integral operators, like those considered in Phong and Stein [15] (see remarks below), but we prefer the twisted convolution case due to simplicity of explanation and connection with classical objects. Twisted convolutions have an interest in their own right, combining properties of both singular integral operators and pseudodifferential ones. They appear naturally playing the same role in the Weil formalism as the usual convolution for the Fourier transform. One can find more details and further features in the paper [9] (see also [19] and the bibliography given there).

Note that $L_p - L_p$ boundedness for twisted convolutions with a kernel whose Fourier transform is Miyachi's symbol $a(|\xi|)m_\alpha(\xi)$ was considered in [16]. This operator differs from the operator with compactly supported kernel by a very "nice" operator and by Cowling ([1]) should have the same boundedness $L_p - L_p$

properties as the usual convolution operator. Actually, the "global" part is where the twisted convolution properties enter.

We use the following notation: $Ff(\xi) \equiv \widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{ix\cdot\xi}d\mu(x)$ is the Fourier transform and $F^{-1}f(x) = (2\pi)^{-n}Ff(-x)$ is the inverse Fourier transform; throughout the paper we assume $a(|\xi|)$ to be a smooth function, described above; $[a]$ denotes the integral part of the real number a; $L_p \equiv L_p(\mathbb{R}^n)$; $S \equiv S(\mathbb{R}^n)$ is the Schwartz class of rapidly decreasing smooth functions and S' is the space of tempered distributions; C_0^∞ is the space consisting of functions from S with compact support; L_∞^0 is the subspace of L_∞ of functions vanishing at infinity.

2. Integral representations for the operators A^α, B^α and some properties of the kernel $\Omega_\alpha(x)$

For $f \in S$ and $(n+1)/2 < \operatorname{Re}\alpha < n$ we define the convolution operator as (1.1), where the kernel $\Omega_\alpha(x)$ is given by

$$\Omega_\alpha(x) = (2\pi)^{-n/2}|x|^{\alpha-n}\int_0^\infty t^{n/2-\alpha}e^{it/|x|}\mathcal{J}_{n/2-1}(t)dt. \tag{2.1}$$

Here we have applied the Bochner formula for the Fourier transform of a radial function, and $\mathcal{J}_\nu(x)$ denotes the first kind Bessel function. Due to the estimate ([21])

$$\left| \, t^{-(a+ib)}\mathcal{J}_{a+ib}(t) \, \right| \le c_a e^{c|b|}(1+t)^{-a-1/2}, \quad 0 < t < \infty,$$

the integral (2.1) converges absolutely for $(n+1)/2 < \operatorname{Re}\alpha < n$, and, moreover, using formulae 6.699.1, 6.699.2 from [3], we obtain

$$\Omega_\alpha(x) = i^{n-\alpha}\frac{\Gamma(n-\alpha)}{2^{n-1}\pi^{n/2}\Gamma(n/2)}F\left(\frac{n-\alpha}{2}, \frac{n+1-\alpha}{2}; \frac{n}{2}; |x|^2\right), \tag{2.2}$$

for $|x| < 1$, and this is understood in the usual way for $|x| > 1$ (see [3], for instance). Here $F(a,b;c;d)$ is the Gauss hypergeometric function. Actually, the right hand side of (2.2) for $|x| < 1$ (and its analytic continuation, for $|x| > 1$) represent almost everywhere the function $\Omega_\alpha(x)$ for all α, $0 \le \operatorname{Re}\alpha < n$. This can be seen by applying the analytic continuation principle.

The above representation implies that the kernel $\Omega_\alpha(x)$ is continuous on $\mathbb{R}^n \setminus S^{n-1}$ and has the following asymptotic:

$$\Omega_\alpha(x) \sim c\,(1-|x|)^{\operatorname{Re}\alpha-(n+1)/2}, \quad (n-1)/2 < \operatorname{Re}\alpha < (n+1)/2, \tag{2.3}$$

$$\Omega_\alpha(x) \sim c\,\ln|1-|x||, \quad \alpha = (n+1)/2, \tag{2.4}$$

as $|x| \longrightarrow 1$, and

$$\Omega_\alpha(x) \sim c\,|x|^{\operatorname{Re}\alpha-n}, \quad 0 \le \operatorname{Re}\alpha < n, \tag{2.5}$$

as $|x| \longrightarrow \infty$. For $(n+1)/2 < \operatorname{Re}\alpha < n$ or $\operatorname{Re}\alpha = (n+1)/2$, $\operatorname{Im}\alpha \neq 0$ the kernel $\Omega_\alpha(x)$ is bounded.

Thus, for $0 \leq \operatorname{Re}\alpha \leq (n+1)/2$ the integral (1.1) no longer makes sense. We will understand the operator A^α for non-integral α in the sense of analytic continuation. From the representation (1.1) for a $(n+1)/2 < \operatorname{Re}\alpha < n$ interchanging the order of integration we have

$$\begin{aligned} A^\alpha f(x) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} |y|^{\alpha-n}\ f(x-y)\ d\mu(y) \int_0^\infty t^{n/2-\alpha}\ e^{it/|y|}\ \mathcal{J}_{n/2-1}(t)\ dt. \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} |y|^{\alpha-n}\ e^{i/|y|}\ d\mu(y) \int_0^\infty t^{n/2} \mathcal{J}_{n/2-1}(t)\ f(x-ty)\ dt. \end{aligned}$$

Due to the formula $\frac{d}{dt}\Big[t^\nu \mathcal{J}_\nu(t)\Big] = t^\nu \mathcal{J}_{\nu-1}(t)$ integrating by parts and neglecting the boundary terms, we get

$$\int_0^\infty t^{n/2} \mathcal{J}_{n/2-1}(t) f(x-yt)\, dt = (-1)^k \int_0^\infty t^{n/2+k} \mathcal{J}_{n/2+k-1}(t) \left[\left(\frac{1}{t}\frac{d}{dt}\right)^k f(x-yt)\right] dt.$$

Thus, for α in the range $\max\{0, (n+1)/2-k\} \leq \operatorname{Re}\alpha \leq (n+1)/2-k+1$ we define for $f \in S$:

$$A^\alpha f(x) = \int_{\mathbb{R}^n} \Omega_\alpha^k(y) \left[\left(\frac{1}{s}\frac{d}{ds}\right)^k f(x-sy)\right]_{s=1} d\mu(y)$$

where $s \in (0,\infty)$, and

$$\begin{aligned} \Omega_\alpha^k(x) &= (-1)^k (2\pi)^{-n/2} |x|^{\alpha-n} \int_0^\infty t^{n/2-\alpha-k}\ e^{it/|x|}\ \mathcal{J}_{n/2+k-1}(t) dt \\ &= i^{n+2k-\alpha} \frac{\Gamma(n-\alpha)}{2^{n-1}\pi^{n/2}\Gamma(n/2)} F\left(\frac{n-\alpha}{2}, \frac{n-\alpha+1}{2}; \frac{n}{2}+k; |x|^2\right), \end{aligned}$$

for $|x| < 1$, understanding it as above for $|x| > 1$.

We can give the integral representation for the twisted convolution operator $B^\alpha f = \Omega_\alpha \times f$ in the same manner, i.e., considering (1.2) for $(n+1)/2 < \operatorname{Re}\alpha < n$, and setting

$$\begin{aligned} B^\alpha f(x) &= \int_{\mathbb{R}^n} \Omega_\alpha^k(y) \left[\left(\frac{1}{s}\frac{d}{ds}\right)^k e^{is\operatorname{Im}\langle x,y\rangle}\ f(x-sy)\right]_{s=1} d\mu(y) \\ &= \int_{\mathbb{R}^n} \Omega_\alpha^k(y)\ e^{it\operatorname{Im}\langle x,y\rangle} \left[\left(\frac{i}{s}\operatorname{Im}\langle x,y\rangle I + \frac{1}{s}\frac{d}{ds}\right)^k f(x-sy)\right]_{s=1} d\mu(y), \end{aligned}$$

for $f \in S$, $\max\{0, (n+1)/2 - k\} \le \operatorname{Re}\alpha \le (n+1)/2 - k + 1$.

In the rest of this section we study behaviour of some special functions. Introduce

$$\Omega_{\alpha,0}(x) = \left(F^{-1} a(|\xi|)|\xi|^{-\alpha} e^{i|\xi|}\right)(x), \quad \Omega_{\alpha,\infty}(x) = \left(F^{-1}(1 - a(|\xi|))|\xi|^{-\alpha} e^{i|\xi|}\right)(x).$$

The function $\Omega_{\alpha,0}(x)$ preserves the local singularities of the kernel $\Omega_\alpha(x)$ and is rapidly decreasing at infinity. More precisely, due to the result of [11], the function $\Omega_{\alpha,0}(x)$ is smooth on $\mathbb{R}^n \setminus S^{n-1}$, $\Omega_{\alpha,0}(x) \sim \Omega_\alpha(x)$, when $|x| \to 1$ (see above for the asymptotic of $\Omega_\alpha(x)$ when $|x| \to 1$), and for any $M > 0$

$$\Omega_{\alpha,0}(x) \le c\, |x|^{-M}, \quad |x| \longrightarrow \infty.$$

For $(n+1)/2 < \operatorname{Re}\alpha < n$ or $\operatorname{Re}\alpha = (n+1)/2$, $\operatorname{Im}\alpha \ne 0$ the function $\Omega_{\alpha,0}(x)$ is bounded.

Actually, the above was established in [11] for $\alpha \in \mathbb{R}$ only. But it is not hard to extend it for complex α, taking into account properties of the function $\Omega_\alpha(x)$ and of the function $\Omega_{\alpha,\infty}(x)$ which we now describe.

Contrary to the above situation, the function $\Omega_{\alpha,\infty}(x)$ is smooth at the finite points, but preserves singularity at infinity:

$$\Omega_{\alpha,\infty}(x) \sim c\, |x|^{\operatorname{Re}\alpha - n}, \quad |x| \longrightarrow \infty. \tag{2.6}$$

Moreover, any derivative of the kernel $\Omega_{\alpha,\infty}(x)$ has the same properties: it is smooth at finite points and its asymptotic at infinity is obtained by formal differentiation of (2.6). Let us prove the last assertion, following ideas of [11]. Consider $\mathcal{D}^m = \frac{\partial^m}{\partial x_{k_1} \partial x_{k_2} \ldots \partial x_{k_m}}$, where $1 \le k_m \le n$ are not necessarily distinct. Let us expand ([20], p. 139, [17])

$$(-i)^m \xi_{k_1} \xi_{k_2} \ldots \xi_{k_m} = \sum_{j=1}^{l} |\xi|^{2j} P_j(\xi),$$

where $l = [m/2]$ and $P_j(\xi)$ is a homogeneous (of order zero) harmonic polinomial of degree $m - 2j$, $\xi \in \mathbb{R}^n$. We obviously have

$$\mathcal{D}^m\, \Omega_{\alpha,\infty}(x) = \sum_{j=1}^{l} \left(F^{-1} e^{i|\xi|} |\xi|^{2j-\alpha}\, a(|\xi|)\, P_j(\xi)\right)(x) \tag{2.7}$$

Using the rule for Fourier transform of a product of a radial function with a solid spherical harmonic ([20], p. 158) we have

$$\left(F^{-1} e^{i|\xi|} |\xi|^{2j-\alpha}\, a(|\xi|)\, P_j(\xi)\right)(x) = \omega_\alpha^j(|x|)\, P_j(x),$$

where

$$\omega_\alpha^j(r) = (2\pi)^{1-n} i^{2j-m}\, r^{\alpha-n-2m+2j} \int_0^\infty t^{n/2+m-\alpha}\, a(t/r)\, e^{it/r}\, \mathcal{J}_{n/2+m-2j-1}(t)\, dt. \tag{2.8}$$

The expression in (2.8) can be viewed (up to a constant multiplier) as radial function of $n+2m-4j$ variables, being the inverse Fourier transform of the function $e^{i|\eta|}|\eta|^{2j-\alpha}a(|\eta|)$, $\eta \in \mathbb{R}^{n+2m-4j}$, i.e.,

$$\omega_\alpha^j(r) = c\, \left(F^{-1}|\eta|^{2j-\alpha}e^{i|\eta|}a(|\eta|)\right)(|\xi|)\Big|_{|\xi|=r}.$$

For this reason, by (2.6)

$$\omega_\alpha^j(|x|) \sim c\, |x|^{\operatorname{Re}\alpha-n-2m+2j}, \quad |x| \longrightarrow \infty,$$

which with (2.7) proves our claim.

3. Boundedness for convolution operators

Theorem 3.1. *Let $0 \le \operatorname{Re}\alpha < n$, $1 \le p \le q \le \infty$. The operator A^α is bounded*

1. *from L_p to L_q, $1 < p \le q < \infty$ if and only if $1/q \le 1/p - \operatorname{Re}\alpha/n$ and either*

   $$\frac{1}{p}+\frac{1}{q} \le 1, \ \frac{1}{p}-\frac{n}{q} \le \operatorname{Re}\alpha - \frac{n-1}{2}, \quad or \quad \frac{1}{p}+\frac{1}{q} \ge 1, \ \frac{n}{p}-\frac{1}{q} \le \operatorname{Re}\alpha + \frac{n-1}{2}.$$

2. *from L_1 to L_q, $1 \le q < \infty$, if and only if $(n+1)/2 - \operatorname{Re}\alpha < 1/q < (n-\operatorname{Re}\alpha)/n$.*
3. *from L_1 to L_∞ if and only if $\operatorname{Re}\alpha > (n+1)/2$ or both $\operatorname{Re}\alpha = (n+1)/2$ and $\operatorname{Im}\alpha \ne 0$.*
4. *from L_p to L_∞, $1 < p < \infty$ if and only if $\operatorname{Re}\alpha/n < 1/p < \operatorname{Re}\alpha - (n-1)/2$.*

The operator A^α is of weak $(1, 2/(n+1-2\operatorname{Re}\alpha))$ type for $n/2 \le \operatorname{Re}\alpha < (n+1)/2$, i.e.,

$$\operatorname{mes}\{x : |A^\alpha f(x)| > \lambda\} \le c\, \lambda^{-q}\, \|f\|_1^q, \quad 1/q = (n+1)/2 - \operatorname{Re}\alpha. \tag{3.1}$$

It is also of weak $(1, n/(n-\operatorname{Re}\alpha))$ type for $n/2 \le \operatorname{Re}\alpha < n$.

Proof. Claims *1 – 4* were proved in [7] for $\alpha \in \mathbb{R}$, $(n-1)/2 < \operatorname{Re}\alpha < n$, and were extended for all $\alpha \in \mathbb{C}$, $0 \le \operatorname{Re}\alpha < n$ in paper [5]. To complete these results, we prove the weak estimates.

Represent $A^\alpha = A_0^\alpha + A_\infty^\alpha$, where $A_0^\alpha f = \Omega_{\alpha,0} * f$ and $A_\infty^\alpha f = \Omega_{\alpha,\infty} * f$. Due to behaviour of $\Omega_{\alpha,\infty}(x)$, and by standard Martsinkievich arguments, the operator A_∞^α is of weak $(1, n/(n - \operatorname{Re}\alpha))$ type for $0 \le \operatorname{Re}\alpha < n$. Consequently, the latest applies to A^α when $n/2 \le \operatorname{Re}\alpha < n$, as can be seen by routine estimates.

In the same way, having established weak $(1, 2/(n+1-2\operatorname{Re}\alpha))$ boundedness of A_0^α for $(n-1)/2 < \operatorname{Re}\alpha < (n+1)/2$, the same will be true for A^α when $n/2 \le \operatorname{Re}\alpha < (n+1)/2$. To this end it suffices to show (3.1) for the operator $f \to \omega_\alpha * f$ under the condition $\|f\|_1 = 1$ and with 2λ instead of λ in the left side. Here

$$\omega_\alpha(x) = \begin{cases} |1-|x||^{\operatorname{Re}\alpha-(n+1)/2}, & \text{for} \quad |x| < 2, \\ 0, & \text{otherwise.} \end{cases}$$

We have

$$\operatorname{mes}\{x : |(\omega_\alpha * f)(x)| > 2\lambda\} \le \operatorname{mes}\{x : |(\omega_{\alpha,1} * f)(x)| > \lambda\} + \operatorname{mes}\{x : |(\omega_{\alpha,2} * f)(x)| > \lambda\}$$

where we denoted

$$\omega_{\alpha,1}(x) = \begin{cases} \omega_{\alpha,1}(x) & \text{for} \quad |1-|x|| < \delta, \\ 0, & \text{otherwise,} \end{cases} \qquad \omega_{\alpha,2}(x) = \omega_\alpha(x) - \omega_{\alpha,1}(x).$$

We are going to choose δ below. First,

$$\operatorname{mes}\{x : |(\omega_{\alpha,1} * f)(x)| > \lambda\} \le 1/\lambda \ \|\omega_{\alpha,1}\|_1 \|f\|_1 = 1/\lambda \ \|\omega_{\alpha,1}\|_1,$$

and,

$$\begin{aligned} \|\omega_{\alpha,1}\|_1 &= \int_{\substack{|1-|x||<\delta \\ |x|<2}} |1-|x||^{\operatorname{Re}\alpha-(n+1)/2} d\mu(x) \\ &= |S^{n-1}| \int_{\substack{|1-r|<\delta \\ 0<r<2}} r^{n-1} |1-r|^{\operatorname{Re}\alpha-(n+1)/2} dr \\ &\le 2^n |S^{n-1}| \int_0^{\min\{\delta,2\}} r^{\operatorname{Re}\alpha-(n+1)/2} dr = c_{n,\alpha} \ [\min\{\delta,2\}]^{\operatorname{Re}\alpha-(n-1)/2}. \end{aligned}$$

Next,

$$\|\omega_{\alpha,2} * f\|_\infty \le \ \|\omega_{\alpha,2}\|_\infty \|f\|_1 = \|\omega_{\alpha,2}\|_\infty = \delta^{\operatorname{Re}\alpha-(n+1)/2}.$$

Thus, setting $\delta^{\operatorname{Re}\alpha-(n+1)/2} = \lambda$, we have

$$\operatorname{mes}\{x : |(\omega_{\alpha,2} * f)(x)| > \lambda\} = 0,$$

and finally,

$$\begin{aligned} \operatorname{mes}\{x : |(\omega_\alpha * f)(x)| > 2\lambda\} &= \operatorname{mes}\{x : |(\omega_{\alpha,1} * f)(x)| > \lambda\} \\ &\le 2c_{n,\alpha} \ \lambda^{-1/((n+1)/2-\operatorname{Re}\alpha)} = c \ \lambda^{-q} \ \|f\|_1^q, \end{aligned}$$

where $1/q = (n+1)/2 - \operatorname{Re}\alpha$, $(n-1)/2 < \operatorname{Re}\alpha < (n+1)/2$ and c depends only on n, α. $\square$

4. Boundedness for twisted convolution operators

First we take into account the following general, simple result.

Lemma 4.1. *Let a twisted convolution operator* $Tf = \Omega \times f$ *be bounded from* L_p *to* L_q*, and* $p > q$*. Then* $T = 0$ *for* $p < \infty$*, and in the case* $p = \infty$ *the restriction of* T *onto* L^0_∞ *is the null operator.*

Proof. We follow method of Hörmander ([4]). Let $\tau_h f(x) = f(x-h)$. While operators like T do not commute with translations, they do satisfy the identity

$$\tau_h Tf(x) = \left(Te^{-\mathrm{Im}\,\langle h,\cdot\rangle}\tau_h f\right)(x). \tag{4.1}$$

Now, for $f \in L_p$, $1 \le p < \infty$,

$$\begin{aligned} \|f + \tau_h f\|_p &\longrightarrow 2^{1/p}\, \|f\|_p, \quad h \to \infty, \\ \|f + e^{-\mathrm{Im}\,\langle h,\cdot\rangle}\tau_h f\|_p &\longrightarrow 2^{1/p}\, \|f\|_p, \quad h \to \infty. \end{aligned}$$

This is obvious for compactly supported functions $f \in C_0^\infty$ and extends for all $f \in L_p$ by density of C_0^∞ in L_p, $1 \le p < \infty$. Suppose that

$$\|Tf\|_q \le c_{p,q}\, \|f\|_p, \qquad f \in L_p, \qquad 1 \le q < p < \infty. \tag{4.2}$$

Replacing the operator T with $\tau_{-h} T \tau_h$ we get an operator of the same type and with the same norm. Due to (4.1),

$$\|Tf + \tau_h Tf\|_q = \|T(f + \tau_h f)\|_q \le c_{p,q}\|f + e^{-\mathrm{Im}\,\langle h,\cdot\rangle}\tau_h f\|_p.$$

Letting $h \to \infty$, we get

$$\|Tf\|_q \le 2^{1/p-1/q} c_{p,q}\|f\|_p.$$

For $p > q$ this estimate is better than (4.2), which contradicts our assumption. The same arguments apply for $f \in L^0_\infty$, when $p = \infty$. □

Having established this, we restrict ourselves to $1 \le p \le q \le \infty$, as in the case of translation invariant operators.

Theorem 4.2. *Let* $0 \le \mathrm{Re}\,\alpha < n$*. The twisted convolution operator* $B^\alpha f = \Omega_\alpha \times f$ *is bounded from* L_p *to* L_q*,* $1 \le p \le q \le \infty$*,*

1. *if*

$$\frac{\mathrm{Re}\,\alpha}{n} \le \frac{1}{p} + \frac{1}{q} \le 1, \quad \frac{1}{p} - \frac{n}{q} \le \mathrm{Re}\,\alpha - \frac{n-1}{2},$$

unless

$$1/p = \operatorname{Re}\alpha/n, \quad 1/q = 0, \qquad n/2 \le \operatorname{Re}\alpha < n$$

or

$$1/p = \operatorname{Re}\alpha - (n-1)/2, \quad 1/q = 0, \qquad n/2 \le \operatorname{Re}\alpha < (n+1)/2$$

2. if

$$1 \le \frac{1}{p} + \frac{1}{q} \le 2 - \frac{\operatorname{Re}\alpha}{n}, \ \ \frac{n}{p} - \frac{1}{q} \le \operatorname{Re}\alpha + \frac{n-1}{2}$$

unless

$$1/p = 1, \quad 1/q = 1 - \operatorname{Re}\alpha/n, \quad n/2 \le \operatorname{Re}\alpha < n$$

or

$$1/p = 1, \quad 1/q = (n+1)/2 - \operatorname{Re}\alpha, \qquad n/2 \le \operatorname{Re}\alpha < (n+1)/2.$$

The operator B^α *is of weak* $(1, 1/((n+1)/2 - \Re\alpha))$ *type for* $n/2 \le \operatorname{Re}\alpha \le (n+1)/2$, *and is of weak* $(1, n/(n - \operatorname{Re}\alpha))$ *type for* $n/2 \le \operatorname{Re}\alpha < n$.

Proof. The proof of this theorem is quite long and technical in spite of the considerations mentioned in the Introduction. Full details will be given elsewhere. Here we only prove boundedness of B^α on L_2. This proof is a modification of arguments of Stein and Phong [15] for our twisted case and is of particular interest.

It turns out that, that the operator $B_0^\alpha f = \Omega_{\alpha,0} \times f$ differs from the operator with compact support by a very nice operator, which is bounded from L_p to L_q for any $1 \le p \le p$. Due to Cowling [1] the twisted convolution operator with compactly supported kernel is bounded on L_p, $1 \le p \le \infty$ if and only if the usual convolution operator (with the same kernel) is. Therefore, we have only to prove L_2 boundedness for the operator $B_\infty^\alpha f = \Omega_{\alpha,\infty} \times f$.

The case $0 \le \operatorname{Re}\alpha < n/2$ follows from the well known result (see [19], p. 556, [22], p. 14), since $\Omega_{\alpha,\infty}(x) \in L_2$ for such α.

Let $n/2 \le \operatorname{Re}\alpha < n$, $\Omega^\varepsilon_{\alpha,\infty}(x) = (1 - a(\varepsilon|x|))\Omega_{\alpha,\infty}(x)$, and $B^\alpha_{\infty,\varepsilon} = \Omega^\varepsilon_{\alpha,\infty} \times f$. We have

$$\begin{aligned} &(B^\alpha_{\infty,\varepsilon})^* B^\alpha_{\infty,\varepsilon} f(y) = \\ = \ &\int_{\mathbb{R}^n} \overline{\Omega^\varepsilon_{\alpha,\infty}(x-y)} e^{-i\operatorname{Im}\langle y,x\rangle} d\mu(x) \int_{\mathbb{R}^n} f(u) \Omega^\varepsilon_{\alpha,\infty}(x-u) e^{-i\operatorname{Im}\langle u,x\rangle} d\mu(u) \\ = \ &\int_{\mathbb{R}^n} f(u) d\mu(u) \int_{\mathbb{R}^n} \overline{\Omega^\varepsilon_{\alpha,\infty}(x-y)} \Omega^\varepsilon_{\alpha,\infty}(x-u) e^{-i\operatorname{Im}\langle y,x\rangle} e^{-i\operatorname{Im}\langle u,x\rangle} d\mu(x) \\ = \ &\int_{\mathbb{R}^n} f(u) e^{i\operatorname{Im}\langle u,y-u\rangle} d\mu(u) \int_{\mathbb{R}^n} \overline{\Omega^\varepsilon_{\alpha,\infty}(x+u-y)} \Omega^\varepsilon_{\alpha,\infty}(x) e^{i\operatorname{Im}\langle x,y-u\rangle} d\mu(x) \end{aligned}$$

Interchanging the order of integration above we may use Fubini's theorem, since

$$\int_{\mathbb{R}^n} |f(u)| \ d\mu(u) \int_{\mathbb{R}^n} \left| \ \overline{\Omega^\varepsilon_{\alpha,\infty}(x+u-y)} \ \Omega^\varepsilon_{\alpha,\infty}(x) \right| d\mu(x) \le c\|f\|_2,$$

for almost all $y \in \mathbb{R}^n$. Here to estimate the inner integral we apply ([17], [15]):

$$\int_{\mathbb{R}^k} \frac{d\mu(\tau)}{(1+|\tau|)^a(1+|\eta-\tau|)^b} \le c \begin{cases} (1+|\eta|)^{-\min\{a,b,a+b-k\}}, & \text{if} \quad \max\{a,b\} \neq k, \\ (1+|\eta|)^{k-a-b}\ln(2+|\eta|), & \text{if} \quad \max\{a,b\} = k, \end{cases} \tag{4.3}$$

provided $a, b \ge 0$, $a+b > m$. Thus,

$$(B^\alpha_{\infty,\varepsilon})^* B^\alpha_{\infty,\varepsilon} f(y) = \int_{\mathbb{R}^n} f(u)\ k_{\alpha,\varepsilon}(y-u)\ e^{i\mathrm{Im}\,\langle u,y\rangle}\ d\mu(u), \quad f \in L_2,$$

where we denote

$$k_{\alpha,\varepsilon}(u) = \int_{\mathbb{R}^n} \overline{\Omega^\varepsilon_{\alpha,\infty}(x-u)}\ \Omega^\varepsilon_{\alpha,\infty}(x)\ e^{i\mathrm{Im}\,\langle x,u\rangle}\ d\mu(x). \tag{4.4}$$

Regarding x, u as elements of $\mathbb{C}^{n/2}$, let us write $x = (x_1, x_2, \ldots, x_n) = (z_1, z_2, \ldots, z_{n/2}) \equiv z$ and $u = (w_1, w_2, \ldots, w_{n/2}) \equiv w$. Thus, we can introduce

$$\nabla_z = \left\{ \frac{\partial}{\partial \overline{z}_1}, \frac{\partial}{\partial \overline{z}_2}, \ldots, \frac{\partial}{\partial \overline{z}_{n/2}} \right\},$$

where as usual $\frac{\partial}{\partial \overline{z}_k}$ denotes the anti-holomorphic derivative in z_k. It is not hard to see that

$$\langle \nabla_z \mathrm{Im}\,\langle x,u\rangle, u\rangle = \frac{i}{2}\langle u,u\rangle = \frac{i}{2}|u|^2.$$

Thus,

$$e^{i\mathrm{Im}\,\langle x,u\rangle} = -\frac{2}{|u|^2}\langle \nabla_z e^{i\mathrm{Im}\,\langle x,u\rangle}, u\rangle.$$

Substituting this in (4.4), integrating by parts and neglecting the boundary terms, we have

$$\begin{aligned} k_{\alpha,\varepsilon}(u) &= -\frac{2}{|u|^2} \int_{\mathbb{R}^n} \overline{\Omega^\varepsilon_{\alpha,\infty}(x-u)}\Omega^\varepsilon_{\alpha,\infty}(x)\langle \nabla_z e^{i\mathrm{Im}\,\langle x,u\rangle}, u\rangle\ d\mu(x) \\ &= -\frac{2}{|u|^2} \left\langle \int_{\mathbb{R}^n} \overline{\Omega^\varepsilon_{\alpha,\infty}(x-u)}\Omega^\varepsilon_{\alpha,\infty}(x)\nabla_z e^{i\mathrm{Im}\,\langle x,u\rangle}\ d\mu(x), u \right\rangle \\ &= \frac{2}{|u|^2} \left\langle \int_{\mathbb{R}^n} \left(\nabla_z \overline{\Omega^\varepsilon_{\alpha,\infty}(x-u)}\Omega^\varepsilon_{\alpha,\infty}(x)\right) e^{i\mathrm{Im}\,\langle x,u\rangle}\ d\mu(x), u \right\rangle \\ &= \sum_{k=1}^{n/2} \overline{w}_k \frac{2}{|u|^2} \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial \overline{z}_k} \overline{\Omega^\varepsilon_{\alpha,\infty}(x-u)}\Omega^\varepsilon_{\alpha,\infty}(x)\right) e^{i\mathrm{Im}\,\langle x,u\rangle}\ d\mu(x). \end{aligned} \tag{4.5}$$

Applying the same procedure to each integral in the summation in (4.5), at the step "m" we arrive at

$$\begin{aligned} k_{\alpha,\varepsilon}(u) &= \frac{2^m}{|u|^{2m}} \sum_{\nu_1+\nu_2+\ldots+\nu_{n/2}=m} \overline{w}_{k_1}^{\nu_1} \overline{v}_{k_2}^{\nu_2} \ldots \overline{w}_{k_{n/2}}^{\nu_{n/2}} \\ &\times \int_{\mathbb{R}^n} \left(\frac{\partial^m}{\partial \overline{z}_{k_1}^{\nu_1} \partial \overline{z}_{k_2}^{\nu_2} \ldots \partial \overline{z}_{k_{n/2}}^{\nu_{n/2}}} \overline{\Omega_{\alpha,\infty}^{\varepsilon}(x)}\ \Omega_{\alpha,\infty}^{\varepsilon}(x) \right) e^{i\mathrm{Im}\,\langle x,u\rangle}\ d\mu(x). \end{aligned}$$

Obviously, the derivatives of $\Omega_{\alpha,\infty}^{\varepsilon}(x)$ admit the same estimate at infinity as those of $\Omega_{\alpha,\infty}(x)$ (see the claim after the formula (2.6)), and moreover, uniformly in ε. Thus, using (4.3) we obtain

$$|k_{\alpha,\varepsilon}(x)| \le c\ |x|^{-m}\ (1+|x|)^{2\mathrm{Re}\,\alpha-n-m}, \quad 0 \le \mathrm{Re}\,\alpha < n.$$

with c independent on ε. Fix m such that $\mathrm{Re}\,\alpha - n/4 < m < n$. Since $(B_{\infty,\varepsilon}^{\alpha})^* B_{\infty,\varepsilon}^{\alpha}$ itself is a twisted convolution operator and its kernel is in L_2, we immediately obtain the boundedness

$$\|(B_{\infty,\varepsilon}^{\alpha})^* B_{\infty,\varepsilon}^{\alpha} f\|_2 \le \|k_{\alpha,\varepsilon}\|_2 \|f\|_2 \le c\|f\|_2.$$

By limiting arguments we obtain the same estimate for $(B_{\infty}^{\alpha})^* B_{\infty}^{\alpha}$ and, hence, boundedness of B_{∞}^{α} on L_2. □

5. Concluding remarks

One natural extension of the previous theorem is the consideration of $L_p - L_q$ boundedness properties for the general operator

$$T_{\Omega_\alpha} f(x) = \int_{\mathbb{R}^n} \Omega_\alpha(y)\ f(y-x)\ e^{iP(x,y)}\ d\mu(y), \tag{5.1}$$

where $P(x,y)$ is a real bilinear form. Boundedness in L_p of the operator T_K (K instead of Ω_α in (5.1)) where $K(x)$ is a Calderon-Zigmund type kernel was considered by Phong and Stein in [15]. It is also of particular interest to investigate $L_p - L_q$ boundedness for such operators T_K. The operators T_{Ω_α} and T_K have similar global properties of their kernels, but these are locally different.

Another extension of Theorem 4.2 can be given considering more general operators B_b^{α} having kernels $\Omega_{\alpha,b}(x) = (F^{-1} b(\xi) m_\alpha(\xi))(x)$ with some special $b(\xi)$. For example, with radial $b(\xi) = b(|\xi|)$ in a Holder class of power-like stabilizing functions at infinity. Or, with homogeneous $b(\xi) = b(\xi/|\xi|)$ satisfying an ellipticity condition $\inf_{\xi\neq 0} |b(\xi)| > 0$, and some smoothness condition. Obviously, it does not always make sense to consider an arbitrary characteristic in the symbol: the kernel's properties will be rather different. But it is essential, that for the classes mentioned we still have assymptotics and properties of $\Omega_{\alpha,b}(x)$ like the ones of

$\Omega_\alpha(x)$ above (see [6], [7] for details, and [18] for the definition of the mentioned Holder class).

In this connection we mention that Theorem 3.1 (the usual convolution case) was proved in [5] (except the weak estimates) for more general operators $A^\alpha_b f = \Omega_{\alpha,b} * f$, with radial $b(|\xi|)$ satisfying: b and $1/b$ both in $M^p_p,\ 1 < p < \infty$, and in the case $\operatorname{Re}\alpha = (n+1)/2,\ p = 1,\ q = \infty$ additionally $b,\ 1/b \in M^1_1$. The weak estimates proved in Theorem 3.1 apply for such general operators as well, with the only assumption $|b^{'}(t)| \le c/t$, $t > 0$, because in this case the corresponding kernels are estimated with $\le$ instead of $\sim$ in (2.3)-(2.5) (see [6] for details).

The case of homogeneous $b(\xi)$ was studied in [7], where it was shown that the operator A^α_b has exactly the same boundedness properties as A^α, provided $(n-1)/2 < \alpha < n$, $b(\xi) \in C^m(S^{n-1})$, $m > 3n/2 - 1$ and the ellipticity condition is satisfied. This was extended in [14] for $0 < \alpha < n$, where non-elliptic characteristics were considered as well.

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