

SECTION A:

MECHANICS

- A1. Rockets are propelled by the momentum reaction of the exhaust gases expelled from the tail. Since these gases arise from the reaction of the fuels carried in the rocket the mass of the rocket is not constant, but decreases as the fuel is expended. Show that the equation of motion for a rocket projected vertically upward in a uniform gravitational field, neglecting atmospheric resistance, is

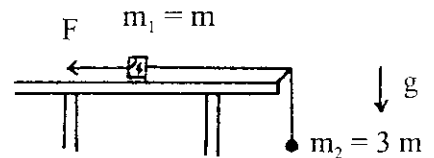
$$m \frac{dv}{dt} = -v' \frac{dm}{dt} - mg,$$

where m is the mass of the rocket, v is the instantaneous speed of the rocket with respect to an inertial reference system, and v' is the velocity of the escaping gases relative to the rocket. Integrate this equation to obtain v as a function of m , assuming a constant time rate of loss of mass. Show, for a rocket starting initially from rest, with v' equal to 6800 ft/sec and a mass loss per second equal to 1/60th of the initial mass, that in order to reach the escape velocity the ratio of the weight of the fuel to the weight of the empty rocket must be almost 300!

- A2. A layer of dust is formed h feet thick (h much smaller than the earth's radius R) by the fall of meteors reaching the earth from all directions. Assuming that the density of earth is D and the density of dust is d . Find the change in the length of the day (T/T_0 , where T_0 is the period of a day without dust and T is the period of a day with the dust layer) due to the addition of the dust layer on the earth surface. Express your result with quantities h , R , D , and d . You can assume that the earth is perfect sphere and keep first order in h .

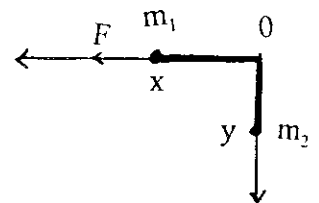
(Hint: Use angular momentum conservation.)

- A3. An unstretchable and massless string of length a is tied to blocks #1 and 2 (which could be considered as particles), and arranged as shown in the diagram on the right.



Put the origin of the x-y coordinates at the corner, as shown, so that

$$x + y = a$$



1. Find the force F that keeps the system in equilibrium. Use Newton's law.
2. If the force F exceeds the value in Part 1, find the acceleration, \ddot{x} , of the system using
 - (1) Newton's law;
 - (2) Lagrange's eqs;
 - (3) Hamilton's eqs.

SECTION B:

E & M

- B1. A point charge q is located in the xy plane near two grounded conducting planes intersecting at right angles as shown in Figure 1. The z axis lies along the line of intersection of the planes. Find and justify the image charges that, together with q , will give the potential at all points in the vacuum region $x \geq 0, y \geq 0, -\infty \leq z \leq \infty$. Find $\phi(x, y, z)$ in the vacuum region. Find $E_y(x, y, z)$. Verify that E_y vanishes on the conducting plane for which it is a tangential component. Find the surface charge density σ_f induced on the plane for which E_y is an appropriate component to use. What is the sign of σ_f ? (Hint: recall multiple image formation in plane mirrors from geometrical optics.)

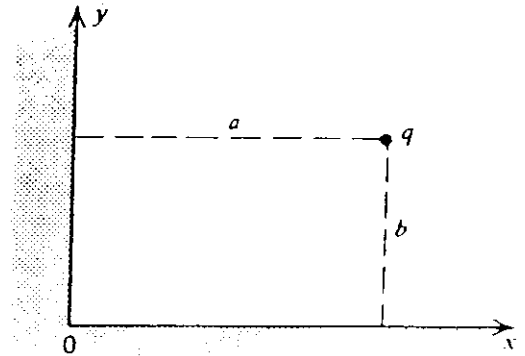
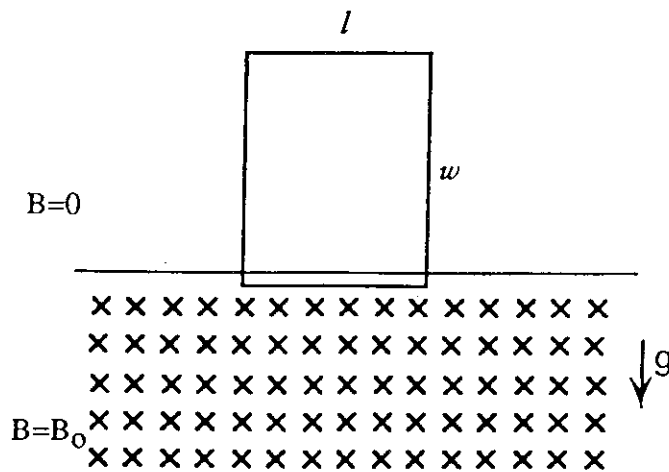


Figure 1

- B2. A rectangular loop of wire with dimension l and w is released at time $t = 0$ from rest just above a region in which the magnetic field is B_0 . The loop has mass m and resistance R , and negligible inductance. Find the current and velocity of the loop as a function of time when the upper edge of the loop is still in the zero field region.



- B3. A sphere of radius a and magnetic permeability μ lies in a constant external magnetic field \vec{H}_0 . Find \vec{B} and \vec{H} vectors both inside and outside the sphere. What is the induced magnetic moment of the sphere? [Hint: To identify the magnetic moment examine fields/potentials far from the sphere].

SECTION C:

STATISTICAL MECHANICS & THERMODYNAMICS

- C1. One hundred non interacting magnetic moments m at temperature T are in a magnetic field B . Each can be parallel or antiparallel to B .

$$m = 1 \times 10^{-23} \text{ Am}^2$$

$$T = 1 \text{ K}$$

$$B = 1 \text{ Tesla}$$

$$(k = 1.38 \times 10^{-23} \text{ J/K})$$

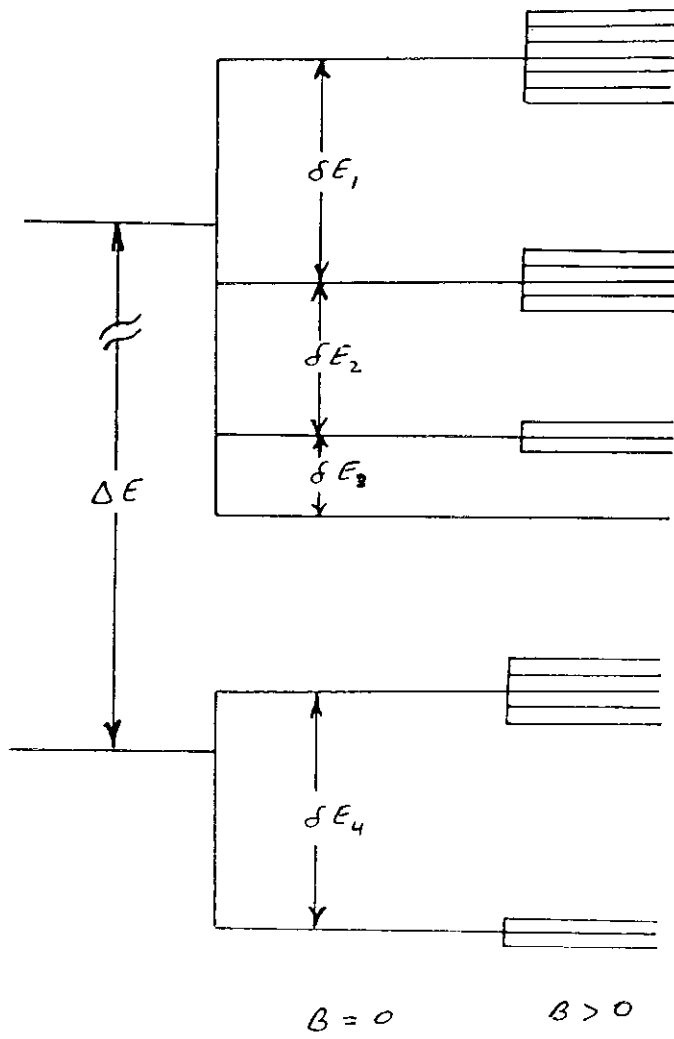
1. At equilibrium what is the energy of this system?
 2. How many moments, on average, are parallel to B ?
 3. What is the entropy of the system?
 4. What is the Helmholtz free energy F ?
 5. Show that at equilibrium, the free energy is minimum. [Hint: Write the F as a function of the number of moments parallel to B and consider this number as a continuous variable.]
- C2. Consider an ideal gas of identical molecules of mass M in equilibrium at temperature T in a gravitational field g .
- (a) What is the Hamiltonian for a single molecule at a height h above sea level? Consider now a subsystem consisting of a thin slice of the gas at a fixed height h . This slice can exchange particles with the rest of the system. Assume this slice has volume ΔV .
 - (b) Find the grand partition function for the subsystem, $\Xi(h, \alpha, \Delta V, T)$ ($\alpha =$ affinity).
 - (c) Find the pressure $P(h, f, T)$ at height h .
 - (d) Find the average number of particles, $N(h, f, T, \Delta, V)$ and determine the equation of state, $P(h, \Delta V, T, N)$ at height h .
 - (e) What is the functional dependence of α on h ? Explain.

Some integrals you may (or may not) want to use:

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

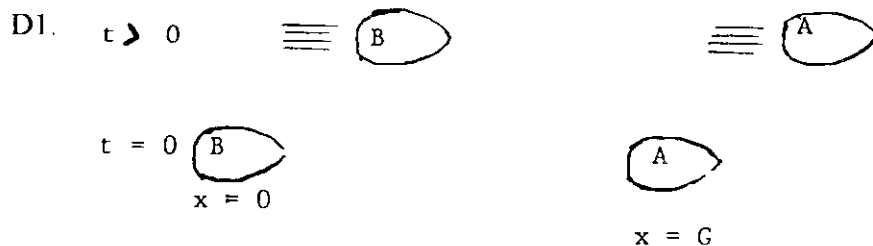
$$\int_0^{\infty} x^n e^{-ax} dx = n! a^{-(n+1)}$$

- C3. N weakly coupled particles obeying Maxwell-Boltzmann statistics may each exist in one of the 3 non-degenerate energy levels of energies $-E$, 0 , $+E$. The system is in contact with a thermal reservoir at temperature T .
- (a) What is the entropy of the system at $T = 0$ K?
 - (b) What is the maximum possible entropy of the system?
 - (c) What is the minimum possible energy of the system?
 - (d) What is the partition function of the system?
 - (e) What is the most probable energy of the system?
 - (f) If $C(T)$ is the heat capacity of the system, what is the value of $\int_0^{\infty} \frac{C(T)}{T} dT$?



SECTION D:

MODERN PHYSICS



Suppose two identical rocketships are at rest pointing toward the positive x -axis with ship B at $x = 0$ behind ship A at $x = G$. At time $t = 0$ in the lab frame (x, t) the ships simultaneously fire their engines which cause each to experience a constant x -acceleration of $g(\text{m s}^{-2})$.

Assuming the burn goes indefinitely:

- (a) Compute the velocity u or relativity parameter $\beta = \frac{u}{c}$ ($c = 3 \times 10^8 \text{ m s}^{-1}$) as a function of the proper time τ elapsed for either ship since ignition, $\beta(\tau) = \tanh\left(\frac{g\tau}{c}\right)$
- (b) Compute the lab time t as a function of proper time τ , $t(\tau) = \frac{c}{g} \sinh\left(\frac{g\tau}{c}\right)$.
- (c) Compute the lab positions x_A and x_B respectively, of each ship as a function of their respective proper time τ and as algebraic functions of lab time t .

$$x^A(\tau) = \frac{c^2}{g} \cosh \frac{g\tau}{c} \qquad x^B(\tau) = \frac{c^2}{g} \cosh \frac{g\tau}{c} - G$$

$$x^A(t) = \sqrt{G^2 + (ct)^2} \qquad x^B(t) = \sqrt{G^2 + (ct)^2} - G$$

- (d) Given that their initial separation was $G = c^2/g$, compute their instantaneous x -separation at proper time τ as measured using the x -coordinate of each of the following inertial frames. (Answer may be given as function of τ or $\beta = \beta(\tau)$.)

1. The lab inertial frame: $\Delta^{\text{lab}}x = G = \frac{c^2}{g} = \text{constant}$.

2. The instantaneous inertial frame attached to ship A: $\Delta^A x = G = \frac{c^2}{g} = \text{constant}$.

3. The instantaneous inertial frame attached to ship B:

$$\Delta^b x = G (2 \cosh (gt/c) - 1) = G (2 (1 - \beta^2)^{-1/2} - 1) \quad \text{for } G = c^2/g.$$

Assuming the rockets have long since exhausted their (identical) fuel supplies: In terms of τ burnout or final velocity parameter β burnout, compute the x-separations ΔX according to frames 1., 2., and 3. in part (d).

$$1. \quad \Delta x^{\text{lab}} = G = c^2/g \quad \text{constant}$$

$$2. \quad \Delta x^A = G \cos h \left(\frac{g\tau}{c} \text{ Burnout} \right) = \frac{G}{\sqrt{1 - \beta_{\text{Final Burnout}}^2}}$$

$$3. \quad \Delta x^B = \text{same as } \Delta x^A$$

D2. The unperturbed Hamiltonian for hydrogen atom

$$H_0 = \frac{p^2}{2m} + V(r)$$

commutes with L^2 , L_z , s^2 , and s_z . Show that when the spin-orbit interaction is included, that is,

$$H = H_0 + \zeta(r) \vec{L} \cdot \vec{s}$$

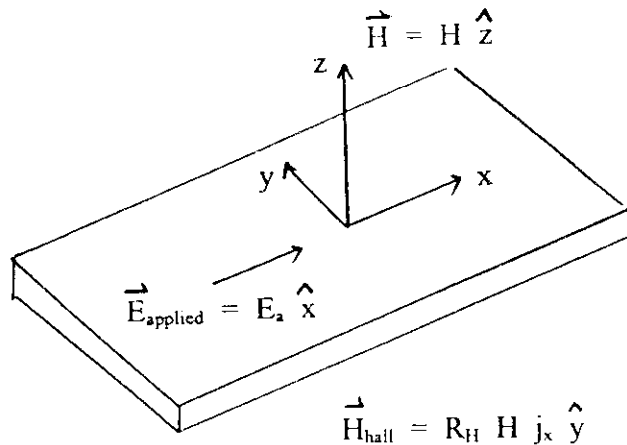
then the total Hamiltonian does not commute with L_z and S_z , but does commute with L^2 , s^2 , J^2 , and J_z .

D3. Some energy levels of an atom are shown below (not shown to scale). δE is of the order of $10^{-3} \times \Delta E$. Column I in the diagram shows the energy levels if energies of the order of δE were not resolved, while Column II shows the energy levels if the energy levels are fully resolved. Column III shows the splitting of the energy levels if the atom was placed in a low (a few Gauss) magnetic field. Identify each energy level with all the quantum numbers associated with it. Note that the nearest another set of levels with the same quantum numbers is separated by about $10^4 \times \Delta E$.

- D4. Elemental sodium (Na) occurs naturally as a body-centered cubic (*bcc*) crystalline structure with cubic (i.e. cube edge) lattice constant $a = 4.225 \text{ \AA}$. It is an alkali metal, and as such its electronic properties are reasonably described by the free electron (Sommerfeld) model. In addition, the relaxation time approximation can be applied within the free electron model to obtain reasonable values for the d.c. conductivity (σ) and the Hall coefficient (R_H). The relaxation time approximation consists in incorporating a damping term in the form

$$-\frac{\vec{p}}{\tau}$$

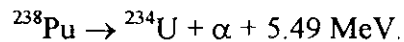
into the equation of motion for the electrons. Here τ is the time between electron scattering events which “relax” the electron momentum. In the context of these two approximations, derive the forms of σ and R_H . Use the derived forms to give σ and R_H for Na. Leave σ in the form of a coefficient times the relaxation time. The Hall field is defined as the electric field, transverse to the applied electric field direction, which results from the steady-state buildup of a charge on the surfaces of the conductor. We take the applied field along the x , so that the Hall field is along the y direction. R_H is then defined as follows:



$j_x =$ current density along \hat{x} direction

(Given: $e = 1.6 \times 10^{-19} \text{ C}$; $m = 9.11 \times 10^{-31} \text{ Kg}$)

- D5. Many spacecraft headed to the outer planets use ^{238}Pu as a power source for their equipment because the sun's energy is too feeble at great distances. The decay is



with a mean life of 128 years. (The mean life of ^{234}U is much longer, 250,000 years, and so this species may be considered stable for present purposes.) If the equipment on a probe requires 1 kW to operate properly, and is to have a useful operating lifetime of 50 years, what minimum mass of ^{238}Pu is required?

- D6. a) How many leptons are there, what are they called, and how are they grouped into families?
- b) Describe in detail the concept of lepton number and describe what lepton number each lepton has.
- c) Specify any additional particles needed to conserve lepton number in the following weak reactions (Note: Baryon # and charge must be conserved):

1) $\mu^- \rightarrow e^- + ?$

2) $\tau^+ \rightarrow e^- + ?$

3) $\nu_\mu + n \rightarrow ?$

4) $\bar{\nu}_e + \text{P} \rightarrow ?$

5) ${}^A_Z\text{X} \rightarrow {}^A_{Z-1}\text{Y} + ?$

6) $e^- + {}^A_Z\text{X} \rightarrow ?$

SECTION E:

QUANTUM MECHANICS

- E1. Find the condition that a particle in the potential given by

$$V(x) = \begin{cases} +\infty & \text{if } x < 0 \\ -V_0 & \text{if } 0 < x < a \\ 0 & \text{if } a < x, \end{cases}$$

has one and only one bound state.

- E2. Consider a free particle in one dimension. Assume that at the time $t=0$ its wavefunction is given by

$$\psi(x,0) = \frac{1}{L^{1/2}\pi^{1/4}} \exp\left[-\frac{x^2}{2L^2}\right]$$

(a) Find $\langle \Delta x^2 \rangle$, $\langle \Delta p^2 \rangle$ and $\langle \Delta E^2 \rangle$ (where E is the energy) at the time $t = 0$.

(b) Calculate $\langle \Delta x^2 \rangle$, $\langle \Delta p^2 \rangle$ and $\langle \Delta E^2 \rangle$ as functions of time.

Hint: the easiest way to do this is to work in the Heisenberg picture.

- E3. A spin $\frac{1}{2}$ particle, with magnetic moment $\bar{\mu} = \mu_0 \bar{\sigma}$, is in a uniform magnetic field

\bar{B} . \bar{B} is in an arbitrary direction (θ, ϕ). Neglect the orbital motion of the particle.

At time $t = 0$, the particle's polarization is in the $+z$ direction, i.e. $\sigma_z = 1$ (or

$\langle \hat{\sigma} \rangle_{t=0} = \bar{e}_3$). Find the polarization of the particle at time t , i.e. to calculate

$\langle \hat{\sigma}_x \rangle_t$, $\langle \hat{\sigma}_y \rangle_t$, and $\langle \hat{\sigma}_z \rangle_t$.