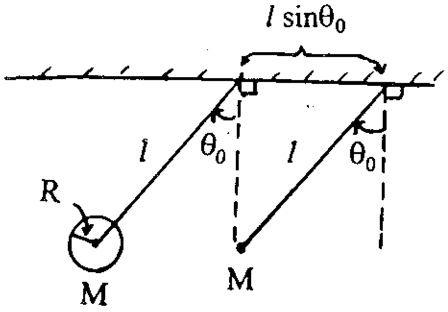
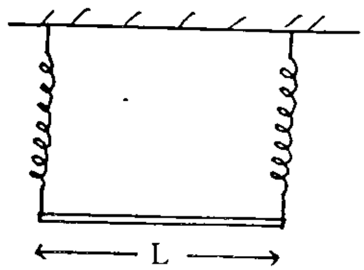
Section A :

Mechanics

A1. Two frictionless pendulums are constructed and hung side by side as shown. The pendulums consist of rigid massless rods of length l attached on the left to a cylindrical disk of radius R and mass M, and on the right to a point-mass M. The pendulums are displaced by a small angle θ_0 and simultaneously released at rest. At approximately what time t do the bobs collide ?



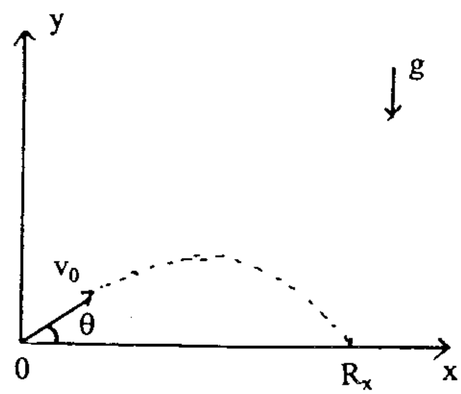
A2. A uniform thin cylindrical rod of length L and mass M is supported at its ends by two massless springs with spring constant k. The equilibrium position of the bar is horizontal at height $y = y_0$. Using the nomal coordinates q_1 corresponding to vertical motion of the center of mass of the bar, and q_2 corresponding to angular motion of the bar about its center of mass, for small-amplitude motion about equilibrium where we can assume that the springs move only vertically, find the eigenfrequencies of the normal modes and describe the corresponding normal mode motions.
(The moment of inertia of the bar, about the center of mass, is equal to $(1/12)\, M L^2$).



A3. A particle with mass m is thrown into the air with initial velocity v_0 at θ to the horizontal. For given v_0, find the θ such as the range R_x is maximum via

(1) Newton's second law;
(2) Lagrange's equations;
(3) Hamilton's equations;
(4) Poisson's brackets;
(5) Hamilton-Jacobi equations.

What are the conserved quantities ? Why ?



Section B :

E & M

B1. The potential on the plane $z = 0$ is specified to be $\Phi = V$ inside a circle of radius a centerd at the origin, and $\Phi = 0$ outside the circle. It is desired to find the potential in the half space $z \geq 0$.

(1) Write down the appropriate Green function $G(\vec{r}, \vec{r}')$.
(2) Write down an integral expression for the potential at a point P specified in terms of cylindrical co-ordinates (ρ, φ, z). Show all limits of integration and integrands explicitly.
(3) Evaluate the leading term for $\rho^2 + z^2 \gg a^2$.

B2. Two thin concentric conducting spheres of radii a and b $(a < b)$ are held at potentials V_1 and V_2. The space between the shells is filled with a dielectric whose dielectric permittivity varies as $\varepsilon = \varepsilon_0 + \varepsilon_1 \cos^2\vartheta$. Determine the capacitance of this arrangement of conductors.

B3. A plane electromagnetic wave $E_1 = \hat{e}_x E_0 e^{i(kz-\omega t)}$ traveling in vacuum is incident normally on the plane surface of a semi-infinite block of an ideal dielectric of permittivity ε and permeability $\mu = 1$. Derive expressions for the fraction of energy reflected and transmittied as functions of ε and the frequency ω of the wave.

Section C :

Thermodynamics & Statistical Mechnics

C1. Consider the two ideal gas systems shown in the figure, each of which have the same total energy and the same number of particles, N. In system A the particles are confined by a partition to occupy one half of the container whose volume is the same as system B. In system B the particles occupy the total volume of the container.

a. Derive an expression for the entropy, S, in terms of N (number of particles), T (temperature), and V (volume).

b. Find the difference of the entropy between systems A and B.

Figure:

system A system B

C2. The state equation of a new matter is

$$p = AT^3/V \ ,$$

where p, V and T are the pressure, volume and temperature, respectively, A is a constant. The internal energy of the matter is

$$U = BT^n \ln(V/V_0) + f(T) \ ,$$

where B, n and V_0 are all constants, $f(T)$ only depends on the temperature. Find B and n.

C3. A plane surface is in contact with an ideal gas. The surface consists of K sites in a square array. Each site can bond one atom, with binding energy ε.

a) Calculate the fraction of sites occupied as a function of gas pressure, temperature, and binding energy.

b) Discuss your results in a).

Hint: The grand canonical partition function for the adsorbed atoms is

$$\Lambda = \left[1 + \exp - \left(\alpha - \frac{\varepsilon}{kT}\right)\right]^K$$

Section D :

Modern Physics

D1. Various groups observing high energy astrophysical phenomena have reported evidence that suggests that strongly interacting neutral particles are coming to Earth from Cygnus X-3, a compact object some 150,000 light years away. Estimate the minimum lifetime a particle of mass 2 GeV and energy 1000 TeV would have to have reached us from Cygnus X-3. What is the minimum energy a neutron would have to have to do so ?
(Given $\tau_{neutron} \cong 900$ sec).

D2. You have derived the energy levels of a hydrogenic atom (single electron bound to a nucleus of charge Ze) using Schrodinger equation, and relativistic mass and spin-orbit corrections to the energy can be derived using perturbation theory. In the limit of nearly circular orbits these corrections can be written as

$$\Delta E_{rel} \propto |E_n| (Z\alpha)^2/n^2 ,$$

and $$\Delta E_{s\text{-}o} \propto |E_n| (Z\alpha)^2/n^3 ,$$

where E_n is the energy level corresponding to principal quantum number n and $\alpha = e^2/(4\pi\varepsilon_0\hbar c)$ is the fine structure constant.

Show that the same results can be obtained very simply by using the results of the Bohr model.

D3. The lowest frequency rotational transition in the ground vibrational state of CO molecule is observed to be at $\bar{\upsilon} = 3.842\ \text{cm}^{-1}$. The masses of the carbon and oxygen atoms are $M_c = 12.00$ amu and $M_o = 16.00$ amu , respectively, where 1 amu = $1.661\text{X}10^{-27}$ kg. Note that $h = 6.626\text{X}10^{-34}$ J·s and $c = 2.998\text{X}10^{8}$ m/s.

(1) Determine bond length of the CO molecule.

(2) Assuming the molecule to be a classical rigid rotator, determine its frequency of rotation corresponding to the J = 1 state.

(3) This is equivalent to a classical electric dipole rotating at the frequency determined in (2). What is the frequency of radiation of the dipole ? How does it compare with the actually observed frequency ?

D4. Elemental potassium (K) occurs naturally as a body-centered cubic (*bcc*) crystalline structure with cubic (i.e. cube edge) lattice constant $a = 5.23$ Å. The electronic structure of K, which is monovalent (i.e. one conduction electron per atom), is well described by the free electron (Sommerfeld) model.

(a.) Draw, as best you can, the conventional (i.e., not the primitive) unit cell for the *bcc* lattice. Label the cubic lattice parameter a in the drawing. How many atoms are there per unit cell in this lattice?

(b.) In the context of this model, calculate the Fermi wavevector magnitude (in cm^{-1}) (k_F) and the Fermi energy (in eV) (E_F) for K (If you do not remember the form of the dependence of k_F and E_F on materials parameters, you can derive these dependences by using solutions of the Schroedinger equation with periodic boundary conditions in k space).

(Given $\hbar = 1.054\text{X}10^{-27}$ erg sec , $m = 9.11\text{X}10^{-28}$ g, and 1 erg = $6.24\text{X}10^{11}$ eV)

D5. The deuteron has spin equal to one and even parity, while the pion has spin zero. The deuteron can capture a negative pion at rest to form two neutrons:

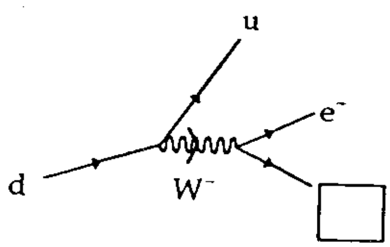
$$\pi^- + d \rightarrow n + n$$

Show how the observation of this capture allows the determination of the parity of the pion.

D6. Consider the following Feynman diagrams showing various examples of weak interactions. In these diagrams "u" and "d" denote the up and down quarks, respectively.

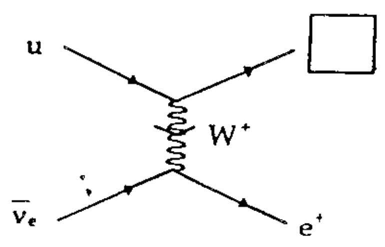
(i) Neutron β Decay

$n \rightarrow p + e^- +$



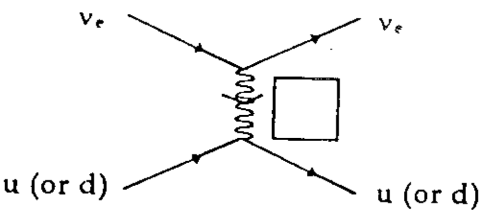
(ii) Neutrino-Proton Scattering

$\bar{\nu}_e + p \rightarrow n + e^+$



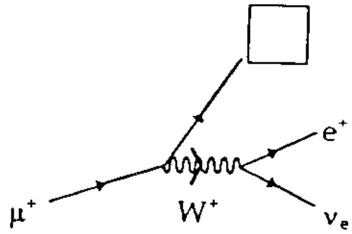
(iii) Neutrino Elastic Scattering

$\nu_e + n \rightarrow n + \nu_e$



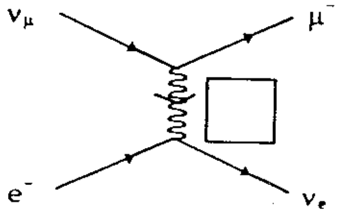
(iv) Muon Decay

$\mu^+ \rightarrow e^+ + \nu_e +$

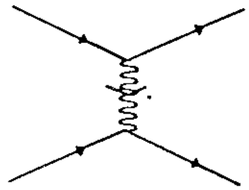


(v) Neutrino-Electron Scattering

$\nu_\mu + e^- \rightarrow \mu^- + \nu_e$



(vi) Electron Capture



(a) In diagrams i–vi above, either a quark, a lepton, or a gluon has been omitted. Complete the interaction (if necessary) and write the correct missing particle in the appropriate box for each diagram. State the quantum-number conservation law or laws used to decide your answer in each case:

i. ____

ii. ____

iii. ____

iv. ____

v. ____

(b) For the blank diagram labeled "Electron Capture" (f above) write down the interaction and fill in the Feynman diagram.

(c) Circle all letters corresponding to diagrams that do not conserve quark flavor.

i   ii   iii   iv   v   vi

(d) Can the $W^\pm$ and Z^0 particles absorb or emit gluons directly? Explain your answer.

Section E

Quantum Mechanics

E1. The Hamiltonian of a system can be written as

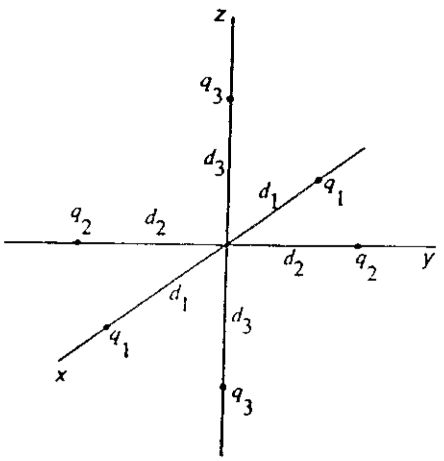
$$\hat{H} = \frac{5}{3}\hat{a}^{+}\hat{a} + \frac{2}{3}(\hat{a}^{2} + \hat{a}^{+2}),$$

where $\hat{a} = \frac{1}{\sqrt{2}}(\hat{q} + i\hat{p})$ *and* $\hat{a}^{+} = \frac{1}{\sqrt{2}}(\hat{q} - i\hat{p})$, *with* $[\hat{q}, \hat{p}] = i$.

(a) Find the eigenvalues of $\hat{H}$.

(b) Find the ground state eigenfunction $\psi_0(q)$ of $\hat{H}$.

E2. Consider a hydrogen atom surrounded by three pairs of point charges, as shown in the figure.



In the limit $r \ll d_1$, $r \ll d_2$, and $r \ll d_3$, the perturbing potential due to the three pairs of charges can be written as

$$V = V_c + 3(\beta_1 x^2 + \beta_2 y^2 + \beta_3 z^2) - (\beta_1 + \beta_2 + \beta_3)r^2$$

with

$$\beta_i = -\frac{eq_i}{d_i^3}$$

(in Gaussian units), and $V_c = -2e(q_1/d_1 + q_2/d_2 + q_3/d_3)$ (the charge of the electron is $-e$).

(a) Calculate the energy shift of the ground state to first order in perturbation theory. (Hint: use symmetry arguments to determine the relationships that must hold between all the various integrals, before you actually attempt to evaluate any of them! The final answer is *very* simple.)

(b) Consider the 4 × 4 matrix M that represents the perturbation V in the basis of the four degenerate excited states with $n = 2$. How many elements of this matrix are zero? (Hint: parts b through d are much simpler in cartesian coordinates!)

(c) Express the nonzero elements of M in terms of V_c and of the following two integrals:

$$A = \langle \psi_{210} | z^2 | \psi_{210} \rangle$$

and

$$B = \langle \psi_{210} | x^2 | \psi_{210} \rangle = \langle \psi_{210} | y^2 | \psi_{210} \rangle$$

(notation: $|\psi_{nml}\rangle$ is the hydrogen eigenstate with quantum numbers n, m, l.) Use symmetry arguments to show how all the other nonzero integrals can be reduced to these two. You do not need to try and evaluate A and B.

(d) Diagonalize M and discuss into how many levels the fourfold degenerate system splits in the three cases of (i) cubic symmetry ($\beta_1 = \beta_2 = \beta_3$), (ii) orthorhombic symmetry ($\beta_1 = \beta_2 \neq \beta_3$), and general tetragonal symmetry (all β_i different).

E3. A particle with orbital angular momentum **L** and spin **S** ($s = 1/2$) evolves under the influence of the Hamiltonian

$$H = \frac{\omega}{\hbar} \mathbf{L} \cdot \mathbf{S}$$

(a) Find the eigenvalues of H, and express the eigenstates in the basis of the eigenstates of $\mathbf{J}^2$ (where **J** is the total angular momentum), $\mathbf{L}^2$, $\mathbf{S}^2$ (you may forget about this one, since it is a constant), and J_z.

(b) Assume that at the time $t = 0$ the state of the particle is $|l = 1, m_l = 0, m_s = -\frac{1}{2}\rangle$. Find its state at any later time.

(c) With the initial condition given in (b), calculate $\langle J_z \rangle$ as a function of time.

Note: You may find it useful to know that the angular momentum ladder operators have the property $L_{\pm}|lm\rangle = \sqrt{l(l+1) - m(m \pm 1)}\,|l, m \pm 1\rangle$.